LEFT-INVARIANT OPTIMAL CONTROL PROBLEMS ON THE HEISENBERG GROUP AND THE ASSOCIATED HAMILTON-POISSON SYSTEMS: CLASSIFICATION, STABILITY AND INTEGRATION

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Abstract

This thesis examines the left-invariant control affine systems of full rank, evolving on the three-dimensional Heisenberg group H_3. Such systems are classified under state space equivalence, detached feedback equivalence and strongly detached feedback equivalence; a complete list of equivalence representatives is obtained. The equivalence of cost-extended control systems corresponding to left-invariant optimal control problems on H_3 with fixed terminal time, affine dynamics, and affine quadratic cost is also considered. To left-invariant optimal control problems on H_3 with quadratic cost, one may, via the Pontryagin Maximum Principle, associate a quadratic Hamilton-Poisson system on the (minus) Lie-Poisson space $\mathfrak{h}_{3-}^*$. Homogeneous and inhomogeneous quadratic Hamilton-Poisson systems are investigated. These systems are classified up to an affine isomorphism. Furthermore, the stability nature of the equilibria of the systems are analysed and explicit expressions for all integral curves are determined.

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Contents

Introduction

Mathematical control theory (including geometric and optimal control theory) examines the underlying mathematical principles, theory and problems concerning the analysis and design of control systems and has far reaching application, most notably in engineering. Further applications are found in the areas of robotics, economics, chemistry, biology, ecology and medicine.

A control system is a family of differential equations, evolving on some state space, parametrized by control parameters. In fixing a control, the system becomes a single nonautonomous ordinary differential equation, the solution of which is completely determined by its initial state and called an admissible trajectory of the control system. Each control (in general) provides a different admissible trajectory. The set of all admissible trajectories form the attainable set of an initial state. To characterize such attainable sets is the first basic problem in control theory: *the Controllability Problem.* As soon as the possibility of reaching a state is realised we try to do it in the best way: *the Optimality Problem* [6].

Optimal control theory began in 1697 with the solution to the brachystochrone problem by Johann Bernoulli [44]. The essential foundation of optimal control theory is that of the Pontryagin Maximum Principle, developed by L.S. Pontryagin and his group in the late 1950s, which gives the fundamental necessary conditions for optimality in optimal control problems. In its original form the Pontryagin Maximum Principle had serious limitations; it is geometric control theory (which began in the late 1960s with the study of (nonlinear) control systems using concepts and methods from differential geometry (cf. [6], [25], [43])) that forms a theoretical foundation for extensions of the maximum principle to optimal control problems on arbitrary differential manifolds [25]. From a geometric view point, a differential equation is a vector field and so a control system is viewed as a parametrised family of vector fields on a manifold.

Of particular interest are invariant control systems on Lie groups in which case the vector fields are invariant. Invariant control systems on Lie groups were first considered in 1972 by Brockett [20] and by Jurdjevic and Sussmann [27]. These systems still model a large class of problems while allowing the use of Lie theory.

Recent attention has been drawn to invariant control affine systems on low dimensional Lie groups, as well as the optimal control problems associated to such systems. These systems have been used to model specific problems such as Euler's elastic problem [[42], [26]], the control of spacecraft [38] and the motion of a free rigid body [11]. A more general approach has also been considered: The classification of three-dimensional Lie groups under state space equivalence and detached feedback equivalence ([4], [7], [12], [13], [14], [16]) as well as the consideration of the associated optimal control problems. In particular, this has been done by considering cost-extended control systems corresponding to certain invariant optimal control problems on Lie groups and the classification of such systems ([19], [17]). An invariant optimal control problem on a Lie group may be lifted to a family of Hamilton-Poisson systems on the cotangent bundle. For several classes of such systems, the Pontryagin Maximum principle reduces the family of Hamilton-Poisson systems on the cotangent bundle to a single Hamiltonian system on the dual Lie algebra ([17], [19], [6]). (Here the Poisson structure is the Lie-Poisson structure.)

A number of quadratic Hamilton-Poisson systems have been considered by several authors ([2], [3], [40], [34], [9], [10]). The integration of quadratic Hamilton-Poisson systems is of great significance in the investigation of the extremal controls of invariant optimal control problems as (as noted by the authors of [17] and [19]) the problem of determining the extremal controls for a large class of invariant optimal control problems reduces to the study of the integral curves of a quadratic Hamilton-Poisson system on the dual Lie-Poisson space. The equivalence of Hamilton-Poisson systems on Lie-Poisson spaces has also been investigated. For instance, the orthogonal equivalence of systems on $\mathfrak{so}(3)^*_-$ ([22]), linear equivalence of Hamilton-Poisson systems ([19], [17]) and the classification of a class of Homogeneous quadratic systems on $\mathfrak{se}(2)^*_-$ ([5]) and on $\mathfrak{se}(1,1)^*_-$([8]).

In this thesis we consider (left-)invariant optimal control problems on the (three-dimensional) Heisenberg group H_3. The central focus is on the classification of such systems; on finding explicit expressions for the integral curves of the associated Hamilton-Poisson systems and determining the stability nature of the equilibria for each of these systems. We outline the topics covered.

Chapter 1 introduces the Heisenberg group H_3 and investigates several of its (topological and algebraic) properties, the results of which are well known but essential in a thorough investigation of the optimal control problems on H_3. Topologically, H_3 is simply connected and connected (important properties in control theory). We determine the Heisenberg Lie algebra $\mathfrak{h}_3$ and the group of Lie algebra automorphisms $\mathsf{Aut}(\mathfrak{h}_3)$ (The group of Lie algebra automorphisms plays a crucial role in the classification of systems in later chapters.). We also calculate the adjoint and coadjoint of orbits H_3.

In chapter 2 we classify all left-invariant control affine systems on H_3 under state space equivalence, detached feedback equivalence and strongly detached feedback equivalence. In each case the classification may be done at the level of the Lie algebra and is carried out in this way. Class representatives are selected and the controllable systems on H_3 are determined.

Chapter 3 covers the left-invariant optimal control problems with affine quadratic cost and fixed terminal time associated to the systems of chapter 2. This is done by considering the associated cost-extended control systems and the equivalence of these systems. We use the associated Hamiltonian on the cotangent bundle to determine the optimal controls of the optimal control problem.

Chapter 4 is devoted to the classification of Quadratic Hamilton-Poisson systems on the Lie-Poisson space $\mathfrak{h}^*_{3-}$. We begin by considering equivalence up to linear isomorphisms of the homogeneous systems and note that for such systems this is the same as equivalence up to affine isomorphisms which we term affine equivalence. Based on the classification of homogeneous systems we arrive at the classification of inhomogeneous systems under affine equivalence.

Finally, in chapter 5 we investigate the class representatives of the Hamilton-Poisson systems of chapter 4. We determine the stability nature of the equilibria for each of the systems and find explicit expressions for the integral curves of the systems, all of which are expressible in terms of elementary functions. The extremal controls for a class of optimal control problems on H_3 may be obtained from these integral curves.

Appendix A contains a summary of the Lie theory and control theory relevant to this thesis. Appendix B contains a table collecting the results of the classification of the left invariant control affine systems of chapter 2. Mathematica 8 has been used to assist with calculations; the code may be found in appendix C.

Original Contributions

To the best of our knowledge, the following contributions are original:

Chapter 2. The classification of all full-rank left-invariant control affine systems on H_3 under state space equivalence (lemmas 2.2.11, 2.2.12, 2.2.13 and 2.2.14 and propositions 2.2.15, 2.2.16 and 2.2.17 and corollary 2.2.18); the characterisation of strongly detached detached feedback equivalence and the classification of all full-rank left-invariant control affine systems on H_3 under strongly detached feedback equivalence (section 2.4); the comparison of state space, detached feedback and strongly detached feedback equivalence class representatives (propositions 2.5.1 and 2.5.2);

Chapter 3. The classification of controllable two and three-input homogeneous cost-extended systems on H_3 under cost-equivalence (propositions 3.1.9 and 3.1.13) and the calculation of the extremal controls of these systems (propositions 3.1.10 and 3.1.14).

Chapter 4. The calculation of the linear Poisson symmetries for each homogeneous Hamilton-Poisson system (proposition 4.4.2); a complete classification of all inhomogeneous quadratic Hamilton-Poisson systems on $\mathfrak{h}_{3-}^*$ under affine equivalence (theorems 4.4.4, 4.4.5, 4.4.6 and 4.4.7).

Chapter 5. A complete stability analysis of all quadratic Hamilton-Poisson systems on $\mathfrak{h}_{3-}^*$ (propositions 5.2.1, 5.2.2, 5.2.3, 5.2.4, 5.2.5, 5.3.1, 5.3.2, 5.3.3, 5.3.4, 5.3.5, 5.3.6, 5.3.8, 5.3.9 and 5.3.10); the calculation of explicit expressions for all integral curves of all quadratic Hamilton-Poisson systems on $\mathfrak{h}_{3-}^*$ (propositions 5.2.6 and 5.3.7).

Notation

Lie groups (with the exception of $\mathcal{H}_3$) are written using upper-case characters in a sans serif typeface (e.g., G). Lie algebras (with the exception of h_3) are written using lower-case letters in Fraktur typeface (e.g., $\mathfrak{g}$). The following notation will also be used:

$\mathbf{1}$	identity element of a Lie group.
$C^\infty(\mathsf{M})$	the set of (smooth) real-valued functions on a smooth manifold M.
$\langle \cdot, \cdot \rangle$	natural pairing $\mathfrak{g} \times \mathfrak{g}$, $(p, X) \mapsto p(X)$ between $\mathfrak{g}^*$ and $\mathfrak{g}$.
$T\phi$	tangent map (differential) of a smooth map ϕ between manifolds; the tangent map at x is denoted $T_x\phi$.
$\langle S \rangle$	linear span of a subset $S \subseteq \mathfrak{g}$ of elements $B_1, \ldots, B_\ell \in \mathfrak{g}$.
$\mathsf{Lie}(S)$	Lie algebra generated by $S \subseteq \mathfrak{g}$, i.e., the smallest Lie subalgebra containing S.
$\mathrm{d}F$	linearisation of $F \in C^\infty(\mathsf{M})$; the linearisation at x is denoted $\mathrm{d}F(x)$.
DX	linearisation of the vextor field X; the linearisation at a point x is denoted $DX(x)$.

Chapter 1

The Heisenberg Group H_3

This chapter introduces the three-dimensional Heisenberg group H_3. It investigates various algebraic and topological properties of H_3 (particularly those properties pertaining to control theory). We begin by introducing two isomorphic forms of the Heisenberg group and the isomorphism between them and show that H_3 is a (matrix) Lie group.

H_3 is shown to be diffeomorphic to $\mathbb{R}^3$. In so doing, the topological properties of H_3, namely that H_3 is connected, simply connected and non-compact, are established. We then draw our attention to the algebraic properties of H_3; first calculating its Lie algebra $\mathfrak{h}_3$ and the group of Lie algebra automorphisms $\mathsf{Aut}(\mathfrak{h}_3)$. ($\mathsf{Aut}(\mathfrak{h}_3)$ is used in chapter 2 in the classification of left-invariant control affine systems.) We proceed to calculate the adjoint and coadjoint representations of the group H_3 on the Lie algebra $\mathfrak{h}_3$ and its dual Lie algebra $\mathfrak{h}_3^*$, respectively. We show that the centres $Z(\mathsf{H}_3)$ and $Z(\mathfrak{h}_3)$ are non-trivial and show that H_3 and $\mathfrak{h}_3$ are nilpotent, completely solvable, solvable, not simple, not semisimple and that H_3 is unimodular. The exponential map $\exp : \mathfrak{h}_3 \to \mathsf{H}_3$ (which is shown to be a diffeomorphism) is determined.

Lastly, we calculate the adjoint and coadjoint orbits of H_3 by making use of the previously calculated adjoint and coadjoint representations.

1.1 Lie Group H_3

The 3-dimensional Heisenberg group denoted $\mathcal{H}_3$ is the set $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ with group law

$$(x_1, x_2, x_3)(y_1, y_2, y_3) = (x_1 + y_1 + \frac{1}{2}(x_2 y_3 - x_3 y_2), x_2 + y_2, x_3 + y_3). \tag{1.1}$$

For convenience we define the polarised 3-dimensional Heisenberg group as

$$\mathsf{H}_3 = \left\{ \begin{bmatrix} 1 & x_2 & x_1 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{bmatrix} | x_1, x_2, x_3 \in \mathbb{R}^3 \right\}$$

where the group law is standard matrix multiplication.

1.1.1 Proposition. *H_3 is a matrix Lie group.*

PROOF. For brevity we let

$$m(x_1, x_2, x_3) = \begin{bmatrix} 1 & x_2 & x_1 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then $m(x_1, x_2, x_3)^{-1} = m(-x_1 + x_2x_3, -x_2, -x_3) \in \mathsf{H}_3$. Also $m(x_1, x_2, x_3)m(y_1, y_2, y_3) = m(x_1 + y_1 + x_2y_3, x_2 + y_2, x_3 + y_3) \in \mathsf{H}_3$. That is H_3 is a subgroup of $\mathsf{GL}(3, \mathbb{R})$. We have left to show that H_3 is closed in $\mathsf{GL}(3, \mathbb{R})$. Let $(g_n)_{n\in\mathbb{N}}$ be a sequence in H_3 where $g_n = m(x_{1n}, x_{2n}, x_{3n})$ with $\lim_{n\to\infty} g_n = g$. If $g \notin \mathsf{GL}(3, \mathbb{R})$ we have nothing to show. Suppose $g \in \mathsf{GL}(3, \mathbb{R})$ with $\lim_{n\to\infty} x_{1n} = x_1$, $\lim_{n\to\infty} x_{2n} = x_2$ and $\lim_{n\to\infty} x_{3n} = x_3$. Then $g = m(x_1, x_2, x_3)$ where $x_1, x_2, x_3 \in \mathbb{R}$ since $\mathbb{R}$ is closed. That is $g \in \mathsf{H}_3$ and H_3 is a matrix Lie group. (The calculations were performed in Mathematica, see section C.1.1.)

1.1.2 PROPOSITION. *The Heisenberg group* H_3 *is diffeomorphic to* $\mathbb{R}^3$.

PROOF. Define the map $h_3 : \mathbb{R}^3 \to \mathsf{H}_3$, $(x_1, x_2, x_3) \mapsto \begin{bmatrix} 1 & x_2 & x_1 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{bmatrix}$. Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ then

$$h_3(x) = h_3(y) \iff \begin{bmatrix} 1 & x_2 & x_1 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & y_2 & y_1 \\ 0 & 1 & y_3 \\ 0 & 0 & 1 \end{bmatrix} \iff x = y.$$

Hence h_3 is both well-defined and injective. Suppose $g = \begin{bmatrix} 1 & x_2 & x_1 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{bmatrix} \in \mathsf{H}_3$ then $x = (x_1, x_2, x_3)$ is an element of $\mathbb{R}^3$ such that $h_3(x) = g$. Hence h_3 is surjective. Now $\mathbb{R}^3$ and H_3 are both 3 dimensional smooth manifolds and h_3 is a bijective map. It therefore follows from the inverse function theorem on manifolds that H_3 is diffeomorphic to $\mathbb{R}^3$ (e.g., [30]).

1.1.3 PROPOSITION. $\mathcal{H}_3$ *and* H_3 *are isomorphic.*

PROOF. For brevity let

$$m(x_1, x_2, x_3) = \begin{bmatrix} 1 & x_2 & x_1 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{bmatrix}.$$

Define the mapping $\phi : \mathcal{H}_3 \to \mathsf{H}_3$, $(x_1, x_2, x_3) \mapsto m(x_1 + \frac{1}{2}x_2x_3, x_2, x_3)$. ϕ is both injective and well defined. Indeed

$$\begin{aligned} & m(x_1 + \frac{1}{2}x_2x_3, x_2, x_3) = m(y_1 + \frac{1}{2}y_2y_3, y_2, y_3) \\ \iff & \begin{cases} x_1 + \frac{1}{2}x_2x_3 = y_1 + \frac{1}{2}y_2y_3 \\ x_2 = y_2 \\ x_3 = y_3 \end{cases} \\ \iff & (x_1, x_2, x_3) = (y_1, y_2, y_3). \end{aligned}$$

Also ϕ is surjective. For all $m(x_1, x_2, x_3) \in \mathsf{H}_3$, $(x_1 - \frac{1}{2}x_2x_3, x_2, x_3) \in \mathcal{H}_3$ such that $\phi(x_1 - \frac{1}{2}x_2x_3, x_2, x_3) = m(x_1, x_2, x_3)$. Now ϕ is a group homomorphism

$$\begin{aligned}\phi((x_1, x_2, x_3), (y_1, y_2, y_3)) &= \phi(x_1 + y_1 + \frac{1}{2}(x_2y_3 - x_3y_2), x_2 + y_2, x_3 + y_3)\\ &= m(x_1 + y_1 + \frac{1}{2}(x_2y_3 - x_3y_2) + \frac{1}{2}(x_2 + y_2)(x_3 + y_3), x_2 + y_2, x_3 + y_3)\\ &= m(x_1 + y_1 + x_2y_3 + \frac{1}{2}(x_2x_3 + y_2y_3), x_2 + y_2, x_3 + y_3)\\ &= m(x_1 + \frac{1}{2}x_2x_3, x_2, x_3)m(y_1 + \frac{1}{2}y_2y_3, y_2, y_3)\\ &= \phi(x_1, x_2, x_3)\phi(y_1, y_2, y_3).\end{aligned}$$

Finally, since $h_3 : \mathbb{R}^3 \to \mathsf{H}_3$ is a diffeomorphism we have that ϕ is a diffeomorphism. Hence $\phi : \mathcal{H}_3 \to \mathsf{H}_3$ is a Lie group isomorphism.

1.1.1 Topological properties

In section 1.1 we showed that H_3 is diffeomorphic to $\mathbb{R}^3$. It therefore inherits the topological properties of $\mathbb{R}^3$. We collect these properties in proposition 1.1.4.

1.1.4 Proposition. H_3 *is connected, simply-connected and non-compact.*

Proof. From proposition 1.1.2 we have that H_3 and $\mathbb{R}^3$ are diffeomorphic; moreover the topological properties of connectedness, simply-connectedness and non-compactness are preserved by diffeomorphisms. The result follows.

1.1.2 Algebraic properties

In this section we first determine the Lie algebra of H_3 and its automorphism group. We then prove several properties of both H_3 and its Lie algebra.

1.1.5 Proposition. *The Lie algebra of* H_3 *is*

$$\mathfrak{h}_3 = \left\{ \begin{bmatrix} 0 & x_2 & x_1 \\ 0 & 0 & x_3 \\ 0 & 0 & 0 \end{bmatrix}, x_1, x_2, x_3 \in \mathbb{R} \right\}.$$

Proof. For brevity let

$$m(x_1, x_2, x_3) = \begin{bmatrix} 1 & x_2 & x_1 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad M(x_1, x_2, x_3) = \begin{bmatrix} 0 & x_2 & x_1 \\ 0 & 0 & x_3 \\ 0 & 0 & 0 \end{bmatrix}.$$

Consider the map $g : (-\epsilon, \epsilon) \to \mathsf{H}_3, t \mapsto m(x_1(t), x_2(t), x_3(t))$ with $x_1(0), x_2(0), x_3(0) = 0$. Then $g(0) = \mathbf{1}$ from which it follows that $\dot{g}(0) \in \mathsf{T}_\mathbf{1}\mathsf{H}_3 = \mathfrak{h}_3$ but $\dot{g}(0) = M(\dot{x}_1(0), \dot{x}_2(0), \dot{x}_3(0))$. Conversely suppose $X = m(x_1, x_2, x_3)$ then $g : \mathbb{R} \to \mathsf{H}_3, t \mapsto g(t)$ where $g(t) = m(x_1t, x_2t, x_3t)$ is a curve in H_3 such that $\dot{g}(0) = M(x_1, x_2, x_3)$.

Define the **standard basis** of $\mathfrak{h}_3$ as

$$E_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The commutator relations of the standard basis are given by $[E_2, E_3] = E_1$ and $[E_1, E_2] = [E_1, E_3] = 0$. We denote the dual basis by $(E_i^*)_{i=1}^3$ where each E_i^* is defined by $\langle E_i^*, E_j \rangle = \delta_{ij},\ i, j = 1, 2, 3$. Elements of $\mathfrak{h}_3$ will be written as column vectors and elements of the Heisenberg dual Lie algebra $\mathfrak{h}_3^*$ will be written as row vectors.

1.1.6 Lemma. *Let* $\mathbf{x} = (x_1, x_2, x_3)$, $\mathbf{y} = (y_1, y_2, y_3) \in \mathbb{R}^3$. $\mathbb{R}^3$ *equipped with the operation* $\owedge : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3, (\mathbf{x}, \mathbf{y}) \mapsto (-x_3 y_2 + x_2 y_3, 0, 0)$ *is a Lie algebra.*

Proof. Let $\mathbf{x} = (x_1, x_2, x_3)$, $\mathbf{y} = (y_1, y_2, y_3), \mathbf{z} = (z_1, z_2, z_3) \in \mathbb{R}^3$. Now $\owedge$ forms a Lie bracket on $\mathbb{R}^3$ since it is:

1. skew symmetric:
$$\mathbf{x} \owedge \mathbf{y} = (x_2 y_3 - x_3 y_2, 0, 0) = -\mathbf{y} \owedge \mathbf{x}.$$

2. bilinear:

   We show that it is linear in the first argument, bilinearity then follows from skew symmetry.
$$\begin{aligned} (\mathbf{x} + \mathbf{y}) \owedge \mathbf{z} &= (x_1 + y_1, x_2 + y_2, x_3 + y_3) \owedge z \\ &= ((x_2 + y_2) z_3 - (x_3 + y_3) z_2, 0, 0) \\ &= (x_2 z_3 + y_2 z_3 - x_3 z_2 - y_3 z_2, 0, 0) \\ &= ((x_2 z_3 - x_3 z_2) + y_2 z_3 - y_3 z_2, 0, 0) \\ &= \mathbf{x} \owedge \mathbf{z} + \mathbf{y} \owedge \mathbf{z}. \end{aligned}$$

   Also
$$\begin{aligned} \lambda(\mathbf{x} \owedge \mathbf{y}) &= \lambda(x_2 y_3 - x_3 y_2, 0, 0) \\ &= (\lambda x_2 y_3 - \lambda x_3 y_2, 0, 0) \\ &= \lambda \mathbf{x} \owedge \mathbf{y}. \end{aligned}$$

3. Satisfies the Jacobi identity:
$$\begin{aligned} &\mathbf{x} \owedge (\mathbf{y} \owedge \mathbf{z}) + \mathbf{y} \owedge (\mathbf{z} \owedge \mathbf{x}) + \mathbf{z} \owedge (\mathbf{x} \owedge \mathbf{y}) \\ &\quad = \mathbf{x} \owedge (y_2 z_3 - y_3 z_2, 0, 0) \\ &\qquad + \mathbf{y} \owedge (z_2 x_3 - z_3 x_2) + \mathbf{z} \owedge (x_2 y_3 - x_3 y_2, 0, 0) \\ &\quad = (0, 0, 0). \end{aligned}$$

We will denote the Lie bracket on $\mathbb{R}^3$ defined by $\owedge$ as $\mathbb{R}^3_{\owedge}$. If we let **i**,**j**,**k** denote the usual basis on $\mathbb{R}^3$ then
$$\mathbf{j} \owedge \mathbf{k} = \mathbf{i}, \quad \mathbf{i} \owedge \mathbf{j} = \mathbf{i} \owedge \mathbf{k} = \mathbf{0}, \quad .$$

1.1.7 PROPOSITION. *$\mathfrak{h}_3$ and $\mathbb{R}^3_{\circledwedge}$ are isomorphic Lie algebras.*

PROOF. We show that

$$\varphi : \mathfrak{h}_3 \to \mathbb{R}^3_{\circledwedge}, \quad \begin{bmatrix} 0 & x_2 & x_1 \\ 0 & 0 & x_3 \\ 0 & 0 & 0 \end{bmatrix} \mapsto (x_1, x_2, x_3)$$

is a Lie algebra isomorphism from $\mathfrak{h}_3 \to \mathbb{R}^3$. Let $X, Y \in \mathfrak{h}_3$ where $X = x_1E_1 + x_2E_2 + x_3E_3$ and $Y = y_1E_1 + y_2E_2 + y_3E_3$ and let $\alpha, \beta \in \mathbb{R}$. Firstly φ is a linear map:

$$\begin{aligned} \varphi(\alpha X + \beta Y) &= \varphi((\alpha x_1 + \beta y_1)E_1 + (\alpha x_2 + \beta y_2)E_2 + (\alpha x_3 + \beta y_3)E_3) \\ &= (\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2, \alpha x_3 + \beta y_3) \\ &= \alpha(x_1, x_2, x_3) + \beta(y_1, y_2, y_3) \\ &= \alpha\varphi(X) + \beta\varphi(Y). \end{aligned}$$

Secondly φ preserves the Lie bracket. It is enough to show that the Lie brackets of the basis elements are preserved. Note that $\varphi(E_1) = \mathbf{i}, \varphi(E_2) = \mathbf{j}$ and $\varphi(E_3) = \mathbf{k}$. Then:

$$\begin{aligned} \varphi([E_1, E_2]) &= \varphi(0) = \mathbf{0} = \mathbf{i} \circledwedge \mathbf{j} = [\varphi(E_1), \varphi(E_2)], \\ \varphi([E_1, E_3]) &= \varphi(0) = \mathbf{0} = \mathbf{i} \circledwedge \mathbf{k} = [\varphi(E_1), \varphi(E_3)], \\ \varphi([E_2, E_3]) &= \varphi(E_1) = \mathbf{i} = \mathbf{j} \circledwedge \mathbf{k} = [\varphi(E_2), \varphi(E_3)]. \end{aligned}$$

And so we have that φ is a linear map from $\mathfrak{h}_3$ to $\mathbb{R}^3_{\circledwedge}$ that preserves the Lie bracket. Also $\dim \mathfrak{h}_3 = \dim \mathbb{R}^3$. Hence $\mathfrak{h}_3$ and $\mathbb{R}^3$ are isomorphic Lie algebras.

By representing our Lie algebra on $\mathbb{R}^3$ we have that the basis elements are given as

$$E_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad E_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad E_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

with commutator relations:

$$[E_1, E_2] = 0, \qquad [E_2, E_3] = E_1 \qquad [E_1, E_3] = 0.$$

1.1.8 PROPOSITION. *In terms of the standard basis $(E_i)_{i=1}^3$ the automorphism group of $\mathfrak{h}_3$ is*

$$\mathsf{Aut}(\mathfrak{h}_3) = \left\{ \begin{bmatrix} v_2w_3 - v_3w_2 & v_1 & w_1 \\ 0 & v_2 & w_2 \\ 0 & v_3 & w_3 \end{bmatrix} | v_1, v_2, v_3, w_1, w_2, w_3 \in \mathbb{R}, v_2w_3 - v_3w_2 \neq 0 \right\}.$$

PROOF. (The calculations were performed in Mathematica, see section C.1.1.) Let $\psi \in \mathsf{Aut}(\mathfrak{h}_3)$. We represent ψ as

$$\psi = \begin{bmatrix} \psi_{11} & \psi_{12} & \psi_{13} \\ \psi_{21} & \psi_{22} & \psi_{23} \\ \psi_{31} & \psi_{32} & \psi_{33} \end{bmatrix}$$

in terms of the standard basis $(E_i)_{i=1}^3$. Now, ψ preserves the centre of $\mathfrak{h}_3$ i.e. ψ preserves the span of E_1 (refer to proposition 1.1.10) and so

$$\begin{aligned} &\psi E_1 = \lambda E_1 \quad \text{for some } \lambda \in \mathbb{R} \\ \implies &\begin{bmatrix} \psi_{11} & \psi_{12} & \psi_{13} \\ \psi_{21} & \psi_{22} & \psi_{23} \\ \psi_{31} & \psi_{32} & \psi_{33} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \psi_{11} \\ \psi_{21} \\ \psi_{31} \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ \implies &\psi_{21}, \psi_{31} = 0. \end{aligned}$$

Also ψ preserves the Lie bracket. From which it follows that

$$\begin{aligned} &\psi\,[E_2, E_3] = [\psi E_2, \psi E_3] \\ \implies &\psi E_1 = [\psi E_2, \psi E_3] \\ \implies &\begin{bmatrix} \psi_{11} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -\psi_{23}\psi_{32} + \psi_{22}\psi_{33} \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

which yields

$$\psi = \begin{bmatrix} -\psi_{23}\psi_{32} + \psi_{22}\psi_{33} & \psi_{12} & \psi_{13} \\ 0 & \psi_{22} & \psi_{23} \\ 0 & \psi_{32} & \psi_{33} \end{bmatrix}.$$

Since ψ is invertible we have that $\det \psi = (-\psi_{23}\psi_{32} + \psi_{22}\psi_{33})^2 \neq 0$. Setting $\psi_{12} = v_1, \psi_{22} = v_2, \psi_{32} = v_3$, $\psi_{13} = w_1, \psi_{23} = w_2$ and $\psi_{33} = w_3$ gives

$$\psi = \begin{bmatrix} v_2w_3 - v_3w_2 & v_1 & w_1 \\ 0 & v_2 & w_2 \\ 0 & v_3 & w_3 \end{bmatrix}, \quad v_1, v_2, v_3, w_1, w_2, w_3 \in \mathbb{R}, \quad v_2w_3 - v_3w_2 \neq 0.$$

Finally we verify that every automorphism of the above form preserves the Lie bracket. Let $A = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ and $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ be arbitrary elements of $\mathfrak{h}_3$ then

$$\begin{aligned} \psi \cdot [A, B] &= \begin{bmatrix} v_2w_3 - v_3w_2 & v_1 & w_1 \\ 0 & v_2 & w_2 \\ 0 & v_3 & w_3 \end{bmatrix} \left[\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \right] \\ &= \psi \cdot \begin{bmatrix} -a_3b_2 + a_2b_3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} (-a_3b_2 + a_2b_3)(-v_3w_2 + v_2w_3) \\ 0 \\ 0 \end{bmatrix}. \end{aligned}$$

On the other hand,

$$\begin{aligned} [\psi \cdot A, \psi \cdot B] &= \left[\begin{bmatrix} a_2v_1 + a_3w_1 + a_1(-v_3w_2 + v_2w_3) \\ a_2v_2 + a_3w_2 \\ a_2v_3 + a_3w_3 \end{bmatrix}, \begin{bmatrix} b_2v_1 + b_3w_1 + b_1(-v_3w_2 + v_2w_3) \\ b_2v_2 + b_3w_2 \\ b_2v_3 + b_3w_3 \end{bmatrix} \right] \\ &= \begin{bmatrix} (-a_3b_2 + a_2b_3)(-v_3w_2 + v_2w_3) \\ 0 \\ 0 \end{bmatrix} = \psi \cdot [A, B]. \end{aligned}$$

The result follows.

1.1.9 Proposition. *Let*

$$g = \begin{bmatrix} 1 & x_2 & x_1 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{bmatrix} \in \mathsf{H}_3, \quad \textit{and} \quad X = \begin{bmatrix} 0 & x_2 & x_1 \\ 0 & 0 & x_3 \\ 0 & 0 & 0 \end{bmatrix} \in \mathfrak{h}_3.$$

In terms of the standard basis $(E_i)_{i=1}^3$*, the adjoint representations of* H_3 *and* $\mathfrak{h}_3$ *are*

$$\mathrm{Ad}_g = \begin{bmatrix} 1 & -x_3 & x_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \textit{and} \quad \mathrm{ad}_X = \begin{bmatrix} 0 & -x_3 & x_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Proof. (Mathematica was used to facilitate calculations, see section C.1.1.) The i^{th} column of the matrix Ad_g is given by the image of E_i under Ad_g written in coordinates. Now

$$\mathrm{Ad}_g E_1 = gE_1g^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = E_1$$
$$\mathrm{Ad}_g E_2 = gE_2g^{-1} = \begin{bmatrix} 0 & 1 & -x_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = -x_3E_1 + E_2$$
$$\mathrm{Ad}_g E_3 = gE_3g^{-1} = \begin{bmatrix} 0 & 0 & x_2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = x_2E_1 + E_3.$$

From which it follows that

$$\mathrm{Ad}_g = \begin{bmatrix} 1 & -x_3 & x_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Now suppose

$$g(\cdot) : (-\epsilon, \epsilon) \to \mathsf{H}_3, \ t \mapsto \begin{bmatrix} 1 & w_2(t) & w_1(t) \\ 0 & 1 & w_3(t) \\ 0 & 0 & 1 \end{bmatrix}$$

with $g(0) = \mathbf{1}$ and $\dot{g}(0) = X$. Then as ad is the linearisation of Ad (proposition A.1.14), we have that

$$\mathrm{ad}_X = \left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_{t=0} \mathrm{Ad}_{g(t)} = \begin{bmatrix} 0 & -\dot{w}_3(0) & \dot{w}_2(0) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -x_3 & x_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

1.1.10 Proposition. *The centres of* H_3 *and* $\mathfrak{h}_3$ *are given by*

$$Z(\mathsf{H}_3) = \left\{ \begin{bmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} | z \in \mathbb{R} \right\} \quad \textit{and} \quad Z(\mathfrak{h}_3) = \left\{ \begin{bmatrix} 0 & 0 & z \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} | z \in \mathbb{R} \right\}.$$

Proof. For brevity let

$$m(x_1,x_2,x_2)=\begin{bmatrix}1 & x_2 & x_1\\ 0 & 1 & x_3\\ 0 & 0 & 1\end{bmatrix} \quad \text{and} \quad M(x_1,x_2,x_3)=\begin{bmatrix}0 & x_2 & x_1\\ 0 & 0 & x_3\\ 0 & 0 & 0\end{bmatrix}.$$

Suppose $g\in Z(\mathsf{H}_3)$ then $ghg^{-1}=h$, $\forall h\in\mathsf{H}_3$. Let $g=m(x_1,x_2,x_3)$ and $h=m(y_1,y_2,y_3)$, arbitrary. Then

$$\begin{aligned} ghg^{-1}=h &\iff m(x_2x_3+y_1-x_3(x_2+y_2)+x_2y_3,y_2,y_3)=m(y_1,y_2,y_3)\\ &\iff y_2x_3=x_2y_3,\ \forall y_2,y_3\in\mathbb{R}\\ &\iff x_2,x_3=0. \end{aligned}$$

Hence $Z(\mathsf{H}_3)=\{m(0,0,z)|z\in\mathbb{R}\}$. (See section C.1.1 for the supporting Mathematica code.) Now $Z(\mathfrak{h}_3)$ is the Lie algebra of $Z(\mathsf{H}_3)$ (see proposition A.1.7). Let

$$g(\cdot):(-\epsilon,\epsilon)\to\mathsf{H}_3,\ t\mapsto m(w(t),0,0)$$

such that $g(0)=\mathbf{1}$ then $\dot g(0)=M(\dot w(0),0,0)$. The result follows.

1.1.11 Definition. Let A,B be subgroups of the group G. Then (A,B) denotes the subgroup of G generated by all elements $ghg^{-1}h^{-1}$ for $g\in A$, $h\in$ B. i.e.

$$(A,B)=\left\{\prod_{i=1}^{n}g_ih_ig_i^{-1}h_i^{-1}|g_i\in A,h_i\in B\right\}.$$

1.1.12 Proposition. H_3 *is nilpotent of class 2 (refer to section A.1.6).*

Proof. For brevity let

$$m(x_1,x_2,x_3)=\begin{bmatrix}1 & x_2 & x_1\\ 0 & 1 & x_3\\ 0 & 0 & 1\end{bmatrix}.$$

We compute the elements of the descending central series

$$\mathsf{H}_3=C^0\mathsf{H}_3\trianglerighteq C^1\mathsf{H}_3\trianglerighteq C^2\mathsf{H}_3\cdots.$$

where $C^{n+1}\mathsf{H}_3=(\mathsf{H}_3,C^n\mathsf{H}_3)$ (using Mathematica (see section C.1.1)). We have that $C^0\mathsf{H}_3=\mathsf{H}_3=\{m(x_1,x_2,x_3)|x_1,x_2,x_3\in\mathbb{R}\}$. Consider $g,h\in\mathsf{H}_3$ with $g=m(x_1,x_2,x_3)$ and $h=m(y_1,y_2,y_3)$. Then $g^{-1}=m(-x_1+x_2x_3,-x_2,-x_3)$ and $h^{-1}=m(-y_1+y_2y_3,-y_2,-y_3)$and $ghg^{-1}h^{-1}=m(-x_3y_2+x_2y_3,0,0)$ from which it follows that

$$C^1\mathsf{H}_3=\{m(z_1,0,0)|z_1\in\mathbb{R}\}.$$

Let $g=m(x_1,x_2,x_3)\in\mathsf{H}_3$ and let $h'=m(z,0,0)\in C^1\mathsf{H}_3$. Then $g^{-1}=m(-x_1+x_2x_3,-x_2,-x_3)$, $h'^{-1}=m(-z,0,0)$ and $gh'g^{-1}h'^{-1}=m(0,0,0)=\mathbf{1}$, i.e., $C^2\mathsf{H}_3=\{\mathbf{1}\}$. Hence H_3 is nilpotent of class 2.

1.1.13 Corollary. $\mathfrak{h}_3$ *is nilpotent*

Proof. H_3 is a connected and simply-connected Lie group (proposition 1.1.4) with Lie algebra $\mathfrak{h}_3$. From which the result follows (see section A.1).

1.1.14 Proposition. $\mathfrak{h}_3$ *is completely solvable*

Proof. Recall that
$$\mathrm{ad}_X = \begin{bmatrix} 0 & -x_3 & x_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
for $X = \begin{bmatrix} 0 & x_2 & x_1 \\ 0 & 0 & x_3 \\ 0 & 0 & 0 \end{bmatrix} \in \mathfrak{h}_3$. The eigenvalues of ad_X are all zero and therefore real for every $X \in \mathfrak{h}_3$. Hence $\mathfrak{h}_3$ is completely solvable. The eigenvalues were calculated using Mathematica (see section C.1.1).

1.1.15 Proposition. H_3 *is solvable.*

Proof. Although this follows from proposition 1.1.12 we give a direct proof here. For brevity let
$$m(x_1, x_2, x_3) = \begin{bmatrix} 1 & x_2 & x_1 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{bmatrix}.$$

We compute the elements of the sequence of normal subgroups
$$\mathsf{H}_3 \rhd \mathsf{H}_3^{(1)} \rhd \mathsf{H}_3^{(2)} \rhd \cdots .$$
where $\mathsf{H}_3^{(1)} = (\mathsf{H}_3, \mathsf{H}_3)$ and $\mathsf{H}_3^{(k+1)} = (\mathsf{H}_3^{(k)}, \mathsf{H}_3^{(k)})$ using Mathematica (see section C.1.1). We show that the sequence terminates at $\mathbf{1}$ and hence that H_3 is solvable. We have that $\mathsf{H}_3 = \{m(x_1, x_2, x_3) | x_1, x_2, x_3 \in \mathbb{R}\}$. Consider $g, g' \in \mathsf{H}_3$ with $g = m(x_1, x_2, x_3)$ and $g' = m(y_1, y_2, y_3)$. Then $g^{-1} = m(-x_1 + x_2x_3, -x_2, -x_3)$, $g'^{-1} = m(-y_1 + y_2y_3, -y_2, -y_3)$ and $gg'g^{-1}g' = m(-x_3y_2 + x_2y_3, 0, 0)$. Accordingly
$$\mathsf{H}_3^{(1)} = \{m(z, 0, 0) | z \in \mathbb{R}\}.$$

Let $h = m(z_1, 0, 0)$, $h' = m(z_2, 0, 0) \in \mathsf{H}_3^{(1)}$. Then $h^{-1} = m(-z_1, 0, 0)$, $h'^{-1} = m(-z_2, 0, 0)$ and $hh'h^{-1}h' = m(0, 0, 0) = \mathbf{1}$. Hence $H_3^{(2)} = \{\mathbf{1}\}$. The result follows.

1.1.16 Corollary. $\mathfrak{h}_3$ *is solvable.*

Proof. This follows from H_3 being a connected and simply-connected Lie group. (See section A.1.)

1.1.17 Proposition. H_3 *is exponential.*

Proof. $\mathfrak{h}_3$ is completely solvable (proposition 1.1.14). Hence H_3 is exponential (proposition A.1.18).

1.1.18 **Proposition.** H_3 *and* $\mathfrak{h}_3$ *are not simple.*

Proof. $Z(\mathfrak{h}_3)$ is nontrivial and therefore a nontrivial ideal of $\mathfrak{h}_3$.

1.1.19 **Proposition.** H_3 *and* $\mathfrak{h}_3$ *are not semisimple.*

Proof. This follows directly from $\mathfrak{h}_3$ not being solvable. A direct proof is however given. We show that $Z(\mathfrak{h}_3)$ is a nontrivial solvable ideal of $\mathfrak{h}_3$. For brevity let

$$M(x_1, x_2, x_3) = \begin{bmatrix} 0 & x_2 & x_1 \\ 0 & 0 & x_3 \\ 0 & 0 & 0 \end{bmatrix}.$$

Let $\mathfrak{g} = Z(\mathfrak{h}_3)$=$\{M(z, 0, 0) | z \in \mathbb{R}\}$. Let $X = M(z_1, 0, 0)$, $Y = M(z_2, 0, 0) \in Z(\mathfrak{h}_3)$. Then $[X, Y] = M(0, 0, 0)$. We therefore have that

$$[\mathfrak{g}, \mathfrak{g}] = \{\mathbf{0}\}.$$

That is $Z(\mathfrak{h}_3)$ is a solvable ideal. Hence H_3 and $\mathfrak{h}_3$ are not semisimple. (Mathematica was used to perform the calculations (see section C.1.1).)

1.1.20 **Proposition.** H_3 *is unimodular*

Proof. Recall that $\mathrm{ad}_X = \begin{bmatrix} 0 & -x_3 & x_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ for $X = x_1 E_1 + x_2 E_2 + x_3 E_3 \in \mathfrak{h}_3$. Clearly $\mathrm{tr}\,\mathrm{ad}_X = 0$. By proposition A.1.20, H_3 is unimodular.

1.1.21 **Proposition.** *The exponential map* $\exp : \mathfrak{h}_3 \to \mathsf{H}_3$ *is given by*

$$X \mapsto \begin{bmatrix} 1 & x_2 & x_1 + \frac{1}{2}x_2 x_3 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{bmatrix}$$

where $X = x_1 E_1 + x_2 E_2 + x_3 E_3$.

Proof. We determine $\exp X$ by making use of the series expansion of the matrix exponential, i.e.,

$$\exp X = \sum_{n=0}^{\infty} \frac{X^n}{n!}.$$

So

$$\begin{aligned} \exp X &= \mathbf{1} + \begin{bmatrix} 0 & x_2 & x_1 \\ 0 & 0 & x_3 \\ 0 & 0 & 0 \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} 0 & 0 & x_2 x_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{1}{3!}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \cdots \\ &= \begin{bmatrix} 1 & x_2 & x_1 + \frac{1}{2}x_2 x_3 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{bmatrix} \in \mathsf{H}_3. \end{aligned}$$

1.1.22 **Proposition.** *The exponential map* $\exp : \mathfrak{h}_3 \to \mathsf{H}_3$ *is a diffeomorphism.*

Proof. H_3 is exponential (proposition 1.1.17). Hence $\exp : \mathfrak{h}_3 \to \mathsf{H}_3$ is a diffeomorphism.

1.2 Adjoint and Coadjoint Orbits

In this section we calculate the adjoint and coadjoint orbits of H_3. The supporting Mathematica code can be found in section C.1.2.

We begin by briefly recalling the necessary theory of adjoint and coadjoint representations. For further details see subsection A.1.5. Let G be a matrix Lie group with Lie algebra $\mathfrak{g}$. Let $\mathfrak{g}^*$ denote the dual Lie algebra of $\mathfrak{g}$. The **adjoint representation** of G on $\mathfrak{g}$ is the map $\mathrm{Ad} : \mathsf{G} \to \mathsf{GL}(\mathfrak{g}), g \mapsto \mathrm{Ad}_g$ where

$$\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}, \quad X \mapsto gXg^{-1}.$$

The **coadjoint representation** of G on $\mathfrak{g}^*$ is the map $\mathrm{Ad}^* : \mathsf{G} \to \mathsf{GL}(\mathfrak{g}^*), g \mapsto \mathrm{Ad}^*_{g^{-1}}$. Here $\mathrm{Ad}^*_{g^{-1}} : \mathfrak{g}^* \to \mathfrak{g}^*$ is the dual map of $\mathrm{Ad}_{g^{-1}}$, i.e.,

$$\left\langle \mathrm{Ad}^*_{g^{-1}} p, X \right\rangle = \left\langle p, \mathrm{Ad}_{g^{-1}} X \right\rangle.$$

The **adjoint orbit** through an element $X \in \mathfrak{g}$ is then the similarity classes $\mathfrak{Orb}(X) = \left\{ \mathrm{Ad}_{g^{-1}} X \mid g \in \mathsf{G} \right\}$. Comparably, the **coadjoint orbit** through $p \in \mathfrak{g}^*$ is the set $\mathfrak{Orb}(p) = \left\{ \mathrm{Ad}^*_{g^{-1}} p \mid g^{-1} \in \mathsf{G} \right\}$.

The following two propositions determine the adjoint and coadjoint orbits of H_3.

1.2.1 PROPOSITION. *The Adjoint orbits of* H_3 *through* $X = x_1E_1 + x_2E_2 + x_3E_3 \in \mathfrak{h}_3$ *are of 2 types, namely*

1. *a single point* $\mathfrak{Orb}(X) = x_1E_1$, *when* $x_2 = x_3 = 0$.
2. *the straight line* $\mathfrak{Orb}(X) = \{(-x_2y_3 + x_1 + y_2x_3)E_1 + x_2E_2 + x_3E_3 \mid y_2, y_3 \in \mathbb{R}\}$, *when* x_2 *and* x_3 *are not simultaneously equal to zero.*

PROOF. Let

$$g = \begin{bmatrix} 1 & y_2 & y_1 \\ 0 & 1 & y_3 \\ 0 & 0 & 1 \end{bmatrix} \in \mathsf{H}_3.$$

Then from proposition 1.1.9 we have that the adjoint representation is given by

$$\mathrm{Ad}_g = \begin{bmatrix} 1 & -y_3 & y_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

It follows that

$$\begin{aligned} \mathrm{Ad}_g X = \begin{bmatrix} 1 & -y_3 & y_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \begin{bmatrix} x_1 + x_3y_2 - x_2y_3 \\ x_2 \\ x_3 \end{bmatrix} \\ &= (x_1 + x_3y_2 - x_2y_3)E_1 + x_2E_2 + x_3E_3. \end{aligned}$$

Hence when $x_2 = x_3 = 0$ we have $\mathfrak{Orb}(X) = x_1E_1$ and when x_2 and x_3 are not both zero we have $\mathfrak{Orb}(X) = \{(-x_2y_3 + x_1 + y_2x_3)E_1 + x_2E_2 + x_3E_3 | \, y_2, y_3 \in \mathbb{R}\}$.

The orbits are shown in figures 1.1

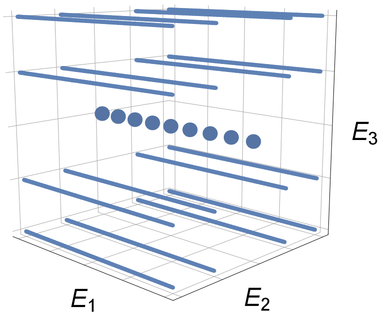

Figure 1.1: Adjoint orbits

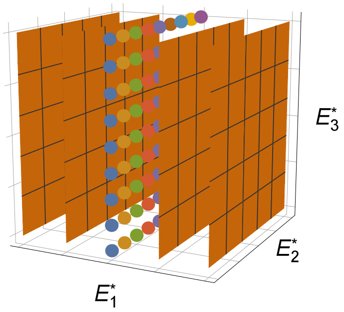

Figure 1.2: Coadjoint orbits

1.2.2 PROPOSITION. *The coadjoint orbits of* H_3 *through* $p = p_1E_1^* + p_2E_2^* + p_3E_3^* \in \mathfrak{h}_3^*$ *are of 2 types, namely*

1. *the single point* $\mathfrak{Orb}(p) = p_2E_2^* + p_3E_3^*$*, when* $p_1 = 0$.
2. *the plane* $\mathfrak{Orb}(p) = \{p_1E_1^* + (p_1y_3 + p_2)E_2^* + (-p_1y_2 + p_3)E_3^* | y_2, y_3 \in \mathbb{R}\}$*, when* $p_1 =$ *constant* $\neq 0$.

PROOF. Let $g = \begin{bmatrix} 1 & y_2 & y_1 \\ 0 & 1 & y_3 \\ 0 & 0 & 1 \end{bmatrix} \in \mathsf{H}_3$ and $X = \begin{bmatrix} 0 & x_2 & x_1 \\ 0 & 0 & x_3 \\ 0 & 0 & 0 \end{bmatrix} \in \mathfrak{h}_3$. Then $g^{-1} = \begin{bmatrix} 1 & -y_2 & -y_1 + y_2y_3 \\ 0 & 1 & -y_3 \\ 0 & 0 & 1 \end{bmatrix}$. Hence from proposition 1.1.9 we have that

$$\mathrm{Ad}_{g^{-1}} = \begin{bmatrix} 1 & y_3 & -y_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

And so

$$\begin{aligned} \mathrm{Ad}_{g^{-1}}X = \begin{bmatrix} 1 & y_3 & -y_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \begin{bmatrix} x_1 - x_3y_2 + x_2y_3 \\ x_2 \\ x_3 \end{bmatrix} \\ &= x_1 - x_3y_2 + x_2y_3E_1 + x_2E_2 + x_3E_3. \end{aligned}$$

Let $p = p_1 E_1^* + p_2 E_2^* + p_3 E_3^* \in \mathfrak{h}_3^*$ then

$$\begin{aligned}\left\langle \mathrm{Ad}^*_{g^{-1}}(p), X \right\rangle &= \left\langle p, \mathrm{Ad}_{g^{-1}}(X) \right\rangle \\ &= \langle p_1 E_1^* + p_2 E_2^* + p_3 E_3^*, (x_2 y_3 + x_1 - y_2 x_3) E_1 + x_2 E_2 + x_3 E_3 \rangle \\ &= p_1 (x_2 y_3 + x_1 - y_2 x_3) + p_2 x_2 + p_3 x_3 \\ &= \langle p_1 E_1^* + (p_1 y_3 + p_2) E_2^* + (-p_1 y_2 + p_3) E_3^*, X \rangle \,.\end{aligned}$$

Therefore $\mathrm{Ad}^*_{g^{-1}}(p) = p_1 E_1^* + (p_1 y_3 + p_2) E_2^* + (-p_1 y_2 + p_3) E_3^*$.

The coadjoint orbits are shown in figure 1.2.

Chapter 2

Classification of Control Systems

In this chapter we classify all full-rank left-invariant control affine systems evolving on H_3 under *state space equivalence, detached feedback equivalence, and strongly detached feedback equivalence.* Each equivalence relation is defined for left-invariant control affine systems on connected Lie groups and various properties of the equivalence relations are shown. Each equivalence relation establishes a one-to-one correspondence between the trajectories of equivalent systems. In each case we show that the classification may be done at the level of Lie algebras and carry the classification out in this way. We also look at the relationship between the three equivalences.

The most natural equivalence relation of a control system is equivalence up to coordinate changes in the state space. This type of equivalence is termed state space equivalence. Left-invariant control affine systems on the Euclidean group $\mathsf{SE}(2)$ and on the semi-Euclidean group $\mathsf{SE}(1,1)$ have been classified under state space equivalence in [1] and [7], respectively.

A weaker equivalence relation is that of detached feedback equivalence in which case invariant feedback transformations of the controls are permitted. In doing so, detached feedback equivalent systems still have the same set of trajectories (up to a diffeomorphism in the state space) but these trajectories are parametrised differently by admissible controls. Detached feedback equivalence is a generalisation of feedback equivalence [18]. The classification under detached feedback equivalence of all left-invariant control affine systems evolving on three-dimensional Lie groups as well as on the orthogonal group $\mathsf{SO}(4)$ have been done in ([12],[13],[14],[15]) and [4], respectively.

Strongly detached feedback equivalence has been introduced as a slightly stronger equivalence relation than detached feedback equivalence. Strongly detached feedback equivalence has the added condition that the drift be preserved.

2.1 Invariant Control Systems

We briefly recall the theory of left-invariant control affine systems as discussed in section A.2. An ℓ-input **left-invariant control affine system** is a pair $\Sigma = (\mathsf{G}, \Xi)$ where G is a (real, finite dimensional) Lie group and the **dynamics** $\Xi : \mathsf{G} \times \mathbb{R}^\ell \to T\mathsf{G}$ are left invariant. Such a system is of the form

$$\dot{g} = \Xi(g,u) = g\Xi(\mathbf{1},u) = gA + u_1 gB_1 + \ldots + u_\ell gB_\ell, \quad g \in \mathsf{G},\ u_1,\ldots,u_\ell \in \mathbb{R},\ A, B_1 \ldots, B_\ell \in \mathfrak{g} \tag{2.1}$$

with $B_1, \ldots, B_\ell$ linearly independent. An **admissible control** $u(\cdot) : [0,T] \to \mathbb{R}^\ell$ is a piecewise continuous map. The **trajectory** corresponding to an admissible control $u(\cdot) : [0,T] \to \mathbb{R}^\ell$ is an absolutely continuous curve $g(\cdot) : [0,T] \to \mathsf{G}$ such that $\dot{g}(t) = \Xi(g(t), u(t))$ for almost every $t \in [0,T]$. The pair $(g(\cdot), u(\cdot))$ will be called a **controlled trajectory**. A is the **drift** of the system. The **trace** of the system is $\Gamma = A + \Gamma^0 = A + \langle B_1, \ldots, B_\ell \rangle$. A system is called **homogeneous** if $A \in \Gamma^0$ and **inhomogeneous** otherwise. For convenience, an ℓ-input homogeneous system will be referred to as a $(\ell, 0)$ system and an inhomogeneous system will be referred to as a $(\ell, 1)$ system. A system is of **full rank** if its trace generates $\mathfrak{g}$ i.e. $\mathsf{Lie}(\Gamma) = \mathfrak{g}$ (a necessary condition for controllability). Hereafter, we shall always assume that the systems under consideration have full rank.

2.1.1 PROPOSITION. *Consider the left-invariant control affine systems*

$$\dot{g} = gA + u_1 gB_1 + \ldots + u_\ell gB_\ell \quad \text{where } A, B_1, \ldots, B_\ell \in \mathfrak{g}.$$

Suppose $\mathfrak{g}$ *is a three-dimensional Lie algebra. Then*

1. *Any* $(1,1)$ *system with trace* $\Gamma = A + \langle B \rangle$ *has full rank if and only if* A, B *and* $[A,B]$ *are linearly independent.*
2. *Any* $(2,0)$ *system with trace* $\Gamma = \langle B_1, B_2 \rangle$ *has full rank if and only if* B_1, B_2 *and* $[B_1, B_2]$ *are linearly independent.*
3. *Any* $(2,1)$ *system has full rank.*

PROOF. 1. For such a system we have that A and B are linearly independent. Suppose $[A,B] \in \langle A, B \rangle$ then $\mathsf{Lie}(\Gamma) = \mathsf{Lie}(\{A,B\}) = \langle A, B \rangle \neq \mathfrak{g}$. For the converse, suppose $A, B, [A,B]$ are linearly independent then $\dim(\mathsf{Lie}(\Gamma)) = \dim(\mathsf{Lie}\,\{A, B, [A,B]\}) = \dim(\mathfrak{g})$. That is $\mathsf{Lie}(\Gamma) = \mathfrak{g}$.

2. (Similar to above). B_1 and B_2 are linearly independent. Suppose $[B_1, B_2] \in \langle B_1, B_2 \rangle$. Then $\mathsf{Lie}(\Gamma) = \mathsf{Lie}(\{B_1, B_2\}) = \langle B_1, B_2 \rangle \neq \mathfrak{g}$. Conversely, if B_1, B_2 and $[B_1, B_2]$ are linearly independent then $\dim(\mathsf{Lie}(\Gamma)) = \dim(\mathsf{Lie}\,\{B_1, B_2, [B_1, B_2]\}) = \dim(\mathfrak{g})$. Therefore $\mathsf{Lie}(\Gamma) = \mathfrak{g}$.

3. For such a system A, B_1 and B_2 are linearly independent, and so $\dim(\mathsf{Lie}(\Gamma)) = \dim(\mathsf{Lie}(\{A, B_1, B_2\})) = \dim(\mathfrak{g})$. That is $\mathsf{Lie}(\Gamma) = \mathfrak{g}$ and the system is of full rank.

2.1.2 PROPOSITION. *Let* $\psi \in \mathsf{Aut}(\mathfrak{g})$. Γ *has full rank if an only if* $\psi \cdot \Gamma$ *has full rank.*

PROOF. Suppose $\Gamma \subseteq \mathfrak{g}$. Then $\mathsf{Lie}(\Gamma)$ is a subalgebra of $\mathfrak{g}$ containing Γ. Now $\psi : \mathfrak{g} \to \mathfrak{g}$ is a Lie algebra automorphism and therefore maps subalgebras to subalgebras. Hence $\psi \cdot \mathsf{Lie}(\Gamma)$ is a subalgebra containing $\psi \cdot \Gamma$. The smallest subalgebra containing $\psi \cdot \Gamma$ is $\mathsf{Lie}(\psi \cdot \Gamma)$. Hence $\mathsf{Lie}(\psi \cdot \Gamma) \subseteq \psi \cdot \mathsf{Lie}(\Gamma)$. On the other hand Lie $(\psi \cdot \Gamma)$ is a subalgebra containing $\psi \cdot \Gamma$. Now ψ^{-1} is a Lie algebra automorphism and so $\psi^{-1}\mathsf{Lie}(\psi \cdot \Gamma)$ is a subalgebra containing $\psi^{-1}(\psi \cdot \Gamma) = \Gamma$. The smallest Lie algebra containing Γ is $\mathsf{Lie}(\Gamma)$ i.e. $\mathsf{Lie}(\Gamma) \subseteq \psi^{-1}\mathsf{Lie}(\psi \cdot \Gamma)$ so $\psi \cdot \mathsf{Lie}(\Gamma) \subseteq \mathsf{Lie}(\psi \cdot \Gamma)$. Hence $\psi \cdot \mathsf{Lie}(\Gamma) = \mathsf{Lie}(\psi \cdot \Gamma)$. Suppose Γ has full rank then $\mathsf{Lie}(\Gamma) = \mathfrak{g}$. Now $\psi \in \mathsf{Aut}(\mathfrak{g})$. $\mathsf{Lie}(\psi \cdot \Gamma) = \psi \cdot \mathsf{Lie}(\Gamma) = \psi \cdot \mathfrak{g} = \mathfrak{g}$. Conversely suppose $\psi \cdot \Gamma$ (where ψ is a Lie algebra automorphism) has full rank. Then $\mathsf{Lie}(\psi \cdot \Gamma) = \mathfrak{g}$ and ψ^{-1} is a Lie algebra automorphism. Hence $\mathsf{Lie}(\Gamma) = \psi^{-1}\mathsf{Lie}(\psi \cdot \Gamma) = \psi^{-1}\mathfrak{g} = \mathfrak{g}$. That is Γ is full rank if and only if $\psi \cdot \Gamma$ is full rank.

2.2 State Space Equivalence

In this section we consider state space equivalence of the left-invariant control affine systems discussed in section 2.1

2.2.1 Definition. Let G be a connected Lie group and let $\Sigma = (\mathsf{G}, \Xi)$ and $\Sigma' = (\mathsf{G}, \Xi')$ be left-invariant control affine systems with the same input space $\mathbb{R}^\ell$. Then Σ and Σ' are **state space equivalent** (or S-equivalent for short) if there exists a diffeomorphism $\phi : \mathsf{G} \to \mathsf{G}$ such that the diagram

$$\begin{tikzcd}
\mathsf{G} \times \mathbb{R}^\ell \arrow[r, "\phi \times id_{\mathbb{R}^\ell}"] \arrow[d, "\Xi"'] & \mathsf{G} \times \mathbb{R}^\ell \arrow[d, "\Xi'"] \\
T\mathsf{G} \arrow[r, "T\phi"'] & T\mathsf{G}
\end{tikzcd}$$

commutes. That is

$$T_g\phi \cdot \Xi(g,u) = \Xi'(\phi(g),u), \quad \text{for } g \in \mathsf{G} \text{ and } u \in \mathbb{R}^\ell.$$

First we show that S-equivalence is indeed an equivalence relation.

2.2.2 Proposition. *S-equivalence is an equivalence relation.*

Proof. Consider three systems $\Sigma = (\mathsf{G}, \Xi)$, $\Sigma' = (\mathsf{G}, \Xi')$ and $\Sigma'' = (\mathsf{G}, \Xi'')$.

1. S-equivalence is reflexive:
   Let $\phi = \mathrm{id}_{\mathsf{G}}$. Then $T_g\phi = \mathrm{id}_{\mathrm{T_gG}}$. So $T_g\phi \cdot \Xi(g,u) = \Xi(g,u) = \Xi(\phi(g),u)$. That is Σ is S-equivalent Σ.

2. S-Equivalence is symmetric:
   Suppose Σ is S-equivalent to Σ'. Then there exists a diffeomorphism $\phi : \mathsf{G} \to \mathsf{G}$ such that

   $$T_g\phi \cdot \Xi(g,u) = \Xi'(\phi(g),u), \quad \forall g \in \mathsf{G}, u \in \mathbb{R}^\ell.$$

   Now $T_g\phi$ is a linear isomorphism and is therefore invertible so we have $(T_g\phi)^{-1}$ such that

   $$\begin{aligned} (T_g\phi)^{-1} \cdot T_g\phi \cdot \Xi(g,u) &= (T_g\phi)^{-1} \cdot \Xi(\phi(g),u) \\ \iff \Xi(g,u) &= (T_g\phi)^{-1} \cdot \Xi'(\phi(g),u), \quad \forall g \in \mathsf{G}, u \in \mathbb{R}^\ell. \end{aligned}$$

   Also $(T_g\phi)^{-1} = T_{\phi(g)}\phi^{-1}$ (Lemma A.1.11), so we have

   $$T_{\phi(g)}\phi^{-1} \cdot \Xi'(\phi(g),u) = \Xi(g,u) = \Xi(\phi^{-1}(\phi(g)),u), \quad \forall\, \phi(g) \in \mathsf{G},\, u \in \mathbb{R}^\ell.$$

   Hence if Σ is S-equivalent to Σ' then Σ' is S-equivalent to Σ.

3. S-equivalence is transitive.
   Suppose Σ is S-equivalent to Σ' and that Σ' is S-equivalent to Σ''. Then there exist diffeomorphisms ϕ_1 and ϕ_2 such that

   $$\begin{cases} T_g\phi_1 \cdot \Xi(g,u) = \Xi'(\phi_1(g),u) \\ T_g\phi_2 \cdot \Xi'(g,u) = \Xi''(\phi_2(g),u), \quad \forall g \in \mathsf{G}, u \in \mathbb{R}^\ell. \end{cases}$$

Let $\phi = \phi_2 \circ \phi_1$. Then for every $g \in \mathsf{G}$ and $u \in \mathbb{R}^\ell$

$$\begin{aligned} T_g\phi \cdot \Xi(g,u) &= T_{\phi_1(g)}\phi_2 \cdot T_g\phi_1 \cdot \Xi(g,u) \quad \text{(proposition A.1.10.)} \\ &= T_{\phi_1(g)}\phi_2 \cdot \Xi'(\phi_1(g),u) \\ &= \Xi''(\phi_2 \circ \phi_1(g),u) \\ &= \Xi''(\phi(g),u). \end{aligned}$$

Hence if Σ is S-equivalent to Σ' and Σ' is S-equivalent to Σ'' then Σ is S-equivalent to Σ''.

S-equivalence is thus an equivalence relation.

Next we show that S-equivalence establishes a one-to-one correspondence between the controlled trajectories of equivalent systems.

2.2.3 Proposition. *The controlled trajectories of two S-equivalent systems* $\Sigma = (\mathsf{G},\Xi)$ *and* $\Sigma' = (\mathsf{G},\Xi')$ *are in a one-to-one correspondence.*

Proof. Let $(g(\cdot),u(\cdot))$ be a controlled trajectory of Σ. Since Σ and Σ' are S-equivalent we have that there exists a diffeomorphism $\phi : \mathsf{G} \to \mathsf{G}$ such that $T_g\phi \cdot \Xi(g,u) = \Xi'(\phi(g(\cdot)),u(\cdot))$, $\forall g \in \mathsf{G}$, $u \in \mathbb{R}^\ell$. We will show that $(\phi(g),u)$ is the unique controlled trajectory of Σ' corresponding to $(g(\cdot),u(\cdot))$ of Σ. Indeed, for almost every t

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\phi(g(t)) &= T_g\phi \cdot \dot{g}(t) \\ &= T_g\phi \cdot \Xi(g(t),u(t)) \\ &= \Xi'(\phi(g(t)),u(t)). \end{aligned}$$

Hence $(\phi(g(\cdot)),u(\cdot))$ is a controlled trajectory of Σ'. We have left to show that controlled trajectories are mapped both injectively and surjectively from Σ to Σ'. Firstly, suppose $\phi(g_1(\cdot)) = \phi(g_2(\cdot))$ where $g_1(\cdot)$ and $g_2(\cdot)$ are trajectories of Σ. Then since ϕ is a diffeomorphism and therefore invertible we have that $g_1(\cdot) = \phi^{-1}\phi(g_1(\cdot)) = \phi^{-1}\phi(g_2(\cdot)) = g_2(\cdot)$. Next, let $(g'(\cdot),u(\cdot))$ be a controlled trajectory of Σ'. Then $(\phi^{-1}(g'(\cdot)),u(\cdot))$ is the corresponding controlled trajectory of Σ mapped to $(g'(\cdot),u(\cdot))$ by $\phi \times \mathrm{Id}_{\mathbb{R}^\ell}$. Hence the controlled trajectories of Σ and Σ' are in a one-to-one correspondence.

Below are two technical lemmas used in showing that the classification under S-equivalence may be done at the level of Lie algebras.

2.2.4 Lemma. *Let* ϕ *be a diffeomorphism and let* $L_g : \mathsf{G} \to \mathsf{G}, h \mapsto gh$ *be the left translation by* $g \in \mathsf{G}$. *Then*

$$T\phi \circ TL_g = TL_{\phi(g)} \circ T\phi.$$

Proof.

$$\begin{aligned} \phi(L_g(h)) &= \phi(gh) \\ &= \phi(g)\phi(h) \\ &= L_{\phi(g)}\phi(h) \\ \implies \phi \circ L_g &= L_{\phi(g)} \circ \phi \\ \implies T\phi \circ TL_g &= TL_{\phi(g)} \circ T\phi. \quad \text{(Proposition } A.1.10.) \end{aligned}$$

2.2.5 LEMMA. *Let $\phi : \mathsf{G} \to \mathsf{G}$ be a diffeomorphism. Then*

$$\begin{aligned} & T_g\phi \cdot \Xi(g,u) = \Xi'(\phi(g),u) \\ \iff & T_g(L_{\phi(1)^{-1}} \circ \phi) \cdot \Xi(g,u) = \Xi'(L_{\phi(1)^{-1}} \circ \phi(g),u). \end{aligned}$$

PROOF.

$$\begin{aligned} & T_g(L_{\phi(1)^{-1}} \circ \phi) \cdot \Xi(g,u) = \Xi'(L_{\phi(1)^{-1}} \circ \phi(g),u) \\ \iff & T_{\phi(g)}L_{\phi(1)^{-1}} \circ T_g\phi \cdot \Xi(g,u) = T_{\phi(g)}L_{\phi(1)^{-1}} \cdot \Xi'(\phi(g),u) \\ \iff & T_g\phi \cdot \Xi(g,u) = \Xi'(\phi(g),u). \end{aligned}$$

In the proposition which follows we verify that a controllable system cannot be S-equivalent to a system which is not controllable and vice-versa.

2.2.6 PROPOSITION. *If $\Sigma = (\mathsf{G}, \Xi)$ and $\Sigma' = (\mathsf{G}, \Xi')$ are S-equivalent then Σ is controllable if and only if Σ' is controllable.*

PROOF. Without loss of generality we may assume that $\phi(\mathbf{1}) = \mathbf{1}$ (lemma 2.2.5). Let $\mathcal{A}$ and $\mathcal{A}'$ denote the attainable sets of Σ and Σ' respectively. Suppose Σ is controllable i.e. $\mathcal{A} = G$. We show that this implies $\mathcal{A}' = \mathsf{G}$ and hence that Σ' is controllable. Firstly,

$$\mathcal{A}' = \left\{ g'(T) \mid g'(\cdot) : [0,T] \to \mathsf{G} \text{ is a trajectory of } \Sigma',\ g'(0) = \mathbf{1} \right\}.$$

Now since Σ and Σ' are S-equivalent (with respect to the diffeomorphism ϕ say) we have that the trajectories of Σ and Σ' are in one-to-one correspondence. That is for the trajectory $g'(\cdot) : [0,T] \to \mathsf{G}$, $g'(\cdot) = \phi(g(\cdot))$ for exactly one trajectory $g(\cdot) : [0,T] \to \mathsf{G}$. Hence

$$\begin{aligned} \mathcal{A}' &= \{ \phi(g(T)) \mid g(\cdot) : [0,T] \to \mathsf{G} \text{ is a trajectory of } \Sigma,\ g(0) = \mathbf{1} \} \\ &= \phi(\mathcal{A}) = \mathsf{G}. \end{aligned}$$

By interchanging the roles of Σ and Σ' in the above argument one is able to show that Σ is controllable if Σ' is controllable. The result follows.

2.2.7 THEOREM. (CF. [18]) *For a simply connected Lie group G, $\Sigma = (\mathsf{G}, \Xi)$ and $\Sigma' = (\mathsf{G}, \Xi')$ are state space equivalent if and only if there exists $\psi \in Aut(\mathfrak{g})$ such that*

$$\psi \cdot \Xi(\mathbf{1}, u) = \Xi'(\mathbf{1}, u).$$

PROOF. Assume Σ and Σ' are state space equivalent. Then there exists a diffeomorphism $\phi : \mathsf{G} \to \mathsf{G}$ such that

$$T_g\phi \cdot \Xi(g,u) = \Xi'(\phi(g),u), \quad \forall\, g \in \mathsf{G}, u \in \mathbb{R}^\ell,$$

from which it follows that $T_{\mathbf{1}}\phi \cdot \Xi(\mathbf{1},u) = \Xi'(\phi(\mathbf{1}),u)$. We may assume $\phi(\mathbf{1}) = \mathbf{1}$ since if this not the case we can replace ϕ with the diffeomorphism $L_{\phi(\mathbf{1})^{-1}} \circ \phi : \mathsf{G} \to \mathsf{G}$ (refer to lemma 2.2.5.) Clearly $(L_{\phi(\mathbf{1})^{-1}} \circ \phi)(\mathbf{1}) = \mathbf{1}$ so

$$T_{\mathbf{1}}\phi \cdot \Xi(\mathbf{1},u) = \Xi'(\phi(\mathbf{1}),u) = \Xi'(\mathbf{1},u), \quad \forall\, u \in \mathbb{R}^\ell.$$

It remains to be shown that $T_{\mathbf{1}}\phi$ is a Lie algebra automorphism. $\phi : \mathsf{G} \to \mathsf{G}$ is a diffeomorphism so from proposition A.1.9 $T_{\mathbf{1}}\phi$ is a linear automorphism; also $T_{\mathbf{1}}\phi$ preserves the Lie bracket: Using the notation where the left invariant vector field $\Xi_u := \Xi(\cdot, u), u \in \mathbb{R}^\ell$ and noting that the pushforward by ϕ preserves the Lie bracket of vector fields we have

$$\phi_*[\Xi_u, \Xi_v] = [\phi_*\Xi_u, \phi_*\Xi_v].$$

This, together with left invariance gives

$$\begin{aligned} \phi_*[\Xi_u, \Xi_v](\phi(\mathbf{1})) &= [\phi_*\Xi_u(\phi(\mathbf{1})), \phi_*\Xi_v(\phi(\mathbf{1}))] \\ \iff T_{\mathbf{1}}\phi \cdot [\Xi_u, \Xi_v](\mathbf{1}) &= [T_{\mathbf{1}}\phi \cdot \Xi_u(\mathbf{1}), T_{\mathbf{1}}\phi \cdot \Xi_v(\mathbf{1})] \\ \iff T_{\mathbf{1}}\phi \cdot [\Xi_u(\mathbf{1}), \Xi_v(\mathbf{1})] &= [T_{\mathbf{1}}\phi \cdot \Xi_u(\mathbf{1}), T_{\mathbf{1}}\phi \cdot \Xi_v(\mathbf{1})]. \end{aligned}$$

Similarly $T_{\mathbf{1}}\phi \cdot [\Xi_u(\mathbf{1}), [\Xi_v(\mathbf{1}), \Xi_w(\mathbf{1})]] = [T_{\mathbf{1}}\phi \cdot \Xi_u(\mathbf{1}), [T_{\mathbf{1}}\phi \cdot \Xi_v(\mathbf{1}), T_{\mathbf{1}}\phi \cdot \Xi_w(\mathbf{1})]]$ and similarly for higher order commutators. We have that $\Gamma = \{\Xi_u(\mathbf{1}) : u \in \mathbb{R}^\ell\}$ and Lie $\Gamma = \mathfrak{g}$ hence the elements $\Xi_u(\mathbf{1}), u \in \mathbb{R}^l$ generate $\mathfrak{g}$ and so $T_{\mathbf{1}}\phi$ is a Lie algebra automorphism.

Conversely suppose that $\exists\ \psi \in \mathsf{Aut}(\mathfrak{g})$ such that

$$\psi \cdot \Xi(\mathbf{1}, u) = \Xi'(\mathbf{1}, u).$$

Since G is simply connected there exists a diffeomorphism $\phi : \mathsf{G} \to \mathsf{G}$ such that $T_{\mathbf{1}}\phi = \psi$ (refer to theorem A.1.9). Therefore we have

$$\begin{aligned} T_g\phi \cdot \Xi(g, u) &= T_g\phi \cdot T_{\mathbf{1}}L_g \cdot \Xi(\mathbf{1}, u) && \text{by left invariance} \\ &= T_{\mathbf{1}}L_{\phi(g)} \cdot T_{\mathbf{1}}\phi \cdot \Xi(\mathbf{1}, u) && \text{by lemma 2.2.4} \\ &= T_{\mathbf{1}}L_{\phi(g)} \cdot \Xi'(\mathbf{1}, u) && \text{by assumption} \\ &= \Xi'(\phi(g), u) && \text{by left invariance.} \end{aligned}$$

Hence Σ and Σ' are state space equivalent.

2.2.8 Proposition. *Let Σ and Σ' be S-equivalent. Then Σ has full rank if and only if Σ' has full rank.*

Proof. Recall that $\Gamma = \mathrm{im}\Xi(\mathbf{1}, \cdot)$. Now from theorem 2.2.7 we have that Σ is S-equivalent to Σ' if and only if $\psi \cdot \Xi(\mathbf{1}, u) = \Xi'(\mathbf{1}, u)$ for all $u \in \mathbb{R}^\ell$. The result follows from proposition 2.1.2

2.2.9 Proposition. *For a simply connected Lie group G with Lie algebra $\mathfrak{g}$, two systems $\Sigma = (\mathsf{G}, \Xi)$ and $\Sigma' = (\mathsf{G}, \Xi')$ (where $\Xi(\mathbf{1}, u) = A + u_1B_1 + \ldots + u_\ell B_\ell$ and $\Xi'(\mathbf{1}, u) = A' + u_1B'_1 + \ldots + u_\ell B'_\ell$) are state space equivalent if and only if*

$$\begin{cases} \psi(A) = A' \\ \psi(B_i) = B'_i. \end{cases}$$

Proof. From theorem 2.2.7 we have that two systems $\Sigma = (\mathsf{G}, \Xi)$ and $\Sigma' = (\mathsf{G}, \Xi')$ are state space equivalent if and only if $\psi \cdot \Xi(\mathbf{1}, u) = \Xi'(\mathbf{1}, u)$ that is if $\psi \cdot (A + u_1B_1 + \ldots u_lB_l) = A' + u_1B'_1 + u_\ell B'_\ell$ for all $u \in \mathbb{R}^\ell$. Since ψ is linear we have

$$\psi(A) + \psi(u_1B_1) + \ldots + \psi(u_\ell B_\ell) = A' + u_1B'_1 + \ldots + u_\ell B'_\ell \quad \text{for all } u \in \mathbb{R}^\ell.$$

$$\begin{array}{lll}
\text{Choose } (u_1,\dots,u_\ell)=(0,0,\dots,0) \implies & \psi(A)=A' & \\
\text{Choose } (u_1,\dots,u_\ell)=(1,0,\dots,0) \implies & \psi(A+B_1)=A'+B_1' \implies \psi(B_1)=B_1' & \\
\quad\vdots & & \\
\text{Choose } (u_1,\dots,u_\ell)=(0,0,\dots,1) \implies & \psi(A+B_\ell)=A'+B_\ell' \implies \psi(B_\ell)=B_\ell'. &
\end{array}$$

The proposition which follows shows that a homogeneous system cannot be S-equivalent to an inhomogeneous system and vice-versa

2.2.10 PROPOSITION. *If Σ and Σ' are S-equivalent the systems are either both homogeneous or both inhomogeneous.*

PROOF. If Σ and Σ' are S-equivalent there exists a Lie algebra automorphism $\psi : \mathfrak{g} \to \mathfrak{g}$ such that $\psi \cdot A = A'$ and that $\psi \cdot B_i = B_i'$ for $i = 1, \dots \ell$ and hence that $A = \psi^{-1}A'$, $B_i = \psi^{-1} \cdot B_i'$ for $i = 1, \dots \ell$. Therefore $0 \in \Gamma$ if and only if $0 \in \Gamma'$.

2.2.1 Classification under S-equivalence

We use proposition 2.2.7 in order to classify the full-rank left-invariant control affine systems on H_3 (2.1) under S-equivalence. When convenient, a system specified by

$$\Xi(\mathbf{1},u)=\sum_{i=1}^{3}a_iE_i+u_1\sum_{i=1}^{3}b_iE_i+u_2\sum_{i=1}^{3}c_iE_i+u_3\sum_{i=1}^{3}d_iE_i$$

will be represented as

$$\left[\begin{array}{c|ccc} a_1 & b_1 & c_1 & d_1\\ a_2 & b_2 & c_2 & d_2\\ a_3 & b_3 & c_3 & d_3\end{array}\right].$$

In doing so the evaluation of $\psi \cdot \Xi(\mathbf{1}, u)$ becomes a matrix multiplication. Recall from proposition 1.1.8 that in terms of the standard basis $(E_i)_{i=1}^3$ the automorphism group of $\mathfrak{h}_3$ is

$$Aut(\mathfrak{h}_3)=\left\{\begin{bmatrix} v_2w_3-v_3w_2 & v_1 & w_1\\ 0 & v_2 & w_2\\ 0 & v_3 & w_3\end{bmatrix} \,|v_1,v_2,v_3,w_1,w_2,w_3\in\mathbb{R}, v_2w_3-v_3w_2\neq 0\right\}.$$

We begin with four lemmas which greatly simplify the classification of systems.

2.2.11 LEMMA. *2 vectors* $A = \begin{bmatrix} a_1\\ a_2\\ a_3\end{bmatrix}$ *and* $B = \begin{bmatrix} b_1\\ b_2\\ b_3\end{bmatrix} \in \mathfrak{h}_3$ *generate* $\mathfrak{h}_3$ *if and only if* $a_2b_3 - a_3b_2 \neq 0$.

PROOF. $\mathfrak{h}_3$ is 3 dimensional and so in order for two vectors A and B to generate the entire Lie algebra $\mathfrak{h}_3$, A, B and $[A, B]$ must be linearly independent. Now

$$[A,B]=\begin{bmatrix} a_2b_3-b_2a_3\\ 0\\ 0\end{bmatrix}.$$

Hence vectors A and B generate $\mathfrak{h}_3$ if and only if

$$\begin{vmatrix} a_1 & b_1 & a_2b_3 - b_2a_3 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & 0 \end{vmatrix} = (a_2b_3 - a_3b_2)^2 \neq 0.$$

2.2.12 Lemma. *For any two vectors* $A = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ *and* $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \in \mathfrak{h}_3$ *which generate* $\mathfrak{h}_3$ *there exists* $\psi \in \mathsf{Aut}(\mathfrak{h}_3)$ *such that*

$$\psi \cdot \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Proof. From lemma 2.2.11 $a_2b_3 - b_2a_3 \neq 0$. Consequently

$$\psi_1 = \begin{bmatrix} -a_3b_2 + a_2b_3 & 0 & 0 \\ 0 & b_3 & -b_2 \\ 0 & -a_3 & a_2 \end{bmatrix} \qquad \big(\det \psi_1 = (a_3b_2 - a_2b_3)^2 \neq 0\big)$$

is an automorphism such that

$$\psi_1 \cdot \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix} = \begin{bmatrix} -a_1a_3b_2 + a_1a_2b_3 & -a_3b_1b_2 + a_2b_1b_3 \\ -a_3b_2 + a_2b_3 & 0 \\ 0 & -a_3b_2 + a_2b_3 \end{bmatrix}.$$

Also

$$\psi_2 = \begin{bmatrix} \frac{1}{(a_3b_2 - a_2b_3)^2} & \frac{-a_1}{(a_3b_2 - a_2b_3)^2} & \frac{-b_1}{(a_3b_2 - a_2b_3)^2} \\ 0 & \frac{1}{-a_3b_2 + a_2b_3} & 0 \\ 0 & 0 & \frac{1}{-a_3b_2 + a_2b_3} \end{bmatrix} \qquad \left(\det \psi_2 = \frac{1}{(a_3b_2 - a_2b_3)^4} \neq 0\right)$$

is an automorphism such that

$$\psi_2 \cdot \begin{bmatrix} -a_1a_3b_2 + a_1a_2b_3 & -a_3b_1b_2 + a_2b_1b_3 \\ -a_3b_2 + a_2b_3 & 0 \\ 0 & -a_3b_2 + a_2b_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Hence $\psi = \psi_2 \circ \psi_1$ is an automorphism such that $\psi \cdot \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$.

2.2.13 Lemma. *The only* $\psi \in \mathsf{Aut}(\mathfrak{h}_3)$ *which preserves* $\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ *is* $\psi = \mathbf{1}$.

Proof. Let $\psi = \begin{bmatrix} v_2w_3 - v_3w_2 & v_1 & w_1 \\ 0 & v_2 & w_2 \\ 0 & v_3 & w_3 \end{bmatrix} \in \mathsf{Aut}(\mathfrak{h}_3)$, arbitrary. We have that

$$\begin{aligned} \begin{bmatrix} v_2w_3 - v_3w_2 & v_1 & w_1 \\ 0 & v_2 & w_2 \\ 0 & v_3 & w_3 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \implies \begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \\ v_3 & w_3 \end{bmatrix} &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

from which it follows that

$$\begin{bmatrix} v_2w_3 - v_3w_2 & v_1 & w_1 \\ 0 & v_2 & w_2 \\ 0 & v_3 & w_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

2.2.14 Lemma. *If two vectors* $A = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ *and* $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ *do not generate* $\mathfrak{h}_3$ *then there images* $\psi \cdot A$ *and* $\psi \cdot B$ *do not generate* $\mathfrak{h}_3$.

Proof. Let ψ be an arbitrary element of $\mathsf{Aut}(\mathfrak{h}_3)$. Then

$$\begin{aligned} \psi \cdot \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix} &= \begin{bmatrix} v_2w_3 - v_3w_2 & v_1 & w_1 \\ 0 & v_2 & w_2 \\ 0 & v_3 & w_3 \end{bmatrix} \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix} \\ &= \begin{bmatrix} a_2v_1 + a_3w_1 - a_1v_3w_2 + a_1v_2w_3 & b_2v_1 + b_3w_1 - b_1v_3w_2 + b_1v_2w_3 \\ a_2v_2 + a_3w_2 & b_2v_2 + b_3w_2 \\ a_2v_3 + a_3w_3 & b_2v_3 + b_3w_3 \end{bmatrix} \end{aligned}$$

Since A and B do not generate $\mathfrak{h}_3$ we have that $a_2b_3 - b_2a_3 = 0$ (lemma 2.2.11). Using this fact we show that $(a_2v_2 + a_3w_2)(b_2v_3 + b_3w_3) - (b_2v_2 + b_3w_2)(a_2v_3 + a_3w_3) = 0$ and hence that $\psi \cdot A$ and $\psi \cdot B$ do not generate $\mathfrak{h}_3$.

$$\begin{aligned} &(a_2v_2 + a_3w_2)(b_2v_3 + b_3w_3) - (b_2v_2 + b_3w_2)(a_2v_3 + a_3w_3) \\ =&(a_2b_2v_2v_3 + a_2b_3v_2w_3 + a_3b_2w_2v_3 + a_3b_3w_2w_3) - (b_2a_2v_2v_3 + b_2a_3v_2w_3 + b_3a_2w_2v_3 + b_3a_3w_2w_3) \\ =&(a_2b_3 - b_2a_3)v_2w_3 + (a_3b_2 - b_3a_2)w_2v_3 \\ =&0. \end{aligned}$$

2.2.15 Proposition. *Any (1,1) system is state space equivalent to the system*

$$\Sigma^{(1,1)} : E_2 + uE_3.$$

Proof. The system must satisfy the full-rank condition. The result therefore follows immediately from lemma 2.2.12.

2.2.16 Proposition. *Any (2,0) system is state space equivalent to exactly one of the following systems*

$$\Sigma^{(2,0)}_{\gamma_1,\gamma_2} : \gamma_1 E_2 + \gamma_2 E_3 + u_1 E_2 + u_2 E_3, \quad \gamma_1, \gamma_2.$$

Here each parameter parametrises a distinct family of class representatives.

Proof. Such a system has $\Xi(\mathbf{1}, u) = \sum_{i=1}^{3} a_i E_i + u_1 \sum_{i=1}^{3} b_i E_i + u_2 \sum_{i=1}^{3} c_i E_i,\ u_1, u_2 \in \mathbb{R}$, where $\sum_{i=1}^{3} a_i E_i \in \left\langle \sum_{i=1}^{3} b_i E_i, \sum_{i=1}^{3} c_i E_i \right\rangle$. In order for the full rank condition to be satisfied $\sum_{i=1}^{3} b_i E_i$ and $\sum_{i=1}^{3} c_i E_i$ must generate $\mathfrak{h}_3$ so from lemma 2.2.12 the system is state space equivalent to

$$\left[\begin{array}{c|cc} a'_1 & 0 & 0 \\ a'_2 & 1 & 0 \\ a'_3 & 0 & 1 \end{array}\right]$$

for some $a'_1, a'_2, a'_3 \in \mathbb{R}$. From linear dependence we have that $a'_1 = 0$. Hence any $(2,0)$ system is state space equivalent to

$$\Sigma^{(2,0)}_{\gamma_1,\gamma_2} : \gamma_1 E_2 + \gamma_2 E_3 + u_1 E_2 + u_2 E_3 \quad u_1, u_2 \in \mathbb{R}$$

where $\gamma_1 = a'_2$ and $\gamma_2 = a'_3$. From lemma 2.2.13 and proposition 2.2.9 we have that for $(\gamma_1, \gamma_2) \neq (\gamma'_1, \gamma'_2)$ the systems are distinct.

2.2.17 Proposition. *Any (2,1) system is state space equivalent to exactly one of the following systems:*

$$\Sigma^{(2,1)}_{1,\alpha,\gamma_1,\gamma_2} : \alpha E_1 + \gamma_1 E_2 + \gamma_2 E_3 + u_1 E_2 + u_2 E_3, \qquad -b_3 c_2 + b_2 c_3 \neq 0 \tag{2.2}$$

$$\Sigma^{(2,1)}_{2,\alpha,\gamma_1} : E_2 + u_1 E_3 + u_2(\alpha E_1 + \gamma_1 E_3), \qquad -b_3 c_2 + b_2 c_3 = 0, -a_3 b_2 + a_2 b_3 \neq 0 \tag{2.3}$$

$$\Sigma^{(2,1)}_{3,\alpha} : E_2 + u_1 \alpha E_1 + u_2 E_3, \qquad -b_3 c_2 + b_2 c_3 = 0, -a_3 b_2 + a_2 b_3 = 0 \tag{2.4}$$

$$\alpha, \gamma_1, \gamma_2, u_1, u_2 \in \mathbb{R}, \alpha \neq 0.$$

Here each parameter parametrises a distinct family of class representatives.

Proof. Such a system is given has $\Xi(\mathbf{1}, u) = \sum_{i=1}^{3} a_i E_i + u_1 \sum_{i=1}^{3} b_i E_i + u_2 \sum_{i=1}^{3} c_i E_i,\ u_1, u_2 \in \mathbb{R}$, where $\sum_{i=1}^{3} a_i E_i \notin \left\langle \sum_{i=1}^{3} b_i E_i, \sum_{i=1}^{3} c_i E_i \right\rangle$, which is represented by the matrix

$$\left[\begin{array}{c|cc} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{array}\right].$$

We consider the 4 cases:

1. $-b_3 c_2 + b_2 c_3 \neq 0$ in which case columns 2 and 3 generate $\mathfrak{h}_3$ (lemma 2.2.11).

2. $-b_3 c_2 + b_2 c_3 = 0,\ -a_3 b_2 + a_2 b_3 \neq 0$ in which case columns 1 and 2 generate $\mathfrak{h}_3$ (columns 2 and 3 do not)(lemma 2.2.11).

3. $-b_3 c_2 + b_2 c_3 = 0,\ -a_3 b_2 + a_2 b_3 = 0,\ -a_3 c_2 + a_2 c_3 \neq 0$ in which case columns 1 and 3 generate $\mathfrak{h}_3$ (columns 2 and 3 as well as 1 and 2 do not)(lemma 2.2.11).

4. $-b_3c_2 + b_2c_3 = 0$, $-a_3b_2 + a_2b_3 = 0$, $-a_3c_2 + a_2c_3 = 0$ in which case columns 2 and 3, 1 and 2 and 1 and 3 do not generate $\mathfrak{h}_3$ (lemma 2.2.11).

Each case is covered below.

1. $(-b_3c_2 + b_2c_3 \neq 0)$. Columns 2 and 3 generate $\mathfrak{h}_3$.

   From lemma 2.2.12 we have that the system is state space equivalent to

   $$\Sigma^{(2,1)}_{1,\alpha,\gamma_1,\gamma_2} : \left[\begin{array}{c|cc} \alpha & 0 & 0 \\ \gamma_1 & 1 & 0 \\ \gamma_2 & 0 & 1 \end{array}\right]$$

   where $\alpha \neq 0$ since the system is inhomogeneous. From lemma 2.2.13 and proposition 2.2.9 we have that for $(\alpha, \gamma_1, \gamma_2) \neq (\alpha', \gamma_1', \gamma_2')$ the systems are distinct.

2. $(-b_3c_2 + b_2c_3 = 0,\ -a_3b_2 + a_2b_3 \neq 0)$. Columns 1 and 2 generate $\mathfrak{h}_3$.
   From lemma 2.2.12 we have that the system is state space equivalent to

   $$\left[\begin{array}{c|cc} 0 & 0 & c_1' \\ 1 & 0 & c_2' \\ 0 & 1 & c_3' \end{array}\right]$$

   for some $c_1', c_2', c_3' \in \mathbb{R}$. Now $c_1' \neq 0$ from linear independence and from lemma 2.2.14 we have that columns 2 and 3 do not generate $\mathfrak{h}_3$ hence $c_2' = 0$ (lemma 2.2.11). The system is therefore state space equivalent to

   $$\Sigma^{(2,1)}_{2,\alpha,\gamma_1} : \left[\begin{array}{c|cc} 0 & 0 & \alpha \\ 1 & 0 & 0 \\ 0 & 1 & \gamma_1 \end{array}\right]$$

   where $\alpha = c_1' \neq 0$, $\gamma_1 = c_3'$. From lemma 2.2.13 and proposition 2.2.9 we have that for $(\alpha, \gamma_1) \neq (\alpha', \gamma_1')$ the systems are distinct.

3. $(-b_3c_2 + b_2c_3 = 0,\ -a_3b_2 + a_2b_3 = 0,\ -a_3c_2 + a_2c_3 \neq 0)$. Columns 1 and 3 generate $\mathfrak{h}_3$. From lemma 2.2.12 we have that the system is state space equivalent to

   $$\left[\begin{array}{c|cc} 0 & b_1' & 0 \\ 1 & b_2' & 0 \\ 0 & b_3' & 1 \end{array}\right]$$

   for some $b_1', b_2', b_3' \in \mathbb{R}$. From lemma 2.2.14 we have that columns 1 and 2 as well as columns 2 and 3 do not generate $\mathfrak{h}_3$ hence $b_2', b_3' = 0$. From linear independence $b_1' \neq 0$. The system is therefore state space equivalent to

   $$\Sigma^{(2,1)}_{3,\alpha} : \left[\begin{array}{c|cc} 0 & \alpha & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

   where $\alpha = b_1' \neq 0$. From lemma 2.2.13 and proposition 2.2.9 we have that for $\alpha \neq \alpha'$ the systems are distinct.

4. $(-b_3c_2 + b_2c_3 = 0, -a_3b_2 + a_2b_3 = 0, -a_3c_2 + a_2c_3 = 0)$. We represent the system as

$$\left[\begin{array}{c|cc} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{array}\right].$$

Now

$$\begin{aligned} \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} &= -a_3b_2c_1 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a_2b_1c_3 + a_1b_2c_3 \\ &= (-a_3b_2 + a_2b_3)c_1 + (a_3c_2 - a_2c_3)b_1 + (-b_3c_2 + b_2c_3)a_1 \\ &= 0. \end{aligned}$$

Hence columns 1, 2 and 3 are linearly dependent and the system is not a $(2,1)$ system.

Finally, we verify that the systems 2.2, 2.3, 2.4 are not equivalent.

Suppose $\Sigma^{(2,1)}_{1,\alpha,\gamma_1,\gamma_2}$ is S-equivalent to $\Sigma^{(2,1)}_{2,\alpha',\gamma_1'}$. Then there exists $\psi \in \mathsf{Aut}(\mathfrak{h}_3)$ such that $\psi \cdot \Xi^{(2,1)}_{1,\alpha,\gamma_1,\gamma_2}(\mathbf{1}, u) = \Xi^{(2,1)}_{2,\alpha',\gamma_1'}(\mathbf{1}, u)$, i.e.,

$$\begin{aligned} \begin{bmatrix} v_2w_3 - v_3w_2 & v_1 & w_1 \\ 0 & v_2 & w_2 \\ 0 & v_3 & w_3 \end{bmatrix} \left[\begin{array}{c|cc} \alpha & 0 & 0 \\ \gamma_1 & 1 & 0 \\ \gamma_2 & 0 & 1 \end{array}\right] &= \left[\begin{array}{c|cc} \gamma_1v_1 + \gamma_2w_1 + \alpha(-v_3w_2 + v_2w_3) & v_1 & w_1 \\ \gamma_1v_2 + \gamma_2w_2 & v_2 & w_2 \\ \gamma_1v_3 + \gamma_2w_3 & v_3 & w_3 \end{array}\right] \\ &= \left[\begin{array}{c|cc} 0 & 0 & \alpha' \\ 1 & 0 & 0 \\ 0 & 1 & \gamma_1' \end{array}\right]. \end{aligned}$$

But this gives $v_2w_3 - v_3w_2 = 0$. These systems are therefore not equivalent.

Suppose $\Sigma^{(2,1)}_{1,\alpha,\gamma_1,\gamma_2}$ is S-equivalent to $\Sigma^{(2,1)}_{3,\alpha'}$. Then there exists a $\psi \in \mathsf{Aut}(\mathfrak{h}_3)$ such that $\psi \cdot \Xi^{(2,1)}_{1,\alpha,\gamma_1,\gamma_2}(\mathbf{1}, u) = \Xi'^{(2,1)}_{3,\alpha'}(\mathbf{1}, u)$, i.e.,

$$\begin{bmatrix} v_2w_3 - v_3w_2 & v_1 & w_1 \\ 0 & v_2 & w_2 \\ 0 & v_3 & w_3 \end{bmatrix} \left[\begin{array}{c|cc} \alpha & 0 & 0 \\ \gamma_1 & 1 & 0 \\ \gamma_2 & 0 & 1 \end{array}\right] = \left[\begin{array}{c|cc} \gamma_1v_1 + \gamma_2w_1\alpha(-v_3w_2 + v_2w_3) & v_1 & w_1 \\ \gamma_1v_2 + \gamma_2w_2 & v_2 & w_2 \\ \gamma_1v_3 + \gamma_2w_3 & v_3 & w_3 \end{array}\right] = \left[\begin{array}{c|cc} 0 & \alpha' & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{array}\right].$$

Again the determinant $v_2w_3 - v_3w_2 = 0$ and these systems are not equivalent.

Lastly, suppose $\Sigma^{(2,1)}_{2,\alpha,\gamma_1}$ is S-equivalent to $\Sigma^{(2,1)}_{3,\alpha}$. Then there exists $\psi \in \mathsf{Aut}(\mathfrak{h}_3)$ such that $\psi \circ \Xi^{(2,1)}_{2,\alpha,\gamma_1}(\mathbf{1}, u) = \Xi'^{(2,1)}_{3,\alpha'}(\mathbf{1}, u)$, i.e.,

$$\begin{bmatrix} v_2w_3 - v_3w_2 & v_1 & w_1 \\ 0 & v_2 & w_2 \\ 0 & v_3 & w_3 \end{bmatrix} \left[\begin{array}{c|cc} 0 & 0 & \alpha \\ 1 & 0 & 0 \\ 0 & 1 & \gamma_1 \end{array}\right] = \left[\begin{array}{c|cc} v_1 & w_1 & \gamma_1w_1 + \alpha(-v_3w_2 + v_2w_3) \\ v_2 & w_2 & \gamma_1w_2 \\ v_3 & w_3 & \gamma_1w_3 \end{array}\right] = \left[\begin{array}{c|cc} 0 & \alpha' & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{array}\right].$$

Once again the determinant $v_2w_3 - v_3w_2 = 0$ and these systems are not equivalent.

2.2.18 Corollary. *Any (3,0) system is state space equivalent to exactly one of the following systems*

$$\gamma_1E_1+\gamma_2E_2+\gamma_3E_3+u_1(\alpha E_1+\gamma_4E_2+\gamma_5E_3)+u_2E_2+u_3E_3, \qquad -c_3d_2+c_2d_3\neq 0 \tag{2.5}$$

$$\gamma_1E_1+\gamma_2E_2+\gamma_3E_3+u_1E_2+u_2E_3+u_3(\alpha E_1+\gamma_4E_3), \qquad \begin{aligned}&-c_3d_2+c_2d_3=0,\\ &-b_3c_2+b_2c_3\neq 0\end{aligned} \tag{2.6}$$

$$\gamma_1E_1+\gamma_2E_2+\gamma_3E_3+u_1E_2+u_2\alpha E_1+u_3E_3, \qquad \begin{aligned}&-c_3d_2+c_2d_3=0,\\ &-b_3c_2+b_2c_3=0.\end{aligned} \tag{2.7}$$

where $\alpha,\gamma_1,\gamma_2,\gamma_3,\gamma_4,\gamma_5\in\mathbb{R}$, $\alpha\neq 0$. *Here each parameter parametrises a distinct family of class representatives.*

Table 2.1 summarises the results of propositions 2.2.15, 2.2.16, 2.2.17 and corollary 2.2.18.

2.3 Detached Feedback Equivalence

This section introduces detached feedback equivalence of left-invariant control affine systems on a connected Lie group and shows that the classification may be done at the level of the Lie algebra.

2.3.1 Definition. Let G be a connected Lie group and let $\Sigma=(\mathsf{G},\Xi)$ and $\Sigma'=(\mathsf{G},\Xi')$ be left invariant control affine systems with the same input space $\mathbb{R}^\ell$. Then Σ and Σ' are **detached feedback equivalent** (or DF-equivalent for short) if there exist diffeomorphisms $\phi:\mathsf{G}\to\mathsf{G}$ and $\varphi:\mathbb{R}^\ell\to\mathbb{R}^\ell$ such that the diagram

$$\begin{tikzcd}
\mathsf{G}\times\mathbb{R}^\ell \arrow[r, "\phi\times\varphi"] \arrow[d, "\Xi"'] & \mathsf{G}\times\mathbb{R}^\ell \arrow[d, "\Xi'"] \\
T\mathsf{G} \arrow[r, "T\phi"'] & T\mathsf{G}'
\end{tikzcd}$$

commutes. That is,

$$T_g\phi\cdot\Xi(g,u)=\Xi'(\phi(g),\varphi(u)) \quad \forall g\in\mathsf{G},\, u\in\mathbb{R}^\ell.$$

2.3.2 Proposition. *DF-equivalence is an equivalence relation.*

Proof. Consider three systems $\Sigma=(\mathsf{G},\Xi)$, $\Sigma'=(\mathsf{G},\Xi')$ and $\Sigma''=(\mathsf{G},\Xi'')$.

1. DF-equivalence is reflexive:
   Let $\phi=\mathrm{id}_\mathsf{G}$ and $\varphi=\mathrm{id}_{\mathbb{R}^\ell}$. Then $T_g\phi=\mathrm{id}_{\mathrm{T_g}\mathsf{G}}$ for every $g\in\mathsf{G}$. Hence

   $$T_g\phi\cdot\Xi(g,u)=\Xi(g,u)=\Xi(\phi(g),\varphi(u)).$$

   Hence, Σ is DF-equivalent to itself.

2. DF-equivalence is symmetric:
   Suppose Σ is DF-equivalent to Σ', then there exists a diffeomorphism $\phi:\mathsf{G}\to\mathsf{G}$ such that

   $$T_g\phi\cdot\Xi(g,u)=\Xi'(\phi(g),\varphi(u)), \quad \forall g\in\mathsf{G}, u\in\mathbb{R}^\ell.$$

Also, $T_g\phi$ is a linear isomorphism and is therefore invertible so we have the linear isomorphism $(T_g\phi)^{-1}$ such that

$$\begin{aligned}(T_g\phi)^{-1} \cdot T_g\phi \cdot \Xi(g,u) &= (T_g\phi)^{-1} \cdot \Xi(\phi(g), \varphi(u)) \\ \iff \Xi(g,u) &= (T_g\phi)^{-1} \cdot \Xi'(\phi(g), \varphi(u)).\end{aligned}$$

Now $(T_g\phi)^{-1} = T_{\phi(g)}\phi^{-1}$(Lemma A.1.11) so we have

$$T_{\phi(g)}\phi^{-1} \cdot \Xi'(\phi(g), \varphi(u)) = \Xi(g,u) = \Xi(\phi^{-1}(\phi(g)), \varphi^{-1}(\varphi(u))), \quad \forall \phi(g) \in \mathsf{G}, \phi(u) \in \mathbb{R}^\ell.$$

Hence, if Σ is DF-equivalent to Σ', then Σ' is DF-equivalent to Σ.

3. DF-equivalence is transitive:
Suppose Σ is DF-equivalent to Σ' and that Σ' is DF-equivalent to Σ''. Then there exist diffeomorphisms ϕ_1 and ϕ_2 and φ_1 and φ_2 such that

$$\begin{cases} T_g\phi_1 \cdot \Xi(g,u) = \Xi'(\phi_1(g), \varphi_1(u)) \\ T_g\phi_2 \cdot \Xi'(g,u) = \Xi''(\phi_2(g), \varphi_2(u)), \quad \forall g \in \mathsf{G}, u \in \mathbb{R}^\ell. \end{cases}$$

Let $\phi = \phi_2 \circ \phi_1$ and $\varphi = \varphi_2 \circ \varphi_1$. Then for every $g \in \mathsf{G}$ and $u \in \mathbb{R}^\ell$

$$\begin{aligned} T_g\phi \cdot \Xi(g,u) &= T_{\phi_1(g)}\phi_2 \cdot T_g\phi_1 \cdot \Xi(g,u) \quad \text{Lemma A.1.10} \\ &= T_{\phi_1(g)}\phi_2 \cdot \Xi'(\phi_1(g), \varphi_1(u)) \\ &= \Xi''(\phi_2 \circ \phi_1(g), \varphi_2 \circ \varphi_1(u)) \\ &= \Xi''(\phi(g), \varphi(u)). \end{aligned}$$

Hence, if Σ is DF-equivalent to Σ' and Σ' is DF-equivalent to Σ'', then Σ is DF-equivalent to Σ''.

DF-equivalence is thus an equivalence relation.

2.3.3 PROPOSITION. *If Σ and Σ' are DF-equivalent, then the controlled trajectories of Σ and Σ' are in a one-to-one correspondence.*

PROOF. Suppose $(g(\cdot), u(\cdot))$ is a controlled trajectory of Σ we have that

$$\dot{g}(t) = \Xi(g(t), u(t)).$$

Since Σ and Σ' are DF-equivalent, there exist diffeomorphisms $\phi : \mathsf{G} \to \mathsf{G}'$ and $\varphi : \mathbb{R}^\ell \to \mathbb{R}^\ell$ such that

$$T_g\phi \cdot \Xi(g,u) = \Xi'(\phi(g), \varphi(u)) \quad \forall g \in \mathsf{G}, u \in \mathbb{R}^\ell.$$

We will show that $(\phi(g(\cdot), \varphi(u(\cdot)))$ is the unique controlled trajectory of Σ' corresponding to the control pair $(g(\cdot), u(\cdot))$ of Σ. Firstly, for almost every t

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\phi(g(t)) &= T_{g(t)}\phi \cdot \dot{g}(t) \\ &= T_{g(t)}\phi \cdot \Xi(g(t), u(t)) \\ &= \Xi'(\phi(g(t)), \varphi(u(t))) \end{aligned}$$

and so $(\phi(g(\cdot), \varphi(u(\cdot)))$ is a controlled trajectory of Σ'. Let $(g_1(\cdot), u_1(\cdot))$ and $(g_2(\cdot), u_2(\cdot))$ be controlled trajectory of Σ. Suppose $\phi(g_1(\cdot)) = \phi(g_2(\cdot))$ and that $\varphi(u_1(\cdot)) = \varphi(u_2(\cdot))$ then by applying ϕ^{-1} and φ^{-1} we have that $g_1(\cdot) = g_2(\cdot)$ and $u_1(\cdot) = u_2(\cdot)$. Hence trajectory-control pairs are mapped injectively from Σ to Σ'. Let $(g'(\cdot), u'(\cdot))$ be a controlled trajectory of Σ' then $(\phi^{-1}(g'(\cdot)), \varphi^{-1}(u'(\cdot)))$ is the corresponding controlled trajectory of Σ mapped to $(g'(\cdot), u'(\cdot))$ by $\phi \times \varphi$. Hence controlled trajectories are mapped surjectively from Σ to Σ' and the controlled trajectories are in a one-to-one correspondence.

In order to be able to prove that a system which is controllable cannot be DF-equivalent to a system which is not controllable and vice-versa and also to prove that the classification under DF-equivalence may be done at the level of Lie algebras we require two technical lemmas.

2.3.4 LEMMA. *Let ϕ be a diffeomorphism and let $L_g : \mathsf{G} \to \mathsf{G}, h \mapsto gh$ be the left translation by $g \in \mathsf{G}$. Then*

$$T\phi \circ TL_g = TL_{\phi(g)} \circ T\phi.$$

PROOF.

$$\begin{aligned} \phi(L_g(h)) &= \phi(gh) \\ &= \phi(g)\phi(h) \\ &= L_{\phi(g)}\phi(h) \\ \implies \phi \circ L_g &= L_{\phi(g)} \circ \phi \\ \implies T\phi \circ TL_g &= TL_{\phi(g)} \circ T\phi \quad \text{(Proposition}A.1.10.). \end{aligned}$$

2.3.5 LEMMA. *Let $\phi : \mathsf{G} \to \mathsf{G}$ and $\varphi : \mathbb{R}^\ell \to \mathbb{R}^\ell$ be diffeomorphisms. Then*

$$\begin{aligned} T_g\phi \cdot \Xi(g,u) &= \Xi'(\phi(g), \varphi(u)) \\ \iff T_g(L_{\phi(1)^{-1}} \circ \phi) \cdot \Xi(g,u) &= \Xi'(L_{\phi(1)^{-1}} \circ \phi(g), \varphi(u)). \end{aligned}$$

PROOF.

$$\begin{aligned} T_g(L_{\phi(1)^{-1}} \circ \phi) \cdot \Xi(g,u) &= \Xi'(L_{\phi(1)^{-1}} \circ \phi(g), \varphi(u)) \\ \iff T_{\phi(g)}L_{\phi(1)^{-1}} \circ T_g\phi \cdot \Xi(g,u) &= T_{\phi(g)}L_{\phi(1)^{-1}} \cdot \Xi'(\phi(g), \varphi(u)) \quad \text{Lemma 2.3.4 and left invariance} \\ \iff T_g\phi \cdot \Xi(g,u) &= \Xi'(\phi(g), \varphi(u)). \end{aligned}$$

2.3.6 PROPOSITION. *If $\Sigma = (\mathsf{G}, \Xi)$ and $\Sigma' = (\mathsf{G}, \Xi')$ are DF-equivalent then Σ is controllable if and only if Σ' is controllable.*

PROOF. Without loss of generality we may assume $\phi(\mathbf{1}) = \mathbf{1}$ (lemma 2.3.5). Let $\mathcal{A}$ and $\mathcal{A}'$ denote the attainable sets of Σ and Σ' respectively. Suppose Σ is controllable i.e. $\mathcal{A} = G$. We show that this implies $\mathcal{A}' = \mathsf{G}$ and hence that Σ' is controllable. Firstly,

$$\mathcal{A}' = \left\{ g'(T) \mid g'(\cdot) : [0,T] \to \mathsf{G} \text{ is a trajectory of } \Sigma',\ g'(0) = \mathbf{1} \right\}.$$

Now since Σ and Σ' are DF-equivalent (with respect to diffeomorphisms ϕ and φ say) we have that the trajectories of Σ and Σ' are in one-to-one correspondence. That is for the trajectory $g'(\cdot) : [0,T] \to \mathsf{G}$, $g'(\cdot) = \phi(g(\cdot))$ for exactly one trajectory $g(\cdot) : [0,T] \to \mathsf{G}$. Hence

$$\begin{aligned}\mathcal{A}' &= \{\phi(g(T)) \mid g(\cdot) : [0,T] \to \mathsf{G} \text{ is a trajectory of } \Sigma,\ g(0) = \mathbf{1}\}\\ &= \phi(\mathcal{A}) = \mathsf{G}.\end{aligned}$$

By interchanging the roles of Σ and Σ' in the above argument one is able to show that Σ is controllable if Σ' is controllable. The result follows.

2.3.7 Theorem. (cf. [18]) *For a simply connected Lie group* G*, two systems* Σ *and* Σ' *are detached feedback equivalent if and only if there exists* $\psi \in \mathsf{Aut}(\mathfrak{g})$ *such that* $\psi \cdot \Gamma = \Gamma'$.

Proof. Suppose Σ and Σ' are detached feedback equivalent. Then there exist diffeomorphisms $\phi : \mathsf{G} \to \mathsf{G}$ and $\varphi : \mathbb{R}^\ell \to \mathbb{R}^\ell$ such that $T_g\phi \cdot \Xi(g,u) = \Xi'(\phi(g), \varphi(u))$ for every $g \in \mathsf{G}$ and $u \in \mathbb{R}^\ell$. From which it follows that $T_{\mathbf{1}}\phi \cdot \Xi(\mathbf{1},u) = \Xi'(\phi(\mathbf{1}), \varphi(u))$. We may assume $\phi(\mathbf{1}) = \mathbf{1}$ since if this is not the case we can replace ϕ with the diffeomorphism $L_{\phi(\mathbf{1})^{-1}} \circ \phi : \mathsf{G} \to \mathsf{G}$ (ref. to lemma 2.3.5). Clearly $(L_{\phi(\mathbf{1})^{-1}} \circ \phi)(\mathbf{1}) = \mathbf{1}$.

We show that $T_{\mathbf{1}}\phi \cdot \Gamma = \Gamma'$. Firstly, using the notation where the left invariant vector field $\Xi_u := \Xi(\cdot, u)$, $\Gamma = \{\Xi_u(\mathbf{1}) | u \in \mathbb{R}^\ell\}$. We have that $T_{\mathbf{1}}\phi \cdot \Xi_u(\mathbf{1}) = \Xi'_{\varphi(u)}(\mathbf{1}) \in \Gamma'$ for all $u \in \mathbb{R}^\ell$ and so $T_{\mathbf{1}}\phi \cdot \Gamma \subseteq \Gamma'$. Also, ϕ is a diffeomorphism hence by theorem A.1.9 we have that $T_{\mathbf{1}}\phi$ is a linear isomorphism. Therefore $\dim(T_{\mathbf{1}}\phi \cdot \Gamma) = \dim(\Gamma)$ and $T_{\mathbf{1}}\phi \cdot \Gamma = \Gamma'$.

It remains to be shown that $T_{\mathbf{1}}\phi$ is a Lie algebra automorphism. We already have that $T_{\mathbf{1}}\phi$ is a linear isomorphism $T_{\mathbf{1}}\phi : \mathfrak{g} \to \mathfrak{g}$ and is therefore a linear automorphism. We have left to show is that $T_{\mathbf{1}}\phi$ preserves the Lie bracket. Noting that the pushforward by ϕ preserves the Lie bracket of vector fields we have

$$\phi_*[\Xi_u, \Xi_v] = [\phi_*\Xi_u, \phi_*\Xi_v].$$

This, together with left invariance gives

$$\begin{aligned} &\phi_*[\Xi_u, \Xi_v](\phi(\mathbf{1})) = [\phi_*\Xi_u, \phi_*\Xi_v](\phi(\mathbf{1}))\\ \iff\ &\phi_*[\Xi_u, \Xi_v](\phi(\mathbf{1})) = [\phi_*\Xi_u(\phi(\mathbf{1})), \phi_*\Xi_v(\phi(\mathbf{1}))]\\ \iff\ &T_{\mathbf{1}}\phi \cdot [\Xi_u, \Xi_v](\mathbf{1}) = [T_{\mathbf{1}}\phi \cdot \Xi_u(\mathbf{1}), T_{\mathbf{1}}\phi \cdot \Xi_v(\mathbf{1})]\\ \iff\ &T_{\mathbf{1}}\phi \cdot [\Xi_u(\mathbf{1}), \Xi_v(\mathbf{1})] = [T_{\mathbf{1}}\phi \cdot \Xi_u(\mathbf{1}), T_{\mathbf{1}}\phi \cdot \Xi_v(\mathbf{1})].\end{aligned}$$

Similarly $T_{\mathbf{1}}\phi \cdot [\Xi_u(\mathbf{1}), [\Xi_v(\mathbf{1}), \Xi_w(\mathbf{1})]] = [T_{\mathbf{1}}\phi \cdot \Xi_u(\mathbf{1}), [T_{\mathbf{1}}\phi \cdot \Xi_v(\mathbf{1}), T_{\mathbf{1}}\phi \cdot \Xi_w(\mathbf{1})]]$ and similarly for higher order commutators. Now $\Gamma = \{\Xi_u(\mathbf{1}) : u \in \mathbb{R}^\ell\}$ and $\mathsf{Lie}(\Gamma) = \mathfrak{g}$. The elements $\Xi_u(\mathbf{1}),\ u \in \mathbb{R}^\ell$ therefore generate $\mathfrak{g}$ and so $T_{\mathbf{1}}\phi$ is a Lie algebra automorphism.

Conversely, suppose ψ is a Lie algebra automorphism such that $\psi \cdot \Gamma = \Gamma'$. Then there exists a Lie group automorphism $\phi : \mathsf{G} \to \mathsf{G}$ such that $T_{\mathbf{1}}\phi = \psi$ (cf. [23]). Also, $\Xi'(\mathbf{1}, \cdot) : \mathbb{R}^\ell \to \mathfrak{g}$ is an injective mapping; by restricting the codomain $\mathfrak{g}$ to Γ' i.e. $\Xi'(\mathbf{1}, \cdot) : \mathbb{R}^\ell \to \Gamma' = \mathrm{im}\Xi'(\mathbf{1}, \cdot)$ we obtain a bijective and therefore invertible map. Suppose the inverse is given by $\xi' : \Gamma' \to \Xi'(\mathbf{1}, \cdot)$. Let $\varphi : \mathbb{R}^\ell \to \mathbb{R}^\ell$ be the diffeomorphism defined by $\varphi(u) = \xi'(T_{\mathbf{1}}\phi \cdot \Xi(\mathbf{1}, u))$. Then $T_{\mathbf{1}}\phi \cdot \Xi(\mathbf{1}, u) =$

$(\xi')^{-1}(\varphi(u)) = \Xi'(\mathbf{1}, \varphi(u))$ (or $\psi \cdot \Xi(\mathbf{1}, u) = \Xi'(\mathbf{1}, \varphi(u))$) for every $u \in \mathbb{R}^\ell$. Therefore we have

$$\begin{aligned} T_g\phi \cdot \Xi(g,u) &= T_g\phi \cdot g\Xi(\mathbf{1},u) \\ &= T_g\phi \cdot T_{\mathbf{1}}L_g \cdot \Xi(\mathbf{1},u) \\ &= T_{\mathbf{1}}L_{\phi(g)} \cdot T_{\mathbf{1}}\phi \cdot \Xi(\mathbf{1},u) \quad \text{Lemma 2.3.4} \\ &= T_{\mathbf{1}}L_{\phi(g)} \cdot \psi \cdot \Xi(\mathbf{1},u) \\ &= T_{\mathbf{1}}L_{\phi(g)} \cdot \Xi'(\mathbf{1},\varphi(u)) \\ &= \Xi'(\phi(g), \varphi(u)). \end{aligned}$$

Hence Σ and Σ' are detached feedback equivalent.

2.3.8 LEMMA. *For a simply connected Lie group* G*, two systems* $\Sigma = (\mathsf{G}, \Xi)$ *and* $\Sigma' = (\mathsf{G}, \Xi')$ *are detached feedback equivalent if and only if* $\psi \cdot A \in \Gamma'$ *and* $\psi \cdot \Gamma^0 = \Gamma'^0$.

PROOF. Suppose Σ and Σ' are detached feedback equivalent then there exists $\psi \in \mathsf{Aut}(\mathfrak{g})$ such that $\psi \cdot \Gamma{=}\Gamma'$(refer to theorem 2.3.7) so

$$\begin{aligned} &\psi \cdot A \in \psi \cdot \Gamma = \Gamma' \\ &\text{i.e. } \psi \cdot A = A' + B' \quad \text{for some } B' \text{ in } \Gamma'^0. \end{aligned}$$

Also

$$\begin{aligned} &\psi \cdot \Gamma = \psi \cdot (A + \Gamma^0) = \psi \cdot A + \psi \cdot \Gamma^0 = A' + B' + \psi \cdot \Gamma^0 = \Gamma' = A' + \Gamma'^0 \\ &\implies B' + \psi \cdot \Gamma^0 = {\Gamma'}^0 \implies \psi \cdot \Gamma^0 = {\Gamma'}^0 \quad (\text{since } B' \in {\Gamma'}^0). \end{aligned}$$

Conversely, suppose $\psi \cdot A \in \Gamma'$ and $\psi \cdot \Gamma^0 = \Gamma'^0$. Then

$$\psi \cdot \Gamma = \psi \cdot (A + \Gamma^0) = \psi \cdot A + \psi \cdot \Gamma^0 = A' + B' + \Gamma'^0 = A' + \Gamma'^0 = \Gamma'.$$

The result follows from theorem 2.3.7.

2.3.9 PROPOSITION. *If* Σ *and* Σ' *are DF-equivalent the systems are both homogeneous or both inhomogeneous.*

PROOF. Suppose the traces of Σ and Σ' are given by Γ and Γ' respectively. Σ and Σ' being DF-equivalent implies that there exists a Lie algebra automorphism $\psi : \mathfrak{g} \to \mathfrak{g}$ such that $\psi \cdot \Gamma = \Gamma'$ and $\Gamma = \psi^{-1}\Gamma'$. Therefore if $0 \in \Gamma$, then $0 \in \Gamma'$. Also if $0 \in \Gamma'$, $0 \in \Gamma$.

The following proposition enables us to use the classification of inhomogeneous systems to obtain a classification of homogeneous systems under DF-equivalence.

2.3.10 PROPOSITION. *Let* Σ *be a full rank* $(\ell, 0)$ *control affine system with trace* $\Gamma = \langle B_1, \ldots, B_\ell \rangle$*. Suppose* $\{\Sigma_i^{(\ell-1,1)} | i \in I\}$ *is an exhaustive collection of equivalence representatives of DF-equivalent* $(\ell-1,1)$ *control affine systems with traces given by* $\Gamma_i = A^i + \left\langle B_1^i, \ldots, B_{\ell-1}^i \right\rangle$*. Then* Σ *is DF-equivalent to a system with trace* $\langle \Gamma_i \rangle$ *for at least one* $i \in I$*.*

Proof. Recall that two systems Σ and Σ' having traces Γ and Γ' respectively are DF-equivalent iff $\exists\psi \in \mathsf{Aut}(\mathfrak{g})$ s.t. $\psi \cdot \Gamma = \Gamma'$. Let $\Gamma = \langle A, B_1, \ldots, B_{\ell-1}\rangle$. Then $\{A, \{B_i\}_{i=\overline{1,\ell-1}}\}$ are linearly independent and $A + \langle B_1, \ldots, B_{\ell-1}\rangle$ is an $(\ell-1, 1)$ affine subspace of $\mathfrak{g}$. Now $\Gamma = \langle A + \langle B_1, \ldots, B_{\ell-1}\rangle\rangle$. Also since $\{\Sigma_i^{(\ell-1,1)} | i \in I\}$ is a complete list of equivalence representatives of DF-equivalent $(\ell-1, 1)$ systems there exists $\psi \in \mathsf{Aut}(\mathfrak{g})$ such that $\psi \cdot (A + \langle B_1, \ldots, B_{\ell-1}\rangle) = \Gamma_i$ for some $i \in I$. Therefore

$$\psi \cdot \Gamma = \psi \cdot \langle A + \langle B_1, \ldots, B_\ell - 1\rangle\rangle = \langle \psi \cdot (A + \langle B_1, \ldots, B_\ell - 1\rangle)\rangle = \langle \Gamma_i\rangle .$$

Hence Σ is DF-equivalent to a system with trace $\Gamma' = \langle \Gamma_i\rangle$ for some $i \in I$.

2.3.1 Classification under DF-equivalence

2.3.11 Proposition. *Any (1,1) system is DF-equivalent to the system*

$$\Sigma^{(1,1)} = E_2 + uE_3.$$

Proof. Such a system has trace $\Gamma = A + \Gamma^0 = \sum_{i=1}^3 a_iE_i + \left\langle \sum_{i=1}^3 b_iE_i \right\rangle$. From the Full rank condition we have that the set $\left\{\sum_{i=1}^3 a_iE_i, \sum_{i=1}^3 b_iE_i, \left[\sum_{i=1}^3 a_iE_i, \sum_{i=1}^3 b_iE_i\right]\right\}$ is linearly independent i.e. $\left\{\sum_{i=1}^3 a_iE_i, \sum_{i=1}^3 b_iE_i, (a_2b_3 - b_2a_3)E_1\right\}$ is linearly independent. From which it follows that

$$\begin{vmatrix} a_1 & b_1 & a_2b_3 - a_3b_2 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & 0 \end{vmatrix} = (a_2b_3 - a_3b_2)^2 \neq 0.$$

Hence

$$\psi_1 = \begin{bmatrix} -a_3b_2 + a_2b_3 & 0 & 0 \\ 0 & b_3 & -b_2 \\ 0 & -a_3 & a_2 \end{bmatrix}, \quad \left(\det(\psi_1) = (a_3b_2 - a_2b_3)^2 \neq 0\right)$$

is an automorphism such that
$\psi_1 \cdot \Gamma = a_1\left(a_2b_3 - a_3b_2\right)E_1 + (a_2b_3 - a_3b_2)E_2 + \langle b_1\left(a_2b_3 - a_3b_2\right)E_1 + (a_2b_3 - a_3b_2)E_3\rangle = a_1(a_2b_3 - a_3b_2)E_1 + (a_2b_3 - a_3b_2)E_2 + \langle b_1E_1 + E_3\rangle = \Gamma'$ and

$$\psi_2 = \begin{bmatrix} \frac{1}{-a_3b_2+a_2b_3} & \frac{a_1}{a_3b_2-a_2b_3} & \frac{b_1}{a_3b_2-a_2b_3} \\ 0 & \frac{1}{-a_3b_2+a_2b_3} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \det\left(\psi_2\right) = \frac{1}{(a_3b_2 - a_2b_3)^2} \neq 0$$

is an automorphism such that $\psi_2 \cdot \Gamma' = E_2 + \langle E_3\rangle$. The result follows from theorem 2.3.7.

2.3.12 Proposition. *Any* $(2,0)$ *system is DF-equivalent to the system*

$$\Sigma^{(2,0)} = u_1 E_2 + u_2 E_3.$$

Proof. Such a system has trace $\Gamma = \left\langle \sum_{i=1}^{3} a_i E_i, \sum_{i=1}^{3} b_i E_i \right\rangle$. From proposition 2.3.10 and proposition 2.3.11 we have that such a system is DF-equivalent to the system with trace $\Gamma = \langle E_2, E_3 \rangle$. The result follows.

The lemma which follows is used in determining the defining conditions for the $(2,1)$ systems.

2.3.13 Lemma. *Let* $\Gamma = \sum_{i=1}^{3} a_i E_i + \left\langle \sum_{i=1}^{3} b_i E_i, \sum_{i=1}^{3} c_i E_i \right\rangle$ *be the trace of a* $(2,1)$ *system then*

$$b_2 c_3 - b_3 c_2 = 0 \Leftrightarrow E_1 \in \Gamma^0.$$

Proof. Suppose $b_2 c_3 - b_3 c_2 = 0$. From linear independence b_1 and c_1 are not simultaneously equal to zero. Suppose $b_3 = 0$ which implies $b_2 = 0$ or $c_3 = 0$ and from linear independence the result follows. Similarly for $c_3 = 0$. Suppose $b_3, c_3 \neq 0$ then

$$\begin{aligned}
&\langle b_1 E_1 + b_2 E_2 + b_3 E_3, c_1 E_1 + c_2 E_2 + c_3 E_3 \rangle \\
&= \langle b_1 c_3 E_1 + b_2 c_3 E_2 + c_3 b_3 E_3, c_1 b_3 E_1 + b_3 c_2 E_2 + b_3 c_3 E_3 \rangle \\
&= \langle (b_1 c_3 - c_1 b_3) E_1, c_1 b_3 E_1 + b_3 c_2 E_2 + b_3 c_3 E_3 \rangle \\
\Longrightarrow\ & E_1 \in \Gamma^0 \text{ (from linear independence).}
\end{aligned}$$

The converse follows from linear independence.

2.3.14 Proposition. *Any (2,1) system is DF-equivalent to exactly one of the following systems*

$$\begin{aligned}
\Sigma_1^{(2,1)} &= E_1 + u_1 E_2 + u_2 E_3, && E_1 \notin \Gamma^0 \\
\Sigma_2^{(2,1)} &= E_2 + u_1 E_1 + u_2 E_3, && E_1 \in \Gamma^0.
\end{aligned}$$

Proof. Such a system has trace $\Gamma = \sum_{i=1}^{3} a_i E_i + \left\langle \sum_{i=1}^{3} b_i E_i, \sum_{i=1}^{3} c_i E_i \right\rangle$. We consider the 2 cases $E_1 \notin \Gamma^0$ and $E_1 \in \Gamma^0$.

1. $E_1 \notin \Gamma^0$. It follows from lemma 2.3.13 that $b_2 c_3 - b_3 c_2 \neq 0$. Hence

$$\psi_1 = \begin{bmatrix} b_3 c_2 - b_2 c_3 & b_1 c_3 & -b_1 c_2 \\ 0 & -c_3 & c_2 \\ 0 & -b_3 & b_2 \end{bmatrix}, \quad \big(\det(\psi_1) = (b_3 c_2 - b_2 c_3)^2 \neq 0\big)$$

is an automorphism such that
$\psi \cdot \Gamma = (b_1(a_2 c_3 - a_3 c_2) + a_1(b_3 c_2 - b_2 c_3)) E_1 + (a_3 c_2 - a_2 c_3) E_2 + (a_3 b_2 - a_2 b_3) E_3$

$+\left\langle (b_3c_2 - b_2c_3)E_2, c_1(b_3c_2 - b_2c_3)E_1 + (b_2c_3 - b_3c_2)E_3 \right\rangle = a_1'E_1 + \langle E_2, c_1E_1 + E_3\rangle = \Gamma'$, where $a_1' = b_1(a_2c_3 - a_3c_2) + a_1(b_3c_2 - b_2c_3) - c_1 \neq 0$ from linear independence. Therefore

$$\psi_2 = \begin{bmatrix} \frac{1}{a_1'} & 0 & -\frac{c_1}{a_1'} \\ 0 & \frac{1}{a_1'} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \left(\det(\psi_2) = \frac{1}{(a_1')^2} \neq 0\right)$$

is an automorphism such that $\psi_2 \cdot \Gamma' = E_1 + \left\langle \frac{1}{a_1'}E_2, E_3 \right\rangle = E_1 + \langle E_2, E_3\rangle$. The system is therefore DF-equivalent to the system $\Sigma_1^{(2,1)}$ (refer to theorem 2.3.7).

2. $E_1 \in \Gamma^0$. From lemma 2.3.13 $b_2c_3 - b_3c_2 = 0$. From linear independence, it follows that

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = c_1(a_2b_3 - a_3b_2) + b_1(a_3c_2 - a_2c_3) + a_1(b_2c_3 - b_3c_2)$$

$$= c_1(a_2b_3 - a_3b_2) + b_1(a_3c_2 - a_2c_3) \neq 0.$$

   (a) Suppose $a_2b_3 - a_3b_2 = 0 \implies a_3c_2 - a_2c_3 \neq 0$. Then

$$\psi_1 = \begin{bmatrix} -a_3c_2 + a_2c_3 & a_3c_1 & -a_2c_1 \\ 0 & -c_3 & c_2 \\ 0 & a_3 & -a_2 \end{bmatrix}, \quad \left(\det(\psi_1) = (a_3c_2 - a_2c_3)^2 \neq 0\right)$$

   is an automorphism such that
   $\psi_1 \cdot \Gamma = a_1(a_2c_3 - a_3c_2)E_1 + (a_3c_2 - a_2c_3)E_2 + \langle b_1(a_2c_3 - a_3c_2)E_1, (a_3c_2 - a_2c_3)E_3\rangle = (a_3c_2 - a_2c_3)E_2 + \langle E_1, E_3\rangle = \Gamma'$. Also

$$\psi_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{a_3c_2 - a_2c_3} & 0 \\ 0 & 0 & a_3c_2 - a_2c_3 \end{bmatrix}, \quad (\det(\psi_2) = 1 \neq 0)$$

   is an automorphism such that $\psi_2 \cdot \Gamma' = E_2 + \langle E_1, (a_3c_2 - a_2c_3)E_3\rangle = E_2 + \langle E_1, E_3\rangle$ and from theorem 2.3.7 the system is DF-equivalent to the system $\Sigma_2^{(2,1)}$.

   (b) Suppose $-a_3b_2 + a_2b_3 \neq 0$. Then

$$\psi_1 = \begin{bmatrix} a_3b_2 - a_2b_3 & -a_3b_1 & a_2b_1 \\ 0 & -b_3 & b_2 \\ 0 & -a_3 & a_2 \end{bmatrix}, \quad \left(\det(\psi_1) = (a_3b_2 - a_2b_3)^2 \neq 0\right)$$

is an automorphism such that $\psi_1 \cdot \Gamma = a_1(a_3b_2 - a_2b_3)E_1 + (a_3b_2 - a_2b_3)E_2$ $+ \langle (a_2b_3 - a_3b_2)E_3, (c_1(a_3b_2 - a_2b_3) + b_1(a_2c_3 - a_3c_2))E_1 + (a_2c_3 - a_3c_2)E_3\rangle$ $= (a_3b_2 - a_2b_3)E_2 + \langle E_3, E_1\rangle = \Gamma'$. And

$$\psi_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{a_3b_2-a_2b_3} & 0 \\ 0 & 0 & a_3b_2 - a_2b_3 \end{bmatrix}, \quad (\det(\psi_2) = 1 \neq 0)$$

is an automorphism such that $\psi_2 \cdot \Gamma' = E_2 + \langle (a_3b_2 - a_2b_3)E_3, E_1\rangle = E_2 + \langle E_1, E_3\rangle$ and the system is DF-equivalent to the system $\Sigma_2^{(2,1)}$ (theorem 2.3.7).

We have left to verify that our two systems are not equivalent. Recall from lemma 2.3.8 that two systems with trace $\Gamma = A + \Gamma^0$ and $\Gamma' = A' + \Gamma'^0$ are DF- equivalent if and only if there exists an Lie algebra automorphism ψ such that $\psi \cdot \Gamma^0 = \Gamma'^0$ and $\psi \cdot A \in \Gamma'$. The traces of systems $\Sigma_1^{(2,1)}$ and $\Sigma_2^{(2,1)}$ are given by $\Gamma_1 = A_1 + \Gamma_1^0 = E_1 + \langle E_2, E_3\rangle$ and $\Gamma_2 = A_2 + \Gamma_2^0 = E_2 + \langle E_1, E_3\rangle$ respectively. We apply an arbitrary automorphism

$$\psi = \begin{bmatrix} v_2w_3 - v_3w_2 & v_1 & w_1 \\ 0 & v_2 & w_2 \\ 0 & v_3 & w_3 \end{bmatrix}$$

to Γ_1^0 and set it equal to Γ_2^0 i.e. $\psi \cdot \Gamma_1^0 = \langle v_1E_1 + v_2E_2 + v_3E_3, w_1E_1 + w_2E_2 + w_3E_3\rangle = \Gamma_2^0 = \langle E_1, E_3\rangle$ which yields $v_2 = w_2 = 0$ and $\det\psi = 0$. The two systems are therefore not DF-equivalent.

2.3.15 Remark. Any (3,0) system is DF-equivalent to the system

$$\Sigma^{(3,0)} = u_1E_1 + u_2E_2 + u_3E_3.$$

2.3.16 Proposition. *Any full-rank left-invariant controllable control affine system on* H_3 *is DF-equivalent to exactly one of the following systems*

$$\Sigma^{(2,0)} = u_1E_2 + u_2E_3, \qquad \Sigma_1^{(2,1)} = E_1 + u_1E_2 + u_2E_3, \qquad \Sigma^{(3,0)} = u_1E_1 + u_2E_2 + u_3E_3.$$

Proof. A system Σ evolving on a completely solvable, connected and simply connected Lie group G is controllable if and only if $\mathsf{Lie}(\Gamma^0) = \mathfrak{g}$ (proposition A.2.3). The result follows from propositions 2.3.11, 2.3.12, 2.3.14, 2.3.15.

Table 2.2 summarises the results of propositions 2.3.11, 2.3.12, 2.3.14 and remark 2.3.15.

2.4 Strongly Detached Feedback Equivalence

2.4.1 DEFINITION. Let G be a connected Lie group and let $\Sigma = (\mathsf{G}, \Xi)$ and $\Sigma' = (\mathsf{G}, \Xi')$ be left invariant control affine systems with the same input space $\mathbb{R}^\ell$. Then Σ and Σ' are **strongly detached feedback equivalent** (or SDF-equivalent for short) if there exist diffeomorphisms $\phi : \mathsf{G} \to \mathsf{G}$ and $\varphi : \mathbb{R}^\ell \to \mathbb{R}^\ell$ such that φ is a linear map and that the diagram

$$\begin{tikzcd}
\mathsf{G} \times \mathbb{R}^\ell \arrow[r, "\phi \times \varphi"] \arrow[d, "\Xi"'] & \mathsf{G} \times \mathbb{R}^\ell \arrow[d, "\Xi'"] \\
T\mathsf{G} \arrow[r, "T\phi"'] & T\mathsf{G}'
\end{tikzcd}$$

commutes, i.e.,

$$T_g\phi \cdot \Xi(g,u) = \Xi'(\phi(g), \varphi(u)), \quad \forall g \in \mathsf{G}, u \in \mathbb{R}^\ell.$$

2.4.2 PROPOSITION. *SDF-equivalence is an equivalence relation.*

PROOF. Consider three systems $\Sigma = (\mathsf{G}, \Xi)$, $\Sigma' = (\mathsf{G}, \Xi')$ and $\Sigma'' = (\mathsf{G}, \Xi'')$.

1. SDF-equivalence is reflexive:
   Let $\phi = \mathrm{id}_{\mathsf{G}}$ and $\varphi = \mathrm{id}_{\mathbb{R}^\ell}$. Then $T_g\phi = \mathrm{id}_{\mathrm{T}_g\mathsf{G}}$. Hence

   $$T_g\phi \cdot \Xi(g,u) = \Xi(g,u) = \Xi(\phi(g), \varphi(u)).$$

   That is, Σ is SDF-equivalent to itself.

2. SDF-equivalence is symmetric:
   Suppose Σ is SDF-equivalent to Σ', then there exist diffeomorphisms $\phi : \mathsf{G} \to \mathsf{G}$ and $\varphi : \mathbb{R}^\ell \to \mathbb{R}^\ell$ where φ is a linear map such that

   $$T_g\phi \cdot \Xi(g,u) = \Xi'(\phi(g), \varphi(u)).$$

   Now $T_g\phi$ is a linear isomorphism and is therefore invertible so we have the linear isomorphism $(T_g\phi)^{-1}$ such that

   $$\begin{aligned}(T_g\phi)^{-1} \cdot T_g\phi \cdot \Xi(g,u) &= (T_g\phi)^{-1} \cdot \Xi(\phi(g), \varphi(u)) \\ \iff \Xi(g,u) &= (T_g\phi)^{-1} \cdot \Xi'(\phi(g), \varphi(u)).\end{aligned}$$

   Now $(T_g\phi)^{-1} = T_{\phi(g)}\phi^{-1}$ (lemma A.1.11) so we have

   $$T_{\phi(g)}\phi^{-1} \cdot \Xi'(\phi(g), \varphi(u)) = \Xi(g,u) = \Xi(\phi^{-1}(\phi(g)), \varphi^{-1}, \quad \forall \phi(g) \in \mathsf{G}, \phi(u) \in \mathbb{R}^\ell.$$

   Hence if Σ is SDF-equivalent to Σ' then Σ' is SDF-equivalent to Σ.

3. SDF-equivalence is transitive:
   Suppose Σ is SDF-equivalent to Σ' and that Σ' is SDF-equivalent to Σ''. Then there exist diffeomorphisms ϕ_1 and ϕ_2 and φ_1 and φ_2 where φ_1 and φ_2 are linear maps such that

   $$\begin{cases} T_g\phi_1 \cdot \Xi(g,u) = \Xi'(\phi_1(g), \varphi_1(u)) \\ T_g\phi_2 \cdot \Xi'(g,u) = \Xi''(\phi_2(g), \varphi_2(u)). \end{cases}$$

Let $\phi = \phi_2 \circ \phi_1$ and $\varphi = \varphi_2 \circ \varphi_1$. Then for every $g \in \mathsf{G}$ and $u \in \mathbb{R}^\ell$

$$\begin{aligned} T_g\phi \cdot \Xi(g,u) &= T_{\phi_1(g)}\phi_2 \cdot T_g\phi_1 \cdot \Xi(g,u) \\ &= T_{\phi_1(g)}\phi_2 \cdot \Xi'(\phi_1(g), \varphi_1(u)) \\ &= \Xi''(\phi_2 \circ \phi_1(g), \varphi_2 \circ \varphi_1(u)) \\ &= \Xi''(\phi(g), \varphi(u)). \end{aligned}$$

Hence if Σ is SDF-equivalent to Σ' and Σ' is SDF-equivalent to Σ'' then Σ is SDF-equivalent to Σ''.

SDF-equivalence is thus an equivalence relation.

2.4.3 Proposition. *The controlled trajectories of SDF-equivalent systems* $\Sigma = (\mathsf{G}, \Xi)$ *and* $\Sigma' = (\mathsf{G}, \Xi')$ *are in a one-to-one correspondence.*

Proof. Let $(g(\cdot), u(\cdot))$ be a controlled trajectory of Σ. Since Σ and Σ' are SDF-equivalent we have that there exist diffeomorphisms $\phi : \mathsf{G} \to \mathsf{G}$ and $\varphi : \mathbb{R}^\ell \to \mathbb{R}^\ell$ such that $T_g\phi \cdot \Xi(g,u) = \Xi'(\phi(g), \varphi(u))$ and $T_g\phi \cdot \Xi(g,0) = \Xi'(\phi(g), 0)$. We will show that $(\phi(g(\cdot)), \varphi(u(\cdot)))$ is the unique controlled trajectory of Σ' corresponding to $(g(\cdot), u(\cdot))$ of Σ. Firstly, for almost every t

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{dt}}\phi(g(t)) &= T_g\phi \cdot \dot{g}(t) \\ &= T_g\phi \cdot \Xi(g(t), u(t)) \\ &= \Xi'(\phi(g(t)), \varphi(u(t))). \end{aligned}$$

Hence $(\phi(g(\cdot)), \varphi(u(\cdot)))$ is a controlled trajectory of Σ'. We show that controlled trajectories are mapped both injectively and surjectively from Σ to Σ'. Suppose $\phi(g_1(\cdot)) = \phi(g_2(\cdot))$ and $\varphi(u_1(\cdot)) = \varphi(u_2(\cdot))$ where $(g_1(\cdot), u_1(\cdot))$ and $(g_2(\cdot), u_2(\cdot))$ are controlled trajectories of Σ then since ϕ and φ are diffeomorphisms and therefore invertible we have that $g_1(\cdot) = \phi^{-1}\phi(g_1(\cdot)) = \phi^{-1}\phi(g_2(\cdot)) = g_2(\cdot)$ and $u_1(\cdot) = \varphi^{-1}\varphi(u_1(\cdot)) = \varphi^{-1}\varphi(u_2(\cdot)) = u_2(\cdot)$. Next let $(g'(\cdot), u'(\cdot))$ be a controlled trajectory in Σ' then $(\phi^{-1}(g'(\cdot)), \varphi^{-1}u'(\cdot))$ is the corresponding controlled trajectory in Σ mapped to $(g'(\cdot), u'(\cdot))$ by $\phi \times \varphi$. Hence the controlled trajectories of Σ and Σ' are in a one-to-one correspondence.

2.4.4 Lemma. *Let* $\phi : \mathsf{G} \to \mathsf{G}$ *and* $\varphi : \mathbb{R}^\ell \to \mathbb{R}^\ell$ *be diffeomorphisms and let* $L_g : \mathsf{G} \to \mathsf{G}$, $h \mapsto gh$ *denote the left translation by* g. *Then*

1.

$$\begin{aligned} & T_g\phi \cdot \Xi(g,u) = \Xi'(\phi(g), \varphi(u)), \quad \forall g \in \mathsf{G}, u \in \mathbb{R}^\ell \\ \iff & T_g(L_{\phi(\mathbf{1})^{-1}} \circ \phi) \cdot \Xi(g,u) = \Xi'(L_{\phi(\mathbf{1})^{-1}} \circ \phi(g), \varphi(u)), \quad \forall g \in \mathsf{G}, u \in \mathbb{R}^\ell. \end{aligned}$$

2.

$$\begin{aligned} & T_g\phi \cdot \Xi(g,0) = \Xi'(\phi(g), 0), \quad \forall g \in \mathsf{G} \\ \iff & T_g(L_{\phi(\mathbf{1})^{-1}} \circ \phi) \cdot \Xi(g,0) = \Xi'(L_{\phi(\mathbf{1})^{-1}} \circ \phi(g), 0), \quad \forall g \in \mathsf{G}. \end{aligned}$$

Proof. 1. From theorem A.1.10 and left invariance:

$$\begin{aligned}
&T_g(L_{\phi(\mathbf{1})^{-1}} \circ \phi) \cdot \Xi(g,u) = \Xi'(L_{\phi(\mathbf{1})^{-1}} \circ \phi(g), \varphi(u)) \\
\iff\ & T_{\phi(g)} L_{\phi(\mathbf{1})^{-1}} \cdot T_g\phi \cdot \Xi(g,u) = T_{\phi(g)} L_{\phi(\mathbf{1})^{-1}} \cdot \Xi'(\phi(g), \varphi(u)) \\
\iff\ & T_g\phi \cdot \Xi(g,u) = \Xi'(\phi(g), \varphi(u)).
\end{aligned}$$

2.

$$\begin{aligned}
&T_g(L_{\phi(\mathbf{1})^{-1}} \circ \phi) \cdot \Xi(g,0) = \Xi'(L_{\phi(\mathbf{1})^{-1}} \circ \phi(g), 0) \\
\iff\ & T_{\phi(g)} L_{\phi(\mathbf{1})^{-1}} \cdot T_g\phi \cdot \Xi(g,0) = \Xi'(\phi(\mathbf{1})^{-1}\phi(g), 0) \quad \text{(theorem A.1.10)} \\
\iff\ & T_{\phi(g)} L_{\phi(\mathbf{1})^{-1}} \cdot T_g\phi \cdot \Xi(g,0) = T_{\phi(g)} L_{\phi(\mathbf{1})^{-1}} \cdot \Xi'(\phi(g), 0) \quad \text{(left invariance)} \\
\iff\ & T_g\phi \cdot \Xi(g,0) = \Xi'(\phi(g), 0).
\end{aligned}$$

2.4.5 Proposition. *If $\Sigma = (\mathsf{G}, \Xi)$ and $\Sigma' = (\mathsf{G}, \Xi')$ are SDF-equivalent then Σ is controllable if and only if Σ' is controllable.*

Proof. Without loss of generality, we may assume $\phi(\mathbf{1}) = \mathbf{1}$ (lemma 2.4.4). Let $\mathcal{A}$ and $\mathcal{A}'$ denote the attainable sets of Σ and Σ' respectively. Suppose Σ is controllable i.e. $\mathcal{A} = G$. We show that this implies $\mathcal{A}' = \mathsf{G}$ and hence that Σ' is controllable. Firstly,

$$\mathcal{A}' = \left\{ g'(T) \mid g'(\cdot) : [0,T] \to \mathsf{G} \text{ is a trajectory of } \Sigma',\ g'(0) = \mathbf{1} \right\}.$$

Now since Σ and Σ' are SDF-equivalent (with respect to diffeomorphisms ϕ and φ say) we have that the trajectories of Σ and Σ' are in one-to-one correspondence. That is for the trajectory $g'(\cdot) : [0,T] \to \mathsf{G}$, $g'(\cdot) = \phi(g(\cdot))$ for exactly one trajectory $g(\cdot) : [0,T] \to \mathsf{G}$. Hence

$$\begin{aligned}
\mathcal{A}' &= \{ \phi(g(T)) \mid g(\cdot) : [0,T] \to \mathsf{G} \text{ is a trajectory of } \Sigma,\ g(0) = \mathbf{1} \} \\
&= \phi(\mathcal{A}) = \mathsf{G}.
\end{aligned}$$

By interchanging the roles of Σ and Σ' in the above argument one is able to show that Σ is controllable if Σ' is controllable. The result follows.

2.4.6 Lemma. *Let $\phi : \mathsf{G} \to \mathsf{G}$ be a diffeomorphism and let $L_g : \mathsf{G} \to \mathsf{G}$, $h \mapsto gh$ denote the left translation by g. Then*

$$T\phi \circ TL_g = TL_{\phi(g)} \circ T\phi.$$

Proof.

$$\begin{aligned}
\phi(L_g(h)) &= \phi(gh) \\
&= \phi(g)\phi(h) \\
&= L_{\phi(g)}\phi(h) \\
\implies \phi \circ L_g &= L_{\phi(g)} \circ \phi \\
\implies T\phi \circ TL_g &= TL_{\phi(g)} \circ T\phi, \quad \text{(theorem A.1.10)}.
\end{aligned}$$

2.4.7 THEOREM. *For a simply connected Lie group* G, *two systems* $\Sigma = (\mathsf{G}, \Xi)$ *and* $\Sigma' = (\mathsf{G}, \Xi')$ *having traces* $\Gamma = A + \Gamma^0$ *and* $\Gamma' = A' + \Gamma^{0'}$ *respectively are SDF-equivalent if and only if* $\exists\ \psi \in \mathsf{Aut}(\mathfrak{g})$ *such that*

$$\psi \cdot \Gamma = \Gamma' \quad \textit{and} \quad \psi \cdot A = A'.$$

PROOF. Suppose Σ and Σ' are SDF-equivalent. Then there exist diffeomorphisms $\phi : \mathsf{G} \to \mathsf{G}$ and $\varphi : \mathbb{R}^\ell \to \mathbb{R}^\ell$ such that $T_g\phi \cdot \Xi(g, u) = \Xi'(\phi(g), \varphi(u))$ and $T_g\phi \cdot \Xi(g, 0) = \Xi'(\phi(g), 0)$ for every $g \in \mathsf{G}$ and $u \in \mathbb{R}^\ell$. So we have

$$\begin{aligned} T_{\mathbf{1}}\phi \cdot \Xi(\mathbf{1}, u) &= \Xi'(\phi(\mathbf{1}), \varphi(u)), \quad \forall\, u \in \mathbb{R}^\ell \\ \textit{and} \quad T_{\mathbf{1}}\phi \cdot \Xi(\mathbf{1}, 0) &= \Xi'(\phi(\mathbf{1}), 0). \end{aligned}$$

We may assume that $\phi(\mathbf{1}) = \mathbf{1}$ since if this is not the case we can replace ϕ with the diffeomorphism $L_{\phi(\mathbf{1})^{-1}} \circ \phi : \mathsf{G} \to \mathsf{G}$. Refer to lemma 2.4.4. Clearly $L_{\phi(\mathbf{1})^{-1}} \circ \phi(\mathbf{1}) = \mathbf{1}$. Therefore

$$\begin{aligned} T_{\mathbf{1}}\phi \cdot \Xi(\mathbf{1}, u) &= \Xi'(\mathbf{1}, \varphi(u)), \quad \forall\, u \in \mathbb{R}^\ell \\ \text{and} \quad T_{\mathbf{1}}\phi \cdot \Xi(\mathbf{1}, 0) &= \Xi'(\mathbf{1}, 0). \end{aligned}$$

Since $\Xi(\mathbf{1}, 0) = A$ and $\Xi'(\mathbf{1}, 0) = A'$, $T_{\mathbf{1}}\phi \cdot A = A'$. Next we show that $T_{\mathbf{1}}\phi \cdot \Gamma = \Gamma'$. Firstly, using the notation where the left invariant vector field $\Xi_u := \Xi(\cdot, u)$, $\Gamma = \{\Xi_u(\mathbf{1}) | u \in \mathbb{R}^\ell\}$ and for all $u \in \mathbb{R}^\ell$, $T_{\mathbf{1}}\phi \cdot \Xi_u(\mathbf{1}) = \Xi'_{\varphi(u)}(\mathbf{1}) \in \Gamma'$ and so $T_{\mathbf{1}}\phi \cdot \Gamma \subseteq \Gamma'$. Also, ϕ is a diffeomorphism. Hence by theorem A.1.9 we have that $T_{\mathbf{1}}\phi$ is a linear isomorphism. Therefore $\dim(T_{\mathbf{1}}\phi \cdot \Gamma) = \dim(\Gamma')$ and $T_{\mathbf{1}}\phi \cdot \Gamma = \Gamma'$.

It remains to be shown that $T_{\mathbf{1}}\phi$ is a Lie algebra automorphism. We already have that $T_{\mathbf{1}}\phi$ is a linear isomorphism $T_{\mathbf{1}}\phi : \mathfrak{g} \to \mathfrak{g}$ and is therefore a linear automorphism. Also, $T_{\mathbf{1}}\phi$ preserves the Lie bracket. Noting that the pushforward by ϕ preserves the Lie bracket of vector fields we have

$$\phi_*[\Xi_u, \Xi_v] = [\phi_*\Xi_u, \phi_*\Xi_v].$$

This, together with left invariance gives

$$\begin{aligned} &\phi_*[\Xi_u, \Xi_v](\phi(\mathbf{1})) = [\phi_*\Xi_u, \phi_*\Xi_v](\phi(\mathbf{1})) \\ \iff\ &\phi_*[\Xi_u, \Xi_v](\phi(\mathbf{1})) = [\phi_*\Xi_u(\phi(\mathbf{1})), \phi_*\Xi_v(\phi(\mathbf{1}))] \\ \iff\ &T_{\mathbf{1}}\phi \cdot [\Xi_u, \Xi_v](\mathbf{1}) = [T_{\mathbf{1}}\phi \cdot \Xi_u(\mathbf{1}), T_{\mathbf{1}}\phi \cdot \Xi_v(\mathbf{1})] \\ \iff\ &T_{\mathbf{1}}\phi \cdot [\Xi_u(\mathbf{1}), \Xi_v(\mathbf{1})] = [T_{\mathbf{1}}\phi \cdot \Xi_u(\mathbf{1}), T_{\mathbf{1}}\phi \cdot \Xi_v(\mathbf{1})]. \end{aligned}$$

Similarly $T_{\mathbf{1}}\phi \cdot [\Xi_u(\mathbf{1}), [\Xi_v(\mathbf{1}), \Xi_w(\mathbf{1})]] = [T_{\mathbf{1}}\phi \cdot \Xi_u(\mathbf{1}), [T_{\mathbf{1}}\phi \cdot \Xi_v(\mathbf{1}), T_{\mathbf{1}}\phi \cdot \Xi_w(\mathbf{1})]]$ and similarly for higher order commutators. Now $\Gamma = \{\Xi_u(\mathbf{1}) | u \in \mathbb{R}^\ell\}$ and $\mathsf{Lie}(\Gamma) = \mathfrak{g}$. The elements $\Xi_u(\mathbf{1})$, $u \in \mathbb{R}^\ell$ therefore generate $\mathfrak{g}$ and so $T_{\mathbf{1}}\phi$ is a Lie algebra automorphism.

Conversely, suppose ψ is a Lie algebra isomorphism such that $\psi \cdot \Gamma = \Gamma'$ and that $\psi \cdot A = A'$. Then there exists a Lie group automorphism $\phi : \mathsf{G} \to \mathsf{G}$ such that $T_{\mathbf{1}}\phi = \psi$ (cf. [23]). Let ξ' be the inverse of the map $\mathbb{R}^\ell \to \Gamma^{0'}, u \mapsto \Xi'(1, u) - A'$. Let $\varphi : \mathbb{R}^\ell \to \mathbb{R}^\ell$ be the linear map defined by $\varphi(u) = \xi'(T_{\mathbf{1}}\phi \cdot (\Xi(\mathbf{1}, u) - A))$. Then $T_{\mathbf{1}}\phi \cdot \Xi(\mathbf{1}, u) - T_{\mathbf{1}}\phi \cdot A = (\xi')^{-1}(\varphi(u)) = \Xi'(\mathbf{1}, \varphi(u)) - A'$ (or

$T_{\mathbf{1}}\phi \cdot \Xi(\mathbf{1}, u) = \Xi'(\mathbf{1}, \varphi(u)))$ for every $u \in \mathbb{R}^\ell$. Therefore we have

$$\begin{aligned} T_g\phi \cdot \Xi(g, u) &= T_g\phi \cdot g\Xi(\mathbf{1}, u) \\ &= T_g\phi \cdot T_{\mathbf{1}}L_g \cdot \Xi(\mathbf{1}, u) \\ &= T_{\mathbf{1}}L_{\phi(g)} \cdot T_{\mathbf{1}}\phi \cdot \Xi(\mathbf{1}, u) \quad \text{lemma 2.4.6} \\ &= T_{\mathbf{1}}L_{\phi(g)} \cdot \Xi'(\mathbf{1}, \varphi(u)) \\ &= \Xi'(\phi(g), \varphi(u)). \end{aligned}$$

Also, since φ is linear

$$T_g\phi \cdot \Xi(g, 0) = \Xi'(\phi(g), \varphi(0)) = \Xi'(\phi(g), 0)$$

Hence Σ and Σ' are SDF-equivalent.

2.4.8 Proposition. *For a simply connected Lie group, two systems* $\Sigma = (\mathsf{G}, \Xi)$ *and* $\Sigma' = (\mathsf{G}, \Xi')$ *are SDF-equivalent if and only if*

$$\begin{cases} \psi(A) = A' \\ \psi \cdot \langle B_1, \ldots, B_\ell \rangle = \langle B_1', \ldots, B_\ell' \rangle . \end{cases}$$

Proof. Suppose Σ and Σ' are SDF-equivalent then there exists $\psi \in \mathsf{Aut}(\mathfrak{g})$ such that $\psi \cdot \Gamma = \Gamma'$ and $\psi \cdot A = A'$(proposition 2.4.7) so

$$\begin{aligned} \psi \cdot \Gamma &= \psi(A + \Gamma^0) \\ \iff \Gamma' &= \psi(A) + \psi(\Gamma^0) \\ \iff A' + \Gamma'^0 &= A' + \psi(\Gamma^0) \\ \implies \psi(\Gamma^0) &= \Gamma'^0 \\ \iff \psi \cdot \langle B_1, \ldots, B_\ell \rangle &= \left\langle B_1', \ldots, B_\ell' \right\rangle . \end{aligned}$$

Conversely, suppose $\psi \cdot A = A'$ and $\psi \cdot \Gamma^0 = \Gamma'^0$ then

$$\begin{aligned} \psi(\Gamma) &= \psi(A + \Gamma^0) \\ &= \psi(A) + \psi(\Gamma^0) \\ &= A' + \Gamma'^0 \\ &= \Gamma'. \end{aligned}$$

The result follows from theorem 2.4.7.

2.4.9 Proposition. *If* Σ *and* Σ' *are SDF-equivalent the systems are both homogeneous or both inhomogeneous.*

Proof. This follows immediately from the fact that $\psi \cdot A = A'$.

2.4.1 Classification under SDF-equivalence

When convenient, an element $A = \sum_{i=1}^{3} a_i E_i$ of $\mathfrak{h}_3$ will be represented as a column vector, i.e., $A = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$. In doing so the evaluation of $\psi \cdot A$ becomes a matrix multiplication. The lemmas which follow assist in the classification of full-rank left-invariant control affine systems on H_3.

2.4.10 Lemma. *2 vectors* $A = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ *and* $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \in \mathfrak{h}_3$ *generate* $\mathfrak{h}_3$ *if and only if* $a_2 b_3 - a_3 b_2 \neq 0$.

Proof. $\mathfrak{h}_3$ is 3 dimensional and so in order for two vectors A and B to generate the entire Lie algebra $\mathfrak{h}_3$, A, B and $[A, B]$ must be linearly independent. Now

$$[A, B] = \begin{bmatrix} a_2 b_3 - b_2 a_3 \\ 0 \\ 0 \end{bmatrix}.$$

Hence vectors A and B generate $\mathfrak{h}_3$ if and only if

$$\begin{vmatrix} a_1 & b_1 & a_2 b_3 - b_2 a_3 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & 0 \end{vmatrix} = (a_2 b_3 - a_3 b_2)^2 \neq 0.$$

2.4.11 Lemma. *For any two vectors* $A = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ *and* $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \in \mathfrak{h}_3$ *which generate* $\mathfrak{h}_3$ *there exists* $\psi \in \mathsf{Aut}(\mathfrak{h}_3)$ *such that*

$$\psi \cdot A = E_2 \quad \textit{and} \quad \psi \cdot B = E_3.$$

Proof. From lemma 2.4.10 $a_2 b_3 - b_2 a_3 \neq 0$. Consequently

$$\psi_1 = \begin{bmatrix} -a_3 b_2 + a_2 b_3 & 0 & 0 \\ 0 & b_3 & -b_2 \\ 0 & -a_3 & a_2 \end{bmatrix} \quad \big(\det \psi_1 = (a_3 b_2 - a_2 b_3)^2 \neq 0\big)$$

is an automorphism such that

$$\psi_1 \cdot \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} -a_1a_3b_2 + a_1a_2b_3 \\ -a_3b_2 + a_2b_3 \\ 0 \end{bmatrix} \quad \text{and} \quad \psi \cdot \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} -a_3b_1b_2 + a_2b_1b_3 \\ 0 \\ -a_3b_2 + a_2b_3 \end{bmatrix}.$$

Also

$$\psi_2 = \begin{bmatrix} \frac{1}{(a_3b_2-a_2b_3)^2} & \frac{-a_1}{(a_3b_2-a_2b_3)^2} & \frac{-b_1}{(a_3b_2-a_2b_3)^2} \\ 0 & \frac{1}{-a_3b_2+a_2b_3} & 0 \\ 0 & 0 & \frac{1}{-a_3b_2+a_2b_3} \end{bmatrix} \qquad \left(\det \psi_2 = \frac{1}{(a_3b_2 - a_2b_3)^4} \neq 0\right)$$

is an automorphism such that

$$\psi_2 \cdot \begin{bmatrix} -a_1a_3b_2 + a_1a_2b_3 \\ -a_3b_2 + a_2b_3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \psi_2 \cdot \begin{bmatrix} -a_3b_1b_2 + a_2b_1b_3 \\ 0 \\ -a_3b_2 + a_2b_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

2.4.12 Lemma. *If two vectors* $A = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ *and* $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ *do not generate* $\mathfrak{h}_3$ *then there images* $\psi \cdot A$ *and* $\psi \cdot B$ *do not generate* $\mathfrak{h}_3$*.*

Proof. Let ψ be an arbitrary element of $\mathsf{Aut}(\mathfrak{h}_3)$. Then

$$\psi \cdot A = \begin{bmatrix} v_2w_3 - v_3w_2 & v_1 & w_1 \\ 0 & v_2 & w_2 \\ 0 & v_3 & w_3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a_2v_1 + a_3w_1 - a_1v_3w_2 + a_1v_2w_3 \\ a_2v_2 + a_3w_2 \\ a_2v_3 + a_3w_3 \end{bmatrix}.$$

Also

$$\psi \cdot B = \begin{bmatrix} v_2w_3 - v_3w_2 & v_1 & w_1 \\ 0 & v_2 & w_2 \\ 0 & v_3 & w_3 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} b_2v_1 + b_3w_1 - b_1v_3w_2 + b_1v_3w_2 + b_1v_2w_3 \\ b_2v_2 + b_3w_2 \\ b_2v_3 + b_3w_3 \end{bmatrix}.$$

Since A and B do not generate $\mathfrak{h}_3$ we have that $a_2b_3 - b_2a_3 = 0$ (lemma 2.4.10). Using this fact we show that $(a_2v_2 + a_3w_2)(b_2v_3 + b_3w_3) - (b_2v_2 + b_3w_2)(a_2v_3 + a_3w_3) = 0$ and hence that $\psi \cdot A$ and

$\psi \cdot B$ do not generate $\mathfrak{h}_3$.

$$\begin{aligned}&(a_2v_2 + a_3w_2)(b_2v_3 + b_3w_3) - (b_2v_2 + b_3w_2)(a_2v_3 + a_3w_3)\\ =&(a_2b_2v_2v_3 + a_2b_3v_2w_3 + a_3b_2w_2v_3 + a_3b_3w_2w_3) - (b_2a_2v_2v_3 + b_2a_3v_2w_3 + b_3a_2w_2v_3 + b_3a_3w_2w_3)\\ =&(a_2b_3 - b_2a_3)v_2w_3 + (a_3b_2 - b_3a_2)w_2v_3\\ =&0.\end{aligned}$$

2.4.13 Proposition. *Any (1,1) system is strongly detached feedback equivalent to the following system*

$$\Sigma^{(1,1)} : E_2 + uE_3. \tag{2.8}$$

Proof. The system must satisfy the full rank condition. The result therefore follows immediately from lemma 2.4.11.

2.4.14 Proposition. *Any* $(2,0)$ *system is strongly detached feedback equivalent to exactly one of the following systems*

$$\Sigma_1^{(2,0)} : E_2 + u_1E_2 + u_2E_3 \tag{2.9}$$

$$\Sigma_2^{(2,0)} : u_1E_2 + u_2E_3. \tag{2.10}$$

Proof. The trace of such a system is given by $\Gamma = \sum_{i=1}^3 a_iE_i + \left\langle \sum_{i=1}^3 b_iE_i, \sum_{i=1}^3 c_iE_i \right\rangle$. Now since the system is full rank vectors $\sum_{i=1}^3 b_iE_i$ and $\sum_{i=1}^3 c_iE_i$ must generate $\mathfrak{h}_3$ so from lemma 2.4.11 we have that there exist $\psi \in \mathsf{Aut}(\mathfrak{h}_3)$ such that $\psi \cdot \Gamma = \sum_{i=1}^3 a_i'E_i + \langle E_2, E_3\rangle$ for some $a_1', a_2', a_3' \in \mathbb{R}$. From linear dependence we have that

$$\begin{vmatrix} a_1' & 0 & 0\\ a_2' & 1 & 0\\ a_3' & 0 & 1\end{vmatrix} = 0 \iff a_1' = 0.$$

So we have that the system is strongly detached feedback equivalent to

$$\Sigma : a_2'E_2 + a_3'E_3 + u_1E_2 + u_2E_3$$

which is a system with trace $\Gamma' = a_2'E_2 + a_3'E_3 + \langle E_2, E_3\rangle$. At this point we split the case into 3 cases namely

1. Case 1: $a_2' \neq 0$. Then

$$\psi = \begin{bmatrix} 1 & 0 & 0\\ 0 & \frac{1}{a_2'} & 0\\ 0 & -a_3' & a_2'\end{bmatrix} \qquad (\det \psi = 1 \neq 0)$$

   is an automorphism such that $\psi\cdot(a_2'E_2 + a_3'E_3) = E_2$ and $\psi\cdot\langle E_2, E_3\rangle = \left\langle \frac{1}{a_2'}E_2 - a_3'E_3, a_2'E_3 \right\rangle = \langle E_2, E_3\rangle$. Hence from proposition 2.4.8 the system is SDF-equivalent to $\Sigma_1^{2,0}$.

2. Case 2: $a_2' = 0, a_3' \neq 0$. Then

$$\psi = \begin{bmatrix} -\frac{1}{a_3'} & 0 & 0 \\ 0 & 0 & \frac{1}{a_3'} \\ 0 & 1 & 0 \end{bmatrix} \qquad (\det \psi = \frac{1}{\mathrm{a}_3'^2} \neq 0)$$

   is an automorphism such that $\psi \cdot a_3' E_3 = E_2$ and $\psi \cdot \langle E_2, E_3 \rangle = \left\langle E_3, \frac{1}{a_3'} E_2 \right\rangle = \langle E_2, E_3 \rangle$ and the system is SDF-equivalent to $\Sigma_1^{(2,0)}$ (proposition 2.4.8).

3. Case 3: $a_2' = a_3' = 0$. In this case we have that the system is SDF-equivalent to $\Sigma_2^{(2,0)}$.

We verify that $\Sigma_1^{(2,0)}$ is not SDF-equivalent to $\Sigma_2^{(2,0)}$. Suppose there exists $\psi \in \mathsf{Aut}(\mathfrak{h}_3)$ such that $\psi \cdot E_2 = 0$ and $\psi \cdot \langle E_2, E_3 \rangle = \langle E_2, E_3 \rangle$. Now, since ψ is a linear map $\psi \cdot E_2 \neq 0$, a contradiction.

2.4.15 **Proposition.** *Any (2,1) system is strongly detached feedback equivalent to exactly one of the following systems*

$$\begin{aligned} \Sigma_1^{(2,1)} &: E_1 + E_2 + u_1 E_2 + u_2 E_3 \\ \Sigma_2^{(2,1)} &: E_1 + u_1 E_2 + u_2 E_3 \\ \Sigma_3^{(2,1)} &: E_2 + u_1 E_3 + u_2 E_1. \end{aligned}$$

Proof. Such a system has trace given by $\Gamma = \sum_{i=1}^3 a_i E_i + \left\langle \sum_{i=1}^3 b_i E_i, \sum_{i=1}^3 c_i E_i \right\rangle$, where the set $\left\{ \sum_{i=1}^3 a_i E_i, \sum_{i=1}^3 b_i E_i, \sum_{i=1}^3 c_i E_i \right\}$ are linearly independent. We divide this case up into 4 cases:

1. Columns 2 and 3 generate $\mathfrak{h}_3$ in which case $-b_3 c_2 + b_2 c_3 \neq 0$.
2. Columns 1 and 2 generate $\mathfrak{h}_3$ (Columns 2 and 3 do not) in which case $-b_3 c_2 + b_2 c_3 = 0$, $-a_3 b_2 + a_2 b_3 \neq 0$.
3. Columns 1 and 3 generate $\mathfrak{h}_3$ (Columns 2 and 3 as well as 1 and 2 do not) in which case $-b_3 c_2 + b_2 c_3 = 0$, $-a_3 b_2 + a_2 b_3 = 0$, $-a_3 c_2 + a_2 c_3 \neq 0$.
4. Columns 2 and 3, 1 and 2 and 1 and 3 do not generate $\mathfrak{h}_3$ in which case $-b_3 c_2 + b_2 c_3 = 0$, $-a_3 b_2 + a_2 b_3 = 0$, $-a_3 c_2 + a_2 c_3 = 0$.

Each case is covered below.

1. $(-b_3 c_2 + b_2 c_3 \neq 0)$ columns 2 and 3 generate $\mathfrak{h}_3$.
   From lemma 2.4.11 we have that this system is SDF-equivalent to the system with trace $\Gamma' = \sum_{i=1}^3 a_i' E_i + \langle E_2, E_3 \rangle$. From linear independence we have that

$$\left| \begin{array}{c|cc} a_1' & 0 & 0 \\ a_2' & 1 & 0 \\ a_3' & 0 & 1 \end{array} \right| = a_1' \neq 0.$$

So

$$\psi = \begin{bmatrix} \frac{1}{a_1'} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{a_1'} \end{bmatrix} \qquad (\det \psi = \frac{1}{(a_1')^2} \neq 0)$$

is an automorphism such that $\psi \cdot (a_1'E_1 + a_2'E_2 + a_3'E_3) = E_1 + a_2'E_2 + \frac{a_3'}{a_1'}E_3 = E_1 + \gamma_1 E_2 + \gamma_2 E_3$ and $\psi \cdot \langle E_2, E_3 \rangle = \left\langle E_2, \frac{1}{a_1'}E_3 \right\rangle = \langle E_2, E_3 \rangle$ for $\gamma_1 = a_2', \gamma_2 = \frac{a_3'}{a_1'}$. We further divide this case into 3 cases.

(a) $\gamma_2 \neq 0$

$$\psi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\gamma_2} \\ 0 & -\gamma_2 & \gamma_1 \end{bmatrix} \qquad (\det \psi = 1 \neq 0)$$

is an automorphism such that $\psi \cdot (E_1 + \gamma_1 E_2 + \gamma_2 E_3) = E_1 + E_2$ and $\psi \cdot \langle E_2, E_3 \rangle = \left\langle -\gamma_2 E_3, \frac{1}{\gamma_2}E_2 + \gamma_1 E_3 \right\rangle = \langle E_2, E_3 \rangle$ and the system is SDF-equivalent to $\Sigma_1^{(2,1)}$ by proposition 2.4.8.

(b) $\gamma_2 = 0$, $\gamma_1 \neq 0$

$$\psi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\gamma_1} & 0 \\ 0 & 0 & \gamma_1 \end{bmatrix} \qquad (\det \psi = 1 \neq 0)$$

is an automorphism such that $\psi \cdot (E_1 + \gamma_1 E_2) = E_1 + E_2$ and $\psi \langle E_2, E_3 \rangle = \left\langle \frac{1}{\gamma_1}E_2, \gamma_1 E_3 \right\rangle = \langle E_2, E_3 \rangle$ and the system is SDF-equivalent to $\Sigma_1^{(2,1)}$ (proposition 2.4.8).

(c) $\gamma_1 = \gamma_2 = 0$. In this case we have that the system is SDF-equivalent to $\Sigma_2^{(2,1)}$.

2. $(-b_3c_2 + b_2c_3 = 0, -a_3b_2 + a_2b_3 \neq 0)$ columns 1 and 2 generate $\mathfrak{h}_3$.
From lemma 2.4.11 we have that such a system is SDF-equivalent to the system with trace $\Gamma' = E_2 + \left\langle E_3, \sum_{i=1}^{3} c_i'E_i \right\rangle$. From linear independence we have that

$$\begin{vmatrix} 0 & 0 & c_1' \\ 1 & 0 & c_2' \\ 0 & 1 & c_3' \end{vmatrix} = c_1' \neq 0.$$

Also from lemma 2.4.12 we have that columns 2 and 3 do not generate $\mathfrak{h}_3$, hence $c_2' = 0$. The system is therefore SDF-equivalent to the system with trace $\Gamma'' = E_2 + \langle E_3, c_1'E_1 + c_3'E_3 \rangle = E_2 + \langle E_1, E_3 \rangle$. That is the system is SDF-equivalent to $\sum_3^{(2,1)}$.

3. $(-b_3c_2 + b_2c_3 = 0, -a_3b_2 + a_2b_3 = 0, -a_3c_2 + a_2c_3 \neq 0)$ columns 1 and 3 generate $\mathfrak{h}_3$.
From lemma 2.4.11 we have that such a system is SDF-equivalent to the system with trace $\Gamma' = E_2 + \left\langle \sum_{i=1}^{3} b_i E_i, E_3 \right\rangle$. From linear independence we have that

$$\begin{vmatrix} 0 & b_1' & 0 \\ 1 & b_2' & 0 \\ 0 & b_3' & 1 \end{vmatrix} = -b_1' \neq 0.$$

Also from lemma 2.4.12 we have that columns 2 and 3 do not generate $\mathfrak{h}_3$ therefore $b_2' = 0$. The system is therefore SDF-equivalent to the system with trace $\Gamma'' = E_2 + \langle b_1' E_1 + b_3' E_3, E_3 \rangle = E_2 + \langle E_1, E_3 \rangle$. That is the system is SDF-equivalent to $\Sigma_3^{(2,1)}$.

4. $(-b_3c_2 + b_2c_3 = 0, -a_3b_2 + a_2b_3 = 0, -a_3c_2 + a_2c_3 = 0)$ no 2 columns generate $\mathfrak{h}_3$.
Now

$$\begin{aligned} \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} &= -a_3b_2c_1 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a_2b_1c_3 + a_1b_2c_3 \\ &= c_1(-a_3b_2 + a_2b_3) + b_1(a_3c_2 - a_2c_3) + a_1(-b_3c_2 + b_2c_3) \\ &= 0. \end{aligned}$$

Therefore the columns are not linearly independent and this is not a $(2,1)$ system.

We have left to verify that the systems are not equivalent. Let

$$\psi = \begin{bmatrix} v_2w_3 - v_3w_2 & v_1 & w_1 \\ 0 & v_2 & w_2 \\ 0 & v_3 & w_3 \end{bmatrix} \in \mathsf{Aut}(\mathfrak{h}_3), \text{ arbitrary.}$$

Suppose $\Sigma_1^{(2,1)}$ and $\Sigma_2^{(2,1)}$ are SDF-equivalent. Then $\psi \cdot (E_1 + E_2) = E_1$ and $\psi \cdot \langle E_2, E_3 \rangle \in \langle E_2, E_3 \rangle$. Now $\psi \cdot (E_1 + E_2) = (-w_2v_3 + v_2w_3 + v_1)E_1 + v_2E_2 + v_3E_3$. Therefore $v_2 = v_3 = 0$ and $\det \psi = 0$ a contradiction. Suppose $\Sigma_1^{(2,1)}$ and $\Sigma_3^{(2,1)}$ are SDF-equivalent. Then $\psi \cdot (E_1 + E_2) = E_2$ and $\psi \cdot \langle E_2, E_3 \rangle \in \langle E_3, E_1 \rangle$. Now $\psi \cdot \langle E_2, E_3 \rangle = \langle v_1E_1 + v_2E_2 + v_3E_3, w_1E_1 + w_2E_2 + w_3E_3 \rangle$. Therefore $v_2 = w_2 = 0$ and $\det \psi = 0$ a contradiction. Finally suppose $\Sigma_2^{(2,1)}$ and $\Sigma_3^{(2,1)}$ are SDF-equivalent. Then $\psi \cdot E_1 = E_2$ and $\psi \cdot \langle E_2, E_3 \rangle \in \langle E_3, E_1 \rangle$. Now $\psi \cdot E_1 = (v_2w_3 - v_3w_2)E_1$ which cannot equal E_2, a contradiction. Our 3 systems are therefore distinct.

2.4.16 Proposition. *Any* $(3,0)$ *system is strongly detached feedback equivalent to exactly one of the following systems*

$$\begin{aligned}
\Sigma_1^{(3,0)} &: E_2 + u_1E_1 + u_2E_2 + u_3E_3\\
\Sigma_2^{(3,0)} &: E_1 + u_1E_1 + u_2E_2 + u_3E_3\\
\Sigma_3^{(3,0)} &: u_1E_1 + u_2E_2 + u_3E_3.
\end{aligned}$$

Proof. Such a system is has trace given by $\Gamma = \sum_{i=1}^{3} a_iE_i + \left\langle \sum_{i=1}^{3} b_iE_i, \sum_{i=1}^{3} c_iE_i, \sum_{i=1}^{3} d_iE_i \right\rangle$. From linear independence we have that the system is SDF-equivalent to the system with trace $\Gamma' = \sum_{i=1}^{3} a_i'E_i + \langle E_1, E_2, E_3\rangle$. We use proposition 2.4.8 in carrying out the classification and consider the four cases below.

1. $a_2' \neq 0$. Then

$$\psi = \begin{bmatrix} 1 & -\frac{a_1'}{a_2'} & 0 \\ 0 & \frac{1}{a_2'} & 0 \\ 0 & -a_3' & a_2' \end{bmatrix} \qquad (\det\psi = 1 \neq 0)$$

   is an automorphism such that $\psi\cdot(a_1'E_1 + a_2'E_2 + a_3'E_3) = E_2$ and $\psi\cdot\langle E_1, E_2, E_3, \rangle = \left\langle E_1, -\frac{a_1'}{a_2'}E_1 + \frac{1}{a_2'}E_2 - a_3'E_3, a_2'E_3 \right\rangle = \langle E_1, E_2, E_3\rangle$. The system is therefore SDF-equivalent to $\Sigma_1^{(3,0)}$.

2. $a_2' = 0$, $a_3' \neq 0$. Then

$$\psi = \begin{bmatrix} -1 & 0 & \frac{a_1'}{a_3'} \\ 0 & 0 & \frac{1}{a_3'} \\ 0 & a_3' & 0 \end{bmatrix} \qquad (\det\psi = 1 \neq 0)$$

   is an automorphism such that $\psi\cdot(a_1'E_1 + a_3'E_3) = E_2$ and $\psi\cdot\langle E_1, E_2, E_3\rangle = \left\langle E_1, a_3'E_3, \frac{a_1'}{a_3'}E_1 + \frac{1}{a_3'}E_2 \right\rangle = \langle E_1, E_2, E_3\rangle$. Therefore the system is SDF-equivalent to $\Sigma_1^{(3,0)}$.

3. $a_2' = a_3' = 0$, $a_1' \neq 0$. Then

$$\psi = \begin{bmatrix} \frac{1}{a_1'} & 0 & 0 \\ 0 & \frac{1}{a_1'} & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \left(\det\psi = \frac{1}{a_1'^2} \neq 0\right)$$

   is an automorphism such that $\psi\cdot(a_1'E_1) = E_1$ and $\psi\cdot\langle E_1, E_2, E_3\rangle = \left\langle \frac{1}{a_1'}E_1, \frac{1}{a_1'}E_2, E_3 \right\rangle = \langle E_1, E_2, E_3\rangle$. Therefore the system is SDF-equivalent to $\Sigma_2^{(3,0)}$.

4. $a'_1 = a'_2 = a'_3 = 0$. The system is SDF-equivalent to $\Sigma_3^{(3,0)}$.

Finally, we verify that the systems $\Sigma_1^{(3,0)}$, $\Sigma_2^{(3,0)}$, $\Sigma_3^{(3,0)}$ are not equivalent. Suppose $\Sigma_2^{(3,0)}$ is SDF-equivalent to $\Sigma_1^{(3,0)}$. Then there exists $\psi \in \mathsf{Aut}(\mathfrak{h}_3)$ such that $\psi \cdot E_1 = (v_2w_3 - w_2v_3)E_1 = E_2$ which cannot happen. A contradiction. Also since ψ is linear $\Sigma_3^{(3,0)}$ cannot be SDF-equivalent to either $\Sigma_1^{(3,0)}$ or $\Sigma_2^{(3,0)}$. The three systems are therefore distinct.

Table 2.3 summarises the results of propositions 2.4.13, 2.4.14, 2.4.15 and 2.4.16.

2.5 Full Classification

2.5.1 Proposition. *1. Any* $(1,1)$ *system which is DF-equivalent to the system*

$$E_2 + u_1E_3$$

is SDF-equivalent to the system

$$E_2 + u_1E_3.$$

2. Any $(2,0)$ *system which is DF-equivalent to the system*

$$u_1E_2 + u_2E_3$$

is SDF-equivalent to exactly one of the following systems

$$E_2 + u_1E_1 + u_2E_3 \quad \textit{or} \quad u_1E_2 + u_2E_3.$$

3. Any $(2,1)$ *system which is DF-equivalent to the system*

$$E_1 + u_1E_2 + u_2E_3$$

is SDF-equivalent to exactly one of the following systems

$$E_1 + E_2 + u_1E_2 + u_2E_3 \quad \textit{or} \quad E_1 + u_1E_2 + u_2E_3.$$

Any $(2,1)$ *system that is DF-equivalent to the system*

$$E_2 + u_1E_1 + u_2E_3$$

is SDF-equivalent to the system

$$E_2 + u_1E_1 + u_2E_3.$$

4. Any $(3,0)$ *system that is DF-equivalent to the system*

$$u_1E_1 + u_2E_2 + u_3E_3$$

is SDF-equivalent to exactly one of the following systems

$$\begin{aligned} &E_2 + u_1E_1 + u_2E_2 + u_3E_3 && \textit{or}\\ &E_1 + u_1E_1 + u_2E_2 + u_3E_3 && \textit{or}\\ &u_1E_1 + u_2E_2 + u_3E_3. \end{aligned}$$

PROOF. We use proposition 2.4.8 in the classification.

1. Since the system is DF-equivalent to the system $E_2 + u_1 E_3$ and of full-rank we have from proposition 2.4.11 that the system is SDF-equivalent to the system $E_2 + u_1 E_3$.

2. Since the system is DF-equivalent to the system $u_1 E_2 + u_2 E_3$ and a (2,0) system we immediately have that the system is SDF-equivalent to the system with trace $\Gamma = a_2 E_2 + a_3 E_3 + \langle b_2 E_2 + b_3 E_3, c_2 E_2 + c_3 E_3 \rangle$. The system is full-rank therefore columns 2 and 3 generate $\mathfrak{h}_3$ and so from lemma 2.4.11 we have that the system is SDF-equivalent to the system with trace $\Gamma' = \gamma_1 E_2 + \gamma_2 E_3 + \langle E_2, E_3 \rangle$. At this point we split the case into three cases.

   (a) Case 1: $\gamma_2 \neq 0$. Then

   $$\psi = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & \frac{1}{\gamma_2} \\ 0 & \gamma_2 & -\gamma_1 \end{bmatrix} \qquad (\det \psi = 1 \neq 0)$$

   is an automorphism such that $\psi \cdot (\gamma_1 E_2 + \gamma_2 E_3) = E_2$ and $\psi \cdot \langle E_2, E_3 \rangle = \left\langle \gamma_2 E_3, \frac{1}{\gamma_2} E_2 + -\gamma_1 E_3 \right\rangle = \langle E_2, E_3 \rangle$. The system is therefore SDF-equivalent to the system $E_2 + u_1 E_1 + u_2 E_3$.

   (b) Case 2: $\gamma_2 = 0, \gamma_1 \neq 0$. Then

   $$\psi = \begin{bmatrix} \frac{1}{\gamma_1} & 0 & 0 \\ 0 & \frac{1}{\gamma_1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \left(\det \psi = \frac{1}{\gamma_1^2} \neq 0\right)$$

   is an automorphism such that $\psi \cdot \gamma_1 E_2 = E_2$ and $\psi \cdot \langle E_2, E_3 \rangle = \left\langle \frac{1}{\gamma_1} E_2, E_3 \right\rangle = \langle E_2, E_3 \rangle$ and the system is SDF-equivalent to $E_2 + u_1 E_2 + u_2 E_3$.

   (c) Case 3: $\gamma_1 = \gamma_2 = 0$ In this case we have that the system is SDF-equivalent to $u_1 E_2 + u_2 E_3$.

   We verify that $\Sigma_1^{(2,0)}$ is not SDF-equivalent to $\Sigma_2^{(2,0)}$. Suppose there exists $\psi \in \mathsf{Aut}(\mathfrak{h}_3)$ such that $\psi \cdot E_2 = 0$ and $\psi \cdot \langle E_2, E_3 \rangle = \langle E_2, E_3 \rangle$. Now, since ψ is a linear map $\psi \cdot E_2 \neq 0$, a contradiction.

3. From the system being DF-equivalent to the system $E_1 + u_1 E_2 + u_2 E_3$ we have that the system is SDF-equivalent to the system $\sum_{i=1}^{3} a_i E_i + \langle b_2 E_2 + b_3 E_3, c_2 E_2 + c_3 E_3 \rangle$ with

   $$\begin{vmatrix} a_1 & 0 & 0 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 (b_2 c_3 - b_3 c_2) \neq 0$$

from linear independence. Hence

$$\psi = \begin{bmatrix} b_3c_2 - b_2c_3 & 0 & 0 \\ 0 & c_3 & -c_2 \\ 0 & b_3 & -b_2 \end{bmatrix} \qquad (\det \psi = (\mathrm{b_3c_2 - b_2c_3})^2 \neq 0)$$

is an automorphism such that $\psi \cdot (a_1E_1 + a_2E_2 + a_3E_3) = a_1(b_3c_2 - b_2c_3)E_1 - (a_3c_2 + a_2c_3)E_2 - (a_3b_2 + a_2b_3)E_3 = a_1'E_1 + a_2'E_2 + a_3'E_3$ for some $a_1', a_2', a_3' \in \mathbb{R}$ and $\psi \cdot \langle b_2E_2 + b_3E_3, c_2E_2 + c_3E_3\rangle = \langle(-b_3c_2 + b_2c_3)E_2, (b_3c_2 - b_2c_3)E_3\rangle = \langle E_2, E_3\rangle$. From linear independence

$$\begin{vmatrix} a_1' & 0 & 0 \\ a_2' & 1 & 0 \\ a_3' & 0 & 1 \end{vmatrix} = a_1' \neq 0.$$

So

$$\psi = \begin{bmatrix} \frac{1}{a_1'} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{a_1'} \end{bmatrix} \qquad (\det \psi = \frac{1}{(\mathrm{a}_1')^2} \neq 0)$$

is an automorphism such that $\psi \cdot (a_1'E_1 + a_2'E_2 + a_3'E_3) = E_1 + a_2'E_2 + \frac{a_3'}{a_1'}E_3 = E_1 + \gamma_1E_2 + \gamma_2E_3$, $\gamma_1 = a_2', \gamma_2 = \frac{a_3'}{a_1'}$, and $\psi \cdot \langle E_2, E_3\rangle = \left\langle E_2, \frac{1}{a_1'}E_3\right\rangle = \langle E_2, E_3\rangle$. We divide this case into three cases.

(a) $\gamma_2 \neq 0$

$$\psi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\gamma_2} \\ 0 & -\gamma_2 & \gamma_1 \end{bmatrix} \qquad (\det \psi = 1 \neq 0)$$

is an automorphism such that $\psi \cdot (E_1 + \gamma_1E_2 + \gamma_2E_3) = E_1 + E_2$ and $\psi \cdot \langle E_2, E_3\rangle = \left\langle -\gamma_2E_3, \frac{1}{\gamma_2}E_2 + \gamma_1E_3\right\rangle = \langle E_2, E_3\rangle$ and the system is SDF-equivalent to $E_1 + E_2 + u_1E_2 + u_2E_3$.

(b) $\gamma_2 = 0$, $\gamma_1 \neq 0$. Then

$$\psi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\gamma_1} & 0 \\ 0 & 0 & \gamma_1 \end{bmatrix} \qquad (\det \psi = 1 \neq 0)$$

is an automorphism such that $\psi \cdot (E_1 + \gamma_1E_2) = E_1 + E_2$ and $\psi\, \langle E_2, E_3\rangle = \left\langle \frac{1}{\gamma_1}E_2, \gamma_1E_3\right\rangle = \langle E_2, E_3\rangle$ and the system is SDF-equivalent to $E_1 + E_2 + u_1E_2 + u_2E_3$.

(c) $\gamma_1 = \gamma_2 = 0$. In this case we have that the system is SDF-equivalent to $E_1 + u_1 E_2 + u_2 E_3$.

We have left to verify that the systems are not equivalent. Suppose $E_1 + E_2 + u_1 E_2 + u_2 E_3$ and $E_1 + u_1 E_2 + u_2 E_3$ are SDF-equivalent. Then there exists $\psi \in \mathsf{Aut}(\mathfrak{h}_3)$ such that $\psi \cdot (E_1 + E_2) = E_1$ and $\psi \cdot \langle E_2, E_3\rangle \in \langle E_2, E_3\rangle$. Now $\psi \cdot (E_1 + E_2) = (-w_2 v_3 + v_2 w_3 + v_1)E_1 + v_2 E_2 + v_3 E_3$. Therefore $v_2 = v_3 = 0$ and $\det \psi = 0$ a contradiction.

Suppose the system is DF-equivalent to the system $E_2 + u_1 E_1 + u_2 E_3$ then the system is SDF-equivalent to the system $\sum_{i=1}^{3} a_i E_i + \langle b_1 E_1 + c_3 E_3, c_1 E_1 + c_3 E_3\rangle$. From linear independence

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & 0 & 0 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_2(b_3 c_1 - b_1 c_3) \neq 0.$$

Clearly b_3 and c_3 are not simultaneously equal to zero.

(a) Assume $b_3 \neq 0$. Then

$$\psi = \begin{bmatrix} -b_3 & \frac{-a_3 b_1 + a_1 b_3}{a_2} & b_1 \\ 0 & -b_3 & 0 \\ 0 & -\frac{a_3}{a_2} & 1 \end{bmatrix} \qquad (\det \psi = \mathrm{b}_3^2 \neq 0)$$

is an automorphism such that $\psi \cdot (a_1 E_1 + a_2 E_2 + a_3 E_3) = -a_2 b_3 E_2 = a_2' E_2$, $a_2' = -a_2 b_3 \neq 0$ and $\psi \cdot \langle b_1 E_1 + b_3 E_3, c_1 E_1 + c_3 E_3\rangle = \langle b_3 E_3, (-b_3 c_1 + b_1 c_3)E_1 + c_3 E_3\rangle = \langle E_1, E_3\rangle$. Now

$$\psi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{a_2'} & 0 \\ 0 & 0 & a_2' \end{bmatrix}, \qquad (\det \psi = 1 \neq 0)$$

is an automorphism such that $\psi \cdot a_2' E_2 = E_2$ and $\psi \cdot \langle E_1, E_3\rangle = \langle E_1, a_2' E_3\rangle = \langle E_1, E_3\rangle$ and the system is SDF-equivalent to $E_2 + u_1 E_1 + u_2 E_3$.

(b) Assume $b_3 = 0$ then $c_3 \neq 0$. Then

$$\psi = \begin{bmatrix} -c_3 & \frac{-a_3 c_1 + a_1 c_3}{a_2} & c_1 \\ 0 & -c_3 & 0 \\ 0 & -\frac{a_3}{a_2} & 1 \end{bmatrix} \qquad (\det \psi = \mathrm{c}_3^2 \neq 0)$$

is an automorphism such that $\psi \cdot (a_1 E_1 + a_2 E_2 + a_3 E_3) = -a_2 c_3 E_2 = a_2' E_2$, $a_2' = -a_2 c_3 \neq$

0 and $\psi \cdot \langle b_1E_1, c_1E_1 + c_3E_3\rangle = \langle -b_1c_3E_1, c_3E_3\rangle = \langle E_1, E_3\rangle$. Then as in (a)

$$\psi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{a_2'} & 0 \\ 0 & 0 & a_2' \end{bmatrix}, \quad (\det \psi = 1 \neq 0)$$

is an automorphism such that $\psi \cdot a_2'E_2 = E_2$ and $\psi \cdot \langle E_1, E_3\rangle = \langle E_1, a_2'E_3\rangle = \langle E_1, E_3\rangle$ and the system is SDF-equivalent to $E_2 + u_1E_1 + u_2E_3$.

4. Such a system is SDF-equivalent to the system that has trace $\Gamma = \sum_{i=1}^3 a_iE_i + \left\langle \sum_{i=1}^3 b_iE_i, \sum_{i=1}^3 c_iE_i, \sum_{i=1}^3 d_iE_i \right\rangle$. From linear independence the system is SDF-equivalent to the system with trace $\Gamma' = \sum_{i=1}^3 \gamma_iE_i + \langle E_1, E_2, E_3\rangle$. We split this case up into the following cases.

   (a) $\gamma_2 \neq 0$. Then

   $$\psi = \begin{bmatrix} 1 & -\frac{\gamma_1}{\gamma_2} & 0 \\ 0 & \frac{1}{\gamma_2} & 0 \\ 0 & -\gamma_3 & \gamma_2 \end{bmatrix} \quad (\det \psi = 1 \neq 0)$$

   is an automorphism such that $\psi \cdot (\gamma_1E_1 + \gamma_2E_2 + \gamma_3E_3) = E_2$ and $\psi \cdot \langle E_1, E_2, E_3, \rangle = \left\langle E_1, -\frac{\gamma_1}{\gamma_2}E_1 + \frac{1}{\gamma_2}E_2 - \gamma_3E_3, \gamma_2E_3 \right\rangle = \langle E_1, E_2, E_3\rangle$. The system is therefore SDF-equivalent to $E_2 + u_1E_1 + u_2E_2 + u_3E_3$.

   (b) $\gamma_2' = 0$, $\gamma_3' \neq 0$. Then

   $$\begin{bmatrix} -1 & 0 & \frac{\gamma_1}{\gamma_3} \\ 0 & 0 & \frac{1}{\gamma_3} \\ 0 & \gamma_3 & 0 \end{bmatrix} \quad (\det \psi = 1 \neq 0)$$

   is an automorphism such that $\psi \cdot (\gamma_1E_1 + \gamma_3E_3) = E_2$ and $\psi \cdot \langle E_1, E_2, E_3\rangle = \left\langle E_1, \gamma_3E_3, \frac{\gamma_1}{\gamma_3}E_1 + \frac{1}{\gamma_3}E_2 \right\rangle = \langle E_1, E_3, E_2\rangle = \langle E_1, E_2, E_3\rangle$. Therefore the system is SDF-equivalent to $E_2 + u_1E_1 + u_2E_2 + u_3E_3$.

   (c) $\gamma_2 = \gamma_3 = 0$, $\gamma_1 \neq 0$. Then

   $$\begin{bmatrix} \frac{1}{\gamma_1} & 0 & 0 \\ 0 & \frac{1}{\gamma_1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (\det \psi = \frac{1}{\gamma_1^2} \neq 0)$$

   is an automorphism such that $\psi \cdot (\gamma_1E_1) = E_1$ and $\psi \cdot \langle E_1, E_2, E_3\rangle = \left\langle \frac{1}{\gamma_1}E_1, \frac{1}{\gamma_1}E_2, E_3 \right\rangle = \langle E_1, E_2, E_3\rangle$. Therefore the system is SDF-equivalent to $E_1 + u_1E_1 + u_2E_2 + u_3E_3$

(d) $\gamma_1 = \gamma_2 = \gamma_3 = 0$. The system is SDF-equivalent to $u_1E_1 + u_2E_2 + u_3E_3$.

We have left to verify that three systems are not SDF-equivalent. Suppose $E_2 + u_1E_1 + u_2E_2 + u_3E_3$ is SDF-equivalent to $E_1 + u_1E_1 + u_2E_2 + u_3E_3$. Then there exists $\psi \in \mathsf{Aut}(\mathfrak{h}_3)$ such that $\psi \cdot E_1 = (v_2w_3 - w_2v_3)E_1 = E_2$ which cannot happen. A contradiction. Also since ψ is linear $u_1E_1 + u_2E_2 + u_3E_3$ cannot be SDF-equivalent to either of the other two systems. The three systems are therefore distinct.

2.5.2 Proposition. *1. Any (1,1) system that is SDF-equivalent to the system*

$$E_2 + u_1E_3$$

is S-equivalent to the system

$$E_2 + u_1E_3.$$

2. Any $(2,0)$ *system that is SDF-equivalent to the system*

$$E_2 + u_1E_2 + u_2E_3$$

is S-equivalent to exactly one of the following systems

$$\gamma_1E_2 + \gamma_2E_3 + u_1E_2 + u_2E_3, \quad \gamma_1^2 + \gamma_2^2 \neq 0.$$

Here each parameter parametrises a distinct family of class representatives.
Any (2,0) system that is SDF-equivalent to the system

$$u_1E_2 + u_2E_3$$

is S-equivalent to the system

$$u_1E_2 + u_2E_3.$$

3. Any (2,1) system that is SDF-equivalent to the system

$$E_1 + E_2 + u_1E_2 + u_2E_3$$

is S-equivalent to exactly one of the systems

$$\alpha E_1 + \gamma_1E_2 + \gamma_2E_3 + u_1E_2 + u_2E_3, \quad \alpha \neq 0,\ \gamma_1^2 + \gamma_2^2 \neq 0$$

where each parameter yields a distinct S-equivalence class.
Any (2,1) system that is SDF equivalent to the system

$$E_1 + u_1E_2 + u_2E_3$$

is S-equivalent to the system

$$\alpha E_1 + u_1E_2 + u_2E_3.$$

Here each parameter parametrises a distinct family of class representatives.
Any $(2,1)$ *system that is SDF-equivalent to the system*

$$E_2 + u_1E_1 + u_2E_3$$

is S-equivalent to exactly one of the following systems

$$\begin{aligned}&E_2 + u_1\alpha E_1 + u_2E_3\\ &E_2 + u_1E_3 + u_2(\alpha E_1 + \gamma_1E_3) \quad \alpha \neq 0.\end{aligned}$$

Here each parameter parametrises a distinct family of class representatives.

4. *Any* $(3,0)$ *system that is SDF-equivalent to the system*

$$E_2 + u_1E_1 + u_2E_2 + u_3E_3$$

is S-equivalent to one of the following systems

$$\begin{aligned}&\gamma_1E_1 + \gamma_2E_2 + \gamma_3E_3 + u_1(\alpha E_1 + \gamma_4E_2 + \gamma_5E_2) + u_2E_2 + u_3E_3\\ &\gamma_1E_1 + \gamma_2E_2 + \gamma_3E_3 + u_1\alpha E_1 + u_1E_2 + u_2E_3\\ &\gamma_1E_2 + \gamma_2E_2 + \gamma_3E_3 + u_1E_2 + u_2E_3 + u_3(\alpha E_1 + \gamma_4E_3)\\ &\gamma_1E_2 + \gamma_2E_2 + \gamma_3E_3 + u_1E_2 + u_2\alpha E_1 + u_3E_3\\ &\gamma_2^2 + \gamma_3^2 \neq 0, \gamma_4^2 + \gamma_5^2 \neq 0, \alpha \neq 0.\end{aligned}$$

Here each parameter parametrises a distinct family of class representatives.
Any $(3,0)$ *system that is SDF-equivalent to the system*

$$E_1 + u_1E_1 + u_2E_2 + u_3E_3$$

is S-equivalent to exactly one of the following systems

$$\begin{aligned}&\alpha_1E_1 + u_1(\alpha_2E_1 + \gamma_1E_2 + \gamma_2E_2) + u_2E_2 + u_3E_3\\ &\alpha_1E_1 + u_1\alpha_2E_1 + u_1E_2 + u_2E_3\\ &\alpha_1E_1 + u_1E_2 + u_1E_3 + u_2(\alpha_2E_1 + \gamma_1E_3)\\ &\alpha_1E_1 + u_1E_2 + u_2\alpha_2E_1 + u_3E_3\\ &\alpha_1, \alpha_2 \neq 0.\end{aligned}$$

Each parameter yields a distinct S-equivalence class.
Any $(3,0)$ *system that is SDF-equivalent to the system*

$$u_1E_1 + u_2E_2 + u_3E_3$$

is S-equivalent to exactly one of the following systems

$$\begin{aligned}&u_1(\alpha E_1 + \gamma_1E_2 + \gamma_2E_2) + u_2E_2 + u_3E_3\\ &u_1\alpha E_1 + u_1E_2 + u_2E_3\\ &u_1E_2 + u_2E_3 + u_3(\alpha E_1 + \gamma_1E_3)\\ &u_1E_2 + u_2\alpha E_1 + u_3E_3\\ &\alpha \neq 0.\end{aligned}$$

Here each parameter parametrises a distinct family of class representatives.

PROOF. 1. Since the system is SDF-equivalent to the system $E_2 + u_1 E_3$ and of full-rank we have from proposition 2.2.12 that the system is S-equivalent to the system $E_2 + u_1 E_3$.

2. Immediately we have that the system is S-equivalent to

$$\left[\begin{array}{c|cc} a_1 & 0 & 0 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{array}\right]$$

with $a_1 = 0$ from linear dependence. Also a_2 and a_3 are not simultaneously equal to zero since $\psi(a_2 E_2 + a_3 E_3) = E_2$. And so from lemma 2.2.12 we have that the system is S-equivalent to the $\gamma_1 E_2 + \gamma_2 E_3 + u_1 E_2 + u_2 E_3$ where $\gamma_1^2 + \gamma_2^2 \neq 0$. From lemma 2.2.13 we have that each γ_1 and γ_2 yield non-equivalent classes.

Consider the case where the system is SDF-equivalent to the system $u_1 E_2 + u_2 E_3$. We immediately have that the system is S-equivalent to the system

$$\left[\begin{array}{c|cc} 0 & 0 & 0 \\ 0 & b_2 & c_2 \\ 0 & b_3 & c_3 \end{array}\right]$$

since $\psi \cdot (a_1 E_1 + a_2 E_2 + a_3 E_3) = 0$. Columns 2 and 3 must generate $\mathfrak{h}_3$ and so from lemma 2.2.12 we have that the system is S-equivalent to the system $u_1 E_2 + u_2 E_3$.

3. Suppose the system is SDF-equivalent to the system $E_1 + E_2 + u_1 E_2 + u_2 E_3$ then we immediately have that the system is S-equivalent to the system

$$\begin{bmatrix} a_1 & 0 & 0 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

Now

$$\begin{vmatrix} a_1 & 0 & 0 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1(b_2 c_3 - b_3 c_2) \neq 0.$$

And so

$$\psi_1 = \begin{bmatrix} b_3 c_2 - b_2 c_3 & 0 & 0 \\ 0 & c_3 & -c_2 \\ 0 & b_3 & -b_2 \end{bmatrix} \quad (\det \psi_1 = (\mathrm{b_3 c_2 - b_2 c_3})^2 \neq 0)$$

is an automorphism such that

$$\psi_1 \cdot \left[\begin{array}{c|cc} a_1 & 0 & 0 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{array}\right] = \left[\begin{array}{c|cc} a_1(b_3c_2 - b_2c_3) & 0 & 0 \\ -a_3c_2 + a_2c_3 & -b_3c_2 + b_2c_3 & 0 \\ -a_3b_2 + a_2b_3 & 0 & b_3c_2 - b_2c_3 \end{array}\right].$$

Also

$$\psi_2 = \begin{bmatrix} -\frac{1}{(b_3c_2-b_2c_3)^2} & 0 & 0 \\ 0 & \frac{1}{-b_3c_2+b_2c_3} & 0 \\ 0 & 0 & \frac{1}{b_3c_2-b_2c_3} \end{bmatrix}, \quad \left(\det \psi_2 = -\frac{1}{(b_3c_2 - b_2c_3)^3(-b_3c_2 + b_2c_3)} \neq 0\right)$$

is an automorphism such that

$$\begin{aligned} \psi_2 \cdot \left[\begin{array}{c|cc} a_1(b_3c_2 - b_2c_3) & 0 & 0 \\ -a_3c_2 + a_2c_3 & -b_3c_2 + b_2c_3 & 0 \\ -a_3b_2 + a_2b_3 & 0 & b_3c_2 - b_2c_3 \end{array}\right] &= \left[\begin{array}{c|cc} \frac{a_1}{-b_3c_2+b_2c_3} & 0 & 0 \\ \frac{a_3c_2-a_2c_3}{b_3c_2-b_2c_3} & 1 & 0 \\ \frac{a_3b_2-a_2b_3}{-b_3c_2+b_2c_3} & 0 & 1 \end{array}\right] \\ &= \left[\begin{array}{c|cc} a_1' & 0 & 0 \\ a_2' & 1 & 0 \\ a_3' & 0 & 1 \end{array}\right] \end{aligned}$$

where $a_1' \neq 0$ from linear independence. Since the system is SDF-equivalent to $E_1 + E_2 + u_1E_2 + u_2E_3$ we have that

$$\psi \cdot \begin{bmatrix} a_1' \\ a_2' \\ a_3' \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

for some $\psi \in \mathsf{Aut}(\mathfrak{h}_3)$. That is

$$\begin{aligned} &\begin{bmatrix} -v_3w_2 + v_2w_3 & v_1 & w_1 \\ 0 & v_2 & w_2 \\ 0 & v_3 & w_3 \end{bmatrix} \begin{bmatrix} a_1' \\ a_2' \\ a_3' \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \\ \iff &\begin{bmatrix} a_2'v_1 + a_3'w_1 - a_1'v_3w_2 + a_1'v_2w_3 \\ a_2'v_2 + a_3'w_2 \\ a_2'v_3 + a_3'w_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \end{aligned}$$

which on solving yields

$$\begin{cases} a_1' = -\frac{-v_3w_1+v_3w_2+v_1w_3-v_2w_3}{(v_3w_2-v_2w_3)^2} \\ a_2' = \frac{w_3}{-v_3w_2+v_2w_3} \\ a_3' = \frac{v_3}{v_3w_2-v_2w_3} \end{cases}$$

and we see that a_2' and a_3' are not simultaneously equal to zero since if they were $\det(\psi) = 0$ and the system is S-equivalent to $\alpha E_1 + \gamma_1 E_2 + \gamma_2 E_3 + u_1 E_2 + u_2 E_3$ with $\alpha \neq 0$, $\gamma_1^2 + \gamma_2^2 \neq 0$. From lemma 2.2.13 we have that α, γ_1 and γ_2 yield non-equivalent classes.

Suppose the system is SDF-equivalent to the system $E_1 + u_1 E_2 + u_2 E_3$ then we immediately have that the system is S-equivalent to the system

$$\left[\begin{array}{c|cc} a_1 & 0 & 0 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{array}\right].$$

We may proceed as above until the point where we have shown that the system is S-Equivalent to the system

$$\left[\begin{array}{c|cc} a_1' & 0 & 0 \\ a_2' & 1 & 0 \\ a_3' & 0 & 1 \end{array}\right]$$

with $a_1' \neq 0$ from linear independence. Since the system is SDF-equivalent to the system $E_1 + u_1 E_2 + u_2 E_3$ we have that

$$\psi \cdot \begin{bmatrix} a_1' \\ a_2' \\ a_3' \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

for some $\psi \in \mathsf{Aut}(\mathfrak{h}_3)$ in other words

$$\begin{bmatrix} v_2w_3 - v_3w_2 & v_1 & w_1 \\ 0 & v_2 & w_2 \\ 0 & v_3 & w_3 \end{bmatrix} \cdot \begin{bmatrix} a_1' \\ a_2' \\ a_3' \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\iff \begin{bmatrix} a_2'v_1 + a_3'w_1 - a_1'v_3w_2 + a_1'v_2w_3 \\ a_2'v_2 + a_3'w_2 \\ a_2'v_3 + a_3'w_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

which on solving yields

$$\begin{cases} a_1' = -\frac{1}{v_3w_2-v_2w_3} \\ a_2' = 0 \\ a_3' = 0 \end{cases}$$

and so the system is S-equivalent to the system $\alpha E_1 + u_1E_2 + u_2E_3$, $\alpha \neq 0$ and from lemma 2.2.13 we have that different α yield distinct systems.

Suppose the system is SDF-equivalent to the system $E_2+u_1E_1+u_2E_3$. We immediately have that the system is S-equivalent to

$$\left[\begin{array}{c|cc} a_1 & b_1 & c_1 \\ a_2 & 0 & 0 \\ a_3 & b_3 & c_3 \end{array}\right].$$

From linear independence we have that

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & 0 & 0 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_2(b_3c_1 - b_1c_3) \neq 0.$$

Clearly b_3 and c_3 are not simultaneously equal to zero.

(a) Suppose $b_3 = 0$ then $c_3 \neq 0$ and

$$\psi_1 = \begin{bmatrix} -c_3 & \frac{-a_3c_1+a_1c_3}{a_2} & c_1 \\ 0 & -c_3 & 0 \\ 0 & -\frac{a_3}{a_2} & 1 \end{bmatrix} \qquad (\det \psi_1 = c_3^2 \neq 0)$$

is an automorphism such that

$$\psi_1 \cdot \left[\begin{array}{c|cc} a_1 & b_1 & c_1 \\ a_2 & 0 & 0 \\ a_3 & 0 & c_3 \end{array}\right] = \left[\begin{array}{c|cc} 0 & -b_1c_3 & 0 \\ -a_2c_3 & 0 & 0 \\ 0 & 0 & c_3 \end{array}\right]$$

and

$$\psi_2 = \begin{bmatrix} -\frac{1}{a_2c_3^2} & 0 & 0 \\ 0 & -\frac{1}{a_2c_3} & 0 \\ 0 & 0 & \frac{1}{c_3} \end{bmatrix} \qquad (\det \psi_2 = \frac{1}{a_2^2c_3^4} \neq 0)$$

is an automorphism such that

$$\psi_2 \cdot \begin{bmatrix} 0 & -b_1c_3 & 0 \\ -a_2c_3 & 0 & 0 \\ 0 & 0 & c_3 \end{bmatrix} = \begin{bmatrix} 0 & \frac{b_1}{a_2c_3} & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$\frac{b_1}{a_2c_3} \neq 0$ from linear independence and the system is S-equivalent to the system $E_2 + u_1\alpha E_1 + u_2E_3$, $\alpha \neq 0$ and from lemma 2.2.13 we have that α yield distinct equivalent classes.

(b) Assume $b_3 \neq 0$. Then

$$\psi = \begin{bmatrix} \frac{1}{a_2b_3} & \frac{a_3b_1-a_1b_3}{a_2^2b_3^2} & -\frac{b_1}{a_2b_3^2} \\ 0 & \frac{1}{a_2} & 0 \\ 0 & -\frac{a_3}{a_2b_3} & \frac{1}{b_3} \end{bmatrix}$$

is an automorphism such that

$$\psi \cdot \left[\begin{array}{c|cc} a_1 & b_1 & c_1 \\ a_2 & 0 & 0 \\ a_3 & b_3 & c_3 \end{array}\right] = \left[\begin{array}{c|cc} 0 & 0 & \frac{b_3c_1-b_1c_3}{a_2b_3^2} \\ 1 & 0 & 0 \\ 0 & 1 & \frac{c_3}{b_3} \end{array}\right]$$

and the system is S-equivalent to $E_2 + u_1E_3 + u_2(\alpha E_1 + \gamma_1 E_3)$, $\alpha \neq 0$, $\gamma_1 \in \mathbb{R}$ from lemma 2.2.13 we have that α and γ_1 yield distinct equivalent classes.

4. Suppose the system is SDF-equivalent to the system $E_2 + u_1E_1 + u_2E_2 + u_3E_3$ then from 3 above we immediately have that the system is S-equivalent to one of the following systems

$$\begin{aligned} &a_1E_1 + a_2E_2 + a_3E_3 + u_1(\alpha E_1 + \gamma_4 E_2 + \gamma_5 E_2) + u_2E_2 + u_3E_3 \\ &a_1E_2 + a_2E_2 + a_3E_3 + u_1\alpha E_1 + u_2E_2 + u_3E_3 \\ &a_1E_2 + a_2E_2 + a_3E_3 + u_1E_2 + u_2E_3 + u_3(\alpha E_1 + \gamma_4 E_3) \\ &a_1E_1 + a_2E_2 + a_3E_3 + u_1E_2 + u_2\alpha E_1 + u_3E_3, \\ &a_1, a_2, a_3, \alpha, \gamma_4, \gamma_5 \in \mathbb{R}, \gamma_4^2 + \gamma_5^2 \neq 0, \alpha \neq 0. \end{aligned}$$

Also since the system is SDF-equivalent to the system $E_2 + u_1E_1 + u_2E_2 + u_3E_3$ we have that

$$\psi \cdot \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

for some $\psi \in \mathsf{Aut}(\mathfrak{h}_3)$ and so we have that

$$\begin{bmatrix} v_2w_3 - v_3w_2 & v_1 & w_1 \\ 0 & v_2 & w_2 \\ 0 & v_3 & w_3 \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\iff \begin{bmatrix} a_2v_1 + a_3w_1 - a_1v_3w_2 + a_1v_2w_3 \\ a_2v_2 + a_3w_2 \\ a_2v_3 + a_3w_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

which on solving yields

$$\begin{cases} a_1 = -\frac{-v_3w_1+v_1w_3}{(v_3w_2-v_2w_3)^2} \\ a_2 = \frac{w_3}{-v_3w_2+v_2w_3} \\ a_3 = \frac{v_3}{v_3w_2-v_2w_3} \end{cases}.$$

From which we have that a_2 and a_3 are not simultaneously equal to zero since if they were $\det\psi = 0$ and so we have that the system is S-equivalent to one of the following systems.

$$\begin{aligned} &\gamma_1E_1 + \gamma_2E_2 + \gamma_3E_3 + u_1(\alpha E_1 + \gamma_4E_2 + \gamma_5E_2) + u_2E_2 + u_3E_3 \\ &\gamma_1E_1 + \gamma_2E_2 + \gamma_3E_3 + u_1\alpha E_1 + u_2E_2 + u_3E_3 \\ &\gamma_1E_2 + \gamma_2E_2 + \gamma_3E_3 + u_1E_2 + u_2E_3 + u_3(\alpha E_1 + \gamma_4E_3) \\ &\gamma_1E_2 + \gamma_2E_2 + \gamma_3E_3 + u_1E_2 + u_2\alpha E_1 + u_3E_3 \\ &\alpha, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5 \in \mathbb{R},\ \gamma_2^2 + \gamma_3^2 \neq 0, \gamma_4^2 + \gamma_5^2 \neq 0\ \alpha \neq 0. \end{aligned}$$

and from 3 above we have that each α,γ_i yield distinct classes.

Consider the case where the system is SDF-equivalent to the system $E_1 + u_1E_1 + u_2E_2 + u_3E_3$ then from 3 above we immediately have that the system is S-equivalent to one of the systems

$$\begin{aligned} &a_1E_1 + a_2E_2 + a_3E_3 + u_1(\alpha E_1 + \gamma_4E_2 + \gamma_5E_2) + u_2E_2 + u_3E_3 \\ &a_1E_1 + a_2E_2 + a_3E_3 + u_1\alpha E_1 + u_2E_2 + u_3E_3 \\ &a_1E_2 + a_2E_2 + a_3E_3 + u_1E_2 + u_2E_3 + u_3(\alpha E_1 + \gamma_4E_3) \\ &a_1E_2 + a_2E_2 + a_3E_3 + u_1E_2 + u_2\alpha E_1 + u_3E_3, \quad a_1, a_2, a_3, \alpha, \gamma_4, \gamma_5 \in \mathbb{R}, \alpha \neq 0. \end{aligned}$$

Now

$$\psi \cdot \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

for some $\psi \in \mathsf{Aut}(\mathfrak{h}_3)$. In other words we have that

$$\begin{aligned}
&\begin{bmatrix} v_2w_3 - v_3w_2 & v_1 & w_1 \\ 0 & v_2 & w_2 \\ 0 & v_3 & w_3 \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\
\iff &\begin{bmatrix} a_2v_1 + a_3w_1 - a_1v_3w_2 + a_1v_2w_3 \\ a_2v_2 + a_3w_2 \\ a_2v_3 + a_3w_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}$$

which on solving yields

$$\begin{cases} a_1 = -\frac{1}{v_3w_2 - v_2w_3} \\ a_2 = 0 \\ a_3 = 0. \end{cases}$$

Hence the system is S-equivalent to exactly one of the following systems

$$\begin{aligned}
&\alpha_1E_1 + u_1(\alpha_2E_1 + \gamma_1E_2 + \gamma_2E_2) + u_2E_2 + u_3E_3 \\
&\alpha_1E_1 + u_1\alpha_2E_1 + u_2E_2 + u_3E_3 \\
&\alpha_1E_1 + u_1E_2 + u_1E_3 + u_2(\alpha_2E_1 + \gamma_1E_3) \\
&\alpha_1E_1 + u_1E_2 + u_2\alpha_2E_1 + u_3E_3 \\
&\alpha_1, \alpha_2, \gamma_1, \gamma_2 \in \mathbb{R},\ \alpha_1, \alpha_2 \neq 0.
\end{aligned}$$

From 3 above the systems are distinct.

Suppose the system is SDF-equivalent to the system $u_1E_1 + u_2E_2 + u_3E_3$ then from 3 above we have that the system is S-equivalent to one of the following systems

$$\begin{aligned}
&a_1E_1 + a_2E_2 + a_3E_3 + u_1(\alpha E_1 + \gamma_4E_2 + \gamma_5E_2) + u_2E_2 + u_3E_3 \\
&a_1E_1 + a_2E_2 + a_3E_3 + u_1\alpha E_1 + u_2E_2 + u_3E_3 \\
&a_1E_2 + a_2E_2 + a_3E_3 + u_1E_2 + u_2E_3 + u_3(\alpha E_1 + \gamma_4E_3) \\
&a_1E_2 + a_2E_2 + a_3E_3 + u_1E_2 + u_2\alpha E_1 + u_3E_3 \\
&a_1, a_2, a_3, \gamma_4, \gamma_5, \alpha \in \mathbb{R}, \alpha \neq 0.
\end{aligned}$$

Now

$$\psi \cdot \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

for some $\psi \in \mathsf{Aut}(\mathfrak{h}_3)$ but ψ is a linear map, therefore $a_1 = a_2 = a_3 = 0$ and we have that the system is S-equivalent to exactly one of the following systems

$$\begin{aligned}
&u_1(\alpha E_1 + \gamma_1 E_2 + \gamma_2 E_2) + u_2 E_2 + u_3 E_3 \\
&u_1 \alpha E_1 + u_2 E_2 + u_3 E_3 \\
&u_1 E_2 + u_2 E_3 + u_3(\alpha E_1 + \gamma_1 E_3) \\
&u_1 E_2 + u_2 \alpha E_1 + u_3 E_3 \\
\alpha \neq 0.&
\end{aligned}$$

We collect the results of propositions 2.5.1 and 2.5.2 in the table B.1.

Table 2.1: Full classification: state space equivalence

System	Equivalence classes ($\alpha \neq 0, \gamma_i \in \mathbb{R}$)	Defining conditions
$(1,1)$	$\Sigma^{(1,1)} : E_2 + u_1 E_3$	
$(2,0)$	$\Sigma^{(2,0)}_{\gamma_1,\gamma_2} : \gamma_1 E_2 + \gamma_2 E_3 + u_1 E_2 + u_2 E_3$	
$(2,1)$	$\Sigma^{(2,1)}_{1,\alpha,\gamma_1,\gamma_2} : \alpha E_1 + \gamma_1 E_2 + \gamma_2 E_3 + u_1 E_2 + u_2 E_3$	$-b_3 c_2 + b_2 c_3 \neq 0$
	$\Sigma^{(2,1)}_{2,\alpha,\gamma_1} : E_2 + u_1 E_3 + u_2(\alpha E_1 + \gamma_1 E_3)$	$-b_3 c_2 + b_2 c_3 = 0, -a_3 b_2 + a_2 b_3 \neq 0$
	$\Sigma^{(2,1)}_{3,\alpha} : E_2 + u_1 \alpha E_1 + u_2 E_3$	$-b_3 c_2 + b_2 c_3 = 0, -a_3 b_2 + a_2 b_3 = 0,$ $-a_3 c_2 + a_2 c_3 \neq 0$
$(3,0)$	$\gamma_1 E_1 + \gamma_2 E_2 + \gamma_3 E_3 + u_1(\alpha E_1 + \gamma_4 E_2 + \gamma_5 E_3) + u_2 E_2 + u_3 E_3$	$-c_3 d_2 + c_2 d_3 \neq 0$
	$\gamma_1 E_1 + \gamma_2 E_2 + \gamma_3 E_3 + u_1 E_2 + u_2 E_3 + u_3(\alpha E_1 + \gamma_4 E_3)$	$-c_3 d_2 + c_2 d_3 = 0, -b_3 c_2 + b_2 c_3 \neq 0$
	$\gamma_1 E_1 + \gamma_2 E_2 + \gamma_3 E_3 + u_1 E_2 + u_2 \alpha E_1 + u_3 E_3$	$-c_3 d_2 + c_2 d_3 = 0, -b_3 c_2 + b_2 c_3 = 0$ $-a_3 c_2 + a_2 c_3 \neq 0$

Table 2.2: Full classification: detached feedback

System	Equivalence classes
$(1,1)$	$\Gamma^{(1,1)} = E_2 + \langle E_3 \rangle$
$(2,0)$	$\Gamma^{(2,0)} = \langle E_2, E_3 \rangle$
$(2,1)$	$\Gamma_1^{(2,1)} = E_1 + \langle E_2, E_3 \rangle$ $\Gamma_2^{(2,1)} = E_2 + \langle E_1, E_3 \rangle$
$(3,0)$	$\Gamma^{(3,0)} = \langle E_1, E_2, E_3 \rangle$

Table 2.3: Full classification: strongly detached feedback

System	Equivalence classes
$(1,1)$	$\Sigma^{(1,1)} : E_2 + uE_3$
$(2,0)$	$\Sigma_1^{(2,0)} : E_2 + u_1E_2 + u_3E_3$ $\Sigma_2^{(2,0)} : u_1E_2 + u_2E_3$
$(2,1)$	$\Sigma_1^{(2,1)} : E_1 + E_2 + u_1E_2 + u_2E_3$ $\Sigma_2^{(2,1)} : E_1 + u_1E_2 + u_2E_3$ $\Sigma_3^{2,1} : E_2 + u_1E_3 + u_2E_1$
$(3,0)$	$\Sigma_1^{(3,0)} : E_2 + u_1E_1 + u_2E_2 + u_3E_3$ $\Sigma_2^{(3,0)} : E_1 + u_1E_1 + u_2E_2 + u_3E_3$ $\Sigma_3^{(3,0)} : u_1E_1 + u_2E_2 + u_3E_3$

Chapter 3

Classification of Cost-Extended Control Systems

After classifying all full-rank left-invariant control affine systems (specifically under detached feedback equivalence) on H_3 in chapter 2, a natural next step is to consider the associated left-invariant optimal control problems with specified cost and boundary conditions (see section A.3). In this chapter we consider optimal control problems with fixed terminal time and affine quadratic cost on H_3:

$$\begin{aligned} \dot{g}(t) =&\Xi(g(t),u(t)), \quad g(\cdot):[0,T]\to\mathsf{H}_3 \quad u(\cdot):[0,T]\to\mathbb{R}^\ell, & (3.1)\\ g(0)=g_0,& \quad g(T)=g_1, \quad g_0,g_1\in\mathsf{H}_3 \text{ fixed}, \ T>0 \text{ fixed}, & (3.2)\\ \mathcal{J}(u(\cdot)) =& \int_0^T \chi(u(t))dt \to \min. & (3.3)\end{aligned}$$

Here $\chi:\mathbb{R}^\ell\to\mathbb{R}, u\mapsto(u-\mu)^\top Q(u-\mu)$, $\mu\in\mathbb{R}^\ell$ and Q is a positive definite $\ell\times\ell$ matrix. In order to investigate such a problem, we associate a cost-extended control system (Σ,χ) with boundary data: $g(0)=g_0$, $g(T)=g_1$, T. Cost-extended control systems have been considered in [17]. We also look at the associated Hamiltonian function on $T^*\mathsf{H}_3\cong\mathsf{H}_3\times\mathfrak{h}_3^*$ (section A.3) and determine the optimal controls. It is the Pontryagin Maximum Principle (PMP for short) which provides necessary conditions for optimality. Cost-equivalence establishes a one-to-one correspondence between the optimal controlled trajectories and also establishes a one-to-one correspondence between the normal extremal curves.

3.1 Cost-Extended Control Systems

Suppose we have a left-invariant optimal control problem specified by (i) a left invariant control affine system (chapter 2) $\Sigma=(\mathsf{G},\Xi)$, (ii) an affine quadratic cost function $\chi:\mathbb{R}^\ell\to\mathbb{R}$, and (iii) boundary data: $g(0)=g_0$, $g(T)=g_1$, and fixed terminal time $T>0$. Formally,

$$\begin{aligned} \dot{g}(t) =&\Xi(g(t),u(t)), \quad g(\cdot):[0,T]\to\mathsf{G} \quad u(\cdot):[0,T]\to\mathbb{R}^\ell, & (3.4)\\ g(0)=g_0,& \quad g(T)=g_1, \quad g_0,g_1\in\mathsf{G} \text{ fixed}, \ T>0 \text{ fixed}, & (3.5)\\ \mathcal{J}(u(\cdot)) =& \int_0^T \chi(u(t))dt \to \min. & (3.6)\end{aligned}$$

Where $\chi : \mathbb{R}^\ell \to \mathbb{R}, u \mapsto (u-\mu)^\top Q(u-\mu)$, $\mu \in \mathbb{R}^\ell$ and Q is a positive definite $\ell \times \ell$ matrix. To the optimal control problem 3.4–3.5–3.6 we associate a **cost-extended control system** which is the pair (Σ, χ) specified by the boundary data. A means of classifying such systems is by cost-equivalence.

3.1.1 DEFINITION. Two cost-extended systems on a connected Lie group G with the same input space $\mathbb{R}^\ell$, $(\Sigma = (\mathsf{G}, \Xi), \chi)$ and $(\Sigma' = (\mathsf{G}, \Xi'), \chi')$, are **cost equivalent** (shortly C-equivalent) if there exists a diffeomorphism $\Phi = (\phi, \varphi) : (\Sigma, \chi) \to (\Sigma', \chi')$:

$$\Phi : \mathsf{G} \times \mathbb{R}^\ell \to \mathsf{G} \times \mathbb{R}^\ell, \quad (g, u) \mapsto (\phi(g), \varphi(u))$$

where $\phi : \mathsf{G} \to \mathsf{G}$ is a diffeomorphism and $\varphi : \mathbb{R}^\ell \to \mathbb{R}^\ell$ is an affine isomorphism such that the diagrams

$$\begin{tikzcd} \mathsf{G} \times \mathbb{R}^\ell \arrow[r, "\Phi"] \arrow[d, "\Xi"'] & \mathsf{G} \times \mathbb{R}^{\ell'} \arrow[d, "\Xi'"] \\ T\mathsf{G} \arrow[r, "T\phi"'] & T\mathsf{G} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathbb{R}^\ell \arrow[r, "\varphi"] \arrow[d, "\chi"'] & \mathbb{R}^\ell \arrow[d, "\chi'"] \\ \mathbb{R} \arrow[r, "\delta_r"'] & \mathbb{R} \end{tikzcd}$$

commute (δr is a dilation by r.). That is

$$T_g\phi \cdot \Xi(g, u) = \Xi'(\phi(g), \varphi(u)) \quad \text{and} \quad r\chi(u) = \chi'(\varphi(u))$$

for some $r > 0$.

3.1.2 PROPOSITION. *C-equivalence is an equivalence relation.*

PROOF. Let (Σ, χ), (Σ', χ') and (Σ'', χ'') be cost-extended systems on G. Cost equivalence is

1. Reflexive.
   $\Phi = (\phi, \varphi)$ where $\phi = \mathrm{Id}_\mathsf{G} : \mathsf{G} \to \mathsf{G}$, $g \mapsto g$ $(T_g\phi = \mathrm{id}_\mathfrak{g})$ and $\varphi = \mathrm{Id}_{\mathbb{R}^\ell} : \mathbb{R}^\ell \to \mathbb{R}^\ell, u \mapsto u$ is a diffeomorphism such that $T_g\phi \cdot \Xi(g, u) = \Xi(g, u) = \Xi(\phi(g), \varphi(u))$ and $\chi(\varphi(u)) = \chi(u)$. Hence (Σ, χ) is C-equivalent to (Σ, χ).

2. Symmetric.
   Suppose (Σ, χ) is C-equivalent to (Σ', χ') then there exists a diffeomorphism $\Phi : \mathsf{G} \times \mathbb{R}^\ell \to \mathsf{G} \times \mathbb{R}^\ell, (g, u) \mapsto (\phi(g), \varphi(u))$ such that $T_g\phi \cdot \Xi(g, u) = \Xi'(\phi(g), \varphi(u))$ and $r\chi(u) = \chi'(\varphi(u))$ for some $r > 0$. We have that

$$\begin{aligned} T_{\phi(g)}\phi^{-1} \cdot \Xi'(\phi(g), \varphi(u)) &= (T_g\phi)^{-1} \cdot \Xi'(\phi(g), \varphi(u)) \quad \text{(lemma } A.1.11) \\ &= (T_g\phi)^{-1} \cdot T_g\phi \cdot \Xi(g, u) \\ &= \Xi(g, u) = \Xi(\phi^{-1}\phi(g), \varphi^{-1}\varphi(u)). \end{aligned}$$

   Let $r' > 0$ then

$$\begin{aligned} r'\chi'(\varphi(u)) &= r'r\chi(u) \\ &= r''\chi(\varphi^{-1}\varphi(u)) \quad \text{for}\, r'' = rr' > 0 \end{aligned}$$

   Hence $\Phi^{-1} : \mathsf{G} \times \mathbb{R}^\ell \to \mathsf{G} \times \mathbb{R}^\ell$, $(g, u) \mapsto (\phi^{-1}(g), \varphi^{-1}(u))$ is a diffeomorphism such that $T_{\phi(g)}\phi^{-1} \cdot \Xi'(\phi(g), \varphi(u)) = \Xi(\phi^{-1}\phi(g), \varphi^{-1}\varphi(u))$ and $r'\chi'(\varphi(u)) = r''\chi(\varphi^{-1}\varphi(u))$ for $r'' > 0$. Hence (Σ', χ') is C-equivalent to (Σ, χ).

3. Transitive.
   Suppose (Σ, χ) is C-equivalent to (Σ', χ') and (Σ', χ') is C-equivalent to (Σ'', χ'') then there exist diffeomorphisms ϕ_1 and ϕ_2 and φ_1 and φ_2 such that

$$\begin{cases} T_g\phi_1 \cdot \Xi(g,u) = \Xi'(\phi_1(g), \varphi_1(u)) \\ T_g\phi_2 \cdot \Xi'(g,u) = \Xi''(\phi_2(g), \varphi_2(u)) \end{cases} \quad \text{and} \quad \begin{cases} r\chi(u) = \chi'(\varphi_1(u)) \\ \chi'(u) = r'\chi''(\varphi_2(u)) \quad \text{for} \quad r, r' > 0. \end{cases}$$

   Let $\phi = \phi_2 \circ \phi_1$ and $\varphi = \varphi_2 \circ \varphi_1$ then

$$\begin{aligned} T_g\phi \cdot \Xi(g,u) &= T_{\phi_1(g)}\phi_2 \cdot T_g\phi_1 \cdot \Xi(g,u) \quad \text{Proposition A.1.10} \\ &= T_{\phi_1(g)}\phi_2 \cdot \Xi'(\phi_1(g), \varphi_1(u)) \\ &= \Xi''(\phi_2 \circ \phi_1(g), \varphi_2 \circ \varphi_1(u)) \\ &= \Xi''(\phi(g), \varphi(u)). \end{aligned}$$

   Also

$$\begin{aligned} r\chi(u) &= \chi'(\varphi_1(u)) \\ &= r'\chi''(\varphi_2(\varphi_1(u))) \\ &= r'\chi''(\varphi(u)) \\ \iff r''\chi(u) &= \chi''(\varphi(u)) \quad \text{for}\, r'' = \frac{r}{r'} > 0. \end{aligned}$$

   Hence (Σ, χ) is C-equivalent to (Σ'', χ'').

Thus C-equivalence is an equivalence relation.

The controlled trajectory $(g(\cdot), u(\cdot))$ is a **virtually optimal control trajectory** (shortly VOCT) of (Σ, χ) if it is a solution of the associated optimal control problem. Also, $(g(\cdot), u(\cdot))$ is a **extremal controlled trajectory** (shortly ECT) of (Σ, χ) if it satisfies the necessary conditions of the maximum principle (Refer to section A.3.4.). Propositions 3.1.3 and 3.1.4 show that the VOCTs as well as the ECTs of C-equivalent systems are in one-to-one correspondence.

3.1.3 PROPOSITION. *If (Σ, χ) is C-equivalent to (Σ', χ') then their VOCTs are in one-to-one correspondence.*

PROOF. (Refer to section A.3.4.) Let $(g(\cdot), u(\cdot))$ be a VOCT of (Σ, χ) we will show that $(\phi \circ g(\cdot), \varphi \circ u(\cdot))$ is the unique VOCT of (Σ', χ') corresponding to $(g(\cdot), u(\cdot))$. First we show that $(\phi \circ g(\cdot), \varphi \circ u(\cdot))$ is the unique trajectory-control pair of Σ' corresponding to $(g(\cdot), u(\cdot))$ of Σ. Since (Σ, χ) and (Σ', χ') are C-equivalent there exist isomorphisms $\phi : \mathsf{G} \to \mathsf{G}$ and $\varphi : \mathbb{R}^\ell \to \mathbb{R}^\ell$ such that $T_g\phi \cdot \Xi(g,u) = \Xi'(\phi(g), \varphi(u))$ and $r\chi(u) = \chi'(\varphi(u))$ for some $r > 0$. For almost every t we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{dt}}\phi(g(t)) &= T_{g(t)}\phi \cdot \dot{g}(t) \\ &= T_{g(t)}\phi \cdot \Xi(g(t), u(t)) = \Xi'(\phi(g(t)), \varphi(u(t))). \end{aligned}$$

That is $(\phi(g(\cdot)), \varphi(u(\cdot)))$ is a trajectory-control pair of Σ'. Let $(g_1(\cdot), u_1(\cdot))$ and $(g_2(\cdot), u_2(\cdot))$ be trajectory-control pairs of Σ. Suppose $\phi(g_1(\cdot)) = \phi(g_2(\cdot))$ and that $\varphi(u_1(\cdot)) = \varphi(u_2(\cdot))$ then we have

$g_1(\cdot) = \phi^{-1}(\phi(g_1(\cdot))) = \phi^{-1}(\phi(g_2(\cdot))) = g_2(\cdot)$ and $u_1(\cdot) = \varphi^{-1}(\varphi(u_1(\cdot))) = \varphi^{-1}(\varphi(u_2(\cdot))) = u_2(\cdot)$. Next we show that $(\phi(g(\cdot)), \varphi(u(\cdot)))$ is the unique VOCT of (Σ', χ'). Since $(g(\cdot), u(\cdot))$ is a VOCT we have that

$$\mathcal{J}(u(\cdot)) = \int_0^T \chi(u(t)) \leq \int_0^T \chi(v(t))dt = \mathcal{J}(v(t)), \quad \forall v(t) \in \mathbb{R}^\ell.$$

Now

$$\begin{aligned}\mathcal{J}'(\varphi(u(\cdot))) &= \int_0^T (\chi' \circ \varphi)(u(t)) \\ &= r\int_0^T \chi(u(t))dt \leq r\int_0^T \chi(v(t))dt \quad \text{for some } r > 0 \text{ and } \forall v(t) \in \mathbb{R}^\ell.\end{aligned}$$

That is $\phi(g(\cdot), \varphi(u(\cdot)))$ is a VOCT and VOCTs are mapped injectively.

Next we show that VOCTs are mapped surjectively. Suppose $(g'(\cdot), u'(\cdot))$ is a VOCT of (Σ', χ') we show that there exists a VOCT $(g(\cdot), u(\cdot))$ of (Σ, χ) such that $(g'(\cdot), u'(\cdot)) = (\phi \circ g(\cdot), \varphi \circ u(\cdot))$. Since ϕ is an isomorphism, there exists $g \in \mathsf{G}$ such that $\phi(g) = g'(0)$. Hence there exists a trajectory-control pair $(g(\cdot), \varphi^{-1}(u'(\cdot)))$ of Σ such that $g(0) = g$. We show that Φ maps $(g(\cdot), \varphi^{-1}(u'(\cdot)))$ to $(g'(\cdot), u'(\cdot))$. We have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\phi(g(t)) &= T_{g(t)}\phi \cdot \dot{g}(t) \\ &= T_{g(t)}\phi \cdot \Xi(g(t), \varphi^{-1}(u'(t))) \\ &= \Xi'(\phi(g(t)), (\varphi \circ \varphi^{-1})u'(t)) \\ &= \Xi'(\phi(g(t)), u'(t)).\end{aligned}$$

Now $\phi(g(\cdot))$ and $g'(\cdot)$ solve the same Cauchy problem and are therefore equal. Next we show that $(g(\cdot), \varphi^{-1}(u'(\cdot)))$ is a VOCT of (Σ, χ). Suppose $(g(\cdot), \varphi^{-1}(u'(\cdot)))$ is not a VOCT of (Σ, χ). Then there exists another controlled trajectory $(h(\cdot), v(\cdot))$ such that $h(0) = g(0)$, $h(T) = g(T)$, and

$$\mathcal{J}(v(\cdot)) = \int_0^T \chi(v(t))\mathrm{d}t < \int_0^T \chi(\varphi^{-1}(u'(\cdot))) = \mathcal{J}(\varphi^{-1}(u'(\cdot))).$$

Hence $(\phi \circ h(\cdot), \varphi \circ v(\cdot))$ is a controlled trajectory of (Σ', χ') such that for some $r > 0$

$$\begin{aligned}\mathcal{J}'(\varphi(v(\cdot))) &= \int_0^T (\chi' \circ \varphi)(v(t))\mathrm{d}t = r\int_0^T \chi(v(t))\mathrm{d}t \\ &< r\int_0^T \chi(\varphi^{-1}(u'(t))) = \int_0^T (\chi' \circ \varphi)(\varphi^{-1}(u'(t))\mathrm{d}t = \mathcal{J}'(u'(t)).\end{aligned}$$

This contradicts the fact that $(g'(\cdot), u'(\cdot))$ is a VOCT of (Σ', χ'). Hence $(g(\cdot), \varphi^{-1}(u'(\cdot)))$ is a VOCT of (Σ, χ). VOCTs are therefore mapped both surjectively and injectively and the VOCTs are in one-to-one correspondence.

3.1.4 Proposition. *If (Σ, χ) is C-equivalent (Σ', χ') then their ECTs are in one-to-one correspondence.*

Proof. Let $(g(\cdot), u(\cdot))$ be a ECT of (Σ, χ) we will show that $(\phi \circ g(\cdot), \varphi \circ u(\cdot))$ is the unique ECT of (Σ', χ) corresponding to $(g(\cdot), u(\cdot))$. First we show that $(\phi \circ g(\cdot), \varphi \circ u(\cdot))$ is the unique trajectory-control pair of Σ' corresponding to $(g(\cdot), u(\cdot))$ of Σ. Since (Σ, χ) and (Σ', χ') are C-equivalent there exist isomorphisms $\phi : \mathsf{G} \to \mathsf{G}$ and $\varphi : \mathbb{R}^\ell \to \mathbb{R}^\ell$ such that $T_g\phi \cdot \Xi(g, u) = \Xi'(\phi(g), \varphi(u))$ and $r\chi(u) = \chi'(\varphi(u))$ for some $r > 0$. For almost every t we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}}\phi(g(t)) &= T_{g(t)}\phi \cdot \dot{g}(t)\\ &= T_{g(t)}\phi \cdot \Xi(g(t), u(t)) = \Xi'(\phi(g(t)), \varphi(u(t))).\end{aligned}$$

That is $(\phi(g(\cdot)), \varphi(u(\cdot)))$ is a trajectory-control pair of Σ'. Let $(g_1(\cdot), u_1(\cdot))$ and $(g_2(\cdot), u_2(\cdot))$ be trajectory-control pairs of Σ. Suppose $\phi(g_1(\cdot)) = \phi(g_2(\cdot))$ and that $\varphi(u_1(\cdot)) = \varphi(u_2(\cdot))$ then we have $g_1(\cdot) = \phi^{-1}(\phi(g_1(\cdot))) = \phi^{-1}(\phi(g_2(\cdot))) = g_2(\cdot)$ and $u_1(\cdot) = \varphi^{-1}(\varphi(u_1(\cdot))) = \varphi^{-1}(\varphi(u_2(\cdot))) = u_2(\cdot)$. Next we show that $(\phi(g(\cdot)), \varphi(u(\cdot)))$ is the unique ECT of (Σ', χ'). The Hamiltonian functions associated to (Σ, χ) and (Σ', χ') are given by

$$H_u(g, p) = p(\Xi(u, \mathbf{1})) - \chi(u) \quad \text{and} \quad H'_{u'}(g', p') = p'(\Xi'(u', \mathbf{1})) - \chi'(u'),$$

respectively. Since $(g(\cdot), u(\cdot))$ is a ECT we have that there exists $p(\cdot) : [0, T] \to \mathfrak{g}^*$ such that $\xi(t) = (g(t), p(t))$ satisfies conditions A.5 and A.6 of theorem A.3.6. From the condition $\dot{\xi}(t) = \vec{H}_{u(t)}(\xi(t))$ we have (cf. [25])

$$\dot{g}(t) = \Xi(g(t), u(t)) \quad \text{and} \quad \dot{p}(t) = \mathrm{ad}^*\Xi(u(t), \mathbf{1}) \cdot p(t).$$

Let $g'(\cdot) = \phi \circ g(\cdot)$ and $u'(\cdot) = \varphi \circ u(\cdot)$. Let $p'(\cdot) = r(T_{\phi(\mathbf{1})}\phi^{-1})^* \cdot p(\cdot)$ and $\xi'(t) = (g'(t), p'(t))$ where r is the unique constant associated with Φ. We show that $\xi'(\cdot)$ satisfies conditions A.5 and A.6. First we show that $\xi'(\cdot)$ satisfies A.5. From above we have that $\frac{\mathrm{d}}{\mathrm{dt}}\phi(g(t)) = \Xi'(\phi(g(t)), \varphi(u(t)))$. We have left to show that $\dot{p}'(t) = \mathrm{ad}^*\Xi(\mathbf{1}, u'(t)) \cdot p'(t)$. For $A \in \mathfrak{g}$,

$$\begin{aligned}\mathrm{ad}^*\Xi'(\mathbf{1}, u'(t)) \cdot p'(t) \cdot A &= \mathrm{ad}^*\Xi'(\mathbf{1}, u'(t)) \cdot r(T_{\phi(\mathbf{1})}\phi^{-1})^* \cdot p(t) \cdot A\\ &= r\ (T_{\phi(\mathbf{1})}\phi^{-1})^* \cdot p(t) \cdot ([\Xi'(\mathbf{1}, u'(t)), A])\\ &= rp(t) \cdot (T_{\phi(\mathbf{1})}\phi^{-1} \cdot [\Xi'(\mathbf{1}, u'(t)), A]).\end{aligned}$$

Also

$$\begin{aligned}\dot{p}'(t) \cdot A &= r((T_{\phi(\mathbf{1})}\phi^{-1})^* \cdot \dot{p}(t)) \cdot A\\ &= r\ \dot{p}(t) \cdot (T_{\phi(\mathbf{1})}\phi^{-1} \cdot A)\\ &= r\ (\mathrm{ad}^*\Xi(\mathbf{1}, u(t)) \cdot p(t))(T_{\phi(\mathbf{1})}\phi^{-1} \cdot A)\\ &= r\ p(t) \cdot [\Xi(\mathbf{1}, u(t)), T_{\phi(\mathbf{1})}\phi^{-1} \cdot A]\\ &= r\ p(t) \cdot (T_{\phi(\mathbf{1})}\phi^{-1} \cdot [\Xi'(\mathbf{1}, u'(t)), A]).\end{aligned}$$

Accordingly

$$\dot{p}(t) \cdot A = \mathrm{ad}^*\Xi(1, u'(t)) \cdot p'(t) \cdot A.$$

Hence $\dot{p}'(t) = \mathrm{ad}^*\Xi(\mathbf{1}, u'(t)) \cdot p'(t)$.

Next we show that $\xi'(\cdot)$ satisfies A.6. Suppose $\xi'(t)$ does not satisfy A.6 i.e. there exists $v' \in \mathbb{R}^\ell$ and $\tilde{t} \in [0, T]$ such that $H_{u'(\tilde{t})}(g'(\tilde{t}), p'(\tilde{t})) < H_{v'(\tilde{t})}(g'(\tilde{t}), p'(\tilde{t}))$. Then

$$\begin{aligned}
&H_{u'(\tilde{t})}(g'(\tilde{t}), p'(\tilde{t})) < H_{v'(\tilde{t})}(g'(\tilde{t}), p'(\tilde{t})) \\
\iff\ & p'(\tilde{t})(\Xi'(\mathbf{1}, u'(\tilde{t}))) - \chi'(u'(\tilde{t})) < p'(\tilde{t})(\Xi'(\mathbf{1}, v'(\tilde{t}))) - \chi'(v'(\tilde{t})) \\
\iff\ & r(T_{\phi(\mathbf{1})}\phi^{-1})^* \cdot p(\tilde{t}) \cdot (\Xi'(\mathbf{1}, u'(\tilde{t}))) - \chi'(u'(\tilde{t})) < r(T_{\phi(\mathbf{1})}\phi^{-1})^* \cdot p(\tilde{t}) \cdot (\Xi'(\mathbf{1}, v'(\tilde{t})) - \chi'(v'(\tilde{t}))) \\
\iff\ & rp(\tilde{t}) \cdot (T_{\phi(\mathbf{1})}\phi^{-1} \cdot \Xi'(\mathbf{1}, u'(\tilde{t}))) - r\chi \circ \varphi^{-1}(u'(\tilde{t})) < rp(\tilde{t}) \cdot (T_{\phi(\mathbf{1})}\phi^{-1} \cdot (\Xi'(\mathbf{1}, v'(\tilde{t})))) - r\chi \circ \varphi^{-1}(v'(\tilde{t})) \\
\iff\ & rp(\tilde{t}) \cdot (\Xi(\mathbf{1}, u(\tilde{t}))) - r\chi \circ \varphi^{-1} \circ \varphi(u(\tilde{t})) < rp(\tilde{t}) \cdot (\Xi(\mathbf{1}, \varphi^{-1}(v'(\tilde{t})))) - r\chi \circ \varphi^{-1}(v'(\tilde{t})) \\
\iff\ & p(\tilde{t}) \cdot (\Xi(\mathbf{1}, u(\tilde{t}))) - \chi(u(\tilde{t})) < p(\tilde{t}) \cdot \Xi(\mathbf{1}, \varphi^{-1}(v'(\tilde{t}))) - \chi \circ \varphi^{-1}(v'(\tilde{t})) \\
\iff\ & H_{u(\tilde{t})}(g(\tilde{t}), p(\tilde{t})) < H_{\varphi^{-1}(v'(\tilde{t}))}(g(\tilde{t}), p(\tilde{t})).
\end{aligned}$$

Which contradicts the fact that $\xi(\cdot)$ satisfies A.6. That is ECTs are mapped injectively.

Next we show that ECTs are mapped surjectively. suppose $(g'(\cdot), u'(\cdot))$ is an ECT of (Σ', χ') we show that there exists an ECT $(g(\cdot), u(\cdot))$ of (Σ, χ) such that $(g'(\cdot), u'(\cdot)) = (\phi \circ g(\cdot), \varphi \circ u(\cdot))$. Since ϕ is an isomorphism, there exists $g \in \mathsf{G}$ such that $\phi(g) = g'(0)$. Hence there exists a trajectory-control pair $(g(\cdot), \varphi^{-1}(u'(\cdot)))$ of Σ such that $g(0) = g$. We show that Φ maps $(g(\cdot), \varphi^{-1}(u'(\cdot)))$ to $(g'(\cdot), u'(\cdot))$. We have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\phi(g(t)) &= T_{g(t)}\phi \cdot \dot{g}(t) \\
&= T_{g(t)}\phi \cdot \Xi(g(t), \varphi^{-1}(u'(t))) \\
&= \Xi'(\phi(g(t)), (\varphi \circ \varphi^{-1})u'(t)) \\
&= \Xi'(\phi(g(t)), u'(t)).
\end{aligned}$$

Now $\phi(g(\cdot))$ and $g'(\cdot)$ solve the same Cauchy and are therefore equal. Next we show that $(g(\cdot), \varphi^{-1}(u'(\cdot))) = (g(\cdot), u(\cdot))$ is an ECT of (Σ, χ). Since $(g'(\cdot), u'(\cdot))$ is an ECT, there exists $p'(\cdot) : [0, T] \to \mathfrak{g}^*$ such that $\xi'(t) = (g'(t), p'(t))$ satisfies conditions A.5 and A.6 of the maximum principle (theorem A.3.6). From condition A.5 we have that

$$\dot{g}'(t) = \Xi(g'(t), u'(t)) \quad \text{and} \quad \dot{p}'(t) = \mathrm{ad}^*\Xi'(\mathbf{1}, u'(t)) \cdot p'(t).$$

Let $p(\cdot) = \frac{1}{r}(T_{\mathbf{1}}\phi)^* \cdot p'(\cdot)$ and $\xi(t) = (g(t), p(t))$ (where $r > 0$ is the unique constant associated to Φ). First we show that $\xi(\cdot)$ satisfies A.5. We have that $\dot{g}(t) = \Xi(g(t), \varphi^{-1}(u'(t)))$ and so we have left to show that $\dot{p}(t) = \mathrm{ad}^*\Xi(\mathbf{1}, u(t)) \cdot p(t)$. For $A \in \mathfrak{g}$,

$$\begin{aligned}
\mathrm{ad}^*\Xi(1, u(t)) \cdot p(t) \cdot A &= \mathrm{ad}^*\Xi(\mathbf{1}, u(t)) \cdot \frac{1}{r}(T_{\mathbf{1}}\phi)^* \cdot p'(t) \cdot A \\
&= \frac{1}{r}\ (T_{\mathbf{1}}\phi)^* \cdot p'(t) \cdot ([\Xi(\mathbf{1}, u(t)), A]) \\
&= \frac{1}{r}p'(t) \cdot (T_{\mathbf{1}}\phi \cdot [\Xi(\mathbf{1}, u(t)), A]).
\end{aligned}$$

Also

$$\begin{aligned}
\dot p(t)\cdot A &= \frac{1}{r}((T_{\mathbf{1}}\phi)^*\cdot \dot p'(t))\cdot A\\
&= \frac{1}{r}\,\dot p'(t)\cdot(T_{\mathbf{1}}\phi\cdot A)\\
&= \frac{1}{r}\,(\mathrm{ad}^*\Xi'(\mathbf{1},u'(t))\cdot p'(t))(T_{\mathbf{1}}\phi\cdot A)\\
&= \frac{1}{r}\,p'(t)\cdot[\Xi'(\mathbf{1},u'(t)),T_{\mathbf{1}}\phi\cdot A]\\
&= \frac{1}{r}\,p'(t)\cdot(T_{\mathbf{1}}\phi\cdot[\Xi(\mathbf{1},u(t)),A]).
\end{aligned}$$

And so $\dot p(t) = \mathrm{ad}^*\Xi(\mathbf{1},u(t))\cdot p(t)\cdot A$.

Finally, we show that $\xi(\cdot)$ satisfies A.6. Suppose this is not the case, i.e., there exists $v\in\mathbb{R}^\ell$ and $\tilde t\in[0,T]$ such that $H_{u(\tilde t)}(g(\tilde t),p(\tilde t)) < H_{v(\tilde t)}(g(\tilde t),p(\tilde t))$. Then

$$\begin{aligned}
& H_{u(\tilde t)}(g(\tilde t),p(\tilde t)) < H_{v(\tilde t)}(g(\tilde t),p(\tilde t))\\
\iff{}& p(\tilde t)(\Xi(\mathbf{1},u(\tilde t))) - \chi(u(\tilde t)) < p(\tilde t)(\Xi(\mathbf{1},v(\tilde t))) - \chi(v(\tilde t))\\
\iff{}& \frac{1}{r}(T_{\mathbf{1}}\phi)^*\cdot p'(\tilde t)\cdot(\Xi(\mathbf{1},u(\tilde t))) - \chi(u(\tilde t)) < \frac{1}{r}(T_{\mathbf{1}}\phi)^*\cdot p'(\tilde t)\cdot(\Xi(\mathbf{1},v(\tilde t)) - \chi(v(\tilde t)))\\
\iff{}& \frac{1}{r}p'(\tilde t)\cdot(T_{\mathbf{1}}\phi\cdot\Xi(\mathbf{1},u(\tilde t))) - \frac{1}{r}\chi'\circ\varphi(u(\tilde t)) < \frac{1}{r}p'(\tilde t)\cdot(T_{\mathbf{1}}\phi\cdot(\Xi(\mathbf{1},v(\tilde t)))) - \frac{1}{r}\chi'\circ\varphi(v(\tilde t))\\
\iff{}& \frac{1}{r}p'(\tilde t)\cdot(\Xi'(\mathbf{1},u'(\tilde t))) - \frac{1}{r}\chi'(u'(\tilde t)) < \frac{1}{r}p'(\tilde t)\cdot(\Xi'(\mathbf{1},\varphi(v(\tilde t))) - \frac{1}{r}\chi'(\varphi(v(\tilde t)))\\
\iff{}& p'(\tilde t)\cdot(\Xi'(\mathbf{1},u'(\tilde t))) - \chi'(u'(\tilde t)) < p'(\tilde t)\cdot(\Xi'(\mathbf{1},\varphi(v(\tilde t)))) - \chi'(\varphi(v(\tilde t)))\\
\iff{}& H_{u'(\tilde t)}(g'(\tilde t),p'(\tilde t)) < H_{\varphi(v(\tilde t))}(g'(\tilde t),p'(\tilde t)).
\end{aligned}$$

Which is a contradiction to the fact that $\xi'(\cdot)$ satisfies A.6.

Hence ECTs of C-equivalent systems are in one-to-one correspondence.

3.1.5 Proposition. (cf. [17]) *Two cost-extended control systems* (Σ,χ) *and* (Σ',χ') *on a simply connected Lie group* G *are C-equivalent if and only if there exists a Lie algebra automorphism* $\psi:\mathfrak{g}\to\mathfrak{g}$ *and an affine isomorphism* $\varphi:\mathbb{R}^\ell\to\mathbb{R}^\ell$ *such that* $\psi\cdot\Gamma=\Gamma'$ *and* $\chi'\circ\varphi = r\chi$ *for some* $r>0$.

Proof. This follows directly from proposition 2.3.7.

3.1.6 Corollary. (cf. [17]) *If* (Σ,χ) *and* (Σ',χ') *are C-equivalent, then* Σ *and* Σ' *are DF-equivalent.*

3.1.7 Definition. For an ℓ-input left-invariant control affine system Σ on a simply connected Lie group G, let $\mathcal{T}_\Sigma$ denote the group of feedback transformations leaving Σ invariant. Formally,

$$\mathcal{T}_\Sigma = \left\{\varphi\in\mathsf{Aff}(\mathbb{R}^\ell) : \exists\psi\in\mathsf{Aut}(\mathfrak{g}), \psi\cdot\Xi(\mathbf{1},u) = \Xi(\mathbf{1},\varphi(u))\right\}.$$

3.1.8 Theorem. (cf. [17]) (Σ,χ) *and* (Σ,χ') *on a simply connected Lie group* G *are C-equivalent if and only if there exists* $\varphi\in\mathcal{T}_\Sigma$ *such that* $\chi' = r\chi\circ\varphi$ *for some* $r>0$.

Proof. Suppose (Σ, χ) and (Σ, χ') are C-equivalent. Then there exists $\psi \in \mathsf{Aut}(\mathfrak{g})$ and an affine isomorphism $\bar{\varphi} : \mathbb{R}^\ell \to \mathbb{R}^\ell$ such that $\psi \cdot \Xi(\mathbf{1}, u) = \Xi'(\mathbf{1}, \bar{\varphi}(u))$ as well as a constant $r > 0$ such that $\chi' \circ \bar{\varphi} = r\chi$(proposition3.1.5). Therefore, $\varphi = \bar{\varphi}^{-1} \in \mathcal{T}_\Sigma$ and $\chi' = r\chi \circ \varphi$.

Conversely, suppose there exists some $\bar{\varphi} \in \mathcal{T}_\Sigma$ such that $\chi' = r\chi \circ \bar{\varphi}^{-1}$ for some $r > 0$, i.e., $\chi' \circ \bar{\varphi} = r\chi$. Since $\bar{\varphi} \in \mathcal{T}_\Sigma$ there exist $\psi \in \mathsf{Aut}(\mathfrak{g})$ such that $\psi \cdot \Xi(\mathbf{1}, u) = \Xi(\mathbf{1}, \bar{\varphi}(u))$. The result follows from proposition 3.1.5.

3.1.1 Classification under C-equivalence

Recall from proposition 1.1.8 that the automorphism group of $\mathsf{Aut}(\mathfrak{h}_3)$ is given by

$$\mathsf{Aut}(\mathfrak{h}_3) = \left\{ \begin{bmatrix} v_2 w_3 - v_3 w_2 & v_1 & w_1 \\ 0 & v_2 & w_2 \\ 0 & v_3 & w_3 \end{bmatrix} \middle| v_1, v_2, v_3, w_1, w_2, w_3 \in \mathbb{R}, v_2 w_3 \neq v_3 w_2 \right\}.$$

3.1.9 Proposition. *Any controllable two-input homogeneous cost-extended system on* H_3 *is C-equivalent to exactly one of the following systems*

$$(\bar{\Sigma}^{(2,0)}, \bar{\chi}_1) : \begin{cases} \bar{\Xi}(\mathbf{1}, u) = u_1 E_2 + u_2 E_3 \\ \bar{\chi}_1 = u_1^2 + u_2^2, \end{cases}$$

$$(\bar{\Sigma}^{(2,0)}, \bar{\chi}_2) : \begin{cases} \bar{\Xi}(\mathbf{1}, u) = u_1 E_2 + u_2 E_3 \\ \bar{\chi}_2 = (u_1 - 1)^2 + u_2^2. \end{cases}$$

Proof. Let (Σ, χ) be a controllable two-input homogeneous system. We have that any two-input homogeneous system is DF-equivalent to $\bar{\Xi}(\mathbf{1}, u) = u_1 E_2 + u_2 E_3$, therefore by corollary 3.1.6 we have that the system is C-equivalent to $(\bar{\Sigma}^{(2,0)}, \chi_0)$ for some cost function χ_0. We determine $\mathcal{T}_{\bar{\Sigma}^{(2,0)}}$. Since H_3 is simply connected we have that $\mathsf{dAut}(\mathsf{H}_3) = \mathsf{Aut}(\mathfrak{h}_3)$. Now

$$\begin{bmatrix} v_2 w_3 - v_3 w_2 & v_1 & w_1 \\ 0 & v_2 & w_2 \\ 0 & v_3 & w_3 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \\ v_3 & w_3 \end{bmatrix}$$

and

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{bmatrix}.$$

Here $\begin{bmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{bmatrix} = \varphi$. Since there must exist $\psi \in \mathsf{Aut}(\mathfrak{h}_3)$ such that $\psi \cdot \Xi(\mathbf{1}, u) = \Xi'(\mathbf{1}, \varphi(u))$ it follows that $\varphi_{11}\varphi_{22} - \varphi_{12}\varphi_{21} \neq 0$ with no further restrictions and we have that $\mathcal{T}_{\bar{\Sigma}^{(2,0)}} = \mathsf{GL}(2, \mathbb{R})$.

Now $\chi_0 : u \mapsto (u-\mu)^\intercal Q(u-\mu)$ for some positive definite matrix $Q = \begin{bmatrix} a_1 & b \\ b & a_2 \end{bmatrix}$. We transform χ_0 by composing it with elements of $\mathcal{T}_{\bar{\Sigma}^{(2,0)}}$.

From positive definiteness of Q we have that $a_1 > 0$, $a_2 > 0$ and $a_1 a_2 - b^2 > 0$ and so

$$\varphi_1 = \begin{bmatrix} \frac{1}{\sqrt{a_1 - \frac{b^2}{a_2}}} & 0 \\ -\frac{b}{a_2\sqrt{a_1 - \frac{b^2}{a_2}}} & \frac{1}{\sqrt{a_2}} \end{bmatrix}, \quad \left(\det \varphi_1 = \frac{1}{\sqrt{a_2\left(a_1 - \frac{b^2}{a_2}\right)}} \neq 0 \right)$$

is an element of $\mathsf{GL}(2,\mathbb{R})$ such that

$$\chi_1(u) = (\chi_0 \circ \varphi_1)(u) = (u-\mu')^\intercal \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (u-\mu')$$

for some $\mu' \in \mathbb{R}^2$. If $\mu' = \mathbf{0}$, then (Σ, χ) is C-equivalent to $(\bar{\Sigma}^{(2,0)}, \bar{\chi}_1)$ (by theorem 3.1.8). Suppose $\mu' \neq \mathbf{0}$. There exists $\alpha > 0$ and $\theta \in \mathbb{R}$ such that $\mu'_1 = \alpha\cos\theta$ and $\mu'_2 = \alpha\sin\theta$. Hence, $\varphi_2 = \begin{bmatrix} \alpha\cos\theta & -\alpha\sin\theta \\ \alpha\sin\theta & \alpha\cos\theta \end{bmatrix} \in \mathcal{T}_{\Sigma^{(2,0)}}$ and

$$\chi_2(u) = \frac{1}{\alpha^2}(\chi_1 \circ \varphi_2)(u) = \left(u - \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)^\top \left(u - \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = (u_1 - 1)^2 + u_2^2.$$

Therefore (Σ, χ) is C-equivalent to $(\bar{\Sigma}^{(2,0)}, \bar{\chi}_2)$ (by theorem 3.1.8).

The systems $(\bar{\Sigma}^{(2,0)}, \bar{\chi}_1)$ and $(\bar{\Sigma}^{(2,0)}, \bar{\chi}_2)$ are distinct. Indeed, suppose $(\bar{\Sigma}^{(2,0)}, \bar{\chi}_1)$ is C-equivalent to $(\bar{\Sigma}^{(2,0)}, \bar{\chi}_2)$. Then there exists $\varphi \in \mathcal{T}_\Sigma$ such that $\bar{\chi}_2 = r\bar{\chi}_1 \circ \varphi$ for some $r > 0$. Let $\varphi = \begin{bmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{bmatrix} \in \mathcal{T}_\Sigma$. Then

$$\begin{aligned} & \bar{\chi}_2 = r\bar{\chi}_1 \circ \varphi \\ \iff & (u_1 - 1)^2 + u_2^2 = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}^\top \begin{bmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{bmatrix}^\top \begin{bmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ \iff & u_1^2 - 2u_1 + 1 + u_2^2 = r\left(u_1^2\varphi_{11}^2 + 2u_1u_2\varphi_{11}\varphi_{12} + u_2^2\varphi_{12}^2 + u_1^2\varphi_{21}^2 + 2u_1u_2\varphi_{21}\varphi_{22} + u_2^2\varphi_{22}^2\right). \end{aligned}$$

Setting $(u_1, u_2) = (0,0)$ yields $1 = 0$. Hence, $(\bar{\Sigma}^{(2,0)}, \bar{\chi}_1)$ is not C-equivalent to $(\bar{\Sigma}^{(2,0)}, \bar{\chi}_2)$.

3.1.10 Proposition. *1. For the system*

$$(\bar{\Sigma}^{(2,0)}, \bar{\chi}_1) : \begin{cases} \bar{\Xi}(\mathbf{1}, u) = u_1 E_2 + u_2 E_3 \\ \bar{\chi}_\alpha = u_1^2 + u_2^2 \end{cases}$$

the extremal control is given by $u_1 = p_2$ *and* $u_2 = p_3$, *where* $H(p) = \frac{1}{2}\left(p_2^2 + p_3^2\right)$.

2. For the system

$$(\bar{\Sigma}, \bar{\chi}_\alpha) : \begin{cases} \bar{\Xi}(\mathbf{1}, u) = u_1 E_2 + u_2 E_3 \\ \bar{\chi}_2 = (u_1 - 1)^2 + u_2^2, \end{cases}$$

the extremal control is given by $u_1 = p_2 + 1$ *and* $u_2 = p_3$, *where* $H(p) = p_2 + \frac{1}{2}\left(p_2^2 + p_3^2\right)$.

Proof. 1. The associated Hamiltonian function on $T^*\mathsf{H}_3$ is given by

$$H_u^{\frac{1}{2}}(g,p) = -\frac{1}{2}u_1^2 - \frac{1}{2}u_2^2 + u_1 p_2 + u_2 p_3.$$

Setting the partial derivatives of $H_u^{\frac{1}{2}}(g,p)$ with respect to u_1 and u_2 equal to zero yields

$$\begin{cases} u_1 = p_2 \\ u_2 = p_3. \end{cases}$$

On substituting back into $H_u^{\frac{1}{2}}(g,p)$ we have

$$H(p) = \frac{1}{2}\left(p_2^2 + p_3^2\right).$$

2. The associated Hamiltonian function on $T^*\mathsf{H}_3$ is given by

$$H_u^{\frac{1}{2}}(g,p) = -\frac{1}{2}(u_1 - 1)^2 - \frac{1}{2}u_2^2 + u_1 p_2 + u_2 p_3.$$

Setting the partial derivatives of $H_u^{\frac{1}{2}}(g,p)$ with respect to u_1 and u_2 equal to zero yields

$$\begin{cases} u_1 = p_2 + 1 \\ u_2 = p_3. \end{cases}$$

On substituting back into $H_u^{\frac{1}{2}}(g,p)$ we have

$$H(p) = p_2 + \frac{1}{2}\left(p_2^2 + p_3^2\right).$$

3.1.11 Proposition. *Any controllable two-input inhomogeneous cost-extended system on* H_3 *is C-equivalent to exactly one of*

$$(\bar{\Sigma}^{(2,1)}, \bar{\chi}_\alpha) : \begin{cases} \bar{\Xi}(\mathbf{1}, u) = E_1 + u_1 E_2 + u_2 E_3 \\ \bar{\chi}_\alpha(u) = (u_1 - \alpha)^2 + u_2^2. \end{cases}$$

Here each α *parametrises a distinct family of class representatives.*

Proof. Let (Σ, χ) be a controllable two-input inhomogeneous system. We have that any such system is DF-equivalent to $\bar{\Xi}(\mathbf{1}, u) = E_1 + u_1 E_2 + u_2 E_3$, therefore by corollary 3.1.6 we have that the system is C-equivalent to $(\bar{\Sigma}^{(2,1)}, \chi_0)$ for some cost function χ_0. We determine $\mathcal{T}_{\bar{\Sigma}^{(2,1)}}$. Since H_3 is simply connected we have that $d\mathsf{Aut}(\mathsf{H}_3) = \mathsf{Aut}(\mathfrak{h}_3)$. Now

$$\begin{bmatrix} v_2w_3 - v_3w_2 & v_1 & w_1 \\ 0 & v_2 & w_2 \\ 0 & v_3 & w_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} v_2w_3 - v_3w_2 & v_1 & w_1 \\ 0 & v_2 & w_2 \\ 0 & v_3 & w_3 \end{bmatrix}$$

and

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{bmatrix}.$$

Here $\begin{bmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{bmatrix} = \varphi$. Since there must exist $\psi \in \mathsf{Aut}(\mathfrak{h}_3)$ such that $\psi \cdot \Xi(\mathbf{1}, u) = \Xi'(\mathbf{1}, \varphi(u))$ it follows that $\varphi_{11}\varphi_{22} - \varphi_{12}\varphi_{21} = 1$ with no further restrictions and we have that $\mathcal{T}_{\bar{\Sigma}^{(2,1)}} = \mathsf{SL}(2, \mathbb{R})$. Now $\chi_0 : u \mapsto (u - \mu)^\intercal Q (u - \mu)$ for some positive definite matrix $Q = \begin{bmatrix} a_1 & b \\ b & a_2 \end{bmatrix}$. We transform χ_0 by composing it with elements of $\mathcal{T}_{\bar{\Sigma}^{(2,1)}}$.

From positive definiteness of Q we have that $a_1 > 0$, $a_2 > 0$ and $a_1 a_2 - b^2 > 0$ and so

$$\varphi_1 = \begin{bmatrix} 1 & -\frac{b}{a_1} \\ 0 & 1 \end{bmatrix}, \quad (\det \varphi_1 = 1)$$

is an element of $\mathsf{SL}(2, \mathbb{R})$ such that

$$\chi_1(u) = (\chi_0 \circ \varphi_1)(u) = (u - \mu')^\intercal \begin{bmatrix} a_1 & 0 \\ 0 & a_2 - \frac{b^2}{a_1} \end{bmatrix} (u - \mu')$$

for $\mu' = \varphi_1^{-1}(\mu)$. Also

$$\varphi_2 = \begin{bmatrix} \frac{(a_1a_2 - b^2)^{1/4}}{\sqrt{a_1}} & 0 \\ 0 & \frac{\sqrt{a_1}}{(a_1a_2 - b^2)^{1/4}} \end{bmatrix}, \quad (\det \varphi_2 = 1)$$

is an element of $\mathsf{SL}(2, \mathbb{R})$ such that

$$\chi_2(u) = \frac{1}{\sqrt{a_1 a_2 - b_2^2}} (\chi_1 \circ \varphi_2)(u) = (u - \mu')^\intercal \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (u - \mu')$$

for $\mu' = \varphi_2^{-1}(\varphi_1^{-1}(\mu))$. $\mathsf{O}(2)$ preserves $\mathbf{1}$. Suppose $\mu' = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$ for some $\alpha_1, \alpha_2 \in \mathbb{R}$. Then

$$\varphi_3 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \in \mathsf{O}(2)$$

with

$$\varphi_3^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

Now

$$\varphi_3^{-1} \cdot \mu' = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} \alpha_1\cos\theta + \alpha_2\sin\theta \\ -\alpha_1\sin\theta + \alpha_2\cos\theta \end{bmatrix}.$$

Solving for θ such that $-\alpha_1\sin\theta + \alpha_2\cos\theta = 0$ we get

$$\theta = \arccos\left(\frac{\alpha_1}{\sqrt{\alpha_1^2 + \alpha_2^2}}\right)$$

If $\alpha_1, \alpha_2 = 0$ there is no need for simplification. In the case where either α_1 or $\alpha_2 \neq 0$ we have that $\left|\frac{\alpha_1}{\sqrt{\alpha_1^2+\alpha_2^2}}\right| \leq 1$ and so $\arccos\left(\frac{\alpha_1}{\sqrt{\alpha_1^2+\alpha_2^2}}\right)$ is defined and there exists θ such that $\varphi_2^{-1} \cdot \mu' = \begin{bmatrix} \alpha \\ 0 \end{bmatrix}$ for some $\alpha \in \mathbb{R}$ (In fact $\alpha = \sqrt{\alpha_1^2 + \alpha_2^2} \geq 0$). Therefore $\bar{\chi}_\alpha = \frac{1}{\sqrt{a_1a_2 - b_2^2}}\chi_0(\varphi_1 \circ \varphi_2 \circ \varphi_3)$ and $\bar{\chi}_\alpha(u) = (u_1 - \alpha)^2 + u_2^2$. We verify that $r\bar{\chi}_\alpha \circ \varphi \neq \bar{\chi}_{\alpha'}$ for any $\alpha \neq \alpha'$, $r > 0$, and $\varphi \in \mathcal{T}_{\bar{\Sigma}^{(2,1)}}$. Let

$$\varphi = \begin{bmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{bmatrix}.$$

Then

$$\begin{aligned} & \bar{\chi}_{\alpha'}(u) = r\bar{\chi}_\alpha \circ \varphi(u) \quad \forall u \in \mathbb{R}^2 \\ \iff & (u_1 - \alpha')^2 + u_2^2 = ru_1^2\varphi_{11}^2 - 2ru_1\alpha\varphi_{11}^2 + r\alpha^2\varphi_{11}^2 + 2ru_1u_2\varphi_{11}\varphi_{12} - 2ru_2\alpha\varphi_{11}\varphi_{12} \\ & \qquad + ru_2^2\varphi_{12}^2 + ru_1^2\varphi_{21}^2 - 2ru_1\alpha\varphi_{21}^2 + r\alpha^2\varphi_{21}^2 + 2ru_1u_2\varphi_{21}\varphi_{22} \\ & \qquad - 2ru_2\alpha\varphi_{21}\varphi_{22} + ru_2^2\varphi_{22}^2, \quad \forall u_1, u_2 \in \mathbb{R}. \end{aligned}$$

Setting $(u_1, u_2) = (\alpha', 0)$ yields

$$r\alpha^2\varphi_{11}^2 - 2r\alpha\alpha'\varphi_{11}^2 + r\alpha'^2\varphi_{11}^2 + r\alpha^2\varphi_{21}^2 - 2r\alpha\alpha'\varphi_{21}^2 + r\alpha'^2\varphi_{21}^2 = 0.$$

Which on solving yields $r = 0$ or $(\varphi_{11}, \varphi_{21}) = (0, 0)$ when $\alpha \neq \alpha'$. Hence the systems $(\bar{\Sigma}^{(2,1)}, \bar{\chi}_{\alpha'})$ and $(\bar{\Sigma}^{(2,1)}, \bar{\chi}_\alpha)$ are not C-equivalent.

3.1.12 **Proposition.** *For the system*

$$(\bar{\Sigma}^{(2,1)}, \bar{\chi}_\alpha) : \begin{cases} \bar{\Xi}(\mathbf{1}, u) = E_1 + u_1 E_2 + u_2 E_3 \\ \bar{\chi}_\alpha = (u_1 - \alpha)^2 + u_2^2, \quad \alpha > 0 \end{cases}$$

the extremal control is given by $u_1 = \alpha + p_2$ *and* $u_2 = p_3$*, where* $H(p) = p_1 + \alpha p_2 + \frac{1}{2}\left(p_2^2 + p_3^2\right)$.

Proof. The associated Hamiltonian function on $T^*\mathsf{H}_3$ is given by

$$H_u^{\frac{1}{2}}(g,p) = -\frac{1}{2}(u_1 - \alpha)^2 - \frac{1}{2}u_2^2 + p_1 + u_1 p_2 + u_2 p_3.$$

Setting the partial derivatives of $H_u^{\frac{1}{2}}(g,p)$ with respect to u_1 and u_2 equal to zero:

$$\begin{cases} \alpha + p_2 - u_1 = 0 \\ p_3 - u_2 = 0 \end{cases}$$

yields

$$\begin{cases} u_1 = \alpha + p_2 \\ u_2 = p_3. \end{cases}$$

On substituting back into $H_u^{\frac{1}{2}}(g,p)$ we have

$$H(p) = p_1 + \alpha p_2 + \frac{1}{2}\left(p_2^2 + p_3^2\right).$$

3.1.13 **Proposition.** *Any controllable three-input homogeneous cost-extended system on* H_3 *is C-equivalent to exactly one of*

$$(\bar{\Sigma}^{(3,0)}, \bar{\chi}_{\alpha_1,\alpha_2}) : \begin{cases} \bar{\Xi}(\mathbf{1}, u) = u_1 E_1 + u_2 E_2 + u_3 E_3 \\ \bar{\chi}_{\alpha_1,\alpha_2}(u) = (u_1 - \alpha_1)^2 + (u_2 - \alpha_2)^2 + u_3^2. \end{cases}$$

Here each $\alpha_1, \alpha_2 \geq 0$ *and each parameter parametrises a distinct family of class representatives.*

Proof. Let (Σ, χ) be a controllable three-input homogeneous system. We have that any such system is DF-equivalent to $\bar{\Xi}(\mathbf{1}, u) = u_1 E_1 + u_1 E_2 + u_2 E_3$, therefore by corollary 3.1.6 we have that the system is C-equivalent to $(\bar{\Sigma}^{(3,0)}, \chi_0)$ for some cost function χ_0. We determine $\mathcal{T}_{\bar{\Sigma}^{(3,0)}}$. Since H_3 is simply connected we have that $d\mathsf{Aut}(\mathsf{H}_3) = \mathsf{Aut}(\mathfrak{h}_3)$. Now

$$\begin{bmatrix} v_2 w_3 - v_3 w_2 & v_1 & w_1 \\ 0 & v_2 & w_2 \\ 0 & v_3 & w_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} v_2 w_3 - v_3 w_2 & v_1 & w_1 \\ 0 & v_2 & w_2 \\ 0 & v_3 & w_3 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} \\ \varphi_{21} & \varphi_{22} & \varphi_{23} \\ \varphi_{31} & \varphi_{23} & \varphi_{33} \end{bmatrix} = \begin{bmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} \\ \varphi_{21} & \varphi_{22} & \varphi_{23} \\ \varphi_{31} & \varphi_{23} & \varphi_{33} \end{bmatrix}.$$

Here $\begin{bmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} \\ \varphi_{21} & \varphi_{22} & \varphi_{23} \\ \varphi_{31} & \varphi_{23} & \varphi_{33} \end{bmatrix} = \varphi$. Since there must exist $\psi \in \mathsf{Aut}(\mathfrak{h}_3)$ such that $\psi \cdot \Xi(\mathbf{1}, u) = \Xi'(\mathbf{1}, \varphi(u))$ it follows that

$$\mathcal{T}_{\tilde{\Sigma}^{(3,0)}} = \left\{ \begin{bmatrix} v_2 w_3 - v_3 w_2 & v_1 & w_1 \\ 0 & v_2 & w_2 \\ 0 & v_3 & w_3 \end{bmatrix} : \begin{array}{c} v_1, v_2, v_3, w_1, w_2, w_3 \in \mathbb{R}, \\ v_2 w_3 - v_3 w_2 \neq 0 \end{array} \right\} \cong \mathsf{Aut}(\mathfrak{h}_3).$$

Now $\chi_0 : u \mapsto (u - \mu)^\intercal Q (u - \mu)$ for some positive definite matrix $Q = \begin{bmatrix} a_1 & b_1 & b_2 \\ b_1 & a_2 & b_3 \\ b_2 & b_3 & a_3 \end{bmatrix}$. We transform χ_0 by composing it with elements of $\mathcal{T}_{\tilde{\Sigma}^{(3,0)}}$. From positive definiteness we have that $a_1 > 0, a_1 a_3 - b_2^2 > 0$. Therefore

$$\varphi_1 = \begin{bmatrix} a_1 & \frac{-a_3 b_1 + b_2 b_3}{a_1 a_3 - b_2^2} & -b_2 \\ 0 & 1 & 0 \\ 0 & \frac{b_1 b_2 - a_1 b_3}{a_1 a_3 - b_2^2} & a_1 \end{bmatrix}, \quad (\det \varphi_1 = a_1^2 \neq 0)$$

is an element of $\mathcal{T}_{\tilde{\Sigma}^{(3,0)}}$ such that

$$\begin{aligned} \chi_1(u) &= (\chi_0 \circ \varphi_1)(u) \\ &= (u - \mu')^\top \begin{bmatrix} a_1^3 & 0 & 0 \\ 0 & -\frac{a_3 b_1^2 + b_2 (a_2 b_2 - 2 b_1 b_3) + a_1 \left(-a_2 a_3 + b_3^2\right)}{a_1 a_3 - b_2^2} & 0 \\ 0 & 0 & a_1 \left(a_1 a_3 - b_2^2\right) \end{bmatrix} (u - \mu') \end{aligned}$$

for $\mu' = \varphi_1^{-1}(\mu)$. Also

$$\varphi_2 = \begin{bmatrix} -\frac{\left(a_1 a_3 - b_2^2\right)^{3/2}}{\left(a_1 \left(a_1 a_3 - b_2^2\right)\right)^{3/2}} & 0 & 0 \\ 0 & 0 & \frac{\sqrt{a_1 a_3 - b_2^2}}{a_1} \\ 0 & \frac{1}{\sqrt{a_1 \left(a_1 a_3 - b_2^2\right)}} & 0 \end{bmatrix}, \quad (\det \varphi_2 = \frac{1}{a_3^2} \neq 0)$$

is an element of $\mathcal{T}_{\bar{\Sigma}^{(3,0)}}$ such that

$$\chi_2(u) = (\chi_0 \circ \varphi_1 \circ \varphi_2)(u) = (u - \mu'')^\top \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \beta \end{pmatrix} (u - \mu'')$$

for $\mu'' = \varphi_2^{-1}(\varphi_1^{-1}(\mu))$ and $\beta = -\frac{a_3 b_1^2 + b_2(a_2 b_2 - 2b_1 b_3) + a_1\left(-a_2 a_3 + b_3^2\right)}{a_1^2} > 0$ from positive definiteness of Q. Also,

$$\varphi_3 = \begin{bmatrix} \sqrt{\beta} & 0 & 0 \\ 0 & \sqrt{\beta} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (\det \varphi_3 = \beta \neq 0)$$

is an element of $\mathcal{T}_{\bar{\Sigma}^{(3,0)}}$ such that

$$\chi_3(u) = \frac{1}{\beta}(\chi_2 \circ \varphi_3)(u) = \frac{1}{\beta}(u - \mu''')^\top \begin{bmatrix} \beta & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \beta \end{bmatrix} (u - \mu''') = (u - \mu''')^\top (u - \mu''')$$

for $\mu''' = \varphi_3^{-1}(\varphi_2^{-1}(\varphi_1^{-1}(u)))$. Let $\mu_1''' = \alpha_1$. There exists $\alpha_2 \geq 0$ and $\theta \in \mathbb{R}$ such that $\mu_2''' = \alpha_2 \cos\theta$ and $\mu_3''' = \alpha_2 \sin\theta$ and

$$\varphi_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

is an element of $\mathcal{T}_{\bar{\Sigma}^{(3,0)}}$ such that

$$\chi_4(u) = (\chi_3 \circ \varphi_4)(u) = \left(u - \varphi_4^{-1}\begin{bmatrix}\alpha_1\\ \alpha_2\cos\theta\\ \alpha_2\sin\theta\end{bmatrix}\right)^\top \left(u - \varphi_4^{-1}\begin{bmatrix}\alpha_1\\ \alpha_2\cos\theta\\ \alpha_2\sin\theta\end{bmatrix}\right)$$

$$= \left(u - \begin{bmatrix}1 & 0 & 0\\ 0 & \cos\theta & \sin\theta\\ 0 & -\sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\alpha_1\\ \alpha_2\cos\theta\\ \alpha_2\sin\theta\end{bmatrix}\right)^\top \left(u - \begin{bmatrix}1 & 0 & 0\\ 0 & \cos\theta & \sin\theta\\ 0 & -\sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\alpha_1\\ \alpha_2\cos\theta\\ \alpha_2\sin\theta\end{bmatrix}\right)$$

$$= \left(u - \begin{bmatrix}\alpha_1\\ \alpha_2\\ 0\end{bmatrix}\right)^\top \left(u - \begin{bmatrix}\alpha_1\\ \alpha_2\\ 0\end{bmatrix}\right) = (u_1-\alpha_1)^2 + (u_2-\alpha_2)^2 + u_3^2.$$

We may assume $\alpha_1 \geq 0$. Since if $\alpha_1 < 0$ then

$$\varphi_5 = \begin{bmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{bmatrix}$$

is an element of $\mathcal{T}_{\bar{\Sigma}^{(3,0)}}$ which serves to change the sign of α_1. Indeed,

$$(\chi_4 \circ \varphi_5)(u) = \left(u - \varphi_5^{-1}\begin{bmatrix}\alpha_1\\ \alpha_2\\ 0\end{bmatrix}\right)^\top \varphi_5^\top\varphi_5 \left(u - \varphi_5^{-1}\begin{bmatrix}\alpha_1\\ \alpha_2\\ 0\end{bmatrix}\right)$$

$$= \left(u - \begin{bmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{bmatrix}\begin{bmatrix}\alpha_1\\ \alpha_2\\ 0\end{bmatrix}\right)^\top \left(u - \begin{bmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{bmatrix}\begin{bmatrix}\alpha_1\\ \alpha_2\\ 0\end{bmatrix}\right)$$

$$= \left(u - \begin{bmatrix}-\alpha_1\\ \alpha_2\\ 0\end{bmatrix}\right)^\top \left(u - \begin{bmatrix}-\alpha_1\\ \alpha_2\\ 0\end{bmatrix}\right).$$

Hence by theorem 3.1.8, (Σ, χ) is C-equivalent to $(\bar{\Sigma}^{(3,0)}, \bar{\chi}_{\alpha_1,\alpha_2})$. Finally, we verify that for $(\alpha_1, \alpha_2) \neq (\alpha_1', \alpha_2')$, that the systems $(\bar{\Sigma}^{(3,0)}, \bar{\chi}_{\alpha_1,\alpha_2})$ and $(\bar{\Sigma}^{(3,0)}, \bar{\chi}_{\alpha_1',\alpha_2'})$ are distinct. Suppose $(\bar{\Sigma}^{(3,0)}, \bar{\chi}_{\alpha_1,\alpha_2})$ is C-equivalent to $(\bar{\Sigma}^{(3,0)}, \bar{\chi}_{\alpha_1',\alpha_2'})$. Then there exists $\varphi \in \mathcal{T}_{\bar{\Sigma}^{(3,0)}}$ such that $\bar{\chi}_{\alpha_1',\alpha_2'} = r\bar{\chi}_{\alpha_1,\alpha_2} \circ \varphi$ for some $r > 0$. Let

$$\varphi = \begin{bmatrix} v_2w_3 - v_3w_2 & v_1 & w_1 \\ 0 & v_2 & w_2 \\ 0 & v_3 & w_3 \end{bmatrix}.$$

Then

$$\begin{aligned}
\bar{\chi}_{\alpha_1',\alpha_2'} &= r\bar{\chi}_{\alpha_1,\alpha_2} \circ \varphi \\
\iff (u_1 - \alpha_1')^2 + (u_2 - \alpha_2')^2 + u_3^2 &= r(u_2^2\left(v_1^2 + v_2^2 + v_3^2\right) + u_3^2\left(w_1^2 + w_2^2 + w_3^2\right) \\
&\quad + (u_1v_3w_2 - u_1v_2w_3 + \alpha_1)^2 + \alpha_2^2 + 2u_2(u_3(v_1w_1 + v_2w_2 + v_3w_3) \\
&\quad - v_1(u_1v_3w_2 - u_1v_2w_3 + \alpha_1) - v_2\alpha_2) \\
&\quad - 2u_3(w_1(u_1v_3w_2 - u_1v_2w_3 + \alpha_1) + w_2\alpha_2)).
\end{aligned}$$

On solving, we have that $\alpha_1 = \alpha_1', \alpha_2' = \alpha_2$ and $r = 1$. Hence $(\bar{\Sigma}^{(3,0)}, \bar{\chi}_{\alpha_1,\alpha_2})$ and $(\bar{\Sigma}^{(3,0)}, \bar{\chi}_{\alpha_1',\alpha_2'})$ are not C-equivalent.

3.1.14 PROPOSITION. *For the system*

$$(\bar{\Sigma}^{(3,0)}, \bar{\chi}_{\alpha_1,\alpha_2}) : \begin{cases} \bar{\Xi}(\mathbf{1}, u) = u_1E_1 + u_2E_2 + u_3E_3 \\ \bar{\chi}_{\alpha_1,\alpha_2}(u) = (u_1 - \alpha_1)^2 + (u_2 - \alpha_2)^2 + u_3^2, \end{cases}$$

the extremal control is given by $u_1 = p_1 + \alpha_1$, $u_2 = p_2 + \alpha_2$ *and* $u_3 = p_3$ *where* $H(p) = \alpha_1p_1 + \alpha_2p_2 + \frac{1}{2}(p_1^2 + p_2^2 + p_3^2)$.

PROOF. The associated Hamiltonian function on $T^*\mathsf{H}_3$ is given by

$$H_u^{\frac{1}{2}}(g, p) = -\frac{1}{2}\left((u_1 - \alpha_1)^2 + (u_2 - \alpha_2)^2 + u_3^2\right) + u_1p_1 + u_2p_2 + u_3p_3.$$

Setting the partial derivatives of $H_u^{\frac{1}{2}}(g, p)$ with respect to u_1, u_2 and u_3 equal to zero:

$$\begin{cases} p_1 - u_1 + \alpha_1 = 0 \\ p_2 - u_2 + \alpha_2 = 0 \\ p_3 - u_3 = 0 \end{cases}$$

yields

$$\begin{cases} u_1 = p_1 + \alpha_1 \\ u_2 = p_2 + \alpha_2 \\ u_3 = p_3. \end{cases}$$

On substituting back into $H_u^{\frac{1}{2}}(g, p)$ we have

$$H(p) = \alpha_1p_1 + \alpha_2p_2 + \frac{1}{2}(p_1^2 + p_2^2 + p_3^2).$$

Chapter 4

Classification of Quadratic Hamilton-Poisson Systems

In chapter 3 we looked at the optimal control system 3.1-3.2-3.3 associated with left-invariant control affine systems on H_3 (chapter 2) with affine quadratic cost $\chi : \mathbb{R}^\ell \to \mathbb{R}, u \mapsto (u-\mu)^\top Q(u-\mu)$, where $\mu \in \mathbb{R}^\ell$ and Q is a positive definite $\ell \times \ell$ matrix. We did this by considering the associated cost-extended control system.

In this chapter we consider the same optimal control problem but with the cost function being a positive definite quadratic form on $\mathbb{R}^\ell$ given by $\chi : \mathbb{R}^\ell \to \mathbb{R}, u \mapsto u^\top Q u$ where $Q \in \mathbb{R}^{\ell\times\ell}$ is a symmetric positive definite matrix. Our optimal control problem is therefore

$$\dot{g}(t) = \Xi(g(t), u(t)), \quad g(\cdot) : [0,T] \to \mathsf{H}_3 \quad u(\cdot) : [0,T] \to \mathbb{R}^\ell, \tag{4.1}$$

$$g(0) = g_0, \quad g(T) = g_1, \quad g_0, g_1 \in \mathsf{H}_3 \text{ fixed}, \ T > 0 \text{ fixed}, \tag{4.2}$$

$$\mathcal{J}(u(t)) = \int_0^T u^\top(t) Q u(t)\, \mathrm{d}t \to \min. \tag{4.3}$$

We investigate the above optimal control problem by looking at its associated quadratic Hamilton-Poisson system.

To the optimal control problem 4.1–4.2–4.3 we associate a (lifted) Hamilton-Poisson system on $\mathfrak{h}_3^*$ via the Pontryagin Maximum Principle (refer to section A.3). From theorem A.3.7 the extremal controls of 3.1–3.2–3.3 are linearly related to the integral curves of the quadratic Hamilton-Poisson system $(\mathfrak{h}_{3-}^*, H_{A,\mathcal{Q}})$ where

$$H_{A,\mathcal{Q}}(p) = p(A) + \mathcal{Q}(p). \tag{4.4}$$

Here $A \in \mathfrak{h}_3$ and $\mathcal{Q}$ is a positive semidefinite quadratic form on $\mathfrak{h}_{3-}^*$. The problem of determining the extremal controls of 4.1–4.2–4.3 is thus reduced to the problem of finding the integral curves of the Hamilton-Poisson system 4.4.

This chapter's primary concern is with the classification of such systems. We classify the homogeneous systems (systems where $A = 0$) under affine equivalence. (This has been done in [17].) Based on the equivalence of homogeneous systems, we classify the inhomogeneous systems ($A \neq 0$) under affine equivalence. Affine equivalence establishes a one-to-one correspondence between the integral curves of equivalent systems.

4.1 Preliminaries

We briefly recall the necessary concepts and notation of Hamiltonian formalism and Lie-Poisson spaces as discussed in sections A.3.2 and A.3.3 and specifically look at the Lie-Poisson space $\mathfrak{h}_3^*$.

Let $\mathfrak{g}$ be a Lie algebra. The **minus Lie-Poisson structure** on its dual Lie algebra $\mathfrak{g}^*$ is given by

$$\begin{aligned}\{F, G\}(p) &= -\left\langle \mathrm{ad}^*_{\mathrm{d}F(p)} p, \mathrm{d}G(p) \right\rangle \\ &= -\left\langle p, [\mathrm{d}F(p), \mathrm{d}G(p)] \right\rangle, \quad p \in \mathfrak{g}^*, F, G \in C^\infty(\mathfrak{g}^*).\end{aligned}$$

We have that $\mathrm{d}F(p), \mathrm{d}G(p) \in \mathfrak{g}^{**}$ and are therefore identified with elements of $\mathfrak{g}$. Hence $[\cdot, \cdot]$ denotes the usual Lie algebra bracket. We denote the Lie-Poisson space $(g^*, \{\cdot, \cdot\})$ as $\mathfrak{g}^*_-$. To each $H \in C^\infty(\mathfrak{g}^*)$ we associate a unique Hamiltonian vector field $\vec{H}$ on $\mathfrak{g}^*$ specified by

$$\vec{H}[F] = \{F, H\}, \quad \forall F \in C^\infty(\mathfrak{g}^*).$$

A **Casimir function** is a function $C \in C^\infty(\mathfrak{g}^*)$ such that $\{C, F\} = 0$ for all $F \in C^\infty(\mathfrak{g}^*)$. A **linear Poisson automorphism** is a linear isomorphism $\psi : \mathfrak{g}^* \to \mathfrak{g}^*$ such that $\{F, G\} \circ \psi = \{F \circ \psi, G \circ \psi\}$ for all $F, G \in C^\infty(\mathfrak{g}^*)$. The linear Poisson automorphisms are exactly the dual maps of the Lie algebra automorphisms.

4.1.1 The Heisenberg (minus) Lie-Poisson space

The Heisenberg (minus) Lie-Poisson space is denoted $\mathfrak{h}_{3-}^*$. We will write elements of the Lie algebra $\mathfrak{h}_3$ as column vectors with respect to the standard basis $(E_i)_{i=1}^3$ and elements of $\mathfrak{h}_3^*$ as row vectors with respect to the standard basis $(E_i^*)_{i=1}^3$. From proposition A.3.3 we have that the equations of motion for a Hamiltonian H with respect to the minus Lie-Poisson bracket on $\mathfrak{h}_3^*$ may be written component-wise as

$$\begin{cases} \dot{p}_1 = -\left\langle p, [E_1, \mathrm{d}H(p)] \right\rangle \\ \dot{p}_2 = -\left\langle p, [E_2, \mathrm{d}H(p)] \right\rangle \\ \dot{p}_3 = -\left\langle p, [E_3, \mathrm{d}H(p)] \right\rangle. \end{cases}$$

4.1.1 Proposition. *$C : p \mapsto p_1$ is a Casimir function on $(\mathfrak{h}_3, \{,\})$*

Proof. $E_1 \in Z(\mathfrak{h}_3)$ (proposition 1.1.10) and therefore, by proposition A.3.4, $E_1^{**} = C$ is a Casimir function.

4.1.2 Remark. Recall that for any Casimir C and for any function $f : \mathbb{R} \to \mathbb{R}$, the function $f(C)$ is also a Casimir function. Hence C^2 is also a Casimir function.

4.1.3 Proposition. *The linear Poisson automorphisms of $\mathfrak{h}_3^*$ are*

$$\left\{ \begin{bmatrix} v_2w_3 - v_3w_2 & v_1 & w_1 \\ 0 & v_2 & w_2 \\ 0 & v_3 & w_3 \end{bmatrix}, v_1, v_2, v_3, w_1, w_2, w_3 \in \mathbb{R},\ v_2w_3 - v_3w_2 \neq 0 \right\}$$

Proof. The Lie algebra automorphisms of $\mathfrak{h}_3$ are

$$\mathsf{Aut}(\mathfrak{h}_3) = \left\{ \begin{bmatrix} v_2w_3 - v_3w_2 & v_1 & w_1 \\ 0 & v_2 & w_2 \\ 0 & v_3 & w_3 \end{bmatrix}, v_1, v_2, v_3, w_1, w_2, w_3 \in \mathbb{R},\ v_2w_3 - v_3w_2 \neq 0 \right\}.$$

Now from proposition A.3.5 we have that the linear Poisson automorphisms of $\mathfrak{h}_3^*$ are exactly the dual maps of $\mathsf{Aut}(\mathfrak{h}_3)$. Because $\mathsf{Aut}(\mathfrak{h}_3)$ are invertible matrices we have that there dual maps are merely there transposes but since we (by convention) write elements of the Lie algebra as column vectors and elements of the dual algebra as row vectors the result follows.

4.2 Quadratic Hamilton-Poisson Systems

4.2.1 Definition. A **quadratic Hamilton-Poisson system** is a pair $(\mathfrak{g}^*_-, H_{A,\mathcal{Q}})$. Here $\mathfrak{g}^*_-$ is a Lie-Poisson space and $H_{A,\mathcal{Q}}$ is a Hamilton function

$$H_{A,\mathcal{Q}}(p) : \mathfrak{g}^* \to \mathbb{R}, \quad p \mapsto p(A) + \mathcal{Q}(p).$$

$A \in \mathfrak{g}$ and $\mathcal{Q}$ is a quadratic form on $\mathfrak{g}^*$.

When the Lie-Poisson space is understood, we shall identify a Hamilton-Poisson system with its Hamiltonian function. For us $\mathcal{Q}$ will be a positive semidefinite quadratic form on $\mathfrak{g}^*$ and

$$H_{A,\mathcal{Q}}(p) = pA + \frac{1}{2}pQp^\top$$

where $Q \in \mathbb{R}^{n\times n}$ is the symmetric positive semidefinite matrix associated to $\mathcal{Q}$. If $A = 0$, the system is called **homogeneous**. In which case we will drop the A in $H_{A,\mathcal{Q}}$ and denote such a system by $H_\mathcal{Q}$. If $A \neq 0$, the system is called **inhomogeneous**. Let $H_{A,\mathcal{Q}} = p(A) + \mathcal{Q}(p) = L_A(p) + H_\mathcal{Q}$ and note that $L_A(p)$ is linear. Then the Hamiltonian vector field of $H_{A,\mathcal{Q}}$ at a point $p \in \mathfrak{g}^*$ is $\vec{H}_{A,\mathcal{Q}}(p) = \vec{L}_A(p) + \vec{H}_\mathcal{Q}(p)$. Indeed, from proposition A.3.3 and since $\mathrm{ad}^* : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}^*)$ is a linear map we have that

$$\begin{aligned}\vec{H}_{A,\mathcal{Q}}(p) &= \mathrm{ad}^*_{\mathrm{d}(L_A+H_Q)(p)}(p) \\ &= \mathrm{ad}^*_{\mathrm{d}L_A(p)+\mathrm{d}H_\mathcal{Q}(p)}(p) \\ &= \mathrm{ad}^*_{\mathrm{d}L_A(p)}(p) + \mathrm{ad}^*_{\mathrm{d}H_{\mathcal{Q}(p)}}(p) \\ &= \vec{L}_A(p) + \vec{H}_\mathcal{Q}(p).\end{aligned}$$

We introduce a natural equivalence relation on quadratic Hamilton-Poisson systems namely affine equivalence which establishes a one-to-one correspondence between the integral curves and equilibria of equivalent systems.

4.2.2 Definition. Two quadratic Hamilton-Poisson systems $H_{A,\mathcal{Q}}$ and $H_{B,\mathcal{R}}$ on $\mathfrak{g}^*_-$ are **affinely equivalent**(or A-equivalent) if there exists an affine isomorphism $\Psi : \mathfrak{g}^* \to \mathfrak{g}^*$, $p \mapsto \Psi_0(p) + q$ such that

$$T\Psi \cdot \vec{H}_{A,\mathcal{Q}} = \vec{H}_{B,\mathcal{R}} \circ \Psi$$

where $T\Psi$ denotes the tangent map of Ψ. Alternatively, $\Psi_0 \cdot \vec{H}_{A,\mathcal{Q}} = \vec{H}_{B,\mathcal{R}} \circ \Psi$.

4.2.3 PROPOSITION. *Affine equivalence is an equivalence relation.*

PROOF. Let $H_{A,\mathcal{Q}}$, $H_{B,\mathcal{R}}$ and $H_{C,\mathcal{S}}$ be quadratic Hamilton-Poisson systems on $\mathfrak{g}^*_-$. A-equivalence is

1. Reflexive.
   $\mathrm{id}_{\mathfrak{g}^*} : \mathfrak{g}^* \to \mathfrak{g}^*$, $p \mapsto p$ is an affine isomorphism such that $T\mathrm{id}_{\mathfrak{g}^*} \cdot \vec{H}_{A,\mathcal{Q}} = \vec{H}_{A,\mathcal{Q}} \circ \mathrm{id}_{\mathfrak{g}^*}$. Hence $H_{A,\mathcal{Q}}$ is A-equivalent to $H_{A,\mathcal{Q}}$.

2. Symmetric.
   Suppose $H_{A,\mathcal{Q}}$ is A-equivalent to $H_{B,\mathcal{R}}$ then there exists an affine isomorphism $\Psi : \mathfrak{g}^* \to \mathfrak{g}^*, p \mapsto \Psi_0(p) + q$ such that $\Psi_0 \cdot \vec{H}_{A,\mathcal{Q}} = \vec{H}_{B,\mathcal{R}} \circ \Psi$ from which it follows that $\Psi^{-1} : \mathfrak{g}^* \to \mathfrak{g}^*$, $p \mapsto \Psi_0^{-1}(p) - \Psi_0^{-1}(q)$ is an affine isomorphism such that $\Psi_0^{-1} \cdot \vec{H}_{B,\mathcal{R}} = \vec{H}_{A,\mathcal{Q}} \circ \Psi^{-1}$. Hence $H_{B,\mathcal{R}}$ is A-equivalent to $H_{A,\mathcal{Q}}$.

3. Transitive.
   Suppose $H_{A,\mathcal{Q}}$ is A-equivalent to $H_{B,\mathcal{R}}$ and $H_{B,\mathcal{R}}$ is A-equivalent to $H_{C,\mathcal{S}}$ then there exist affine isomorphisms $\Psi : \mathfrak{g}^* \to \mathfrak{g}^*$, $p \to \Psi_0(p) + q$ and $\Psi' : \mathfrak{g}^* \to \mathfrak{g}^*$, $p \mapsto \Psi_0'(p) + q'$ such that $\Psi_0 \cdot \vec{H}_{A,\mathcal{Q}} = \vec{H}_{B,\mathcal{R}} \circ \Psi$ and $\Psi_0' \cdot \vec{H}_{B,\mathcal{R}} = \vec{H}_{C,\mathcal{S}} \circ \Psi'$. Therefore $\Psi' \circ \Psi : \mathfrak{g}^* \to \mathfrak{g}^*$, $p \mapsto \Psi_0'(\Psi_0(p) + q) + q' = (\Psi_0' \cdot \Psi_0)(p) + \Psi_0'(q) + q'$ is an affine isomorphism such that $\Psi_0' \cdot \Psi_0 \cdot \vec{H}_{A,\mathcal{Q}} = \Psi_0' \cdot \vec{H}_{B,\mathcal{R}} \circ \Psi = \vec{H}_{C,\mathcal{S}} \circ \Psi' \circ \Psi$. Hence $H_{A,\mathcal{Q}}$ is A-equivalent to $H_{C,\mathcal{S}}$.

Thus A-equivalence is an equivalence relation.

4.2.4 PROPOSITION. *Let $H_{A,\mathcal{Q}}$ and $H_{B,\mathcal{R}}$ be two systems on $\mathfrak{g}^*_-$. If $H_{A,\mathcal{Q}}$ and $H_{B,\mathcal{R}}$ are A-equivalent then the integral curves of $H_{A,\mathcal{Q}}$ are in one-to-one correspondence with those of $H_{B,\mathcal{R}}$.*

PROOF. Suppose $H_{A,\mathcal{Q}}$ and $H_{B,\mathcal{R}}$ are A-equivalent then there exists $\Psi : \mathfrak{g}^* \to \mathfrak{g}^*$, $p \mapsto \Psi_0(p) + q$ such that $\Psi_0 \cdot \vec{H}_{A,\mathcal{Q}} = \vec{H}_{B,\mathcal{R}} \circ \Psi$. Let $p(\cdot)$ be an integral curve of $\vec{H}_{A,\mathcal{Q}}$ i.e. $\dot{p}(t) = \vec{H}_{A,\mathcal{Q}}(p(t))$. We show that $\Psi(p(\cdot))$ is the unique integral curve of $\vec{H}_{B,\mathcal{R}}$ corresponding to $p(\cdot)$. Indeed

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}}\Psi(p(t)) &= T_{p(t)}\Psi \cdot \dot{p}(t) \\ &= \Psi_0 \cdot \vec{H}_{A,\mathcal{Q}}(p(t)) \\ &= \vec{H}_{B,\mathcal{R}}(\Psi(p(t))).\end{aligned}$$

Therefore $\Psi(p(\cdot))$ is an integral curve of $\vec{H}_{B,\mathcal{R}}$. Suppose $\Psi(p_1(\cdot)) = \Psi(p_2(\cdot))$ where $p_1(\cdot)$ and $p_2(\cdot)$ are integral curves of $H_{A,\mathcal{Q}}$ then $p_1(\cdot) = \Psi^{-1}(\Psi(p_1(\cdot))) = \Psi^{-1}(\Psi(p_2(\cdot))) = p_2(\cdot)$. Hence integral curves are mapped injectively. Now suppose $p'(\cdot)$ is an integral curve of $\vec{H}_{B,\mathcal{R}}$ i.e., $\dot{p}'(t) = \vec{H}_{B,\mathcal{R}}(p'(t))$. We show that $\Psi^{-1}(p'(\cdot))$ is an integral curve in $H_{A,\mathcal{Q}}$ corresponding to $p'(\cdot)$. We have that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}}\Psi^{-1}(p'(t)) &= T_{p'(t)}\Psi^{-1} \cdot \dot{p}'(t) \\ &= \Psi_0^{-1} \cdot \vec{H}_{B,\mathcal{R}}(p'(t)) \\ &= \vec{H}_{A,\mathcal{Q}}(\Psi^{-1}(p'(t))).\end{aligned}$$

Hence $\Psi^{-1}(p'(\cdot))$ is an integral curve in $H_{A,\mathcal{Q}}$ corresponding to $p'(\cdot)$.

4.2.5 PROPOSITION. *Let* $H_{A,\mathcal{Q}}$ *and* $H_{B,\mathcal{R}}$ *be two systems on* $\mathfrak{g}^*_-$. *If* $H_{A,\mathcal{Q}}$ *and* $H_{B,\mathcal{R}}$ *are A-equivalent then the equilibrium points of* $H_{A,\mathcal{Q}}$ *are in one-to-one correspondence with those of* $H_{B,\mathcal{R}}$.

PROOF. Suppose $H_{A,\mathcal{Q}}$ and $H_{B,\mathcal{R}}$ are A-equivalent then there exists $\Psi : \mathfrak{g}^* \to \mathfrak{g}^*$, $p \mapsto \Psi_0(p) + q$ such that $\Psi_0 \cdot \vec{H}_{A,\mathcal{Q}} = \vec{H}_{B,\mathcal{R}} \circ \Psi$. Now suppose p_e is an equilibrium point of $\vec{H}_{A,\mathcal{Q}}$ i.e. $\vec{H}_{A,\mathcal{Q}}(p_e) = 0$. Then $\vec{H}_{B,\mathcal{R}}(\Psi(p_e)) = \Psi_0 \cdot \vec{H}_{A,\mathcal{Q}}(p_e) = 0$ (since Ψ_0 is a linear map). That is $\Psi(p_e)$ is an equilibrium point of $\vec{H}_{A,\mathcal{Q}}$. Suppose $\Psi(p_e) = \Psi(q_e)$ where p_e and q_e are equilibrium points of $\vec{H}_{A,\mathcal{Q}}$. Then $p_e = \Psi^{-1}(\Psi(p_e)) = \Psi^{-1}(\Psi(q_e)) = q_e$. Equilibrium points are therefore mapped injectively from $H_{A,\mathcal{Q}}$ to $H_{B,\mathcal{R}}$. Suppose p'_e is an equilibrium point of $\vec{H}_{B,\mathcal{R}}$ i.e. $\vec{H}_{B,\mathcal{R}}(p'_e) = 0$. Then $\vec{H}_{A,R}(\Psi^{-1}(p'_e)) = \Psi_0^{-1} \cdot H_{B,\mathcal{R}}(p'_e) = 0$. That is $\Psi^{-1}(p'_e)$ is an equilibrium point of $\vec{H}_{A,\mathcal{Q}}$ and the equilibrium points of $H_{A,\mathcal{Q}}$ are mapped surjectively to those of $H_{B,\mathcal{R}}$. We have that the equilibria of $\vec{H}_{A,\mathcal{Q}}$ and $\vec{H}_{B,\mathcal{R}}$ are in one-to-one correspondence.

4.2.6 PROPOSITION. (CF. ([17],[19])) *Let* $H_{A,\mathcal{Q}}$ *be a quadratic Hamilton-Poisson system on* $\mathfrak{g}^*_-$. *Then* $H_{A,\mathcal{Q}}$ *is A-equivalent to*

1. $H_{A,\mathcal{Q}} \circ \Psi$, *for any linear Poisson automorphism* $\Psi : \mathfrak{g}^* \to \mathfrak{g}^*$.
2. $H_{A,\mathcal{Q}} + C$, *for any Casimir function* $C : \mathfrak{g}^* \to \mathbb{R}$.
3. $H_{A,r\mathcal{Q}}$, *for any* $r \neq 0$.

PROOF. 1. Let $F \in C^\infty(\mathfrak{g}^*)$, arbitrary. Then, for a linear Poisson automorphism $\Psi : \mathfrak{g}^* \to \mathfrak{g}^*$,

$$\begin{aligned}(\vec{H}_{A,\mathcal{Q}} \circ \Psi)[F] &= \vec{H}_{A,\mathcal{Q}}[F] \circ \Psi = \{F, H_{A,\mathcal{Q}}\} \circ \Psi \\ &= \{F \circ \Psi, H_{A,\mathcal{Q}} \circ \Psi\} = \overrightarrow{H_{A,Q} \circ \Psi}[F \circ \Psi] \\ &= (\Psi \cdot \overrightarrow{H_{A,\mathcal{Q}} \circ \Psi})[F].\end{aligned}$$

Which holds for all $F \in C^\infty(\mathfrak{g}^*)$ (since F was arbitrary). Hence $\Psi \cdot \overrightarrow{H_{A,\mathcal{Q}} \circ \Psi} = \vec{H}_{A,\mathcal{Q}} \circ \Psi$. That is $H_{A,\mathcal{Q}}$ is A-equivalent to $H_{A,\mathcal{Q}} \circ \Psi$.

2. Again let $F \in C^\infty(\mathfrak{g}^*)$, arbitrary. Then, for a Casimir function $C : \mathfrak{g}^* \to \mathbb{R}$,

$$\begin{aligned}\overrightarrow{H_{A,\mathcal{Q}} + C}[F] &= \{F, H_{A,\mathcal{Q}} + C\} \\ &= \{F, H_{A,\mathcal{Q}}\} + \{F, C\} \\ &= \{F, H_{A,\mathcal{Q}}\} = \vec{H}_{A,\mathcal{Q}}[F].\end{aligned}$$

That is $\overrightarrow{H_{A,\mathcal{Q}} + C} = \vec{H}_{A,\mathcal{Q}}$. Therefore $H_{A,\mathcal{Q}}$ is A-equivalent to $H_{A,\mathcal{Q}} + C$.

3. We show that $\Psi : p \mapsto \frac{1}{r}p$ is a linear isomorphism such that $\Psi \cdot \vec{H}_{A,\mathcal{Q}}(p) = \vec{H}_{A,r\mathcal{Q}} \circ \Psi$ for all $p \in \mathfrak{g}^*$. Firstly

$$\begin{aligned}\Psi \cdot \vec{H}_{A,\mathcal{Q}}(p) &= \frac{1}{r}\vec{H}_{A,\mathcal{Q}}(p) = \frac{1}{r}\vec{L}_A(p) + \frac{1}{r}\vec{H}_Q(p) \\ &= \frac{1}{r}\vec{L}_A + \frac{1}{r}\mathrm{ad}^*_{\mathrm{d}H_Q(p)}p.\end{aligned}$$

On the other hand since $\mathrm{d}H_{r\mathcal{Q}}(p) = r\mathrm{d}H_Q(p) = \mathrm{d}H_Q(rp)$ and also since $\vec{L}_A$ and ad^* are linear we have that

$$\begin{aligned}\vec{H}_{A,r\mathcal{Q}} \circ \Psi(p) &= \vec{H}_{A,r\mathcal{Q}}\left(\frac{1}{r}p\right) = \vec{L}_A\left(\frac{1}{r}p\right) + \vec{H}_{r\mathcal{Q}}\left(\frac{1}{r}p\right)\\ &= \frac{1}{r}\vec{L}_A(p) + \mathrm{ad}^*_{dH_{r\mathcal{Q}\left(\frac{1}{r}p\right)}}\left(\frac{1}{r}p\right)\\ &= \frac{1}{r}\vec{L}_A + \mathrm{ad}^*_{\mathrm{d}H_Q(p)}\left(\frac{1}{r}p\right)\\ &= \frac{1}{r}\vec{L}_A + \frac{1}{r}\mathrm{ad}^*_{\mathrm{d}H_Q(p)}p.\end{aligned}$$

That is $\Psi \cdot \vec{H}_{A,\mathcal{Q}}(p) = \vec{H}_{A,r\mathcal{Q}} \circ \Psi$.

4.3 Classification of Homogeneous Systems under A-Equivalence

We begin with a technical lemma.

4.3.1 Lemma. *Let Q be a symmetric positive semidefinite matrix and let $\Psi : p \mapsto p\psi$ be a linear Poisson automorphism. Then $\psi Q \psi^\top$ is a positive semidefinite matrix.*

Proof. By definition Q is positive semidefinite if $x^\top Q x \geq 0$, $\forall x \in \mathbb{R}^n$. Let $x \in \mathbb{R}^n$ then

$$\begin{aligned}x^\top \psi Q \psi^\top x &= (\psi^\top x)^\top Q(\psi^\top x)\\ &= y^\top Q y \quad (\text{for some } y \in \mathbb{R}^n)\\ &\geq 0 \quad (\text{from positive semidefiniteness of Q}).\end{aligned}$$

Therefore $\psi Q \psi^\top$ is positive semidefinite if Q is positive semidefinite.

We will use proposition 4.2.6 in carrying out the classification.

4.3.2 Theorem. *Any homogeneous quadratic Hamilton-Poisson system $H_\mathcal{Q}$ on $\mathfrak{h}^*_{3-}$ is A-equivalent to exactly one of the following systems*

$$H_0(p) = 0, \qquad H_1(p) = \frac{1}{2}p_2^2, \qquad H_2(p) = \frac{1}{2}(p_2^2 + p_3^2).$$

Proof. Let $H_{\mathcal{Q}(p)} = \frac{1}{2}pQp^\top$ be an arbitrary Hamilton-Poisson system, where

$$Q = \begin{bmatrix} a_1 & b_1 & b_2 \\ b_1 & a_2 & b_3 \\ b_2 & b_3 & a_3 \end{bmatrix}, \quad a_1, a_2, a_3 \geq 0$$

Since Q is positive semidefinite we have that all principal minors of Q are non-negative from which it follows that

$$\begin{aligned} a_2a_3 - b_3^2 &\geq 0 \\ a_1a_3 - b_2^2 &\geq 0 \\ a_1a_2 - b_1^2 &\geq 0. \end{aligned}$$

We consider the cases $a_3 = 0$ and $a_3 \neq 0$ separately. Suppose $a_3 = 0$; Immediately we have $b_3 = b_2 = 0$. If $a_2 = 0$ then $b_1 = 0$ and

$$\begin{aligned} H_Q(p) - \frac{1}{2}a_1C^2(p) &= \frac{1}{2}pQp^\top - \frac{1}{2}a_1p_1^2 \\ &= \frac{1}{2}\begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix}\begin{bmatrix} a_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} - \frac{1}{2}a_1p_1^2 \\ &= 0 = H_0(p). \end{aligned}$$

Suppose $a_2 \neq 0$. Then

$$\Psi_1 : p \mapsto p\psi_1, \quad \psi_1 = \begin{bmatrix} \sqrt{a_2} & -\frac{b_1}{\sqrt{a_2}} & 0 \\ 0 & \frac{1}{\sqrt{a_2}} & 0 \\ 0 & 0 & a_2 \end{bmatrix}, \quad (\det \psi_1 = a_2 \neq 0)$$

is a linear Poisson automorphism such that

$$\psi_1Q\psi_1^\top = \begin{bmatrix} a_1a_2 - b_1^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore $H_Q \circ \Psi_1(p) - \frac{1}{2}(a_1a_2 - b_1^2)C^2(p) = H_1(p)$. Suppose $a_3 \neq 0$. Then

$$\Psi_2 : p \mapsto p\psi_2, \quad \psi_2 = \begin{bmatrix} -a_3 & 0 & b_2 \\ 0 & 0 & 1 \\ 0 & a_3 & -b_3 \end{bmatrix}, \quad (\det \psi_2 = a_3^2 \neq 0)$$

is a linear Poisson automorphism such that

$$\psi_2Q\psi_2^\top = \begin{bmatrix} a_3\left(a_1a_3 - b_2^2\right) & 0 & a_3(-a_3b_1 + b_2b_3) \\ 0 & a_3 & 0 \\ a_3(-a_3b_1 + b_2b_3) & 0 & a_3\left(a_2a_3 - b_3^2\right) \end{bmatrix} = \begin{bmatrix} a_1' & 0 & b_2' \\ 0 & a_3 & 0 \\ b_2' & 0 & a_3' \end{bmatrix}.$$

If $a_3' = 0$, then from positive semidefiniteness (see lemma 4.3.1) $b_2' = 0$ and

$$\Psi_3 : p \mapsto p\psi_3; \quad \psi_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{a_3}} & 0 \\ 0 & 0 & \sqrt{a_3} \end{bmatrix}, \quad (\det \psi_3 = 1 \neq 0)$$

is a linear Poisson automorphism such that

$$\psi_3 \psi_2 Q \psi_2^\top \psi_3^\top = \begin{bmatrix} a_1' & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore $H_Q \circ (\Psi_2 \circ \Psi_3)(p) - \frac{1}{2} a_1' C^2(p) = H_1(p)$. Suppose $a_3' \neq 0$. Then

$$\Psi_4 : p \mapsto p\psi_4; \quad \psi_4 = \begin{bmatrix} -\frac{1}{\sqrt{a_3}\sqrt{a_3'}} & 0 & \frac{b_2'}{\sqrt{a_3}\left(a_3'\right)^{3/2}} \\ 0 & 0 & \frac{1}{\sqrt{a_3'}} \\ 0 & \frac{1}{\sqrt{a_3}} & 0 \end{bmatrix}, \quad (\det \psi_4 = \frac{1}{a_3 a_3'} \neq 0)$$

is a linear Poisson automorphism such that

$$\psi_4 \psi_2 Q \psi_2^\top \psi_4^\top = \begin{bmatrix} \frac{a_1' a_3' - \left(b_2'\right)^2}{a_3 \left(a_3'\right)^2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Therefore $H_Q \circ (\Psi_2 \circ \Psi_4)(p) - \frac{a_1' a_3' - \left(b_2'\right)^2}{2 a_3 \left(a_3'\right)^2} C^2(p) = H_2(p)$.

We have left to verify that our three systems are not A-equivalent. (The supporting Mathematica code may be found in section C.2.) Clearly $H_0(p)$ is not A-equivalent to either $H_1(p)$ or $H_2(p)$. Suppose there exists an affine isomorphism $\Psi : p \mapsto \Psi_0(p) + q$, $\Psi_0(p) = p\psi$ where $\psi = [\psi_{ij}]$ such that $(\Psi_0 \cdot \vec{H}_2)(p) = (\vec{H}_1 \circ \Psi)(p)$ for all p. Now

$$\psi \cdot \vec{H}_2(p) = \begin{bmatrix} \psi_{13} p_1 p_2 - \psi_{12} p_1 p_3 \\ \psi_{23} p_1 p_2 - \psi_{22} p_1 p_3 \\ \psi_{33} p_1 p_2 - \psi_{32} p_1 p_3 \end{bmatrix}^\top$$

and

$$\vec{H}_1 \circ \psi(p) = \begin{bmatrix} 0 \\ 0 \\ (\psi_{11}p_1 + \psi_{12}p_2 + \psi_{13}p_3 + q_1)(\psi_{21}p_1 + \psi_{22}p_2 + \psi_{23}p_3 + q_2) \end{bmatrix}^\top$$

and so we have

$$\begin{cases} \psi_{13}p_1p_2 - \psi_{12}p_1p_3 = 0 \\ \psi_{23}p_1p_2 - \psi_{22}p_1p_3 = 0 \\ \psi_{33}p_1p_2 - \psi_{32}p_1p_3 = (\psi_{11}p_1 + \psi_{12}p_2 + \psi_{13}p_3 + q_1)(\psi_{21}p_1 + \psi_{22}p_2 + \psi_{23}p_3 + q_2) \end{cases}$$

for all p. By equating coefficients of p_1p_2 and p_1p_3 in the first two components we have $\psi_{13} = \psi_{12} = \psi_{23} = \psi_{22} = 0$. With these substitutions $\det\psi$=0. The two systems are therefore not A-equivalent.

4.3.3 REMARK. Let $H_\mathcal{Q}$ be a homogeneous quadratic Hamilton-Poisson system on $\mathfrak{h}_{3-}^*$ then there exists a linear Poisson automorphism Ψ and $k \in \mathbb{R}$ such that $H_\mathcal{Q} \circ \Psi + kC^2 = H_i$ for $i = \{0,1,2\}$ where $H_0(p) = 0$, $H_1(p) = \frac{1}{2}p_2^2$, $H_2(p) = \frac{1}{2}(p_2^2 + p_3^2)$.

4.4 Classification of Inhomogeneous Systems under A-Equivalence

4.4.1 PROPOSITION. *Let $(\mathfrak{h}_{3-}^*, H_{A,\mathcal{Q}})$ be an inhomogeneous quadratic Hamilton-Poisson system on $\mathfrak{h}_{3-}^*$. Then $H_{A,\mathcal{Q}}$ is A-equivalent to the system $L_B + H_i$ for $i = \{0,1,2\}$ and some $B \in \mathfrak{h}_3$.*

PROOF. From theorem 4.3.2 we have that there exists a linear Poisson automorphism $\Psi : p \mapsto \psi p$, $k \in \mathbb{R}$ and exactly one $i \in \{0,1,2\}$ such that $H_\mathcal{Q} \circ \Psi + kC = H_i$ as a result we have that

$$H_{A,\mathcal{Q}} \circ \Psi + kC = L_A \circ \Psi + H_\mathcal{Q} \circ \Psi + kC = L_{\psi \cdot A} + H_i = L_B + H_i$$

where $B = \psi \cdot A$ and so $H_{A,\mathcal{Q}}$ is A-equivalent to $L_B + H_i$ for $i = \{0,1,2\}$.

We carry out the classification of inhomogeneous systems by classifying the inhomogeneous systems associated with each H_i and then verifying that none of these systems are equivalent. A **linear Poisson symmetry** for a Hamilton-Poisson system $H_\mathcal{Q}$ on $\mathfrak{h}_{3-}^*$ is a linear Poisson automorphism $\Psi : p \mapsto p\psi$ such that $H_\mathcal{Q} \circ \Psi = H_{r\mathcal{Q}} + kC, r \neq 0, k \in \mathbb{R}$ for some Casimir C. We begin by determining the linear Poisson symmetries for each H_i. These maps leave each H_i invariant up to a dilation or the addition of a Casimir. We are therefore able to use these maps to simplify the linear part L_A of the inhomogeneous systems while leaving the homogeneous part invariant.

4.4.2 PROPOSITION. *The linear Poisson symmetries of each H_i for $i \in \{0,1,2\}$ are the linear Poisson* *automorphisms* $\Psi^{(i)} : p \mapsto p\psi^{(i)}$ *where for each* H_i, $\psi^{(i)}$ *is of the form given below*

$$H_0 : \psi^{(0)} = \begin{bmatrix} v_2w_3 - v_3w_2 & v_1 & w_1 \\ 0 & v_2 & w_2 \\ 0 & v_3 & w_3 \end{bmatrix}, \qquad H_1 : \psi^{(1)} = \begin{bmatrix} v_2w_3 & 0 & w_1 \\ 0 & v_2 & w_2 \\ 0 & 0 & w_3 \end{bmatrix},$$

$$H_2 : \psi^{(2)} = \begin{bmatrix} \mp v_2^2 \mp v_3^2 & 0 & 0 \\ 0 & v_2 & \pm v_3 \\ 0 & v_3 & \mp v_2 \end{bmatrix}.$$

Proof. Now

$$H_0(p) = pQ_0p^\top, \quad H_1(p) = \frac{1}{2}pQ_1p^\top, \quad H_2(p) = \frac{1}{2}pQ_2p^\top$$

where

$$Q_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad Q_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let

$$\psi = \begin{bmatrix} v_2w_3 - v_3w_2 & v_1 & w_1 \\ 0 & v_2 & w_2 \\ 0 & v_3 & w_3 \end{bmatrix}.$$

1. H_0
   Clearly all Linear Poisson automorphisms $\Psi : p \mapsto p\psi$ of $\mathfrak{h}_{3-}^*$ preserve H_0.

2. H_1
   let $\Psi : p \mapsto p\psi$ be an arbitrary linear Poisson automorphism of $\mathfrak{h}_{3-}^*$ then

$$(H_1 \circ \Psi)(p) = \frac{1}{2}p \begin{bmatrix} v_1^2 & v_1v_2 & v_1v_3 \\ v_1v_2 & v_2^2 & v_2v_3 \\ v_1v_3 & v_2v_3 & v_3^2 \end{bmatrix} p^\top$$

From which it follows that $v_1 = v_3 = 0$ and $v_2w_3 \neq 0$. In which case we have

$$H_1 \circ \Psi(p) = \frac{1}{2}p \begin{pmatrix} 0 & 0 & 0 \\ 0 & v_2^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} p^\top = \frac{1}{2}v_2^2p_2^2 = v_2^2H_1(p)$$

which is a dilation of $H_1(p)$ by v_2^2.

3. H_2

$$H_2 \circ \Psi(p) = \frac{1}{2} p \begin{bmatrix} v_1^2 + w_1^2 & v_1v_2 + w_1w_2 & v_1v_3 + w_1w_3 \\ v_1v_2 + w_1w_2 & v_2^2 + w_2^2 & v_2v_3 + w_2w_3 \\ v_1v_3 + w_1w_3 & v_2v_3 + w_2w_3 & v_3^2 + w_3^2 \end{bmatrix} p^\top$$

In order for $H_2(p)$ to be preserved we have the conditions

$$\begin{cases} v_1v_2 + w_1w_2 = 0 \\ v_1v_3 + w_1w_3 = 0 \\ v_2v_3 + w_2w_3 = 0 \\ v_2^2 + w_2^2 = v_3^2 + w_3^2 \neq 0 \end{cases}$$

Which on solving yields $v_1 = w_1 = 0, w_2 = \pm v_3, w_3 = \mp v_2$ with these substitutions

$$\begin{aligned} H_2 \circ \Psi(p) &= \frac{1}{2} p \begin{bmatrix} 0 & 0 & 0 \\ 0 & (v_2^2 + w_2^2) & 0 \\ 0 & 0 & (v_2^2 + w_2^2) \end{bmatrix} p^\top \\ &= \frac{1}{2}\left(v_2^2 + w_2^2\right)\left(p_2^2 + p_3^2\right) \\ &= \left(v_2^2 + w_2^2\right) H_2 \circ \Psi(p) \end{aligned}$$

which is a dilation of $H_2(p)$ by a factor of $\left(v_2^2 + w_2^2\right)$ and so we have

$$\psi^{(2)} = \begin{bmatrix} \mp v_2^2 \mp v_3^2 & 0 & 0 \\ 0 & v_2 & \pm v_3 \\ 0 & v_3 & \mp v_2 \end{bmatrix}.$$

4.4.3 Corollary. *Any inhomogeneous quadratic Hamilton-Poisson system $(\mathfrak{h}_{3-}^*, H_{A,Q})$ is A-equivalent to the system $L_B + H_i$ for $B = \psi A$ for some $\psi \in \psi^{(i)}$ and $i = \{0, 1, 2\}$.*

4.4.1 Inhomogeneous systems associated with H_0

4.4.4 Theorem. *Any inhomogeneous quadratic Hamilton-Poisson system on $\mathfrak{h}_{3-}^*$ of the form $H_{A,Q} = L_A + H_0$ is A-equivalent to exactly one of the systems*

$$H_0^{(0)}(p) = 0, \qquad H_1^{(0)}(p) = p_2.$$

PROOF. Let $A = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in \mathfrak{h}_3$. From corollary 4.4.3 we have that such a system is A-equivalent to the system $L_B + H_0$ where $B = \psi \cdot A$ for some $\psi \in \psi^{(0)}$. We consider three cases

1. $a_3 = a_2 = 0$.

   Then $A = \begin{bmatrix} a_1 \\ 0 \\ 0 \end{bmatrix}$ and $L_A(p) + H_0(p) = a_1 p_1$. Hence by subtracting the Casimir $C(p) = a_1 p_1$ we have that the system is A-equivalent to the system $H_0^{(0)}(p) = 0$ (proposition 4.2.6.).

2. $a_3 = 0$, $a_2 \neq 0$.

   Then $A = \begin{bmatrix} a_1 \\ a_2 \\ 0 \end{bmatrix}$ and

$$\psi = \begin{bmatrix} \frac{1}{a_2} & 0 & 0 \\ 0 & \frac{1}{a_2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \psi^{(0)}, \quad \det(\psi) = \frac{1}{a_2^2} \neq 0$$

   is a linear Poisson symmetry such that

$$\psi \cdot A = \begin{bmatrix} \frac{a_1}{a_2} \\ 1 \\ 0 \end{bmatrix}.$$

   Hence $L_{\psi \cdot A}(p) + H_0(p) = \frac{a_1}{a_2} p_1 + p_2$ and by subtracting the Casimir $C(p) = \frac{a_1}{a_2} p_1$ we have that the system is A-equivalent to the system $H_1^{(0)}(p) = p_2$. (proposition 4.2.6).

3. $a_3 \neq 0$.

Then $A = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ and

$$\psi = \begin{bmatrix} -\frac{1}{a_3} & 0 & 0 \\ 0 & 0 & \frac{1}{a_3} \\ 0 & 1 & -\frac{a_2}{a_3} \end{bmatrix} \in \psi^{(0)}, \quad \left(\det(\psi) = \frac{1}{a_3^2} \neq 0\right)$$

is a linear Poisson symmetry such that

$$\psi \cdot A = \begin{bmatrix} -\frac{a_1}{a_3} \\ 1 \\ 0 \end{bmatrix}.$$

Hence $L_{\psi \cdot A}(p) + H_0(p) = -\frac{a_1}{a_3}p_1 + p_2$ and by adding the Casimir $C(p) = \frac{a_1}{a_3}p_1$ we have that the system is A-equivalent to the system $H_1^{(0)}(p) = p_2$. (proposition 4.2.6). Also, the two systems are clearly not A-equivalent.

4.4.2 Inhomogeneous systems associated with H_1

4.4.5 Theorem. *Any inhomogeneous quadratic Hamilton-Poisson system on $\mathfrak{h}_{3-}^*$ of the form $H_{A,Q} = L_A + H_1$ is A-equivalent to exactly one of the following systems*

$$H_0^{(1)}(p) = \frac{1}{2}p_2^2, \quad H_1^{(1)}(p) = p_2 + \frac{1}{2}p_2^2, \quad H_2^{(1)}(p) = p_3 + \frac{1}{2}p_2^2.$$

Proof. Let $A = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in \mathfrak{h}_3$, $a_1, a_2, a_3 \in \mathbb{R}$, arbitrary. From corollary 4.4.3 we have that such a system is A-equivalent to system $L_B + H_1$ where $B = \psi \cdot A$ for some $\psi \in \psi^{(1)}$. We consider the three cases below.

1. $a_2 = a_3 = 0$.

   Then $A = \begin{bmatrix} a_1 \\ 0 \\ 0 \end{bmatrix}$ and $L_A(p) + H_1(p) = a_1 p_1 + \frac{1}{2}p_2^2$. Hence by subtracting the Casimir $a_1 C(p) = a_1 p_1$ we have that the system is A-equivalent to the system $H_0^{(1)}(p) = \frac{1}{2}p_2^2$ (proposition 4.2.6.).

2. $a_3 = 0, a_2 \neq 0$.

   Then $A = \begin{bmatrix} a_1 \\ a_2 \\ 0 \end{bmatrix}$. Hence

$$\psi = \begin{bmatrix} \frac{1}{a_2} & 0 & 0 \\ 0 & \frac{1}{a_2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \psi^{(1)}, \quad \left(\det(\psi) = \frac{1}{a_2^2} \neq 0\right)$$

   is a linear Poisson symmetry such that

$$\psi \cdot A = \begin{bmatrix} \frac{a_1}{a_2} \\ 1 \\ 0 \end{bmatrix}.$$

   Therefore $L_{\psi \cdot A}(p) + H_1(p) = \frac{a_1}{a_2} p_1 + p_2 + \frac{1}{2} p_2^2$. By subtracting the Casimir $\frac{a_1}{a_2} C(p) = \frac{a_1}{a_2} p_1$ we have that the system is A-equivalent to the system $H_1^{(1)}(p) = p_2 + \frac{1}{2} p_2^2$.

3. $a_3 \neq 0$.

   Then $A = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ and

$$\psi = \begin{bmatrix} \frac{1}{a_3} & 0 & 0 \\ 0 & 1 & -\frac{a_2}{a_3} \\ 0 & 0 & \frac{1}{a_3} \end{bmatrix} \in \psi^{(1)}, \quad \left(\det(\psi) = \frac{1}{a_3^2} \neq 0\right)$$

   is a linear Poisson symmetry such that

$$\psi \cdot A = \begin{bmatrix} \frac{a_1}{a_3} \\ 0 \\ 1 \end{bmatrix}.$$

   Hence $L_{\psi \cdot A}(p) + H_1(p) = \frac{a_1}{a_3} p_1 + p_3 + \frac{1}{2} p_2^2$. Therefore by subtracting the Casimir $\frac{a_1}{a_3} C(p) = \frac{a_1}{a_3} p_1$ we have that the system is A-equivalent to the system $H_2^{(1)}(p) = p_3 + \frac{1}{2} p_2^2$.

We verify that the three systems are distinct. (The supporting Mathematica code may be found in section C.2.) Suppose $H_0^{(1)}$ is A-equivalent to $H_1^{(1)}$. Then there exists an affine isomorphism $\Psi : p \mapsto \Psi_0(p) + q$, $\Psi_0(p) = p\psi$ where $\psi = [\psi_{ij}]$ such that

$$\Psi_0 \cdot \vec{H}_0^{(1)}(p) = \vec{H}_1^{(1)} \circ \Psi(p),$$

i.e.,

$$\begin{bmatrix} \psi_{13}p_1p_2 \\ \psi_{23}p_1p_2 \\ \psi_{33}p_1p_2 \end{bmatrix}^\top = \begin{bmatrix} 0 \\ 0 \\ (-\psi_{11}p_1 + \psi_{12}p_2 + \psi_{13}p_3 + q_1)(-1 - \psi_{21}p_1 - \psi_{22}p_2 - \psi_{23}p_3 - q_2) \end{bmatrix}^\top$$

for all $p_1, p_2, p_3 \in \mathbb{R}$. By comparing the coefficients of p_1p_2 in the first two rows we have that $\psi_{13} = \psi_{23} = 0$ and by comparing the coefficients of p_1 and p_2 in the third rows we have that $\psi_{11} = \psi_{12} = 0$. Hence $\det \psi = 0$ and the systems $H_0^{(1)}$ and $H_1^{(1)}$ are not A-equivalent. Next suppose $H_0^{(1)}$ is A-equivalent to $H_2^{(1)}$. Then there exists an affine isomorphism $\Psi : p \mapsto \Psi_0(p) + q$, $\Psi_0(p) = p\psi$ where $\psi = [\psi_{ij}]$ such that

$$\Psi_0 \cdot \vec{H}_0^{(1)}(p) = \vec{H}_2^{(1)} \circ \Psi(p),$$

i.e.,

$$\begin{bmatrix} \psi_{13}p_1p_2 \\ \psi_{23}p_1p_2 \\ \psi_{33}p_1p_2 \end{bmatrix}^\top = \begin{bmatrix} 0 \\ -\psi_{11}p_1 - \psi_{12}p_2 - \psi_{13}p_3 - q_1 \\ (\psi_{11}p_1 + \psi_{12}p_2 + \psi_{13}p_3 + q_1)(\psi_{21}p_1 + \psi_{22}p_2 + \psi_{23}p_3 + q_2) \end{bmatrix}^\top$$

for all $p_1, p_2, p_3 \in \mathbb{R}$. Comparing the coefficients of p_1, p_2 and p_3 in the second row yields $\psi_{11} = \psi_{12} = \psi_{13} = 0$. Hence $\det \psi = 0$ and the system $H_0^{(1)}$ is not A-equivalent to $H_2^{(1)}$. Finally suppose $H_1^{(1)}$ is A-equivalent to $H_2^{(1)}$ then there exists an affine isomorphism $\Psi : p \mapsto \Psi_0(p) + q$, $\Psi_0(p) = p\psi$ where $\psi = [\psi_{ij}]$ such that

$$\Psi_0 \cdot \vec{H}_2^{(1)}(p) = \vec{H}_1^{(1)} \circ \Psi(p)$$

that is

$$\begin{bmatrix} -\psi_{12}p_1 + \psi_{13}p_1p_2 \\ -\psi_{22}p_1 + \psi_{23}p_1p_2 \\ -\psi_{32}p_1 + \psi_{33}p_1p_2 \end{bmatrix}^\top = \begin{bmatrix} 0 \\ 0 \\ (-\psi_{11}p_1 + \psi_{12}p_2 + \psi_{13}p_3 + q_1)(-1 - \psi_{21}p_1 - \psi_{22}p_2 - \psi_{23}p_3 - q_2) \end{bmatrix}^\top$$

for all $p_1, p_2, p_3 \in \mathbb{R}$. Comparing coefficients of p_1 and p_1p_2 in the first two rows yields $\psi_{12} = \psi_{13} = \psi_{22} = \psi_{23} = 0$. Therefore

$$\det(\psi) = \begin{vmatrix} \psi_{11} & 0 & 0 \\ \psi_{21} & 0 & 0 \\ \psi_{31} & \psi_{32} & \psi_{33} \end{vmatrix} = 0.$$

Hence $H_1^{(1)}$ is not A-equivalent to $H_2^{(1)}$. The three systems are therefore not A-equivalent.

4.4.3 Inhomogeneous systems associated with H_2

4.4.6 THEOREM. *Any inhomogeneous quadratic Hamilton-Poisson system on* $\mathfrak{h}_{3-}^*$ *of the form* $H_{A,Q} = L_A + H_2$ *is A-equivalent to the following system*

$$H_0^{(2)}(p) = \frac{1}{2}(p_2^2 + p_3^2), \qquad H_1^{(2)}(p) = p_2 + \frac{1}{2}(p_2^2 + p_3^2).$$

PROOF. Let $A = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in \mathfrak{h}_3$, $a_1, a_2, a_3 \in \mathbb{R}$, arbitrary. From corollary 4.4.3 we have that such a system is A-equivalent to system $L_B + H_2$ where $B = \psi \cdot A$ for some $\psi \in \psi^{(2)}$. We consider the three cases below.

1. $a_2 = a_3 = 0$.

   Then $A = \begin{bmatrix} a_1 \\ 0 \\ 0 \end{bmatrix}$ and $L_A(p) + H_2(p) = a_1 p_1 + \frac{1}{2}(p_2^2 + p_3^2)$. Hence by subtracting the Casimir $a_1 C(p) = a_1 p_1$ we have that the system is A-equivalent to the system $H_0^{(2)}(p) = \frac{1}{2}(p_2^2 + p_3^2)$ (proposition 4.2.6.).

2. $a_3 = 0$, $a_2 \neq 0$.

   In this case $A = \begin{bmatrix} a_1 \\ a_2 \\ 0 \end{bmatrix}$. Hence

$$\psi = \begin{bmatrix} \frac{1}{a_2^2} & 0 & 0 \\ 0 & \frac{1}{a_2} & 0 \\ 0 & 0 & \frac{1}{a_2} \end{bmatrix} \in \psi^{(2)}, \qquad \left(\det(\psi) = \frac{1}{a_2^4} \neq 0\right)$$

   is a linear Poisson symmetry such that

$$\psi \cdot A = \begin{bmatrix} \frac{a_1}{a_2^2} \\ 1 \\ 0 \end{bmatrix}.$$

And so, $L_{\psi\cdot A}(p) + H_2(p) = \frac{a_1}{a_2^2}p_1 + p_2 + \frac{1}{2}(p_2^2 + p_3^2)$. Hence by subtracting the Casimir $\frac{a_1}{a_2^2}C(p) = \frac{a_1}{a_2^2}p_1$ we have that the system is A-equivalent to the system $H_1^{(2)}(p) = p_2 + \frac{1}{2}(p_2^2 + p_3^2)$.

3. $a_3 \neq 0$.

   In which case $A = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ and

$$\psi_1 = \begin{bmatrix} \frac{a_2^2+a_3^2}{a_3^4} & 0 & 0 \\ 0 & \frac{a_2}{a_3^2} & \frac{1}{a_3} \\ 0 & -\frac{1}{a_3} & \frac{a_2}{a_3^2} \end{bmatrix}, \quad \left(\det(\psi) = \frac{a_2^2 + a_3^4}{a_3^8} \neq 0\right)$$

   and

$$\psi_2 = \begin{bmatrix} \frac{1}{\left(1+\frac{a_2^2}{a_3^4}\right)^2} & 0 & 0 \\ 0 & \frac{1}{1+\frac{a_2^2}{a_3^2}} & 0 \\ 0 & 0 & \frac{1}{1+\frac{a_2^2}{a_3^2}} \end{bmatrix}, \quad \left(\det(\psi_2) = \frac{1}{\left(1+\frac{a_2^2}{a_3^2}\right)^4} \neq 0\right)$$

   are linear Poisson symmetries such that

$$\psi_2 \cdot \psi_1 \cdot A = \begin{bmatrix} \frac{a_1}{a_2^2+a_3^2} \\ 1 \\ 0 \end{bmatrix}.$$

   Hence $L_{\psi_2\psi_1\cdot A}(p) + H_2(p) = \frac{a_1}{a_2^2+a_3^2}p_1 + p_2 + \frac{1}{2}(p_2^2 + p_3^2)$. Therefore by subtracting the Casimir $\frac{a_1}{a_2^2+a_3^2}C(p) = \frac{a_1}{a_2^2+a_3^2}p_1$ we have that the system is A-equivalent to the system $H_1^{(2)}(p) = p_2 + \frac{1}{2}(p_2^2 + p_3^2)$.

We have left to verify that the systems are not equivalent. (The supporting Mathematica code may be found in section C.2.) Suppose $H_0^{(2)}$ is A-equivalent to the system $H_1^{(2)}$. Then there exists an affine isomorphism $\Psi : p \mapsto \Psi_0(p) + q$, $\Psi_0(p) = p\psi$ where $\psi = [\psi_{ij}]$ such that

$$\Psi_0 \cdot H_0^{(2)}(p) = H_1^{(2)} \circ \Psi(p),$$

i.e.,

$$\begin{bmatrix} \psi_{13}p_1p_2 - \psi_{12}p_1p_3 \\ \psi_{23}p_1p_2 - \psi_{22}p_1p_3 \\ \psi_{33}p_1p_2 - \psi_{32}p_1p_3 \end{bmatrix}^\top = \begin{bmatrix} 0 \\ (-\psi_{11}p_1 + \psi_{12}p_2 + \psi_{13}p_3 + q_1)(\psi_{31}p_1 + \psi_{32}p_2 + \psi_{33}p_3 + q_3) \\ (-\psi_{11}p_1 + \psi_{12}p_2 + \psi_{13}p_3 + q_1)(-1 - \psi_{21}p_1 - \psi_{22}p_2 - \psi_{23}p_3 - q_2) \end{bmatrix}^\top$$

for all $p_1, p_2, p_3 \in \mathbb{R}$. By equating the coefficients of p_1p_2, p_1p_3 in the first rows we have that $\psi_{13} = \psi_{12} = 0$ and by equating the coefficients of p_1 in the third rows we have that $\psi_{11} = 0$. Hence $\det\psi = 0$ and $H_0^{(2)}$ is not A-equivalent to the system $H_1^{(2)}$.

4.4.4 Inhomogeneous systems

In this subsection we verify that the systems of theorems 4.4.4, 4.4.5, 4.4.6 are distinct.

4.4.7 Theorem. *Any inhomogeneous quadratic Hamilton-Poisson system on* $\mathfrak{h}_{3-}^*$ *is A-equivalent to exactly one of the following systems.*

$$H_0^{(0)}(p) = 0, \qquad H_1^{(0)}(p) = p_2, \qquad H_0^{(1)}(p) = p_2^2,$$
$$H_1^{(1)}(p) = p_2 + \frac{1}{2}p_2^2, \qquad H_2^{(1)}(p) = p_3 + \frac{1}{2}p_2^2, \qquad H_0^{(2)}(p) = \frac{1}{2}(p_2^2 + p_3^2)$$
$$H_1^{(2)}(p) = p_2 + \frac{1}{2}(p_2^2 + p_3^2).$$

Proof. (The supporting Mathematica code may be found in section C.2.) Clearly no system is A-equivalent to the system $H_0^{(0)}$. Suppose $H_0^{(1)}$ is A-equivalent to $H_1^{(0)}$. Then there exists an affine isomorphism $\Psi : p \mapsto \Psi_0(p) + q$, $\Psi_0(p) = p\psi$ where $\psi = [\psi_{ij}]$ such that

$$\Psi_0 \cdot \vec{H}_0^{(1)}(p) = \vec{H}_1^{(0)} \circ \Psi(p),$$

i.e.,

$$\begin{bmatrix} \psi_{13}p_1p_2 \\ \psi_{23}p_1p_2 \\ \psi_{33}p_1p_2 \end{bmatrix}^\top = \begin{bmatrix} 0 \\ 0 \\ \psi_{11}p_1 + \psi_{12}p_2 + \psi_{13}p_3 + q_1 \end{bmatrix}^\top$$

for all $p_1, p_2, p_3 \in \mathbb{R}$. Comparing the coefficients of p_1,p_2 and p_3 in the third row gives $\psi_{11} = \psi_{12} = \psi_{13} = 0$. Hence $\det\psi = 0$ and $H_0^{(1)}$ and $H_1^{(0)}$ are not A-equivalent. Suppose $H_1^{(1)}$ is A-equivalent to $H_1^{(0)}$. Then there exists an affine isomorphism $\Psi : p \mapsto \Psi_0(p) + q$, $\Psi_0(p) = p\psi$ where $\psi = [\psi_{ij}]$ such that

$$\Psi_0 \cdot \vec{H}_1^{(1)}(p) = \vec{H}_1^{(0)} \circ \Psi(p),$$

i.e.,

$$\begin{bmatrix} -\psi_{13}p_1\,(-1-p_2) \\ -\psi_{23}p_1\,(-1-p_2) \\ -\psi_{33}p_1\,(-1-p_2) \end{bmatrix}^\top = \begin{bmatrix} 0 \\ 0 \\ \psi_{11}p_1+\psi_{12}p_2+\psi_{13}p_3+q_1 \end{bmatrix}^\top$$

for all $p_1, p_2, p_3 \in \mathbb{R}$. By comparing the coefficients of p_1, p_2, p_3 and p_1p_2 in the third rows we have that $\psi_{11} = -\psi_{33} = 0$ and that $\psi_{12} = \psi_{13} = 0$. Hence $\det\psi = 0$ and the system $H_1^{(1)}$ is not A-equivalent to the system $H_1^{(0)}$. Suppose $H_2^{(1)}$ is A-equivalent to the system $H_1^{(0)}$. Then there exists an affine isomorphism $\Psi : p \mapsto \Psi_0(p) + q$, $\Psi_0(p) = p\psi$ where $\psi = [\psi_{ij}]$ such that

$$\Psi_0 \cdot \vec{H}_2^{(1)}(p) = \vec{H}_1^{(0)} \circ \Psi(p),$$

i.e.,

$$\begin{bmatrix} -\psi_{12}p_1+\psi_{13}p_1p_2 \\ -\psi_{22}p_1+\psi_{23}p_1p_2 \\ -\psi_{32}p_1+\psi_{33}p_1p_2 \end{bmatrix}^\top = \begin{bmatrix} 0 \\ 0 \\ \psi_{11}p_1+\psi_{12}p_2+\psi_{13}p_3+q_1 \end{bmatrix}^\top$$

for all $p_1, p_2, p_3 \in \mathbb{R}$. Comparing the coefficients of p_1 and p_1p_2 in the first and second rows yields $\psi_{12} = \psi_{13} = \psi_{22} = \psi_23 = 0$. Hence $\det\psi = 0$ and the systems $H_2^{(1)}$ and $H_1^{(0)}$ are not A-equivalent. Suppose $H_0^{(2)}$ is A-equivalent to the system $H_1^{(0)}$. Then there exists an affine isomorphism $\Psi : p \mapsto \Psi_0(p) + q$, $\Psi_0(p) = p\psi$ where $\psi = [\psi_{ij}]$ such that

$$\Psi_0 \cdot \vec{H}_0^{(2)}(p) = \vec{H}_1^{(0)} \circ \Psi(p),$$

i.e.,

$$\begin{bmatrix} \psi_{13}p_1p_2-\psi_{12}p_1p_3 \\ \psi_{23}p_1p_2-\psi_{22}p_1p_3 \\ \psi_{33}p_1p_2-\psi_{32}p_1p_3 \end{bmatrix}^\top = \begin{bmatrix} 0 \\ 0 \\ \psi_{11}p_1+\psi_{12}p_2+\psi_{13}p_3+q_1 \end{bmatrix}^\top$$

for all $p_1, p_2, p_3 \in \mathbb{R}$. Comparing the coefficients of p_1, p_2 and p_3 in the third rows we have that $\psi_{11} = \psi_{12} = \psi_{13} = 0$. Hence $\det\psi = 0$ and the systems $H_1^{(0)}$ and $H_0^{(2)}$ are not A-equivalent. Suppose $H_1^{(2)}$ is A-equivalent to $H_1^{(0)}$. Then there exists an affine isomorphism $\Psi : p \mapsto \Psi_0(p) + q$, $\Psi_0(p) = p\psi$ where $\psi = [\psi_{ij}]$ such that

$$\Psi_0 \cdot \vec{H}_1^{(2)}(p) = \vec{H}_1^{(0)} \circ \Psi(p),$$

i.e.,

$$\begin{bmatrix} -\psi_{13}p_1(-1-p_2)-\psi_{12}p_1p_3 \\ -\psi_{23}p_1(-1-p_2)-\psi_{22}p_1p_3 \\ -\psi_{33}p_1(-1-p_2)-\psi_{32}p_1p_3 \end{bmatrix}^\top = \begin{bmatrix} 0 \\ 0 \\ \psi_{11}p_1+\psi_{12}p_2+\psi_{13}p_3+q_1 \end{bmatrix}^\top$$

for all $p_1, p_2, p_3 \in \mathbb{R}$. Comparing the coefficients of p_1, p_2, p_3 and p_1p_2 in the third row we have that $\psi_{11} = \psi_{33} = 0$ and $\psi_{12} = \psi_{13} = 0$. Hence $\det\psi = 0$ and the systems $H_1^{(0)}$ and $H_1^{(2)}$ are not A-equivalent. In theorem 4.4.5 we showed that $H_0^{(1)}$ is not A-equivalent to either $H_1^{(1)}$ or $H_2^{(1)}$. Suppose $H_0^{(1)}$ and $H_0^{(2)}$ are A-equivalent. Then there exists an affine isomorphism $\Psi : p \mapsto \Psi_0(p)+q$, $\Psi_0(p) = p\psi$ where $\psi = [\psi_{ij}]$ such that

$$\Psi_0 \cdot \vec{H}_0^{(2)}(p) = \vec{H}_0^{(1)} \circ \Psi(p),$$

i.e.,

$$\begin{bmatrix} \psi_{13}p_1p_2-\psi_{12}p_1p_3 \\ \psi_{23}p_1p_2-\psi_{22}p_1p_3 \\ \psi_{33}p_1p_2-\psi_{32}p_1p_3 \end{bmatrix}^\top = \begin{bmatrix} 0 \\ 0 \\ (\psi_{11}p_1+\psi_{12}p_2+\psi_{13}p_3+q_1)(\psi_{21}p_1+\psi_{22}p_2+\psi_{23}p_3+q_2) \end{bmatrix}^\top$$

for all $p_1, p_2, p_3 \in \mathbb{R}$. Comparing the coefficients of p_1p_2 and p_1p_3 in the first two rows we have that $\psi_{13} = \psi_{12} = \psi_{23} = \psi_{22} = 0$. Hence $\det\psi = 0$ and $H_0^{(2)}$ and $H_0^{(1)}$ are not A-equivalent. Suppose $H_0^{(1)}$ is A-equivalent to $H_1^{(2)}$. Then there exists an affine isomorphism $\Psi : p \mapsto \Psi_0(p) + q$, $\Psi_0(p) = p\psi$ where $\psi = [\psi_{ij}]$ such that

$$\Psi_0 \cdot \vec{H}_1^{(2)}(p) = \vec{H}_0^{(1)} \circ \Psi(p),$$

i.e.,

$$\begin{bmatrix} -\psi_{13}p_1(-1-p_2)-\psi_{12}p_1p_3 \\ -\psi_{23}p_1(-1-p_2)-\psi_{22}p_1p_3 \\ -\psi_{33}p_1(-1-p_2)-\psi_{32}p_1p_3 \end{bmatrix}^\top = \begin{bmatrix} 0 \\ 0 \\ (\psi_{11}p_1+\psi_{12}p_2+\psi_{13}p_3+q_1)(\psi_{21}p_1+\psi_{22}p_2+\psi_{23}p_3+q_2) \end{bmatrix}^\top$$

for all $p_1, p_2, p_3 \in \mathbb{R}$. comparing the coefficients of p_1 and p_1p_3 in the first two rows yields $\psi_{13} = \psi_{12} = \psi_{23} = \psi_{22} = 0$. Hence $\det\psi = 0$ and the systems $H_0^{(1)}$ and $H_1^{(2)}$ are not A-equivalent. In theorem 4.4.5 we showed that $H_1^{(1)}$ and $H_2^{(1)}$ are not A-equivalent.Suppose $H_1^{(1)}$ is A-equivalent to $H_0^{(2)}$. Then there exists an affine isomorphism $\Psi : p \mapsto \Psi_0(p) + q$, $\Psi_0(p) = p\psi$ where $\psi = [\psi_{ij}]$ such that

$$\Psi_0 \cdot \vec{H}_0^{(2)}(p) = \vec{H}_1^{(1)} \circ \Psi(p),$$

i.e.,

$$\begin{bmatrix}\psi_{13}p_1p_2-\psi_{12}p_1p_3\\ \psi_{23}p_1p_2-\psi_{22}p_1p_3\\ \psi_{33}p_1p_2-\psi_{32}p_1p_3\end{bmatrix}^\top=\begin{bmatrix}0\\0\\-\left(\psi_{11}p_1+\psi_{12}p_2+\psi_{13}p_3+q_1\right)\left(-1-\psi_{21}p_1-\psi_{22}p_2-\psi_{23}p_3-q_2\right)\end{bmatrix}^\top$$

for all $p_1, p_2, p_3 \in \mathbb{R}$. Comparing the coefficients of p_1p_2 and p_1p_3 in the first two rows yields $\psi_{13} = \psi_{12} = \psi_{23} = \psi_{22} = 0$. Hence $\det\psi = 0$ and the systems $H_1^{(1)}$ and $H_0^{(2)}$ are not A-equivalent. Suppose $H_1^{(1)}$ and $H_1^{(2)}$ are A equivalent. Then there exists an affine isomorphism $\Psi : p \mapsto \Psi_0(p)+q$, $\Psi_0(p) = p\psi$ where $\psi = [\psi_{ij}]$ such that

$$\Psi_0 \cdot \vec{H}_1^{(2)}(p) = \vec{H}_1^{(1)} \circ \Psi(p),$$

i.e.,

$$\begin{bmatrix}-\psi_{13}p_1\left(-1-p_2\right)-\psi_{12}p_1p_3\\ -\psi_{23}p_1\left(-1-p_2\right)-\psi_{22}p_1p_3\\ -\psi_{33}p_1\left(-1-p_2\right)-\psi_{32}p_1p_3\end{bmatrix}^\top$$
$$=\begin{bmatrix}0\\0\\-\left(\psi_{11}p_1+\psi_{12}p_2+\psi_{13}p_3+q_1\right)\left(-1-\psi_{21}p_1-\psi_{22}p_2-\psi_{23}p_3-q_2\right)\end{bmatrix}^\top$$

for all $p_1, p_2, p_3 \in \mathbb{R}$. Comparing the coefficients of p_1 and p_1p_3 in the first two rows gives $\psi_{13} = \psi_{12} = \psi 23 = \psi_{22} = 0$. Therefore $\det\psi = 0$ and the systems $H_1^{(1)}$ and $H_1^{(2)}$ are not A-equivalent. Suppose $H_2^{(1)}$ is A-equivalent to $H_0^{(2)}$. Then there exists an affine isomorphism $\Psi : p \mapsto \Psi_0(p) + q$, $\Psi_0(p) = p\psi$ where $\psi = [\psi_{ij}]$ such that

$$\Psi_0 \cdot \vec{H}_0^{(2)}(p) = \vec{H}_2^{(1)} \circ \Psi(p),$$

i.e.,

$$\begin{bmatrix}\psi_{13}p_1p_2-\psi_{12}p_1p_3\\ \psi_{23}p_1p_2-\psi_{22}p_1p_3\\ \psi_{33}p_1p_2-\psi_{32}p_1p_3\end{bmatrix}^\top=\begin{bmatrix}0\\-\psi_{11}p_1-\psi_{12}p_2-\psi_{13}p_3-q_1\\ \left(\psi_{11}p_1+\psi_{12}p_2+\psi_{13}p_3+q_1\right)\left(\psi_{21}p_1+\psi_{22}p_2+\psi_{23}p_3+q_2\right)\end{bmatrix}^\top$$

for all $p_1, p_2, p_3 \in \mathbb{R}$. Comparing the coefficients of p_1, p_2 and p_3 in the second rows gives $\psi_{11}, \psi_{12} = \psi_{13} = 0$. Therefore $\det\psi = 0$ and the systems $H_2^{(1)}$ and $H_0^{(2)}$ are not A-equivalent. Suppose $H_2^{(1)}$

and $H_1^{(2)}$ are A-equivalent. Then there exists an affine isomorphism $\Psi : p \mapsto \Psi_0(p) + q$, $\Psi_0(p) = p\psi$ where $\psi = [\psi_{ij}]$ such that

$$\Psi_0 \cdot \vec{H}_1^{(2)}(p) = \vec{H}_2^{(1)} \circ \Psi(p),$$

i.e.,

$$\begin{bmatrix} -\psi_{13}p_1\left(-1-p_2\right) - \psi_{12}p_1p_3 \\ -\psi_{23}p_1\left(-1-p_2\right) - \psi_{22}p_1p_3 \\ -\psi_{33}p_1\left(-1-p_2\right) - \psi_{32}p_1p_3 \end{bmatrix}^\top = \begin{bmatrix} 0 \\ -\psi_{11}p_1 - \psi_{12}p_2 - \psi_{13}p_3 - q_1 \\ \left(\psi_{11}p_1 + \psi_{12}p_2 + \psi_{13}p_3 + q_1\right)\left(\psi_{21}p_1 + \psi_{22}p_2 + \psi_{23}p_3 + q_2\right) \end{bmatrix}^\top$$

for all $p_1, p_2, p_3 \in \mathbb{R}$. Comparing the coefficients of p_1, p_2, p_3 and p_1p_2 in the second rows yields $\psi_{11} = \psi_{23} = 0$ and $\psi_{12} = \psi_{13} = 0$. Hence $\det \psi = 0$ and the systems $H_2^{(1)}$ and $H_1^{(2)}$ are not A-equivalent. Finally, theorem 4.4.6 shows that systems H_0^2 and H_1^2 are not A-equivalent.

Chapter 5

Stability and Integration of Hamilton-Poisson Systems

This chapter examines the stability nature of the equilibria as well as the integration of the Hamilton-Poisson systems discussed in chapter 4. (Stability and integration of quadratic Hamilton-Poisson systems on the (minus) Lie-Poisson spaces $\mathfrak{se}(1,1)^*_-$ and $\mathfrak{so}(3)^*_-$ have been investigated in [8] and [5], respectively.) For each equilibrium point we investigate spectral as well as Lyapunov stability (see section A.4). Spectral instability may be used to show Lyapunov instability; it turns out however that all equilibrium points are spectrally stable. We therefore show Lyapunov instability directly by finding a suitable integral curve. Lyapunov stability is shown by the (continuous) energy-Casimir method (section A.4). In all cases, the Hamilton-Poisson systems could be integrated in terms of elementary functions. The supporting Mathematica code for this chapter may be found in section C.3.

5.1 Preliminaries

In order to determine stability we will need a means of measuring distance. We will therefore need a norm function. Since $\mathfrak{h}_3^*$ is finite dimensional all norms are equivalent. We will make use of the Euclidean norm (for $p = p_1E_1^* + p_2E_2^* + p_3E_3^*$, $||p|| = \sqrt{p_1^2 + p_2^2 + p_3^2}$).

5.2 Homogeneous Systems

5.2.1 Stability and integration of $H_1(p) = \frac{1}{2}p_2^2$

The equations of motion of the system $H_1(p) = \frac{1}{2}p_2^2$ are

$$\begin{cases} \dot{p}_1 = 0 \\ \dot{p}_2 = 0 \\ \dot{p}_3 = p_1p_2. \end{cases}$$

The integration of these equations is immediate. On solving we get the integral curve $p(t) = (c_1, c_2, c_3 + c_1c_2t)$ for $c_1, c_2, c_3 \in \mathbb{R}$. From above, we see that the equilibrium states of $\vec{H}_1$ occur when $p_1p_2 = 0$, i.e., the equilibrium states of $\vec{H}_1$ are $e_1^{\eta,\mu} = (0,\eta,\mu)$ and $e_2^{\nu,\mu} = (\nu,0,\mu)$ where $\eta,\nu,\mu \in \mathbb{R}$, with $\nu \neq 0$ (since this is covered by the case $e_1^{\eta,\mu}$).

5.2.1 PROPOSITION. *The equilibrium points* $e_1^{\eta,\mu}$ *and* $e_2^{\nu,\mu}$ *are spectrally stable.*

PROOF. Firstly the linearization of the vector field $\vec{H}_1$ at p is

$$D\vec{H}_1(p) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ p_2 & p_1 & 0 \end{bmatrix}$$

And so

$$D\vec{H}_1(e_1^{\eta,\mu}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \eta & 0 & 0 \end{bmatrix}$$

which has all eigenvalues as zero. Hence $e_1^{\eta,\mu}$ is spectrally stable. Also

$$D\vec{H}_1(e_1^{\nu,\mu}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \nu & 0 \end{bmatrix}$$

which has all eigenvalues as zero. Hence $e_2^{\nu,\mu}$ is spectrally stable.

5.2.2 PROPOSITION. *The equilibrium points* $e_1^{\eta,\mu}$ *and* $e_2^{\nu,\mu}$ *are Lyapunov unstable.*

PROOF.

1. $e_1^{\eta,\mu}$. We consider the cases $\eta = 0$ and $\eta \neq 0$ separately. Consider the case $\eta \neq 0$. Fix a neighbourhood U of $e_1^{\eta,\mu}$ and let $V \subseteq U$ be any neighbourhood contained in U. In order to prove instability we need an integral curve with initial point $p(0) \in V$ but with $p(t_1) \notin U$ for some t_1. Consider the integral curve $p(t) = (\delta, \eta, \mu + \delta\eta t)$. Clearly $p(0) = (\delta, \eta, \mu)$. Now since $||p(0) - e_1^{\eta,\mu}|| = \delta$ we can find a δ sufficiently small such that $p(0) \in V$. Now $\lim_{t\to\infty} ||p(t)||^2 = \lim_{t\to\infty}(\delta^2 + \eta^2 + (\mu + \delta\eta t)^2) = \delta^2 + \eta^2 + \lim_{t\to\infty}(\mu + \delta\eta t)^2 = \infty$. Hence there exists $t_1 > 0$ such that $p(t_1) \notin U$.
   Now consider $e_1^{0,\mu}$. Again fix a neighbourhood U of $e_1^{0,\mu}$. Let $V \subseteq U$ be any neighbourhood contained in U. Then the integral curve $p(t) = (\frac{1}{\sqrt{2}}\delta, \frac{1}{\sqrt{2}}\delta, \mu + \frac{1}{2}\delta^2 t)$ has $p(0) = (\frac{1}{\sqrt{2}}\delta, \frac{1}{\sqrt{2}}\delta, \mu)$ and $||p(0) - e_1^{0,\mu}|| = \delta$. Therefore there exists a δ such that $p(0) \in V$. But $\lim_{t\to\infty} ||p(t)|| = \lim_{t\to\infty}((\frac{1}{\sqrt{2}}\delta)^2 + (\frac{1}{\sqrt{2}}\delta)^2 + (\mu + \frac{1}{2}\delta^2 t)^2) = (\frac{1}{\sqrt{2}}\delta)^2 + (\frac{1}{\sqrt{2}}\delta)^2 + \lim_{t\to\infty}(\mu + \frac{1}{2}\delta^2 t)^2 = \infty$. Hence $\exists\, t_1 > 0$ such that $p(t_1) \notin U$ and the equilibrium point is unstable.

2. $e_2^{\nu,\mu}$ Fix a neighbourhood U of $e_2^{\nu,\mu}$ and let $V \subseteq U$ be any neighbourhood contained in U. Once again, in order to prove instability we need an integral curve with initial point $p(0) \in V$

but with $p(t_1) \notin U$ for some t_1. Consider the integral curve $p(t) = (\nu, \delta, \mu + \delta\nu t)$. Clearly $p(0) = (\nu, \delta, \mu)$. Now since $||p(0) - e_1^{\nu,\mu}|| = \delta$ we can find a δ sufficiently small such that $p(0) \in V$. Now $\lim_{t\to\infty} ||p(t)||^2 = \lim_{t\to\infty}(\nu^2 + \delta^2 + (\mu + \delta\nu t)^2) = \nu^2 + \delta^2 + \lim_{t\to\infty}(\mu + \delta\nu t)^2 = \infty$. Hence there exists $t_1 > 0$ such that $p(t_1) \notin U$ and $e^{\nu,\mu}$ is an unstable equilibrium point.

5.2.2 Stability and integration of $H_2(p) = \frac{1}{2}(p_2^2 + p_3^2)$

Stability

The equations of motion of the system $H_2(p) = \frac{1}{2}(p_2^2 + p_3^2)$ are

$$\begin{cases} \dot{p}_1 = 0 \\ \dot{p}_2 = -p_1 p_3 \\ \dot{p}_3 = p_1 p_2. \end{cases}$$

The equilibrium points of the system are $e_1^{\eta,\mu} = \{0, \eta, \mu\}$ and $e_2^{\mu} = \{\mu, 0, 0\}$.

5.2.3 Proposition. *$e^{\eta,\mu} = \{0, \eta, \mu\}$ and $e^{\mu} = \{\mu, 0, 0\}$ are spectrally stable.*

Proof. The linearization of the vector field $\vec{H}_2$ is

$$\mathsf{D}\vec{H}_2(p) = \begin{bmatrix} 0 & 0 & 0 \\ -p_3 & 0 & -p_1 \\ p_2 & p_1 & 0 \end{bmatrix}.$$

Hence

$$\mathsf{D}\vec{H}_2(e_1^{\eta,\mu}) = \begin{bmatrix} 0 & 0 & 0 \\ -\mu & 0 & 0 \\ \eta & 0 & 0 \end{bmatrix}$$

which has eigenvalues all zero. Hence the equilibrium point $e^{\eta,\mu}$ is spectrally stable. Also

$$\mathsf{D}\vec{H}_2(e_2^{\mu}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mu \\ 0 & -\mu & 0 \end{bmatrix}$$

which has eigenvalues $\{0, -\mu i, -\mu i\}$ which clearly has non-positive real parts. The equilibrium point e_2^{μ} is therefore spectrally stable.

5.2.4 Proposition. *The equilibrium point $e_1^{\eta,\mu}$ is Lyapunov stable.*

Proof. Let $H_2(p) = \frac{1}{2}(p_2^2 + p_3^2)$ be our Hamiltonian and $C(p) = p_1^2$ our Casimir function. Then

$$\begin{aligned} H^{-1}(H(e^{\eta,\mu})) \cap C^{-1}(C(e^{\eta,\mu})) &= H^{-1}(H(0,\eta,\mu)) \cap C^{-1}(C(0,\eta,\mu)) \\ &= H^{-1}\left(\frac{1}{2}\eta^2 + \frac{1}{2}\mu^2\right) \cap C^{-1}(0) \\ &= (\alpha,\eta,\mu) \cap (0,\beta,\gamma), \quad \alpha,\beta,\gamma \in \mathbb{R} \\ &= (0,\eta,\mu). \end{aligned}$$

Hence by theorem A.4.4 $e^{\eta,\mu}$ is stable.

5.2.5 Proposition. *The equilibrium point e_2^μ is Lyapunov stable.*

Proof. We make use of the continuous Energy-Casimir method (refer to theorem A.4.4). Let $H_2(p) = \frac{1}{2}(p_2^2 + p_3^2)$ be our Hamiltonian and $C(p) = p_1^2$ our Casimir function. Now

$$\begin{aligned} H^{-1}(H(e^{\mu})) \cap C^{-1}(C(e^{\mu})) &= H^{-1}(H(\mu,0,0)) \cap C^{-1}(C(\mu,0,0)) \\ &= H^{-1}(0) \cap C^{-1}(\mu^2) \\ &= (\alpha,0,0) \cap (\mu,\gamma,\beta), \quad \alpha,\gamma,\beta \in \mathbb{R} \\ &= (\mu,0,0). \end{aligned}$$

Hence e^μ is Lyapunov stable.

Integration

5.2.6 Proposition. *If $p(\cdot) : (-\epsilon,\epsilon) \to \mathfrak{h}_3^*$ is an integral curve of $H_2 = \frac{1}{2}(p_2^2 + p_3^2)$ with $p_1(0) = c$ for some $c \in \mathbb{R}$ then*

$$\begin{cases} p_1(t) = c \\ p_2(t) = p_2(0)\cos(ct) - p_3(0)\sin(ct) \\ p_3(t) = p_3(0)\cos(ct) + p_2(0)\sin(ct). \end{cases}$$

Proof. Since $\dot{p}_1 = 0$ and $p_1(0) = c$ for some $c \in \mathbb{R}$ we have that $p_1(t) = c$. And so we have

$$\begin{cases} \dot{p}_2(t) = -cp_3(t) \\ \dot{p}_3(t) = cp_2(t). \end{cases}$$

Now let $P(t) = \begin{bmatrix} p_2(t) \\ p_3(t) \end{bmatrix}$. The solution to the equation

$$\dot{P}(t) = \begin{bmatrix} 0 & -c \\ c & 0 \end{bmatrix} P(t)$$

is

$$\begin{aligned} P(t) &= P(0)\exp\left(\begin{bmatrix} 0 & -c \\ c & 0 \end{bmatrix} t\right) \\ &= \begin{bmatrix} \cos(ct) & -\sin(ct) \\ \sin(ct) & \cos(ct) \end{bmatrix} \begin{bmatrix} p_2(0) \\ p_3(0) \end{bmatrix} \\ &= \begin{bmatrix} p_2(0)\cos(ct) - p_3(0)\sin(ct) \\ p_3(0)\cos(ct) + p_2(0)\sin(ct) \end{bmatrix}. \end{aligned}$$

The result follows.

5.3 Inhomogeneous Systems

5.3.1 Stability and integration of $H_1^{(0)}(p) = p_2$

Integration

The equations of motion of the system $H_1^{(0)}(p) = p_2$ are

$$\begin{cases} \dot{p}_1 = 0 \\ \dot{p}_2 = 0 \\ \dot{p}_3 = p_1. \end{cases}$$

The integration is trivial and we have that $p(t) = (c_1, c_2, c_1 t + c_3)$ for $c_1, c_2, c_3 \in \mathbb{R}$.

Stability

From the equations of motion it follows that the equilibrium point of $\vec{H}_1^{(0)}$ is $e^{\eta,\mu} = (0, \eta, \mu)$.

5.3.1 PROPOSITION. *The equilibrium point $e^{\eta,\mu}$ is spectrally*

PROOF. The linearisation of the vector field $\vec{H}_1^{(0)}$ at p is

$$D\vec{H}_1^{(0)}(p) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} = D\vec{H}_1^{(0)}(e^{\eta,\mu}).$$

which has eigenvalues all zero. Hence $e^{\eta,\mu}$ is spectrally stable.

5.3.2 PROPOSITION. *The equilibrium point $e_1^{\eta,\mu}$ is Lyapunov unstable.*

Proof. Fix a neighbourhood U of $e_1^{\eta,\mu}$ and let $V \subseteq U$ be any neighbourhood contained in U. In order to prove instability we need an integral curve with initial point $p(0) \in V$ but with $p(t_1) \notin U$ for some t_1. Consider the integral curve $p(t) = (\delta, \eta, \delta t + \mu)$. Clearly $p(0) = (\delta, \eta, \mu)$. Now since $||p(0) - e_1^{\eta,\mu}|| = \delta$ we can find a δ sufficiently small such that $p(0) \in V$. Now $\lim_{t\to\infty} ||p(t)||^2 = \lim_{t\to\infty}(\delta^2 + \eta^2 + (\delta t + \mu)^2) = \delta^2 + \eta^2 + \lim_{t\to\infty}(\delta t + \mu)^2 = \infty$. Hence there exists $t_1 > 0$ such that $p(t_1) \notin U$ and the equilibrium point $e_1^{\eta,\mu}$ is Lyapunov unstable.

5.3.2 Stability and integration of $H_1^{(1)}(p) = p_2 + \frac{1}{2}p_2^2$

Integration

The equations of motion of the system $H_1^{(1)}(p) = p_2 + \frac{1}{2}p_2^2$ are

$$\begin{cases} \dot{p}_1 = 0 \\ \dot{p}_2 = 0 \\ \dot{p}_3 = p_1 + p_1 p_2 \end{cases}$$

The integration is trivial and we have that $p(t) = (c_1, c_2, (c_1 + c_1c_2)t + c_3)$ for $c_1, c_2, c_3 \in \mathbb{R}$.

Stability

From the equations of motion we have that the equilibrium points of $\vec{H}_1^{(1)}$ are $e_1^{\eta,\mu} = (0, \eta, \mu)$ and $e_2^{\nu,\mu} = (\nu, -1, \mu)$ where $\eta, \nu, \mu \in \mathbb{R}$, with $\nu \neq 0$ since this is covered by the case $e_1^{\eta,\mu}$.

5.3.3 Proposition. *The equilibrium points $e_1^{\eta,\mu} = (0, \eta, \mu)$ and $e_2^{\nu,\mu} = (\nu, -1, \mu)$ are spectrally stable*

Proof. The linearisation of the vector field $\vec{H}_1^{(1)}$ at p is

$$D\vec{H}_1^{(0)}(p) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 + p_2 & p_1 & 0 \end{bmatrix}.$$

Hence

$$D\vec{H}_1^{(0)}(e_1^{\eta,\mu}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 + \eta & 0 & 0 \end{bmatrix}$$

which has eigenvalues all zero. Hence $e_1^{\eta,\mu}$ is spectrally stable. Also

$$D\vec{H}_1^{(0)}(e_1^{\nu,\mu}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

which has eigenvalues all zero. Hence $e_2^{\nu,\mu}$ is spectrally stable.

5.3.4 PROPOSITION. *The equilibrium points $e_1^{\eta,\mu}$ and $e_2^{\nu,\mu}$ are Lyapunov unstable.*

PROOF.

1. $e^{\eta,\mu}$. We consider the cases $\eta = 0$ and $\eta \neq 0$ separately. Consider the case $\eta \neq 0$. Fix a neighbourhood U of $e_1^{\eta,\mu}$ and let $V \subseteq U$ be any neighbourhood contained in U. In order to prove instability we need an integral curve with initial point $p(0) \in V$ but with $p(t_1) \notin U$ for some t_1. Consider the integral curve $p(t) = (\delta, \eta, (\delta + \delta\nu)t + \mu)$. Clearly $p(0) = (\delta, \eta, \mu)$. Now since $||p(0) - e_1^{\eta,\mu}|| = \delta$ we can find a δ sufficiently small such that $p(0) \in V$. Now $\lim_{t\to\infty} ||p(t)||^2 = \lim_{t\to\infty}(\delta^2 + \eta^2 + ((\delta + \delta\nu)t + \mu)^2) = \delta^2 + \eta^2 + \lim_{t\to\infty}((\delta + \delta\nu)t + \mu)^2 = \infty$. Hence there exists $t_1 > 0$ such that $p(t_1) \notin U$.
   Now consider $e_1^{0,\mu}$. Again fix a neighbourhood U of $e_1^{0,\mu}$. Let $V \subseteq U$ be any neighbourhood contained in U. Then the integral curve $p(t) = (\frac{1}{\sqrt{2}}\delta, \frac{1}{\sqrt{2}}\delta, (\frac{1}{\sqrt{2}}\delta + \frac{1}{2}\delta^2)t + \mu)$ has $p(0) = (\frac{1}{\sqrt{2}}\delta, \frac{1}{\sqrt{2}}\delta, \mu)$ and $||p(0) - e_1^{0,\mu}|| = \delta$. Therefore there exists a δ such that $p(0) \in V$. But $\lim_{t\to\infty} ||p(t)|| = \lim_{t\to\infty}((\frac{1}{\sqrt{2}}\delta)^2 + (\frac{1}{\sqrt{2}}\delta)^2 + (\mu + \frac{1}{2}\delta^2 t)^2) = (\frac{1}{\sqrt{2}}\delta)^2 + (\frac{1}{\sqrt{2}}\delta)^2 + \lim_{t\to\infty}(\frac{1}{\sqrt{2}}\delta + \frac{1}{2}\delta^2)t + \mu)^2 = \infty$. Hence $\exists\, t_1 > 0$ such that $p(t_1) \notin U$ and the equilibrium point is unstable.

2. $e_2^{\nu,\mu}$. Fix a neighbourhood U of $e_2^{\nu,\mu}$ and let $V \subseteq U$ be any neighbourhood contained in U. Consider the integral curve $p(t) = (\nu, -1 + \delta, \nu\delta t + \mu)$. Clearly $p(0) = (\nu, \delta, \mu)$. Now since $||p(0) - e_2^{\nu,\mu}|| = \delta$ we can find a δ sufficiently small such that $p(0) \in V$. Now $\lim_{t\to\infty} ||p(t)||^2 = \lim_{t\to\infty}(\nu^2 + (-1 + \delta)^2 + (\nu\delta t + \mu)^2) = \nu^2 + \delta^2 + \lim_{t\to\infty}(\nu\delta t + \mu)^2 = \infty$. Hence there exists $t_1 > 0$ such that $p(t_1) \notin U$ and $e_2^{\nu,\mu}$ is an unstable equilibrium point.

5.3.3 Stability and integration of $H_2^{(1)}(p) = p_3 + \frac{1}{2}p_2^2$

Integration

The equations of motion of the system $H_2^{(1)}(p) = p_3 + \frac{1}{2}p_2^2$ are

$$\begin{cases} \dot{p}_1 = 0 \\ \dot{p}_2 = -p_1 \\ \dot{p}_3 = p_1 p_2. \end{cases}$$

The integration is trivial and we have that $p(t) = (c_1, -c_1 t + c_2, -\frac{1}{2}c_1^2 t^2 + c_1 c_2 t + c_3)$ for $c_1, c_2, c_3 \in \mathbb{R}$.

Stability

It follows from the equations of motion that the equilibrium point of $\vec{H}_2^{(1)}$ is $e^{\eta,\mu} = (0, \eta, \mu)$ where $\eta, \mu \in \mathbb{R}$.

5.3.5 PROPOSITION. *The equilibrium point $e^{\eta,\mu} = (0, \eta, \mu)$ is spectrally stable.*

PROOF. The linearisation of the vector field $\vec{H}_2^{(1)}$ at p is

$$D\vec{H}_1^{(0)}(p) = \begin{bmatrix} 0 & 0 & 0 \\ -p_1 & 0 & 0 \\ p_2 & p_1 & 0 \end{bmatrix}.$$

Hence

$$D\vec{H}_1^{(0)}(e_1^{\eta,\mu}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \eta & 0 & 0 \end{bmatrix}$$

which has eigenvalues all zero. Hence $e_1^{\eta,\mu}$ is spectrally stable.

5.3.6 PROPOSITION. *The equilibrium point $e^{\eta,\mu}$ is Lyapunov unstable.*

PROOF. Fix a neighbourhood U of $e_1^{\eta,\mu}$ and let $V \subseteq U$ be any neighbourhood contained in U. Consider the integral curve $p(t) = (\delta, -\delta t + \eta, -\frac{1}{2}\delta^2 t^2 + \delta\eta t + \mu)$. Clearly $p(0) = (\delta, \eta, \mu)$. Since $||p(0) - e_1^{\eta,\mu}|| = \delta$ we can find a δ sufficiently small such that $p(0) \in V$. Now $\lim_{t\to\infty} ||p(t)||^2 = \lim_{t\to\infty}(\delta^2 + (\delta t + \eta)^2 + (-\frac{1}{2}\delta^2 t^2 + \delta\eta t + \mu)^2) = \delta^2 + \lim_{t\to\infty}(\delta t + \eta)^2 + \lim_{t\to\infty}(-\frac{1}{2}\delta^2 t^2 + \delta\eta t + \mu)^2 = \infty$. Hence there exists $t_1 > 0$ such that $p(t_1) \notin U$

5.3.4 Stability and integration of $H_1^{(2)}(p) = p_2 + \frac{1}{2}(p_2^2 + p_3^2)$

Integration

The equations of motion of the system $H_1^{(2)}(p) = p_2 + \frac{1}{2}(p_2^2 + p_3^2)$ are

$$\begin{cases} \dot{p}_1 = 0 \\ \dot{p}_2 = -p_1 p_3 \\ \dot{p}_3 = p_1(1 + p_2) \end{cases}$$

5.3.7 PROPOSITION. *If $p(\cdot) : (-\epsilon, \epsilon) \to \mathfrak{h}_3^*$ is an integral curve of $H_1^{(2)}(p) = p_2 + \frac{1}{2}(p_2^2 + p_3^2)$ with $p_1(0) = c$ for some $c \in \mathbb{R}$ then*

$$\begin{cases} p_1(t) = c \\ p_2(t) = p_2(0)\cos(ct) - p_3(0)\sin(ct) - 1 \\ p_3(t) = p_3(0)\cos(ct) + p_2(0)\sin(ct). \end{cases}$$

PROOF. Since $\dot{p}_1 = 0$ and $p_1(0) = c$ for some $c \in \mathbb{R}$ we have that $p_1(t) = c$. And so we have

$$\begin{cases} \dot{p}_2(t) = -cp_3(t) \\ \dot{p}_3(t) = c(1 + p_2)(t). \end{cases}$$

Now let $P(t) = \begin{bmatrix} p_2(t) \\ p_3(t) \end{bmatrix}$. We then have that

$$\dot{P}(t) = \begin{bmatrix} 0 & -c \\ c & 0 \end{bmatrix} P(t) + \begin{bmatrix} 0 \\ c \end{bmatrix}.$$

Let $A = \begin{bmatrix} 0 & -c \\ c & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ c \end{bmatrix}$. Also let H(t)=P(t) The solution to the equation

$$\dot{H}(t) = AH(t)$$

is

$$\begin{aligned} H(t) &= H(0)\exp\left(\begin{bmatrix} 0 & -c \\ c & 0 \end{bmatrix} t\right) \\ &= \begin{bmatrix} \cos(ct) & -\sin(ct) \\ \sin(ct) & \cos(ct) \end{bmatrix} \begin{bmatrix} p_2(0) \\ p_3(0) \end{bmatrix} \\ &= \begin{bmatrix} p_2(0)\cos(ct) - p_3(0)\sin(ct) \\ p_3(0)\cos(ct) + p_2(0)\sin(ct) \end{bmatrix}. \end{aligned}$$

Let $M = -A^{-1}B = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$. Then

$$P(t) = H(t) + A = \begin{bmatrix} p_2(0)\cos(ct) - p_3(0)\sin(ct) - 1 \\ p_3(0)\cos(ct) + p_2(0)\sin(ct) \end{bmatrix}.$$

The result follows.

Stability

The equations of motion of the system $H_1^{(2)}(p) = p_2 + \frac{1}{2}(p_2^2 + p_3^2)$ are

$$\begin{cases} \dot{p}_1 = 0 \\ \dot{p}_2 = -p_1 p_3 \\ \dot{p}_3 = p_1(1 + p_2) \end{cases}$$

from which it follows that the equilibrium points of $\vec{H}_1^{(2)}$ are $e_1^{\eta,\mu} = (0, \eta, \mu)$ and $e_2^{\nu} = (\nu, -1, 0)$, $\eta, \mu, \nu \in \mathbb{R}$ and $\nu \neq 0$ since this case is covered by $e_1^{\eta,\mu} = (0, \eta, \mu)$.

5.3.8 Proposition. *The equilibrium points* $e_1^{\eta,\mu} = (0, \eta, \mu)$ *and* $e_2^{\nu} = (\nu, -1, 0)$ *are spectrally stable.*

Proof. The linearisation of the vector field $\vec{H}_2^{(1)}$ at p is

$$D\vec{H}_1^{(0)}(p) = \begin{bmatrix} 0 & 0 & 0 \\ -p_3 & 0 & -p1 \\ 1 + p_2 & p_1 & 0 \end{bmatrix}.$$

Hence

$$D\vec{H}_1^{(0)}(e_1^{\eta,\mu}) = \begin{bmatrix} 0 & 0 & 0 \\ -\mu & 0 & 0 \\ 1+\eta & 0 & 0 \end{bmatrix}$$

which has eigenvalues all zero. Hence $e_1^{\eta,\mu}$ is spectrally stable. Also

$$D\vec{H}_1^{(0)}(e_2^{\nu}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\nu \\ 0 & \nu & 0 \end{bmatrix}$$

which has eigenvalues $0, -i\nu$ and $i\nu$ which have all real parts zero. The equilibrium point e_2^{ν} is therefore spectrally stable.

5.3.9 PROPOSITION. *The equilibrium point $e_1^{\eta,\mu}$ is Lyapunov stable.*

PROOF. Let $H_2^1(p) = p_2 + \frac{1}{2}(p_2^2 + p_3^2)$ be our Hamiltonian and $C(p) = p_1^2$ our Casimir function. Then

$$\begin{aligned} H^{-1}(H(e_1^{\eta,\mu})) \cap C^{-1}(C(e_1^{\eta,\mu})) &= H^{-1}(H(0,\eta,\mu)) \cap C^{-1}(C(0,\eta,\mu)) \\ &= H^{-1}\left(\eta + \frac{1}{2}\eta^2 + \frac{1}{2}\mu^2\right) \cap C^{-1}(0) \\ &= (\alpha,\eta,\mu) \cap (0,\beta,\gamma), \quad \alpha,\beta,\gamma \in \mathbb{R} \\ &= (0,\eta,\mu) = e_1^{(\eta,\mu)}. \end{aligned}$$

Hence by theorem A.4.4 $e^{\eta,\mu}$ is stable.

5.3.10 PROPOSITION. *The equilibrium point e_2^{ν} is Lyapunov stable.*

PROOF. Let $H_2^1(p) = p_2 + \frac{1}{2}(p_2^2 + p_3^2)$ be our Hamiltonian and $C(p) = p_1^2$ our Casimir function. Then

$$\begin{aligned} H^{-1}(H(e_2^{\nu})) \cap C^{-1}(C(e_2^{\nu})) &= H^{-1}(H(\nu,-1,0)) \cap C^{-1}(C(\nu,-1,0)) \\ &= H^{-1}\left(-1 + \frac{1}{2}(-1^2 + 0^2)\right) \cap C^{-1}(\nu^2) \\ &= H^{-1}\left(-\frac{1}{2}\right) \cap C^{-1}(\nu^2) \\ &= (\alpha,-1,0) \cap (\nu,\beta,\gamma), \quad \alpha,\beta,\gamma \in \mathbb{R} \\ &= (\nu,-1,0) = e_2^{\nu}. \end{aligned}$$

Hence by theorem A.4.4 e_2^{ν} is stable.

Conclusion

In this thesis we examined the left-invariant control affine systems on the (three-dimensional) Heisenberg group H_3; characterizing such systems under suitable equivalence relations and investigating the associated left-invariant optimal control problems with fixed terminal time, affine dynamics and either affine quadratic cost (looking at the associated cost-extended systems) or quadratic cost (viewing the associated Hamilton-Poisson systems).

Chapter 1 viewed H_3 showing that it is a matrix Lie group diffeomorphic to $\mathbb{R}^3$ and investigated several of its topological and algebraic properties. These properties have already been shown by other sources (see for example [21]) but are crucial in the investigation of control systems on H_3. In particular, we showed that H_3 is connected, simply-connected and non-compact. Proposition A.2.2 shows that connectedness is a necessary property of a group for the systems evolving on it to be controllable. Also, H_3 being simply-connected simplified obtaining a global (and not just local) classification of the systems on it . We determined the Heisenberg Lie algebra $\mathfrak{h}_3$ and its automorphism group $\mathsf{Aut}(\mathfrak{h}_3)$ which was used in the classification of systems in later chapters. $\mathsf{Aut}(\mathfrak{h}_3)$ being six-dimensional allowed for a great deal of manipulation in the classification of these systems. We showed that the centres of H_3 and $\mathfrak{h}_3$ are non-trivial which enabled the (almost) immediate determination of a Casimir function on $\mathfrak{h}_{3-}^*$ (see proposition A.3.4). The adjoint representations of H_3 and $\mathfrak{h}_3$ were calculated which were later used in determining the adjoint and coadjoint orbits of H_3. H_3 and $\mathfrak{h}_3$ were shown to be nilpotent, completely solvable, solvable, unimodular, exponential, not simple and not semisimple. It was also shown that the exponential map $\exp : \mathfrak{h}_3 \to \mathsf{H}_3$ is a diffeomorphism.

In chapter 2 we considered all full-rank left-invariant control affine systems evolving on H_3; organising these systems under three natural equivalence relations: state space equivalence, detached feedback equivalence and strongly detached feedback equivalence. The results obtained for state space equivalence showed that such a classification requires a fairly large number of parameters. Detached feedback equivalence led to exactly five class representatives, only three of which pertained to controllable systems (suggesting that such an equivalence was perhaps more feasible). These equivalence classes were

$$u_1E_2 + u_2E_3, \qquad E_1 + u_1E_2 + u_2E_3, \qquad u_1E_1 + u_2E_2 + u_3E_3.$$

Strongly detached feedback equivalence, was a slightly stronger equivalence relation than that of detached feedback equivalence and led to nine class representatives. By restricting to those which were controllable systems seven class representatives were obtained (none of which required parameters). Namely the systems

$$\begin{array}{lll} E_2 + u_1E_2 + u_2E_3, & u_1E_2 + u_2E_3, & E_1 + E_2 + u_1E_2 + u_2E_3, \\ E_1 + u_1E_2 + u_2E_3, & E_2 + u_1E_1 + u_2E_2 + u_3E_3, & E_1 + u_1E_1 + u_2E_2 + u_3E_3, \\ & u_1E_1 + u_2E_2 + u_3E_3. & \end{array}$$

Chapter 3 examined the cost-extended control systems corresponding to the class of left-invariant optimal control problems on H_3 with fixed terminal time, affine dynamics, and affine quadratic cost. The optimal control problems were formally given by

$$\begin{aligned} &\dot{g} = g(A + u_1B_1 + \cdots + u_\ell B_\ell), \quad g \in \mathsf{H}_3, u \in \mathbb{R}^\ell \\ &g(0) = g_0, \quad g(T) = g_1 \\ &\mathcal{J} = \int_0^T \chi(u(t))\mathrm{d}t \to \min. \end{aligned}$$

Here $\chi : \mathbb{R}^\ell \to \mathbb{R}, u(t) \mapsto (u(t) - \mu)Q(u(t) - \mu)$, $\mu \in \mathbb{R}^\ell$, Q a positive definite $\ell \times \ell$ matrix. The associated controllable cost-extended systems (Σ, χ) were classified under cost-equivalence. We realised that if (Σ, χ) and (Σ', χ') were cost-equivalent then Σ and Σ' were detached feedback equivalent (corollary 3.1.6) and therefore used the classification of chapter 2 to carry out this classification obtaining the following class representatives (containing one single parameter family and one two parameter family):

$$(\bar{\Sigma}^{(2,0)}, \bar{\chi}_1) : \begin{cases} \bar{\Xi}(\mathbf{1}, u) = u_1E_2 + u_2E_3 \\ \bar{\chi}_1 = u_1^2 + u_2^2, \end{cases}$$

$$(\bar{\Sigma}^{2,0}, \bar{\chi}_2) : \begin{cases} \bar{\Xi}(\mathbf{1}, u) = u_1E_2 + u_2E_3 \\ \bar{\chi}_2 = (u_1 - 1)^2 + u_2^2. \end{cases}$$

$$(\bar{\Sigma}^{(2,1)}, \bar{\chi}_\alpha) : \begin{cases} \bar{\Xi}(\mathbf{1}, u) = E_1 + u_1E_2 + u_2E_3 \\ \bar{\chi}_\alpha(u) = (u_1 - \alpha)^2 + u_2^2, \quad \alpha > 0, \end{cases}$$

$$(\bar{\Sigma}^{(3,0)}, \bar{\chi}_{\alpha_1,\alpha_2}) : \begin{cases} \bar{\Xi}(\mathbf{1}, u) = u_1E_1 + u_1E_2 + u_2E_3 \\ \bar{\chi}_{\alpha_1,\alpha_2}(u) = (u_1 - \alpha_1)^2 + (u_2 - \alpha_2)^2 + u_3^2, \quad \alpha_1, \alpha_2 \geq 0. \end{cases}$$

We determined the associated Hamiltonian on $T^*\mathsf{H}_3$ for each system from which we obtained the extremal controls.

Chapter 4 studied the quadratic Hamilton-Poisson systems on $\mathfrak{h}_{3-}^*$ of the form $H_{A,\mathcal{Q}}(p) = p(A) + \mathcal{Q}(p)$ where $A \in \mathfrak{h}_3$ and $\mathcal{Q}$ is a positive semidefinite quadratic form on $\mathfrak{h}_{3-}^*$. We first classified the homogeneous systems $H_\mathcal{Q}$ under linear equivalence and as a result had the classification of these systems under affine equivalence. Three class representatives were found namely

$$H_0(p) = 0, \qquad H_1(p) = \frac{1}{2}p_2^2, \qquad H_2(p) = \frac{1}{2}(p_2^2 + p_3^2).$$

These class representative were used in determining the affine equivalence class representatives of the inhomogeneous systems of which there are seven:

$$\begin{aligned} &H_0^{(0)}(p) = 0, \quad H_1^{(0)}(p) = p_2, \quad H_1(p) = \frac{1}{2}p_2^2, \quad H_1^{(1)}(p) = p_2 + \frac{1}{2}p_2 + \frac{1}{2}p_2^2 \\ &H_2^{(1)}(p) = p_3 + \frac{1}{2}p_2^2, \quad H_0^{(2)}(p) = \frac{1}{2}(p_2^2 + p_3^2), \quad H_1^{(2)}(p) = p_2 + \frac{1}{2}(p_2^2 + p_3^2). \end{aligned}$$

(An inhomogeneous system may in fact be affinely equivalent to a homogeneous system.) The Hamilton-Poisson systems considered are exactly those systems arising from the study of optimal control problems of the form

$$\dot{g} = g(A + u_1B_1 + \cdots + u_\ell B_\ell), \quad g \in \mathsf{H}_3, u \in \mathbb{R}^\ell, \tag{5.1}$$

$$g(0) = g_0, \quad g(T) = g_1, \quad T > 0 \text{ fixed}, \tag{5.2}$$

$$\mathcal{J} = \int_0^T \chi(u(t))\mathrm{d}t \to \min \tag{5.3}$$

where $\chi(u) = u^\top Qu$ and $Q \in \mathbb{R}^\ell$ is positive semidefinite. The extremal controls of the above optimal control problems are linearly related to the integral curves of the Hamilton-Poisson systems investigated (theorem A.3.7). Accordingly, finding the extremal control trajectories of such an optimal control system may be reduced to the study of the Hamilton-Poisson system $(\mathfrak{g}^*_-, H_{A,\mathcal{Q}})$.

Chapter 5 was concerned with the stability and integration of the Hamilton-Poisson systems of chapter 4. This chapter investigated the (spectral and Lyapunov) stability nature of the equilibria of all homogeneous and inhomogeneous systems showing that all systems are spectrally stable. Lyapunov instability was shown directly by finding a suitable integral curve. The (continuous) *energy-Casimir method* was used to prove Lyapunov stability. Explicit expressions were found for the integral curves of all the associated Hamiltonian vector fields. (In all cases the integral curves were expressible in terms of elementary functions.) In doing so, we essentially determined the extremal controls (up to an affine isomorphism) of all optimal control problems of the form 5.1–5.2–5.3.

As mentioned in chapter 2, left-invariant control affine systems on the Euclidean group $\mathsf{SE}(2)$ and on the semi-Euclidean group $\mathsf{SE}(1,1)$ have been classified under state space equivalence in [1] and [7], respectively; all such systems evolving on three-dimensional Lie groups as well as on the orthogonal group $\mathsf{SO}(4)$ have been classified under detached feedback equivalence in ([12],[13],[14],[15]) and [4], respectively. The classification under strongly detached feedback equivalence (with exception to this thesis) has, as of yet, not been considered. It appears feasible to classify all left-invariant control affine systems evolving on three dimensional Lie groups under strongly detached feedback equivalence but not necessarily under state space equivalence due to the strength of the equivalence relation. The classification of such systems on higher dimensional Lie groups under detached feedback equivalence is realizable. Cost-equivalence of a number of cost-extended control systems on several Lie groups has been done in [17]. The classification of cost-extended systems on the Lie groups not considered is worth while. The equivalence, stability and integration of quadratic Hamilton-Poisson systems on the (minus) Lie-Poisson spaces $\mathfrak{se}(1,1)^*_-$ and $\mathfrak{so}(3)^*_-$ have been investigated in [8] and [5], respectively. It would again be feasible to consider the equivalence, stability and integration of quadratic Hamilton-Poisson on further (minus) Lie-Poisson spaces.

Appendix A

Review of Prerequisites

A.1 Lie Theory

In this section we review the basic notions of Lie theory. In particular, we look at Lie groups, Lie algebras and the relationship between the two. This section draws a great deal on [23] but also on [28], [24], [41], [31], [35].

A.1.1 Lie groups

A **Lie group** G is a smooth manifold equipped with a group structure, such that the group operations on G are smooth i.e. the multiplication and inversion maps $\mu : \mathsf{G} \times \mathsf{G},\ (g, h) \mapsto gh$ and $\iota : \mathsf{G} \times \mathsf{G},\ g \mapsto g^{-1}$ are smooth. A (real, finite-dimensional) **matrix Lie group** is a closed subgroup (closed under products, inverses and nonsingular limits) of the general linear group $\mathsf{GL}(n, \mathbb{R})$ of $n \times n$ invertible matrices. An abstract subgroup H of G is called a **Lie subgroup** of G if it is an immersed submanifold of G and is called a **closed Lie subgroup** of G if is an embedded submanifold of G. H is normal if it is normal as an abstract group of G.

We will denote the **centre** of a Lie group G as $Z(\mathsf{G})$. $Z(\mathsf{G})$ is a normal subgroup of G and is defined as $Z(\mathsf{G}) = \big\{g \in \mathsf{G} : ghg^{-1}h^{-1} = \mathbf{1} \text{ for every } h \in \mathsf{G}\big\}$.

A Lie group **homomorphism** is a smooth map $\phi : \mathsf{G} \to \mathsf{G}'$ between Lie groups G and G' such that $\phi(g_1 g_2) = \phi(g_1)\phi(g_2)$ for every $g_1, g_2 \in \mathsf{G}$. A Lie group homomorphism which is bijective is a Lie group **isomorphism** and a Lie group isomorphism with $\mathsf{G} = \mathsf{G}'$ is a Lie group **automorphism**. A Lie group isomorphism is a diffeomorphism which preserves the group structure.

A.1.2 Lie algebras

A (real, n-dimensional) **Lie algebra** $\mathfrak{g}$ is a n-dimensional vector space equipped with a bilinear, skew symmetric map $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, the Lie bracket, satisfying the Jacobi identity: $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ for every $X, Y, Z \in \mathfrak{g}$. A subset $\mathfrak{h} \subseteq \mathfrak{g}$ is called a **Lie subalgebra** if it is a Lie algebra in its own right. An **ideal** $\mathfrak{h}$ of $\mathfrak{g}$ is a subalgebra of $\mathfrak{g}$ closed under Lie brackets with arbitrary members of $\mathfrak{g}$. That is for every $X \in \mathfrak{h}$ and $Y \in \mathfrak{g}$, we have $[X, Y] \in \mathfrak{h}$.

We will denote the centre of $\mathfrak{g}$ by $Z(\mathfrak{g})$. $Z(\mathfrak{g})$ is an ideal of $\mathfrak{g}$ and is defined as $Z(\mathfrak{g}) = \{X \in \mathfrak{g} : [X, Y] = 0 \text{ for every } Y \in \mathfrak{g}\}$

The Lie algebra generated by a subset Γ of $\mathfrak{g}$ will be denoted as $\mathsf{Lie}(\Gamma)$. $\mathsf{Lie}(\Gamma)$ is the smallest Lie algebra containing Γ that is

$$\mathsf{Lie}(\Gamma) = \operatorname{span}\left\{A_1, [A_1, A_2], [A_1, [A_2, A_3]], \cdots, [A_1, [A_2, \ldots, [A_{k-1}, A_k]] \cdots]] : A_i \in \Gamma, k \in \mathbb{N}\right\}.$$

A Lie algebra **homomorphism** is a linear map $\psi : \mathfrak{g} \to \mathfrak{g}'$ between Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$ that preserves the Lie bracket: $\psi \cdot [X, Y] = [\psi \cdot X, \psi \cdot Y]$ for every $X, Y \in \mathfrak{g}$. A bijective Lie algebra homomorphism is a Lie algebra **isomorphism** and an isomorphism with $\mathfrak{g} = \mathfrak{g}'$ is a Lie algebra **automorphism**. The group of all automorphisms of $\mathfrak{g}$ is denoted $\mathsf{Aut}(\mathfrak{g})$.

A.1.3 The relationship between Lie groups and Lie algebras

Let G be a Lie group. The tangent space of G at the identity is given by $T_{\mathbf{1}}\mathsf{G} = \{\dot{g}(0) | g(\cdot) \text{ is a smooth curve in } \mathsf{G},\ g(0) = \mathbf{1}\}$. $T_{\mathbf{1}}\mathsf{G}$ is a vector space. $T_{\mathbf{1}}\mathsf{G}$ equipped with the Lie bracket $[X, Y] = XY - YX,\ \forall X, Y \in T_{\mathbf{1}}\mathsf{G}$ is the Lie algebra of G and is denoted $\mathfrak{g}$. It turns out that $T_{\mathbf{1}}\mathsf{G}$ and the tangent space at $g \in \mathsf{G}$ i.e. $T_g\mathsf{G}$ are isomorphic Lie algebras. For any $g \in \mathsf{G}$ a **left translation** is a map $L_g : \mathsf{G} \to \mathsf{G},\ L_g(h) = gh$. L_g is in fact a diffeomorphism. Now each vector field X at $g \in \mathsf{G}$ is an element of $T_g\mathsf{G}$; also we have the following result

A.1.1 PROPOSITION. (CF. [31]) *If $Y \in T_g\mathsf{G}$ then there exists a vector field X such that $X(g) = Y$.*

A vector field is **left-invariant** if $T_hL_g \cdot X(h) = X(gh)$ for every $g, h \in \mathsf{G}$. In matrix notation this is equivalent to the condition $gX(h) = X(gh)$. As a result we have that every left-invariant vector field is of the form $X(g) = gA$ for some $A \in \mathfrak{g}$. We have the following result.

A.1.2 PROPOSITION. (CF. [41]) *Let G be a matrix Lie group with Lie algebra $\mathfrak{g}$. Let $X(g) = gA$ and $Y(g) = gB$ be left invariant vector fields on G. Then $[X, Y](g) = g[A, B]$.*

That is the Lie bracket of two Left-invariant vector fields is itself left-invariant. Left-invariant vector fields on a Lie group G equipped with the Lie bracket of vector fields: $[X, Y][f] = X[Y[f]] - Y[X[f]], f \in C^\infty(\mathsf{G})$ form a Lie algebra which is isomorphic to the Lie algebra $\mathfrak{g} = T_1\mathsf{G}$. The isomorphism is defined as

$$\text{left-invariant vector field } X(g) = gA \iff X(\mathbf{1}) = A \in \mathfrak{g}.$$

It is via the **exponential map** $\exp : \mathfrak{g} \to \mathsf{G}$ that we are able to relate the Lie algebra $\mathfrak{g}$ to the Lie group G. The exponential for an $n \times n$ matrix X is defined as the power series

$$\exp X = \sum_{k=1}^{\infty} \frac{X^k}{k!}, \quad X \in \mathfrak{g}.$$

The series for $\exp X$ is absolutely convergent. We have the following results.

A.1.3 PROPOSITION. ([23]) *There exists a neighbourhood U of $0 \in \mathfrak{g}$ and a neighbourhood V of $\mathbf{1} \in \mathsf{G}$ such that the exponential mapping takes U diffeomorphically onto V.*

A.1.4 PROPOSITION. ([24]) *If G is a connected matrix Lie group, then every element g of G can be written in the form*

$$g = e^{X_1} e^{X_2} \ldots e^{X_m}$$

for some $X_1, X_2, X_3 \in \mathfrak{g}$.

We review the relationships between normal subgroups and ideals as well as the link between Lie group homomorphisms and Lie algebra homomorphisms.

A.1.5 Theorem. (cf. [23]) *Let* G *be a Lie group with Lie algebra* $\mathfrak{g}$. *If* H *is a Lie subgroup of* G, *then the Lie algebra of* H *is a subalgebra of* $\mathfrak{g}$.

A.1.6 Theorem. (cf. [23]) *Suppose* G *is a connected Lie group with Lie algebra* G. *A connected Lie subgroup* H *of* G *is normal if and only if the Lie algebra of* H *is an ideal of* $\mathfrak{g}$.

A.1.7 Proposition. (cf. [23]) *The centre* $\mathsf{Z}(\mathsf{G})$ *of a connected Lie group* G *is a normal close Lie subgroup whose Lie algebra is the centre* $Z(\mathfrak{g})$ *of* $\mathfrak{g}$.

Tangent maps

A.1.8 Theorem. *A Lie group homomorphism* ϕ *from a connected Lie group* G *to a Lie group* H *is uniquely determined by its tangent map at the identity.*

A.1.9 Theorem. (cf. [45],) *Let* G *and* G' *be Lie groups with Lie algebras* $\mathfrak{g}$ *and* $\mathfrak{g}'$ *respectively.*

1. *If* $\phi : \mathsf{G} \to \mathsf{G}'$ *is a Lie group homomorphism, then* $T_{\mathbf{1}}\phi : \mathfrak{g} \to \mathfrak{g}'$ *is a Lie algebra homomorphism and for* $X \in \mathfrak{g}$
$$\phi(\exp X) = \exp[T_{\mathbf{1}}\phi(X)].$$
2. *If* $\phi : \mathsf{G} \to \mathsf{G}$ *is a diffeomorphism then* $T_{\mathbf{1}}\phi : \mathfrak{g} \to \mathfrak{g}'$ *is a linear isomorphism.*
3. *If* G *is simply connected, then for every Lie algebra homomorphism* $\psi : \mathfrak{g} \to \mathfrak{g}'$ *there exists a (unique) Lie group homomorphism* $\phi : \mathsf{G} \to \mathsf{G}'$ *such that* $T_{\mathbf{1}}\phi = \psi$.

A.1.10 Proposition. ([45]) *Let* $\phi_1 : \mathsf{G}_1 \to \mathsf{G}_2$ *and* $\phi_2 : \mathsf{G}_2 \to \mathsf{G}_3$ *be Lie group diffeomorphisms. Then*
$$T(\phi_1 \circ \phi_2) = T\phi_1 \circ T\phi_2.$$

A.1.11 Lemma. *Let* ϕ *be a diffeomorphism then*
$$(T_g\phi)^{-1} = T_{\phi(g)}\phi^{-1}.$$

Proof. Let $X_g \in T_g\mathsf{G}$ then
$$T_{\phi(g)}\phi^{-1} \cdot T_g\phi \cdot X_g = T_{\phi(g)}\phi^{-1} \cdot X_{\phi(g)} = X_g.$$
The result follows.

A.1.4 Compactness, connectedness and simply connectedness of Lie groups

A Lie group is said to be **compact** if it is compact as a topological space. That is, for every open cover of G there exists a finite subcover. Explicitly, we have that for every open cover $\{U_\alpha\}_{\alpha \in A}$ of G there exists a finite subset J of A such that $\{U_i\}_{i \in J}$ covers G. A topological space X is **disconnected** if there exist two disjoint non-empty open sets H and K (i.e. $H \neq \emptyset$, $K \neq \emptyset$ and $H \cap K = \emptyset$) in X such that $X = H \cup K$. A topological space is **connected** if it is not disconnected. A Lie group G is disconnected\connected if it is disconnected\connected as a topological space. A Lie group G is **path-connected** if for any two points $g_0, g_1 \in \mathsf{G}$ there exists a smooth curve $g(\cdot) : [0,1] \to \mathsf{G}$ such that $g(0) = g_0$ and $g(1) = g_1$. For a Lie group G connectedness and path-connectedness are equivalent. Indeed, we have

A.1.12 Proposition. ([23]) *A Lie group* G *is connected if and only if it is path connected.*

G is **simply connected** if it is path connected and for any two paths $g(\cdot) : [0,1] \to \mathsf{G}$ and $h(\cdot) : [0,1] \to \mathsf{G}$ with $g(0) = h(0)$ and $g(1) = h(1)$ there exists a **deformation** of $g(\cdot)$ to $h(\cdot)$. A deformation of a path $g(\cdot) : [0,1] \to \mathsf{G}$ to a path $h(\cdot) : [0,1] \to \mathsf{G}$ is a continuous function $H : \mathsf{G} \times [0,1] \to \mathsf{G}$ such that $H(\cdot, 0) = g(\cdot)$ and $H(\cdot, 1) = h(\cdot)$. Every Lie algebra has a unique simply connected Lie group associated with it. Indeed, we have the following result.

A.1.13 Theorem. ([23]) *A simply connected Lie group is determined up to an isomorphism by its Lie algebra.*

A.1.5 Adjoint and coadjoint representations

An **action** of a Lie group G on a manifold M is a smooth mapping $\phi : \mathsf{G} \times M \to M$ such that

1. $\phi(\mathbf{1})X = X$, for all $X \in M$; and
2. $\phi(g_1)\phi(g_2)X = \phi(g_1 g_2)X$, for all $g_1, g_2 \in \mathsf{G}$ and $X \in M$.

The **orbit** of an element $X \in M$ is given as $\mathfrak{O}(X) = \{\Phi(g)X \,|\, g \in \mathsf{G}\}$. The orbits form a partition of M. Let V be a vector space over $\mathbb{R}$ and let $\mathsf{GL}(V)$ be the group of invertible linear transformations of V. A **representation** of a Lie group G is a pair (V, ϕ) where ϕ is a Lie group homomorphism $\phi : \mathsf{G} \to \mathsf{GL}(V)$. Any representation of a Lie group G defines a natural action over its vector space. That is $\mathsf{G} \times V \to V, (g, X) \mapsto \phi(g)X$, where $\phi(g) \in \mathsf{GL}(V)$. ϕ being a homomorphism ensures that this is indeed a Lie group action. The representation of G on $\mathfrak{g}$ given by $\mathrm{Ad} : \mathsf{G} \to \mathsf{GL}(\mathfrak{g})$, $g \mapsto \mathrm{Ad}\, g$ where

$$\mathrm{Ad}\, g : \mathfrak{g} \to \mathfrak{g}, \quad X \mapsto gXg^{-1}$$

is called the **adjoint representation** of G. Ad is a Lie group homomorphism. We will write Ad g as Ad_g. The **adjoint orbits** of G are similarity classes $\mathfrak{Orb}(X) = \left\{gXg^{-1} \,|\, g \in \mathsf{G}\right\}$ and define an **equivalence relation** on $\mathfrak{g}$; $X \sim Y$ if $Y \in \mathfrak{Orb}(X)$ i.e. there exists $g \in \mathsf{G}$ such that $Y = \mathrm{Ad}_g X$. Again let V be a vector space over $\mathbb{R}$ and let $\mathfrak{gl}(V)$ be the Lie algebra of $\mathsf{GL}(V)$. A representation of a Lie algebra $\mathfrak{g}$ is a pair (V, ψ) where ψ is a Lie algebra homomorphism $\psi : \mathfrak{g} \to \mathfrak{gl}(V)$. The representation of $\mathfrak{g}$ on $\mathfrak{g}$ given by $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$, $X \mapsto \mathrm{ad}\, X$ where

$$\mathrm{ad}\, X : \mathfrak{g} \to \mathfrak{g}, Y \mapsto [X, Y]$$

is called the **adjoint representation** of $\mathfrak{g}$. We will denote ad X as ad_X. There is a natural relation between $\mathrm{Ad} : \mathsf{G} \to \mathsf{GL}(\mathfrak{g})$ and $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(V)$. We have the result.

A.1.14 Proposition. ([45]) *The tangent map at the identity of the adjoint representation of the Lie group* G *is the adjoint representation of the Lie algebra* $\mathfrak{g}$*. That is* $T_{\mathbf{1}}\mathrm{Ad} = \mathrm{ad}$.

We now look at the dual maps of Ad and ad. Let $\mathfrak{g}^*$ denote the dual Lie algebra of $\mathfrak{g}$. The **coadjoint** representation of G on $\mathfrak{g}^*$ is defined as $\mathrm{Ad}^* : \mathsf{G} \to \mathsf{GL}(\mathfrak{g}^*), g \mapsto \mathrm{Ad}^*\ g^{-1}$. We denote $\mathrm{Ad}^* g^{-1}$ as $\mathrm{Ad}^*_{g^{-1}}$. Here $\mathrm{Ad}^*_{g^{-1}} : \mathfrak{g}^* \to \mathfrak{g}^*$ is the dual map of $\mathrm{Ad}_{g^{-1}}$. That is

$$\left\langle \mathrm{Ad}^*_{g^{-1}} p, X \right\rangle = \left\langle p, \mathrm{Ad}_{g^{-1}} X \right\rangle.$$

The coadjoint orbit through $p \in \mathfrak{g}^*$ is $\mathfrak{Orb}(p) = \left\{\mathrm{Ad}_{g^{-1}}(p) | g^{-1} \in \mathsf{G}\right\}$. The tangent map at the identity of the coadjoint representation of G on $\mathfrak{g}^*$ is $\mathrm{ad}^* : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}^*), X \mapsto \mathrm{ad}^* - X$. Here $\mathrm{ad}^* - X : \mathfrak{g}^* \to \mathfrak{g}^*$. We denote $\mathrm{ad}^* - X$ as ad^*_{-X}. ad^*_{-X} is the dual map of ad_{-X}. That is

$$\left\langle \mathrm{ad}^*_{-X} p, Y \right\rangle = \left\langle p, \mathrm{ad}_{-X} Y \right\rangle .$$

A.1.6 Classes of Lie groups and Lie algebras

We review several different classes of connected Lie Groups and Lie algebras. The references for this section are [45], [23], [28] and [24].

Let A, B be subgroups of the group G. Then (A, B) denotes the subgroup of G generated by all elements $xyx^{-1}y^{-1}$ for x $\in A, y \in B$. i.e.

$$(A, B) = \left\{ \prod_{i=1}^{n} x_i y_i x_i^{-1} y_i^{-1} | x_i \in A, y_i \in B \right\}.$$

If A and B are normal subgroups of G then (A, B) is itself a normal subgroup of G. Let $C^0\mathsf{G} = \mathsf{G}$ and let $C^{n+1}\mathsf{G} = (\mathsf{G}, C^n\mathsf{G})$. The **descending central series** of normal subgroups is

$$\mathsf{G} = C^0\mathsf{G} \trianglerighteq C^1\mathsf{G} \trianglerighteq C^2\mathsf{G} \trianglerighteq \cdots .$$

If there exists a positive integer n such that the descending central series terminates with $C^n\mathsf{G} = \{\mathbf{1}\}$, G is **nilpotent** of class n. The **derived** series of Lie group G is given by the sequence of normal subgroups

$$\mathsf{G} \triangleright \mathsf{G}^{(1)} \triangleright \mathsf{G}^{(2)} \triangleright \cdots$$

were $\mathsf{G}^{(1)} = (\mathsf{G}, \mathsf{G})$ and $\mathsf{G}^{(k+1)} = (\mathsf{G}^{(k)}, \mathsf{G}^{(k)})$. G is **solvable** if the derived series is finite and terminates at $\{\mathbf{1}\}$. That is, $\mathsf{G}^{(n)} = \{\mathbf{1}\}$ for some n. Then G is solvable of class n.

A Lie algebra can also be classified as nilpotent and/or solvable. Let $C^0\mathfrak{g} = \mathfrak{g}$ and $C^{n+1}\mathfrak{g} = [\mathfrak{g}, C^n\mathfrak{g}]$, the ideal of $\mathfrak{g}$ generated by all Lie brackets $[X, Y]$ with $X \in \mathfrak{g}$ and $Y \in C^n\mathfrak{g}$. The **descending central series** of ideals of $\mathfrak{g}$ is given by

$$\mathfrak{g} = C^0\mathfrak{g} \triangleright C^1\mathfrak{g} \triangleright \cdots .$$

A Lie algebra $\mathfrak{g}$ is **nilpotent** if there exists a positive integer n such that the descending central series terminates with $C^n\mathfrak{g} = \{0\}$. $\mathfrak{g}$ is nilpotent of class n. The **derived** series of a Lie algebra $\mathfrak{g}$ is given by the sequence of ideals

$$\mathfrak{g} \triangleright \mathfrak{g}^{(1)} \triangleright \mathfrak{g}^{(2)} \cdots .$$

where $\mathfrak{g}^{(n+1)} = [\mathfrak{g}^{(n)}, \mathfrak{g}^{(n)}]$, the ideal of $\mathfrak{g}^{(n)}$ generated by all Lie brackets $[X, Y]$ with $X, Y \in \mathfrak{g}^{(n)}$. A Lie algebra $\mathfrak{g}$ is **solvable** if there exists a positive integer n such that the derived central series is finite and terminates with $\mathfrak{g}^n = \{0\}$. $\mathfrak{g}$ is solvable of class n. For connected Lie groups we have the following results.

A.1.15 Theorem. ([45]) *Let* G *be a connected Lie group with Lie algebra* $\mathfrak{g}$. $\mathfrak{g}$ *is solvable if and only if* G *is solvable.*

A.1.16 Theorem. ([45]) *Let* G *be a connected Lie group with Lie algebra* $\mathfrak{g}$. G *is nilpotent if and only if* $\mathfrak{g}$ *is nilpotent.*

A Lie group G is **completely solvable** if for any $g \in \mathsf{G}$ all eigenvalues of the adjoint operator Ad_g are real. A Lie algebra $\mathfrak{g}$ is completely solvable if for any $X \in \mathfrak{g}$ the adjoint operator ad_X has only real eigenvalues. For connected Lie groups we have the following result

A.1.17 THEOREM. *Let* G *be a connected Lie group with Lie algebra* $\mathfrak{g}$. G *is completely solvable if and only if* $\mathfrak{g}$ *is completely solvable.*

A Lie group G is called **exponential** if the exponential map $\exp : \mathfrak{g} \to \mathsf{G}$ is a diffeomorphism. A Lie algebra $\mathfrak{g}$ is exponential if the simply connected Lie group with Lie algebra $\mathfrak{g}$ is exponential. We have the following implications concerning the classes of Lie algebras (adapted from [23], [28],[36]).

A.1.18 PROPOSITION. *For a Lie algebra* $\mathfrak{g}$, *the following chain of implications holds.*

$$\mathfrak{g} \textit{ is nilpotent} \implies \mathfrak{g} \textit{ is completely solvable} \implies \mathfrak{g} \textit{ is exponetial} \implies \mathfrak{g} \textit{ is solvable.}$$

A Lie algebra $\mathfrak{g}$ is **simple** if $[\mathfrak{g}, \mathfrak{g}] \neq 0$ and $\mathfrak{g}$ has no proper ideals. That is 0 and $\mathfrak{g}$ are the only ideals of $\mathfrak{g}$. A Lie group G is simple if its Lie algebra is simple. A Lie algebra $\mathfrak{g}$ is **semisimple** if it has no nontrivial solvable ideals and a Lie group G is semisimple if its Lie algebra is semisimple. We have the following result.

A.1.19 PROPOSITION. *If* $\mathfrak{g}$ *is semisimple then it is not solvable.*

Lastly, G is **unimodular** if the Left Haar measure on G is also right-invariant. Below is a characterisation for connected unimodular Lie groups.

A.1.20 PROPOSITION. ([33]) *A connected Lie group* G *is unimodular if and only if the trace of the adjoint operator* $\mathrm{ad}X$ *is zero for all* $X \in \mathfrak{g}$.

A.2 Left-Invariant Control Affine Systems

The main references for this section are [6],[25] and [41].

A **left-invariant control affine system** is a pair $\Sigma = (\mathsf{G}, \Xi)$, where the **state space** G is a (real, finite dimensional) Lie group. Such a system consists of a family of left-invariant vector fields on G in which the vector fields are affinely parametrised by the controls and the **dynamics** $\Xi : \mathsf{G} \times \mathbb{R}^\ell \to T\mathsf{G}$ are left invariant. Formally our systems are of the form

$$\begin{aligned}\dot{g} &= \Xi(g, u) = gA + u_1 gB_1 + \ldots + u_2 gB_\ell \\ &= g\,\Xi(\mathbf{1}, u) = g(A + u_1B_1 + \ldots + u_2B_\ell) \quad g \in \mathsf{G},\ u \in \mathbb{R}^\ell,\ A, B_1 \ldots, B_\ell \in \mathfrak{g}\end{aligned}$$

The product $g\,\Xi(\mathbf{1}, u)$ stands for the image of the element $\Xi(\mathbf{1}, u) \in \mathfrak{g}$ under the tangent map of the left translation $T_{\mathbf{1}}L_g : \mathfrak{g} \to T_g\mathsf{G}$.

Each **admissible control** is a piecewise continuous curve $u(\cdot) : [0, T] \to \mathbb{R}^\ell$ of some control set $\mathbb{R}^\ell$. The **trajectory** corresponding to an admissible control $u(\cdot) : [0, T] \to \mathbb{R}^\ell$ is an absolutely continuous curve $g(\cdot) : [0, T] \to \mathsf{G}$ such that $\dot{g}(t) = \Xi(g(t), u(t))$ for almost every $t \in [0, T]$. The Carathéodory existence and uniqueness theorem of ordinary differential equations guarantees the existence of local trajectories. Also for left-invariant systems the left translation of a trajectory is itself a trajectory.

Here the **parametrisation** map $\Xi(\mathbf{1},\cdot):\mathbb{R}^\ell\to\mathfrak{g}$ is an injective map i.e. $B_1,\dots B_\ell$ are linearly independent.

The **trace** of the system is the image set $\Gamma=\mathrm{im}\Xi(\mathbf{1},\cdot)$ which is the affine subspace of $\mathfrak{g}$ given by $\Gamma=A+\Gamma^0=A+\langle B_1,\dots,B_\ell\rangle$. A system is called **homogeneous** if $A\in\Gamma^0$ and **inhomogeneous** otherwise. For convenience, an ℓ-input homogeneous system will be denoted by $(\ell,0)$ and a inhomogeneous system will be denoted by $(\ell,1)$. A system is of **full rank** if its trace generates $\mathfrak{g}$ i.e. $\mathrm{Lie}(\Gamma)=\mathfrak{g}$.

The **attainable set** (from the identity $\mathbf{1}\in\mathsf{G}$), denoted $\mathcal{A}$, is the set of all terminal points $g(T)$ of all trajectories $g(\cdot):[0,T]\to\mathsf{G}$ with $g(0)=\mathbf{1}$. Formally,

$$\mathcal{A}=\{g(T):g(\cdot):[0,T]\to\mathsf{G}\text{ is a trajectory}, g(0)=\mathbf{1}\}.$$

A system Σ is **controllable** if for every $g_0,g_1\in\mathsf{G}$, there exists a $T>0$ and a trajectory $g:[0,T]\to\mathsf{G}$ such that $g(0)=g_0$ and $g_1=g(T)$. We have the following result.

A.2.1 Theorem. ([41]) *Σ is controllable if and only if $\mathcal{A}=\mathsf{G}$.*

We also have the following necessary conditions for a system to be controllable.

A.2.2 Proposition. ([41]) *Suppose Σ is controllable. Then G is connected and Σ has full rank.*

Concerning a completely solvable, connected and simply connected Lie group G we have the following characterisation of controllability

A.2.3 Proposition. ([41]) *A system Σ evolving on a completely solvable, connected and simply connected Lie group G is controllable if and only if $\mathsf{Lie}(\Gamma^0)=\mathfrak{g}$.*

A.3 Optimal Control Problems on Lie Groups

In this section we briefly recall the key concepts in dealing with optimal control problems on Lie groups. The main references for this section are [6],[25] and [29]. Also, section A.3.1 draws on [31] and sections A.3.2 and A.3.3 draw on [32] and [25].

A.3.1 Trivialisation of the cotangent bundle of a Lie group

Recall that for a Lie group G the left translation $L_g:\mathsf{G}\to\mathsf{G}, L_g(h)=gh$ is a diffeomorphism and that its tangent map is linear isomorphism of the form $T_hL_g:T_h\mathsf{G}\to T_{gh}\mathsf{G}$. The dual of T_hL_g is the linear isomorphism given by $(T_hL_{g^{-1}})^*:T_h^*\mathsf{G}\to T_{gh}^*\mathsf{G}, \xi(\cdot)\mapsto(\xi\circ T_hL_{g^{-1}})(\cdot)$.

Now the diffeomorphism $\Phi:\mathsf{G}\times\mathfrak{g}^*\to T^*\mathsf{G}, (g,p)\mapsto(T_{\mathbf{1}}L_{g^{-1}})^*\cdot p$ provides a trivialisation of the cotangent bundle. That is $\Phi:\mathsf{G}\times\mathfrak{g}^*\to T^*\mathsf{G}$ is a diffeomorphism such that

1. the diagram

$$\begin{tikzcd}
\mathsf{G}\times\mathfrak{g}^* \arrow[rr, "\Phi"] \arrow[dr, "\mathrm{pr}_1"'] & & T^*\mathsf{G} \arrow[dl, "\pi"'] \\
& \mathsf{G}. &
\end{tikzcd}$$

commutes. Here $\pi:T^*\mathsf{G}\to\mathsf{G}$ is the natural projection $\pi:T_g^*\mathsf{G}\to\{g\}$ and $\mathrm{pr}_1:\mathsf{G}\times\mathfrak{g}^*$ is the projection onto the first coordinate. Indeed $\pi\circ\Phi(g,p)=g=\mathrm{pr}_1(g,p)$.

2. the restriction $\Phi(g,\cdot) : \mathfrak{g}^* \to T_g^*\mathsf{G}, p \mapsto \Phi(g,p)$ is a linear isomorphism. This follows from the fact that $T_\mathbf{1}L_{g^{-1}} : T_\mathbf{1}^*\mathsf{G} \to T_g^*\mathsf{G}$ is a linear isomorphism.

This enables us to identify $T^*\mathsf{G}$ with $\mathsf{G} \times \mathfrak{g}^*$. In so doing functions on $T^*\mathsf{G}$ become functions on $\mathsf{G}\times\mathfrak{g}^*$. For a function F on $T^*\mathsf{G}$, $\mathrm{d}F(p)$ will be the differential of the restriction of F to g=constant.

A.3.2 Hamiltonian formalism on Poisson manifolds

Let M be a smooth manifold. A **Poisson structure** on M is bilinear, skew symmetric map $C^\infty(\mathsf{M}) : C^\infty\mathsf{M} \times C^\infty(\mathsf{M}) \to C^\infty(\mathsf{M})$ satisfying

1. the Jacobi identity, i.e., $\{F,\{G,H\}\} + \{G,\{H,F\}\} + \{H,\{F,G\}\} = 0$
2. $\{\cdot,\cdot\}$ is a derivation in each factor, i.e., $\{FG,H\} = \{F,H\}G + F\{G,H\}$

for all F, G and $H \in C^\infty(\mathsf{M})$. A manifold M equipped with a Poisson bracket $\{\cdot,\cdot\}$ is called a **Poisson Manifold**. A Poisson manifold is therefore the pair $(\mathsf{M},\{\cdot,\cdot\})$.

To each Hamiltonian function $H \in C^\infty\mathsf{M}$ we associate a **Hamiltonian vector field** $\vec{H}$ specified by its action on smooth functions:

$$\vec{H}[F] = \{F,H\}, \quad \forall F \in C^\infty(\mathsf{M}).$$

(The Hamiltonian vector field is unique.) The triplet $(\mathsf{M},\{\cdot,\cdot\},H)$ is a **Hamilton-Poisson system**.

An **integral curve** of a Hamiltonian vector field $\vec{H}$ on M is an absolutely continuous curve $\xi(\cdot)$ that satisfies the equations of motion, i.e., $\dot{\xi}(t) = \vec{H}(\xi(t))$. From the Carathéodory existence and uniqueness theorem for ordinary differential equations, we have that there exists a unique solution to the initial value problem

$$\dot{\xi}(\cdot)(t) = \vec{H}(\xi(t)), \quad \xi(0) = \xi_0 \in \mathsf{M}.$$

A **Casimir function** $C \in C^\infty(\mathsf{M})$ is a function which Poisson commutes with every other function on M, that is, $\{C,F\} = 0$ for all $F \in C^\infty(\mathsf{M})$. Equivalently, C generates trivial dynamics, i.e., $\vec{C} = 0$. For any Casimir C and for any function $f : \mathbb{R} \to \mathbb{R}$, the function $f(C)$ is also a Casimir function. A Casimir is a constant of motion. We have the following result.

A.3.1 PROPOSITION. ([32]) *Let C be a Casimir function. Then C is constant along the integral curves of all Hamiltonian vector fields.*

Another constant of motion is provided by the Hamiltonian function.

A.3.2 PROPOSITION. ([32]) *Let $p(\cdot)$ be an integral curve of the Hamiltonian vector field $\vec{H}$. Then $H(p(t)) = H$, i.e., $H(p(t))$ is constant in time.*

This result is known as the **conservation of energy**. Both Casimir functions and Hamiltonian functions play an important role in the analysis of the stability of systems, as we shall see in section A.4.

A.3.3 The Lie-Poisson space $\mathfrak{g}^*_-$

Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{g}^*$ its dual Lie algebra. $\mathfrak{g}^*$ equipped with the **(minus) Lie-Poisson bracket**(or **(minus) Lie-Poisson structure**) defined by

$$\begin{aligned}\{F, G\}_-(p) &= -\left\langle \mathrm{ad}^*_{dF(p)} p, dG(p)\right\rangle \\ &= -\left\langle p, [dF(p), dG(p)]\right\rangle, \quad F, G \in C^\infty(\mathfrak{g}^*)\end{aligned}$$

where $[\cdot,\cdot]$ is the Lie bracket on $\mathfrak{g}$ is a Poisson Manifold. ($dF(p)$ and $dG(p)$ are linear functions on $\mathfrak{g}^*$ and are therefore elements of $\mathfrak{g}^{**} \cong \mathfrak{g}$.) The pair $(\mathfrak{g}^*, \{\cdot,\cdot\}_-)$ is the (minus) **Lie-Poisson space** which we will denote as $\mathfrak{g}^*_-$. We have that

A.3.3 Proposition. ([32]) *The equations of motion for a Hamiltonian H with respect to the minus Lie-Poisson bracket on $\mathfrak{g}^*$ are given by*

$$\dot{p} = \mathrm{ad}^*_{dH(p)}(p).$$

That is, in coordinates the Hamiltonian vector field of $H : \mathfrak{g}^* \to \mathbb{R}$ is given by $\vec{H} = \mathrm{ad}^*_{dH(p)}(p)$ and the equations of motion may be written component-wise as $\dot{p}_i = -\left\langle p, [E_i, dH(p)]\right\rangle, i = 1, \ldots, n$ where $(E_i)_{i=1}^n$ is a basis for $\mathfrak{g}$.

For Casimir functions (refer to subsection A.3.2) on $\mathfrak{g}^*$ we have the following result.

A.3.4 Proposition. *For any $X \in Z(\mathfrak{g}), X^{**} : \mathfrak{g}^* \to \mathbb{R}$, $Y^* \mapsto Y^*(X)$ is a Casimir function.*

Proof. For any $F \in C^\infty(\mathsf{M})$ we have that

$$\{X^{**}, F\}(p) = -p \cdot [X, dF(p)] = -p \cdot 0 = 0.$$

A **linear Poisson automorphism** is a linear automorphism $\Psi : \mathfrak{g}^* \to \mathfrak{g}^*$ which preserves the Lie-Poisson bracket, i.e., $\{F, G\} \circ \Psi = \{F \circ \Psi, G \circ \Psi\}$, for every $F, G \in C^\infty(\mathfrak{g}^*)$. The Linear Poisson automorphisms are directly related to the Lie algebra automorphisms as shown in the proposition below.

A.3.5 Proposition. ([32]) *A linear map $\psi : \mathfrak{g} \to \mathfrak{g}$ is a Lie algebra automorphism if and only if its dual $\psi^* : \mathfrak{g}^* \to \mathfrak{g}^*$ is a linear Poisson automorphism.*

That is the linear Poisson automorphisms are exactly the dual maps of the Lie algebra automorphisms.

A.3.4 The optimal control problem

For us, an optimal control problem consists in minimising some cost functional over the trajectories of a control system subject to boundary conditions over a fixed time. Let Σ be a left-invariant control affine system as described in section A.2. An **invariant optimal control problem** associated to Σ is specified (i) the control system Σ, (ii) the **cost functional** $\mathcal{J}(u(t)) = \int_0^T \chi(u(t))dt$, where the **cost function** $\chi : \mathbb{R}^\ell \to \mathbb{R}$ is a positive definite quadratic form and (iii) the boundary data, i.e., $g(0) = 0$, $g(T) = g_1$ for some fixed time $T > 0$. We seek to minimise the cost functional $\mathcal{J}$ among all possible admissible controls $u(\cdot) : [0, T] \to \mathbb{R}^\ell$ for which the corresponding trajectory

$g(\cdot) : [0,T] \to \mathsf{G}$ satisfies the boundary conditions $g(0) = g_0$ and $g(T) = g_1$. The pair $(g(\cdot), u(\cdot))$ is called the **controlled trajectory**. An optimal control problem may therefore be written as

$$\dot{g}(t) = \Xi(g(t), u(t)), \quad g(\cdot) : [0,T] \to \mathsf{G}, \quad u(\cdot) : [0,T] \to \mathbb{R}^{\ell}, \tag{A.1}$$

$$g(0) = g_0, \quad g(T) = g_1, \quad g_0, g_1 \in \mathsf{G} \text{ fixed}, \ T > 0 \text{ fixed}, \tag{A.2}$$

$$\mathcal{J}(u(\cdot)) = \int_0^T \chi(u(t))dt \to \min. \tag{A.3}$$

In general a solution to the optimal control problem is not guaranteed.

The Hamiltonian lift

Each left-invariant vector field Ξ on G may be canonically lifted to a Hamiltonian vector field as follows. To each vector field Ξ on G we associate a Hamiltonian H_{Ξ} which is a function on $T^*\mathsf{G}$ defined by

$$H_{\Xi}(g,p) = (g,p) \cdot \Xi(g) = p \cdot (T_g L_{g^{-1}} \cdot \Xi(g)).$$

The vector field $\vec{H}_{\Xi}$ is called the **Hamiltonian lift**.

This idea may be extended to a family of cost extended Hamiltonians. Consider an optimal control problem as given by A.1–A.2 –A.3. We associate a family of **cost extended Hamiltonians** $\{H_u^{\nu} | \nu \in \mathbb{R}, u \in \mathbb{R}^{\ell}\}$ where each $H_u^{\nu} : T^*\mathsf{G} \to \mathbb{R}$ is defined as

$$\begin{aligned} H_u^{\mu}(\xi) &= \langle \xi, \Xi_u(g) \rangle + \nu\chi(u) \\ &= \langle p, \Xi_u(\mathbf{1}) \rangle + \nu\chi(u), \quad \xi = (g,p) \in T^*\mathsf{G}. \end{aligned} \tag{A.4}$$

The Hamiltonian functions $H_u^{\nu}(g,p) = H_u^{\nu}(p)$ are G-invariant.

Pontryagin Maximum Principle

It is the Pontryagin Maximum Principle (PMP for short) which provides necessary conditions for the optimality of the optimal control problems. An optimal control problem is lifted via the PMP to a family of invariant Hamiltonian functions on the cotangent bundle $T^*\mathsf{G}$ which is then reduced to a single Hamiltonian system on the minus Lie Poisson space $\mathfrak{g}^*_-$. We state PMP in terms of the above Hamiltonian functions A.4

A.3.6 Theorem. (cf. [29]) *Suppose $(\tilde{g}(t), \tilde{u}(t))$, $t \in [0,T]$ is a solution to the optimal control problem A.1–A.2–A.3. Then there exists a nontrivial pair: $(\nu, \xi(\cdot)) \neq 0$, $\nu \in \mathbb{R}$, $\xi(\cdot) : [0,T] \to T^*\mathsf{G}$ with $\xi(t) \in T^*_{\tilde{g}(t)}\mathsf{G}, t \in [0,T]$ such that the following conditions hold for almost every $t \in [0,T]$:*

$$\dot{\xi}(t) = \vec{H}_{\tilde{u}(t)}(\xi(t)) \tag{A.5}$$

$$H^{\nu}_{\tilde{u}(t)}(\xi(t)) = \max_{u \in \mathbb{R}^{\ell}} H^{\nu}_u(\xi(t)) = \text{constant}, \quad \nu \leq 0. \tag{A.6}$$

An optimal trajectory $g(t) : [0,T] \to \mathsf{G}$ is then the projection of the integral curve $\xi(\cdot) : [0,T] \to T^*\mathsf{G}$ of the (time-varying) Hamiltonian vector field $\vec{H}^{\nu}_{\tilde{u}(t)}$.

If $\nu \neq 0$, the integral curve $\xi(\cdot)$ is called a **normal extremal** as is its corresponding trajectory control pair $(g(\cdot), u(\cdot))$. If $\nu = 0$, $\xi(\cdot)$ and its corresponding trajectory control pair $(g(\cdot), u(\cdot))$ are called **abnormal extremals**.

Suppose the maximised Hamiltonian is smooth. Then the PMP eliminates the control parameter u from the family of Hamiltonians and reduces the problem to the study of solutions of a single G-invariant Hamiltonian function on $T^*\mathsf{G} \cong \mathsf{G} \times \mathfrak{g}^*$. The G-Invariance of H allows a reduction of the Poisson structure of $T^*\mathsf{G}$ to a Poisson structure (the (minus) Lie-Poisson structure (see subsectionA.3.3)) on the dual space $\mathfrak{g}^*$. The extremal controls are related to the integral curves of the vector field $\vec{H}$. Hence the investigation of the extremal controls may be reduced to the study of the Hamilton-Poisson system $(\mathfrak{g}^*_-, H)$.

Consider an optimal control problem A.1–A.2–A.3 with dynamics $\Xi(\mathbf{1}, u) = A+u_1B_1+\ldots+u_\ell B_\ell$ and cost function $\chi(u) = (u-\mu)^\top Q(u-\mu)$ where $Q \in \mathbb{R}^\ell$ is symmetric positive definite. Let $\mathbf{B}$ be the $n \times \ell$ matrix where the i^{th} column of $\mathbf{B}$ is the coordinate vector of B_i in the basis of $\mathfrak{g}$. Then $\Xi(\mathbf{1}, u) = A + \mathbf{B}u$. By an application of the PMP we have the following result.

A.3.7 Theorem. ([17],[19]) *Any normal extremal trajectory control pair* $(g(\cdot), u(\cdot))$ *is given by* $\dot{g}(t) = \Xi(g(t), u(t))$, $u(t) = Q^{-1}\mathbf{B}^\top p(t)$. *Here* $p(\cdot)$ *is an integral curve of the Hamilton-Poisson system on* $\mathfrak{g}^*_-$ *specified by*

$$H(p) = p(A + B\mu) + \frac{1}{2}p\mathbf{B}Q^{-1}\mathbf{B}^\top p^\top.$$

Clearly when $\mu = 0$ we have

$$H(p) = p(A) + \frac{1}{2}p\mathbf{B}Q^{-1}\mathbf{B}^\top p^\top.$$

A.3.8 Remark. Since Q is a positive definite matrix and $\mathbf{B}$ does not have full rank in general, $\mathbf{B}Q\mathbf{B}^\top$ is a positive semidefinite matrix.

Accordingly, finding the extremal control trajectories of such an optimal control system may be reduced to the study of the Hamilton-Poisson system $(\mathfrak{g}^*_-, H_{A,\mathcal{Q}})$ where

$$H_{A,\mathcal{Q}} = p(A) + \mathcal{Q}(p).$$

Here $A \in \mathfrak{g}$ and $\mathcal{Q}$ is a positive semidefinite quadratic form on $\mathfrak{g}^*_-$. The equations of motion for the controlled trajectory $(g(\cdot), u(\cdot))$ on $T^*\mathsf{G}$ take the form

$$\begin{cases} \dot{p}(t) = \vec{H}(p(t)) \\ \dot{g}(t) = \Xi(g(t), u(t)). \end{cases}$$

A.4 Stability of Dynamical Systems

This section introduces (nonlinear) stability and spectral stability on Poisson manifolds (section A.3.2). The definitions and theory draw on [32] and [37].

Let $(\mathsf{M}, \{\cdot,\cdot\})$ be a Poisson manifold and let X be a smooth vector field. An **equilibrium point** of X is a point p_e such that $X(p_e) = 0$. Hence the unique integral curve $\xi(\cdot)$ starting at p_e is constant. That is to say $\xi(t) = p_e$ for all t. The stability analysis of equilibrium points involves looking at the trajectories starting near the equilibrium point. An equilibrium state p_e is said to be stable when trajectories which start near p_e remain near p_e. Formally

A.4.1 DEFINITION. Let $DX(p_e)$ denote the linearization of X at p_e. We say that an equilibrium point p_e of a vector field X is

1. **Lyapunov stable** if for every neighbourhood U of p_e there exists a neighbourhood $V \subseteq U$ of p_e such that for any integral curve $\xi(\cdot)$ starting in V i.e. $\xi(0) \in V$ then $\xi(t) \in U$, $\forall t \geq 0$.
2. **Lyapunov unstable** if it is not Lyapunov stable. That is there exists a neighbourhood U of p_e such that for every neighbourhood $V \subseteq U$ of p_e there exists an integral curve $\xi(\cdot)$ with $\xi(0) \in V$ such that $\xi(t_1) \notin U$ for some $t_1 \geq 0$.
3. **spectrally stable** if all eigenvalues of $DX(p_e)$ have non-positive real parts.
4. **spectrally unstable** if it is not spectrally stable.

We have the following result linking spectral stability and Lyapunov stability

A.4.2 PROPOSITION. ([32]) *If an equilibrium point is Lyapunov stable then it is spectrally stable.*

Which is useful in proving Lyapunov instability. Otherwise Lyapunov instability can be shown directly by finding a suitable integral curve. The **energy-Casimir method** gives sufficient conditions for an equilibrium point of a Hamiltonian vector field to be Lyapunov stable.

A.4.3 THEOREM. (ENERGY-CASIMIR METHOD, CF. [37]) *Suppose* p_e *is an equilibrium point of a Hamiltonian vector field* $\vec{H}$ *on* M *corresponding to a Hamiltonian function* $H \in C^\infty(\mathsf{M})$ *and that there exists a Casimir function* C *such that* $\mathrm{d}(H+C)(p_e) = 0$ *and* $\mathrm{d}^2(H+C)(p_e)$ *is (positive or negative) definite then* p_e *is Lyapunov stable.*

A useful extension of this method is the **continuous energy-Casimir method**.

A.4.4 THEOREM. (CONTINUOUS ENERGY-CASIMIR METHOD, CF. [37]) *Let* p_e *be an equilibrium point corresponding to the Hamiltonian vector field* $\vec{H}$ *of the Hamiltonian* H. *Then if*

$$H^{-1}(H(p_e)) \bigcap_{i=1,k} C_i^{-1}C_i(p_e) = p_e$$

for any local Casimir functions C_i *around the point* p_e *then* p_e *is Lyapunov stable.*

Appendix B

Tabulation of results

The table B.1 collects the results of chapter 2.

Table B.1: Classification of full-rank left-invariant control affine systems on H_3

System	DF-equivalence classes	SDF-equivalence classes	S-equivalence classes $(\alpha \neq 0, \gamma_i \in \mathbb{R})$
$(1,1)$	$E_2 + u_1 E_3$	$E_2 + u_1 E_3$	$E_2 + u_1 E_3$
$(2,0)$	$u_1 E_2 + u_2 E_3$	$E_2 + u_1 E_2 + u_2 E_3$	$\gamma_1 E_2 + \gamma_2 E_3 + u_1 E_2 + u_2 E_3, \quad \gamma_1^2 + \gamma_2^2 \neq 0$
		$u_1 E_2 + u_2 E_3$	$u_1 E_2 + u_2 E_3$
$(2,1)$	$E_1 + u_1 E_2 + u_2 E_3$	$E_1 + E_2 + u_1 E_2 + u_2 E_3$	$\alpha E_1 + \gamma_1 E_2 + \gamma_2 E_3 + u_1 E_2 + u_2 E_3, \quad \gamma_1^2 + \gamma_2^2 \neq 0$
		$E_1 + u_1 E_2 + u_2 E_3$	$\alpha E_1 + u_1 E_2 + u_2 E_3$
	$E_2 + u_1 E_1 + u_2 E_3$	$E_2 + u_1 E_1 + u_2 E_3$	$E_2 + u_1 \alpha E_1 + u_2 E_3$
			$E_2 + u_1 E_3 + u_2(\alpha E_1 + \gamma_1 E_3)$
$(3,0)$	$u_1 E_1 + u_2 E_2 + u_3 E_3$	$E_2 + u_1 E_1 + u_2 E_2 + u_3 E_3$	$\gamma_1 E_1 + \gamma_2 E_2 + \gamma_3 E_3 + u_1(\alpha E_1 + \gamma_4 E_2 + \gamma_5 E_2 + u_2 E_2 + u_3 E_3)$
			$\gamma_1 E_1 + \gamma_2 E_2 + \gamma_3 E_3 + u_1 \alpha E_1 + u_2 E_2 + u_3 E_3$
			$\gamma_1 E_2 + \gamma_2 E_2 + \gamma_3 E_3 + u_1 E_2 + u_2 E_3 + u_3(\alpha E_1 + \gamma_4 E_3)$
			$\gamma_1 E_2 + \gamma_2 E_2 + \gamma_3 E_3 + u_1 E_2 + u_2 \alpha E_1 + u_3 E_3, \ \gamma_2^2 + \gamma_3^2 \neq 0, \gamma_4^2 + \gamma_5^2 \neq 0.$
		$E_1 + u_1 E_1 + u_2 E_2 + u_3 E_3$	$\alpha E_1 + u_1(\alpha E_1 + \gamma_1 E_2 + \gamma_2 E_2) + u_2 E_2 + u_3 E_3$
			$\alpha E_1 + u_1 \alpha E_1 + u_2 E_2 + u_3 E_3$
			$\alpha E_1 + u_1 E_2 + u_1 E_3 + u_2(\alpha E_1 + \gamma_1 E_3)$
			$\alpha E_1 + u_1 E_2 + u_2 \alpha E_1 + u_3 E_3$
		$u_1 E_1 + u_2 E_2 + u_3 E_3$	$u_1(\alpha E_1 + \gamma_1 E_2 + \gamma_2 E_2) + u_2 E_2 + u_3 E_3$
			$u_1 \alpha E_1 + u_2 E_2 + u_3 E_3$
			$u_1 E_2 + u_2 E_3 + u_3(\alpha E_1 + \gamma_1 E_3)$
			$u_1 E_2 + u_2 \alpha E_1 + u_3 E_3$

Appendix C

Mathematica Code

Wolfram Mathematica 8 was used to produce the graphs of the adjoint and coadjoint orbits and to assist in calculations. This section includes the code for the graphs of the adjoint and coadjoint orbits as well as the code for calculating the Hamiltonian vector fields of chapter 4. The input is given in **bold** while the output is not.

C.1 Heisenberg group

The code in this section was used in the calculations of chapter 1

C.1.1 Lie Group H_3

H_3 is a Lie group.

$$\mathbf{Inverse}\left[\begin{pmatrix} \mathbf{1} & \mathbf{x2} & \mathbf{x1} \\ \mathbf{0} & \mathbf{1} & \mathbf{x3} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix}\right] \mathbf{//MatrixForm}$$

$$\begin{pmatrix} 1 & -\mathrm{x2} & -\mathrm{x1}+\mathrm{x2x3} \\ 0 & 1 & -\mathrm{x3} \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{1} & \mathbf{x2} & \mathbf{x1} \\ \mathbf{0} & \mathbf{1} & \mathbf{x3} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix} . \begin{pmatrix} \mathbf{1} & \mathbf{y2} & \mathbf{y1} \\ \mathbf{0} & \mathbf{1} & \mathbf{y3} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix} \mathbf{//MatrixForm}$$

$$\begin{pmatrix} 1 & \mathrm{x2}+\mathrm{y2} & \mathrm{x1}+\mathrm{y1}+\mathrm{x2y3} \\ 0 & 1 & \mathrm{x3}+\mathrm{y3} \\ 0 & 0 & 1 \end{pmatrix}$$

Automorphism Group of $\mathfrak{h}_3$

$$\boldsymbol{\psi} = \begin{pmatrix} \mathbf{v2w3}-\mathbf{v3w2} & \mathbf{v1} & \mathbf{w1} \\ \mathbf{0} & \mathbf{v2} & \mathbf{w2} \\ \mathbf{0} & \mathbf{v3} & \mathbf{w3} \end{pmatrix};$$

$$\mathbf{cc[A\_,B\_]{:=}A.B-B.A;}$$

$$\mathbf{A}=\begin{pmatrix}\mathbf{0} & \mathbf{a2} & \mathbf{a1}\\ \mathbf{0} & \mathbf{0} & \mathbf{a3}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\end{pmatrix};\mathbf{B}=\begin{pmatrix}\mathbf{0} & \mathbf{b2} & \mathbf{b1}\\ \mathbf{0} & \mathbf{0} & \mathbf{b3}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\end{pmatrix};\mathbf{AA}=\begin{pmatrix}\mathbf{a1}\\ \mathbf{a2}\\ \mathbf{a3}\end{pmatrix};\mathbf{BB}=\begin{pmatrix}\mathbf{b1}\\ \mathbf{b2}\\ \mathbf{b3}\end{pmatrix};$$

$\mathbf{cc[A,B]//MatrixForm}$

$$\begin{pmatrix}0 & 0 & -\mathrm{a3b2}+\mathrm{a2b3}\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

$$\boldsymbol{\psi}.\begin{pmatrix}\mathbf{-a3b2+a2b3}\\ \mathbf{0}\\ \mathbf{0}\end{pmatrix}\mathbf{//MatrixForm}$$

$$\begin{pmatrix}(-\mathrm{a3b2}+\mathrm{a2b3})(-\mathrm{v3w2}+\mathrm{v2w3})\\ 0\\ 0\end{pmatrix}$$

$\{\boldsymbol{\psi}\mathbf{.AA//MatrixForm}, \boldsymbol{\psi}\mathbf{.BB//MatrixForm}\}$

$$\left\{\begin{pmatrix}\mathrm{a2v1}+\mathrm{a3w1}+\mathrm{a1}(-\mathrm{v3w2}+\mathrm{v2w3})\\ \mathrm{a2v2}+\mathrm{a3w2}\\ \mathrm{a2v3}+\mathrm{a3w3}\end{pmatrix},\begin{pmatrix}\mathrm{b2v1}+\mathrm{b3w1}+\mathrm{b1}(-\mathrm{v3w2}+\mathrm{v2w3})\\ \mathrm{b2v2}+\mathrm{b3w2}\\ \mathrm{b2v3}+\mathrm{b3w3}\end{pmatrix}\right\}$$

$$\mathbf{cc}\left[\begin{pmatrix}\mathbf{0} & \mathbf{a2v2+a3w2} & \mathbf{a2v1+a3w1+a1(-v3w2+v2w3)}\\ \mathbf{0} & \mathbf{0} & \mathbf{a2v3+a3w3}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\end{pmatrix},\right.$$

$$\left.\begin{pmatrix}\mathbf{0} & \mathbf{b2v2+b3w2} & \mathbf{b2v1+b3w1+b1(-v3w2+v2w3)}\\ \mathbf{0} & \mathbf{0} & \mathbf{b2v3+b3w3}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\end{pmatrix}\right]\mathbf{//MatrixForm//Simplify}$$

$$\begin{pmatrix}0 & 0 & (\mathrm{a3b2}-\mathrm{a2b3})(\mathrm{v3w2}-\mathrm{v2w3})\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

Adjoint representations

$$\mathbf{g}=\begin{pmatrix}\mathbf{1} & \mathbf{x2} & \mathbf{x1}\\ \mathbf{0} & \mathbf{1} & \mathbf{x3}\\ \mathbf{0} & \mathbf{0} & \mathbf{1}\end{pmatrix};$$

$$\left\{\mathbf{g}.\begin{pmatrix}\mathbf{0} & \mathbf{0} & \mathbf{1}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\end{pmatrix}\mathbf{.Inverse[g]//MatrixForm}, \mathbf{g}.\begin{pmatrix}\mathbf{0} & \mathbf{1} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\end{pmatrix}\mathbf{.Inverse[g]//MatrixForm},\right.$$

$$\left.\mathbf{g}.\begin{pmatrix}\mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{1}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\end{pmatrix}\mathbf{.Inverse[g]//MatrixForm}\right\}$$

$$\left\{\begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},\begin{pmatrix}0 & 1 & -\mathrm{x3}\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},\begin{pmatrix}0 & 0 & \mathrm{x2}\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}\right\}$$

Centres

$$\begin{pmatrix} \mathbf{1} & \mathbf{x2} & \mathbf{x1} \\ \mathbf{0} & \mathbf{1} & \mathbf{x3} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix} . \begin{pmatrix} \mathbf{1} & \mathbf{y2} & \mathbf{y1} \\ \mathbf{0} & \mathbf{1} & \mathbf{y3} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix} \mathbf{.Inverse}\left[\begin{pmatrix} \mathbf{1} & \mathbf{x2} & \mathbf{x1} \\ \mathbf{0} & \mathbf{1} & \mathbf{x3} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix}\right] \mathbf{//MatrixForm}$$

$$\begin{pmatrix} 1 & \mathrm{y2} & \mathrm{x2x3} + \mathrm{y1} - \mathrm{x3}(\mathrm{x2} + \mathrm{y2}) + \mathrm{x2y3} \\ 0 & 1 & \mathrm{y3} \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{Solve}\left[\begin{pmatrix} \mathbf{1} & \mathbf{y2} & \mathbf{y1} \\ \mathbf{0} & \mathbf{1} & \mathbf{y3} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix} == \begin{pmatrix} \mathbf{1} & \mathbf{y2} & \mathbf{x2x3} + \mathbf{y1} - \mathbf{x3}(\mathbf{x2} + \mathbf{y2}) + \mathbf{x2y3} \\ \mathbf{0} & \mathbf{1} & \mathbf{y3} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix}, \{\mathbf{x1}, \mathbf{x2}, \mathbf{x3}\}\right]$$

Solve::svars : Equations may not give solutions for all “solve” variables. ⟩⟩ $\left\{\left\{\mathrm{x3} \to \frac{\mathrm{x2y3}}{\mathrm{y2}}\right\}\right\}$

Nilpotency

$$\mathbf{g} = \begin{pmatrix} \mathbf{1} & \mathbf{x2} & \mathbf{x1} \\ \mathbf{0} & \mathbf{1} & \mathbf{x3} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix}; \mathbf{h} = \begin{pmatrix} \mathbf{1} & \mathbf{y2} & \mathbf{y1} \\ \mathbf{0} & \mathbf{1} & \mathbf{y3} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix}; \mathbf{h}' = \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{z} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix};$$

$\{\mathbf{Inverse[g]//MatrixForm}, \mathbf{Inverse[h]//MatrixForm}\}$

$$\left\{\begin{pmatrix} 1 & -\mathrm{x2} & -\mathrm{x1} + \mathrm{x2x3} \\ 0 & 1 & -\mathrm{x3} \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -\mathrm{y2} & -\mathrm{y1} + \mathrm{y2y3} \\ 0 & 1 & -\mathrm{y3} \\ 0 & 0 & 1 \end{pmatrix}\right\}$$

$\mathbf{g.h.Inverse[g].Inverse[}h\mathbf{]//MatrixForm//Simplify}$

$$\begin{pmatrix} 1 & 0 & -\mathrm{x3y2} + \mathrm{x2y3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\mathbf{Inverse[h']//MatrixForm}$

$$\begin{pmatrix} 1 & 0 & -z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\mathbf{g.h'.Inverse[g].Inverse[h']//MatrixForm//Simplify}$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Complete Solvability

$$\mathbf{Eigenvalues}\left[\begin{pmatrix} \mathbf{0} & -\mathbf{x3} & \mathbf{x2} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}\right]$$

$\{0, 0, 0\}$

Solvable

$$\mathbf{g}=\begin{pmatrix}\mathbf{1}&\mathbf{x2}&\mathbf{x1}\\\mathbf{0}&\mathbf{1}&\mathbf{x3}\\\mathbf{0}&\mathbf{0}&\mathbf{1}\end{pmatrix};\mathbf{g'}=\begin{pmatrix}\mathbf{1}&\mathbf{y2}&\mathbf{y1}\\\mathbf{0}&\mathbf{1}&\mathbf{y3}\\\mathbf{0}&\mathbf{0}&\mathbf{1}\end{pmatrix};\mathbf{h}=\begin{pmatrix}\mathbf{1}&\mathbf{0}&\mathbf{z1}\\\mathbf{0}&\mathbf{1}&\mathbf{0}\\\mathbf{0}&\mathbf{0}&\mathbf{1}\end{pmatrix};\mathbf{h'}=\begin{pmatrix}\mathbf{1}&\mathbf{0}&\mathbf{z2}\\\mathbf{0}&\mathbf{1}&\mathbf{0}\\\mathbf{0}&\mathbf{0}&\mathbf{1}\end{pmatrix};$$

g.g′.Inverse[g].Inverse[g′]//MatrixForm//Simplify

$$\begin{pmatrix}1&0&-\mathrm{x3y2}+\mathrm{x2y3}\\0&1&0\\0&0&1\end{pmatrix}$$

h.h′.Inverse[h].Inverse[h′]//MatrixForm//Simplify

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

Semisimple

$$\mathbf{Z1}=\begin{pmatrix}\mathbf{0}&\mathbf{0}&\mathbf{z1}\\\mathbf{0}&\mathbf{0}&\mathbf{0}\\\mathbf{0}&\mathbf{0}&\mathbf{0}\end{pmatrix};\mathbf{Z2}=\begin{pmatrix}\mathbf{0}&\mathbf{0}&\mathbf{z2}\\\mathbf{0}&\mathbf{0}&\mathbf{0}\\\mathbf{0}&\mathbf{0}&\mathbf{0}\end{pmatrix};$$

cc[A_, B_]:=A.B − B.A;
cc[Z1, Z2]//MatrixForm

$$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$$

C.1.2 Adjoint and coadjoint orbits

Adjoint orbits

$$\begin{pmatrix}\mathbf{1}&\mathbf{-y3}&\mathbf{y2}\\\mathbf{0}&\mathbf{1}&\mathbf{0}\\\mathbf{0}&\mathbf{0}&\mathbf{1}\end{pmatrix}.\begin{pmatrix}\mathbf{x1}\\\mathbf{x2}\\\mathbf{x3}\end{pmatrix}\mathbf{//MatrixForm}$$

$$\begin{pmatrix}\mathrm{x1}+\mathrm{x3y2}-\mathrm{x2y3}\\\mathrm{x2}\\\mathrm{x3}\end{pmatrix}$$

The code below was used in producing the graphs of adjoint orbits shown in section 1.2

adjoint1 = ListPointPlot3D[Table[{x, 0, 0}, {x, −1, 1, 0.25}], PlotStyle → PointSize[Large]];
adjoint2 = ParametricPlot3D[Table[{x, y, z}, {y, {−1, −.5, .5, 1}}, {z, {−1, −.5, .5, 1}}], {x, −1, 1}];
Opts = {Axes→ True, BoxRatios → {1, 1, 1},PlotRange → 1.05{{−1, 1}, {−1, 1}, {−1, 1}}, Boxed → False, ImageSize → Small, AxesLabel→ {$\mathbf{E_1^*}$, $\mathbf{E_2^*}$, $\mathbf{E_3^*}$},LabelStyle → Directive [Medium], AxesEdge → {−{1, 1}, {1, −1}, {1, 1}}, FaceGrids→ {{−1, 0, 0}, −{0, −1, 0}, {0, 0, −1}},Ticks → None}; Viewv = {0, 0, 1}; View = 2{1, −2, 0.5};
Opts1 = Flatten[Append[Opts, {ViewVertical → Viewv, ViewPoint → View}]];
Show[adjoint1, adjoint2]

Coadjoint orbits

$$\mathbf{g} = \begin{pmatrix} \mathbf{1} & \mathbf{y2} & \mathbf{y1} \\ \mathbf{0} & \mathbf{1} & \mathbf{y3} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix}; \textbf{Inverse}[\mathbf{g}]//\textbf{MatrixForm}$$

$$\begin{pmatrix} 1 & -\mathrm{y2} & -\mathrm{y1}+\mathrm{y2y3} \\ 0 & 1 & -\mathrm{y3} \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{1} & \mathbf{y3} & -\mathbf{y2} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix} . \begin{pmatrix} \mathbf{x1} \\ \mathbf{x2} \\ \mathbf{x3} \end{pmatrix} //\textbf{MatrixForm}$$

$$\begin{pmatrix} \mathrm{x1} - \mathrm{x3y2} + \mathrm{x2y3} \\ \mathrm{x2} \\ \mathrm{x3} \end{pmatrix}.$$

The code below was used to obtain the graphs of the coadjoint orbits shown in section 1.2
coadjoint1 = ParametricPlot3D[**Table**[{**x, y, z**}, {**x**, {−**1**, −.**5**, .**5**, **1**}}], {**y**, −**1**, **1**}, {**z**, −**1**, **1**}, **Mesh** → 4];
coadjoint2 = ListPointPlot3D[**Table**[{**0, y, z**}, {**y**, −**1**, **1**, **0.25**}, {**z**, −**1**, **1**, **0.25**}], **PlotStyle** → **PointSize**[**Large**]];
Opts = {**Axes**→ **True**, **BoxRatios** → {**1**, **1**, **1**},**PlotRange** → **1.05**{{−**1**, **1**}, {−**1**, **1**}, {−**1**, **1**}}, **Boxed** → **False**, **ImageSize** → **Small**, **AxesLabel**→ {$\mathbf{E}_1^*$, $\mathbf{E}_2^*$, $\mathbf{E}_3^*$},**LabelStyle** → **Directive** [**Medium**], **AxesEdge** → {−{**1**, **1**}, {**1**, −**1**}, {**1**, **1**}}, **FaceGrids**→ {{−**1**, **0**, **0**}, −{**0**, −**1**, **0**}, {**0**, **0**, −**1**}},**Ticks** → **None**}; **Viewv** = {**0**, **0**, **1**}; **View** = **2**{**1**, −**2**, **0**.**5**};
Opts1 = **Flatten**[**Append**[**Opts**, {**ViewVertical** → **Viewv**, **ViewPoint** → **View**}]];
Show[**coadjoint1**, **coadjoint2**, **Opts1**]

C.2 Equivalence of Quadratic Hamilton-Poisson Systems

For $p \in \mathfrak{h}_3^*$, let $\Psi : p \mapsto \Psi_0(p) + q$, $\Psi_0(p) = p\psi$ where $\psi = [\psi_{ij}]$. In this section we determine

$$\psi_0 \cdot \vec{H}(p) \quad \text{and} \quad \vec{H} \circ \Psi(p)$$

for the quadratic Hamilton-Poisson systems on $\mathfrak{h}_{3-}^*$. The results of which are used in chapter 4 in the verification that the systems are not A-equivalent. The Mathematica code follows.
BF[**M_**]:=**MatrixForm**[**M**];
FS:=FullSimplify;
cc[**A_**, **B_**]**:=A.B-B.A;**

$$\mathbf{M}[\{\mathbf{x\_,y\_,z\_}\}]\mathbf{:=} \begin{bmatrix} \mathbf{0} & \mathbf{y} & \mathbf{x} \\ \mathbf{0} & \mathbf{0} & \mathbf{z} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix};$$

E1=M[{**1,0,0**}]; **E2=M**[{**0,1,0**}]; **E3=M**[{**0,0,1**}];
Minv[**MM_**]**:=Module**[{ss, $\mathbf{z_1}, \mathbf{z_2}, \mathbf{z_3}$},
ss = **Solve**[**MM** = =$\mathbf{z_1}$**E1** + $\mathbf{z_2}$**E2** + $\mathbf{z_3}$**E3**, {**z1**, **z2**, **z3**}];{$\mathbf{z_1}, \mathbf{z_2}, \mathbf{z_3}$}/.ss[[1]]]; **Minv**[**M**[{**x**, **y**, **z**}]];
Base = {**E1**, **E2**, **E3**};
pp = {$\mathbf{p_1}, \mathbf{p_2}, \mathbf{p_3}$};
qq = {$\mathbf{q_1}, \mathbf{q_2}, \mathbf{q_3}$};

$\mathbf{PB[F\_,G\_]:=-\,pp.Minv@cc[M@D[F,\{pp\}],M@D[G,\{pp\}]];}$
$\mathbf{Hvec[H\_]:=Table(PB(p_i,H),\{i,3\});}$
$\mathbf{Hvec[H\_,v\_]:=Hvec[}H\mathbf{]/.\{p_1\rightarrow p_1,p_2\rightarrow p_2,p_3\rightarrow p_3\}/.\{p_1\rightarrow v[[1]],p_2\rightarrow v[[2]],p_3\rightarrow v[[3]]\};}$

Hamilton Poisson systems.
$\mathbf{H0{=}\ 0;H1{=}\ \frac{1}{2}p_2^{\wedge}2;H2{=}\ \frac{1}{2}(p_2^{\wedge}2+p_3^{\wedge}2);H01{=}\ p_2;H11{=}\ p_2+\frac{1}{2}p_2^{\wedge}2;}$
$\mathbf{H12{=}\ p_3+\frac{1}{2}p_2^{\wedge}2;H21{=}\ p_2+\frac{1}{2}(p_2^{\wedge}2+p_3^{\wedge}2);}$

$$\boldsymbol{\psi}{=}\begin{bmatrix}\boldsymbol{\psi 11} & \boldsymbol{\psi 12} & \boldsymbol{\psi 13}\\ \boldsymbol{\psi 21} & \boldsymbol{\psi 22} & \boldsymbol{\psi 23}\\ \boldsymbol{\psi 31} & \boldsymbol{\psi 32} & \boldsymbol{\psi 33}\end{bmatrix};$$

$\mathbf{\{MatrixForm[\boldsymbol{\psi}.Hvec[H0]],MatrixForm[\boldsymbol{\psi}.Hvec[H1]],MatrixForm[\boldsymbol{\psi}.Hvec[H2]],}$
$\mathbf{MatrixForm[\boldsymbol{\psi}.Hvec[H01]],MatrixForm[\boldsymbol{\psi}.Hvec[H11]],MatrixForm[\boldsymbol{\psi}.Hvec[H12]],}$
$\mathbf{MatrixForm[\boldsymbol{\psi}.Hvec[H21]]\}}$

$$\left\{\begin{bmatrix}0\\0\\0\end{bmatrix},\begin{bmatrix}\psi 13p_1p_2\\ \psi 23p_1p_2\\ \psi 33p_1p_2\end{bmatrix},\begin{bmatrix}\psi 13p_1p_2-\psi 12p_1p_3\\ \psi 23p_1p_2-\psi 22p_1p_3\\ \psi 33p_1p_2-\psi 32p_1p_3\end{bmatrix},\begin{bmatrix}\psi 13p_1\\ \psi 23p_1\\ \psi 33p_1\end{bmatrix},\right.$$
$$\left.\begin{bmatrix}-\psi 13p_1-1-p_2\\ -\psi 23p_1-1-p_2\\ -\psi 33p_1-1-p_2\end{bmatrix},\begin{bmatrix}-\psi 12p_1+\psi 13p_1p_2\\ -\psi 22p_1+\psi 23p_1p_2\\ -\psi 32p_1+\psi 33p_1p_2\end{bmatrix},\begin{bmatrix}-\psi 13p_1-1-p_2-\psi 12p_1p_3\\ -\psi 23p_1-1-p_2-\psi 22p_1p_3\\ -\psi 33p_1-1-p_2-\psi 32p_1p_3\end{bmatrix}\right\}$$

$\mathbf{\{vecH0=MatrixForm[Hvec[H0,\boldsymbol{\psi}.pp+qq]],vecH1=MatrixForm[Hvec[H1,\boldsymbol{\psi}.pp+qq]],}$
$\mathbf{vecH2=MatrixForm[Hvec[H2,\boldsymbol{\psi}.pp+qq]],vecH01=MatrixForm[Hvec[H01,\boldsymbol{\psi}.pp+}$
$\mathbf{qq]],vecH11=MatrixForm[Hvec[H11,\boldsymbol{\psi}.pp{+}qq]],vecH12=MatrixForm[Hvec[H12,\boldsymbol{\psi}.pp{+}}$
$\mathbf{qq]],vecH21=MatrixForm[Hvec[H21,\boldsymbol{\psi}.pp+qq]]\}}$

$$\left\{\begin{bmatrix}0\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\ \psi 11p_1+\psi 12p_2+\psi 13p_3+q_1\psi 21p_1+\psi 22p_2+\psi 23p_3+q_2\end{bmatrix},\right.$$
$$\begin{bmatrix}0\\ -\psi 11p_1+\psi 12p_2+\psi 13p_3+q_1\psi 31p_1+\psi 32p_2+\psi 33p_3+q_3\\ \psi 11p_1+\psi 12p_2+\psi 13p_3+q_1\psi 21p_1+\psi 22p_2+\psi 23p_3+q_2\end{bmatrix},\begin{bmatrix}0\\0\\ \psi 11p_1+\psi 12p_2+\psi 13p_3+q_1\end{bmatrix},$$
$$\begin{bmatrix}0\\0\\ -\psi 11p_1+\psi 12p_2+\psi 13p_3+q_1-1-\psi 21p_1-\psi 22p_2-\psi 23p_3-q_2\end{bmatrix},$$
$$\begin{bmatrix}0\\ -\psi 11p_1-\psi 12p_2-\psi 13p_3-q_1\\ \psi 11p_1+\psi 12p_2+\psi 13p_3+q_1\psi 21p_1+\psi 22p_2+\psi 23p_3+q_2\end{bmatrix},$$
$$\left.\begin{bmatrix}0\\ -\psi 11p_1+\psi 12p_2+\psi 13p_3+q_1\psi 31p_1+\psi 32p_2+\psi 33p_3+q_3\\ -\psi 11p_1+\psi 12p_2+\psi 13p_3+q_1-1-\psi 21p_1-\psi 22p_2-\psi 23p_3-q_2\end{bmatrix}\right\}.$$

C.3 Stability and Integration of Hamilton-Poisson Systems

Below is the code used in determining the stability nature of the equilibria of the Hamilton-Poisson systems as well as the code used in finding explicit expressions for the integral curves of the systems.

$H_1(p) = \frac{1}{2}p_2^2$

$$\mathbf{Hvec}\left[\tfrac{\mathbf{1}}{\mathbf{2}}\mathbf{p_2}\hat{}\mathbf{2}\right]\mathbf{//FullSimplify//MatrixForm}$$

$$\begin{pmatrix} 0 \\ 0 \\ p_1p_2 \end{pmatrix}$$

$$\mathbf{D}\left[\{\mathbf{0},\mathbf{0},\mathbf{p_1p_2}\},\{\{\mathbf{p_1},\mathbf{p_2},\mathbf{p_3}\}\}\right]\mathbf{//MatrixForm}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ p_2 & p_1 & 0 \end{pmatrix}$$

$$\left\{\mathbf{Eigenvalues}\left[\begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \boldsymbol{\eta} & \mathbf{0} & \mathbf{0} \end{pmatrix}\right],\mathbf{Eigenvalues}\left[\begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\nu} & \mathbf{0} \end{pmatrix}\right]\right\}$$

$$\{\{0,0,0\},\{0,0,0\}\}$$

$H_2(p) = \frac{1}{2}(p_2^2 + p_3^2)$

$$\mathbf{Hvec}\left[\tfrac{\mathbf{1}}{\mathbf{2}}\left(\mathbf{p_2}\hat{}\mathbf{2}+\mathbf{p_3}\hat{}\mathbf{2}\right)\right]\mathbf{//FullSimplify//MatrixForm}$$

$$\begin{pmatrix} 0 \\ -p_1p_3 \\ p_1p_2 \end{pmatrix}$$

$$\mathbf{D}\left[\{\mathbf{0},-\mathbf{p_1p_3},\mathbf{p_1p_2}\},\{\{\mathbf{p_1},\mathbf{p_2},\mathbf{p_3}\}\}\right]\mathbf{//MatrixForm}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ -p_3 & 0 & -p_1 \\ p_2 & p_1 & 0 \end{pmatrix}$$

$$\left\{\mathbf{Eigenvalues}\left[\begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\boldsymbol{\mu} & \mathbf{0} & \mathbf{0} \\ \boldsymbol{\eta} & \mathbf{0} & \mathbf{0} \end{pmatrix}\right],\mathbf{Eigenvalues}\left[\begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{\mu} \\ \mathbf{0} & -\boldsymbol{\mu} & \mathbf{0} \end{pmatrix}\right]\right\}$$

$H_1^{(0)} = p_2$

$$\mathbf{Hvec}[\mathbf{p_2}]\mathbf{//FullSimplify//MatrixForm}$$

$$\begin{pmatrix} 0 \\ 0 \\ p_1 \end{pmatrix}$$

$$\mathbf{D}\left[\{\mathbf{0},\mathbf{0},\mathbf{p_1}\},\{\{\mathbf{p_1},\mathbf{p_2},\mathbf{p_3}\}\}\right]\mathbf{//MatrixForm}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\mathbf{Eigenvalues}\left[\begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} \end{pmatrix}\right]$$

$$\{0,0,0\}$$

$H_1^{(1)}(p) = p_2 + \frac{1}{2}p_2^2$

$$\mathbf{Hvec}\left[\mathbf{p_2 + \tfrac{1}{2}p_2\hat{\ }2}\right]\mathbf{//FullSimplify//MatrixForm}$$

$$\begin{pmatrix} 0 \\ 0 \\ p_1(1+p_2) \end{pmatrix}$$

$$\mathbf{MatrixForm, D}\left[\{\mathbf{0, 0, p_1 + p_1p_2}\}, \{\{\mathbf{p_1, p_2, p_3}\}\}\right]\mathbf{//MatrixForm}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1+p_2 & p_1 & 0 \end{pmatrix}$$

$$\left\{\mathbf{Eigenvalues}\left[\begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{1}+\boldsymbol{\eta} & \mathbf{0} & \mathbf{0} \end{pmatrix}\right], \mathbf{Eigenvalues}\left[\begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}\right]\right\}$$

$$\{\{0,0,0\},\{0,0,0\}\}$$

$H_2^{(1)}(p) = p_3 + \frac{1}{2}p_2^2$

$$\mathbf{Hvec}\left[\mathbf{p_3 + \tfrac{1}{2}p_2\hat{\ }2}\right]\mathbf{//FullSimplify//MatrixForm}$$

$$\begin{pmatrix} 0 \\ -p_1 \\ p_1p_2 \end{pmatrix}$$

$$\mathbf{D}\left[\{\mathbf{0, -p_1, p_1p_2}\}, \{\{\mathbf{p_1, p_2, p_3}\}\}\right]\mathbf{//MatrixForm}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ p_2 & p_1 & 0 \end{pmatrix}$$

$$\mathbf{Eigenvalues}\left[\begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \boldsymbol{\eta} & \mathbf{0} & \mathbf{0} \end{pmatrix}\right]$$

$$\{0,0,0\}$$

$H_1^{(2)}(p) = p_2 + \frac{1}{2}(p_2^2 + p_3^2)$

$$\mathbf{Hvec}\left[\mathbf{p_2 + \tfrac{1}{2}\left(p_2\hat{\ }2 + p_3\hat{\ }2\right)}\right]\mathbf{//FullSimplify//MatrixForm}$$

$$\begin{pmatrix} 0 \\ -p_1p_3 \\ p_1(1+p_2) \end{pmatrix}$$

$$\mathbf{D}\left[\{\mathbf{0, -p_1p_3, p_1(1+p_2)}\}, \{\{\mathbf{p_1, p_2, p_3}\}\}\right]\mathbf{//MatrixForm}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ -p_3 & 0 & -p_1 \\ 1+p_2 & p_1 & 0 \end{pmatrix}$$

$$\left\{\mathbf{Eigenvalues}\left[\begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\boldsymbol{\mu} & \mathbf{0} & \mathbf{0} \\ \mathbf{1}+\boldsymbol{\eta} & \mathbf{0} & \mathbf{0} \end{pmatrix}\right], \mathbf{Eigenvalues}\left[\begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\boldsymbol{\nu} \\ \mathbf{0} & \boldsymbol{\nu} & \mathbf{0} \end{pmatrix}\right]\right\}$$

$$\{\{0,0,0\},\{0,-i\nu,i\nu\}\}$$

Integration of $\frac{1}{2}(p_2^2+p_3^2)$

$$\mathbf{A} = \begin{pmatrix} \mathbf{0} & -\mathbf{c1} \\ \mathbf{c1} & \mathbf{0} \end{pmatrix};$$

$$\mathbf{MatrixExp[At]//MatrixForm}$$

$$\begin{pmatrix} \mathrm{Cos}[ct] & -\mathrm{Sin}[ct] \\ \mathrm{Sin}[ct] & \mathrm{Cos}[ct] \end{pmatrix}$$

Integration of $p_2+\frac{1}{2}(p_2^2+p_3^2)$

$$\mathbf{A} = \begin{pmatrix} \mathbf{0} & -\mathbf{c1} \\ \mathbf{c1} & \mathbf{0} \end{pmatrix}; \mathbf{B} = \begin{pmatrix} \mathbf{0} \\ \mathbf{c1} \end{pmatrix}; \mathbf{M} = -\mathbf{Inverse[A].B}$$

$$\{\{-1\},\{0\}\}$$

$$\mathbf{MatrixExp[At]//MatrixForm}$$

$$\begin{pmatrix} \mathrm{Cos}[\mathrm{c1}t] & -\mathrm{Sin}[\mathrm{c1}t] \\ \mathrm{Sin}[\mathrm{c1}t] & \mathrm{Cos}[\mathrm{c1}t] \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{Cos}[\mathbf{c1}t] & -\mathbf{Sin}[\mathbf{c1}t] \\ \mathbf{Sin}[\mathbf{c1}t] & \mathbf{Cos}[\mathbf{c1}t] \end{pmatrix} . \begin{pmatrix} \mathbf{p20} \\ \mathbf{p30} \end{pmatrix}$$

$$\mathbf{//MatrixForm}$$

$$\begin{pmatrix} \mathrm{p20Cos}[\mathrm{c1}t] - \mathrm{p30Sin}[\mathrm{c1}t] \\ \mathrm{p30Cos}[\mathrm{c1}t] + \mathrm{p20Sin}[\mathrm{c1}t] \end{pmatrix}$$

$$\mathbf{Pt} = \begin{pmatrix} \mathbf{p20Cos}[\mathbf{c1}t] - \mathbf{p30Sin}[\mathbf{c1}t] \\ \mathbf{p30Cos}[\mathbf{c1}t] + \mathbf{p20Sin}[\mathbf{c1}t] \end{pmatrix} + \mathbf{M//MatrixForm}$$

$$\begin{pmatrix} -1 + \mathrm{p20Cos}[\mathrm{c1}t] - \mathrm{p30Sin}[\mathrm{c1}t] \\ \mathrm{p30Cos}[\mathrm{c1}t] + \mathrm{p20Sin}[\mathrm{c1}t] \end{pmatrix}$$

$$\mathbf{Reduce[p1p3 == 0\&\&p1 + p1p2 == 0, \{p1, p2, p3\}]}$$

$$(\mathrm{p2} == -1\&\&\mathrm{p3} == 0) \| \mathrm{p1} == 0$$

$$\mathbf{eqns = \{p2'[t] == -cp3[t], p3'[t] == c(1 + p2[t])\};}$$

$$\mathbf{sol = DSolve[eqns, \{p2, p3\}, t]//FullSimplify}$$

$$\left\{\left\{\mathrm{p2} \to \mathrm{Function}\left[\{t\}, C[1]\mathrm{Cos}[ct] - \mathrm{Cos}[ct]^2 - C[2]\mathrm{Sin}[ct] - \mathrm{Sin}[ct]^2\right], \mathrm{p3} \to \mathrm{Function}[\{t\}, C[2]\mathrm{Cos}[ct] + C[1]\mathrm{Sin}[ct]]\right\}\right\}$$

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