RHODES UNIVERSITY

DEPARTMENT OF MATHEMATICS

COMPARISON OF DIFFERENT NOTIONS OF COMPACTNESS IN THE FUZZY TOPOLOGICAL SPACE

by

E.Z. MORAPELI

A thesis submitted in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE in Mathematics.

January 1989

COMPARISON OF DIFFERENT NOTIONS OF COMPACTNESS IN THE FUZZY TOPOLOGICAL SPACE

E.Z. MORAPELI

ERRATA PAGE

Title page	"partial fulfillment" instead of "fulfillment".
Abstract page	line 2↑ "stronger" instead of "weaker".
Content page	CHAPTER III η is replaced by –.
Preface page	line 6↑ "weaker" is replaced by "stronger".
page iv	"advise" is replaced by "advice".
page 14	line 9↓ we include reference [3].
page 19	line 1↑ "see [17] Theorem 11.1" is replaced by "see [17] Theorem 11.1 and remark after Theorem 0.2.7".
page 21	line 5↑ "[8]" instead of "[19]".
page 22	line 11↑ "that any α–net..." is replaced by "that any net..."
page 22	line 7↑ "limit of the α–net..." should read "limit of the constant α–net".
page 25	line 1↓ "prefilter–base" should be "prefilterbase".
page 31	line 2↓ "$\mathfrak{S}$" should read "=".
page 32	line 10↓ "[16]" should be "[15]".
	line 11↓ "the one using fuzzy ..." should read "the one of Pu and Liu using fuzzy nets ([17])..."
	line 12↓ "[16]" should read "[15]".
page 37	line 2↑ "p_m" should read "P_m".
page 41	line 2↑ we omit Definitions 0.2.12, 0.2.13, and 0.2.15.
page 42	Omit line 7↓ (In view of Theorem.....).
page 69	lines 10↓ and 11↓ "into I have ..." should read "into I assume".
Reference page	No.15 "The relations between...." should read "The relation between ...".

ABSTRACT

Various notions of compactness in a fuzzy topological space have been introduced by different authors. The aim of this thesis is to compare them. We find that in a T_2 space (in the sense that no fuzzy net converges to two fuzzy points with different supports) all these notions are equivalent for the whole space. Furthermore, for N–compactness and f–compactness (being the only notions that are defined for an arbitrary fuzzy subset) we have equivalence under a ~~weaker~~ stronger condition, namely, a T_2 space in the sense that every prime prefilter has an adherence that is non–zero in at most one point.

CONTENTS

PREFACE

In this thesis, we compare all those notions of compactness, in a fuzzy topological space, that have been introduced so far ([1], [3], [10], [12], [19]). Whilst we restrict ourselves to fuzzy sets with values in the closed unit interval, we would like to point out that many of our results can be extended to fuzzy sets with values in completely distributive lattice ([3], [27]).

In this comparison, category theory techniques are not employed.

In Section 1 of Chapter 0 we give the definition of a fuzzy topology and some of its basic properties. We would like to mention here that we do not require our fuzzy topology to include the constants ([10]). In Section 2 we define what is meant by a fuzzy point and discuss its neighbourhood structures, namely the Q–neighbourhoods and the R–neighbourhoods. We finish this section by giving different types of a fuzzy Hausdorff (T_2) space and the implications that exist amongst them.

In Section 1, Chapter I we define what is meant by a fuzzy net and its subnet. We then give some of their properties that will be used in the sequel. In Section 2 we give definitions of a prefilter and prime prefilter and discuss their properties. Section 3 of this chapter deals with the relationship between the prefilter and the fuzzy net theories. We, in fact, show how a prefilter can be constructed from a given fuzzy net and vice versa. We complete this section by briefly mentioning Lowen's Convergence Theory using prefilters ([13]) and his definition of a T_2 space [14], to conclude that his (Lowen) notion of a T_2 space is much ~~weaker~~ stronger than the ones introduced in Chapter 0.

In Chapter II Section 1 we give two definitions of compactness, namely, α–compactness and α^*–compactness. We find out that the two are only defined for the whole space, being a very restrictive concept. We discuss some of their properties and show that α–compactness is a "good extension" whilst α^*–compactness is not.

In Section 2 we show that for α–compactness, the Alexander subbase lemma and the

Tychonoff Theorem holds and show by means of counter–examples that this is not the case for α^*–compactness.

In Chapter III we give yet another two definitions of compactness, namely the strong fuzzy compactness (section 1) and the ultra–fuzzy compactness (section 2). These are also defined for the whole space. We show that the two are good extensions and that the Tychonoff Theorem holds for both of them.

Section 1 of Chapter IV gives the first notion of compactness defined for an arbitrary fuzzy subset, namely, N–compactness. Amongst other properties of N–compact fuzzy sets, we show that N–compact sets attain their maxima. This property is used in the characterisations of N–compactness by prefilters. We also show that N–compactness is a good extension. In Section 2, we give characterisations of N–compactness by prefilters and prime prefilters. These characterisations enable us to deduce the Tychonoff Theorem much more easily than it was done using fuzzy nets given ([19]).

In Section 1 of Chapter V we introduce yet another notion of compactness defined for an arbitrary fuzzy subset, namely f–compactness. We give some of its properties and in particular, show that f–compactness is also a good extension. In Section 2 we show that the Tychonoff Theorem holds for f–compactness. Secondly we show that for f–compactness, it really does not matter whether or not our fuzzy topology includes constants.

In Chapter VI we compare for the whole space, all these notions of compactness that we found to be good extensions. We find a condition that gives the equivalence of all these notions. Lastly, we show that for a Hausdorff space in the sense of [14], N–compactness and f–compactness are equivalent for an arbitrary fuzzy set.

The following concepts and small contributions are original ones which in some cases were achieved with the assistance of Dr. Chadwick:

Definition 0.2.3, Theorem 0.2.7, Theorem 0.2.10, Corollary 0.2.11, Theorem I.1.13, Theorem I.3.6, Theorem II.1.9, Lemma II.1.10, Lemma III.1.3, Theorem III.1.4, Theorem II.2.6, Lemma III.2.7, Proposition IV.1.2, Theorem IV.1.4, Lemma IV.1.11, Theorem IV.1.12, Theorem IV.2.2, Theorem IV.2.3, Theorem VI.1.21.

ACKNOWLEDGMENTS

Firstly, I would like to thank my supervisor Professor W. Kotzé for his invaluable support and supervision in writing this thesis. I would also like to express my sincere gratitude to Dr. J. Chadwick for his ~~advise~~ advice without which this thesis would have been difficult to complete. Many thanks to Rhodes Mathematics Department staff whose support and encouragement particularly in the regular "fuzzy" seminars are highly appreciated. I should like to mention Dr. V. Murali in particular who had been my mathematics lecturer in many of my under–graduate studies.

My thanks to Mrs. S. Smith for her meticulous, painstaking work and above all, understanding in typing this thesis.

Lastly, I would like to thank my family for their support and sacrifices in making all this possible. I love you.

CHAPTER 0

FUZZY TOPOLOGICAL SPACES

0.1 *PRELIMINARIES*

Throughout this work X will be a base space consisting of at least one point, and we denote by I the closed unit interval [0,1]. The fuzzy sets on X, i.e. the elements of I^X, will, in general, be denoted by the symbols μ,ν,λ,σ, etc, the exceptions being that if $c \in I$, the constant fuzzy set with value c will sometimes be denoted by c. If no confusion is likely to arise and if $A \subseteq X$, then the fuzzy set that is 1 on A and 0 elsewhere (i.e. the characteristic function of A) will be denoted by 1_A. We will always denote the fuzzy set that is 0 everywhere on X by 0_X.

If $\mu,\nu \in I^X$ we will write $\mu \leq \nu$ if for each $x \in X$, $\mu(x) \leq \nu(x)$ provided no confusion is likely to arise.

If $\mu \in I^X$ then the complement, μ', of μ was defined in [26] to be $1_X - \mu$. It is clear that this complementation is a pseudo–complementation in the sense that $\mu \wedge \mu' = 0_X$ and $\mu \vee \mu' = 1_X$ do not hold in general.

It can also be verified that the de Morgan's laws hold in I^X i.e. if $\mu_i \in I^X$ for each $i \in J$ then

(i) $(\bigvee_{i\in J} \mu_i)' = \bigwedge_{i=J} \mu_i'$ and

(ii) $(\bigwedge_{i\in J} \mu_i)' = \bigvee_{i\in J} \mu_i'$

We first give

0.1.1 DEFINITION [2]

A fuzzy topology on X is a collection δ of fuzzy sets in X which satisfies the following conditions:

(i) $0_X, 1_X \in \delta$

(ii) If $\mu, \nu \in \delta$ then $\mu \wedge \nu \in \delta$

(iii) If J is an index set and $\nu_j \in \delta$ for each $j \in J$ then $\bigvee_{j \in J} \nu_j \in \delta$.

0.1.2 DEFINITION [2]

If δ is a fuzzy topology on X, then the pair (X,δ) is called the fuzzy topological space or fts for short.

In [10] Lowen changes condition (i) of definition 0.1.1 namely $0_X, 1_X \in \delta$ to (i)′ $c \in \delta$ for all $c \in I$, so that his definition of a fuzzy topology is the one with conditions (i)′ together with (ii) and (iii) of definition 0.1.1.

He gives several reasons why he requires this alternative definition, the most important one being that with this change, the constant functions between fuzzy topological spaces are continuous (as with "crisp" topological spaces), whereas with definition 0.1.1 this is not the case in general.

0.1.3 DEFINITION [2]

If (X,δ) is a fts, then each $\mu \in \delta$ is called a δ–open fuzzy set and μ' is called a δ–closed fuzzy set.

If no confusion is likely to arise, we will simply refer to a δ–open (δ–closed) fuzzy set as an open (closed) fuzzy set.

In the sequel, we will often denote the collection $\{\mu' : \mu \in \delta\}$ by δ'. Thus, if δ is a fuzzy topology, then δ' is the collection of closed fuzzy sets.

By analogy with ordinary topology, the indiscrete fuzzy topology contains only 0_X and 1_X, while the discrete fuzzy topology contains all the fuzzy sets on X.

0.1.4 DEFINITION [2]

Let δ_1 and δ_2 be two topologies on X. If the inclusion relation $\delta_1 \subseteq \delta_2$ holds, then we say δ_2 is finer than δ_1, or δ_1 is coarser than δ_2. Hence if δ_2 is finer than δ_1 the δ_1–open (δ_1–closed) fuzzy sets are also the δ_2–open (δ_2–closed) fuzzy sets.

In the sequel, we will sometimes write sup μ and inf μ to denote $\sup_{x \in X} \mu(x)$ and $\inf_{x \in X} \mu(x)$ respectively. Sup μ is sometimes called the height of μ, or the maximum of μ, while inf μ is sometimes called the minimum of μ.

0.1.5 DEFINITION [17]

Let (X,δ) be a fts. A subfamily β of δ is called a base for δ if for each $\nu \in \delta$ there exists $\beta_\nu \subseteq \beta$ such that $\nu = \vee\beta_\nu = \sup\{\lambda : \lambda \in \beta_\nu\}$.
A subfamily $\mathcal{S}$ of δ is called a subbase for δ if the family
$\beta = \{ \wedge \mathcal{F} : \mathcal{F}$ is a finite subfamily of $\mathcal{S}\}$ is a base for δ.

It is clear from definition 0.1.5 that if $\mathcal{S}$ is a subbase for a fuzzy topology δ on X, then for each $\nu \in \delta$ there exists $\lambda_{j_1}, \ldots, \lambda_{j_{n_j}} \in \mathcal{S}$ such that

$$\nu = \bigvee_{j \in J} (\lambda_{j_1} \wedge \ldots \wedge \lambda_{j_{n_j}})$$

In the sequel, we will often write $\psi \in 2^{(\beta)}$, if ψ is a finite subfamily of β and [n] ($n \in \mathbb{N}$) will always denote the set $\{1,2,\ldots,n\}$.

0.1.6 DEFINITION [17]

Let (X,δ) be a fts and $A \subseteq X$. If $\delta_A = \{\mu_A : \mu \in \delta\}$, where μ_A is the restriction of μ to A then δ_A is clearly a fuzzy topology on A and (A,δ_A) is called a subspace of (X,δ).

0.1.7 DEFINITION [2]

Let μ, $\nu_j \in I^X$ ($j \in J$). Then the collection $\{\nu_j : j \in J\}$ is called a cover for μ if $\mu \leq \vee\{\nu_j : j \in J\}$. If $J_0 \subseteq J$ such that $\mu \leq \vee\{\nu_j : j \in J_0\}$, then the collection $\{\mu_j : j \in J_0\}$ is called a subcover of $\{\mu_j : j \in J\}$. A cover is called open if each member of it is an open fuzzy set.

0.1.8 DEFINITION [2]

Given $f : X \rightarrow Y$, we define the fuzzy set $f^{\leftarrow}(\nu)$ ($\nu \in I^Y$) in X by $f^{\leftarrow}(\nu)(x) = \nu(f(x))$ for all $x \in X$. Converseley, if $\mu \in I^X$ then the fuzzy set $f^{\rightarrow}(\mu)$ in Y is defined by

$$f^{\rightarrow}(\mu)(y) = \begin{cases} \sup\{\mu(x) : x \in \{f^{\leftarrow}(y)\}\}, & \text{if } \{f^{\leftarrow}(y)\} \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

for all $y \in Y$, where $\{f^{\leftarrow}(y)\} = \{x \in X : f(x) = y\}$.

The following can easily be verified.

0.1.9 PROPOSITION

Let $f : X \rightarrow Y$. Then

(a) $f^{\leftarrow}(\nu') = (f^{\leftarrow}(\nu))'$, for any $\nu \in I^Y$.

(b) $f^{\rightarrow}(\mu') \geq (f^{\rightarrow}(\mu))'$ for any $\mu \in I^X$.

(c) If $\nu_1, \nu_2 \in I^Y$ such that $\nu_1 \leq \nu_2$ then $f^{\leftarrow}(\nu_1) \leq f^{\leftarrow}(\nu_2)$.

(d) If $\mu_1, \mu_2 \in I^X$ such that $\mu_1 \leq \mu_2$ then $f^{\rightarrow}(\mu_1) \leq f^{\rightarrow}(\mu_2)$.

(e) $\nu \geq f^{\rightarrow}(f^{\leftarrow}(\nu))$, for any $\nu \in I^Y$.

(f) $\mu \leq f^{\leftarrow}(f^{\rightarrow}(\mu))$, for any $\mu \in I^X$.

(g) If f: $X \rightarrow Y$ and $g : Y \rightarrow Z$ and $\lambda \in I^Z$, then $(g \circ f)^{\leftarrow}(\lambda) = f^{\leftarrow}(g^{\leftarrow}(\lambda))$, where $g \circ f$ is the composition of f and g.

0.1.10 DEFINITION [2]

Let (X,δ) and (Y,σ) be fts and f: $(X,\delta) \longrightarrow (Y,\sigma)$. Then f is called fuzzy continuous, if for each $\nu \in \sigma$, $f^{\leftarrow}(\nu) \in \delta$.

In the sequel we will often say a function is continuous instead of fuzzy continuous.

0.1.11 PROPOSITION

Let $f : (X,\delta_1) \longrightarrow (Y, \delta_2)$ and $g : (Y,\delta_2) \longrightarrow (Z,\delta_3)$.

If both f and g are continuous then the composition $g \circ f$ is a continuous function from (X,δ_1) to (Z,δ_3), i.e. if $\lambda \in \delta_3$ then $(g \circ f)^{\leftarrow}(\lambda) \in \delta_1$.

0.1.12 PROPOSITION

Let $f : X \longrightarrow Y$ and $\{\nu_j : j \in J\} \subset I^Y$. Then

(i) $f^{\leftarrow}(\sup_{j\in J} \nu_j) = \sup_{j\in J} f^{\leftarrow}(\nu_j)$

(ii) $f^{\leftarrow}(\inf_{j\in J} \nu_j) = \inf_{j\in J} f^{\leftarrow}(\nu_j)$.

0.1.13 DEFINITION [17]

Let (X,δ) be a fts and $\mu \in I^X$. Then $\sup \{\nu \in \delta : \nu \leq \mu\}$ is called the interior of μ and denoted by μ^0, and $\inf \{\nu \in \delta' : \nu \geq \mu\}$ is called the closure of μ, denoted by $\bar{\mu}$.

It is obvious from Definition 0.1.13 that μ^0 is the largest open fuzzy set contained in μ and $\bar{\mu}$ is the smallest closed fuzzy set containing μ.

0.1.14 PROPOSITION

Let (X,δ) be a fts and $\mu \in I^X$. Then

(i) $\mu^0 = 1_X - \overline{(1_X - \mu)}$

(ii) $\bar{\mu} = 1_X - (1_X - \mu)^0$.

The proof is similar to that of the corresponding result in "crisp" topology.

Given a topology τ on X, we can generate a fuzzy topology,

$$\omega(\tau) = \{\mu \in I^X : \mu^{\leftarrow} < \alpha, 1] \in \tau \text{ for all } \alpha \in [0,1>\} \text{ on X.}$$

That $\omega(\tau)$ is indeed a fuzzy topology (also in the Lowen sense) on X can easily be seen. It is also clear that the $\omega(\tau)$ open fuzzy sets are precisely all the lower semicontinuous functions μ: X $\longrightarrow$ [0,1] and the $\omega(\tau)$– closed fuzzy sets are the upper semicontinuous functions ν : X $\longrightarrow$ [0,1].

(X,$\omega(\tau)$) will often be called the (natural) fuzzy topological space generated by the crisp topological space (X,τ).

On the other hand, starting with a fuzzy topology δ on X, we can generate a crisp topology i(δ) on X as the weak topology on X with $i_\alpha(\delta) = \{\mu^{\leftarrow} < \alpha, 1] : \mu \in \delta\}$, $\alpha \in [0,1>$ as base.
In fact, for each $\alpha \in [0,1>$, $i_\alpha(\delta)$ itself is a topology on X.
We observe that if $\alpha \in <0,1]$ then $i_\alpha(\delta) = \{\nu^{\leftarrow} [0,\alpha> : \nu \in \delta' \}$.

<u>0.1.15</u> <u>THEOREM</u>

If E is a set, and $(F_j, \tau_j)_{j\in J}$ is a family of topological spaces and $\{f_j\}_{j\in J}$ is a family of functions $f_j : E \longrightarrow F_j$ then $\sup_{j\in J} f^{\leftarrow} (\omega(\tau_j)) = \omega (\sup f_j^{\leftarrow} (\tau_j))$,
where $f^{\leftarrow}_j (\tau_j) = \{f^{\leftarrow}_j (A) : A \in \tau_j\}$. In particular, for each $j \in J$, $\omega(f_j^{\leftarrow} (\tau_j)) = f_j^{\leftarrow} (\omega(\tau_j))$.

<u>PROOF</u>

See [11] Theorem 1.4

<u>0.1.16</u> <u>THEOREM</u>

If E is a set $(F_j, \delta_j)_{j\in J}$ is a family of fts and $\{f_j\}_{j\in J}$ is a family of functions
$f_j : E \longrightarrow F_j$ then $\sup_{j\in J} f_j^{\leftarrow} (i(\delta_j)) = i (\sup_{j\in J} f_j^{\leftarrow} (\delta_j))$, where $f^{\leftarrow}_j (\delta_j) = \{f^{\leftarrow}_j(\mu) : \mu \in \delta_j\}$
In particular, for each $j \in J$, $i(f_j^{\leftarrow} (\delta_j)) = f_j^{\leftarrow}(i((\delta_j))$

PROOF

See [11] Theorem 1.5

Suppose we are given a topology τ on X, we can also generate a fuzzy topology $\underline{\omega}(\tau)$ on X as follows:

Let $\underline{\omega}(\tau) = \{1_A : A \in \tau\}$

It is clear that $\underline{\omega}(\tau)$ is a fuzzy topology on X and hence $(X,\underline{\omega}(\tau))$ is a fts.

If $1_A \in \underline{\omega}(\tau)$ then $A \in \tau$ and since for each $\alpha \in [0,1>$, $1_A^{\leftarrow} <\alpha,1] = A \in \tau$, we have that $1_A \in \omega(\tau)$. Therefore

0.1.17 PROPOSITION

If (X,τ) is a topological space, then $\underline{\omega}(\tau) \subseteq \omega(\tau)$.

In the sequel we sometimes use 1_A and A, where $A \subseteq X$, interchangeably i.e. we will sometimes say "A is fuzzy open" instead of 1_A is fuzzy open.

The next result can easily be proved.

0.1.18 PROPOSITION

Suppose (X,τ) is a topological space; then

(i) $i\,(\omega(\tau)) = \tau$

(ii) $i\,(\underline{\omega}(\tau)) = \tau$

Let $\{X_j : j \in J\}$ be a non–empty collection of non–empty sets and let $X = \prod_{j\in J} X_j$ be the product set.

Thus, the elements of X are those functions

$$f : J \to \bigcup_{j\in J} X_j \text{ such that for all } j \in J,\ f(j) \in X_j.$$

For each $j \in J$, we define the projection maps, Π_j, such that $\Pi_j : X \to X_j$ as follows:

$$\Pi_j(f) = f(j) \text{ for all } f \in X.$$

It is clear from the fact that each X_j is non–empty, we have that Π_j is surjective for each $j \in J$.

Suppose $\mu \in I^X$ and $j \in J$; then $\Pi_j^{\to}(\mu) : X_j \to I$ is the fuzzy set on X_j defined as follows:

For each $t \in X_j$, $\Pi_j^{\to}(\mu)(t) = \sup\{\mu(f) : \Pi_j(f) = t\}$ (as defined in 0.1.8)

$= \sup\{\mu(f) : f(j) = t\}$ (since $\Pi_j(f) = f(j)$)

Since each Π_j is surjective we have that $\{\mu(f) : f(j) = t\} \neq \emptyset$, hence $\sup\{\mu(f) : f(j) = t\}$ exists.

Conversely, suppose $\nu \in I^{X_j}$, then $\Pi_j^{\leftarrow}(\nu) : X \to I$ is a fuzzy set on X defined as follows:

For each $f \in X$, $\Pi_j^{\leftarrow}(\nu)(f) = \nu(\Pi_j(f))$ (as defined in 0.1.8)

$= \nu(f(j))$.

The fact that for each $j \in J$, Π_j is a function from X onto X_j, together with Proposition 0.1.9 give us the next simple result that will often be used in the sequel.

0.1.19 PROPOSITION

(a) $\Pi_j^{\leftarrow}(\Pi_j^{\to}(\mu) \geq \mu$, for all $\mu \in I^X$.

(b) $\Pi_j^{\to}(\Pi_j^{\leftarrow}(\nu)) = \nu$, for all $\nu \in I^{X_j}$

(c) $\sup_{f\in X} \mu(f) = \sup_{f\in X} \Pi_j^{\to}(\mu)(f(j)) = \sup_{t\in X_j} \Pi_j^{\to}(\mu)(t)$, for all $\mu \in I^X$.

(d) $\mu_1, \mu_2 \in I^X$ and $\mu_1 \leq \mu_2$ implies $\Pi_j^{\rightarrow}(\mu_1) \leq \Pi_j^{\rightarrow}(\mu_2)$

(e) $\nu_1, \nu_2 \in I^{X_j}$ and $\nu_1 \leq \nu_2$ implies $\Pi_j^{\leftarrow}(\nu_1) \leq \Pi_j^{\leftarrow}(\nu_2)$.

(f) If $\{\nu_\ell : \ell \in L\}$ is an indexed family of fuzzy sets in X_j, then

$$\Pi_j^{\leftarrow}(\inf_{\ell \in L} \nu_\ell) = \inf_{\ell \in L} \Pi_j^{\leftarrow}(\nu_\ell) \text{ and } \Pi_j^{\leftarrow}(\sup_{\ell \in L} \nu_\ell) = \sup_{\ell \in L} \Pi_j^{\leftarrow}(\nu_\ell).$$

(g) $\Pi_j^{\rightarrow}(\bar{\mu}) \leq \overline{\Pi_j^{\rightarrow}(\mu)}$, for all $\mu \in I^X$.

(h) $\Pi_j^{\leftarrow}(\bar{\nu}) \leq \overline{\Pi_j^{\leftarrow}(\nu)}$, for all $\nu \in I^{X_j}$.

Suppose now that each X_j is equipped with a fuzzy topology δ_j, then the product fuzzy topology $\prod_{j \in J} \delta_j$, is defined as the weakest fuzzy topology on the product set, $\prod_{j \in J} X_j$, such that all the projection maps, Π_j, are continuous.

A base for $\prod_{j \in J} \delta_j$ is given by the collection of all fuzzy sets in $\prod_{j \in J} X_j$ of the form $\min_{j \in J_0} \Pi_j^{\leftarrow}(\nu_j)$, where J_0 is a finite subset of J and for each $j \in J_0$, $\nu_j \in \delta_j$.

Equivalently, a closed base is given by the collection of all fuzzy sets of the form $\max_{j \in J_0} \Pi_j^{\leftarrow}(\lambda_j)$, where J_0 is a finite subset of J and for each $j \in J_0$, $\lambda_j \in \delta_j'$. By this we mean that, if $\delta = \prod_{j \in J} \delta_j$, then every δ–closed fuzzy set is an infimum of sets of the form $\max_{j \in J_0} \Pi_j^{\leftarrow}(\lambda_j)$.

Let $\nu_j \in I^{X_j}$ $(j \in J)$, then the product $\prod_{j \in J} \nu_j$ is the fuzzy set on $X = \prod_{j \in J} X_j$ defined by

$$(\Pi_{j \in J}^{\rightarrow} \nu_j)(f) = \inf_{j \in J} \nu_j(f(j)) = \inf_{j \in J} \Pi_j^{\leftarrow}(\nu_j)(f).$$

0.2 *CONCEPT OF A FUZZY POINT AND ITS NEIGHBOURHOOD STRUCTURE.*

0.2.1 DEFINITION [24]

A fuzzy set μ in X is called a fuzzy point if it takes the value 0 for all $y \in X$ except at exactly one point, say, $x \in X$ i.e. $\mu \in I^X$ is a fuzzy point if

$$\mu(y) = \begin{cases} \lambda \ (0 < \lambda \leq 1) & \text{if } y = x \\ 0 & \text{otherwise} \end{cases}$$

We then denote μ by x_λ, where the point $x \in X$ is called the support of μ and λ is the value of μ.

It is sometimes more convenient to denote x_λ by $\lambda 1_x$, which is self–explanatory.

We also use symbols such as p, e, q, etc to denote fuzzy points.

X^* will always denote the collection of all fuzzy points in X. If p is a fuzzy point then V(p) denotes the value of p. e.g. If $p = \alpha 1_x$, then $V(p) = \alpha$.

0.2.2 DEFINITION [17]

A fuzzy point x_α is said to be contained in a fuzzy set μ (or, to belong to μ) denoted by $x_\alpha \in \mu$ if $\alpha \leq \mu(x)$. By $x_\alpha \notin \mu$, we mean that the relation $\alpha \leq \mu(x)$ does not hold i.e. $\alpha > \mu(x)$.

It is clear from Definition 0.2.2 that every fuzzy set is the supremum of all its fuzzy points. It is also clear from the definition of μ', where $\mu \in I^X$, that if $p \in \mu$ then $p \notin \mu'$ does not hold in general.

In [19] Wang gave definition of what he called a remote–neighbourhood, as follows: if (X,δ) is a fts, p a fuzzy point, then ν is called an R–nbd of p if $p \notin \nu$ and $\nu \in \delta'$.

In this work, we give a more generalised form of that definition as

0.2.3 DEFINITION

Let (X,δ) be a fts and $\mu \in I^X$. Then μ is called an R–nbd of a fuzzy point p if there is $\nu \in \delta'$ such that $p \notin \nu$ and $\mu \leq \nu$.

The collection of all R–nbds of a fuzzy point p is called the R–nbd system of p.

It is clear from definition 0.2.3 that 0_X is an R–nbd of every fuzzy point in X, whereas 1_X is not an R–nbd of any fuzzy point in X.

0.2.4 DEFINITION [17]

A fuzzy point x_λ is said to be quasi–coincident with a fuzzy set μ, denoted by $x_\lambda \, q \, \mu$, if $\lambda > \mu'(x)$ i.e. $\lambda + \mu(x) > 1$.

It is obvious from definitions 0.2.2 and 0.2.4 that if $x_\lambda \, q \, \mu$ then $x_\lambda \notin \mu'$, and conversely.

Furthermore

0.2.5 DEFINITION [17]

A fuzzy set μ is said to be quasi–coincident with a fuzzy set ν, denoted by $\mu \, q \, \nu$ if there exists $x \in X$ such that $\mu(x) > \nu'(x)$, i.e. $\mu(x) + \nu(x) > 1$.

Since $\mu(x) > \nu'(x)$ implies $\nu(x) > \mu'(x)$, we have that $\mu \, q \, \nu$ implies $\nu \, q \, \mu$. Hence if μ is quasi–coincident with ν then μ and ν are quasi–coincident (with each other).

We can now give

0.2.6 DEFINITION [17]

Let (X,δ) be a fts and $\mu \in I^X$. Then μ is called a Q–neighbourhood, (Q–nbd, for short) of a fuzzy point p, if there exists $\nu \in \delta$ such that p q ν and $\nu \leq \mu$.

The collection of all Q–nbds of a fuzzy point p is called a Q–nbd system of p.

The next result shows that there exists a duality between the R–nbd and the Q–nbd. The straightforward proof is omitted.

0.2.7 THEOREM

Let (X,δ) be a fts and $\mu \in I^X$. Then μ is an R–nbd of a fuzzy point p if and only if μ' is a Q–nbd of p.

In [17] Pu and Liu gave extensions of some important results in general topology using the Q–nbds, so, because of Theorem 0.2.7 we will give without proof some of the results in [17]. But, before we do this, we would like to point out that the definitions that we will be using (ones given in terms of R–nbds) are equivalent to the ones in [17] (ones in terms of Q–nbds). This is indeed the case, and we give an illustration:

We will now give two definitions of an adherence point of a fuzzy set, and show that the two definitions are in fact equivalent. The first one is given in terms of Q–nbds whilst the second one is in terms of R–nbds.

0.2.8 (a) DEFINITION [17]

A fuzzy point e is called an adherence point of a fuzzy set μ if every Q–nbd of e is quasi–coincident with μ.

0.2.8 (b) DEFINITION [19]

A fuzzy point e is called an adherence point of a fuzzy set μ if for each R–nbd ν of e we have that $\mu \leq \nu$, does not hold.

0.2.9 PROPOSITION

Definitions 0.2.8 (a) and 0.2.8 (b) are equivalent.

PROOF

Suppose x_λ is an adherence point of μ in terms of definition 0.2.8 (a) and let ν be an R–nbd of x_λ.

By Theorem 0.2.7 we have that ν' is a Q–nbd of x_λ. Hence by our assumption ν' q μ. i.e. there exists some $y \in X$ such that $\mu(y) > \nu(y)$ which in turn implies $\mu \leq \nu$ does not hold.

Conversely, suppose x_λ is an adherence point of μ in terms of definition 0.2.8 (b) and let ν be a Q–nbd of x_λ.

By Theorem 0.2.7 we have that ν' is an R–nbd of x_λ. Hence by our assumption, $\mu \leq \nu'$ does not hold.

That is, there exists a $y \in X$ such that $\mu(y) > \nu'(y)$ which clearly implies that μ q ν.

From now on we will be using Definition 0.2.8(b).

0.2.10 THEOREM

A fuzzy point $p \in \bar{\mu}$ if and only if μ is not an R–nbd of p.

PROOF

"$\Longrightarrow$" Suppose $p \in \bar{\mu} = \inf \{\nu \in \delta' : \nu \geq \mu\}$.

Hence, for all $\nu \in \delta'$ with $\mu \leq \nu$ we have that $p \in \nu$, so that by definition 0.2.3, μ cannot be an R–nbd of p.

"$\Longleftarrow$" Suppose μ is not an R–nbd of p.

Hence for all $\nu \in \delta'$ with $\mu \leq \nu$ we have that $p \in \nu$. In particular $p \in \bar{\mu}$.

0.2.11 COROLLARY

$\bar{\mu} = \vee \{p : p \text{ is an adherence point of } \mu\}$.

PROOF

Follows at once from the fact that μ is not an R–nbd of p if and only if for each R–nbd ν of p we have that $\mu \nleq \nu$ if and only if p is an adherence point of μ, together with Theorem 0.2.10.

Next we give different definitions of a Hausdorff (T_2) space and investigate the implications that might exist amongst them. For further details we refer to [8], [9],[16],[17] and [18].

0.2.12 DEFINITION [19]

Let (X,δ) be a fts. Then (X,δ) is T_2 if for any two fuzzy points x_α and y_c with different supports, there exists R–nbds, ν_x and ν_y of x_α and y_c respectively, such that $\nu_x \vee \nu_y = 1$.

0.2.13 DEFINITION [17]

Let (X,δ) be a fts. Then (X,δ) is T_2 if for any two fuzzy points x_α and y_c with different supports, there exists Q–nbds, ν_x and ν_y of x_α and y_c respectively, such that $\nu_x \wedge \nu_y = 0$.

0.2.14 DEFINITION [18]

Let (X,δ) be a fts. Then (X,δ) is T_2 if for any two fuzzy points x_α and y_c with different supports, there exists $\nu_x, \nu_y \in \delta$ such that $x_\alpha \in \nu_x$, $y_c \in \nu_y$ and $\nu_x \wedge \nu_y = 0$.

0.2.15 DEFINITION [16]

Let (X,δ) be a fts. Then (X,δ) is T_2 if for any two fuzzy points x_α and y_c with different supports there exists $\nu_x, \nu_y \in \delta$ such that $\nu_x(x) = 1 = \nu_y(y)$ and $\nu_x(z) + \nu_y(z) \leq 1$ for all $z \in X$.

0.2.16 DEFINITION [3]

Let (X,δ) be a fts. Then (X,δ) is T_2 if for any two "crisp" points, $x, y \in X$, $x \neq y$ there exists $\nu_x, \nu_y \in \delta$ such that $\nu_x(x) = 1 = \nu_y(y)$ and $\nu_x \wedge \nu_y = 0$.

In [8] and [9] a fts satisfying definitions 0.2.14, 0.2.13 and 0.2.15 are called fuzzy Hausdorff, q–fuzzy Hausdorff and fuzzy Hausdorff (M & B) respectively.

We can now give

0.2.17 THEOREM

(i) Definition 0.2.12 is equivalent to Definition 0.2.13

(ii) Definition 0.2.14 implies Definition 0.2.13 but not conversely.

(iii) Definition 0.2.14 implies Definition 0.2.15 but not conversely.

(iv) Definition 0.2.16 implies Definition 0.2.15 but not conversely.

(v) Definition 0.2.16 is equivalent to Definition 0.2.14.

(vi) No implications exists between Definition 0.2.13 and 0.2.15.

PROOF

(i) Follows at once from Theorem 0.2.7

(ii) See [9] Theorem 5.5 (2)

(iii) See [8] Theorem 3.5 (1) (a).

(iv) Straightforward.

(v) Straightforward.

(vi) See [8] Theorem 3.5 (1) (b) together with Examples 3.4 and 3.6 (for counter–examples).

We also have

0.2.18 THEOREM

(i) (X,τ) is Hausdorff if and only if $(X,\omega(\tau))$ is Hausdorff with respect to definition 0.2.14.

(ii) If (X,δ) is Hausdorff with respect to definitions 0.2.13 and 0.2.14 then $(X,i(\delta))$ is Hausdorff but not necessarily conversely.

(iii) (X,τ) is Hausdorff if and only if $(X,\omega(\tau))$ is Hausdorff with respect to definition 0.2.13.

(iv) (X,τ) is Hausdorff if and only if $(X,\omega(\tau))$ is Hausdorff with respect to definition 0.2.15.

(v) If (X,δ) is Hausdorff with respect to definition 0.2.15 then $(X,i(\delta))$ is Hausdorff but not necessarily conversely.

PROOF

(i) See [18] Theorem 3.2

(ii) See [9] Theorem 4.4 and Theorem 5.3

(iii) See [9] Theorem 5.2

(iv) See [9] Theorem 3.1

(v) See [8] Theorem 3.2

As a consequence of Theorems 0.2.17 (iv) and 0.2.18 (v) we have the following.

0.2.19 THEOREM

If (X,δ) is Hausdorff with respect to definition 0.2.16 then $(X, i(\sigma))$ is Hausdorff but not necessarily conversely.

CHAPTER I

FUZZY NET THEORY AND PREFILTER THEORY

I.1 *FUZZY NETS AND FUZZY SUBNETS*

I.1.1 DEFINITION [17]

Let D be a non–void set and $\geq$ be an order relation. If for every m,n $\in$ D there exists p $\in$ D such that p $\geq$ m and p $\geq$ n then $\geq$ is said to direct D and the pair (D,$\geq$) is called a directed set.

A simple example of a directed set is ($\mathbb{N}$, $\geq$), where $\mathbb{N}$ is the set of natural numbers and $\geq$ is the usual order on $\mathbb{R}$, the set of real numbers.

I.1.2 DEFINITION [17]

Let (D,$\geq$) be a directed set and X be any set. Let X* be the collection of all fuzzy points in X. Then a function S : D $\longrightarrow$ X* is called a fuzzy net in X. In other words, a fuzzy net is a pair (S,$\geq$) such that S is a function from D into X* and $\geq$ directs the domain of S.
For n $\in$ D, we will often denote S(n) by S_n, and hence a fuzzy net S is often denoted by $\{S_n : n \in D\}$. If no confusion is likely to arise, we will, in the sequel, write "net" instead of "fuzzy net".

I.1.3 DEFINITION [19]

Let (X,δ) be a fts. For a fuzzy net $S = \{S_n : n \in D\}$, let λ_n be the value of S_n, i.e. $V(S_n) = \lambda_n$. Then we obtain a "crisp" net $\{\lambda_n : n \in D\}$ in the half open interval <0,1]. $\{\lambda_n : n \in D\}$ will be called the value net of S and denoted by V(S). If V(S) converges to a real number $\alpha \in$ <0,1], then we say S is an α–net. In particular, if $\lambda_n = \alpha$ holds for all n $\in$ D, then we say that S is a constant α–net. It is clear that any constant α–net is itself an α–net but not conversely.

I.1.4 EXAMPLE

Let $D = \mathbb{N}$, $X = \mathbb{R}$ and for each $n \in D$, let

$$S_n(x) = \begin{cases} \frac{1}{2} + \sin((-1)^n/2n), & \text{if } x = \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$$

Put $S = \{S_n : n \in D\}$.

Since $V(S_n)$ converges to $\frac{1}{2}$ (as $n \to \infty$), we have that S is an $\frac{1}{2}$–net.

I.1.5 DEFINITION [17]

A fuzzy net $T = \{T_m : m \in E\}$ in X is called a fuzzy subnet of a fuzzy net $S = \{S_n : n \in D\}$ if there is a function $F : E \to D$ such that

(1) $T = S \circ F$ i.e. for each $m \in E$, $T_m = S_{F(m)}$.

(2) For each $n \in D$, there exists $m \in E$ such that, if $k \in E$ and $k \geq m$ then $F(k) \geq n$.

It is clear from definition I.1.5 that a fuzzy subnet of an α–net is itself an α–net.

I.1.6 DEFINITION [19]

Let $S = \{S_n : n \in D\}$ be a fuzzy net in X and $\mu \in I^X$. Then we say S is in μ if $S_n \in \mu$, for each $n \in D$.

If for each $n \in D$ there exists $m \in D$ such that $m \geq n$ and $S_m \in \mu$, we say S is frequently in μ. We also say S is eventually in μ if there exists $N \in D$ such that $S_n \in \mu$ for each $n \geq N$. The statements that: S is outside, frequently outside, and eventually outside μ, are defined analogously.

I.1.7 PROPOSITION [1]

Let $S = (S_n : n \in D\}$ be a fuzzy net in X and $\mu \in I^X$, then

(i) S is frequently in μ if and only if S has a subnet in μ.

(ii) S is frequently outside μ if and only if S is not eventually in μ.

(iii) S in eventually outside μ if and only if S is not frequently in μ.

The above Proposition will be used in the sequel often without further explanation.

I.1.8 DEFINITION [19]

Let (X,δ) be a fts. Then a fuzzy point e in X is called a limit of a fuzzy net $S = \{S_n : n \in D\}$ in X (or S converges to e, in symbols $S \to e$) if for each R–nbd, ν, of e, we have eventually $S_n \not\in \nu$, (i.e. there exists $N \in D$ such that $n \geq N$ implies $S_n \not\in \nu$). It is easy to see that if $S \to e$ and T is a subnet of S then $T \to e$ also.

I.1.9 DEFINITION [19]

Let (X,δ) be a fts. Then the fuzzy point e in X is called a cluster point of a fuzzy net $S = \{S_n : n \in D\}$ in X (or S clusters at e, in symbols $S \infty e$), if for each R–nbd, ν, of e we have frequently $S_n \not\in \nu$, (i.e. for each $n \in D$, there exists $m \in D$ such that $m \geq n$ and $S_m \not\in \nu$).

I.1.10 THEOREM

In a fts (X,δ) a fuzzy point $e \in \bar{\mu}$ if and only if there is a fuzzy net in μ converging to e.

PROOF

See [17] Theorem 11.1. and the remark after Theorem 0.2.7.

I.1.11 COROLLARY

Let (X,δ) be a fts and $\mu \in I^X$. Then μ is closed if and only if whenever $S = \{S_n : n \in D\}$ is a fuzzy net in μ with $S \rightarrow e$ we have $e \in \mu$.

PROOF

Follows at once from the fact that μ is closed if and only if $\mu = \bar{\mu}$, together with Theorem I.1.10.

I.1.12 PROPOSITION

In a fts (X,δ), if a fuzzy net S converges to a fuzzy point x_λ, then for every $\alpha \in <0,\lambda]$, S also converges to x_α.

PROOF

Suppose $S = \{S_n : n \in D\}$ is a fuzzy net in (X,δ) such that $S \rightarrow x_\lambda$ and let $\alpha \in <0,\lambda]$.
For any R–nbd, ν, of x_α there exists $\sigma \in \delta'$ such that $x_\alpha \notin \sigma$ and $\nu \leq \sigma$.
Since $\alpha \leq \lambda$ and $x_\alpha \notin \sigma$ we have that $\lambda \geq \alpha > \sigma(x)$. So that $x_\lambda \notin \sigma$.
But $\nu \leq \sigma$. Therefore ν is an R–nbd of x_λ.
From $S \rightarrow x_\lambda$, we have that $S_n \notin \nu$ eventually.
Hence $S \rightarrow x_\alpha$.

The next Theorem is valid for a T_2 fts using Definitions 0.2.12, 0.2.13, 0.2.14 and 0.2.16.

I.1.13 THEOREM

A fts (X,δ) is T_2 if and only if no fuzzy net in (X,δ) converges to two fuzzy points with different supports.

PROOF

We give a proof for a space that is T_2 in the sense of Definition 0.2.12.

For sufficiency we suppose that $S = \{S_n : n \in D\}$ is a fuzzy net converging to two fuzzy points x_α, y_c with different supports. Then for any R–nbds ν_x and ν_y of x_α and y_c respectively, we have that $S_n \notin \nu_x$ and $S_n \notin \nu_y$ eventually. Hence $S_n \notin \nu_x \vee \nu_y$ eventually, which clearly implies that $\nu_x \vee \nu_y \neq 1$.

Therefore (X,δ) is not T_2.

To prove the necessity part, we suppose there exist fuzzy points e_1 and e_2, with different supports such that for any R–nbds ν_1, λ_1 of e_1 and e_2 respectively, $\nu_1 \vee \lambda_1 \neq 1$. Therefore there exists $x \in X$ such that $\nu_1(x) < 1$ and $\lambda_1(x) < 1$. Let $\nu_1(x) \vee \lambda_1(x) = \alpha$, so that $\alpha < 1$. Choose β such that $\alpha < \beta < 1$. Then $\beta > \nu_1(x)$ and $\beta > \lambda_1(x)$. Hence $x_\beta \notin \nu_1$ and $x_\beta \notin \lambda_1$. We assume without loss of generality that $\nu_1, \lambda_1 \in \delta'$. Thus ν_1, λ_1 are R–nbds of x_β.

For each ν_1, λ_1 denote x_β by $S(\nu_1, \lambda_1)$, and let Ω_{e_1} and Ω_{e_2} be the neighbourhood systems of e_1 and e_2 respectively, which are directed by the inclusion of fuzzy sets. Consider the product set $(\Omega_{e_1} \times \Omega_{e_2}, \geq)$ where "$\geq$" is defined as:

$(\nu_1 \times \lambda_1) \geq (\nu_2 \times \lambda_2)$ if and only if $\nu_2 \leq \nu_1$ and $\lambda_2 \leq \lambda_1$.

It can easily be shown that $(\Omega_{e_1} \times \Omega_{e_2}, \geq)$ is a directed set and hence

$S = \{S(\nu_1, \lambda_1) : (\nu_1, \lambda_1) \in (\Omega_{e_1} \times \Omega_{e_2}, \geq)\}$ is a fuzzy net in X which clearly converges to both e_1 and e_2.

For the other definitions we refer to Theorem 0.2.17 (v) together with [19] Theorem 4.2.4.

However, for a T_2 fts given in Definition I.1.15 we have

I.1.14 THEOREM

If no fuzzy net in (X,δ) converges to two fuzzy points with different supports, then (X,δ) is T_2, but not conversely.

The next result will frequently be used in the sequel.

I.1.15 THEOREM

In a fts (X,δ) a fuzzy point e is cluster point of a fuzzy net S if and only if S has a subnet T converging to e.

PROOF

See [17] Theorem 13.2.

I.1.16 DEFINITION [19]

Suppose $S = \{S_n : n \in D\}$ and $T = \{T_n : n \in D\}$ are two fuzzy nets with the same domain D and for each $n \in D$, S_n and T_n are fuzzy points with the same support; then we say that S and T are similar nets. Suppose S is an α–net converging to x_α ($\alpha \in <0,1]$). If for each $c \in <0,1]$ the constant c–net similar to S converges to x_c, then x_α is called a transitive limit of S.

It is clear from definition I.1.16 that any ~~α–net~~ net is similar to itself.

We can now give

I.1.17 PROPOSITION

Let (X,δ) be a fts. If for some $\alpha \in <0,1]$ and some $x \in X$, the fuzzy point x_α is a transitive limit of the constant α–net $S = \{S_n : n \in D\}$, then the "crisp" point x is a limit of the "crisp" net $\{x^n : n \in D\}$ in $(X,i(\delta))$, where x^n is the support of S_n $(n \in D)$.

PROOF

Let U be an open neighbourhood of x in $(X, i(\delta))$. Therefore by definition of $i(\delta)$, there are $\nu_j \in \delta'$ and $c_j \in <0,1]$ $(j \in [k])$ so that

$$x \in \bigcap_{j\in[k]} U_j \subseteq U,$$

where $U_j = \{t \in X : \nu_j(t) < c_j\}$, $j \in [k]$.

Therefore, $\nu_j(x) < c_j$, for each $j \in [k]$ i.e. $x_{c_j} \notin \nu_j$, $j \in [k]$.

This implies that ν_j is an R–nbd of x_{c_j}, for each $j \in [k]$.

By the assumption of the theorem, the constant c_j–net $\{x^n_{c_j} : n \in D\}$ $(j \in [k])$ converges to x_{c_j}.

Hence there exists an $N_j \in D$ such that $x^n_{c_j} \notin \nu_j$, holds for each $n \geq N_j$ $(j \in [k])$. That is, $\nu_j(x^n) < c_j$ or $x^n \in U_j$, for each $n \geq N_j$. Choose $N \in D$ so that $N \geq N_j$ for each $j \in [k]$.

Then $x^n \in \bigcap_{j \in [k]} U_j \subseteq U$, holds for each $n \geq N$.

Hence x is a limit of the "crisp" net $\{x^n : n \in D\}$.

The next Proposition is valid for Definitions 0.2.12, 0.2.13, 0.2.14, 0.2.16.

I.1.18 PROPOSITION

Let (X,δ) be a T_2 fts and S a fuzzy net in X. If $S \rightarrow x_\alpha$ and $S \infty y_c$, then $x = y$.

PROOF

Suppose S is a fuzzy net such that $S \rightarrow x_\alpha$ and $S \infty y_c$.

By Theorem I.1.15, S has a subnet T converging to y_c, i.e. $T \rightarrow y_c$.

Since $S \rightarrow x_\alpha$ and T is a subnet of S, we have that $T \rightarrow x_\alpha$.

By Theorem I.1.13, the fact that (X,δ) is T_2 implies that $x = y$.

The following concept will be used in the sequel.

I.1.19 DEFINITION [1]

Let $S = \{S_n : n \in D\}$ be a fuzzy net; then the characteristic, c(S), of S is defined by

$$c(S) = \inf_{n \in D} \sup_{m \geq n} V(S_m).$$

It is clear from definitions I.1.19 and I.1.3 that if S is an α–net, then $c(S) = \alpha$. Also, if S is any net, then there exists a subnet T of S such that T is a c(S)–net provided $c(S) > 0$.

I.2. *PREFILTERS AND PRIME PREFILTERS*

I.2.1 DEFINITION [13]

A prefilter on X is a non–empty subset $\mathcal{F}$ of I^X such that

(i) $0_X \notin \mathcal{F}$

(ii) If $\nu_1, \nu_2 \in \mathcal{F}$ then $\nu_1 \wedge \nu_2 \in \mathcal{F}$.

(iii) If $\nu_1 \in \mathcal{F}$ and $\nu \in I^X$ such that $\nu_1 \leq \nu$, then $\nu \in \mathcal{F}$.

I.2.2 DEFINITION [13]

A prefilterbase on X is a non–empty subset $\mathcal{B}$ of I^X such that

(i) $0_X \notin \mathcal{B}$.

(ii) If $\beta_1, \beta_2 \in \mathcal{B}$ then there exists $\beta \in \mathcal{B}$ such that $\beta \leq \beta_1 \wedge \beta_2$.

It is clear from the above definitions that any prefilter is a prefilterbase.

Suppose $\mathcal{B}$ is a prefilterbase on X.

Put $<\mathcal{B}> = \{\nu \in I^X : \beta \leq \nu \text{ for some } \beta \in \mathcal{B}\}$.

We claim that $<\mathcal{B}>$ is a prefilter on X. In fact, if $\nu_1, \nu_2, \in <\mathcal{B}>$ then there exists $\beta_1, \beta_2 \in \mathcal{B}$ such that $\beta_1 \leq \nu_1$ and $\beta_2 \leq \nu_2$.

Thus, $\beta_1 \wedge \beta_2 \leq \nu_1 \wedge \nu_2$.

Now since $\mathcal{B}$ is a prefilterbase, $\beta_1, \beta_2 \in \mathcal{B}$ implies there exists $\beta \in \mathcal{B}$ such that $\beta \leq \beta_1 \wedge \beta_2$

Therefore $\beta \leq \nu_1 \wedge \nu_2$, for some $\beta \in \mathcal{B}$. That is $\nu_1 \wedge \nu_2 \in <\mathcal{B}>$.

Since $\mathcal{B} \neq \emptyset$ and $\mathcal{B} \subseteq <\mathcal{B}>$, we have that $<\mathcal{B}> \neq \emptyset$. Also, $0_X \notin \mathcal{B}$ implies $0_X \notin <\mathcal{B}>$.

Lastly, let $\nu_1 \in <\mathcal{B}>$, $\nu \in I^X$ such that $\nu_1 \leq \nu$, then there exists $\beta \in \mathcal{B}$ such that $\beta \leq \nu_1$.

Clearly $\beta \leq \nu$ and hence $\nu \in <\mathcal{B}>$.

Therefore $<\mathcal{B}>$ is a prefilter on X.

It is therefore clear that given any prefilter base, $\mathcal{B}$, we can construct a prefilter $<\mathcal{B}>$ called the prefilter generated by $\mathcal{B}$. We say that $\mathcal{B}$ is a base for $<\mathcal{B}>$.

I.2.3 EXAMPLE

Let $\mu \in I^X$, $\mu \neq 0$. Then it is clear that $\{\mu\}$ forms a prefilterbase on X, and that $\{\nu \in I^X : \nu \geq \mu\}$ is the prefilter generated by $\{\mu\}$.

As a special case, we take $\mu = \alpha 1_x$, a fuzzy point in X, then $\{\nu \in I^X : \alpha 1_x \in \nu\}$ is the prefilter generated by $\{\mu\}$.

I.2.4 DEFINITION [13]

Let $\mathcal{B}$ be a prefilterbase. The characteristic of $\mathcal{B}$ is $c(\mathcal{B}) = \inf_{\nu \in B} \sup \nu$.

It is clear that if $\mathcal{B}$ is a prefilterbase, then $c(\mathcal{B}) = c(<\mathcal{B}>)$, once we note that

$\inf_{\lambda \in <\mathcal{B}>} \sup \lambda = \inf_{\nu \in \mathcal{B}} \sup \nu$.

It is also easy to see that if $\mathcal{B}$ is a base for the prefilter $\mathcal{F}$, then $c(\mathcal{F}) = c(\mathcal{B})$.

It is sometimes useful to extend this concept slightly:

If $\mu \in I^X$ and $\mathcal{B}$ is a prefilterbase then

$\mathcal{B}_1 = \{\mu \wedge \nu : \nu \in \mathcal{B}\}$ may or may not be a prefilterbase.

(e.g. if $\mu = 0_X$ then for each $\nu \in \mathcal{B}$, $\mu \wedge \nu = 0_X$ and hence $0_X \in \mathcal{B}_1$, so that $\mathcal{B}$ is not a prefilterbase).

We define $c(\mathcal{B},\mu)$ to be $c(\mathcal{B}_1)$ if $\mathcal{B}_1$ is a prefilterbase and zero otherwise.

In all cases $c(\mathcal{B},\mu) = \operatorname{Inf}_{\nu \in \mathcal{B}} \sup \mu \wedge \nu$:

If $\mu \wedge \nu = 0$ for some $\nu \in \mathcal{B}$, then $c(\mathcal{B}, \mu) = 0$. On the other hand, if $\mu \wedge \nu \neq 0$ for every $\nu \in \mathcal{B}$ then the collection $\mathcal{B}_1 = \{\mu \wedge \nu. \nu \in \mathcal{B}\}$ is a prefilter base. We write $(\mathcal{B},\mu)$ for the prefilter it generates.

Hence $c(\mathcal{B},\mu)$ is the characteristic of the prefilter generated by $\mathcal{B}_1$, if it exists.
It is easy to see that if $\mathcal{B}$ is a base for a prefilter $\mathcal{F}$, then for any $\mu \in I^X$, $c(\mathcal{F}, \mu) = c(\mathcal{B},\mu)$, whether or not the prefilter $(\mathcal{B},\mu)$ exists, and if it does exist we have $(\mathcal{F}, \mu) = (\mathcal{B}, \mu)$.

I.2.5 DEFINITION [19]

Let $\mathcal{F}$ be a prefilter on X such that $c(\mathcal{F}) = \alpha$. Then $\mathcal{F}$ is called an α–prefilter.

I.2.6 DEFINITION [13]

Let (X,δ) be a fts and $\mathcal{B}$ a prefilterbase on X. Then the adherence of $\mathcal{B}$ is defined as

$$\mathrm{Adh}\,\mathcal{B} = \inf_{\nu\in\mathcal{B}} \bar{\nu}.$$

It is clear that if $\mathcal{B}$ is a prefilterbase, then we have

$$\mathrm{Adh}\,\mathcal{B} = \mathrm{Adh}\,<\mathcal{B}>.$$

I.2.7 DEFINITION [13]

Let (X,δ) be a fts and $\mathcal{B}$ a prefilterbase. Then a fuzzy point $\alpha 1_x$ is called an adherence point of $\mathcal{B}$, if $\alpha 1_x \in \mathrm{Adh}\,\mathcal{B}$, i.e. $\alpha 1_x \in \bar{\nu}$, for all $\nu \in \mathcal{B}$.

The next result is a useful identity for prefilters.

I.2.8 PROPOSITION [1]

Let $\mu_1, \mu_2 \in I^X$ and let $\mathcal{F}$ be a prefilter on X.
Then $c(\mathcal{F}, \mu_1 \vee \mu_2) = \max\,\{c(\mathcal{F},\mu_1), c(\mathcal{F},\mu_2)\}$.

PROOF

For any $\nu \in \mathcal{F}$, we have $\mu_1 \wedge \nu \leq (\mu_1 \vee \mu_2) \wedge \nu$ and hence $\sup\,(\mu_1 \wedge \nu) \leq \sup\,((\mu_1 \vee \mu_2) \wedge \nu)$, so that we have

$$c(\mathcal{F},\mu_1) = \inf_{\nu\in\mathcal{F}} \sup\,(\mu_1 \wedge \nu) \leq \inf_{\nu\in\mathcal{F}} \sup\,((\mu_1 \vee \mu_2) \wedge \nu) = c(\mathcal{F}, \mu_1 \vee \mu_2)$$

Similarly $c(\mathcal{F}, \mu_2) \leq (\mathcal{F}, \mu_1 \vee \mu_2)$

Then $\max \{c(\mathcal{F}, \mu_1), c(\mathcal{F}, \mu_2)\} \leq c(\mathcal{F}, \mu_1 \vee \mu_2)$

On the otherhand, let $\nu_1, \nu_2 \in \mathcal{F}$.

Therefore, $\sup (\mu_1 \wedge \nu_1) \vee \sup (\mu_2 \wedge \nu_2)$

$$\geq \sup (\mu_1 \wedge (\nu_1 \wedge \nu_2)) \vee \sup (\mu_2 \wedge (\nu_1 \wedge \nu_2))$$

$$= \sup [(\mu_1 \wedge (\nu_1 \wedge \nu_2)) \vee (\mu_2 \wedge (\nu_1 \wedge \nu_2))]$$

$$= \sup [(\mu_1 \vee \mu_2) \wedge (\nu_1 \wedge \nu_2)] \geq c(\mathcal{F}, \mu_1 \vee \mu_2)$$

and hence $\inf_{\nu_1, \nu_2 \in \mathcal{F}} (\sup (\mu_1 \wedge \nu_1) \vee \sup (\mu_2 \wedge \nu_2)) \geq c(\mathcal{F}, \mu_1 \vee \mu_2)$

But $\inf_{\nu_1, \nu_2 \in \mathcal{F}} (\sup (\mu_1 \wedge \nu_1) \vee \sup (\mu_2 \wedge \nu_2))$

$$= \inf_{\nu_1 \in \mathcal{F}} \sup (\mu_1 \wedge \nu_1) \vee \inf_{\nu_2 \in \mathcal{F}} \sup (\mu_2 \wedge \nu_2)$$

$$= c(\mathcal{F}, \mu_1) \vee c(\mathcal{F}, \mu_2)$$

$$= \max \{c(\mathcal{F}, \mu_1), c(\mathcal{F}, \mu_2)\}$$

so that, $\max \{c(\mathcal{F}, \mu_1), c(\mathcal{F}, \mu_2)\} \geq c(\mathcal{F}, \mu_1 \vee \mu_2)$

Suppose $f: X \to Y$, $\mathcal{F}$ is a prefilter on Y and $\mu \in I^X$.

Then $c(\mathcal{F}, f^{\rightarrow}(\mu)) = \inf_{\nu \in \mathcal{F}} \sup_{y \in Y} (f^{\rightarrow}(\mu)(y) \wedge \nu(y)) = \inf_{\nu \in \mathcal{F}} \sup_{x \in X} f^{\rightarrow}(\mu)(f(x)) \wedge \nu(f(x))$

$= \inf_{\nu \in \mathcal{F}} \sup_{x \in X} f^{\leftarrow}(f^{\rightarrow}(\mu)(x) \wedge f^{\leftarrow}(\nu)(x) = \inf_{\nu \in \mathcal{F}} \sup_{t \in X} \mu(t) \wedge f^{\leftarrow}(\nu)(t) = \inf_{\nu \in \mathcal{F}} \sup \mu \wedge f^{\leftarrow}(\nu)$.

Remark

Let $\mathcal{B} = \{f^{\leftarrow}(\nu) : \nu \in \mathcal{F}\}$.

If $c(\mathcal{F}, f^{\rightarrow}(\mu)) > 0$, then $\mathcal{B}$ is a prefilterbase on X such that $c(\mathcal{F}, f^{\rightarrow}(\mu)) = c(\mathcal{B}, \mu)$.

For the case when $c(F, f^{\leftarrow}(\mu)) = 0$, the result holds, trivially.

I.2.9 PROPOSITION

Let (X,δ) be a fts, $\mathcal{F}$ be a prefilter consisting of closed fuzzy sets and $\mu \in I^X$. Then $\sup(\mu \wedge \mathrm{Adh}\,\mathcal{F}) \leq c(\mathcal{F},\mu)$.

PROOF

Let $\mathcal{B}$ be a base for $\mathcal{F}$ consisting of closed fuzzy sets.

Then $\sup \mu \wedge \mathrm{Adh}\,\mathcal{F} = \sup \mu \wedge \mathrm{Adh}\,\mathcal{B} = \sup \inf_{\nu \in \mathcal{B}} \mu \wedge \bar{\nu} = \sup \inf_{\nu \in \mathcal{B}} \mu \wedge \nu$.

But for any $x \in X$, $\inf_{\nu \in \mathcal{B}} (\mu \wedge \nu)(x) \leq \inf_{\nu \in \mathcal{B}} \sup \mu \wedge \nu$.

$$= c(\mathcal{B},\mu) = c(\mathcal{F},\mu).$$

Hence $\sup \inf_{\nu \in \mathcal{B}} \mu \wedge \nu \leq c(\mathcal{F},\mu)$.

So that, $\sup(\mu \wedge \mathrm{Adh}\,\mathcal{F}) \leq c(\mathcal{F},\mu)$.

I.2.10 DEFINITION [13]

Let $\mathcal{F}$ be a prefilter on X. Then $\mathcal{F}$ is called a prime prefilter, if whenever $\nu_1, \nu_2 \in I^X$ and $\nu_1 \vee \nu_2 \in \mathcal{F}$, we have either $\nu_1 \in \mathcal{F}$ or $\nu_2 \in \mathcal{F}$.

It is clear that if $\mathcal{F}$ is a prime prefilter and

$\nu_1 \vee \nu_2 \vee \dots \vee \nu_n \in \mathcal{F}$, where $\nu_i \in I^X$ for each $i \in [n]$, then $\nu_i \in F$ for some $i \in [n]$.

I.2.11 LEMMA [1,13]

Let $\mathcal{F}$ be a prefilter. Then there exists a prime prefilter $\mathcal{G}$ such that $\mathcal{F} \subseteq \mathcal{G}$ and $c(\mathcal{F}) = c(\mathcal{G})$.

PROOF

We assume $c(\mathcal{F}) > 0$.

The collection $B = \{\nu^{\leftarrow}\langle\epsilon,1] : \nu \in \mathcal{F},\ 0 < \epsilon < c(\mathcal{F})\}$ is a filterbase. Let U be a ultra–filter finer than B.

Then $\mathcal{B} = \{\nu \wedge 1_A : \nu \in \mathcal{F},\ A \in U\}$ is a prefilterbase.

Let $\mathcal{G} = \langle\mathcal{B}\rangle$.

Clearly $\mathcal{F} \subseteq \mathcal{G}$ and hence $c(\mathcal{F}) \geq c(\mathcal{G})$.

On the other hand, let $0 < \epsilon < c(\mathcal{F})$.

For any $\nu \in \mathcal{F}$, $A \in U$ we have $A \cap \nu^{\leftarrow}\langle\epsilon,1] \neq \emptyset$, and hence $\sup \nu \wedge 1_A > \epsilon$.

It follows that, $c(\mathcal{G}) = c(\mathcal{B}) = \inf_{\substack{A \in U \\ \nu \in \mathcal{F}}} \sup \nu \wedge 1_A \geq \epsilon$.

Since this holds for any ϵ such that $0 < \epsilon < c(\mathcal{F})$, we have that $c(\mathcal{G}) \geq c(\mathcal{F})$.

Therefore $c(\mathcal{G}) = c(\mathcal{F})$

It remains to prove that $\mathcal{G}$ is prime.

Let $\nu_1 \vee \nu_2 \in \mathcal{G}$. Then there exists $\nu \in \mathcal{F}$, $A \in U$ such that $\nu \wedge 1_A \leq \nu_1 \vee \nu_2$.

Let $A_1 = \{x \in X : (\nu \wedge 1_A)(x) \leq \nu_1(x)\}$ and $A_2 = \{x \in X : (\nu \wedge 1_A)(x) \leq \nu_2(x)\}$.

Then $A_1 \cup A_2 = X$, so either $A_1 \in U$ or $A_2 \in U$. Suppose $A_1 \in U$; then $\nu \wedge 1_{A \cap A_1} \in \mathcal{B}$ and $\nu \wedge 1_{A \cap A_1} \leq \nu_1$, so that $\nu_1 \in \mathcal{G}$. Similarly $A_2 \in U$ leads to $\nu_2 \in \mathcal{G}$.

Hence $\mathcal{G}$ is prime.

If $c(\mathcal{F}) = 0$ the proof is similar but one considers instead the filterbase

$B = \{\nu^{\leftarrow}\langle 0,1] : \nu \in \mathcal{F}\}$.

Let $\{X_j : j \in J\}$ be a family of non–void set and $X = \prod_{j \in J} X_j$. Let $\mathcal{F}$ be a prefilter on X and $\Pi_j : X \to X_j$ $(j \in J)$ be the projection maps.

Since for each $\nu \in \mathcal{F}$, $\nu \neq 0$, we have that $\Pi_j^{\rightarrow}(\nu) \neq 0$, for each $j \in J$.

Let $\nu_1, \nu_2 \in \mathcal{F}$.

Then $\Pi_j^{\rightarrow}(\nu_1 \wedge \nu_2) \leq \Pi_j^{\rightarrow}(\nu_1) \wedge \Pi_j^{\rightarrow}(\nu_2)$ and hence the collection $\{\Pi_j^{\rightarrow}(\nu) : \nu \in \mathcal{F}\}$ is a prefilterbase on X_j $(j \in J)$.

We denote the prefilter generated by $\{\Pi_j^{\rightarrow}(\nu) : \nu \in \mathcal{F}\}$ on X_j $(j \in J)$ by $\mathcal{F}_j$.

We can now give

<u>I.2.12</u> <u>THEOREM</u> [1,13]

For each $j \in J$, let $\mu_j \in I^{X_j}$ and let $\mu = \prod_{j \in J} \mu_j$.

If $\mathcal{F}$ is a prefilter on X, then $\operatorname{Inf}_{j \in J} c(\mathcal{F}_j, \mu_j) \geq c(\mathcal{F}, \mu)$.

<u>PROOF</u>

Suppose $\mathcal{F}$ is a prefilter on X, and let $t \in X_j$.

Then $\Pi_j^{\rightarrow}(\mu)(t) = \sup\{\mu(f) : f(j) = t\}$

$\leq \sup\{\mu_j(f(j)) : f(j) = t\} = \mu_j(t)$.

Now, for any $\nu \in \mathcal{F}$, we have $\mu \wedge \nu \leq \mu$ and $\mu \wedge \nu \leq \nu$.

By Proposition 0.1.19 (d) we have $\Pi_j^{\rightarrow}(\mu \wedge \nu) \leq \Pi_j^{\rightarrow}(\mu)$ and $\Pi_j^{\rightarrow}(\mu \wedge \nu) \leq \Pi_j^{\rightarrow}(\nu)$ for each $j \in J$.

Therefore $\Pi_j^{\rightarrow}(\mu \wedge \nu) \leq \Pi_j^{\rightarrow}(\mu) \wedge \Pi_j^{\rightarrow}(\nu)$, for each $j \in J$.

But $\Pi_j^{\rightarrow}(\mu)(t) \leq \mu_j(t)$, for all $j \in J$, $t \in X_j$ implies that $\Pi_j^{\rightarrow}(\mu) \leq \mu_j$, for all $j \in J$.

Thus, $\Pi_j^{\rightarrow}(\mu \wedge \nu) \leq \Pi_j^{\rightarrow}(\mu) \wedge \Pi_j^{\rightarrow}(\nu) \leq \mu_j \wedge \Pi_j^{\rightarrow}(\nu)$, for all $j \in J$.

Using Proposition 0.1.19 (c) we obtain

$c(\mathcal{F},\mu) = \inf_{\nu \in \mathcal{F}} \sup_{f \in X} (\mu \wedge \nu)(f) = \inf_{\nu \in \mathcal{F}} \sup_{t \in X_j} \Pi_j^{\rightarrow}(\mu \wedge \nu)(t)$.

$\leq \inf_{\nu \in \mathcal{F}} \sup_{t \in X_j} (\mu_j \wedge \Pi_j^{\rightarrow}(\nu))(t) = \inf_{\nu \in \mathcal{F}} \sup \mu_j \wedge \Pi_j^{\rightarrow}(\nu) = c(\mathcal{F}_j, \mu_j)$. Since this is true for each $j \in J$, we conclude that $\inf_{j \in J} c(\mathcal{F}_j, \mu_j) \geq c(\mathcal{F}, \mu)$.

I.2.13 THEOREM [1,13]

Let $\mathcal{F}$ be a prime prefilter on X. Then (i) $\text{Adh}\,\mathcal{F} = \prod_{j\in J} \text{Adh}\,\mathcal{F}_j$, and

(ii) for each $j \in J$, $\mathcal{F}_j$ is prime.

PROOF

(i) We first note that $\prod_{j\in J} \text{Adh}\,\mathcal{F}_j = \inf_{j\in J} \Pi_j^{\leftarrow} (\text{Adh}\,\mathcal{F}_j)$.

For each $j \in J$ and $\nu \in \mathcal{F}$ we have:

$\Pi_j^{\leftarrow} (\text{Adh}\,\mathcal{F}_j) = \Pi_j^{\leftarrow} (\inf_{\nu\in\mathcal{F}} \overline{\Pi_j^{\rightarrow}(\nu)}) = \inf_{\nu\in\mathcal{F}} \Pi_j^{\leftarrow} (\overline{\Pi_j^{\rightarrow}(\nu)})$

$\geq \inf_{\nu\in\mathcal{F}} \Pi_j^{\leftarrow} (\Pi_j^{\rightarrow}(\bar{\nu}))$, where the last inequality is true since the continuity of Π_j

implies $\Pi_j^{\rightarrow}(\bar{\nu}) \leq \overline{\Pi_j^{\rightarrow}(\nu)}$.

It then follows that $\Pi_j^{\leftarrow} (\text{Adh}\,\mathcal{F}_j) \geq \inf_{\nu\in\mathcal{F}} \bar{\nu} = \text{Adh}\,\mathcal{F}$.

Taking the infimum over all $j \in J$ we get $\prod_{j\in J} \text{Adh}\,\mathcal{F}_j \geq \text{Adh}\,\mathcal{F}$.

On the other hand, let $\nu_0 \in \mathcal{F}$ and let λ be a basic closed fuzzy set with $\nu_0 \leq \lambda$.

Then λ has the form $\lambda = \max_{1\leq i\leq n} \Pi_{j_i}^{\leftarrow} (\lambda_{j_i})$, where λ_{j_i} is closed in X_{j_i} $(i \in [n])$.

Since $\mathcal{F}$ is prime and $\lambda \in \mathcal{F}$, we have $\Pi_{j_k}^{\leftarrow} (\lambda_{j_k}) \in \mathcal{F}$ for some $k \in [n]$.

Hence $\lambda_{j_k} = \Pi_{j_k}^{\rightarrow} (\Pi_{j_k}^{\leftarrow}(\lambda_{j_k})) \in \mathcal{F}_k$ and we have $\inf_{j\in J} \Pi_j^{\leftarrow} (\text{Adh}\,\mathcal{F}_j) \leq \Pi_{j_k}^{\leftarrow} (\text{Adh}\,\mathcal{F}_{j_k}) \leq \Pi_{j_k}^{\leftarrow} (\bar{\lambda}_{j_k})$

$= \Pi_{j_k}^{\leftarrow} (\lambda_{j_k}) \leq \lambda$.

This holds for all basic closed fuzzy sets λ such that $\nu_0 \leq \lambda$.

So we have $\inf_{j\in J} \Pi_j^{\leftarrow} (\text{Adh}\,\mathcal{F}_j) \leq \bar{\nu}_0$.

Since $\nu_0 \in \mathcal{F}$ was chosen arbitrarily, we have $\prod_{j\in J} (\text{Adh}\,\mathcal{F}_j) = \inf_{j\in J} \Pi_j^{\leftarrow} (\text{Adh}\,\mathcal{F}_j) \leq \text{Adh}\,\mathcal{F}$.

The proof of (i) is complete.

(ii) Suppose $\lambda_1, \lambda_2 \in I^{X_j}$, $\lambda_1 \vee \lambda_2 \in \mathcal{F}_j$. We can choose $\nu \in \mathcal{F}$ such that $\Pi^{\rightarrow}_j(\nu) \leq \lambda_1 \vee \lambda_2$.
Therefore $\Pi^{\leftarrow}_j(\lambda_1) \vee \Pi^{\leftarrow}_j(\lambda_2) = \Pi^{\leftarrow}_j(\lambda_1 \vee \lambda_2) \geq \Pi^{\leftarrow}_j(\Pi^{\rightarrow}_j(\nu)) \geq \nu$ and hence
$\Pi^{\leftarrow}_j(\lambda_1) \vee \Pi^{\leftarrow}_j(\lambda_2) \in \mathcal{F}$.

Since $\mathcal{F}$ is prime, either $\Pi^{\leftarrow}_j(\lambda_1) \in \mathcal{F}$ or $\Pi^{\leftarrow}_j(\lambda_2) \in \mathcal{F}$.
Hence either $\lambda_1 = \Pi^{\rightarrow}_j(\Pi^{\leftarrow}_j(\lambda_1)) \in \mathcal{F}_j$ or $\lambda_2 = \Pi^{\rightarrow}_j(\Pi^{\leftarrow}_j(\lambda_2)) \in \mathcal{F}_j$.
Therefore $\mathcal{F}_j$ is prime.

I.3 *THE RELATIONSHIP BETWEEN THE PREFILTER AND THE FUZZY NET THEORIES.*

In [13] Lowen introduced the theory of convergence using prefilters, he then proved in [15] that his theory and the one using fuzzy nets [17] are equivalent in the sense that one completely determines the other. For further details we refer to [13], [15], and [17].

In Section I.1 and Section I.2 we gave the definitions of a fuzzy net and a prefilter on X, respectively, and discussed some of their properties.
The purpose of this section is to see how the two theories are related. We will show how a prefilter can be constructed from a given fuzzy net and vice versa ([15]).
We proceed as follows:
Given a fuzzy net $S = \{S_n : n \in D\}$ in X let $\nu_N = \sup_{n \geq N} S_n$ $(N \in D)$ and $\mathcal{B} = \{\nu_N : N \in D\}$.
Then clearly $\mathcal{B} \neq \emptyset$ and $0_X \notin \mathcal{B}$.
Now, let $\nu_{N_1}, \nu_{N_2} \in \mathcal{B}$.
Since $(D, \geq)$ is a directed set, $N_1, N_2 \in D$ implies that there is an $N \in D$ with $N \geq N_1$ and $N \geq N_2$.

Therefore, $\nu_{N_1} \wedge \nu_{N_2} = \sup_{n \geq N_1} S_n \wedge \sup_{m \geq N_2} S_m$

$$\geq \sup_{n \geq N} S_n \wedge \sup_{m \geq N} S_m = \nu_N.$$

Since $\nu_N \in \mathcal{B}$ we have that $\mathcal{B}$ is a prefilter base.

Let $P(S) = \{\nu \in I^X : \sigma \leq \nu \text{ for some } \sigma \in \mathcal{B}\}$ i.e. $P(S) = \langle\mathcal{B}\rangle$..

Then P(S) is a prefilter, called the prefilter associated with the fuzzy net S.

On the other hand, starting with a prefilter $\mathcal{F}$ on X, let $D = \{(x,\mu) : \mu \in \mathcal{F} \text{ and } \mu(x) > 0\}$ and define an order relation "$\geq$" on D as follows: If $(x_1, \mu_1), (x_2, \mu_2) \in D$, then $(x_1,\mu_1) \geq (x_2, \mu_2)$ if and only if $\mu_1 \leq \mu_2$.

It can easily be verified that "$\geq$" is a partial order on D.

We claim that $(D,\geq)$ is a directed set. In fact, if $(x_1, \mu_1), (x_2, \mu_2) \in D$, let $\mu_3 = \mu_1 \wedge \mu_2$ and choose x_3 such that $\mu_3(x_3) > 0$. Then the fact that $(x_1, \mu_1), (x_2, \mu_2) \in D$ implies that $\mu_1, \mu_2 \in \mathcal{F}$, $\mu_3 = \mu_1 \wedge \mu_2 \in \mathcal{F}$. From $\mu_3(x_3) > 0$ we therefore have that $(x_3\ \mu_3) \in D$. It is obvious from the definition of μ_3 that we have $\mu_3 \leq \mu_2$ and $\mu_3 \leq \mu_1$.

Therefore $(x_3, \mu_3) \geq (x_1, \mu_1)$ and $(x_3, \mu_3) \geq (x_2, \mu_2)$.

Let $N(\mathcal{F}) : D \longrightarrow X^*$ be given by

$$N(\mathcal{F})\ (x,\mu) = \mu(x)\ 1_x.$$

Then $N(\mathcal{F}) = \{\mu(x)1_x : (x,\mu) \in D\}$ is a fuzzy net in X, called the fuzzy net associated with the prefilter $\mathcal{F}$.

<u>I.3.1</u> <u>PROPOSITION</u>

Let X be a non–void set, $\mathcal{F}$ be a prefilter on X and S a fuzzy net in X. Then $P(N(\mathcal{F})) = \mathcal{F}$.

PROOF

By definition, $P(N(\mathcal{F}))$ is the prefilter generated by the prefilterbase

$\mathcal{B} = \{\nu(x,\mu) : (x,\mu) \in D\}$ where for each $(x,\mu) \in D$, $\nu(x,\mu) = \sup \{\mu_1(x_1)1_{x_1} : (x_1,\mu_1) \geq (x,\mu)\}$

$= \sup \{ \mu_1(x_1)\, 1_{x_1} : \mu_1 \leq \mu \text{ and } \mu_1(x_1) > 0\}$

$= \sup \{ \mu_1(x_1)\, 1_{x_1} : \mu_1 \leq \mu \text{ and } x_1 \in \mu_1^{\leftarrow} \langle 0,1]\}$

$$= \sup_{\substack{\mu_1 \leq \mu \\ \mu_1 \in \mathcal{F}}} \ \sup_{x_1 \in \mu_1^{\leftarrow}\langle 0,1]} \mu_1(x_1)\, 1_{x_1} = \sup_{\substack{\mu_1 \in \mathcal{F} \\ \mu_1 \leq \mu}} \mu_1 = \mu .$$

Hence, $P(N(\mathcal{F}))$ $= \{\nu \in I^X : \sigma \leq \nu \text{ for some } \sigma \in \mathcal{B}\}$

$= \{\nu \in I^X : \mu \leq \nu \text{ for some } \mu \in \mathcal{F}\}$

$= \mathcal{F}$.

The next theorem shows that the adherence points of the prefilter associated with a given fuzzy net are precisely the cluster points of that net and vice versa.

I.3.2 THEOREM

Let (X,δ) be a fts, $\mathcal{F}$ a prefilter on X and S a fuzzy net in X. Then

(i) $\alpha 1_x \in \text{Adh } P(S)$ if and only if $S \infty \alpha 1_x$

(ii) $\alpha 1_x \in \text{Adh } \mathcal{F}$ if and only if $N(\mathcal{F}) \infty \alpha 1_x$.

PROOF

(i)

"$\Longrightarrow$" suppose $\alpha 1_x \in \text{Adh } P(S)$ and σ is an R–nbd of $\alpha 1_x$.

Then by definition 0.2.3 there exists $\mu \in \delta'$ such that $\alpha 1_x \notin \mu$ and $\mu \leq \sigma$.

Hence $\mu = \bar{\mu}$ and $\alpha 1_x \notin \bar{\mu}$.

From $\alpha 1_x \in \text{Adh } P(S)$ we get $\mu \notin P(S)$. Hence, for each $N \in D$, $\mu \not\geq \nu_N = \sup_{n \geq N} S_n$, i.e. for each $N \in D$ there is $n \geq N$ such that $S_n \notin \mu$. Hence, for each $N \in D$ there is $n \geq N$ such that $S_n \notin \sigma$. Which implies that $S \notin \sigma$, frequently.

Since σ was chosen arbitrarily we have that $S \infty \alpha 1_x$.

"$\Leftarrow$" Suppose $S \infty \alpha 1_x$ and let $\nu \in P(S)$.

Then there exists some $N \in D$ such that $\nu \geq \nu_N$.

By definition of ν_N we have that $S_n \in \nu_N$ for each $n \geq N$.

Thus $S_n \in \nu$ for each $n \geq N$, i.e. S is eventually in ν.

Suppose $\alpha 1_x \notin \bar{\nu}$.

Then the fact that $\bar{\nu}$ is closed would imply that $\bar{\nu}$ is an R–nbd of $\alpha 1_x$.

Since $\alpha 1_x$ is a cluster point of S, we have that $S \notin \bar{\nu}$ frequently.

By Proposition I.1.7 (ii) S is not eventually in $\bar{\nu}$ which clearly contradicts the fact that S is eventually in ν. Therefore $\alpha 1_x \in \bar{\nu}$.

Since this is true for each $\nu \in P(S)$ we have that $\alpha 1_x \in \text{Adh } P(S)$.

(ii) $\alpha 1_x \in \text{Adh } \mathcal{F}$ if and only if $\alpha 1_x \in \text{Adh } P(N(\mathcal{F}))$ (by Proposition I.3.1) if and only if $N(\mathcal{F}) \infty \alpha 1_x$ (by (i) above).

We also have a simple

<u>I.3.3</u> <u>THEOREM</u>

Let (X,δ) be a fts, $\mathcal{F}$ a prefilter on X and $S = \{S_n : n \in D\}$ a fuzzy net in X. Then

(i) $c(P(S)) = c(S)$

(ii) $c(N(\mathcal{F})) = c(\mathcal{F})$.

PROOF

(i) It is clear from Section I.1 that $c(\mathcal{B}) = c(<\mathcal{B}>) = c(P(S))$ where $\mathcal{B} = \{\nu_N : N \in D\}$ and for each $N \in D$, $\nu_N = \sup_{n \geq N} S_n$.

Therefore we only need to show that $c(\mathcal{B}) = c(S)$.

Now, $c(\mathcal{B}) = \inf_{\nu \in \mathcal{B}} \sup \nu = \inf_{N \in D} \sup \nu_N = \inf_{N \in D} \sup (\sup_{n \geq N} S_n) = \inf_{N \in D} \sup_{n \geq N} \vee (S_n) = c(S)$.

The proof of (i) is complete.

The proof of (ii) is similar and hence omitted.

In Section 0.2 we gave several definitions of a T_2 fts, and discussed the relationships between them (Theorem 0.2.17). In Section I.1 we showed that for definitions 0.2.12, 0.2.13, 0.2.14, and 0.2.16, the T_2 property is equivalent to the statement; "No fuzzy net converges to two fuzzy points with different supports" (Theorem I.1.13).

In [13] Lowen gave a definition of a limit of a prefilter as follows:

I.3.4 DEFINITION [13]

Let $\mathcal{F}$ be a prefilter and $P_m(\mathcal{F})$ be the set of all minimal prime prefilters finer than $\mathcal{F}$, then we define the limit of $\mathcal{F}$ by $\lim \mathcal{F}(x) = \inf_{\mathcal{B} \in P_m(\mathcal{F})} \operatorname{Adh} \mathcal{B}$.

Lowen proposed a concept of a T_2 space which amounts to:

I.3.5 DEFINITION [14]

A fts (X, δ) is T_2 if and only if each prime prefilter has an adherence that is non–zero in at most one point.

The following result is true for definition 0.2.12, 0.2.13, 0.2.14, 0.2.15 and 0.2.16.

I.3.6 THEOREM

Let (X,δ) be fts. If for every prime prefilter $\mathcal{F}$ on X, the adherence of $\mathcal{F}$ is non–zero in at most one point then (X,δ) is T_2.

(Thus T_2 in the sense of Lowen implies T_2 in the sense of Definitions 0.2.12, 0.2.13, 0.2.14, 0.2.15, 0.2.16).

PROOF

Suppose the hypothesis of the theorem is true and let S be a fuzzy net in X such that $S \to x_\alpha$ and $S \to y_c$ for some $x_\alpha, y_c \in X^*$.

We only need to show that x = y. The result then follows from Theorems 0.2.13 and 0.2.14.

From $S \to x_\alpha$, we have by [15] Theorem 4.10 is that $x_\alpha \in \lim P(S)$.

Similarly $y_c \in \lim P(S)$.

By Definition 1.3.4 we have $\lim P(S) = \inf_{\mathcal{G} \in P_m(P(S))} \text{Adh}\, \mathcal{G}$.

Hence $x_\alpha, y_c \in \text{Adh}\, \mathcal{G}$, for each $\mathcal{G} \in P_m\,(P(S))$.

By the hypothesis of the Theorem, x = y.

CHAPTER II

α AND α^*COMPACTNESS

INTRODUCTION

In this chapter we introduce two types of fuzzy compactness, namely, α–compactness and α^*–compactness ($\alpha \in I$). In Section 1 we discuss their properties and show that α–compactness is a good extension, while α^*–compactness is not. In the last section we obtain a Tychonoff Product Theorem for α–compactness and show by means of an example that for α^*–compactness, this is not true in general.

II.1 *α AND α^*–COMPACTNESS.*

II.1.1 DEFINITION [3].

Let (X,δ) be a fts and $\alpha \in I$. The collection $\mathcal{U} \subseteq \delta$ will be called an α–shading (resp. α^*–shading) of X if, for each $x \in X$ there exists $\mu \in \mathcal{U}$ with $\mu(x) > \alpha$ (resp. $\mu(x) \geq \alpha$). A subcollection $\mathcal{F}$ of an α–shading (resp. α^*–shading) $\mathcal{U}$ of X that is also an α–shading (resp. α^*–shading) is called an α–subshading (resp. α^*–subshading) of $\mathcal{U}$.

II.1.2 DEFINITION [3].

(α) Let (X, δ) be a fts and $\alpha \in [0,1\rangle$. Then (X,δ) is said to be α–compact if each α–shading of X has a finite α–subshading.

(α^*) Let (X, δ) be a fts and $\alpha \in \langle 0,1]$. Then (X, δ) is said to be α^*–compact if each α^*–shading of X has a finite α^*–subshading.

It is clear from the above definitions that

(i) If X is finite then every fts (X, δ) is both α and α^*–compact.

(ii) If δ is finite then every fts (X, δ) is both α and α^*–compact.

However, [3] Example 2.2 shows that in general no implications exists among these various notions of compactness.

II.1.3 DEFINITION [3].

Let $\alpha \in I$, and $\mathcal{U} \subseteq I^X$. Then $\mathcal{U}$ is called α–centered, if for all $\mu_1, \ldots\ldots, \mu_n \in \mathcal{U}$, there exists $x \in X$ such that $\mu_k(x) \geq \alpha'$, for all $k \in [n]$.

Also, $\mathcal{U}$ is called α^*–centered, if for all $\mu_1, \ldots\ldots, \mu_n \in \mathcal{U}$, there exists $x \in X$ such that $\mu_k(x) > \alpha'$, for all $k \in [n]$.

II.1.4 THEOREM

Let (X, δ) be a fts.

(α) (X, δ) is α–compact if and only if for every α–centered system $\mathcal{F}$ of closed fuzzy sets in X, there exists $x \in X$ such that $\nu(x) \geq \alpha'$, for all $\nu \in \mathcal{F}$.

(α^*) (X, δ) is α^*–compact if and only if, for every α^*–centred system $\mathcal{F}$ of closed fuzzy sets in X, there exists $x \in X$ such that $\nu(x) > \alpha'$ for all $\nu \in \mathcal{F}$.

PROOF (α)

$\Longrightarrow$: Suppose $\mathcal{F}$ is an α–centered system of closed fuzzy sets in X such that, for each $x \in X$ there is $\nu \in \mathcal{F}$ with $\nu(x) < \alpha'$. We will show that (X, δ) is not α–compact.

Let $\mathcal{U} = \{\nu' : \nu \in \mathcal{F}\}$, so that $\mathcal{U} \subseteq \delta$.

We claim that $\mathcal{U}$ is an α–shading of X with no finite α–subshading. In fact, if $\nu_1', \ldots\ldots, \nu_n' \in \mathcal{U}$, then the fact that $\mathcal{F}$ is α–centered implies that there is an $x \in X$ such that $\nu_k(x) \geq \alpha'$, for all $k \in [n]$; i.e. $\nu_k'(x) \leq \alpha$, for all $k \in [n]$. It is evident from the definition of $\mathcal{F}$ that $\mathcal{U}$ is an α–shading of X.

Therefore (X, δ) is not α–compact.

$\Leftarrow$ Suppose $\mathcal{U}$ is an α–shading of X with no finite α–subshading.

Then $\mathcal{F} = \{\mu' : \mu \in \mathcal{U}\}$ is a collection of closed fuzzy sets in X.

We claim that $\mathcal{F}$ is α–centered. In fact, if $\mu'_1, \ldots, \mu'_n \in \mathcal{F}$ then $\{\mu_1, \ldots, \mu_n\} \subseteq \mathcal{U}$ is not an α–shading of X. That is, there exists an $x \in X$ such that $\mu_k(x) \leq \alpha$ for all $k \in [n]$. Hence $\mu'_k(x) \geq \alpha'$, for all $k \in [n]$.

But the fact that $\mathcal{U}$ is an α–shading of X implies that, for each $x \in X$ there is $\mu \in \mathcal{U}$ with $\mu(x) > \alpha$.

Hence $\mu' \in \mathcal{F}$ and $\mu'(x) < \alpha'$.

The proof of (α) is complete.

The proof of (α^*) is similar and is hence omitted.

As a consequence of Theorem II.1.4 we give

<u>II.1.5</u> <u>COROLLARY</u>

Let (X, δ) be a fts.

(α) (X, δ) is α–compact if and only if for every α–centered system $\mathcal{F}$ of fuzzy sets in X, there is an $x \in X$ such that $\bar{\lambda}(x) \geq \alpha'$ holds for all $\lambda \in \mathcal{F}$.

(α^*) (X, δ) is α^*–compact if and only if for every α^*–centered system $\mathcal{F}$ of fuzzy sets in X, there is an $x \in X$ such that $\bar{\lambda}(x) > \alpha'$ holds for all $\lambda \in \mathcal{F}$.

Next we show that α and α^*–compactness are inherited by closed "crisp" subsets. We would like to point out here that the fact that this is only true for "crisp" subsets makes these notions very restrictive. Nonetheless we give

<u>II.1.6</u> <u>THEOREM</u>

Let (X, δ) be an fts and $F \subseteq X$ closed.

(α) If (X, δ) is α–compact, then F is α–compact as a subspace of (X, δ).

(α^*) If (X, δ) is α^*–compact, then F is α^*–compact as a subspace of (X, δ).

PROOF (α)

Suppose (X,δ) is α–compact and $F \subseteq X$.

Then by definition 0.1.6, (F,δ_F) is a subspace of (X,δ), where $\delta_F = \{\mu \wedge 1_F : \mu \in \delta\}$.

We will show that (F,δ_F) is α–compact.

Let $\mathcal{U} \subseteq \delta_F$ be an α–shading of F.

We claim that $\mathcal{B} = \{\mu \vee 1_{F'} : \mu \in \mathcal{U}\}$ is an α–shading of X.

In fact, if $x \in X$ then either (i) $x \in F$ or (ii) $x \in F'$.

(i) Case $x \in F$

Then the fact that $\mathcal{U}$ is an α–shading of F implies that there exists $\mu \in \mathcal{U}$ with $\mu(x) > \alpha$.

Therefore $\mu \vee 1_{F'} \in \mathcal{B}$ and $(\mu \vee 1_{F'})(x) = \mu(x) \vee 1_{F'}(x) = \mu(x) \vee 0 = \mu(x) > \alpha$.

(ii) Case $x \in F'$

Then $1_{F'}(x) = 1$, so that for each $\mu \in \mathcal{U}$, $(\mu \vee 1_{F'})(x) = \mu(x) \vee 1_{F'}(x) = \mu(x) \vee 1 = 1 > \alpha$, and clearly $\mu \vee 1_{F'} \in \mathcal{B}$.

Therefore $\mathcal{B}$ is an α–shading of X.

By α–compactness of X we have that $\mathcal{B}$ has a finite α–subshading

$\{\mu_1 \vee 1_{F'}, \ldots, \mu_n \vee 1_{F'} : \mu_k \in \mathcal{U}, k \in [n]\}$.

We claim that $\{\mu_1, \ldots, \mu_n : \mu_k \in \mathcal{U}, k \in [n]\}$ is an α–subshading of $\mathcal{U}$. In fact, if $x \in F$ then $x \in X$, and hence there exists $k \in [n]$ such that $(\mu_k \vee 1_{F'})(x) > \alpha$.

i.e. $\alpha < \mu_k(x) \vee 1_{F'}(x) = \mu_k(x) \vee 0 = \mu_k(x)$.

Therefore $\{\mu_1, \ldots, \mu_n : \mu_k \in \mathcal{U}\}$ is a finite α–subshading of $\mathcal{U}$.

Hence (F, δ_F) is compact.

The proof of $(\alpha)^*$ is similar and is hence omitted.

Next we show that in a T_2 space (Definition ~~0.2.12, 0.2.13~~, 0.2.14, ~~0.2.15~~, 0.2.16), α–compact and α^*–compact subsets are closed.

II.1.7 THEOREM

Let S be a crisp subspace of a T_2 fts (X, δ).

(α) If S is α–compact, then S is closed in X.

(α^*) If S is α^*–compact, then S is closed in X.

PROOF (α).

Suppose S is α–compact. Let $x \in S'$.

We will show that there exists $\mu \in \delta$ such that $\mu(x) = 1$ and $\mu \leq 1_{S'}$.

~~(In view of Theorem 0.2.17 we only need to consider a T_2 space given by Definition 0.2.16)~~

For each $y \in S$ there exists $\mu_y^x, \nu_y^x \in \delta$ such that

$\mu_y^x(x) = \nu_y^x(y) = 1$ and $\mu_y^x \wedge \nu_y^x = 0$.

Hence $\Lambda_x = \{\nu_y^x \wedge 1_S : y \in S\}$ is an α–shading of S.

By α–compactness of S, Λ_x has a finite α–subshading $\{\nu_{y_1}^x \wedge 1_S, \ldots, \nu_{y_n}^x \wedge 1_S\}$.

Let $\mu^x = \mu_{y_1}^x \wedge \ldots \wedge \mu_{y_n}^x$.

Clearly, $\mu^x \in \delta$ and $\mu^x(x) = \mu_{y_1}^x(x) \wedge \ldots \wedge \mu_{y_n}^x(x) = 1$.

$$\begin{aligned} \text{Also } \mu^x \wedge (\nu_{y_1}^x \vee \ldots \vee \nu_{y_n}^x) &= (\mu_{y_1}^x \wedge \ldots \wedge \mu_{y_n}^x) \wedge (\nu_{y_1}^x \vee \ldots \vee \nu_{y_n}^x) \\ &\leq (\mu_{y_1}^x \wedge \nu_{y_1}^x) \vee \ldots \vee (\mu_{y_n}^x \wedge \nu_{y_n}^x) \\ &= 0 \end{aligned}$$

So that $\mu^x \wedge (\nu_{y_1}^x \vee \ldots \vee \nu_{y_n}^x) = 0$.

For each $z \in S$, there exists a $y_k \in S$ with $\nu_{y_k}^x(z) > \alpha \geq 0$, for some $k \in [n]$.

Then $\mu^x(z) = 0$ and hence $\mu^x \leq 1_{S'}$.

Therefore $1_{S'} = \sup \{\mu^x : x \in S'\}$ which implies that $1_{S'} \in \delta$.

Hence S is closed in X.

The proof of (α^*) is similar and is hence omitted.

Next we show that α and α^*–compactness are invariant under continuous mappings.

II.1.8 THEOREM

Let (X, δ) and (Y,σ) be fts, and $f : (X,\delta) \rightarrow (Y, \sigma)$ be a continuous mapping.

(α) If (X, δ) is α–compact, then $f^{\rightarrow}(X)$ is α–compact as a subspace of (Y, σ).

(α^*) If (X, δ) is α^*–compact, then $f^{\rightarrow}(X)$ is α^*–compact as a subspace of (Y, σ).

PROOF (α)

Suppose X is α–compact.

We assume, without loss of generality that f is a surjection.

Let $\mathcal{U} \subseteq \sigma$ be an α–shading of Y.

We show that $f^{\leftarrow}(\mathcal{U}) = \{f^{\leftarrow}(\mu): \mu \in \mathcal{U}\}$ is an α–shading of X.

Let $x \in X$, then $f(x) \in Y$, so that there exists $\mu \in \mathcal{U}$ such that $\mu(f(x)) > \alpha$. i.e. $f^{\leftarrow}(\mu)(x) > \alpha$.

Since f is continuous, we have that $f^{\leftarrow}(\mathcal{U}) \subseteq \delta$ and hence $f^{\leftarrow}(\mathcal{U})$ is an α–shading of X.

By α–compactness of X we have that $f^{\leftarrow}(\mathcal{U})$ has a finite α–subshading

$\{f^{\leftarrow}(\mu_1),\ldots, f^{\leftarrow}(\mu_n)\}$.

We claim that $\{\mu_1,\ldots, \mu_n\}$ is an α–subshading of $\mathcal{U}$. In fact, if $y \in Y$ then the fact that f is onto implies that there exists an $x \in X$ such that $f(x) = y$.

Hence there exists some $k \in [n]$ such that $f^{\leftarrow}(\mu_k)(x) > \alpha$.

That is, $\mu_k(y) = \mu_k(f(x)) = f^{\leftarrow}(\mu_k)(x) > \alpha$.

Therefore Y is α–compact.

The proof of (α^*) is similar and is hence omitted.

Lastly, we show that α–compactness is a "good extension" in the following sense:

II.1.9 THEOREM

Let $(X, \mathcal{T})$ be a topological space, $\alpha \in [0,1>$ and $K \subseteq X$. Then the following statements are equivalent.

(a) K is compact in $(X, \mathcal{T})$.

(b) K is α–compact as a subset of $(X, \omega(\mathcal{T}))$

(c) K is α–compact as a subset of $(X, \underline{\omega}(\mathcal{T}))$.

Before we prove Theorem II.1.9 we first need a simple

II.1.10 LEMMA

Let δ_1 and δ_2 be two fuzzy topologies on X such that $\delta_2 \subseteq \delta_1$.
Then if $K \subseteq X$ is α–compact (resp. α^*–compact) as a subspace of (X,δ_1) then it is α–compact (resp α^*–compact) as a subspace of (X,δ_2).

PROOF of Theorem II.1.9

(a) $\Rightarrow$ (b) : Suppose K is compact in $(X, \mathcal{T})$ and let $\mathcal{U} \subseteq \omega(\mathcal{T})_A = \{\mu_A : \mu \in \omega(\mathcal{T})\}$ be an α–shading of K.

Therefore for each $x \in K$ there exists $\mu \in \mathcal{U}$ such that $\mu(x) > \alpha$. i.e. $x \in \mu^{\leftarrow} <\alpha,1]$.

This together with the fact that $\mu \in \{\nu_A : \nu \in \omega(\mathcal{T})\}$ implies $\mu^{\leftarrow}<\alpha, 1] \in \mathcal{T}$, so that $\{\mu^{\leftarrow}<\alpha,1] : \mu \in \mathcal{U}\}$ is an open cover for K.

By compactness of K, there exists $\{\mu_i^{\leftarrow} <\alpha,1] : \mu_i \in \mathcal{U}, i \in [n]\}$ covering K. It is clear that $\{\mu_1, ..., \mu_n\}$ is an α–shading of K.

Therefore K is α–compact as a subspace of $(X, \omega(\mathcal{T}))$.

(b) $\Rightarrow$ (c) : Follows from the fact that $\underline{\omega}(\mathcal{T}) \subseteq \omega(\mathcal{T})$ together with Lemma II.1.10.

(c) $\Rightarrow$ (a) : Suppose K is α–compact as a subspace of $(X, \underline{\omega}(\mathcal{T}))$ and let $M \subseteq \mathcal{T}$ be a cover for K.

Therefore, for each $x \in K$, there exists $U \in M$ such that $x \in U$.

This implies $1_{U \cap K}(x) = 1 > \alpha$.

Hence $\{1_{U \cap K} : U \in M\} = \{1_U \wedge 1_K : U \in M\}$ is an α–shading for K.

By α–compactness we have $\{1_{U_1}, ..., 1_{U_n}\}$ α–shading K. It is clear that $\{U_1, ... U_n\}$ is an open cover for K.

Therefore K is compact in $(X, \mathcal{T})$.

α^*–compactness is not a good extension as seen in the next

II.1.11 EXAMPLE [12].

If X is infinite and $\mathcal{T}$ is the finite complement topology on X, then $(X, \omega(\mathcal{T}))$ is not α^*–compact for any $\alpha \in <0,1]$; indeed if α is $<0,1]$, choose a family $\{A_n : n \in \mathbb{N}\}$ of subsets of X such that for each $n \in \mathbb{N}$, A_n has a countable complement and such that $\{A_n : n \in \mathbb{N}\}$ covers X, but no finite subfamily does.

For each $n \in \mathbb{N}$ put $A'_n = \{x^n_1, x^n_2, ...\}$ and define a fuzzy set ν_n by $\nu_n(x) = \alpha$, for all $x \in A_n$ and $\nu_n(x^n_j) = \alpha - \frac{\alpha}{j}$, $j = 1, 2, ...$.

Now, let $n \in \mathbb{N}$. If $\beta < \alpha$ then there exists $j_0 \in \mathbb{N}$ such that $\alpha - \frac{\alpha}{j} > \alpha' \geq \alpha - \frac{\alpha}{j_0}$, for all $j > j_0$, and hence $\nu_n^{\leftarrow}[0,\beta] = \{x^n_1, ..., x^n_{j_0}\}$

On the other hand if $\beta \geq \alpha$ then $\nu_n^{\leftarrow}[0,\beta] = X$.

The fact that τ is the finite complement topology on X implies that $\nu_n \in \omega(\tau)$. Since this is true for each $n \in \mathbb{N}$ we have that for all $n \in \mathbb{N}$, $\nu_n \in \omega(\tau)$.

Let $x \in X$. Then the fact that $\{A_n : n \in \mathbb{N}\}$ is a cover for X implies that there exists an $n \in \mathbb{N}$ such that $x \in A_n$, and thus $\nu_n(x) = \alpha$.

This shows that $\{\nu_n : n \in \mathbb{N}\}$ is an α^*–shading of X in $\omega(\tau)$. But for any $K \in 2^{(\mathbb{N})}$, let $\mathcal{U} = \{\nu_k : k \in K\}$.

We claim that $\mathcal{U}$ is not an α^*–shading of X. In fact if $y \in X \setminus \bigcup_{k \in K} A_k$ then for all $k \in K$ there exists $j_k \in \mathbb{N}$ such that $y = x^k_{j_k}$ and hence $\nu_k(y) = \alpha - \frac{\alpha}{j_k} < \alpha$.

Therefore (X, $\omega(\tau)$) is not α^*–compact.

On the other hand we have that $(X, \mathcal{T})$ is compact.

II.2 *THE TYCHONOFF THEOREM*

We first show that for α–compactness, we have an extension of the Alexander Subbase lemma in general topology and show by means of an example that this is not true in general for α^*–compactness.

We will need

II.2.1 DEFINITION [3]

Let (X, δ) be a fts. The collection $\mathcal{U} \subseteq \delta$ is said to have the α–finite union property (α–FUP in short) if, for any finite sub–collection $\{\mu_1,..., \mu_n\} \subseteq \mathcal{U}$, there exists an $x \in X$ such that $\mu_1(x) \vee .... \vee \mu_n(x) \leq \alpha$.

The next Lemma will be used in the sequel.

II.2.2 LEMMA

Let (X, δ) be a fts and $\alpha \in [0,1>$. Then X is α–compact if and only if no $\mathcal{U} \subseteq \delta$ having the α–FUP is an α–shading of X.

PROOF

"$\Longrightarrow$" Suppose X is α–compact and let $\mathcal{U} \subseteq \delta$ have the α–FUP.

Suppose $\mathcal{U}$ is an α–shading of X. Then by α–compactness of X there exists $\{\mu_1,...., \mu_n\} \subseteq \mathcal{U}$ such that, for each $x \in X$ we have $\mu_k(x) > \alpha$, for some $k \in [n]$.

But then $\mu_1(x) \vee ..... \vee \mu_n(x) > \alpha$. Contradicting our assumption that $\mathcal{U}$ has the α–FUP.

"$\Longleftarrow$" Suppose no $\mathcal{U} \subseteq \delta$ having the α–FUP is an α–shading of X, and let $\mathcal{U} \subseteq \delta$ be an α–shading of X. By hypothesis of the theorem, $\mathcal{U}$ does not have the α–FUP.

Suppose $\mathcal{U}$ has no finite α–subshading. Then for every $\{\mu_1, ..., \mu_n\} \subseteq \mathcal{U}$ there exists a $y \in X$ with $\mu_k(y) \leq \alpha$, for all $k \in [n]$.

Then we have that $\mu_1(y) \vee .... \vee \mu_n(y) \leq \alpha$, contradicting the fact that $\mathcal{U}$ does not have the α–FUP.

II.2.3 THEOREM (ALEXANDER SUBBASE LEMMA)

Let (X, δ) be a fts, $\mathcal{S}$ be a subbase for the fuzzy topology δ and $\alpha \in [0,1>$. If every α–shading of X consisting of members of $\mathcal{S}$ has a finite α–subshading, then X is α–compact.

PROOF

Suppose $\mathcal{U} \subseteq \delta$ has the α– FUP and let

$$\mathcal{F} = \{ \mathcal{B} : \mathcal{U} \subseteq \mathcal{B} \subseteq \delta \text{ and } \mathcal{B} \text{ has the } \alpha\text{– FUP}\}.$$

Then $(\mathcal{F},\subseteq)$ is a non–empty partially ordered set that is inductive.

Appealing to Zorn's Lemma we have that $\mathcal{F}$ has a maximal element, $\mathcal{D}$.

To prove the thoerem, it suffices to show that $\mathcal{U}$ is not an α–shading of X. We can then apply Lemma II.2.2 to conclude that X is α–compact.

Since $\mathcal{U} \subseteq \mathcal{D} \subseteq \delta$ we need only show that $\mathcal{D}$ is not an α–shading of X.

We claim that no subcollection of $\mathcal{S}$ that has the α–FUP is an α–shading of X:– In fact if $\mathcal{C} \subseteq \mathcal{S}$ has the α– FUP and is also an α–shading of X, then by the hypothesis of the theorem, there would exist $\sigma_1,.... \sigma_n \in \mathcal{C}$ such that for all $x \in X$, $\sigma_k(x) > \alpha$, for some $k \in [n]$. Hence $\sigma_1(x) \vee .... \vee \sigma_n(x) > \alpha$ for all $x \in X$. This clearly contradicts the fact that $\mathcal{C}$ has the α–FUP.

Since $\mathcal{D}$ has the α–FUP, $\mathcal{D} \wedge \mathcal{S}$ is a subcollection of $\mathcal{S}$ that has the α–FUP. Hence $\mathcal{D} \wedge \mathcal{S}$ is not an α–shading of X. Which implies there exists $p \in X$ such that $\mu(p) \leq \alpha$, for all $\mu \in \mathcal{D} \wedge \mathcal{S}$.

We will show that $\nu(p) \leq \alpha$, for all $\nu \in \mathcal{D}$, so that $\mathcal{D}$ is not an α–shading of X (as we wanted to show).

In order to accomplish this, it is sufficient to show that for each $\sigma \in \mathcal{D}$, there exists $\mathcal{S}_\nu \subseteq \mathcal{D} \wedge \mathcal{S}$ with $\nu \leq \vee \mathcal{S}_\nu$. It will then follow that $\sigma(p) \leq \mu(p)$, for some $\mu \in \mathcal{S}_\sigma$, whence $\nu(p) \leq \alpha$.

We first show that $\delta \setminus \mathcal{D}$ satisfies the following two conditions: (i) if $\mu, \nu \in \delta\setminus\mathcal{D}$ then $\mu \wedge \nu \in \delta\setminus\mathcal{D}$ (ii) if $\mu \in \delta\setminus\mathcal{D}$ and $\nu \in \delta$ with $\mu \leq \nu$ then $\nu \in \delta\setminus\mathcal{D}$:–

(i) Let $\mu, \nu \in \delta \setminus \mathcal{D}$.

Then $\mathcal{D} \vee \{\mu\}$ and $\mathcal{D} \vee \{\nu\}$ do not have the α–FUP, by the maximality of $\mathcal{D}$.

Therefore, there exists $\sigma_1, ..., \sigma_{m+n} \in \mathcal{D}$ such that for each $x \in X$,

$\sigma_1(x) \vee .... \vee \sigma_m(x) \vee \mu(x) > \alpha$ and $\sigma_{m+1}(x) \vee ..... \vee \sigma_{m+n}(x) \vee \nu(x) > \alpha$.

Put $\sigma = \sigma_1 \vee ..... \vee \sigma_{m+n}$.

Then $(\sigma \vee \mu)(x) > \alpha$ and $(\sigma \vee \nu)(x) > \alpha$.

Thus, $(\sigma \vee (\mu \wedge \nu))(x) > \alpha$, so that $\mu \wedge \nu \notin \mathcal{D}$ which implies that $\mu \wedge \nu \in \delta \setminus\mathcal{D}$.

(ii) Suppose now that $\mu \in \delta \setminus\mathcal{D}$ and $\nu \in \delta$ with $\mu \leq \nu$.

Again $\mathcal{D} \vee \{\mu\}$ does not have the α–FUP.

Hence there exists $\sigma_1 ...., \sigma_1 \in \mathcal{D}$ such that for all $x \in X$,

$$(\alpha_1 \vee ..... \vee \sigma_n \vee \mu)(x) > \alpha.$$

But then, $(\alpha_1 \vee ..... \vee \sigma_n \vee \nu)(x) > \alpha$, for all $x \in X$.

Hence $\nu \in \delta \setminus\mathcal{D}$.

Now let $\sigma \in \mathcal{D}$ and $\mu_1, ..., \mu_n \in \delta$ such that $\mu_1 \wedge ..... \wedge \mu_n \leq \sigma$.

We claim that $\mu_k \in \mathcal{D}$ for some $k \in [n]$:–

In fact, if for all $k \in [n]$, $\mu_k \in \delta \setminus\mathcal{D}$, then by (i) we have that $\mu_1 \wedge ..... \wedge \mu_n \in \delta \setminus\mathcal{D}$, so that (ii) implies $\sigma \in \delta \setminus\mathcal{D}$, which contradicts the fact that $\sigma \in \mathcal{D}$.

To complete the proof let $\sigma \in \mathcal{D}$.

Then $\sigma \in \delta$ and $\sigma \neq 1$ (otherwise, if $\sigma = 1$ then the fact that $\mathcal{D}$ has the α–FUP would imply that there exists an $x \in X$ such that $\sigma(x) \leq \alpha$, which would then imply that $1 \leq \alpha$, a contradiction, since $\alpha \in [0,1>$).

Therefore by definition of a subbase for δ we have that: $\sigma = \bigvee_{j=J} \left[\lambda_{j_1} \wedge \ldots\ldots \wedge \lambda_{j_{n_j}}\right]$

where $\lambda_{j_k} \in \mathcal{S}$, for each $j \in J$ and $k \in [n_j]$.

Therefore for each $j \in J$ we have that $\lambda_{j_1} \wedge \ldots\wedge \lambda_{j_{n_j}} \leq \sigma$.

Hence there exists some $k_j \in [n_j]$ such that $\lambda_{j_{k_j}} \in \mathcal{D}$.

Thus, $\sigma \leq \bigvee_{j\in J} \lambda_{j_{k_j}}$.

The result therefore follows from the fact that each $\lambda_{j_{k_j}} \in \mathcal{D} \wedge \mathcal{S}$.

The Alexander subbase lemma is not true in general for α^*–compactness, as seen in:

<u>II.2.4</u> <u>EXAMPLE</u>

Let X be an infinite set. For each $x \in X$ and each $n \in \mathbb{N}$ define a fuzzy set $\sigma_{n_x} : X \longrightarrow I$ as follows:–

$$\sigma_{n_x}(y) = \begin{cases} \alpha - \frac{\alpha}{n}, & \text{if } y = x \\ 0 & \text{otherwise} \end{cases}$$

where $\alpha \in \langle 0,1]$ is fixed.

Let $\mathcal{S} = \{\sigma_{n_x} : n \in \mathbb{N}, x \in X\}$.

Then $\mathcal{S}$ forms a subbase for a unique fuzzy topology, $\delta(\mathcal{S})$ on X.

Since $\alpha \neq 0$, it is clear that for all $\sigma_{n_x} \in \mathcal{S}$ we have that $\sigma_{n_x}(y) < \alpha$, for all $y \in X$.

Therefore by definition there is no α–shading consisting of members of $\mathcal{S}$.

We will however show that X is not α^*–compact.

For each $x \in X$, let $\mu_x = \bigvee_{n=1}^{\infty} \sigma_{n_x}$.

We claim that $\{\mu_x : x \in X\}$ is an α–shading of X.

In fact, if $p \in X$, choose $\mu_p \in \{\mu_x : x \in X\}$. Then $\mu_p(p) = \bigvee_{n=1}^{\infty} \sigma_{n_p}(p) = \bigvee_{n=1}^{\infty} (\alpha - \frac{\alpha}{n}) \geq \alpha$.

But if $\{\mu_{x_1}, ..., \mu_{x_n}\}$ is any finite subcollection of $\{\mu_x : x \in X\}$, choose $y \in X \setminus \{x_1,.... x_n\}$.

Then for all $k \in [n]$ we have $\mu_{x_k}(y) = \bigvee_{n=1}^{\infty} \sigma_{n_{x_k}}(y) = 0 < \alpha$.

Therefore X is not α^*–compact.

The next result shows that the Tychonoff theorem holds for α–compactness.

II.2.5 THEOREM (TYCHONOFF THEOREM)

The product space of the indexed family $\{(X_j, \delta_j) : j \in J\}$ of fts is α–compact, if and only if, for each $j \in J$, (X_j, δ_j) is α–compact.

PROOF

The direct implication follows from Theorem II.1.8 and the fact that the projection maps are continuous and onto.

To prove the converse we apply Theorem II.2.3 to the subbase $\mathcal{S} = \{\Pi_j^{\leftarrow}(\mu_j): j \in J$ and $\mu_j \in \delta_j\}$ of the product fuzzy topology $\delta(\mathcal{S})$ on $X = \prod_{j=J} X_j$.

We will show that every α–shading of X consisting of members of $\mathcal{S}$ has a finite α–subshading, or equivalently, no $\mathcal{C} \subseteq \mathcal{S}$ having the α–FUP is an α–shading of X.

Suppose $\mathcal{C} \subseteq \mathcal{S}$ has the α– FUP and for each $j \in J$ let $\mathcal{U}_j = \{\mu \in \delta_j : \Pi_j^{\leftarrow}(\mu) \in \mathcal{C}\}$.

We claim that each $\mathcal{U}_j$ has the α– FUP.

In fact, if $\mu_1,...., \mu_n \in \mathcal{U}_j$, then the fact that $\mathcal{C}$ has the α–FUP implies that there exists an $f \in X$ such that $\Pi_j^{\leftarrow}(\mu_1)(f) \vee,...., \vee \Pi_j^{\leftarrow}(\mu_n)(f) \le \alpha$, i.e. $\mu_1(x_j) \vee .... \vee \mu_n(x_j) \le \alpha$, where $x_j = f(j)$.

Since X_j is α–compact for each $j \in J$, we have that $\mathcal{U}_j$ cannot be an α–shading of X_j.

Therefore for each $j \in J$, there exists $p_j \in X_j$ such that $\mu(p_j) \le \alpha$, for all $\mu \in \mathcal{U}_j$.

Define $p = \prod_{j=J} p_j \in X$ and for each $j \in J$ define $\mathcal{C}_j = \{\Pi_j^{\leftarrow}(\mu) : \mu \in \mathcal{U}_j\}$. Then $\mathcal{C} = \bigvee_{j \in J} \mathcal{C}_j$.

For each $j \in J$ and $\Pi_j^{\leftarrow}(\mu) \in \mathcal{C}_j$ we have $\Pi_j^{\leftarrow}(\mu)(p) = \mu(\Pi_j(p) = \mu(p_j) \leq \alpha$.

Therefore $\mathcal{C}$ is not an α–shading of X.

Next we show that for α^*–compactness, the Tychonoff Theorem does not hold. We will in fact give an example to show that α^*–compactness is not invariant under finite products.

II.2.6 EXAMPLE

Let X be any non–empty set, Y be a non–degenerate bounded closed interval of real numbers and $\alpha \in <0.1]$.

For each $\beta \in <0,1>$, define the fuzzy set

$$\nu_\beta : X \longrightarrow I \text{ by}$$
$$\nu_\beta(x) = \alpha(1-\beta), \text{ for all } x \in X.$$

Let $\mathcal{T}$ denote the usual topology on Y and for each $U \in \mathcal{T}$ define the fuzzy set

$$\sigma_U : X \longrightarrow I \text{ by}$$
$$\sigma_U(x) = \begin{cases} \alpha & \text{if } x \in U \\ 0 & \text{other wise} \end{cases}$$

Let δ_X be the fuzzy topology on X with subbase $\{\nu_\beta : \beta \in <0,1>\}$ and δ_Y be the fuzzy topology on Y with subbase $\{\sigma_U : U \in \mathcal{T}\}$.

Then $\delta_X = \{C \leq \alpha : C \text{ is a constant fuzzy set}\} \cup \{1_X\}$.

We claim that (X, δ_X) is α^*–compact.

In fact, if $\mathcal{U} \subseteq \delta_X$ is an α^*–shading of X, then for any fixed $x \in X$ there exists $\mu_x \in \mathcal{U}$ such that $\mu_x(x) \geq \alpha$.

Since $\mu_x \in \mathcal{U} \subseteq \delta_X$, we have by defintion of δ_X that $\mu_x = C$, where C is a constant fuzzy set $\leq \alpha$. Clearly $\{C\}$ is a finite α^*–subshading. Therefore (X, δ_X) is α^*–compact.

By the Heine–Borel Theorem, it is clear that $(Y, \mathcal{T})$ is compact.

We claim that (Y, δ_Y) is α^*–compact. In fact, if $\mathcal{U} \subseteq \delta_Y$, is an α^*–shading of Y, then for each $y \in Y$ there exists $\mu \in \mathcal{U}$ such that $\mu(y) \geq \alpha$, that is, $y \in \mu^{\leftarrow}[\alpha,1]$. Clearly, $\mu \in \mathcal{U} \subseteq \delta_Y$ implies $\mu^{\leftarrow}[\alpha,1] \in \mathcal{T}$.

Hence the collection $\{\mu^{\leftarrow}[\alpha,1] : \mu \in \mathcal{U}\}$ is an open cover for Y.

By compactness of Y, $\{\mu^{\leftarrow}[\alpha,1] : \mu \in \mathcal{U}\}$ has a finite subcover, $\{\mu_1^{\leftarrow}[\alpha,1],\dots,\mu_n^{\leftarrow}[\alpha,1]\}$.

It is easy to verify that $\{\mu_1, \dots, \mu_n\}$ is an α^*–subshading of $\mathcal{U}$.

Therefore (Y, δ_Y) is α^*–compact.

We will show that the product space $(X \times Y, \delta_X \times \delta_Y)$ is not α^*–compact.

For each $y \in Y$, let $B(y,\beta)$ be an open ball in Y of radius β and define a fuzzy set

$$\theta_y : X \times Y \longrightarrow I \text{ by}$$

$\theta_y = \vee \{\nu_\beta \times \sigma_{B(y,\beta)} : 0 < \beta < 1\}$ where $\nu_\beta \wedge \sigma_y = \Pi_1^{\leftarrow}(\nu_\beta) \wedge \Pi_2^{\leftarrow}(\sigma_U)$, $\Pi_1 : X \times Y \longrightarrow X$ and $\Pi_2 : X \times Y \longrightarrow Y$ are projection maps.

Since ν_β is fuzzy open in (X, δ_X) and $\sigma_{B(y,\beta)}$ is fuzzy open in (Y, δ_Y) we have that θ_y is fuzzy open in $(X \times Y, \delta_X \times \delta_Y)$.

It is clear from the definition of ν_β and that of σ_U that on $X \times \{y\}$, $\theta_y \geq \alpha$.

We claim $\{\theta_y : y \in Y\}$ is an α–shading of $X \times Y$.

In fact, if $z = (x,y) \in X \times Y$, then $\theta_y(z) \geq \alpha$.

That is, for each $x \in X$; $y \in Y$, there exists a θ_y such that $\theta_y(x,y) \geq \alpha$.

Therefore $\{\theta_y : y \in Y\}$ is an α^*–shading of $X \times Y$.

But for any $n \in \mathbb{N}$, $\{\theta_{y_1}, \dots\dots, \theta_{y_n}\}$ is not an α^*–shading of $X \times Y$.

In fact, if $y \in Y \setminus \{y_1,\dots, y_n\}$ then on $X \times \{y\}$ the fuzzy set $\theta_{y_k} = 0 < \alpha$, for all $k \in [n]$.

Therefore $\{\theta_y : y \in Y\}$ has no finite α–subshading.

That is, $(X \times Y, \delta_X \times \delta_Y)$ is not α^*–compact.

CHAPTER III

STRONG FUZZY AND ULTRA–FUZZY COMPACTNESS.

INTRODUCTION

In this chapter we examine yet another two definitions of fuzzy compactness. We discuss their properties and show that the two are good extensions and that the Tychonoff Theorem holds for these notions. We also give their characterisations in terms of fuzzy nets.

III.1. *STRONG FUZZY COMPACTNESS*

Motivated by the definition of α–compactness and its properties, we adopt the following definition.

III.1.1 DEFINITION [12].

A fts (X, δ) will be called strong fuzzy compact if it is α–compact for each $\alpha \in [0,1>$.

It is clear, from the above definition, that most of the results which we have for α–compactness are carried through to strong fuzzy compactness.

We will for example show that strong fuzzy compactness is invariant under continuous mappings.

III.1.2 THEOREM

Let (X, δ) and (Y,σ) be two fts, f : (X, δ) $\longrightarrow$ (Y, σ) a continuous mapping. If (X, δ) is strong fuzzy compact, then $f^{\rightarrow}(X)$ is strong fuzzy compact as a subspace of (Y, σ).

PROOF

Suppose (X, δ) is strong fuzzy compact.

Therefore (X, δ) is α–compact for each $\alpha \in [0,1>$.

By Theorem II.1.8 we have that $f^{\rightarrow}(X)$ is α–compact for each $\alpha \in [0,1>$. Thus, $f^{\rightarrow}(X)$ is strong fuzzy compact, as a subspace of (Y,σ).

III.1.3 LEMMA

Let δ_1 and δ_2 be two fuzzy topologies on X such that $\delta_2 \subseteq \delta_1$. If K $\subseteq$ X is strong fuzzy compact as a subspace of (X,δ_1) then it is strong fuzzy compact as a subspace of (X,δ_2).

Strong fuzzy compactness is a good extension, as seen in the next

III.1.4 THEOREM

Let $(X, \mathcal{T})$ be a topological space and K $\subseteq$ X. Then the following statements are equivalent.

(a) K is compact in $(X, \mathcal{T})$

(b) K is strong fuzzy compact as a subspace of $[X, \omega(\mathcal{T})]$

(c) K is strong fuzzy compact as a subspace of $[X, \underline{\omega}(\mathcal{T})]$.

d) K is β–compact as a subspace of $[X,\omega(\mathcal{T})]$ for some $\beta \in [0,1>$.

PROOF

Apply Theorem II.1.9 for each level $\alpha \in [0,1>$.

III.1.5 THEOREM

Let (X, δ) be a fts; then (X, δ) is strong fuzzy compact if and only if one of the following statements hold:

(1) For each $\alpha \in [0,1>$, if $\mathcal{U} \subseteq \delta$ and $\vee\, \mathcal{U} > \alpha$ then $\mathcal{U}$ has a finite subfamily ψ with $\vee\, \psi > \alpha$.

(2) For each $\alpha \in <0,1]$, if $\mathcal{U} \subseteq \delta'$ and $\wedge\, \mathcal{U} < \alpha$ then $\mathcal{U}$ has a finite subfamily ψ with $\wedge\, \psi < \alpha$

PROOF

Follows at once from the fact that (1) and (2) are equivalent, together with the definition of strong fuzzy compactness.

Up to now, we have been characterising strong fuzzy compactness in terms of α–shadings. Next we give a characterisation of strong fuzzy compactness by fuzzy nets.

III.1.6 THEOREM

The space (X, δ) is strong fuzzy compact if and only if every constant α–net has a cluster point with value α, for each $\alpha \in <0,1]$.

PROOF

"$\Rightarrow$" Let $S = \{S_n : n \in D\}$ be a constant α–net without a cluster point of value α in X. Therefore, for each $x \in X$, $\alpha 1_x$ is not a cluster point of S, i.e. there exists an $N_x \in D$ and an R–nbd, ν_x, of $\alpha 1_x$ with $S_n \notin \nu_x$, for all $n \geq N_x$.

Suppose $x^1, ..., x^k \in X$.

Then there exists an $N \in D$ such that $S_n \notin \nu_{x^i}$, for all $i \in [k]$, and all $n \geq N$.

By definition of an R–nbd we have that for each $x \in X$ there exists $\sigma_x \in \delta'$ such that $\alpha 1_x \notin \sigma_x$ and $\nu_x \leq \sigma_x$.

Put $B = \{\sigma_x : x \in X\}$.

Since each $\sigma_x \in \delta'$, we have that $B \subseteq \delta'$.

From $\alpha 1_x \notin \sigma_x$, we have that $\sigma_x(x) < \alpha$, for each $x \in X$.

Therefore, $\wedge B < \alpha$.

On the other hand we have that for any $\psi = \{\sigma_{x^1}, ..., \sigma_{x^k}\} \subseteq B$, $S_n \notin \wedge \psi$ for some n. It follows that $\wedge \psi < \alpha$ is not true.

Therefore by Theorem III.1.5 we have that (X, δ) is not strong fuzzy compact.

"$\Leftarrow$" Suppose (X, δ) is not strong fuzzy compact.

Then by Theorem III.1.5, there is an $\alpha \in \langle 0,1]$ and $\mathcal{B} \subseteq \delta'$ such that $\wedge \mathcal{B} < \alpha$, but for any $\psi \in 2^{(\mathcal{B})}$, $\wedge \psi < \alpha$ does not hold i.e. there is an $x \in X$ such that $\lambda(x) \geq \alpha$, for all $\lambda \in \psi$.

Therefore there exists a fuzzy point $\alpha 1_x \in \wedge \psi$.

Let us write $\alpha 1_x$ as S_ψ and in $2^{(\mathcal{B})}$, we introduce an order as follows:

$$\psi_1 \leq \psi_2 \text{ if and only if } \psi_1 \subseteq \psi_2.$$

Then $2^{(\mathcal{B})}$ is directed by '$\leq$' and so $S = \{S_\psi : \psi \in 2^{(\mathcal{B})}\}$ is a constant α–net which satisfies the condition that for each (closed) fuzzy set $\nu \in \psi$, $S_\psi \in \nu$.

Let $y \in X$.

Then $\wedge \mathcal{B} < \alpha$ implies there is $\nu \in \mathcal{B}$ such that $\nu(y) < \alpha$ ....(*)

Since $\nu \in \mathcal{B}$ implies that ν is a closed fuzzy set and since $\alpha 1_y \notin \nu$ by (*) we have that ν is an R–nbd of $\alpha 1_y$.

Put $\psi_0 = \{\nu\}$.

Then for any $\psi \geq \psi_0$ we have $S_\psi \in \wedge \psi \leq \wedge \psi_0 = \nu$.

Hence, $\alpha 1_y$ is not a cluster point of S.

Since $y \in X$ was arbitrary, it then follows that the constant α–net S has no cluster point of value α in X.

Lastly we show that for strong fuzzy compactness, the Tychonoff Theorem holds.

III.1.7 THEOREM (TYCHONOFF THEOREM)

The product space of the indexed family $\{(X_j, \delta_j) : j \in J\}$ of fts is strong fuzzy compact if and only if for each $j \in J$, (X_j, δ_j) is strong fuzzy compact.

PROOF

Apply Theorem II.2.5 at each level $\alpha \in [0,1\rangle$.

III.2 *ULTRA-FUZZY COMPACTNESS*

III.2.1 DEFINITION [12]

Let (X, δ) be a fts. Then (X, δ) is called ultra–fuzzy compact if $[X, i(\delta)]$ is compact.

III.2.2 PROPOSITION

Let (X, δ) be a fts.

(1) If X is finite, then (X, δ) is ultra–fuzzy compact.

(2) If δ is finite then (X, δ) is ultra–fuzzy compact.

Next, we show that ultra–fuzzy compactness is inherited by closed crisp subsets (c.f. Theorem II.1.6).

III.2.3 THEOREM

Let (X, δ) be a fts and $F \subseteq X$ be closed. If X is ultra–fuzzy compact, then F is ultra–fuzzy compact as a subspace of (X, δ).

PROOF

Suppose X is ultra–fuzzy compact and let $\mathcal{T} = \{A \cap F : A \in i(\delta)\}$.

Then $(F, \mathcal{T})$ is a subspace of $[X, i(\delta)]$.

Let $\mathcal{U} \subseteq \mathcal{T}$ be an open cover for F i.e. $F = \bigcup_{B \in \mathcal{U}} B$.

We will show that F has a finite subcover.

From $B \in \mathcal{U}$, we have that there exists an $A_B \in i(\delta)$ such that $B = A_B \cap F$.

Hence $X = F \cup F' = \bigcup_{B \in \mathcal{U}} (A_B \cap F) \cup F' = \bigcup_{B \in \mathcal{U}} (A_B \cup F') \cap (F \cup F') = \bigcup_{B \in \mathcal{U}} (A_B \cup F')$.

Since $F' \in i(\delta)$ we have that $H = \{A_B \cup F' : A_B \in i(\delta)\}$ is an open cover for X.

By compactness of X, we have that H has a finite subcover

$\{A_{B_1} \cup F', \ldots, A_{B_n} \cup F' : A_{B_k} \in i(\delta), k \in [n]\}$, that is $X = \bigcup_{k=1}^{n} (A_{B_k} \cup F')$.

Hence, $F = X \cap F = \bigcup_{k=1}^{n} (A_{B_k} \cup F') \cap F = \bigcup_{k=1}^{n} (A_{B_k} \cap F) \cup (F' \cap F)$

$= \bigcup_{k=1}^{n} (A_{B_k} \cap F) = \bigcup_{k=1}^{n} B_k$ where $B_k = A_{B_k} \cap F \in \mathcal{U}$, $k \in [n]$.

In a Hausdorff space (definitions 0.2.12, 0.2.13, 0.2.14, 0.2.15, 0.2.16), ultra–fuzzy compact subsets are closed.

III.2.4 THEOREM

Let (X, δ) be a Hausdorff fts, and $F \subseteq X$. If F is ultra–fuzzy compact as a subspace, then it is closed in X.

PROOF

Suppose F is ultra–fuzzy compact as a subspace of (X, δ). Then F is compact as a subspace of $[X, i(\delta)]$. By Theorem 0.2.18 $[X,i(\delta)]$ is Hausdorff.
Therefore F is closed.

Next we show that ultra–fuzzy compactness is invariant under continuous mappings.

III.2.5 THEOREM

Let (X, δ) and (Y, σ) be two fts, and $f: (X,\delta) \to (Y, \sigma)$ be a continuous mapping. If (X,δ) is ultra–fuzzy compact, then $f^{\to}(Y)$ is ultra–fuzzy compact as a subspace of (Y, σ).

PROOF

We assume, without loss of generality that f is surjective.
Suppose (X, δ) is ultra–fuzzy compact and let $\mathcal{U} \subseteq i(\sigma)$ be an open cover for Y.
By Theorem 0.1.16 we have that $f^{\leftarrow}(i(\sigma)) = i(f^{\leftarrow}(\sigma)) \subseteq i(\delta)$ and hence $f^{\leftarrow}(\mu) \subseteq f^{\leftarrow}(i(\sigma))$ $\subseteq i(\delta)$, so that, $f^{\leftarrow}(\mathcal{U})$ is an open cover for X.
By compactness of $[X,i(\delta)]$ we have that $f^{\leftarrow}(\mathcal{U})$ has a finite subcover
$\{f^{\leftarrow}(A_1),\ldots, f^{\leftarrow}(A_n): A_k \in \mathcal{U}, k \in [n]\}$

We claim that $\{A_1, ..., A_n : A_k \in \mathcal{U}, k \in [n]\}$ is an open cover for Y. In fact, if $y \in Y$ then the fact that f is surjective implies that there exists an $x \in X$ such that $y = f(x)$. Hence $x \in f^{\leftarrow}(A_k)$, for some $k \in [n]$.

That is, $y = f(x) \in A_k$, for some $k \in [n]$.

Therefore Y is compact, which implies Y is ultra–fuzzy compact.

The ultra–fuzzy compactness is a good extension as shown in the next

III.2.6 THEOREM

Let $(X, \mathcal{T})$ be a topological space and $K \subseteq X$. Then the following statements are equivalent:

(a) K is compact in $(X, \mathcal{T})$

(b) K is ultra–fuzzy compact as a subspace of $[X, \omega(\mathcal{T})]$

(c) K is ultra–fuzzy compact as a subspace of $[X, \underline{\omega}(\mathcal{T})]$

Before we prove Theorem III.2.6, we first need a simple

III.2.7 LEMMA

Let δ_1 and δ_2 be two topologies on X such that $\delta_2 \subseteq \delta_1$. If $K \subseteq X$ is ultra–fuzzy compact as a subspace of (X, δ_1) then it is ultra–fuzzy compact as a subspace of (X, δ_2).

PROOF of Theorem III.2.6

(a) $\Longrightarrow$ (b) : Follows at once from Proposition 0.1.18 (i)

(b) $\Longrightarrow$ (c) : Follows from Lemma III.2.7

(c) $\Longrightarrow$ (a) : Follows from Proposition 0.1.18 (ii).

The Tychonoff Theorem is valid for ultra–fuzzy compactness.

III.2.8 THEOREM (TYCHONOFF THEOREM)

The product space, (X, δ) of the indexed family $\{(X_j, \delta_j) : j \in J\}$ of fts is ultra–fuzzy compact if and only if each (X_j, δ_j) is ultra–fuzzy compact.

PROOF

Since the projection maps are continuous and onto the necessity part of the Theorem follows at once from Theorem III.2.5.

To prove the sufficiency part, suppose for each $j \in J$, (X_j, δ_j) is ultra–fuzzy compact.

Therefore, $(X_j, \prod_{j \in J} i(\delta_j))$ is compact.

By Theorem 0.1.16, we have that

$$\prod_{j \in J} i(\delta_j) = i\,(\prod_{j \in J} \delta_j) = i(\delta).$$

Thus, $[X, i(\delta)]$ is compact. That is (X, δ) is ultra–fuzzy compact.

Lastly, we give a characterisation of ultra–fuzzy compactness by fuzzy nets.

III.2.9 THEOREM

Let (X, δ) be a fts. Then (X, δ) is ultra–fuzzy compact if and only if each α–net ($\alpha \in <0,1]$) has a convergent subnet with a transitive limit of value α.

PROOF

"$\Longrightarrow$" Suppose X is ultra–fuzzy compact and let $\alpha \in <0,1]$.

Now $[X, i(\delta)]$ is compact.

Let $S = \{S_n : n \in D\}$ be an α–net in X and for each $n \in D$ let x^n denote the support of S_n.

Therefore $f = \{x^n : n \in D\}$ is a crisp net in X.

By compactness of X, f has a subnet $\vartheta = \{x^{n(m)}, m \in E\}$ converging to some $x \in X$.

Clearly $T = \{S_{n(m)} : m \in E\}$ is a subnet of S.

We will show that $\alpha\, 1_x$ is a transitive limit of T.

Since a constant c–net is a c–net, we will in fact show that if $T' = \{T_{n(m)} : m \in E\}$ is a c–net similar to T, then $T' \to c1_x$, for each $c \in <0,1]$.

Let ν be an R–nbd of $c\, 1_x$ where $c \in <0,1]$.

By definition of an R–nbd, there exists $\sigma \in \delta'$ such that $c\, 1_x \not\leq \sigma$ and $\nu \leq \sigma$ i.e. $\sigma(x) = c_1 < c$ and $\nu(x) \leq \sigma(x)$. Choose c_0 such that $c_1 < c_0 < c$.

Put $U = \{t \in X : \sigma(t) < c_0\}$.

Since $i(\delta) = \sup \{i_\beta(\delta) : \beta \in <0,1]\}$ where $i_\beta(\delta) = \{\lambda^{\leftarrow}[0,\beta> : \lambda \in \delta'\}$, we have that $x \in U$ and $U \in i(\delta)$.

Hence there is an $M_1 \in E$ such that $x^{n(m)} \in U$ for each $m \geq M_1$.

Since T′ is a constant c–net, we have that $V(T_{n(m)}) = c > C_0$ for all $m \in E$.

Therefore for each $m \geq M_1$ we have that $\sigma(x^{n(m)}) < c_0 < V(T_{n(m)})$.

Hence $T_{n(m)} \notin \sigma$ eventually.

From $\nu \leq \sigma$, we have that $T_{n(m)} \notin \nu$ eventually.

Therefore $T' \longrightarrow c\,1_x$.

"$\Longleftarrow$" suppose each α–net ($\alpha \in <0,1]$) has a convergent subnet with a transitive limit of value α and let $f = \{x^n : n \in D\}$ be a crisp net in $(X, i(\delta))$.

For a fixed $\alpha \in <0,1]$, take a constant α–net $S = \{\alpha 1_{x^n} : n \in D\}$.

By the assumption, δ has a subnet $T = \{\alpha 1_{x^{n(m)}} : m \in E\}$ having a transitive limit $\alpha 1_x$.

Hence by Proposition I.1.17, x is a limit of the crisp net $g = \{x^{n(m)} : m \in E\}$ in $[X, i(\delta)]$.

Since g is a subnet of the fuzzy net f, we have that $f \infty x$.

Hence $[X, i(\delta)]$ is compact.

Therefore (X, δ) is ultra–fuzzy compact.

CHAPTER IV

N–COMPACTNESS

INTRODUCTION

In this chapter we define a notion of fuzzy compactness called N–compactness. Whereas all of the notions of compactness discussed so far are only defined for the whole space, N–compactness is defined for arbitrary fuzzy subsets. In the first section we show that: N–compactness is inherited by closed subsets, the continuous images of N–compact sets are N–compact, N–compactness is invariant under finite suprema, N–compactness is a good extension and N–compact sets attain their maxima. In the last section, we give a characterisation of N–compactness using prefilters and show that for N–compactness the Tychonoff Theorem holds.

IV.1. *N–COMPACTNESS*

IV.1.1 DEFINITION [19]

Let (X,δ) be a fts and $\mu \in I^X$. Then μ is called N–compact if each α–net ($\alpha \in \langle 0,1]$) in μ has a cluster point in μ with value α.

In particular if $\mu = 1_X$ is N–compact, we will say (X,δ) (or simply X) is N–compact.

Obvious from definition IV.1.1 is the fact that in every fts, the fuzzy set 0_X is always N–compact, furthermore we have

IV.1.2 PROPOSITION

In every fts, each fuzzy point is N–compact.

PROOF

Let (X,δ) be an fts. An α–net in a fuzzy point x_λ amounts to a crisp net in $[0,\lambda]$ with limit α.

IV.1.3 EXAMPLE

Let X be any set and $\delta = \delta' = \{\sigma \in I^X : \sigma(x) = c$, for all $x \in X$, $c \in [0,1]\}$. Then (X,δ) is N–compact. In fact we will show that for each $x \in X$ and each $\alpha \in \langle 0,1]$, the fuzzy point $p = \alpha 1_x$ is a cluster point of every α–net $S = \{S_n : n \in D\}$ in X.

Let ν be an R–nbd of p. Then there exists a $C \in [0,1]$ such that $\nu \leq C$ and $p \notin C$.

Hence $\alpha 1_x \notin C$ i.e. $\alpha > C$. Since S is an α–net, $V(S_n) \to \alpha$. Which implies $V(S_n) > C$ eventually.

Hence $S_n \notin C$ eventually, so that $S_n \notin \nu$ eventually. Thus, $S_n \notin \nu$ frequently.

Therefore $S \infty p$.

N–compactness is closed under finite suprema.

IV.1.4 THEOREM

Let (X,δ) be a fts and $\mu,\nu \in I^X$. If both μ and ν are N–compact, so is $\mu \vee \nu$.

PROOF

Suppose both μ and ν are N–compact and let $S = \{S_n : n \in D\}$ be a α–net ($\alpha \in \langle 0,1]$) in $\mu \vee \nu$. Therefore there exists a fuzzy subnet T of S in either μ or ν.

We assume without loss of generality that T is in μ.

Since S is an α–net so is T. Hence by N–compactness of μ, there is a fuzzy point $p \in \mu$ such that $V(p) = \alpha$ and $T \infty p$. Hence $p \in \mu \vee \nu$ and $S \infty p$.

Therefore $\mu \vee \nu$ is N–compact.

The above theorem is valid only for a finite supremum as shown in the next

IV.1.5 EXAMPLE

Let (X,δ) be a fts and $\mu \in I^X$.

Clearly $\mu = \sup \{p : p \in \mu\}$ and by Proposition IV.1.2 each $p \in \mu$ is N–compact.

Since this is true for any fuzzy set μ, we can choose μ such that μ is not N–compact.

We do however have, as a consequence of Theorem IV.1.4

IV.1.6 PROPOSITION

Every fuzzy set with a finite support is N–compact.

IV.1.7 THEOREM

Let (X,δ) be a fts. If $\mu \in I^X$ is N–compact, then

(a) $\mu \wedge \sigma$ is N–compact for each closed fuzzy set σ.

(b) $\mu \wedge C$ is N–compact for each constant fuzzy set C.

PROOF

Suppose μ is N–compact and $\alpha \in <0,1]$.

(a) Let $\sigma \in \delta'$ and $S = \{S_n : n \in D\}$ be an α–net in $\mu \wedge \sigma$.

Then S is in both μ and σ.

By N–compactness of μ, there exists a fuzzy point $p \in \mu$ such that $V(p) = \alpha$ and $S \infty p$.

We claim that $p \in \sigma$. Otherwise, if $p \notin \sigma$, the fact that $\sigma \in \delta'$ would imply that σ is an R–nbd of p. Hence from $S \infty p$ we have that $S_n \notin \sigma$, frequently, which clearly contradicts the fact that S is in σ.

(b) Let C be a constant fuzzy set and $S = \{S_n : n \in D\}$ be an α–net in $\mu \wedge C$.

Then as in (a) there exists a fuzzy point $p \in \mu$ such that $V(p) = \alpha$ and $S \infty p$.

Put $p = \alpha 1_x$.

We claim that $p \in C$. Otherwise if $p \notin C$ then $\alpha > C(x) = C$.

Since S is an α–net, we have that $V(S_n) > C$ eventually, i.e. $S_n \notin C$, eventually, which clearly contradicts the fact that S is in C.

As a corollary to Theorem IV 1.7, we have that N–compactness is inherited by closed subsets.

IV.1.8 COROLLARY

Let (X,δ) be a fts and $\mu \in I^X$. If μ is N–compact, then each closed fuzzy subset of μ is N–compact as well.

The following two results are immediate consequences of Definition IV 1.1 and Theorem I.1.15.

IV.1.9 THEOREM

Let (X,δ) be a fts and $\mu \in I^X$. Then μ is N–compact if and only if each α–net ($\alpha \in \langle 0,1]$) contained in μ has a subnet converging to some fuzzy point in μ with value α.

IV.1.10 THEOREM

Let (X,δ) be a fts and $\mu \in I^X$. Then μ is N–compact if and only if each fuzzy net contained in μ has a cluster point in μ with value $\alpha \in \langle 0,1]$, whenever its value net has the crisp cluster point α.

The following simple result will be needed in the sequel.

IV.1.11 LEMMA

Let δ_1 and δ_2 be two fuzzy topologies on X such that $\delta_2 \subseteq \delta_1$ and $\mu \in I^X$. If μ is N–compact in (X,δ_1) then it is N–compact in (X,δ_2).

From now on let us agree on the following notation:

If $\mu \in I^X$ and $t \in I$, we will write $\mu_t = \{x \in X : \mu(x) \geq t\}$.

It is thus clear that if $0 \leq t < \sup \mu$ then $\mu_t \neq \emptyset$.

IV.1.12 THEOREM

Let $(X,\mathcal{T})$ be a topological space and $\mu \in I^X$. Then in the following statements, we have the implications.

(a) $\Longrightarrow$ (b) $\Longrightarrow$ (c).

(a) μ is N–compact in $[X,\omega(\mathcal{T})]$.

(b) μ is N–compact in $[X,\underline{\omega}(\mathcal{T})]$.

(c) For each $0 < \beta \leq 1$, every net in μ_β has a cluster point in μ_β, i.e. μ_β is compact for each $0 < \beta \leq 1$.

PROOF

(a) $\Longrightarrow$ (b): Follows from Lemma IV.1.11.

(b) $\Longrightarrow$ (c): Suppose μ is N–compact in $[X, \underline{\omega}(\mathcal{T})]$, $0 < \beta \leq 1$, and

$f = \{x^n : n \in D\}$ is a crisp net in μ_β.

Hence for each $n \in D$, we have $x^n \in \mu_\beta$ i.e. $\mu(x^n) \geq \beta$.

Therefore $x^n_\beta \in \mu$, for each $n \in D$.

Put $S = \{x^n_\beta : n \in D\}$.

Therefore S is a constant β–net in μ and hence, by N–compactness of μ. there exists $x_\beta \in \mu$ such that $S \infty x_\beta$.

From $x_\beta \in \mu$ we have that $\mu(x) \geq \beta$ i.e. $x \in \mu_\beta$. So that f has a cluster point in μ_β and hence μ_β is compact.

The reverse implications in Theorem IV 1.12 are not true in general as seen in the next example ([1]).

IV.1.13 EXAMPLE

Let $X = I$ and $\mathcal{T} = \{\emptyset, X, \{1\}\}$. Let $\mu : X \longrightarrow I$ be given by $\mu(x) = x$, for each $x \in X$.

Put $F = [0,1\rangle$.

Therefore $F \in \mathcal{T}'$ and hence 1_F is closed in $\underline{\omega}(\mathcal{T})$.

Since $\sup(\mu \wedge 1_F) = 1$ and for each $x \in X$, $(\mu \wedge 1_F)(x) < 1$ we have that $\mu \wedge 1_F$ cannot attain its maximum.

It will later be shown (Theorem IV 1.15) that an N–compact fuzzy set attains its maximum and hence μ is not N–compact by Theorem IV 1.7(a).

On the other hand, $\mu_t = \{x \in X : \mu(x) \geq t\} = \{x \in X : x \geq t\} = \{x \in X : t \leq x \leq 1\} = [t,1]$ is compact in $(X,\mathcal{T})$ for each $t \in I$.

We do however have that N–compactness is a good extension:

IV.1.14 THEOREM

Let $(X,\mathcal{T})$ be a topological space and $K \subseteq X$. Then the following statements are equivalent.

(a) K is compact in $(X,\mathcal{T})$.

(b) K is N–compact in $(X, \omega(\mathcal{T}))$

(c) K is N–compact in $(X, \underline{\omega}(\mathcal{T}))$.

PROOF

If $\mu = 1_K$ in Theorem IV 1.12, we have that $\mu_\alpha = \mu_\beta = K$ and hence condition (c) in Theorem IV 1.12 reduces to the statement that "*each net in K has a cluster point in K*"; which is equivalent to the compactness of K. So, in view of this, we are only left to show that (a) $\Longrightarrow$ (b).

To do this, suppose K is a compact in $(X,\mathcal{T})$ and let $S = \{S_n : n \in D\}$ be an α–net ($\alpha \in \langle 0,1]$) in 1_K. Let x^n denote the support of S_n, for each $n \in D$.

Put $f = \{x^n : n \in D\}$.

Then f is a crisp set in K and hence by the compactness of K we have an $x \in K$ such that $f \infty x$.

We claim that $S \infty x_\alpha$. In fact, if ν is a R–nbd of x_α then there exists σ closed in $(X,\omega(\mathcal{T}))$ such that $x_\alpha \notin \sigma$ and $\nu \leq \sigma$.

Then clearly $\alpha > \sigma(x)$ and hence we can choose β such that $\sigma(x) < \beta < \alpha$.

Since σ is closed in $(X, \omega(\mathcal{T}))$, it is upper–semicontinuous on X, so that

$G = \{t \in X : \sigma(t) < \beta\}$ is open in $(X,\mathcal{T})$ and $x \in G$.

From $f \infty x$, we have that $x^n \in G$ frequently.

That is, for each $N_1 \in D$ there is an $n \in D$ such that $n \geq N_1$ and $x^n \in G$.

Also, since S is an α–net, there is an $N_2 \in D$ such that $V(S_n) > \beta$ for each $n \geq N_2$.

Therefore, there is an $n_0 \in D$ such that $n_0 \geq N_1$, $n_0 \geq N_2$ and $\sigma(x^{n_0}) < \beta < V(S_n)$.

Hence $S_n \notin \sigma$, frequently, which implies that $S_n \notin \nu$, frequently.

Clearly $x_\alpha \in 1_K$.

Therefore K is N–compact in $(X, \omega(\mathcal{T}))$.

Next we show that N–compact sets attain their maxima.

IV.1.15 THEOREM

Let (X,δ) be a fts and $\mu \in I^X$. If μ is N–compact, then there exists an $x \in X$ such that

$$\mu(x) = \sup \mu.$$

PROOF

Suppose μ is N–compact and let $\sup \mu = \alpha$. If $\alpha = 0$, we are done.

Suppose now that $\alpha \in \langle 0,1]$.

For each $k \in \mathbb{N}$, choose $x^k \in X$ such that $\mu(x^k) > \alpha - \frac{1}{k}$ ....*

Denote by S_k the fuzzy point with support x^k and value $\mu(x^k)$. From * we get an α–net $S = \{S_k : k \in \mathbb{N}\}$ in μ. By N–compactness of μ, S has a cluster point $\alpha 1_x \in \mu$.

Hence $\alpha \leq \mu(x)$.

On the other hand, we have by definition of α, that $\alpha \geq \mu(x)$.

Therefore $\mu(x) = \sup \mu$.

Theorem IV 1.15 together with Corollary IV 1.8 give:

IV.1.16 COROLLARY

Let (X,δ) be an N–compact fts. Then each closed fuzzy set in X has a maximum and each open fuzzy set in X (as a function) has a minimum

and the following well–known classical result:–

IV.1.17 COROLLARY

Let $(X,\mathcal{T})$ be a compact topological space. Then the lower semicontinuous (upper semicontinuous) functions from $(X,\mathcal{T})$ into I ~~have~~ assume their minima (maxima).

More especially, the continuous functions from $(X,\mathcal{T})$ into I ~~have~~ assume both maxima and minima.

Lastly, we show that N–compactness is invariant under continuous mappings.

IV.1.18 THEOREM

Let (X,δ) and (Y,σ) be fts, $f : (X,\delta) \to (Y,\sigma)$ be a continuous mapping and $\theta \in I^X$. Then if θ is N–compact in (X,δ), $f^{\rightarrow}(\theta)$ is N–compact in (Y,σ).

Before we prove Theorem IV 1.18 we first need a simple

IV.1.19 LEMMA

Let (X,δ) and (Y,σ) be fts, $f : (X,\delta) \to (Y,\sigma)$ be a mapping, $\nu \in I^Y$ and $\theta = f^{\leftarrow}(\nu)$.

Then for any fuzzy point p in X satisfying $p \notin \theta$, the image f(p) is a fuzzy point in Y satisfying $f(p) \notin \nu$.

Proof of Theorem IV 1.18

Let $T = \{T_n : n \in D\}$ be an α–net ($\alpha \in <0,1]$) in $\nu = f^{\rightarrow}(\theta)$.

For each $n \in D$, let y^n be the support of T_n.

Then $f^{\rightarrow}(\theta)(y^n) = \sup \{\theta(x^n) : x^n \in \{f^{\leftarrow}(y^n)\}\}$.

Hence, for each $k \in \mathbb{N}$, there exists $x^n \in \{f^{\leftarrow}(y^n)\} \subseteq X$ such that $\theta(x^n) > \nu(y^n) - \frac{1}{k}$.

Furthermore, since $T_n \in \nu$, for all $n \in D$, we have that $\nu(y^n) \geq V(T_n)$.

Therefore $\theta(x^n) > V(T_n) - \frac{1}{k}$.

For each $n \in D$ and each $k \in \mathbb{N}$ choose a fuzzy point $S(n,k) \in \theta$ such that

$V(T_n) - \frac{1}{k} \leq V(S(n,k)) \leq V(T_n)$; $f(S(n,k)) \in T$ and $f(S(n,k))$ has support y^n...........*

Put $E = \{(n,k)\ n \in D, k \in \mathbb{N}\}$ and define an order on E as follows:

$(n_1,k_1) \geq (n_2, k_2)$ if and only if $n_1 \geq n_2$ and $k_1 \geq k_2$.

It can be shown that E is directed by "$\geq$" and hence $S = \{S(n,k) : (n,k) \in E\}$ is a fuzzy net in θ. It follows from the fact that T is an α–net together with * that S is in fact an α–net in θ. By N–compactness of θ, there exists a fuzzy point $\alpha 1_x \in \theta$ such that $S \infty \alpha 1_x$.

Let $y = f(x)$.

Then $\alpha\, 1_y$ is a fuzzy point in Y such that $\alpha 1_y \in f^{\rightarrow}(\theta)$.

We claim that $T \infty \alpha 1_y$. In fact, if λ is an R–nbd of $\alpha 1_y$ then there exists $\beta \in \sigma'$ such that $\alpha 1_y \notin \beta$ and $\lambda \leq \beta$. Since f is continuous, $\mu = f^{\leftarrow}(\beta) \in \delta'$.

From $\alpha 1_y \notin \beta$ it follows that $\alpha > \beta(y)$ and thus $\mu(x) = (f^{\leftarrow}(\beta))(x) = \beta(f(x)) = \beta(y) < \alpha$ i.e. $\alpha 1_x \notin \mu$. Hence the fact that $\mu \in \delta'$ implies that μ is an R–nbd of $\alpha 1_x$.

For each $N \in D$, $K \in \mathbb{N}$, $S \infty \alpha 1_x$ implies there exists $(n,k) \in E$ such that $(n,k) \geq (N,K)$ implies $S(n,k) \notin \mu$.

By Lemma IV 1.19, $f(S(n,k)) \notin \beta$ and hence $f(S(n,k)) \notin \lambda$, from $\lambda \leq \beta$.

By * we have that $T_n \notin \lambda$. So that $T \infty \alpha\, 1_y$.

Therefore ν is N–compact.

IV.2 CHARACTERISATIONS OF N–COMPACTNESS BY PRE FILTERS AND THE TYCHONOFF THEOREM

In this section we give characterisation of N–compactness by means of prefilters and deduce the Tychonoff Theorem from them. We find that the proof using prefilters is much shorter than the one using fuzzy nets ([19]).

IV.2.1 THEOREM

Let (X,δ) be a fts and $\mu \in I^X$. Then μ is N–compact if and only if each α–prefilter $\mathcal{F}$ ($\alpha \in$ <0,1]) has an adherence point $\alpha 1_x \in \mu$ whenever $\mu \in \mathcal{F}$.

PROOF

"$\Longrightarrow$" Suppose μ is N–compact and let $\mathcal{F}$ be an α–prefilter ($\alpha \in$ <0,1]) on X with $\mu \in \mathcal{F}$.

Hence $\alpha = c(\mathcal{F}) = \inf\limits_{\nu \in \mathcal{F}} \sup \nu$.

Since $\mu \in \mathcal{F}$, we have $\sup(\mu \wedge \nu) \geq \alpha$ for each $\nu \in \mathcal{F}$. Hence for each $\nu \in \mathcal{F}$ and each $k \in \mathbb{N}$ we can choose a fuzzy point $\lambda 1_y \in \mu \wedge \nu$ such that $\lambda > \alpha - \frac{1}{k}$.

Furthermore, λ can be chosen such that $\lambda < \alpha + \frac{1}{k}$.

Therefore, for each $\nu \in \mathcal{F}$ and each $k \in \mathbb{N}$ we can find a fuzzy point $\lambda 1_y \in \mu \wedge \nu$ such that

$\alpha - \frac{1}{k} < \lambda < \alpha + \frac{1}{k}$ ....*

Choose and fix such a $\lambda 1_y$ and denote it by $S(\nu,k)$.

Put $D = \{(\nu, k) : \nu \in \mathcal{F} \text{ and } k \in \mathbb{N}\}$ and define an order on D as follows:

$(\nu_1,k_1) \geq (\nu_2, k_2)$ if and only if $\nu_1 \leq \nu_2$ and $k_1 \geq k_2$.

It is easy to see that "$\geq$" directs D and hence $S = \{S(\nu,k): (\nu,k) \in D\}$ is a fuzzy net in μ.

From * it follows that S is in fact, an α–net in μ.

By N–compactness of μ, there exists a fuzzy point $\alpha 1_x \in \mu$ such that $S \infty \alpha 1_x$.

To show that $\alpha 1_x \in \text{Adh}\,\mathcal{F}$ let σ be an R–nbd of $\alpha 1_x$ and $\nu \in \mathcal{F}$, $K \in \mathbb{N}$.

Then from $S \infty \alpha 1_x$, there exists $(\nu_1,k_1) \in D$ such that $(\nu_1, k_1) \geq (\nu,K)$ implies $S(\nu_1, k_1) \notin \sigma$.

But $S(\nu_1, k_1) \in \nu_1 \wedge \mu$ and $\nu_1 \leq \nu_1$ so that $S(\nu_1, k_1) \in \nu$.

Therefore $\nu \nleq \sigma$, which implies $\alpha 1_x \in \mathrm{Adh}\,\mathcal{F}$.

"$\Leftarrow$" Suppose the hypotheses of the theorem is true and let $S = \{S_n : n \in D\}$ be a α–net ($\alpha \in <0,1]$) in μ. For each $N \in D$ let $\nu_N = \sup\{S_n : n \geq N\}$ and put $\mathcal{F} = \{\nu \in I^X : \nu \geq \nu_N$ for some $N \in D\}$.

It is easy to see that $\mathcal{F}$ is a prefilter on X and since $\mu \geq \sup\{S_n : n \in D\}$ we have that $\mu \in \mathcal{F}$.

We claim that $\mathcal{F}$ is an α–prefilter. In fact, if $\epsilon > 0$ the fact that S is an α–net implies that there exists $N \in D$ such that $| V(S_n) - \alpha | < \epsilon$, for each $n \geq N$.

Hence $-\epsilon + \alpha < V(S_n) < \epsilon + \alpha$, for each $n \geq N$.

From definition of ν_N and the fact that $V(S_n) < \epsilon + \alpha$ for each $n \geq N$ we have that $\sup(\nu_N) \leq \epsilon + \alpha$.

So that $\inf_{\nu \in \mathcal{F}} \sup \nu \leq \epsilon + \alpha$ ....(1)

On the other hand, for each $\nu \in \mathcal{F}$ there is an $n \in D$ such that $\nu_n \leq \nu$.

Choose $n_1 \in D$ such that $n_1 \geq n$ and $n_1 \geq N$. Therefore $\nu_{n_1} \leq \nu_n \leq \nu$. From the fact that $V(S_n) > -\epsilon + \alpha$, for each $n \geq N$, we have that $V(\nu_{n_1}) > -\epsilon + \alpha$.

It follows that $\inf_{\nu \in \mathcal{F}} \sup \nu \geq -\epsilon + \alpha$ ....(2)

From (1) and (2) together with the fact that $\epsilon > 0$ was arbitrary we have that $c(\mathcal{F}) = \inf_{\nu \in \mathcal{F}} \sup \nu = \alpha$, i.e... $\mathcal{F}$ is an α–prefilter.

By the hypothesis of the Theorem we have that there exists a fuzzy point $\alpha 1_x \in \mu \wedge \mathrm{Adh}\,\mathcal{F}$.

We claim that $S \propto \alpha 1_x$. In fact, if σ is an R–nbd of $\alpha 1_x$, the fact that $\alpha 1_x \in \mathrm{Adh}\,\mathcal{F}$ implies that $\nu \leq \sigma$ does not hold for each $\nu \in \mathcal{F}$.

Hence for each $N \in D$, $\nu_N \leq \sigma$ does not hold, i.e. there exists $n \geq N$ such that $S_n \notin \sigma$ so that $S_n \notin \sigma$ frequently. Hence $S \propto \alpha 1_x$.

Theorem IV 2.1 characterises N–compactness by means of α–prefilters ($\alpha \in <0,1]$). Next we give a characterisation of N–compactness by arbitrary prefilters.

IV.2.2 THEOREM

Let (X, δ) be a fts and $\mu \in I^X$. Then μ is N–compact if and only if for each prefilter $\mathcal{F}$ on X the following two conditions are satisfied.

(a) $\sup (\mu \wedge \text{Adh}\,\mathcal{F}) \geq c(\mathcal{F}, \mu)$

(b) $\mu \wedge \text{Adh}\,\mathcal{F}$ attains its maximum.

PROOF

"$\Longrightarrow$" Suppose μ is N–compact and let $\mathcal{F}$ be a prefilter on X.

(a) If $\mu \wedge \nu = 0$ for some $\nu \in \mathcal{F}$, then $c(\mathcal{F},\mu) = 0 \leq \sup (\mu \wedge \text{Adh}\,\mathcal{F})$.

Suppose $c(\mathcal{F},\nu) \neq 0$. So that by definition $\inf\limits_{\nu \in \mathcal{F}} \sup \mu \wedge \nu \neq 0$ and hence $\sup \mu \wedge \nu \neq 0$ for each $\nu \in \mathcal{F}$. Thus $\mu \wedge \nu \neq 0$ for each $\nu \in \mathcal{F}$. Therefore $c(\mathcal{F}, \mu) = c(\mathcal{F})$.

Let $c(\mathcal{F},\mu) = \alpha$.

Then $\alpha \in <0,1]$ and $(\mathcal{F}, \mu)$ is therefore an α–prefilter on X such that $\mu \in (\mathcal{F},\mu)$.

Since μ is N–compact, Theorem IV 2.1 guarantees the existence of a fuzzy point $\alpha 1_x \in \mu$ such that $\alpha 1_x \in \text{Adh}\,(\mathcal{F},\mu)$.

We claim that $\alpha 1_x \in \text{Adh}\,\mathcal{F}$. In fact, if $\nu \in \mathcal{F}$ and σ is an R–nbd of $\alpha 1_x$, then the fact that $\alpha 1_x \in \text{Adh}\,(\mathcal{F},\mu)$ implies there exists $y \in X$ such that $(\mu \wedge \nu)(y) > \sigma(y)$, which clearly implies that $\nu(y) > \sigma(y)$. Thus $\mathcal{F} \infty\, \alpha 1_x$ i.e. $\alpha 1_x \in \text{Adh}\,\mathcal{F}$.

Therefore $\alpha 1_x \in \mu \wedge \text{Adh}\,\mathcal{F}$ i.e. $\alpha \leq (\mu \wedge \text{Adh}\,\mathcal{F})(x)$.

Hence $c(\mathcal{F},\mu) = \alpha \leq (\mu \wedge \text{Adh}\,\mathcal{F})(x) \leq \sup (\mu \wedge \text{Adh}\,\mathcal{F})$.

(b) By definition $\text{Adh}\,\mathcal{F} = \inf\limits_{\nu \in \mathcal{F}} \bar{\nu}$.

Hence $\text{Adh}\,\mathcal{F} \in \delta'$.

Since μ is N–compact, Theorem IV 1.7 (a) implies that $\mu \wedge \text{Adh}\,\mathcal{F}$ is N–compact.

Hence by Theorem IV 1.15, $\mu \wedge \text{Adh}\,\mathcal{F}$ attains it's maximum.

"$\Leftarrow$" Suppose the two conditions (a) and (b) are satisfied and let $\mathcal{F}$ be an α–prefilter on X ($\alpha \in$ <0,1]) with $\mu \in \mathcal{F}$.

From (b) there exists an $x \in X$ such that $(\mu \wedge \text{Adh}\,\mathcal{F})(x) = \sup\,(\mu \wedge \text{Adh}\,\mathcal{F}) \geq c(\mathcal{F},\mu)$ by (a) and $c(\mathcal{F},\mu) = c(\mathcal{F})$ since $\mu \in \mathcal{F} = \alpha$. Therefore $\alpha 1_x \in \mu \wedge \text{Adh}\,\mathcal{F}$.

By Theorem IV 2.1, we have that μ is N–comapct.

Next we give a characterisation of N–compactness by the prime prefilters.

IV.2.3 THEOREM

Let (X, δ) be a fts and $\mu \in I^X$. Then μ is N–compact if and only if each prime prefilter $\mathcal{F}$ on X satisfies the following two conditions.

(a) $\sup\,(\mu \wedge \text{Adh}\,\mathcal{F}) \geq c(\mathcal{F}, \mu)$

(b) $\mu \wedge \text{Adh}\,\mathcal{F}$ attains it's maximum.

PROOF

"$\Rightarrow$" Follows at once from the fact that a prime prefilter is a prefilter together with Theorem IV 2.2

"$\Leftarrow$" Suppose the two conditions are satisfied and let $\mathcal{F}$ be an α–prefilter on X with $\mu \in \mathcal{F}$.

By Lemma I.2.11 there exists a prime prefilter $\mathcal{G}$ on X finer than $\mathcal{F}$ and such that $c(\mathcal{G}) = c(\mathcal{F}) = \alpha$.

By (b) $\mu \wedge \text{Adh}\,\mathcal{G}$ attains its maximum.

Therefore there exists some $y \in X$ such that $(\mu \wedge \text{Adh}\,\mathcal{G})(y) = \sup\,(\mu \wedge \text{Adh}\,\mathcal{G})$ $\geq c(\mathcal{G}, \mu)$ by (a) $= c(\mathcal{G})$ since $\mu \in \mathcal{G} = \alpha$. Thus $\alpha 1_y \in \mu \wedge \text{Adh}\,\mathcal{G}$.

Since $\mathcal{G}$ is finer than $\mathcal{F}$ we have that $\text{Adh}\,\mathcal{G} \leq \text{Adh}\,\mathcal{F}$, which then implies $\alpha 1_y \in \mu \wedge \text{Adh}\,\mathcal{F}$.

Hence μ is N–compact by Theorem IV 2.1

For N–compactness, the Tychonoff Theorem holds.

IV.2.4 THEOREM (TYCHONOFF THEOREM).

The product space (X,δ) of the indexed family $\{(X_j, \delta_j) : j \in J\}$ of fts, is N–compact if and only if for each $j \in J$, (X_j,δ_j) is N–compact.

Since the projection maps are both continuous and onto, the necessity part of Theorem IV 2.4 is a consequence of Theorem IV 1.18.

The sufficiency is a consequence of a much stronger Theorem IV 2.5 below.

IV.2.5 THEOREM

Let J be an index set and suppose for each $j \in J$, $\mu_j \in I^{X_j}$ is N–compact. Then the product set $\prod_{j\in J} \mu_j$ is N–compact in $\prod_{j\in J} X_j$.

PROOF

We only need to show that the conditions in Theorem IV 2.3 are satisfied for each prime prefilter $\mathcal{F}$ on X.

Suppose each μ_j $(j \in J)$ is N–compact in X_j and let $\mu = \prod_{j\in J} \mu_j$ and $X = \prod_{j\in J} X_j$.

Let $\mathcal{F}$ be a prime prefilter on X. By N–compactness of μ_j we have that

$\sup (\mu_j \wedge \operatorname{Adh} \mathcal{F}_j) \geq c(\mathcal{F}_j,\mu_j)$ $(j \in J)$ ....(1)

For each $j \in J$, choose $t_j \in X_j$ such that $(\mu_j \wedge \operatorname{Adh} \mathcal{F}_j)(t_j) = \sup (\mu_j \wedge \operatorname{Adh} \mathcal{F}_j)$ .....(2)

Define $h \in X$ by $h(j) = t_j$, $j \in J$.

(a) $\operatorname{Sup} (\mu \wedge \operatorname{Adh} \mathcal{F}) \geq (\mu \wedge \operatorname{Adh} \mathcal{F})(h) = \mu(h) \wedge (\operatorname{Adh} \mathcal{F})(h)$

$= \inf_{j\in J} \mu_j(h(j)) \wedge \prod_{k\in J} \operatorname{Adh} \mathcal{F}_k)(h)$ by Theorem I.2.13 (i)

$= \inf_{j\in J} \mu_j (h(j)) \wedge \inf_{k\in J} (\operatorname{Adh} \mathcal{F}_k)\, h(k) = \inf_{j\in J} (\mu_j \wedge \operatorname{Adh} \mathcal{F}_j)(t_j)$

$= \inf_{j \in J} \sup (\mu_j \wedge \text{Adh}\,\mathcal{F}_j)$ (by (2)) $\geq \inf_{j \in J} c(\mathcal{F}_j,\mu_j)$ (by (1)) $\geq c(\mathcal{F},\mu)$ by Theorem I.2.12.

(b) As in (a) above, we have that $(\mu \wedge \text{Adh}\,\mathcal{F})(h) = \inf_{j \in J} \sup (\mu_j \wedge \text{Adh}\,\mathcal{F}_j)$

$= \sup \inf_{j \in J} (\mu_j \wedge \text{Adh}\,\mathcal{F}_j)$

$= \sup (\inf_{j \in J} \mu_j \wedge \inf_{j \in J} \text{Adh}\,\mathcal{F}_j)$ (using the fact that I^X is completely distributive)

$= \sup (\mu \wedge \text{Adh}\,\mathcal{F})$

Therefore μ is N–compact in X.

The converse of Theorem IV 2.5 is not true in general as seen in

IV.2.6 EXAMPLE

Let (X,δ) and (Y,σ) be fts, $\mu \in I^X$ and $\nu \in I^Y$. If $\mu = 0$ then clearly $\mu \times \nu = 0$ which is N–compact in $(X \times Y, \delta \times \sigma)$ for any fuzzy set $\nu \in I^Y$, even if ν is not N–compact.

CHAPTER V

f–COMPACTNESS

INTRODUCTION

In this chapter we introduce yet another notion of fuzzy compactness for an arbitrary fuzzy set, namely, the f–compactness. In the first section we show that f–compactness is inherited by closed fuzzy subsets, invariant under continuous mappings, invariant under finite suprema, a good extension and we also give a characterisation of f–compactness by fuzzy nets. In the last section we show that for f–compactness, the Tychonoff theorem holds. Lastly, we show that for f–compactness it really does not matter whether our topology includes the constants ([10]) or not (definition O.1.1).

V.1. *f–COMPACTNESS.*

V.1.1 DEFINITION [1]

Let (X,δ) be a fts and $\mu \in I^X$. Then μ is called f–compact if for every prefilter $\mathcal{F}$ on X we have $\sup(\mu \wedge \text{Adh}\,\mathcal{F}) \geq c(\mathcal{F},\mu)$.

In particular, if $\mu = 1_X$ is f–compact, we will say (X,δ) (or simply X) is f–compact as a fts, i.e. (X,δ) is f–compact if for every prefilter $\mathcal{F}$ on X we have $\sup(\text{Adh}\,\mathcal{F}) \geq c(\mathcal{F})$.

V.1.2 PROPOSITION

Let (X,δ) be a fts and $\mu \in I^X$. Then the following are equivalent:

a) μ is f–compact.

b) For every prefilterbase $\mathcal{B}$ on X, we have $\sup(\mu \wedge \text{Adh}\,\mathcal{B}) \geq c(\mathcal{B},\mu)$.

c) For every prefilter $\mathcal{F}$ on X with $\mu \in \mathcal{F}$, we have $\sup(\mu \wedge \text{Adh}\,\mathcal{F}) \geq c(\mathcal{F})$.

PROOF

(a) $\Rightarrow$ (b): Follows at once from the equalities Adh $\mathcal{B}$ = Adh $<\mathcal{B}>$, $c(\mathcal{B},\mu) = c(<\mathcal{B}>,\mu)$ for any prefilterbase $\mathcal{B}$ on X.

(b) $\Rightarrow$ (a): Follows at once from definition V.1.1 together with the fact that any prefilter is a prefilterbase.

(a) $\Rightarrow$ (c): Let $\mathcal{F}$ be any prefilter on X with $\mu \in \mathcal{F}$. Then $c(\mathcal{F},\mu) = c(\mathcal{F})$ and hence (c) follows from the f–compactness of μ.

(c) $\Rightarrow$ (a): Assuming (c) let $\mathcal{F}$ be any prefilter on X.

If $c(\mathcal{F},\mu) = 0$, then $\sup(\mu \wedge \text{Adh}\,\mathcal{F}) \geq c(\mathcal{F},\mu)$.

Suppose now that $c(\mathcal{F},\mu) > 0$. Then $(\mathcal{F},\mu)$ is a prefilter and $\mu \in (\mathcal{F},\mu)$.

Moreover, $\mathcal{F} \subseteq (\mathcal{F},\mu)$. So that Adh $\mathcal{F} \geq$ Adh $(\mathcal{F},\mu)$. Applying (c) we get $\sup(\mu \wedge \text{Adh}\,\mathcal{F}) \geq \sup(\mu \wedge \text{Adh}\,(\mathcal{F},\mu)) \geq c(\mathcal{F},\mu)$.

Hence μ is f–compact.

V.1.3 THEOREM

Let (X,δ) be a fts and $\mu \in I^X$. Then μ is f–compact if and only if for each prefilter $\mathcal{F}$ on X such that $\mu \in \mathcal{F}$ and every $\epsilon \in \,<0,c(\mathcal{F})>$ there is a fuzzy point p such that $p \in \mu \wedge \text{Adh}\,\mathcal{F}$ and $V(p) = c(\mathcal{F}) - \epsilon$.

PROOF

"$\Rightarrow$" Suppose μ is f–compact and let $\mathcal{F}$ be a prefilter on X such that $\mu \in \mathcal{F}$ and $\epsilon \in \,<0,c(\mathcal{F})>$.

By Proposition V.1.2 (c) we have that $\sup(\mu \wedge \text{Adh}\,\mathcal{F}) \geq c(\mathcal{F}) > c(\mathcal{F}) - \epsilon$.

Choose $x_0 \in X$ such that $(\mu \wedge \text{Adh}\,\mathcal{F})(x_0) > c(\mathcal{F}) - \epsilon$ and let p be the fuzzy point with support x_0 and value $c(\mathcal{F}) - \epsilon$; then $V(p) = c(\mathcal{F}) - \epsilon$ and $p \in \mu \wedge \text{Adh}\,\mathcal{F}$.

"$\Leftarrow$" Let $\mathcal{F}$ be any prefilter on X such that $\mu \in \mathcal{F}$ and $c(\mathcal{F}) > 0$, let $\epsilon \in \,<0,c(\mathcal{F})>$ and choose a fuzzy point $p \in \mu \wedge \text{Adh}\,\mathcal{F}$ such that $V(p) = c(\mathcal{F}) - \epsilon$.

Therefore sup $(\mu \wedge \mathrm{Adh}\,\mathcal{F}) \geq$ sup p $= c(\mathcal{F}) - \epsilon$. Since this is true for each $\epsilon \in <0,c(\mathcal{F})>$, we obtain sup $(\mu \wedge \mathrm{Adh}\,\mathcal{F}) \geq c(\mathcal{F})$.

The result holds trivially when $c(\mathcal{F}) = 0$.

Therefore μ is f–compact.

It is easy to see that the following result is true.

V.1.4 PROPOSITION

In every fts, each fuzzy point is f–compact.

Next we show that f–compactness is closed under finite suprema.

V.1.5 THEOREM

Let (X,δ) be a fts and $\mu, \nu \in I^X$. If both μ and ν are f–compact, so is $\mu \vee \nu$.

PROOF

Let $\mathcal{F}$ be any prefilter on X. Put $\sigma = \mu \vee \nu$, where both μ and ν are f–compact. Then $\sigma \geq \mu$ and $\sigma \geq \nu$, hence sup $(\mu \wedge \mathrm{Adh}\,\mathcal{F}) \leq$ sup $(\sigma \wedge \mathrm{Adh}\,\mathcal{F})$ and sup $(\nu \wedge \mathrm{Adh}\,\mathcal{F}) \leq$ sup $(\sigma \wedge \mathrm{Adh}\,\mathcal{F})$.

Therefore sup $(\sigma \wedge \mathrm{Adh}\,\mathcal{F}) \geq$ sup $(\mu \wedge \mathrm{Adh}\,\mathcal{F}) \vee$ sup $(\nu \wedge \mathrm{Adh}\,\mathcal{F})$

$\geq \max\{c(\mathcal{F},\mu),c(\mathcal{F},\mu)\}$ by f–compactness of μ,ν

$= c(\mathcal{F}, \mu \vee \nu)$ by Proposition I.2.8.

Hence sup$((\mu \vee \nu) \wedge \mathrm{Adh}\,\mathcal{F}) \geq c(\mathcal{F}\ \mu \vee \nu)$, that is, $\mu \vee \nu$ is f–compact.

It is clear that the above Theorem is only valid for a finite suprema; to see this, we replace N–compactness by f–compactness and Proposition IV.1.3 by Proposition V.1.4 in Example IV.1.6.

We do however have

V.1.6 THEOREM

Let (X,δ) be a fts and $\mu \in I^X$. If δ is finite, then μ is f–compact.

PROOF

Suppose δ is finite and let $\mathcal{F}$ be any prefilter on X.

Suppose $\{\bar{\nu} : \nu \in \mathcal{F}\} = \{\bar{\nu}_1, ..., \bar{\nu}_n : \nu_k \in \mathcal{F}, k \in [n]\}$

Let $\nu_0 = \nu_1 \wedge .... \wedge \nu_n$.

Then by definition of a prefilter we have that $\nu_0 \in \mathcal{F}$.

Hence $\sup (\mu \wedge \text{Adh}\,\mathcal{F}) = \sup (\mu \wedge \min_{i \leq i \leq n} \bar{\nu}_i) \geq \sup (\mu \wedge \nu_0) \geq c(\mathcal{F}, \mu)$.

Therefore μ is f–compact.

Next we give

V.1.7 THEOREM

Let (X,δ) be a fts. If $\mu \in I^X$ is f–compact, then

(a) $\mu \wedge \nu$ is f–compact, for each closed fuzzy set ν.

(b) $\mu \wedge$ a is f–compact, for each constant fuzzy set a.

PROOF

Suppose μ is f–compact.

(a) Let $\nu \in I^X$ be closed.

For all $\sigma \in I^X$ we have that $\nu \wedge \sigma \leq \nu \wedge \bar{\sigma}$, and since ν is closed we have $\nu \wedge \bar{\sigma}$ closed, and hence $\overline{\nu \wedge \sigma} \leq \nu \wedge \bar{\sigma}$

Let $\mathcal{F}$ be any prefilter on X such that $\mu \wedge \nu \in \mathcal{F}$.

For each $\sigma \in \mathcal{F}$, we have that $\sup (\nu \wedge \sigma) \geq \sup (\mu \wedge \nu \wedge \sigma) > 0$, in fact $\nu \geq \mu \wedge \nu$ which implies $\nu \in \mathcal{F}$ so that $(\mathcal{F}, \nu) = \mathcal{F}$.

We have $c[(\mathcal{F},\nu),\mu] = \inf_{\sigma\in\mathcal{F}} (\mu\wedge\nu\wedge\sigma) = c(\mathcal{F})$ (since $\mu\wedge\nu\in\mathcal{F}$).

By f–compactness of μ, we have $\sup(\mu\wedge\nu\wedge \mathrm{Adh}\,\mathcal{F}) = \sup_{\sigma\in\mathcal{F}}(\mu\wedge\nu\wedge\bar{\sigma}) \geq \sup_{\sigma\in\mathcal{F}} \overline{\mu\wedge\nu\wedge\sigma}$

$= \sup[\mu\wedge \mathrm{Adh}\,(\mathcal{F},\nu)] \geq c[(\mathcal{F},\nu),\mu] = c(\mathcal{F})$.

Hence $\mu\wedge\nu$ is f–compact.

(b) Again, assuming μ is f–compact, let $a\in I$ and $\mathcal{F}$ be any prefilter on X.

We have $\sup(\mu\wedge a\wedge \mathrm{Adh}\,\mathcal{F}) = a\wedge\sup(\mu\wedge \mathrm{Adh}\,\mathcal{F}) \geq a\wedge c(\mathcal{F},\mu) = a\wedge \inf_{\nu\in\mathcal{F}}(\sup(\mu\wedge\nu))$

$= \inf_{\nu\in\mathcal{F}}(a\wedge\sup(\mu\wedge\nu)) = \inf_{\nu\in\mathcal{F}}\sup(a\wedge\mu\wedge\nu) = c(\mathcal{F},\mu\wedge a)$.

Hence $\mu\wedge a$ is f–compact.

As a corollary to Theorem V.1.7 we have that f–compactness is inherited by closed fuzzy subsets.

V.1.8 COROLLARY

Let (X,δ) be a fts and $\mu\in I^X$. If μ is f–compact, then each closed fuzzy subset of μ is f–compact as well.

V.1.9 PROPOSITION

Let (X,δ) be a fts and $\mu\in I^X$. If μ has a finite support, then μ is f–compact.

V.1.10 PROPOSITION

Let (X,δ) be a fts, $\mu\in I^X$ and $\mathcal{F}$ a prefilter on X which has a base consisting of closed fuzzy sets. Then if μ is f–compact we have $\sup(\mu\wedge \mathrm{Adh}\,\mathcal{F}) = c(\mathcal{F},\mu)$.

PROOF

Follows at once from definition IV.1.1 together with Proposition I.2.9.

V.1.11 THEOREM

Let (X,δ) be a fts and $\mu \in I^X$. Then μ is f–compact if and only if for every prefilterbase $\mathcal{B}$ consisting of closed fuzzy sets, we have $\sup (\mu \wedge \operatorname{Adh} \mathcal{B}) = c(\mathcal{B},\mu)$ if and only if $\sup\limits_{\nu\in\mathcal{B}} \inf \mu \wedge \nu = \inf\limits_{\nu\in\mathcal{B}} \sup \mu \wedge \nu$.

PROOF

"$\Longrightarrow$" suppose μ is f–compact and let $\mathcal{B}$ be a prefilterbase on X consisting of closed fuzzy sets.

Then by Proposition V.1.10 and the fact that $\operatorname{Adh} <\mathcal{B}> = \operatorname{Adh} \mathcal{B}$ and $c(<\mathcal{B}>,\mu) = c(\mathcal{B},\mu)$ the result follows.

"$\Longleftarrow$" Let $\mathcal{F}$ be any prefilter on X, and put $\mathcal{B} = \{\bar{\nu} : \nu \in \mathcal{F})$.

Therefore $\sup (\mu \wedge \operatorname{Adh} \mathcal{B}) = c(\mathcal{B},\mu)$.

But $\operatorname{Adh} \mathcal{B} = \operatorname{Adh} \mathcal{F}$ and since $\mathcal{B} \subseteq \mathcal{F}$ we have that $c(\mathcal{B}, \mu) \geq (\mathcal{F},\mu)$.

Therefore, $\sup (\mu \wedge \operatorname{Adh} \mathcal{F}) = \sup (\mu \wedge \operatorname{Adh} \mathcal{B}) = c(\mathcal{B},\mu) \geq c(\mathcal{F},\mu)$.

Thus μ is f–compact.

Next we give a characterisation of f–compactness in terms of fuzzy nets.

V.1.12 THEOREM

Let (X,δ) be a fts and $\mu \in I^X$. Then μ is f–compact if and only if for every fuzzy net S in μ and every $\epsilon \in <0,c(S)>$, there exists a cluster point p of S such that $p \in \mu$ and $V(p) = c(S) - \epsilon$.

PROOF

"$\Longrightarrow$" Suppose μ is f–compact and let $S = \{S_n : n \in D\}$ be a fuzzy net in μ.

For each $n \in D$, put $\nu_n = \sup\limits_{m \geq n} S_m$.

Then $B = \{\nu_n : n \in D\}$ is a prefilterbase on X, and since $S_m \leq \mu$, for each $m \in D$ we have $\nu_n \leq \mu$ for each $n \in D$.

Hence $c(\mathcal{B},\mu) = \inf\limits_{n\in D} \sup \nu_n \wedge \mu = \inf\limits_{n\in D} \sup \nu_n = \inf\limits_{n\in D} \sup\limits_{m\leq n} V(S_m) = c(S)$.

By f–compactness of μ we have sup $(\mu \wedge \text{Adh}\,\mathcal{B}) \geq c(S)$. Let $\epsilon \in <0,c(S)>$ and choose $x_0 \in X$ such that $(\mu \wedge \text{Adh}\,\mathcal{B})\,(x_0) > c(S) - \epsilon$.

Let p be the fuzzy point with support x_0 and value $c(S) - \epsilon$. Therefore $p \in \mu$,

$V(p) = c(S) - \epsilon$ and $p \in \bar{\nu}_n$ for all $n \in D$.

We claim that p is a cluster point of S. Otherwise, if p is not a cluster point of S, there exists an R–nbd λ of p such that $p \in \lambda$ eventually.

That is, there exists $n \in D$ with $S_m \in \geq$ for all $m \geq n$.

But λ being an R–nbd of p implies there exists $\sigma \in \delta'$ such that $p \notin \sigma$ and $\lambda \leq \sigma$. So that,

$S_m \in \lambda$ for all $m \geq n$ implies $\nu_n = \sup_{m \geq n} \leq \lambda$.

Hence $p \in \bar{\nu}_n \leq \bar{\lambda} \leq \sigma$, which contradicts the fact that $p \notin \sigma$.

Therefore p is a cluster point of S.

"$\Leftarrow$" Suppose the hypothesis of the Theorem is true and let $\mathcal{F}$ be any prefilter on X.

Let $\epsilon \in <0,c(\mathcal{F})>$ and for each $\nu \in \mathcal{F}$, choose $x_\nu \in X$ such that $(\mu \wedge \nu)\,(x_\nu) > c(\mathcal{F}) - \epsilon/2$.

Let S_ν denote the fuzzy point with support x_ν and value $c(\mathcal{F}) - \epsilon/2$.

Therefore $S_\nu \in \mu \wedge \nu$ for all $\nu \in \mathcal{F}$.

If F is directed in the natural way, we have that $S = \{S_\nu : \nu \in \mathcal{F}\}$ is a fuzzy net in μ with characteristic $c(\mathcal{F}) - \epsilon/2$.

By the hypothesis, there exists a cluster point $p \in \mu$ of S such that $V(p) = (c(\mathcal{F})-\epsilon/2)-\epsilon/2 = c(\mathcal{F})-\epsilon$.

It remains to show that $p \in \text{Adh}\,\mathcal{F}$ so that by Theorem V.1.3 we have that μ is f–compact.

To do this, let $\nu \in \mathcal{F}$. If $\nu^* \in \mathcal{F}$ and $\nu^* \leq \nu$ then $S_{\nu^*} \leq \mu \wedge \nu^* \leq \mu \wedge \nu$.

Therefore S is eventually in ν and we conclude that $p \in \bar{\nu}$.

Since $\nu \in \mathcal{F}$ was arbitrary, we have $p \in \text{Adh}\,\mathcal{F}$.

Therefore μ is f–compact.

Making use of Theorem V.1.12 we can establish some of our earlier results more simply in terms of fuzzy nets.

As an illustration, we give a result concerning nested families of closed fuzzy sets.

V.1.13 THEOREM

Let (X,δ) be a fts, $\mu \in I^X$ be f–compact and D be a directed set. Suppose $\{\lambda_n : n \in D\}$ is a family of closed fuzzy sets such that

(i) $\lambda_n \leq \mu$ for all $n \in D$.

(ii) $n_1 \leq n_2$ implies $\lambda_{n_2} \leq \lambda_{n_1}$.

(iii) $\inf_{n \in D} \sup \lambda_n > 0$.

Then there exists a fuzzy point p such that $p \in \lambda_n$ for all $n \in D$.

PROOF

Put $\alpha = \inf_{n \in D} \sup \lambda_n$ and let $0 < \epsilon < \alpha$.

For each $n \in D$ choose $x_n \in X$ with $\lambda_n(x_n) > \alpha - \epsilon/2$.

Let p_n be the fuzzy point with support x_n and value $\alpha - \epsilon/2$. Then $S = \{p_n : n \in D\}$ is a fuzzy net in μ with $c(S) = \alpha - \epsilon/2$.

By f–compactness of μ, there exists a cluster point p of S such that $p \in \mu$ and $V(p) = (\alpha - \epsilon/2) - \epsilon/2 = \alpha - \epsilon$.

If $n_1, n_2 \in D$ and $n_1 \geq n_2$ then $\lambda_{n_1} \leq \lambda_{n_2}$. So that $V(p_{n_1}) = \alpha - \epsilon/2 < \lambda_{n_1}(x_{n_1}) \leq \lambda_{n_2}(x_{n_1})$.

Hence $p_{n_1} \in \lambda_n$ for all $n_1 \geq n$.

Since p is a cluster point of S, we conclude that $p \in \bar{\lambda}_n = \lambda_n$. Otherwise, if $p \notin \lambda_n$, for some $n \in D$ then the fact that λ_n is a closed fuzzy set would imply that λ_n is an R–nbd of p. From $S \infty p$ we therefore have that $p_n \notin \lambda_n$ frequently. That is, there exists some $n_1 \geq n$ such that $p_{n_1} \notin \lambda_n$. This contradicts the fact that $p_{n_1} \in \lambda_n$ for all $n_1 \geq n$.

Therefore p is the required fuzzy point.

V.1.14 LEMMA

Let δ_1 and δ_2 be two fuzzy topologies on X such that $\delta_2 \subsetneq \delta_1$. If $\mu \in I^X$ is f–compact in δ_1 then it is f–compact in δ_2.

V.1.15 THEOREM

Let (X,τ) be a topological space and $\mu \in I^X$. Then the following statements are equivalent.

(a) μ is f–compact in $\omega(\tau)$.

(b) μ is f–compact in $\underline{\omega}(\tau)$.

(c) If $0 < s < t \leq 1$, then every net in μ_t has a cluster point in μ_s.

PROOF

(a) $\Longrightarrow$ (b) : Follows at once from Lemma V.1.14.

(b) $\Longrightarrow$ (c) : Suppose μ is f–compact in $\underline{\omega}(\tau)$ and let $Q = \{x_\lambda : \lambda \in D\}$ be a net in μ_t, $0 < s < t \leq 1$.

Let p_λ denote the fuzzy point with support x_λ and value t. Then $S = \{p_\lambda : \lambda \in D\}$ is a fuzzy net in μ and $c(S) = t$.

By f–compactness of μ in $\underline{\omega}(\tau)$ together with Theorem V.1.12 we have that S has a cluster point p in μ with value s.

Let x denote the support of p.

Then $x \in \mu_s$.

We claim that $Q \propto x$. In fact, if G is open and $x \in G$ then $1_{G'}$ is closed in $\underline{\omega}(\tau)$ and $p \notin 1_{G'}$. Hence $1_{G'}$ is an R–nbd of p. From $S \propto p$ we have that $p_\lambda \notin 1_{G'}$ frequently. So that $x_\lambda \in G$ frequently i.e. x is a cluster point of Q.

(c) $\Longrightarrow$ (a): Suppose condition (c) is satisfied and $S = \{S_n : n \in D\}$ is a fuzzy net in μ, $\epsilon \in < 0, c(S)>$.

By extracting a subnet if necessary, we may assume that $V(S_n) \geq c(S) - \epsilon/2$, for each $n \in D$.

Let x_n be the support of S_n for each $n \in D$; then for all $n \in D$, $x_n \in \mu_t$ where $t = c(S) - \epsilon/2$.

Put $s = c(S) - \epsilon$.

By condition (c) we have a cluster point $x \in \mu_s$ of a crisp net $\{x_n : n \in D\}$ in μ_t. So that $\mu(x) \geq s = c(S) - \epsilon$. Let p be the fuzzy point with support x and value $c(S) - \epsilon$. Then $p \in \mu$.

Let λ be an R–nbd of p.

Then there exists σ closed in $\omega(\tau)$ such that $p \notin \sigma$ and $\lambda \leq \sigma$; hence $\sigma(x) < c(S) - \epsilon$. Choose $a \in [0,1]$ such that $\sigma(x) < a < c(S) - \epsilon$. Since σ is closed in $\omega(\tau)$, it is upper semicontinuous. Therefore $G = \{y \in X : \sigma(y) < a\}$ is τ–open and $x \in G$. Since x is a cluster point of $\{x_n : n \in D\}$ we have that $x_n \in G$ frequently. But then $\sigma(x_n) < a < c(S) - \epsilon < V(Sn)$ frequently. Hence $S_n \notin \sigma$ frequently, which implies $S_n \notin \lambda$ frequently.

Therefore p is a cluster point of S.

f–compactness is a good extension as seen in the next

<u>V.1.16</u> <u>THEOREM</u>

Let (X,τ) be a topological space and $K \subseteq X$. Then the following statements are equivalent.

(a) K is compact.

(b) K is f–compact in $\omega(\tau)$.

(c) K is f–compact in $\underline{\omega}(\tau)$.

<u>PROOF</u>

If $\mu = 1_K$ then condition (c) of Theorem V.1.15 is equivalent to the compactness of K, and hence Theorem V.1.16 is a special case of Theorem V.1.15.

Lastly we show that f–compactness is invariant under continuous mappings.

<u>V.1.17</u> <u>THEOREM</u>

Let (X,δ_1) and (Y,δ_2) be fts, $f : (X,\delta_1) \rightarrow (Y,\delta_2)$ be a continuous mapping and $\mu \in I^X$. If μ is f–compact then the image $f^{\rightarrow}(\mu)$ is f–compact as well.

PROOF

Let $\mathcal{F}$ be a prefilter on Y and suppose $\mu \in I^X$ is f–compact.

If $c(\mathcal{F}, f^{\to}(\mu)) = 0$; then we are done.

Suppose now that $c(\mathcal{F}, f^{\to}(\mu)) > 0$ and let $\mathcal{B} = \{f^{\leftarrow}(\nu) : \nu \in \mathcal{F}\}$.

Then $\mathcal{B}$ is a prefilterbase on X and $c(\mathcal{F}, f^{\to}(\mu)) = c(\mathcal{B}, \mu)$ by the remark preceding Proposition I.2.9.

Thus, $\sup\limits_{y \in Y} (f^{\to}(\mu) \wedge \mathrm{Adh}\,\mathcal{F})(y) = \sup\limits_{y \in Y} \inf\limits_{\nu \in \mathcal{F}} (f^{\to}(\mu) \wedge \bar{\nu})(y) = \sup\limits_{y \in Y} \inf f^{\to}(\mu)(y) \wedge \bar{\nu}(y) = \sup\limits_{x \in X}$
$\inf\limits_{\nu \in Y} f^{\to}(\mu)(f(x)) \wedge \bar{\nu}(f(x)) = \sup\limits_{x \in X} \inf\limits_{\nu \in F} f^{\to}(\mu)(f(x)) \wedge f^{\leftarrow}(\bar{\nu})(x)$

$\geq \sup\limits_{x \in X} \inf\limits_{\nu \in \mathcal{F}} \mu(x) \wedge \overline{f^{\leftarrow}(\nu)}(x)$ (by continuity of f)

$= \sup\limits_{x \in X} (\mu \wedge \mathrm{Adh}\,\mathcal{B})(x) = c(\mathcal{B}, \mu) = c(\mathcal{F}, f^{\to}(\mu))$

Therefore $f^{\to}(\mu)$ is f–compact.

V.2 *THE TYCHONOFF THEOREM*

We first give a characterisation of f–compactness using prime prefilters.

V.2.1 THEOREM

Let (X, δ) be a fts and $\mu \in I^X$. Then μ is f–compact if and only if for every prime prefilter $\mathcal{F}$ on X we have $\sup(\mu \wedge \mathrm{Adh}\,\mathcal{F}) \geq c(\mathcal{F}, \mu)$.

PROOF

"$\Longrightarrow$" Follows at once from the fact that a prime prefilter is a prefilter together with definition V.1.1.

"$\Longleftarrow$" Suppose the hypothesis of the theorem is true and let $\mathcal{F}$ be any prefilter on X.

If $c(\mathcal{F}, \mu) = 0$, then we are done.

Suppose now that $c(\mathcal{F}, \mu) > 0$. Then the prefilter $(\mathcal{F}, \mu)$ exists.

By Lemma I.2.11 we have that there exists a prime prefilter $\mathcal{G}$ on X such that $(\mathcal{F},\mu) \subseteq \mathcal{G}$ and $c(\mathcal{G}) = c(\mathcal{F},\mu)$. Clearly $\mu \in \mathcal{G}$ and hence $c(\mathcal{G}) = c(\mathcal{G},\mu) = c(\mathcal{F},\mu)$. Also $\mathcal{F} \subseteq \mathcal{G}$ and hence $\mathrm{Adh}\,\mathcal{G} \leq \mathrm{Adh}\,\mathcal{F}$.

Therefore $\sup(\mu \wedge \mathrm{Adh}\,\mathcal{F}) \geq \sup(\mu \wedge \mathrm{Adh}\,\mathcal{G}) = c(\mathcal{G},\mu) = c(\mathcal{F},\mu)$.

V.2.2 THEOREM

Let J be an index set and for each $j \in J$, let (X_j, δ_j) be a fts. Then if each $\mu_j \in I^{X_j}$ is f–compact in X_j, the product set $\prod_{j\in J} \mu_j$ is f–compact in $\prod_{j\in J} X_j$.

PROOF

Suppose each μ_j is f–compact in X_j, and let $\mu = \prod_{j\in J} \mu_j$ and $X = \prod_{j\in J} X_j$.

Let $\mathcal{F}$ be any prime prefilter on X.

By f–compactness of each μ_j, we have that for each $j \in J$, $\sup(\mu_j \wedge \mathrm{Adh}\,\mathcal{F}_j) \geq c(\mathcal{F}_j, \mu_j)$.

Let $\epsilon > 0$. For each $j \in J$, choose $t_j \in X_j$ such that

$$(\mu_j \wedge \mathrm{Adh}\,\mathcal{F}_j)(t_j) > c(\mathcal{F}_j, \mu_j) - \epsilon \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (*)$$

Define $g \in X$ by $g(j) = t_j$.

Then $\sup(\mu \wedge \mathrm{Adh}\,\mathcal{F}) \geq (\mu \wedge \mathrm{Adh}\,\mathcal{F})(g) = \mu(g) \wedge (\mathrm{Adh}\,\mathcal{F})(g) = \inf_{j\in J} \mu_j(g(j)) \wedge (\mathrm{Adh}\,\mathcal{F})(g)$

$= \inf_{j\in J} \mu_j(g(j)) \wedge (\prod_{k\in J} \mathrm{Adh}\,\mathcal{F}_k)(g)$ (by Theorem I.2.13)

$= \inf_{j\in J} \mu_j(t_j) \wedge \inf_{k\in J} (\mathrm{Adh}\,\mathcal{F}_k)(t_k)$

$\geq \left[\inf_{j\in J} c(\mathcal{F}_j, \mu_j) - \epsilon\right] \wedge \left[\inf_{k\in J} c(\mathcal{F}_k, \mu_k) - \epsilon\right]$ (by(*)) $\geq c(\mathcal{F},\mu)$ (by Theorem 1.2.12)

Since $\epsilon > 0$ was arbitrary we have that $\sup(\mu \wedge \mathrm{Adh}\,\mathcal{F}) \geq c(\mathcal{F},\mu)$.

By Theorem V.2.1 we have that μ is f–compact.

The converse of Theorem V.2.2 is not true in general as seen in the next

V.2.3 EXAMPLE

Let $\{(X_j, \delta_j) : j \in J\}$ be an indexed family of fts.

For each $j \in J$, let $\mu_j \in I^{X_j}$.

suppose $\mu_{j_0} = 0$, $j_0 \in J$.

Then the product set $\prod_{j\in J} \mu_j = 0$ is clearly f–compact.

But for each $j \in J \setminus \{j_0\}$, μ_j may be selected arbitrarily, and hence they need not be f–compact.

We do however have that the converse of Theorem V.2.2 is valid for fuzzy sets of the same height.

V.2.4 THEOREM

Let J be an index set and (X_j, δ_j) be a fts for each $j \in J$. Suppose for each $j \in J$, $\mu_j \in I^{X_j}$ and $\sup \mu_j = a$, where $a \in I$. If $\prod_{j\in J} \mu_j$ is f–compact, then μ_j is f–compact for each $j \in J$.

PROOF

Suppose $\mu = \prod_{j\in J} \mu_j$ is f–compact in $X = \prod_{j\in J} X_j$ and let $k \in J$.

By Theorem V.1.17 and the fact that the projection maps Π_j are continuous we have that $\Pi_k^{\rightarrow}(\mu)$ is f–compact.

We only need to show that $\Pi_k^{\rightarrow}(\mu) = \mu_k$.

For each $t \in X_k$, $\Pi_k^{\rightarrow}(\mu)(t) = \sup\{\mu(f) : f \in X, f(k) = t\} = \sup\{\inf_{j\in J} \mu_j(f(j)) : f(k) = t\} \leq \mu_k(t)$.

Thus $\Pi_k^{\rightarrow}(\mu) \leq \mu_k$.

On the other hand, let $t \in X_k$ and $\epsilon > 0$.

For each $j \in J$, $j \neq k$ choose $t_j \in X_j$ such that $\mu_j(t_j) > a - \epsilon$.

Define $f \in X$ by $f(j) = t_j$ for $j \neq k$ and $f(k) = t$.

Then $\Pi_k^{\rightarrow}(\mu)(t) \geq \mu(f) = \inf_{j\in J} \mu_j(f(j)) = \inf_{\substack{j\in J \\ j\neq k}} \mu_j(t_j) \wedge \mu_k(t) \geq (a-\epsilon) \wedge \mu_k(t)$.

Since this is true for each $\epsilon > 0$ we have $\Pi_k^{\rightarrow}(\mu)(t) \geq a \wedge \mu_k(t) = \mu_k(t)$. That is $\Pi_k^{\rightarrow}(\mu) \geq \mu_k$.

Therefore $\Pi_k^{\rightarrow}(\mu) = \mu_k$.

Since this is true for each $k \in J$, the result follows.

For f–compactness, the Tychonoff Theorem holds:

V.2.5 THEOREM (THE TYCHONOFF THEOREM)

The product space of the indexed family $\{(X_j, \delta_j) : j \in J\}$ of fts is f–compact if and only if each (X_j, δ_j) $(j \in J)$ is f–compact.

PROOF

The necessity follows at once from the fact that the projection maps are both continuous and onto together with Theorem V.1.17. The sufficiency is a consequence of Theorem V.2.2.

Since the fuzzy topology that we have been using (Definition 0.1.1) is the one by Chang [2], we next show that for f–compactness, we can also use the fuzzy topology by Lowen [10]. In short, we will show that for f–compactness, it does not matter whether we use one definition or the other i.e. if our fuzzy topology is augmented by including the constants, in an obvious way, f–compact sets are not affected.

We adopt the following notation:

Let (X,δ) be a fts and denote by δ^c, the weakest fuzzy topology on X which includes both δ and the constant fuzzy sets. A closed base for δ^c is then given by the collection $\{c \vee \lambda : c$ a constant fuzzy set and λ a closed fuzzy set in $(X,\delta)\}$.

V.2.6 THEOREM

Let (X,δ) be a fts, δ^c be the fuzzy topology generated by δ together with the constant fuzzy sets and let $\mu \in I^X$. Then μ is f–compact in δ if and only if it is f–compact in δ^c.

PROOF

"$\Leftarrow$" Follows at once from the fact that $\delta \subseteq \delta^c$ together with Proposition V.1.14.

"$\Rightarrow$" Suppose μ is f–compact in δ and let $\mathcal{F}$ be any prefilter on X such that $\mu \in \mathcal{F}$.

Choose a prime prefilter $\mathcal{G}$ on X such that $\mathcal{F} \subseteq$ G and $c(\mathcal{F}) = c(\mathcal{G})$.

Since $\mu \in \mathcal{F}$, we have that $\mu \in \mathcal{G}$ and hence $c(\mathcal{G},\mu) = c(\mathcal{G})$.

For any fuzzy set ν the δ^c–closure, $\bar{\nu}$, of ν is given by $\bar{\nu} = \inf\{c \vee \nu : \nu \leq c \vee \lambda$, c a constant fuzzy set and λ closed in $\delta^c\}$.

Let $\nu \in \mathcal{G}$. If $\nu \leq c \vee \lambda$, c constant, λ closed in δ, then $c \vee \lambda \in \mathcal{G}$. Now $\mathcal{G}$ is a prime prefilter implies that either $c \in \mathcal{G}$ or $\lambda \in \mathcal{G}$.

Thus, if $c \in \mathcal{G}$ we have $c \vee \lambda \geq c = \sup c \geq c(\mathcal{G})$.

On the other hand if $\lambda \in \mathcal{G}$, then $c \vee \geq \lambda \geq \mathrm{Adh}_\delta \mathcal{G}$, where $\mathrm{Adh}_\delta \mathcal{G}$ denotes the δ–adherence of $\mathcal{G}$.

Combining the two cases we have $c \vee \lambda \geq c(\mathcal{G}) \wedge \mathrm{Adh}_\delta \mathcal{G}$, for all $c \vee \lambda$ such that $\nu \leq c \vee \lambda$, c a constant and λ–closed in δ. Hence $\bar{\nu} \geq c(\mathcal{F}) \wedge \mathrm{Adh}_\delta \mathcal{G}$. Since this holds for all $\nu \in \mathcal{G}$, we have $\mathrm{Adh}_\delta \mathcal{G} \geq c(\mathcal{G}) \wedge \mathrm{Adh}_\delta \mathcal{G}$. Hence $\mathrm{Adh}_{\delta^c} \mathcal{G} \geq c(\mathcal{G}) \wedge \mathrm{Adh}_\delta \mathcal{G}$.

Finally, $\sup \mu \wedge \mathrm{Adh}_{\delta^c} \mathcal{F} \geq \sup\ \mu \wedge (c(\mathcal{G}) \wedge \mathrm{Adh}_\delta \mathcal{G}) = c(\mathcal{G}) \wedge \sup(\mu \wedge \mathrm{Adh}_\delta \mathcal{G}) \geq c(\mathcal{G})$ by the f–compactness of μ in δ. Since $c(\mathcal{G}) = c(\mathcal{F})$, f–compactness of μ in δ^c follows.

CHAPTER VI

A COMPARISON OF DIFFERENT NOTIONS OF COMPACTNESS IN A FTS.

INTRODUCTION

In this chapter we will mostly restrict ourselves to the whole space and compare all those notions which we found to be good extensions. We give the implications that might exist amongst them and lastly, we find out that for a T_2 (Definitions I.2.12, I.2.13, I.2.14, I.2.16) space all these notions are equivalent.

VI.1. *COMPARISON OF DIFFERENT NOTIONS OF COMPACTNESS IN A FTS.*

VI.1.1 PROPOSITION

(X,δ) is α–compact if and only if $(X, i_\alpha(\delta))$ is compact, where $\alpha \in [0,1\rangle$.

PROOF

This follows at once from the fact that the collection $\beta \subseteq \delta$ is an α–shading of X if and only if the collection $\{\mu^{\leftarrow}\langle\alpha,1] : \mu \in \beta\}$ is an open cover for X.

VI.1.2. THEOREM

Suppose (X,δ) is ultra–fuzzy compact; then it is strong fuzzy compact.

PROOF

Follows from the fact that for all $\alpha \in [0,1\rangle$, $i(\delta) \supseteq i_\alpha(\delta)$ together with Proposition VI.1.1

VI.1.3 PROPOSITION

Suppose (X,δ) is strong fuzzy compact and $\alpha \in [0,1\rangle$; then it is α–compact.

We also have

VI.1.4 THEOREM

Suppose (X,δ) is N–compact, then it is strong fuzzy compact.

PROOF

Follows at once from the fact that a constant α–net is an α–net together with Theorem III.1.6 and definition IV.1.1.

The converse of Theorem VI.1.4 is not true in general as seen in the next

VI.1.5 EXAMPLE

Let $X = <0,1>$ and define a fuzzy set $\lambda : X \longrightarrow I$ by $\lambda(x) = x$ for each $x \in X$.

Let δ be a fuzzy topology generated by all the constant fuzzy sets and λ.

Therefore $\delta = \{\text{constant fuzzy sets}, \lambda, \mu_x\}$ where for each $x \in X$ the fuzzy set $\mu_x : X \longrightarrow I$ is given by:

$$\mu_x(y) = \begin{cases} y & \text{if } y \leq x \\ x & \text{if } y > x. \end{cases}$$

It can easily be seen that if $\alpha \in [0,1>$, then (X,δ) is α–compact.

Since this is true for each $\alpha \in [0,1>$, we have that (X,δ) is strong fuzzy compact.

On the other hand we have that $\lambda \in \delta$, hence $1 - \lambda \in \delta'$.

Since $\sup \{(1-\lambda)(x) : x \in X\} = 1$, but for each $x \in X$, $(1-\lambda)(x) < 1$, we have that $1-\lambda$ does not attain its maximum. Therefore Corollary IV.1.16 implies that (X, δ) is not N–compact.

An obvious question to ask is: under what conditions will the converse of Theorem VI.1.4 be true?

This is answered in the next

VI.1.6 THEOREM

Suppose (X,δ) is strong fuzzy compact, then it is N–compact if and only if each closed fuzzy set attains its maximum.

PROOF

The necessity part follows at once from Corollary IV.1.16. To prove the sufficiency suppose (X,δ) is strong fuzzy compact but not N–compact. Then there exists $\alpha \in <0,1]$ and an α–net $S = \{S_n : n \in D\}$ without any cluster point of value α.

Therefore, for each $x \in X$ there is an R–nbd ν_x of $\alpha 1_x$ such that $S_n \in \nu_x$, eventually.

Thus, there exists an $N_x \in D$ such that $S_n \in N_x$ holds for all $n \geq N_x$. By definition of an R–nbd, for each $x \in X$, there is $\sigma_x \in \delta'$ such that $\alpha 1_x \notin \sigma_x$ and $\nu_x \leq \sigma_x$.

Let $\beta = \{\sigma_x : x \in X\}$.

Therefore $\beta \subseteq \delta'$ and since $\sigma_x(x) < \alpha$ we have that $\wedge\, \beta < \alpha$. The fact that (X,δ) is strong fuzzy compact together with Theorem III.1.5 implies that β has a finite subfamily $\psi = \{\sigma_x 1, ..., \sigma_x k\}$ with $\wedge\, \psi < \alpha$.

Let $\sigma = \wedge\, \psi$.

Then $\sigma \in \delta'$, and we will show that σ has no maximum.

In fact, if $N \in D$ so that $N \geq N_{x^i}$ for all $i \in [k]$, then $S_n \in \sigma$ holds for all $n \geq N$.

By definition of an α–net, for any $\epsilon > 0$ there is an $x \in X$ such that $\sigma(x) > \alpha - \epsilon$.

On the other hand, since $\sigma < \alpha$, $\sigma(x) < \alpha$ holds for all $x \in X$.

Hence σ has no maximum.

VI.1.7 THEOREM

Suppose (X,δ) is ultra–fuzzy compact; then it is N–compact.

PROOF

This follows at once from the fact that a transitive limit is a limit, together with Theorems III.2.9 and IV.1.9.

That the converse is not true in general is seen in the next

VI.1.8 EXAMPLE

Let $X = \mathbb{N}$.

For each $\alpha \in <0,1>$, there exists an $n \in \mathbb{N}$ such that

$$\frac{n-1}{n} < \alpha \leq \frac{n}{n-1}.$$

Choose $\alpha_i \in \left[\frac{n-1}{n}, \alpha\right\rangle$ $(i \in [n])$ and put

$$\lambda(\alpha, \alpha_1, \ldots, \alpha_n)(x) = \begin{cases} \alpha & \text{if } \alpha > n \\ \alpha_i & \text{if } x = i, i \in [n]. \end{cases}$$

Let δ be the family consisting of all the complements $1-\lambda$ of these fuzzy sets as well as the fuzzy sets 0_X and 1_X.

It can easily be seen that δ is a fuzzy topology on X and hence (X,δ) is a fts.

Since $i(\delta)$ is the collection of all subsets of X, we have that $i(\delta)$ is a discrete topology on X and hence $(X,i(\delta))$ is not compact. Which implies (X,δ) is not ultra–fuzzy compact.

On the other hand, let S be an α–net in (X,δ). If $\frac{n-1}{n} < \alpha \leq \frac{n}{n+1}$, then for $m > n$ the fuzzy point with support m and value α is a limit point of S.

If $\alpha = 1$, then every fuzzy point with value α is a limit of S.

Therefore (X,δ) is N–compact.

VI.1.9 THEOREM

Suppose (X,δ) is N–compact, then it is f–compact.

PROOF

Follows at once from Theorem IV.2.2 together with definition V.1.1.

The converse is not true in general:

VI.1.10 EXAMPLE

Let $X = <0,1>$ and $\lambda : X \longrightarrow I$ be a fuzzy set defined by $\lambda(x) = x$ for each $x \in X$.

Let $\delta = \{0, \lambda, 1\}$.

It is easy to see that the fuzzy set $1-\lambda$ does not attain its maximum and since $1-\lambda \in \delta'$, we have by Corollary IV.1.16 that (X,δ) cannot be N–compact.

But since δ is finite, we have by Theorem V.1.6 that (X,δ) is f–compact.

We do however have

VI.1.11 THEOREM

Let (X,δ) be a fts and $\mu \in I^X$ be f–compact. Then μ is N–compact if and only if $\mu \wedge \lambda$ attain its maximum for every closed fuzzy set λ.

PROOF

"$\Rightarrow$" Suppose μ is N–compact and let λ be any closed fuzzy set in X; so that by Theorem IV.1.7 $\mu \wedge \lambda$ is N–compact. Then by Theorem IV.1.15 we have that $\mu \wedge \lambda$ attains its maximum.

"$\Leftarrow$" Suppose μ is f–compact and $\mu \wedge \lambda$ attains its maximum for each closed fuzzy set λ. Let $\mathcal{F}$ be any prefilter on X such that $\mu \in \mathcal{F}$ and $c(\mathcal{F}) > 0$. Therefore by f–compactness of μ we have that $\sup (\mu \wedge \text{Adh}\,\mathcal{F}) \geq c(\mathcal{F}) > 0$.

But Adh $\mathcal{F}$ is a closed fuzzy set, so that the assumption of the theorem implies that $\mu \wedge \text{Adh}\,\mathcal{F}$ attains its maximum, at some $x_0 \in X$.

That is, $(\mu \wedge \text{Adh}\,\mathcal{F})(x_0) = \sup (\mu \wedge \text{Adh}\,\mathcal{F}) \geq c(\mathcal{F}) > 0$. Thus $c(\mathcal{F})\, 1_{x_0} \in \mu \wedge \text{Adh}\,\mathcal{F}$.

Therefore by Theorem IV.2.1 μ is N–compact.

Taking $\mu = 1_X$ in Theorem VI.1.11 we have the following

VI.1.12 COROLLARY

Suppose (X,δ) is f–compact; then it is N–compact if and only if each closed fuzzy set attains its maximum.

In view of Theorem VI.1.6 and Corollary VI.1.2, an obvious question is: Is there any implication between the strong fuzzy compactness and the f–compactness for the whole space? Before we answer the question, we need a few results.

We firstly need

VI.1.13 DEFINITION [10].

Let (X,δ) be a fts. Then (X,δ) is fuzzy compact if for every family $\beta \subseteq \delta$ and for each $\alpha \in I$ such that $\sup_{\mu\in\beta} \mu \geq \alpha$ and for each $\epsilon \in <0,\alpha]$ there exists a finite subfamily β_0 of β such that

$$\sup_{\mu\in\beta_0} \mu \geq \alpha - \epsilon.$$

In [10] Lowen requires the fuzzy topology to include the constants and hence many results which he has for fuzzy compactness are only valid for that type of a topology. Since in our fuzzy topology (definition O.1.1) we did not require the constants to be included, we would like to point out here that we will only use those results that are also valid in our case.

In [13] Lowen gives a characterisation of fuzzy compactness as

VI.1.14 THEOREM

Let (X,δ) be a fts. Then (X,δ) is fuzzy compact if and only if for each prefilter $\mathcal{F}$ on X we have $\sup \mathrm{Adh}\,\mathcal{F} \geq c(\mathcal{F})$.

PROOF

See [13] Theorem 5.1

Motivated by this, in [1] Chadwick gives the definition of f–compactness (Definition V.1.1) and proves that f–compactness reduces to fuzzy compactness where fuzzy compactness is meaningful. He in fact, gives the following result:

VI.1.15 THEOREM

Let A be a subset of X and let $\Delta_A = \{\sigma_A : \sigma \in \Delta\}$ be the fuzzy topology on A consisting of the restrictions to A of members of Δ, where (X,Δ) is a fts. Then (A,Δ_A) is fuzzy compact if and only if 1_A is f–compact.

PROOF

See [1] Proposition 2.2

In particular, if we take A = X then we have

VI.1.16 THEOREM

A fts (X,δ) is f–compact if and only if it is fuzzy compact.

In [12] Lowen gives the following result:

VI.1.17 THEOREM

If (X,δ) is strong fuzzy compact, then it is fuzzy compact.

In view of Theorem V.2.6, our question is then answered in the next

VI.1.18 THEOREM

If (X,δ) is strong fuzzy compact, then it is f–compact.

PROOF

Follows at once from Theorem VI.1.16 and Theorem VI.1.17 in view of Theorem V.2.6.

The converse of Theorem VI.2.17 (equivalently, Theorem VI.1.18) is not true in general as shown in the next example ([12]).

VI.1.19 EXAMPLE

Let $X = I$, for all $x \in X \cap \mathbb{Q}$, let $x = p/q$, where p and q have no common factor.

Put $\nu_x^s = s/q + 1/q\ 1_x$, for all $s \in \mathbb{N}$, $0 \leq s \leq q-1$.

Let $\sigma_1 = \{1_x : x \in X, \text{x irrational}\}$

$\sigma_2 = \{\nu_x^s : x \in X \cap \mathbb{Q},\ x = p/q,\ s \in \mathbb{N},\ 0 \leq s \leq q-1\}$

Let δ denote the fuzzy topology on X generated by

$\sigma = \{\alpha : \alpha \text{ a constant fuzzy set}\} \cup \sigma_1 \cup \sigma_2$.

Then in [12] Lowen shows that

(i) (X,δ) is fuzzy compact (equivalently f–compact).

(ii) For each $\alpha \in [0,1>$, $i_\alpha(\delta)$ is a discrete topology on X.

It is thus clear from (ii) that (X,δ) is not α–compact for some $\alpha \in [0,1>$. Hence (X,δ) is not strong fuzzy compact.

Putting together the foregoing arguments we have the following implications:

$$\begin{array}{ccc} \text{Ultra–fuzzy compactness} & \Longrightarrow & \alpha\text{–compactness} \\ \Downarrow & & \Uparrow \\ \text{N–compactness} & \Longrightarrow & \text{strong fuzzy compactness} \\ \Downarrow & & \Downarrow \\ \text{f–compactness} & \Longleftrightarrow & \text{fuzzy compactness} \end{array}$$

We have showed by means of counter–examples that the arrows above are not reversible in general. It is therefore of interest to know a condition (if any) which would guarantee the converses of the above implications. We seek a condition that will give the equivalence between strong fuzzy, ultra–fuzzy, fuzzy and N–compactness.

To do this, we only need to find a condition that would "force" fuzzy compactness to imply ultra–fuzzy compactness (since then, all the above notions would be equivalent).

This will be done in the following way:

<u>STEP 1</u>: Fuzzy compact $\Rightarrow$ strong fuzzy compact.

<u>STEP 2</u>: Strong fuzzy compact $\Rightarrow$ ultra–fuzzy compact:

As seen in the next

VI.1.20 THEOREM

Let (X,δ) be a T_2 fts in the sense that no fuzzy net converges to two distinct fuzzy points (see Theorem I.1.13). Then fuzzy compactness, strong fuzzy compactness, N–compactness and ultra–fuzzy compactness are equivalent.

PROOF

STEP 1

Suppose (X,δ) is a T_2 fuzzy compact fts and let $S = \{x^n_\alpha : n \in D\}$ be a constant α–net in (X,δ) $(\alpha \in <0,1])$. Since a constant α–net is an α–net we have by Theorem V.1.12 that for each $\epsilon \in <0,\alpha>$, S has a cluster point $x_{\alpha-\epsilon}$.

By Theorem I.1.15 we have that S has a subnet T_ϵ converging to $x_{\alpha-\epsilon}$. T_ϵ is a constant α–net as well, hence for each $k \in \mathbb{N}$ we can apply the fuzzy compactness again to have $y_{\alpha-\epsilon/k}$ such that $T_\epsilon \infty\, y_{\alpha-\epsilon/k}$.

By Propositon I.1.18 we have that $x = y$.

Therefore for each $k \in \mathbb{N}$, $x_{\alpha-\epsilon/k}$ is a cluster point of S. Suppose ν is an R–nbd of the fuzzy point x_α. Hence there exists $\sigma \in \delta'$ such that $x_\alpha \notin \sigma$ and $\nu \leq \sigma$. We have $\alpha > \sigma(x) \geq \nu(x)$. By choosing $k \in \mathbb{N}$ suitably large we have $\alpha - \epsilon/k > \sigma(x) \geq \nu(x)$, which implies ν is an R–nbd of the fuzzy point $x_{\alpha-\epsilon/k}$.

We claim that x_α is a cluster point of S. In fact, if ν is an R–nbd of x_α then for some $k \in \mathbb{N}$ (suitably large) we have that ν is an R–nbd of $x_{\alpha-\epsilon/k}$.

Since $x_{\alpha-\epsilon/k}$ is a cluster point of S we have that $S \notin \nu$, frequently. Which implies x_α is a cluster point of S.

Therefore by Theorem III.1.6 we have that (X,δ) is strong fuzzy compact.

STEP 2.

Suppose (X,δ) is a T_2 strong fuzzy compact fts and let $f = \{x^n : n \in D\}$ be a crisp net in X. We can construct a 1–net $S = \{x^n_1 : n \in D\}$ in (X,δ).

By strong fuzzy compactness of (X,δ) we have that S has a cluster point x_1.

Suppose $T = \{x^{n(m)}_1 : m \in E\}$ is a subnet of S converging to x_1.

By Propostion I.1.17 we only need to show that x_1 is a transitive limit of T. In fact, if R = $\{x_\alpha^{n(m)} : m \in E\}$ is a constant α–net ($\alpha \in <0,1]$) similar to T, then by strong fuzzy compactness of (X,δ) it follows that R has a cluster point y_α.

It is clear that y_α is also a cluster point of T, and hence by Proposition I.1.18 we have that x = y. Hence x_α is a unique cluster point of R with value α.

We claim that $R \rightarrow x_\alpha$. Otherwise, there would exist on R–nbd, ν, of x_α such that $R \in \nu$, frequently. That is, for each $M \in E$ there is an $m \geq M$ for which $x_\alpha^{n(m)} \in \nu$.

Pick R* a subnet of R in ν.

Then clearly R* is a constant α–net as well, and thus by strong fuzzy compactness of (X,δ) we have that R* has a cluster point y_α. Since ν is an R–nbd of x_α, there exists $\sigma \in \delta'$ such that $x_\alpha \notin \sigma$ and $\nu \leq \sigma$. Since σ is a closed fuzzy set, we see that $y_\alpha \in \sigma$. But $x_\alpha \notin \sigma$ would imply that R has two distinct cluster points with value α, a contradiction.

Hence $R \rightarrow x_\alpha$.

Therefore x_1 is a transitive limit of T.

Since N–compactness and f–compactness are the only notions defined for an arbitrary fuzzy set, an obvious question is:

"Under what condition will N–compactness and f–compactness be equivalent for an arbitrary fuzzy set in X?"

This is answered in the following

<u>VI.1.21</u> <u>THEOREM</u>

Let (X,δ) be a fts. If (X,δ) is T_2 in the sense that every prime prefilter has an adherence that is non–zero in at most one point (see Theorem I.3.6) and $\mu \in I^X$, then μ is f–compact if and only if μ is N–compact.

PROOF

"$\Longrightarrow$" Suppose μ is f–compact and let $\mathcal{F}$ be a prime prefilter on X. We only need to show that $\mu \wedge \text{Adh}\,\mathcal{F}$ attain its maximum; since by f–compactness of μ together with Theorem IV.2.3 we will have that μ is N–compact.

If Adh $\mathcal{F}$ is zero everywhere on X then the result holds trivially. Otherwise, the fact that (X,δ) is T_2 implies that there exists $x_0 \in X$ such that $(\text{Adh}\,\mathcal{F})\,(x_0) \neq 0$ and for each $y \in X$, $y \neq x_0$; $(\text{Adh}\,\mathcal{F})(y) = 0$. Hence $\sup(\mu \wedge \text{Adh}\,\mathcal{F}) = (\mu \wedge \text{Adh}\,\mathcal{F})\,(x_0)$.

Therefore $\mu \wedge \text{Adh}\,\mathcal{F}$ attains its maximum (at x_0).

"$\Longleftarrow$" Follows from definition V.1.1 together with IV.2.2.

REFERENCES

1. J.J. CHADWICK, A generalized form of compactness in Fuzzy Topological spaces. (submitted to J. Math. Anal.Appl.)

2. C.L. CHANG, Fuzzy Topological spaces, J. Math.Anal. Appl. 24(1968), 182–190.

3. T.E. GANTNER & R.C. STEINLAGE, Compactness in Fuzzy Topological spaces, J. Math. Anal. Appl. 62(1978), 547–562.

4. J.A. GOGUEN, L–fuzzy sets, J. Math. Anal. Appl. 18(1967), 145–174.

5. J.A. GOGUEN, The Fuzzy Tychonoff Theorem, J.Math. Anal. Appl. 43 (1973), 734–742.

6. B. HUTTON, Products of Fuzzy Topological spaces, Topology Appl. 11(1980), 59–67.

7. J.L. KELLEY, General Topology, Von Nostrand, Princeton, New Jersey, 1955.

8. W. KOTZÉ, The Fuzzy Hausdorff Property, Glasnik Mat. 20 (40) (1985), 409–417.

9. W. KOTZÉ, Quasi–coincidence and Quasi–fuzzy Hausdorff, J. Math. Anal. Appl. 116 (2) (1986), 465–472.

10. R. LOWEN, Fuzzy Topological Spaces and Fuzzy Compactness, J. Math. Anal. Appl. 56 (1976), 621–633.

11. R. LOWEN, Initial and Final Topologies and the Fuzzy Tychonoff Theorem.,J. Math. Anal. Appl. 58 (1977), 11–21.

12. R. LOWEN, A Comparison of different Compactness notions in Fuzzy Topological spaces, J. Math. Anal. Appl. 64 (1978), 446–454.

13. R. LOWEN, Convergence in Fuzzy Topological Spaces, General Topology and Appl. 10 (1979), 147–160.

14. R. LOWEN, Compact Hausdorff Fuzzy Topological spaces are Topological, Topology Appl. 12 (1981), 66–74.

15. R. LOWEN, The Relations between Filter and net convergence in Fuzzy Topological spaces, Fuzzy Math. 12 (1983), 41–52.

16. S.R. MALGHAN & S.S. BENCHALLI, On Fuzzy Topological Spaces, Glasnik Mat. Ser III 16 (1981), 313–325.

17 PU PAO–MING & LIU YING–MING, Fuzzy Topology, I, Neighbourhood structure of a Fuzzy point and Moore–Smith convergence, J. Math. Anal. Appl. 76 (1980), 571–599.

18. R. SRIVASTAVA, S.N. LAL, & A.K. SRIVASTAVA, Fuzzy Hausdorff Topological spaces, J. Math. Anal. Appl. 81 (1981), 497–506.

19. WANG GUOJUN, A new fuzzy compactness defined by Fuzzy nets, J. Math. Anal. Appl. 94 (1983), 1–23.

20. R.H. WARREN, Continuity of mappings of Fuzzy Topological spaces, Notices Amer. Math. Soc. 21(4), (1974), A–451.

21. R.M. WARREN, Neighbourhoods, bases and continuity in Fuzzy Topological spaces, Rocky Mt. J. Math. 8(3) (1978), 459–470.

22. M.D. WEISS, Fixed points, separation and induced topology for Fuzzy sets, J. Math. Anal. Appl. 50(1975), 142–150.

23. C.K. WONG, Covering properties of Fuzzy Topological Spaces, J. Math. Anal. Appl. 43 (1973), 697–704.

24. C.K. WONG, Fuzzy Topology: Product and Quotient Theorems, J. Math. Anal. Appl. 45 (1974), 512–521.

25. C.K. WONG, Fuzzy points and local properties of Fuzzy Topology, J. Math. Anal. Appl. 46 (1974), 315–328.

26. L.A. ZADEH, Fuzzy sets, Information and Control 8 (1965), 338–353.

27. ZHAO DONGSHENG, The N–compactness in L–fuzzy topological spaces, J. Math. Anal. Appl. 128 (1987), 64–79.

COMPARISON OF DIFFERENT NOTIONS OF COMPACTNESS IN THE FUZZY TOPOLOGICAL SPACE

E.Z. MORAPELI

ERRATA PAGE

Title page	"partial fulfillment" instead of "fulfillment".
Abstract page	line 2↑ "stronger" instead of "weaker".
Content page	CHAPTER III η is replaced by –.
Preface page	line 6↑ "weaker" is replaced by "stronger".
page iv	"advise" is replaced by "advice".
page 14	line 9↓ we include reference [3].
page 19	line 1↑ "see [17] Theorem 11.1" is replaced by "see [17] Theorem 11.1 and remark after Theorem 0.2.7".
page 21	line 5↑ "[8]" instead of "[19]".
page 22	line 11↑ "that any α–net..." is replaced by "that any net..."
page 22	line 7↑ "limit of the α–net..." should read "limit of the constant α–net".
page 25	line 1↓ "prefilter–base" should be "prefilterbase".
page 31	line 2↓ "$\gneqq$" should read "=".
page 32	line 10↓ "[16]" should be "[15]".
	line 11↓ "the one using fuzzy ..." should read "the one of Pu and Liu using fuzzy nets ([17])..."
	line 12↓ "[16]" should read "[15]".
page 37	line 2↑ "p_m" should read "P_m".
page 41	line 2↑ we omit Definitions 0.2.12, 0.2.13, and 0.2.15.
page 42	Omit line 7↓ (In view of Theorem.....).
page 69	lines 10↓ and 11↓ "into I have ..." should read "into I assume".
Reference page	No.15 "The relations between...." should read "The relation between ...".