Visual representations of linear algebraic expressions: a case study in a Grade 9 after-school mathematics club

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Abstract

Visualisation is commonly used as a tool in introducing algebra through visual or kinaesthetic sequences designed to prompt learners' development of a general rule for moving from a term's position to its output value. Fluency in both the concepts and the conventions of elementary algebra are essential to learners, as algebra forms the language in which many advanced mathematical ideas are encoded. Moreover, algebraic fluency is often associated with an ability to think abstractly about arithmetic processes.

In many classrooms, however, research has shown that learners often focus on fluency in algebraic conventions rather than concepts, learning how to manipulate expressions without understanding the algorithms they are taught to follow. This trend can be linked to several causes, including teacher-centred mathematics classrooms in which learners are – whether implicitly or explicitly – encouraged to copy formulae and methods in order to 'get it right in a test' without necessarily grasping the underlying logical relationships.

This case study, therefore, aimed to determine whether there was value in using visual, kinaesthetic models to broaden and deepen learners' use of algebra. To that end, in the context of an extra-curricular mathematics club that aimed to decentre the teachers and demand innovative ideas of the participants, six pairs of Grade 9 learners were tasked with creating visual representations of a linear algebraic expression using coloured building cubes.

The responses to this task over the course of five assignments were many and varied and almost universally displayed a sustained internal logic that the learners were able to explain and develop. Most pairs began with a visual list of terms arranged in sets of towers, pyramids or, in one case, a spiral. At the end of the study, all but one of the pairs had settled on a Visual Expression, in which various colours were used to represent elements of the algebra such as the values of the coefficient, the variable and the constant term. The participants' representations grew in complexity over the course of the study and the conformity of the final responses showed that the club was a collaborative space in which learners shared ideas. However, the structure of the Visual Expressions and their own confessions of nerves about 'getting it wrong' in the interviews suggest that the participants were stuck in a mindset that led them to seek out and idealise the representation closest to the original algebra, even though that representation revealed little about the structural relationship underlying the expression.

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- God, who gives me breath and purpose.

Declaration of Originality

I, Sindisiwe Herbert (student number 09M3337), declare that this thesis, *An analysis of visual representations of linear algebraic expressions created by selected Grade 9 learners: a case study in an after-school mathematics club*, is my own work and written in my own words. Where I have drawn on or quoted the words and ideas of others, I have acknowledged the authors by using the reference practices as set out by Rhodes University Education Department's Guide to referencing.

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Signed

5 February 2023

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Date

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List of Acronyms

CAPS	Curriculum Assessment Policy Statements
FET	Further Education and Training
NRF	National Research Fund
RUESC	Rhodes University Ethical Standards Committee

Chapter 1

Introduction

Algebra and visualisation are rarely considered partners in mathematics. Algebra, at least the elementary algebra studied at school level, is associated with strict logic and clear, predetermined procedures (Clements, 2014), most often used in solving equations or simplifying expressions whose visual aspect consists entirely of letters, numbers and operation symbols. On the other hand, visual imagery is associated with creativity and artistic ability. In mathematics, visualisation is a major component of geometry, graphing and trigonometry, but is often ignored outside of these contexts, despite much research demonstrating its importance in mathematics education (Presmeg, 2006b). Despite these conceptions, visualisation can form an important part of the introduction and extension of algebraic ideas.

1.1 Background

Despite general agreement that visualisation plays an important role in mathematics and mathematics education, its definition often depends on the background and goals of the person doing the defining (Clements, 2014). For the purposes of this study, visualisation is considered to be both an internal "mode of … thinking" and an external means of communicating and developing mathematical ideas and relationships (Nardi, 2014, p. 198).

Visualisation was a cornerstone of mathematics for centuries but it fell out of favour in the 1800s with the formalising of analysis (Tall, 1991). Indeed, Noss et al. (1997) suggest that mathematicians often assume that visual, concrete contexts are necessary for introducing concepts, but are inferior to more abstract representations like algebra that seem to get closer to the underlying ideas. This is an assumption worth interrogating.

1.2 Visualisation in algebra – the gap

The link between algebra and visualisation is often found in visual patterning activities such as those depicted in **Error! Reference source not found.**. Learners are provided with a visual context and asked to determine the number of items (dots, squares or matchsticks) in a distant or even abstract, general term. Studies involving this kind of activity are well represented in the literature. English and Warren (1998) and Pegg and Redden (1990), among others, describe the benefits and pitfalls of introducing algebra using these kinds of activities with young teenagers. Hershkowitz et al., (2001) describe an activity designed to promote discussion and deepen the algebraic understanding of trainee teachers. Samson (2007) interrogates the degree to which question design has an impact on learners' approaches to these problems.

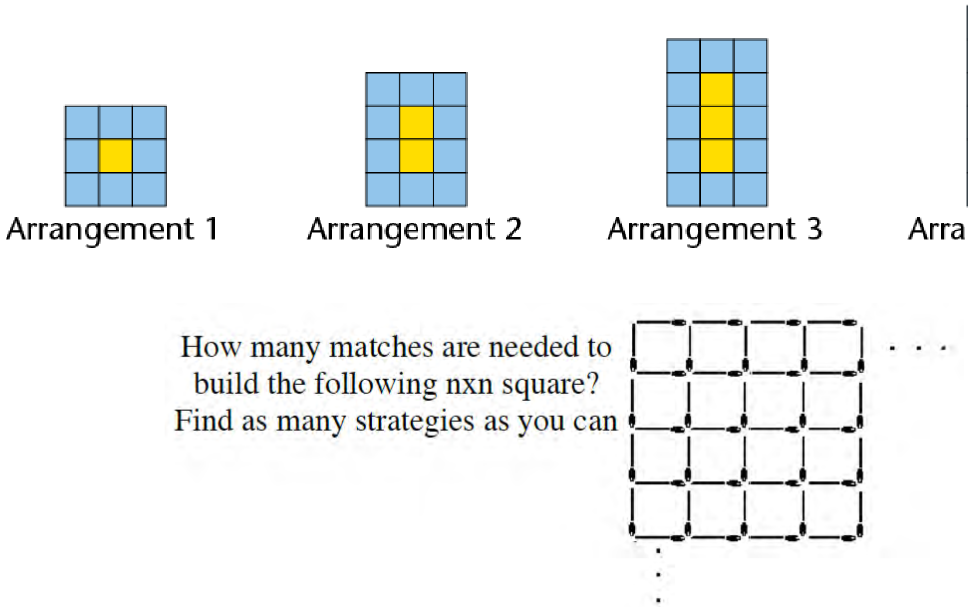


Figure 1.1: Common visual patterning activities

There is, however, a gap in the literature. It is clear that visual patterning activities can be used as a strong foundation for algebraic thinking. Thus, activities in which learners are asked to find a general algebraic expression to describe a visual sequence are worth incorporating into a teacher's classroom practice. The question is, would there be value in performing the process the other way around – building a visualisation to represent the algebra?

1.3 Building block expressions – the study and the research questions

To this end, a study was conceived in which learners would use physical objects to build a visual representation of a linear algebraic expression in the form of a sequence. This kind of creative, learner-centred work is not an explicit part of the national South African mathematics curriculum and, for this and various other reasons, is rarely seen in classrooms. Indeed, many learners experience and come to expect a focus on "skills and information before meaning" in mathematics classrooms where the teacher is seen as the holder and active distributor of knowledge and the learners are passive recipients (Graven, 2011, p. 164). Consequently, the study was designed to fit into a slightly different space – a maths club, inspired by the work of Stott and Graven (2013) – in which classroom norms would be explicitly challenged and learners would feel free to take their time and actively explore mathematical ideas. The study took place over four club sessions during which the learners built representations of linear algebraic expressions in pairs. It concluded with each pair participating in a task-based interview and a discussion about their various creations.

As the study progressed, two particular research questions emerged:

- What aspects of linear algebraic expressions did participants represent visually?
- In what ways did the visualisations develop over the course of the club, both within and between groups?

Answering the first question would create a summative description of what exactly the participants created over the course of the study. Answering the second question would provide a comparison between early and late visualisations as well as between different groups' creations. It would also allow a degree of interpretation as to what the visualisations implied about learners' understanding of the task and the mathematical relationship at the heart of linear algebraic expressions.

1.4 Document Overview

In order to describe the study and answer these questions, this thesis first reviews the relevant literature that informed and inspired it in Chapter Chapter 2: Literature Review. This covers definitions and reasons for the importance of visualisation and algebra, along with an overview of studies linking the two that have already been conducted in Sections 2.1 and 2.2. Furthermore, the theoretical framework is examined in Section 2.3 and arguments for the implementation of a maths club are presented in Section 2.4.

From here, the document moves into Chapter 3: Methodology, which describes the methodology of the study: the orientation, method, selection of participants, research design and instruments, along with a brief look at the plan for analysing the data (Sections 3.1 to 3.6). Finally, Sections 3.7 and 3.8 discuss the ethics, validity and reliability of the study.

Chapter 4: Results, Analysis & Discussion covers the results of the study and their analysis. It begins in Section 4.1 with a description of each club session: an explanation of the task and of each pair's response to it. Section 4.2 then covers each interview, describing the participants' response to the interview task and then their discussion of each of their creations. Overall, this provides a chronological overview of the participants' creations and then a look at each pair's overall output, which allows insight into the progression of ideas and conceptual focus on both a global and an individual scale. The chapter then attempts to answer the two research questions in Sections 4.3 and 4.4 before moving on to a discussion of the results in Section4.5.

Chapter 5: Conclusion concludes the document with a document summary in Section 5.1 and a summary of findings in Section5.2, followed by a look at the study's limitations, potential areas for further research and recommendations to teachers and researchers based on the outcomes of the research in Sections 5.3 to 5.5. Finally, Section 5.6 describes my journey through the study and serves as a personal note on my growth as a researcher, scholar and a teacher.

Chapter 2

Literature Review

Algebra and visualisation are often regarded as polar opposites in mathematics – one associated with strict logic and algorithm (Clements, 2014) and the other with great (but sometimes deceptive) "leaps of insight" (Tall, 1991, p. 1). Tall (1991, p. 1) notes that "Euclidean geometry was held as the archetypal theory of logical deduction" for thousands of years, but the discovery of conclusions that had been drawn based on subtle visual assumptions rather than axioms and definitions led to a subsequent reliance on analysis and algebra in proving theorems. Hence, visualisation has long been regarded by many in the mathematical community being of little value (Dreyfus, 1991). However, this opinion has begun to change in recent decades (Arcavi, 2003). Campbell et al. (1995, p. 177) claim that

> There is increasing recognition that intuitive and imagery based processes play an important role in all levels of mathematical problem solving from those of the child in the early stages of mathematical development ... to the novel creations of highly gifted mathematicians and scientists.

Despite this, even the strongest advocates for improving the status of visualisation in mathematics and mathematics education stress that to be truly constructive, visual methods of learning and teaching must be coupled with "rigorous analytical thought" (Nardi, 2014, p. 213).

Consequently, this study aims to explore some of the possibilities of "visualising algebra" in order to attempt this merging of analytical algebraic and intuitive visual learning. It does so through the vehicle of sequences, which are often used in the introduction of algebra (English & Warren, 1998; Pegg & Redden, 1990). Numerical and pictorial sequences are an important part of the South African curriculum (Department of Basic Education, 2011), but my experience as a high school teacher has led me to believe that many learners fail to grasp the structure underlying these sequences, preferring to memorise algorithms and formulae that will provide them with the general term.

This chapter provides a review of the literature around visualisation, algebra and sequences, as well as the theoretical framework used to ground the study. I begin by discussing visualisation and the many studies that have shown its importance in mathematics education.

The chapter then moves on to explore algebra and sequences, their importance in mathematics, their place in the South African curriculum and learners' strategies in generalising them. I also describe the ways in which visual and pictorial sequences have been studied, identifying the "gap" into which my research falls. This leads into a review of knowledge objectification (Radford, 2008), the main theoretical framework of the study and a look at semiotics in mathematics education. Finally, I consider the concept of after-school mathematics clubs, particularly in the South African context, and how the use of such a club fits into the theoretical framework of the study.

2.1 Visualisation

Visualisation is notoriously difficult to define. Clements (2014) describes how researchers' differing backgrounds have led to highly varied definitions. Psychologists using "factor analytic techniques" (Clements, 2014, p. 179) have tended to identify mathematical strategies as either analytical/verbal or spatial/visual. These approaches are often portrayed as opposite ends of a spectrum, with many researchers acknowledging that many people implement a mixture of the two strategies. Contrastingly, mathematics educators have often taken a constructivist position using frameworks like Peircean Semiotics (Presmeg, A semiotic view of the role of imagery and inscriptions in mathematics teaching and learning, 2006a) and Bruner's modes of mental representation (Tall, 1994) to discuss the ways in which learners visualise in mathematics. Mathematicians seem to have a less strictly 'mental' definition of visualisation, taking it to mean "the process of producing or using geometrical or graphical representations of mathematical concepts, principles or problems, whether hand drawn or computer generated" (Clements, 2014, p. 181). Despite this proliferation of possible meanings, it is useful to have a basic idea of the concept from which to proceed. This section therefore will provide several relevant perspectives before providing a working definition and then proceeding to review studies on the topic.

2.1.1. Defining Visualisation

Arcavi (2003, p. 217) presents a very comprehensive definition of visualisation, that:

> Visualisation is the ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information,

thinking about and developing previously unknown ideas and advancing understandings.

Presmeg (1986, p. 42) notes that along with "shape, pattern or form ... verbal, numerical or mathematical symbols may be arranged spatially to form [a] kind of imagery." Given the diversity of visual images and their uses which have already been described, one can appreciate the difficulty of succinctly defining 'visualisation'. However, for the purposes of this study it will be sufficient to adopt Hershkowitz's dual understanding of visualisation as

- a mode of mathematical thinking [and]
- a group of signs and relationships among them ("a language"), by which mathematical thinking, including the visual one, might be developed, limited, expressed and communicated to oneself and to others (Nardi, 2014, p. 198).

This definition relies heavily on semiotics, which is discussed in Section 2.3.1. Hershkowitz notes that "neither of these two perspectives [that is, visualisation as a mode of thinking or as a language] ... has a meaningful existence by itself: visual thinking needs "a language" (either visual or another language) to be expressed; and visual language, when it does not represent a thought, is just a group of signs without a meaning" (Nardi, 2014, p. 198). Consequently, this study aims to analyse not only the visual representations created by participants, but also the ways in which these representations are created, discussed and shared between the learners.

2.1.2. The Importance of Visualisation

There is a great deal of research into visualisation in mathematics education that has taken place in the last four decades, summarised by Presmeg (2006b) and Clements (2014). Presmeg (2006b, p. 206) notes that "renewed interest in the topic of visualisation research in mathematics education started to become apparent from 1988 onwards". This research has included young children (Razel & Eylon, 1990), pre-service teachers (Hershkowitz et al., 2001), high school learners and teachers (Presmeg, 1986). It has covered such diverse content as calculus (Tall, 1991), functions (Thomas, 2003), geometry, shape and sequence (Razel & Eylon, 1990) and (increasingly) the use of computers in providing visual environments for learning (Noss et al., 1997). The work of several researchers whose insights have been particularly useful for this study are summarised below.

Norma Presmeg (1986) developed a model with which to describe a person's preference for visual methods in the mathematics classroom, producing an instrument for measuring this preference in both teachers and learners. She also identified five types of visual imagery:

- concrete, pictorial imagery,
- pattern imagery,
- memory images of formulae,
- kinaesthetic imagery, and
- dynamic imagery.

She recorded each type's frequency of use. In this research, Presmeg (1986) identified both advantages and disadvantages to learners' use of visual methods in learning mathematics. Moreover, she found that teachers who employed a combination of visual and analytical methods in the classroom were best able to assist learners who naturally thought in visual ways. Since this initial research, Presmeg (2006b) has participated in and encouraged further studies into visualisation in mathematics education. In 2006 she compiled a list of thirteen questions "of major significance for research on visualisation in mathematics education" (Presmeg, 2006b, p. 227), which include questions on "compartmentalisation" and making "connections between visual and symbolic inscriptions" that inspired this study.

Rina Hershkowitz and her colleagues at the Weizman Institute of Science spent many years researching the Agam Programme for Visual Cognition (Razel & Elyon, 1990; Hershkowitz & Markovits, 1992), which worked with 3- to 5-year-olds to develop their visual vocabulary and thus their ability to think visually. They explored shapes, patterns, ratio and other mathematical ideas in tactile, kinaesthetic ways and the research has shown that the programme helped children with problem solving, arithmetic and writing readiness (Razel & Eylon, 1990).

diSessa et al. describe a study in which eight Grade 6 learners "invented graphing as a means of representing motion" (1991, p. 117). The study, which took place in an extra-curricular mathematics and physics class, began with several weeks of "programming computer simulations of various 'real life' [scenarios of] motion" (diSessa et al., 1991, p. 121). Learners were then asked to create a static representation of a particular scenario on paper. They each invented a representation and through discussion with each other and their teacher refined these ideas, eventually arriving at a representation essentially identical to a Cartesian graph of time vs. speed. The authors admit that this actually amounts to a *re*invention of graphing (with which some of the learners already had basic experience, despite the fact that it was not a part of the school curriculum). However, they stress the "meta-representational expertise" that learners gained in creating and critiquing various "motion pictures" (diSessa et al., 1991). The learners did more than merely learn how to visualise motion: they also analysed the visualisation itself. In this context, says Arcavi (2003, p. 238), learners' use of visual tools "develop and stabilise" as they interact with each other – "ways of seeing emerge in a social practice as it evolves."

All of this research indicates that visualisation plays a very important role in mathematics and mathematics education. Indeed, as Hershkowitz et al. (2001, p. 255) claim,

> visualisation can be much more than the intuitive support of higher level reasoning: it also may constitute the essence of rigorous mathematics; [moreover] visualisation can be central not only in areas which are obviously associated with visual images (such as geometry), but also in formal symbolic arguments (such as high school algebra).

Despite academic interest, general disdain for visual reasoning in mathematical proofs and mathematics classrooms persisted for many years. Dreyfus (1991, p. 34) describes a common attitude that "visualisation may be a useful and efficient learning aid for many topics in high school and college mathematics, but nevertheless an aid, a crutch, a step, sometimes a necessary and important step, but only a step on the way to the real mathematics." However, Presmeg (2006a) claims that research implies there has been a shift towards acceptance and even encouragement of visual methods in both school and university settings. This has been helped by the advancement of computers, which now provide many opportunities for using dynamic visual tools in the teaching and learning of mathematics. The use of these tools in teaching algebra specifically will be considered in Section 2.2.

There are also well documented difficulties associated with visualisation. Presmeg (1986) lists several common problems that occurred during her research:

- visual methods are often time consuming;
- assumptions are often made based on the way things look;
- concrete, pictorial imagery can hinder flexible thinking; and
- diagrams which look different to the "standard" are not recognised.

However, Presmeg (2006a, p. 23) also stresses that "students can *learn* to use their mathematical imagery effectively" if their teachers emphasise the importance of linking images with "rigorous analytical thought" (Nardi, 2014, p. 213). Pattern imagery and dynamic imagery can be particularly powerful in this way.

2.2 Algebra, Sequences and Visualisation

2.2.1. Algebra

Much of the algebra taught in 21st century classrooms is thousands of years old, but Usiskin (2004, p. 147) notes that "the language we use in today's algebra is relatively new as mathematics goes, dating back only to the French mathematician Viète in 1591 and the systematisation of the content by Euler in 1770." Since the introduction of this "language" of letters and symbols, algebra has become "the unifying thread which interlaces almost all of mathematics" (Herstein, 1975, p. 1).

Throughout this thesis the term 'algebra' is used in the sense of *elementary* algebra, "a generalised arithmetic of numbers and quantities" (Carraher et al., 2006, p. 88) as taught at a school level, rather than the study of groups, rings and fields that make up *abstract* algebra. The South African Curriculum and Assessment Policy Statement (referred to hereafter as the CAPS document) defines algebra as "the language for investigating and communicating most of mathematics, [which] can be extended to the study of functions and other relationships between variables" (Department of Basic Education, 2011, p. 10).

One of the defining features of algebra is its ability to concisely capture general rules and relationships between quantities. It is this quality that makes the ability to use algebraic symbols a requirement for learning any branch of mathematics, along with most sciences and any study that requires the use of statistics. In fact, Lannin (2005, p. 233) claims that "statements of generality and discovering generality are at the very core of mathematical activity." However important, the ability to generalise does not come easily to learners. In reflecting on her 1986 study Presmeg says, "all of the difficulties experienced by the 54 high school students that I interviewed … related in one way or another to problems of generalisation" (Presmeg, 2006a, p. 23). Algebra is therefore difficult to master, requiring an ability to generalise as well as knowledge of all the conventions that make up the 'language' of algebraic notation. Even when a learner is able to "think algebraically … there is a gap between students' ability to express generality verbally and their ability to employ algebraic notation comfortably" (Zazkis & Liljedahl, 2002, p. 400). There are conventions in mathematical notation that are not at all intuitive, such as the convention that multiplication takes place before addition. For example, "subtract one and then multiply by two' must be given algebraically with a bracket, since $(x-1)\times 2 \neq x-1\times 2$. If learners are not already aware of and used to working with these conventions, they may struggle to 'translate' their thinking into algebraic notation or to accurately read and process algebraic expressions. Moreover, English and Warren (1998, p. 168) point out that "'two groups of three,' 'two multiplied by three,' 'add three to itself,' 'multiply three by two,' 'three plus three,' and 'three groups of two' can all be expressed symbolically as '3×2.'" Learners may struggle to recognise that all of these 'different' operations can be reduced to essentially the same thing.

Despite algebra's clear importance to the mathematical community, it is a stumbling block for many learners. Many never gain fluency in algebraic manipulation. Moreover, as Pegg and Redden (1990, p. 386) explain, even where "the majority of students appear to be able to carry out routine exercises, we are increasingly concerned that many students appear to perform rules in isolation – divorced from reality and without much understanding." This is particularly problematic in the 21st century, where computers can now easily perform complicated calculations. According to Friedlander and Arcavi (2012, p. 609),

> Although today's computerised environments may have decreased the need to master algebraic skills, procedural competence is still a central component in any mathematical activity. However, technological tools have shifted the emphasis from performing operations on complex algebraic expressions to understanding their role and meaning. Consequently, learning rules and procedures should be linked to a deeper understanding of their meaning and to a flexible choice of solution methods.

Beyond simply doing complicated calculations for us, computers have allowed for new, dynamic ways of seeing mathematical objects, including algebraic expressions. Yerushalmy (2000) describes a study in which learners were introduced to algebra with the aid of graphing software and a curriculum was designed around the concept of functions. She argues that this approach allowed learners to become problem solvers, "taking responsibility while dealing with new problems rather than rehearsing known procedures" (Yerushalmy, 2000, p. 144). Noss et al. (1997) carried out a study with learners who interacted with a software environment called Mathsticks, in which they could build patterns of matchsticks. The software allowed for interaction with the graphical representations of matchsticks (which could be dragged into place) or with the symbolic code that could be manipulated to build a pattern. Interaction with the graphics would cause code to self-generate and running code would cause graphics to be placed, which emphasised the direct link between the two. Learners were encouraged to write code that would produce a required pattern of any length, encouraging generalised 'algebraic' thinking.

In fact, diSessa (2018) argues that computational literacy should be developed in the same way that algebraic literacy has been, defining literacy as "a particular representational form for wide-spread learning, use, and subsequent value" (diSessa, 2018, p. 7) that stabilises after many years of social and intellectual interaction.

However, there are barriers to computer-based learning in many South African schools. Although most schools have at least one computer laboratory, this lab may not have access to the internet and teachers often experience scheduling difficulties when accessing the computer lab. For this reason, the current study has kept away from the many excellent applications that offer visual learning opportunities and opted for a more tactile approach.

2.2.2. Patterns and Sequences

Patterns are the foundation of mathematics (Liljedahl, 2004). The CAPS document describes mathematics as "a human activity that involves observing, representing and investigating patterns and quantitative relationships in physical and social phenomena and between mathematical objects themselves" (Department of Basic Education, 2011, p. 8).

The word 'pattern' as used in these contexts is necessarily broad, including everything from the organisation of subatomic particles to the distribution of galaxies and beyond. In the context of the South African curriculum, 'pattern' is usually used in the much narrower sense of numeric or pictorial sequences: lists of numbers or pictures which have some recurring relationship. This study considers only this narrower idea of a pattern, which, despite its limitations, is still a fundamental part of the learning of mathematics.

In the Senior (Grades 7 to 9) and Further Education and Training (FET, Grades 10 to 12) Phases, the South African curriculum covers linear (or arithmetic), quadratic and geometric sequences (introduced in Grades 7, 10 and 12 respectively), in addition to visual representations of many of these. As this study was designed to be carried out with Grade 9 learners, it focuses on linear sequences – those that have a constant difference between consecutive terms and can be represented algebraically by $T_x = dx + c$, where d and c are usually integers and x is a natural number. **Error! Reference source not found.** shows an example of this type of sequence.

$$3; \quad 7; \quad 11; \quad 15; \quad 19; \ \ldots$$

$$+4 \quad +4 \quad +4 \quad +4$$

(constant 1st difference of 4)

$$T_n = 4n - 1$$

Figure 2.1: A linear sequence

Sequences are often used in the introduction of algebra, and the South African curriculum makes this link explicit – the Senior Phase (Grade 7 to 9) CAPS document requires "investigation of numerical and geometric [i.e. visual] patterns *to establish the relationships between variables*" (Department of Basic Education, 2011, p. 21) (emphasis mine). Pegg and Redden (1990) describe the results of a study in which learners were required to build and expand concrete sequences before learning to describe and discuss rules for the sequences. Finally, the learners were introduced to variables and encouraged to use algebraic notation to represent these generalisations. They claim that learners introduced to algebra in this way "seem more positive in their approach to algebra [and] better able to develop algebraic generalisations from practical investigations" (Pegg & Redden, 1990, p. 391).

English and Warren (1998) similarly recommend this approach over the use of equations when introducing the concept of variables, as equations erroneously imply that the variable has a single value to be determined. However, they warn that the method requires "a facility with number … flexible, articulate thinking and an understanding of equivalence" (English & Warren, 1998, p. 168) that are often underdeveloped in learners being introduced to algebra. Learners in English & Warren's study often took an approach to solving patterning problems that did not lend itself to generalisation. Like Stacey (1989), they identified several strategies commonly used in generalising linear sequences. Stacey's (1989) methods are listed here, with English and Warren's (1998) equivalent categories given in brackets:

- *Counting* method, in which the constant difference (d) is noted and used to count up to the term total (T_n) [*additive* strategy];
- *Difference* method, in which the constant difference (d) is again noted and multiplied by the term's position (n) to give $T_n = dn$;

- *Whole-object* method, in which, for example, the total for the third term (T_3) is multiplied by ten to find the total for the thirtieth term (T_{30}) [*ratio* strategy]; and
- *Linear* method, in which the learner seeks to identify the link between the term's position (n) and its total (T_n), recognising that (usually) both multiplication and addition are necessary [*functional relationship* strategy].

Stacey (1989) and Zazkis and Liljedahl (2002) note that checking the differences between consecutive terms is a very common strategy. This leads almost directly into the counting method, which is useful for continuing a sequence or finding a term not too distant, such as T_{10}. However, it is difficult to "count" all the way up to T_{100} and Stacey (1989) found that learners were likely to change their strategy when dealing with larger terms. The additive strategy is also unhelpful in finding an algebraic formula for T_n. If a learner's perception of a sequence is dominated by the difference between terms, they might find it very difficult to "abandon" this first impression and move from recursive, term-to-term reasoning to the "algebraic thinking" of the linear method (Zazkis & Liljedahl, 2002). As discussed earlier, even when a learner is able to generalise a sequence, they may struggle to put this understanding into algebraic notation if they are not fluent in their use of algebraic conventions.

2.2.3. Visual Sequences

Visual sequences such as the one shown in **Error! Reference source not found.** (Siyavula Education, 2014, p. 87) are common in the South African curriculum and classroom. The CAPS document refers to these as "geometric patterns" (Department of Basic Education, 2011), although they should not be confused with the geometric (exponential) sequences introduced in Grade 12 (Department of Basic Education, 2011a). These types of problems usually require learners to find a general (usually algebraic) representation that represents the image, along with other tasks such as drawing the next term or predicting the number of items in a distant term. These questions are common around the world and there has been much research into these types of exercise. Pegg and Redden (1990), English and Warren (1998) and Stacey (1989) all focus on concrete and visual patterns in their studies of patterns and algebra.

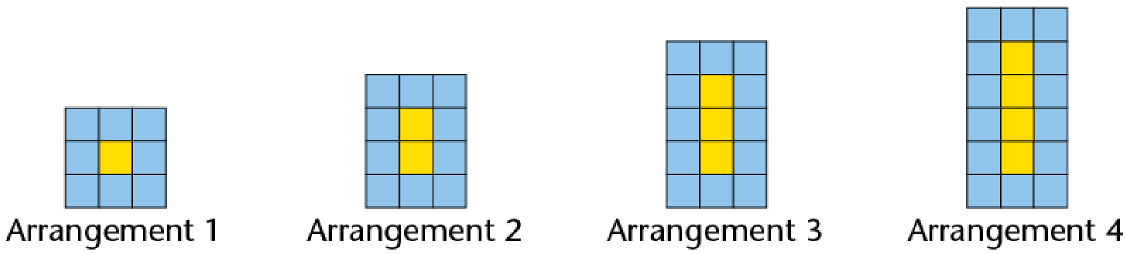


Figure 2.2: A visual sequence from a South African textbook

As with any sequence, there are a variety of approaches to generalising visual sequences. Samson (2011) built on the systems of Stacey (1989) and English and Warren (1998), among others, to develop a detailed classification of learners' choice of strategy when generalising visual sequences that distinguished between visual and non-visual methods. He found "a broad diversity of styles" – from learners who "extracted the numbers from the given diagrams" and thereafter ignored the pictures, to those "whose generalisation strategies all made explicit use of the given diagrams" (Samson, 2011, p. 96). Working further with those learners who showed a preference for visual methods of generalising, he found that each of these learners was able to decompose and generalise pictorial patterns in a great variety of ways.

Similarly, Hershkowitz et al. (2001) describe "the case of the matches" in which they presented teachers with an $n \times n$ square made of matches (**Error! Reference source not found.**) and asked them to calculate the number of matches needed to build it. The researchers analyse the great variety of approaches used in "counting" the matches and coming up with a general (algebraic) rule. Hershkowitz continued this research in a different study considering whether pre-service teachers could "reverse" the process and find a visual decomposition of the square for a given algebraic expression (Olsher & Hershkowitz, 2015). Participants were first asked, as in the first study, to find an algebraic expression to represent the matchstick square. After engaging with the image in this way they were provided with an unsimplified algebraic expression and asked to create a "broken down" version of the image that illustrated a strategy that could have been used to find the algebraic expression. For example, **Error! Reference source not found.** shows a possible visual representation of $4 + 3(n-1) \cdot 2 + 2(n-1)(n-1)$. The leading 4 gives the number of matches in the top left square. $3(n-1)$ gives the number of matches in the left column broken into "U" shapes of three matches each – and this is multiplied by two to account for the top row. $2(n-1)(n-1)$ shows the remaining array of matches broken into backwards "L" shapes of two matches each. The authors note that trying to reunite the structure of the algebra with the original image is far more difficult than the original task of breaking the image down and finding an algebraic expression.

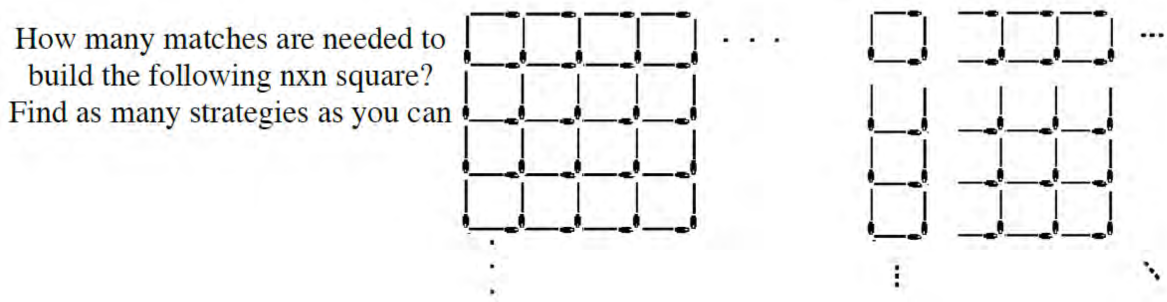


Figure 2.3: Hershkowitz's matches problem and one possible strategy for generalisation

2.2.4. Creating Visual Sequences

Despite the proliferation of research into the generalisation of sequences, including pictorial/visual sequences described above, I have found little research into the creation of visual sequences by learners. Pegg and Redden (1990, p. 387) describe an exercise in which learners were "asked to create a concrete [visual] representation" of patterns such as 4; 7; 10; ... and 1; 4; 9; .... This was given as an extension after several "standard" exercises in which the learners were asked to generalise a concrete representation. The researchers report that "this activity proved a challenging one that the more capable learners found interesting" (Pegg & Redden, 1990, p. 388).

Orton (2009, p. 19) directly addresses this question when discussing the creation of sequence tasks, which are sometimes given as a sequence of shapes:

> Such tasks hardly ever seemed to be set the other way round, namely, to present a given number pattern and ask for a drawing of an illustrative sequence of shapes. On reflection, we decided this was potentially an extremely difficult task, and that this was presumably why we couldn't find many references to it in the literature.

The rarity of asking learners to move from an abstract representation of a pattern (especially an algebraic one) to a visual representation may well stem from the potential difficulty of the task. However, Noss et al. (1997, p. 204) describe a commonly held belief "that learning mathematics begins with the concrete, and 'ascends' ... to the abstract." In other words, it is assumed that the abstract representations afforded by algebra are superior to the concrete representations of a list of numbers or a picture and once the abstract is mastered, the concrete is no longer necessary. The CAPS document seems to adhere to this assumption, stating that, "In Patterns, Functions and Algebra, learners' conceptual development progresses *from* [among other things] an understanding of number *to* an understanding of variables, where the variables are numbers of a given type ... in generalised form" (Department of Basic Education, 2011, p. 21) (emphasis mine).

As discussed earlier, the problem of generalisation is one of utmost importance in mathematics and the reason why algebra is so useful. However, although the CAPS document calls for "*investigation* of numerical and geometric patterns to establish the relationships between variables" (Department of Basic Education, 2011, p. 10) (emphasis mine), many learners are taught to find algebraic representations of patterns by applying well known algorithms and formulae rather than recognising the underlying relationship that the algebra encodes. This is cause for concern, considering Clements's claim that "students brought up on a heavy regime of algorithms that make little or no sense to them ... think in very wooden, mechanical ways about algebra" (2014, p. 188). Thus, I believe it is worth questioning the assumption that algebra is superior to and should follow from visual representations. It is entirely possible that the process of creating a (specific) visual representation for a sequence could help learners understand the (general) algebraic representation at a deeper level. Indeed, English and Warren (1998) recommend the relation of algebraic expressions to concrete contexts, since learners "frequently divorce their algebraic work from a problem's context, giving such answers as '$x = 123{,}56$ tickets'" (1998, p. 168). An early recognition that a pattern could not include 5,21 or -4 matches could help learners notice this kind of inconsistency between question context and answer.

2.3 Theoretical Framework

In *What Is Mathematics, Really?* Reuben Hersh argues for a socio-historical understanding of mathematics:

> If mathematical objects were an other-worldly, nonhuman reality (Platonism), or symbols and formulas whose meaning is irrelevant (formalism), it would be a mystery how we can teach it or learn it. Its teachability is the heart of the humanist conception of mathematics. (Hersh, 1997, p. 238)

This 'humanist conception' – the idea that mathematics is a system of ideas and processes that have been developed through human interaction with the environment and especially, with other humans – is the basis of Radford's "Knowledge Objectification" (2013), the theory by which this study is most informed.

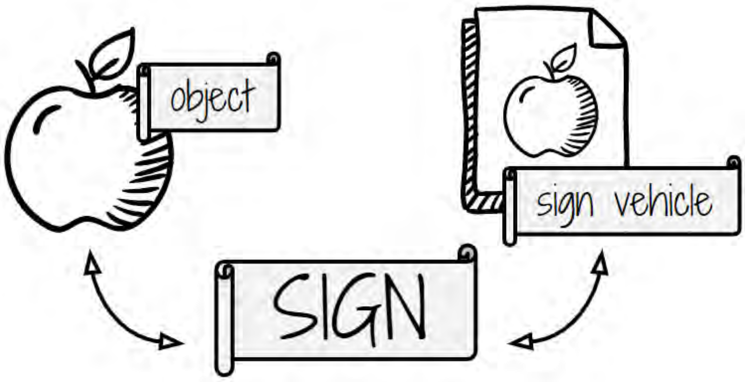


Figure 2.4: An illustration of Presmeg's semiotic vocabulary

2.3.1. Semiotics and Compartmentalisation

It is important to note that mathematical objects can only be accessed through the use of signs. A group of four blocks can be used to signify the quantity that can also be represented by the numeral 4, but neither sign is the number itself – numbers are strictly conceptual objects. Semiotics is thus an inescapable part of any attempt to study mathematics or mathematics education. Presmeg (2006a) describes a modified version of Peircean semiotics, the vocabulary of which I will use throughout this thesis. A *sign vehicle* is a word, drawing or other means by which we represent (or signify) an *object* (tangible, or, as in the case with mathematics, abstract). The interpreted relationship between the object and the sign vehicle is what Presmeg defines as the *sign*. **Error! Reference source not found.** considers the basic example of an apple (the object) and a drawing of an apple (the sign vehicle), with the sign itself being the link between the two. This link has been variously conceptualised as a *representational device* or *mediating tool* (Presmeg, 2016). However, according to Radford, a sign is better understood as a relationship that allows the individual to "organise and reorganise their interactions with other individuals and their deeds in the historical world" (Presmeg et al., 2018). This sociocultural understanding is based on the later work of Vygotsky and is an integral part of the theory of objectification discussed in Section 2.3.2.

Registers are different signs (or even different semiotic systems) used to represent (or relate) the same mathematical object. For example, Figure 2.5 on the next page shows five different registers used to represent a parabola, four of which a South African learner should be able to use fluently by the end of Grade 10 (Department of Basic Education, 2011a). Duval (1999) claims that the issue of converting from one register to another is a "crucial problem for the learning of mathematics”. Dreyfus (2002) suggests that learning a mathematical concept consists of four stages:

- using a single representation,
- using more than one representation in parallel,
- making links between parallel representations,
- integrating representations and flexible switching between them. (Dreyfus, 2002)

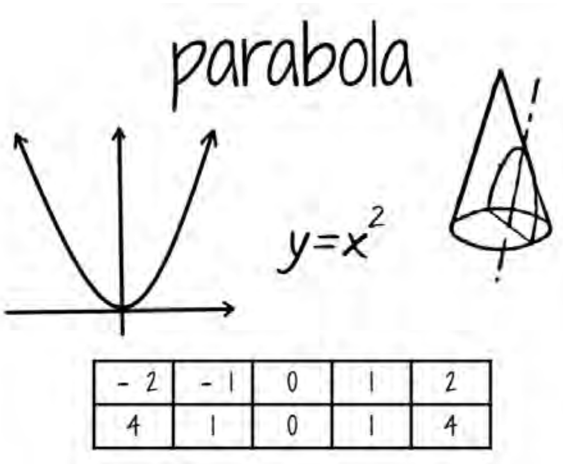


Figure 2.5: Several registers representing a parabola

Learners' common inability to recognise the relationship between different registers, which Presmeg (2006b) calls “compartmentalisation”, is a serious issue in mathematics education, addressed in three of Presmeg’s twelve “big research questions” (2006b). Duval writes that progress in mathematics is not possible without “the coordination between the registers of representation” (Duval, 1999, p. 11). However, it is possible to overstate the importance of register switching. Dreyfus (2002) and Duval (1999) imply that a person’s ability to move between representations is an indication of the degree of their understanding. This is challenged by Presmeg and Nenduradu (2005), who describe the case study of a pre-service teacher whose “facility in moving amongst representational registers was not matched by conceptual understanding of the underlying mathematical ideas” (2005, p. 105).

Santi (2011) notes similar cases in which “unexpected behaviours on the part of students ... defy Duval’s claim that *conversion* is the most difficult cognitive function that alone ensures a correct conceptualisation of mathematical objects.” (2011, p. 285). He argues that “a more comprehensive notion of mathematical knowledge and signs, that takes into account the role of mathematical activity, is necessary to tackle the issue of meaning in learning environments” (Santi, 2011, p. 285) and claims that objectification is just such a “notion”: “an effective lens when we analyse the nature of the mathematical activity mediated by semiotic tools. It allows us to interpret meaning as a relationship between a cultural dimension, in which the mathematical object lives, and the personal one" (Santi, 2011, p. 307).

2.3.2. Objectification

This study is deeply informed by Radford's theory of knowledge objectification (Radford et al., 2018) (Radford, 2008) (Radford, 2013). This is a sociocultural theory, descended from the cultural-historical and activity theories of Vygotsky and Leont'ev (Radford, 2013) (Radford & Roth, 2011). Knowledge is defined as consisting of culturally and historically generated "processes of reflection and action" (Radford, 2013, p. 10). According to this definition, knowledge exists neither outside of the human whose task is merely to discover it (as proposed by realism), nor uniquely in the mind of each individual person (as proposed by constructivism) but is instead shared between people. Consequently, mathematics is seen as a human construct, and mathematical objects are understood as "fixed patterns of reflexive human activity incrusted in ... social practice [and] mediated by artefacts" (Radford, 2008, p. 222). As described above and illustrated in Figure 2.5, a parabola is a mathematical relationship that can be accessed via its multiple registers. This type of relationship and its various registers were created and refined and are now recognised and used by the mathematical community.

The primary difficulty of learning mathematics, then, is the difficulty of developing a relationship with and an ability to use both mathematical objects and their various signs, in order to participate in society. Radford defines this process as objectification, a term that has several layers of meaning. In one sense, objectification is the process of an individual solidifying a historical-cultural object in their own mind, turning it from something vague into an almost-tangible concept that can be "examined" from different angles and used in different contexts – what Radford describes as "actualisation" (Radford, 2013). Presmeg summarises this as "the ability to treat a general structural relationship as an object in its own right, represent it with a sign vehicle, and interpret and work further with this sign" (Presmeg, 2006a, p. 22). In another sense, Radford (2013) describes knowledge as something "other" to the individual – something which opposes or *objects* them. Here, "objectification is precisely the process of recognition of that which objects us – systems of ideas, cultural meanings, forms of thinking, etc." (Radford, 2013, p. 23). In this sense, objectification (the process of knowing) produces subjectification (the process of becoming). As Radford rather poetically states, "Learning ... is not just about knowing something but also about becoming someone" (Radford, 2008, p. 215).

Radford describes a painting lesson on colour and tone given to Vincent van Gogh, in which he imagines that "pointing and words" were the signs (or more specifically, the "*semiotic means of objectification*") that "may have made visible, for the first time to ... van Gogh, something new – something that had escaped him until then" (Radford, 2002, p. 14). In mathematics education the semiotic means of objectification are myriad – not just the symbols required to even begin accessing mathematical objects, but all of the words, gestures, graphs, artefacts (blocks, calculators, number lines, rulers) and linguistic devices (metaphors, metonymies) used in our classrooms and in society more broadly (Radford, 2002). The lesson itself, in which van Gogh "learned to see", was what the theory of objectification calls "joint labour" (Presmeg, 2016, p. 16), a space in which two or more people are performing an *activity* together through which they can learn new ways to "see" the world. Learning therefore consists of *processes of objectification* – "those social processes of progressively becoming critically aware of an encoded form of thinking and doing—something we gradually take note of and at the same time endow with meaning" (Radford, 2013, p. 26). The teacher's role is to bring particular pieces of "knowledge" to the learners' *attention* (Samson, 2011) so that they can eventually discern the general structure behind a specific activity.

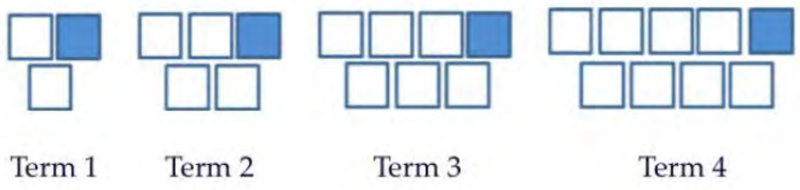


Figure 2.6: A visual sequence presented to Grade 2 learners

Radford (2013) describes a study with Grade 2 learners who were asked to generalise a visual pattern, shown above in Figure 2.6. The theory of objectification identifies many layers and complexities in this common, 'simple' activity. First, when we ask learners to engage with this kind of activity,

> we expect the students to enter into a relationship with a historically constituted form of knowledge about arithmetic sequences. More specifically, we expect the students to become aware of an algebraic form of perceiving, reflecting and investigating sequences that goes back to ancient times. (Radford, 2013, p. 13)

Even if we do not communicate the details of this history to learners it is important to bear in mind the fact that these ideas and methods were developed slowly, over the course of thousands of years, and we should not assume that they are simple or easy to learn. When generalising the given pattern, a mathematician would immediately attend to the spatial features of the terms, *noticing* that the number of white squares in each row matches the term number. This perception “usually moves so fast that [they] virtually do not even notice the complex work behind it” (Radford, 2013, p. 24). These salient aspects are not always “seen” by learners who lack experience of this kind of problem. The Grade 2 learners in the study focused on the number of squares without “realising yet that the spatiality of the terms provides … clues that are interesting from an algebraic viewpoint” (Radford, 2013, p. 25).

The process of moving from *not noticing* particular aspects of a situation to *recognising* both their existence and their significance is the process of objectification – and discussing which aspects of linear expressions and visualisation were and which were not recognised by the participants will form an integral part of the analysis of this study.

2.4 After-school Mathematics Clubs

Many South African learners have negative relationships with mathematics. In fact, Graven (2011) argues that the intense feelings of alienation and hopelessness that learners often experience in mathematics classes are comparable to those produced by emotional abuse. In a typical South African classroom there is pressure on both the teacher and learners to complete the curriculum and perform well in tests, which often leads to anxiety, disengagement and a culture of “teaching for/to assessments” (Stott & Graven, 2013, p. 2). In this classroom culture, “learners tend to equate mathematical success with teacher dependence, compliance, and careful listening rather than relating it to independent thinking, problem solving, or making sense of mathematics” (Heyd-Metzuyanim & Graven, 2016). Graven therefore began to explore “the potential of extracurricular maths clubs in providing supportive communities ... disrupting passive learning culture and deliberately working with learners to become confident mathematical participators” (Graven, 2011, p. 161).

The work done in this study (Graven, 2011) led to further research by Stott and Graven (2013) into similar clubs with Grade 3 learners. These clubs are defined as follows:

> Informal, after-school clubs focused on developing a supportive learning community where learners can develop their mathematical proficiency, make sense of their mathematics and where they can engage and participate actively in mathematical activities (Stott & Graven, 2013, p. 2).

Graven and Stott's club environments are spaces that embody the spirit of objectification, where the pressure of the curriculum abates and teachers and learners can attend to activities with the sole intention of exploring and engaging with the human activity that forms the landscape of mathematics.

2.5 Conclusion

Despite their differences, this chapter has illustrated that research shows many parallels and intersections between algebra and visualisation in mathematics education. The strength of algebra's capacity for rigorous generalisation helps to prevent visualisation's propensity for allowing quick but potentially erroneous assumptions. Conversely, the dense barrier to entry created by algebra's many conventions can be mitigated with visualisation strategies.

Definitions of visualisation vary based on the background of those defining it, but this study relies on Hershkowitz's definition of visualisation as both "a mode of mathematical thinking" and a language for communicating and developing this thinking (Nardi, 2014, p. 198). The importance of the types of thinking and communicating allowed by visual methods has been illustrated by multiple studies into visualisation in mathematics education, from Presmeg's ongoing research into modes of visual and mathematical thinking (Presmeg, 2006b) to diSessa et al. (1991), Noss et al. (1997) and Yerushalmy's (2000) computer-based interventions. More specifically, the usefulness of visual methods in learning algebra and working with sequences has been well documented. Because patterns are a commonly recommended means of introducing algebraic thinking (Pegg & Redden, 1990) (English & Warren, 1998), the three topics combined well into the structure of this study. Despite the prevalence of generalising pictorial patterns in the introduction of algebra – both in studies and in classrooms – very few researchers have explored the potential benefits of creating a visual pattern based on an algebraic expression. Consequently, this was the area chosen for the research undertaken here, which is centred on the creation of visual and kinaesthetic representations of algebraic expressions created by Grade 9 learners.

The study was deeply informed by Radford's theory of knowledge objectification, a sociocultural theory that defines knowledge as "processes of reflection and action" (Radford, 2013, p. 10) created by an individual's historical and cultural grounding. These processes are communicated via and partially embodied in signs such as the numeral '4' and the word 'four', which represent but are not equivalent to the intangible mathematical object that is the number itself. Semiotics is therefore essential to this study, especially in that the study is concerned with at least three different registers (semiotic systems) representing the same mathematical relationship (Duval, 1999) : a strictly conventional algebraic register, a fairly standard register of representing sequences with lists of numbers and the unconventional, evolving register of the visual expressions created by the learners. The study had several aims, among which was expanding the participants' ability to switch between different registers that represent the same mathematical object, which (though not necessarily an indicator that it has been achieved) is at least one part of conceptual understanding (Presmeg & Nenduradu, 2005).

Another aim of the study was to foster a more positive relationship between participants and mathematics, as Graven (2011) has found that many South African learners' experiences with the subject border on the abusive. To this end, and also with the intention of creating a social environment in line with the tenets of Radford's objectification, the research was carried out in the context of an extramural club, inspired by Stott and Graven (2013).

The aims of the study are summarised by the research questions:

- What aspects of linear algebraic expressions did participants represent visually?
- In what ways did the visualisations develop over the course of the club, both within and between groups?

The first question deals with participants' ability to represent algebra and numerical sequences in a new, visual format. Answering this question would provide a surface-level view of learners' developing understanding of the mathematical relationship that connects the algebra to the sequence and their new, visual creations. The second question deals with the mathematical community that was developed during the course of the study. Answering it would provide a deeper perspective on learners' individual knowledge and how this knowledge was disseminated among the pairs of participants. The next chapter will go into the details of the design and implementation of the study that aimed to answer these questions.

Chapter 3

Research Methodology

Following the reading done in preparation for this study, it was clear that this study should take place in an environment where learners are encouraged to *become* mathematicians – people who actively engage with the local, global and historical mathematical community. For this reason, a Grade 9 Maths Club was initiated at a local school, with two intentions: providing a space for research and creating a community that would continue after the research was done in an effort to enrich the learners' relationships with mathematics. As in Stott and Graven's initial club (2013), there were around twelve participants (although this changed from week to week, depending on individual learners' other engagements and level of commitment to the club) and two mentors (myself and a Grade 9 teacher from the school). Additionally, we followed Stott and Graven's lead in making sessions approximately an hour long and including activities such as "playing mathematical games, using manipulatives ... and problem solving" (Stott & Graven, 2013).

In the context of this club, participants were asked to create visual representations of a sequence, presented to them as an algebraic expression of the form $dx \pm c$ (where d and c are natural numbers) using colourful building cubes. This activity deals with the issue of changing between mathematical registers, but differs from typical classroom approaches in asking participants to create their own semiotic system for the representation of a particular mathematical object. Participants worked in pairs and then presented their ideas to the group, in order to develop a representation that was both personal (representing their own understanding of the algebra) and meaningful to the club and the wider mathematical community. Through this activity, it was hoped that participants would deepen their understanding of algebra and sequences and their appreciation of mathematics as a collaborative human endeavour.

3.1 Orientation

This study is oriented in the interpretive paradigm. According to Bassey (1999), the purpose of interpretive research is "to advance knowledge by describing and interpreting the phenomena of the world in attempts to get shared meanings with others." Moreover, Cohen et al. (2007, p. 21) describe the interpretive paradigm as "characterised by a concern for the individual" This orientation lines up theoretically with Radford's knowledge objectification, which is focused on how individuals in a community come to understand the "shared meanings" that constitute mathematics.

The study was inspired to a great degree by diSessa et al. (1991) whose orientation is "deeply constructivist" (1991, p. 118) and dependent on the idea that "knowledge flows from and emerges in activity, so the design of activity should be the first priority of those who would aid learners" (1991, p. 119). In this paper the researchers outline the process and results of a series of elective sessions in which eight learners, through presentation, discussion and critique, collectively created a visual representation of motion that was essentially equivalent to graphing on a Cartesian plane. The influence of the diSessa et al. (1991) study on my own research means that it draws on those same, deeply constructivist roots. However, constructivism is a highly "individualistic stance" (Radford, 2019, p. 3064) that fails to acknowledge the participation of a community of fellow learners, teachers, parents and the wider world in an individual's pursuit of knowledge. Therefore, Radford's Theory of Objectification, similarly grounded in "actions … carried out through the body, the human senses, and through the use of physical objects and cultural artefacts" (Radford, 2019, p. 3063) was chosen as the theoretical grounding for the design of the activity that formed the heart of the study.

3.2 Method

The research project took the form of a qualitative case study. Cohen et al. (2007, p. 253) define a case study as an examination of "a specific instance" that "provides a unique example of real people in real situations, enabling readers to understand ideas more clearly than simply by presenting them with abstract theories or principles."

The case was the after-school club described above, located at a public boys' school in South Africa. The unit of analysis was the visual representations created by the club members. How the participants were chosen and the manner in which the representations were created and analysed are described further on.

3.3 Participants

Participants were selected through non-probability convenience sampling (Cohen et al., 2007, p. 113), a method that prioritises practical considerations over ensuring the equal probability of selecting any given participant. This allowed for ease of planning and carrying out the study, since previous participation in the club was considered to be a good indicator of learners' commitment, ability to attend at the necessary time and interest in mathematics outside of the classroom.

As described above, the twelve study participants were members of a mathematics club at a public boys' school in South Africa. The school was selected because, having previously taught there, I had good relationships with the management, teachers and learners. Moreover, I was still involved in teaching and facilitating other extra-mural maths activities at the school. Club participation was voluntary and open to all Grade 9 learners, since this is a critical grade in the South African schooling system, when learners decide whether they should continue to study pure mathematics or choose the less abstract, more practical subject mathematical literacy. Working with Grade 9 learners was useful for the study, as the South African curriculum dictates that by the end of Grade 8 learners should already have worked with linear sequences and finding algebraic representations. The learners thus had some basic knowledge that would be useful to them during the research activities.

3.4 Research Design

The research design was divided into two phases:

3.4.1. Phase One - Club Sessions

The first phase of this study took place during club sessions, the structure of which were based on Radford's (2013) description of four stages in classroom activities:

1. a mentor introduces the activity for the session;
2. participants work on the activity in pairs;
3. the mentors visit groups to ask questions, give feedback, etc.; and
4. pairs present their ideas in a general discussion.

The study took place over four of these sessions, during which participants were asked to produce visual representations of a given linear algebraic expression using different coloured cubes. The given expression was different every week, but the process followed was the same. During the introduction of the activity we discussed how the expression could be viewed as a sequence by substituting natural numbers. For example, $2x + 1$ has an output of 3 when $x = 1$; 5 when $x = 2$ and 7 when $x = 3$. So the sequence 'representation' of $2x + 1$ is $3; 5; 7; \ldots$ Participants were asked to illustrate the terms of this sequence using coloured building blocks in order to represent the algebraic expression visually. The mentors visited pairs to observe and ask questions, but were careful not to give advice or lead participants to a particular representation. Pairs then took turns presenting their representations to the group for critique and discussion. As an addition to the usual club format, at the end of each session, participants completed a guided journal entry, as shown in Figure 3.1.

Date: ____________________

Expression: ____________________

Draw your visual representation (include at least Terms 1 to 4):

Explain your representation:

____

____

How could you improve your representation?

____

____

Figure 3.1: The journal entry that participants were asked to complete at the end of each session.

Participants were asked to work in pairs for two reasons. First, since the inception of the club the learners had worked in small groups (pairs or trios, depending on the task) in order to encourage the development of their ability to communicate mathematical ideas and to build a community within the club. Second, having learners work in groups of three would have resulted in fewer representations for analysis and would have made interviews more difficult. The intention of the interviews was to analyse the way in which they worked together to create this representation – this form of encoded knowledge. During the club sessions that did not form part of the research, these small groups were usually formed using the visibly random strategy advised by Liljedahl (2016). Forming groups randomly has several benefits that aligned with the goals of the club, such as breaking down social barriers and increasing student engagement with problem-solving tasks (Liljedahl, 2016). However, in the sessions that informed this research the learners were allowed to choose their own partners, since they would be working together for four weeks. Learners were kept in the same pairs for the duration of the research so that the development of their representations could easily be traced in order to answer the first research question.

Like the groupwork, the end-of-session discussion was intended to build community but it became apparent in the first session that it was difficult for learners to describe what they had created. Although this would have been a valuable skill to develop, we did not have the time and preferred to have them spend more time working on their representations.

3.4.2. Phase Two - Interviews

The second phase of this study was comprised of informal, task-based interviews (Koichu & Harel, 2007). One interview was conducted with each pair of participants who worked together during the club sessions. Interviews were scheduled at the convenience of the participants.

The interview schedule is given below. Actions of and notes to the interviewer are given in italics.

Interview Schedule

Task

Please build $4x - 3$ to term 3 or 4. "Think aloud" (Clement, 2000, p. 565), discuss and explain while you build!

Questions about interview representation:

a) Explain your representation.
b) What inspired you to build it this way?
c) What would term 10 look like? *After their answer:* How do you know… ?
d) Why did you choose … ?
e) What are the strengths of your representation?
   If they struggle with ideas:
   Is it easy to see how x is growing? Is it easy to see how the answer is growing?
f) Does your representation have any weaknesses?
g) *After I build a term from a different expression with the same representation.*
   Can you find the algebra that matches this?

h) Could you use this representation to show ANY expression?

After (and depending on) their answer: What about $2x + 5$? $-2x + 5$? x^2?

Questions about each earlier representation (looking at photos):

Remember that Week 1 $= 2x$, *Week 2* $= 3x + 1$, *Week 3* $= 2x + 3$, *Week 4* $= 3x - 2$.

- See questions (a) through (g) above.

Comparing representations:

- Which representation was your best? Why?
- Did your approach or your thinking change from Week 1 to today?

General questions:

- Did you enjoy this activity? Did it get boring doing it for 4 weeks?
- Was it difficult trying to record what you did in your journal?
- Do you think you learned anything from this activity?

3.5 Research Instruments

Two research instruments were used in this study: the collection of photographs taken of each representation at the end of club sessions and the interview recordings and transcripts. Each photograph was accompanied by a journal entry (except for the final photos of the interview representations). Further insight was gained from informal observations noted at the end of each club session and video recordings of each interview.

The photographs were used as the primary research instrument because they were clear, unbiased records of the representations produced by each pair. Moreover, they served the research questions, which sought to identify features of the representations and how these features developed over the course of the club sessions. However, given that the research questions required an analysis of the intent behind each representation, the photographs needed to be supplemented by instruments that would allow insight into the participants' processes of thought and collaboration.

The journals were intended to be records, along with photographs, of the learners' work, in addition to spaces for reflection that would encourage them to think through their process of creation, compare their representations to others' and consider how they could improve. As was the case with the end-of-session discussions, it became clear that the participants found it difficult to articulate what they had created and how the representations worked. Unlike the discussions, however, the journal entries were continued with the intention of helping the participants to learn some metacognitive skills.

The journals were not used extensively during analysis. Instead, insight into participants' intentions for their representations was gained from discussion – both discussions that occurred during club sessions, recorded in observations written up after each session and the discussions about previous representations that were conducted during the interviews.

Interviews were conducted with the intention of gaining insight into learners' processes of thought and co-operation. Recording each club session would have been logistically difficult (considering that there were six pairs of learners working simultaneously) and would have produced far more data than what was necessary for the purposes of the study. Hence, any important points that were discussed during club sessions were noted and the majority of the necessary discussion happened during the interviews. Excerpts from the transcripts of the interviews are woven into the analysis to enrich and authenticate the data.

3.6 Analysis

The primary purpose of the analysis was to describe the representations along with the environment and interactions that gave rise to them. The analysis then sought to discover "commonalities, differences and similarities" (Cohen et al., 2007, p. 461) between the representations and thus, in a small way, to move towards conclusions on the depth of participants' understanding of linear algebraic expressions. Initially, influenced by Samson (2007), I intended to classify the representations according to the participants' intentions when building, much as Stacey (1989) classified students' strategies in generalising patterns. This, however, proved impossible to do accurately, as the participants found it very difficult to articulate why they had approached the task in any particular way.

In light of these intentions and limitations, the research sought to answer two emergent questions:

- What aspects of linear algebraic expressions did participants represent visually? (description)

- In what ways did the visualisations develop over the course of the club, both within and between groups? (comparison and interpretation)

Chenail (2012) discusses the importance of creating a coherent structure for the themes, categories and codes within which qualitative researchers present their data. The most straightforward way to initially categorise the data in this study was either by date of creation (which would allow for a ‘horizontal’ examination that compared what all pairs were doing with the same prompt on the same day), or by creator, which would allow for a ‘vertical’ examination (that compared the outputs of each pair over time). (See Figure 3.2.) Because both forms of categorisation would provide useful context for answering the research questions, the analysis began with an examination of each club session and then moved on to a description and discussion of each pair’s interview. Because the interviews included a set of retrospective questions, this part of the analysis allowed for a comparison of all the representations produced by a particular pair and how these developed.

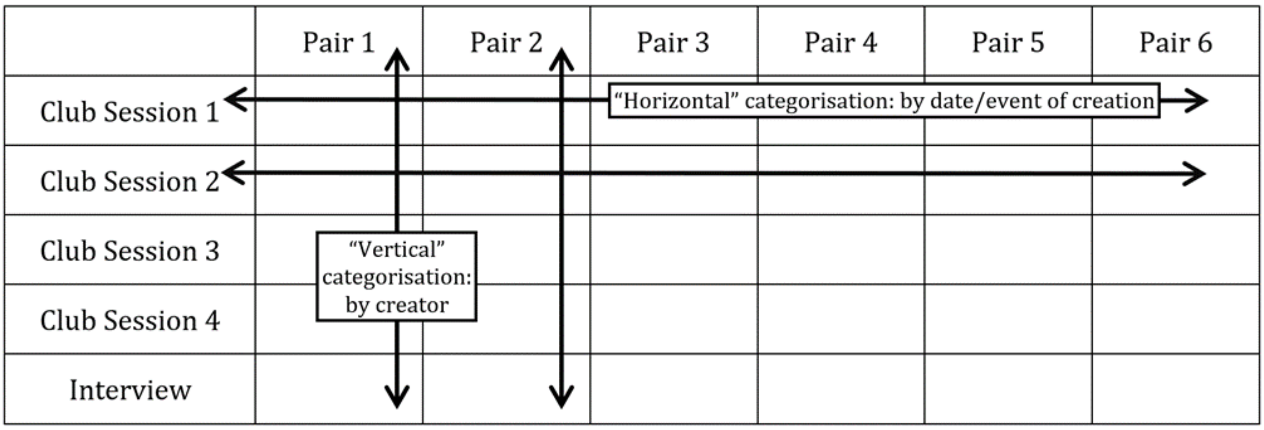


Figure 3.2: Initial categories for presentation of the data

After this introduction to the data and in order to answer the first question, a list was made of the features of linear algebraic expressions that were found to be represented in the participants’ creations:

- constant difference between consecutive terms,
- increasing nature of expression,
- total value of each term,
- value of x in each term,
- value of coefficient (multiplier),
- value of constant term (additive), and
- concept of multiplication.

The photographs of each representation were examined and the representations were classified according to which features they displayed. The journals, interviews and observations were used to inform whether the features were judged to have been included purposefully or not.

The second question was answered by comparing all of the participants' creations and determining general trends in the creative processes of the group as a whole within three categories:

- use of colour,
- representation of large terms / movement from T_x to $dx + c$, and
- use of area and grouping.

Again, the journals, interviews and observations were used to give deeper insight into these processes. As with the research questions, the categories used for comparing and classifying the various representations emerged from the data. Overall, along with a classification of the representations created during the research, this analysis provided some insight into the learners' understanding of linear expressions and their perception of the intention behind the task of creating a visualisation for algebraic expressions.

Much of the description of the study's results is written in the first person. Although this is unusual and often frowned upon in academic writing, the nature of the study meant that I was directly involved in all parts of the data gathering and it would have been both artificial and awkward to write this section entirely in the third person and/or the passive voice.

3.7 Ethics

Formal permission was obtained from the principal of the school (Appendix B: Permission Letter & Form to School Principal) as well as the Eastern Cape Department of Education (Appendix C: Permission Letter to Eastern Cape Department of Education) before any research was conducted. As described above, assent and consent were then sought from the learners and their guardians respectively.

The mathematics club in which the research took place was open to all Grade 9 learners in the school and functioned as a formal extra-curricular activity that was advertised along with other club-like activities (such as debating) and was listed on the school calendar. Along with this, the Grade 9 mathematics teacher informed all of her students about the club at the beginning of the school year and invited them to attend. It was made clear that the club would not function as a support for curricular mathematics but would cover topics outside of the curriculum in order to explore maths more fully than is possible in the classroom. Invitations to participate in the study were given to all learners who had attended club sessions in the month before the research sessions began. (See Appendix A: Invitation & Consent Form to Guardians and Participants.) The purpose and process of the research was explained in person and letters addressed to learners and their parents/guardians were then handed out. These letters included a form that learners and their guardians were asked to sign and return, indicating the learners' assent and their guardians' consent to participation in the study. Those who preferred not to be a part of the research or who did not complete the reply forms continued to take part in the club activities that constituted the study, but their representations were not included in the study data and they were not interviewed. Participants were allowed to withdraw from the study at any stage, either by withdrawing from the club or by indicating that they no longer wished to have their representations considered as part of the data.

Club sessions were carefully designed to function differently from most in-school lessons; however, they were conducted after school, in a classroom, and regarded as an official extra-mural activity. The intention of these choices was to create a space that was familiar to learners but provided a distance from the maths lessons they were familiar with in order to emphasise the fact that the club was a creative space that had no bearing on marks and only a tangential relationship to the national curriculum. Conducting the research during an extra-mural activity rather than regular lessons was also a good way to ensure that learners' regular classroom activities were not disrupted.

The club facilitators, as adults and teachers, were in a position of power over the participants. Many of the learners who took part in the club were taught by the teacher-facilitator at the time of the study, which meant that she had access to and control over their marks. I had been a teacher at the school and was therefore known to the participants, although I had never taught these particular learners in a formal classroom setting. A serious concern was that learners would consider the club activities as having a direct bearing on their academic results via the influence of the facilitators. It was therefore emphasised that this was not the case, and the club was presented as a space where creativity was valued over accuracy and 'correctness'. Care was taken to encourage and praise thinking that differed from convention, in order to decentralise the facilitators and break down the notion that they were the locus of knowledge in the room. Moreover, the teacher-facilitator was not present at the interviews, in order to further distance the research from any anxiety associated with regular school activities.

As in any social setting, participation in the club carried the risk of arguments, embarrassment and offence. With experience in classroom management, the club facilitators were aware of these risks and ready to mediate any social difficulties that arose during club sessions. There were very few issues of this kind. During the research sessions, there was one pair of learners who disagreed on their approach to building a representation. They were not forced to work together, but by the time the interview took place they had agreed to try working together again and had found an arrangement that seemed to satisfy both of them. Their interactions are described in the next chapter. All group presentations were given in pairs in order to relieve some of the pressure of public speaking and critique. These presentations were carefully monitored, but no participants made comments that were unkind or unhelpful. Rather, many participants neglected to pay attention to other pairs' presentations, and this, together with the time-consuming nature of the presentations, led to a decision to stop holding full group discussions after the first session.

Participants were given the option to choose their own pseudonyms, which many did. They were asked to choose real first names, such as Clark rather than Superman, but were otherwise allowed freedom of choice. Only one chosen pseudonym was changed – from a surname to a related first name. Some participants elected not to choose their own pseudonyms and these were therefore selected for them. Pseudonyms can be considered markers of gender, as all participants were male and chose or were given typically male pseudonyms. However, pseudonyms are not markers of race or culture and were chosen or assigned without reference to these attributes.

3.8 Validity & Reliability

The study was carefully designed according to the literature, with the pre-existence of the club and a small pilot study providing useful indicators as to what could be expected to work with respect to both the club structure and the patterning activity.

Although it is impossible to claim that a study based on qualitative methods is *reliable* in the sense that it is fully replicable, it would certainly be possible to carry out a study similar to this one and to analyse the resulting data in the same way, answering 'What aspects of linear algebraic expressions did participants represent visually?' This level of reliability is provided by the fact that the primary source of data was the photographs of the representations, supported by both journal entries and member-checking discussions (based on the photos) held during the interviews. In this sense, the study satisfies Cohen et al.'s requirement that "in qualitative research reliability can be regarded as a fit between what researchers record as data and what actually occurs in the natural setting that is being researched" (2007, p. 149).

The fact that all participants' representations were accurately recorded in photographs and that any elements of the representation that were unclear were clarified in the interviews meant that consistency could be maintained when analysing the representations of the different groups. Such an analysis performed on a subsequent study would lend itself to comparing the variables in this study, such as facilitator involvement or the age and prior knowledge of the participants to those in the follow-up.

In answering the second question, 'In what ways did the visualisations develop over the course of the club, both within and between groups?', it was possible to use the analysis from the first question to show that the visualisations had unquestionably developed at least in the number of different aspects of linear expressions that they were able to depict. However, it was necessary to go beyond merely showing that a vague change had taken place and consequently three particular aspects were highlighted: use of colour, representation of the individual values within the algebraic expression and use of area or grouping to express the concept of multiplication. In each of these areas general trends were established among the participants. This was the most subjective area of analysis, as it sought not just to enumerate the changes but to attempt an explanation of what these changes showed about the participants' understanding of the task's demands and the concepts behind it. Care was taken to reduce subjectivity through references to the changes described in answer to the first question and, where necessary, references to or quotes from the interviews that explained a change in the process or thinking of a particular pair of learners.

3.9 Conclusion

This study, oriented in the interpretive paradigm (Bassey, 1999) and inspired by the theories of constructivism (diSessa et al., 1991) and knowledge objectification (Radford, 2019), was designed as a qualitative case study (Cohen et al., 2007). Located in an after-school mathematics club at a public boys' school in South Africa, it sought to answer two questions:

- What aspects of linear algebraic expressions did participants represent visually?
- In what ways did the visualisations develop over the course of the club, both within and between groups?

In order to answer these questions, data were collected from participants in a maths club over the course of four club sessions (Stott & Graven, 2013) and six informal, task-based interviews (Koichu & Harel, 2007). Learners were invited to participate in these activities because of their attendance at the maths club, which was open to all Grade 9 learners at the school, and permissions or consent were obtained from all relevant parties. There were club attendees who chose not to participate in the research but continued to take part in the club activities and participants were allowed to withdraw from the study and/or the club if they wished to. Holding the research during an extra-curricular activity ensured that learners' regular mathematics classes were not disrupted and allowed the creation of an environment in which the participants held more power than in a traditional classroom. However, the facilitators still held a position of power and steps were taken to mitigate the risks that this posed to participants. There was also a risk of social distress, which was carefully managed by the facilitators when it arose during the club sessions. Participants' pseudonyms were chosen with reference to participants' gender but not their race or culture.

The study and its analysis were designed according to the literature, with the added benefit of the club's pre-existence and a small pilot study to guide the structure of club sessions and the patterning task. With the creative nature of the activity and the particular input of the individuals involved, it would not be possible to recreate the study exactly. However, a study that followed the same progression of tasks and used the same tools for analysis would be possible and would provide an interesting comparison. The data consisted of researcher observations, guided participant journal entries, recordings and transcripts of the interviews and, most importantly, a set of photographs that served as an accurate record of each visualisation that was created. These were very reliable and the interview functioned as a form of member-checking that allowed for consistency of analysis between pairs of participants.

The twelve participants, selected via non-probability convenience sampling (Cohen et al., 2007), chose partners and worked through most of the study within the same pairs. During each club session they were asked to represent a particular algebraic expression and its associated sequence (for example, $2x$ and “2; 4; 6; 8;…”) using coloured building blocks. After the fourth club session, individual interviews were conducted with each of the six pairs of participants. The interview schedule began with the task of building a visual representation of an algebraic expression, just as they had done during the club sessions, with the request that the participants discuss and “think aloud” (Clement, 2000) as they worked. The remainder of the interview consisted of questions, first about the representation just built and then (with photographs as a reference) about each representation created during the club sessions. The questions were designed to elicit participants’ engagement with their visualisations, with the underlying mathematical relationship and with the algebraic expression that also represented this relationship.

Analysis of the data began with a thorough review of each club session and interview. The examination of the club sessions served as a kind of ‘horizontal’ analysis, describing and discussing each pair’s work from Session One before moving on to Sessions Two, Three and Four. This allowed for an examination of the similarities and differences between the visualisations produced by each pair during a single session. Conversely, the description of individual interviews functioned as a ‘vertical’ analysis, since the interviews contained retrospective discussions about each visualisation made by the pair in question. This made it possible to track the changes that each pair made to their creations over the course of four club sessions and one interview. Inspired by Samson (2007) and Stacey (1989) and in order to answer the first research question, the visual representations were then classified according to the aspects of linear algebraic expressions that they displayed. Finally, general trends in the development of participants’ visualisations were discussed, answering the second question and providing a window into the learners’ perception of the activity’s objective and their understanding of the mathematics underlying it. All of this will be covered in Chapter 4.

Chapter 4

Results, Analysis & Discussion

The results of this study are presented as a series of descriptions, first of each club session and then of each interview. Each overview of the four club sessions covers the facilitators' instructions and then the activities and output of the six participating pairs of learners. The interview descriptions serve as both a vignette of each interview and summary of each pair's collection of visual representations. The chapter then moves on to an analysis of the data focussing on answering the two research questions, namely:

- What aspects of linear algebraic expressions did participants represent visually?
- In what ways did the visualisations develop over the course of the club, both within and between groups?

To begin answering the first question, the chapter describes the seven identified aspects of expressions that were represented in at least some of the visualisations and then presents a table summarising which aspects were present in each of the visualisations. The next section provides an overview of the general developments over the course of the study and then highlights three particularly notable aspects of change that were present in the work of every group. Finally, I discuss the results and how they can be interpreted in light of the preceding research highlighted in Chapter 2.

This chapter contains many references to the various parts of a linear expressions. The following definitions are used with the notation $T_x = dx + c$, where:

- T_x is the *output* value of the term (which learners often called "the answer"),
- x is the *variable*, showing the input value of the term (that is, the position of the term within the sequence),
- d is the coefficient or multiplier (equal to the *difference* between terms) and
- c is the *constant term*, equal to the output of the function when x is zero (the hypothetical zero-th term).

4.1 Club Sessions

The club members, not all of whom participated in the research, chose partners to work with throughout the study. The pairs who agreed to take part in the research were Ranveer & Alex, Philip & Kabelo, Kylian & Warren, Deon & Mike, Jeff & Dev and Sam & Loyiso.

4.1.1. Session One

Session One began with learners choosing their partners. The activity was then explained in detail, as this was the first time that learners were asked to do this task. We discussed linear expressions and I wrote '$2x$' on the board. I explained how $2x$ could represent a sequence if you substituted consecutive positive integers for x and I asked what the sequence would be. After input from the learners, I wrote '2; 4; 6; 8; …' on the board. Finally, I told the learners that I wanted them to represent the expression and sequence using the connecting cubes provided. Each pair picked a colour or two and got to work. Deon and Mike did not work well together, and each produced a different representation. Warren was late and missed the explanation, which affected his and Kylian's work.

As expected, the creations in this session were mostly straightforward representations of '2; 4; 6; 8; …' The most common representation was one that I called 'Towers', variations of which were produced by Jeff & Dev, Ranveer & Alex and Deon. In this representation each column represents a term and they are arranged in ascending order. Figure 4.1 shows Jeff & Dev's yellow and green representation. They switched from yellow to green when they ran out of yellow blocks.

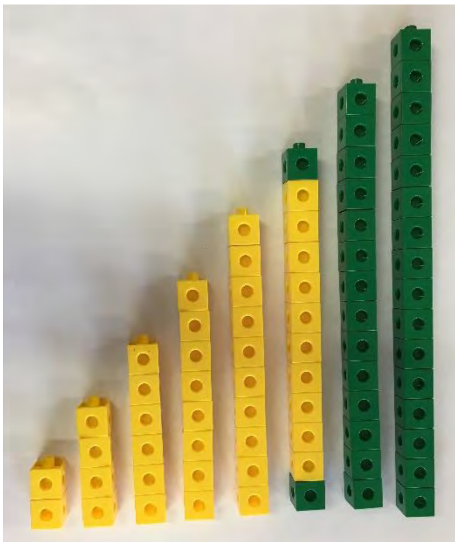


Figure 4.1: Jeff & Dev's yellow and green Towers

Although Ranveer & Alex began the first session with Towers, they noticed that many other pairs were building similar designs and modified their Towers into Rectangles, shown in Figure 4.2. This design excited me, as I noticed that the rectangles showed 2 (the 'multiplier' of the linear expression) by their consistent widths and x by their increasing heights. However, it became clear in the following weeks and during the interview that this was not an intentional aspect of the design. Ranveer & Alex's representation ascends from right to left, a trait common to many of their creations. When asked why they chose to orient their design in this way they pointed out that they had often worked facing each other across the table, so they had each seen the representations ascending in a different direction and thus had not considered orientation important when the photos were taken.

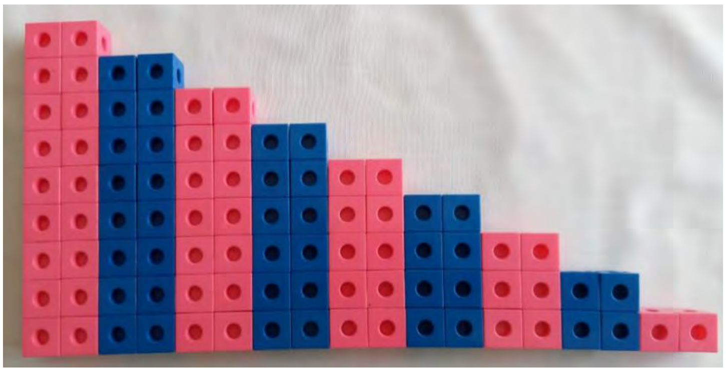


Figure 4.2: Alex & Ranveer's Rectangles

Deon created a black and white version of Towers that also ascended from right to left (see Figure 4.3). In the interview, he said there was no reason for this choice of orientation. Like Ranveer & Alex, he used different colours to differentiate between terms as he connected the columns, rather than keeping them separate. Deon added one block too many to his sixth term, which neither he nor I noticed until the interview, at which point he immediately pointed out that the last term was wrong.

Figure 4.3: Deon's black and white Towers

Mike disagreed with Deon's "boring" approach and made a spiralling representation on his own (shown in Figure 4.4), starting with the first term at the centre of the spiral and using direction to show the differences between terms. Note that the total length of each section represents the term, so each corner block is a part of two terms. In the interview, Mike stated that he had made a mistake and would, in hindsight, have changed colour from black to white for the third term.

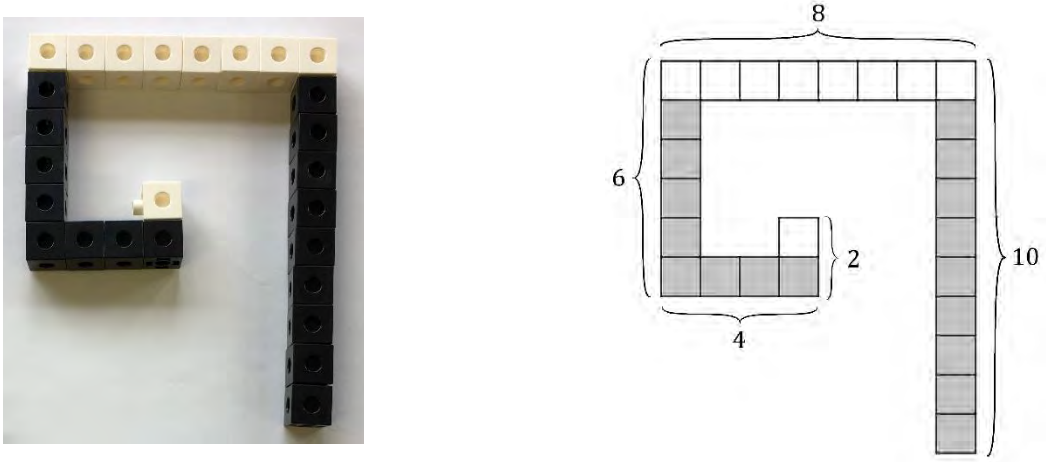


Figure 4.4: Mike's Spiral

Philip & Kabelo created a "Pyramid" representation, shown in Figure 4.5. As in the Towers representation, the Pyramid shows each term with a new column but aligns the columns at the centre rather than the bottom so that the terms grow both upwards and downwards. Like Jeff & Dev, Philip & Kabelo did not use colour purposefully, using orange until they ran out of blocks and then changing to red.

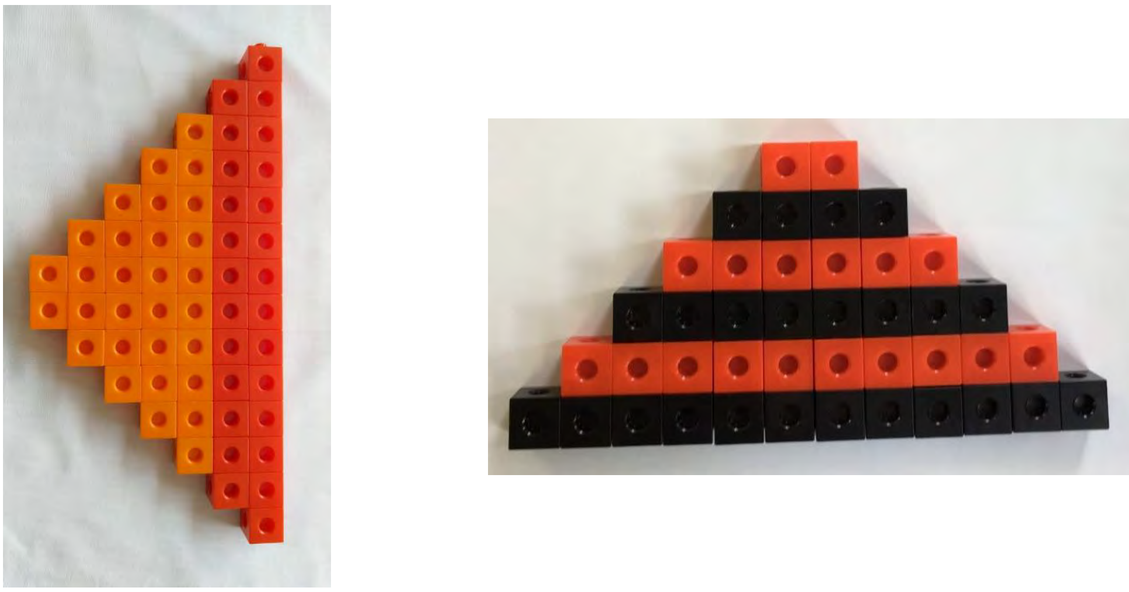


Figure 4.5: Pyramids built by Philip & Kabelo (L) and Sam & Loyiso (R)

Sam & Loyiso also built a Pyramid (see Figure 4.5), orienting it to increase from top to bottom rather than left to right as with Philip & Kabelo. As Ranveer & Alex and Deon did, they used alternating colours to differentiate the terms.

Kylian started work on his own without understanding what exactly he had been asked to do. As Warren was late and missed the explanation, he was also confused and followed Kylian's lead in building what appeared to be a robot. When he was asked how his creation represented the algebraic expression, Kylian pointed out parts of the robot made of 2, 4 and 6 blue blocks as shown in Figure 4.6, but there was no consistency in his use of blue to represent the sequence and he wrote in the research journal that he *"thought you had to build a real-life object."*

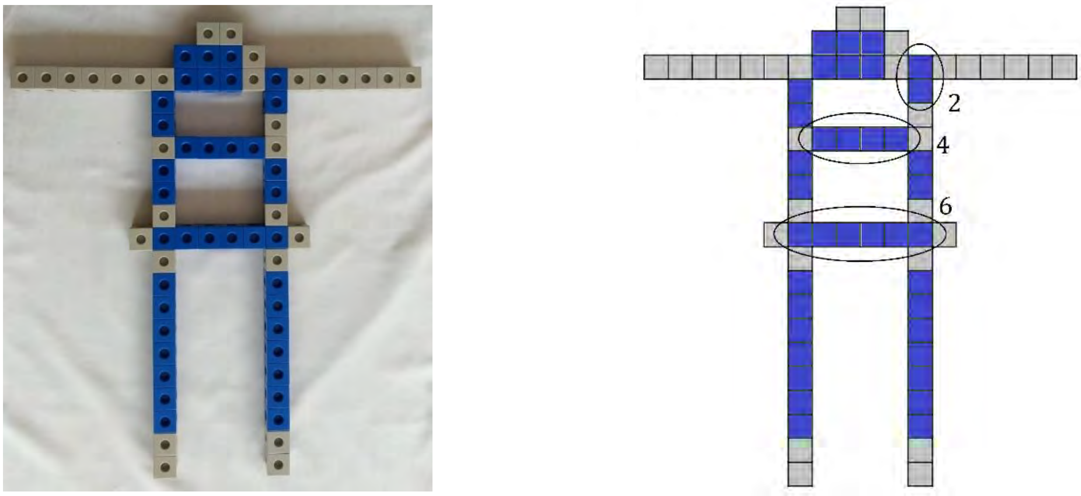


Figure 4.6: Kylian & Warren's Robot

After the pairs had been given time to work on their representations, they were asked to stand in front of the classroom to show and explain their creations to the rest of the club. Learners found explaining their representations difficult and given the size of the club it was hard for those presenting to keep the attention of other pairs, who fiddled with blocks and continued to tweak their own representations instead of listening. When the learners were prompted to ask questions and provide suggestions, Warren made some particularly useful comments about colour and orientation that I used to emphasise possible improvements to be included in the journal entries. Journals were handed out and explained and once they had been filled in the session was closed.

4.1.2. Session Two

For the second session, I asked the learners to build a representation of $3x + 1$ (that is, 4; 7; 10; 13; …), introducing a positive additive to slightly increase the complexity of the expression, as I felt most of the learners had understood the task and produced good representations in Session One. I reminded them of the possible improvements discussed at the end of Session One. Philip & Kabelo, Jeff & Dev and Deon worked on improving their initial representations, while others worked on new ideas. Loyiso and Warren were absent, leaving Sam and Kylian to work on their own. Deon and Mike continued to work separately.

Jeff & Dev improved on their Towers from Session One (see Figure 4.7). They changed their use of colour to differentiate between terms, as many pairs had done in the first session. Dev had the idea of including grey towers below the main towers to indicate the value of x for each term. This proved to be pivotal for the development of the 'visual expression' representation that Jeff & Dev created in Session Three, which was eventually adopted in some form by all but one pair of learners.

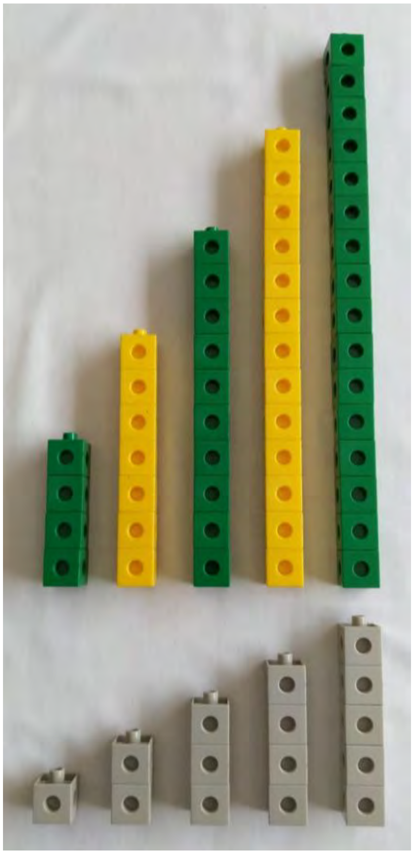
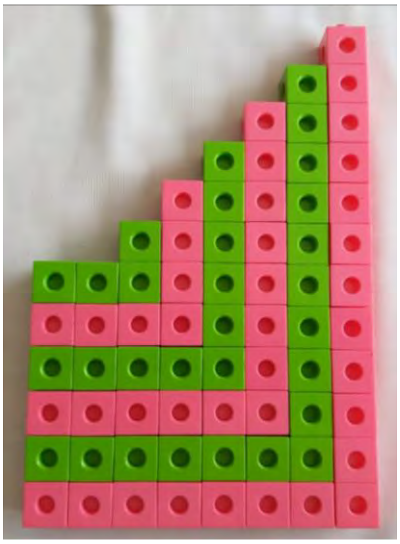


Figure 4.7 Jeff & Dev's Session Two Towers

I had hoped that Alex & Ranveer would continue to work on their Session One rectangles, but they opted to create a completely different representation and produced the 'L s' shown in Figure 4.8. As in their Rectangles, they alternated colours between terms and used the number of blocks in each term to show the term's total value.



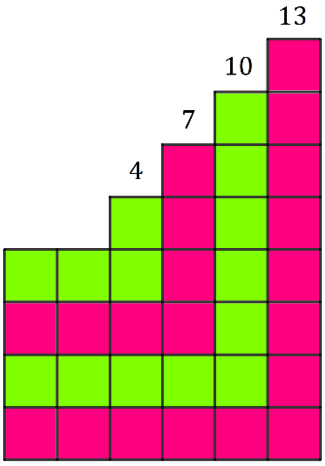


Figure 4.8: Alex & Ranveer's Session Two 'Ls

Philip & Kabelo also updated their representation from the first session. Like Jeff & Dev, they changed from an arbitrary to a purposeful use of colour. They also 'fanned' out the design artistically, becoming the first to use the three-dimensional nature of the cubes in their representation, shown in Figure 4.9.

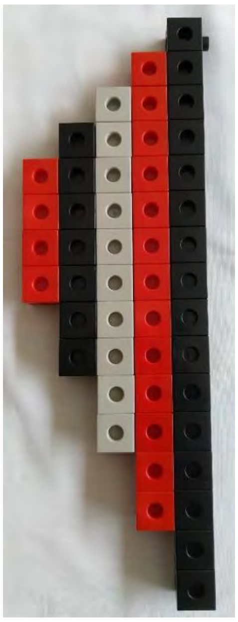


Figure 4.9: Philip & Kabelo's Session Two Pyramid, displayed 2D (L) and fanned (R)

Deon reproduced his Towers almost exactly from the first session, merely changing the colours from black and white to purple and blue, as seen in Figure 4.10. He made another small 'mistake' (which, again, we only noticed in the interview) and started with $T_0 = 1$ instead of $T_1 = 4$.

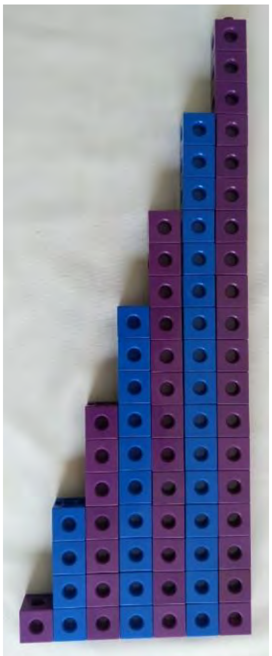
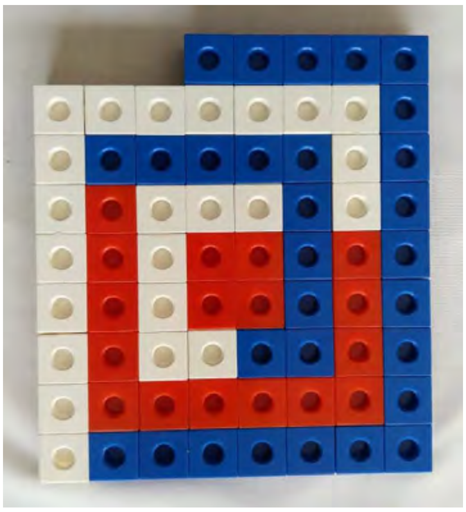


Figure 4.10: Deon's second session Towers

Mike continued to work on his spiral idea but changed most of the implementation. He used colour rather than direction changes to differentiate between terms, which meant that the corner blocks were now part of only one term, and he removed the spaces between the spiral arms (Figure 4.11).



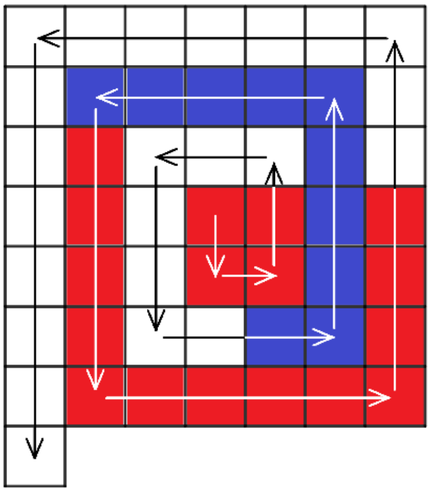


Figure 4.11: Mike's Session Two Spiral

Sam, who worked with Mike for the session, appeared to take inspiration from Mike's spiral, beginning his own spiral and then changing it into a kind of 'zig-zag', or irregular pyramid, shown in Figure 4.12. He used a grey block (included in the first term) to indicate the beginning of the representation and alternated between red and white to distinguish terms. During the interview, he explained that in the first term the grey block represented x and the three red blocks $\times\ 3$, though it was unclear whether this was intentional at the time of building or just interpreted that way after having built multiple Visual Expressions in the intervening weeks.

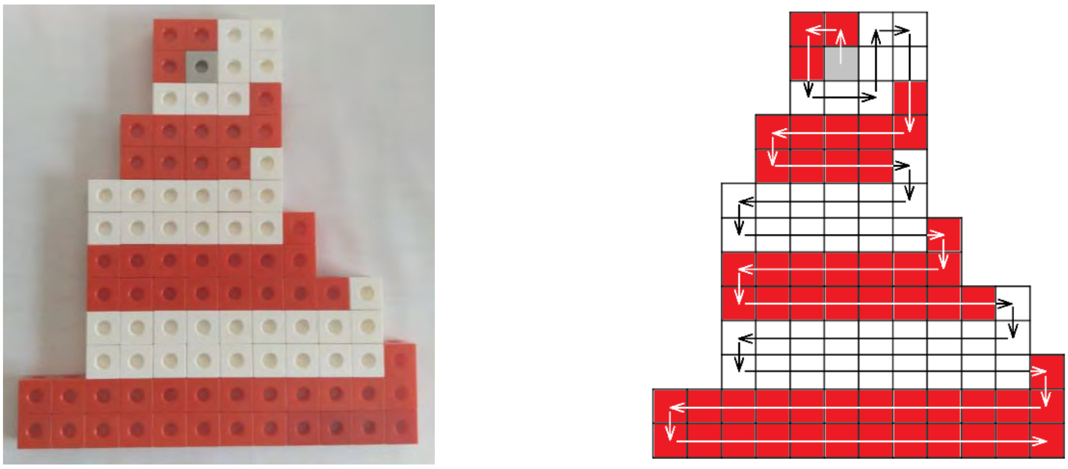


Figure 4.12: Sam's Spiral/Zig-Zag

Kylian, without input from Warren, still seemed to be confused by the task and created a design like others' towers but included a seemingly arbitrary horizontal divider (see Figure 4.13). His columns showed $T_1 = 4$, $T_2 = 7$ and $T_4 = 13$. There was no column of ten blocks to represent T_3 and there were other black blocks that did not form part of the pattern.

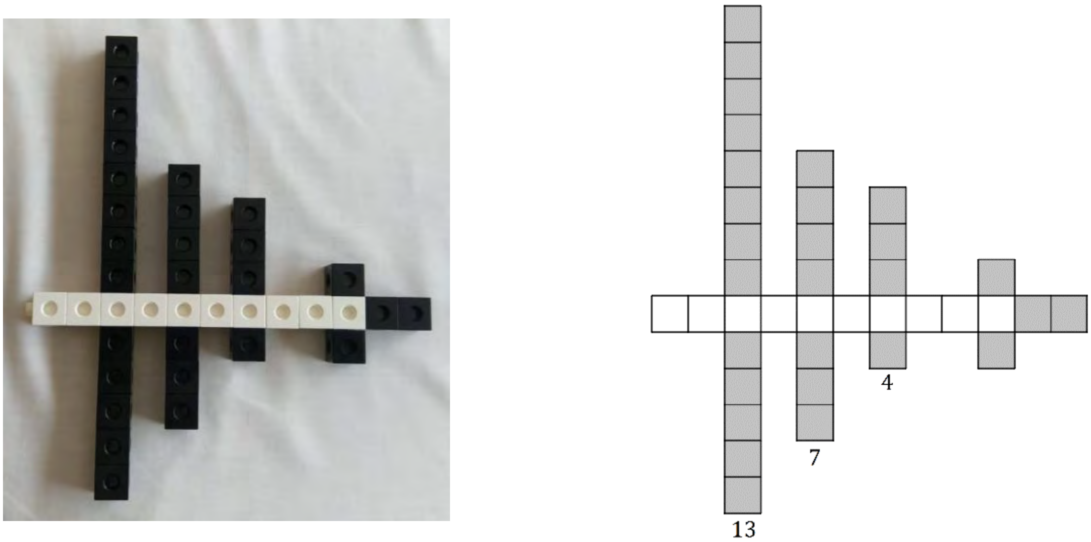


Figure 4.13: Kylian's Session Two representation

After each pair had completed their representations, I asked them to show me the eighth term, which most of them could do. They completed a journal entry and the session ended. It was a chaotic session with no time for presentations, but I challenged the learners to work on a representation in the next session that would allow me to find the algebraic expression when looking at *only* the eighth term.

4.1.3 Session Three

I started the session by reminding learners about my challenge to build their representation in such a way that I could find the expression if I looked at just the eighth term. I asked them to represent $2x + 3$ (5; 7; 9; 11; 13; 15; 17; 19; ...), choosing to stick with the same form as the expression from Session Two. I instructed them to make sure they built at least T_1, T_2 and T_8 and suggested that they find some way to show the value of x in each term, as Jeff & Dev had done in Session Two. This led to the development of a type of representation that I called the “Visual Expression” by Jeff & Dev (Figure 4.14), Sam & Loyiso (Figure 4.17: Sam & Loyiso’s first Visual Expression showing $T_1 - T_4$ (L) with an explanation of T_2 (R)) and, to an extent, Kylian & Warren (Figure 4.18). A Visual Expression uses different colours to represent different parts of the algebraic expression. If the algebraic expression is given in the form $T_x = dx + c$, each term includes d cubes of one colour to represent $\times\, d$, c cubes of another colour to represent $+\, c$ and a third colour to represent x. The d and c blocks stay constant from term to term but the x blocks increase by one from each term to the next. Most Visual Expressions also included T_x blocks of a fourth colour to represent ‘the answer’. Deon & Mike finally worked together but were frustrated by the challenge and broke their representation apart before I photographed it, as did Philip & Kabelo.

Jeff & Dev built on their idea of including an indication of the value of x by introducing blocks in different colours to represent each part of the expression. Figure 4.14 shows their representation, in which, to the left of the red blocks, the number of blue blocks shows the value of x for the term, the three green blocks represent the addition of three in the expression $2x + 3$ and the two black blocks represent multiplication by two.

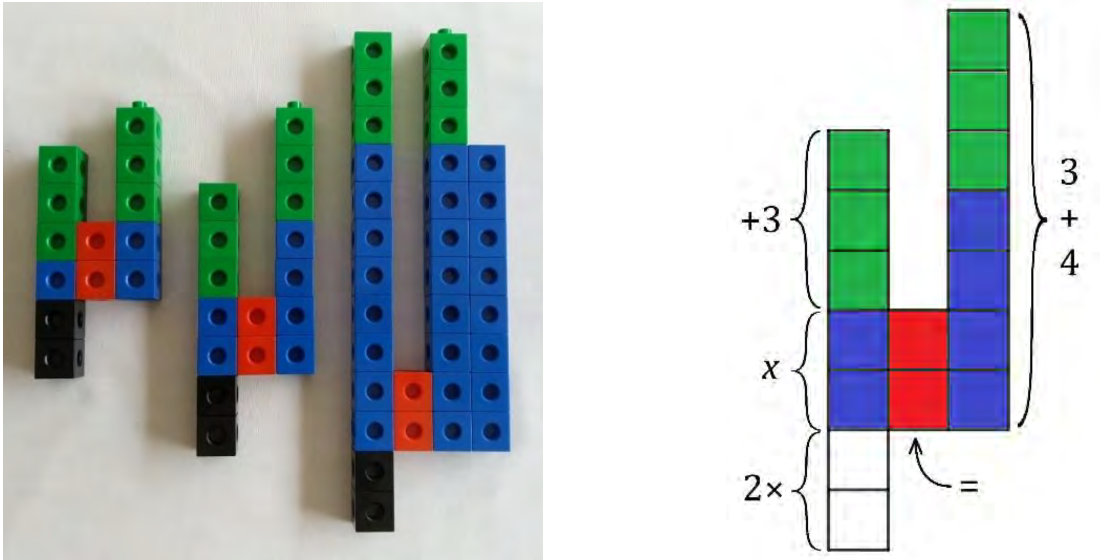


Figure 4.14: Jeff & Dev's first Visual Expression showing T_1, T_2 and T_8 (L) and an explanation of T_2 (R)

They began by building only the column on the left, but Jeff then added an *"equals sign"* in the form of the two red blocks, and they included the output value of the term, or as they called it, *"the answer"* on the right. They continued to use green for the additional three blocks and used blue to represent the value of $2x$ for each term.

In T_1 and T_2, Jeff placed the blue 'answer' blocks in a single column. However, in T_8 he built these blocks into a 2×8 array. When I asked him why, he explained that it was just a space-saving measure since a column of sixteen blocks would be inconveniently tall. I pointed out that in T_8 the x-column and the answer rectangle had the same height of eight blocks and that this was perhaps useful, as it showed the multiplication of $2 \times x$ in the width and height of the rectangle. We then modified T_1 and T_2 to match T_8, as shown in Figure 4.15.

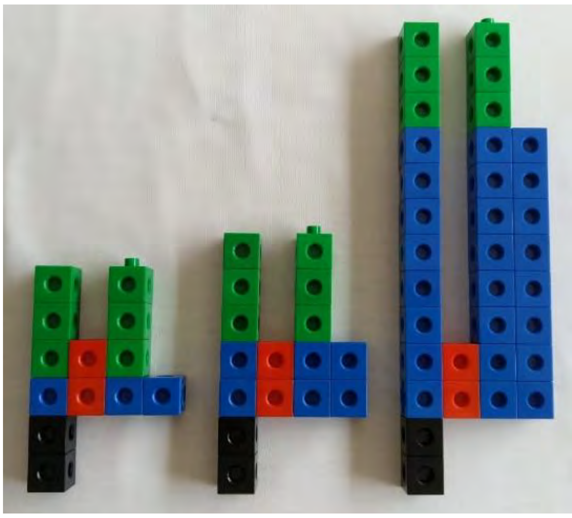


Figure 4.15: Jeff & Dev's modified Visual Expression

Alex & Ranveer did something new again, creating a spiral very similar to Mike's Session One representation. However, they used orange blocks to show both the separation of terms and the increase of two from term to term, as shown in Figure 4.16. Term three, for example, had seven white blocks to show the total number of blocks of Term Two and two orange blocks to show the increase from T_2 to T_3, making for a total of nine blocks in T_3. Rather than build T_8 separately they built every term from T_1 to T_8.

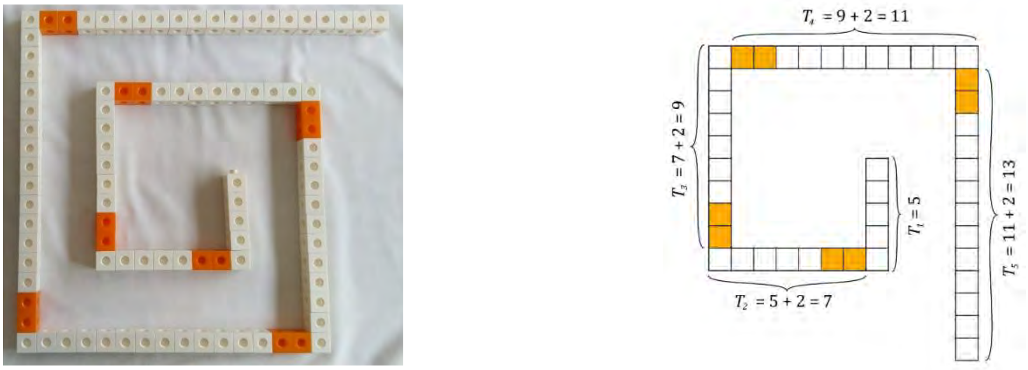


Figure 4.16: Alex & Ranveer's session three Spiral showing the increase of two between terms

Philip & Kabelo did not produce a final representation in this session, breaking down their attempts before I could photograph them. In their journal they drew Jeff & Dev's representation. Deon & Mike finally tried to work together but also did not present a representation for me to photograph. In their journal, they drew very simple Towers without any colour differences.

Sam & Loyiso, like Jeff & Dev, built a Visual Expression (see Figure 4.17). They chose not to include the total value of the term, simply representing $2x + 3$ with two pink blocks, x as red blocks and three brown blocks.

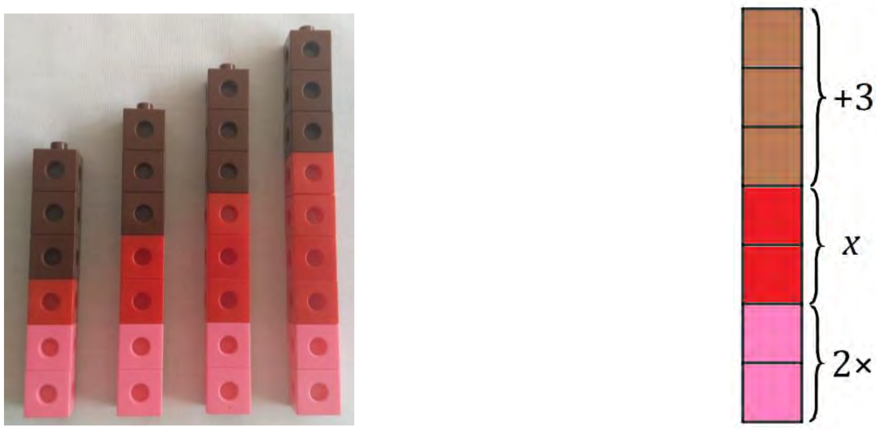
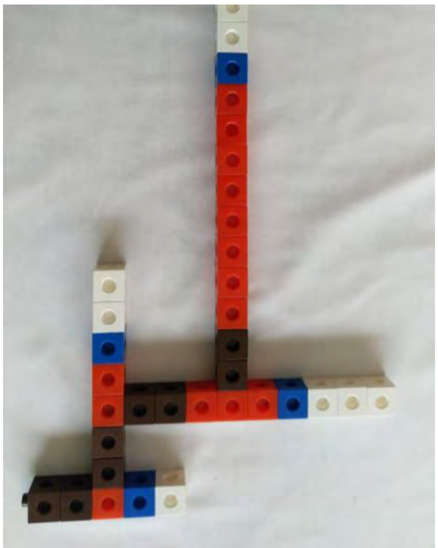


Figure 4.17: Sam & Loyiso's first Visual Expression showing $T_1 - T_4$ (L) with an explanation of T_2 (R)

Kylian & Warren finally seemed to grasp what I was asking them to do and produced a kind of Visual Expression, shown in Figure 4.18. They showed the value of x with both red and white blocks, showed the value of the coefficient of x with two brown blocks and added one blue block to make up the total value of the term. Each term grew out from the previous term's red blocks. I noticed that the two different colours representing x could be understood as two groups of x, showing the multiplication by two, and that the brown and blue blocks together (since they stayed constant) showed the addition of three. Warren was excited when I pointed this out, and his reaction implied that it was not a purposeful feature of the design.



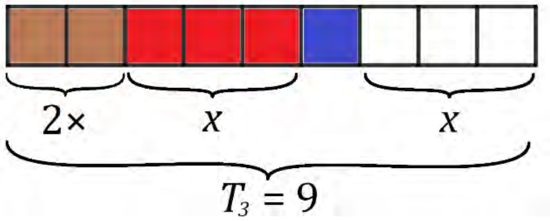


Figure 4.18: Kylian & Warren's Session Three representation showing T_1, T_2, T_3, the beginning of T_8 (L) and an explanation of T_3 (R)

After the pairs had finished building, I held a quick presentation of the four completed representations, explaining them myself rather than having the learners present their own. I pointed out the strengths of the Visual Expression and the clear representation of the increase by two in Alex & Ranveer's spiral. They then filled in the session's journal entry and completed the session.

4.1.3. Session Four

In Session Four I asked the learners to build a representation of $3x - 2$ (1; 4; 7; 10; 13; 16; 19; 22 …), changing from a positive constant term to a negative constant term to see how they would represent subtraction. Deon & Mike, Sam & Loyiso and Kylian, all produced versions of the Visual Expression, building on either their own or other pairs' ideas from Session Three. Warren was absent again. Jeff & Dev, Alex & Ranveer and Philip & Kabelo all created Rectangles, some of which included various aspects of the Visual Expression. The similarities between these three representations were the result of my greater input in this session, which guided these pairs to create something closer to my idea (rather than their own ideas) of the ideal representation.

Jeff & Dev modified their representation from Session Three, using white blocks to show x, orange blocks to show $\times 3$ and yellow blocks to show -2, as shown in Figure 4.19. Because I had encouraged them to build rectangles with a base of two blocks in Session Three, they put their black 'answer' blocks into two columns. In the interview, Jeff stated that it was *"easier if ... we had two rows because we'd be counting upwards in twos, not in ones."* I explained again how a rectangle with base three might show multiplication by three and they modified their representation to show x on the left and 3 at the bottom with a $3 \times x$ rectangle of black blocks showing the answer, two of which were replaced with yellow to show the subtraction.

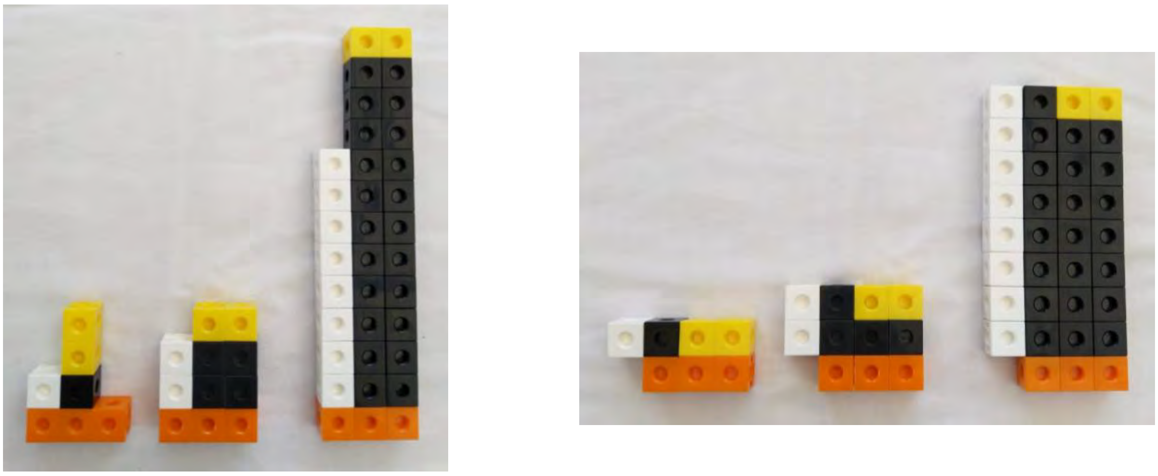


Figure 4.19: Jeff & Dev's second Visual Expression before (L) and (R) after modifications

Alex & Ranveer could not think of a way to represent T_8 in such a way that the expression would be evident, since I had pointed out in Session Three that when looking at only their eighth term I would need more information to be able to find the algebraic expression. I encouraged them to go back in their journals and try to update their Rectangles representation from Session One. They started with rectangles with a base of two, as they had done in Session One. I asked if there was a better choice of width and Alex built the widths to equal x. This led them to build 'L' shapes again, with the blocks not part of the base simply stacked in a column. We agreed that this still was not entirely informative, and Alex realised that he could show $\times 3$ by changing the single column to multiple columns of three, as shown in Figure 4.20. Once he had explained how the representation worked to Ranveer they worked together to build each term up to T_8.

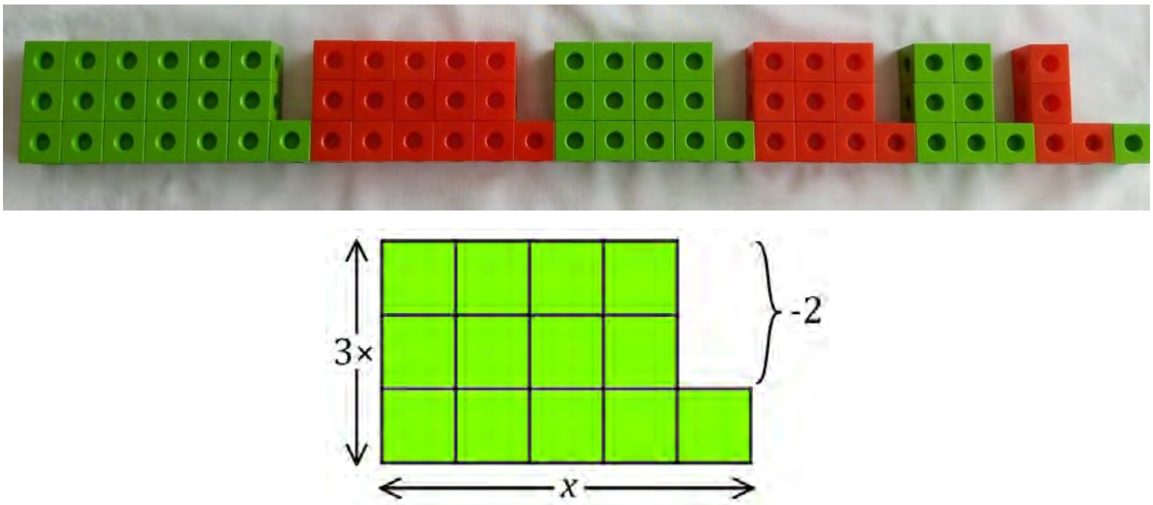


Figure 4.20: Alex & Ranveer's revised Rectangles (Top) with an explanation of T_5 (Bottom)

Philip & Kabelo also struggled to come up with a coherent representation that would show the expression with a single term. They had three blue blocks to show $\times 3$, two white blocks to show -2 and eight purple blocks to show $x = 8$. (They told me in the interview that they chose white to blend in with the background and show that the blocks had been "*removed*"). I asked them how they could show multiplication by three visually and with some thought they came up with a rectangle, as shown in Figure 4.21. They initially included the three blue blocks in the 'answer' using three blue blocks, nineteen purple blocks and two white blocks to show the total of $24 - 2 = 22$, but could not represent the first term this way, so I encouraged them to add another purple row and consider only the purple blocks to be a part of the answer, which they did.

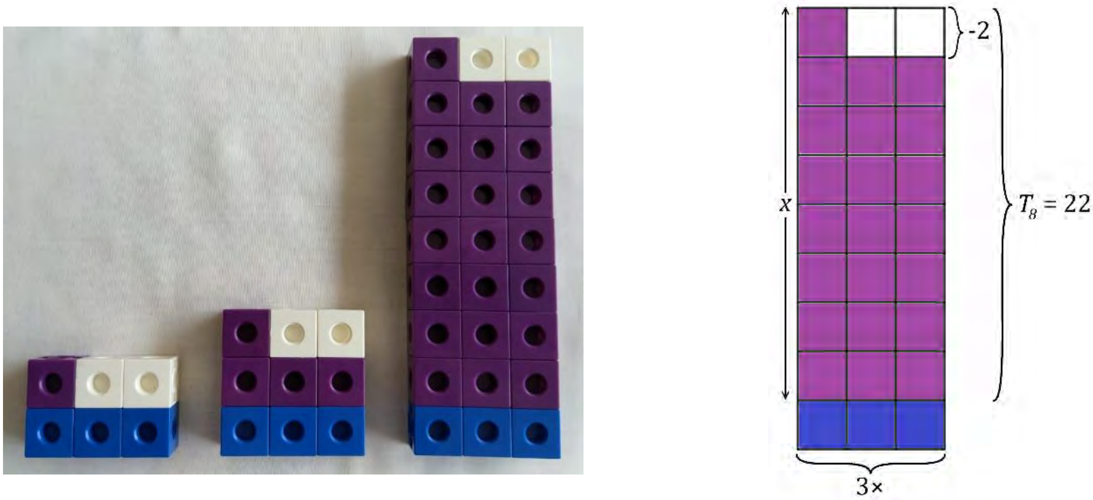


Figure 4.21: Philip & Kabelo's Rectangles showing T_1, T_2 and T_8 (L) and an explanation of T_8 (R)

In contrast to previous weeks, Deon & Mike had a very productive and cooperative session, creating Towers (oriented from top to bottom instead of left to right) with a Visual Expression on the left, in which red represented $\times 3$, orange represented x and black represented -2 (Figure 4.22).

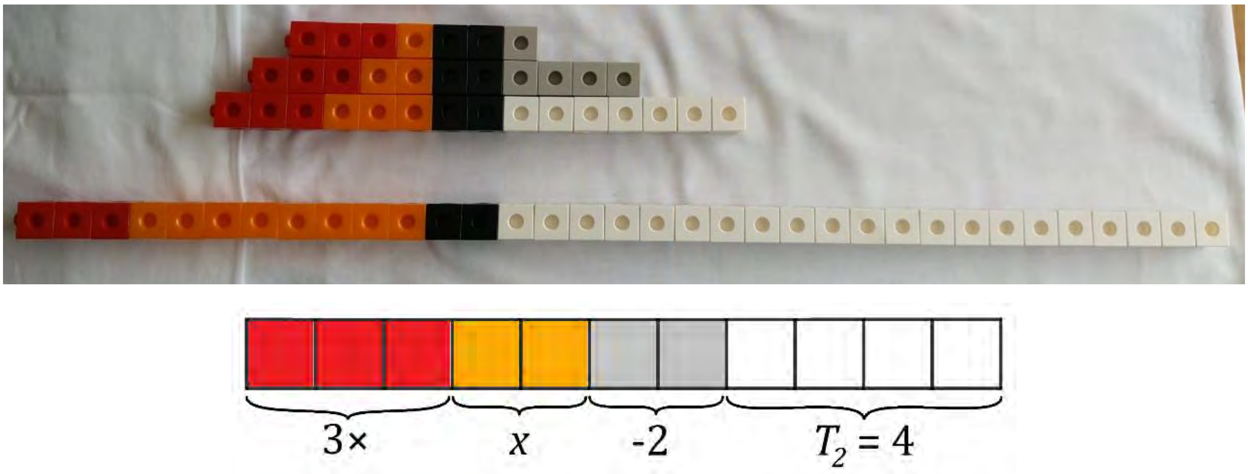


Figure 4.22: Deon & Mike's first Visual Expression showing T_1, T_2, T_3 and T_8 (Top) and an explanation of T_2 (Bottom)

Sam & Loyiso built another Visual Expression, this time choosing to include the total value of the term as a grey column at the end of each term (Figure 4.23). They used white for the final T_8 column simply because they had run out of grey blocks.

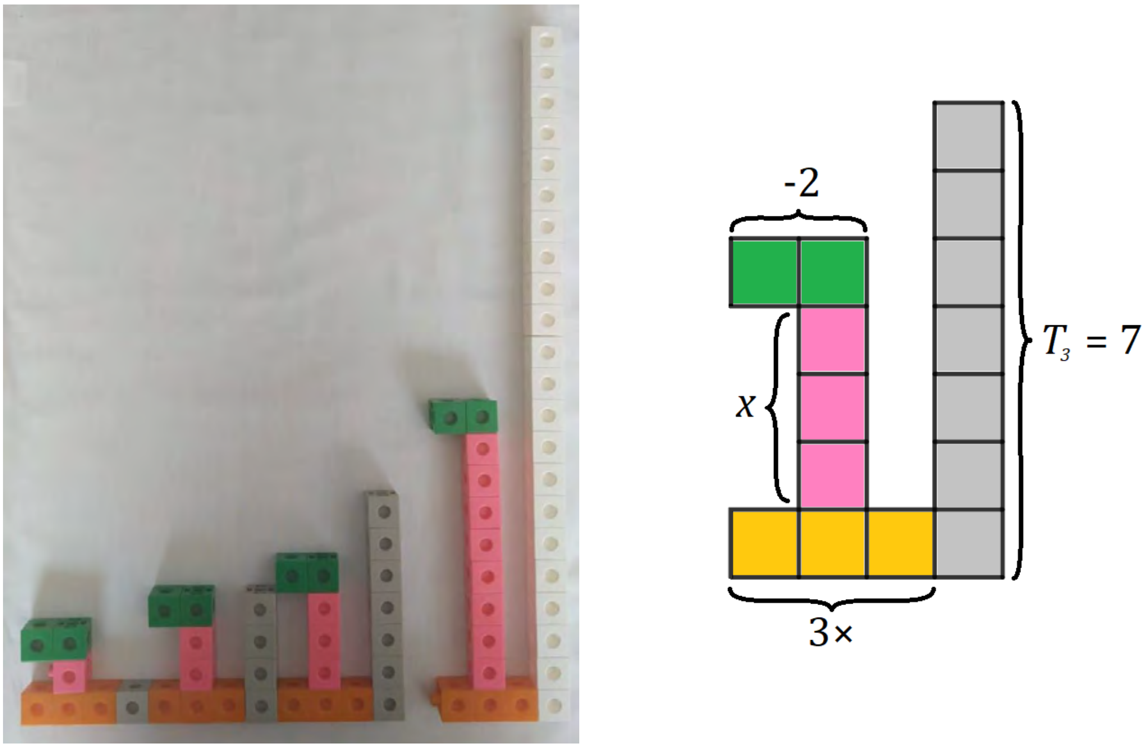


Figure 4.23: Sam & Loyiso's second Visual Expression showing T_1, T_2, T_3 and T_8 (L) and an explanation of T_3 (R)

As Warren was absent, Kylian worked alone and created a wild Visual Expression (Figure 4.24). He used yellow blocks to represent the value of x, grey blocks to represent $\times 3$ and brown blocks to represent the final value of the term. Each term contained two blue and two red blocks, which both represented -2. The representation grew in unusual and inconsistent ways, but his use of colour was consistent and made sense once explained.

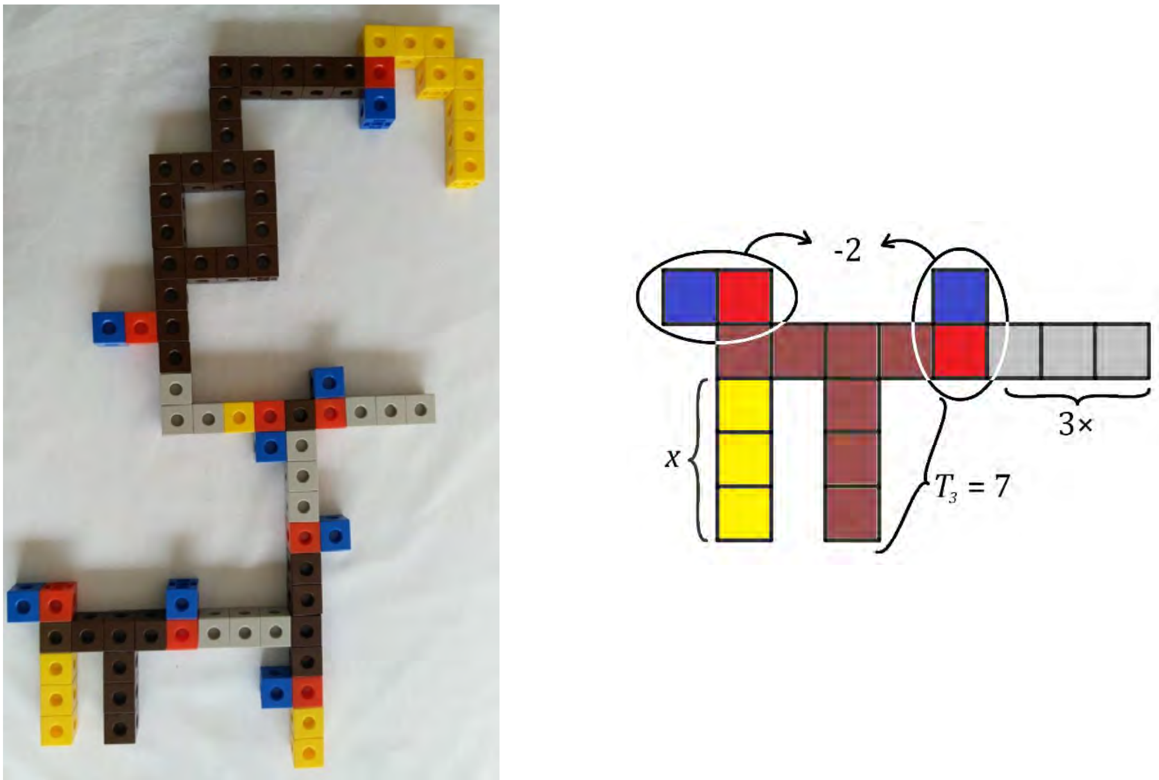


Figure 4.24: Kylian's Session Four representation showing T_1, T_2, T_3 and T_8 (L) and an explanation of T_3 (R)

As this was the final session, we did not hold a discussion and after each pair had completed their last journal entry, we scheduled interviews and the learners were allowed to leave.

4.2 Interviews

The interviews were conducted two to three weeks after the fourth club session, depending on the availability of the participants. During each of the six sessions, I asked the learners to create a representation of $4x - 3$ (1; 5; 9; 13; 17; ...), similar to the expression $3x - 2$ from session four. After they had completed their visualisation I asked them questions about it (see the interview schedule in Section 3.4.2) and then moved on to reviewing and comparing all five of their representations by looking through the compiled photographs on my laptop.

4.2.1. Jeff & Dev: The Trailblazers

As soon as they started discussing what they would build, Jeff & Dev were already committed to building a Visual Expression. Dev asked *"so what colour's going to represent x?"* The idea of representing the value of x with a particular colour was one that Dev first came up with in Session Two and led directly into the pair's development of the Visual Expression. After agreeing on colours, they proceeded to build what Jeff called the *"straightforward"* representation shown in Figure 4.25. Initially, they had it in a different order, with the white x blocks before the yellow $\times 4$ blocks. Jeff then swapped them, saying *"Wouldn't it be easier if we actually did this? 'cause now you can say four x minus 1."* We discussed that it was not important mathematically because, as Dev pointed out, *"Order doesn't matter in multiplication."* But Jeff insisted that it was better to have the visualisation conform to the conventions of algebraic expressions, in which multiplication is written with the coefficient (multiplier) before the variable/s.

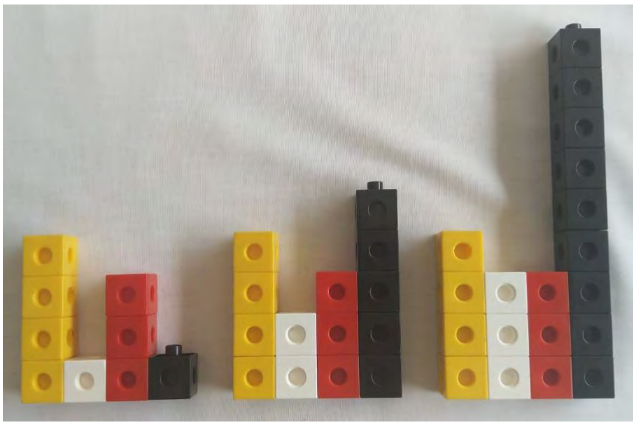
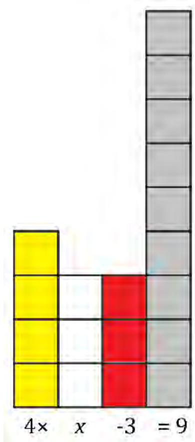



Figure 4.25: Jeff & Dev's final Visual Expression showing T_1, T_2 and T_3 (L) and an explanation of T_3 (R)

Jeff & Dev 's representations (Figure 4.26) showed a clear progression of ideas from session One through to the interview. Initially, they built a basic version of Towers that Jeff explained as *"increasing by two every single time."* The pair was unimpressed by their first representation in review: Jeff called it *"poorly made"* and criticised its lack of an indication of the value of x. However, we agreed that it clearly showed the constant increase between terms and that its orientation from left to right could be interpreted to show that it was increasing (with a coefficient of 2) rather than decreasing (with a coefficient of -2). In the second session, they modified their Towers to include grey blocks that showed the value of x and changed their use of colour to alternate between terms. Their third representation was a turning point, when they began using different colours to represent each part of the expression, coming up with the Visual Expression. During the interview, I asked the pair to find the algebraic expression that they had been representing in Session Three. They quickly and correctly surmised or remembered that the red meant '=', the blue on the left stood for x and the blue and green blocks to the right of the red showed the answer, but then got stuck. Dev said it was *"confusing"*. Jeff suggested $3x + 2$, which I asked them to check against the 'answers' they had built into their representation. Finding that substituting 2 into $3x + 2$ gave them eight blocks instead of the seven shown in the representation. Jeff at first thought that they had made a mistake when building the representation, but then figured out that the expression was $2x + 3$. They did not have this problem with their visualisation from Session Four, as Dev quickly found the expression $3x - 2$. Despite this, Jeff stated that this fourth representation was *"very hard to understand"*. He explained that, in the fourth representation, *"everything's all over. Here [in the interview representation] it's moving left to right straight"*. Dev agreed that their final representation (Figure 4.25) was the best and clearest of their designs.

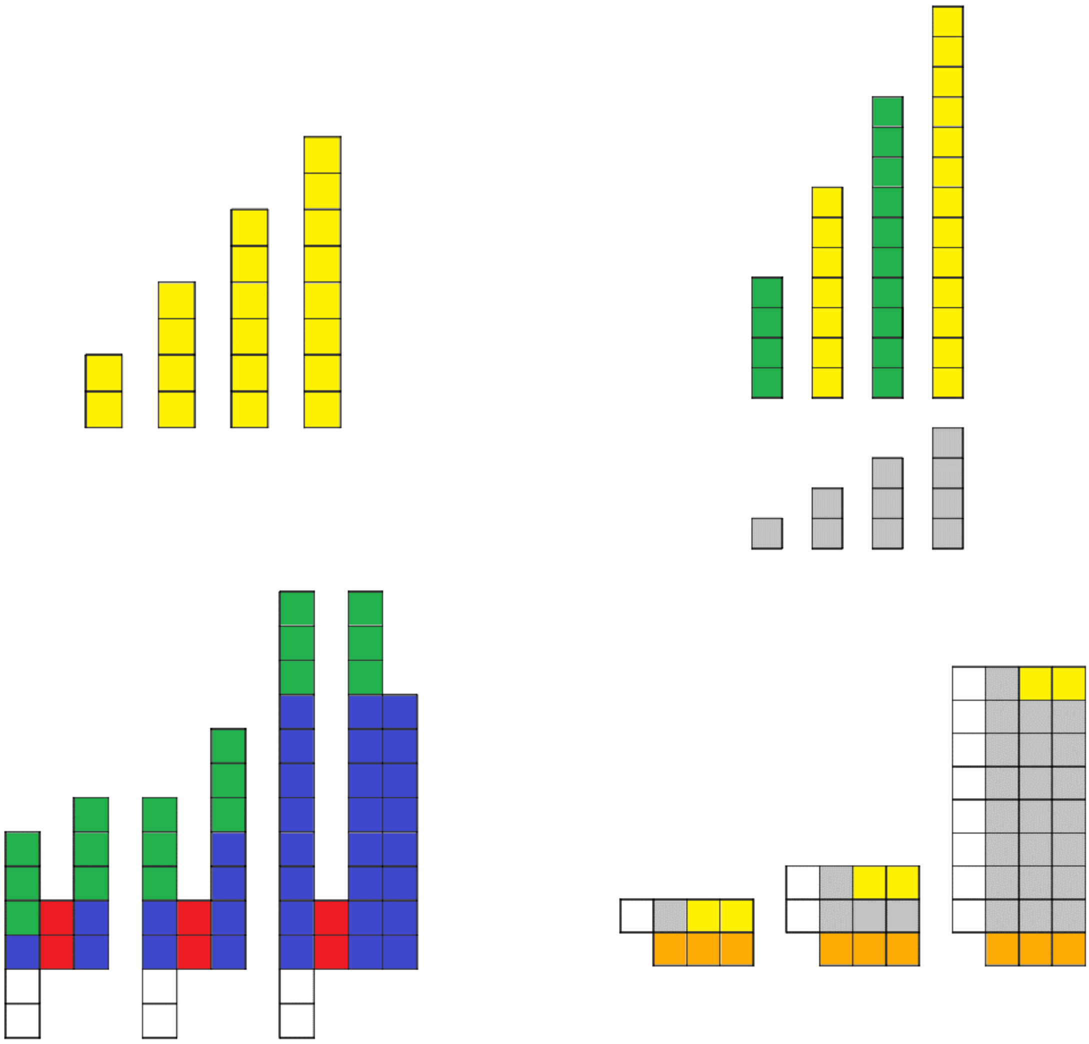


Figure 4.26: Illustrations of Jeff & Dev's first four representations

4.2.2. Ranveer & Alex: The Independents

Ranveer & Alex were an unusual pair in that they were the only ones who did not end up producing some version of the Visual Expression in their interview. Rather than using different colours to represent different parts of the algebraic expression, they used alternating colours to distinguish between even and odd terms for all but one of their designs. At the beginning of the interview Alex asked, *"Is it going to be easy, ma'am?"* and Ranveer said, *"I hope so,"* which led me to believe that they expected the interview to be or contain a kind of test with right and wrong answers. I explained that I merely wanted to know about their *"thoughts and ideas"* rather than getting answers.

Figure 4.27: Alex's arrangement of Term 3

As they began building their final representation in the interview, they first substituted 1, 2, 3 and 4 into the expression and made Towers of 1, 5, 9 and 13. Assuming they were not finished, I asked, *"How do you want to put it together?"* Alex took Term 3 and put it together as in Figure 4.27. He explained that the difference of three between the two columns showed the subtraction of three in the expression. However, Ranveer had noticed something different – that Alex's first column had three blocks, which corresponded with x, the term number. Ranveer arranged Term 2 and Term 4 to match his idea of what the terms should look like and Alex, disagreeing, then rearranged them according to his original intention. After some further disagreements on how the surrounding terms should be built, I suggested that Ranveer build separate representations so that we could see and compare them both. The two resulting representations are shown in Figure 4.28. Alex's representation has a constant difference of three between the columns in each term. Ranveer's representation shows the term number in the right-hand column of each term.

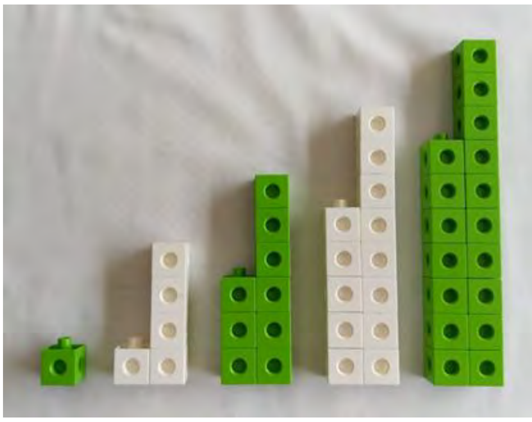
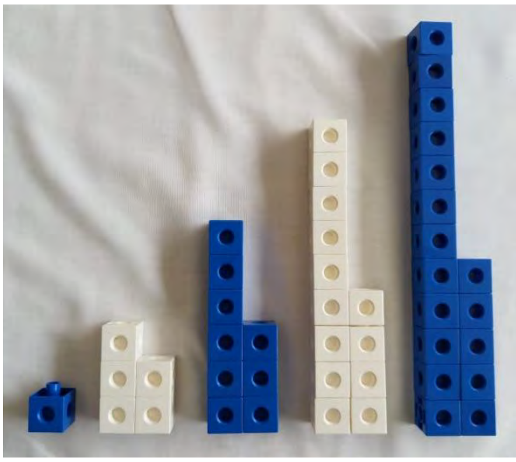
 

Figure 4.28: Alex (L) and Ranveer's (R) representations of $4x - 3$

Both the features they noticed in this fifth representation were present in Alex & Ranveer's representation from Session 4. As shown in Figure 4.29, their representation of $3x - 2$ both displayed a difference of two between columns *and* the value of x (in the width of the term). Each of the boys seems to have noticed and remembered something different about their representation from Session 4 and used this in their interview representations.

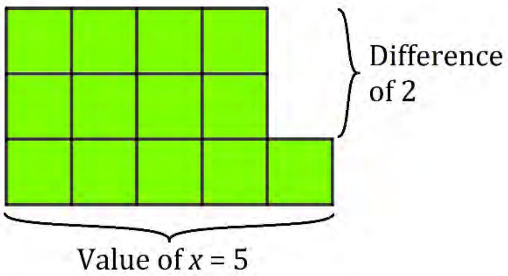


Figure 4.29: Term 5 of Alex & Ranveer's representation of $3x - 2$ *from Session 4*

I asked Alex at one point about the differences between consecutive terms in his representation. At that point, Ranveer started to stack each successive term behind the last (as shown in Figure 4.30), which he explained later was to explicitly show the differences between terms. He was the only participant to use the third dimension for mathematical (rather than aesthetic) purposes.

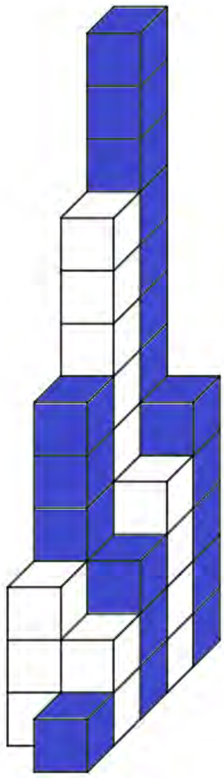


Figure 4.30: A 3D illustration of Ranveer's representation, showing the differences between terms

Alex and Ranveer's representations (Figure 4.31) showed no clear progression of ideas from week to week. Rather than building on and improving previous ideas, they chose to do something new in each session until I specifically prompted them in Session Four to build on their representation from Session One. When we reviewed their first Rectangles representation during the interview, Ranveer immediately pointed out that *"you could see the increases by two each time … and you can see that we multiplied … by two."* Although we had discussed the link between the constant difference and the coefficient of x many times, Ranveer did not seem to consider "increasing by two" and "multiplying by two" to be the same. He also pointed out that *"you can know what term it is"* by the height of the rectangle. The second, 'L s' representation caused some confusion, with Alex claiming that *"it was kind of Ranveer's design"*. We agreed that there was some difficulty in comparing consecutive terms and that the first design was clearer. Alex explained that their third, Spiral representation used orange blocks to separate terms and to show the increase of two between consecutive terms, which was an improvement of the 'Ls'. Both Alex and Ranveer believed that their fourth representation, the 'updated Rectangles' was their best of all their designs, including the interview design. Alex quickly pointed out the *"gaps"* that showed subtraction of two and Ranveer said that the rectangles showed the *"term number multiplied by three, so they were going to be three high."*

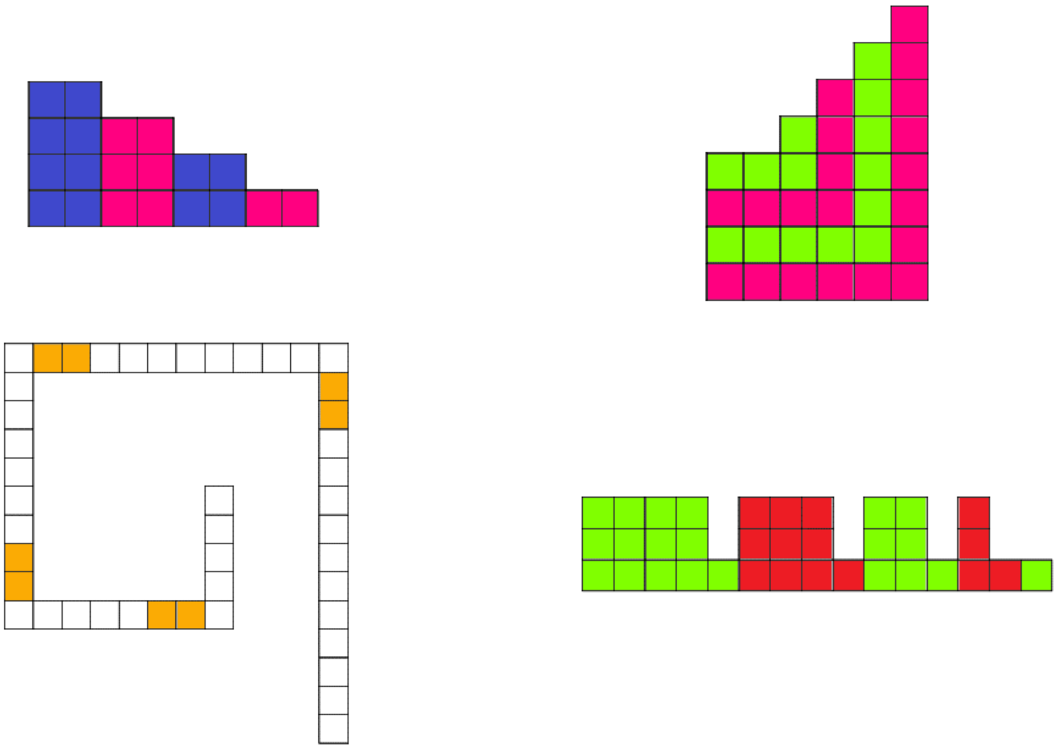


Figure 4.31: Illustrations of Alex & Ranveer's first four representations

4.2.3. Philip & Kabelo: The Consistents

For their interview task, Philip and Kabelo seem to have intended to recreate their visualisation from Session Four (Figure 4.21), even selecting the same colours to work with. However, they did not remember much of how they had built their last representation and essentially started from scratch. Like Alex and Ranveer, they started by substituting 1, 2 and 3 into the expression. Philip put out one white block for the first term and five for the second term. Kabelo said, "*Why don't we build it in a way that shows that we're taking two, multiplying by four and subtracting three, instead of just using a solid colour?*" Philip agreed. As T_1 was 1, they chose to start with the second term, building the Visual Expression shown in Figure 4.32. The purple blocks represented x, the blue blocks represented $\times 4$ and the white blocks represented -3.

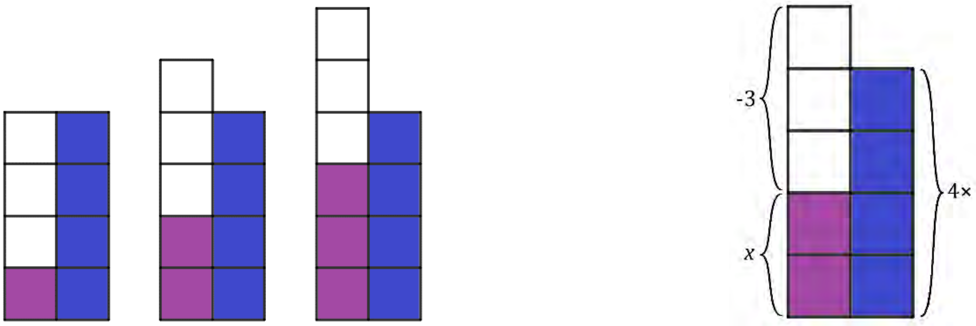


Figure 4.32: Philip & Kabelo's first interview representation showing T_1, T_2 and T_3 (L) and an explanation of T_2 (R)

Kabelo expected the total number of blocks in their representation of the second term to equal the output value of five, so the resulting nine blocks prompted them to some thought. Philip pointed out that there were five blocks in the left column, so Kabelo concluded that counting the white and purple blocks together should always give the term number. However, Philip quickly built the matching first term and disproved Kabelo's hypothesis, as the first column had four blocks and not one. Still trying to find the output value in each construction, they built the third term but were unsatisfied. Kabelo then began to rearrange Term 2 as shown in Figure 4.33 and added a final column of blue blocks to represent the value of T_x. Now, Philip said, to get to the next term you would add one to the purple blocks and four to the right-hand column of blue blocks.

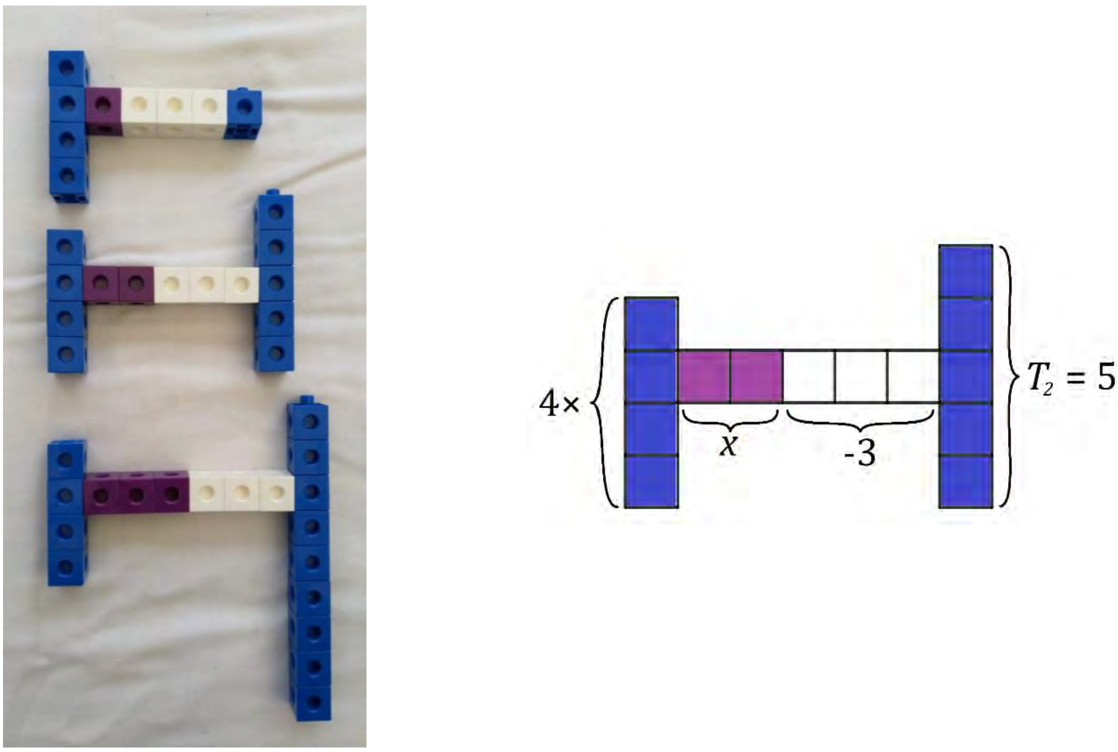


Figure 4.33: Philip & Kabelo's final interview representation showing T_1, T_2 and T_3 (L) and an explanation of T_2 (R)

Philip and Kabelo's representations (Figure 4.34) showed some progression over the course of the study, with most changes to their designs inspired by direct prompts from the facilitators. As they seemed very satisfied with their Pyramid, there was little change in their design from Session One to Session Two, except for the inclusion of different colours to better distinguish between the terms. In Session Three there was no progress, as they struggled to adapt their pyramid to my request that they represent the eighth term without the support of all the preceding terms. However, they took some inspiration from other pairs, particularly Jeff & Dev, whose representation they drew in their journal. Session Four ended with a complete departure from the Pyramid, as they built a Rectangle similar to those of Jeff & Dev and Alex & Ranveer. When we reviewed their representations, Philip had no difficulty explaining the rectangle: "*The number we multiplied was three, so blue is three. Three multiplied by one … subtract two, which is the white and then you would … just check the purple [for the answer].*" The pair agreed that their Rectangle representation was "*more compact*" than the Visual Expression from their interview but did not select a favourite.

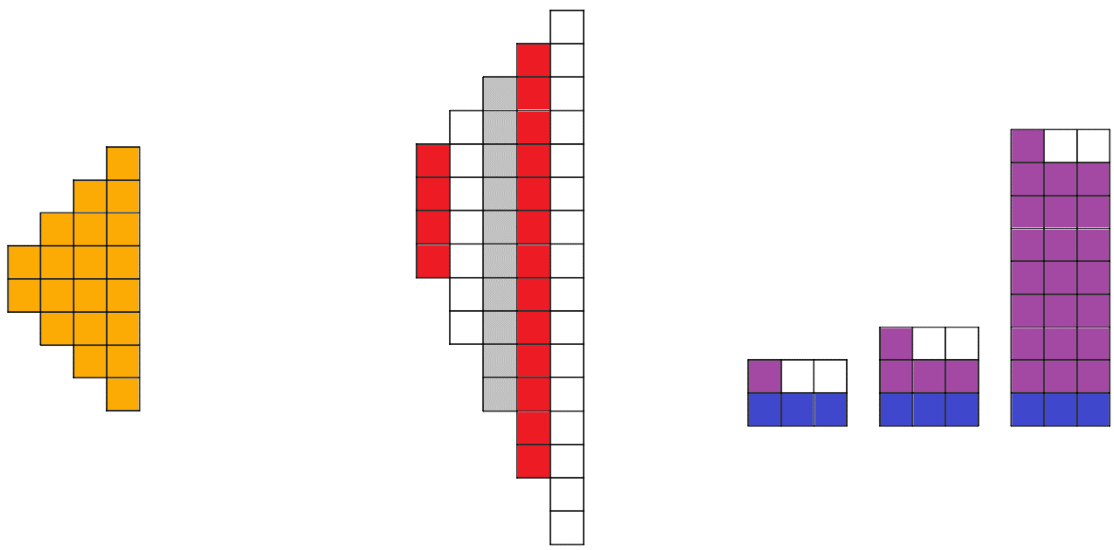


Figure 4.34: Illustrations of Philip & Kabelo's first three representations

4.2.4. Deon & Mike: The Divergents

As soon as I asked them to build a representation of $4x - 3$, Mike began building the Visual Expression shown in Figure 4.35 without any discussion or input from Deon. The visualisation is identical to their Session Four representation and Mike built the first three terms and the tenth term without any difficulty. "*It's like writing*," he said, "*but with the blocks*." Throughout the interview Deon deferred to Mike, usually waiting for him to answer questions, and often shaking his head, exclaiming softly as though he believed he could not adequately respond, even when he was asked directly.

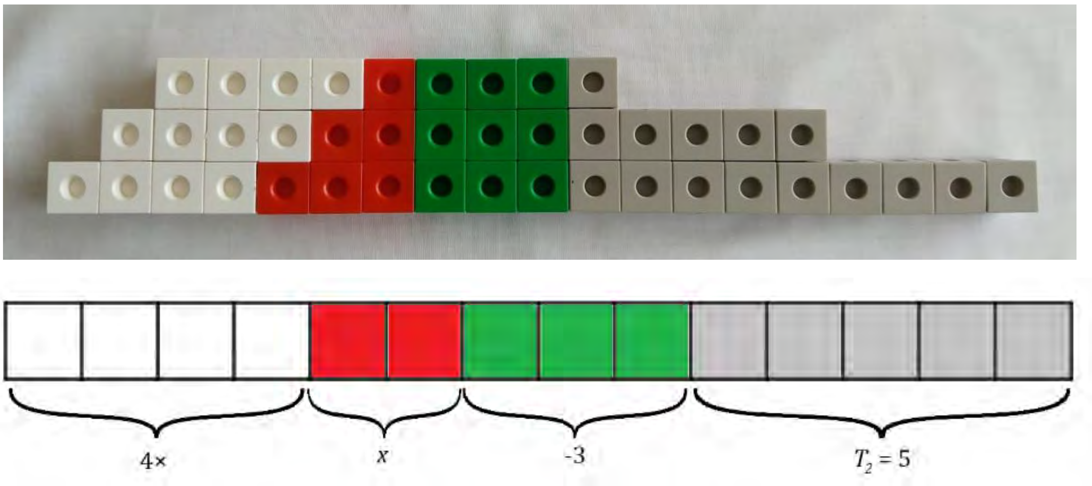


Figure 4.35: Deon & Mike's interview representation, showing T_1, T_2 and T_3 (Top) and an explanation of T_2 (Bottom)

Deon and Mike's representations did show some progress over the course of the study. In the first two sessions, Mike was focused on doing something that was different from everyone else and moved from a fairly straightforward representation in the first session to one that needed a very clear explanation in the second (Figure 4.37). Deon, on the other hand, was happy to stick with the same representation from Session One to Two, feeling it was straightforward and clear (Figure 4.36). When they began to work together (Figure 4.38) they seem to have combined their differing perspectives, as their final visualisation was essentially the same as Deon's first two representations with a Visual Expression added, combining Deon's clarity with Mike's ambitious ideas.

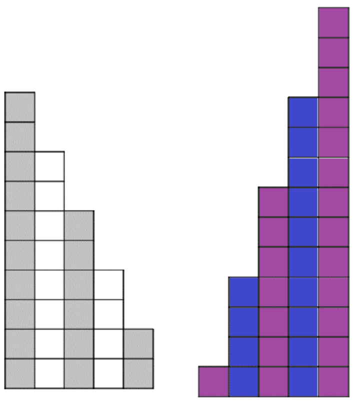


Figure 4.36: Deon's first two representations

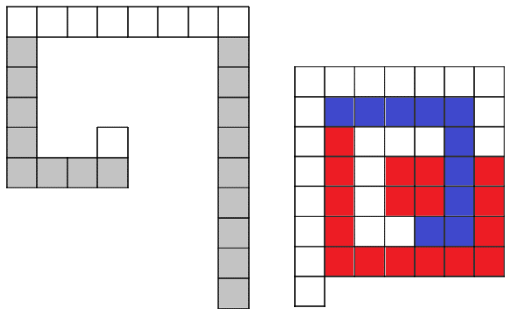


Figure 4.37: Mike's first two representations

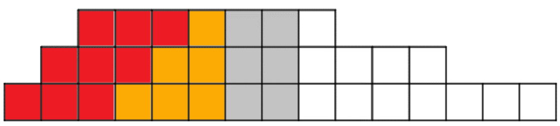


Figure 4.38: Deon & Mike's Session Four representation

4.2.5. Sam & Loyiso: The Innovators

Sam and Loyiso built separate representations during the interview rather than working together. They both settled on Visual Expressions without any deliberation. Initially, Sam left out the value of the constant term and Loyiso left out the value of the 'answer', but after I asked them to describe their visualisations they each added an extra column to create their final representations, shown in Figure 4.39. Sam explained that "*the brown and the white next to each other represent multiplication*", reflecting the algebraic convention that $ab = a \times b$. Moreover, he chose a horizontal orientation for the blue blocks (representing -3) to reflect the shape of the subtraction sign. At first, he said that he would not be able to represent addition in the same way unless he was adding five and could thus build a $+$ sign. However, over the course of the interview, as we discussed ways in which we could represent different expressions, such as $-2x + 4$. Sam decided that he could consistently use horizontal rows to represent negative values and vertical columns to represent positive values, demonstrating with various terms and expressions. He also suggested the use of one block to represent many in combatting the issue of large, fast-growing visualisations. This pair was the most creative in their suggestions for representing other types of expression (including quadratic polynomials) with their Visual Expressions.

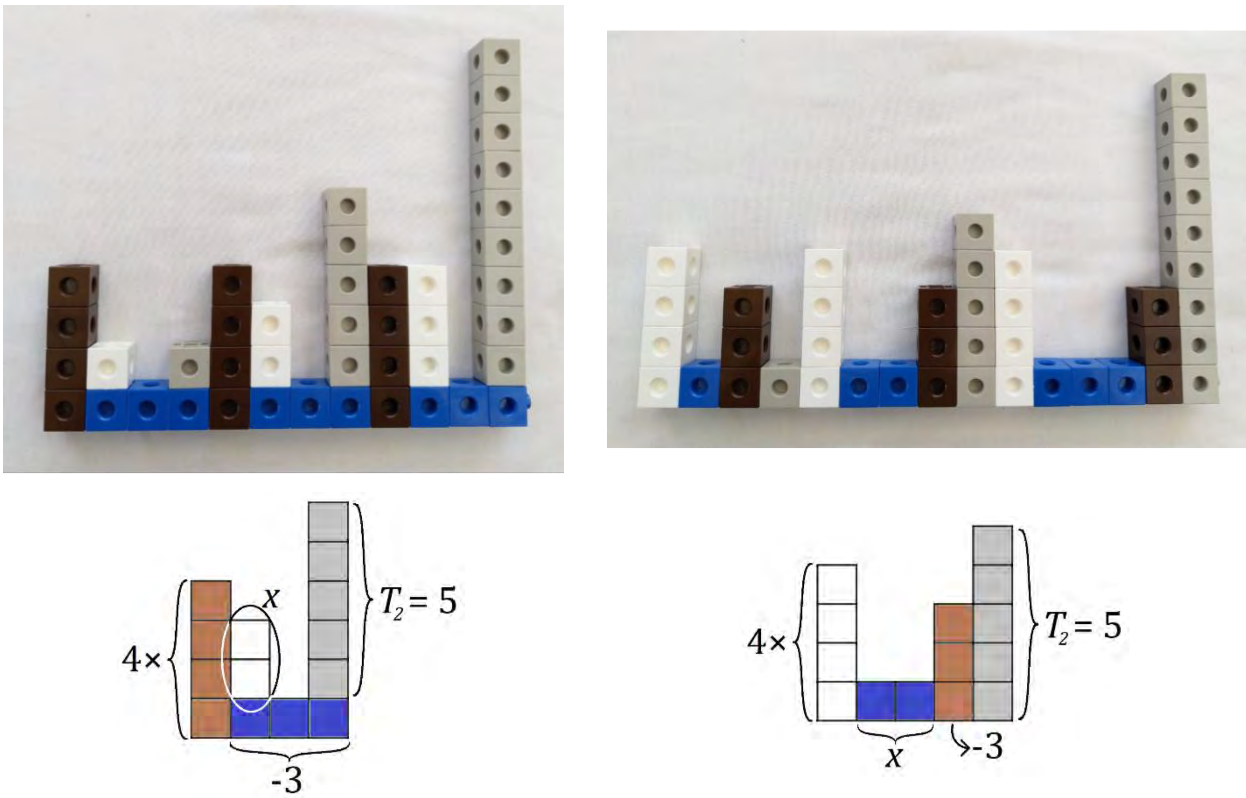


Figure 4.39: Sam (L) and Loyiso's (R) interview representations with explanations of each T_2 below.

Sam and Loyiso's representations and ideas evolved a great deal from Session One to the interview (Figure 4.40), even changing during the interview itself. The pair at first struggled to make sense of the Pyramid when we reviewed their representations, only figuring it out after I told them that it represented $2x$. There was no clear progression from Session One to Session Two, during which Loyiso was absent and Sam built a less regular Pyramid/Spiral that did not show the differences between terms as well as their first representation had done. Sam explained that the first term showed $x = 1$ in grey and $\times 3$ in red, but that this was not intentional at the time of building, but an interpretation he made later due to the Visual Expressions he had been building between Session Two and the interview. Whether or not Session Two contained the beginnings of the Visual Expression, in Session Three the pair built the first, clearest Visual Expression of the group, showing the values of d, x and c for each term but not including the value of T_x. In Session Four they built another Visual Expression, this time including the value of the 'answers'. Despite their initial difficulty understanding it during the review, Loyiso maintained at the end of the interview that this first representation was their best due to its simplicity. Sam agreed initially but later decided that he preferred their creation from Session Four, saying that it was "*clear how you got the answer*".

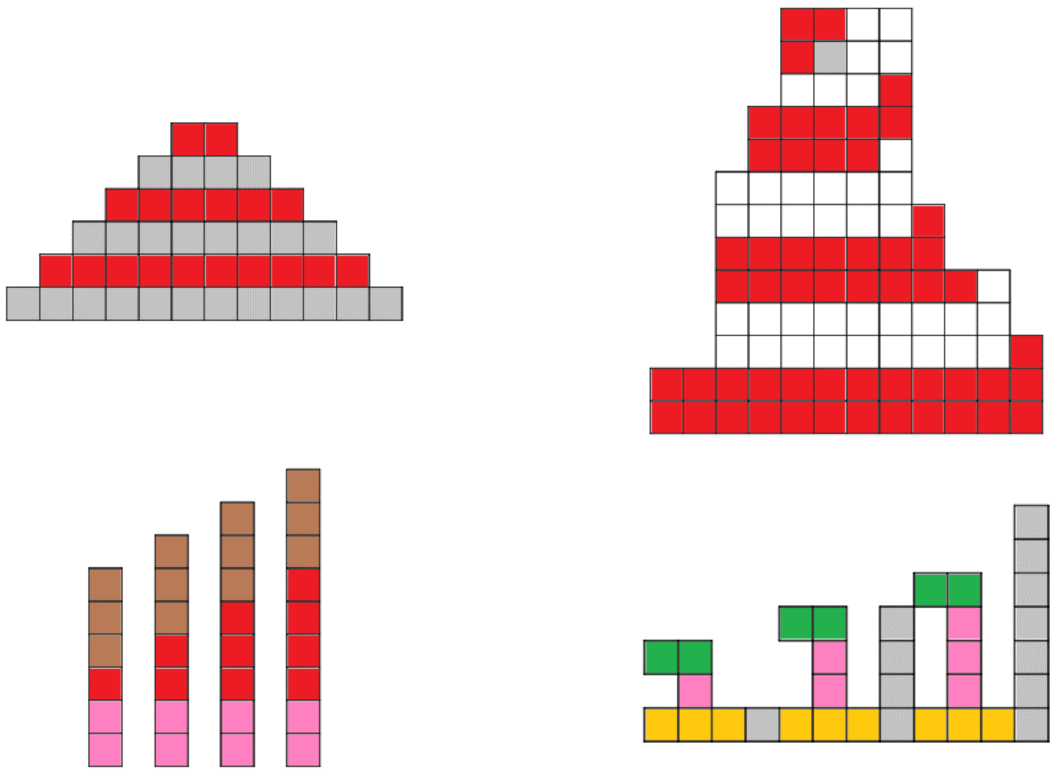


Figure 4.40: Sam & Loyiso's first four representations

4.2.6. Kylian & Warren: The Wildcards

Kylian and Warren were an unpredictable pair, never producing a consistent representation because of Kylian's insistence on pushing boundaries and Warren's common absenteeism. Although they built a friendship while they worked together on the study, they did not display much teamwork during the interview. After I asked them to build a representation of $4x - 3$, Warren requested paper so that he could work out the values of T_x with substitution. Kylian started building the snaky Visual Expression shown on the right in Figure 4.41. He used red to represent $\times\ 4$, blue to represent x, green to represent -3 and white to represent T_x. As Kylian had built the representation without any input from Warren, I asked if Warren wanted to change anything about it. He thought for a while before saying, "*This is really clever, actually,*" but decided to simplify the idea by grouping the terms into Towers (Figure 4.41, left) rather than Kylian's meandering design. Kylian protested that this was "*boring*" and we agreed to keep them as two separate representations rather than trying to merge them into something that would satisfy both parties.

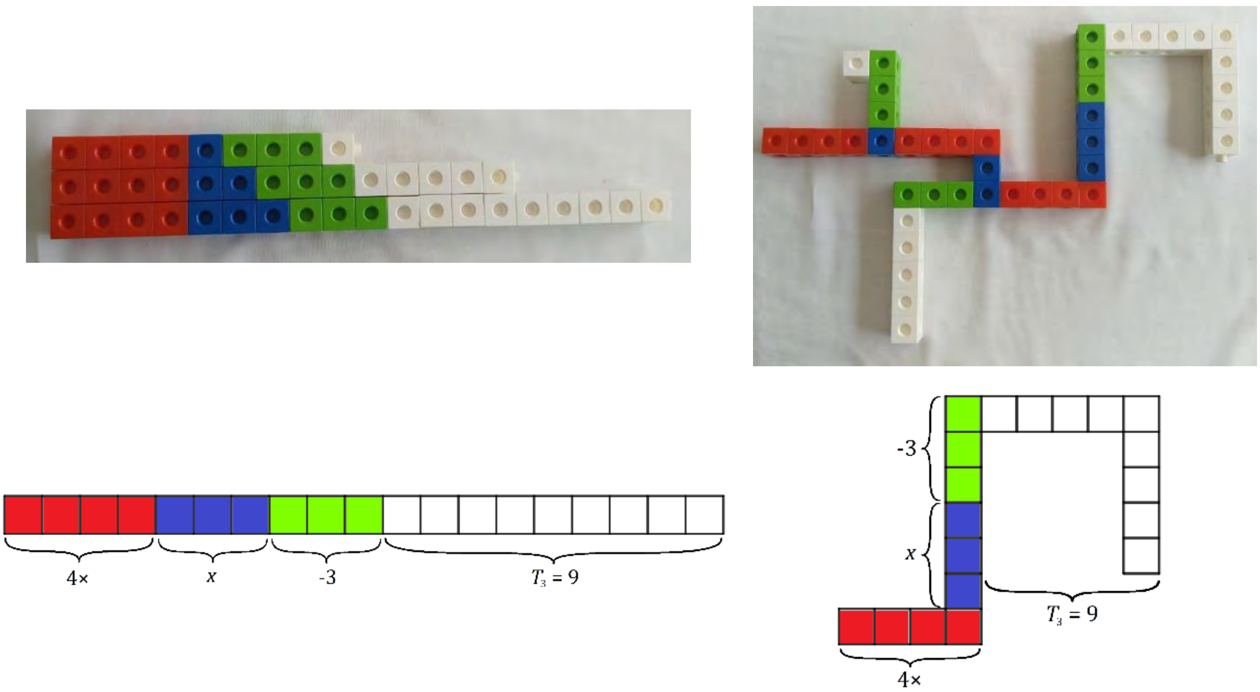


Figure 4.41: Warren (L) and Kylian's (R) interview representations with explanations of T_3 below.

Kylian and Warren's representations showed dramatic changes from Session One to the interview (Figure 4.42). In our review of their previous representations it became clear that Kylian had not understood what I expected of him in the first and even the second session. Of their first representation, Warren said, "*Kylian thought that you had to build a robot or something.*" Kylian himself struggled to explain the visualisations from weeks one and two which, due to Warren's lateness and absence, were almost entirely Kylian's creations. Their third representation, however, was logically consistent and showed some very interesting features, including the use of colour to indicate different parts of the expression (red and white showing x, black showing the coefficient of 2). They explained the photo without much difficulty, but after comparing the photo to their journal entry explained that they changed it after the photo was taken to include three blue blocks instead of one. This would have changed the total number of blocks in each term so that they no longer displayed the value of T_x, and showed a move towards the Visual Expression that favoured showing each part of the expression over showing how the final value of T_x was reached. Their fourth representation, again built by Kylian alone, was complicated enough that upon studying the photo Warren joked, "*I think Kylian needs me,*" implying that he had a rationalising effect on the pair's outputs. However, Kylian was able to explain that it was a Visual Expression, describing the meaning of each colour and showing the growth of each term from the last.

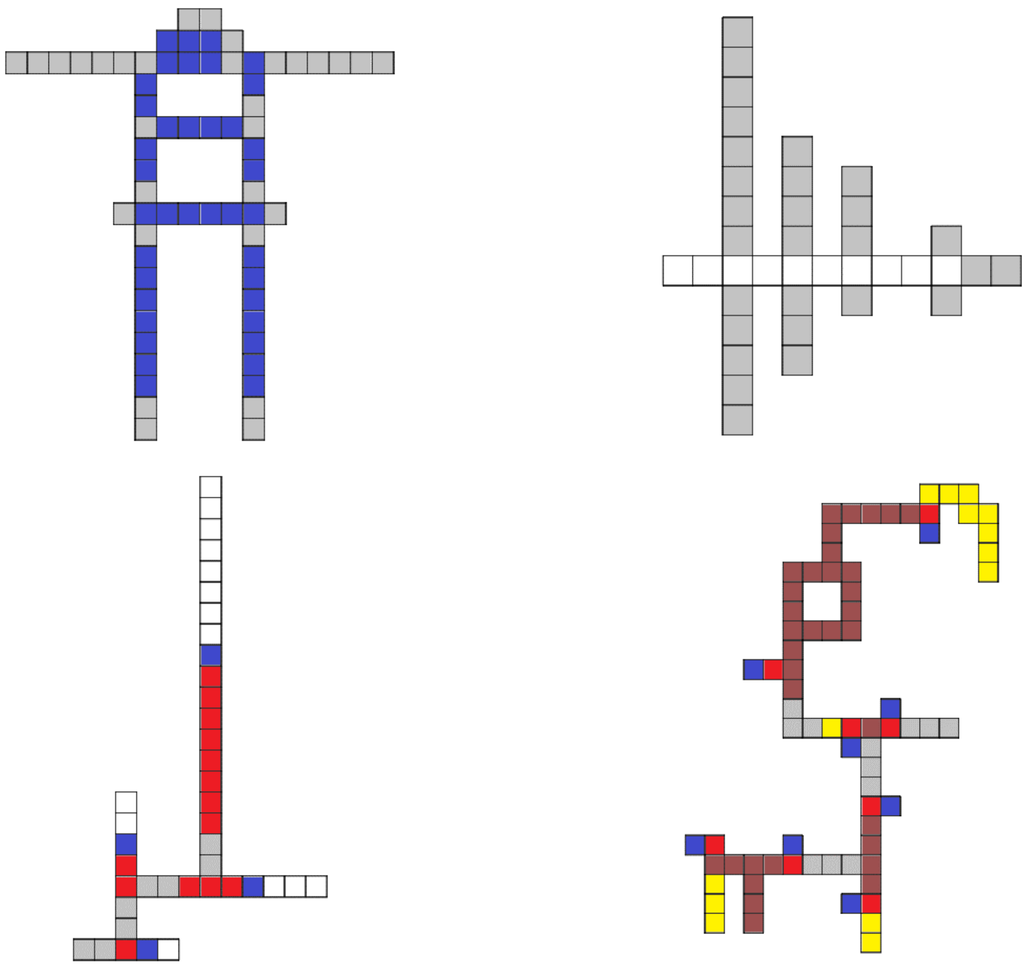


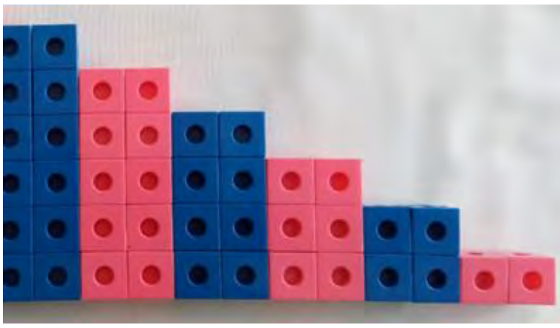
Figure 4.42: Kylian and Warren's first four representations

4.2.7. Common Types of Representation

Pictures and descriptions of the most common representations are given in the table below. As a reminder, general linear expressions are referred to with the notation $T_x = dx + c$, where:

- T_x is the *output* value of the term,
- x is the *variable*, showing the input value of the term (that is, the position of the term within the sequence),
- d is the coefficient or multiplier (equal to the *difference* between terms) and
- c is the *constant term*.

Table 4.1: Descriptions of the most common visual representations

	**Towers** The values of T_1, T_2, T_3 ... are arranged in columns, aligned at the bottom. This visualisation was the most common one in Sessions One and Two. The example here represents $2x$.
	**Pyramid** The values of T_1, T_2, T_3 ... are arranged in rows (or columns), aligned at the centre. This visualisation was the second most common one in Sessions One and Two. The example here represents $2x$.
	**Rectangles** The values of T_1, T_2, T_3 ... are arranged in rectangles with dimensions $d \times x$. One pair produced this kind of visualisation in Session One and three pairs in Session Four. The example here represents $2x$.

	**Visual Expression** The values of d, x and c are shown using different colours. The values of T_1, T_2, T_3 … may or may not be included. Three pairs produced this type of visualisation in Session Three and by the time of the interviews it had been adopted by all but one of the pairs. The example here represents $2x + 3$.

4.3 Representing Aspects of Linear Expressions

The first research question that this study seeks to answer is, "What aspects of linear algebraic expressions were participants able to represent visually?" Seven features of these expressions were identified as being at least occasionally present in the visualisations created by the participants. Each feature is described below, along with the criteria used to judge its presence or absence in any given representation.

4.3.1. Constant Difference Between Consecutive Terms

A linear expression is defined by a constant 'gradient' – that is, given a collection of input values with a consistent difference between them (for example, the natural numbers 1; 2; 3; … which have a consistent difference of 1), the corresponding output values of the expression will also have a consistent difference. They are named 'linear' expressions because this constant gradient means that any linear expression will form a straight line when represented on a Cartesian plane. In the context of a linear sequence, there will be a 'constant' (unchanging) difference between consecutive terms. Consider the sequence $2; 5; 8; 11; \ldots$ represented algebraically by $T_x = 3x - 1$. Each term is three greater than the preceding term. This defining feature was the first aspect considered when classifying the various visual representations created by the participants. It was judged to be present when consecutive terms were adjacent to one another and the placement of new blocks was consistent, such as in the various Towers and Pyramid representations. In Alex & Ranveer's third representation the terms were not adjacent but the increase in each term was shown explicitly with the use of a different colour. The difference between terms was considered not to have been shown when terms were not adjacent or when increases were not added consistently, as was always the case with Kylian's representations.

4.3.2. Increasing Nature of the Expression

Linear expressions can be classified as respectively increasing or decreasing, depending on whether the gradient is positive or negative. All of the expressions that the participants were asked to represent were increasing expressions, although decreasing expressions were discussed during the interviews. The increasing nature of the expression was judged to have been shown when each term was clearly defined and it was easy to see that each consecutive term was larger than the last. On a Cartesian plane consecutive numbers are always read from left to right. Some pairs adhered to this convention and oriented their representations accordingly, but most pairs did not consider orientation particularly important, and so it was not used as a deciding factor in classification.

4.3.3. Output Value of Each Term

When a linear sequence is represented as a list, the numbers in the list are the output values of the related algebraic expression: $T_1; T_2; T_3; \dots$ where $T_x = dx + c$ and x is a natural number. Considering the fact that the activity was introduced by explicitly discussing just such a list, it is not surprising that this was the most common of all the features represented visually. The only pair that purposefully omitted the term values was Sam & Loyiso in their first Visual Expression (in Session Three) and initially also in the their interview representations. The total value of each term, T_x, was judged to have been represented if each term included T_x blocks, either in total or in a particular colour.

4.3.4. Value of x in Each Term

As discussed above, for the purposes of the activity we essentially considered the algebraic expression synonymous with its related sequence. For this reason, after an initial discussion, the value of the variable was assumed to be one of the natural numbers. Initially all of the representations used $x = \{1; 2; 3; 4; \dots\}$ and left the value of these inputs to assumption. In the second session, Jeff & Dev began the trend of including blocks in a different colour to indicate the value of x in any given term. This became particularly useful and more widespread when the participants were asked to build the eighth term in Session Three. The value of x was judged to be present when it was explicitly shown in each term with blocks in a separate row or colour, or when pointed out as being shown by one side of a rectangle.

4.3.5. Value of Coefficient and Constant Term

As with the value of x, the value of the coefficient and the constant term in the linear expression were only purposefully included in later visualisations, particularly the Visual Expression. They were almost always presented with blocks in a separate row and/or colour, except in Ranveer & Alex's fourth representation, which showed the coefficient as one side of a rectangle and the (negative) constant term with two 'missing' blocks.

4.3.6. Concept of Multiplication

Multiplication is a shortcut for repeated addition, and this can be conceptualised in different ways. The fact that the constant difference between consecutive terms of a linear sequence is equal to the coefficient of x in the equivalent expression is no coincidence – it is a consequence of the relationship between adding and multiplying. The repeated addition of d to each subsequent term can be 'summarised' in the general term as multiplication of d by x. This can be seen in Ranveer & Alex's first Rectangles visualisation (Figure 4.43). It was built with the sole intention of increasing the total of each subsequent term by two, but inadvertently showed the value of x in the height and the value of d in the width of each rectangle. With some encouragement they returned to this representation in Session Four, this time purposefully using the dimensions of the rectangles to show the values that are multiplied in the algebraic expression before removing blocks to represent subtraction.

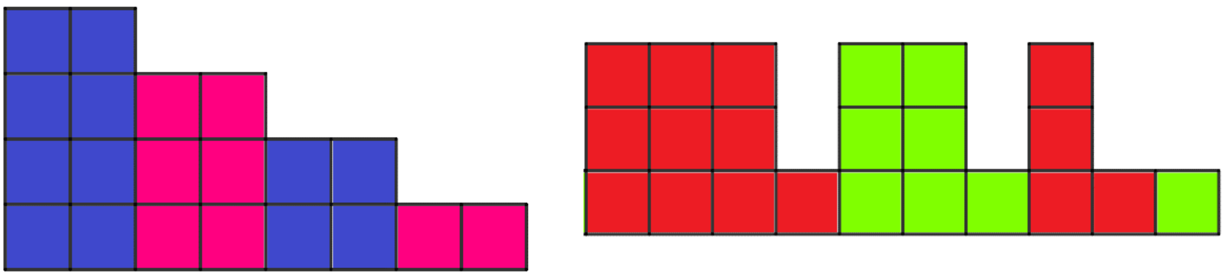


Figure 4.43: Ranveer & Alex's first (L) and fourth (R) representations, showing the use of area to represent multiplication

Jeff & Dev and Philip & Kabelo, also with some guidance, built similar rectangles in Session Four. However, as none of these pairs built rectangles in their interviews it seems that representing multiplication in this way in Session Four did not make an impact on how they thought about the problem of representing linear expressions visually. When I did not encourage them towards creating Rectangles they mostly gravitated to the idea of the Visual Expression.

Kylian & Warren showed the conceptual idea of multiplication in Session Three by representing x with both red and white blocks (Figure 4.44). This showed multiplication of x by two by including two groups of x in each term. However, as with Ranveer & Alex's first representation, this aspect of the representation seems to have been a by-product rather than an intentional feature, since they were surprised when I pointed it out and did not include it in any further visualisations.

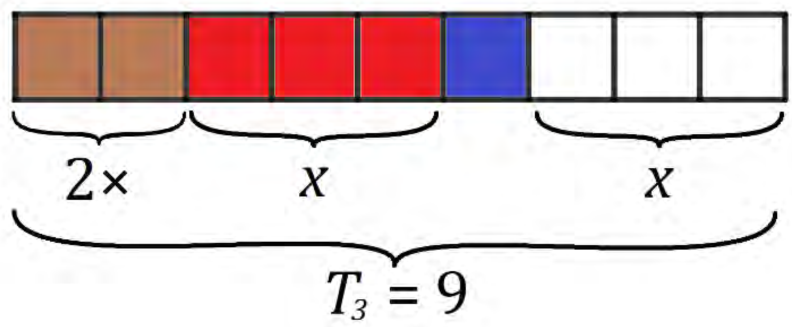


Figure 4.44: Term 3 of Kylian & Warren's Session Three representation

This aspect was judged to be present when the participants purposefully built their representations to include it, even when they were guided towards that kind of representation. If it was present 'accidentally' it was classified as present but not intended.

4.3.7. Table of Representations

The table below summarises the aspects of linear expressions displayed in each visualisation. A tick indicates that the feature was judged to be present. In the case of the first representation, the expression illustrated ($2x$) did not include a constant term – so this feature is indicated as 'not applicable' (n/a) in the table. Finally, there were cases in which a feature was included without the clear intention of the participants, indicated in the table with a 'p'. The totals indicated at the bottom include every representation, even those that were variations within a pair. (For example, Sam and Loyiso's very similar interview visualisations were counted separately because that is how they were presented in the interview.) The totals also include the occasions where aspects were present ('p') without the explicit intention of the creators.

Table 4.2: Aspects of linear expressions represented in each visualisation

	Visualisation from session:	Constant diff. bet. consecutive terms	Increasing nature of expression	Total value of each term	Value of *x* in each term	Value of coefficient (multiplier)	Value of constant term (additive)	Concept of Multiplication
Jeff & Dev	1	✓	✓	✓			n/a	
	2	✓	✓	✓	✓			
	3		✓	✓	✓	✓	✓	p
	4	✓	✓	✓	✓	✓	✓	✓
	Interview		✓	✓	✓	✓	✓	
Ranveer & Alex	1	✓	✓	✓	p	p	n/a	p
	2		✓	✓				
	3	✓	✓	✓			✓	
	4	✓	✓	✓	✓	✓	✓	✓
	Int Alex		✓	✓			✓	
	Int Ranveer	✓	✓	✓	✓			
Deon & Mike	1 Deon	✓	✓	✓			n/a	
	1 Mike		✓	✓				
	2 Deon	✓	✓	✓				
	2 Mike		✓	✓				
	4	✓	✓	✓	✓	✓	✓	
	Interview	✓	✓	✓	✓	✓	✓	
Philip & Kabelo	1	✓	✓	✓			n/a	
	2	✓	✓	✓				
	4	✓	✓	✓	✓	✓	✓	✓
	Interview		✓	✓	✓	✓	✓	
Sam & Loyiso	1	✓	✓	✓			n/a	
	2 Sam		✓	✓				
	3		✓		✓	✓	✓	
	4		✓	✓	✓	✓	✓	
	Int Sam		✓	✓	✓	✓	✓	
	Int Loyiso		✓	✓	✓	✓	✓	
Kylian & Warren	1			✓			n/a	
	2 Kylian		✓					
	3			✓	✓	✓		p
	4 Kylian			✓	✓	✓	✓	
	Int Kylian			✓	✓	✓	✓	
	Int Warren	✓	✓	✓	✓	✓	✓	
Totals	33 representations	16/33	29/33	31/33	19/33	17/33	17/27	6/33

4.3.8. Table and Graphs of the Number of Features Represented

The table below records how many of each identified feature were present in each session. This information is then displayed in line graphs that show the general trends in representation of each feature. In each line graph the number of features present in a session's representations is shown as a fraction of the total number of representations created in that session.

Table 4.3: Number of features of linear expressions represented in each session

Session	**1**	**2**	**3**	**4**	**Interview**	**Total**	**%**
Number of Representations Created	**7**	**7**	**4**	**6**	**9**	**33**	
Constant diff. bet. consecutive terms	5	3	1	4	3	16	48%
Increasing nature of expression	6	7	3	5	8	29	88%
Total value of each term	7	6	3	6	9	31	94%
Value of x in each term	1	1	3	6	8	19	58%
Value of coefficient (multiplier)	1	0	3	6	7	17	52%
Value of constant term (additive)	n/a	0	3	6	8	17 (/26)	65%
Concept of Multiplication	1	0	2	3	0	6	18%
Fraction of total possible features represented	$\frac{21}{42}$	$\frac{17}{49}$	$\frac{18}{28}$	$\frac{36}{42}$	$\frac{43}{63}$		
Percentage of total possible features represented	**50%**	**35%**	**64%**	**86%**	**68%**		

Note that the total number of possible features is always the number of representations created multiplied by the number of features being counted (this is six in Session One and seven in all subsequent sessions).

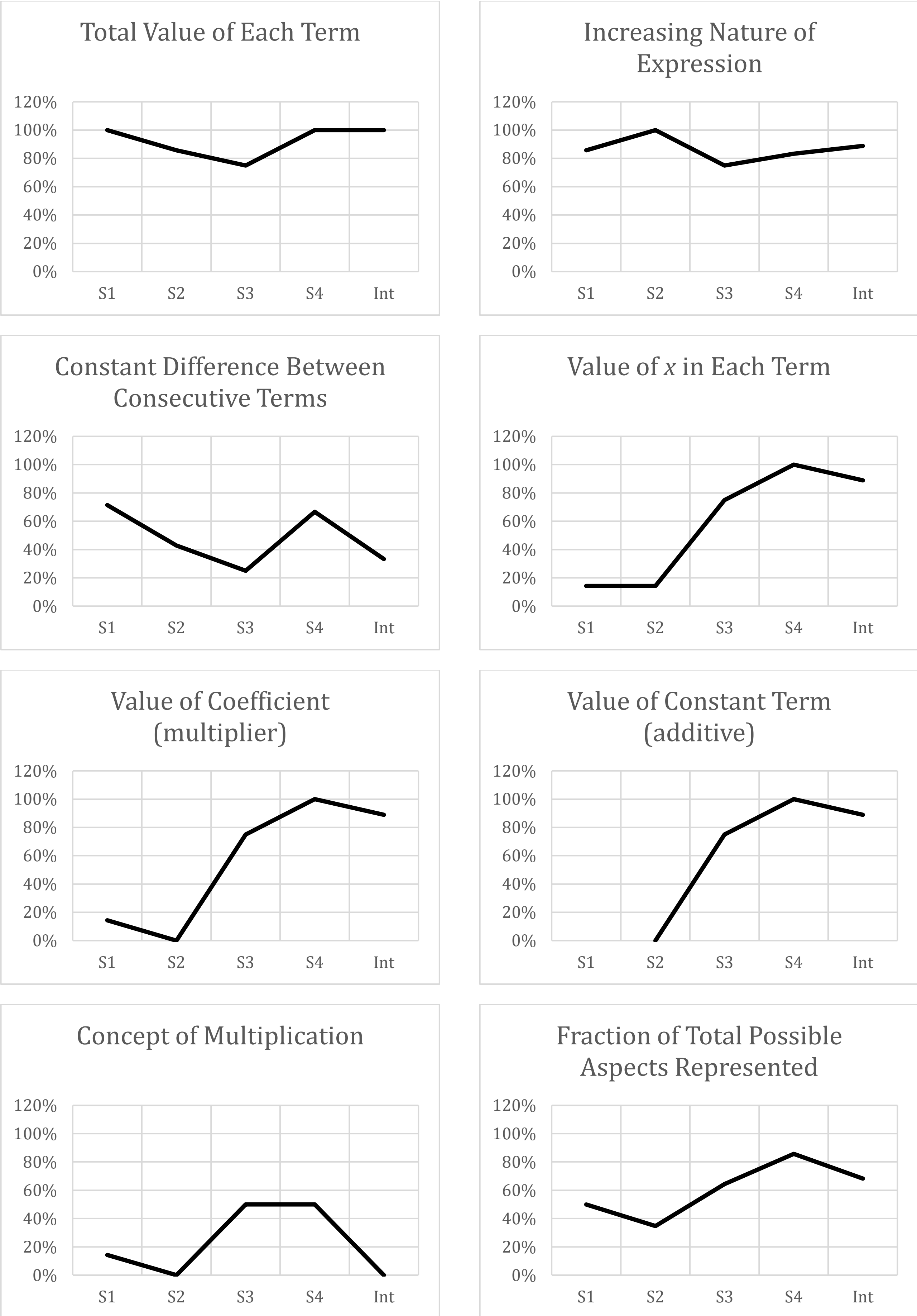


Figure 4.45: Graphs showing general trends in the representation of each feature

4.3.9. Summary of Represented Features

As seen in the tables (2.2 and 2.3) and graphs (Figure 4.45) in Sections 4.3.7 and 4.3.8, the participants represented many aspects of linear expressions in their visualisations. By far, the most commonly represented aspects were the total value of each term (present in 94% of the total 33 visualisations created) and the increasing nature of the expressions (present in 88% of the visualisations). These two features were represented consistently over the five sessions.

The constant difference between terms (48%), the value of x in each term (58%), and the values of the coefficient (52%) and constant term (65%) were all present in around half of the visualisations. The constant difference was represented less often over time as participants moved from stacking terms next to each other to displaying more features and thus distancing consecutive terms from each other. The values of the variable (x), the coefficient and the constant term were represented more often towards the end of the study when participants had spent more time thinking about the task and learning from each other's creations.

Finally, representations of the concept of multiplication (as repeated addition, an array or an area) were few and far between, occurring in only 18% of the participants' visualisations. Inclusion of this feature increased in Sessions Three and Four but fell away completely during the interviews, showing that it was not an aspect of the expressions that made a lasting impression on the participants.

The fraction of total possible features represented in each session ranged from a minimum of 17 out of 49 (that is, 35%) in Session Two, to a maximum of 36 out of 42 (86%) in Session Four. This shows that the number of features represented and the complexity of these features increased over the course of the study. These changes will be discussed further in answer to the second research question.

4.4 Development of Visualisations

The second research question that this study seeks to answer is, "In what ways did the visualisations develop over the course of the club, both within and between groups?" Each pair showed significant development of their visualisations from the first session through to the interview. Some, particularly Jeff & Dev, progressed in a very linear fashion, building on and adding to their previous work to create something more complex in each session. Others, like Ranveer & Alex, did something completely new every week unless specifically encouraged to return to a previous idea.

The focus of this section will be on how the participants' representations changed and, more subjectively, what these changes suggest about the processes of objectification that took place during the club. Some of these developments can already be seen in Figure 4.45, which detailed the changes in the number of features represented over the course of the study. Most of the notable developments in the visualisations were the direct result of an observation or challenge raised by the facilitators, encouraging the participants to *notice* aspects of sequences, algebraic expressions or visual conventions that they had yet to acknowledge. Radford describes a Grade 2 student who did not yet realise that "the spatiality of the terms [in a given visual sequence] provides us with clues that are interesting from an algebraic viewpoint" (Radford, 2013, p. 25). Similarly, the participants in this study started with a focus on the numerical output of the given expression and sequence. Over the course of the study they noticed and incorporated new aspects of this type of mathematical object into their visualisations. Because of these changes in their representations I will argue for which aspects of algebraic expressions the participants *learned to see* and which aspects remained beyond their conscious grasp by the end of the study.

4.4.1. Use of Colour

The first common change in the visualisations was seen in most participants' use of colour, from the entire representation being built in a single colour (or the change in colour only happening in response to running out of the first colour) to the use of different colours in distinguishing terms and eventually different colours representing specific parts of the algebraic expression. This change began as early as the second session, after the usefulness of colour as a tool in differentiating terms was pointed out at the end of the first session.

4.4.2. Representing Large Terms and Moving from T_x to $dx + c$

The most notable shift in the visualisations was a change of focus from a representation of the value of the outputs alone to a representation of the entire algebraic expression. Initially, and unsurprisingly given our focus on interpreting linear algebraic expressions as sequences, every pair represented the expression with a visual 'list': Towers or other groups of blocks numbering T_1, T_2, T_3 and so on. Jeff & Dev were the first to include another aspect of the expression, showing the value of x in the second session. The third session marked a particular turning point in this regard, prompted by the suggestion that participants try to create an eighth term that could stand on its own in representing the expression. With their focus changed from representing the first few terms to a single, larger term many of the participants found their previous strategies lacking and worked to change or create an entirely new visualisation. Jeff & Dev and Sam & Loyiso created the first full Visual Expressions in this session. While Jeff & Dev set their representation up as a set of equations showing $2x + 3 = T_x$ (complete with two red blocks to symbolise the equals sign), Sam & Loyiso parted entirely with the idea of including the output in their visualisation and only showed $2x + 3$ (see Figure 4.46).

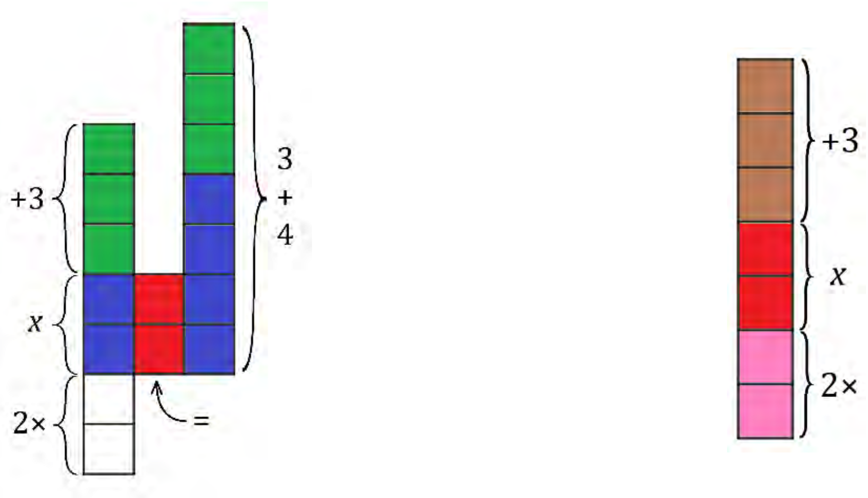


Figure 4.46: The second terms of Jeff & Dev (L) and Sam & Loyiso's (R) first Visual Expressions

This change indicated a different way of approaching the problem. When the participants were made aware of the fact that a single, long tower of blocks (for example) would not tell them much about the algebra that gave rise to the term without a comparison to the surrounding terms, they realised that they could do more with the given medium than they had thought possible. The Visual Expression became the most popular of any representation and all but one pair produced a variation of this visualisation in the interviews. There are several possible reasons behind this wide adoption of a single idea. First, it easily met all of the challenges posed by the problem of representing a distant term and of moving back to the algebraic expression if given just one term of the visualisation. Because of this, it satisfied the demands of all the questions I posed to prompt further development during the club sessions. Second, the participants were already fairly fluent in their use of basic algebra and aware of conventions like "$3x$" meaning 'multiply 3 by x'. Because of this, once they had come up with or seen an example of the Visual Expression it was easy for them to conceptualise and adapt to any given expression. During the interview, Jeff expressly changed the order of the colours representing '4 ×' and 'x' in order to adhere to the algebraic convention of writing a variable and coefficient with the number first and the letter second. Sam also built his interview representation with visual references to written algebra, particularly the placement of three blocks horizontally to represent the '−' in '−3'.

4.4.3. Use of Area and Grouping

Addition is not a particularly interesting operation to visualise. To show the result of adding $2 + 5$ one can simply put two and five blocks together to make seven blocks in total. Apart from possibilities in using colour or representing inputs and outputs as in the Visual Expression discussed above, there is not a huge variation in possible visualisations. Multiplication, however, is defined as repeated addition, which can be conceptualised and thus visualised in different ways. The concept of multiplying $d \times x$ is often introduced to primary school students visually and kinaesthetically through the addition of d groups of x objects, as in Figure 4.47. Only Kylian & Warren, in Session Three, used what could be seen as grouping (Section 4.1.3), although conversation revealed that this was not a purposeful feature of the visualisation.

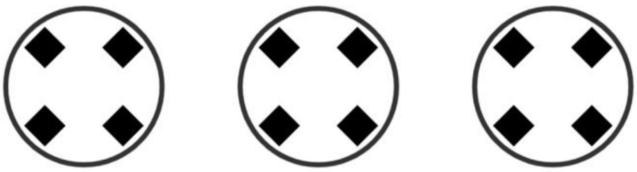


Figure 4.47: 3×4 *represented as three groups of four squares.*

Multiplication as grouping can be taken a step further by arranging the groups into rows to form an array (or rectangle) with dimensions $d \times x$, as in Figure 4.48. An array is particularly useful in at least two ways:

- it shows the commutative property of multiplication, i.e. that $d \times x = x \times d$ (the array can be viewed as d rows of x objects or x columns of d objects) and
- it relates directly to the idea of area (the number of 1×1 squares filling a 2D space).

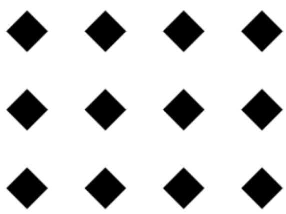


Figure 4.48: 3×4 *represented as an array of three rows of four squares or four columns of three squares.*

Different pairs of participants used Rectangles at various points in the study. Alex & Ranveer created $2 \times x$ Rectangles in the first session, though their only explicit intention was to do something different from other pairs. The next pair to use Rectangles was Jeff & Dev in Session Three, who simply wanted to compress the height of their eighth term. During the fourth session, with some encouragement, three pairs used Rectangles (Figure 4.49), this time explicitly to represent multiplication with area, as this idea was pointed out to them and discussed. However, the interviews made it clear that the usefulness of area in representing multiplication was not *recognised* by the participants. Philip & Kabelo attempted to recreate their Session Four visualisation during the interview, even choosing to work with the same colours, but could not remember the details and ended up creating a Visual Expression completely different from their creation in Session Four. Alex & Ranveer also seemed to attempt a recreation, each recalling different aspects of their previous representation, as discussed in Section 4.2.2, but neither returning to the use of area. Jeff & Dev moved away from area purposefully, opting for a more streamlined version of the Visual Expression, which Jeff called "*straightforward … like writing the equation down and you can see it exactly – what to put in the equation and the answer.*"

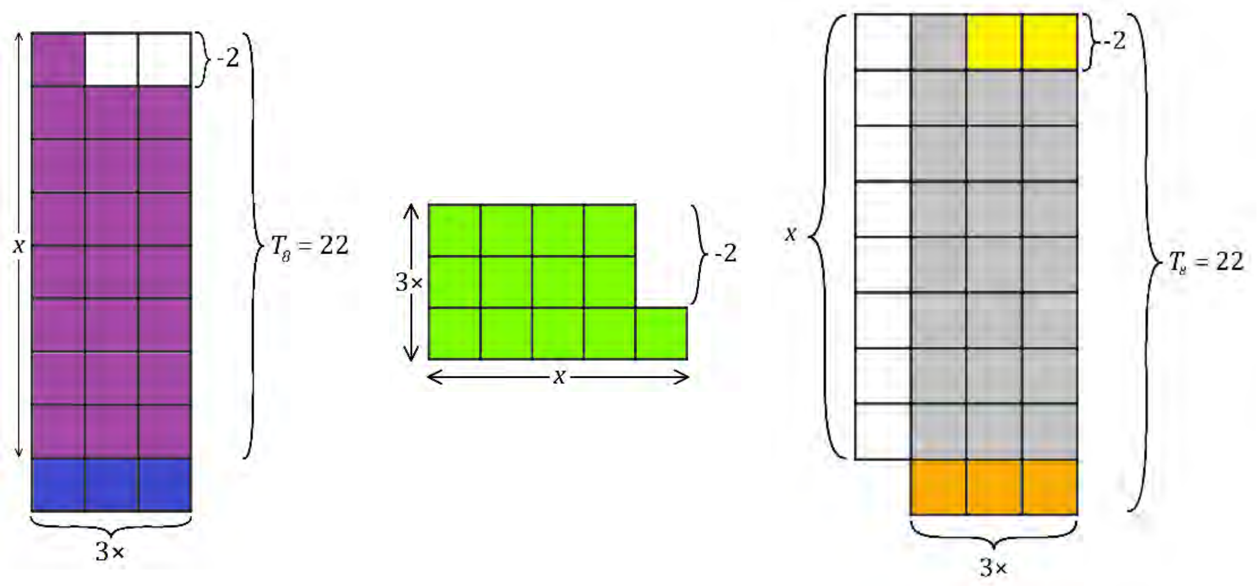


Figure 4.49: The three rectangular representations from Session Four

The challenge to represent a distant term was intended to push the participants into thinking about the meaning behind the algebra and thus, presumably, representations involving either groups or rectangles. However, the Visual Expression that dominated the interview representations indicated that, rather than grappling with the concrete definition of multiplication, participants remained attached to the abstract notion of lists of numbers that only had meaning in a particular context – given the rules attached to them by the community of the maths club.

4.5 Discussion

The summaries presented in the preceding sections make it clear that over the course of the study the participants' representations grew more sophisticated, including an increased number of aspects of linear expressions and an increased complexity in the concepts being conveyed. Moreover, as in the study by diSessa et al. (1991), the fact that the participants learned from and built on each other's ideas is evident in the fact that almost all of them ended up with a variation of the same representation at the end of the study. As Arcavi explains: "When a classroom is considered as a … community of practice, learning is no longer viewed only as instruction and exercising, but also becomes a form of participation in a disciplinary practice" (2003). Although we succeeded in many aspects of creating this kind of community within the maths club and there were definitely many aspects of "explorative participation" present in the club environment, elements of "ritual participation" (Heyd-Metzuyanim & Graven, 2016) remained. The most glaring difficulty was our inability to decentre the facilitators from the process of leaning mathematics. Despite hosting the club outside of school hours and never holding tests or other assessments, the participants showed a keen desire to gain the facilitators' approval throughout the club sessions and many pairs admitted to nerves before the interview because they worried that the task would be difficult or that they would "get it wrong". This demonstrated the participants' continued assumption that the teacher is an expert, even when the task at hand is the creation of something novel. As evidenced by Heyd-Metzuyanim & Graven, changing this mindset is complicated and would take extended immersion in an "educational environment that consistently highlighted other meta-rules for mathematical activity [than pleasing the teacher], meta-rules that highlight the value of describing the world mathematically as a goal in itself" (2016, p. 370).

Though it was not the focus of this study and was not measured in any quantifiable way, it is likely that working with visual, kinaesthetic patterns positively impacted the participants' ability to read visual cues. Most pairs were able to correctly interpret their own representations when they were reviewed several weeks after their creation and expressed an appreciation for some of the changes that they incorporated into their visualisations. Although they worked with much younger students, Hershkowitz & Markovits found that concepts such as “directions, colours and size relationships” (all of which were present in these representations) form “the basis for more advanced concepts … that serve as building blocks in scientific and mathematical thinking” (1992, p. 38).

Along with foregrounding these visual “building blocks”, the patterning activity engaged with signs and semiotics in several ways. First, the participants were required by the activity to interact with established registers and semiotic norms of the mathematical community, such as the practice of writing numerical coefficients before variables. The activity prompted conversations about these conventions, why they exist and whether they are necessary. By engaging with signs in this deeper way participants had to think explicitly about the components of the algebraic expressions they were trying to represent, rather than simply repeatedly performing algorithms that they may not have understood (Pegg & Redden, 1990). Second, the learners created new sign vehicles to represent algebraic expressions, eventually narrowing their various ideas down into versions of the Visual Expression. As discussed earlier, the fact that most of the participants chose essentially the same representation for their final product shows that during the activity they were not only engaging with standard mathematical notations but also learning to read and understand each other’s non-standard registers before incorporating their favourite ideas into their own creations. All of this work amounted to a large amount of register switching and by the time of the interviews many of the participants displayed a notable fluency in working with these various representations. Although merely showing an appreciation of and ability to translate between multiple representations of the same object is an inadequate marker of mathematical competence (Santi, 2011, p. 285), it does show some degree of competence beyond rote rule-following.

Representing expressions kinaesthetically also forced the participants to interact with algebraic expressions as objects in their own right, not merely representing a process (“multiply by two and then add one”) but also a product (Tall & Thomas, 1991). This semiotic interaction might have been extended by interacting more deeply with others’ representations as well as their own and not merely representing but also performing operations on linear expressions. For example, would Kylian and Warren have correctly interpreted Jeff and Dev’s Session Four Visual Expression? What would Sam and Loyiso have done if I had asked them to show me how to *add* $2x - 1$ and $3x - 2$ using their final representation? Further questions for suggested research will be discussed in the concluding chapter of the thesis.

4.6 Conclusion

Through the description of each club session and interview, this chapter explored the results of the associated study, providing an in-depth view of the participants and their responses to the tasks given to them in the maths club. The focus then moved to synthesis of the data, collating the aspects of linear expressions successfully represented in the learners' visualisations and discussing their development.

This development eventually resulted in all but one of the pairs of participants choosing the Visual Expression for their final representation. This seems to suggest three things: first, the learners were participating in a mathematical community in which they were happy to collaborate and learn from each other's creations. Second, they believed (either consciously or sub-consciously) that they were striving for a single correct representation, pre-conceived by the facilitators or, indeed, dictated by 'mathematics' itself. Third, as they were already relatively fluent in basic algebraic conventions, the participants chose, of all the options they created over the course of the study, the visual representation closest to what they knew to be the standard (and therefore the 'correct') means of representing this kind of relationship (that is, the algebra), without digging more deeply into the relationship itself and the meaning of multiplying an input value by a positive integer and then adding or subtracting another integer.

By the end of the study, in Radford's terms, the participants showed evidence of objectification processes, noticing and asking questions about things like the order of operations in a linear expression and the relationship between the constant difference in a sequence and the coefficient in an expression. It was clear, however, that objectification is not a linear process, as there were aspects of algebraic expressions that learners pointed out in one context but failed to see in another and features that they displayed in earlier club sessions but forgot about or chose not to use in the interviews. For example, Ranveer & Alex built Rectangles in the first club session and returned to them in the fourth session. We discussed the use of area in representing multiplication but it was not an idea that stayed with them in the weeks between Session Four and the interview, when they built different kinds of representations for their final task. Later in the interview, however, when we reviewed their first representation, Ranveer immediately pointed out the relationship between the width of the rectangles and the value of the coefficient in the expression. This was a concept that he was still in the process of fully understanding and was yet to fully objectify.

Chapter 5

Conclusion

This research focused on the use of visual representations in Grade 9 learners' engagement with linear algebraic expressions. Over the course of four club sessions and an interview, six pairs of participants were asked to represent various linear expressions, such as $2x$ and $3x - 2$, using colourful building blocks. Their responses to this task were creative, varied and almost always logically sound. This final chapter provides a summary of the document, a discussion of the study's findings, a note on the limitations inherent in the design of the research and a list of recommendations both for further research and for educators interested in using visualisation to help their learners with algebra. It concludes with a personal note on my growth as a researcher, scholar and a teacher.

5.1 Document Summary

This thesis began with an overview and introduction in Chapter 1, which briefly described the background of the study and the gap that it sought to fill, before laying out the research questions to be answered.

The background was expanded in Chapter 2, which examined the underlying literature, laying out the necessary definitions and closely exploring the field of visualisation in mathematics education, particularly with respect to its applications in teaching and learning algebra. The chapter concluded with a look at the theory of knowledge objectification that underpinned the study and explained why it was important, in the context of a South African public school, to run the study in a maths club rather than a classroom.

In Chapter 3 the document began to focus on the study itself, outlining the interpretive orientation, the choice of a qualitative case study and the means of selecting participants through non-probability convenience sampling. It then laid out the research design, which was split into the club sessions and the interviews. The final sections of the chapter outlined the analysis process and addressed the ethics, validity and reliability of the study.

Chapter 4 provided an in-depth description and discussion of the study's results and their analysis. Following the structure of the study itself, the chapter began with an account of each club session with an explanation of the various representations created every week. It then moved on to a description of each interview, which included a task and then a conversation about the pair's previous representations. The analysis of the data was laid out so as to answer the two research questions, and the implications of these answers were then discussed.

This chapter will further explore these implications before contextualising them in terms of the limitations of the study, the areas in which the research could be continued and the recommendations that can be made for mathematics educators based on these results.

5.2 Summary of Findings

The participants created a number of different, innovative ways in which to represent a linear algebraic expression with coloured building cubes. Importantly, almost all of the visualisations possessed a sustained internal logic that the participants were able to explain and on which they were able to expand. This showed that, regardless of any other conclusions that could be drawn, the learners were engaged in mathematical thought and activity throughout the course of the study. The findings are reviewed more closely below in a discussion of each of the two research questions.

5.2.1. Discussion of First Research Question

The first of the questions was, "What aspects of linear algebraic expressions were participants able to represent visually?" The participants represented the *total value of each term* and the *increasing nature of the expressions* in 88% to 94% of their representations. More subtle aspects, like the *constant difference between terms* and the values of the *variable*, *coefficient* and *constant term* were present in 48% to 63% of the visualisations, and often the frequency of their inclusion increased over time. Visual depictions of the *concept of multiplication* were only present in six out of 33 visualisations. The total number of features included showed an overall increase from the first session to the interviews and the general trend was a movement away from simple, list-like representations towards Visual Expressions that closely mimicked the algebra.

5.2.2. Discussion of Second Research Question

The popularity of these Visual Expressions towards the end of the study revealed a great deal about the mindset of the participants and their choices, which provides answers to the second research question: "In what ways did the visualisations develop over the course of the club, both within and between groups?" Each pair progressed from a list-like visualisation towards a final representation, in their own way. Jeff & Dev consistently added something new and inventive to their previous week's representation, becoming one of the first pairs to invent and then simplify the Visual Expression. Ranveer & Alex aimed to do something new and different in each session, until they were specifically asked to work on and adapt one of their previous creations. At this point they developed a very sophisticated and yet simple set of rectangles that they only partially remembered and yet definitely favoured during their interview. Philip & Kabelo worked in completely the opposite manner to Ranveer and Alex, in that they liked their first representation so much that they did not change it until they were specifically prompted to do so. Inspired by other groups' Visual Expressions, their final two products were a set of rectangles and then a Visual Expression. Deon & Mike worked separately in the first two sessions, but finally brought their spirals and towers together into a plain and easily-read Visual Expression. Sam & Loyiso were the originators of many new ideas, from one of the first Visual Expressions to the representation of negative numbers with horizontal lines and positive numbers with vertical lines. Kylian & Warren made the most 'progress' of the learners involved in the study, since they started with what seemed to be very little understanding of the task and moved towards a representation that, while convoluted, was a clear and legible Visual Expression.

Overall, as partly indicated in the answer to the first question, there were general changes in the move from visual lists of output values to more complex structures that showed multiple features of the expressions. The other clear change was a more purposeful use of colour, with many pairs in the first session only changing colours when they ran out of their first option compared to every pair using colour to indicate either alternating terms or different features of each term in their interview representations. A notable change that did not take place was any long-term adoption of a representation that used area or grouping to purposefully display the mechanics of multiplication. These choices suggest that the learners were learning from and collaborating with each other between groups, greatly influenced by a conscious or sub-conscious desire to 'get it right' and thus more interested in creating something similar to the algebraic expression than they were in more fully representing the relationship that lay beneath the algebra.

4.4 Limitations

As is evident in the discussion above, this research was limited by the fact that it was designed as a small, interpretive case study with participants chosen through non-probability convenience sampling rather than a large, quantitative study with measurable outcomes and randomly chosen participants. Because of this, the conclusions drawn cannot be generalised. The same study carried out in a different environment with different learners would almost certainly result in completely different results. However, such a study would form an interesting counterpoint to this one and the similarities and differences found would be highly informative.

4.5 Areas for further research

As implied above, even repeating this study exactly would further the research in this area, since different participants would undoubtedly produce different visual representations and a different kind of club community. However, there are also changes that could be made, small or large, that would lead to different and therefore valuable results. Something as minor as changing the material used to create the visualisations from building blocks to stones or bottle caps would change the nature of the representations created. Carrying out the study in a more privileged, less privileged, co-educational or all-girls school would provide different and interesting insights. However, a few suggestions for more expansive changes are detailed below.

First, a similar study could be carried out on a larger scale, with more participants. With some adjustments and a willing teacher, the tasks could be carried out over a single week in a more traditional classroom setting. This would make it easier for participants to edit their previous visualisations and build on their observations, as there would be far less time separating the sessions. However, in a typical South African classroom it might be even more difficult than we found it to move learners out of a teacher-centred mindset.

A second suggestion is for a study which uses the same materials and an aim for a final result that could also be used to represent linear algebraic expressions but employs different prompts along the way. One example would be to use a more traditional route and start by explicitly representing arithmetic operations like adding and multiplying before moving on to discuss how the visualisations created could be applied to algebraic expressions.

A study very similar to this one in other respects could be extended in at least two ways. First, participants could be challenged to engage more deeply with other groups' representations to further extend their fluency with the medium and to ensure a more collaborative environment. Would Jeff & Dev easily been able to read Philip & Kabelo's Rectangles from Session Four? A second possible extension would be continuing to ask the participants to engage with their visualisations in starting to perform operations with their expressions. What would it look like if we built $2x - 1$ and $3x + 2$ with the same visualisation and then added the expressions? What would it look like to subtract the expressions? When would it be possible to do this adding and when would it be impossible? These kinds of prompts would continue to build learners' fluency with their visualisations and would require them to engage further with the algebraic conventions around adding and subtracting like terms.

Finally, a study of any kind carried out in a similar mathematics club, particularly with learners in high school, would be a valuable contribution to our understanding of how these more creative and relaxed spaces might be used in decreasing mathematical trauma and helping learners realise that the true nature of mathematics is not found in drill and routine but in exploration, creativity and understanding. Similarly, a study that focused on South African teachers' experiences of participating in or running a maths club would provide valuable insight into their understanding of what mathematics is and how the ideal mathematical environment should be run.

4.6 Recommendations

Of all the things learned in this study, perhaps the most important is the fact that the learners involved showed a huge capacity for creativity in mathematics, even though they were habituated to teacher-centred routines and right-or-wrong answers. Their creativity was visible in the great number of different approaches they took over the course of the study. This creativity could be of use to learners even in the most analytical areas of mathematics – like algebra – and high school teachers, despite the limitations on their classroom time, would do well to make space for this kind of creativity in their lessons. For example, even asking learners to create mini-tests for their peers would require them to engage more deeply with the content and think creatively about different ways to ask about the concepts being reviewed. More specifically to this study, visual and kinaesthetic approaches to this kind of creativity, particularly in areas of study like algebra, are vital tools in the introduction and use of very abstract, convention-heavy concepts. Visual or metaphorical systems like sets of blocks are extremely useful in helping learners understand how the system works and why the conventions exist in their current form. Foregrounding the fact that mathematics was created by people and that conventions are agreements between mathematicians and not fundamental to the structure of the concepts, can help mathematics feel less alien and more accessible.

Another important lesson from this study is the fact that the learners involved were strongly inclined towards staying 'safe' in their creations, trying to be sure that they got the 'right' answer. This was visible in the fact that every pair except one ended up with a similar representation at the end of the study and that this representation was the closest of all of the options to the algebra that we were trying to represent. Many of the learners also expressed worries during the interview that they might get their representation 'wrong'. Convincing learners to take risks in their creativity takes time and continued exposure to an environment where they are expected to conform to different rules of engagement. The creativity expected of learners in the given tasks and the already established nature of the club as a relaxed space without tests where multiple answers were acceptable, helped to encourage engagement that was less rigid and more exploratory. However, the collaborative nature of the club may also have led to more conformity in trying to get to the 'ideal' solution. Club environments can go a long way to opening up learners' ownership of mathematical ideas and spaces but they are not magic. Moving from a teacher-centred to a learner-centred classroom environment takes time, patience and consistency.

4.7 A final word

I have been greatly enriched by the experience of running and writing about this study. I have always enjoyed algebra: the logical system of the syntax, the compact means of conveying a mathematical idea and the ability to compare seemingly disparate contexts and seeing if they are actually describing the same relationship. When I began to plan for a project focused on visualisation in mathematics education I found it difficult to connect with my natural inclinations in the subject. Once I decided to research whether I could design a study connecting visualisation to algebra, however, it became clear that the connections were vast and deep and already present, to a small degree, in my teaching. The process of working on the study with the participants further broke down my assumptions about whether algebra could be a visual subject.

When designing the study I hoped for creative responses from the participants, but many of their visualisations were far more interesting and innovative than I was expecting. These springs of imagination that emerged from a study of what might well be considered the most dry area of mathematics studied at a high school level showed me that I have sacrificed more than I realised when I prioritised speed over creativity in my classroom.

I have learned to turn a scholar's eye to my own teaching and have found that, having been taught in a very rigid, teacher-centred manner, I tend towards this in my own practice. It was more difficult than I had anticipated during the study to allow the learners space to take the lead in sessions, especially when they clearly expected me to lead them neatly to a single, correct answer. This has made me even more determined to break out of this habit in my regular lessons and encourage learners to explore the full landscape of mathematics rather than merely being led down a single, monotonous path to a predetermined destination.

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Appendices

Appendix A: Invitation & Consent Form to Guardians and Participants

SARChI MATHEMATICS EDUCATION CHAIR
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Date

Dear Parent/Guardian and «First_Name» «Last_Name»

«First_Name» «Last_Name» has been a part of the Grade 9 Maths Club run by myself and [redacted] since the beginning of the second term. We have been exploring mathematics topics such as general problem solving, trigonometry and probability from 13:30 to 14:30 every Tuesday afternoon.

The main purpose of this club is to help participants think about maths visually and more innovatively, in order to improve their understanding of important concepts that will be necessary for their success in Grade 10 and beyond. I am a student at Rhodes University working towards a masters degree in mathematics education and I would like this keen group of young mathematicians to partner with me in my research project. The title of my research project is: An analysis of visual representations of linear algebraic expressions created by selected Grade 9 learners: a case study in an after-school mathematics club. This research is intended to deepen the participants' understanding of algebra and to give me insight into any understanding they have gained during the project.

Here is the plan for my research:

- For four weeks, the club will focus its attention on algebra and sequences.
- The boys will work in pairs (as they usually do).

- The work they produce will be photographed. (These photographs will be used as data in the research.)
- Every week, each pair will complete a journal entry describing the work they did in the session. (These journals will be used as data in the research.)
- [redacted] and I will discuss the club session (as we usually do) to determine how successful it was and how it could be improved the following week.
- After the four weeks have been completed, I will arrange interviews with each pair of participants,
- scheduled at their convenience. (More information on the interviews is given below.)

Each participant will be referred to by a pseudonym in the final thesis and in any papers written on this research. However, it will not be possible to make participants completely anonymous, as their interview responses and copies of their work may be included in the final write-up.
During the interviews, the participants will complete mathematical tasks very similar to the ones we will have worked on in the club. They will explain their thinking as they work on the task, and I will ask them questions related to the task and the preceding club sessions. All questions will be related to mathematics and the participants' thought processes. Nothing of a personal nature will be asked or expected of the participants.

The interviews will be video recorded from above, so that the participants' faces are obscured but their work and interactions with each other can be observed. The videos will be transcribed to include both spoken words and descriptions of body language and actions. Along with the transcription, still photos from the video may be used as data. Any pictures used will be edited, if necessary, to ensure anonymity.

Participants may continue to participate fully in the club without contributing to the research. This would mean that they do all the tasks that the other boys do, but their work and journals will not be used for research and they will not be interviewed. Further, anyone who agrees to participate in the research will be allowed to stop participating at any time with no penalty. As before, participants may also leave the club at any time, as long as they inform either myself or [redacted].

The data collected during this project will be used in my MEd thesis and possibly in journal papers written about this research. After the research has been completed and the results have been analysed, I will provide participants with a short report summarising the outcomes of the research. Once the final MEd thesis has been completed I would be happy to provide you with an electronic (PDF) copy if you request one.

I hope that spending time working on club activities will improve participants' understanding of mathematics in general, which may positively impact their results on school tests and exams. However, there are no marks given for work done in the club, and participation in the club and/or research will not be directly related to their school work.

This research has been approved by the Rhodes University Ethical Standards Committee. For more information on the committee and Rhodes University's ethics policy you can visit www.ru.ac.za/researchgateway/ethics/. Should there be any unethical behaviour in the running of the project, please send comments or complaints to Mr Siyanda Manqele, Rhodes University's ethics coordinator, on 046 603 7727 or at s.manqele@ru.ac.za. Futher, permission to carry out this research at [redacted] has been obtained from [redacted] and from the Eastern Cape Department of Education. There will be an information session at time on date, which I would encourage you to attend. If you are not able to attend and have any questions, please feel free to email me at [redacted] or contact me on [redacted]. I would be happy to arrange a meeting at your convenience to further discuss this research.
Please discuss this with «First_Name» and together decide whether he will be participating in the research.

Then complete the attached form and bring it to the information session or return it to [redacted] by date.

Kind regards,
Sindisiwe Herbert

I, «First_Name» «Last_Name», agree to the following (please tick one):

☐ I will participate in the research described. My work may be photographed and collected to be used as data. I am willing to be interviewed and to have the interview video recorded and transcribed. Data collected (including still pictures taken from the video) may be included in the final thesis and any journal papers written on this research.

☐ I will not be participating in the research described. I will, however, continue to participate in the club, but will not have my work used as data and will not be interviewed.

Signed: ____

Date: ____

I, ______________________ ______________________, parent/guardian of «First_Name» «Last_Name», agree to the following (please tick one):

☐ «First_Name» may participate in the research described. His work may be photographed and collected to be used as data. He may be interviewed, and this interview may be video recorded and transcribed. Data collected (including still pictures taken from the video) may be included in the final thesis and any journal papers written on this research.

☐ «First_Name» will not be participating in the research described. He will continue to participate in the club, but his work will not be used as data and he will not be interviewed.

Signed: ____

Date: ____

Appendix B: Permission Letter & Form to School Principal

SARChI MATHEMATICS EDUCATION CHAIR
EDUCATION DEPARTMENT
19 Somerset Street, Grahamstown, 6139, South Africa
PO Box 94, Grahamstown, 6140, South Africa
t: +27 (0) 46 603 7211
f: +27 (0) 46 603 8084
www.ru.ac.za

Date

The Principal
[redacted]

Dear [redacted]

Permission to perform research with selected [redacted] learners

As you know, [redacted] and I began a Grade 9 Mathematics Club on Tuesday afternoons at the beginning of the second term. The intention of this club is to help participants think about maths visually and more innovatively, in order to improve their understanding of important concepts that will be necessary for their mathematical success in Grade 10 and beyond. As we have discussed in person before, I would like this keen group of young mathematicians to partner with me in the research I am doing in order to complete my masters in mathematics education at Rhodes University. The title of my research project is: *An analysis of visual representations of linear algebraic expressions created by selected Grade 9 learners: a case study in an after-school mathematics club.* The research is intended to deepen the participants' understanding of algebra and to give me insight into any understanding participants have gained during the project.

Here is the plan for my research:

- For four weeks, the club will focus its attention on algebra and sequences.

- The boys will work in pairs (as they usually do).
- The work they produce will be photographed. (These photographs will be used as data in the research.)
- Every week, each pair will complete a journal entry describing the work they did in the session. (These journals will be used as data in the research.)
- [redacted] and I will discuss the club session (as we usually do) to determine how successful it was and how it could be improved the following week.
- After the four weeks have been completed, I will arrange interviews with each pair of participants, scheduled at their convenience. (More information on the interviews is given below.)

Each participant will be referred to by a pseudonym in the final thesis and any papers written on this research. However, it will not be possible to make participants completely anonymous, as their interview responses and copies of their work may be included in the final write-up. During the interviews, the participants will complete mathematical tasks similar to the ones we will have worked on in the club. They will explain their thinking as they work on the task, and I will ask them questions related to the task and the preceding club sessions. All questions will be related to mathematics and the participants' thought processes. Nothing of a personal nature will be asked or expected of the participants.

The interviews will be video recorded from above, so that the participants' faces are obscured but their work and interactions with each other can be observed. The videos will be transcribed to include both spoken words and descriptions of body language and actions. Along with the transcription, still photos from the video may be used as data. Any pictures used will be edited, if necessary, to ensure anonymity.

Participants may continue to participate fully in the club without contributing to the research. This would mean that they do all the tasks that the other boys do, but their work and journals will not be used for research and they will not be interviewed. Further, anyone who agrees to participate in the research will be allowed to stop participating at any time with no penalty. As before, participants may also leave the club at any time, as long as they inform either myself or [redacted].

Before any research is conducted, informed assent will be obtained from the learners and informed consent will be obtained from their parents/guardians. A copy of the letter and form that will be sent home is attached. An information session for parents/guardians will be held in order to explain the research and answer any questions.

I hope that spending time working on club activities will improve participants' understanding of mathematics in general, which may positively impact their results on school tests and exams. However, there are no marks given for work done in the club, and participation in the club and/or research will not be directly related to their school work. There are no anticipated personal risks to the participants.

The data collected during this project will be used in my MEd thesis and possibly in journal papers written about this research. After the research has been completed and the results have been analysed, I will provide you and the participants with a short report summarising the outcomes of the research. Once the final MEd thesis has been completed I would be happy to provide you with an electronic (PDF) copy if you request one.

This research has been approved by the Rhodes University Ethical Standards Committee (RUESC), provisional to your permission being given. For more information on the committee and Rhodes University's ethics policy you can visit www.ru.ac.za/researchgateway/ethics/. Should there be any unethical behaviour in the running of the project, please send comments or complaints to Mr Siyanda Manqele, Rhodes University's ethics coordinator, on 046 603 7727 or at s.manqele@ru.ac.za.

If you have any questions or wish to discuss the research further, please feel free to email me at sindisiwe.herbert@gmail.com or call me on [redacted]. If you are satisfied that this research will be beneficial to the learners concerned and are willing to allow me to conduct this study, please complete the attached form. This form will be forwarded to RUESC, who will then give me final approval to proceed with the research.

Thank you for your time.

Kind regards,
Sindisiwe Herbert

Consent Form

I, ____, Principal of [redacted], give Sindisiwe Herbert permission to conduct a research project with selected [redacted] learners on [redacted] property. The learners' work (produced during sessions of the Grade 9 Maths Club) may be photographed and collected to be used as data. The learners may be interviewed, and these interviews may be video recorded and transcribed. The data (including still pictures taken from the video) may be included in the final thesis and any journal papers written on this research.

Signed: ____

Date: ____

Appendix C: Permission Letter to Eastern Cape Department of Education

RHODES UNIVERSITY
Where leaders learn

SARChI MATHEMATICS EDUCATION CHAIR
EDUCATION DEPARTMENT
19 Somerset Street, Grahamstown, 6139, South Africa
PO Box 94, Grahamstown, 6140, South Africa
t: +27 (0) 46 603 7211
f: +27 (0) 46 603 8084
www.ru.ac.za

Date

The Director

Strategic Planning Policy Research & Secretariat Services

Eastern Cape Department of Education

Private Bag X0032

Bhisho

5605

To Whom It May Concern:

Permission to perform research with selected learners at [redacted], [redacted]

I am a student at Rhodes University reading for a Masters degree in mathematics education. The title of my research project is: *An analysis of visual representations of linear algebraic expressions created by selected Grade 9 learners: a case study in an after-school mathematics club*. Please find the relevant application form and research proposal attached for full details on the study.

Having taught at [redacted] in 2017 and 2018, and still being involved in several extra-mural activities at the school, I decided to request permission to carry out my research there. One of the extramurals with which I assist is a Grade 9 Maths Club that takes place for an hour every Tuesday afternoon. It is with this group of learners that I wish to work. The main purpose of this club is to help participants think about maths visually and more innovatively, in order to improve their understanding of important concepts that will be necessary for their success in Grade 10 and beyond. The research is intended to deepen the participants' understanding of algebra and to give me insight into any understanding participants have gained during the project.

Here is the plan for my research:

- For four weeks, the club will focus its attention on algebra and sequences.
- The boys will work in pairs (as they usually do).
- The work they produce will be photographed. (These photographs will be used as data in the research.)
- Every week, each pair will complete a journal entry describing the work they did in the session. (These journals will be used as data in the research.)
- After the four weeks have been completed, I will arrange interviews with each pair of participants, scheduled at their convenience. (More information on the interviews is given below.)

Each participant will be referred to by a pseudonym in the final thesis and any papers written on this research. However, it will not be possible to make participants completely anonymous, as their interview responses and copies of their work may be included in the final write-up.
During the interviews, the participants will complete mathematical tasks very similar to the ones we will have worked on in the club. They will explain their thinking as they work on the task, and I will ask them questions related to the task and the preceding club sessions. All

questions will be related to mathematics and the participants' thought processes. Nothing of a personal nature will be asked or expected of the participants.

The interviews will be video recorded from above, so that the participants' faces are obscured but their work and interactions with each other can be observed. The videos will be transcribed to include both spoken words and descriptions of body language and actions. Along with the transcription, still photos from the video may be used as data. Any pictures used will be edited, if necessary, to ensure anonymity.

Participants may continue to participate fully in the club without contributing to the research. This would mean that they do all the tasks that the other boys do, but their work and journals will not be used for research and they will not be interviewed. Further, anyone who agrees to participate in the research will be allowed to stop participating at any time with no penalty. As before, participants may also leave the club at any time.

Before any research is conducted, permission from the principal will be sought. Then informed assent will be obtained from the learners and informed consent will be obtained from their parents/guardians. A copy of the letter and form that will be sent home is attached. An information session for parents/guardians will be held in order to explain the research and answer any questions.

I hope that spending time working on club activities will improve participants' understanding of mathematics in general, which may positively impact their results on school tests and exams. However, there are no marks given for work done in the club, and participation in the club and/or research will not be directly related to their school work. There are no anticipated personal risks to the participants.

This research has been approved by the Rhodes University Ethical Standards Committee, provisional to your permission being given. For more information on the committee and Rhodes University's ethics policy you can visit www.ru.ac.za/researchgateway/ethics/. Should there be any unethical behaviour in the running of the project, please send comments or complaints to Mr Siyanda Manqele, Rhodes University's ethics coordinator, on 046 603 7727 or at s.manqele@ru.ac.za.

The data collected during this project will be used in my MEd thesis and possibly in journal papers written about this research. After the research has been completed and the results have been analysed, I will present the participants with a short report summarising the outcomes of the research. As required, I will provide the Department with both an electronic (PDF) and a hard copy of the final thesis and any papers that are written on this research.

Thank you for your time.

Kind regards,
Sindisiwe Herbert