A RE-EXAMINATION OF THE CARTER SOLUTIONS OF EINSTEIN'S FIELD EQUATIONS

by

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INTRODUCTION

The study of geodesics in space-time is essential to a comprehensive understanding of the physics of the field. Global properties, e.g. the singularity structure and completeness of space-time, can be related to the geodesic properties, thus it is through the solutions of the geodesic equation of motion that many of the global properties of space-time can be obtained in an easily interpretable form. However, it is usually very difficult to integrate the geodesic equations for the particle motion in the presence of a gravitational field.

Solutions to the Einstein field equation are difficult to obtain. However, many exact solutions have been found for space-times having high symmetry properties. In 1963 Kerr discovered a metric that is stationary, axi-symmetric and has two Killing vectors, one timelike and one spacelike. The space-time is asymptotically flat and is of Petrov type D. The Kerr solutions are the only known family of exterior exact solutions that could represent the field outside a rotating massive source. In its earliest forms the solution appeared to have singularities at $r_o = a \cos \theta$ and $r_{\pm} = m \pm \sqrt{m^2 - a^2}$ where m represents the mass of the body and 'a' the angular momentum of the rotating body and r was interpreted as the radius in polar coordinates. In order to determine whether these singularities are real (physical) or coordinate (geometric) singularities, one has to investigate the solutions further by coordinate extension and or the solving the geodesic equations. In 1968 Carter found that the Hamilton-Jacobi equation for the geodesics in the Kerr metric separates in a special coordinate system and thus the geodesic equations could be solved. At the same time he produced solutions for the coordinate extension through which the geodesics could be continued. This shows that the singularities at $r_{\pm}$ were coordinate singularities and not physical ones.

In the wake of the discovery, the use of the separability condition of the Hamilton-Jacobi equation in related topics and other fields of study (e.g. astrophysical calculations and quantum mechanics) revived. Finally, in 1977 Collinson and Fugére presented the solutions of the complete set of Einstein's vacuum equation which have separable Hamilton-Jacobi equations.

In this thesis we will demonstrate the method for integrating the geodesic equation. Also we will set up general forms for the geodesic equations in the integrated form and general forms for the first integrals in each of the four canonical cases of the generalised Carter metric. Further, the standard form of solution of the types of differential equation that are given by the equations of motion are given in detail.

Although Collinson and Fugére [1977] have proved necessary and sufficient conditions for a space-time which is a solution to the Einstein field in vacuo to have a separable Hamilton-Jacobi equation, further they have given a full set of solutions to the equation, we have based our work on the separability conditions as presented by Carter since these conditions give insight into the geometry and physics of the solutions. In some sense the work of Collinson and Fugére completes the treatment of the separable space-times. However, a complete geometrical and physical explanation of the role of separability has not yet been given so it was felt worthwhile to rework the path followed by Carter where an attempt has been made to tie the separability into the space-time structure. In addition, this thesis indicates that further work on the separability conditions in more general space-times is needed.

In Chapter 1 we set up the space-time structure required. We introduce the Grassman algebra on which the exterior algebra is based. This enables us to define the exterior differentiation and the exterior calculus is presented and Cartan's equations of structure are derived.

In Chapter 2 we formulate the Hamilton-Jacobi theory, due to Rund, in the context of general relativity theory. In Chapter 3 we present a simple example of how the Hamilton-Jacobi theory may be used to integrate the geodesic equations.

Chapter 4 brings us to the derivation of the general Carter metric. The symmetry and separability conditions, under which the Hamilton-Jacobi equation separates, are discussed and interpreted. General forms of the separable Carter metric are given at the end of the chapter.

Finally, in Chapter 5 we integrate the four canonical cases of the general Carter metric and give the geodesic equations in their general form. The method is explicit and easy to follow.

In the body of the thesis equations are labelled by three numbers; the chapter number, section number and equation number in that section. When referring to an equation only the last two numbers are used if the equation is in the same chapter. References are according to the Havard system and are listed alphabetically by name and year of publication.

CHAPTER 1

THE SPACE-TIME STRUCTURE AND DIFFERENTIAL GEOMETRY

1.1 SPACE-TIME STRUCTURE

Space-time is the set of all events. The mathematical model for space-time which will be used, is the pair (M,g), where M is a four-dimensional, connected, Hausdorff, differentiable manifold of class C^∞ and g is a class C^2 Lorentz metric of positive signature +2 on M.

This structure allows differentiable functions of classes up to and including C^∞ to be defined on M. In the coordinate chart (U,ϕ) on M denote the local coordinates in a neighbourhood U by $\{x^\mu\}$, $\mu = 1,2,3,4$. At each point p of M coordinate transformation between two sets of local coordinates, in the intersection of the charts (U,ϕ) and (U',ϕ'), $\{x^\mu\}$ and $\{x^{\mu'}\}$ around p is given by,

$$x^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu} x^\mu \quad (\mu,\mu' = 1,2,3,4)^{1)} \qquad \ldots(1.1.1)$$

where $\left|\frac{\partial x^{\mu'}}{\partial x^\mu}\right| \neq 0$.

For further details on the space-time structure see Trautman (1964) or Hawking & Ellis (1973).

1.2 TANGENT VECTORS AND VECTOR FIELDS

Let D be the set of all real valued C^∞ functions defined on M. Consider any point p of M. A tangent vector u_p to M at p is defined as a linear mapping of D to the real line $\mathbb{R}$ which obeys the Leibnitz

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1) The Einstein summation convention will be used throughout this thesis unless otherwise specified. The Greek indices will take the values 1, 2, 3, 4 unless otherwise specified

rule. That is, for any pair of functions f, g $\in$ D we have that

(i) $u_p(f+g) = u_p(f) + u_p(g)$

(ii) $u_p(fg) = u_p(f)g(p) + f(p)u_p(g)$ . ...(1.2.1)

The set of all tangent vectors to M at p forms a vector space T_p over $\mathcal{R}$, i.e. for any pair of tangent vectors u_p, v_p to M at p, m, n $\in \mathcal{R}$,

$$(mu_p + nv_p)(f) = (mu_p)(f) + (nv_p)(f)$$

$$= m(u_p f) + n(v_p f) \; . \qquad ...(1.2.2)$$

The vector space T_p is called the tangent space to M at p.

Let $\{x^\mu\}$, $\mu = 1,2,3,4$ be the local coordinates in a neighbourhood U of p then there exists a C^∞ real valued function f from M to $\mathcal{R}$ and a function F from $\mathcal{R}^n$ to $\mathcal{R}$ such that

$$f(q) = F(x^1(q),\, x^2(q),\, x^3(q),\, x^4(q)) \qquad ...(1.2.3)$$

for all q in U, and then from the definition of the tangent vector it follows that the operation of the tangent vector u_p on F is

$$u_p[F(x^1(p),\, \ldots,\, x^4(p))] = \sum_{\mu=1}^{4} (\frac{\partial F}{\partial x^\mu})(p)u_p(x^\mu) \; . \qquad ...(1.2.4)$$

This shows that a tangent vector can be represented as a differential operator. In particular the mappings e_μ ($\mu = 1,2,3,4$) defined by

$$e_\mu(f) := (\frac{\partial F}{\partial x^\mu})(p) = (\frac{\partial f}{\partial x^\mu})_p \qquad ...(1.2.5)$$

satisfy the definition of a tangent vector at p and will be shown to form a linearly independent basis of T_p.

Suppose u_p is an arbitrary tangent vector at p in a coordinate patch (U,ϕ) of M. Define the real numbers

$$u_p^\mu := u_p(x^\mu) \qquad ...(1.2.6)$$

From (2.4) it follows that,

$$u_p(f) = \left(\frac{\partial f}{\partial x^\mu}\right)_p u_p(x^\mu) = e_\mu(f)u^\mu_p \quad \text{using (2.5) and (2.6)}$$

$$= u^\mu_p e_\mu(f) \; . \qquad \ldots(1.2.7)$$

Since f is arbitrary it follows that, for any tangent vector u_p,

$$u_p = u^\mu_p e_\mu \; . \qquad \ldots(1.2.8)$$

Thus any vector can be expressed as a linear combination of the e_μ. In order to complete the proof that the $\{e_\mu\}$ form a basis for T_p all that is required is to show that they are linearly independent.

Assume that there exist real numbers m^μ, not all zero, such that

$$m^\mu e_\mu = 0$$

then,

$$0 = m^\mu e_\mu(x^\nu) = m^\mu \frac{\partial}{\partial x^\mu}(x^\nu) = m^\mu \delta^\nu_\mu = m^\nu \qquad \ldots(1.2.9)$$

where δ^ν_μ denotes the kronecker delta. Thus all the m^ν are zero, contrary to the assumption, and hence the linearly independence condition is proved.

Thus $\{e_\mu\}$ forms a linearly independent basis of the vector space T_p. We call such a basis a natural basis since it has an essential property that for any pair of vectors in the basis,

$$e_\mu e_\nu = e_\nu e_\mu \, . \qquad \ldots(1.2.10)$$

In the Lie bracket notation, equation (2.10) implies that

$$[e_\mu, e_\nu] := e_\mu e_\nu - e_\nu e_\mu = 0 \; . \qquad \ldots(1.2.11)$$

This, however, is not true for a general basis since, in general, for an arbitrary basis $\{E_i\}$ (i,j,k = 1,2,3,4) it can be shown that

$$[E_i, E_j] = \Omega_{i\ j}^{\ k}E_k \neq 0 \qquad \ldots(1.2.12)$$

where $\Omega_{i\ j}^{\ k}$ is called the object of anholonomity. The form of $\Omega_{i\ j}^{\ k}$ is easily obtained as follows:

Consider a real valued function $f \in D$. The Lie bracket defined in (2.11) gives,

$$[E_i, E_j](f) = E_i(E_j(f)) - E_j(E_i(f)) \qquad \ldots(1.2.13)$$

Since $\{E_i\}$ and $\{E_j\}$ are general bases, we may write

$$E_i = e_i^{\alpha}E_{\alpha} = e_i^{\alpha}\frac{\partial}{\partial x^{\alpha}} \qquad \ldots(1.2.14)$$

and

$$E_j = e_j^{\beta}E_{\beta} = e_j^{\beta}\frac{\partial}{\partial x^{\beta}} \quad \text{(by (2.5))} \qquad \ldots(1.2.15)$$

in which each vector E_i of the new basis is expressed in terms of its components in the natural basis. It follows from (2.13) that

$$e_i^{\alpha}\frac{\partial}{\partial x^{\alpha}}(e_j^{\beta}\frac{\partial f}{\partial x^{\beta}}) - e_j^{\beta}\frac{\partial}{\partial x^{\beta}}(e_i^{\alpha}\frac{\partial f}{\partial x^{\alpha}})$$

$$= e_i^{\alpha}\frac{\partial e_i^{\beta}}{\partial x^{\alpha}}\frac{\partial f}{\partial x^{\beta}} - e_j^{\beta}\frac{\partial e_i^{\alpha}}{\partial x^{\beta}}\frac{\partial f}{\partial x^{\alpha}}$$

$$= (e_i^{\alpha}\frac{\partial e_j^{\beta}}{\partial x^{\alpha}} - e_j^{\alpha}\frac{\partial e_i^{\beta}}{\partial x^{\alpha}})\frac{\partial f}{\partial x^{\beta}}$$

by changing of dummy suffices $\alpha \to \beta$.

From (2.12) we have $\Omega_{i\ j}^{\ k}E_k = \Omega_{i\ j}^{\ k}e_k^{\beta}E_{\beta}$ and hence

$$\Omega_{i\ j}^{\ k}e_k^{\beta} = (e_i^{\alpha}\frac{\partial e_i^{\beta}}{\partial x^{\alpha}} - e_j^{\alpha}\frac{\partial e_i^{\beta}}{\partial x^{\alpha}}) \qquad \ldots(1.2.16)$$

or, $\quad [E_i, E_j](f) = \Omega_{i\ j}^{\ k}E_k(f)$,

from which (2.12) follows.

Let T_p be a tangent vector space to the differentiable manifold M at p. A vector field u over M is defined as an assignment of a tangent vector $u_p \in T_p$ to each point p of M.

Suppose $f \in D$ and u is a vector field on M, then u(f) is a real valued function on M defined by

$$u[f(p)] = u_p(f) \in R \qquad \ldots(1.2.17)$$

In general, u(f) will not be differentiable and hence does not belong to D. However, if $u(f) \in D$ for all $f \in D$ then u is called a C^∞ differentiable vector field on M. The set of all differentiable vector fields on M will be denoted by $\chi = \chi(M)$.

1.3 THE DUAL SPACE TO THE TANGENT SPACE

The dual space T_p^* to the tangent space T_p is called the cotangent space to M at p and the elements of T_p^* are called differential forms or covectors. In particular a differential form at p is a linear mapping of the elements of T_p to R. Thus if ω_p is a differential form at p, u_p, $v_p \in T_p$, m, $n \in R$, then by definition,

$$\omega_p(mu_p + nv_p) = m\omega_p(u_p) + n\omega_p(v_p) \in R \qquad \ldots(1.3.1)$$

A field of differential forms ω is defined to be an assignment of a form $\omega_p \in T_p^*$ to each point $p \in M$. If u is a vector field on M and ω a form field on M then $\omega(u)$ can be regarded as a real valued function on M, defined by

$$\omega(u)(p) = \omega_p(u_p) \in R \quad \forall\ p \in M \qquad \ldots(1.3.2)$$

If $\omega(u) \in D$ for all $u \in \chi$ then ω is said to be a C^∞ differentiable form field on M and the set of all differentiable form fields on M will be denoted by $\chi^*(M)$.

1.4 TENSORS

Let T be an N-dimensional vector space with dual space T* and let $\{e_\mu\}$ and $\{e^\mu\}$ be their respective bases. Form the N^{r+s}-dimensional vector space by taking the tensor product (Trautman 1964, p.83).

$$\underbrace{T \otimes T \otimes \ldots \otimes T}_{r \text{ factors}} \otimes \underbrace{T^* \otimes T^* \otimes \ldots \otimes T^*}_{s \text{ factors}} \qquad \ldots(1.4.1)$$

which shall be denoted by $T\binom{r}{s}$. Then any $X \in T\binom{r}{s}$ is a real valued, multilinear function on the Cartesian product space,

$$\underbrace{T^* \times T^* \times \ldots \times T^*}_{r \text{ factors}} \times \underbrace{T \times \ldots \times T}_{s \text{ factors}} . \qquad \ldots(1.4.2)$$

Any $X \in T\binom{r}{s}$ can be written in the form

$$X = X^{i_1 \ldots i_r}{}_{j_1 \ldots j_s} e_{i_1 \ldots i_r}{}^{j_1 \ldots j_s} \qquad \ldots(1.4.3)$$

where

$$e_{i_1 \ldots i_r}{}^{j_1 \ldots j_s} = e_{i_1} \otimes \ldots \otimes e_{i_r} \otimes e^{j_1} \otimes \ldots \otimes e^{j_s}$$

are the basis elements of $T\binom{r}{s}$ (Trautman 1964, p.84). A tensor $X \in T\binom{r}{s}$ is said to be of contravariant valence r and covariant valence s, or an (r,s) tensor. Thus $T\binom{r}{s}$ is the tensor space of all (r,s) tensors. Elements of T are (1,0) tensors and elements of T* are (0,1) tensors. A real number will be defined to be an (0,0) tensor.

Linear transformations of $\{e_\mu\}$ and $\{e^\mu\}$ induce transformations in the bases elements of the tensor space and hence in the component of elements of $T\binom{r}{s}$. If

$$e_{i'_1 \ldots i'_r}{}^{j'_1 \ldots j'_s} = A^{i_1}_{i'_1} \ldots A^{i_r}_{i'_r} B^{j'_1}_{j_1} \ldots B^{j'_s}_{j_s} e_{i_1 \ldots i_r}{}^{j_1 \ldots j_s} \qquad \ldots(1.4.4)$$

then,

$$X^{i'_1 \ldots i'_r}{}_{j'_1 \ldots j_s} = B^{i'_1}_{i_1} \ldots B^{i'_r}_{i_r} A^{j_1}_{j'_1} \ldots A^{j_s}_{j'_s} X^{i_1 \ldots i_r}{}_{j_1 \ldots j_s} \qquad \ldots(1.4.5)$$

follows from the definitions.

Two tensors of the same type - (type (r,s), say) - can be added to give another tensor of the same type but tensors of different types cannot be added as they belong to different vector spaces. The components of the sum of two tensors is just the sum of the respective components.

Any (r,s) tensor can be multiplied by a real number to yield a (r,s) tensor; the real number simply multiplies the components of a tensor.

The product of X (an (r,s) tensor) and Y (a (p,q) tensor) is the (r+p,s+q) tensor, X ⊗ Y, the components of which are given by

$$Z^{i_1 \ldots i_{r+p}}{}_{j_1 \ldots j_{s+q}} = X^{i_1 \ldots i_r}{}_{j_1 \ldots j_s} \, Y^{i_{r+1} \ldots i_{r+p}}{}_{j_{s+1} \ldots j_{s+q}} \qquad \ldots(1.4.6)$$

An operation known as contraction can be performed on any (r,s) tensor where $r,s \geq 1$. Contraction of a tensor reduces it from type (r,s) to type (r-1,s-1) and is performed by simply identifying one of the contravariant indices with one of the covariant indices and summing over this pair of repeated indices according to the summation convention.

$$\text{If} \quad X(u^*,v^*) = X(v^*,u^*) \quad \text{with } X \in T\binom{2}{0} \qquad \ldots(1.4.7)$$

for all u^*, $v^* \in T$, then X is a symmetric (2,0) tensor. Similarly, if

$$X(u^*,v^*) = -X(v^*,u^*) \qquad \ldots(1.4.8)$$

for all u^*, $v^* \in T$, then X is a skew-symmetric (2,0) tensor. If the basis vectors

$$u^* = e^{\mu}, \quad v^* = e^{\nu}$$

are used, then

$$X^{\mu\nu} = X^{\nu\mu} \qquad \ldots(1.4.9)$$

for the symmetric case, and

$$X^{\mu\nu} = -X^{\nu\mu} \qquad \ldots(1.4.10)$$

for the skew-symmetric case.

An (r,0) tensor, $r \geq 2$, is symmetric in the p^{th} and q^{th} places if

$$X(u^*_1 \ldots u^*_p \ldots u^*_q \ldots u^*_v) = X(u^*_1 \ldots u^*_q \ldots u^*_p \ldots u^*_v)$$

for all $u^*_1 \ldots u^*_v \in T^*$. The corresponding definition of skew symmetry is obvious.

A completely symmetric (skew-symmetric) tensor is one which is symmetric (skew-symmetric) in all pairs of indices. Such tensors must be either entirely covariant or entirely contravariant.

The metric tensor field g of the space-time is a class C^2 symmetric (0,2) tensor field. With respect to the natural basis $\{e_\mu\}$, the components of g are

$$g_{\mu\nu} := g(e_\mu, e_\nu) = g(e_\nu, e_\mu) \ , \qquad \ldots(1.4.11)$$

as the metric g is required to be of signature +2 in a four dimensional space and it defines a unique "length" to a vector X_p, $p \in M$, by

$$|X_p| = +\sqrt{g_p(X_p, X_p)} \ , \qquad \ldots(1.4.12)$$

the model of space-time has a pseudo Riemannian geometric structure, and thus the "length" of a vector may be 0, >0, or <0.

As singularly structures are not to be considered in this thesis, the metric will be assumed to be non-degenerate throughout M. This means that the matrix of components of g, defined by (4.11) is non-singular and thus a unique inverse matrix $g^{\mu\nu}$ such that

$$g^{\mu\nu} g_{\nu\alpha} = \delta^\mu_\alpha \qquad \ldots(1.4.13)$$

exists. In this way a unique symmetric dual tensor field $\tilde{g}$ is defined.

The dual tensor field is of type (2,0). The tensor fields g and its dual are non-degenerate and may be used to give a unique isomorphism between covariant and contravariant indices. In this way tensor indices may be raised or lowered, i.e. $g \in T\binom{0}{2}$ can be used to map $u \in T\binom{r}{s}$ to $v \in T\binom{r-1}{s+1}$ and $\tilde{g} \in T\binom{2}{0}$ can be used to map $w \in T\binom{r}{s}$ to $z \in T\binom{r+1}{s-1}$. The mappings are unique and uniquely invertible and thus the raising or lowering of indices may be regarded as merely changing the representation of the same geometric object.

1.5 GRASSMAN ALGEBRA

Let T be a vector space over R on which a multiplication process is defined such that the product of any two elements in T is also an element of T. Consider a, b, c, ... elements of T and denote the vector multiplication by @ so that a @ b ∈ T, for all a, b ∈ T. Then T is called an algebra over R if the following properties hold

(i) a @ (b@c) = (a@b) @ c (associative)

(ii) a @ (b+c) = a @ b + a @ c
(a+b) @ c = a @ c + b @ c distributive

(iii) α(b@c) = αb @ c = b @ αc where $\alpha \in R$

If a @ b = b @ a then T is said to be a commutative algebra. In general the algebra will not be commutative, so a @ b ≠ b @ a.

We shall now define the exterior or Grassman algebra. Let T_p and T_q be two vector spaces of dimension N_p and N_q respectively. The direct sum of T_p and T_q (denoted by $T_p \dotplus T_q$) is defined to be the vector space of dimension $N_p + N_q$. Elements of $T_p \dotplus T_q$ are the set of all ordered pairs of vectors (u_p, u_q) where $u_p \in T_p$ and $u_q \in T_q$.

Addition and multiplication by a real number in $T_p \dot{+} T_q$ are defined by

$$(u_p,u_q) + (v_p,v_q) = (u_p+v_p,\ u_q+v_q) \qquad \ldots(1.5.1)$$

$$m(u_p,u_q) = (mu_p,\ mu_q)\ ,\quad m \in R \qquad \ldots(1.5.2)$$

Let $\{e_\mu\}$ and $\{e_\nu\}$ $(\mu = 1,\ldots,N_p,\ \nu = 1,\ldots,N_q)$ be bases of T_p and T_q respectively. Then the set of $N_p + N_q$ elements $(e_\mu,0)$, $(0,e_\nu)$ form a basis of $T_p \dot{+} T_q$. The concept of the direct sum can be extended without difficulty to arbitrary many vector spaces if the so-called weak direct sum of vector spaces is used. This process avoids the problems of convergence and divergence associated with infinite sequences. The weak direct sum of an infinite number of vector spaces is an infinite dimensional vector space consisting of ordered multiplets $(u_1,u_2,\ldots)$ where only a finite number of the components u's are non-zero. According to the definition any element of the weak direct sum of vector spaces can be expressed in the form

$$(u_1,u_2,\ldots) = (u_1,0,\ldots) + (0,u_2,0,\ldots) + (0,0,u_3,0,\ldots) + \text{etc.} \qquad \ldots(1.5.3)$$

where there are only a finite number of terms on the right hand side.

Consider an N-dimensional vector space T and the dual space T^* and all the possible tensor product spaces and form the weak direct sum A(T) defined by

$$A(T) = R \dot{+} T \dot{+} T^* \dot{+} T\otimes T \dot{+} T\otimes T^* \dot{+} T^*\otimes T \dot{+} T^*\otimes T^* \dot{+} T\otimes T\otimes T \dot{+} \qquad \ldots(1.5.4)$$

The multiplication process in A(T) is the tensor product. Thus A(T) is a tensor algebra of the vector space T over R.

The algebra A(T) has important subalgebras of skew-symmetric contravariant or covariant tensors which are defined by

$$\wedge(T) = R \dotplus T \dotplus [T\otimes T] \dotplus [T\otimes T\otimes T] \dotplus \ldots \qquad \ldots(1.5.5)$$

and

$$\wedge(T^*) = R \dotplus T^* \dotplus [T^*\otimes T^*] \dotplus [T^*\otimes T^*\otimes T^*] \dotplus \ldots \qquad \ldots(1.5.6)$$

These are the exterior or Grassman algebras of T and T^* respectively.

The multiplication is performed by calculating the tensor product and then taking the skew-symmetric part of the result.

Let T be an N-dimensional vector space consisting of (1,0) tensors, u, v, ..., ω. The exterior product in T is defined by

$$u_\wedge v = \tfrac{1}{2}(u\otimes v - v\otimes u) = -v_\wedge u \,, \qquad \ldots(1.5.7)$$

which belongs to [T⊗T] since it is a linear combination of elements of [T⊗T]. It is therefore a member of ∧(T). The exterior product satisfies the following conditions.

(i) $u_\wedge u = 0$

(ii) $u_\wedge(v+\omega) = u_\wedge v + u_\wedge\omega$

$(u+v)_\wedge\omega = u_\wedge\omega + v_\wedge\omega \qquad \ldots(1.5.8)$

(iii) $(u_\wedge v) = mu_\wedge v + u_\wedge mv \quad m \in R$.

If $\{e_\mu\}$ is a basis of T then equation (5.7) can be written as

$$\begin{aligned} u_\wedge v &= \tfrac{1}{2}(u^\mu v^\nu e_\mu\otimes e_\nu - v^\mu u^\nu e_\mu\otimes e_\nu) \\ &= \tfrac{1}{2}(u^\mu v^\nu - u^\nu v^\mu)e_\mu\otimes e_\nu \\ &= u^\mu v^\nu \tfrac{1}{2}(e_\mu\otimes e_\nu - e_\nu\otimes e_\mu) \\ &= u^\mu v^\nu e_{\mu\wedge} e_\nu \,, \qquad \ldots(1.5.9) \end{aligned}$$

or

$$u_\wedge v = \sum_{\mu<\nu}\sum u^\mu v^\nu e_{\mu\wedge} e_\nu + \sum_{\mu>\nu}\sum u^\mu v^\nu e_{\mu\wedge} e_\nu \,, \qquad \ldots(1.5.10)$$

since the cases $\mu = \nu$ are all zero according to (5.8(i)). Hence,

$$u_\wedge v = \sum_{\mu<\nu}\sum (u^\mu v^\nu e_{\mu\wedge} e_\nu + u^\nu v^\mu e_{\nu\wedge} e_\mu)$$

$$= \sum_{\mu<\nu}\sum (u^\mu v^\nu - u^\nu v^\mu) e_{\mu\wedge} e_\nu \quad , \qquad \ldots(1.5.11)$$

and we write

$$u_\wedge v = X^{\mu\nu} e_{\mu\wedge} e_\nu \ , \qquad \ldots(1.5.12)$$

where

$$X^{\mu\nu} := u^\mu v^\nu - u^\nu v^\mu =: -X^{\nu\mu} . \qquad \ldots(1.5.13)$$

The set $\{e_{\mu\wedge} e_\nu, \mu<\nu\}$ is a linearly independent set and it forms a basis of the subspace $[T\otimes T]$ consisting of all skew-symmetric (2,0) tensors. This space is denoted by $\wedge^2(T)$. The extension to $\wedge^r(T)$, called the spaces of r-vectors can be made very easily.

From the dual space T^* we can construct the space $\wedge^r(T^*)$. This space is the subspace of $T^* \otimes \ldots \otimes T^*$ (r times) which consists of all skew-symmetric (0,r) tangent tensors. Any differentiable element $\omega \in \wedge^r(T^*)$ is called an r-form and is written in the natural basis as

$$\omega = \omega_{i_1 \ldots i_r} dx^{i_1} \wedge \ldots \wedge dx^{i_r} \qquad \ldots(1.5.14)$$

where $\omega_{i_1 \ldots i_r}$ are real-valued differentiable functions. Thus an r-form is an assignment of a differentiable skew-symmetric (0,r) tensor to each point $p \in M$. The set of all r-forms on M is denoted by $F^r(M)$.

1.6 EXTERIOR DIFFERENTIATION

We put $F^o = F^o(M) = D$ and form the weak direct sum

$$F = F^o \dotplus F^1 \dotplus F^2 \dotplus \ldots \dotplus F^r \qquad \ldots(1.6.1)$$

where F^r denotes the r-forms on M. In this way F assigns the exterior or Grassman algebra $\wedge(T^*_p)$ to each $p \in M$. Note that with the exterior product F is an algebra over $\mathbb{R}$ but not over the set of (0,0) tensor fields on M, D, as F^r are vector spaces (of infinite dimension) over $\mathcal{R}$ but not over D.

We denote exterior differentiation by d and the operator is defined to satisfy the following conditions,

(i) d is an $\mathcal{R}$-linear mapping of F into itself with the property that $d(F^r) \in F^{r+1}$, i.e. if $\omega, \pi \in F^r$, m, n $\in \mathcal{R}$, then

$$d(m\omega+n\pi) = md\omega + nd\pi \in F^{r+1} \qquad \text{...(1.6.2)}$$

(ii) For any function $f \in F^o$, df is the total differential as defined by

$$df_p(u_p) = u_p(f) \qquad \text{...(1.6.3)}$$

(iii) If $\omega \in F^r$, $\pi \in F^s$, then

$$d(\omega_\wedge\pi) = d\omega_\wedge\pi + (-1)^r\omega_\wedge d\pi \qquad \text{...(1.6.4)}$$

(iv) $d^2 = 0$ $\qquad$ ...(1.6.5)

The existence and uniqueness of the operator d which satisfies the conditions (6.4 (i), (ii), (iii) and (iv)) is shown in Flanders (1963, pp.20-22). To illustrate the method, we will show that for $X \in F^p$ and $Y \in F^q$,

$$X_\wedge Y = (-1)^{pq}Y_\wedge X \qquad \text{...(1.6.6)}$$

is consistent with condition (iii). Assume (6.6) holds, then

$$\begin{aligned} d(X_\wedge Y) &= d[(-1)^{pq}X_\wedge Y] \\ &= (-1)^{pq}d(Y_\wedge X) \\ &= (-1)^{pq}[dY_\wedge X + (-1)^pY_\wedge dX] \\ &= (-1)^{pq}[(-1)^{(p+1)q}X_\wedge dY + (-1)^p(-1)^{p(q+1)}dX_\wedge Y] \end{aligned}$$

$$= (-1)^{2(pq+q)} X_\wedge dY + (-1)^{2(pq+p)} dX_\wedge Y$$

$$= dX_\wedge Y + (-1)^p X_\wedge dY \ .$$

Now assume,

$$d(X_\wedge Y) = dX_\wedge Y + (-1)^p X_\wedge dY$$

$$= (-1)^{pq+1} Y_\wedge dX + (-1)^p (-1)^{pq+1} dY_\wedge X \ , \text{ using (6.6)}$$

$$= (-1)^{pq} d(X_\wedge Y) \ .$$

1.7 CARTAN'S STRUCTURE EQUATIONS

In the space-time (M,g) consider a coordinate neighbourhood U in which there exists a set of four linearly independent differentiable vector fields $\{e_i\}$, i = 1,2,3,4 which forms an orthonormal basis at each point p ∈ U. In general the vector fields will not correspond to a natural basis but will rather form an anholonomic system [Schouten 1954]. For the conditions under which such a tetrad basis exists in a space-time, see Geroch (1968) and Wolf (1972).

From the definition of the tetrad it follows that at each point p,

$$g_{\mu\nu} e^\mu_i e^\nu_j = \eta_{ij} \qquad \ldots(1.7.1)$$

where $g_{\mu\nu}$, e^μ_i, e^ν_j, are the components of the various quantities in the coordinate neighbourhood U and η_{ij} is the Minkowski metric, i.e.

$$\eta_{ij} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} . \qquad \ldots(1.7.2)$$

For an arbitrary small displacement dP of point p in U, we write

$$dP = \omega^i e_i \qquad \ldots(1.7.3)$$

where ω^i are differential one forms defined on U and ω^i and e_i are evaluated at p. The sense in which (7.3) is to be understood is similar to that of the metric which, in this case, would be

$$ds^2 = (dP)^2 = g_{\mu\nu}\omega^i e^\mu_i \omega^j e^\nu_j$$

$$= -(\omega^o)^2 + (\omega^1)^2 + (\omega^2)^2 + (\omega^3)^2 .$$

The $\{\omega^i\}$ form an orthonormal basis for the 1-forms [Flanders 1963, p.128]. The equation (7.3) is the Cartan first equation of structure.

For the second equation of structure we consider the change in the $\{e_i(P)\}$ as a result of the displacement and write,

$$de_i = \omega^j_i e_j \qquad \text{...(1.7.4)}$$

where ω^j_i and e_j are evaluated at p and ω^j_i are one forms. The equations (7.4) do not fix the ω^j_i. The following two conditions will be shown to determine the one forms $\{\omega^j_i\}$ uniquely;

(i) $g_{\alpha\beta}[(d\ e_i)^\alpha e^\beta_k + e^\alpha_i(d\ e_k)^\beta] = 0$, ...(1.7.5)

where $(d\ e_i)^\alpha = \omega^j_i e^\alpha_j$, and

(ii) $d(dP) = 0$ ...(1.7.6)

From (7.6) it follows that

$$0 = d(\omega^i e_i) = d\omega^i e_i - \omega^i{}_\wedge d\ e_i$$

$$= d\omega^i e_i - \omega^i{}_\wedge \omega^j_i e_j$$

$$= (d\omega^j - \omega^i{}_\wedge \omega^j_i) e_j$$

where we have changed the dummy suffixes i to j in the first term. Hence,

$$d\omega^j = \omega^i{}_\wedge \omega^j_i \ , \qquad \text{...(1.7.7)}$$

since the e_j are independent.

Before completing the proof of the uniqueness of the $\{\omega^i_j\}$ restricted by conditions (i) and (ii) it is necessary to introduce some new quantities and prove two lemmas.

The $\{\omega^j_i\}$ are a set of one forms and therefore we may find quantities Γ^i_{jk} such that

$$\omega^j_i := \Gamma_i{}^j{}_k\omega^k \ , \qquad \ldots(1.7.8)$$

since the $\{\omega^i\}$ are a basis for the 1-forms on M. The quantities $\Gamma_i{}^j{}_k$ are affine connection coefficients. We define

$$g_{\alpha\beta}e^\alpha_i e^\beta_j := g_{ij} \ , \qquad \ldots(1.7.9)$$

then it follows from (7.5) that

$$g_{\alpha\beta}\omega^j_i e^\alpha_j e^\beta_k + g_{\alpha\beta}e^\alpha_i\omega^\ell_k e^\beta_\ell = 0 \ ,$$

$$g_{ik}\omega^j_i + g_{i\ell}\omega^\ell_k = 0$$

and hence

$$\omega_{ik} + \omega_{ki} = 0 \qquad \ldots(1.7.10)$$

The $d\omega^i$ are 2-forms and therefore can be expressed in terms of the 2-form basis $\{\omega^i{}_\wedge\omega^j\}$ by introducing the quantities $\{\gamma^i_{jk}\}$ with,

$$d\omega^i = \gamma^i_{jk}\omega^j{}_\wedge\omega^k \qquad \ldots(1.7.11)$$

$$\gamma^i_{jk} = \gamma^i_{kj} = 0 \qquad \ldots(1.7.12)$$

The $\{\gamma^i_{jk}\}$ defined this way are shown by Israel [1970, p.13] to be the Ricci rotation coefficient for the tetrad field defined by the $\{e_i\}$. Since the $\{\omega^i\}$ are known the γ^i_{jk} are completely determined by (7.11) and (7.12).

<u>Lemma 1</u>. The conditions (i) $\omega_i^j + \omega_j^i = 0$ and (ii) $d\omega^i = \omega^j{}_\wedge\omega_j^i$ are equivalent to

$$\Gamma_i{}^j{}_k + \Gamma_j{}^i{}_k = 0 \qquad \text{...(1.7.13)}$$

and

$$\Gamma_j{}^i{}_k - \Gamma_k{}^i{}_j = \gamma^i_{jk} \qquad \text{...(1.7.14)}$$

<u>Proof</u>. From

$$\omega_i^j + \omega_j^i = 0$$

and the definition (7.8), we have

$$\Gamma_i{}^j{}_k\omega^k + \Gamma_j{}^i{}_k\omega^k = 0,$$

the $\{\omega^k\}$ form a basis for the 1-forms and are linearly independent and therefore

$$\Gamma_i{}^j{}_k + \Gamma_j{}^i{}_k = 0 \qquad \text{...(1.7.15)}$$

From (7.7) and (7.11) it follows that

$$\gamma^i_{jk}\omega^i{}_\wedge\omega^k = d\omega^i = \omega^j{}_\wedge\omega^i_j$$

$$= \omega^i{}_\wedge\Gamma_j{}^i{}_k\omega^k$$

$$= (\Gamma_j{}^i{}_k - \Gamma_k{}^i{}_j)\omega^j{}_\wedge\omega^k \;, \qquad \text{...(1.7.16)}$$

so that

$$[\gamma^i{}_{jk} - (\Gamma^i{}_{jk} - \Gamma^i{}_{kj})]\omega^j{}_\wedge\omega^k = 0$$

and hence

$$\Gamma_j{}^i{}_k - \Gamma_k{}^i{}_j = \gamma^i{}_{jk} \qquad \text{...(1.7.17)}$$

because the $\{\omega^j{}_\wedge\omega^k\}$ are independent.

<u>Lemma 2</u>. Given functions $\gamma^i{}_{jk}$ with

$$\gamma^i{}_{jk} + \gamma^i{}_{kj} = 0 \;,$$

the system of linear equations

$$\Gamma_{i\ k}^{\ j} + \Gamma_{j\ k}^{\ i} = 0$$

$$\Gamma_{j\ k}^{\ i} - \Gamma_{k\ j}^{\ i} = \gamma^{i}_{\ jk}$$

has a unique solution

$$\Gamma_{i\ k}^{\ j} = \tfrac{1}{2}(\gamma^{k}_{\ ij} - \gamma^{j}_{\ ki} - \gamma^{i}_{\ jk}) \qquad \text{...(1.7.18)}$$

<u>Proof</u>. Suppose $\Gamma_{i\ k}^{\ j}$ is any solution, then we can derive,

$$\begin{aligned}\Gamma_{i\ k}^{\ j} = -\Gamma_{j\ k}^{\ i} &= -\Gamma_{k\ j}^{\ i} - \gamma^{i}_{\ jk} \\ &= \Gamma_{i\ j}^{\ k} - \gamma^{i}_{\ jk} \\ &= \Gamma_{j\ i}^{\ k} + \gamma^{k}_{\ ij} - \gamma^{i}_{\ jk} \\ &= -\Gamma_{k\ i}^{\ j} + \gamma^{k}_{\ ij} - \gamma^{i}_{\ jk} \\ &= -\Gamma_{i\ k}^{\ j} - \gamma^{j}_{\ ki} + \gamma^{k}_{\ ij} - \gamma^{i}_{\ jk}\end{aligned}$$

which gives, on taking $-\Gamma_{i\ k}^{\ j}$ to the left hand side,

$$2\Gamma_{i\ k}^{\ j} = \gamma^{k}_{\ ij} - \gamma^{j}_{\ ki} - \gamma^{i}_{\ jk} \qquad \text{...(1.7.19)}$$

We may now state:

<u>Theorem 1</u>. Given a basis $\omega^1, \ldots, \omega^n$ of 1-forms there exists one and only one solution set ω_i^j of one forms that satisfy the equations

$$\omega_i^j + \omega_j^i = 0 \qquad \text{...(1.7.20)}$$

$$d\omega^i = \omega^j \wedge \omega^i \qquad \text{...(1.7.21)}$$

<u>Proof</u>. In Lemma 1 it is shown that conditions (i) and (ii) in the forms (7.7) and (7.10) are equivalent to the conditions (7.13) and (7.14).

Thus by Lemma 2, conditions (i) and (ii) imposed on $\omega^i_{\cdot j}$ yield a unique set of functions $\{\Gamma^i_{\cdot jk}\}$ and hence via the definitions (7.8) a unique set of 1-forms $\{\omega^j_{\cdot i}\}$.

The Cartan equations of structure are (7.2), (7.4) and (7.10) with the integrability conditions (7.7). We investigate further integrability conditions by finding the second derivative of equation (7.4).

$$\begin{aligned} d(d\, e_i) &= d(\omega^j_{\cdot i} e_j) \\ &= d\omega^j_{\cdot i} e_j - \omega^j_{\cdot i} \wedge d\, e_j \\ &= d\omega^j_{\cdot i} e_j - \omega^j_{\cdot i} \wedge \omega^k_{\cdot j} e_k \\ &= (d\omega^k_{\cdot i} - \omega^j_{\cdot i} \wedge \omega^k_{\cdot j}) e_k \end{aligned} \qquad \ldots(1.7.22)$$

where we have changed dummy suffices j to k in the first term. We define the two forms,

$$\theta^j_{\cdot i} := d\omega^k_{\cdot i} - \omega^j_{\cdot i} \wedge \omega^k_{\cdot j} \qquad \ldots(1.7.23)$$

which will be called the curvature 2-forms of the manifold. In terms of the basis $\{\omega^i\}$ we may write

$$\theta^j_{\cdot i} = \tfrac{1}{2} R_{i\ k\ell}^{\cdot j} \omega^k \wedge \omega^\ell \qquad \ldots(1.7.24)$$

where $R_{i\ k\ell}^{\cdot j}$ is called the Riemann curvature tensor which has the following properties [Flanders 1963, p.131].

$$R_{i\ k\ell}^{\cdot j} + R_{i\ \ell k}^{\cdot j} = 0 \qquad \ldots(1.7.25)$$

$$R_{i\ k\ell}^{\cdot j} + R_{j\ k\ell}^{\cdot i} = 0 \qquad \ldots(1.7.26)$$

$$R_{i\ k\ell}^{\cdot j} = R_{k\ ij}^{\ \ell} \qquad \ldots(1.7.27)$$

In Flanders (1963) it is shown that $R_{i\ k\ell}^{\cdot j}$ is the Riemann curvature tensor associated with the metric. A further integrability condition is obtained by exterior differentiation of $d\omega^i = \omega^j{}_\wedge \omega^i_{\cdot j}$,

$$\begin{aligned} 0 = d(d\omega^i) &= d(\omega^j{}_\wedge \omega^i_{\cdot j}) \\ &= d\omega^j{}_\wedge \omega^i_{\cdot j} - \omega^j{}_\wedge d\omega^i_{\cdot j} \\ &= \omega^k{}_\wedge \omega^j_{k\wedge} \omega^i_{\cdot j} - \omega^j{}_\wedge (\theta^k_{\cdot i} + \omega^k_{\cdot i\wedge} \omega^i_{\cdot j}) \\ &= -\omega^j{}_\wedge \theta^k_{\cdot j} \; . \end{aligned}$$

Hence, $$\omega^j{}_\wedge \theta^k_{\cdot j} = 0 \; . \qquad \ldots (1.7.28)$$

If (7.24) is taken into account then (7.28) implies

$$0 = \omega^i{}_\wedge \theta^j_{\cdot i} = \tfrac{1}{2}\, \omega^i{}_\wedge R_{i\ k\ell}^{\ j} \omega^k{}_\wedge \omega^\ell$$

which is equivalent to the condition

$$R_{i\ k\ell}^{\ j} + R_{i\ \ell j}^{\ k} + R_{i\ jk}^{\ \ell} = 0 \; . \qquad \ldots (1.7.29)$$

CHAPTER 2

THE HAMILTON-JACOBI THEORY IN GENERAL RELATIVITY

2.1 INTRODUCTION

In terms of Einstein's theory of General Relativity the gravitational field is described by the pseudo-Riemannian geometry of space-time. The field equations for the metric g which determined the geometry, are

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = K\, T_{\mu\nu} \qquad \text{...(2.1.1)}$$

where K is a constant,

$$R_{\mu\nu} = R_{\mu\ \nu\rho}^{\ \rho} , \qquad \text{...(2.1.2)}$$

$$R = R^{\mu}_{\mu} , \qquad \text{...(2.1.3)}$$

and $T_{\mu\nu}$ is the energy momentum tensor of the matter and or fields other than gravitation. Further it is assumed that the motions of particles in a gravitational field are geodesics of the corresponding space-time.

The study of geodesics is important in that many of the global properties of space-time can be derived since they follow from the geodesic properties, e.g. completeness. The physical consequences of global properties also become easier to understand. For example, if the motion of the particle is described by an incomplete geodesic then it would imply that a particle reaches some sort of "edge" in space-time. This aspect will not be followed up here (for details see Hawking and Ellis 1973).

The equations of motion of particles in gravitation fields are usually written as

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma_{\nu\ \rho}^{\ \mu} + \frac{dx^\nu}{d\lambda}\frac{dx^\rho}{d\lambda} = 0 , \qquad \text{...(2.1.4)}$$

in General Relativity, where λ is an affine parameter and $\Gamma_{\mu\ \rho}^{\ \nu}$ are the Christoffel symbols of the space-time, defined as

$$\Gamma_{\mu\ \rho}^{\ \nu} := \tfrac{1}{2}g^{\nu\delta}[g_{\mu\delta,\rho} + g_{\rho\delta,\mu} - g_{\mu\rho,\delta}] \ , \qquad \ldots(2.1.5)$$

where we have by definition

$$g_{\mu\delta,\rho} := \frac{\partial g_{\mu\delta}}{\partial x^{\rho}} \qquad \ldots(2.1.6)$$

Alternatively, these geodesic equations can be approached via the calculus of variations because they are the Euler-Lagrange equations for a suitable Lagrangian to be defined below.

The purpose of presenting the equations of motion via the theory of the calculus of variations is that there are several advantages to be gained from this approach. Firstly, the equations are easier to derive and the calculations of the Christoffel symbols are avoided. In a four-dimensional space-time there are forty Christoffel symbols to be calculated. Secondly, there is already a well-developed integration theory for integration of the Euler-Lagrange equation via, for instance, the Hamilton-Jacobi theory. Finally, the equations can be developed in a manifestly invariant form.

More often than not the Euler-Lagrange equations themselves are the easiest to solve, however, in some cases in general relativity it turns out that the Hamilton-Jacobi equation is the easiest to solve.

The Hamilton-Jacobi theory as presented here is based on that of Rund (1966) to which the reader is referred when more details are required. The approach will not be via differential forms since forms do not lend themselves to a discussion of the Hamilton-Jacobi equation itself. The exterior calculus is used later where the field equations themselves are discussed in more detail.

2.2 THE LAGRANGIAN FORMALISM

According to Rund (1966) the Lagrangian which describes a particle in a gravitational field is

$$L^2 := g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} \ , \qquad \ldots(2.2.1)$$

where $\dot{x}^{\mu} = \frac{dx^{\mu}}{d\tau}$, τ being any arbitrary parameter. The Lagrangian function is defined as a square in order to preserve the homogeneity properties and hence invariance of the theory so that the equations are invariant for any choice of the parameter τ. The components of the canonical momentum are defined by

$$y_{\mu} := L\,\frac{\partial L}{\partial \dot{x}^{\mu}} \qquad \ldots(2.2.2)$$

The metric is non-singular and therefore (2.2) may be solved to give

$$\dot{x}^{\mu} = g^{\mu\nu}y_{\nu} \ , \qquad \ldots(2.2.3)$$

and it follows that

$$y_{\alpha} = g_{\alpha\nu}\dot{x}^{\nu} \ . \qquad \ldots(2.2.4)$$

Rund defines the Hamiltonian function as

$$H^2 := g^{\mu\nu}y_{\mu}y_{\nu} \ , \qquad \ldots(2.2.5)$$

from which, together with (2.4) it follows that,

$$H^2 = L^2 \ . \qquad \ldots(2.2.6)$$

Thus, there exists a reciprocal symmetry between the Lagrangian and the Hamiltonian. Note, however, that the Hamiltonian is a function of the x^{μ} and y_{μ}, $H(x^{\mu}, y_{\mu})$, and the Lagrangian is a function of the

x^μ and $\dot{x}^\mu$, $L(x^\mu, \dot{x}^\mu)$. In view of the definition of H^2 as given by (2.5) we stipulate that the signs of L and H must coincide. The parameter is no longer arbitrary because it implies that

$$\tau = s. \qquad \text{...(2.2.7)}$$

Thus,

$$H = L = \pm 1, \qquad \text{...(2.2.8)}$$

or more explicitly,

$$H(x^\mu, y_\mu) = L(x^\mu, \psi^\mu(x,y)) \qquad \text{...(2.2.9)}$$

The inherent symmetry between the Lagrangian and Hamiltonian formalism of the homogeneous theory is best observed if the parameter is left arbitrary, since the homogeneity properties of the formalism will be destroyed and errors can also occur when partial derivatives are taken of the functions.

2.3 THE HAMILTON-JACOBI EQUATION

Let a one parameter family of hypersurfaces, which cover the region U of M simply, be represented by

$$S(x^\mu) = \Sigma , \qquad \text{...(2.3.1)}$$

where Σ denotes the parameter of the family. Let the congruence of curves, which are transversal to the family of hypersurfaces, be determined by

$$y_\mu = \frac{\partial S}{\partial x^\mu} . \qquad \text{...(2.3.2)}$$

It is shown in Rund (1966) that the Hamilton-Jacobi equation is given by

$$H(x^\mu, \frac{\partial S}{\partial x^\mu}) = \pm 1 . \qquad \text{...(2.3.3)}$$

If a solution S can be found to the Hamilton-Jacobi equation then a field $y_\mu(x^\mu)$ is determined via (2.2) and hence via the equations (2.3) a field $\dot{x}^\mu$ of tangent vectors to the geodesics are determined. It is shown in Rund (1966, p.26) that this vector field yields a field of first integrals of the Euler-Lagrange equations,

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x}^\mu}\right) - \frac{\partial L}{\partial x^\mu} = 0 \qquad \ldots(2.3.4)$$

After carrying out all the formal analysis we may, without loss of generality, choose $\tau = s$ and also take

$$F = \tfrac{1}{2}L^2 = \tfrac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu , \qquad \ldots(2.3.5)$$

and then, since $\frac{dF}{ds} = 0$, the Euler-Lagrange equations can be written as

$$\frac{d}{ds}\left(\frac{\partial F}{\partial x'^\mu}\right) - \frac{\partial F}{\partial x^\mu} = 0 , \qquad \ldots(2.3.6)$$

where

$$x'^\mu = \frac{dx^\mu}{ds} \quad \text{and} \quad y_\mu = \frac{\partial F}{\partial x'^\mu} . \qquad \ldots(2.3.7)$$

For details see Matravers (1972).

Since the null geodesics can be shown to satisfy analogous equations to those above [Matravers 1972] it follows that the equation

$$g^{\mu\nu}\frac{\partial S}{\partial x^\mu}\frac{\partial S}{\partial x^\nu} = \lambda , \quad \lambda = 0,1,-1 , \qquad \ldots(2.3.8)$$

holds for geodesics in the region U in space-time. The value of λ depends on whether the geodesic is null, spacelike or timelike. The equations (3.6), (3.8) together with (3.2) and (3.7) can be regarded as the extension of the Hamilton-Jacobi theory into space-time.

The justification for this approach is that since the geodesic equations in space-time are difficult to solve in general, we now have an indirect method of finding the solutions via the Hamilton-Jacobi equation.

The most essential importance of this theory in general relativity in recent times is brought to the fore by the separability of the Hamilton-Jacobi equation in the Kerr family of solutions of the Einstein field equations [Carter 1968] and subsequently by others [Bardeen 1972, Matravers 1972, 1976, Bonanos 1976] who have used it to suggest simple forms for the metric which give rise to integrable forms of the Einstein field equations.

Other methods of integrating the same geodesic equations such as that of Walker and Penrose (1970) are less useful in leading directly to simple forms of the metric. However, this aspect has not yet been fully explored. Further, the Hamilton-Jacobi theory offers some geometric interpretations in that, according to the choice of λ, the solutions yield are parameter families of null, spacelike or timelike hypersurfaces which are transversal to the corresponding geodesics [Matravers 1972].

Only limited classes of space-times will permit such structures because they imply the existence of congruences of null geodesics which are rotation free null congruences through the region [Pirani 1964].

CHAPTER 3

THE CASE OF A SPHERICALLY SYMMETRIC VACUUM SPACE-TIME

3.1 INTRODUCTION

In order to illustrate the procedures which have been employed by Carter and others to the use of the separability of the Hamilton-Jacobi equation in producing solutions to Einstein's field equations, a spherically symmetric space-time in a vacuum will be treated here. Clearly no new results will emerge but the methods will be explained in this simple case without involving the complications of calculation which arise in the cases to be discussed subsequently. The method is as follows. A separable metric is postulated, the Einstein field equations in vacuo will be solved and then the Hamilton-Jacobi equation will be integrated. This case will be used as a reference where calculations and details are omitted in later sections.

3.2 THE METRIC AND THE SOLUTION OF EINSTEIN'S FIELD EQUATIONS

In order to demonstrate the detailed calculations involved in obtaining solutions to Einstein's field equations, using the exterior calculus, a general metric for a spherically symmetric mass source in vacuo will be considered. The line element has the general form, [Papapetrou 1974, p.66],

$$ds^2 = -f(r)^2 dt^2 + \frac{1}{f^2}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad ...(3.1.1)$$

This metric yields a trivially separable Hamilton-Jacobi equation.

The calculations will provide a clear example of methods which will be discussed later of the use of orthonormal frames (tetrads) and how the curvature tensor can be obtained. This simple case is used because even in this case, despite the simplifications, the calculations are somewhat long. As an indication of this time saving that results from using tetrads in this method, there are only four non-zero connection forms $\omega^{\mu}{}_{\nu}$ which have to be calculated in the space defined by (1.1) as opposed to the nine non-zero Christoffel symbols in coordinate form and only six curvature forms $\theta^{\mu}{}_{\nu}$ in contrast to the twenty independent components of the Riemann curvature tensor $R_{\mu\nu\alpha\beta}$.

Denote the covariant basis 1-forms by ω^{o}, ω^{1}, ω^{2} and ω^{3} so that with respect to the orthonormal tetrad basis the metric (1.1) may be written in the form,

$$ds^2 = g_{\mu\nu}\omega^{\mu}\omega^{\nu} = -(\omega^{o})^2 + (\omega^{1})^2 + (\omega^{2})^2 + (\omega^{3})^2 \quad \ldots(3.1.2)$$

in which,

$$\begin{aligned} \omega^{o} &= f(r)dt && (a) \\ \omega^{1} &= f^{-1}dr && (b) \\ \omega^{2} &= r\, d\theta && (c) \\ \omega^{3} &= r\, \sin\theta d\phi && (d)\,. \end{aligned} \quad \ldots(3.1.3)$$

The exterior derivatives of the one-forms (1.3) (a), (b), (c) and (d) are

$$\begin{aligned} d\omega^{o} &= df(r)_{\wedge}dt + f(r)d.dt \\ &= \frac{df}{dr}\, dr_{\wedge}dt \qquad (d.dt = 0) \\ &= f'\omega^{1}{}_{\wedge}\omega^{o} \qquad f' = \frac{df}{dr} \end{aligned}$$

$$\begin{aligned} d\omega^{1} &= f'f^{-2}dr_{\wedge}dr + f^{-1}d.dr \\ &= 0\,, \end{aligned}$$

where we have used the fact that

$$dr_{\wedge}dr = 0 \quad \text{and} \quad d.dr = 0$$

$$d\omega^2 = d[r\ d\theta]$$
$$= dr_\wedge d\theta$$
$$= \frac{f}{r}\omega^1{}_\wedge\omega^2$$

$$d\omega^3 = d[r\ \sin\theta d\phi]$$
$$= \sin\theta dr_\wedge d\phi + r\ \cos\theta d\theta_\wedge d\phi$$
$$= \frac{f}{r}\omega^1{}_\wedge\omega^3 + \frac{\cot\theta}{r}\omega^2{}_\wedge\omega^3 .$$

So,

$$d\omega^o = f'\omega^1{}_\wedge\omega^o \qquad (a)$$
$$d\omega^1 = 0 \qquad (b)$$
$$d\omega^2 = \frac{f}{r}\omega^1{}_\wedge\omega^2 \qquad (c) \qquad \ldots(3.1.4)$$
$$d\omega^3 = \frac{f}{r}\omega^1{}_\wedge\omega^2 + \frac{\cot\theta}{r}\omega^2{}_\wedge\omega^3 \qquad (d) \quad .$$

To obtain the connection forms $\omega^\mu{}_\nu$, the equations (1.4) have to be compared with the equations

$$d\omega^\mu = -\omega^\mu{}_{\nu\wedge}\omega^\nu \qquad \ldots(3.1.5)$$

derived in §1. The solutions for $\omega^\mu{}_\nu$ are uniquely defined if they satisfy equations (1.5) and

$$\omega_{\mu\nu} + \omega_{\nu\mu} = 0 \ , \qquad \ldots(3.1.6)$$

as was shown in Theorem 1 of Chapter 1.7.

Consider equation (1.4)(a). According to (1.5) it must be of the form

$$d\omega^o = -\omega^o_{1\wedge}\omega^1 - \omega^o_{2\wedge}\omega^2 - \omega^o_{3\wedge}\omega^3 \ , \qquad \ldots(3.1.7)$$

From the first terms on the right hand sides of the equations (1.4)(a) and (1.7) a possible choice of ω^o_1 could be

$$\omega^{o}_{1} = f'\omega^{o} , \qquad \ldots(3.1.8)$$

from which we have

$$\omega^{o}_{1} = -\omega_{o1} = \omega_{1o} = \omega^{1} , \qquad \ldots(3.1.9)$$

where the metric tensor $\eta_{\alpha\beta}$ defined by (1.1) is used to lower indices and $\eta^{\alpha\beta}$ to raise them,

$$\eta_{\alpha\beta} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \eta^{\alpha\beta} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$\eta_{\alpha\beta}\eta^{\beta\nu} = \delta^{\nu}_{\alpha} .$$

Equation (1.6) has been used in getting the terms in (1.9).

If the choice (1.8) is correct, then

$$\omega^{o}_{2} = \omega^{o}_{3} = 0 . \qquad \ldots(3.1.10)$$

Equation (1.8) is consistent with (1.4)(b),

$$0 = d\omega^{1} = -\omega^{1}_{o\wedge}0^{o} - \omega^{1}_{2\wedge}\omega^{2} - \omega^{1}_{3\wedge}\omega^{3} \qquad \ldots(3.1.11)$$

if $\omega^{1}_{2} = A\omega^{2}$ and $\omega^{1}_{3} = B\omega^{3}$ since $\omega^{o}{}_{\wedge}\omega^{o} = 0$, and where A and B are to be determined. Equation (1.4)(c) with

$$d\omega^{2} = -\omega^{2}_{o\wedge}\omega^{o} - \omega^{2}_{1\wedge}\omega^{1} - \omega^{2}_{3\wedge}\omega^{3} \qquad \ldots(3.1.12)$$

suggests

$$\omega^{2}_{1} = r^{-1}f\omega^{2} , \qquad \ldots(3.1.13)$$

that is, $A = -r^{-1}f$. Similarly

$$d\omega^{3} = -\omega^{2}_{o\wedge}\omega^{o} - \omega^{3}_{1\wedge}\omega^{1} - \omega^{3}_{2\wedge}\omega^{2} \qquad \ldots(3.1.14)$$

suggests

$$\omega^{3}_{1} = r^{-1}f\omega^{3} \qquad \ldots(3.1.15)$$

from which it follows that $B = -r^{-1}f$ and (1.14) is satisfied if

$$\omega^{2}_{\;3} = -\frac{\cot\theta}{r}\omega^{3} \qquad \ldots(3.1.16)$$

We have then

$$\begin{aligned}
\omega^{o}_{\;1} &= \omega^{1}_{\;o} = f'\omega^{o} && \text{(a)}\\
\omega^{2}_{\;3} &= -\omega^{3}_{\;2} = -\frac{\cot\theta}{r}\omega^{3} && \text{(b)}\\
\omega^{3}_{\;1} &= -\omega^{1}_{\;3} = r^{-1}f\omega^{3} && \text{(c)}\\
\omega^{1}_{\;2} &= -\omega^{2}_{\;1} = -r^{-1}f\omega^{2} && \text{(d)}
\end{aligned} \qquad \ldots(3.1.17)$$

which satisfy (1.5) and (1.6) and are therefore the unique connection forms required.

To find the curvature forms the equation (1.7.23) is used. The explicit form of $\theta^{o}_{\;1}$ for the metric (1.1) is

$$\begin{aligned}
\theta^{o}_{\;1} &= d\omega^{o}_{\;1} + \omega^{o}_{\;2}\wedge\omega^{2}_{\;1} + \omega^{o}_{\;3}\wedge\omega^{3}_{\;1}\\
&= d[f'\omega^{o}] + 0 + 0\\
&= f''f\omega^{1}\wedge\omega^{o} + f'f'\omega^{1}\wedge\omega^{o}\\
&= -f''f\omega^{o}\wedge\omega^{1} - (f')^{2}\omega^{o}\wedge\omega^{1}\\
&= 2A\omega^{o}\wedge\omega^{1}
\end{aligned}$$

where we have put

$$A = -\tfrac{1}{4}(f^{2})'' .$$

Similar calculations performed for each of the curvature forms yield, after the indices have been lowered using $\eta_{\alpha\beta}$:

$$\theta_{01} = -2A\omega^0 \wedge \omega^1$$

$$\theta_{02} = C\omega^0 \wedge \omega^2$$

$$\theta_{03} = C\omega^0 \wedge \omega^3 \qquad \ldots(3.1.18)$$

$$\theta_{23} = 2B\omega^2 \wedge \omega^3$$

$$\theta_{31} = -C\omega^3 \wedge \omega^1$$

$$\theta_{12} = -C\omega^1 \wedge \omega^2$$

where

$$A = -\tfrac{1}{4}(f^2)''$$

$$B = \tfrac{1}{2}[-f^2 + 1]/r^2 \qquad \ldots(3.1.19)$$

$$C = [\tfrac{1}{2}r(f^2)']/r^2.$$

The curvature tensor $R_{\alpha\beta\mu\nu}$ is obtained by comparison with

$$\theta^\mu_\nu = \tfrac{1}{2}R^\mu{}_{\nu\alpha\beta}\omega^\alpha \wedge \omega^\beta \qquad \ldots(3.1.20)$$

and yields

$$R_{0101} = -2A$$

$$R_{2323} = 2B \qquad \ldots(3.1.21)$$

$$R_{2020} = R_{0303} = C$$

$$R_{3131} = R_{1212} = -C\,.$$

The Ricci forms $R_{\mu\nu}$ are obtained by contraction,

$$R_{11} = R_1{}^0{}_{10} + R_1{}^2{}_{12} + R_1{}^3{}_{13}$$

$$= -R_{1010} = R_{1212} + R_{1313}$$

$$= 2A - C - C$$

$$= 2A - 2C \qquad \ldots(3.1.22)$$

$$R_{22} = R_{2}{}^{0}{}_{20} + R_{2}{}^{1}{}_{21} + R_{2}{}^{3}{}_{23}$$

$$= -R_{2020} + R_{2121} + R_{2323}$$

$$= -C - C + 2B$$

$$= -2C + 2B \qquad \text{...(3.1.23)}$$

$$R_{33} = R_{3}{}^{0}{}_{30} = R_{3}{}^{1}{}_{31} + R_{3}{}^{2}{}_{32}$$

$$= -R_{3030} + R_{3131} + R_{3232}$$

$$= -C - C + 2B$$

$$= -2C + 2B \qquad \text{...(3.1.24)}$$

$$R_{00} = R_{0}{}^{1}{}_{01} + R_{0}{}^{2}{}_{02} + R_{0}{}^{3}{}_{03}$$

$$= R_{0101} + R_{0202} + R_{0303}$$

$$= -2A + C + C$$

$$= -2A + 2C \ , \qquad \text{...(3.1.25)}$$

and the Ricci scalar is furnished by

$$R = R^{\mu}{}_{\mu} = R^{\mu\nu}{}_{\mu\nu} = 4(A+B-2C) , \qquad \text{...(3.1.26)}$$

Substitute these values into the Einstein field equations,

$$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R \ , \qquad \text{...(3.1.27)}$$

and the non vanishing components are

$$G_{11} = R_{11} - \tfrac{1}{2}g_{11}R$$

$$= 2A - 2C - \tfrac{1}{2}\cdot 4(A+B-C)$$

$$= 2(C-B) \qquad \text{...(3.1.28)}$$

$$-G_{00} = 2(C-B) \qquad \text{...(3.1.29)}$$

and $$G_{22} = G_{33} = 2(C-A) \ . \qquad \text{...(3.1.30)}$$

The empty space Einstein equations thus require that

$$A = B = C \ .$$

Put $F = f^2$ then B = C implies that

$$\tfrac{1}{2}(-F+1)/r^2 = [\tfrac{1}{2}r(F)']/r^2$$

or

$$-F + 1 = rF'$$

which intergrates to give

$$\ell n(F-1) = -\ell n\ r + \ell n\ A\ ,$$

where A is an integration constant.

$$F - 1 = A/r$$

$$F = 1 + A/r\ . \qquad \ldots(3.1.31)$$

All that remains is to check the consistency of the solution. From equation (1.19)

$$\begin{aligned} B &= \tfrac{1}{2}[-F + 1]/r^2 \\ &= \tfrac{1}{2}[-1 - A/r + 1]/r^2 \\ &= -A/2r^3 \end{aligned}$$

$$\begin{aligned} A &= -\tfrac{1}{4}[F]'' \\ &= -\tfrac{1}{4}[1 + A/r] \\ &= \tfrac{1}{4}[a/r^2]' \\ &= -A/2r^3\ . \end{aligned}$$

Thus the solution (1.3.1) is consistent with the requirement A = B = C. Thus the metric (1.1) has the form,

$$ds^2 = -(1+\frac{A}{r})dt^2 + (1+\frac{A}{r})^{-1}dr^2 + r^2(d\theta^2+\sin^2\theta d\phi^2) \quad \ldots(3.1.32)$$

which is in the standard Schwarzschild form, as expected.

Take the general metric as

$$ds^2 = -F(r)dt^2 + \frac{1}{F(r)}dt^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2 \qquad \ldots(3.1.33)$$

then the metric tensor has the form,

$$g_{\mu\nu} = \begin{bmatrix} -F & 0 & 0 & 0 \\ 0 & F^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{bmatrix} \qquad \ldots(3.1.34)$$

and the contravariant components are

$$g^{\mu\nu} = \begin{bmatrix} -F^{-1} & 0 & 0 & 0 \\ 0 & F & 0 & 0 \\ 0 & 0 & r^{-2} & 0 \\ 0 & 0 & 0 & r^{-2}\sin^{-2}\theta \end{bmatrix} . \qquad \ldots(3.1.35)$$

The general form for the Hamilton-Jacobi equation is given by

$$g^{\mu\nu} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu} = \lambda \qquad \lambda = 1,-1,0 \qquad \ldots(3.1.36)$$

where the value of λ depends on whether the geodesics are spacelike, timelike or null. For the metric (1.33) the equation (1.36) has the form

$$-F^{-1}\left(\frac{\partial S}{\partial t}\right)^2 + F\left(\frac{\partial S}{\partial r}\right)^2 + r^{-2}\left(\frac{\partial S}{\partial \theta}\right)^2 + r^{-2}\sin^{-2}\theta\left(\frac{\partial S}{\partial \phi}\right)^2 = \lambda . \qquad \ldots(3.1.37)$$

The Lagrangian for the geodesics is

$$L = [g_{\alpha\beta}x'^\alpha x'^\beta]^{\frac{1}{2}} , \quad x'^\alpha = \frac{dx^\alpha}{d\tau} , \qquad \ldots(3.1.38)$$

where τ is an arbitrary parameter, and hence from (1.33) we get

$$L = [-F(r)\dot{t}^2 + \frac{1}{F(r)}\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2]^{\frac{1}{2}} .$$

We define the canonical momenta by

$$y_\alpha := L \frac{\partial L}{\partial x'^\alpha} = \frac{1}{2} \frac{\partial L^2}{\partial x'^\alpha} \qquad \ldots(3.1.39)$$

and hence we can solve for

$$x'^\alpha = \chi^\alpha(x^\beta, y_\beta) , \qquad \ldots(3.1.40)$$

since $|g_{\alpha\beta}| \neq 0$.

The coordinates t and ϕ do not arise in the Lagrangian and, therefore, are ignorable and thus there are first integrals

$$\frac{\partial L}{\partial \dot{t}} \quad \text{and} \quad \frac{\partial L}{\partial \dot{\phi}} \qquad \ldots(3.1.41)$$

of the Euler Lagrange equations for the geodesics. From the definition of the momenta these imply that

$$P_t := \frac{\partial L}{\partial \dot{t}} = \text{const} \qquad \ldots(3.1.42)$$

$$P_\phi := \frac{\partial L}{\partial \dot{\phi}} = \text{const} \qquad \ldots(3.1.43)$$

and since,

$$P_\phi = \frac{\partial S}{\partial \phi} \quad \text{and} \quad P_t = \frac{\partial S}{\partial t}$$

on these geodesic curves, we may write

$$\frac{\partial S}{\partial t} = \text{const} = -E\ , \qquad \ldots(3.1.44)$$

and $$\frac{\partial S}{\partial t} = \text{const} = \Phi \qquad \ldots(3.1.45)$$

on these curves.

The Hamilton-Jacobi equation becomes

$$-F^{-1}E^2 + F(\frac{\partial S}{\partial r}) + r^{-2}(\frac{\partial S}{\partial \theta})^2 + r^{-2}\sin^{-2}\theta\Phi^2 = \lambda\ . \qquad \ldots(3.1.46)$$

Multiply through by r^2 to get

$$-\frac{r^2}{F}E^2 + r^2F(\frac{\partial S}{\partial r})^2 + (\frac{\partial S}{\partial \theta})^2 + \sin^{-2}\theta\Phi^2 = \lambda r^2 \qquad \ldots(3.1.47)$$

which is separable and there exists a solution to the form

$$S = S_1(r) + S_2(\theta) + S_3(\phi) + S_4(t)\ . \qquad \ldots(3.1.48)$$

Thus equation (1.47) can be reduced to

$$r^2F(\frac{dS_1}{dr})^2 - \frac{r^2}{F}E^2 - \lambda r^2 = K, \qquad \ldots(3.1.49)$$

and $$\left(\frac{dS_2}{d\theta}\right)^2 + \sin^{-2}\theta\Phi^2 = -K \, , \qquad \ldots(3.1.50)$$

where K is the separation constant,

$$\frac{dS_3}{d\phi} = \Phi \qquad \ldots(3.1.51)$$

and $$\frac{dS_4}{dt} = -E \, . \qquad \ldots(3.1.52)$$

Equation (1.49) may be written as

$$\left(\frac{dS_1}{dr}\right)^2 = \frac{K}{r^2F} + \frac{E^2}{F^2} + \frac{\lambda}{F}$$

or $$\frac{dS_1}{dr} = \left(\frac{K}{r^2F} + \frac{E^2}{F^2} + \frac{\lambda}{F}\right)^{\frac{1}{2}} \qquad \ldots(3.1.53)$$

and (1.50) gives

$$\left(\frac{dS_2}{d\theta}\right)^2 = -K - \sin^{-2}\theta\Phi^2$$

or $$\frac{dS_2}{d\theta} = (-K - \sin^{-2}\theta\Phi^2)^{\frac{1}{2}} \qquad \ldots(3.1.54)$$

From (1.51), (1.52), (1.53) and (1,54) it follows that

$$S(r,\theta,\phi,t) = \int^r \left(\frac{K}{r^2F} + \frac{E^2}{F^2} + \frac{\lambda}{F}\right)^{\frac{1}{2}} dr + \int^\theta (-K - \sin^{-2}\theta\Phi^2)^{\frac{1}{2}} - Et + \Phi\phi \qquad \ldots(3.1.55)$$

which is a solution to the Hamilton-Jacobi equation in the form of quadratures. In fact, the solution is of little practical value because the first integral is difficult to evaluate and in any case the explicit form of S itself is of only limited value. The alternative approach adopted by Carter (1969) and others has been to perform qualitative analysis of the first order equation which result from (1.51), (1.52), (1.53) and (1.54). These may be obtained as follows,

$$\frac{dS_1}{dt} = P_t = \frac{\partial L}{\partial \dot{t}} = -F\dot{t} \qquad \ldots(3.1.56)$$

$$\frac{dS_1}{d\theta} = P_\theta = \frac{\partial L}{\partial\dot{\theta}} = r^2\dot{\theta} \qquad ...(3.1.57)$$

$$\frac{dS_2}{d\phi} = P_\phi = \frac{\partial L}{\partial\dot{\phi}} = r^2\sin^2\theta\dot{\phi} \qquad ...(3.1.58)$$

$$\frac{dS_4}{dr} = P_r = \frac{\partial L}{\partial\dot{r}} = F^{-1}\dot{r} \qquad ...(3.1.59)$$

on the geodesics. The equations (1.56), (1.57), (1.58) and (1.59) together with the explicit forms for the left hand side as given in (1.51), (1.52), (1.53) and (1.54) give four first order ordinary differential equations for the geodesics

$$\begin{aligned}
\dot{t} &= -(\frac{K}{r^2F} + \frac{E^2}{F^2} + \frac{\lambda}{F})^{\frac{1}{2}}/F \\
\dot{\theta} &= (-K - \sin^{-2}\theta\Phi^2)^{\frac{1}{2}}/r^2 \\
\dot{\phi} &= \Phi/r^2\sin^2\theta \\
\dot{r} &= -EF \ .
\end{aligned} \qquad ...(3.1.60)$$

The analysis will not be taken further as the equations are well known [Papapetrou 1974, pp. 73, 74]. However, the method employed here has important ramifications which will be explored in the next section of this thesis.

CHAPTER 4

THE DERIVATION OF THE GENERAL CARTER METRIC

4.1 INTRODUCTION

Separability of the Hamilton-Jacobi equation in various contexts has been extensively studied in the past (Liouville 1846, Stäckel 1893). See Woodhouse [1974] for a more extensive bibliography. In recent years increased interest in the study of the behaviour of geodesics in pseudo Riemannian spaces and especially in curved space-times in general relativity, the fact that the Einstein field equations are difficult to solve unless certain restrictions are placed on the metric have together focussed attention on the separability of the Hamilton-Jacobi equation. The interest can be said to have initiated in a paper by Carter [1968a]. He shows that the Hamilton-Jacobi equation for the geodesics in the Kerr solutions can be solved by separation of the variables.

In a second paper Carter [1969b] determined all solutions to Einstein's field equations, with or without the cosmological constant, which satisfy certain simple conditions, in addition separability of the Einstein-Maxwell equation. The work of Carter has been followed by various people.

Matravers [1972] introduced the Hamilton-Jacobi theory as presented by Rund (1966) in general relativity and used a generalization of the separability results of Carter to integrate the geodesic equations for all vacuum type D cases. Bardeen [1973] used the property in astrophysical calculations of the geodesic trajectories in space-time.

Matravers [1975] studied Petrov type N exact solutions of Einstein equations which yield separable Hamilton-Jacobi equations. Bonanos [1976] applied Carter's separability conditions to the null geodesic and examined the Einstein's equation with a perfect fluid source. Finally, Collinson and Fugère [1977] presented the solutions to Einstein's vacuum equations which have separable Hamilton-Jacobi equations.

Walker and Penrose [1970] provided another method for integrating the geodesics of the charged Kerr solution. They found that the Kerr solution admits a pair of non-trivial valence 2 Killing tensors (which are shown in Matravers [1972] to correspond to the separated sections of the Hamilton-Jacobi equation). These tensors were obtained via the twistor equation and it was hoped to obtain interpretations of the separability through this result. Further work in this direction is described in Hughston et al. [1972, 1973]. They looked at space-time with Killing tensors, in particular the Kerr metric, whereas Woodhouse[1974] studied the converse case. He showed the relationship between Killing tensors and separable systems for the geodesic equations in Riemannian and Lorentzian manifolds.

We now give a discussion of the separability conditions of Carter which apply to a space-time with no matter present but which has electromagnetic fields.

4.2 SYMMETRY AND SEPARABILITY CONDITIONS

Conditions under which the Hamilton-Jacobi and Schrodinger equations in space-time can be solved by separating the variable have been studied [Robertson 1927, Eisenhart 1934]. However, these equations were considered only to separate in spaces in which all the coordinates are mutually orthogonal. This leads to the simple case in which the metric tensor is diagonal and a set of four first integrals can easily be found.

In order to study a slightly more general case the above suggests that the conditions imposed should be chosen with care. Clearly if the restrictions are too restrictive then no solution may exist or the class of metrics will be trivially separated. However, if too few restrictions were imposed then separability alone seemed too weak a condition to allow the integration of the Einstein-Maxwell field equations. In vacuo Collison and Fugére [1977] have shown that separability of the Hamilton-Jacobi equation alone is sufficient to allow integration of the Einstein field equations. That the same holds true for the Einstein-Maxwell equation remains to be shown.

For the Einstein-Maxwell field the following conditions were proposed by Carter [1968b], on the basis of experience with the Kerr metric [Kerr 1963, Carter 1968a] for which he had previously integrated the geodesic equations.

The following are Carter's conditions [Carter 1968b].

I. "The space and the electromagnetic field are under a two parameter Abelian symmetry group" [Carter 1968b].

The significance of a two parameter Abelian symmetry group is that a coordinate system $\{x^\mu\}$ can be introduced in the manifold admitting two commuting Killing vectors x^1 and x^2 such that these two vectors are constant on the surfaces of the transitivity, while the coordinates x^3 and x^4 are dragged along by the group operations and are thus ignorable. The coordinates x^3 and x^4 are termed ignorable in that they do not appear in the Lagrangian nor the Hamiltonian nor the metric tensor.

II. "The symmetry group is invertible with non-null surfaces of transitivity" [Carter 1968b].

"A group is invertible at a point P in a neighbourhood U if there is an isometry that leaves P fixed and simultaneously inverts the sense of the two commuting vectors in the surfaces of transitivity at P, but does not change the sense of directions orthogonal to the surfaces of transitivity at P." [Carter 1967]. The reader is referred to the reference for more details. In the same reference it is also shown that the invertability implies that the group is orthogonally transitive. The invertibility property of the isometry group thus implies that there are no cross terms of the metric tensor between the ignorable and non-ignorable coordinates.

III. "The Hamilton-Jacobi equation is soluble by separation of variables in the simplest possible way, i.e. the solution S takes the form of a sum of terms each depending on one variable only, and on substitution of this form, and if necessary after multiplication of the whole equation by a suitable separating factor, the Hamilton-Jacobi equation breaks up into a sum of terms each depending on only one of the non-ignorable coordinates." [Carter 1968b].

Condition III states that the solution to the Hamilton-Jacobi equation, S, must be expressible as

$$S = S_1(x^1) + S_2(x^2) + k_1 x^3 + k_2 x^4 \qquad \ldots(4.2.1)$$

where k_1 and k_2 are constants and S_1 and S_2 are differentiable functions.

From Condition I it follows that $g^{\mu\nu}$ is only a function of x^1, x^2. From Condition II there are no cross terms of the type $g^{1\nu}$ and $g^{2\nu}$, $\nu \neq 1,2$. When Condition III is imposed it is clear that the Hamilton-Jacobi equation has the form:

$$g^{11}(x^1,x^2)\left(\frac{\partial S_1}{\partial x^1}\right)^2 + 2g^{12}(x^1,x^2)\left(\frac{\partial S_1}{\partial x^1}\right)\left(\frac{\partial S_2}{\partial x^2}\right) + g^{22}(x^1,x^2)\left(\frac{\partial S_2}{\partial x^2}\right)^2$$

$$+ g^{33}(x^1,x^2)\left(\frac{\partial S_3}{\partial x^3}\right)^2 + 2g^{34}(x^1,x^2)\left(\frac{\partial S_3}{\partial x^3}, \frac{\partial S_4}{\partial x^4}\right)$$

$$+ g^{44}(x^1,x^2)\left(\frac{\partial S_4}{\partial x^4}\right)^2 = \lambda . \qquad \ldots(4.2.2)$$

This equation must by Condition III be separable itself or have an integrating factor which makes it separable. Thus, if an integrating factor is necessary, it must have the form,

$$U = U_1(x^1) + U_2(x^2) \qquad \ldots(4.2.3)$$

in terms of the non-ignorable coordinates. The Conditions I, II and III can be exploited to restrict the undetermined metric further.

In order to remove the cross term $\left(\frac{\partial S_1}{\partial x^1}\right)\left(\frac{\partial S_2}{\partial x^2}\right)$ in (2.2) it is sufficient to set

$$g^{12} = g^{21} = 0 . \qquad \ldots(4.2.4)$$

Such mixed terms between $\frac{\partial S_1}{\partial x^1}$ and $\frac{\partial S_2}{\partial x^2}$ can only be avoided if after choosing coordinates such that (2.4) holds, then

$$Ug^{11} = A(x^1) \qquad \ldots(4.2.5)$$

$$Ug^{22} = B(x^2) . \qquad \ldots(4.2.6)$$

For the terms involving x^μ, x^ν ($\mu,\nu = 3,4$) to separate as stipulated in Condition III, it is sufficient that

$$\begin{aligned} Ug^{33} &= M(x^1) + N(x^2) \\ Ug^{34} &= V(x^1) + W(x^2) \\ Ug^{43} &= Q(x^1) + P(x^2) \\ Ug^{44} &= R(x^1) + T(x^2) \end{aligned} \qquad \ldots(4.2.7)$$

IV. "The separation required by Condition III takes place in such a way that the terms containing derivations with respect to the ignorable coordinates, separates as the sum of two squares each depending on only one of the non-ignorable coordinates." [Carter 1968b].

We shall now prove a lemma in order to show that this condition holds.

Lemma 3. Condition IV holds if and only if the matrix defined in (2.7) can be written in the form

$$\underset{i}{M}^{rs} = \epsilon_i \underset{i}{V}^{r}(x^{i}) \underset{i}{V}^{s}(x^{i}) \qquad \text{...(4.2.8)}$$

where i can take the values 1, 2; r and s the values 3, 4 and ϵ_i = +1 if i = 1 and −1 if i = 2.

Proof. Assume $\underset{i}{M}^{rs}$ has the form (2.8), then writing

$$\underset{1}{V}^{3} = M, \quad \underset{1}{V}^{4} = N, \quad \underset{2}{V}^{3} = R, \quad \underset{2}{V}^{4} = T$$

we have

$$\underset{1}{M}^{33} = M^2, \quad \underset{1}{M}^{34} = MN, \quad \underset{1}{M}^{43} = MN, \quad \underset{1}{M}^{44} = N^2$$

$$\underset{2}{M}^{33} = R^2, \quad \underset{2}{M}^{43} = RT, \quad \underset{2}{M}^{34} = RT, \quad \underset{2}{M}^{44} = T^2,$$

i.e.,

$$\epsilon_i \underset{i}{M}^{rs} \frac{\partial}{\partial x^r}\frac{\partial}{\partial x^s} = \left(M\frac{\partial}{\partial x^1} + N\frac{\partial}{\partial x^2}\right)^2 - \left(R\frac{\partial}{\partial x^1} + T\frac{\partial}{\partial x^2}\right)^2 \qquad \text{...(4.2.9)}$$

as required. Thus condition (2.8) is sufficient to ensure that Condition IV holds.

If Condition IV holds then there exist four functions M, N, R, T such that the right hand side of (2.9) is defined and the reverse of the argument above leads to the form (2.8) for $\underset{i}{M}^{rs}$.

<u>Corollary</u>. The determinants of the two matrices $\underset{1}{M}^{rs}$ and $\underset{2}{M}^{rs}$ vanish identically separately.

This result follows immediately if the determinant are written out in full.

This lemma proved for the contravariant case shows on inversion that the Conditions I to IV are necessary and sufficient for us to write down the covariant metric in the canonical form

$$ds^2 = U\{\frac{(dx^1)^2}{A} + \frac{(dx^2)^2}{B} + \frac{B(Hdx^3 - Idx^4)^2}{W^2} - \frac{A(Jdx^3 - Kdx^4)^2}{W^2}\} \qquad ...(4.2.10)$$

where $W = HK - JI$ . ...(4.2.11)

H and I are functions of x^1 only and J and K are functions of x^2 only, x^1 and x^2 being the non-ignorable coordinates of the system.

IIIS. "The Schrodinger equation is separable in a manner analogous to that required in the Hamilton-Jacobi equation in Condition III." [Carter 1968b].

In order to restrict the field equations to facilitate the integration process, Carter makes an additional assumption that the "Schrodinger" equation

$$i\frac{\partial\Psi}{\partial\tau} = (|g|^{\frac{1}{2}}i\frac{\partial}{\partial x^\alpha}|g|^{\frac{1}{2}} - \bar{e}A_\alpha)\tfrac{1}{2}g^{\alpha\beta}(i\frac{\partial}{\partial x^\beta} - \bar{e}A_\beta)\Psi \qquad ...(4.2.12)$$

where $|g| = \det(g_{\alpha\beta})$ separates in the Stäckel sense in the same coordinate system as that in which the Hamilton-Jacobi equation separates.

The Schrodinger equation (2.12), which is a 5-dimensional analogue of the Klein-Gordon equation, reduces to the equation,

$$(|g|^{-\frac{1}{2}}i\frac{\partial}{\partial x^\alpha}|g|^{\frac{1}{2}} - \bar{e}A_\alpha)g^{\alpha\beta}(i\frac{\partial}{\partial x^\beta} - \bar{e}A_\beta)\Psi + \bar{m}^2\psi = 0 , \quad \ldots(4.2.13)$$

to an equation in 4-dimensional space-time when an essentially arbitrary reduction procedure is employed.

Carter [1968b] uses the freedom of choice of parameter to reduce the 5-dimensional Hamilton-Jacobi equation which he introduces to a form which coincides with that given in this thesis. The corresponding reduction of the proposed Klein-Gordon equation produces in a flat Minkowski space-time, the Klein-Gordon equation which corresponds to the 4-dimensional Klein-Gordon equation, proposed by Rund [1966]

$$\frac{\partial^2\psi}{(\partial x^1)^2} + \frac{\partial^2\psi}{(\partial x^2)^2} + \frac{\partial^2\psi}{(\partial x^3)^2} - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = (\frac{me}{\hbar})^2\psi . \quad \ldots(4.2.14)$$

This shows that the 5-dimensional formulation is unnecessary in the Hamilton-Jacobi case and may be unnecessary in the Klein-Gordon case. To check this, more work needs to be done on the compatibility of the equations proposed by Carter and also the formalism of Rund [1966] to see if it could be generalized to obtain, in a similar way, a covariant form of the Klein-Gordon equation for curved space-time. For the purpose of the discussion here we will use equation (2.12) since it is a covariant equation and provides the statement of a restriction on the metric which Carter used. In fact, the Schrodinger separability condition has been shown to be redundant and the separability follows by virtue of the field equations [Debever 1969]. The proof which will be given later in the chapter is based on one given by Bonanos [1976].

The separability Condition IIIS can be shown to require, for the metric in the form (2.12), that the following condition,

$$U = W \quad \ldots(4.2.15)$$

holds [Carter 1968b]. In fact, the condition U = W follows from the 1, 2 component of the empty field equations as will now be shown.

In the metric (2.10) put

$$T^2 = \frac{W}{U} \; . \qquad \ldots(4.2.16)$$

Define the set of one forms

$$\begin{aligned}
\omega^1 &= \frac{1}{T}\sqrt{\frac{W}{A}}\; dx^1 \\
\omega^2 &= \frac{1}{T}\sqrt{\frac{W}{B}}\; dx^2 \\
\omega^3 &= \frac{1}{T}\sqrt{\frac{B}{W}}\; (Hdx^3 - Idx^4) \\
\omega^4 &= \frac{1}{T}\sqrt{\frac{A}{W}}\; (Jdx^3 - Kdx^4)
\end{aligned} \qquad \ldots(4.2.17)$$

so that the metric now has the form

$$ds^2 = (\omega^1)^2 + (\omega^2)^2 + (\omega^3)^2 - (\omega^4)^2 \; .$$

By the methods demonstrated earlier for the Schwarzschild metric the line tensor can be computed for the orthonormal fram defined by ω^μ. The field equation $R_{12} = 0$ yields, [Bonanos, 1976]

$$\frac{2\dfrac{\partial^2 T}{\partial x^1 \partial x^2}}{T} + \frac{\frac{1}{2}\dfrac{\partial^2 W}{\partial x^1 \partial x^2}}{W} - \frac{\dfrac{\partial T}{\partial x^\lambda}\dfrac{\partial W}{\partial x^\mu} + \dfrac{\partial T}{\partial x^\mu}\dfrac{\partial W}{\partial x^\lambda}}{TW} = 0 \; . \qquad \ldots(4.2.18)$$

For the general case where $U = U(x^1,x^2)$ we will show that the derivation of the Schrodinger separability condition from the (1,2) vacuum field equation no longer applies.

We now use the following notation

$$\frac{\partial X}{\partial x^\lambda} := X_\lambda \quad \text{and} \quad \frac{\partial^2 X}{\partial x^\lambda \partial x^\mu} := X_{\lambda\mu} \; .$$

With the condition that

$$U_{\mu\lambda} \neq 0 \qquad \ldots(4.2.19)$$

may hold, we obtain from (2.16)

$$T_\lambda = \frac{1}{2(UW)^{\frac{1}{2}}} W_\lambda - \tfrac{1}{2}(\frac{W}{U^3})^{\frac{1}{2}} U_\lambda \qquad \text{...(4.2.20)}$$

and

$$T_{\lambda\mu} = -\frac{1}{4(WU^3)^{\frac{1}{2}}} W_\lambda U_\mu - \frac{1}{4(UW^3)^{\frac{1}{2}}} W_\lambda W_\mu + \frac{1}{2(UW)^{\frac{1}{2}}} W_{\mu\lambda}$$

$$-\frac{1}{4(U^3W)^{\frac{1}{2}}} U_\lambda W_\mu + \tfrac{3}{4}(\frac{W}{U^5})^{\frac{1}{2}} U_\lambda U_\mu - \tfrac{1}{2}(\frac{W}{U^3})^{\frac{1}{2}} U_{\lambda\mu} . \qquad \text{...(4.2.21)}$$

Substituting (2.16), (2.19), (2.20) and (2.21) into (2.18) we finally arrive at the result

$$\frac{3}{2W} W_{\mu\lambda} - \frac{3}{2W^2} W_\mu W_\lambda + \frac{3}{2U^2} U_\lambda U_\mu - \frac{1}{U} U_{\lambda\mu} = 0 \qquad \text{...(4.2.22)}$$

i.e.,

$$\frac{W_{\mu\lambda}}{W} - \frac{W_\mu W_\lambda}{W^2} + \frac{U_\lambda U_\mu}{U^2} = \frac{2}{3} \frac{U_{\mu\lambda}}{U} . \qquad \text{...(4.2.23)}$$

The equation

$$(\ell n \frac{W}{U})_{\lambda\mu} \qquad \text{...(4.2.24)}$$

with the condition (2.19) yields

$$(\ell n \frac{W}{U})_{\lambda\mu} = \frac{W_{\mu\lambda}}{W} - \frac{W_\mu W_\lambda}{W^2} + \frac{U_\mu U_\lambda}{U^2} - \frac{U_{\lambda\mu}}{U}$$

$$= \frac{U_{\lambda\mu}}{3U} \neq 0, \quad \text{unless } U_{\lambda\mu} = 0 . \qquad \text{...(4.2.25)}$$

In Condition III Carter explicitly assumes U separates and thus $U_{\mu\lambda} = 0$, hence

$$(\ell n \frac{W}{U})_{\lambda\mu} = 0 \qquad \text{...(4.2.26)}$$

which in the canonical case I = K = 1 can be shown to be equivalent to the Schrodinger separability condition [Bonanos 1976].

From the definition of W,

$$W = HK - JI \,, \qquad \ldots(4.2.27)$$

we obtain, after substituting I = K = 1,

$$W = H - J \qquad \ldots(4.2.28)$$

This makes W a sum of two functions, one of each variable. Thus we have from (2.26)

$$0 = \left[\ell n \frac{W_1(x^1) + W_2(x^2)}{U_1(x^1) + U_2(x^2)}\right]_{1,2}$$

$$= \left[\frac{U}{W}\left(\frac{W_1(x^1)}{U} - \frac{WU_1(x^1)}{U^2}\right)\right]_2 = \left[\frac{W_1(x^1)'}{W} - \frac{U_1(x^1)'}{U}\right]_2$$

$$= -\frac{W_1(x^1)'W_2(x^2)'}{W^2} + \frac{U_1(x^1)'U_2(x^2)'}{U^2}$$

which is only satisfied if W = U, if we disregard constant multipliers.

In the general case the Schrodinger separability condition does not follow from the (1,2) field equation since $U_{\lambda\mu} \neq 0$. Thus even in the canonical case we cannot have W = U, as required for the Schrodinger separability, because this would imply $\ell n(\frac{W}{U}) = 0$, i.e. $U_{\lambda\mu} = 0$, which is a contradiction.

Carter's separability conditions have been further modified by Matravers [1975] in order to study some exact solutions that yield separable Hamilton-Jacobi equations for null geodesics. A noteworthy conclusion is reached in that the only exact solutions to the Einstein empty space field equations for the Hamilton-Jacobi equation, subject to the form and conditions given in the paper, are of plane-fronted gravitational waves with parallel rays.

Collinson and Fugére [1977a] studied the various possible cases of separability which arise from the Hamilton-Jacobi equations in empty spacetimes. This supports the results by Matravers in that solutions are found to be either of Petrov type D or Petrov type N with non-diverging rays. In a second paper Collinson and Fugère [1977b] proved the following theorem, giving the necessary and sufficient conditions for separation.

"The Hamilton-Jacobi equation can only separate in vacuum space-times of Petrov type D or provided the separable coordinate is space-like, type II." [Collinson and Fugère 1977].

These results complete the search for spacetimes that admit separable Hamilton-Jacobi equations.

In a different approach in the study of separable space-times, Woodhouse [1974] uses an alternative formulation of the same separability conditions. He investigates the relationship between separable systems and the existence of constants of motion for the geodesics. The technique that Woodhouse uses, employs the Lie algebra of Killing tensors. The reader is referred to Sommers [1972] for further details of the connection between Killing tensors and constants of motion. Woodhouse divides the separability in two types and has extended the work of Eisenhart [1934] and has been able to produce necessary and sufficient conditions for a space which admits a valence 2 Killing tensor to admit coordinates in which the Hamilton-Jacobi equation separates. This work supplies an insight into the relationship between Killing tensors and separability, since separability implies that there exists a Killing tensor [Matravers 1972], however, the converse is only true under restricted conditions which are described by Woodhouse [1974].

4.3 GENERAL FORMS OF SEPARABLE CARTER METRICS

As was shown in §4.2, the general Carter metric (2.10) is defined by the orthonormal tetrad of one-forms,

$$\omega^o = \frac{1}{T}\sqrt{\frac{A}{W}}\,(Jdx^3 - Kdx^4) \qquad \ldots(4.3.1)$$

$$\omega^1 = \frac{1}{T}\sqrt{\frac{W}{A}}\,dx^1 \qquad \ldots(4.3.2)$$

$$\omega^2 = \frac{1}{T}\sqrt{\frac{W}{B}}\,dx^2 \qquad \ldots(4.3.3)$$

$$\omega^3 = \frac{1}{T}\sqrt{\frac{B}{W}}\,(Hdx^3 - Idx^4) \qquad \ldots(4.3.4)$$

where $T^2 = \frac{W}{U}$. A, H, I are functions of the x^1 coordinate only and B, J, K are functions of the x^2 coordinate only. Thus the covariant continuous metric tensor for lowering and raising the indices is

$$\eta_{ij} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \eta^{ij} . \qquad \ldots(4.3.5)$$

Now that the tetrad of one-forms (3.1), (3.2), (3,3) and (3.4) has been set up so that the metric has the form (3.1.2), the procedure of Carter can be employed to obtain the curvature tensor and hence the field equations. The theory is given in §1.7. The details of the calculations will not be given as these are analogous to that shown in §3.

Since there are a number of misprints in the Carter paper [1969] the calculations are difficult to follow through in detail, although the method is clear. The experience of the author of this thesis suggests that the reader who wishes to use parts of the calculations, as quoted in Carter [1969], make an independent check of the equations required.

As in §3, the method yields a set of field equations for the A, B, J, K, H, I and W, which can be separated into 4 canonical forms. The different canonical forms depend upon the number of separable coordinates and the number of ignorable coordinates admitted by the metric. Since in each case the integrating factor U separates in the same way, we shall classify the solutions according to the way U separates.

On solving the resulting sets of differential equations obtained from the Einstein empty space field equation for the various canonical cases of separability, we obtain the corresponding sets of solution which are presented below. For more details on the calculations the reader is referred to Carter [1968b, p.282] and Collinson and Fugère [1977, p.747].

[A] $$ds^2 = U\{-\frac{A(dx^3 - Kdx^4)^2}{W^2} + \frac{(dx^1)^2}{A} + \frac{(dx^2)^2}{B} + \frac{B(dx^3 - Idx^4)^2}{W}\},$$

where U separates as $U = U_1(x^1) + U_2(x^2)$.

[B] $$ds^2 = U\{-\frac{A(Ldx^3 - Mdx^4)^2}{W^2} + \frac{(dx^1)^2}{A} + \frac{(dx^2)^2}{B} + B(dx^3)^2\},$$

where U separates as $U = U_1(x^1) + k_1$, k_1 a constant.

[$\tilde{B}$] $$ds^2 = U\{\frac{B(Ndx^3 - Qdx^4)^2}{W^2} + \frac{(dx^1)^2}{A} + \frac{(dx^2)^2}{B} - A(dx^4)^2\},$$

where U separates as $U = U_2(x^2) + k_2$, k_2 a constant.

[C] $$ds^2 = \frac{(dx^1)^2}{A} + \frac{(dx^2)^2}{B} + B(dx^3)^2 - A(dx^4)^2,$$

where U = 1.

As an indication of the validity of the method, it will be shown that the Schwarzschild metric follows from [A] when certain canonical conditions are met. From the method [A], and the conditions,

$$U = W = r^2$$

$$I = 0$$

$$K = 1$$

$$A = r^2 - 2mr \qquad \ldots(4.3.6)$$

$$B = 1$$

$$x^1 = r, \quad x^2 = \theta, \quad x^3 = \phi, \quad x^4 = t$$

we obtain

$$ds^2 = \frac{r^2 - 2mr}{r^2} dt^2 + \frac{r^2}{r^2 - 2mr} dr^2 + r^2 d\theta^2 + \frac{r^4 \sin^2\theta d\phi^2}{r^2}$$

$$= (1 - \frac{2m}{r}) dt^2 + \frac{1}{(1 - \frac{2m}{r})} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \qquad \ldots(4.3.7)$$

CHAPTER 5

INTEGRATION OF THE GEODESIC EQUATIONS FOR GENERAL FORMS OF CARTER METRICS

5.1 INTEGRATION OF GEODESIC EQUATIONS

In this chapter the geodesic equations, derived from the general Carter metrics, integrated in their general form will be considered. The form of the general Carter metric that will be used is the one given in Carter [1969 equation 77, p.293]. Except for the change in notation and minor formal changes we prefer to work with the metric in the form

$$ds^2 = \frac{U}{A}(dx^1)^2 + \frac{U}{B}(dx^2)^2 + \frac{1}{U}(BH^2-AJ^2)(dx^3)^2 + \frac{2}{U}(AJK-BHJ)$$

$$dx^3dx^4 + \frac{1}{U}(BI^2-AK^2)(dx^4)^2 , \qquad \ldots(5.1.1)$$

where H, I are functions of the x^1 coordinate only and J, K are functions of the x^2 coordinate only. The Lagrangian for the motion of a particle is defined by

$$L = [g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta]^{\frac{1}{2}} \qquad \ldots(5.1.2)$$

where $\dot{x}^\alpha = \frac{dx^\alpha}{d\tau}$ etc., and hence following the procedure given in §3.1, we have

$$L = [\frac{U}{A}(\dot{x}^1)^2 + \frac{U}{B}(\dot{x}^2)^2 + \frac{1}{U}(BH^2-AJ^2)(\dot{x}^3)^2 + \frac{2}{U}(AJK-BHI)\dot{x}^3\dot{x}^4$$

$$+ \frac{1}{U}(BI^2-AK^2)(\dot{x}^4)^2]^{\frac{1}{2}} . \qquad \ldots(5.1.3)$$

The coordinates x^3 and x^4 do not occur in the Lagrangian (1.3) and thus in the Euler-Lagrange equation

$$\frac{d}{d\tau}(\frac{\partial L}{\partial \dot{x}^\alpha}) - \frac{\partial L}{\partial x^\alpha} = 0 \qquad \ldots(5.1.4)$$

we get, by using (3.1.42) and (3.1.43),

$$\frac{d}{d\lambda}(P_{x^3}) = 0 \qquad \text{...(5.1.5)}$$

and

$$\frac{d}{d\lambda}(P_{x^4}) = 0 \ , \qquad \text{...(5.1.6)}$$

where λ is the affine parameter on the geodesic.

The remaining Euler-Lagrange equations, that is, those for x^1 and x^2 do not yield such simple integrals. In fact,

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x}^1}\right) - \frac{\partial L}{\partial \dot{x}^1} = 0 \qquad \text{...(5.1.7)}$$

and

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x}^2}\right) - \frac{\partial L}{\partial \dot{x}^2} = 0 \ , \qquad \text{...(5.1.8)}$$

where τ is any arbitrary parameter, yield non-linear differential equations with no obvious integration procedure.

However, fortunately the metrics were chosen so that the Hamilton-Jacobi equation can be solved by separation of variables. Taking the contravariant form of the metric (1.1) and after some manipulation, the Hamilton-Jacobi equation is

$$\frac{A}{U}\left(\frac{\partial S}{\partial x^1}\right)^2 + \frac{B}{U}\left(\frac{\partial S}{\partial x^2}\right)^2 + \frac{1}{U}\left(\frac{K^2}{B} - \frac{I^2}{A}\right)\left(\frac{\partial S}{\partial x^3}\right)^2 + \frac{2}{U}\left(\frac{IH}{A} - \frac{KJ}{B}\right)$$

$$\frac{\partial S}{\partial x^3}\frac{\partial S}{\partial x^4} - \frac{1}{U}\left(\frac{H^2}{A} - \frac{J^2}{B}\right)\left(\frac{\partial S}{\partial x^4}\right)^2 = \lambda \ . \qquad \text{...(5.1.9)}$$

Since on the geodesics

$$P_\alpha = \frac{\partial S}{\partial x^\alpha} \qquad \text{...(5.1.10)}$$

it follows that

$$\frac{\partial S}{\partial x^3} = \Phi \qquad \text{...(5.1.11)}$$

and

$$\frac{\partial S}{\partial x^4} = -E \ , \qquad \text{...(5.1.12)}$$

and hence we may write for the solution of the Hamilton-Jacobi equation

$$S = S(x_1, x_2) + \Phi\phi - Et . \qquad \ldots(5.1.13)$$

Now it is clear from the form of (1.9) that the Hamilton-Jacobi equation can be written as

$$A\left(\frac{\partial S}{\partial x^1}\right)^2 - \frac{I^2}{A}\left(\frac{\partial S}{\partial x^3}\right)^2 + \frac{2IH}{A}\frac{\partial S}{\partial x^3}\frac{\partial S}{\partial x^4} - \frac{H^2}{A}\left(\frac{\partial S}{\partial x^4}\right)^2 + B\left(\frac{\partial S}{\partial x^2}\right)^2$$

$$+ \frac{K^2}{B}\left(\frac{\partial S}{\partial x^3}\right)^2 - \frac{2KJ}{B}\frac{\partial S}{\partial x^3}\frac{\partial S}{\partial x^4} + \frac{J^2}{B}\left(\frac{\partial S}{\partial x^4}\right)^2$$

$$= \lambda[U_1(x^2) + U_2(x^2)] \qquad \ldots(5.1.14)$$

or

$$A\left(\frac{\partial S}{\partial x^1}\right)^2 - \frac{I^2}{A}\left(\frac{\partial S}{\partial x^3}\right)^2 + \frac{2IH}{A}\frac{\partial S}{\partial x^3}\frac{\partial S}{\partial x^4} - \frac{H^2}{A}\left(\frac{\partial S}{\partial x^4}\right)^2 - \lambda U_1(x^1)$$

$$+ B\left(\frac{\partial S}{\partial x^2}\right)^2 + \frac{K^2}{B}\left(\frac{\partial S}{\partial x^3}\right)^2 - \frac{2KJ}{B}\frac{\partial S}{\partial x^3}\frac{\partial S}{\partial x^4} + \frac{J}{B}\left(\frac{\partial S}{\partial x^4}\right)^2 - \lambda U_2(x^2) = 0 \qquad \ldots(5.1.15)$$

i.e., the equation separates and has a solution of the form

$$S = S_1(x^1) + S_2(x^2) + \Phi x^3 - Ex^4 \qquad \ldots(5.1.16)$$

which, when substituted in equation (1.15), yields

$$A\left(\frac{\partial S_1}{\partial x^1}\right)^2 - \frac{I^2}{A}\Phi^2 - \frac{2IH}{A}\Phi E - \frac{H^2}{A}E^2 - \lambda U_1(x^1) = M \qquad \ldots(5.1.17)$$

and

$$B\left(\frac{\partial S_2}{\partial x^2}\right)^2 + \frac{K^2}{B}\Phi^2 + \frac{2KJ}{B}\Phi E + \frac{I^2}{B}E - \lambda U_2(x^2) = -M \qquad \ldots(5.1.18)$$

where M is the constant of integration. From (1.17) and (1.18) it follows that

$$\frac{dS_1}{dx^1} = \sqrt{R} \qquad \ldots(5.1.19)$$

where

$$R = \frac{M}{A} + \frac{I^2\Phi^2}{A^2} + \frac{2IH}{A^2} + \Phi E + \frac{H^2}{A^2}E^2 + \frac{\lambda U_1(x^1)}{A} \qquad \ldots(5.1.20)$$

and $$\frac{dS_2}{dx^2} = \sqrt{\theta} \qquad \ldots(5.1.21)$$

where

$$\theta = -\frac{M}{B} - \frac{K^2\Phi^2}{B^2} - \frac{2KJ}{B^2}\Phi E - \frac{J^2E^2}{B^2} + \frac{\lambda U_2(x^2)}{B} . \qquad \ldots(5.1.22)$$

Thus the full set of first integrals of the Hamilton-Jacobi equation are

$$P_{x^1} = \frac{dS_1}{dx^1} = \sqrt{R}$$

$$P_{x^2} = \frac{dS_2}{dx^2} = \sqrt{\theta} \qquad \ldots(5.1.23)$$

$$P_{x^3} = \frac{dS_3}{dx^3} = \Phi$$

$$P_{x^4} = \frac{dS_4}{dx^4} = -E .$$

Since the right hand side of each of these equations is a function of one variable only, they can be reduced to quadratures, i.e.,

$$S = \int^{x^1} \sqrt{R}\, dx^1 + \int^{x^2} \sqrt{\theta}\, dx^2 + \Phi x^3 - Ex^4 . \qquad \ldots(5.1.24)$$

From (4.1.24) the explicit equation for the geodesic can be obtained by the usual procedures, that is, differentiate with respect to the constants of the motion as in Carter [1968]. However, this procedure is not particularly useful because the S is only in the form of quadratics and therefore the equations are likely to be in the form of integral equations.

A more practical procedure is to consider qualitative analysis of the equations (1.23). For this purpose it is useful to rewrite the left hand side in terms of the derivatives of x^1, x^2, x^3 and x^4

with respect to the affine parameter.

From the definitions of the P_{x^1}, P_{x^2}, P_{x^3}, P_{x^4} and (3.1.39) we have

$$L\frac{\partial L}{\partial \dot{x}^1} = \frac{U}{A}\dot{x}^1 \qquad \ldots(5.1.25)$$

$$L\frac{\partial L}{\partial \dot{x}^2} = \frac{U}{B}\dot{x}^2 \qquad \ldots(5.1.26)$$

$$L\frac{\partial L}{\partial \dot{x}^3} = \frac{1}{U}(BH^2-AJ^2)\dot{x}^3 + \frac{1}{U}(AJK-BHI)\dot{x}^4 \qquad \ldots(5.1.27)$$

$$L\frac{\partial L}{\partial \dot{x}^4} = \frac{1}{U}(AJK-BHI)\dot{x}^3 + \frac{1}{U}(BI^2-AK^2)\dot{x}^4. \qquad \ldots(5.1.28)$$

The first integrals of the geodesic equations can thus be obtained directly from (1.23) by utilizing the relations

$$L\frac{\partial L}{\partial \dot{x}^1} = P_{x^1} = \frac{dS_1}{dx^1} \qquad \ldots(5.1.29)$$

$$L\frac{\partial L}{\partial \dot{x}^2} = P_{x^2} = \frac{dS_2}{dx^2} \qquad \ldots(5.1.30)$$

$$L\frac{\partial L}{\partial \dot{x}^3} = P_{x^3} = \frac{dS_3}{dx^3} \qquad \ldots(5.1.31)$$

$$L\frac{\partial L}{\partial \dot{x}^4} = P_{x^4} = \frac{dS_4}{dx^4} \ . \qquad \ldots(5.1.32)$$

From (1.29) we get

$$\frac{U}{A}\dot{x}^1 = \sqrt{R}, \text{ and hence } \dot{x}^1 = \frac{A}{U}\sqrt{R} \ , \qquad \ldots(5.1.33)$$

and (1.30) gives

$$\frac{U}{B}\dot{x}^2 = \sqrt{\theta}, \text{ and hence } \dot{x}^2 = \frac{B}{U}\sqrt{\theta} \ . \qquad \ldots(5.1.34)$$

Taking (1.31) we get,

$$\frac{1}{U}(BH^2-AJ^2)\dot{x}^3 + \frac{1}{U}(AJK-BHI)\dot{x}^4 = \Phi \ , \qquad \ldots(5.1.35)$$

and (1.32) gives

$$\frac{1}{U}(AJK-BHI)\dot{x}^3 + \frac{1}{U}(BI^2-AK^2)\dot{x}^4 = -E , \qquad \ldots(5.1.36)$$

From the last two equations we may solve for $\dot{x}^3$ and $\dot{x}^4$ to get

$$\dot{x}^3 = \frac{U[\phi(BI^2-AK^2) + E(AJK-BHI)]}{(BI^2-AK^2)(BH^2-AJ^2) - (AJK-BHI)^2} \qquad \ldots(5.1.37)$$

$$\dot{x}^4 = \frac{U[\phi(AJK-BHI) + E(BH^2-AJ^2)]}{(AJK-BHI)^2 - (BH^2-AJ^2)(BI^2-AK^2)}$$

5.2 APPLICATION OF THE INTEGRATION FORMULA TO CARTER'S FOUR CASES

We shall now apply the method of integration to all the four families of metrics of Carter [1969, p.282].

$$[A] \quad ds^2 = \frac{U}{A}(dx^1)^2 + \frac{U}{B}(dx^2)^2 + \frac{B[dx^3 - Idx^4]^2}{U} - \frac{A[dx^3 - Kdx^4]^2}{U} \qquad \ldots(5.2.1)$$

where $U = U_1(x^1) + U_2(x^2)$

$$L = [\frac{U}{A}(\dot{x}^1)^2 + \frac{U}{B}(\dot{x}^2)^2 + \frac{1}{U}(B-A)(\dot{x}^3)^2 + \frac{2}{U}(AK-BI)\dot{x}^3\dot{x}^4$$

$$+ \frac{1}{U}(BI^2-AK^2)(\dot{x}^4)^2]^{\frac{1}{2}} . \qquad \ldots(5.2.2)$$

The metric tensor of (2.1) is

$$g_{\mu\nu} = \begin{bmatrix} \frac{U}{A} & 0 & 0 & 0 \\ 0 & \frac{U}{B} & 0 & 0 \\ 0 & 0 & \frac{B-A}{U} & \frac{AK-BI}{U} \\ 0 & 0 & \frac{AK-BI}{U} & \frac{BI^2-AK^2}{U} \end{bmatrix} \qquad \ldots(5.2.3)$$

and the contravariant metric tensor is:

$$g^{\mu\nu} = \begin{bmatrix} \frac{A}{U} & 0 & 0 & 0 \\ 0 & \frac{B}{U} & 0 & 0 \\ 0 & 0 & \frac{1}{U}(\frac{K}{A}-\frac{I}{B}) & \frac{1}{U}(\frac{J}{A}-\frac{K}{B}) \\ 0 & 0 & \frac{1}{U}(\frac{I}{A}-\frac{K}{B}) & \frac{1}{BU}-\frac{1}{UA} \end{bmatrix} . \qquad \ldots(5.2.4)$$

Thus the Hamilton-Jacob equation reads

$$A(\frac{\partial S}{\partial x^1})^2 + B(\frac{\partial S}{\partial x^2})^2 + (\frac{K^2}{B}-\frac{I^2}{A})(\frac{\partial S}{\partial x^3})^2 + 2(\frac{I}{A}-\frac{K}{B})\frac{\partial S}{\partial x^3}\frac{\partial S}{\partial x^4}$$

$$+ (\frac{1}{A}-\frac{1}{B})(\frac{\partial S}{\partial x^4})^2 = \lambda(U_1(x^2) + U_2(x^2)) \qquad \ldots(5.2.5)$$

which, with $\frac{\partial S_3}{\partial x^3} = \Phi$ and $\frac{\partial S_4}{\partial x^4} = -E$, separate as

$$A(\frac{\partial S_1}{\partial x^1})^2 - \frac{I^2}{A}\Phi^2 - \frac{2I}{A}\Phi E - \frac{1}{A}E^2 - \lambda U_1(x^1) = N \qquad \ldots(5.2.6)$$

$$B(\frac{\partial S_2}{\partial x^2})^2 - \frac{K^2}{B}\Phi^2 + \frac{2K}{B}\Phi E - \frac{1}{B}E^2 - \lambda U_2(x^2) = -N . \qquad \ldots(5.2.7)$$

These two equations then give

$$\frac{dS_1}{dx^1} = \sqrt{R_1} \qquad \ldots(5.2.8)$$

where

$$R_1 = \frac{N}{A} + \frac{J^2}{A^2}\Phi^2 + \frac{2I}{A^2}\Phi E + \frac{E^2}{A^2} + \frac{\lambda U_1(x^1)}{A} \qquad \ldots(5.3.9)$$

and

$$\frac{dS_2}{dx^2} = \sqrt{\theta_1} \qquad \ldots(5.2.10)$$

where

$$\theta_1 = -\frac{N}{B} - \frac{K^2\Phi^2}{B^2} - \frac{2K\Phi E}{B^2} - \frac{E^2}{B^2} + \frac{\lambda U_2(x^2)}{B} . \qquad \ldots(5.2.11)$$

Thus, the solution of the Hamilton-Jacobi equation is,

$$S = \int^{x^1} \sqrt{R_1}\, dx^1 + \int^{x^2} \sqrt{\theta_1}\, dx^2 + \Phi x^3 - Ex^4 \qquad \ldots(5.2.12)$$

$$L \frac{\partial L}{\partial \dot{x}^1} = \frac{U}{A} \dot{x}^1 \qquad \ldots(5.2.13)$$

$$L \frac{\partial L}{\partial \dot{x}^2} = \frac{U}{B} \dot{x}^2 \qquad \ldots(5.2.14)$$

$$L \frac{\partial L}{\partial \dot{x}^3} = \frac{1}{U}(AK-BI)\dot{x}^4 + \frac{1}{U}(B-A)\dot{x}^3 \qquad \ldots(5.2.15)$$

$$L \frac{\partial L}{\partial \dot{x}^4} = \frac{1}{U}(AK-BI)\dot{x}^3 + \frac{1}{U}(BI^2-AK^2)\dot{x}^4 . \qquad \ldots(5.2.16)$$

Hence

$$\dot{x}^1 = \frac{A}{U} \sqrt{R_1} \qquad \ldots(5.2.17)$$

$$\dot{x}^2 = \frac{B}{U} \sqrt{\theta_1} \qquad \ldots(5.2.18)$$

$$\dot{x}^3 = \frac{U[\Phi(BI^2-AK^2) + E(AK-BI)]}{(BI^2-AK^2)(B-A) - (AK-BI)^2} \qquad \ldots(5.2.19)$$

$$\dot{x}^4 = \frac{U[\Phi(AK-BI) + E(B-A)]}{(AK-BI)^2 - (BI^2-AK^2)(B-A)} . \qquad \ldots(5.2.20)$$

[B] $$ds^2 = \frac{U}{A}(dx^1)^2 + \frac{U}{B}(dx^2)^2 - \frac{A}{U}(dx^3)^2 - \frac{2AL}{U} dx^3 dx^4 + (UB - \frac{AL^2}{U})(dx^4)^2 \qquad \ldots(5.2.21)$$

where $U = U(x^1)$

$$L = [\frac{U}{A}(\dot{x}^1)^2 + \frac{U}{B}(\dot{x}^2)^2 - \frac{A}{U}(\dot{x}^3)^2 - \frac{2AL}{U} \dot{x}^3\dot{x}^4 + (UB - \frac{AL^2}{U})\dot{x}^4]^{\frac{1}{2}} \qquad \ldots(5.2.22)$$

The metric tensor of (2.21) is

$$g_{\mu\nu} = \begin{bmatrix} \frac{U}{A} & 0 & 0 & 0 \\ 0 & \frac{U}{B} & 0 & 0 \\ 0 & 0 & -\frac{A}{U} & -\frac{AL}{U} \\ 0 & 0 & -\frac{AL}{U} & UB-\frac{AL^2}{U} \end{bmatrix} \qquad \ldots(5.2.23)$$

and the contravariant metric tensor is

$$g^{\mu\nu} = \begin{bmatrix} \frac{A}{U} & 0 & 0 & 0 \\ 0 & \frac{B}{U} & 0 & 0 \\ 0 & 0 & \frac{1}{U}\frac{L^2}{B}-\frac{1}{UA} & \frac{L}{UB} \\ 0 & 0 & \frac{L}{UB} & \frac{1}{UB} \end{bmatrix} . \qquad \ldots(5.2.24)$$

Thus the Hamilton-Jacobi equation reads

$$\frac{A}{U}\left(\frac{\partial S}{\partial x^1}\right)^2 + \frac{B}{U}\left(\frac{\partial S}{\partial x^2}\right)^2 + \left(\frac{1}{U}\frac{L^2}{B}-\frac{1}{UA}\right)\left(\frac{\partial S}{\partial x^3}\right)^2 + 2\,\frac{L}{UB}\frac{\partial S}{\partial x^3}\frac{\partial S}{\partial x^4}$$

$$-\frac{1}{UB}\left(\frac{\partial S}{\partial x^4}\right)^2 = \lambda \qquad \ldots(5.2.25)$$

which, with $\frac{\partial S_3}{\partial x^3} = \Phi$ and $\frac{\partial S_4}{\partial x^4} = -E$, separates as

$$A\left(\frac{\partial S_1}{\partial x^1}\right)^2 - \frac{1}{A}\,\Phi^2 - \lambda U(x^1) = P \qquad \ldots(5.2.26)$$

$$B\left(\frac{\partial S_2}{\partial x^2}\right)^2 + \frac{L^2}{B}\,\Phi^2 - 2\,\frac{L}{B}\,\Phi E - \frac{1}{B}\,E^2 = -P\ . \qquad \ldots(5.2.27)$$

These two equations then give,

$$\frac{dS_1}{dx^1} = \sqrt{R_2} \qquad \ldots(5.2.28)$$

where $R_2 = \frac{P}{A} + \frac{\Phi^2}{A^2} + \frac{\lambda U(x^1)}{A}$ $\qquad \ldots(5.2.29)$

and $$\frac{dS_2}{dx^2} = \sqrt{\theta_2} \qquad \ldots(5.2.30)$$

where $$\theta_2 = -\frac{P}{B} - \frac{L^2}{B^2}\Phi^2 + \frac{2L}{B^2}\Phi E + \frac{E^2}{B^2} \; . \qquad \ldots(5.2.31)$$

Thus, the solution of the Hamilton-Jacobi equation is,

$$S = \int^{x^1} \sqrt{R_2}\, dx^1 + \int^{x^2} \sqrt{\theta_2}\, dx^2 + \Phi x^3 - Ex^4 \qquad \ldots(5.2.32)$$

$$L \frac{\partial L}{\partial \dot{x}^1} = \frac{U}{A}\dot{x}^1 \qquad \ldots(5.2.33)$$

$$L \frac{\partial L}{\partial \dot{x}^2} = \frac{U}{B}\dot{x}^2 \qquad \ldots(5.2.34)$$

$$L \frac{\partial L}{\partial \dot{x}^3} = -\frac{A}{U}\dot{x}^3 - \frac{AL}{U}\dot{x}^4 \qquad \ldots(5.2.35)$$

$$L \frac{\partial L}{\partial \dot{x}^4} = -\frac{AL}{U}\dot{x}^3 + (UB - \frac{AL^2}{U})\dot{x}^4 \; . \qquad \ldots(5.2.36)$$

Hence

$$\dot{x}^1 = \frac{A}{U}\sqrt{R_2} \qquad \ldots(5.2.37)$$

$$\dot{x}^2 = \frac{B}{U}\sqrt{\theta_2} \qquad \ldots(5.2.38)$$

$$\dot{x}^3 = \frac{U[-E(AL) - \Phi(UB - AL^2)]}{-(UB-AL^2)A + A^2L^2} \qquad \ldots(5.2.39)$$

$$\dot{x}^4 = \frac{U[-\Phi AL + E(-A)]}{A(UB-AL^2) - A^2L^2} \; . \qquad \ldots(5.2.40)$$

[$\tilde{B}$] $$ds^2 = \frac{U}{A}(dx^1)^2 + \frac{U}{B}(dx^2)^2 + \frac{B}{U}(dx^3)^2 - \frac{2BN}{U}dx^3dx^4 + (\frac{BN}{U} - UA)(dx^4)^2 \qquad \ldots(5.2.41)$$

where $U = U(x^2)$

$$L = [\frac{U}{A}(\dot{x}^1)^2 + \frac{U}{B}(\dot{x}^2)^2 + \frac{B}{U}(\dot{x}^3)^2 - \frac{2BN}{U}\dot{x}^3\dot{x}^4 + (\frac{BN}{U} - UA)(\dot{x}^4)^2]^{\frac{1}{2}} \qquad \ldots(5.2.42)$$

The metric tensor of (2.41) is

$$g_{\mu\nu} = \begin{bmatrix} \frac{U}{A} & 0 & 0 & 0 \\ 0 & \frac{U}{B} & 0 & 0 \\ 0 & 0 & \frac{B}{U} & -\frac{B}{U}N \\ 0 & 0 & -\frac{B}{U}N & \frac{BN}{U} - UA \end{bmatrix} \qquad \ldots(5.2.43)$$

and the contravariant metric tensor is

$$g^{\mu\nu} = \begin{bmatrix} \frac{A}{U} & 0 & 0 & 0 \\ 0 & \frac{B}{U} & 0 & 0 \\ 0 & 0 & \frac{1}{U}(\frac{1}{B} \quad \frac{N^2}{A}) & \frac{N}{UA} \\ 0 & 0 & \frac{N}{UA} & -\frac{1}{UA} \end{bmatrix} . \qquad \ldots(5.2.44)$$

Thus the Hamilton-Jacobi equation reads

$$\frac{A}{U}(\frac{\partial S}{\partial x^1})^2 + \frac{B}{U}(\frac{\partial S}{\partial x^2})^2 + \frac{1}{U}(\frac{1}{B} - \frac{N^2}{A})(\frac{\partial S}{\partial x^3})^2 + \frac{2N}{UA}\frac{\partial S}{\partial x^3}\frac{\partial S}{\partial x^4}$$

$$- \frac{1}{UA}(\frac{\partial S}{\partial x^4}) = \lambda \qquad \ldots(5.2.45)$$

which, with $\frac{\partial S_3}{\partial x^3} = \Phi$ and $\frac{\partial S_3}{\partial x^4} = -E$, separates as

$$A(\frac{\partial S_1}{\partial x^1})^2 - \frac{N^2}{A}\Phi^2 - \frac{2N}{A}\Phi E - \frac{1}{A}E^2 = Q \qquad \ldots(5.2.46)$$

$$B(\frac{\partial S_2}{\partial x^2})^2 + \frac{1}{B}\Phi^2 - \lambda U(x^2) = -Q . \qquad \ldots(5.2.47)$$

These last two equations then give

$$\frac{dS_1}{dx^1} = \sqrt{R_3}$$

where $R_3 = \frac{Q}{A} + \frac{N^2}{A^2}\Phi^2 + \frac{2N}{A^2}\Phi E + \frac{E^2}{A^2}$ ...(5.2.49) and $$\frac{dS_2}{dx^2} = \sqrt{\theta_3}$$ ...(5.2.50)

where $\theta_3 = -\frac{Q}{B} - \frac{\Phi^2}{B^2} + \frac{\lambda U(x^2)}{B}$.

Thus,

$$S = \int^{x^1} \sqrt{R_3}\, dx^1 + \int^{x^2} \sqrt{\theta_3}\, dx + x - Ex$$ ...(5.2.52)

$$L \frac{\partial L}{\partial \dot{x}^1} = \frac{U}{A} \dot{x}^1$$ ...(5.2.53)

$$L \frac{\partial L}{\partial \dot{x}^2} = \frac{U}{B} \dot{x}^2$$ ...(5.2.54)

$$L \frac{\partial L}{\partial \dot{x}^3} = \frac{B}{U} \dot{x}^3 - \frac{BN}{U} \dot{x}^4$$ ...(5.2.55)

$$L \frac{\partial L}{\partial \dot{x}^4} = -\frac{BN}{U} \dot{x}^3 + (\frac{BN}{U} - UA)\dot{x}^4$$ ...(5.2.56)

Hence,

$$\dot{x}^1 = \frac{A}{U} \sqrt{R_3}$$ ...(5.2.57)

$$\dot{x}^2 = \frac{B}{U} \sqrt{\theta_3}$$ ...(5.2.58)

$$\dot{x}^3 = \frac{U[\Phi(BN-U^2A) - EBN]}{B(BN-U^2A) - B^2N^2}$$ ...(5.2.59)

$$\dot{x} = \frac{U[\Phi BN - EB]}{B(BN-U^2A) - B^2N^2}$$ . ...(5.2.60)

[C] $$ds^2 = \frac{1}{A}(dx^1)^2 + \frac{1}{B}(dx^2)^2 + B(dx^3)^2 - A(dx^4)^2$$ ...(5.2.61)

$$U = 1$$

$$L = [\frac{1}{A}(\dot{x}^1)^2 + \frac{1}{B}(\dot{x}^2)^2 + B(\dot{x}^3)^2 - A(\dot{x}^4)^2]^{\frac{1}{2}},$$ ...(5.2.62)

The metric tensor of (2.61) is:

$$g_{\mu\nu} = \begin{bmatrix} \frac{1}{A} & 0 & 0 & 0 \\ 0 & \frac{1}{B} & 0 & 0 \\ 0 & 0 & B & 0 \\ 0 & 0 & 0 & -A \end{bmatrix} \qquad \ldots(5.2.63)$$

and the contravariant metric tensor is

$$g^{\mu\nu} = \begin{bmatrix} A & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & \frac{1}{B} & 0 \\ 0 & 0 & 0 & -\frac{1}{A} \end{bmatrix} . \qquad \ldots(5.2.64)$$

Thus the Hamilton-Jacobi equation reads

$$A\left(\frac{\partial S}{\partial x^1}\right)^2 + B\left(\frac{\partial S}{\partial x^2}\right)^2 + \frac{1}{B}\left(\frac{\partial S}{\partial x^3}\right)^2 - \frac{1}{A}\left(\frac{\partial S}{\partial x^4}\right)^2 = \lambda \qquad \ldots(5.2.65)$$

which, with $\frac{\partial S_3}{\partial x^3} = \Phi$ and $\frac{\partial S_4}{\partial x^4} = -E$, separates as

$$A\left(\frac{\partial S_1}{\partial x^1}\right)^2 + \frac{1}{A} E^2 = \lambda_1 \qquad \ldots(5.2.66)$$

$$B\left(\frac{\partial S_2}{\partial x^2}\right)^2 + \frac{1}{B} \Phi^2 = -\lambda_1 . \qquad \ldots(5.2.67)$$

These last two equations then give

$$\frac{dS_1}{dx^1} = \sqrt{\frac{\lambda_1}{A} - \frac{E^2}{A^2}} \qquad \ldots(5.2.68)$$

and

$$\frac{dS_2}{dx^2} = \sqrt{-\frac{\lambda_1}{B} - \frac{\Phi^2}{B^2}} . \qquad \ldots(5.2.69)$$

Thus,

$$S = \int^{x^1} \sqrt{\frac{\lambda_1}{A} + \frac{E^2}{A^2}}\, dx^1 + \int^{x^2} \sqrt{-\frac{\lambda_1}{B} - \frac{\Phi^2}{B^2}}\, dx^2 + \Phi x^3 - E x^4 \qquad \ldots(5.2.70)$$

$$L \frac{\partial L}{\partial \dot{x}^1} = \frac{1}{A} \dot{x}^1 \qquad \ldots(5.2.71)$$

$$L \frac{\partial L}{\partial \dot{x}^2} = \frac{1}{B} \dot{x}^2 \qquad \text{...(5.2.72)}$$

$$L \frac{\partial L}{\partial \dot{x}^3} = B\dot{x}^3 \qquad \text{...(5.2.73)}$$

$$L \frac{\partial L}{\partial \dot{x}^4} = -A\dot{x}^4 \quad . \qquad \text{...(5.2.74)}$$

Hence,

$$\dot{x}^1 = A\sqrt{\frac{\lambda_1}{A} + \frac{E^2}{A^2}} \qquad \text{...(5.2.75)}$$

$$\dot{x}^2 = B\sqrt{-\frac{\lambda_1}{B} - \frac{\Phi^2}{A^2}} \qquad \text{...(5.2.76)}$$

$$\dot{x}^3 = \frac{\Phi}{B} \qquad \text{...(5.2.77)}$$

$$\dot{x}^4 = \frac{E}{A} \quad . \qquad \text{...(5.2.78)}$$

This completes the analysis of the Carter metrics in so far as a general treatment is concerned. Further work would require a detailed analysis of the qualitative behaviour of each system of differential equations to discover the detailed properties of the space-time. Some work has already been done along these lines by Bardeen (1972) and Carter (1968a) and Carter (1972 "Les Houches") for the Kerr class of metrics and Kinnersley and Walker [1970] for the Kundt and Ehlers type C metric. However, complete analytical treatments of the full set of solutions has not yet been completed.

BIBLIOGRAPHY

BARDEEN, J.M. (1972), "Timelike and null geodesics in the Kerr metric", Les Houches Summer School, pp.215-239

BONANOS, S. (1976), "Hamilton-Jacobi Separable, Anisymmetric, Perfect-fluid Solutions of Einstein's Equations", Commun. Math. Phys. 49, pp.53-61

CARTER, B (1968a), "Global structure of the Kerr family of gravitational fields", Phys. Rev. 174, pp.1559-71

——— (1968b), "Hamilton-Jacobi and Schrodinger separable solutions of Einstein's equations", Comm. Math. Phys. 10, pp.280-310

——— (1969), "Killing horizons and orthogonally transitive groups in space-time", J. Math. Phys. 10, pp.70-81

COLLINSON, C.D. and FUGÈRE, J. (1977a), "Empty space-times with separable Hamilton-Jacobi equation", J. Phys. A. Math. Gen., Vol. 10, pp.745-53

——— (1977), "Conditions for the separation of the Hamilton-Jacobi equation", J. Phys. A. Math. Gen., Vol. 10, No. 11, pp.1877-85

DEBEVER, R. (1969), "Sur les espaces de Brandon Carter", Acad. Roy. de Belgique. Cl. des. sc. Bull v. 55 (5th sc), pp.8-16

EISENHART, L.P. (1934), "Separable systems of Stäckel", Ann. Math. 35, pp.284-305

FLANDERS, H. (1963), "Differential Forms", Academic Press, N.Y., London

GEROCH, R. (1968), "Spinor Structure of space-time in G.R. I", J. M. Phys. 9, p.1739

——— (1970), "Spinor Structure of space-time in G.R. II", J.M. Phys. 11, p.343

HAWKING, S.W. and ELLIS, G. (1973), "The large scale structure of space-time", Cambridge University Press, Cambridge, London

HUGHSTON, L.P., PENROSE, R., SOMMERS, P., WALKER, M. (1972), Comm. Math. Phys. 27, p.303

KINNERSLEY, W. and WALKER, M. (1970), "Uniformly accelerated charged mass in General Relativity", Phys. Rev. D. 2, pp.1359-70

KERR, R.P. (1963), Phys. Rev. Letter II, p.237

MATRAVERS, D.R. (1972), "The Hamilton-Jacobi theory in General Relativity theory and certain Petrov type - D Metvis", Utilitas Mathematics, Vol. 2, pp.55-70

MATRAVERS, D.R. (1976), "A study of some exact solutions to Einstein's Field equations which yield Separable Hamilton-Jacobi equations", G.R.G. Vol. 7, No. 12, pp.937-47

PAPAPETROU, A. (1974), "Lectures on General Relativity", D. Reidel Publishing Co., Dordrecht, Holland

PIRANI, F.A.E. (1964), "Introduction to gravitational radiation theory", Brandeis Summer Institute in Theoretical Physics, pp.249-73

RUND, H. (1966), "The Hamilton-Jacobi theory in the calculas of variations", D. Van Nostrad Co. Ltd., London

SOMMERS, P. (1972), "On Killing tensors and constants of motion", J. Math. Phys. 14, pp.787-90

TRAUTMAN, A. (1964), "Foundations and current problems of General Relativity", Brandeis Summer Institute in Theoretical Physics, pp.1-248

WALKER, M. and PENROSE, R. (1970), "On Quadratic first integrals of the geodesic Equations for type 22 spacetimes", Comm. Math. Phys. 18, pp.265-74

WOLF, J.A. (1972a), "On the geometry and classification of absolute parallisms II", J. Differential Geometry, 7, pp.19-44

WOODHOUSE, N.M.J. (1975), "Killing Tensors and the separation of the Hamilton-Jacobi equation", Comm. Math. Phys. 44, pp.9-38