LESNIEWSKI'S LOGIC

ASPECTS OF HIS

PROTOTHETIC, ONTOLOGY AND MEREOLOGY

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CHAPTER 1

INTRODUCTION

Stanislaw Lesniewski (1886 - 1939) was professor of Philosophy of Mathematics at the University of Warsaw from 1919 until his death . He played a leading role in the Warsaw school of logic and had a lasting influence on many of its members. The calibre of his work led Jordan to make the following appraisal: "The most thorough, original, and philosophically significant attempt to provide a logically secure foundation for the whole of mathematics comes from Lesniewski."[1)]

Lesniewski, did, however, not develop his system merely to add alternative calculi to those already in existence. He aimed at the construction of "logically secure" systems which would be considered reliable for deductive needs and for scientific investigation. Ultimately, he wanted to provide a foundation of mathematics which would solve the problems presented by the paradoxes discovered in other systems. It was indeed his intial analysis of Russell's paradox (this paradox arises when considering the class of classes which are not elements of themselves) that induced Lesniewski to construct his system of mereology. In this system he describes the "collective" interpretation of "class", having concluded that a confusion of the "collective" and "distributive" notions of "class" was to a large degree responsible for the paradox. These notions are described in chapter 5, as is his mereology.

Lesniewski constructed his first description of mereoloy in colloquial language and in the absence of a secure logical foundation. In order to effectively distinguish between the collective and distributive notions of class, further description of the distributive notion was necessary. He therefore formalized the distributive concepts in his theory of ontology. Henceforth "ontology" will be used specifically to refer to this theory of Lesniewski.

Finally, the construction of protothetic (a system of propositional logic) provided a sound logical foundation for Lesniewski's ontology and mereology. Protothetic, together with his prescribed rules of procedure and his grammar of semantic categories, also facilitated the formalization of his systems in a logically rigorous manner.

Mereology thus incorporates and is based on protothetic and ontology. To depict this fact, the systems will be dealt with in their hierarchic deductive order rather than in the historical order of their development. Therefore protothetic is considered in chapter 3, and ontology in chapter 4. The grammar of semantic categories is a prerequisite for protothetic and ontology, and will be introduced in chapter 2. We will return to the problems which initiated the development of Lesniewski's systems in the final chapter. A solution to Russell's paradox in terms of Lesniewski's systems will therefore be outlined in chapter 6.

Despite its quality, Lesniewski's work for a long time remained relatively unknown outside Poland. There are a number of reasons for this. Lesniewski, being a perfectionist, was loath to publish his theories in uncompleted form. When he did, however, publish, he condensed all his terminological explanations and his directives for protothetic and ontology into 24 pages of Latin abbreviations, non-standard technical terms and non-standard logical symbols. Although, he intended elaborating on these later, they were in fact the only descriptions published in his lifetime. A high degree of precision and conciseness was obtained in these published accounts; but this very conciseness had the effect of complicating matters for the reader and discouraged further investigation - readers baulking at the effort of deciphering his work. These published outlines, however, make up what Luschei calls the "cornerstone of Lesniewski's foundation" (pg. 44 of [14]).

Further complicating matters is the fact that the above articles were published in a Polish journal with differently abbreviated French and Polish names. The articles themselves were, in addition, provided with Polish titles despite being written in German.

Additional reasons for Lesniewski's work being relatively unknown are that none of his articles were published in English and that even his few extant publications are difficult to acquire. The major reason would, however, seem to be the devastation brought about by the Second World War. Only a few of Lesniewski's papers survived the war. His lecture notes and unpublished manuscripts, for example, were destroyed in the uprising of 1944. In this way most of his valuable research into logic and the foundations of mathematics was lost.

The post-war reconstruction efforts and investigations of researchers such as Sobocinski, Lejewski, Luschei, Slupecki and more recently Clay, have made Lesniewski's work more accessible. Woodger's English translation of Tarski's earlier papers has also helped in this regard. Many publications of the above-mentioned researchers, are, however, not available locally.

All of the researchers acknowledge that one of Lesniewski's most fundamental achievements was the development of his deductive systems (protothetic, ontology and mereology). An attempt is made in the chapters which follow to give a brief account of a few salient features of these systems.

NOTES:

1) Jordan, Z: "The Development of Mathematical Logic and of Logical Positivism in Poland between the two Wars", London, 1945. The quotation is from page 24 and is cited by Luschei in [14], 3.10

CHAPTER 2

BACKGROUND AND EXPLANATORY NOTES

Before proceeding with the investigation of Lesniewski's deductive theories, it seems necessary to outline the milieu within which it will be carried out. This chapter therefore aims at providing background information, introducing prerequisite theories and clarifying the terminology and notation to be used.

2.1 The Nature of Deductive Systems

This question has a long history. Kilmister in [9] considers Aristotle's view that a "deductive science" could be regarded as a set of sentences S, satisfying the following conditions:"

(i) each sentence in S refers to a set of real objects;
(ii) each sentence in S is true;
(iii) S contains the logical consequences of any two members of S;
(iv) S contains some undefined terms whose meaning is obvious and in terms of which any other terms of S can be defined.
(v) S contains some unproved sentences, whose truth is obvious and from which other sentences of S can be proved."

These conditions imply much of what is still regarded as desirable in a deductive system. However, as Kilmister points out, with the advance of scientific knowledge and the experimental method it became necessary to distinguish between what he terms "empirical science", in which experimental verification could replace conditions (iii), (iv) and (v), and "rational science", in which condition (i) was no longer required. In this context mathematics would be regarded as "rational science". In Lesniewski's systems condition (i) would, however, be desirable (cf. §2.2). The characterization of Kilmister's "rational science" corresponds closely to what might otherwise be termed an "uninterpreted" deductive system, and that of the original "deductive science" (where condition (i) is included) to an "interpreted" deductive system. The Lesniewskian systems were intended to be of the latter type.

The "undefined terms" of condition (iv) are often called "primitive terms" and are introduced into a system by the "unproved sentences", commonly termed axioms, of condition (v). The "other sentences" which may be proved from these axioms are in turn the theorems, derived according to given rules of procedure (inference).

An anologous approach to deductive systems is to consider the systems as languages (cf. [25]) in which all the "sentences" have been so arranged, that from a small number of initial ones (the axioms), the rest may be derived via applications of given directives for their construction and directives according to which one sentence may be regarded as a consequence of others. Such a language may be termed a calculus.

Lesniewski aimed at reducing the primitive basis (axioms and procedural directives) of his deductive systems to a single axiom of minimal length and a minimal number of directives satisfying his principles. Furthermore in his systems both axioms and theorems - thusfar called "sentences" of the language - are termed "theses".

Lesniewski (cited in §4 of [14]) describes his approach to deductive systems as follows:

"...... in formulating deductive theories I try to express a series of thoughts on this or that subject in a series of meaningful propositions, and to deduce the individual propositions from others according to principles of inference I take as 'intuitively' binding. And I know of no more effective method to acquaint the reader with my 'logical intuitions' than to 'formalize' the deductive theory presented which, 'formalized', in no way ceases to consist solely of meaningful propositions that are for me intuitively valid."

From this description two further issues arise:- the question of "intuitionist" versus "formalist" approaches to mathematics (this will be considered in §2.2) and once again the question of "interpreted" versus "uninterpreted" theories. Lesniewski's intention that his theories be interpreted in a specific way, follows this view of interpreted "mathematical sciences" as being "deductive theories conceived to comprehend the diverse reality of the world in ideally exact laws." On the other hand, he considered uninterpreted formal systems to be "free from contradiction and prolific in generating theorems, but lacking any intuitive scientific connection with reality."

2.2 Philosophical Background:

A fundamental problem in the philosophy of mathematics is the explanation of the relation between mathematics and reality. Different approaches to this problem are mirrored in the work and judgements of mathematicians and philosophers, including Lesniewski, his supporters and his detractors. In considering evaluations of Lesniewski's work, it is therefore wise to bear in mind the philosophical bias from which they are made. A number of approaches are relevant here.

The so-called "Platonists" regard abstract entities as having an existence independent of the mind. They see "ideal" objects as representing a number of individual objects and possessing the characteristics common to them all. On the other hand, according to the "nominalist" approach, abstract entities are denied existence and only entities known from experience are admitted. Between these approaches is that of the "conceptualists" who reduce all entities to an "existence in the mind."

Two further opposing approaches require attention. These are "intuitionism" and "formalism". Intuitionists would admit the existence only of mathematical entities which are "constructible" out of what is intuitively well-founded. The formalist, however, does not concern himself with the existence or non-existence of the entities with which he is working, but merely with manipulating these entities according to given rules. This formalist approach has been compared to a "game played with symbols."

In considering Lesniewski's work in relation to this wide range of approaches, it is necessary to note two facts at the outset. These are: (a) that the "labels" outlined above are somewhat arbitrary and may mean different things to different users; and (b) that Lesniewski preferred not to be labelled in this way.

Luschei sees Lesniewski as having nominalist tendencies, but as expressing these "constructively" and "linguistically". The reason for this view is that Lesniewski did not deny the existence of abstract entities; but instead constructed the logical basis for his systems in such a way that within the accompanying language, it is possible to describe aspects of reality without referring to abstract entities. Lesniewski furthermore rejected platonic ideas of "ideal" objects and would consider an "ideal object" as representing only itself. He regarded his systems as "flexible growing organisms" with potentials, anologous to that of natural languages, for indefinite growth. They were, however, finite at any particular stage. Each of Lesniewski"s systems were, in fact, successfully based on a single axiom and a finite number of directives, which determined at the outset, the rules according to which the systems could be extended.

Lesniewski's systems do not presuppose the existence of anything and are rigorously formalized. On the other hand, Lesniewski severely criticized the "pure formalist's game" and considered a "sense of reality" to be vital in logic. He therefore stressed the fact that interpretation was of paramount importance in his own systems. He was pragmatic to a certain degree - in Luschei's words: "to the extent one must be to survive". He, however, at the same time considered it necessary to counter-balance utilitarian needs with philosophical clarity.

Clearly, it would be difficult to classify Lesniewski's work in terms of any one of the various approaches. He, himself, writes: "I should see no contradiction in asserting that I construct my system in a radically 'formalist' fashion just because I am a confirmed 'intuitionist'." It therefore seems advisable to let such a "criss-cross" [1)] classification of Lesniewski's rigorously formalized intuitive logic suffice.

The chapters which follow, largely steer clear of the dangerous area of opinions based on differing philosophical approaches. The dangers lurking in this area may be illustrated by an example. In strongly criticizing what he considered to be "careless formalism", Lesniewski antagonised certain formalists. Having criticized the directives for

Chiwistek's propositional calculus and also having pointed out a contradiction in this calculus, he elicits the following comment from Chiwistek: "...... throughout Lesniewski's life there was, on the whole, no time when he understood the concept of class" (cited in § 3.10 of [14]). Luschei, however, comments that "Chiwistek demonstrated chiefly that he was too pure a formalist to appreciate the force of Frege's and Lesniewski's arguments..." (§3.10 of [14]).

Lesniewski's work may, however, also be evaluated by considering whether his systems achieve what he set out to do. In chapter 6 we therefore consider his solution to the problem which sparked off the development of his deductive systems - Russell's paradox.

2.3 Notation

The notation of [1] is adopted, except that the universal quantifier will be represented by square brackets, "[...] ", containing the variables which are to occur in its scope. Furthermore, brackets, used as punctuation symbols, will, where necessary, be given subscripts to denote how they are paired. For example, $(_1 \ldots (_2 \ldots (\ldots) \ldots)_2 \ldots)_1$. The only aim in using these subscripts is to avoid confusion.

The Peano-Russell symbolism, which is used in many of the references, differs from that adopted here chiefly in that it uses "dot-notation" instead of parentheses for punctuation; uses the symbol " $\equiv$" instead of " $\leftrightarrow$" for co-implication; and uses a dot instead of "$\wedge$" for conjunction.

As Lesniewski made no categorial (cf. §2.5) distinction between "proper" and "common" names, this distinction will also not be made in the notation used here for nominal variables. Small letters of the Latin alphabet will be used throughout.

2.4 Terminology

In most cases the meaning of terms will be explained when these terms are first used. Moreover, a knowledge of terminology used by texts such as [1] and [18] is assumed. Comments are, however, made here about a few terms.

(a) Individuals: These are described by Whitehead and Russell as things "destitute of complexity". The term's obvious connotations of "singleness" and "specificness", should also be noted.

(b) Variables: Symbols which do not have a fixed "meaning" are called variables. The different meanings which the variables may assume are referred to as its values. Variables may thus be used to represent (assume as values), inter alia, different propositions, functions and names. They will then respectively be referred to as: propositional variables, functorial variables and nominal variables.

(c) Constants: Constants do have a definite denotation or "meaning". They are introduced into Lesniewski's systems either by means of axioms (in which case they are termed primitive), or by means of definitions. Their meaning is determined explicitly by the definitions or implicitly by the axioms. In this sense we also speak of constant names, constant functors, and constant functions (cf. §4.2.2, §3.2.1. and §4.2.3.)

(d) Propositions: For an extended discussion of propositions, see § 5.3 of [14] . Lesniewski termed individual indicative clauses or sentences of a language: propositions. Propositions were in addition required to have a definite meaning and to be either true or false. If in a deductive system each proposition is, as required here, either true or false, the system may be termed bi-valued (or bi-valent). This term follows from the fact that the truth-value of a proposition may be said to be truth if it is true, and falsehood if it is false.

(e) Functors, functions and arguments: Functors are discussed more fully in §2.5. We use a simple example to illustrate the sense in which these terms are used. Suppose the variable "p" represents a proposition. Then consider: "¬p". In this expression "¬p" is termed a function, "¬" a functor, and "p" the argument of the functor. A functor plus its argument(s) therefore jointly make up the function. A more precise description of the functor used in the above example would be: proposition-forming functor of one propositional argument (cf. §2.5). Similarly, the function in the example may be termed a propositional function.

(f) Thesis: A thesis may be regarded as a provable proposition. Lesniewski termed both axioms and theorems: "theses" (cf. §2.1).

(g) Definens and definiendum: Definiendum is used to refer to what is being defined in a definition and definiens to that which it is defined as being equivalent to (cf. §3.2.3. and §4.2.2).

2.5 Semantic Categories:

In their "Principia Mathematica", Whitehead and Russell develop a "theory of logical types". This theory seemed useful to them in the first instance because of its "ability to solve certain contradictions..." (pg. 37 of [24]). Among these contradictions which the theory is able to "solve", is Russell's paradox. A simplified description of the way in which this theory provides a solution to Russell's paradox is supplied in [9]. In terms of this description, individuals will be objects of type 0, sets of individuals will be objects of type 1, sets of objects of type 1 will be of type 2, etc. It is furthermore only valid to assert that x is a member of y if y is of a type exactly one greater than than the type of x. In view of the fact that the totality of a collection is of a type different from that of the members of the collection, this ensures that the Russell paradox is avoided. 2)

The formal aspect of the role performed by the notion of "semantic category" in the construction of a logical system, is anologous to that of Whitehead and Russell's "type". Its origin and content, however, correspond more closely to the idea of "part of speech" in the grammar of natural languages. In fact, Lesniewski believed his theory of semantic categories to have linguistic justification independent of the role it plays in the avoidance of paradoxes.

The theory postulates a potentially infinite hierarchy of semantic categories. There are two fundamental categories: the category of names, and the category of statements (propositions). In the category of names no distinction is made between "proper" and "common" names referring to a single object (cf. §4.2.1.).

We now note that, roughly speaking, a functor may be considered as an operator which operates on members of categories to produce members of other (or the same) categories. These functors may be classified into categories and we proceed to consider these categories. The categories of functors are determined by:

(a) the number of arguments the functor has;
(b) the categories of which these arguments are representative; and
(c) the category of the referent [3] of the expression which results from the application of the functor to its argument(s).

Examples of categories of functors are:

(1) the category of those functors which operate on a single argument representing members of the category of names, and which produce expressions whose referents are members of the category of statements (propositions);

(2) the category of those functors which operate on two arguments, both representing members of the category of names, and which produce expressions whose referents are members of the category of statements;

(3) the category of those functors which operate on one argument representing members of the category of statements (propositions) and which produce expressions, the referents of which are also members of the category of statements; and

(4) the category of those functors which operate on one argument representing members of the category described in (1), and which produce expressions, the referents of which are also members of the category described in (1).

In the notation of Adjukiewicz as used in [7], the symbol "n" designates the category of names and the symbol "s" the category of statements (propositions). The different categories of functors are designated by appropriate combinations of these symbols and the symbol "/". The categories of examples (1) to (4) would respectively, be designated by: "s/n", "s/nn", "s/s" and "s/n//s/n". Thus, in the designation of a category of functors, the symbol(s) which appear to the right of the largest group of adjacent strokes ("/"), designate, for each argument of the functors, the category whose members are represented by the argument. The symbol(s) which appear(s) to the left of the largest group of adjacent strokes, designate the category of the referents of the expression produced by the operation of the functors.

Grzegorczyk (cf. [8]) claims that Lesniewski constructed the theory of semantic categories "when studying the semantic categories of expressions of everyday speech". Examples from natural languages to illustrate the above notions therefore seem appropriate. Examples of members of the category of names (designated by "n") are: "Jack" and "mathematician". The sentence: "Jack is a mathematician", is a member of the category of statements (designated by "s"). Therefore, "..... is a mathematician" is an example of a functor which is a member of the category of functors designated by "s/n". For, clearly it produces the statement: "Jack is a mathematician", when operating on the argument "Jack". An example of a member of the category designated by "s/s", is: "It is not the case that....". Here, the functor could, for example, operate on: "Jack is a mathematician", to produce: "It is not the case that Jack is a mathematician."

We give one further example from English. The adverb: "diligently", in the sentence: "Joe diligently worked", may be seen as an example of a member of the category of functors designated by "s/n//s/n". In this case, the functor: "diligently worked" (an example of a member of the category designated by "s/n"), is produced from the argument: "worked" (also member of the category designated by "s/n").

Consider, finally, the following examples from set theory (these examples are taken from [7]):

Category designation	Examples
n	x, y
s	x ε y
s/n	..ε y
s/nn	.... ε .....
s/nn//s/nn	the "/" in: .....ε̸ ....
s/s	˥ (........)

Further examples may be found in chapter 3 of [7]. Examples in Lesniewski's own systems will emerge as these systems are discussed [4].

Lesniewski in fact provided extensive directives for the construction of a logical grammar of semantic categories (cf. chapter 7 of [14]). In terms of this grammar, new semantic categories may be added indefinitely, at any stage of the construction of a logical language. This must be done according to his directives (cf. § 3.2.3.). It is, however, seldom necessary in practice to go beyond the first few levels of the language's construction. At each of these levels, the language contains only a finite number of categories. This will in fact be the situation where reference is made to semantic categories in later chapters.

NOTES:

1) This rather apt term is due to Luschei.

2) Kilmister points out that this solution is not accepted by all mathematicians. Cf. also chapter 6.

3) The term "referent" in: "...referent of the expression", is used to mean: "that to which the expression refers."

4) It will frequently be necessary to refer to functors of the category "s/s" and "s/ss". The functors will, respectively, be termed: "proposition-forming functors of one propositional argument" and proposition-forming functors of two propositional arguments." (cf. § 2.4 (e)).

CHAPTER 3

PROTOTHETIC

3.1 Introductory Remarks

Protothetic may be roughly characterized as a bi-valued system of propositional logic. It forms the basis of Lesniewski's logic and is presupposed by both his ontology and mereology. It was, however, only after he had completed the construction of his ontology and mereology, that Lesniewski tackled the problem of developing a strong system of propositional logic to serve as a basis for the formalization of his systems. He expounded his theory of protothetic in the following papers: { 5 } , { 8 } and { 9 } .

Protothetic goes beyond even the so-called extended sentential calculus [1], and its characteristics are such that Lejewski (cf. [11]) concludes that it is: "... the most comprehensive logic of propositions ever constructed." This chapter will, however, only outline a few of the more important features of the system.

A few explanatory remarks are necessary before proceeding with the description of protothetic. These are:

(a) that in the description below, "S.C." will refer to a traditional system of propositional calculus such as that expounded in [23] or [1];

(b) that only bound variables occur in Lesniewski's theses (he regarded free variables as superflous);

(c) that because Lesniewski did not introduce the existential quantifier into this system, the symbol " ∃" will only be used below as an abbreviation for: " ⌉[....] ⌉".

3.2 Description

3.2.1 Enriching the Propositional Calculus

One way of formalizing metalogical statements (i.e. statements about a logical system) is the extension of the original system to accommodate the metalogical material. The S.C. can be enriched in this way by the addition of a new type of variable. The introduction of such a new type of variable in Lesniewski's protothetic, enriches it beyond the limits of the extended S.C.

As is the case in the extended S.C., a quantifier is introduced in protothetic. Lesniewski chose to introduce only the universal quantifier. It should also be noted that Lesniewski considered quantifiers as single units, in the sense that a formula in protothetic will have the form: $[x_1, x_2, \ldots\ldots, x_n]\ \emptyset\ (x_1, \ldots\ldots, x_n)$ and not the form: $[x_1]\ [x_2]\ \ldots\ldots\ [x_n]\ \emptyset\ (x_1, x_2, \ldots\ldots, x_n)$. Furthermore, there are no vacuous quantifiers in protothetic.

Lesniewski then goes further and introduces what have been termed "functorial variables" (cf. §2.4). These variables as their name suggests, stand for functors. Thus, for example, if "f" is such a variable and "p" a propositional variable, then: f(p) may represent any function of "p", such as: $\neg p, p \leftrightarrow q$ etc. In the simplest case, Lesniewski added functorial variables which may take as values only the constant (definite) functors: negation, falsum, verum and assertium. These are all proposition forming functors of one propositional argument. They will, respectively, be denoted by the following symbols: "$\neg$", "fl", "vr", "as". If "p" is a propositional variable, then the truth-values (cf. §2.4 (d)) of the functions formed from these functors and their arguments, are as depicted in table 1. In the table, the symbol "0" indicates falsehood and the symbol "1" indicates truth.

p	$\neg$p	fl(p)	vr(p)	as(p)
0	1	0	1	0
1	0	0	1	1

In view of the fact that the truth-value of each of the functions is constant with respect to the truth-value of the argument, the four functors are termed: "constant". Clearly, the different columns of the table exhaust all possible pairings of 0 and 1. It therefore follows that these four functors are the only constant functors of one propositional argument, which exist in a bi-valued system of logic. The addition, to the extended S.C., of functorial variables with their possible values restricted as described here, resulted in what we shall term: the "restricted system of protothetic".

Sobocinski points out in [21] that the restricted system of protothetic could be based on the following axiomatic assumptions:

(i) an axiom system of the complete propositional calculus with the implicator as the only primitive (undefined) functor; and

(ii) the additional axiom,

$$A1 : \ [p,f] \ \{ f([u] \ (u)) \rightarrow(_1 \ f([u] \ (u) \leftrightarrow [u] \ (u)) \rightarrow f(p))_1 \}$$

These assumptions require some explanation. An axiom system such as that described in (i) may be found in appendix 1 of [18]. In (ii) "p" and "u" are propositional variables, and "f" is a functorial variable with its possible values assumed to be restricted as described in the above paragraph. Now consider the proposition: " [u] (u)". It is, in the words of Prior, "doubtful whether the precise sense of this proposition can be expressed without using variables...." (pg. 91 of [18]). Roughly speaking, however, it asserts that "everything is true." Lesniewski's intention was that it should serve as the standard false proposition of his system. He also defines "0" to be equivalent to this standard false proposition (cf. § 3.2.2. and § 3.2.3), and thereby introduces "0" into his system as a constant (cf. § 2.4 (c)). Similarly, "[u] (u) $\leftrightarrow$ [u] (u)" is to be seen as the standard true proposition. Furthermore, "1" is defined as being equivalent to this standard true proposition. Thus, the truth-value of "0" is always <u>falsehood</u> and the truth-value of "1" is always <u>truth</u>. The axiom A1 may now be regarded as asserting that if the truth-value of "f(0)" is truth, and the truth-value of "f(1)" is truth, then the truth-value of "f(p)" is truth for all values assumed by "p" (i.e. for all propositions)[2].

Axiom A1 is termed: the principle of bivalency for propositions. Tarski, however, showed that it could be replaced in the axiom system by the so-called law of extensionality for propositions, viz.

AZ: $[p,q,f]\ \{(p \leftrightarrow q) \rightarrow (f(p) \rightarrow f(q))\}$

In AZ, "p" and "q" are propositional variables and "f" is a functorial variable. In future we shall continue to use symbols such as: "p", "q" and "r", for propositional variables and "f", "g" and "h", for functorial variables.

Different systems of protothetic may be obtained by enriching the restricted system outlined above. This enrichment can be accomplished by altering the restriction on the values the functorial variables may assume. Functorial variables may, for example, be allowed to represent functors of two or more propositional arguments. In Lesniewski's final system of protothetic, functors of any kind allowed by the theory of semantic categories, could be represented.

3.2.2. Coimplication as the only primitive term

Lesniewski wished to construct a system of protothetic in which the coimplicator was the only primitive functor. He, however, only embarked on this project once Tarski had shown it to be feasible.

Tarski in his article: "On the primitive term of logistic" (see [23]), poses the question whether it is "possible to construct a system of logistic in which the sign of equivalence is the only primitive sign". He proceeds to prove results which provide a positive answer to this question. The feasibility of Lesniewski's project was therefore demonstrated.

The system of protothetic with the coimplicator (sign of equivalence) as the only primitive term, was originally based on the following three axioms:

A3: $[p,q,r]\{(_1(p\leftrightarrow r)\leftrightarrow(q\leftrightarrow p))_1\leftrightarrow(r\leftrightarrow q)\}$

A4: $[p,q,r]\ \{(_1 p\leftrightarrow(q\leftrightarrow r))_1\leftrightarrow\ (_2(p\leftrightarrow q)\leftrightarrow r)_2\}$

A5: $[g,p]\ \{\ [f](_1 g(p,p)\leftrightarrow(_2[r]\ (_3\ f(r,r)\leftrightarrow g(p,p))_3\leftrightarrow[r](_4 f(r,r)$

$\leftrightarrow g((_5\ p\leftrightarrow[q]\ (q))_5,\ p))_4)_2)_1\leftrightarrow[q](_6\ g(q,p))_6\ \}$

It is worth noting that in A5, "f" and "g" represent functors of two propositional variables.

The above axiom system complies with Lesniewski's view that the axiom system of a deductive theory should involve as few as possible primitive [3) semantic categories. Indeed in the above axioms only two semantic categories are involved: the category of propositions (e.g. the propositions represented by "p", "r" and "r ↔ q") and the category of proposition forming functors of two propositional variables (e.g. the functors represented by "f" and " ↔"). It is furthermore clear, that the only primitive term occurring in the axiom system, is the coimplicator.

The use of the coimplicator as lone primitive term also makes it possible to state definitions as theses of the system. The vocabulary of protothetic is expanded in this way, without the use of an additional sign for definitional equivalence. The directives governing definitions will be discussed in §3.2.3. At this stage we may, however, mention a few examples. These are: the definition of "O" as " $[u]\ (u)\leftrightarrow O$" (cf. §3.2.1); and the following definitions of the four constant functors for one propositional argument:

(i) $[p]\ \{\ p\ \leftrightarrow as\ (p)\ \}$
(ii) $[p]\ \{(p\ \leftrightarrow o)\ \leftrightarrow\ \neg p\ \}$
(iii) $[p]\ \{(p\ \leftrightarrow p)\ \leftrightarrow\ vr(p)\ \}$
(iv) $[p]\ \{\ \neg(vr(p))\ \leftrightarrow\ fl(p)\ \}$

Lesniewski's method of definition (using the coimplicator) has not escaped criticism. Prior, for example, believes that it is "wrong-headed" and feels that a "grave objection to it is that it makes it difficult to distinguish between the definitions of a system and additional axioms" (chapter 4, § 4 of [18]). On the other hand, in constructing a deductive system based on only one primitive term, it is important to ensure that another undefined term does not creep into the system. The possibility of this happening is increased by using a term, distinct from the chosen primitive term, as a special term of definitional equivalence. Tarski points out that the choice of the coimplicator as a primitive term has the advantage that it makes it possible to guard against the above danger in a strict manner. Furthermore, he feels that it enables one to give the definitions "a form as natural as it is convenient, that is to say the form of equivalences" (article 1 of [23]). Clearly, Lesniewski's method of definition requires further explanation. Comment is therefore deferred until this has been dealt with in the next section.

3.2.3. Rules of Procedure

There is actually only one rule of procedure. It may, however, be divided into five sets of directives. Before discussing these directives, we state the rule in summarised form. Terms used in the rule will also, where necessary, be explained when the directives are discussed. The rule states that a formula [4] F may be added to a system of protothetic as a new proved thesis, if and only if it results from one and only one of the following five conditions being met:

1) F is a well-formed definition.
2) A thesis B from which F can be obtained by a distribution of its (B's) main quantifier, is in the system.
3) Theses B and C from which F can be obtained via modus ponens, are in the system.
4) A thesis B from which F can be obtained by substitution, is in the system.
5) F is a thesis of extensionality for members of semantic categories other than that of propositions. The semantic categories in question must, however, already have meaning in the system.

Each of the sets of directive implied by the above conditions merit individual consideration.

1) Directives for definition : In protothetic a definition will be "well-formed", as required by condition (1) above, if it is constructed according to the directives to be outlined here. Such a well-formed definition may then in terms of the rule be automatically asserted as a proved thesis of protothetic. Strong and precise directives for the construction of defintions are therefore provided, in order to guard against the possibility of defective definitions being included in the system and leading to contradictions. The preciseness of these directives would also seem to some extent to weaken the validity of Prior's "grave objection" (cf. § 3.2.2.).

The directives permit only the definition of simple constant terms (i.e. not quantifiers, etc). They may generally be formulated in the following way [5]:

A definition assumes the form $[a_1, a_2, \ldots, a_n]$ $(\emptyset \leftrightarrow \psi)$, where "$\emptyset$" is the definiens, "$\psi$" the definiendum, the coimplicator the main functor, and "$[a_1, a_2, \ldots, a_n]$" the main quantifier. The following conditions and restrictions are imposed on the definition's constituents:

(a) The main quantifier may in some cases (cf. (c) below) be omitted. A definition from which the main quantifier is absent, is termed an "absolute" protothetical definition. When, as above, the main quantifier is present, the definition is termed a "relative" protothetical definition. In the latter case, the variables occuring in the main quantifier are required to be representative of semantic categories which already have meaning in the system (i.e. categories which are primitive or categories of which at least one member has already been defined as a constant in the system). Each of these variables is furthermore required to occur in both $\emptyset$ and ψ.

(b) The definiens is a formula in which all variables are representative of semantic categories which already have meaning in the system. All constants occuring in the definiens must either have been previously defined or be the primitive functor (the coimplicator).

(c) In the case of absolute protothetical definitions, the definiendum is a new constant without arguments. In the case of relative protothetical definitions, however, the definiendum is a new constant which may have variable arguments [6]. Only variables which also belong to the main quantifier are, however, permitted. The definiendum may not contain quantifiers.

The following two examples (taken from [21]) may serve to illustrate compliance with the above directives:

(i) $[u]\ (u) \leftrightarrow O$

(ii) $[p,q,r,s]\{\ (_1\ p \wedge (_2\ q \wedge (_3 r \ \wedge s)_3\ )_2\ )_1 \leftrightarrow \wedge^1 (p,q,r,s)\}$

In (i), the new constant "O" is defined. Clearly it has no arguments. Hence as there is no main quantifier in (i), it is an example of an absolute definition. In (ii), we assume that " $\wedge$ " (conjunction) has already been defined. The main quantifier is present in (ii) and hence this example is a relative definition. The new constant being defined is: " $\wedge^1$". We may also note that as requied by condition (c) above, the variables representing its arguments all also occur in the main quantifier.

In view of the strong directives for definition, laid down by Lesniewski, Prior's concern (§3.2.2.) about distinguishing between definitions and other theses, does not seem to have much justification. Condition (c), for example, implies that a constant which is being newly defined, is always taken up in the definiendum. As the definiendum occurs on the right hand side of a definition's main functor, this provides one means of recognizing a definition.

The directives for definition facilitate the indefinite extension of the number of semantic categories occuring in a system of protothetic. For, suppose a new constant which belongs to a semantic category not yet represented in the system, is being defined. Then the definition of the new constant automatically provides a new semantic category with meaning in the system. In example (ii) above, the definition of " Λ^1" would add a category of proposition-forming functors of four propositional arguments.

2) Directives for the distribution of quantifiers: The remarks about quantifiers at the start of §3.2.1 should be borne in mind when considering these directives. Now, suppose that there is in the system a thesis B, having the form: $[a_1, a_2, \ldots\ldots, a_n](\emptyset \leftrightarrow \psi)$. Here " $\emptyset$ " and " ψ " are formulas of protothetic. Furthermore, the variables: a_i (i= 1,2,......, n), of the main quantifier each occur either in $\emptyset$ or in ψ or in both of these. The directives provide for a formula F to be asserted as a proved thesis of protothetic if it can be derived from B by the distribution of the main quantifier, or any part of the main quantifier, between the arguments of the main coimplicator. Each of the following formulas may be derived from B in this way:

$$[a_2, a_3, \ldots\ldots, a_n]\{[a_1](\emptyset) \leftrightarrow [a_1](\psi)\}$$

$$[a_1, a_3, a_4, \ldots, a_n]\{[a_2]\{(\emptyset) \leftrightarrow [a_2](\psi)\}$$

$$[a_1, a_2, a_4, a_5, \ldots, a_n]\{[a_3](\emptyset) \leftrightarrow [a_3](\psi)\}$$

etc.

$$[a_3, a_4, \ldots\ldots, a_n]\{[a_1, a_2](\emptyset) \leftrightarrow [a_1, a_2](\psi)\}$$

etc.

$$[a_4, a_5, \ldots\ldots, a_n]\{[a_1, a_2, a_3](\emptyset) \leftrightarrow [a_1, a_2, a_3](\psi)\}$$

etc.

$$[a_1, a_2, \ldots\ldots, a_n](\emptyset) \leftrightarrow [a_1, a_2, \ldots\ldots, a_n](\psi)$$

Clearly, if the main quantifier of B contains n variables (these are distinct), there are 2^{n-1} distinct formulas which may be derived in the above way. If F is any one of these formulas, it may, in terms of the directives, be asserted as a newly proved thesis of protothetic.

3) Modus Ponens (detachment): The directives for modus ponens in protothetic differ slightly from those usually found in the S.C. The directives may be formulated as follows: A new proved thesis F, having the form ψ, may be added to the system if there are in the system theses B and C which, respectively, have the forms $\emptyset \leftrightarrow \psi$ and $\emptyset$. The directives thus allow modus ponens only if the main functor of B is coimplication and not merely implication. Furthermore, they do not permit detachment under quantifiers.

4) Directives for substitution: The directives permit substitution for variables bound by an initial quantifier. The variables in the quantifier are adjusted accordingly. Examples are: from $[f, p]\{f(p) \leftrightarrow f(p)\}$ may be inferred $[p]\{\neg p \leftrightarrow \neg p\}$; and from $[f, p]\{f(p) \leftrightarrow f(_1 f(_2 f(p))_2)_1\}$ we may by substitution, according to the directives, conclude say: $[p]\{\neg p \leftrightarrow \neg(_1 \neg(_2 \neg p)_2)_1\}$.

Furthermore, if a substitution is to be made for a variable which occurs in the main quantifier of a thesis, only the following may be substituted:

(i) a constant which has already been defined in the system, and

(ii) a formula, satisfying the requirement that its constituents represent only members of semantic categories which already have meaning in the system.

As has been pointed out previously, a semantic category is said to have "meaning" in the system if it is primitive or if at least one of its members has been previously defined as a constant in the system. We will term a formula which meets the requirements of (ii), "meaningful".

5) Directives for extensionality: A thesis of extensionality may automatically be asserted as a proved thesis of protothetic. The extensionality thesis for propositions is:

E1: $[p,q] \{ (p \leftrightarrow q) \leftrightarrow [f](_1 f(p) \leftrightarrow f(q))_1 \}$

This thesis differs from the "law of extensionality" for propositions in §3.2.1. The reason is that in § 3.2.1, we were working in a system which had the implicator as sole primitive functor; whereas we are now working in a system which has the coimplicator as sole primitive functor. The thesis E1, above, may be independently derived in the system. For this reason the directives for extensionality do not include E1 or any extensionality thesis for propositions. The directives do, however, provide extensionality theses for members of all other semantic categories which already have meaning in the system. Although they are somewhat more complicated, these extensionality theses have a structure similar to the one for propositions.

We give one example of an extensionality thesis for members of a category of functors of n arguments. Let "f" and "g" be variables representing members of this category of functors of n arguments. Then we may state an appropriate extensionality thesis as follows:

E2: $[f,g] \{ [a_1,a_2,\ldots\ldots, a_n] (_1 f(a_1,a_2,\ldots\ldots, a_n) \leftrightarrow g(a_1,a_2,\ldots, a_n))_1 \leftrightarrow [\emptyset] (_2 \emptyset < f > \leftrightarrow \emptyset < g >)_2 \}$

where " $\emptyset$ " is representative of members of a category of proposition forming functors of one argument from the same category as the one of which "f" and "g" are representative.

3.2.4 Development of a single axiom for protothetic

The axioms of protothetic and ontology, together with the rule of procedure outlined in § 3.2.3., constitute Lesniewski's primitive logical basis. Lesniewski sought to reduce the number and length of axioms in this basis (cf. § 2.1). This problem, as it pertains to protothetic, is considered here.

In simplifying the axiom system, it is necessary to have a means of ensuring that the simplified system remains a complete axiom system of protothetic. This was achieved by checking that the simplified system met the conditions of a metatheorem which asserts completeness provided its conditions are met. The original metatheorem implied by Lesniewski in his work, was later found by Sobocinski (cf. [21]) to be replaceable by a metatheorem with weaker conditions. Sobocinski's metatheorem is formulated explicitly as follows:

An axiom system of protothetic, having rules of procedure inferentially equivalent to that described in §3.2.3., constitutes a complete system if in its field the following conditions are satisfied:

1) The two theses
   $[u]\ (u) \leftrightarrow [u]\ (u)$ and
   $[p,q]\ \{ p \leftrightarrow (_1(q \leftrightarrow p) \leftrightarrow q)_1 \}$
   are provable.

2) The principle of bi-valency for propositions is provable as a metarule stating that if a formula "$\emptyset$ (p)" has a meaning in the system and the formulas " $\emptyset$ ([u](u))" and " $\emptyset$([u] (u) $\leftrightarrow$ [u] (u))" have already been proved, then the formula "[p] ($\emptyset$ (p))" is also a thesis of the system.

We note that the first of the theses of condition (1) is explained in § 3.2.1. The principle of bi-valency for propositions is also stated in §3.2.1 (axiom A1).

By 1923 it was possible to base protothetic on a single axiom. This first single axiom for protothetic could replace the three axioms (A3 - A5) given in § 3.2.2. The new axiom consisted of 290 signs in genuine Lesniewskian symbolism. The problem of reducing its length was to prove difficult. The shortest single axiom Lesniewski was able to construct for protothetic before his death, had a length of 62 signs in his symbolism. It may be transcribed as follows:

$$[p,q] \{ (p \leftrightarrow q) \leftrightarrow [f] ({}_1 f(q, f({}_2\, q, [u]\,(u))_2) \leftrightarrow [f] ({}_3 f(p,r) \leftrightarrow ({}_4 p \leftrightarrow ({}_5\, q \leftrightarrow (r \leftrightarrow p))_5)_4)_3)_1 \}$$

In this axiom "f" represents proposition-forming functors of two propositional arguments.

A shorter single axiom for protothetic was, however, established by Sobocinski in 1945. This axiom is made up of 54 signs in Lesniewskian symbolism. It may be transcribed as follows:

$$[p,q] \{ (p \leftrightarrow q) \leftrightarrow [f] ({}_1 f(p, f({}_2\, p, [u]\,(u)\,)_2) \leftrightarrow [r] ({}_3 f(q,r) \leftrightarrow ({}_4 q \leftrightarrow p)_4)_3)_1 \}$$

This axiom of Sobocinski, together with the rule of procedure described in 3.2.3, constitutes an adequate basis for protothetic. It also displays the economy desired by Lesniewski.

3.3 Comment

The most controversial evaluation of protothetic is probably contained in [8] . Grzegorczyk's comments in this article have been criticized by both Sobocinski and Luschei. Sobocinski's comment (cf. [21]) on Grzegorczyk's evaluation is that "it seems that several of his remarks are too hastily formulated". This comment seems to be justified in view of the examples which follow.

Grzegorczyk argues that: "....the deduction of theorems in protothetic is not interesting for the mathematician. We know beforehand what these theorems may tell us. It is worth while to know that protothetic exists, but in practical deduction we can make use of less elaborate systems." Luschei responds (cf. § 6.5 of [14]) to this argument by drawing an anology with the situation of using only "Basic English" for our communicative needs. A more important point mentioned by Luschei, is that if a simpler system without functorial variables were to be used, Sobocinski's single axiom for protothetic would have to be written as a conjunction of 128 tautologies. Grzegorczyk's rather subjective comment regarding the interest theorems in protothetic have for mathematicians, is also countered by Luschei. Luschei points to an example in Grzegorczyk's own paper which shows that the contention about "beforehand" knowledge on which the comment rests, is false.

In evaluating the above remarks, a matter which should not be overlooked, is the relationship of protothetic to Lesniewski's other systems. It would seem unwise to relegate protothetics to the ranks of the unnecessary, as Grzegorczyk has done, without considering whether any other "less elaborate" deductive system would constitute an adequate and equally effective basis for ontology and mereology.

A further contentious comment of Grzegorczyk is that it is difficult to prove the completeness of protothetic and that "according to some logicians, it is even impossible to prove." A completeness proof for protothetic is, however, facilitated by the metatheorems of Lesniewski and Sobocinski, mentioned in § 3.2.4. A proof of completeness may also be found in an article by Slupecki in "Studia Logica", vol. 1, 1953.

It is worth noting, in conclusion, that protothetic may be shown to be consistent relative to the complete classical propositional logic. This follows as it may be proved that every thesis of protothetic is translatable into a classical tautology. Taking matters a step further, ontology and mereology may again be proved consistent relative to their basis, protothetic.

In this brief overview of protothetic we have only scratched the surface of an extensive theory. The aspects touched on here, do, however, illustrate something of the nature of this basis of Lesniewski's logic.

<u>NOTES</u>:

1) A description of the extended S.C. can be found in article lV of [23] .

2) Cf. the familiar tautology of S.C. (lemma 1.8, chapter 2 of [1]): $\neg \emptyset \rightarrow (\emptyset \rightarrow \psi)$ where "$\emptyset$" and "ψ" are formulas of S.C.

3) A semantic category is said to be primitive if its members are represented by some constituent of an axiom.

4) No explicit definition of a formula of protothetic could be found in the references consulted. For a system of protothetic with coimplication primitive, the following definition seems, however, to correspond with the sense in which the term is used. We let a "string" be a finite set of symbols of the language of protothetic. Then:

   (a) A string of the form" p or $f(p_1, p_2, \ldots, p_n)$ where "p" and "$p_1, p_2, \ldots, p_n$" are propositional variables and "f" is a functorial variable representing a functor of n arguments, is a formula.

   (b) If $\emptyset$ and ψ are formulas, then so are $\emptyset \leftrightarrow \psi$ and $[x]\emptyset$, where "x" is a propositional or functorial variable.

   (c) A string is a formula if and only if it follows from a finite number of applications of (a) and (b).

   The above recursive definition was constructed in a manner anologous to the definitions of formulas of S.C. and P.C. in [1].

5) Lesniewski's directives for definition are very much more comprehensive and extensive than the somewhat superficial outline we provide.

6) This case may give rise to the so-called "mult-link" functors. In addition to their other roles in protothetic, multi-link functors prevent ambiguities of grouping. They, however, present a complication which we have chosen not to introduce in our outline of protothetic. For details see §7.1 of [14] and pp. 58 - 59 of [21].

CHAPTER 4

ONTOLOGY

4.1 Introductory remarks

Lesniewski's mereology presupposed a logic of names. Lesniewski therefore in 1920 constructed an axiomatic foundation for such a logic of names (cf. § 4.2). In this way the theory he later called: "ontology", came into being. As mentioned in chapter 1, we use the term "ontology" to refer specifically to Lesniewski's ontology. Lejewski, in considering the subject matter of ontology, describes it as "a theory of what there is". From this description also emerges the aptness of the name "ontology" for this theory. Indeed, the theory's interpretation was intended to be ontological (in the philosophical sense) and the theory contains concepts of existence and identity.

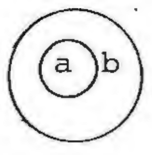
Opinions expressed about ontology differ greatly. The authors of [7] discuss ontology as a "less spectacular attempt" at rectifying shortcomings of traditional set theory by rewriting the underlying logic. On the other hand, Luschei considers the system of logic formed by ontology in conjunction with protothetic to be "...comparable in scope and power to 'Principia Mathematica' as a foundation for classical mathematics..." (Pg. 28 of [14]). Furthermore, Lejewski is of the opinion that ontology is the "most comprehensive logic of names" (cf. [11]).

Formally, ontology may be obtained from protothetic by the addition of one axiom and a few further directives. The new axiom furthermore introduces only one new primitive (undefined) term. A comprehensive exposition of Lesniewski's directives for the construction of ontology can be found in chapter 7 of [14] . Lesniewski's own articles on the subject include: {3}, {4}, {6} and § 11, vol. XXXIV of {2} . The most important of these articles is {2} . The description in §4.2 will largely follow Lejewski (cf. [12]), by introducing the ontological vocabulary with the aid of an appropriate table of diagrams. The axioms and other theses of the system will be considered after this.

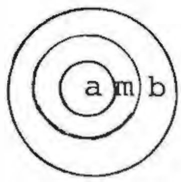
4.2 Description

4.2.1 The Ontological Vocabulary

Members of the semantic category of names play an important part in ontology. Lejewski makes use of an "ontological table" to illustrate what he terms the "semantical status" of some of these names and of combinations of names. By introducing the vocabulary of ontology in this way, it is possible to avoid problems of interpretation which could result from a translation into English of the original descriptions. Furthermore this method follows a successful precedent: Euler's use of circles in explaining the logic of Aristotle (Syllogistic). Euler's circles (cf. chapter 5 of [13]) would, for example, interpret (i) an expression of the form: "all a is b"; and (ii) the syllogism: "if all m is b and all a is m, then all a is b", as:



(i)



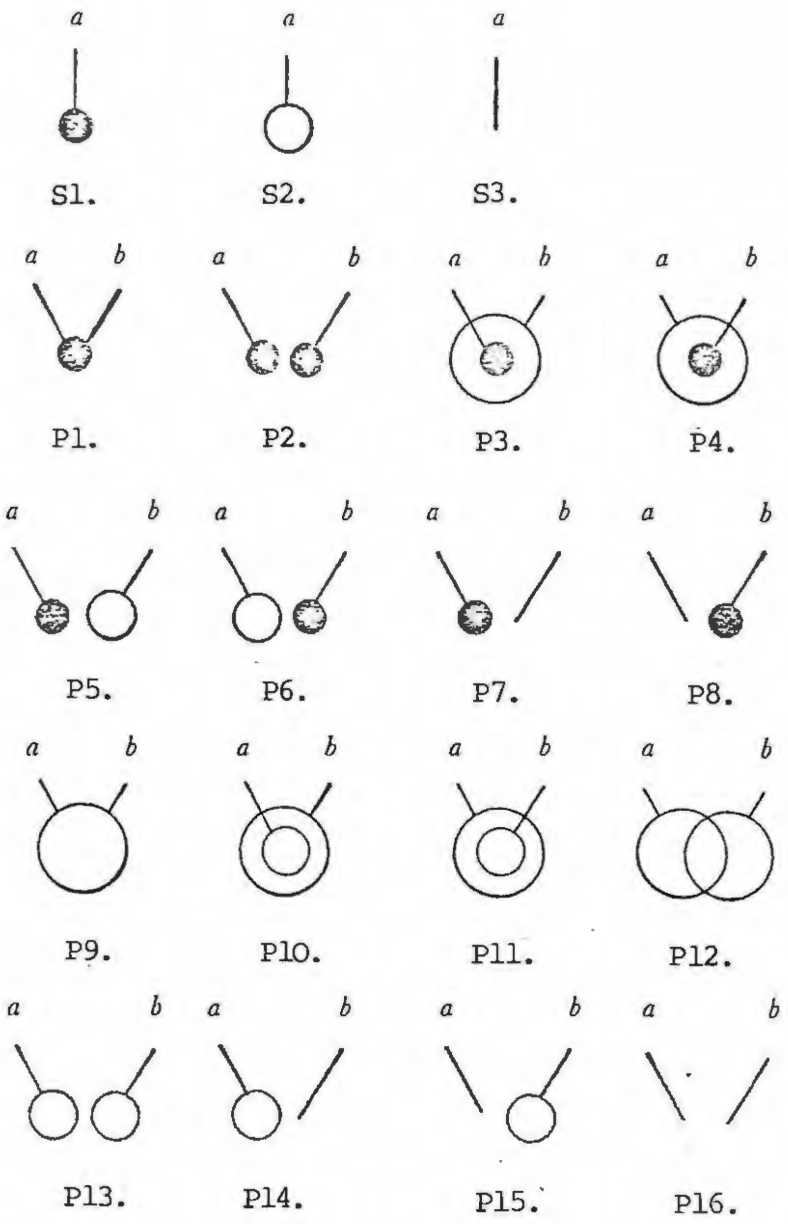
(ii)

We note at this stage that we will use symbols such as: "a", "b", "c", for variables representing names.

The use of Lejewski's ontological table to depict the semantical status of names, constitutes a considerable extension of Euler's ideas. Lejewski's ontological table for single names and pairs of names is depicted below. Diagrams for combinations of three or more names may be developed in a similar way.

In the table diagrams S1 - S3 indicate the semantical status of single names. Diagram S1 depicts what Woodger termed "unshared" names. These are names which refer to only one specific object in a particular context - e.g. (continued overleaf, below diagram)

Ontological Table



"Aristotle", "the point (1,1)", "Gunfire Hill". Diagram S2 depicts the so-called "shared" names. There are names which refer to (are "shared" by) more than one object - e.g. "man", "student", "house". The introduction by Lesniewski of the group of names depicted by diagram S3, has wider implications. These will be discussed at the end of § 4.2.2. Diagram S3 depicts the so-called "fictitious" names. These are names of objects which do not exist in reality. They fall into the category of names on the basis of their behaviour, despite referring to no real existing object. Examples are: "unicorn", "goblin", "Superman". An especially important fictitious name (cf. § 4.2.2.) used by Lesniewski is: "object which does not exist."

Diagrams P1 - P16 depict the semantical status of pairs of names. We give an example for each case.

Number of diagram	Name represented by "a"	Name represented by "b"
P1	Julius	Caesar
P2	John	Jane
P3	John	man
P4	house	Groote Schuur
P5	Rover (name of dog)	vegetable
P6	vegetable	Napoleon
P7	Politician (name of a horse)	unicorn
P8	centaur	Napoleon
P9	automobile	motor-car
P10	dog	animal
P11	fruit	apple
P12	man	student
P13	vegetable	animal
P14	man	minotaur
P15	Superman	American
P16	goblin	unicorn

The ontological table will be put to further use as an aid to introducing a number of proposition-forming functors of one or two nominal arguments. By "nominal" arguments, we mean arguments which are names. We will, however, first introduce the only primitive (undefined) functor of ontology in a formal manner. Other functors which form part of the ontological vocabulary will be discussed in § 4.2.3.

4.2.2 Formal additions; 'Existence' concepts and Fictitious Names

The single axiom subjoined to protothetic also introduces the sole primitive term, " ε ", into the system. The only additional directives are a number of additional extensionality theses and directives for ontological definition. The protothetical directives are also adapted to ontology.

Lesniewski originally based ontology on the following single axiom:

$$\text{B1: } [a,b]\ \{(a \varepsilon b) \leftrightarrow (_1(\exists c)(c \varepsilon a) \wedge [c](_2(c \varepsilon a) \rightarrow (c \varepsilon b))_2 \wedge [c,d]$$

$$(_3(_4(c \varepsilon a) \wedge (d \varepsilon a))_4 \rightarrow (c \varepsilon d))_3)_1\}$$

It should be noted that in the above axiom "$(\exists c)$" is used merely as an abbreviation for "$\neg([c]\neg \ldots .)$". Before discussing the axiom, we consider the primitive functor represented by the symbol "ε". This functor is called the functor of singular inclusion and is the only primitive functor in Lesniewski's original system of ontology. It is provided with meaning by the above axiom and corresponds in natural language to the Polish copula: "jest". Its "meaning" is also approximated by the Latin copula: "est". The English "is", however, can be used in so many ways that ambiguities may creep into an English translation. This is due in part to the fact that the articles: "a" and "the", are obligatory in English copulative constructions. If, despite this problem, we approximate a rendition of " " in English as the copula, "is", or as this copula plus an article, then the expression: "$b \varepsilon c$", may be rendered as: "b is c" or "b is a c".

The ontological table is used to determine the truth-value of a proposition of the type represented by: "$a \varepsilon b$". Such a proposition will be regarded as being true, if and only if the semantical status of the pair of names represented by "a" and "b" is depicted by diagram P1 or P3. This implies that the proposition is true, if and only if "a" represents an unshared name, and the specific single object this name has as referent, is also named by the name which "b" represents. We list a few examples from natural languages.

a	ε	b	diagram	truth-value
Iulius	est	homo	P3	truth
Iulius	est	Caesar	P1	truth
Canis	est	animal	P10	falsehood
Superman	is an	American	P15	falsehood
One	is an	integer	P3	truth
. . .	. . . . .	. . . . . .	.	

We now return to the axiom B1. In terms of the axiom the assertion: "a is a b" ("a ε b"), is true if and only if:

(i) something is an "a"
(ii) every "a" is a "b"
(iii) for every "c" and "d", if the "c" is an "a" and the "d" is an "a", then the "c" is a "d".

Condition (iii) can clearly only be met if there is only one "a". This requirement, together with condition (i), implies that "a" represents an unshared name. Condition (ii) implies that the referent of the name represented by "a", should also be named by the name represented by "b". The axiom therefore also justifies the interpretation we gave "a ε b" in terms of the ontological table.

Lesniewski was successful in achieving his aim of shortening the single axiom of ontology. He was eventually (1929) able to show that the following shorter axiom could serve as the single axiom of ontology:

B2: $[a,b] \{ (a \varepsilon b) \leftrightarrow (\exists c) (_1 (a \varepsilon c) \wedge (c \varepsilon b))_1 \}$

Once again, "$(\exists c)$" is used to abbreviate "$\neg([c] \neg .....)$". We will continue to use this abbreviation.

We now turn to the directives for ontological definition [1). The purpose of ontological definition is the introduction of special constant names (cf. § 2.4 (c)) or nominal functions (i.e. functions made up of name-forming functors and their arguments). If the term to be introduced is "τ ", an ontological definition will give an equivalent for "$a\varepsilon\tau$ ". An ontological definition meeting the conditions outlined below, may, in terms of the directives, immediately be asserted as a new, proved thesis of ontology.

An ontological definition assumes the form:

$$[a, \ldots\ldots] \{ (a\varepsilon\tau) \leftrightarrow (_1(a\varepsilon a) \wedge \emptyset(a))_1 \}$$

In the definition "τ " either represents a name which is being defined as a constant, or it represents a nominal function. In the first case the name (constant) represented by "τ " is required not to occur in any previous thesis. In the second case, the function is required to consist of a functor which is a constant term not occuring in any previous thesis, and of arguments. These arguments must be representative of semantic categories which have already been introduced in previous theses. In the definition, no variable may occur more than once in "$a\varepsilon\tau$ ". Furthermore, "$\emptyset$ (a)" represents a propositional expression in which every constant belongs to protothetic or occurs in at least one previous thesis of ontology. Variables occuring in " $\emptyset$(a)" represent members only of semantic categories which already have meaning in the system. Detailed conditions which an ontological definition is required to meet, may be found on page 272 of [14] . It may be noted that, whereas in the case of protothetical definition the definiendum occurred on the right hand side of the main coimplicator, we have in the case of ontolgical definition adopted the convention that it occur on the left hand side. 2)

The examples which follow, may serve to illustrate ontological definition. They are also of particular significance in an investigation of Lesniewski's approach to notions of "existence".

E1: $[a] \{ (a\varepsilon V) \leftrightarrow (a\varepsilon a) \}$

E2: $[a] \{ (a\varepsilon\Delta) \leftrightarrow (_1(a\varepsilon a) \wedge \neg(a\varepsilon a))_1 \}$

E3: $[a,b] \{ (a\varepsilon N(b)) \leftrightarrow (_1(a\varepsilon a) \wedge \neg(a\varepsilon b))_1 \}$

In E1 and E2, constant names, represented by the constant terms "V" and "Δ", are being defined. We may interpret "$a \varepsilon V$" and "$a \varepsilon \Delta$" as respectively saying: "a is an object" and "a is an object which does not exist". In E3, "N(b)" is a nominal function, with the functor "N" a constant. Here, "$a \varepsilon N(b)$" may be redered as: "a is non-b".

Before considering the way in which Lesniewski deals with "existence", we outline two other approaches. Leśniewski's approach may then be compared with these. The linguist, John Lyons, points out that "to say that a particular word...'refers to an object' implies that its referent is an object which 'exists' (is 'real')....." (pg. 425 of [15]). When saying an object "exists", he means that it should be possible to describe the object's physical properties. He takes the notion of "physical existence" to be the fundamental "meaning" of the term: "existence". He, however, considers it possible to extend the use of the term: "existence". The example, Lyons gives, of such an extension, is useful for our purposes. He states: ".... although there are no such objects (we will assume) as goblins, unicorns or centaurs, we can quite reasonably ascribe to them a fictional or mythical 'existence' in a certain kind of discourse...."

Bertrand Russell explains "existence" in terms of the so-called "theory of descriptions" (pg. 303 of [6]). By "description" he means the designation of a person or object by "some property which is supposed or known to be peculiar to him or it". In the statement: "Scott was the author of *Waverley* ", such a description is constituted by "the author of *Waverley*". In terms of the theory, we may only assert that what is referred to in such a description (the "peculiar property") exists. Hence, we may assert: "The author of *Waverley* exists". On the other hand, Russell points out that "to say 'Scott exists' is bad grammar...". It is "bad grammar" as in Russell's symbolic logic, "existence" does not occur as a predicate. To express the idea that "Scott exists", we would in Russell's logic have to write something like:

$$(\exists x)\ (S(x) \wedge H(x))$$

where S is the property of having the name Scott and H is the property of being human. If we wanted to refer to the "existence" of the Scott who wrote *Waverley*, in particular, we might write:

$(\exists x)\ (S(x) \wedge H(x) \wedge W(x))$

where W is the property of being the author of *Waverley*.

Let us now consider the position in Lesniewski's ontology. Lesniewski introduces a number of so-called "functors of existence" into ontology. These functors all have names as arguments and form propositions which assert the "existence" of these names. One of these functors is needed for our present discussion. This functor is represented by " ex " and is introduced into the system by the following protothetical definition:

$[a]\ \{\neg(_1\ [b]\ \neg(b\,\varepsilon\, a))_1 \leftrightarrow\ ex(a)\ \}$

We note that " ex (a)" may be rendered as "a exists". The definition therefore says that for all "a", it is not the case that all "b's" are not "a", if and only if "a exists". The assertion " ex (a)" is considered to be true if and only if the semantical status of the name represented by "a" is depicted by diagram S1 or S2 of the ontological table. We will introduce other "functors of existence" in § 4.2.3.

In Lesniewski's ontology, the introduction of the constant " V ", makes it possible to say "a εV", i.e. to say "a is an object". Saying this, however, implies that the referent of the name represented by "a", exists. This corresponds to the view of Lyons that if we say a "word" (therefore also a name) refers to an object, we in fact assert that the referent of the word (name) exists. In terms of the ontological definition E1, we may, however, only say "a is an object" if "a is a". It is possible to show in ontology that the requirement that "a be a" is equivalent to a requirement that "ex (a)" be true (cf. pp 295 - 296 of [18]). The assertion "a is a" is therefore true, if and only if the name represented by "a", is a shared or an unshared name. It is not true if the name represented by "a" is a fictitious name. Thus, as the name "Scott" is not a fictitious name, we may assert that "Scott is an object". This assertion implies that the man referred to by the name "Scott" exists. Hence, we may conclude that "Scott (the man) exists". Lesniewski, therefore seems to allow "existence" to be used as a "predicate" in the linguistic sense, e.g. in the statement "Scott is an object", the predicate is: "is an object".

Lesniewski's view of objects which do not exist also seems to come close to Lyons' view that they may be assigned a "fictional existence". Indeed, Lesniewski admits fictitious names (cf. §4.2.1) to his semantic category of names, and in a sense provides them with ontological status by defining such a fictitious name as a constant. Consider, in this regard, the definition E2 of the "object which does not exist". This does not, however, mean that if we wish to express the non-existence of, say, a "unicorn" or the "King of South Africa", we may assert: "The unicorn (King of South Africa) is an object which does not exist." To discover the reason for this, an examination of the definition of the "object which does not exist", is necessary. We will henceforth also refer to the "object which does not exist" as a "non-existent object."

In terms of definition E2, it is true to say: "a is a non-existent object", if and only if two conditions are met:

(i) "a is a"; i.e. "ex(a)" is true; and
(ii) "it is not the case that a is a".

It was pointed out earlier that the first of these conditions can be met if and only if the semantical status of the name represented by "a" is depicted by diagram S1 or S2 of the ontological table. Therefore if "a" represents a fictitious name (the semantical status of such names is depicted by S3), condition (i) is not met. Thus, it is not true to say: "a is a non-existent object", when "a" represents a fictitious name (e.g. "King of South Africa"). Hence, it is false to say the referent of a fictitious name does not exist. Condition (i) above is also the condition which has to be met for it to be true to say that the referent of the name represented by "a" exists. Therefore it is also false to say that the referent of a fictitious name exists. The fact that we may assert neither the "existence", nor the "non-existence" of the referent of a fictitious name, coincides with the idea that the referents of fictitious names have only a "fictional existence". The second condition will not be met if "a" represents a shared or an unshared name (i.e. a name which is not fictitious). This ensures that if the referent of the name represented by "a" is "real" (i.e. "exists"), we cannot assert that "a is a non-existent object". Together, the two conditions imply that it is false to assert of any kind of name that it is a "non-existent object".

The position of fictitious names in Lesniewski"s logic may be summarized by noting that if "a" represents a fictitious name, it will always be false to say "$a\varepsilon b$", no matter what name "b" represents. In this regard, we may in conclusion comment on the function defined in E3. In terms of E3, it is true to assert that "a is non-b" if and only if "it is not the case that a is b" and also "a is a". The latter requirement clealy implies that if "a" represents a fictitious name, then it is always false to assert "$a\varepsilon N(b)$". Thus to say "a is non-b", differs from saying "a is not b" ("$\neg(a\varepsilon b)$"), as the latter statement is always true if "a" represents a fictitious name. Thus, the statement: "unicorn is non-man", is false, but the statement: "it is not the case that unicorn is man", is true.

4.2.3 More Functors

Additional functors may be introduced by means of protothetical definitions in terms of the primitive functor ("ε"). Examples of four types will be considered (cf. also [12] and appendix 1 of [18]). We use "$(\exists c)$" solely as an abbreviation for "$\neg([c]\neg\ldots)$".

1) Functors of inclusion:

   (a) The functor of strong inclusion: This functor may be represented by the symbol "$\sqsubset$" and is introduced by the thesis:
   $[a,b]\{(_1(\exists c)(c\varepsilon a) \wedge [c](_2(c\varepsilon a) \rightarrow (c\varepsilon b))_2)_1 \leftrightarrow (a\sqsubset b)\}$

   The expression "$a\sqsubset b$" may be interpreted as "every a is b". A proposition of this type will be considered true if and only if the semantical status of the names represented by "a" and "b" is depicted by one of the diagrams: P1, P3, P9 or P10, on the ontological table.

(b) The functor of weak inclusion: The symbol "U" will be used for this functor. It is defined by the following thesis:

$$[a,b]\{[c]\,(_1(c\varepsilon a) \rightarrow (c\varepsilon b))_1 \;\leftrightarrow\; (aUb)\}$$

In English the assertion "aUb" may be rendered as "all a is b". It is considered true if and only if the referents of "a" and "b" have their semantical status depicted by one of the diagrams: P1, P3, P8, P9, P10, P15 or P16. Notice that the distinction between this functor and functor (a) depends on the semantical status of "a".

2) Functors of exclusion:

The counterparts of the primitive functor and the above example are considered.

(c) The functor of singular exclusion: This functor may be represented by the symbol "$\not\varepsilon$". It occurs in expressions of the type "$a\not\varepsilon b$" ("a is not b") and may be introduced as follows:

$$[a,b]\{(_1(a\varepsilon a) \wedge \neg(a\varepsilon b))_1 \leftrightarrow (a\not\varepsilon b)\}$$

"$a\not\varepsilon b$" is regarded as true if and only if the semantical status of the referents of "a" and "b" is depicted by one of diagrams: P2, P5 or P7.

(d) The functor of strong exclusion: The introductory thesis is:

$$[a,b]\{(_1(\exists c)\,(c\varepsilon a)\wedge\;[c]\,(_2(c\varepsilon a) \rightarrow \neg(c\varepsilon b))_2)_1 \;\leftrightarrow\; (a\not\subset b)\}$$

The assertion "$a \not\subset b$" may be rendered as "every a is not b" in English and is considered true if the semantical status of the referents of "a" and "b" is depicted by one of diagrams P2, P5, P6, P7, P13 or P14.

(e) The functor of strong exclusion ("$\not\triangle$"): Here the definition is:

$$[a,b]\{[c](_1(c\varepsilon a) \rightarrow \neg(c\varepsilon b))_1 \leftrightarrow (a\not\triangle b)\}$$

The expression "a$\not\triangle$b" is read "no a is b" and is considered true if and only if one of diagrams P2, P5, P6, P7, P8, P13, P14, P15 or P16 illustrates the semantical status of the names represented by "a" and "b".

3) Functors of identity and existence:

One example of a functor of identity and two examples of functors of existence will be provided (cf. also the functor of existence introduced in §4.2.2.)

(f) The functor of singular identity: The symbol "I" will be used for this functor, which may be introduced by the thesis:

$$[a,b]\{(_1(a\varepsilon b)\wedge(b\varepsilon a))_1 \leftrightarrow (aIb)\}$$

The expression "aIb" may be read as "the a is the same object as the b" and is considered true if and only if the semantical status of the names represented by "a" and "b" is depicted by diagram P1.

(g) The functors of existence: The two functors of existence which we have not yet introduced, occur in the functions:

"sol(a)" - "there exists at most one a"
"ob(a)" - "there exists exactly one a"

These functors may, respectively, be defined as follows:

$$[a]\ \{[b,c](_1(_2(b\varepsilon a)\ \wedge\ (c\varepsilon a))_2 \rightarrow (b\varepsilon c))_1 \leftrightarrow \mathrm{sol}(a)\}$$

$$[a]\{\neg(_1[b]\ \neg\ (a\varepsilon b))_1 \leftrightarrow \mathrm{ob}(a)\}$$

The assertion "sol(a)" will be regarded as true if and only if the semantical status of the name represented by "a" is illustrated by diagram S1 or S2; while "ob(a)" will be regarded as true if and only if diagram S1 depicts the semantic status of "a".

4.3 Comment

Grzegorczyk comments on ontology from a historicist point of view. He praises Lesniewski's formulation of directives for the construction of ontology as "an achievement of historical value". He is, however, of the opinion that the history of mathematics took a different course since the time of ontology's development. As a result, he regards ontology as having lost virtually all its value. It would seem, however, that ontolgoy is a theory with much more versatility than that with which Grzegorczyk credits it. Luschei, for example, points out that it is possible to construct within ontology much of arithmetic, number theory, analysis, abstract algebra and other theories of classical mathematics (cf. §6.2.3. of [14]). In addition it would seem possible to adapt parts of this "theory of names" for use in linguistics.

In comparing ontology to set theory, the authors of [17] conclude that ontology should be seen as a rival for, rather than a variant of set theory. They base this conclusion on the fact that "ε", the primitive term of ontology, does not represent class-membership. Despite this conclusion,they do point out that certain theorems of set theory may be translated into ontology, and vice versa. They are unable to decide just how important a rival for set theory ontology is. Their overall impression of ontology is, however, rather negative. In spite of this, the authors do agree that "Lesniewski has convincingly shown that the standard arguments leading to logical antimonies cannot be reproduced, in any of the various plausible rephrasings, in his system; some of the counterparts of these arguments fail to comply with the theory of semantic categories, others require certain steps which are not viable in ontology". Thus as is the case in their mereological interpretation (we discuss this in chapter 6), the problems presented in other systems by these paradoxes, are successfully removed in their ontological interpretation.

NOTES:

1) The directives we outline are for a system of ontology based on B1 as single axiom.
2) We adopt this convention as in virtually all the references the definiendum of ontological definitions occurs on the left hand side of the main coimplicator. Exceptions are references describing Lesniewski's early work on mereology. Here, name-forming functors of mereology are introduced by definitions (of the ontological type) in which the definiendum occurs on the right hand side of the main coimplicator and the definiens on the left hand side (cf. [10] and [22]). See also § 5.2.2.

CHAPTER 5

MEREOLOGY

5.1 Introductory Remarks

The first deductive theory to be developed by Lesniewski was mereology. He started developing the theory in 1914. In 1916, he published his first outline of mereology in {1}. Further details appeared in {2}. Logically and mathematically, mereology presupposes the basis protothetic and ontology later created for it. Mereology may be described as an "extremely general extralogical theory" based on protothetic and ontology.

As pointed out in chapter 1, Lesniewski's in depth analysis of Russell's paradox sparked off the development of mereology. It is therefore not surprising that in mereology the concept : "the ingredient of the whole" (treated as a relation between individuals), is central. Indeed to obtain mereology from its basis of protothetic and ontology, the concepts : "ingredient of" and "class of" need to be introduced. Normally the former is the sole primitive term of the system and the latter is introduced by definition. The interpretation of these concepts is a key concern both in mereology and the description below.

5.2 Description

5.2.1 Distributive versus Collective Interpretation of Class Expressions

A careful differentiation between "collective" and "distributive" interpretations of class expressions is a fundamental requirement in an investigation of aspects of mereology - a theory in which the collective use of such expressions is formalized. A confusion of these distributive and collective conceptions was indeed considered by Lesniewski to be characteristic of certain naive set theories and moreover to be one of the characteristics instrumental in allowing logical paradoxes such as that of Russell to arise. Nagel and Newman express a similar opinion when in commenting on Russell's paradox, they remark :
"This fatal contradiction results from an uncritical use of the apparently pellucid notion of class" (pg. 24 of [17]).

Consider therefore the expression :

"a is an ingredient of the class of b's".
We proceed to distinguish between its distributive and collective interpretations.

1) Distributive interpretation : Interpreted in this way the expression implies merely that "a" is a "b". It thus serves only to "predicate a definitive property of any member element of the corresponding distributive class" (Lesniewski as cited by Luschei in §3.3.2 of [14]). This end may be accomplished in ontology which embodies the distributive use of the copula ("ε") for singular predication (cf. the interpretation of "a ε b").

2) Collective interpretation : Here the expression implies that "a" is part of the object of which every "b" is a part, and of which every part has a part in common with some "b". Lesniewski's term "ingredient" seems particularly appropriate to describe this relationship of a part to a whole which is a specific individual literally consisting of its constituent (collective) parts (its ingredients). The notion of "the class of b's" is therefore also introduced in mereology as a totality which is an individual literally made up of the "b's".

The examples which follow should further clarify the concepts.

i) The distributive class of the provinces of South Africa has only the individual provinces, i.e. Cape, O.F.S., Natal and Transvaal, as members. To say the Cape is an ingredient of this set is thus also equivalent to saying : "the Cape is a province of South Africa". The collective class of the provinces of South Africa, however, in addition has as ingredients any arbitrary part or collection of arbitrary parts of these provinces (e.g. Grahamstown, the Karoo, the export harbours, national parks) and the totality of provinces (South Africa itself). The latter ingredient is the same individual as the collective class itself, as are the totality of all parts of provinces, etc.

ii) A collection such as a team of rugby players may be interpreted as a distributive class in which case its only ingredients will be each of the 15 individual players. Interpreted as a collective class, however, ingredients would also include, for example, the back-line, the forwards, the tight forwards, the halves and the whole team itself.

Additional characteristics of collective classes need to be made specific here. Any collective class of individuals (its ingredients) is literally made up of these and is itself an individual. Thus, it follows that the individual which is the collective class of its ingredients, ceases to exist in the absence of these ingredients. A common example is that of a forest vanishing with its trees. In <u>this</u> sense, it is therefore meaningless to speak of an "empty" collective class.[1)] Furthermore, as any individual is the same individual as the totality (collective class) of itself and every individual is an ingredient of itself, there is no class which is not an ingredient of itself. Hence, in mereology there is no such object as the class of classes which are not ingredients of themselves (cf. chapter 6).

The notions introduced above will be illustrated formally in §5.2.2.

5.2.2 <u>Formalization</u>

Formally, mereology may be built onto its foundation of protothetic and ontology by subjoining the mereological axiom system and adapting the ontological and protothetical directives to it. No further directives are necessary.

Axiom systems for mereology may be based on a variety of primitive functors (e.g. "ingredient of" and "class of"). If, as is most common, the functor: "ingredient of", is taken as the sole primitive term, then axiom C1, below, may serve as an adequate, single characteristic axiom for mereology (cf. [4]). This single axiom is subjoined to the basis provided by protothetic and ontology. We will use the symbol "μ" to denote the primitive term "ingredient of".

$$\text{C1}: \; [a,b]\{(a\varepsilon\mu b) \leftrightarrow (_1(_2(a\varepsilon a) \wedge (b\varepsilon b))_2 \wedge (_3(b\varepsilon\mu b) \rightarrow [c,d](_4(b\varepsilon c) \leftrightarrow [e](_5 (e\varepsilon d) \leftrightarrow [f](_6(f\varepsilon c) \rightarrow (f\varepsilon\mu e))_6 \wedge [f](_7(f\varepsilon\mu e) \rightarrow \neg(_8[g,h]\neg(_9(g\varepsilon c) \wedge (h\varepsilon\mu f) \wedge (h\varepsilon\mu g))_9)_8)_7)_5)_4 \rightarrow (a\varepsilon\mu d))_3)_1\}$$

An ontological definition may then be used for the introduction of the notion "class of". This definition may be derived by considering the implications of the collective interpretation of "class". The symbol "k" will be used to represent the functor "class of". Following Prior (cf [18]) four implications of the collective interpretation of "a" being a "class of b's" may be isolated for our purpose (cf. §5.2.1)

i) The "class of b's" is an individual literally made up of all the "b's" ; i.e.

$(a\varepsilon kb) \rightarrow (a\varepsilon a)$

i.e. if the "a" is a "class of b's" then the "a" is an individual.

ii) In the absence of ingredient "b's", the "class of b's" does not exist. The formalization of this implication asserts that if there is a "class of b's" then there is at least one ingredient "b", viz.

$(a\varepsilon kb) \rightarrow \neg(_1[c]\neg(c\varepsilon b))_1$

iii) If anything is a "b", it is an ingredient of the "class of b's"; i.e.

$(a\varepsilon kb) \rightarrow [c](_1(c\varepsilon b) \rightarrow (c\varepsilon\mu a))_1$

iv) Any individual which is an ingredient of the "class of b's", must have an ingredient in common with some "b" - formalized this yields

$(a\varepsilon kb) \rightarrow [c]\{(c\varepsilon\mu a) \rightarrow \neg(_1[d,e]\neg(_2(d\varepsilon b) \wedge (_3(e\varepsilon\mu d) \wedge (e\varepsilon\mu c))_3)_2)_1\}$

It is also necessary to note that the converse of implication (iii) does not hold. That is, it is not the case that each ingredient of the "class of b's" is necessarily one of the "b's". However, (iv) does allow for this possibility. In this case, the ingredient "c" of the "class of b's" is itself a "b" and hence "d" and "e" would simply be the same individual as "c" - this is possible as in mereology all individuals are ingredients of themselves. We may also note that under a distributive interpretation, the converse of implication (iii) would have been required to hold.

The four implications above are jointly equivalent to the expression : "the a is a class of b's". Their conjunction may therefore be used as the definiens of an appropriate ontological definition, viz.

DC : $[a,b]\{(a\varepsilon kb) \leftrightarrow (_1(a\varepsilon a) \wedge (\exists c)(c\varepsilon b) \wedge [c](_2(c\varepsilon b) \rightarrow (c\varepsilon\mu a))_2 \wedge [c](_3 (c\varepsilon\mu a) \rightarrow (\exists d,e)(_4(d\varepsilon b) \wedge (_5(e\varepsilon\mu d) \wedge (e\varepsilon\mu c))_5)_4)_3)_1\}$

We have used "$(\exists c)$" and "$(\exists d,e)$" as abbreviations for "$\neg([c]\neg \ldots)$" and "$\neg([d,e]\neg \ldots)$", respectively.

It is furthermore provable from this definition and the single axiom of mereology that if an application of the expression "kb" is possible, it is unique. It may therefore be interpreted as "<u>the</u> class of b's".

The single axiom included in the above approach to the formalization of mereology is rather complicated. An outline of an alternate axiom system, containing more axioms, but also yielding mereology when subjoined to ontology, might therefore serve to make the properties of the primitive term, "ingredient of", more apparent. The following set of axioms is used by Clay in [3]:

I $[a,b,c]\{(_1(a\varepsilon\mu b) \wedge (b\varepsilon\mu c))_1 \rightarrow (a\varepsilon\mu c)\}$

II $[a,b]\{(a\varepsilon\mu b) \rightarrow (b\varepsilon b)\}$

DM: $[a,b]\{(a\varepsilon k b) \leftrightarrow (_1(a\varepsilon a) \wedge [d](_2(d\varepsilon b) \rightarrow (d\varepsilon\mu a))_2 \wedge [d](_3(d\varepsilon\mu a) \rightarrow (\exists e,f)(_4(e\varepsilon b) \wedge (f\varepsilon\mu d) \wedge (f\varepsilon\mu e))_4)_3\}$

III: $[a,b,c]\{(_1(a\varepsilon k c) \wedge (b\varepsilon k c))_1 \rightarrow (a I b)\}$

IV : $[a,b]\{(a\varepsilon b) \rightarrow (\exists c)(c\varepsilon k b)\}$

The abbreviations "$(\exists c)$" and "$(\exists e,f)$" are also used here, respectively for "$\neg([c]\neg\ \ldots)$" and "$\neg([e,f]\ \neg\ \ldots)$". These abbreviations will be used throughout this chapter. The functor "I" which appears in III, was introduced as example (f) in §4.2.3 . In the above axiom system, axioms I and II depict properties of the primitive functor. Definition DM introduces the term "k" which is also used in axioms III and IV.

In 1920, Lesniewski constructed an axiom system (comprising four axioms) using "μ" as sole primitive term. Lesniewski's original (1916) axiom system, however, used a functor which is denoted by "pt" in [10] and [22] , as sole primitive functor. The term "pt" is interpreted as "part of". The following thesis shows the relationship between "μ" and "pt".

T1 : $[a,b]\{(a\varepsilon pt(b)) \leftrightarrow (_1\neg(a I b) \wedge (a\varepsilon\mu b))_1\}$

Thus, "a is part of b" if and only if "a is an ingredient of b", but "a" is not identical with "b" itself. Therefore, with the exception of "b" itself, all ingredients of "b" are "part of b".

Lesniewski's original axiom system consists of four axioms. The first two are :

M1 : $[a,b,c]\{(_1(a\varepsilon pt(b)) \wedge (b\varepsilon pt(c)))_1 \rightarrow (a\varepsilon pt(c))\}$

M2 : $[a,b]\{(a\varepsilon pt(b)) \rightarrow (_1 b\varepsilon N(pt(a)))_1\}$

Lesniewski needs to use "μ" and "k" in the remaining two axioms. He therefore first introduces "μ" as a defined term, and then in a second definition defines "k" in terms of "μ". Both these definitions are of the ontological type. At this stage, the convention, however, seems to have been that the definiendum be written on the right hand side of a definition's main coimplicator (cf. note 2, chapter 4). The definition of "μ" in terms of "pt" therefore had the form :

DI : $[a,b]\{(_1(a\varepsilon a) \wedge (_2(aIb) \vee (a\varepsilon pt(b)))_2)_1 \leftrightarrow (a\varepsilon\mu b)\}$

Note that "v" represents "or". The definition of "k" was the same as DC, except that it was given with the definiens on the left hand side and the definiendum on the right hand side of the coimplicator (cf. [10] and [22]). The remaining axioms of Lesniewski's original axiom system are :

M3 : $[a,b,c]\{(_1(a\varepsilon kc) \wedge (b\varepsilon kc))_1 \rightarrow (aIb)\}$

M4 : $[a,b]\{(a\varepsilon b) \rightarrow (\exists c)(c\varepsilon kb)\}$

Another important mereological functor defined by Lesniewski, is introduced by definition DE.

DE : $[a,b]\{(_1(a\varepsilon a) \wedge (\exists c)(c\varepsilon\mu b) \wedge [c](_2(c\varepsilon\mu b) \rightarrow (c\varepsilon N(\mu a)))_2)_1 \leftrightarrow (a\,\varepsilon\, \mathrm{extr}(b))\}$

The definition is given with the definiendum on the right hand side. The expression : "a ε extr(b)", may be read as "a is outside b". The definition asserts that "a is outside b" if and only if "a" and "b" have no common ingredients. Lesniewski proved that "extr" could be used as the sole primitive term of mereology, and constructed an appropriate axiom system in 1921.

A few theses to illustrate further properties of the "ingredient" and "class" concepts, conclude this section :

T2 : $[a,b,c]\{(_1(a\varepsilon\mu b) \wedge (b\varepsilon\mu c))_1 \rightarrow (a\varepsilon\mu c)\}$

T3 : $[a,b,c]\ (_1(a\varepsilon\mu b) \wedge (b\varepsilon c))_1 \rightarrow (a\varepsilon\mu kc)\}$

T4 $[a]\{(a\varepsilon a) \rightarrow (a\varepsilon\mu a)\}$

T5 : $[a,b]\{(a\varepsilon b) \rightarrow (aIka)\}$

T6 : $[a,b]\{(a\varepsilon kb) \rightarrow (aIkb)\}$

T7 : $[a,b,c]\ (_1(a\varepsilon kc) \wedge (b\varepsilon kc) \wedge \neg(a\varepsilon\mu b))_1 \rightarrow (a\varepsilon\ \Delta\ )\}$

5.3 Mereology and Boolean Algebra

Formally, strong similarities exist between mereology and extended systems of Boolean algebra (cf. article XI of [23] for a description of such a system). In fact according to Tarski they differ only in one respect - that the axioms of mereology imply[2)] that "there is no individual corresponding to the Boolean algebraic zero". That is, there is no individual which is an ingredient of every other individual. However, consider Boolean algebraic inclusion as the relation in a model *B* of the extended system of Boolean algebra with domain B. A model for mereology may be obtained by removing the zero element from B and also substituting mereological ingredience for its "correlate", Boolean algebraic inclusion. Conversely, suppose *A* is a model for mereology with domain M. Then, by replacing ingredience with inclusion, adding a new element to M and postulating that this element is included in every other element of M, it is possible to obtain a model for the extended Boolean algebra.

Tarski's comparison of the models as outlined above, does, however, also seem to be open to some of the criticism which will be levelled at Grzegorczyk's proof that these models are the same. Grzegorczyk (cf. [8]) considers a system of axioms for "mereology", with primitive term : "ingr", defined as : 'a ingr b $\leftrightarrow$ a is a part of b or a is identical with b'. The expression "a ingr b" is read as : "a is an ingredient of b". He then compares this axiom system to one for Boolean algebra. He defines "ingr" in terms of Boolean algebraic inclusion between elements other than the zero element of Boolean algebra. This definition of the term "ingr" and a definition of "the set of all those 'non-empty' objects which are to be discussed in mereology" (designated by "Z"), are then added to his axiom system for Boolean algebra. A proof that "elementary mereology" (based on his axiom system) is "an axiomatic theory containing all those and only those theorems of Boolean algebra which are mereological propositions", follows. "Mereological propositions" is the term he uses to describe provable propositions (in the system of Boolean algebra enriched by the definitions of "Z" and "ingr") which contain "Z" and "ingr" as the "only extralogical terms".

The most fundamental criticism which can be made of Grzegorczyk's proof is that the primitive term, "ingr", he introduces, does not coincide with the primitive term of Lesmewski's mereology. In representing "a is an ingredient of b" by "a ingr b", the "is an" and "ingredient of" (represented respectively, by "ε" and "μ" in §5.2.2) are conflated into one. Clay sees this as an elimination of the "is" and hence the notion of distributive class. The vital role of the copula, "is", in the distributive interpretation of "class" expressions, was mentioned in § 5.2.1. Even if "ingr" is seen merely as a conflation of "ε" and "μ", the result is that the distributive and collective notions of class become confused and that what Clay terms the "interplay between the two notions", is lost. This deficiency is serious as it allows the fundamental problems mereology aimed to solve to threaten once more.

Tarski's description (cf. note 1, article XI, §1 of [23]) of the primitive relation of "the part to the whole"[3] as the "correlate of Boolean algebraic inclusion", similarly seems open to an interpretation in which the "is" ("ε") is absorbed into the "part of" relation. This can be seen from Tarski's rendition of the expression "x < y" as "the element x is included in the element y"; thus including "is" in the interpretation of "<", his symbol for inclusion. Although the contrast between inclusion ("<") and ingredience ("εμ") as regards the need for the copula, "is", seems to have been overlooked here, Tarski, himself, is explicit as regards the separation of "is" and "ingredient of" in other articles - e.g. article II of [23].

Clay investigates the relationship between models of mereology and complete Boolean algebra in great detail. He notes firstly that as the logical basis of Boolean algebra is usually not protothetic and ontology as in the case of mereology, an investigation of this kind needs to be placed within a precise context. Thus, for example, in view of the fact that Lesniewski avoided existential postulates, it is necessary to use a definition of Boolean algebra which differs from the normal in that it does not postulate the existence of any non-zero element.

Clay considers the statements that :

a) mereology is a complete Boolean algebra with zero deleted ; and
b) that any complete Boolean algebra with zero deleted, is a system of mereology.

He proves statement (a), but concludes that (b) is false in the sense that it is necessary to be "unreasonably restrictive" in the definition of a Boolean algebra (with zero deleted) for which mereology is able to depict the structure.

5.4 Comment

An important requirement for a deductive theory is consistency. We mentioned in earlier chapters that mereology may be proved consistent relative to ontology and protothetic. According to Sobocinski, Lesniewski in fact constructed an interpretation of mereology in the real number system. This construction did not survive. Clay, in [3] , however, attempts a reconstruction. It differs from Lesniewski's original construction only in that Clay makes use of a subset of the real numbers for the construction of his interpretation, whereas Lesniewski probably used the whole set of real numbers. This difference does, as Clay points out, not effect the question of consistency. Clay introduces the real number system via an axiom system in ontology. This makes it possible to regard the real numbers as objects in ontology. By successfully constructing an interpretation of mereology in terms of these axioms for the real numbers, the axioms of ontology and the rules of procedure for ontology, Clay establishes the relative consistency of the two systems.[4)]

The fact that the above construction is possible and that axioms for the real numbers may be formulated in ontology, is in itself an indicator of the functionality of Lesniewski's systems. Indeed, Lesniewski intended his systems to be used practically in philosophical or scientific work. Mereology provides the conceptual and formal apparatus for an investigation of the foundations of geometry (in the traditional sense) and for the formalization of certain empirical sciences. Consider, for example, Tarski's solution (article II of [23]) to the problem of establishing the foundations of a "geometry of solids"[5)] . In this solution, Tarski bases the geometry of solids on mereology. A further example is Woodger's use of mereology in his attempt at formalizing biology in [25] .[6)]

We may note here that Lesniewski considered his axiomatization of the collective use of "class" expressions to follow in the Cantorian tradition, Consider, for example, Cantor's view that "any set of distinct things can itself be regarded as a unity, a thing in which those things are ingredients or constituent elements" (cited in §4.10 of [14]). Lesniewski and Cantor's conceptions of "class" did not, however, correspond in every respect. Lesniewski required a greater degree of precision in his formulations of "class" expressions. The preciseness of the treatment and of the formalization of the "ingredience" and "class" notions in mereology, makes mereology useful in any investigation in which a clear conceptualization of these notions is essential. The role of mereology in solving the problem presented by Russell's paradox is discussed in chapter 6.

Notes

1) From previous descriptions it may seem that an anology exists between the well-known set theoretical idea of "subset of a set" and the collective interpretation of "ingredient of a set". However, such an anology would break down here, as in set theory the empty set is a subset of every set.

2) This is providing it is assumed that at least two distinct individuals exist. It should, however, be noted that mereology itself makes no existential judgements of this kind.

3) Tarski's "part" is what we have thusfar termed "ingredient".

4) Newman and Nagel's comments on relative consistency proofs, in chapter 2 of [17] , should, however, be considered here.

5) Tarski describes this as a system of geometry "destitute of such geometrical figures as points, lines, and surfaces, and admitting as figures only solids - the intuitive correlates of open (or closed) regular sets of three-dimensional Euclidean geometry".

6) It is worth noting that, although Woodger designates "P" to denote the relation "part of", he interprets "xPy" as "x is part of y". The copula, "is", is therefore not, as required in mereology, clearly separated (by a separate symbol) from the "part of".

CHAPTER 6

RESOLUTION OF RUSSELL's PARADOX, AND FINAL COMMENT

Russell's paradox was probably the most fundamental of the paradoxes discovered in Cantor's ("naive") set theory. Lesniewski came into contact with this paradox in Lukasiewicz's . "O Zasadzie Sprzecznosci u Arystotelesa: Studium krytyczna", Krakow, 1910 ("On the principle of contradiction in Aristotle : a critical study"). Luschei (cf. §4.9 of [14]) renders Lukasiewicz's formalization of the paradox as :
"Most classes are not elements of themselves but, as collections, possess properties quite different from the properties characterizing their own elements. The collection of men is not a man .... Certain classes, however, such as the class of classes, apparently are exceptions to the rule. Since there are non-empty classes, having at least one element, the class of non-empty classes for example is non-empty, and consequently an element of itself. Now consider the class K of classes not elements of themselves : Since a class is an element of class K if and only if not an element of itself, class K is an element of itself if and only if not an element of itself. Is or is not K an element of itself? If it is, then it also is not. So it is not. Yet if it is not, then it also is. Either of the possible alternatives leads to a contradiction that it both is and is not. What is to be done about it?"

What Whitehead and Russell did about it was to construct the "Theory of Logical Types" mentioned in §2.5. The "axiom of reducibility" required by this theory was, however, according to Kilmister (cf. chapter 7 of [9]), not acceptable to many mathematicians.

What Lesniewski did about it lead, as outlined in chapter 1, to the development of the deductive systems described in the previous chapters. In terms of these systems the problem presented by the paradox is seemingly successfully solved.

We pointed out, in §5.2.1, that in mereology, there is no such object as the class of classes which are not ingredients of themselves." In §4.2.2, we mentioned that any statement of the form "aεb" is false if "a" represents a fictitious name. Therefore, if we let "K" represent "the object which is the class of classes which are not ingredients of themselves", the statement : "K is an ingredient of K", is false. It was also pointed out, in §4.2.2, that the denial of a statement of the above type, would always be true.

The argument may be presented in a more formal manner, as follows :

Let the symbol "x" represent the "object which is not an ingredient of itself". We introduce the "object which is not an ingredient of itself" by the following definition :

DN: $[a]\{(a\varepsilon x) \leftrightarrow ({}_1(a\varepsilon a) \land \neg(a\varepsilon\mu a))_1\}$

It is a thesis of mereology that

T4: $[a]\{(a\varepsilon a) \leftrightarrow (a\varepsilon\mu a)\}$

Hence, it follows from DN that

T8: $[a]\{(a\varepsilon x) \leftrightarrow ({}_1(a\varepsilon a) \land \neg(a\varepsilon a))_1\}$

It therefore follows from the discussion of "Δ" in §4.2.2, that the assertion "$a\varepsilon x$" is always false. This agrees with the fact that every individual is an ingredient of itself. It is, however, also a thesis of mereology (Clay's axiom IV) that

T9: $[a,b]\{(a\varepsilon kb) \rightarrow (\exists c)(c\varepsilon b)\}$

Hence, by substitution, we get

T10: $[a]\{(a\varepsilon kx) \rightarrow (\exists c)(c\varepsilon x)\}$

A statement of the type "$a\varepsilon x$" has, however, been shown to be false in all cases (for all "a") and therefore, in particular, also "$c\varepsilon x$" is false. It therefore follows from T10 that the statement "$a\varepsilon kx$" is always false (is false for all "a"). Thus, in particular, "$kx\varepsilon kx$" is also false.

Now, from T4 it follows that

$$(kx\varepsilon kx) \leftrightarrow (kx\varepsilon\mu kx)$$

and so as "$kx\varepsilon kx$ is false, the assertion "$kx\varepsilon\mu kx$" is false. Therefore the assertion : "$\neg(kx\varepsilon\mu kx)$", is true.

It has thus been shown that it is false to assert that the class of "objects which are not ingredients of themselves" is an ingredient of itself, and also that it is true that it is not the case that this class is an ingredient of itself. In particular, this applies to the class K of classes which are "not ingredients of themselves".

Final Comment : As illustrated above, one of the fundamental problems in Cantorian set theory is avoided in Lesniewski's systems. How effective Lesniewski's systems are as a foundation for mathematics, will, however, finally depend on the shape and size of the "buildings" which can be erected on this foundation. Although the possibilities seem promising, there are as yet too few of these "buildings" in use to pass final judgement.

Notes

1) Mereology embodies the collective interpretation of "class" expressions. Under a distributive interpretation, it is possible to deduce that the ontological status of the "class of classes which are not ingredients of themselves", is the same as that of the "object which does not exist." This follows from

i) the fact that a distributive interpretation of the statement : "a is an ingredient of the class of objects which are ingredients of themselves", yields : "a is an object which is not an ingredient of itself". Therefore, in particular, under a distributive interpretation, "K is an ingredient of itself if and only if K is K and K is not an ingredient of itself"

ii) the comments on ontological definition and existence concepts in §4.2.2

APPENDIX

Lesniewski's Published Work

Works referred to in the text as well as most of his other important publications are listed below. The listing is in chronological order. (Publications before 1916 are omitted - most of these were later repudiated by Lesniewski).

{1} "Podstawy ogólnej teorii mnogości I" (Foundations of general theory of manifolds or collective sets I), Moscow (1916).

{2} "O podstawach matematyki" (On foundations of mathematics) in "Przeglad Filozoficzny", vol. xxx (1927); vol. xxxi (1928); vol. xxxii (1929); vol. xxxiii (1930); vol. xxxiv (1931), Warszawa.

{3} "Über Funktionen, deren Felder Gruppen mit Rücksicht auf diese Funktionen sind", in "Fundamenta Mathematicae", vol. xiii, Warszawa (1929).

{4} "Über Functionen, deren Felder Abelsche Gruppen in Bezug auf diese Fuktionen sind", in "Fundamenta Mathematiae", vol. xiv, Warszawa (1929).

{5} "Grundzüge eines neuen Systems der Grundlagen der Mathematik" (§ 1 - 11), in "Fundamenta Mathematicae", vol. xiv, Warszawa (1929).

{6} "Über die Grundlagen der Ontologie", in "Comptes rendus des séances de la Societé des Sciences et des Lettres de Varsovie", Classe III, vol. xxiii, Warszawa (1930).

{7} "Über Definitionen in der sogenannten Theorie der Deduktion", in "Comptes rendus des séances de la Societé des Sciences et des Lettres de Varsovie", Classe III, vol. xxiv, Warszawa (1931).

{8} "Grundzüge eines neuen Systems der Grundlagen der Mathematik" (§12), in "Collectanea Logica", vol. I (1938) - This journal was destroyed with its Warsaw printing house in war. Offprints of this article and {9} survive in Harvard College Library.

{9} "Einleitende Bemerkungen zur Fortsetzung meiner Mitteilung u.d.T. 'Grundzüge eines neuen Systems der Grundlagen der Mathematik'", in "Collectanea Logica", vol. I (1938) - cf. {8}.

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[5] R E Clay : "The relation of weakly discrete to set and equinumerosity in mereology", Notre Dame Journal of Formal Logic, vol. V, number 4, October 1965.

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