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Classical and Quantum Picture of the Interior of Two-Dimensional Black Holes

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Declaration of Authorship

I, Mark Shawa, declare that this thesis titled, 'Classical and Quantum Picture of the Interior of Two-Dimensional Black Holes' and the work presented in it are my own. I confirm that:

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- Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
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"Science does not limit itself merely to what is currently verifiable. But it is interested in questions that are potentially verifiable (or, rather, falsifiable)."

Sam Harris

RHODES UNIVERSITY

Abstract

Faculty of Science
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Classical and Quantum Picture of the Interior of Two-Dimensional Black Holes

by Mark Shawa

A quantum-mechanical description of black holes would represent the final step in our understanding of the nature of space-time. However, any progress towards that end is usually foiled by persistent space-time singularities that exist at the center of black holes. From the four-dimensional point of view, black holes seem to resist quantization. Under highly symmetric conditions, all higher-dimensional black holes are two-dimensional. Unlike their higher-dimensional counterparts, two dimensional black holes may not resist quantization. A non-trivial description of gravity in two dimensions is not possible using Einstein's theory of gravity alone. However, we may still arrive at a consistent description of gravity by introducing a scalar field known as the dilaton. In this thesis, we study both the classical and quantum aspects of the interior of two-dimensional black holes using a generalized dilaton-gravity theory. Classically, we will find that the interior of most two-dimensional black holes is not much different from that of four-dimensional black holes. But by introducing quantized matter into the theory, the fluctuations in space-time will give a different picture of the structure of interior of black holes. Using a low-energy effective field theory, we will show that it is indeed possible to identify quantum modes in the interior of black holes and perform quantum-mechanical calculations near the singularity.

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Contents

Abbreviations

QFT	Quantum Field Theory
QED	Quantum Electrodynamics
JT	Jackiw-Teitelboim
SIG	string inspired gravity
SSG	spherically symmetric gravity
eq	equation
eqs.	equations

To my family, may the force be with you all.

Conventions

- Unless explicitly stated, we will express observables in natural units, that is, we will take $\hbar = c = G = k_B = 1$.
- The diagonal of all metrics will have the sign $(-,+,+,+,\cdots)$.
- We will use the Einstein summation convention, where $a^\mu b_\mu = \sum_i a^i b_i$.
- An overdot will be used to denote a derivative with respect to time. This means $\dot{a} = \frac{da}{dt}$ and the number of dots indicate the number of derivatives.

For geometric quantities, we will use the notation in [1].

- The Christoffel symbol will be defined as

$$\Gamma^\mu_{\nu\sigma} = \frac{1}{2} g^{\mu\rho} \Big(\partial_\nu g_{\rho\sigma} + \partial_\sigma g_{\nu\rho} - \partial_\rho g_{\nu\sigma} \Big),$$

  where $\partial_\alpha = \frac{\partial}{\partial x^\alpha}$.
- We will assume $\Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\gamma\beta}$.
- The Riemann curvature tensor will be given by

$$R^\alpha_{\beta\gamma\sigma} = \partial_\gamma \Gamma^\alpha_{\beta\sigma} - \partial_\sigma \Gamma^\alpha_{\beta\gamma} + \Gamma^\rho_{\beta\sigma} \Gamma^\alpha_{\rho\gamma} - \Gamma^\rho_{\beta\gamma} \Gamma^\alpha_{\rho\sigma}.$$

- The Ricci tensor will be given by

$$R_{\mu\nu} = R^\gamma_{\mu\gamma\nu}.$$

- The curvature scalar or Ricci scalar will be given by

$$R = g^{\mu\nu} R_{\mu\nu}.$$

Chapter 1

Introduction

The turn of the twentieth century saw the invention of three revolutionary scientific ideas: special relativity, general relativity, and quantum theory. The first of these, special relativity, showed that space and time could be placed on equal footing as space-time. General relativity went a step further with the radical notion that gravity is a manifestation of the bending of space-time. For the first time, we learned that space-time is a dynamical field. This idea alone caused a revolution in our understanding of physical reality. Central to general relativity is Einstein's equation which describes how matter influences the geometry of space-time. This equation is usually written as

$$G_{\mu\nu} + \frac{1}{2}\Lambda g_{\mu\nu} = 8\pi\ T_{\mu\nu}, \tag{1.1}$$

where

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu}, \tag{1.2}$$

$G_{\mu\nu}$ is the Einstein tensor, $R_{\mu\nu}$ is the Ricci tensor, R is the Ricci or curvature scalar, Λ is the cosmological constant, $T_{\mu\nu}$ is the stress-energy tensor which describes the matter content as the source of gravity and $g_{\mu\nu}$ is the metric tensor, the solution to the equation. The last of this great scientific revolution, quantum theory, taught us that the universe is fundamentally quantized. This means that at microscopic scales, certain familiar quantities such as energy, momentum, electric charge and possibly space-time are not continuous but are in fact discrete quantities.

As our understanding of the universe improved, we sought a unified picture of all physical theories. It was only natural that quantum theory was combined with special relativity to form Quantum Field Theory (QFT) which later formed the basis of the Standard Model of particle physics. Another notable success was the unification of QFT with

Maxwell's electromagnetism to form Quantum Electrodynamics (QED), a theory that completely describes electromagnetic phenomenon. The Standard Model describes how elementary particles interact and are governed by the fundamental forces excluding gravity. This means that a complete description of elementary particles is still lacking and completion would require the inclusion of gravity into the Standard Model. For that to happen, general relativity has to be reconciled with quantum theory and the resulting theory would be called quantum gravity. Such a theory would necessarily mean the quantization of geometric degrees of freedom of space-time. However, the invention of such a theory has proved to be very difficult mainly due to the persistence of infinities in any attempt at unifying quantum field theory and general relativity. The occurrence of infinities in field theories is not unique to attempts at a quantum theory of gravity, such problems existed during the development of QED but were resolved using regularization and renormalization techniques. These techniques have proved ineffective when it comes to dealing with the issue of quantizing gravity.

There are candidate theories such as string theory that attempt to solve some of the problems associated with the quantization of gravity. This theory predicts that the universe is fundamentally composed of one-dimensional extended objects known as strings [2]. Furthermore, this theory makes the claim that fundamental particles of the Standard Model can be classified as either open or closed strings identifiable with a particular "vibrational mode". For example the graviton, the quantum of the gravitational field, would be a closed string vibrating at a particular frequency. String theory strives towards the unification of all the fundamental forces, this makes it a good candidate for the theory of everything.

One thing to point out is that string theory, although mathematically elegant, has one fatal flaw; it lacks experimental evidence. But this is not entirely the fault of the theory, the energy needed to validate it is far too high for any existing technology.

1.1 Black holes and quantum gravity

The Schwarzschild solution, named after Karl Schwarzschild who discovered it while on the battlefield in World War I, was the first non-trivial solution to Einstein's equations of general relativity. It describes the region of space-time surrounding a spherical object of mass M. In spherical coordinates, it is usually written as

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{1.3}$$

The Schwarzschild metric is a somewhat unique solution of Einstein's equations in the sense that the space-time around a spherical, non-rotating, electrically neutral gravitating body can always be described by eq. (1.3). This last statement is commonly referred to as Birkhoff's theorem. This solution also predicts the existence of regions of space-time with such an intense gravitational field that not even light could escape from them. These regions are known as black holes and are characterized by two important locations; the *horizon*, when $r = 2M$, and behind it the *singularity*, when $r = 0$. At the horizon, the spatial component of the Schwarzschild metric goes to infinity. As it turns out, this infinity is a manifestation of our choice of coordinates, hence it is sometimes referred to as the coordinate singularity. The situation is different at the singularity; both the metric and the curvature scalar diverge, for this reason it is referred to as a physical singularity. Due to these unresolvable divergences, the singularity is regarded as the place at which the laws of physics fail to make any sense.

Furthermore, black holes are not just mathematical artefacts, they are in fact the final state of stellar collapse due to gravity [3]. In addition, any asymptotic observer can describe a black hole using only three parameters; the charge, the mass and the angular momentum. This is the famous *no-hair theorem*, although mathematicians prefer to call it the *no-hair conjecture* as it lacks a rigorous mathematical proof.

A major motivation for studying black holes is that in places sufficiently far from them, gravity is generally a weak force in comparison to the other fundamental forces. Gravity is effective at describing large distance scale interactions but completely ineffective at describing small distance scale interactions, such as those studied in quantum theory. Given the nature of black holes, we find the perfect theoretical environment in which to study strongly interacting gravitational fields for which quantum effects could play a significant role, as described below.

In the 1970s, it was discovered that black holes were governed by mechanical laws analogous to the laws of thermodynamics [4]. It was believed that this very close analogy could provide the key to understanding the quantum mechanical nature of black holes. For instance, given a non-spinning and charge free black hole, the first law of black hole mechanics can be written as

$$dM = \frac{\kappa}{8\pi} dA, \tag{1.4}$$

where M is the mass of the black hole, κ is the surface gravity and A is the area of the horizon. This statement is analogous the first law of thermodynamics

$$dQ = TdS, \tag{1.5}$$

where Q is the energy of the system, T is its temperature and S is the associated thermodynamic entropy. The analogous quantities being $Q \leftrightarrow M$, $T \leftrightarrow \kappa$ and $S \leftrightarrow A$. It was first pointed out by Bekenstein [5] that the analogy goes further when we consider Hawking's black hole area theorem [6] which requires that $dA \geq 0$, which again reminds us of the irreversibility of thermodynamic[1] processes and is usually presented as $dS \geq 0$. The laws of black hole mechanics were given a permanent place in physics when Hawking used QFT in a curved background and found that black holes emit thermal radiation [7, 8]. This kind of radiation is famously known as Hawking radiation. Its temperature is given as

$$T_H = \frac{\kappa}{2\pi}. \tag{1.6}$$

This result combined with eq. (1.4) and realizing that Q and M represent the total energy of the system, showed that black holes have an entropy given by

$$S_{BH} = \frac{A}{4}. \tag{1.7}$$

At the time, this was surprising because it showed that the entropy of the black hole is proportional to its horizon area as opposed to its volume as one would find for most thermodynamic systems. This entropy is widely referred to as Bekenstein-Hawking entropy.

Hawking radiation, temperature and black hole entropy are not simply mathematical consequences but are physical properties of black holes. That being said, the microscopic origin of the entropy is still an open problem in physics. Over the years many people have devised radically different ways of calculating black hole entropy. For example, [9–12] have obtained the same Bekenstein-Hawking value given by eq. (1.7) using very different approaches. It would seem then that Bekenstein-Hawking entropy of a black hole stands as a kind of litmus test for all prospective theories of quantum gravity. This has led to the belief that a consistent theory of quantum gravity, semi-classical or complete, must be able to reproduce the Bekenstein-Hawking entropy formula. Perhaps a complete theory of quantum gravity could better explain the origin of black hole entropy.

1.2 Semi-classical gravity

Coming back to the problem of quantizing gravity, we look at what goes wrong when one attempts to make sense of certain quantities. The problem of persistent singularities

[1] In this thesis we synonymously use the phrases thermodynamical system and statistical mechanical system.

comes down to the problem of finding finite probability amplitudes for certain processes at particular energy scales. When treated perturbatively, it's usually the case that the coupling constants play a role similar to perturbation parameters. In QED, the coupling constant is $e^2/c\hbar$ which is a dimensionless quantity. At first order the divergent parts maybe absorbed into some zeroth order parameters and at higher orders, higher powers of the coupling constant become really small and terms become negligible. To illustrate this, we take some physical quantity $F(\alpha)$, where $\alpha = e^2/c\hbar$, and perturbatively expand it (with the use of Feynman diagrams)

$$F(\alpha) = F_0 + \alpha F_1 + \alpha^2 F_2 + \mathcal{O}(\alpha^3), \tag{1.8}$$

where F_0, F_1 and F_2 are respectively the zeroth (tree level), first (one-loop) and second order (two-loop) contributions to F [13]. When it comes to gravity the Planck length, $\ell_P = (G\hbar/c^3)^{\frac{1}{2}}$, takes the role of the perturbation parameter as there is no dimensionless measure of gravitational coupling. And since it has dimensions of length, complications may arise when working close to Planck scales since higher order perturbation terms can be comparable to lower order terms. When this happens more parameters need to be added at each order in order to absorb the divergent parts.

If we consider a system constituting of two mass (or energy) scales, light and heavy, then we may choose to use an effective field theory approach. Following this approach, we can set some ultraviolet cutoff on the mass scale and remove (or integrate out) the heavy degrees of freedom. The outcome is a perturbative expansion in terms of the ratio of light to heavy masses. This perturbative expansion is relevant to counting loops in Feynman diagrams.

In Feynman diagrams, it can be shown that there is a relationship between the number of loops, internal lines and vertices [14]. Internal lines indicate the presence of high momentum virtual particles in the scattering process. As a result, these internal lines can be shown to contribute positive powers of the ultraviolet cutoff energy into the scattering amplitudes. On the other hand, vertices contribute negative powers of the ultraviolet cutoff energy. In higher-loop diagrams, the contributions of powers of the ultraviolet cutoff energy from the internal lines outweigh those from the vertices. Hence, higher-loop diagrams correspond to higher energies. In trying to perform a perturbative expansion using the ratio of light to heavy masses, one then finds that the expansion is doomed to fail at higher orders in the number of loops. We are thus forced to cutoff the expansion at some order in the ratio of masses.

As we will be working with the one-loop effective action (as is common in dilaton-gravity theories), the effective field theory must also be cutoff at first order for consistency. This will explain why, in chapter 5, we only consider the leading order in the perturbative expansion.

In the construction of a low-energy description, we may consider a small quantum fluctuation in gravity propagating on a classical background. This approach is known as the background field method because the classical geometry acts as the background on which quantum interactions take place [15]. Furthermore, one considers quantizing the matter fields and writes down the Einstein equation in terms of the perturbed metric. Instead of a stress tensor, we would have a quantized stress tensor as the source term. Mathematically, this is represented by taking the expectation value of the classical stress tensor. The semi-classical Einstein equation would then take the form

$$\bar{R}_{\mu\nu} - \frac{1}{2}\bar{R}\bar{g}_{\mu\nu} + \frac{1}{2}\Lambda\bar{g}_{\mu\nu} = 8\pi \langle T_{\mu\nu} \rangle , \tag{1.9}$$

where

$$\bar{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}. \tag{1.10}$$

The metric $g_{\mu\nu}$ is the background geometry that satisfies the vacuum Einstein equation and $h_{\mu\nu}$ is the quantum fluctuation in the geometry that propagates in the classical background, $g_{\mu\nu}$. Looking at eq. (1.10), one may reconstruct eq. (1.9) as

$$G_{\mu\nu} + H_{\mu\nu} = 8\pi \langle T_{\mu\nu} \rangle , \tag{1.11}$$

where $G_{\mu\nu}$ are the contributions from $g_{\mu\nu}$ and $H_{\mu\nu}$ are the contributions that depend on both the quantum field $h_{\mu\nu}$ and the classical field. It is possible to derive eq. (1.11) by applying the variational principle to an appropriate action functional [16]. This involves minimizing the action functional with respect to its constituent fields. Such an action would generally take the form

$$I[\bar{g}_{\mu\nu}] = I_0[g_{\mu\nu}] + \bar{I}[g_{\mu\nu}, h_{\mu\nu}] + W[g_{\mu\nu}], \tag{1.12}$$

where I_0 is the Einstein-Hilbert action which gives $G_{\mu\nu}$, $\bar{I}[g_{\mu\nu}, h_{\mu\nu}]$ is a quantum corrected action that gives $H_{\mu\nu}$ in eq. (1.11) and W is the matter-induced quantum action that gives the quantum stress-energy tensor when varied with respect to $g_{\mu\nu}$. More explicitly, the variation in the last term yields

$$\langle T_{\mu\nu} \rangle = \frac{2}{\sqrt{-g}} \frac{\delta W}{\delta g^{\mu\nu}}. \tag{1.13}$$

The quantized parts of the action eq. (1.12) would naturally contain $\hbar$ and as such can be expanded up to any order in $\hbar$. But since $\hbar^n$ gets smaller as n gets larger, one typically truncates the action to first order because this leads to a theory that can be renormalized [17].

1.3 Lower dimensional quantum gravity

Even if one tries to oversimplify 3+1 dimensional gravity, it still proves to be difficult to quantize. Starting with the work by Callan, Giddings, Harvey, and Strominger [18], also known as the CGHS model after its authors, it became apparent that lower dimensional renormalizable gravity was possible and could be used to study Hawking radiation in lower dimensions. A rigorous proof of renormalizability of lower dimensional gravity was shown by Kitazawa [19]. In most cases, 1+1 dimensional gravity models are exactly solvable classically and it was hoped that they would provide a deeper understanding of the origin of black hole entropy [20].

Things can get problematic in two dimensions. For instance, in two dimensions the Einstein tensor trivially vanishes. So the source free Einstein equation becomes redundant because neither the cosmological constant Λ nor the metric necessarily have to vanish. To illustrate this, $\Lambda \neq 0$ would imply a vanishing metric which implies the absence of gravity. On the other hand, $\Lambda = 0$ would imply an indeterminate metric tensor, which is again unacceptable.

And so to make sense of two-dimensional gravity, a scalar field can be introduced [21]. This scalar field is typically referred to as the *dilaton* and is the key to studying gravity in two dimensions. With the inclusion of the dilaton, two dimensional gravity becomes a scalar-tensor theory. Such a theory is not unusual considering that four-dimensional Einstein gravity is equivalent to the less popular scalar-tensor theory known as the Brans-Dicke theory [22]. In the Brans-Dicke theory, gravitational interactions are mediated by a scalar field and the metric tensor. Brans-Dicke theory differs from general relativity in that it comes with an adjustable parameter and this is the reason general relativity is the preferred theory of gravity. Furthermore, the Brans-Dicke scalar field is the inverse of Newton's gravitational constant promoted to a dynamical field. Dilaton-gravity in two dimensions differs from Brans-Dicke theory since, as we shall see later on, we can trace its origin from Einstein gravity and other higher dimensional theories, all of which are free of adjustable parameters. However, they share certain similarities in the sense that the dilaton non-minimally couples with gravity, which in this case the dilaton can be interpreted as the inverse of Newton's gravitational constant. Furthermore, the dilaton can couple with matter which also allows for the description of Hawking radiation and temperature in two dimensions. To paraphrase Callan et al. [18]; such theories are complicated enough to enable one to ask interesting questions on black hole evaporation yet simple enough to get reasonable answers. In the not so distant past, these theories have received considerable attention because of their connection with string theory [23].

Dilaton-gravity theories do have a few setbacks, some of which we shall elaborate in due course. Though most notably is the loss of information when one reduces a higher dimensional theory to two dimensions. Certain assumptions, usually based on the symmetries of the higher dimensional theory, are made in the process. For instance, spherical symmetry is assumed in the dimensional reduction of Einstein's 3+1 dimensional gravity to what is known as spherically symmetric dilaton-gravity. In addition, the same dimensional reduction process must apply to the matter fields, this means only particular modes of the matter fields can be quantized in the process. These anomalies may weaken the connection between two-dimensional theories and their higher dimensional more realistic counterparts [24]. In addition other anomalies may emerge due to the renormalization process of the matter fields and as will be shown later on, these can have unexpected consequences.

The quantization of dilaton-gravity theories has been studied at great length [25, 26]. In general, there are two ways to go about the quantization of these two-dimensional theories. In one approach one considers quantizing the two-dimensional action directly to come up with a quantum field theory. Then to find the relevant quantum mechanics, one would need to apply a quantum Birkhoff's theorem which would essentially reduce the theory to a one-dimensional theory [27]. The second approach is to consider a Birkhoff's theorem in the two-dimensional action so as to reduce it to a one-dimensional theory followed by the relevant quantization procedure. The former approach has been used extensively to quantize dilaton-gravity models [25–28]. The latter approach will be used in this thesis.

Another aspect to consider is the interior of a black hole. Not many theorists have considered dilaton-gravity theories in the interior of a black hole. Moreover, only a handful of theorists have managed to come up with reasonable quantum theories for the interior if two-dimensional black holes [29–32]. Even then, progress has been restricted to the quantization of the string inspired (SIG) model or CGHS model by [18]. According to [30], when one considers a quantum theory, the space-time around the singularity is homogeneous and well-defined. The wave functions should not experience an infinity at the singularity. This is known as singularity resolution [29, 30].

1.4 Brief overview of contents

The goal of this thesis is to study the classical and quantum mechanical aspects of dilaton-gravity theories in the interior of two-dimensional black holes. We begin in chapter 2 with the reduction of some well-known higher dimensional theories to two-dimensions. This will be followed by a study of the general properties of these reduced theories in relation to black holes. In chapter 3, we will apply the generic theory to the interior of Schwarzschild-like black holes. This will result in a classical picture of two-dimensional black holes. We will then extend the generic theory in chapter 4 by adding quantized matter. There will be a brief discussion on the quantum stress-tensor and the trace-anomaly in two-dimensions. From the trace-anomaly we will arrive at an effective action. For a semi-classical dilaton-gravity theory, the effective action will be added to a quantum corrected classical action. The semi-classical theory will then be applied to the interior of a black hole in chapter 5. Furthermore, we will introduce two types of quantum theories near the singularity of two-dimensional black holes.

Chapter 2

Generic dilaton-gravity

As we discussed in chapter 1, dilaton-gravity theories can be related to higher dimensional theories through the process of dimensional reduction. In this chapter we will show the process of dimensional reduction of some of the known theories and some of the general features of dilaton-gravity theories.

2.1 Reduction from higher dimensional theories

The first two-dimensional dilaton-gravity theory was the Jackiw-Teitelboim (JT) model [33]. Its connection to three-dimensional gravity was first pointed out by [34]. The implied three-dimensional gravity was proposed by [35] after whom it is named BTZ gravity. With matter fields, the three-dimensional action takes the form:

$$I_{(3)} = \int d^3x \sqrt{-g_{(3)}} \left(R_{(3)} + 2\Lambda_{(3)} \right) + \int d^3x \sqrt{-g_{(3)}} \sum_{i=1}^{N} (\nabla f_i)^2. \tag{2.1}$$

where here $B_{(n)}$ denotes that B is an n-dimensional quantity, $R_{(3)}$ is the scalar curvature, $\Lambda_{(3)}$ is the cosmological constant and f_i are the matter fields. This theory also admits a black hole solution which is remarkably similar to black holes in 3+1 dimensions. The dimensional reduction procedure follows by using an axial-symmetric metric ansatz of the form

$$ds^2 = g_{ij} dx^i dx^j = h_{\mu\nu} dx^\mu dx^\nu + \ell^2 \phi^2 (x^\mu) d\theta^2, \tag{2.2}$$

where $i, j = \{0, 1, 2\}$, $\mu, \nu = \{0, 1\}$, $h_{\mu\nu}$ is the two-dimensional metric tensor, θ is the polar coordinate, ℓ is a parameter of length dimension and ϕ is the dilaton. Using the metric eq. (2.2), the Ricci scalar and the determinant are calculated and placed into eq. (2.1). The final step involves integrating out the θ coordinate to give the action (see appendix A for detailed calculation)

$$I_{(2)} = \int d^2x\sqrt{-h}\phi\left(R_{(2)} + 2\Lambda_{(2)}\right) + \int d^2x\sqrt{-h}\phi\sum_{i=1}^{N}(\nabla f_i)^2, \tag{2.3}$$

where $\Lambda_{(2)}$ is the two-dimensional cosmological constant and we have absorbed $2\pi\ell$ into the left hand side. This action is known as the JT model. For a positive cosmological constant, this model has a constant negative scalar curvature. Such space-times are known as anti-de Sitter space-times (AdS).

A widely studied dilaton-gravity theory is the spherically reduced Einstein four-dimensional gravity. The first step is to take the four-dimensional Einstein-Hilbert action with N matter fields

$$I_{(4)} = \int d^4x\sqrt{-g_{(4)}}R_{(4)} + \frac{1}{2}\int d^4x\sqrt{-g_{(4)}}\sum_{i=1}^{N}(\nabla f_i)^2 \tag{2.4}$$

and then impose the following spherically symmetric metric ansatz

$$ds^2_{(4)} = g_{\mu\nu(2)}dx^\mu dx^\nu + \frac{\phi^2 l^2}{2}\left(d\theta^2 + \sin^2\theta d\gamma^2\right), \tag{2.5}$$

where $\mu, \nu = \{0, 1\}$, θ and γ are the polar and azimuthal angles respectively and l has dimensions of length so that $\phi = \phi(x^\mu)$ is a dimensionless scalar field we call the dilaton. It is clear from eq. (2.5) that for spherically symmetric gravity (SSG), the dilaton must be proportional to the radial coordinate. We can calculate the Ricci scalar from the metric eq. (2.5) followed by integrating out the angular modes up to a constant factor. Finally, the Einstein-Hilbert action is reduced to the form (see appendix A for explanation of detailed calculation.)

$$I_{(2)} = \int d^2x\sqrt{-g_{(2)}}\left\{\frac{\phi^2}{4}R_{(2)} + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{l^2}\right\} + \int d^2x\sqrt{-g_{(2)}}\frac{\phi^2}{4}\sum_{i=1}^{N}(\nabla f_i)^2, \tag{2.6}$$

where π and l^2 have been absorbed by the left hand side.

The most popular of all two-dimensional theories is perhaps the SIG model also known as the CGHS model. This model was of great interest in the early 1990s since it was exactly solvable and included a description of Hawking radiation in two-dimensional black holes. The original action for this theory reads

$$I_{(2)} = \int d^2x\sqrt{-g}\left[e^{-2\phi}\left\{R_{(2)} + 4(\nabla\phi)^2 - 4\lambda^2\right\} - \sum_{i=1}^{N}(\nabla f_i)^2\right], \tag{2.7}$$

where λ^2 is the cosmological constant in two-dimensions.

Another two-dimensional theory, particularly popular among string theorists [36], is the Liouville gravity. Although the equations of motion were first introduced by Joseph Liouville (1809 - 1882) for the purpose of understanding the conformal properties of Riemann surfaces, the theory itself arises in non-critical string theory and quantization of two-dimensional gravity. The action for this theory reads as

$$I_{(2)} = \frac{1}{4\pi}\int d^2x\sqrt{-g}\left((b+b^{-1})\phi R + (\nabla\phi)^2 + 4\pi e^{2b\phi}\right), \tag{2.8}$$

where b is a coupling constant and ϕ is the dilaton.

When it comes to black holes in two dimensions, there is a class of dilaton-gravity theories whose solutions admit Schwarzschild-like black holes. Such a class of theories has been extensively studied by Katanaev, Kummer, and Liebl [37, 38]. In a vacuum, where $f_i \to 0$, the action for this class of theories reads as

$$I = \int d^2x\sqrt{-g}\left\{e^{-2\phi}R + 4ae^{-2\phi}(\nabla\phi)^2 + \gamma^2 e^{-2(a+b)\phi}\right\}, \tag{2.9}$$

where a and b are real constants, γ is the inverse length parameter. With the appropriate reparametrizations of the dilaton, we see that for $a = \frac{1}{2}, b = -\frac{1}{2}$ we recover the spherically reduced Einstein theory, for $a = 1, b = 0$ we recover the CGHS theory and for $a = 0, b = 1$ we have the JT theory. Interestingly, Lemos and Sa [39] have studied the case for $b = 1 - a$ while keeping a general. We should point out that their study includes a Brans-Dicke type adjustable parameter that they call ω. By tuning it to different values, they recover results of some already known dilaton-gravity theories. For example, when $\omega = 0$ they recover the solutions of the JT theory. Similarly, for other values of ω they recover solutions of different types of two-dimensional string theories. Another interesting approach is the one given by Mignemi [40] who considers black hole solutions with $a = 1$ while keeping b general.

In general, there probably exists an infinite number of dilaton-gravity theories, Grumiller and Meyer [41] list at least 20 different two-dimensional theories. Luckily, they all take the generic form

$$I = \frac{1}{2}\int d^2x\sqrt{-g}\Big[Z(\phi)R + U(\phi)(\nabla\phi)^2 + 2V(\phi)\Big] + \frac{1}{2}\int d^2x\sqrt{-g}Y(\phi)\sum_{i=1}^{N}(\nabla f_i)^2, \tag{2.10}$$

where ϕ is the dilaton, $Z(\phi)$, $Y(\phi)$, $U(\phi)$ and $V(\phi)$ are model dependent functions of the dilaton. We note that for the time being we have chosen to absorb the inverse length parameter (denoted as γ^2 in eq. (2.9)) into the potential function $V(\phi)$ so as to simplify the next couple of calculations.

2.2 Kinetic-free generic dilaton-gravity

Given the generalized action eq. (2.10), we will eliminate the kinetic term by performing a reparametrization on the dilaton and a conformal rescaling of the metric. This action is permissible because the action functional is presumed to be invariant under diffeomorphisms[1], as is a requirement of covariant laws of physics. The procedure used here follows from the work in [32] (see also [16]). From here on we will consider the vacuum case where we take the matter fields $f_i \to 0$. The first step is to redefine the dilaton as

$$\phi \to \bar{\phi}, \tag{2.11}$$

so that the dilaton dependent functions become:

$$U(\phi)(\nabla\phi)^2 \to (\nabla\bar{\phi})^2, \tag{2.12}$$

$$Z(\phi) \to \bar{Z}(\bar{\phi}), \tag{2.13}$$

$$V(\phi) \to \bar{V}(\bar{\phi}). \tag{2.14}$$

Bearing in mind that we have made a transformation in the dilaton, we can get rid of the bars so that the action eq. (2.10) becomes

$$I = \int d^2x\sqrt{-g}\left\{\frac{1}{2}Z(\phi)R + \frac{1}{2}(\nabla\phi)^2 + V(\phi)\right\}. \tag{2.15}$$

We then redefine the metric and, again, the dilaton:

$$g'_{\mu\nu} = \Omega^2(\phi)g_{\mu\nu}, \tag{2.16}$$

$$\phi' = \frac{1}{2}Z(\phi), \tag{2.17}$$

where $\Omega(\phi)$ is yet to be determined. We also require that $Z(\phi)$ and its derivative with respect to the dilaton do not vanish anywhere. Equation (2.16) is known as a conformal transformation. According to [42, 43], the inverse of the metric, its determinant and

[1] Isomorphic transformations between differentiable manifolds that leave the metric unchanged.

Ricci scalar, respectively, transform as

$$g^{\mu\nu} = \Omega^2 g'^{\mu\nu}, \tag{2.18}$$

$$\sqrt{-g} = \Omega^{-2}\sqrt{-g'}, \tag{2.19}$$

$$R = \Omega^2\left[R' + 2\frac{\Box'\Omega}{\Omega} - 2g'^{\mu\nu}\frac{\nabla'_\mu\Omega\nabla'_\nu\Omega}{\Omega^2}\right], \tag{2.20}$$

where the covariant derivative ∇'_μ and the box operator $\Box' = g'^{\mu\nu}\nabla'_\mu\nabla'_\nu$ are associated with the transformed metric $g'_{\mu\nu}$. Also the covariant derivatives ∇'_μ and ∇_μ have the same effect on scalar fields which means the covariant derivatives of scalars are related through

$$\nabla_\mu\phi = \left[\frac{d\phi'}{d\phi}\right]^{-1}\nabla'_\mu\phi'. \tag{2.21}$$

It is also useful to express the covariant derivative of Ω in terms of the covariant derivative of transformed dilaton ϕ'; to that end we use the chain rule

$$\nabla'_\mu\Omega = \frac{d\Omega}{d\phi}\nabla'_\mu\phi = \frac{d\Omega}{d\phi}\left[\frac{d\phi'}{d\phi}\right]^{-1}\nabla'_\mu\phi'. \tag{2.22}$$

Similarly, we use the product rule to find the covariant derivative of the previous expression as follows

$$\begin{aligned}\nabla'_\nu\nabla'_\mu\Omega = {} & \frac{d^2\Omega}{d\phi^2}\left[\frac{d\phi'}{d\phi}\right]^{-2}\nabla'_\nu\phi'\nabla'_\mu\phi' - \frac{d\Omega}{d\phi}\frac{d^2\phi'}{d\phi^2}\left[\frac{d\phi'}{d\phi}\right]^{-3}\nabla'_\nu\phi'\nabla'_\mu\phi' \\ & + \frac{d\Omega}{d\phi}\left[\frac{d\phi'}{d\phi}\right]^{-2}\left[\nabla'_\nu\frac{d\phi'}{d\phi}\right]\nabla'_\mu\phi' - \frac{d\Omega}{d\phi}\left[\frac{d\phi'}{d\phi}\right]^{-2}\left[\nabla'_\nu\frac{d\phi'}{d\phi}\right]\nabla'_\mu\phi' + \frac{d\Omega}{d\phi}\left[\frac{d\phi'}{d\phi}\right]^{-1}\nabla'_\nu\nabla'_\mu\phi'\end{aligned} \tag{2.23}$$

which in a slightly more compact form is written as

$$\nabla'_\mu\nabla'_\nu\Omega = \left\{\frac{d^2\Omega}{d\phi^2} - \frac{d\Omega}{d\phi}\left[\frac{d\phi'}{d\phi}\right]^{-1}\frac{d^2\phi'}{d\phi^2}\right\}\left[\frac{d\phi'}{d\phi}\right]^{-2}\nabla'_\nu\phi'\nabla'_\mu\phi' + \frac{d\Omega}{d\phi}\left[\frac{d\phi'}{d\phi}\right]^{-1}\nabla'_\nu\nabla'_\mu\phi'. \tag{2.24}$$

We can then substitute the eqs. (2.18) to (2.22) and (2.24) into the action eq. (2.15) and perform an integration by parts on the term with $\nabla'_\mu\nabla'_\nu\phi'$ to get, up to surface terms, the action

$$I = \int d^2x\sqrt{-g'}\left[\phi' R' + \left(\frac{1}{2} - \frac{2}{\Omega}\frac{d\Omega}{d\phi}\frac{d\phi'}{d\phi}\right)\left(\frac{d\phi'}{d\phi}\right)^{-2} g'^{\mu\nu}\nabla'_\mu\phi'\nabla'_\nu\phi' + \frac{V(\phi')}{\Omega^2}\right]. \tag{2.25}$$

It becomes obvious that to eliminate the kinetic term in the action eq. (2.25), we require that

$$\frac{1}{2} - \frac{d\phi'}{d\phi}\frac{d\ln\Omega^2}{d\phi} = 0 \tag{2.26}$$

and

$$\Omega^2 \neq 0. \tag{2.27}$$

Equation (2.26) allows us to define the conformal factor, Ω, as

$$\Omega^2(\phi) = \exp\frac{1}{2}\int^{\phi} d\phi\left(\frac{d\phi'}{d\phi}\right)^{-1}. \tag{2.28}$$

We can drop the primes and redefine the potential as

$$V(\phi') \to \Omega^2 V(\phi). \tag{2.29}$$

The general dilaton-gravity action becomes

$$I = \int d^2x\sqrt{-g}\{\phi R + V(\phi)\}. \tag{2.30}$$

This action is called the kinetic-free generic dilaton-gravity action and through a conformal transformation is equivalent to the action eq. (2.10). This means it naturally incorporates the class of theories with Schwarzschild-like black holes given in eq. (2.9). The only free input in the theory is the potential term and is therefore the only model dependent term. It is worth mentioning again that the dilaton in eq. (2.30) is a transformation of the dilaton of the generic action eq. (2.10). This new form could work to our advantage as we shall see later on.

We will now derive the equations of motion from the action eq. (2.30) using the variational principle. We vary the action eq. (2.30) with respect to the dilaton and the metric as follows:

$$\delta I = \int d^2x\sqrt{-g}\left[\left(R + \frac{dV}{d\phi}\right)\delta\phi + \frac{\phi}{\sqrt{-g}}\delta(\sqrt{-g}R) + \frac{V(\phi)}{\sqrt{-g}}\delta(\sqrt{-g})\right]. \tag{2.31}$$

The variations $g_{\mu\nu}$ and $\sqrt{-g}$ are well known [22]:

$$\delta g^{\mu\nu} = -g^{\mu\rho}g^{\nu\sigma}\delta g_{\rho\sigma} \tag{2.32}$$

$$\delta(\sqrt{-g}) = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}. \tag{2.33}$$

We also expand the variation for the third term in eq. (2.31) as follows:

$$\begin{aligned}\delta(\sqrt{-g}R) &= \delta(\sqrt{-g})R + \sqrt{-g}\delta g^{\mu\nu}R_{\mu\nu} + \sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu} \\ &= \sqrt{-g}\left[R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right]\delta g^{\mu\nu} + \sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu} \\ &= \sqrt{-g}\,\left[G_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}\right]. \end{aligned} \tag{2.34}$$

But the Einstein tensor trivially vanishes in two-dimensions, that is $G_{\mu\nu} = 0$. This leaves the last term which can be evaluated using the Palatini identity [22] as follows:

$$\delta R_{\mu\nu} = \frac{1}{2}g^{\alpha\beta}\Big(\nabla_\mu\nabla_\alpha\delta g_{\nu\beta} + \nabla_\nu\nabla_\alpha\delta g_{\beta\mu} - \nabla_\mu\nabla_\nu\delta g_{\alpha\beta} - \nabla_\alpha\nabla_\beta\delta g_{\mu\nu}\Big). \tag{2.35}$$

Next we will evaluate $g^{\mu\nu}\delta R_{\mu\nu}$ which initially reads as follows:

$$g^{\mu\nu}\delta R_{\mu\nu} = \frac{1}{2}g^{\alpha\beta}g^{\mu\nu}\Big(\nabla_\mu\nabla_\alpha\delta g_{\nu\beta} + \nabla_\nu\nabla_\alpha\delta g_{\beta\mu} - \nabla_\mu\nabla_\nu\delta g_{\alpha\beta} - \nabla_\alpha\nabla_\beta\delta g_{\mu\nu}\Big). \tag{2.36}$$

Now using $\nabla_\mu g^{\alpha\beta} = 0$ and $g^{\alpha\beta}\delta g_{\rho\beta} = -g_{\rho\beta}\delta g^{\alpha\beta}$, eq. (2.36) becomes

$$g^{\mu\nu}\delta R_{\mu\nu} = \frac{1}{2}\Big(- g^{\alpha\beta}g_{\nu\beta}\nabla_\mu\nabla_\alpha\delta g^{\mu\nu} - g^{\alpha\beta}g_{\beta\mu}\nabla_\nu\nabla_\alpha\delta g^{\mu\nu} + g_{\alpha\beta}\Box\delta g^{\alpha\beta} + g_{\mu\nu}\Box\delta g^{\mu\nu}\Big). \tag{2.37}$$

We can clean up the previous expression using $g^{\alpha\beta}g_{\beta\mu} = \delta^\alpha_\mu$ so that finally we get

$$\delta(\sqrt{-g}R) = \sqrt{-g}\big(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu\big)\delta g^{\mu\nu}. \tag{2.38}$$

Finally, after integrating by parts the variation of the action reads:

$$\delta I = \int d^2x\sqrt{-g}\left[\left(R + \frac{dV}{d\phi}\right)\delta\phi + \left(\left[g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu\right]\phi - \frac{1}{2}g_{\mu\nu}V(\phi)\right)\delta g^{\mu\nu}\right]. \tag{2.39}$$

There are surface terms in the derivation of eq. (2.39) and also from the derivation of the action eq. (2.30). As argued in [16], the variations from these surface terms in the action should cancel out the variations that arose during the integration by parts in the derivation of eq. (2.39).

The equations of motion follow by extremizing the action, that is $\delta I = 0$. If we bring out γ^2 from the potential, the equations of motion read

$$R + \gamma^2\frac{dV}{d\phi} = 0 \tag{2.40}$$

and

$$[g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu]\,\phi - \frac{1}{2}g_{\mu\nu}\gamma^2 V(\phi) = 0. \tag{2.41}$$

By multiplying by $g^{\mu\nu}$ and contracting the indices $g^{\mu\nu}g_{\mu\nu} = 2$, we have

$$\Box\phi = \gamma^2 V(\phi), \tag{2.42}$$

which when reinserted into eq. (2.41) gives:

$$\nabla_\mu\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\gamma^2 V(\phi) = 0. \tag{2.43}$$

2.3 Properties of dilaton-gravity models

Dilaton-gravity theories have been found to have a few interesting properties. In trying to describe two-dimensional black holes, the equations of motion play a huge role. From eqs. (2.40) and (2.43), one can derive the local and thermodynamic properties of all generic dilaton-gravity models. These properties can be further extended to find the global properties of the theory.

2.3.1 Local properties

A rigorous study by Louis-Martinez and Kunstatter [44, 45] has shown that a generalized Birkhoff's theorem holds for this theory provided eqs. (2.40) and (2.43) hold (see appendix B for a detailed proof of generalized Birkhoff's theoerem). The generalized Birkhoff's theorem would then require that the space-time be stationary: space-time admits an asymptotically time-like Killing vector. Killing vectors are infinitesimal generators of isometries: transformations that leave the metric unchanged. If X_μ is a Killing vector, then

$$\nabla_\mu X_\nu + \nabla_\nu X_\mu = 0. \tag{2.44}$$

The generalized Birkhoff's theorem is further emphasized in [46], where the solutions to eqs. (2.40) and (2.43) can be written in a Schwarzschild-like metric as

$$ds^2 = -\big(J(\gamma r) - 2M\big)dt^2 + \big(J(\gamma r) - 2M\big)^{-1}dr^2, \tag{2.45}$$

$$\phi = \gamma r, \tag{2.46}$$

where γ is the inverse length, $J(\phi)$ is a function defined as

$$J(\phi) = \int^{\phi} dy V(y) \tag{2.47}$$

and M is a coordinate independent constant that arises as a result of Birkhoff's theorem (see appendix B for derivation of M). It takes the form

$$M = -\frac{1}{2}\left(\frac{1}{\gamma^2}(\nabla\phi)^2 - J(\phi)\right). \tag{2.48}$$

M can also be interpreted as the coordinate independent mass of the spherical object [44]. In addition, the Killing vector of generic dilaton-gravity takes the form

$$k^{\mu} = \frac{\epsilon^{\mu\nu}}{\gamma\sqrt{-g}}\nabla_{\nu}\phi, \tag{2.49}$$

where $\epsilon^{\mu\nu}/\sqrt{-g}$ is the permutation tensor. Using eq. (2.43), it is clear that k^{μ} satisfies the Killing equation eq. (2.44) owing to the fact that $\epsilon^{\mu\nu}$ is antisymmetric in its indices. The square of the magnitude of the Killing vector is given by

$$k_{\mu}k^{\mu} = -\frac{1}{\gamma^2}(\nabla\phi)^2. \tag{2.50}$$

Substituting eq. (2.48) into eq. (2.50) gives

$$k_{\mu}k^{\mu} = 2M - J(\phi). \tag{2.51}$$

Equation (2.51) is of particular importance considering that if eq. (2.43) admits one solution for the dilaton, ϕ_h, such that

$$J(\phi_h) = 2M, \tag{2.52}$$

then the Killing vector becomes null. If in the neighbourhood of ϕ_h the function $J(\phi)$ is monotonic, then the Killing vector changes sign from being time-like to being space-like and thereby indicating the presence of a Killing horizon. One could conclude that the theory indeed contains black hole solutions. However, as emphasized in [46], one cannot simply infer the existence of a black hole from the local properties of the solutions. A more consistent way to describe a black hole would require an assessment of both the global and thermodynamic properties of the solutions.

2.3.2 Thermodynamic properties.

The thermodynamic properties of two-dimensional black holes can be obtained using the results of the previous subsections. We recall that the Hawking temperature of Hawking radiation is

$$T_H = \frac{\kappa}{2\pi}, \tag{2.53}$$

where κ is the surface gravity of the black hole. According to Wald [42], we can compute the surface gravity of a black hole using the formula

$$\kappa^2 = -\frac{1}{2}\nabla^\mu k^\nu \nabla_\mu k_\nu, \tag{2.54}$$

where k^μ is the Killing vector and eq. (2.54) is evaluated at the event horizon. Assuming that the theory can describe black holes, we can substitute the Killing vector associated to this theory as given by eq. (2.49) into Wald's formula. The result is

$$\kappa^2 = \frac{1}{2\gamma^2 g} g^{am} g_{bn} \epsilon^{bc} \epsilon^{np} (\nabla_m \nabla_c \phi)\Big|_{\phi=\phi_h} (\nabla_a \nabla_p \phi)\Big|_{\phi=\phi_h}. \tag{2.55}$$

Then using eq. (2.43) in eq. (2.55) we get

$$\kappa^2 = \frac{1}{8\gamma^2 g} g^{am} g_{bn} g_{mc} g_{ap} \epsilon^{bc} \epsilon^{np} \gamma^4 V^2(\phi_h). \tag{2.56}$$

Then contracting indices and recalling that the determinant of the metric tensor can be written as $g = \frac{1}{2!}\epsilon^{ba}\epsilon^{np} g_{bn} g_{ap}$, the expression for the surface gravity can be written as

$$\kappa = \frac{\gamma}{2} V(\phi_h). \tag{2.57}$$

We could also obtain this expression using the more familiar approach used for Schwarzschild-like metrics. Given an exterior Schwarzschild-like metric

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2, \tag{2.58}$$

the surface gravity is given by

$$\kappa = \frac{1}{2}\frac{df}{dr}\Big|_{r=r_h}, \tag{2.59}$$

where r_h is the value of the radius at the horizon. Then from the metric given by eq. (2.45), we get

$$\frac{d}{dr}(J(\gamma r) - 2M)\bigg|_{r=r_h} = \gamma V(\phi_h), \tag{2.60}$$

from which the surface gravity is given by

$$\kappa = \frac{\gamma}{2}V(\phi_h). \tag{2.61}$$

A straightforward substitution into eq. (2.53) gives the Hawking temperature for generic two-dimensional black holes as

$$T_H = \frac{\gamma}{4\pi}V(\phi_h), \tag{2.62}$$

where ϕ_h is the horizon value for the dilaton. To get the entropy, we recall from chapter 1 that the first law of black hole mechanics is given by

$$\delta M = \frac{\kappa}{8\pi}\delta A, \tag{2.63}$$

where M is the mass of the black hole, κ is the surface gravity and A is the area of the event horizon. This law is analogous to the first law of thermodynamics given by

$$\delta Q = T\delta S, \tag{2.64}$$

where Q is the energy of the system, T is its temperature and S is the entropy. The analogous quantities between the two laws being $Q \leftrightarrow M$, $T \leftrightarrow \kappa$ and $S \leftrightarrow A$. Q and M in fact represent the energy of the system if we consider $Q = Mc^2$ and that c is set to unity. We recall that in two dimensions, M the coordinate independent parameter from eq. (2.48) can be interpreted as the mass of the black hole. Hence, if we consider a small change at the horizon $\phi = \phi_h$, eq. (2.48) becomes

$$\delta M = \frac{1}{2}\delta J(\phi_h), \tag{2.65}$$

where the kinetic term vanishes because the Killing vector is null at the horizon. Using the definition of $J(\phi)$ in eq. (2.47), the right hand side of eq. (2.65) becomes

$$\delta J(\phi_h) = V(\phi_h)\delta\phi_h. \tag{2.66}$$

Putting the last expression into eq. (2.65) and replacing M with Q, we get

$$\delta Q = \frac{1}{2}V(\phi_h)\delta\phi_h. \tag{2.67}$$

We then compare eqs. (2.64) and (2.67) and use the temperature at the horizon, which turns out to be the Hawking temperature T_H, what we get is

$$T_H \delta S = \frac{1}{2} V(\phi_h) \delta \phi_h. \tag{2.68}$$

Substituting eq. (2.62) into eq. (2.68) and simplifying the resulting equation gives

$$\delta S = \frac{2\pi}{\gamma} \delta \phi_h. \tag{2.69}$$

From this we get the entropy of generic two-dimensional black holes as

$$S = \frac{2\pi}{\gamma} \phi_h. \tag{2.70}$$

2.3.3 Global properties

Following the analysis in [46], we will now look at the global structure of space-time in the theory given by eq. (2.30). As has been stated, the only free input in the theory is the potential $V(\phi)$. It would then seem reasonable to presume that if indeed the theory has black holes, then such information is encoded in potential. Since the potential depends on the dilaton, then perhaps special attention must be given to the dilaton and, if necessary, to place a few constrains on it. It turns out that just as in the Brans-Dicke theory, the dilaton takes the role of the coupling constant in the theory described by eq. (2.30). Particularly in the context of gauge theory, as also demonstrated in [47], one considers the coupling constant of the theory to be $g_s = \sqrt{\phi^{-1}}$. This is not surprising considering that even in higher dimensional string theories where the dilaton appears, its expectation value controls the string coupling [2]. Hence, the space-time is divided into a strong coupling region where $\phi \to 0$ and a weak coupling region where $\phi \to \infty$. To ensure that the coupling constant is real, we restrict the range of the dilaton to $0 \leq \phi < \infty$.

Next we impose more conditions on the theory so that eq. (2.45) can be identified as a black hole solution. We consider the following conditions:

I. There is at least one point $\phi_h > 0$ such that $J(\phi_h) \neq 0$ and $V(\phi_h) \neq 0$. In the neighbourhood of ϕ_h where $\phi_{h-1} < \phi_h < \phi_{h+1}$, J is monotonically increasing. That is to say $J(\phi_{h-1}) < J(\phi_h) < J(\phi_{h+1})$.

II. In the weak coupling region where $\phi \to \infty$, the Killing vector is time-like for a finite value of M.

III. In the weak coupling region, the curvature is finite.

Condition I, as we have stated it here, is a slight deviation from the literature in [46]. We firmly require that $J(\phi)$ be a monotonically increasing function if we are to identify a Killing horizon. If for $\phi > \phi_h$ the Killing vector is time-like and for $\phi < \phi_h$ the Killing vector is space-like, then the point $\phi = \phi_h$ is where the Killing vector becomes null. For our purposes this null horizon will coincide with the black hole event horizon.

Condition II places a constraint on the behaviour of the potential through the function $J(\phi)$. To illustrate this, we recall that the mass of a black hole M is finite and supposing we let $J(\phi \to \infty) \to L$, for some constant L, then it is possible for $M > L/2$ so that the Killing vector is space-like in the weak coupling region. This could be problematic since one would like events to be causally connected when one is sufficiently far from a black hole. Hence condition II necessarily requires that $J(\phi) \to \infty$ as $\phi \to \infty$, which implies that $\int^{\infty} V(y)dy = \infty$. This can also be viewed as a statement of the global monotonicity of $J(\phi)$.

Condition III directly constrains the potential if we recall from eq. (2.40) that the curvature scalar goes as $R \propto \frac{dV}{d\phi} = \frac{d^2 J}{d\phi^2}$ and must be finite as $\phi \to \infty$. In addition, we expect R to be finite everywhere except at the singularity. Following this constraint, we take the simplest form for J as

$$J(\phi) \propto \phi^{\alpha+1}, \quad \alpha \leq 1. \tag{2.71}$$

However, there is an additional constraint we must impose on eq. (2.71). From the second law of thermodynamics and the second law of black hole mechanics, we know that

$$\delta S \geq 0 \text{ and } \delta A \geq 0, \tag{2.72}$$

respectively. From the entropy of two-dimensional black holes given by eq. (2.70), we know that

$$S \sim \phi_h. \tag{2.73}$$

We also know that at the horizon, the mass and the function $J(\phi)$ are related through

$$M \sim J. \tag{2.74}$$

So using eqs. (2.71), (2.73) and (2.74), we obtain a relationship between the entropy and the mass of the black hole in the form of

$$S \sim M^{\frac{1}{\alpha+1}}. \tag{2.75}$$

For small variations, this expression comes down to

$$\delta S \sim \frac{1}{\alpha+1} M^{-\frac{\alpha}{\alpha+1}} \delta M \geq 0. \tag{2.76}$$

The first law of black hole mechanics and the entropy relation tell us that

$$\delta M \geq 0. \tag{2.77}$$

As a consequence, eq. (2.76) requires that

$$\alpha > -1. \tag{2.78}$$

From eqs. (2.74) and (2.77), we can deduce that J is a monotonically increasing function

$$\delta J \geq 0. \tag{2.79}$$

This expression also confirms condition I for the region close to the event horizon. We can therefore conclude that the simplest potential that is consistent with conditions I - III and eq. (2.71) is

$$V(\phi) \propto \phi^{\alpha}, \ \ -1 < \alpha \leq 1. \tag{2.80}$$

The lower limit of α ensures $J(\phi)$ is a monotonic increasing function while the upper limit ensures the curvature is finite in the weak coupling region. Using the dilaton reparametrization and conformal rescaling from eqs. (2.16) and (2.17) on eq. (2.9), it can be shown that $\alpha \equiv b$. Hence, the potential given by eq. (2.80) also accounts for black holes studied in [39].

At first glance, the metric solution given by eq. (2.45) does not generally appear to be a black hole solution. This becomes more apparent when we set $\alpha = 0$ or $\alpha = 1$ in eq. (2.80), where the curvature scalar is constant and the entire space-time appears to be free of physical singularities. This conundrum is addressed in [46] by recalling that the space-time is described by both the metric tensor and the dilaton. Since the dilaton is a geometrical quantity that has a lower bound at $\phi = 0$, the space-time comes to an abrupt end at $\phi = 0$. Thus, we consider the point $\phi = 0$ as the end point of the space-time for all possible models in the theory. Furthermore, we are reminded that even in four dimensions the singularity is considered to be the end point of space-time. The natural conclusion would be that the point $\phi = 0$ is the physical singularity of any black hole for any model in this theory. This also reinforces the reasoning that the region near $\phi = 0$ is the strong coupling region since the coupling constant, g_s, diverges there.

Furthermore, it has been argued in [48] that physical observables associated with such a theory are invariant under the dilaton reparametrization and conformal transformations from section 2.2. For instance, we could have still obtained the entropy given by eq. (2.70) and Hawking temperature eq. (2.62) from the solutions of the generic theory eq. (2.10). The notion of a singularity is also gauge invariant in two-dimensional gravity.

2.4 Conformal gauge frame

It is well known that two-dimensional space-times are conformally flat, that is any two-dimensional space-time metric can be written as

$$ds^2 = \Omega^2(t,x)(-dt^2 + dx^2). \tag{2.81}$$

Now supposing the metric solution to the theory given by eq. (2.30) can be written in the conformal gauge as

$$ds^2 = e^{2\rho(x)}(-dt^2 + dx^2), \tag{2.82}$$

then the Killing vector from the metric eq. (2.82) would be $k^\mu = (1,0)$ and the square of its magnitude would be

$$k_\mu k^\mu = -e^{2\rho(x)}. \tag{2.83}$$

Since we are describing the same theory and the magnitude of the killing vector is in fact an invariant, then the expressions from eqs. (2.51) and (2.83) are indeed equivalent

$$e^{2\rho} = J(\phi) - 2M. \tag{2.84}$$

Equation (2.84) tells us that at the Killing horizon where eq. (2.52) holds, the conformal factor behaves as

$$\lim_{x \to x_h} e^{2\rho(x)} = 0, \tag{2.85}$$

where x_h denotes the point when $\phi = \phi_h$. Equation (2.85) holds for any model that admits a black hole. In fact eq. (2.84) can be considered to be a solution to the theory expressed in the conformal gauge. Admittedly, the conformal gauge perspective does not have the descriptive power of the Schwarzschild gauge, but it is the more general way to study two-dimensional gravity. For that reason, we will mostly study dilaton-gravity in the conformal gauge.

In summary, higher dimensional theories of gravity are reducible to two-dimensional scalar-tensor theories that admit Schwarzschild-like black holes. These theories take the form

$$I[\phi, g_{\mu\nu}] = \int d^2x\sqrt{-g}\left[\phi R + \gamma^2\phi^{\alpha}\right], \tag{2.86}$$

where $-1 < \alpha \leq 1$ and γ^2 is a constant of dimensions 1/length^2.

Chapter 3

Classical picture of black hole interior

Much of the work that has been presented so far has been to motivate the use of two-dimensional dilaton-gravity for the study of black holes. Although not explicitly stated, the theory studied in chapter 2 was meant to tell a story of how we could consistently describe the exterior geometry of Schwarzschild-like black holes in dilaton gravity. This chapter will be devoted to studying the interior of these black holes using the theory described in chapter 2. In the spirit of tradition, we will briefly look at the interior of realistic four dimensional Schwarzschild black holes before we proceed with the two-dimensional description.

3.1 The nature of the black hole interior

Since the theoretical discovery of black holes, not much has been said about their interior. As they are experimentally inaccessible, the only thing we know for sure is that behind the event horizon lies the singularity. In fact, the event horizon is appropriately named so as to describe the limit of our understanding of events after the surface of a black hole. We can at least study the behaviour of coordinates from the exterior perspective versus the interior perspective. This is possible if we take an infinitesimal variation in a coordinate of interest and fix the other coordinates followed by measuring the corresponding change of a coordinate invariant quantity such as the line element, ds^2. If we apply this procedure on the Schwarzschild metric given by eq. (1.3) and study the behaviour of the t-coordinate from r-infinity to a point near the horizon, we find that it exhibits a time-like behaviour for all $r > 2M$. For the Schwarzschild metric, the interior and exterior geometries are disconnected. However, we can use an analytic continuation on the Schwarzschild metric and study the behaviour of t for the interior. We then find that t has a space-like behaviour when $r < 2M$. Just the opposite effect happens if we focus on the behaviour of the r-coordinate. While the angular coordinates maintain their properties, this tells us that the temporal and the spatial coordinates exchange characteristics once we observe the metric from either side of the event horizon of a black hole. In a nutshell, space becomes time and time becomes space. This is further emphasized in more recent works by Doran, Lobo, and Crawford [49] where the metric of the interior takes the form

$$ds^2 = -\left[\frac{2M}{t} - 1\right] dr^2 + \left[\frac{2M}{t} - 1\right]^{-1} dt^2 + t^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{3.1}$$

where M is the mass of the black hole concentrated at the singularity. We could study other aspects of the geometry of the interior including the geodesics of in-falling observers. But for our purposes only, it will be sufficient to state that in the interior of a black hole, the metric tensor is a function of t for any choice of coordinate system. In two dimensions we can express the metric of the interior of a black hole in terms of a conformally flat metric. It would then take the form

$$ds^2 = \Omega^2(t)(-dt^2 + dx^2). \tag{3.2}$$

Since we have already established that a description of gravity in two dimensions requires the dilaton, it is only natural that it too would be a function of t only in the interior of a black hole. Drawing inspiration from eqs. (2.45) and (3.1), we have an idea of the metric solution for generic dilaton-gravity theory eq. (2.86) in the interior of a black hole. Such a metric solution expressed in the Schwarzschild gauge would take the form

$$ds^2 = -\big(J(\gamma\tau) - 2M\big)dr^2 + \big(J(\gamma\tau) - 2M\big)^{-1} d\tau^2, \tag{3.3}$$

where τ is the space-like coordinate and again we have the monotonic function $J(\phi) = \int^{\phi} d\zeta\ V(\zeta)$. In the interior, the function $J(\phi)$ ranges as $0 \leq J(\phi) \leq J(\phi_h)$, where at the singularity we have $J(0) = 0$ and at the horizon we have $J(\phi_h) = 2M$. In addition, the dilaton solution takes the form similar to that of the exterior solution

$$\phi = \gamma\tau. \tag{3.4}$$

The Schwarzschild gauge formalism has the advantage of helping us understand the behaviour of dynamical fields for any dilaton-gravity theory. However, the conformal gauge formalism is usually the preferred formalism when one wants to study the quantum aspects of a two-dimensional theory. This is because any two-dimensional manifold is conformally flat, hence the conformal gauge formalism in some sense gives us the most general picture of a two-dimensional theory. Recently, the interior of SIG black holes was studied in [32] using the conformal gauge formalism. However, we will take on a different approach by using the kinetic-free generalized dilaton-gravity action eq. (2.86). The classical picture in this chapter will be extended in chapter 5 when we include quantum corrections to the dilaton and the metric tensor.

3.2 Classical equations of motion for black hole interior

The starting point is the generalized kinetic-free dilaton-gravity action,

$$I[g,\phi] = \int d^2x\sqrt{-g}\{\phi R + \gamma^2\phi^\alpha\}, \text{ where } -1 < \alpha \leq 1 \tag{3.5}$$

and γ is a strictly non-zero real number. We introduce the conformal gauge for the interior of a black hole,

$$ds^2 = e^{2\rho(t)}(-dt^2 + dr^2). \tag{3.6}$$

The curvature scalar for this metric is given by $R = -2\Box\rho = 2e^{-2\rho}\ddot{\rho}$, where $\dot{\rho} = \frac{d\rho}{dt}$ and $\ddot{\rho} = \frac{d^2\rho}{dt^2}$. Since we consider all geometrical fields to be time dependent in the interior, the action given by eq. (3.5) becomes

$$I = L\int dt\{2\phi\ddot{\rho} + \gamma^2 e^{2\rho}\phi^\alpha\}, \tag{3.7}$$

where $L = \int dx$. We then vary eq. (3.7) with respect to each field and obtain the following equations of motion

$$\ddot{\rho} + \frac{\alpha}{2}\gamma^2 e^{2\rho}\phi^{\alpha-1} = 0, \tag{3.8}$$

$$\ddot{\phi} + \gamma^2 e^{2\rho}\phi^\alpha = 0. \tag{3.9}$$

Finding exact generic solutions for these non-linear equations has proved to be difficult if not impossible. Ideally, we would like to find solutions of the type:

$$\phi = \phi(t), \tag{3.10}$$

$$\rho = \rho(t), \tag{3.11}$$

but unfortunately the power law in the potential makes it nearly impossible to achieve such results. It turns out that for some specific values of α, $\alpha = \{0, 1\}$, the equations of motion can be solved exactly. This is only because these models have a finite curvature scalar everywhere and so the dilaton and the conformal factor decouple right from the start. These special models are in fact the SIG and JT models [46]. For a more general treatment, we use the global features of the theory, as discussed in section 2.3, to determine the behaviour of the fields in some regime of interest. We are also reminded that the action eq. (3.5) also yields an extra equation of motion in the form

$$\nabla_\mu \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} \gamma^2 \phi^\alpha = 0. \tag{3.12}$$

In the conformal gauge, we can express the non-trivial equation as

$$\ddot{\phi} - \dot{\rho}\dot{\phi} + \frac{1}{2}\gamma^2 e^{2\rho} \phi^\alpha = 0. \tag{3.13}$$

Unfortunately, even if we tried several manipulations using eqs. (3.8), (3.9) and (3.13), we would still be nowhere near finding the most general solutions of the type (3.10) and (3.11). Still, any kind of solution we find must be consistent with eq. (3.13).

We could still employ one more method given that the action eq. (3.5) admits Schwarzschild-like black holes which have an interior given by eq. (3.3). Using the fact that the line element is an invariant quantity and both metric components depend on a single coordinate, the conformal gauge metric eq. (3.6) and the Schwarzschild gauge metric eq. (3.3) are equivalent if the r-components are related through

$$e^{2\rho(t)} dr^2 = \big(2M - J(\gamma\tau)\big) dr^2 \tag{3.14}$$

and the t-components are related through

$$-e^{2\rho(t)} dt^2 = \big(J(\gamma\tau) - 2M\big)^{-1} d\tau^2. \tag{3.15}$$

From eq. (3.14) we get,

$$e^{2\rho(t)} = \big(2M - J(\gamma\tau)\big). \tag{3.16}$$

To understand eq. (3.15), we are reminded that the dilaton and the temporal coordinate are related through eq. (3.4). Hence, by substituting eq. (3.16) into eq. (3.15) and making the necessary simplifications we get

$$\gamma dt = \frac{d\phi}{\big(2M - J(\phi)\big)}. \tag{3.17}$$

From the power law potential in the action eq. (3.5), we know the explicit form of the function J is

$$J(\phi) = \int^\phi \zeta^\alpha d\zeta = \frac{1}{\alpha+1}\phi^{\alpha+1}. \tag{3.18}$$

Substituting this term back into eq. (3.17), we get

$$\frac{\gamma}{\alpha+1}t = \int \frac{d\phi}{C_0 - \phi^{\alpha+1}}, \tag{3.19}$$

where $C_0 = 2M(\alpha+1)$. After integrating the right hand side of eq. (3.19), the general solution takes the form

$$\frac{\gamma}{\alpha+1}t = \frac{\phi}{C_0^{\frac{\alpha}{\alpha+1}}} F\left(\frac{1}{\alpha+1}, 1, \frac{\alpha+2}{\alpha+1}; \frac{\phi^{\alpha+1}}{C_0}\right), \tag{3.20}$$

where $F(a, b, c; z)$ is a hypergeometric function defined as

$$F(a, b, c; z) = 1 + \frac{ab}{1!c}z + \frac{a(a+1)b(b+1)}{2!c(c+1)}z^2 + \cdots . \tag{3.21}$$

When eq. (3.21) is applied in eq. (3.20), the series may not be convergent for general α. This is clearly problematic since it may not be possible to invert eq. (3.20) so as to express ϕ in terms of t. However, things look more favourable when we consider values $\alpha = -1/2$, $\alpha = 0$ and $\alpha = 1$ because then we can integrate eq. (3.19). But as we shall see later on, using the conformal gauge and trying to find solutions by solving eqs. (3.8) and (3.9) is more beneficial when we consider quantum corrections. So before trying to understand the behaviour of solutions for the general model, we will first look at the exact solutions of some specific models.

3.3 Interior space of solutions

We know that the generic action eq. (3.5) is related to the action eq. (2.9) through a dilaton reparametrization and a conformal transformation. So α must be related to the parameters a, b and so we found in section 2.3.3 that $\alpha \equiv b$.

3.3.1 String inspired gravity (SIG) ($a = 1$ *and* $b = 0$ *in eq.* (2.9))

For this model $\alpha = 0$ and we obtain a constant curvature from eq. (2.40). The field eqs. (3.8) and (3.9) become

$$\ddot{\rho} = 0, \tag{3.22}$$

$$\ddot{\phi} + \gamma^2 e^{2\rho} = 0. \tag{3.23}$$

The differential eqs. (3.22) and (3.23) have solutions

$$\rho = c_0 t + c_1, \tag{3.24}$$

$$\phi = -\frac{\gamma^2}{4c_0^2} e^{2\rho} + d_0 t + d_1, \tag{3.25}$$

where c_0, c_1, d_0 and d_1 are integration constants. These solutions can also be obtained using the Schwarzschild-gauge eqs. (3.16) and (3.19). Next we find the horizon of the black hole using Killing vectors as we did in section 2.3. The square of the magnitude of the Killing vector is proportional to $e^{2\rho}$ and so at the Killing horizon we expect that $e^{2\rho} = 0$. From eq. (3.24), we find that if $c_0 > 0$, then the horizon is located at $t_H = -\infty$. On the other hand, if $c_0 < 0$, then we have the horizon at $t_H = +\infty$. Since we can choose the domain of the time coordinate as the half-line $-\infty \le t \le 0$, as long as $0 \le \phi < \infty$, then we can set $0 < c_0 < \infty$. Also, the dilaton must be finite at the horizon, hence we are forced to set $d_0 = 0$. We can also choose a value for d_1 such that at the singularity, for some finite value $t = t_s$, $\phi(t_s) = 0$ and as a consequence, d_1 takes the value

$$d_1 = \frac{\gamma^2}{4c_0^2} e^{2\rho(t_s)}. \tag{3.26}$$

The value of d_1 in eq. (3.26) therefore fixes the location of the horizon:

$$\phi(-\infty) = d_1 = \frac{\gamma^2}{4c_0^2} e^{2\rho(t_s)}. \tag{3.27}$$

Furthermore, we are free to set the value of c_0 to

$$c_0 = \frac{\gamma e^{\rho(t_s)}}{2}. \tag{3.28}$$

This appropriately sets the position of the horizon and therefore the radius of the black hole as $\phi = 1$. As pointed out in section 2.3, this type of theory has a constant curvature throughout the interior of the black hole. The lack of a curvature singularity is resolved by realizing that the coupling constant diverges for a specific value of the dilaton.

Of crucial importance is the near singularity behaviour of the dilaton for this model. Let us suppose $t_s = 0$ and set $c_1 = 0$. We then have $\phi(0) = 0$ and $d_1 = \frac{\gamma^2}{4c_0^2}$. For small values of t such that $t \to 0$, we have

$$\begin{aligned}\phi(t) &= -\frac{\gamma^2}{4c_0^2}\Big(1 + 2c_0 t + \mathcal{O}(t^2)\Big) + \frac{\gamma^2}{4c_0^2} \\ &= -\frac{\gamma^2}{2c_0} t.\end{aligned} \tag{3.29}$$

The last expression is positive if we recall that either $c_0 > 0$ and $t \leq 0$ or $c_0 < 0$ and $t \geq 0$. The important thing here is that close to the singularity, the dilaton behaves as $\phi \sim t$.

Finally, we note the proper time for an observer to freely fall from the horizon to any point in the interior, including the singularity,

$$\Delta\tau(t_f) = \int_{-\infty}^{t_f} dt\ e^{\rho} = \frac{1}{c_0} e^{\rho(t_f)}, \tag{3.30}$$

which is a finite period of time.

3.3.2 Jackiw-Teitelboim model (JT) ($a = 0$ *and* $b = 1$ *in eq.* (2.9))

This is another constant curvature theory where we set $\alpha = 1$. The equations of motion for this model are given by

$$\ddot{\rho} + \frac{\gamma^2}{2} e^{2\rho} = 0, \tag{3.31}$$

$$\ddot{\phi} + \gamma^2 e^{2\rho}\phi = 0. \tag{3.32}$$

The solutions to eqs. (3.31) and (3.32) are

$$\phi = \tanh(\gamma t), \tag{3.33}$$

$$e^{2\rho} = 2\mathrm{sech}^2(\gamma t). \tag{3.34}$$

These solutions can be obtained using eqs. (3.16) and (3.19). The event horizon is at a location where $e^{2\rho(t)} = 0$, which given eq. (3.34) happens to be when $t = \infty$. At this point the dilaton takes value, $\phi = 1$. Similarly, we know that at the singularity we are supposed to have $\phi = 0$, this happens when $t = 0$. Finally, a falling observer from the horizon to the singularity takes the proper time

$$\Delta\tau(0) = \int_{\infty}^{0} dt\ e^{\rho} = \frac{\pi\sqrt{2}}{\gamma}. \tag{3.35}$$

3.3.3 Spherically symmetric gravity (SSG) ($a = \frac{1}{2}$ *and* $b = -\frac{1}{2}$ *in eq.* (2.9))

We recover Einstein's spherically reduced gravity when we set $\alpha = -1/2$ in eq. (3.5). This model admits black holes reminiscent of realistic four dimensional black holes as evidenced by the diverging curvature scalar when we set $\phi = 0$ in

$$R = \frac{\gamma^2}{2}\phi^{-3/2}. \tag{3.36}$$

The field equations for this model read:

$$\ddot{\rho} - \frac{\gamma^2}{4} e^{2\rho} \phi^{-3/2} = 0, \tag{3.37}$$

$$\ddot{\phi} + \gamma^2 e^{2\rho} \phi^{-1/2} = 0. \tag{3.38}$$

Here we encounter the same difficulties of finding solutions as we did in the general case. Using eqs. (3.16) and (3.19), we find the solutions for this model take the form

$$e^{2\rho} = 2M - 2\phi^{1/2}, \tag{3.39}$$

$$t = -\frac{4}{M} \ln\left(2 - \phi^{1/2}\right) + \frac{2}{M}\left(2 - \phi^{1/2}\right). \tag{3.40}$$

If we insert eq. (3.39) into eq. (3.38), we find a non-linear, non-exact second order differential equations whose solution is eq. (3.40). We still cannot express these solutions in the form

$$\rho = \rho(t), \tag{3.41}$$

$$\phi = \phi(t), \tag{3.42}$$

The best we can do is study the relationship between ϕ and t from eq. (3.40). We know that $\phi = 0$ marks the position of the singularity. In this model, this happens when

$$t = \frac{4}{M} \ln \frac{e}{2}. \tag{3.43}$$

We could always perform a shift $t \to t'$ so that the solution behaves as

$$\lim_{\phi \to 0} t' = 0. \tag{3.44}$$

Ideally, in the shifted coordinate system we would then invert the solution so that

$$\lim_{t' \to 0} \phi(t') = 0. \tag{3.45}$$

3.4 Near singularity behaviour of generalized dilaton-gravity

As we discussed earlier, the general solution of eqs. (3.8) and (3.9) and the integral of eq. (3.19) could be impossible to find. We could at least try to find an approximate solution or the general leading order behaviour of the dilaton and the conformal factor as one approaches the singularity. We are mostly interested in the region near the singularity because later on we shall make an attempt at partially resolving the singularity using quantum mechanics.

From the SIG and JT models, we saw that near the singularity the dilaton behaves as

$$\phi \sim t, \tag{3.46}$$

$$\lim_{t \to 0} \phi(t) = 0. \tag{3.47}$$

For SSG, we saw that we can perform a shift in the t-coordinate so that the dilaton follows the same behaviour as eq. (3.47). So if we are able to invert the general solution so that $\phi = \phi(t)$, we would then expect the dilaton to behave as shown in eq. (3.47). For $-1 < \alpha < 0$ and $0 < \alpha < 1$, the curvature scalar

$$R = -\gamma^2 \alpha \phi^{\alpha - 1} \tag{3.48}$$

diverges at $\phi = 0$, the physical singularity of the theory. This can also be regarded as a motivation for eq. (3.47) if we consider the limit

$$\lim_{\phi \to 0} t = 0, \tag{3.49}$$

from the general solution eq. (3.20). We can then conclude that near the singularity, the leading-order behaviour of the dilaton should be

$$\phi(t) \sim t^b, \tag{3.50}$$

where $b > 0$.

From eq. (3.16), we can find that for $\phi(0) = 0$, $e^{2\rho(0)}$ is always a constant regardless of model. However, the derivatives of ρ depend on t. This can be easily be shown by differentiating eq. (3.16) with respect to t. We can assume that near the singularity, the leading-order behaviour of these quantities should be

$$e^{2\rho} \sim t^0, \tag{3.51}$$

$$\ddot{\rho} \sim t^a, \tag{3.52}$$

where a is a real number. Obviously, the constants a and b must be model dependent, they must in someway be dependent on α. To see their connection with α, we need the leading-order approximations to be consistent with the field equations

$$\ddot{\rho} + \frac{\alpha}{2}\gamma^2 e^{2\rho}\phi^{\alpha-1} = 0, \tag{3.53}$$

$$\ddot{\phi} + \gamma^2 e^{2\rho}\phi^{\alpha} = 0. \tag{3.54}$$

And so we can determine how a and b are related to α by first substituting eqs. (3.50) and (3.52) into eq. (3.53) to get

$$t^a \sim t^{b(\alpha-1)}. \tag{3.55}$$

Using the same procedure on eq. (3.54), we get

$$t^{b-2} \sim t^{\alpha b}. \tag{3.56}$$

Then from eqs. (3.55) and (3.56), we have the system of equations

$$a = b(\alpha - 1), \tag{3.57}$$

$$b - 2 = b\alpha. \tag{3.58}$$

Equation (3.58) also gives us

$$b = \frac{2}{1-\alpha}. \tag{3.59}$$

If $b > 0$, then $\alpha < 1$. This is to be expected, since we require that the dilaton be finite at $t = 0$. Using eqs. (3.58) and (3.59), we get

$$a = -2. \tag{3.60}$$

This means that as long as $\alpha \neq \{0, 1\}$, the ρ field has the following near singularity leading order behaviour

$$\ddot{\rho} \sim t^{-2}. \tag{3.61}$$

Since $e^{2\rho}$ and its inverse are constant at the singularity, eq. (3.61) is physically reasonable since the curvature scalar must diverge[1] at the singularity for all models, with the exception of the SIG and the JT models.

[1]See eq. (3.48) for general divergent behaviour at the singularity

We can also check the approximate near-singularity solution for the SSG model. For $\alpha = -1/2$, the near singularity leading-order solution for the dilaton is

$$\phi = ct^{\frac{4}{3}}, \tag{3.62}$$

where c is some positive constant. Inserting eq. (3.62) into eq. (3.40) and making use of $\ln(1-x) \approx -x$ for small x, we recover eq. (3.43). This confirms that eq. (3.62) is indeed the near singularity leading-order approximation for the spherically symmetric model. We can also check for the consistency of eq. (3.61), by inserting eq. (3.62) into eq. (3.39) followed by two differentiations. A Taylor expansion would reveal that the leading-order term behaves as eq. (3.61).

To summarize this chapter, we have found the classical picture of a black hole interior by studying the behaviour of the the dilaton and geometry in the conformal gauge. Physically, the behaviour of the dilaton gives us a notion of how fast a free-falling observer approaches the singularity. Then from the behaviour of the conformal factor we can determine what an observer encounters during free-fall. We have thus determined that an observer encounters a space-time with infinite curvature at the singularity for all models such that $\alpha \neq \{0, 1\}$. On the other hand, the space-time is appears to be smooth throughout the interior of a black hole for constant curvature models that are such that $\alpha = \{0, 1\}$. Mathematically, we have found exact solutions for constant curvature models. Since we determined that it is impossible to obtain closed form solutions for near singularity divergent curvature models, we sought to investigate the general behaviour of the classical fields near the singularity. We found that the leading-order behaviour of the classical fields, in the conformal gauge, is

$$\phi \sim t^{\frac{2}{1-\alpha}}, \tag{3.63}$$

$$e^{2\rho} \sim t^0, \tag{3.64}$$

$$\ddot{\rho} \sim t^{-2}. \tag{3.65}$$

This behaviour is consistent as long as $\alpha \neq \{0, 1\}$. Furthermore, one could verify that this behaviour is consistent with eq. (3.13) as long as $\dot{\rho} \sim t^{-1}$.

Chapter 4

Quantum corrections in dilaton-gravity

In this chapter, we will look at an extension of the study in chapter 2 by adding one-loop quantum corrections to generic dilaton-gravity. For a while, we will bring the matter fields back into the picture. This will aid us in finding the effective action of dilaton-gravity.

4.1 The role of classical matter

One of the many statements of Einstein's theory of general relativity is that matter or radiation generates a curvature of space-time. So in as much as matter is influenced by gravity, matter also influences gravity by bending the space-time manifold. This means that when we look at Einstein's equation,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}, \tag{4.1}$$

the nature of the stress tensor $T_{\mu\nu}$ also dictates the nature of $g_{\mu\nu}$. A semi-classical approach to studying gravity involves the use of quantized matter. In Einstein's equation, this would be represented by taking the expectation value of the stress-tensor. And just as in the purely classical case, we expect to observe a measurable fluctuation in gravity due to the quantized matter. As we recall from chapter 1, the semi-classical equation would take the form

$$\bar{R}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{R} + \Lambda\bar{g}_{\mu\nu} = 8\pi \langle T_{\mu\nu}\rangle\,, \tag{4.2}$$

where

$$\bar{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}. \tag{4.3}$$

In eq. (4.3), $h_{\mu\nu}$ represents a small quantum fluctuation in space-time due to quantized matter fields. It stands to reason that an explicit knowledge of $\langle T_{\mu\nu}\rangle$, allows us to understand $h_{\mu\nu}$ and vice-versa. Furthermore, we also mentioned that the field $h_{\mu\nu}$ can be treated as any other quantum field in the theory as long it is understood that it propagates on the classical background $g_{\mu\nu}$. Just as in linearized gravity, we can write the action up to first order in $h_{\mu\nu}$ as

$$I^{\text{total}} = I[\bar{g}_{\mu\nu}] + I^m[g_{\mu\nu}], \tag{4.4}$$

where I^{total} is the total action, I is the Einstein-Hilbert action which when varied with respect to $\bar{g}_{\mu\nu}$ gives rise to the left hand side of eq. (4.2) and finally a variation of I^m with respect to $g_{\mu\nu}$ is what gives rise to the the quantum stress tensor. I^m is typically known as the effective action [15].

The same reasoning applies when it comes to two-dimensional gravity. Only this time, one considers the quantum stress tensor as inducing quantum fluctuations in both the dilaton and the space-time geometry. So just as in eq. (4.4), our goal would be to find an action such that

$$I = I_0[\bar{\phi}, \bar{g}_{\mu\nu}] + W[\phi, g_{\mu\nu}, f_i], \tag{4.5}$$

where the barred fields are the quantum-corrected fields, the unbarred fields represent the classical fields, I_0 is the classical action and W is the quantum effective action that when varied yields the quantum stress tensor. To find an action of the type in eq. (4.5), the first step is to recall the classical form of generic dilaton-gravity theories

$$I = \frac{1}{2}\int d^2x\sqrt{-g}\Big[Z(\phi)R + U(\phi)(\nabla\phi)^2 + 2\gamma^2 V(\phi)\Big] + \frac{1}{2}\int d^2x\sqrt{-g}Y(\phi)\sum_{i=1}^{N}(\nabla f_i)^2, \tag{4.6}$$

where Z, U, V and Y are model dependent functions of the dilaton. Let us suppose we are considering higher dimensional Einstein-like theories that are reduced through spherical or angular symmetry using a metric ansatz of the form

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu + l^2 f^2(\phi)d\Omega^2_{n-2}, \tag{4.7}$$

where $l^2 d\Omega^2$ is the metric of the $n-2$ dimensional surface. From the determinant of the metric tensor in eq. (4.7), we get $\sqrt{-g_{(n)}} = \sqrt{-g}\ l^{n-2} f^{n-2}(\phi)$ which leads to $Z(\phi) = Y(\phi) = f^{n-2}(\phi)$ in the generic action eq. (4.6). In addition, it can be shown that the matter action in eq. (4.6) is invariant under conformal transformations. We can then remove the kinetic terms from the generic action (4.6), as we did in section 2.2, using the transformations

$$\phi' = \frac{1}{2} Z(\phi), \tag{4.8}$$

$$g'_{\mu\nu} = \Omega^2(\phi) g_{\mu\nu}, \tag{4.9}$$

where the conformal factor takes the form $\Omega^2(\phi) = \exp \frac{1}{2} \int^{\phi} d\phi (\frac{d\phi'}{d\phi})^{-1}$. We can then drop the primes so that the generic action reduces to

$$I = \int d^2x \sqrt{-g} \big(\phi R + \gamma^2 V(\phi)\big) + \int d^2x \sqrt{-g} \phi \sum_{i=1}^{N} (\nabla f_i)^2. \tag{4.10}$$

This reduced action would be valid for all higher dimensional theories that can be written in the Einstein-Hilbert form and are reduced using spherical or axial symmetry. However, the action would be different if we consider other types of theories that are reduced by some other mechanism. An explicit knowledge of the dimensional reduction procedure would be necessary to understand the dilaton-matter coupling function. For example, in the original SIG model the matter fields are not coupled to the dilaton, see eq. (2.7). However, as discussed in [50], this can have serious consequences because no realistic higher dimensional theory of gravity can reduce to a two-dimensional theory in which the dilaton is not coupled to the matter fields. To account for this redundancy, we will assume that the SIG model takes the form

$$I = \int d^2x \sqrt{-g} \big(\phi R + \gamma^2\big) + \int d^2x \sqrt{-g} j(\phi) \sum_{i=1}^{N} (\nabla f_i)^2, \tag{4.11}$$

where $j(\phi)$ is some dilaton-matter coupling function. We can then draw inspiration from the action functional eqs. (4.10) and (4.11) and infer that the most general classical kinetic-free action that includes matter fields must take the form

$$I = \frac{1}{2} \int d^2x \sqrt{-g} \big(\phi R + \gamma^2 V(\phi)\big) + \int d^2x \sqrt{-g} X(\phi) \sum_{i=1}^{N} (\nabla f_i)^2, \tag{4.12}$$

where V and X are model dependent functions of the dilaton. If we label the matter action as I^M_{cl}, we could then vary it with respect to $g^{\mu\nu}$ to obtain an expression for $T_{\mu\nu}$, the classical stress tensor. More explicitly, this is written as

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta I^M_{\text{cl}}}{\delta g^{\mu\nu}} = X(\phi) \big[2 \nabla_\mu f_i \nabla_\nu f_i - g_{\mu\nu} (\nabla f_i)^2\big]. \tag{4.13}$$

One important property of the classical stress tensor is that it has a trace equal to zero, as can be see when we multiply $g^{\mu\nu}$ with the stress tensor in eq. (4.13) to find

$$T^{\mu}_{\mu} = 0. \tag{4.14}$$

There is another slightly more useful way to show this property. As described in [15], we would like the matter action to be invariant under conformal transformations, so we consider the transformation $\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$ on the action $I^M[\phi, g_{\mu\nu}]$ so that

$$\tilde{I}^M_{\text{cl}}[\phi, \tilde{g}_{\mu\nu}, f_i] = I^M_{\text{cl}}[\phi, g_{\mu\nu}, f_i] + \int d^2x\sqrt{-\tilde{g}}\frac{\delta I^M_{\text{cl}}}{\delta\tilde{g}^{\mu\nu}}\delta\tilde{g}^{\mu\nu}. \tag{4.15}$$

Using the fact that $\delta\tilde{g}^{\mu\nu} = -2\tilde{g}^{\mu\nu}\Omega^{-1}\delta\Omega$ and the definition of the stress tensor from eq. (4.13), eq. (4.15) becomes

$$\tilde{I}^M_{\text{cl}}[\phi, \tilde{g}_{\mu\nu}, f_i] = I^M_{\text{cl}}[\phi, g_{\mu\nu}, f_i] - \int d^2x\sqrt{-\tilde{g}}\; T^{\mu}_{\mu}[\tilde{g}_{\mu\nu}]\; \Omega^{-1}\delta\Omega. \tag{4.16}$$

From this equation, we notice that if the action is to be conformally invariant, then the trace of the stress tensor must be zero. If we compare eq. (4.16) to

$$\tilde{I}^M_{\text{cl}} = I^M_{\text{cl}} + \int d^2x\; \delta I^M_{\text{cl}}, \tag{4.17}$$

we see that the trace of the stress tensor can be written as

$$T^{\mu}_{\mu} = -\frac{\Omega}{\sqrt{-g}}\frac{\delta I^M_{\text{cl}}}{\delta\Omega}\bigg|_{\Omega\to 1}, \tag{4.18}$$

which implies that for a conformally invariant theory, we require that

$$T^{\mu}_{\mu} \propto \frac{\delta I^M_{\text{cl}}}{\delta\Omega} = 0. \tag{4.19}$$

We can then conclude that for a conformally invariant matter action, the stress tensor must be traceless.

4.2 The effective action

Thus far we have discussed the classical stress tensor, our next goal would be to find an effective action that would give us the quantum stress tensor. As discussed in [15], we would expect the relation between the effective action and the quantum stress tensor to follow the canonical form

$$\langle T_{\mu\nu}\rangle = \frac{2}{\sqrt{-g}}\frac{\delta W}{\delta g^{\mu\nu}}, \tag{4.20}$$

where W is the effective action. To obtain the effective action, one can consider quantizing the classical matter action as we would in ordinary quantum field theory. But because these computations are performed in a curved background, divergences inevitably arise. We would then require that the quantized matter action be cured of all divergences using well established renormalization procedures [15]. The renormalized first order quantum matter action is what we would refer to as the effective action. The flow of this process can be represented as

$$I^M_{\text{cl}} \xrightarrow{\text{quantization}} I^M_{\text{quant}} \xrightarrow{\text{renormalization}} W.$$

However, there is an intermediate step that may be of great benefit for the study of dilaton-gravity. It has been shown in [15] that by using the path integral formalism, one can deduce the structure of the quantum stress tensor. This involves the use of the generating functional, Z, associated to a physical system. In quantum field theory, this quantity is of great significance since it represents the transition amplitude from some initial state to some final state. If we consider our two-dimensional system and place a condition that particles are neither created nor destroyed, then the generating functional takes the form

$$Z = \int \mathcal{D}[f_i]\exp\left\{iI^M_{\text{cl}}[\phi, g, f_i]\right\} = \langle \text{final}, 0|\, 0, \text{initial}\rangle\,, \tag{4.21}$$

where $|0, \text{initial}\rangle$ and $|0, \text{final}\rangle$ are the initial and final vacuum states respectively, I^M_{cl} is the matter action and the integration measure $\mathcal{D}[f_i]$ is evaluated over all possible paths in the form:

$$\int \mathcal{D}[f_j] = \prod_{i=1}^{\infty}\left[\int df_{j(i)}\right]. \tag{4.22}$$

To get the quantum stress tensor, we only need to consider the variation in Z such that

$$\delta Z = i\left\langle \text{final}, 0|\delta I^M_{\text{cl}}|0, \text{initial}\right\rangle, \tag{4.23}$$

which when varied with respect to the metric and making use of eq. (4.13) becomes

$$\frac{2}{\sqrt{-g}}\frac{\delta Z}{\delta g^{\mu\nu}} = i\,\langle \text{final}, 0|T_{\mu\nu}|0, \text{initial}\rangle\,. \tag{4.24}$$

This gives us a connection between the path integral and the quantum stress tensor. But more important is the functional integral eq. (4.21). From it we may choose to integrate out the matter fields so that what remains in the exponential is an action dependent on only the metric and the dilaton. This, however, comes at the cost of giving up an explicit description of Hawking radiation when we study two-dimensional black holes. What would follow is the quantization of the classical matter action to the form $I^M_{\text{quant}} = I^M_{\text{quant}}[\phi, g_{\mu\nu}]$. This is then followed by the renormalization of I^M_{quant} into $W[\phi, g_{\mu\nu}]$. And since the renormalization process involves removing divergent quantities from the action, it is highly likely that the final action would not be conformally invariant. This means that under a conformal transformation $g_{\mu\nu} \to \Omega^2 g_{\mu\nu}$, the trace of the quantum stress tensor

$$\langle T^\mu_\mu \rangle = \frac{\Omega}{\sqrt{-g}} \frac{\delta W}{\delta \Omega}\bigg|_{\Omega\to 1}, \tag{4.25}$$

does not necessarily vanish. This phenomenon is known as the *trace anomaly* [51]. It is possible to obtain the effective action from the trace anomaly. The way to go about it is to recall that the effective action is only dependent on geometrical quantities, the dilaton and the metric, so it is only natural to claim that the trace anomaly be a function of geometrical quantities. Fortunately, the task of finding the exact form of the trace anomaly has already been achieved by several authors [15, 50, 52]. From their work, we have learnt that the trace anomaly in $2n$-dimensions contains geometrical scalars with $2n$-derivatives, where n is a natural number. Particularly, from the work of Bousso and Hawking [50], we find that the trace anomaly takes the form

$$\langle T^\mu_\mu \rangle = a_0 R + a_1 (\nabla\phi)^2 + a_2 \Box\phi, \tag{4.26}$$

where a_0, a_1, a_2 are dimensionless quantities. In two-dimensions, R, $(\nabla\phi)^2$ and $\Box\phi$ are the only 2-derivative scalars. They are the two-dimensional analogs of what we would have in four-dimensions where R^2, $R_{ab}R^{ab}$ and $R_{abcd}R^{abcd}$ are 4-derivative scalars. We should also mention that the motivation for Bousso and Hawking [50] was to find a trace anomaly for two-dimensional theories that are reduced from higher dimensional Einstein-Hilbert-like theories. So for a generalized treatment, Nojiri and Odintsov [53] included a general dilaton-matter coupling function and derived the generalized trace anomaly

$$\langle T^\mu_\mu \rangle = aR + b(D)(\nabla D)^2 + c(D)\Box D, \tag{4.27}$$

where $a = \frac{N}{24\pi}$, N is the number of matter fields, D[1] is the dilaton from the generic action eq. (4.6), $b(D)$ and $c(D)$ are dilaton-dependent functions. In general, these functions are not known. One thing to point out is that the derivation of eqs. (4.26) and (4.27) does not depend on the generic or kinetic-free dilaton-gravity actions. With this under consideration, Nojiri and Odintsov [53] originally defined $b(D)$ and $c(D)$ as

$$b(D) = \frac{1}{8\pi}\left[-\frac{N}{Y}\frac{d^2Y}{dD^2} + \frac{N}{2}\left(\frac{1}{D}\frac{dY}{dD}\right)^2\right], \tag{4.28}$$

$$c(D) = -\frac{1}{8\pi}\frac{N}{Y}\frac{dY}{dD}. \tag{4.29}$$

where Y is the dilaton-matter coupling function from eq. (4.6). These functions can be transformed into a more convenient form using

$$c(D) = \frac{\partial\tilde{c}(D)}{\partial D}, \tag{4.30}$$

$$b(D) = \tilde{b}(D) + \frac{\partial^2\tilde{c}(D)}{\partial D^2}. \tag{4.31}$$

These transformations, eqs. (4.30) and (4.31), allow us to write the trace anomaly as

$$\langle T^{\mu}_{\mu}\rangle = aR + \tilde{b}(D)(\nabla D)^2 + \Box\tilde{c}(D). \tag{4.32}$$

In terms of the coupling function, $\tilde{b}(D)$ and $\tilde{c}(D)$ can be written as

$$\tilde{b}(D) = -\frac{N}{16\pi}\left(\frac{d\ln Y}{dD}\right)^2 \tag{4.33}$$

$$\tilde{c}(D) = -\frac{N}{8\pi}\ln Y. \tag{4.34}$$

The trace anomaly and the functions $\tilde{b}$ and $\tilde{c}$ are expressed in terms of fields from the generic theory given by eq. (2.10). However, we are interested in the kinetic-free dilaton-gravity theory given by eq. (2.86). So we shall perform the same transformations we used in section 2.2 on the trace anomaly. This will make it compatible with the kinetic-free theory.

If we consider two-dimensional theories that are reduced from higher-dimensional Einstein-like theories through spherical symmetry, then we can express the $\tilde{b}$ and $\tilde{c}$ functions in terms of ϕ, the dilaton from the kinetic-free action eq. (4.10). We recall from eq. (4.8) that the dilaton, D, is transformed into ϕ through

$$\phi = \frac{1}{2}Z(D). \tag{4.35}$$

[1]So as to not confuse the reader, for now we have chosen D to represent the original dilaton and ϕ to represent the reparametrized dilaton.

We also argued in section 4.1 that when higher dimensional Einstein-like theories are reduced to two-dimensions, $Z(D) = Y(D)$. And so eq. (4.35) would be equivalent to

$$\phi = \frac{1}{2}Y(D), \tag{4.36}$$

where ϕ is the dilaton from the kinetic-free dilaton-gravity theory. For an n-dimensional Einstein-like theory that reduces through the spherically symmetric ansatz

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu + D^2 d\Omega_{n-2}^2, \tag{4.37}$$

where D is the dilaton, we get the dilaton-matter coupling function

$$Y(D) = D^{n-2}. \tag{4.38}$$

Using eqs. (4.36) and (4.38), we can express the D-dependent terms as functions of ϕ. Starting with

$$D = 2^{\frac{1}{n-2}}\phi^{\frac{1}{n-2}}, \tag{4.39}$$

we have expressions like

$$\nabla_\mu D = \frac{1}{n-2}2^{\frac{1}{n-2}}\phi^{\frac{3-n}{n-2}}\nabla_\mu\phi, \tag{4.40}$$

$$(\nabla D)^2 = \frac{1}{(n-2)^2}2^{\frac{2}{n-2}}\phi^{2\frac{3-n}{n-2}}(\nabla\phi)^2. \tag{4.41}$$

The functions $\tilde{b}(D)$ and $\tilde{c}(D)$ are similarly transformed into

$$\tilde{b}(D) = -\frac{N}{16\pi}\left(\frac{1}{Y}\frac{dY}{dD}\right)^2 = -\frac{N}{16\pi}\frac{(n-2)^2}{D^2} = -\frac{N}{16\pi}(n-2)^2 2^{-\frac{2}{n-2}}\phi^{-\frac{2}{n-2}}, \tag{4.42}$$

$$\tilde{c}(D) = -\frac{N}{8\pi}\ln Y = -\frac{N}{8\pi}\ln D^{n-2} = -\frac{N}{8\pi}\ln 2\phi. \tag{4.43}$$

From eqs. (4.41) and (4.42), we get

$$\tilde{b}(D)(\nabla D)^2 = -\frac{N}{16\pi}\phi^{-2}(\nabla\phi)^2. \tag{4.44}$$

We can always label the coefficient of $(\nabla\phi)^2$ as $\tilde{b}(\phi)$. Equation (4.44) can be written as

$$\tilde{b}(D)(\nabla D)^2 = \tilde{b}(\phi)(\nabla\phi)^2, \tag{4.45}$$

where

$$\tilde{b}(\phi) = -\frac{N}{16\pi}\frac{1}{\phi^2}. \tag{4.46}$$

For the sake of clarification, eq. (4.43) can also be written as

$$\tilde{c}(\phi) = -\frac{N}{8\pi}\ln 2\phi. \tag{4.47}$$

We also have to perform a conformal rescaling on the trace anomaly. The following calculations will make the trace anomaly, and eventually the effective action, compatible with the action of the kinetic-free dilaton-gravity theory from section 2.2. However, the JT model does not need any kind of fine tuning of the trace anomaly since the model itself does not contain any kinetic terms in its generic action. As it turns out, for the JT model, eqs. (4.46) and (4.47) are the original forms for $\tilde{b}$ and $\tilde{c}$, respectively. From here on, we shall only consider models that reduce from $n > 3$ dimensions through spherical symmetry. Before we proceed, we are reminded that the trace anomaly in fact lives inside an integral just like the classical trace in eq. (4.16). Hence, we consider a conformal rescaling of the following integral

$$\int d^2x\sqrt{-g}\,\langle T^{\mu}_{\mu}\rangle = \int d^2x\sqrt{-g}\Big[aR + \tilde{b}(\phi)(\nabla\phi)^2 + \Box\tilde{c}(\phi)\Big]. \tag{4.48}$$

The second and third terms on the right hand side of eq. (4.48) are conformally invariant. We only need to perform a transformation on the first term by using the conformal transformation from section 2.2 and the dilaton reparametrization given by eq. (4.35). In terms of D, the conformal transformations read

$$\bar{g}_{\mu\nu} = \Omega^2(D)g_{\mu\nu}, \tag{4.49}$$

$$\sqrt{-g} = \Omega^{-2}\sqrt{\bar{g}}, \tag{4.50}$$

$$R = \Omega^2\left[\bar{R} + 2\frac{\bar{\Box}\Omega}{\Omega} - 2\frac{(\bar{\nabla}\Omega)^2}{\Omega^2}\right]. \tag{4.51}$$

We need to express these transformations in terms of ϕ using

$$\phi = \frac{1}{2}D^{n-2}, \tag{4.52}$$

$$\Omega^2(D) = \exp\left[\frac{1}{2}\int^D dD\left(\frac{d\phi}{dD}\right)^{-1}\right]. \tag{4.53}$$

Using eq. (4.52), it can be shown that, for $n \neq 4$, eq. (4.53) is equivalent to

$$\Omega^2(D) \equiv \Omega^2(\phi) = \exp\left[\frac{(2\phi)^{\frac{4-n}{n-2}}}{(n-2)(4-n)}\right] \tag{4.54}$$

and for $n = 4$, we have

$$\Omega^2(D) \equiv \Omega^2(\phi) = (2\phi)^{\frac{1}{4}}. \tag{4.55}$$

It then follows that the covariant derivatives of Ω can be written as covariant derivatives of ϕ as follows:

$$(\bar{\nabla}\Omega)^2 = \left(\frac{d\Omega}{d\phi}\right)^2 (\bar{\nabla}\phi)^2, \tag{4.56}$$

$$\bar{\Box}\Omega = \frac{d^2\Omega}{d\phi^2}(\bar{\nabla}\phi)^2 + \frac{d\Omega}{d\phi}\bar{\Box}\phi. \tag{4.57}$$

For $n = 4$, we can compute $\sqrt{-g}R$ as follows:

$$\sqrt{-g}R = \sqrt{-\bar{g}}\left[\bar{R} + \frac{1}{4}\frac{\bar{\Box}\phi}{\phi} - \frac{29}{32}\left(\frac{\bar{\nabla}\phi}{\phi}\right)^2\right]. \tag{4.58}$$

Similarly, for $n > 4$, we can show that

$$\sqrt{-g}R = \sqrt{-\bar{g}}\left[\bar{R} + 2d_1^n \phi^{\frac{8-3n}{n-2}}(\bar{\nabla}\phi)^2 + 2d_0^n \phi^{2\frac{3-n}{n-2}}\bar{\Box}\phi\right], \tag{4.59}$$

where

$$d_0^n = \frac{2^{2\frac{3-n}{n-2}}}{(n-2)^2}, \tag{4.60}$$

$$d_1^n = -\frac{n-3}{n-2}2^{\frac{4-n}{n-2}}. \tag{4.61}$$

When $n = 4$, we can drop the bars so that the trace anomaly is written as

$$\langle T_\mu^\mu \rangle = aR + \frac{a}{4\phi}\Box\phi - \frac{29a}{32\phi^2}(\nabla\phi)^2 + \tilde{b}(\phi)(\nabla\phi)^2 + \Box\tilde{c}(\phi). \tag{4.62}$$

Recalling that $a = \frac{N}{24\pi}$, we can decompose the functions $\tilde{b}$ and $\tilde{c}$ and make the necessary simplifications so that the trace anomaly reads

$$\langle T_\mu^\mu \rangle = aR - \frac{11}{4}a\frac{\Box\phi}{\phi} + \frac{19}{32}a\left(\frac{\nabla\phi}{\phi}\right)^2. \tag{4.63}$$

We can then define the functions $m(\phi)$ and $s(\phi)$:

$$m(\phi) = \frac{19}{32}\frac{a}{\phi^2}, \tag{4.64}$$

$$s(\phi) = -\frac{11}{4}\frac{a}{\phi}. \tag{4.65}$$

The new trace anomaly would then read

$$\langle T_\mu^\mu \rangle = aR + m(\phi)(\nabla\phi)^2 + s(\phi)\Box\phi. \tag{4.66}$$

Then using transformations similar to eqs. (4.30) and (4.31), we can transform $m(\phi)$ and $s(\phi)$ as follows

$$s(\phi) = \frac{\partial \tilde{s}}{\partial \phi}, \tag{4.67}$$

$$m(\phi) = \tilde{m}(\phi) + \frac{\partial^2 \tilde{s}}{\partial \phi^2}. \tag{4.68}$$

The new trace anomaly can now be written in the canonical form as

$$\langle T^\mu_\mu \rangle = aR + \tilde{m}(\phi)(\nabla\phi)^2 + \Box\tilde{s}(\phi), \tag{4.69}$$

where

$$\tilde{s}(\phi) = \frac{11}{4} a \ln\phi, \tag{4.70}$$

$$\tilde{m}(\phi) = -\frac{69}{32} a \frac{1}{\phi^2}. \tag{4.71}$$

A similar process can be used to find the trace anomaly when $n > 4$. After the decomposition of $\tilde{b}$ and $\tilde{c}$, the trace anomaly would read

$$\langle T^\mu_\mu \rangle = aR + 2ad^n_1 \frac{1}{\phi^{\frac{3n-8}{n-2}}}(\nabla\phi)^2 + 2ad^n_0 \frac{\Box\phi}{\phi^{2\frac{n-3}{n-2}}} + \frac{3}{2}\frac{a}{\phi^2}(\nabla\phi)^2 - 3a\frac{\Box\phi}{\phi}. \tag{4.72}$$

We can then define the functions $s(\phi)$ and $m(\phi)$ as

$$s(\phi) = \frac{2ad^n_0}{\phi^{2\frac{n-3}{n-2}}} - \frac{3a}{\phi}, \tag{4.73}$$

$$m(\phi) = \frac{2ad^n_1}{\phi^{\frac{3n-8}{n-2}}} + \frac{3}{2}\frac{a}{\phi^2}. \tag{4.74}$$

These then transform according to eqs. (4.67) and (4.68) into

$$\tilde{s}(\phi) = 2ad^n_0 \phi^{\frac{4-n}{n-2}} - 3a\ln\phi, \tag{4.75}$$

$$\tilde{m}(\phi) = 2ad^1_n \frac{1}{\phi^{\frac{3n-8}{n-2}}} + 4ad^n_0 \frac{n-3}{n-2}\frac{1}{\phi^{\frac{3n-8}{n-2}}} - \frac{3a}{2\phi^2}. \tag{4.76}$$

Perhaps the key thing here is that regardless of which parametrization we choose, $\tilde{s}(\phi)$, $\tilde{m}(\phi)$, $\tilde{b}(D)$ and $\tilde{c}(D)$ are all divergent at the singularity. We can finally write the trace anomaly eq. (4.32) as

$$\langle T^\mu_\mu \rangle = aR + \tilde{m}(\phi)(\nabla\phi)^2 + \Box\tilde{s}(\phi), \tag{4.77}$$

where for the JT model

$$\tilde{m}(\phi) = -\frac{3}{2}\frac{a}{\phi^2}, \tag{4.78}$$

$$\tilde{s}(\phi) = -3a\ln 2\phi, \tag{4.79}$$

for SSG, $n = 4$,

$$\tilde{m}(\phi) = -\frac{69}{32}\frac{a}{\phi^2}, \tag{4.80}$$

$$\tilde{s}(\phi) = \frac{11}{4}a\ln\phi \tag{4.81}$$

and for higher dimensional theories, $n > 4$, that reduce through spherical symmetry

$$\tilde{s}(\phi) = 2ad_0^n\phi^{\frac{4-n}{n-2}} - 3a\ln\phi, \tag{4.82}$$

$$\tilde{m}(\phi) = 2ad_n^1\frac{1}{\phi^{\frac{3n-8}{n-2}}} + 4ad_0^n\frac{n-3}{n-2}\frac{1}{\phi^{\frac{3n-8}{n-2}}} - \frac{3a}{2\phi^2}. \tag{4.83}$$

From here on, we can assume all the two-dimensional theories we shall study are reductions of a higher-dimensional Einstein-like theory. This is because we have no way of knowing what $\tilde{m}$ and $\tilde{s}$ are in other cases.

Finally, it has been shown by [50, 53] that one can retrieve the trace anomaly eq. (4.77) from the following non-local effective action

$$W = -\frac{\hbar}{2}\int d^2x\sqrt{-g}\left[\frac{a}{2}R\frac{1}{\Box}R + \tilde{m}(\phi)(\nabla\phi)^2\frac{1}{\Box}R + \tilde{s}(\phi)R\right], \tag{4.84}$$

where $\frac{1}{\Box}$ is the non-local operator defined through the relations $\Box(x^\mu)\frac{1}{\Box}(\tilde{x}^\mu) = \delta(x^\mu - \tilde{x}^\mu)$ and $\Box\frac{1}{\Box}f = \frac{1}{\Box}\Box f = f$, where $\delta(x^\mu - \tilde{x}^\mu)$ is the generalized Dirac delta function and f is a scalar function. We can also artificially remove N and π from the functions $\tilde{m}(\phi)$ and $\tilde{s}(\phi)$ by introducing a new constant κ. If we choose to write the effective action as

$$W = -\frac{\kappa}{2}\int d^2x\sqrt{-g}\left[R\frac{1}{\Box}R + \tilde{m}(\phi)(\nabla\phi)^2\frac{1}{\Box}R + \tilde{s}(\phi)R\right], \tag{4.85}$$

then

$$\kappa = \frac{a\hbar}{2} = \frac{N\hbar}{48\pi}. \tag{4.86}$$

This definition of κ assumes $\hbar \to 0$ faster than $N \to \infty$ so that κ is very small. One only needs to make the transformations $\tilde{m}(\phi) \to \frac{a}{2}\tilde{m}(\phi)$ and $\tilde{s}(\phi) \to \frac{a}{2}\tilde{s}(\phi)$ in eqs. (4.78) to (4.83). Also it is fairly easy to show that we can obtain eq. (4.77) from eq. (4.84), if we used the conformal gauge, $ds^2 = e^{2\rho}(-dt^2 + dx^2)$.

Before we conclude that this is indeed the effective action, one thing needs to be addressed. It has been suggested in [16, 54] that the effective action in eq. (4.85) may still contain some conformally invariant components. Unfortunately, it is impossible to find exact forms for these conformally invariant components. At best, they can only be approximated using DeWitt-Schwinger expansions such as those found in [15]. With the higher order terms of the expansion suppressed on physical grounds, the end product of this process is the conformally invariant component of the effective action [54]

$$W^{\star} = \frac{\kappa}{2}\int d^2x\sqrt{-g}\,\tilde{m}(\phi)(\nabla\phi)^2\ln\mu^2 + \cdots\,, \tag{4.87}$$

where μ^2 is a dimensional remnant of the DeWitt-Schwinger expansion. Since $W^{\star}$ is conformally invariant, it bears no effect on the trace anomaly eq. (4.32). The complete effective action would then be the sum of eqs. (4.85) and (4.87)

$$W = -\frac{\kappa}{2}\int d^2x\sqrt{-g}\left[R\frac{1}{\Box}R + \tilde{m}(\phi)\left(\frac{1}{\Box}R - \ln\mu^2\right)(\nabla\phi)^2 + \tilde{s}(\phi)R\right]. \tag{4.88}$$

4.3 Localized effective action

Although we have the effective action given by eq. (4.88), we can still make it local by introducing extra fields. This is not an uncommon practice, for example in string theory the Nambu-Goto action can be made linear by introducing an auxiliary field. Also, a localized action is easier to deal with and its equations of motion are easier to derive. This first mention of a localized effective action equivalent to eq. (4.88) is found in [16]. This action reads as follows

$$W^{\dagger} = -\frac{\kappa}{2}\int d^2x\sqrt{-g}\left[\left(\psi + \chi + \tilde{s}(\phi)\right)R + \nabla^{\mu}\psi\nabla_{\mu}\chi + \tilde{m}(\phi)\left(\psi - \ln\mu^2\right)(\nabla\phi)^2\right], \tag{4.89}$$

where the pair of auxiliary fields are

$$\Box\psi = R, \tag{4.90}$$

$$\Box\chi = R + \tilde{m}(\phi)(\nabla\phi)^2. \tag{4.91}$$

It can be shown that through eqs. (4.90) and (4.91), the local and non-local actions give the same equations of motion.

We can then conclude that the kinetic-free one-loop quantum-corrected dilaton-gravity action is

$$I = \frac{1}{2}\int d^2x\sqrt{-\bar{g}}\left(\bar{\phi}\bar{R} + \gamma^2 V(\bar{\phi})\right) \\ -\frac{\kappa}{2}\int d^2x\sqrt{-g}\Big[\big(\psi + \chi + \tilde{s}(\phi)\big)R + \nabla^\mu\psi\nabla_\mu\chi + \tilde{m}(\phi)\big(\psi - \ln\mu^2\big)(\nabla\phi)^2\Big], \quad (4.92)$$

with the perturbation parameter $\kappa \propto \hbar$, $\bar{g}$ and $\bar{\phi}$ are the quantum-corrected fields, ϕ and g are the classical fields, V, $\tilde{m}$ and $\tilde{s}$ are model-dependent functions of the dilaton described by eqs. (4.78) to (4.83), ψ and χ are auxiliary functions.

Chapter 5

Quantum-corrected picture of black hole interior

In this chapter we return to the interior of two-dimensional black holes. This time we shall extend the classical picture we formulated in chapter 3 into a quantum mechanical picture. To be more accurate, this will only be a semi-classical picture since we will use the first order quantum corrections from the previous chapter. It is these fields that will help us formulate the semi-classical picture. Following the discussion in section 1.2, a low-energy treatment will be used in the present analysis. This will lead to a partial resolution of the singularity.

The starting point is the semi-classical dilaton-gravity action given by

$$\begin{aligned} I[\phi, g, \bar{\phi}, \bar{g}] = &\frac{1}{2}\int d^2x\sqrt{-\bar{g}}\left(\bar{\phi}\bar{R} + \gamma^2\bar{\phi}^{\alpha}\right) \\ &- \frac{\kappa}{2}\int d^2x\sqrt{-g}\Big[\left(\psi + \chi + \tilde{s}(\phi)\right)R + \nabla^{\mu}\psi\nabla_{\mu}\chi + \tilde{m}(\phi)\left(\psi - \ln\mu^2\right)(\nabla\phi)^2\Big], \end{aligned} \tag{5.1}$$

where $-1 < \alpha \leq 1$, the quantum-corrected fields are the functional arguments of the classical action and the classical fields are the functional arguments of the effective action. In general, the quantum-corrected fields can be expanded as

$$\bar{\phi}(x^{\mu}) = \sum_{i=0}^{N} \kappa^i \phi_i(x^{\mu}), \tag{5.2}$$

$$\bar{g}_{\alpha\beta}(x^{\mu}) = \sum_{i=0}^{N} \kappa^i g^{(i)}_{\alpha\beta}(x^{\mu}), \tag{5.3}$$

where κ^i is the i-th order perturbation parameter, $\phi_i(x^\mu)$ is the i-th order dilaton and $g^{(i)}_{\alpha\beta}$ is the i-th order metric. In our case, the action (5.1) is only valid up to first order, so we only need that the quantum-corrected fields take the form

$$\bar{\phi} = \phi_0 + \kappa\phi_1, \tag{5.4}$$

$$\bar{g}_{\mu\nu} = g^{(0)}_{\mu\nu} + \kappa g^{(1)}_{\mu\nu}. \tag{5.5}$$

Equation (5.4) can easily be applied to eq. (5.1), however, eq. (5.5) may need a bit more care. We proceed by considering the following proposition:

Proposition. In two-dimensional gravity, quantum fluctuations in the geometry are generated by a generalized conformal transformation.

Proof. From the quantum-corrected metric eq. (5.3) and the fact that two-dimensional manifolds are conformally related, we can establish that every i-th order metric, $g^{(i)}_{\mu\nu}$, is uniquely conformally related to the zeroth order metric $g^{(0)}_{\mu\nu}$ through

$$g^{(i)}_{\mu\nu} = \Omega_i^2 g^{(0)}_{\mu\nu}, \tag{5.6}$$

where Ω_i^2 is a conformal factor. So we can write eq. (5.3) as

$$\bar{g}_{\alpha\beta}(x^\mu) = \sum_{i=0}^{N} \kappa^i g^{(i)}_{\alpha\beta}(x^\mu) = (1 + \kappa\Omega_1^2 + \kappa^2\Omega_2^2 + \cdots + \kappa^N\Omega_N^2) g_{\mu\nu}, \tag{5.7}$$

where $g_{\mu\nu} = g^{(0)}_{\mu\nu}$. If we define the generalized conformal transformation as

$$\tilde{\Omega}^2(\kappa, x^\alpha) = 1 + \kappa\Omega_1^2(x^\alpha) + \kappa^2\Omega_2^2(x^\alpha) + \cdots + \kappa^N\Omega_N^2(x^\alpha), \tag{5.8}$$

then the quantum-corrected metric can be written as

$$\bar{g}_{\mu\nu}(x^\alpha) = \tilde{\Omega}^2(\kappa, x^\alpha) g_{\mu\nu}(x^\alpha). \tag{5.9}$$

□

It is clear from eq. (5.8) that the generalized conformal transformation has the property

$$\tilde{\Omega}(0, x^\mu) = 1. \tag{5.10}$$

This can be interpreted as the effect of switching off the quantum corrections. In the form of eq. (5.9), the quantum corrected metric can easily be incorporated into the action eq. (5.1). This would also require that $\tilde{\Omega}(\kappa, x^\alpha)$ be treated as a dynamical field as long as we expand it up to first order κ.

5.1 Quantum-corrected equations of motion for the interior of a black hole

We now have expressions for the quantum-corrected fields, we could simply apply the appropriate conformal transformations onto the action eq. (5.1) and work out the equations of motion by varying it with respect to all the fields, $\phi_0, \phi_1, \psi, \chi, g_{\mu\nu}$ and $\tilde{\Omega}(\kappa, x^\mu)$. Although this allows us to work in a coordinate independent fashion, we can greatly simplify things by working in the conformal gauge. We should recall that all fields are time dependent in the interior of a black hole. Hence, let us define the classical interior metric as

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = e^{2\rho_0(t)}(-dt^2 + dx^2). \tag{5.11}$$

The quantum-corrected metric tensor can be written as

$$\bar{g}_{\mu\nu} = e^{2\kappa\rho_1(t)}g_{\mu\nu}. \tag{5.12}$$

It can be shown that $e^{2\kappa\rho_1(t)}$ carries the properties of the generalized conformal factor $\tilde{\Omega}(\kappa, t)$. From eqs. (5.11) and (5.12), we can write the quantum-corrected line element as

$$ds^2 = \bar{g}_{\mu\nu}dx^\mu dx^\nu = e^{2\bar{\rho}(t)}(-dt^2 + dx^2), \tag{5.13}$$

where

$$\bar{\rho}(t) = \rho_0(t) + \kappa\rho_1(t). \tag{5.14}$$

Thus, the action eq. (5.1), can now be written in the conformal gauge as

$$I = \frac{1}{2}\int d^2x\left[2\bar{\phi}\ddot{\bar{\rho}} + \gamma^2\bar{\phi}^\alpha e^{2\bar{\rho}}\right] - \frac{\kappa}{2}\int d^2x\left[2(\psi + \chi + \tilde{s}(\phi_0))\ddot{\rho}_0 - \dot{\psi}\dot{\chi} - \tilde{m}(\phi_0)(\psi - \ln\mu^2)\dot{\phi}_0^2\right]. \tag{5.15}$$

Using eqs. (5.4) and (5.14), we can expand the last expression into

$$\begin{aligned} I = &\frac{1}{2}\int d^2x\left[2(\phi_0 + \kappa\phi_1)(\ddot{\rho}_0 + \kappa\ddot{\rho}_1) + \gamma^2(\phi_0 + \kappa\phi_1)^\alpha e^{2(\rho_0 + \kappa\rho_1)}\right] \\ &- \frac{\kappa}{2}\int d^2x\left[2(\psi + \chi + \tilde{s}(\phi_0))\ddot{\rho}_0 - \dot{\psi}\dot{\chi} - \tilde{m}(\phi_0)(\psi - \ln\mu^2)\dot{\phi}_0^2\right]. \end{aligned} \tag{5.16}$$

Now we only need zeroth and first order κ terms, so if we represent a product of a set functions as F then

$$F = F_0 + \kappa F_1 + \mathcal{O}(\kappa^2). \tag{5.17}$$

where F_0 are products of functions of order κ^0 and F_1 are products of functions of order κ^1. This means we will remove some terms from eq. (5.16) because higher order terms belong to higher-loop quantum corrections, which we will not use in the present analysis. So the first product in eq. (5.16) will be truncated as follows:

$$(\phi_0 + \kappa\phi_1)(\ddot{\rho}_0 + \kappa\ddot{\rho}_1) = \phi_0\ddot{\rho}_0 + \kappa(\phi_0\ddot{\rho}_1 + \phi_1\ddot{\rho}_0) + \mathcal{O}(\kappa^2). \tag{5.18}$$

The second product involves an exponential and the power law potential. So we use a binomial expansion on the power law potential term as follows:

$$(\phi_0 + \kappa\phi_1)^\alpha = \phi_0^\alpha + \kappa\alpha\phi_0^{\alpha-1}\phi_1 + \mathcal{O}(\kappa^2). \tag{5.19}$$

For the exponential term we will preserve the classical term and expand the quantum correction as follows:

$$e^{2(\rho_0+\kappa\rho_1)} = e^{2\rho_0}(1 + 2\kappa\rho_1) + \mathcal{O}(\kappa^2). \tag{5.20}$$

So the second product in the action eq. (5.16) can be further truncated as follows:

$$e^{2(\rho_0+\kappa\rho_1)}(\phi_0 + \kappa\phi_1)^\alpha = e^{2\rho_0}\left[\phi_0^\alpha + \kappa(2\rho_1\phi_0^\alpha + \alpha\phi_0^{\alpha-1}\phi_1)\right] + \mathcal{O}(\kappa^2). \tag{5.21}$$

Following the previous truncations, eq. (5.16) can now be written as

$$\begin{aligned} I = {} & \frac{1}{2}\int d^2x\left[2\phi_0\ddot{\rho}_0 + e^{2\rho_0}\phi_0^\alpha\right] + \frac{\kappa}{2}\int d^2x\left[2\phi_0\ddot{\rho}_1 + 2\phi_1\ddot{\rho}_0 + \gamma^2 e^{2\rho_0}(2\rho_1\phi_0^\alpha + \alpha\phi_0^{\alpha-1}\phi_1)\right] \\ & - \frac{\kappa}{2}\int d^2x\left[2(\psi + \chi + \tilde{s}(\phi_0))\ddot{\rho}_0 - \dot{\psi}\dot{\chi} - \tilde{m}(\phi_0)(\psi - \ln\mu^2)\dot{\phi}_0^2\right]. \end{aligned} \tag{5.22}$$

The first integral is just the classical action, the middle integral represents the action of first order quantum corrections and the last integral represents the first order quantum effective action.

We can now find the equations of motion from eq. (5.22) by making variations with respect to each field in the action. The first thing to note in eq. (5.22) is that none of the fields are dependent on κ. We can therefore label the first integral as I_0 and separate it from the other two. From the second and third integral, we obtain what we shall call the quantum action, I_q. An accurate representation of the action eq. (5.22) is

$$I = I_0[\phi_0, \rho_0] + \kappa I_q[\phi_0, \rho_0, \phi_1, \rho_1, \psi, \chi]. \tag{5.23}$$

Since we already know what we will get from the classical action, all we need to do now is derive the equations of motion from the quantum action. By varying it with respect to each of the six fields, we get

$$\begin{aligned}\frac{\delta I_q}{\delta \phi_0} = \frac{\kappa}{2}\int d^2x\Big\{&2\ddot{\rho}_1 + 2\gamma^2\alpha e^{2\rho_0}\rho_1\phi_0^{\alpha-1} + \alpha(\alpha-1)\gamma^2 e^{2\rho_0}\phi_0^{\alpha-2}\phi_1 - 2\ddot{\rho}_0\frac{d\tilde{s}}{d\phi_0}\\ &-\psi\dot{\phi}_0^2\frac{d\tilde{m}}{d\phi_0} - 2\dot{\psi}\dot{\phi}_0\tilde{m}(\phi_0) - 2\psi\ddot{\phi}_0\tilde{m}(\phi_0) + \ln(\mu^2)\left(\frac{d\tilde{m}}{d\phi_0}\dot{\phi}_0^2 + 2\tilde{m}(\phi_0)\ddot{\phi}_0\right)\Big\} = 0,\end{aligned} \tag{5.24}$$

$$\frac{\delta I_q}{\delta \rho_0} = \frac{\kappa}{2}\int d^2x\Big\{2\ddot{\phi}_1 + 4\gamma^2\rho_1\phi_0^{\alpha}e^{2\rho_0} + 2\gamma^2\alpha\phi_0^{\alpha-1}\phi_1 e^{2\rho_0} - 2\left(\ddot{\psi} + \ddot{\chi} + \dot{\phi}_0^2\frac{d^2\tilde{s}}{d\phi_0^2} + \ddot{\phi}_0\frac{d\tilde{s}}{d\phi_0}\right)\Big\} = 0, \tag{5.25}$$

$$\frac{\delta I_q}{\delta \phi_1} = \frac{\kappa}{2}\int d^2x\Big\{2\ddot{\rho}_0 + \gamma^2\alpha\phi_0^{\alpha-1}e^{2\rho_0}\Big\} = 0, \tag{5.26}$$

$$\frac{\delta I_q}{\delta \rho_1} = \frac{\kappa}{2}\int d^2x\Big\{2\ddot{\phi}_0 + 2\gamma^2\phi_0^{\alpha}e^{2\rho_0}\Big\} = 0, \tag{5.27}$$

$$\frac{\delta I_q}{\delta \chi} = \frac{\kappa}{2}\int d^2x\Big\{-2\ddot{\rho}_0 - \ddot{\psi}\Big\} = 0, \tag{5.28}$$

$$\frac{\delta I_q}{\delta \psi} = \frac{\kappa}{2}\int d^2x\Big\{-2\ddot{\rho}_0 - \ddot{\chi} + \tilde{m}(\phi_0)\dot{\phi}_0^2\Big\} = 0. \tag{5.29}$$

From eqs. (5.26) and (5.27) we recover the classical equations obtained in section 3.2. Their solutions have already been discussed in section 3.3. So with a prior knowledge of the classical solutions, we can obtain the closed form solutions of ψ and χ by simply making two integrations on both eqs. (5.28) and (5.29).

However, our main interest is to study the near singularity dynamics of the quantum-corrected fields in eqs. (5.24) and (5.25), the other equations are only meant to assist this effort. So we proceed by first moving the purely classical terms in each equation to the right hand side as follows:

$$\begin{aligned}2\ddot{\rho}_1 + 2\gamma^2\alpha e^{2\rho_0}\rho_1\phi_0^{\alpha-1} + \alpha(\alpha-1)\gamma^2 e^{2\rho_0}\phi_0^{\alpha-2}\phi_1 &= 2\ddot{\rho}_0\frac{d\tilde{s}}{d\phi_0}\\ +\psi\dot{\phi}_0^2\frac{d\tilde{m}}{d\phi_0} + 2\dot{\psi}\dot{\phi}_0\tilde{m}(\phi_0) + 2\psi\ddot{\phi}_0\tilde{m}(\phi_0) &- \ln(\mu^2)\left(\frac{d\tilde{m}}{d\phi_0}\dot{\phi}_0^2 + 2\tilde{m}(\phi_0)\ddot{\phi}_0\right),\end{aligned} \tag{5.30}$$

$$2\ddot{\phi}_1 + 4\gamma^2\rho_1\phi_0^{\alpha}e^{2\rho_0} + 2\gamma^2\alpha\phi_0^{\alpha-1}\phi_1 e^{2\rho_0} = 2\left(\ddot{\psi} + \ddot{\chi} + \dot{\phi}_0^2\frac{d^2\tilde{s}}{d\phi_0^2} + \ddot{\phi}_0\frac{d\tilde{s}}{d\phi_0}\right). \tag{5.31}$$

From now on, we shall study these equations and deduce a quantum picture of the interior of a black hole. Before we consider finding solutions to eqs. (5.30) and (5.31), we recall that the functions $\tilde{m}(\phi_0)$ and $\tilde{s}(\phi_0)$ take the form

$$\tilde{m}(\phi_0) = -\frac{3}{\phi_0^2}, \tag{5.32}$$

$$\tilde{s}(\phi_0) = -6\ln 2\phi_0, \tag{5.33}$$

for the JT model. For reduced four-dimensional Einstein gravity we have

$$\tilde{m}(\phi_0) = -\frac{69}{16}\frac{1}{\phi_0^2}, \tag{5.34}$$

$$\tilde{s}(\phi_0) = \frac{11}{2}\ln\phi_0. \tag{5.35}$$

In addition, for all models that are derived from higher dimensional ($n > 4$) Einstein-like theories we have

$$\tilde{s}(\phi_0) = 4d_0^n\phi_0^{\frac{4-n}{n-2}} - 6\ln\phi_0, \tag{5.36}$$

$$\tilde{m}(\phi_0) = 4d_n^1\frac{1}{\phi_0^{\frac{3n-8}{n-2}}} + 8d_0^n\frac{n-3}{n-2}\frac{1}{\phi_0^{\frac{3n-8}{n-2}}} - \frac{3}{\phi_0^2}, \tag{5.37}$$

where

$$d_0^n = \frac{2^{2\frac{3-n}{n-2}}}{(n-2)^2}, \tag{5.38}$$

$$d_1^n = -\frac{n-3}{n-2}2^{\frac{4-n}{n-2}}. \tag{5.39}$$

First we will consider the possibility of finding closed form solutions for specific models before attempting to solve eqs. (5.30) and (5.31).

5.1.1 String inspired model, $\alpha = 0$

As we mentioned in chapter 4, in the original SIG model the dilaton is not coupled to matter and this leads to an interpretation of unrealistic higher dimensional black holes [50]. Subsequently, it has been argued in the same paper that the matter fields must pick up a dilaton coupling from the dimensional reduction process. It has been proposed that the same spherical symmetric reduction mechanism used for Einstein-like theories should be applied on the matter fields of the SIG model. Therefore, even the kinetic-free dilaton-gravity action will have the same dilaton-matter coupling in the matter action as any model descendent from an Einstein-like theory. This means we can still use eqs. (5.34) and (5.35) or eqs. (5.36) and (5.37) for the SIG model. With $\alpha = 0$ into consideration, we make the appropriate substitutions into eqs. (5.30) and (5.31) and get the following equations of motion

$$\ddot{\rho}_1 = \psi\dot{\phi}_0^2\frac{d\tilde{m}}{d\phi_0} + 2\dot{\psi}\dot{\phi}_0\tilde{m}(\phi_0) + 2\psi\ddot{\phi}_0\tilde{m}(\phi_0) - \ln{(\mu^2)}\left(\frac{d\tilde{m}}{d\phi_0}\dot{\phi}_0^2 + 2\tilde{m}(\phi_0)\ddot{\phi}_0\right), \tag{5.40}$$

$$\ddot{\phi}_1 + 2\gamma^2\rho_1 e^{2\rho_0} = 2\left(\tilde{m}(\phi_0)\dot{\phi}_0^2 + \dot{\phi}_0^2\frac{d^2\tilde{s}}{d\phi_0^2} + \ddot{\phi}_0\frac{d\tilde{s}}{d\phi_0}\right), \tag{5.41}$$

where some terms have been dropped since the classical equations dictate that $\ddot{\rho}_0 = 0$. If we think of this theory as having originated from a four-dimensional Einstein like theory, then we can use the matter coupling functions from eqs. (5.34) and (5.35). We can then obtain ρ_1 by simply integrating eq. (5.40) twice. We would then use it to find ϕ_1 in eq. (5.41). The bigger picture is to attempt to partially resolve the singularity using quantum mechanics. With that in mind, at the classical singularity, both equations become divergent on the right hand side. This problem will be resolved later on when we consider the general case for all α.

5.1.2 Jackiw-Teitelboim model, $\alpha = 1$

Considering only the equations of motion for the quantum corrections, we have

$$\ddot{\rho}_1 + \gamma^2 e^{2\rho_0}\rho_1 = 2\ddot{\rho}_0\frac{d\tilde{s}}{d\phi_0} + \psi\dot{\phi}_0^2\frac{d\tilde{m}}{d\phi_0} + 2\dot{\psi}\dot{\phi}_0\tilde{m}(\phi_0) + 2\psi\ddot{\phi}_0\tilde{m}(\phi_0) - \ln{(\mu^2)}\left(\frac{d\tilde{m}}{d\phi_0}\dot{\phi}_0^2 + 2\tilde{m}(\phi_0)\ddot{\phi}_0\right), \tag{5.42}$$

$$\ddot{\phi}_1 + \gamma^2 e^{2\rho_0}\phi_1 + 2\gamma^2\phi_0 e^{2\rho_0}\rho_1 = 2\left(\ddot{\psi} + \ddot{\chi} + \dot{\phi}_0^2\frac{d^2\tilde{s}}{d\phi_0^2} + \ddot{\phi}_0\frac{d\tilde{s}}{d\phi_0}\right), \tag{5.43}$$

where the fields ρ_0, ϕ_0, χ and ψ are

$$\phi_0 = \tanh{(\gamma t)}, \tag{5.44}$$

$$\rho_0 = \ln{\left[\sqrt{2}\,\operatorname{sech}(\gamma t)\right]}, \tag{5.45}$$

$$\psi = -\ln{\left[2\,\operatorname{sech}^2(\gamma t)\right]}, \tag{5.46}$$

$$\chi = \psi + \int^t d\tau \int^\tau dt\left[\tilde{m}(\phi_0)\dot{\phi}_0^{\,2}\right]. \tag{5.47}$$

Admittedly, closed form solutions of differential eqs. (5.42) and (5.43) are extremely difficult to find. Even the quantum-corrected equations for SSG would be equally difficult to solve. We could find the near singularity solutions but the terms on the right hand side become infinite at the singularity. This behaviour is present in all dilaton-gravity models.

5.2 Low-energy effective theory

In chapter 3, we saw that it was impossible to find the closed form solutions for the general classical equations. For that reason, finding closed form solutions for eqs. (5.30) and (5.31) is impossible since we would also require closed forms of the classical fields. In section 3.4, we resolved to study the near singularity behaviour of the classical fields. Unfortunately, that behaviour is what gives rise to the divergent behaviour of eqs. (5.30) and (5.31) in the region close to the singularity. However, the quantum corrections themselves may not be divergent at the singularity. In this section, we will only make the assumption that the position of the singularity remains invariant when quantum corrections are added. To state this more explicitly, we will infer that

$$\bar{\phi}(0) = 0. \tag{5.48}$$

Due to the fact that some quantities appear to be extremely large while others have the potential to be really small, we are forced to consider the use of effective field theory techniques. It is fairly common in particle physics that a situation may arise in which we encounter a system that has both light and heavy degrees of freedom. The heavy degrees of freedom can be regarded, at some approximation, as infinitely massive compared to the light ones. If we only concern ourselves with studying the dynamics of light degrees, then we can use well-known techniques for effectively removing the heavy degrees of freedom from our equations [55, 56]. This technique can also be used in the context of gravity [14]. This kind of renormalization leads to a low-energy effective theory from which we can extract useful physics.

In our case, we can think of the divergent terms as having originated from a Lagrangian with both light and heavy degrees of freedom. From the quantum action, we can write this Lagrangian up to surface terms as

$$\mathcal{L}_q = 2\phi_0\ddot{\rho}_1 + 2\ddot{\phi}_1\rho_0 + \gamma^2\tilde{V} - 2\ddot{\psi}\rho_0 - 2\ddot{\chi}\rho_0 - 2\tilde{s}\ddot{\rho}_0 - \ddot{\chi}\psi + \tilde{m}(\psi - \ln\mu^2)\dot{\phi}_0^2, \tag{5.49}$$

where $\tilde{V} = 2e^{2\rho_0}\phi_0^\alpha\rho_1 + \alpha e^{2\rho_0}\phi_0^{\alpha-1}\phi_1$ is the quantum correction in the power law potential. This Lagrangian can be compared to that of coupled harmonic oscillators containing both light $\{m, l, \omega_l\}$ and heavy $\{M, H, \omega_H\}$ degrees of freedom

$$L_{c.h.o} = \frac{1}{2}m\dot{l}^2 + \frac{1}{2}M\dot{H}^2 - \frac{1}{2}m\omega_l^2 l^2 - \frac{1}{2}M\omega_H^2 H^2 + g(l, H, \dot{l}, \dot{H}), \tag{5.50}$$

where the coupling is determined in $g(l, H, \dot{l}, \dot{H})$. If $M \to \infty$, then the heavy degrees become non-dynamical. In the low-energy limit, we can remove these heavy degrees and remain with the light ones only. Hence, to get the near singularity low-energy effective field theory from the quantum Lagrangian eq. (5.49), we have to remove every dynamical term containing the following infinitely massive terms

$$\tilde{m} \sim \frac{1}{\phi_0^{\frac{2}{}}} \to \infty, \tag{5.51}$$

$$\tilde{s} \sim \ln \phi_0 \to -\infty. \tag{5.52}$$

The derivatives of the fields ψ and χ decouple due to their dependency on only the classical fields through eqs. (5.28) and (5.29), they too can be removed from the Lagrangian. This procedure is analogous to the weak-field limit in linearized gravity where one considers a kind of gauge condition for the background fields. This condition can be written as

$$\dot{\xi}_0 = 0, \tag{5.53}$$

where $\xi = \{\rho, \phi\}$. Hence, up to surface terms the leading-order low-energy effective Lagrangian reads

$$\mathcal{L}_q^{effective} = 2\phi_0\ddot{\rho}_1 + 2\ddot{\phi}_1\rho_0 + \gamma^2\tilde{V}. \tag{5.54}$$

This low-energy effective Lagrangian is valid for all models because we use the same decoupling arguments and renormalization procedure to take care of the infinitely massive terms in the action. From the Lagrangian eq. (5.54), we can obtain the following equations of motion:

$$\ddot{\rho}_1 + \gamma^2\alpha e^{2\rho_0}\phi_0^{\alpha-1}\rho_1 + \frac{\alpha(\alpha-1)}{2}\gamma^2 e^{2\rho_0}\phi_0^{\alpha-2}\phi_1 = 0, \tag{5.55}$$

$$\ddot{\phi}_1 + \gamma^2\alpha e^{2\rho_0}\phi_0^{\alpha-1}\phi_1 + 2\gamma^2 e^{2\rho_0}\phi_0^{\alpha}\rho_1 = 0. \tag{5.56}$$

The near singularity leading order behaviour of ϕ_0 was determined and we ignored any constants simply because we were interested in the character of ϕ_0 in terms of α and t. For models such that $\alpha \neq \{0, 1\}$, we can express the near singularity behaviour of ϕ_0 and $e^{2\rho_0}$ as

$$\phi_0 = \text{constant } \times t^{\frac{2}{1-\alpha}}, \tag{5.57}$$

$$e^{2\rho_0} = \text{constant}. \tag{5.58}$$

If we consider solutions of the type $\rho_1 = \rho_1(\gamma t)$ and $\phi_1 = \phi_1(\gamma t)$, then we can get rid of γ^2 in eqs. (5.55) and (5.56). We can also compensate for any constants we may have lost along the way if we define the following constants:

$$c_0 = \alpha \times \{\text{other constants}\}, \tag{5.59}$$

$$c_1 = \frac{\alpha(\alpha-1)}{2} \times \{\text{other constants}\}, \tag{5.60}$$

$$c_2 = 2 \times \{\text{other constants}\}. \tag{5.61}$$

5.2.1 Solution space for constant curvature models, $\alpha = 0, 1$

For the SIG model ($\alpha = 0$) we get $c_0 = c_1 = 0$ and $c_2 = 2$. The low-energy equations of motion read

$$\ddot{\rho}_1 = 0, \tag{5.62}$$

$$\ddot{\phi}_1 + 2e^{2\rho_0}\rho_1 = 0. \tag{5.63}$$

These equations have the solutions

$$\rho_1 = e_0 t, \tag{5.64}$$

$$\phi_1 = r_0 t e^{2\rho_0} + r_1 e^{2\rho_0} + r_2, \tag{5.65}$$

where e_0, r_0, r_1 and r_2 are integration constants. To be consistent with eq. (5.48), we can let $r_2 = -r_1 e^{2\rho_0(0)}$.

Similarly, for the JT model ($\alpha = 1$), the low-energy equations of motion from eq. (5.54) read

$$\ddot{\rho}_1 + e^{2\rho_0}\rho_1 = 0, \tag{5.66}$$

$$\ddot{\phi}_1 + e^{2\rho_0}\phi_1 + 2e^{2\rho_0}\phi_0\rho_1 = 0. \tag{5.67}$$

Since eq. (5.66) takes the same form as the classical dilaton equation, we can write its solution in terms of ϕ_0 as

$$\rho_1 = \phi_0. \tag{5.68}$$

Using eq. (5.68) and the identity $\operatorname{sech}^2(x) = 1 - \tanh^2(x)$ in the form of $\phi_0^2 = 1 - \frac{1}{2}e^{2\rho_0}$, eq. (5.67) reduces to

$$\ddot{\phi}_1 + e^{2\rho_0}\phi_1 = -\epsilon \tag{5.69}$$

where $\epsilon = 2e^{2\rho_0} - e^{4\rho_0}$ and $\epsilon \approx 0$, because $2e^{2\rho_0}$ and $e^{4\rho_0}$ are numerically equivalent near the singularity. However, the actual solution to eq. (5.69) is

$$\phi_1 = \phi_0 - \frac{1}{2}e^{2\rho_0}. \tag{5.70}$$

Although eq. (5.70) is not consistent with eq. (5.48), in the grand scheme this is not a total loss. What this means is that the singularity in the quantum-corrected picture has been shifted towards the horizon by a value of κ. If we use the numerical approach in which we have to set $\epsilon = 0$, then the solution becomes

$$\phi_1 = \phi_0, \tag{5.71}$$

which is consistent with eq. (5.48).

5.2.2 Solution space for other models, $-1 < \alpha < 0$ and $0 < \alpha < 1$

Using the near singularity classical approximations given by eqs. (5.57) and (5.58), we can write eqs. (5.55) and (5.56) as

$$\ddot{\rho}_1 + c_0 t^{-2}\rho_1 + c_1 t^{b(\alpha-2)}\phi_1 = 0, \tag{5.72}$$
$$\ddot{\phi}_1 + c_0 t^{-2}\phi_1 + c_2 t^{b\alpha}\rho_1 = 0, \tag{5.73}$$

where $b = \frac{2}{1-\alpha}$. To solve this system of equations, we can assume that near the singularity the simplest first order expansion of ϕ_1 can be expressed as

$$\phi_1 = g_0 t^k, \tag{5.74}$$

where $g_0 > 0$ and $k > 0$. This ansatz follows from the assumption that the classical and quantum-corrected singularities should coincide[1]. We can then write the second derivative of eq. (5.74) as

$$\ddot{\phi}_1 = g_1 t^{-2}\phi_1, \tag{5.75}$$

where $g_1 = g_0 k(k-1)$. Substituting eq. (5.75) into eq. (5.73) leads to

$$\phi_1 = g_2 t^{b\alpha+2}\rho_1, \tag{5.76}$$

[1]See eq. (5.48).

where $g_2 = -\frac{c_2}{g_1+c_0}$. But we know that $b\alpha + 2 = b$ and so what follows is a substitution into eq. (5.72) which leads to

$$\ddot{\rho}_1 + c_0 t^{-2}\rho_1 + c_1 g_2 t^{b\alpha-2b+b}\rho_1 = 0. \tag{5.77}$$

The last term in eq. (5.77) conveniently reduces to $c_1 g_2 t^{-2}\rho_1$ since $b\alpha - 2b + b = -2$. Finally, eq. (5.77) comes down to

$$\ddot{\rho}_1 + g_3 t^{-2}\rho_1 = 0, \tag{5.78}$$

where $g_3 = c_0 + c_1 g_2$. If we let $\tau = \ln t$ and denote derivatives with respect to τ with primes, then eq. (5.78) becomes

$$\rho_1'' - \rho_1' + g_3\rho_1 = 0, \tag{5.79}$$

which is clearly solvable. For $g_3 > 0$, the solution of eq. (5.79) can be expressed in terms of t as

$$\rho_1(t) = A\sqrt{t}\cos\left(g_4 \ln t + \epsilon\right), \tag{5.80}$$

where $g_4 = \sqrt{4g_3 - 1}$, A is an integration constant and ϵ is a phase factor. Then using the relation eq. (5.76), we obtain the solution for ϕ_1

$$\phi_1 = Ag_2 t^{b\alpha+\frac{5}{2}}\cos\left(g_4 \ln t + \epsilon\right). \tag{5.81}$$

For $g_3 < 0$, the solution of eq. (5.79) in terms of t reads

$$\rho_1(t) = Ce^{\frac{1}{2}(\sqrt{1-4g_3}+1)\ln t}, \tag{5.82}$$

where C is an integration constant. Substituting this solution into eq. (5.76) leads to

$$\phi_1 = Cg_2 t^{b\alpha+\frac{5}{2}} t^{\frac{1}{2}\sqrt{1-4g_3}} \quad (g_3 < 0). \tag{5.83}$$

From eqs. (5.81) and (5.83), we can show that ϕ_1 is in agreement with eq. (5.48). Since $-1 < \alpha < 0$ and $b = \frac{2}{1-\alpha}$, we find that $b\alpha > -1$. Therefore, $b\alpha + \frac{5}{2} > \frac{3}{2}$ which then implies that eqs. (5.81) and (5.83) are in agreement with eq. (5.48) in the near singularity limit. If we started out with the assumption that $\ddot{\rho}_1 = j_0 t^{-2}\rho_1$, for some constant j_0, we would obtain the same results and ultimately get an equation equivalent to the ansatz eq. (5.75). Furthermore, by using the transformation $\tau = \ln t$, we would also get an equation similar to eq. (5.79). Such an equation would take the form

$$\phi_1'' - \phi_1' - g_1\phi_1 = 0. \tag{5.84}$$

5.3 Near singularity quantum theory

The first step towards a near singularity quantum theory is to construct Hamiltonians that through a process of quantization can be turned into operators that act on quantum states. We know from Hamiltonian mechanics that given a Hamiltonian $\mathcal{H} = \mathcal{H}(\mathbf{q}, \mathbf{p}, t)$, where $(\mathbf{q}, \mathbf{p})$ are the canonical coordinates and t is time, we can obtain the equations of motion using Hamilton's equations. In our case, we have the equations of motion and so the first thing is to find the Hamiltonians that give us the low-energy equations of motion. We could have formulated the Hamiltonian from the action eq. (5.1). However, we would have encountered problems since the fields are coupled and so the construction of a quantum mechanics based on a single type of field would be difficult but not necessarily impossible since we would have had to use a quantum version of Birkhoff's theorem. As it turns out, we simplified matters during our various attempts at solving the equations of motion to find ϕ_1 and ρ_1. We ended up with decoupled equations of motion in each field and it is from these equations that we will find the Hamiltonians. The next subsections will briefly highlight the process of finding the Hamiltonian in each field for each model which will ultimately lead to a quantum picture of the interior of a black hole.

5.3.1 Quantization in the string inspired model, $\alpha = 0$

As has been the tradition, we start by considering the SIG model. From eq. (5.62), it is immediately clear that we get the Hamiltonian of a free particle in one dimension

$$\mathcal{H}_{\rho_1} = \frac{1}{2}\Pi_{\rho_1}^2, \tag{5.85}$$

where Π_{ρ_1} is the canonical momentum to ρ_1. This system is well known, classically and quantum mechanically. To quantize the Hamiltonian eq. (5.85) inside a black hole, we recall that central to quantization is the canonical commutation relation

$$[\hat{\rho}_1, \hat{\Pi}_{\rho_1}] = i, \tag{5.86}$$

where $[\cdot, \cdot]$ is the usual commutator operator, ρ_1 and Π_{ρ_1} have become the operators $\hat{\rho}_1$ and $\hat{\Pi}_{\rho_1}$ respectively. Then from eq. (5.86), we can construct a momentum operator that takes the form

$$\hat{\Pi}_{\rho_1} = -i\frac{d}{d\rho_1}. \tag{5.87}$$

The Hamiltonian operator would then be

$$\hat{\mathcal{H}}_{\rho_1} = -\frac{1}{2}\frac{d^2}{d\rho_1^2}. \tag{5.88}$$

The time-independent Schrödinger equation at the singularity would take the form

$$-\frac{1}{2}\frac{d^2\Psi}{d\rho_1^2} = E\Psi, \tag{5.89}$$

where $\Psi = \Psi(\rho_1)$ is the wave function. Solving the Schrödinger equation yields the wave function

$$\Psi(\rho_1) = Ae^{ik\rho_1} + Be^{-ik\rho_1}, \tag{5.90}$$

where A and B are contributions to the normalization constant and k is an integer. Using eqs. (5.89) and (5.90), one can check that the energy of this system is

$$E = \frac{1}{2}k^2. \tag{5.91}$$

The important thing here is that we have successfully managed to quantize the corrections to gravity for the low-energy SIG model.

If we consider the dilaton sector of this model, we can rewrite the equation of motion (5.63) as

$$\ddot{\phi}_1 + 2e_1te^{2e_0t} = 0, \tag{5.92}$$

where e_0 and e_1 are constants. We end up with a time-dependent Hamiltonian for the ϕ_1 field in the form

$$\mathcal{H}_{\phi_1} = \frac{1}{2}\Pi_{\phi_1}^2 + \left(2e_1te^{2e_0t}\right)\phi_1, \tag{5.93}$$

where Π_{ϕ_1} is the canonical momentum to ϕ_1. If we consider the dominant term only as we get closer to the singularity, then it can be shown that

$$\mathcal{H}_{\phi_1} = \frac{1}{2}\Pi_{\phi_1}^2. \tag{5.94}$$

Then just as we did for the gravitational sector, we can quantize this Hamiltonian. However, there is a caveat associated with this field. The ρ_1 field was unbounded, here

ϕ_1 is bounded from below by 0. As a consequence, the Hamiltonian may not necessarily be self-adjoint. This kind of system has been studied in [32, 57], where they quantize different systems on a half-line. Following the procedure in [32], we have to consider a family of one-parameter self-adjoint extensions of the Hamiltonian. This corresponds to choosing boundary conditions on the wave function, $\Psi(\phi_1)$, at the singularity such that

$$\frac{d\Psi}{d\phi_1}(0) = \lambda\Psi(0), \tag{5.95}$$

where λ is a free parameter. The positive energy states can be shown to take the form

$$\Psi_k(\phi_1) = z_0[(-1+ik\lambda)e^{-ik\phi_1} + (1+ik\lambda)e^{ik\phi_1}], \tag{5.96}$$

where z_0 is a normalization constant and for integer values of k, the energy is given by

$$E_k = k^2. \tag{5.97}$$

5.3.2 Quantization in the Jackiw-Teitelboim model, $\alpha = 1$

The Hamiltonians for this model can be derived from eqs. (5.66) and (5.69). From eq. (5.66), we can write the time-dependent Hamiltonian as

$$\mathcal{H}_{\rho_1} = \frac{1}{2}\Pi^2_{\rho_1} + \frac{1}{2}e^{2\rho_0}\rho_1^2, \tag{5.98}$$

where $e^{2\rho_0} = 2\text{sech}^2(t)$ and $\Pi_{\rho_1} = \dot{\phi}_1$. It can be shown that very close to the singularity, the kinetic part becomes dominant. With that in consideration, we quantize the Hamiltonian

$$\mathcal{H}_{\rho_1} = \frac{1}{2}\Pi^2_{\rho_1}. \tag{5.99}$$

Since ρ_1 is unbounded, we should follow the same quantization procedure as that of a free particle just as we did in section 5.3.1 for ρ_1. Similarly, we can construct a time-dependent Hamiltonian for the ϕ_1 field from eq. (5.69) as

$$\mathcal{H}_{\phi_1} = \frac{1}{2}\Pi^2_{\phi_1} + \frac{1}{2}e^{2\rho_0}\phi_1^2, \tag{5.100}$$

where we have taken $\epsilon = 0$. Again, closer to the singularity the kinetic part becomes dominant and we have the Hamiltonian

$$\mathcal{H}_{\phi_1} = \frac{1}{2}\Pi^2_{\phi_1}. \tag{5.101}$$

The quantum correction to the dilaton is bounded from below such that $\phi_1 \geq 0$ and just as we did for the SIG model, the quantization of this system must be done on a half line. Such a quantum system would be identical to the quantum system found for the SIG model for the ϕ_1 field.

This completes the near singularity quantization of constant curvature models. In both models, we see that the wave functions do not experience an infinity at the classical singularity, $\phi_0 \to 0$. This indicates that in the singularity has been partially resolved.

5.3.3 Divergent curvature models $-1 < \alpha < 0$ and $0 < \alpha < 1$

The remarkable thing about eqs. (5.79) and (5.84) is that they describe a system of damped harmonic oscillators. It is possible to construct Lagrangians and Hamiltonians for such systems. If we let $\varphi(\tau) = \{\phi_1(\tau), \rho_1(\tau)\}$, then the decoupled equations of motion, (5.79) and (5.84), can be written in terms of Euler-Lagrange equations as

$$\frac{d}{d\tau}\frac{\partial \mathcal{L}^{ns}}{\partial \varphi'} - \frac{\partial \mathcal{L}^{ns}}{\partial \varphi} = \frac{\partial \mathcal{S}}{\partial \varphi'}, \tag{5.102}$$

where $\mathcal{L}^{ns}$ is the Lagrangian that describes the near singularity (ns) physics and takes the form

$$\mathcal{L}^{ns} = \frac{1}{2}(\varphi')^2 - \frac{1}{2}g_n\varphi^2, \tag{5.103}$$

where $\varphi(\tau) = \{\phi_1(\tau), \rho_1(\tau)\}$. The term $\frac{\partial \mathcal{S}}{\partial \varphi'}$ in eq. (5.102) is the contribution from the damping term in eqs. (5.79) and (5.84), with $\mathcal{S}$ given by

$$\mathcal{S} = \frac{1}{2}(\varphi')^2. \tag{5.104}$$

The constant, g_n, is either $-g_1$ or g_3 from eqs. (5.79) and (5.84). The right hand side of eq. (5.102) leads to the description of an open system.

In principle, we could write the Hamiltonian of this system as

$$\mathcal{H}^{total} = \mathcal{H}^{ns} + \mathcal{H}_{heat\ bath} + \mathcal{H}_{system-heat\ bath}. \tag{5.105}$$

This type of Hamiltonian is known as the Caldeira-Leggett Hamiltonian [58], it allows us to partition the complete system into a smaller solvable system coupled to a heat bath. The dissipative quantum evolution of such systems using the path integral approach has also been studied at great length by Weiss [59].

For the near singularity quantum corrections, the smaller system takes the form of a simple harmonic oscillator. The corresponding Hamiltonian can be written as

$$\mathcal{H}^{ns} = \frac{1}{2}\Pi_\varphi^2 + \frac{1}{2}g_n\varphi^2, \tag{5.106}$$

where the canonical momentum is defined as $\Pi_\varphi = \varphi'$. The heat bath Hamiltonian can be described as a set of N-harmonic oscillators

$$\mathcal{H}_{heat\ bath} = \frac{1}{2}\sum_{i=1}^{N}(p_i^2 + \omega_i^2 x_i^2), \tag{5.107}$$

where p_i and x_i are momenta and positions, respectively, associated with the heat bath and each ω_i is a constant associated with the heat bath. The system-heat bath coupling can be described using

$$\mathcal{H}_{system-heat\ bath} = \sum_{i=1}^{N}\bigg(- c_i\varphi x_i + \varphi^2\frac{c_i^2}{2\omega_i^2}\bigg), \tag{5.108}$$

where each c_i is a system-heat bath coupling constant. Using a method developed in [60], it can be shown that from the equations of motion resulting from eqs. (5.105) to (5.108), the heat bath degrees can be eliminated so that one obtains a damping term equivalent to one resulting from the partial derivative of eq. (5.104). From this we can conclude that the low-energy near singularity dynamics of the quantum corrections for $\alpha \neq \{0,1\}$ is equivalent or at least analogous to Caldeira-Leggett dynamics. The classical background would probably serve as the analog of the heat-bath.

The quantum evolution of this system can be described by a quantum master equation [60]. To show this, we first define the density matrix of the system as

$$\varrho = \sum_n p_n \left|n\right\rangle \left\langle n\right|, \tag{5.109}$$

where p_n is the probability of finding the system in a state $|n\rangle$. The quantum Liouville master equation is then defined as

$$\frac{\partial\varrho}{\partial\tau} = -i[\hat{\mathcal{H}}^{total}, \varrho]. \tag{5.110}$$

From the master equation, we would have to solve for the density matrix which can be very difficult. While this quantum evolution would be accurate for the corrections to the metric, we should note that it is not completely accurate when we consider the dilaton quantum corrections. Just as we saw for the SIG and JT models, ϕ_1 is such that $\phi_1 \geq 0$. The procedure for developing a proper quantum theory from the ϕ_1 field, must be one that is consistent with quantization on a half-line [57]. Such a quantum theory would require that the wave function, $\Psi(\phi_1)$, be constrained through the relation

$$\Psi(0) = \lambda \frac{d\Psi}{d\phi_1}(0), \tag{5.111}$$

where λ is the free parameter. Given this constraint and considering that we have an open quantum system, which in general contains mixed states, we would expect the quantum dynamics to differ from that of known open quantum systems. We admit that we do not know how such a quantum theory would look like, this is a subject of further research, but at least we know that it is generally possible to formulate a quantum theory near the singularity of two-dimensional black holes. The important thing is that we can arrive at a low-energy near singularity quantum theory that does not contain any divergent quantities and the wave functions do not feel an infinity at the classical singularity. We have shown that there is a path to singularity resolution. This is what we call the quantum picture of a black hole.

Chapter 6

Conclusions

In this thesis, we have studied two-dimensional dilaton-gravity theory with quantum corrections. We started by showing that higher dimensional classical theories of gravity can reduce to a generic two-dimensional scalar-tensor theory. We then removed the kinetic term from the generic theory by using a dilaton reparametrization and a conformal rescaling of the metric. The kinetic-free theory was proved to admit Schwarzschild-like black holes that obey a generalized Birkhoff's theorem. We then used the equations of motion for the kinetic-free theory to express the thermodynamic properties of two-dimensional black holes in terms of dilaton dependent functions. This was followed by a global description of the space-time which finally led to a power law potential for the kinetic-free theory.

We then applied the kinetic-free theory to the interior of a black hole. Using the conformal gauge formalism, we found exact solutions for constant curvature models. Interestingly, our solutions for the SIG model are much simpler than those found in [30, 32]. This is because we had the advantage of working with a theory that does not contain a kinetic term in the dilaton. In addition, our conformal gauge solutions for the JT model do not appear anywhere in the literature. This is probably because not much work has been done on the interior of two-dimensional black holes and exterior solutions are usually expressed in the Schwarzschild gauge. We also established that in the conformal gauge, generalized closed form solutions are impossible to obtain. This led to us finding generic near singularity approximations for the dilaton and gravitational field in terms of conformal time. We have primarily chosen to study the near singularity dynamics simply because only a handful of theorists have focussed on the issue of singularity resolution in two-dimensional gravity and the near horizon dynamics has been studied at great length in the past.

Furthermore, we included a description of the effect of quantized matter on the geometry of space-time. This led to the discussion on the trace-anomaly in two-dimensional gravity. We have modified the trace anomaly so as to make it compatible with kinetic-free dilaton-gravity theory. We briefly discussed the one-loop effective action of two-dimensional gravity. We then used the classical and the effective action to study the interior of a black hole. It should be stressed that the quantum corrections in the dilaton and the metric enter the picture as a result of the effect of quantized matter on the geometrical quantities. If we compare our procedure to linearized gravity, then the quantum correction in the metric could be understood as the analogue of the graviton for our two-dimensional system. As we mentioned earlier, these quantum corrections could be viewed as ordinary quantum fields propagating in the classical background.

Our equations of motion revealed a few near singularity divergent quantities that were entirely dependent on the behaviour of the classical fields. Using effective field theory techniques and decoupling arguments, we removed these non-dynamical terms from the action and remained with a low-energy effective theory centred around the quantum fluctuations in the dilaton and the metric. This low-energy theory was proved to be solvable and from its equations of motion, we found the relevant quantizable Hamiltonians.

Perhaps our biggest result is that we managed to come up with a divergent-free quantum theory at the singularity of a black hole. Strictly speaking, if we considered constant curvature models only, we could have arrived at the same quantum theories using the classical equations. This is what is usually done in two-dimensional gravity; for example the quantization of the SIG model in [30–32]. Strangely, for these models, the classical fields and the quantum corrections have similar dynamical behaviour. In addition, if we take into account that we are also interested in a general quantum theory that includes all possible two-dimensional models, we find it necessary to come up with full quantum theories based on the quantum corrections. For the SIG model, our method still manages to recover the main results in [30, 32]. In addition, we also discovered that the SIG and JT models have the same near singularity quantum dynamics.

If we consider divergent curvature models where $\alpha \neq \{0, 1\}$, quantization of the classical equations would not have been possible since we would inevitably stumble into unavoidable infinities at the singularity. But, as we have stated repeatedly, small corrections to the geometry make it is possible to work out a quantum theory. As we mentioned earlier, we do not have an exact picture of how the quantum dynamics of divergent curvature models would look like. This is due to technical complications that arise while quantizing a system on a half line. Thus given the half line boundary conditions, the complete quantization of an open system should prove to be a challenge in its own right.

In the long run, we would hope to see a theory of quantum dynamics that parallels quantum Markov dynamics [61]. Such dynamics can be easily translated into the language of open quantum walks [62]. This would suggest a direct link with the field of quantum information processing.

Admittedly, we could have studied a lot more on the nature of black holes and the paradoxes associated with them. We believe resolving the singularity, even if done partially, could open doorways to resolving other classical problems associated with black holes. Granted we have been ambitious in our analysis to include all possible two-dimensional semi-classical black holes, but we believe similar or much better results could only be obtained in the context of a complete quantum theory of gravity. It is widely believed that a full quantum theory of gravity must resolve the singularity. Perhaps when such theory is discovered, our results will, to some approximation, correspond to those in the complete two-dimensional quantum theory of gravity.

Appendix A

Dimensional reduction

Here we will show how the dimensional reduction procedure is performed by reducing three-dimensional Einstein gravity to two-dimensional dilaton-gravity. A similar but very lengthy calculation is used in the reduction of four-dimensional Einstein gravity using a spherically symmetric ansatz metric.

In three dimensions we define a manifold $\mathcal{M}_3$ on which we can define a metric $ds^2 = g_{ij}dx^i dx^j$, where $i, j = \{0, 1, 2\}$. We then recall the three-dimensional Einstein-Hilbert action with N matter fields as

$$I_{(3)} = \int_{\mathcal{M}_3} d^3x\sqrt{-g_{(3)}}\big(R_{(3)} + 2\Lambda\big) + \frac{1}{2}\int_{\mathcal{M}_3} d^3x\sqrt{-g_{(3)}}\sum_{i=1}^{N}(\nabla f_i)^2, \tag{A.1}$$

where $g_{(3)} = \det g_{ij}$, R is the curvature scalar, Λ is the cosmological constant and f_i are the matter fields. For the reduction we use the axial-symmetric metric ansatz given by

$$ds^2 = g_{ij}dx^i dx^j = g_{\mu\nu}dx^\mu dx^\nu + \ell^2\phi^2(x^\mu)d\theta^2, \tag{A.2}$$

where the Greek indices run as $\mu, \nu = \{0, 1\}$, ϕ is the dilaton, ℓ is some parameter with length dimension and θ is the angular coordinate such that $0 \le \theta \le 2\pi$. In matrix form, the metric tensor g_{ij} can be written as

$$g_{ij} = \begin{pmatrix} g_{00} & g_{01} & 0 \\ g_{10} & g_{11} & 0 \\ 0 & 0 & \ell^2\phi^2 \end{pmatrix}. \tag{A.3}$$

The determinant of g_{ij} is

$$g_{(3)} = \det g_{ij} = \ell^2\phi^2 g_{(2)}, \tag{A.4}$$

where $g_{(2)} = \det g_{\mu\nu}$. From eq. (A.4) we get

$$\sqrt{-g_{(3)}} = \ell\phi\sqrt{-g_{(2)}}. \tag{A.5}$$

Next we compute the curvature scalar $R_{(3)}$ using the axial-symmetric metric eq. (A.2). From the Riemann tensor,

$$R^d_{abc} = \partial_b\Gamma^d_{ac} - \partial_c\Gamma^d_{ab} + \Gamma^d_{eb}\Gamma^e_{ac} - \Gamma^d_{ec}\Gamma^e_{ab}, \tag{A.6}$$

where $\partial_a = \frac{\partial}{\partial x^a}$, we can compute the Ricci tensor

$$R_{ac} = R^d_{adc}, \tag{A.7}$$

in order to get the Ricci/curvature scalar

$$R = g^{ac}R_{ac}. \tag{A.8}$$

The curvature scalar in the action eq. (A.1) under the metric eq. (A.2) takes the general form

$$R_{(3)} = g^{\mu\nu}R_{\mu\nu} + g^{22}R_{22}. \tag{A.9}$$

The term $R_{\mu\nu}$ requires a bit of care since it contains terms for the two-dimensional Ricci scalar, $R_{(2)}$, and some remainder terms. Explicitly, this term is written as

$$R_{\mu\nu} = \partial_k\Gamma^k_{\mu\nu} - \partial_\nu\Gamma^k_{\mu k} + \Gamma^k_{ek}\Gamma^e_{\mu\nu} - \Gamma^k_{e\nu}\Gamma^e_{\mu k}. \tag{A.10}$$

From this equation we can see that values $k = 0$ and $k = 1$ contribute to $R_{(2)}$, the two-dimensional curvature scalar, while the $k = 2$ terms contribute to the remainder term which we will call $R^{(r)}_{\mu\nu}$. We can then write eq. (A.9) as

$$R_{(3)} = R_{(2)} + g^{\mu\nu}R^{(r)}_{\mu\nu} + g^{22}R_{22}. \tag{A.11}$$

Since all fields are dependent on the first two coordinates x^0 and x^1, we can write $R^{(r)}_{\mu\nu}$ as

$$R^{(r)}_{\mu\nu} = -\partial_\nu\Gamma^2_{\mu 2} + \Gamma^2_{e2}\Gamma^e_{\mu\nu} - \Gamma^2_{e\mu}\Gamma^e_{\nu 2}. \tag{A.12}$$

The non-vanishing Christofell symbols from eq. (A.12) are

$$\begin{aligned}
\Gamma^2_{\mu 2} &= \frac{1}{2} g^{22} \partial_\mu g_{22}, \\
\Gamma^e_{\mu\nu} &= \frac{1}{2} g^{es} \Big(\partial_\mu g_{\nu s} + \partial_\nu g_{\mu s} - \partial_s g_{\mu\nu} \Big), \\
\Gamma^2_{e\mu} &= \frac{1}{2} g^{22} \partial_\mu g_{e2}, \\
\Gamma^e_{\nu 2} &= \frac{1}{2} g^{ei} \partial_\nu g_{i2}.
\end{aligned} \tag{A.13}$$

Using eqs. (A.13), eq. (A.12) takes the form

$$R^{(r)}_{\mu\nu} = -\frac{1}{2} \partial_\nu \Big(g^{22} \partial_\mu g_{22} \Big) + \frac{1}{4} g^{22} \partial^\sigma g_{22} \Big(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu} \Big) - \frac{1}{4} g^{22} g^{22} \partial_\mu g_{22} \partial_\nu g_{22}. \tag{A.14}$$

The next step involves the calculation of R_{22} as follows:

$$R_{22} = \partial_\mu \Gamma^\mu_{22} + \Gamma^k_{ek} \Gamma^e_{22} - \Gamma^k_{e2} \Gamma^e_{2k}. \tag{A.15}$$

After computing the non-vanishing Christofell symbols in eq. (A.15) and making the necessary simplifications, we get

$$R_{22} = -\frac{1}{2} \partial_\mu \Big(g^{\mu\nu} \partial_\nu g_{22} \Big) - \frac{1}{4} g^{\mu\nu} \partial^\sigma g_{22} \Big(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu} \Big) + \frac{1}{4} g^{22} g^{\mu\nu} \partial_\mu g_{22} \partial_\nu g_{22}. \tag{A.16}$$

Using eqs. (A.14) and (A.16), eq. (A.11) can be written as

$$\begin{aligned}
R_{(3)} &= R_{(2)} - \frac{1}{2} \partial^\mu \left(g^{22} \partial_\mu g_{22} \right) - \frac{1}{2} g^{22} \partial_\mu \partial^\mu g_{22}, \\
&= R_{(2)} - \frac{1}{2} \partial^\mu g^{22} \partial_\mu g_{22} - g^{22} \partial^\mu \partial_\mu g_{22}.
\end{aligned} \tag{A.17}$$

Substituting $g_{22} = \ell^2 \phi^2$ and $g^{22} = \ell^{-2} \phi^{-2}$ into eq. (A.17) we get

$$R_{(3)} = R_{(2)} - 2\phi^{-1} \partial^\mu \partial_\mu \phi. \tag{A.18}$$

We then substitute eqs. (A.5) and (A.18) into eq. (A.1) and integrate the angular coordinate to get a two-dimensional action defined on a two-dimensional manifold, $\mathcal{M}_2$, that takes the form

$$I_{(2)} = 2\pi\ell \int_{\mathcal{M}_2} d^2x \sqrt{-g_{(2)}} \left(\phi R_{(2)} + 2\Lambda\phi - 2\partial^\mu \partial_\mu \phi \right) + \pi\ell \int_{\mathcal{M}_2} d^2x \sqrt{-g_{(2)}} \sum_{i=1}^{N} \phi (\nabla f_i)^2. \tag{A.19}$$

Now suppose we focus on the term that contains a second derivative in ϕ from eq. (A.19) and apply the divergence theorem

$$\int_{\mathcal{M}_2} d^2x\sqrt{-g_{(2)}}\partial_\mu\partial^\mu\phi = \int_{\partial\mathcal{M}_2} dx\sqrt{-\gamma}n_\mu\partial^\mu\phi, \tag{A.20}$$

where n_μ is a unit vector normal to the boundary of the manifold, $\partial\mathcal{M}_2$, and γ is the induced metric at the boundary. The integral on the right hand side of eq. (A.20) is evaluated at the boundary of the manifold. From this we conclude that the left hand side, which is also present in the action eq. (A.19), is in fact a boundary term. Such a term does not affect the bulk dynamics of the theory. With this in consideration, we can then write the JT action as

$$I_{(2)} = 2\pi\ell\int d^2x\sqrt{-g_{(2)}}\phi\big(R_{(2)} + 2\Lambda\big) + \pi\ell\int d^2x\sqrt{-g_{(2)}}\sum_{i=1}^{N}\phi(\nabla f_i)^2. \tag{A.21}$$

It can also be shown that the dimensional reduction of four-dimensional Einstein gravity follows by using the same procedure we have used for BTZ gravity. Only this time, if we use the metric ansatz

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu + \ell^2\phi^2 d\Omega_2^2, \tag{A.22}$$

then the four-dimensional curvature scalar reduces to

$$R_{(4)} = R_{(2)} - \frac{2}{\phi^2}\left[\frac{1}{\ell^2} + (\nabla\phi)^2\right] - \frac{4}{\phi}\Box\phi. \tag{A.23}$$

Adding the contribution from $\sqrt{-g_{(4)}} = \ell^2\phi^2\sin\theta\sqrt{-g_{(2)}}$ followed by an integration by parts on the last term. Finally, we integrate the angular modes so that the Hilbert-Einstein action comes down to

$$\int d^4x\sqrt{-g_{(4)}}R_{(4)} = 4\pi\ell^2\int d^2x\sqrt{-g_{(2)}}\left[\phi^2 R_{(2)} - \frac{2}{\ell^2} + 2(\nabla\phi)^2\right]. \tag{A.24}$$

The left hand side represents Einstein's four-dimensional gravity, while the right hand side is spherically reduced two-dimensional gravity.

Appendix B

Birkhoff's theorem in two dimensions

The proof of Birkhoff's theorem presented here follows the one in [45] but includes a few minor modifications. In order to prove that a particular theory is consistent with Birkhoff's theorem, the metric solutions of the theory must be time-independent (static) and the solutions should depend on only one coordinate independent parameter. Since all two-dimensional space-times are conformally flat, it may be beneficial to proceed with the conformal gauge formalism. So we take the conformally flat metric

$$ds^2 = e^{2\rho(t,x)}(-dt^2 + dx^2), \tag{B.1}$$

and introduce light cone coordinates by making the following coordinate transformations

$$\begin{aligned} z_+ &= x + t, \\ z_- &= x - t, \end{aligned} \tag{B.2}$$

for which metric eq. (B.1) is transformed into

$$ds^2 = e^{2\rho(z_+,z_-)}dz_-dz_+. \tag{B.3}$$

In this light cone gauge, eq. (B.3), the field eqs. (2.40) and (2.43) are given by

$$\frac{\partial^2 \rho}{\partial z_+ \partial z_-} - \frac{1}{8} e^{2\rho} \gamma^2 \frac{dV}{d\phi} = 0, \tag{B.4}$$

$$\frac{\partial^2 \phi}{\partial z_+ \partial z_-} - \frac{1}{4} e^{2\rho} \gamma^2 V(\phi) = 0, \tag{B.5}$$

$$\frac{\partial^2 \phi}{\partial z_+^2} - 2 \frac{\partial \rho}{\partial z_+} \frac{\partial \phi}{\partial z_+} = 0, \tag{B.6}$$

$$\frac{\partial^2 \phi}{\partial z_-^2} - 2 \frac{\partial \rho}{\partial z_-} \frac{\partial \phi}{\partial z_-} = 0. \tag{B.7}$$

Equations (B.6) and (B.7), respectively, can be rewritten as

$$e^{2\rho} \frac{\partial}{\partial z_+} \left[e^{-2\rho} \frac{\partial \phi}{\partial z_+} \right] = 0, \tag{B.8}$$

$$e^{2\rho} \frac{\partial}{\partial z_-} \left[e^{-2\rho} \frac{\partial \phi}{\partial z_-} \right] = 0. \tag{B.9}$$

Equations (B.8) and (B.9) tell us that the terms inside the parenthesis are independent of z_+ and z_-, respectively. An integration would then yield

$$e^{-2\rho} \frac{\partial \phi}{\partial z_+} = f(z_-), \tag{B.10}$$

$$e^{-2\rho} \frac{\partial \phi}{\partial z_-} = g(z_+), \tag{B.11}$$

where $f(z_-)$ and $g(z_+)$ are arbitrary functions. From eqs. (B.10) and (B.11) we find

$$f(z_-) \frac{\partial \phi}{\partial z_-} = g(z_+) \frac{\partial \phi}{\partial z_+}. \tag{B.12}$$

In the next step we substitute eqs. (B.10) and (B.11) into eq. (B.5) to get

$$\frac{\partial^2 \phi}{\partial z_+ \partial z_-} - \frac{1}{4} \frac{1}{f(z_-)} \frac{\partial \phi}{\partial z_+} \gamma^2 V(\phi) = 0, \tag{B.13}$$

$$\frac{\partial^2 \phi}{\partial z_+ \partial z_-} - \frac{1}{4} \frac{1}{g(z_+)} \frac{\partial \phi}{\partial z_-} \gamma^2 V(\phi) = 0. \tag{B.14}$$

Looking at eqs. (B.13) and (B.14), we can define a function $J(\phi)$ such that

$$V(\phi) = \frac{dJ}{d\phi}. \tag{B.15}$$

We can further define partial derivatives of J in terms of z_+ and z_- as

$$\frac{\partial J}{\partial z_+} = \frac{\partial \phi}{\partial z_+}\frac{dJ}{d\phi}, \tag{B.16}$$

$$\frac{\partial J}{\partial z_-} = \frac{\partial \phi}{\partial z_-}\frac{dJ}{d\phi}. \tag{B.17}$$

Equations (B.13) and (B.14) can then be written as

$$\frac{\partial}{\partial z_+}\left[\frac{\partial \phi}{\partial z_-} - \frac{1}{4}\frac{1}{f(z_-)}\gamma^2 J\right] = 0, \tag{B.18}$$

$$\frac{\partial}{\partial z_-}\left[\frac{\partial \phi}{\partial z_+} - \frac{1}{4}\frac{1}{g(z_+)}\gamma^2 J\right] = 0. \tag{B.19}$$

From eqs. (B.18) and (B.19) we find that

$$\frac{\partial \phi}{\partial z_-} - \frac{1}{4}\frac{1}{f(z_-)}\gamma^2 J(\phi) = M^-(z_-), \tag{B.20}$$

$$\frac{\partial \phi}{\partial z_+} - \frac{1}{4}\frac{1}{g(z_+)}\gamma^2 J(\phi) = M^+(z_+), \tag{B.21}$$

where $M^-(z_-)$ and $M^+(z_+)$ are some functions. Using eqs. (B.12), (B.20) and (B.21), we find

$$M^+(z_+)g(z_+) = M^-(z_-)f(z_-). \tag{B.22}$$

Furthermore, we also consider the derivatives of eq. (B.22) with respect to z_- and z_+ and find

$$\frac{\partial}{\partial z_+}\left[M^+(z_+)g(z_+)\right] = \frac{\partial}{\partial z_+}\left[M^-(z_-)f(z_-)\right] = 0, \tag{B.23}$$

$$\frac{\partial}{\partial z_-}\left[M^-(z_-)f(z_-)\right] = \frac{\partial}{\partial z_-}\left[M^+(z_+)g(z_+)\right] = 0. \tag{B.24}$$

From eqs. (B.23) and (B.24) we conclude that

$$M^+(z_+)g(z_+) = M^-(z_-)f(z_-) = -\frac{\gamma^2}{2}M, \tag{B.25}$$

where M is a constant and the $-\gamma^2/2$ factor is meant to simplify future calculations and to keep M dimensionless. Following this development, we can then rewrite eqs. (B.20) and (B.21) as

$$f(z_-)\frac{\partial \phi}{\partial z_-} - \frac{1}{4}\gamma^2 J(\phi) = -\frac{\gamma^2}{2}M, \tag{B.26}$$

$$g(z_+)\frac{\partial \phi}{\partial z_+} - \frac{1}{4}\gamma^2 J(\phi) = -\frac{\gamma^2}{2}M. \tag{B.27}$$

Now the claim is that the field eqs. (B.4) to (B.7) are equivalent to the pair of differential eqs. (B.10) and (B.26) or the pair eqs. (B.11) and (B.27). For instance, without going too much into detail, using $\frac{\partial^2}{\partial z_+ \partial z_-}$ on eq. (B.10) and $\frac{\partial^2}{\partial z_+^2}$ on eq. (B.26) followed by a few expansions and simplifications, one can easily recover eq. (B.4). The other field equations can be cleverly worked out with a similar analysis. What is interesting though, is when we replace $f(z_-)$ in eq. (B.26) with its definition as given in eq. (B.10) we get

$$e^{-2\rho}\frac{\partial \phi}{\partial z_+}\frac{\partial \phi}{\partial z_-} - \frac{1}{4}\gamma^2 J(\phi) = -\frac{\gamma^2}{2}M. \tag{B.28}$$

Considering that $g^{+-} = g^{-+} = 2e^{-2\rho}$, we find that the first term in eq. (B.28) is actually $\frac{1}{4}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$ expressed in the light cone gauge. So it turns out that eqs. (B.26) to (B.28) can be written into a single equation

$$-g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi + \gamma^2 J(\phi) = 2\gamma^2 M. \tag{B.29}$$

What is interesting about eq. (B.29) is that one can consider a change in coordinates

$$\begin{aligned} z_+ &\to \bar{z}_+, \\ z_- &\to \bar{z}_-, \end{aligned} \tag{B.30}$$

so that the usual metric tensor transformation rule, under the change of coordinates $x^\mu \to \bar{x}^\mu$,

$$g_{\mu\nu}(\bar{x}) = \frac{\partial x^\rho}{\partial \bar{x}^\mu}\frac{\partial x^\sigma}{\partial \bar{x}^\nu}g_{\rho\sigma}(x), \tag{B.31}$$

in the light cone gauge can be written as

$$e^{2\rho(\bar{z}_+,\bar{z}_-)} = \frac{\partial z_+}{\partial \bar{z}_+}\frac{\partial z_-}{\partial \bar{z}_-}e^{2\rho(z_+,z_-)}. \tag{B.32}$$

Following the coordinate transformations as given in eqs. (B.30), we know that the dilaton is a scalar which is invariant under a change of coordinates. This is shown as

$$\phi(z_+, z_-) = \bar{\phi}(\bar{z}_+, \bar{z}_-), \tag{B.33}$$

which also extends to $J(\phi) = \bar{J}(\bar{\phi})$. Furthermore, we can use eqs. (B.32) and (B.33) to show that

$$\bar{g}^{\mu\nu}\bar{\nabla}_\mu\bar{\phi}\bar{\nabla}_\nu\bar{\phi} = g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi. \tag{B.34}$$

We therefore conclude that eq. (B.29) is invariant under coordinate transformation as given in eq. (B.30). Essentially this means that the solutions of the theory only depend on one coordinate independent parameter, M. We have thus proved one of the two things we set out to prove. The next thing is to prove that there exists a coordinate system in which the solutions are static. Following the coordinate transformations of eq. (B.30) and the metric transformation of eq. (B.32), the functions $f(z_-)$, $g(z_+)$ transform as

$$f(z_-) = \frac{\partial z_-}{\partial \bar{z}_-}\bar{f}(\bar{z}_-), \tag{B.35}$$

$$g(z_+) = \frac{\partial z_+}{\partial \bar{z}_+}\bar{g}(\bar{z}_+), \tag{B.36}$$

where $\bar{f}(\bar{z}_-)$ and $\bar{g}(\bar{z}_+)$ take the canonical forms of eqs. (B.10) and (B.11) in the coordinate system $(\bar{z}_+, \bar{z}_-)$, respectively. We can then multiply eqs. (B.35) and (B.36), respectively, to get

$$f(z_-)g(z_+) = \frac{\partial z_+}{\partial \bar{z}_+}\frac{\partial z_-}{\partial \bar{z}_-}\bar{f}(\bar{z}_-)\bar{g}(\bar{z}_+). \tag{B.37}$$

In the transformed coordinate system, we are free to set $\bar{f}(\bar{z}_-) = \bar{g}(\bar{z}_+) = 1$. Thus, as required by eq. (B.32), which is also a requirement of a diffeomorphism invariant theory, we have

$$\frac{\partial z_+}{\partial \bar{z}_+}\frac{\partial z_-}{\partial \bar{z}_-} \neq 0, \tag{B.38}$$

which also leads to $f(z_-)g(z_+) \neq 0$. This last expression must hold in any coordinate system. Setting $\bar{f}(\bar{z}_-) = \bar{g}(\bar{z}_+) = 1$, we find that for the barred coordinate system $(\bar{z}_+, \bar{z}_-)$ the following hold

$$\frac{\partial \bar{\phi}}{\partial \bar{t}} = \frac{\partial \bar{\phi}}{\partial \bar{z}_+} - \frac{\partial \bar{\phi}}{\partial \bar{z}_-} = 0, \tag{B.39}$$

$$\frac{\partial \bar{\phi}}{\partial \bar{x}} = \frac{\partial \bar{\phi}}{\partial \bar{z}_+} + \frac{\partial \bar{\phi}}{\partial \bar{z}_-} = 2e^{2\rho(\bar{x})}, \tag{B.40}$$

as long as the barred light cone coordinates can be expressed as

$$\bar{z}_+ = \bar{x} + \bar{t}, \tag{B.41}$$

$$\bar{z}_- = \bar{x} - \bar{t}. \tag{B.42}$$

From eqs. (B.39) and (B.40), we see that there is a coordinate system in which the metric is time-independent. This evidently completes the proof of the generalized Birkhoff's theorem in two-dimensional dilaton-gravity.

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