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COMPLETE REGULARITY AND RELATED CONCEPTS IN L–UNIFORM SPACES

by

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ABSTRACT

L will denote a completely distributive lattice with an order reversing involution.

The concept of an L–uniform space is introduced. An extension theorem concerning L–uniformly continuous functions is proved. A characterisation of L–uniformizability, involving L–complete regularity is given. With respect to L–completely regular spaces it is shown that the topological modification of an L–completely regular space is completely regular. Furthermore it is shown that the topologically generated L–topology of a completely regular space is L–completely regular.

CONTENTS

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PREFACE

In this thesis we consider the extension of the notion of a uniform structure on a set to an L–fuzzy environment. From the outset we omit the word fuzzy and L will always denote a complete lattice which for most of the time is required to be completely distributive with an order reversing involution.

L–uniform spaces were defined by Hutton ([6]). A considerable part of this thesis deals with his paper ([6]). In the context of L–uniform spaces L will always be a completely distributive lattice with an order reversing involution.

Chapter 0 is a preliminary chapter. All lattice concepts and results needed in this thesis appear in Section 1. The definition of an L–topology with most of its basic properties is given in Section 2.

In Chapter 1 we begin by considering an alternative approach to the theory of uniform spaces. Briefly, we show how a subset of the Cartesian product of a set, containing the diagonal set, can be treated as a special type of set mapping. We call this approach, in which a uniform structure is viewed as a collection of these mappings, a uniform mapping approach to the theory of uniform spaces. Hutton ([6]) generalised this special type of set mapping to a set mapping of L–sets. This generalisation is dealt with in Section 2. Section 3 contains the definition of an L–uniform structure on a set. Analogues in an L–setting of uniform topology and uniformly continuous functions are considered. In this section we assign to each uniformity an L–uniformity. Subspaces of L–uniform spaces, weak L–uniformities and product L–uniformities are the operations on L–uniform spaces dealt with in Section 4. Hutton's ([6],[7]) extension of the unit interval, the L–unit interval I(L), plays an important role in the characterisation of L–uniformities. The L–unit interval I(L), with an L–uniformity on it which induces its natural L–topology, is discussed at length in Section 5, the final section of this chapter.

Katetov ([8],[9]), using an insertion type lemma, proved that a bounded real–valued, uniformly continuous function defined on a subspace of a uniform space can be extended to the entire space. His lemmas can be easily applied to L–uniform spaces but his constructions of a function which uniformly extends a uniformly continuous function relies to a large extent on the algebraic properties of the real line. Having considered the situation in a more lattice theoretic setting, we have been able, in Chapter II, to prove an L–version of his result, in which the unit interval is replaced by the L–unit interval I(L).

It is worth noting that Kubiak ([12]) was able to prove an L–Tietze extension theorem using the same lemmas. In Section 1 of Chapter II, we state and prove Katetov's lemmas. A function which L–uniformly extends an L–uniformly continuous function defined on a subspace of an L–uniform space is constructed in Section 2.

In Chapter III we consider L–completely regular spaces. We begin, in Section 1, by proving a lemma of Hutton's which motivates his definition of L–complete regularity. Hutton's characterisation of L–uniformizability in terms of L–complete regularity and weak L–topologies generated by collections of I(L)–valued L–continuous functions is given. In Section 2 the usual questions of subspaces and products of L–completely regular spaces are considered.

The final chapter, Chapter IV, deals mainly with the topological modification of L–completely regular spaces. Kubiak ([11]) extended the Lowen ([16]) functors ω and ι to L–topological spaces. The functor ω assigns to each topology an L–topology known as the topologically generated L–topology. Conversely the functor ι assigns to each L–topology a topology known as the topological modification. In both these assignments the upper interval topology on a complete lattice is used. In Section 1 the upper interval topology is defined and a useful base for this topology is found when L is a completely distributive lattice.

In this chapter we show firstly that the functor ω maps the category of completely regular spaces into the category of L–completely regular spaces. Secondly we show that the functor ι maps the category of L–completely regular spaces into the category of completely regular spaces.

In Section 2 these two functors are defined and their properties which are needed in this chapter are proved. We do not use categorical methods to analyse these functors, in fact we treat them rather as mappings than functors.

Working with the unit interval I as a lattice, Katsaras ([10]) constructed for each uniformity an I–uniformity which induces the topologically generated I–topology. We extend this construction to completely distributive lattices in Section 3. This result allows us to prove that the topologically generated L–topology of a completely regular topology is L–completely regular.

Kubiak ([11]) proved that the topological modification of the L–unit interval I(L), is a compact Hausdorff space. This crucial result is proved in Section 4. Using an L–version of an embedding lemma and the extension of the T_0 axiom to L–topological spaces we give a new proof of Liu's ([14]) result that an L–Tychonoff space (an L–completely regular, L–T_0 space) can be embedded in a cube of I(L). This embedding is preserved in the topological modification. This enables us to prove that the topological modification of an L–Tychonoff space is Tychonoff. This material forms part of Sections 5 and 6. Using an L–T_0 modification technique in which we are able to modify an L–completely regular space into an L–Tychonoff space, we prove that the topological modification of an L–completely regular space is completely regular. Section 8 contains a counterexample, showing that for many of the properties of the functors in Section 2 completeness of L is not sufficient.

All topological concepts and results used in this thesis are well known. As such we have not referenced any topological concepts. All of these concepts and results may be found in S. Willard's General Topology, ([3]).

CHAPTER 0

LATTICE AND L–FUZZY CONCEPTS

0.1 LATTICE CONCEPTS

We shall use the following definition of a lattice.

DEFINITION 0.1.1 ([1] : 0.1.6, [2] : I.1.1)

A partially ordered set $(P, \leq)$ consists of a non–empty set P and a relation $\leq$ which is reflexive, antisymmetric and transitive. If all elements of P are comparable under $\leq$, that is $\alpha \leq \beta$ or $\beta \leq \alpha$ for all elements $\alpha, \beta \in P$, we call P a chain. If $\alpha \leq \beta$ and $\alpha \neq \beta$ we write $\alpha < \beta$.

■

We define an opposite relation $\leq^{op}$ on P by the condition that for all $\alpha, \beta \in P$, we have

$$\alpha \leq^{op} \beta \text{ if and only if } \beta \leq \alpha.$$

Thus $(P, \leq^{op})$ is a partially ordered set. We write P^{op} for short. Let Φ be a statement about partially ordered sets. If we replace each occurrence of $\leq$ by $\leq^{op}$ we get the dual of Φ. Observe that Φ holds in $(P,\leq)$ if and only if the dual of Φ holds in $(P, \leq^{op})$. This is known as the Duality principle.

0.1.2 DEFINITION ([1] : 0.1.1, [2] : I.1.2)

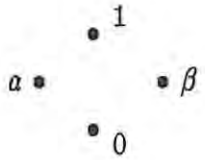
Let $(P, \leq)$ be a partially ordered set, $H \subseteq P$, $\alpha \in P$. Then α is an upper bound of H if $\beta \leq \alpha$ for all $\beta \in H$. An upper bound α of H is the least upper bound of H or supremum of H if for any upper bound β of H, we have $\alpha \leq \beta$. We shall write $\alpha = \sup H$ or $\alpha = \vee H$. The concepts of lower bound and greatest lower bound or infimum are defined dually. We denote the infimum of H by inf H or $\wedge H$. The supremum of the empty set (which if it exists is the same as inf P) is called the bottom element and is denoted by 0. The infimum of the empty (which if it exists is the same as sup P) is called the top element and is denoted by 1.

■

A diagram of $(P,\leq)$ represents the elements of P with dots $\bullet$. The dots representing α and β are joined with a line if $\alpha \leq \beta$ or $\beta \leq \alpha$. If $\alpha \leq \beta$, the dot representing β occurs higher up than the dot representing α. Let $P = \{0,1,\alpha,\beta\}$ and

$$\leq\ = \{(0,0);(0,\alpha);(0,\beta);(0,1);(\beta,1);(1,1);(\alpha,\alpha);(\beta,\beta)\}.$$

Then the diagram of $(P,\leq)$ is given by



0.1.3 DEFINITION ([1]: 0.1.8, [2] : I.1.4)

A partially ordered set $(L,\leq)$ is called a lattice if

$$\alpha \vee \beta = \sup\{\alpha;\beta\},\ \alpha \wedge \beta = \inf\{\alpha,\beta\}$$

exists for all α, $\beta \in L$.

If $\vee A$, $\wedge A$ exists for each $A \subseteq L$ then L is called a complete lattice.

■

We shall often simply write $(L,\leq)$ as L. Concerning complete lattices we have,

0.1.4 PROPOSITION ([1]: 0.2.2)

Let $(L,\leq)$ be a partially ordered set. Then the following are equivalent.

(i) L is a complete lattice.

(ii) For each $H \subseteq L$ we have that $\vee H$ exists.

(iii) For each $H \subseteq L$ we have that $\wedge H$ exists.

(iv) L has a top element and for each non–empty $H \subseteq L$ we have that $\wedge H$ exists.

(v) L has a bottom element and for each non–empty $H \subseteq L$ we have that $\vee H$ exists.

■

We now describe certain subsets of a lattice.

0.1.5 DEFINITION ([1]: 0.1.3)

Let L be a lattice and $A \subseteq L$ be a non–empty subset.

(i) A is a directed set if for each pair α, $\beta \in A$ there exists $\gamma \in A$ such that $\alpha \leq \gamma$, $\alpha \leq \gamma$.

(ii) A is a filter base if A is a directed set in L^{op}.

(iii) A is an upper set if for α, $\beta \in L$, $\alpha \leq \beta$ and $\alpha \in A$ implies that $\beta \in A$. The notion of a lower set is defined dually.

(iv) A is a filter if A is an upper set and a filter base.

(v) For $\alpha \in L$ let $\uparrow \alpha = \{\beta \in L : \alpha \leq \beta\}, \downarrow \alpha = \{\beta \in L : \beta \leq \alpha\}$.

■

If we set $\uparrow^{op} \alpha = \{\beta \in L : \alpha \leq^{op} \beta\}$, then $\uparrow^{op} \alpha = \downarrow \alpha$.
Similary $\downarrow^{op} \alpha = \{\beta \in L : \alpha \leq^{op} \beta\} = \uparrow \alpha$. When we refer to a filter on the underlying set of L we mean a filter of subsets of L.

0.1.6 DEFINITION ([1]: I.3.11)

An element α in a lattice L is called prime if and only if $\gamma \wedge \beta \leq \alpha$ implies $\gamma \leq \alpha$ or $\beta \leq \alpha$. An element is coprime if and only if it is prime in L^{op}. We denote the set of prime elements by PRIME(L), and the set of coprime elements by COP(L).

■

In Chapter IV we shall make extensive use of the way below relation on a complete lattice.

0.1.7 DEFINITION ([1]: I.1.1)

Let L be a complete lattice. We define the relation $<<$ on L by

$\alpha << \beta$ if and only if for directed subsets $D \subseteq L$ satisfying $\beta \leq \vee D$
there exists $\gamma \in D$ such that $\alpha \leq \gamma$.

We say α is way below β if $\alpha << \beta$. For $\alpha \in L$,

$$\Uparrow \alpha = \{\beta \in L : \alpha << \beta\}, \Downarrow \alpha = \{\beta \in L : \beta << \alpha\}.$$

■

0.1.8 PROPOSITION ([1]: I.1.2)

In a complete lattice L we have the following statements for $\alpha, \beta, \lambda, \gamma \in L$:

(i) $\alpha << \beta$ implies $\alpha \leq \beta$.

(ii) $\lambda \leq \alpha << \beta \leq \gamma$ implies $\lambda << \gamma$.

(iii) $\alpha << \gamma$ and $\beta << \gamma$ implies $\alpha \vee \beta << \gamma$.

■

We now consider a few examples of how the way below relation behaves on different types of complete lattices. These examples appear in ([1]:I.1).

(a) Let L be a finite complete lattice. Then every directed subset has a maximum element. Thus for each $\alpha \in L$ we have $\alpha << \alpha$.

(b) Let L be a complete chain. Then $\alpha << \beta$ if and only if $\alpha < \beta$ or if $\alpha = \beta$ then $\vee \{\gamma \in L : \gamma < \beta\} = \beta$.

0.1.9 PROPOSITION

Let L be a complete lattice. Then for $\alpha \in COP(L)$ and $A \subseteq L$, $\alpha << \vee A$ if and only if there exists $\gamma \in A$ such that $\alpha << \gamma$.

PROOF

Suppose that $\alpha << \vee A$ but that for each $\gamma \in A$, $\alpha << \gamma$. Therefore for each $\gamma \in A$ there exists a directed subset D_γ such that $\gamma \leq \vee D_\gamma$ and for each $\lambda \in D_\gamma$, $\alpha \nleq \lambda$. Let S be the set of suprema of all finite subsets of $\bigcup_{\gamma \in A} D_\gamma$. Then S is a directed subset of L and $\vee A \leq \bigvee_{\gamma \in A} (\vee D_\gamma) = \vee S$. Since $\alpha << \vee A$ there exists $\lambda \in S$ such that $\alpha \leq \lambda$. The element λ is of the form $\bigvee_{i=1}^{n} \lambda_{\gamma_i}$, $\lambda_{\gamma_i} \in D_{\gamma_i}$. Therefore $\alpha \leq \lambda_{\gamma_i}$ for some i since α is co–prime. This contradicts the statement that $\alpha \nleq \lambda$, for each $\lambda \in D_{\gamma_i}$. Thus for some $\gamma \in A$, $\alpha << \gamma$. The converse follows trivially from (0.1.8(ii)).

■

0.1.10 DEFINITION ([1]: I.1.6)

A complete lattice L is continuous if for each $\alpha \in$ L,

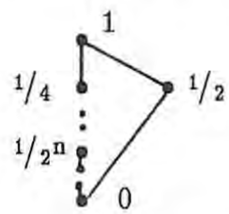
$$\begin{aligned} \alpha &= \vee \{\beta \in L : \beta << \alpha\} \\ &= \vee \twoheaddownarrow \alpha. \end{aligned}$$

■

Observe that in a continuous lattice L if $\alpha \nleq \beta$ then there exists $\gamma \in$ L such that $\gamma << \alpha$ and $\gamma \nleq \beta$.

(c) A strictly increasing infinite chain is a chain (P,$\leq$) in which for each $\alpha \in$ P there exists $\beta \in$ P such that $\alpha < \beta$. Let L be a complete lattice. Then L is said to satisfy the ascending chain condition if L contains no strictly increasing chains. This can be shown to be equivalent to every subset of L having a maximal element. A maximal element of a directed subset is a maximum element. Therefore if L satisfies the ascending chain condition we have $\alpha << \alpha$ for each $\alpha \in$ L. Thus L is continuous.

(d) Let L be the complete lattice described in the diagram below.



Every chain in L contains it supremum. Therefore L satisfies the ascending chain condition. This implies that L is continuous. However L^{op} does not satisfy the ascending chain condition. The set $\{^1/_{2^n} : n \geq 2\}$ is a strictly increasing chain in L^{op}. Further, the only element way below $^1/_2$ in L^{op} is 1. Hence L^{op} is not continuous.

Continuous lattices obey an order density type of property described in the proposition below.

0.1.11 PROPOSITION ([1]:I.1.18)

Let L be a continuous lattice. Then for α, $\beta \in$ L,

$\alpha << \beta$ and $\alpha \neq \beta$ implies that there exists $\gamma \in$ L such that
$\alpha << \gamma << \beta$ and $\alpha \neq \gamma$.

■

0.1.12 COROLLARY ([1]:I.1.19)

Let L be a continuous lattice and $\alpha, \beta \in L$. Then if $\alpha << \beta$ there exists $\gamma \in L$ such that $\alpha << \gamma << \beta$.

■

0.1.13 DEFINITION ([2]: I.1.5, VII. I.10, [1]: I.2.4)

Let L be a lattice.

(i) If for any $\alpha, \beta, \gamma \in L$, the identity $\alpha \wedge (\beta \vee \gamma) = (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$ holds, L is then said to satisfy the distributive law or to be a distributive lattice.

(ii) If L is complete and for any family $\{\alpha_{jk} : j \in J, k \in K(j)\} \subseteq L$, the identity

$$\bigwedge_{j \in J} \bigvee_{k \in K(j)} \alpha_{jk} = \bigvee_{\varphi \in \prod_j K(j)} \bigwedge_{j \in J} \alpha_{j\varphi(j)}$$

holds, then L satisfies the completely distributive law and L is called a completely distributive lattice.

■

0.1.14 PROPOSITION ([2]:I.1.5, VII.1.10)

(i) L is distributive if and only if L^{op} is distributive.

(ii) L is completely distributive if and only if L^{op} is completely distributive.

■

Thus the duals of the identities appearing in (0.1.13) hold in the relevant lattices. Observe that by (d) if L is continuous then L^{op} is not necessarily continuous.

In a completely distributive lattice L, the infinite distributive law

$$\alpha \wedge (\vee A) = \vee \{\alpha \wedge \beta : \beta \in A\} \quad \text{for } A \subseteq L,$$

holds ([2]: VII.1.10).

The next proposition shows that continuous lattices are completely distributive in a directed sense.

0.1.15 PROPOSITION ([1]: I.2.3)

Let L be a complete lattice. Then the two statements below are equivalent.

(i) L is continuous.

(ii) For any family $\{\alpha_{jk} : j \in J, k \in K(j)\} \subseteq L$ where for each $j \in J$, $\{\alpha_{jk} : k \in K(j)\}$ is a directed set, the identity

$$\bigwedge_{j \in J} \bigvee_{k \in K(j)} \alpha_{jk} = \bigvee_{\varphi \in \prod_{j \in J} K(j)} \bigwedge_{j \in J} \alpha_{j\varphi(j)},$$

holds.

■

0.1.16 COROLLARY ([1]:I.2.5)

If L is completely distributive then L is continuous.

■

The concepts of complete distributivity, distributivity and continuity of a lattice are linked in the proposition below.

0.1.17 PROPOSITION ([1]:I.3.15)

Let L be a lattice. Then the following statements are equivalent.

(i) L is completely distributive.

(ii) L is continuous and every element can be written as the supremum of coprimes.

(iii) L is distributive, L and L^{op} are continuous.

■

0.1.18 PROPOSITION ([1]:I.3.14)

The prime elements of a completely distributive lattice L are order generating in the sense that for each $\alpha \in L$,

$$\alpha = \bigwedge \{\beta \in \text{PRIME}(L) : \alpha \leq \beta\}.$$

■

From (0.1.17(ii)) we have that in a completely distributive lattice L, for each $\alpha \in L$, $\alpha = \bigvee (\Downarrow \alpha \cap \text{COP}(L))$. Using (0.1.17) we are able to give examples of completely distributive lattices.

(e) Every element of a complete chain is coprime and by (b) every complete chain is continuous. Therefore by (0.1.17(ii)) a complete chain is completely distributive. Thus the unit interval I with its usual ordering is a completely distributive lattice.

(f) The lattice described in (d) fails to be completely distributive since its opposite lattice is not continuous.

Our next lemma, originally a lemma of Hutton ([6]), is crucial in this thesis and shall be used frequently.

0.1.19 LEMMA

Let L be a completely distributive lattice. For each $\alpha \in$ L, there exists a subset B(α), which satisfies

(i) $\alpha = \vee$ B(α).

(ii) If A $\subseteq$ L and $\alpha \leq \vee$A, then for each $\gamma \in$ B(α) there exits $\beta \in$ A such that $\gamma \leq \beta$.

PROOF

Let $\alpha \in$ L. Set $\mathscr{J}$ = {A $\subseteq$ L : $\alpha \leq \vee$A}. Allowing the elements of $\mathscr{J}$ to index themselves we have by the complete distributive law that

$$\bigwedge_{A \in \mathscr{J}} \bigvee_{\gamma \in A} \gamma = \bigvee_{\varphi \in \prod_{A \in \mathscr{J}} A} \bigwedge_{A \in \mathscr{J}} \varphi(A).$$

Clearly $\alpha \leq \bigwedge_{A \in \mathscr{J}} \bigvee_{\gamma \in A} \gamma$. Since $\{\alpha\}$ satisfies $\alpha \leq \vee \{\alpha\}$, we have that

$\alpha = \bigwedge_{A \in \mathscr{J}} \bigvee_{\gamma \in A} \gamma$. For each $\varphi \in \prod_{A \in \mathscr{J}} A$ put $\alpha_\varphi = \bigwedge_{A \in \mathscr{J}} \varphi(A)$. Then $\alpha = \vee \{\alpha_\varphi : \varphi \in \prod_{A \in \mathscr{J}} A\}$. For every A $\subseteq$ L satisfying $\alpha \leq \vee$A, we have for each $\varphi \in \prod_{A \in \mathscr{J}} A$, that $\varphi(A) \in$ A and that $\alpha_\varphi \leq \varphi(A)$. Therefore $\{\alpha_\varphi : \varphi \in \prod_{A \in \mathscr{J}} A\}$ is a set satisfying (i) and (ii).

■

0.1.20 DEFINITION ([11])

Let L be a complete lattice. A function $'$: $L \longrightarrow L$ satisfying

(i) $\alpha \leq \beta$ implies $\beta' \leq \alpha'$,

(ii) $(\alpha')' = \alpha$,

for $\alpha, \beta \in L$ is called an order reversing involution.

0.1.21 PROPOSITION ([11])

Let $'$ be an order–reversing involution on L. Then $'$ satisfies de Morgan's laws :

(i) $(\bigvee_{\alpha \in A} \alpha)' = \bigwedge_{\alpha \in A} \alpha'$

(ii) $(\bigwedge_{\alpha \in A} \alpha)' = \bigvee_{\alpha \in A} \alpha'$

■

0.2 L–FUZZY CONCEPTS

In this section X,Y will denote non–empty sets and L will denote a complete lattice with an order–reversing involution $'$.

0.2.1 DEFINITION ([5],[19])

A function $\mu : X \longrightarrow L$ is called an L–fuzzy set.

■

From this point onwards we shall omit the adjective fuzzy and simply prefix any concept pertaining to L–fuzzy sets with L. Further, in cases where no confusion will result, we altogether omit L. We shall, where possible, use the Greek letters η, λ, μ, ν to denote L–sets.

The collection L^X of L–sets is itself a complete lattice with an induced partial ordering given by

$$\mu \leq \nu \text{ if and only if } \mu(x) \leq \nu(x) \text{ for each } x \in X$$

and an induced order–reversing involution given by

$$\mu' : X \longrightarrow L, \ \mu'(x) = (\mu(x))'.$$

Since this induced partial ordering on L^X is given pointwise, all lattice theoretic properties of L such as continuity and complete distributivity are inherited by L^X. When dealing with L–concepts we shall virtually always require that L have an order–reversing involution. Therefore when working in a L–frame work we shall always assume that L has an order reversing involution. We denote the bottom element of L^X by $\underline{0}$ and the top element by $\underline{1}$. Let $A \subseteq X$, $\alpha \in L$. The L–set $\alpha 1_A$ is defined by

$$\alpha 1_A(x) = \begin{cases} \alpha & x \in A \\ 0 & \text{otherwise} \end{cases},$$

when $\alpha = 1$, we write 1_A. When $L = \{0,1\} = 2$, we shall use A and 1_A interchangably. We treat 2^X as the power set $\mathscr{P}(X)$ on X and also as the collection of $\{0,1\}$–sets. That is we regard 2^X as a lattice under inclusion with union and intersection as sup and inf. We shall use $2^{(X)}$ to denote the collection of finite subsets of X.

0.2.2 DEFINITION ([19],[5])

Let $f : X \rightarrow Y$. This function induces two L–set mappings. If $\mu \in L^X$, then its image $f^{\rightarrow}(\mu)$ under f is the L–set defined by

$$f^{\rightarrow}(\mu)(y) = \bigvee_{x \in X \,:\, f(x) = y} \mu(x) \quad \text{for } y \in Y.$$

If $\nu \in L^Y$, then its reverse image $f^{\leftarrow}(\nu)$ under f is the L–set given by

$$f^{\leftarrow}(\nu)(x) = \nu(f(x)) \quad \text{for } x \in X.$$

■

When $L = \{0,1\}$, the above definition reduces to the usual definition of the image and reverse image of a subset under a function. In some cases where we deal simultaneously with $2^X, 2^Y$ and L^X, L^Y for some lattice L other than 2, we shall use the notation

$$f(A) = \{y \in Y : y = f(x) \text{ for some } x \in A\},$$

$$f^{-1}(B) = \{x \in X : f(x) \in B\} \text{ for } A \in 2^X, B \in 2^Y.$$

The main properties of these L–set mappings are given in the proposition below.

0.2.3 PROPOSITION

Let $f : X \rightarrow Y$. Further let $\mu, \mu_1 \in L^X$, $\nu, \nu_1 \in L^Y$ and

$\{\mu_j : j \in J\} \subseteq L^X$, $\{\nu_k : k \in K\} \subseteq L^Y$.

(i)	$f^{\rightarrow}(\underline{0}) = \underline{0}$.
(ii)	If $\mu \leq \mu_1$, then $f^{\rightarrow}(\mu) \leq f^{\rightarrow}(\mu_1)$.
(iii)	$f^{\rightarrow}(\bigvee_j \mu_j) = \bigvee_j f^{\rightarrow}(\mu_j)$.
(iv)	$f^{\rightarrow}(\bigwedge_j \mu_j) \leq \bigwedge_j f^{\rightarrow}(\mu_j)$.
(v)	If $\nu \leq \nu_1$, then $f^{\leftarrow}(\nu) \leq f^{\leftarrow}(\nu_1)$.
(vi)	$f^{\leftarrow}(\bigvee_k \nu_k) = \bigvee_k f^{\leftarrow}(\nu_k)$.
(vii)	$f^{\leftarrow}(\bigwedge_k \nu_k) = \bigwedge_k f^{\leftarrow}(\nu_k)$.
(viii)	$\mu \leq f^{\leftarrow}(f^{\rightarrow}(\mu))$, with equality if f is one to one.
(ix)	$f^{\rightarrow}(f^{\leftarrow}(\nu)) \leq \nu$, with equality if f is onto.
(x)	$f^{\leftarrow}(\nu') = f^{\leftarrow}(\nu)'$.
(xi)	$f^{\leftarrow}(f^{\rightarrow}(\mu)') \leq \mu'$.

■

We now introduce the L–concepts of topology and continuity.

0.2.4 DEFINITION ([4],[6])

An L–topology τ on X is a collection of L–sets satisfying the following three conditions :

(LT1) $\underline{0}, \underline{1} \in \tau$.

(LT2) For any subset $\{\mu_j : j \in J\} \subseteq \tau$, we have $\bigvee_j \mu_j \in \tau$.

(LT3) For $\mu, \nu \in \tau$, we have $\mu \wedge \nu \in \tau$.

The pair (X,τ) is called an L–topological space or more simply an L–space. Elements of τ are referred to as L–open sets. An L–set μ is called an L–closed set if $\mu' \in \tau$. Let $Y \subseteq X$. Then for $\mu \in L^X$ let $\mu_{|Y}$ denote the restriction of μ to Y. Then $\tau_{|Y} = \{\mu_{|Y} : \mu \in \tau\}$ is an L–topology on Y and is referred to as the subspace L–topology induced by τ.

■

We use the Greek letters δ, τ to denote L–topologies. If $\delta \subset \tau$ we say that δ is weaker than τ and τ is stronger than δ. Observe that the class of $\{0,1\}$–topologies is precisely the class of topologies in the usual sense.

We reserve the symbol Δ to denote a topology. To every topology Δ on X we can assign an L–topology defined by

$$\chi(\Delta) = \{1_A : A \in \Delta\}.$$

0.2.5 DEFINITION ([4])

Let (X,τ), (Y,δ) be L–spaces. Then $f : X \longrightarrow Y$ is L–continuous if and only if for each $\nu \in \delta$, we have $f^{\leftarrow}(\nu) \in \tau$. If f is a bijection, then we say that f is an L–homeomorphism if f^{-1} and f are L–continuous. On some occasions we will write that f is (τ,δ)–L–continuous. Suppose $e : (X,\tau) \longrightarrow (Y,\delta)$ is one to one and L–continuous. Then e^{-1} if $(\delta \mid_{e(X)},\tau)$–L–continuous, we say that e is an L–embedding.

■

Let Δ be a topology on X. A collection $B \subseteq \Delta$ is a base for Δ if Δ can be recovered from unions of subcollections of B. Any collection $\mathscr{S} \subseteq 2^X$ generates a topology on X in the sense that the collection of finite intersections of elements of $\mathscr{S}$ forms a base for some topology on X. That is any collection of subsets is a subbase for a topology. The validity of these statements depends directly on the fact that 2^X is a completely distributive lattice which allows us to write

$$(\bigcup_j A_j) \cap (\bigcup_k B_k) = \bigcup_{j,k} (A_j \cap B_k).$$

In our more general setting of a complete lattice we can at best do the following.

0.2.6 PROPOSITION ([11])

Let $\mathscr{S} \subseteq L^X$. Then we define the L–topology generated by $\mathscr{S}$ to be the weakest L–topology τ containing $\mathscr{S}$. Then

$$\tau = \bigcap \{\delta \subseteq L^X : \delta \text{ is an L–topology and } \mathscr{S} \subseteq \delta\}.$$

We refer to $\mathscr{S}$ as a subbase of τ. Conversely for an L–topology τ on X, $\mathscr{S} \subseteq \tau$ is a subbase for τ if τ is the weakest L–topology containing $\mathscr{S}$.

■

If L is completely distributive, then L^X is completely distributive. In this improved situation we have

0.2.7 PROPOSITION ([4])

Let L be completely distributive and let τ be an L–topology on X. Then we define $\mathscr{B} \subseteq \tau$ to be a base for τ if each $\mu \in \tau$ can be written as the supremum of a subcollection of τ.

(i) A collection $\mathscr{B} \subseteq L^X$ is a base for some L–topology on X if and only if $\underline{1} = \bigvee_{\mu \in \mathscr{B}} \mu$ and for each $\mu, \nu \in \mathscr{B}, \mu \wedge \nu$ can be written as the supremum of a subcollection of $\mathscr{B}$.

(ii) Let $\mathscr{S} \subseteq L^X$ be a subbase for some L–topology on X. Then

$$\mathscr{B}(\mathscr{S}) = \{ \bigwedge_{\mu \in A} \mu : A \in 2^{(\mathscr{S})}\},$$

is a base for τ.

PROOF

It is enough to recall that since L^X is completely distributive we have that $(\bigvee_j \mu_j) \wedge (\bigvee_k \nu_k) = \bigvee_{j,k} (\mu_j \wedge \nu_k)$. The proof is then an exact copy of the usual topological proof.

■

0.2.8 PROPOSITION ([6])

Let int : $L^X \longrightarrow L^X$, with L completely distributive, satisfy

(IO1) $\text{int}(\underline{1}) = \underline{1}$,

(IO2) $\text{int}(\mu) \leq \mu$,

(IO3) $\text{int}(\mu \wedge \nu) = \text{int}(\mu) \wedge \text{int}(\nu)$,

(IO4) $\text{int}(\text{int}(\mu)) = \text{int}(\mu)$,

for $\mu, \nu \in L^X$. Then

$$\tau = \{\mu \in L^X : \mu = \text{int}(\mu)\}$$

is an L–topology.

■

0.2.9 DEFINITION ([11])

(i) Let $\{\tau_k : k \in K\}$ be a collection of L–topologies on X. Then the weakest L–topology τ containing each τ_k is denoted by $\bigvee_k \tau_k$. This topology has $\bigcup_k \tau_k$ as a subbase.

(ii) Let $f : X \to (Y,\delta)$. The weak L–topology generated by f is defined to be the weakest L–topology making L–continuous and is denoted by $f^{\leftarrow}(\delta)$. This L–topology has $\{f^{\leftarrow}(\nu) : \nu \in \delta\}$ as a subbase.

(iii) Let $\mathscr{A}$ be a collection of functions $f : X \to (Y_f,\delta_f)$. The weak L–topology $\tau_{w(\mathscr{A})}$ is defined to be $\bigvee_f f^{\leftarrow}(\delta_f)$. Therefore $\tau_{w(\mathscr{A})}$ is the weakest L–topology making each f L–continuous.

(iv) Let $\{(X_j,\tau_j) : j \in J\}$ be a collection of L–spaces. The product L–topology τ on $\prod_j X_j$ is defined to be $\bigvee_j \pi_j^{\leftarrow}(\tau_j)$ where $\pi_j : \prod_j X_j \to X_j$ is the j'th projection mapping.

(v) Let $f : (X,\tau) \to Y$. The strong L–topology generated by f is defined to be the strongest L–topology on Y making f L–continuous and is denoted by $f^{\rightarrow}(\tau)$. As in the topological case $f^{\rightarrow}(\tau)$ is given by

$$f^{\rightarrow}(\tau) = \{\nu \in L^Y : f^{\leftarrow}(\nu) \in \tau\}.$$

When f is onto, we refer to $f^{\rightarrow}(\tau)$ as the quotient L–topology.

■

With regard to L–continuity we have the following unpublished result of U. Höhle which appears in Kubiak ([11]).

0.2.10 PROPOSITION

Let (X,τ), (Y,δ) be L–spaces and $\mathscr{S} \subseteq \delta$, a subbase of δ. Then f is L–continuous if and only if for each $\nu \in \mathscr{S}$, $f^{\leftarrow}(\nu) \in \tau$.

PROOF

The forward direction needs no proof. Conversely $f^{\rightarrow}(\tau)$ contains $\mathscr{S}$, which implies that $\delta \subseteq f^{\rightarrow}(\tau)$. Therefore f is L–continuous.

■

The topological result that the composition of two continuous functions is a continuous function extends to L–spaces. That is the composition of two L–continuous functions is an L–continuous function. In view of (0.2.10) we have

0.2.11 PROPOSITION

A function $f : X \longrightarrow \prod_j X_j$ is L–continuous if and only if $\pi_j \circ f$ is L–continuous for each $j \in J$.

PROOF

The forward direction is clear since the composition of two L–continuous functions is L–continuous. Conversely we have that for each $j \in J$, $f^{\leftarrow}(\pi_j^{\leftarrow}(\mu)) \in \tau$ for each $\mu \in \tau_j$. But $\{\pi_j^{\leftarrow}(\mu) : \mu \in \tau_j, j \in J\}$ is a subbase for the product L–topology on $\prod_j X_j$. Therefore by (0.2.10) f is L–continuous.

■

CHAPTER I

L–UNIFORMITIES

INTRODUCTION

In this chapter we examine in some detail Hutton's ([6]) extension of the concept of a uniform structure on a topological space to L–spaces. Each section begins with a paragraph describing its contents.

I.1 : A MAPPING APPROACH TO THE THEORY OF UNIFORM SPACES.

In this section we show how a uniformity $\mathbb{D}$ on X, a non empty set, can be interpreted as a collection of union–preserving, increasing mappings from 2^X to 2^X which map the empty set to itself. We begin, after having recalled some standard definitions and terminology, by defining a lattice structure on this collection of mappings. We then establish a lattice isomorphism between this collection and the subsets of $X \times X$ containing the diagonal set. Next we show how a uniformity $\mathbb{D}$ on X can be expressed in terms of a collection of these mappings. This approach is generalised in sections 2 and 3 to the case where $2 = \{0,1\}$ is replaced by a completely distributive lattice. We conclude by finding equivalent formulations of the concepts of uniform topology and uniform continuity in this mapping approach to the theory of uniform spaces. These formulations will serve as motivation for analogous concepts in our generalised setting.

By diag(X) we mean the diagonal set $\{(x,x) \in X \times X : x \in X\}$ and DIAG(X) denotes the set $\{M \subseteq X \times X : \text{diag}(X) \subseteq M\}$ of subsets of the Cartesian product $X \times X$. We refer to elements of DIAG(X) as surroundings.

I.1.1 DEFINITION

For M, N $\in$ DIAG(X)

(i) The composition of M and N is defined to be

$$M \circ N = \{(x,y) \in X \times X : (x,z) \in M \text{ and } (z,y) \in N \text{ for some } z \in X\}.$$

(ii) The inverse of M is the set

$$M^{-1} = \{(x,y) \in X \times X : (y,x) \in M\}.$$

(iii) For $x \in X$, $M(x)$ denotes the set

$$\{y \in X : (x,y) \in M\}.$$

This is extended to $A \in 2^X$ as follows

$$M(A) = \bigcup_{a \in A} M(a) = \{y \in X : (a,y) \in M \text{ for some } a \in A\}.$$

■

I.1.2 PROPOSITION

Let $M,N \in DIAG(X)$.

(i) $M(\emptyset) = \emptyset$.

(ii) $A \subseteq M(A)$ for $A \in 2^X$.

(iii) $M(\bigcup_{A \in \mathscr{A}} A) = \bigcup_{A \in \mathscr{A}} M(A)$ for $\mathscr{A} \subseteq 2^X$.

(iv) $(M^{-1})^{-1} = M$.

(v) If $M \subseteq N$ then $M(A) \subseteq N(A)$ for $A \in 2^X$.

(vi) For $A, B \in 2^X$, $M(A) \subseteq B$ if and only if $M^{-1}(X/B) \subseteq X/A$.

(vii) $M(X/M^{-1}(A)) \subseteq X/A$ for each $A \in 2^X$.

(viii) $M^{-1}(A) = \bigcap \{B \in 2^X : M(X/B) \subseteq X/A\}$ for $A \in 2^X$.

(ix) $M \circ N(A) = N(M(A))$ for $A \in 2^X$.

PROOF

Parts (i) – (v), (ix) are straightforward.

(vi) Suppose $M(A) \subseteq B$. Let $y \in M^{-1}(X/B)$. Then there exists $x \in X/B$ such that $(x,y) \in M^{-1}$. This implies that $(y,x) \in M$. If $y \in A$ then $x \in M(A) \subseteq B$, which is impossible. Thus $y \in X/A$ and we have $M^{-1}(X/B) \subseteq X/A$. Conversely if $M^{-1}(X/B) \subseteq X/A$ then by what we have just shown,

$$(M^{-1})^{-1}(X/(X/A)) \subseteq X/(X/B),$$

that is $M(A) \subseteq B$.

(vii) Apply (vi) to $M^{-1}(A) \subseteq M^{-1}(A)$ and use the fact that $(M^{-1})^{-1} = M$.

(viii) By (v) $M^{-1}(A) \subseteq B$ if and only if $M(X/B) \subseteq X/A$. Therefore

$$M^{-1}(A) = \cap \{B \in 2^X : M^{-1}(A) \subseteq B\}$$
$$= \cap \{B \in 2^X : M(X/B) \subseteq X/A\}.$$

■

Let $x,y, z \in X$ and $M, N \in \mathrm{DIAG}(X)$. If $(x,y) \in M$ we say that "the distance between x and y is less than M". If $(x,z) \in M$ and $(z,y) \in N$ then $(x,y) \in M \circ N$. This can be taken as an axiomatization of the triangle inequality. The set $M(x)$ can be thought of as "the ball centred at x with radius M".

I.1.3 DEFINITION

A uniformity on X is a subfamily $\mathbb{D}$ of DIAG(X) which satisfies the following conditions :

(U1) If $V \in \mathbb{D}$ and $V \subseteq W \in \mathrm{DIAG}(X)$, then $W \in \mathbb{D}$.

(U2) If $V_1, V_2 \in \mathbb{D}$, then $V_1 \cap V_2 \in \mathbb{D}$.

(U3) For every $V \in \mathbb{D}$ there exists $W \in \mathbb{D}$ such that $W \circ W \subseteq V$.

(U4) For every $V \in \mathbb{D}$ there exists $W \in \mathbb{D}$ such that $W^{-1} \subseteq V$.

The pair $(X,\mathbb{D})$ consisting of X and a uniformity $\mathbb{D}$ on X is called a uniform space.

A family $B \subseteq \mathbb{D}$ is called a base for the uniformity $\mathbb{D}$ if for every $V \in \mathbb{D}$ there exists $W \in B$ such that $W \subseteq V$.

A collection $B \subseteq \mathrm{DIAG}(X)$ is a base for some uniformity on X if and only if the following conditions are satisfied.

(BU1) For every $V_1, V_2 \in B$ there exists $V \in B$ such that $V \subseteq V_1 \cap V_2$.

(BU2) For every $V \in B$ there exists $W \in B$ such that $W \circ W \subseteq V$.

(BU3) For every $V \in B$ there exists $W \in B$ such that $W^{-1} \subseteq V$.

■

I.1.4 DEFINITION

Let $\mathscr{H}_2(X)$ denote the collection of mappings $F : 2^X \to 2^X$ satisfying the following conditions :

(H1) $F(\emptyset) = \emptyset$.

(H2) $A \subseteq F(A)$ for $A \in 2^X$.

(H3) $F(\bigcup_{A \in \mathscr{A}} A) = \bigcup_{A \in \mathscr{A}} F(A)$ for $\mathscr{A} \subseteq 2^X$.

I.1.5 PROPOSITION

For F, G $\in \mathscr{H}_2(X)$ define $F \leq G$ to mean $F(A) \subseteq G(A)$ for each $A \in 2^X$. Then $\mathscr{H}_2(X)$ is a complete lattice.

PROOF

By virtue of the fact that the inclusion relation $\subseteq$ on 2^X is a partial order relation we have that $\leq$ partially orders $\mathscr{H}_2(X)$. Let $\tilde{0} : 2^X \to 2^X$ be defined by $\tilde{0}(A) = A$ for each $A \in 2^X$, and $\tilde{1} : 2^X \to 2^X$ be defined by $\tilde{1}(A) = X$ for $A \neq \emptyset$ and $\tilde{1}(\emptyset) = \emptyset$. Then $\tilde{0}, \tilde{1} \in \mathscr{H}_2(X)$ and are the bottom and top elements of $\mathscr{H}_2(X)$ respectively. To show that $\mathscr{H}_2(X)$ is a complete lattice it suffices to show that every non–empty subset has an infimum and a supremum.

To show that every non–empty subset of $\mathscr{H}_2(X)$ has an infimum, consider an arbitary non–empty subset $\{F_j : j \in J\} \subseteq \mathscr{H}_2(X)$. Define $F : 2^X \to 2^X$ by

$$F(A) = \bigcup_{a \in A} \bigcap_j F_j(\{a\}) \text{ for } A \in 2^X.$$

We check that F satisfies (H1), (H2), (H3). The very definition of F ensures that $F(\emptyset) = \emptyset$. If $x \in X$, then $x \in F_j(\{x\})$ for each $j \in J$, so that $x \in \bigcap_j F_j(\{x\})$. Thus for $A \in 2^X$, $A \subseteq \bigcup_{a \in A} \bigcap_j F_j(\{a\}) = F(A)$. This establishes (H2). Let $\mathscr{A} \subseteq 2^X$, then

$$F(\bigcup_{A \in \mathscr{A}} A) = \bigcup_{a \in \bigcup \mathscr{A}} \bigcap_j F_j(\{a\}) = \bigcup_{A \in \mathscr{A}} \bigcup_{a \in A} \bigcap_j F_j(\{a\}) = \bigcup_{A \in \mathscr{A}} F(A).$$

This proves that F satisfies (H3). Hence $F \in \mathscr{H}_2(X)$. For each j and $A \in 2^X$,

$$F(A) = \bigcup_{a \in A} \bigcap_j F_j(\{a\}) \subseteq \bigcup_{a \in A} F_j(\{a\}) = F_j(A).$$

Therefore $F \leq F_j$ for $j \in J$. This implies that F is a lower bound of $\{F_j : j \in J\}$. To show that F is the greatest lower bound, that is $F = \inf_j F_j$, let $G \in \mathscr{H}_2(X)$ satisfy

$G \leq F_j$ for all $j \in J$.

Then for $A \in 2^X$

$$G(A) = \bigcup_{a \in A} G(\{a\}) \subseteq \bigcup_{a \in A} \bigcap_j F_j(\{a\}) = F(A).$$

Hence $G \leq F$, proving that $F = \inf_j F_j$.

Although for a partially ordered set with a top element it is sufficient to assume the existence of infima of non–empty sets, we shall explicitly show the existence of arbitary suprema in $\mathscr{H}_2(X)$.

Let $\{G_j : j \in J\} \subseteq \mathscr{H}_2(X)$. Define $G : 2^X \longrightarrow 2^X$ by

$$G(A) = \bigcup_j G_j(A) \quad \text{for } A \in 2^X.$$

Again we must check that G satisfies (H1), (H2) and (H3). Since $G_j(\emptyset) = \emptyset$ for each j, we obtain $G(\emptyset) = \emptyset$. Hence G satisfies (H1). Let $A \subseteq 2^X$, then $A \subseteq G_j(A)$ for j. This implies that $A \subseteq \bigcup_j G_j(A) = G(A)$. Thus (H2) is satisfied. Let $\mathscr{A} \subseteq 2^X$, then

$$G(\bigcup_{A \in \mathscr{A}} A) = \bigcup_j G_j(\bigcup_{A \in \mathscr{A}} A) = \bigcup_j \bigcup_{A \in \mathscr{A}} G_j(A) = \bigcup_{A \in \mathscr{A}} \bigcup_j G_j(A) = \bigcup_{A \in \mathscr{A}} G(A)$$

Therefore G satisfies (H3).

Clearly $G_j \leq G$ for each $j \in J$. Suppose that $F \in \mathscr{H}_2(X)$ satisfies $G_j \leq F$ for all $j \in J$. Then for $A \in 2^X$ we have that

$$G(A) = \bigcup_j G_j(A) \leq F(A).$$

Thus G is the least upper bound, that is $G = \sup G_j$.

■

We observe that for $F,G \in \mathscr{H}_2(X)$, $F \vee G$ is given by $(F \vee G)(A) = F(A) \cup G(A)$ but $F \wedge G$ is not given by $F(A) \cap G(A)$ in general. This is shown in the counterexample below.

COUNTEREXAMPLE

Suppose that X consists of three elements x_1, x_2, x_3. Let $F,G \in \mathscr{H}_2(X)$ be such that

$$F(\{x_1\}) = \{x_1\}\ , \qquad G(\{x_1\}) = X,$$

$$F(\{x_2\}) = X, \qquad G(\{x_2\}) = \{x_2\}.$$

Then $$F(\{x_1,x_2\}) \cap G(\{x_1,x_2\}) = X, \quad \text{but}$$

$$F \wedge G(\{x_1,x_2\}) = \bigcup_{i=1}^{2} F(\{x_i\}) \cap G(\{x_i\}) = \{x_1,x_2\}.$$

■

In later sections where 2 is replaced by L, an arbitary completely distributive lattice and $F,G : L^X \rightarrow L^X$, we shall find it necessary to define mappings given by $(F \wedge G)(\mu) = F(\mu) \wedge G(\mu)$ and $(F \vee G)(\mu) = F(\mu) \vee G(\mu)$. Confusion may arise when we define a lattice structure on the generalisation of $\mathscr{H}_2(X)$ to L. To eliminate any possibility of this occurring we shall replace the symbols $\wedge$, $\vee$ on $\mathscr{H}_2(X)$ by Δ, ∇ respectively and maintain these symbols when 2 is replaced by L.

In the next proposition we construct a lattice isomorphism of DIAG(X) onto $\mathscr{H}_2(X)$.

I.1.6 PROPOSITION

The mapping $\Phi : \mathrm{DIAG}(X) \rightarrow \mathscr{H}_2(X)$ given by

$$\Phi(M)(A) = M(A) \ \text{ for } M \in \mathrm{DIAG}(X),\ A \in 2^X$$

is a lattice isomorphism. The inverse of Φ is given by

$$\Phi^{-1}(F) = \bigcup \{\{x\} \times F(\{x\}) : x \in X\}.$$

PROOF

We have from (I.1.2(i)–(iii)) that for each $M \in \mathrm{DIAG}(X)$, $\Phi(M)$ satisfies (H1), (H2), (H3) respectively. Thus Φ is well defined.

To show that Φ defines a one to one correspondence between DIAG(X) and $\mathscr{H}_2(X)$ it is enough to check that

(i) for each $F \in \mathscr{H}_2(X)$ there exists $M_F \in \mathrm{DIAG}(X)$ such that $\Phi(M_F) = F$,

and then that

(ii) for each $N \in DIAG(X)$ we have that $M_{\Phi(N)} = N$.

(i) Let $F \in \mathscr{H}_2(X)$ and define M_F to be the set $\cup \{\{x\} \times F(\{x\}) : x \in X\}$ noting that $M_F(a) = F(\{a\})$ for any $a \in X$. Since $\{x\} \subseteq F(\{x\})$ we have that $diag(X) \subseteq M_F$. Thus $M_F \in DIAG(X)$.

For each $A \in 2^X$,

$$\begin{aligned}\Phi(M_F)(A) &= M_F(A)\\ &= \bigcup_{a \in A} M_F(a)\\ &= \bigcup_{a \in A} F(\{a\})\\ &= F(A)\end{aligned}$$

This implies $\Phi(M_F) = F$.

(ii) Let $N \in DIAG(X)$, then

$$\begin{aligned}M_{\Phi(N)} &= \cup \{\{x\} \times \Phi(N)(\{x\}) : x \in X\}\\ &= \cup \{\{x\} \times N(x) : x \in X\}\\ &= N.\end{aligned}$$

Therefore the inverse mapping of Φ, $\Phi^{-1} : \mathscr{H}_2(X) \to DIAG(X)$ is given by $\Phi^{-1}(F) = M_F$. By (I.1.2(v)) Φ is order preserving. If $F \leq G$ then $F(\{x\}) \subseteq G(\{x\})$. This inequality implies that $M_F \subseteq M_G$. Thus Φ^{-1} is also order preserving. Since Φ and its inverse Φ^{-1} are both order preserving, Φ is a lattice isomorphism.

■

I.1.7 DEFINITION

For $F, G \in \mathscr{H}_2(X)$ define

(a) $F \circ G = \Phi(\Phi^{-1}(G) \circ \Phi^{-1}(F))$.

(b) $F^{-1} = \Phi([\Phi^{-1}(F)]^{-1})$.

I.1.8 PROPOSITION

Let $M, N \in DIAG(X)$, $F \in \mathscr{H}_2(X)$.

(i) $\Phi(M)^{-1} = \Phi(M^{-1})$, $\Phi^{-1}(F^{-1}) = M_F^{-1}$.

(ii) $F^{-1}(A) = \cap \{B \in 2^X : F(X/B) \subseteq X/A\}$ for each $A \in 2^X$.

(iii) $\Phi(M \circ N) = \Phi(N) \circ \Phi(M)$, $\Phi^{-1}(F \circ G) = \Phi^{-1}(G) \circ \Phi^{-1}(F)$.

PROOF

These formulae are easily deduced from the definitions of $F \circ G$, F^{-1} and the fact that $\Phi \circ \Phi^{-1}$, $\Phi^{-1} \circ \Phi$ are the identity mappings on $\mathscr{H}_2(X)$ and DIAG(X) respectively. As an illustration we prove (ii).

Let $A \in 2^X$, then $F^{-1}(A) = \Phi([\Phi^{-1}(F)]^{-1})(A)$. By (I.1.2(viii)) we have that

$$[\Phi^{-1}(F)]^{-1}(A) = \cap \{B \in 2^X : \Phi^{-1}(F)(X/B) \subseteq X/A\}$$

But $\Phi^{-1}(F)(X/B) = F(X/B)$. Hence

$$F^{-1}(A) = \cap \{B \in 2^X : F(X/B) \subseteq X/A\}.$$

■

I.1.9 DEFINITION

Let $\mathbb{D}$ be a uniformity on X. A mapping $F \in \mathscr{H}_2(X)$ is called uniform with respect to $\mathbb{D}$ if $\Phi(V) = F$ for some $V \in \mathbb{D}$.

■

We now show that the uniform mappings of a uniformity describe it just as well as its surroundings do.

I.1.10 PROPOSITION

The collection $\mathscr{D}(\mathbb{D}) = \Phi^{\rightarrow}(\mathbb{D})$ of all uniform maps of a uniform space $(X,\mathbb{D})$ has the properties :

(i) If $D \in \mathscr{D}(\mathbb{D})$ and $D \leq E \in \mathscr{H}_2(X)$ then $E \in \mathscr{D}(\mathbb{D})$.

(ii) If $D_1, D_2 \in \mathscr{D}(\mathbb{D})$, then $D_1 \Delta D_2 \in \mathscr{D}(\mathbb{D})$.

(iii) For each $D \in \mathscr{D}(\mathbb{D})$ there exists $E \in \mathscr{D}(\mathbb{D})$ such that $E \circ E \leq D$.

(iv) For each $D \in \mathscr{D}(\mathbb{D})$ there exists $E \in \mathscr{D}(\mathbb{D})$ such that $E^{-1} \leq D$.

Conversely, given a family $\mathscr{D} \subseteq \mathscr{H}_2(X)$ satisfying (i) – (iv) then

$$\mathbb{D}(\mathscr{D}) = \{V \in \text{DIAG}(X) : V = \Phi^{-1}(D) \text{ for some } D \in \mathscr{D}\} = (\Phi^{-1})^{\rightarrow}(\mathscr{D})$$

is a uniformity on X whose collection of uniform maps is $\mathscr{D}$.

PROOF

Bearing in mind that Φ is a lattice isomorphism and considering the properties of Φ, Φ^{-1} as listed in (I.1.8), properties (i) – (iv) are formulations of (U1) – (U4) in $\mathscr{H}_2(X)$ under the mapping Φ. Conversely (U1) – (U4) are also formulations of (i) – (iv) in DIAG(X) under the mapping Φ^{-1}.

■

We must now find suitable criteria for uniform continuity of a function in terms of this uniform mapping method of generating a uniformity.

I.1.11 DEFINITION

Let $(X,\mathbb{D}_1)$, $(Y,\mathbb{D}_2)$ be uniform spaces and $f : X \longrightarrow Y$. Then f is uniformly continuous if for each $W \in \mathbb{D}_2$ there exists $V \in \mathbb{D}$ such that $V \subseteq (f\times f)^{\leftarrow}(W)$.

■

I.1.12 DEFINITION

For $f : X \longrightarrow Y$, and each $G \in \mathscr{H}_2(Y)$ define $f^{\leftarrow}(G) : 2^X \longrightarrow 2^X$ by

$f^{\leftarrow}(G)(A) = f^{\leftarrow}(G(f^{\rightarrow}(A)))$.

■

I.1.13 PROPOSITION

Let Φ_X, Φ_Y denote the lattice isomorphisms of DIAG(X) onto $\mathscr{H}_2(X)$ and DIAG(Y) onto $\mathscr{H}_2(Y)$. Then for $N \in$ DIAG(Y), $G \in \mathscr{H}_2(Y)$, $f : X \longrightarrow Y$ we have :

(i) $f^{\leftarrow}(G) \in \mathscr{H}_2(X)$.

(ii) $\Phi_X((f\times f)^{\leftarrow}(N)) = f^{\leftarrow}(\Phi_Y(N))$.

(iii) $\Phi_X^{-1}(f^{\leftarrow}(G)) = (f\times f)^{\leftarrow}(\Phi_Y^{-1}(G))$.

PROOF

(i) We check that $f^{\leftarrow}(G)$ satisfies (H1), (H2), (H3).

(H1)
$$\begin{aligned} f^{\leftarrow}(G)(\emptyset) &= f^{\leftarrow}(G(f^{\rightarrow}(\emptyset))) \\ &= f^{\leftarrow}(G(\emptyset)) \\ &= f^{\leftarrow}(\emptyset) \\ &= \emptyset. \end{aligned}$$

(H2) Let $A \in 2^X$, then

$$A \subseteq f^{\leftarrow}(f^{\rightarrow}(A)) \subseteq f^{\leftarrow}(G(f^{\rightarrow}(A))) = f^{\leftarrow}(G)(A).$$

(H3) Let $\mathscr{A} \subseteq 2^X$, then

$$\begin{aligned} f^{\leftarrow}(G)(\bigcup_{A \in \mathscr{A}} A) &= f^{\leftarrow}(G(f^{\rightarrow}(\bigcup_{A \in \mathscr{A}} A))) \\ &= f^{\leftarrow}(G(\bigcup_{A \in \mathscr{A}} f^{\rightarrow}(A))) \\ &= f^{\leftarrow}(\bigcup_{A \in \mathscr{A}} G(f^{\rightarrow}(A))) \\ &= \bigcup_{A \in \mathscr{A}} f^{\leftarrow}(G(f^{\rightarrow}(A))) \\ &= \bigcup_{A \in \mathscr{A}} f^{\leftarrow}(G)(A). \end{aligned}$$

Therefore $f^{\leftarrow}(G) \in \mathscr{H}_2(X)$.

Let $A \in 2^X$.

(ii) $\Phi_X((f\times f)^{\leftarrow}(N))(A) = (f\times f)^{\leftarrow}(N)(A)$. Now

$(f\times f)^{\leftarrow}(N) = \{(x,y) \in X\times X : (f(x), f(y)) \in N\}$. Hence

$$\begin{aligned} (f\times f)^{\leftarrow}(N)(A) &= \{y \in X : (f(a),f(y)) \in N \text{ for some } a \in A\} \\ &= f^{\leftarrow}(\{z \in Y : (f(a),z) \in N \text{ for some } a \in A\}) \\ &= f^{\leftarrow}(N(f^{\rightarrow}(A))) \\ &= f^{\leftarrow}(\Phi_Y(N)(f^{\rightarrow}(A))) \\ &= f^{\leftarrow}(\Phi_Y(N))(A). \end{aligned}$$

(iii)
$$\begin{aligned} \Phi_X^{-1}(f^{\leftarrow}(G))(A) &= f^{\leftarrow}(G)(A) \\ &= f^{\leftarrow}(G(f^{\rightarrow}(A))) \\ &= f^{\leftarrow}(\Phi_Y^{-1}(G)(f^{\rightarrow}(A))) \end{aligned}$$

Then retracing the appropriate steps in (ii), we get that
$\Phi_X^{-1}(f^{\leftarrow}(G))(A) = (f\times f)^{\leftarrow}(\Phi_Y^{-1}(G))$.

■

I.1.14 PROPOSITION

Let $(X,\mathbb{D}_1)$ and $(Y,\mathbb{D}_2)$ be uniform spaces. Then $f : X \rightarrow Y$ is uniformly continuous if and only if for each $E \in \mathscr{E}(\mathbb{D}_2) = \Phi_Y^{\rightarrow}(\mathbb{D}_2)$ there exists $D \in \mathscr{D}(\mathbb{D}_1)$ such that $D \leq f^{\leftarrow}(E)$.

PROOF

Suppose f is uniformly continuous. Let $E \in \mathscr{E}(\mathbb{D}_2)$. Then $\Phi_Y^{-1}(E) \in \mathbb{D}_2$ which implies that there exists $V \in \mathbb{D}$ such that $V \subseteq (f\times f)^{\leftarrow}(\Phi_Y^{-1}(E))$. By (I.1.12(ii)) and since Φ_X is order preserving we have that

$$\Phi_X(V) \leq f^{\leftarrow}(\Phi_Y(\Phi_Y^{-1}(E))) = f^{\leftarrow}(E).$$

But $\Phi_X(V) \in \mathscr{D}(\mathbb{D})$. Therefore there exists $D \in \mathscr{D}(\mathbb{D})$ such that $D \leq f^{\leftarrow}(E)$.

Conversely let $W \in \mathbb{D}_2$ and suppose that for each $E \in \mathscr{E}(\mathbb{D}_2)$ there exists $D \in \mathscr{D}(\mathbb{D}_1)$ such that $D \leq f^{\leftarrow}(E)$. Therefore there exists $D \in \mathscr{D}(\mathbb{D}_1)$ such that $D \leq f^{\leftarrow}(\Phi_Y(W))$. Then by (I.1.12(iii)) and the fact that Φ_X^{-1} is order preserving this inequality implies that

$$\begin{aligned}\Phi_X^{-1}(D) &\subseteq \Phi_X^{-1}(f^{\leftarrow}(\Phi_Y(W))) \\ &= (f\times f)^{\leftarrow}(\Phi_Y^{-1}(\Phi_Y(W))) \\ &= (f\times f)^{\leftarrow}(W).\end{aligned}$$

This establishes the uniform continuity of f.

■

Every uniformity $\mathbb{D}$ on X induces a topology $\tau_{\mathbb{D}}$. More exactly we have the following very standard result.

I.1.15 THEOREM

For every uniformity $\mathbb{D}$ on X the family

$$\tau_{\mathbb{D}} = \{A \in 2^X : \text{for every } a \in A \text{ there exists } V \in \mathbb{D} \text{ such that } V(a) \subseteq A\}$$

is a topology on X.

■

I.1.16 PROPOSITION

For $A \in 2^X$,

$$\begin{aligned}\text{int}_{\tau_{\mathbb{D}}}(A) &= \cup \{B \in 2^X : \text{there exists } V \in \mathbb{D} \text{ such that } V(B) \subseteq A\} \\ &= \cup \{B \in 2^X : \text{there exists } D \in \mathscr{D}(\mathbb{D}) \text{ such that } D(B) \subseteq A\}\end{aligned}$$

PROOF

Let $C = \cup \{B \in 2^X : \text{there exists } V \in \mathbb{D} \text{ such that } V(B) \subseteq A\}$. For every $x \in C$ there exists $B \in 2^X$ and $V \in \mathbb{D}$ such that $x \in B \subseteq V(B) \subseteq A$. Since $V \in \mathbb{D}$ there exists $W \in \mathbb{D}$ such that $W \circ W \subseteq V$. This implies

$$x \in W(x) \subseteq W(W(x)) \subseteq V(B) \subseteq A.$$

Thus $x \in W(x) \subseteq C$. Hence $C \in \tau_{\mathbb{D}}$. Further $C \subseteq A$. Therefore $C \subseteq \text{int}_{\tau_{\mathbb{D}}}(A)$.

Conversely for $x \in \text{int}_{\tau_{\mathbb{D}}}(A)$ there exists an open set O such that $x \in O \subseteq A$. Since O is open there exists $V \in \mathbb{D}$ such that $V(x) \subseteq O$. Thus $x \in V(x) \subseteq A$. This implies $x \in C$. We therefore have $\text{int}_{\tau_{\mathbb{D}}}(A) \subseteq C$. Hence $\text{int}_{\tau_{\mathbb{D}}}(A) = C$.

The fact that $\Phi^{\rightarrow}(\mathbb{D}) = \mathscr{D}(\mathbb{D})$ and $(\Phi^{-1})^{\rightarrow}(\mathscr{D}(\mathbb{D})) = \mathbb{D}$ ensures that

$$C = \cup \{B \in 2^X : \text{there exists } D \in \mathscr{D}(\mathbb{D}) \text{ such that } D(B) \subseteq A\}.$$

■

I.2 : A CLASS OF MAPPINGS FROM L^X TO ITSELF.

In order to generalise the uniform mapping approach to the theory of uniform spaces in a systematic fashion, we must first describe, for an arbitary completely distributive lattice L and set X, a collection of mappings from L^X to itself which retain the essential aspects of $\mathscr{H}_2(X)$. This approach is that of Hutton's ([6]). Slight imperfections of some of his original definitions were observed by Liu ([14]) and corrected. Proofs of the main results in this section are those of Hutton's, with the exception of I.2.5 and I.2.7, which are those of Liu. The results I.2.11 and from I.2.13(ii) onwards are our own.

Throughout this section and all later sections L will denote a fixed completely distributive lattice and X will denote a fixed non–empty set.

I.2.1 DEFINITION

Let $\mathscr{H}_L(X)$ denote the collection of mappings $F : L^X \longrightarrow L^X$ which satisfy

(LH1) $F(\underline{0}) = \underline{0}$,

(LH2) $\mu \leq F(\mu)$ for each $\mu \in L^X$,

(LH3) $F(\bigvee_{\mu \in \mathscr{A}} \mu) = \bigvee_{\mu \in \mathscr{A}} F(\mu)$ for $\mathscr{A} \subseteq L^X$.

■

The set L^X is just the partially ordered set of functions $\mu : X \longrightarrow L$ under the pointwise ordering. As noted previously all lattice theoretic properties of L are inherited by L^X. Consequently L^X is a completely distributive lattice. Therefore $\mathscr{H}_L(X)$ is an instance of the following object.

I.2.2 DEFINITION

Let $\mathscr{L}(L)$ denote the collection of functions $f : L \longrightarrow L$ which satisfy

(a) $f(0) = 0$,

(b) $\alpha \leq f(\alpha)$ for each $\alpha \in L$,

(c) $f(\bigvee_{\alpha \in A} \alpha) = \bigvee_{\alpha \in A} f(\alpha)$ for $A \subseteq L$.

■

In terms of the above definition $\mathscr{H}_L(X) = \mathscr{L}(L^X)$. We also observe that $\mathscr{L}(L)$ is closed under ordinary composition of functions. That is if $f, g \in \mathscr{L}(L)$ then $f \circ g : L \longrightarrow L$ where $f \circ g(\alpha) = f(g(\alpha))$ is an element of $\mathscr{L}(L)$. Also note that (c) implies that if $\alpha \leq \beta$ then $f(\alpha) \leq f(\beta)$ for each $f \in \mathscr{L}(L)$.

I.2.3 DEFINITION

For $f, g : L \longrightarrow L$ define $f \leq g$ to mean that $f(\alpha) \leq g(\alpha)$ for each $\alpha \in L$.

■

This relation $\leq$ is a partial order relation on L^L. Therefore $(\mathscr{L}(L), \leq)$ is a partially ordered set.

The following lemma is important for its various applications. One of its consequences is the fact that $(\mathscr{L}(L), \leq)$ is a complete lattice.

I.2.4 LEMMA

Let $f : L \longrightarrow L$ satisfy

(i) $f(0) = 0$,

(ii) $\alpha \leq f(\alpha)$ for each $\alpha \in L$,

(iii) $f(\alpha) \leq f(\beta)$ for $\alpha, \beta \in L$, $\alpha \leq \beta$.

The function $f^* : L \longrightarrow L$ defined by

$$f^*(\alpha) = \bigwedge_{A \subseteq L \,:\, \vee A = \alpha} \; \bigvee_{\gamma \in A} f(\gamma)$$

is an element of $\mathscr{L}(L)$. Further $f^*(\alpha) = \bigvee_{\beta \in B(\alpha)} f(\beta)$ where $B(\alpha) \subseteq L$ is defined in (0.1.18). The greastest $g \in \mathscr{L}(X)$ satisfying $g \leq f$ is f^*.

PROOF

We first verify that $f^* \in \mathscr{L}(L)$ by showing that f^* satisfies requirements (a), (b), (c) of (I.2.2).

(a) The only set $A \subseteq L$ which satisfies $\vee A = 0$ is the empty set or $\{0\}$. However $\bigvee_{\gamma \in \emptyset} f(\gamma) = \vee\, \emptyset = 0$ and $f(0) = 0$ by (i). Thus $f^*(0) = 0$.

(b) Let $\alpha \in L$ and $A \subseteq L$ such that $\vee A = \alpha$. By (ii) for each $\gamma \in A$ we have $\gamma \leq f(\gamma)$. This implies that $\alpha = \bigvee_{\gamma \in A} \gamma \leq \bigvee_{\gamma \in A} f(\gamma)$. Therefore

$$\alpha \leq \bigwedge_{A \subseteq L \,:\, \vee A = \alpha} \; \bigvee_{\gamma \in A} f(\gamma) = f^*(\alpha).$$

(c) We first show that f^* satisfies (iii) and that $f^*(\alpha) = \bigvee_{\beta \in B(\alpha)} f(\beta)$.

Suppose $\alpha \leq \beta$ and $B \subseteq L$ satisfies $\vee B = \beta$. Let $A = \{\alpha \wedge \gamma : \gamma \in B\}$. Since L is completely distributive

$\vee A = \alpha \wedge (\vee B) = \alpha$.

This implies that

$$f^*(\alpha) \leq \bigvee_{\lambda \in A} f(\lambda) = \bigvee_{\gamma \in B} f(\alpha \wedge \gamma) \leq \bigvee_{\gamma \in B} f(\gamma).$$

This inequality holds for each $B \subseteq L$ satisfying $\vee B = \beta$. Thus $f^*(\alpha) \leq f^*(\beta)$.

By definition of $B(\alpha)$ we have that for each $A \subseteq L$ satisfying $\vee A = \alpha$ and each $\beta \in B(\alpha)$ there exists, $\gamma \in A$ such that $\beta \leq \gamma$. Since f satisfies (iii) this means that for each $\beta \in B(\alpha)$ there exist $\gamma \in A$ such that $f(\beta) \leq f(\gamma) \leq f(\alpha)$. Hence

$$\bigvee_{\beta \in B(\alpha)} f(\beta) \leq \bigvee_{\gamma \in A} f(\gamma).$$

Therefore since $\vee B(\alpha) = \alpha$, we have that

$$f^*(\alpha) = \bigvee_{\beta \in B(\alpha)} f(\beta).$$

Let $\alpha \in L$ be of the form $\alpha = \bigvee_{j \in J} \alpha_j$. To verify (c) we must show that $f^*(\alpha) = \bigvee_{j \in J} f^*(\alpha_j)$. Since f^* satisfies (iii) we have $\bigvee_{j \in J} f^*(\alpha_j) \leq f^*(\alpha)$.

Let $A = \bigcup_j B(\alpha_j)$. Then

$$\begin{aligned} \vee A &= \bigvee_j (\vee B(\alpha_j)) \\ &= \bigvee_j \alpha_j \\ &= \alpha. \end{aligned}$$

Thus

$$f^*(\alpha) \leq \bigvee_{\gamma \in A} f(\gamma) = \bigvee_{j \in J} \bigvee_{\beta \in B(\alpha_j)} f(\beta) = \bigvee_{j \in J} f^*(\alpha_j).$$

Let $g \in \mathscr{L}(L)$ such that $g \leq f$. Suppose $A \subseteq L$ and $\vee A = \alpha$, then

$$g(\alpha) = \bigvee_{\gamma \in A} g(\gamma) \leq \bigvee_{\gamma \in A} f(\gamma) \leq f(\alpha).$$

This implies that $g(\alpha) \leq f^*(\alpha) \leq f(\alpha)$ for each $\alpha \in L$. We therefore conclude that $g \leq f^* \leq f$ and that f^* is the greatest such g.

∎

I.2.5 PROPOSITION

$\mathscr{L}(L)$ is a complete lattice.

PROOF

Let $\tilde{0} : L \longrightarrow L$ be defined by $\tilde{0}(\alpha) = \alpha$ for each $\alpha \in L$, and $\tilde{1} : L \longrightarrow L$ be defined by $\tilde{1}(\alpha) = 1$ for $\alpha \in L/\{0\}$ and $\tilde{1}(0) = 0$. Then $\tilde{0}, \tilde{1} \in \mathscr{L}(L)$ are the bottom and top elements of $\mathscr{L}(L)$ respectively. To show that $\mathscr{L}(L)$ is a complete lattice we must show that every non–empty subset of $\mathscr{L}(L)$ has an infimum and supremum in $\mathscr{L}(L)$. Let $\mathscr{A} = \{f_j : j \in J\}$ be an arbitary non–empty subset of $\mathscr{L}(L)$. Define $f : L \longrightarrow L$ by

$$f(\alpha) = \bigwedge_j f_j(\alpha).$$

We show that f satisfies requirements (i), (ii), (iii) of Lemma I.2.4.

(i) $f(0) = \bigwedge_j f_j(0) = 0$

(ii) Let $\alpha \in L$. Then $\alpha \leq f_j(\alpha)$ for each $j \in J$. This implies $\alpha \leq \bigwedge_j f_j(\alpha) = f(\alpha)$.

(iii) For $\alpha, \beta \in L$ satisfying $\alpha \leq \beta$, we have that $f_j(\alpha) \leq f_j(\beta)$. Thus

$$f(\alpha) = \bigwedge_j f_j(\alpha) \leq \bigwedge_j f_j(\beta) = f(\beta).$$

By the Lemma $f^* \in \mathscr{L}(L)$ and for any $g \in \mathscr{L}(L)$ satisfying $g \leq f_j$ for each $j \in J$, we have that $g \leq f^* \leq f$. Hence $f^* = \inf_j f_j$.

As noted in the proof of (I.1.5) it is sufficient to prove the existence of infima of arbitary non–empty sets of $\mathscr{L}(L)$. We shall nevertheless prove the existence of suprema of arbitary non–empty subsets of $\mathscr{L}(L)$.

Let $\mathscr{B} = \{g_k : k \in K\} \subseteq \mathscr{L}(L)$ be non–empty. Define $g : L \longrightarrow L$ by $g(\alpha) = \bigvee_k g_k(\alpha)$.

Then $g(0) = \bigvee_k g_k(0) = 0$ and $\alpha \leq \bigvee_k g_k(\alpha) = g(\alpha)$ for $\alpha \in L$. For $A \subseteq L$,

$$\begin{aligned} g\Big(\bigvee_{\alpha \in A} \alpha\Big) &= \bigvee_k g_k\Big(\bigvee_{\alpha \in A} \alpha\Big) \\ &= \bigvee_k \bigvee_{\alpha \in A} g_k(\alpha) \\ &= \bigvee_{\alpha \in A} \bigvee_k g_k(\alpha) \\ &= \bigvee_{\alpha \in A} g(\alpha). \end{aligned}$$

This proves that $g \in \mathscr{L}(L)$. Suppose now that $f \in \mathscr{L}(L)$ and $g_k \leq f$ for each $k \in K$. Then for $\alpha \in L$, $g(\alpha) = \bigvee_k g_k(\alpha) \leq f(\alpha)$. Therefore the supremum of $\mathscr{B}$ exists and is given by g.

■

We note that showing the existence of suprema of subsets $\mathscr{L}(L)$ does not require the use of complete distributivity. We can therefore define $\mathscr{L}(L)$ for a complete lattice L and prove that $\mathscr{L}(L)$ is complete. However in this situation we will not have an explicit formula for the infimum of a subset of $\mathscr{L}(L)$.

For this reason we restrict our attention to $\mathscr{L}(L)$ for completely distributive lattices.

I.2.6 DEFINITION

For a finite subset $\{f_i : i = 1,\cdots,n\} \subseteq \mathscr{L}(L)$ we denote

(i) the infimum of $\{f_i : i = 1,\cdots,n\}$ by $\mathop{\Delta}\limits_{i=1}^{n} f_i$, and in the case of n = 2 by $f_1 \Delta f_2$;

(ii) the supremum of $\{f_i : i = 1,\cdots,n\}$ by $\mathop{\nabla}\limits_{i=1}^{n} f_i$, and in the case of n = 2 by $f_1 \nabla f_2$.

I.2.7 PROPOSITION

Let $f, f_1, g, g_1 \in \mathscr{L}(L)$.

(i) $f \Delta g(\alpha) = \bigwedge_{\alpha_1 \vee \alpha_2 = \alpha} f(\alpha_1) \vee g(\alpha_2)$ for $\alpha \in L$.

(ii) If $\alpha \in COP(L)$, then $f \Delta g(\alpha) = f(\alpha) \wedge g(\alpha)$.

(iii) If $f_1 \circ f_1 \leq f$ and $g_1 \circ g_1 \leq g$ then $(f_1 \Delta g_1) \circ (f_1 \Delta g_1) \leq f \Delta g$.

PROOF

(i) Let $h : L \longrightarrow L$ be defined by

$$h(\alpha) = \bigwedge_{\alpha_1 \vee \alpha_2 = \alpha} f(\alpha_1) \vee g(\alpha_2).$$

For $\alpha \in L$

$$\begin{aligned} f \Delta g(\alpha) &= \bigwedge_{A \subseteq L \,:\, \vee A = \alpha} \; \bigvee_{\gamma \in A} f(\gamma) \wedge g(\gamma) \\ &\leq \bigwedge_{\alpha_1 \vee \alpha_2 = \alpha} [f(\alpha_1) \wedge g(\alpha_1)] \vee [f(\alpha_2) \wedge g(\alpha_2)] \\ &\leq \bigwedge_{\alpha_1 \vee \alpha_2 = \alpha} f(\alpha_1) \vee g(\alpha_2) \\ &= h(\alpha). \end{aligned}$$

According to Lemma I.2.4 to show that $h = f \Delta g$, it will suffice to show that

(a) $h \leq f \wedge g$

(b) $h \in \mathscr{L}(L)$.

(a) Let $\alpha \in L$. Then

$$\begin{aligned} h(\alpha) &= \bigwedge_{\alpha_1 \vee \alpha_2 = \alpha} f(\alpha_1) \vee g(\alpha_2) \\ &\leq [f(\alpha) \vee g(0)] \wedge [f(0) \vee g(\alpha)] \\ &= f(\alpha) \wedge g(\alpha) \\ &= f \wedge g(\alpha), \end{aligned}$$

since $f(0) = 0$ and $g(0) = 0$.

(b) It is clear that $h(0) = 0$ and $\alpha \leq h(\alpha)$. Let $\alpha = \bigvee_{j \in J} \alpha_j$. For each j let

$$B_j = \{(\alpha_j^1, \alpha_j^2) \in L \times L : \alpha_j^1 \vee \alpha_j^2 = \alpha_j\}.$$

For $\varphi \in \prod_j B_j$ put

$$\varphi_1 = \bigvee_j \pi_1(\varphi(j)), \; \varphi_2 = \bigvee_j \pi_2(\varphi(j)).$$

Then $\varphi_1 \vee \varphi_2 = \bigvee_{j, k \in J} \pi_1(\varphi(j)) \vee \pi_2(\varphi(k)) = \alpha$. Let $\mathscr{A} = \{(\varphi_1, \varphi_2) : \varphi \in \prod_j B_j\}$ and $\mathscr{B} = \{(\alpha_1, \alpha_2) \in L \times L : \alpha_1 \vee \alpha_2 = \alpha\}$. We have just shown that $\mathscr{A} \subseteq \mathscr{B}$. Conversely suppose that $\alpha_1 \vee \alpha_2 = \alpha$. For each α_j we have that $\alpha_j = \alpha_j \wedge (\alpha_1 \vee \alpha_2) = (\alpha_j \wedge \alpha_1) \vee (\alpha_j \wedge \alpha_2)$. Thus $\bigvee_j [(\alpha_j \wedge \alpha_1) \vee (\alpha_j \wedge \alpha_2)] = \alpha$.

Let $\varphi : J \longrightarrow \bigcup_j B_j$ be defined by $\varphi(j) = (\alpha_j \wedge \alpha_1, \alpha_j \wedge \alpha_2)$. Then $\varphi \in \prod_j B_j$ and $\varphi_1 = \alpha_1$, $\varphi_2 = \alpha_2$. Therefore $\mathscr{B} \subseteq \mathscr{A}$, proving that $\mathscr{A} = \mathscr{B}$.

From the complete distributive law we have

$$\bigvee_j h(\alpha_j) = \bigvee_j \bigwedge_{\alpha_1^j \vee \alpha_2^j = \alpha_j} f(\alpha_1^j) \vee g(\alpha_2^j)$$

$$= \bigwedge_{\varphi \in \prod_j B_j} \bigvee_j [f(\pi_1(\varphi(j))) \vee g(\pi_2(\varphi(j)))]$$

$$= \bigwedge_{\varphi \in \prod_j B_j} f(\bigvee_j \pi_1(\varphi(j))) \vee g(\bigvee_j \pi_2(\varphi(j)))$$

$$= \bigwedge_{\varphi \in \prod_j B_j} f(\varphi_1) \vee g(\varphi_2)$$

$$= \bigwedge_{(\alpha_1,\alpha_2) \in \mathscr{B}} f(\alpha_1) \vee g(\alpha_2)$$

$$= h(\alpha).$$

Therefore $h \in \mathscr{L}(L)$.

(ii) Let $\alpha \in \mathrm{COP}(L)$. If $\alpha_1 \vee \alpha_2 = \alpha$, then either $\alpha_1 = \alpha$ or $\alpha_2 = \alpha$. This implies that $f(\alpha) \wedge g(\alpha) \leq f(\alpha_1) \vee g(\alpha_2)$. Using the formula established in (i) $f(\alpha) \wedge g(\alpha) \leq f \Delta g(\alpha)$. But $f \Delta g(\alpha) \leq f(\alpha) \wedge g(\alpha)$. Thus $f \Delta g(\alpha) = f(\alpha) \wedge g(\alpha)$.

(iii) We have that $f_1 \Delta g_1 \leq f_1$ and $f_1 \Delta g_1 \leq g_1$. Therefore if $f_1 \circ f_1 \leq f$ and $g_1 \circ g_1 \leq g$ we have that $(f_1 \Delta g_1) \circ (f_1 \Delta g_1) \leq f_1 \circ f_1 \leq f$ and $(f_1 \Delta g_1) \circ (f_1 \Delta g_1) \leq g_1 \circ g_1 \leq g$. Hence

$$(f_1 \Delta g_1) \circ (f_1 \Delta g_1) \leq f \Delta g.$$

■

Motivated by (I.1.8(ii)) we define an "inverse" of an element of $\mathscr{L}(L)$.

I.2.8 DEFINITION

For $f \in \mathscr{L}(L)$ define f^{-1} to be the function $f^{-1} : L \longrightarrow L$ given by

$$f^{-1}(\alpha) = \wedge \{\beta \in L : f(\beta') \leq \alpha'\}.$$

If $f^{-1} = f$ we refer to f as a symmetric element of $\mathscr{L}(L)$.

■

I.2.9 PROPOSITION

Let f, g $\in \mathscr{L}(L)$, $\alpha, \beta \in L$

(i) $f(\alpha) \leq \beta$ if and only if $f^{-1}(\beta') \leq \alpha'$.

(ii) $f^{-1} \in \mathscr{L}(L)$.

(iii) $(f^{-1})^{-1} = f$.

(iv) $f \leq g$ if and only if $f^{-1} \leq g^{-1}$.

(v) $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.

PROOF

(i) The forward direction follows directly from the definition of f^{-1}. Conversely suppose $f^{-1}(\beta') \leq \alpha'$. Then $\alpha \leq (f^{-1}(\beta'))'$. Therefore

$$\begin{aligned} f(\alpha) &\leq f((f^{-1}(\beta'))') \\ &= f([\wedge \{\gamma \in L : f(\gamma') \leq \beta\}]') \\ &= f(\vee \{\gamma \in L : f(\gamma) \leq \beta\}) \\ &= \vee \{f(\gamma) : \gamma \in L \text{ and } f(\gamma) \leq \beta\} \\ &\leq \beta. \end{aligned}$$

(ii) By definition of f^{-1} we have that $f^{-1}(0) = 0$. If $f(\beta') \leq \alpha'$, then since $\beta' \leq f(\beta')$ we have that $\beta' \leq \alpha'$ which implies $\alpha \leq \beta$. Thus

$$\alpha \leq \wedge \{\beta \in L : f(\beta') \leq \alpha'\} = f^{-1}(\alpha).$$

This establishes (a), (b) of (I.2.2).

Let $\alpha = \bigvee_{j \in J} \alpha_j$. Then for $\beta \in L$,

$$\begin{aligned} \bigvee_{j \in J} f^{-1}(\alpha_j) \leq \beta \quad &\text{if and only if } f^{-1}(\alpha_j) \leq \beta \text{ for each } j \\ &\text{if and only if } f(\beta') \leq \alpha'_j \text{ for each } j \\ &\text{if and only if } f(\beta') \leq \bigwedge_j \alpha'_j = (\bigvee_j \alpha_j)' \\ &\text{if and only if } f^{-1}(\bigvee_j \alpha_j) \leq \beta. \end{aligned}$$

Thus $\bigvee_j f^{-1}(\alpha_j) = f^{-1}(\bigvee_j \alpha_j)$. This shows that f^{-1} satisfies requirement (c) of (I.2.2). Hence $f^{-1} \in \mathscr{L}(L)$.

(iii) We have that

$$
\begin{aligned}
(f^{-1})^{-1}(\alpha) &= \wedge \{\beta \in L : f^{-1}(\beta') \leq \alpha'\} \\
&= \wedge \{\beta \in L : f(\alpha) \leq \beta\} \\
&= f(\alpha)
\end{aligned}
$$

for each $\alpha \in L$.

(iv) If $f \leq g$ then $g(\beta') \leq \alpha'$ implies $f(\beta') \leq \alpha'$ for $\alpha, \beta \in L$. Thus $f^{-1}(\alpha) \leq g^{-1}(\alpha)$ for each $\alpha \in L$. Therefore $f^{-1} \leq g^{-1}$. Conversely if $f^{-1} \leq g^{-1}$ then by what we have just shown $(f^{-1})^{-1} \leq (g^{-1})^{-1}$. But $(f^{-1})^{-1} = f$ and $(g^{-1})^{-1} = g$. Hence $f \leq g$.

(v) Let $\alpha, \beta \in L$. Then

$$
\begin{aligned}
f \circ g(\beta') \leq \alpha' \quad & \text{if and only if } f(g(\beta')) \leq \alpha' \\
& \text{if and only if } f^{-1}(\alpha) \leq g(\beta')' \\
& \text{if and only if } g(\beta') \leq (f^{-1}(\alpha))' .
\end{aligned}
$$

Therefore $(f \circ g)^{-1}(\alpha) = g^{-1}(f^{-1}(\alpha)) = g^{-1} \circ f^{-1}(\alpha)$.

■

I.2.10 PROPOSITION

For $f, g \in \mathscr{L}(L)$

$$(f \,\Delta\, g)^{-1} = f^{-1} \,\Delta\, g^{-1}$$

PROOF

Let $\alpha \in L$ and let $C(\alpha)$ denote the set $\{\gamma \in L : f \,\Delta\, g(\gamma') \leq \alpha'\}$. Then

$$(f \,\Delta\, g)^{-1}(\alpha) = \wedge\, C(\alpha)$$

and for each $\gamma \in C(\alpha)$

$$f \,\Delta\, g(\gamma') = \bigwedge_{\alpha_1 \vee \alpha_2 = \gamma'} f(\alpha_1) \vee g(\alpha_2)$$

We may write $C(\alpha)$ as

$$
\begin{aligned}
C(\alpha) &= \{\gamma \in L : \alpha \leq \bigvee_{\alpha_1 \vee \alpha_2 = \gamma'} f(\alpha_1)' \wedge g(\alpha_2)'\} \\
&= \{\gamma \in L : \text{for each } \beta \in B(\alpha) \text{ there exists } \alpha_1, \alpha_2 \in L \text{ such that} \\
&\qquad \alpha_1 \vee \alpha_2 = \gamma' \text{ and } \beta \leq f(\alpha_1)' \wedge g(\alpha_2)'\},
\end{aligned}
$$

where $B(\alpha)$ is the set defined in (0.1.18).

For each $\beta \in B(\alpha)$, put

$$H_\beta = \{\gamma \in L : \text{there exist } a_1, a_2 \in L \text{ such that } a_1 \vee a_2 = \gamma' \text{ and } f(a_1) \leq \beta',\ g(a_2) \leq \beta'\}$$

$$F_\beta = \{\gamma \in L : f(\gamma') \leq \beta'\}$$

$$G_\beta = \{\gamma \in L : g(\gamma') \leq \beta'\}.$$

Since $\beta \leq f(\alpha_1)' \wedge g(\alpha_2)'$ is equivalent to $f(\alpha_1) \leq \beta'$ and $g(\alpha_2) \leq \beta'$, it follows that

$$C(\alpha) = \bigcap_{\beta \in B(\alpha)} H_\beta$$

Observe that $f^{-1}(\beta) = \wedge F_\beta$ and $g^{-1}(\beta) = \wedge G_\beta$. By (I.2.9(i)) this implies that

$$f((\wedge F_\beta)') \leq \beta' \text{ and } g((\wedge G_\beta)') \leq \beta'.$$

Hence

$$(\wedge F_\beta) \wedge (\wedge G_\beta) \in H_\beta.$$

Each $\gamma \in H_\beta$ is of the form $\gamma = \gamma_1 \wedge \gamma_2$ where $f(\gamma_1') \leq \beta'$ and $g(\gamma_2') \leq \beta'$. This implies that $\gamma_1 \in F_\beta$ and $\gamma_2 \in G_\beta$. Thus $\wedge F_\beta \leq \gamma_1$ and $\wedge G_\beta \leq \gamma_2$, which implies $(\wedge F_\beta) \wedge (\wedge G_\beta) \leq \gamma_1 \wedge \gamma_2$. Therefore

$$\begin{aligned} \wedge H_\beta &= (\wedge F_\beta) \wedge (\wedge G_\beta) \\ &= f^{-1}(\beta) \wedge g^{-1}(\beta). \end{aligned}$$

If we can show that

$$\wedge C(\alpha) = \bigvee_{\beta \in B(\alpha)} (\wedge H_\beta),$$

we will have

$$\begin{aligned} (f \,\Delta\, g)^{-1}(\alpha) &= \wedge C(\alpha) \\ &= \bigvee_{\beta \in B(\alpha)} (\wedge H_\beta) \\ &= \bigvee_{\beta \in B(\alpha)} f^{-1}(\beta) \wedge g^{-1}(\beta) \\ &= f^{-1} \,\Delta\, g^{-1}(\alpha) \end{aligned}$$

To prove that $\wedge C(\alpha) = \bigvee_{\beta \in B(\alpha)} (\wedge H_\beta)$ we show that

(i) $\bigvee_{\beta \in B(\alpha)} \wedge H_\beta \leq \wedge C(\alpha)$.

(ii) $\wedge C(\alpha) \leq \bigvee_{\beta \in B(\alpha)} \wedge H_\beta$

(i) For each $\beta \in B(\alpha)$ we have $C(\alpha) \subseteq H_\beta$. Thus $\wedge H_\beta \leq \wedge C(\alpha)$ for each $\beta \in B(\alpha)$. Hence $\bigvee_{\beta \in B(\alpha)} \wedge H_\beta \leq \wedge C(\alpha)$.

(ii) For each $\beta \in B(\alpha)$, let

$$\gamma_{1,\beta} = \wedge F_\beta = f^{-1}(\beta)$$

and

$$\gamma_{2,\beta} = \wedge G_\beta = g^{-1}(\beta).$$

From the complete distributive law

$$\begin{aligned} \bigvee_{\beta \in B(\alpha)} (\wedge H_\beta) &= \bigvee_{\beta \in B(\alpha)} \bigwedge_{i=1}^{2} \gamma_{i,\beta} \\ &= \bigwedge_{\varphi \in \{1,2\}^{B(\alpha)}} \bigvee_{\beta \in B(\alpha)} \gamma_{\varphi(\beta),\beta}. \end{aligned}$$

Let $\varphi \in \{1,2\}^{B(\alpha)}$. Put

$$\alpha_i = \bigvee_{\beta \in \varphi^{-1}(i)} \gamma_{i,\beta} \quad \text{for } i = 1,2.$$

Then
$$\alpha_1 = \bigvee_{\beta \in \varphi^{-1}(1)} \gamma_{i,\beta} = \bigvee_{\beta \in \varphi^{-1}(1)} f^{-1}(\beta) = f^{-1}\Big(\bigvee_{\beta \in \varphi^{-1}(1)} \beta\Big),$$

and similarly $g^{-1}\Big(\bigvee_{\beta \in \varphi^{-1}(2)} \beta\Big) = \alpha_2$.

Applying (I.2.9(i)) to the above, we have

$$f(\alpha_1') \leq \Big(\bigvee_{\beta \in \varphi^{-1}(1)} \beta\Big)',$$

and

$$g(\alpha_2') \leq \Big(\bigvee_{\beta \in \varphi^{-1}(2)} \beta\Big)'.$$

Therefore

$$\begin{aligned} f \,\Delta\, g((\alpha_1 \vee \alpha_2)') &\leq f((\alpha_1 \vee \alpha_2)') \wedge g((\alpha_1 \vee \alpha_2)') \\ &\leq f(\alpha_1') \wedge g(\alpha_2') \\ &\leq (\bigvee_{\beta \in \varphi^{-1}(1)} \beta)' \wedge (\bigvee_{\beta \in \varphi^{-1}(2)} \beta)' \\ &= (\bigvee_{\beta \in \varphi^{-1}(\{1,2\})} \beta)' \\ &= \alpha' . \end{aligned}$$

This implies, when using the original definition of $C(\alpha)$, that $\alpha_1 \vee \alpha_2 \in C(\alpha)$. Further $\alpha_1 \vee \alpha_2 = \bigvee_{\beta \in B(\alpha)} \gamma_{\varphi(\beta),\beta}$. Therefore

$$\begin{aligned} \wedge\, C(\alpha) &\leq \bigwedge_{\varphi \in \{1,2\}^{B(\alpha)}} (\bigvee_{\beta \in B(\alpha)} \gamma_{\varphi(\beta),\beta}) \\ &= \bigvee_{\beta \in B(\alpha)} (\wedge\, H_\beta). \end{aligned}$$

■

We now return to the special case of $\mathscr{L}(L^X) = \mathscr{H}_L(X)$.

I.2.11 LEMMA

Let $F, G \in \mathscr{H}_L(X)$.

(i) For $\alpha \in COP(L)$ and $x \in X$ we have

$$F \,\Delta\, G(\alpha\, 1_x) = F(\alpha\, 1_x) \wedge G(\alpha\, 1_x)$$

(ii) $F = G$ if and only if $F(\alpha\, 1_x) = G(\alpha\, 1_x)$ for $\alpha \in COP(L)$, $x \in X$.

PROOF

(i) For $\alpha \in COP(L)$ and $x \in X$ we have have $\alpha\, 1_x \in COP(L^X)$. Therefore (i) is an application of (I.2.7(ii)) in $\mathscr{L}(L^X)$.

(ii) The forward direction is obvious.

Conversely since each $\alpha \in L$ can be written as the supremum of a subset of COP(L), (0.1.17), we can write each $\mu \in L^X$ as

$$\mu = \bigvee_{x \in X} \bigvee_{\alpha \in (\downarrow \mu(x)) \cap COP(L)} \alpha 1_x$$

Thus

$$\begin{aligned} F(\mu) &= \bigvee_{x \in X} \bigvee_{\alpha \in (\downarrow \mu(x)) \cap COP(L)} F(\alpha 1_x) \\ &= \bigvee_{x \in X} \bigvee_{\alpha \in (\downarrow \mu(x)) \cap COP(L)} G(\alpha 1_x) \\ &= G(\mu) \text{ for each } \mu \in L^X. \end{aligned}$$

Hence F = G.

■

Motivated by (I.1.12) we make the following definition.

I.2.12 DEFINITION

Let Y be a non–empty set and $f : X \to Y$. Then for each $G \in \mathscr{H}_L(Y)$ let

$$f^{\leftarrow}(G) : L^X \to L^X,$$

be defined by $f^{\leftarrow}(G)(\mu) = f^{\leftarrow}(G(f^{\rightarrow}(\mu)))$ for $\mu \in L^X$

■

I.2.13 PROPOSITION

Let $G, G_1, G_2 \in \mathscr{H}_L(Y)$ and $f : X \to Y$.

(i) $f^{\leftarrow}(G) \in \mathscr{H}_L(X)$.

(ii) $G_1 \leq G_2$ implies $f^{\leftarrow}(G_1) \leq f^{\leftarrow}(G_2)$.

(iii) $f^{\leftarrow}(G_1 \Delta G_2) = f^{\leftarrow}(G_1) \Delta f^{\leftarrow}(G_2)$.

(iv) $f^{\leftarrow}(G_1 \circ G_2) \leq f^{\leftarrow}(G_1) \circ f^{\leftarrow}(G_2)$.

(v) $[f^{\leftarrow}(G)]^{-1} = f^{\leftarrow}(G^{-1})$.

PROOF

(i) (LH1) : $f^{\leftarrow}(G)(\underline{0}) = f^{\leftarrow}(G(f^{\rightarrow}(\underline{0}))) = f^{\leftarrow}(G(\underline{0})) = f^{\leftarrow}(\underline{0}) = \underline{0}$.

(LH2) : $\mu \leq f^{\leftarrow}(f^{\rightarrow}(\mu)) \leq f^{\leftarrow}(G(f^{\rightarrow}(\mu))) = f^{\leftarrow}(G)(\mu)$ for $\mu \in L^X$.

(LH3) : Let $\mu = \bigvee_j \mu_j \in L^X$, then

$$
\begin{aligned}
f^{\leftarrow}(G)(\mu) &= f^{\leftarrow}(G(f^{\rightarrow}(\bigvee_j \mu_j))) \\
&= f^{\leftarrow}(G(\bigvee_j f^{\rightarrow}(\mu_j))) \\
&= f^{\leftarrow}(\bigvee_j G(f^{\rightarrow}(\mu_j))) \\
&= \bigvee_j f^{\leftarrow}(G(f^{\rightarrow}(\mu_j))) \\
&= \bigvee_j f^{\leftarrow}(G)(\mu_j).
\end{aligned}
$$

This proves that $f^{\leftarrow}(G) \in \mathscr{H}_L(X)$.

(ii) Let $\mu \in L^X$. Then $G_1 \leq G_2$ implies $G_1(f^{\rightarrow}(\mu)) \leq G_2(f^{\rightarrow}(\mu))$. Therefore $f^{\leftarrow}(G_1)(\mu) = f^{\leftarrow}(G_1(f^{\rightarrow}(\mu))) \leq f^{\leftarrow}(G_2(f^{\rightarrow}(\mu))) = f^{\leftarrow}(G_2)(\mu)$.

(iii) By (I.2.11(ii)) it suffices to show that
$f^{\leftarrow}(G_1 \Delta G_2)(\alpha 1_x) = f^{\leftarrow}(G_1) \Delta f^{\leftarrow}(G_2)(\alpha 1_x)$ for each $\alpha \in COP(L)$, $x \in X$.
By (I.2.11(i)) we have that

$$
\begin{aligned}
f^{\leftarrow}(G_1 \Delta G_2)(\alpha 1_x) &= f^{\leftarrow}(G_1 \Delta G_2(\alpha 1_{f(x)})) \\
&= f^{\leftarrow}(G_1(\alpha 1_{f(x)}) \wedge G_2(\alpha 1_{f(x)})) \\
&= f^{\leftarrow}(G_1(\alpha 1_{f(x)})) \wedge f^{\leftarrow}(G_2(\alpha 1_{f(x)})) \\
&= f^{\leftarrow}(G_1)(\alpha 1_x) \wedge f^{\leftarrow}(G_2)(\alpha 1_x) \\
&= f^{\leftarrow}(G_1) \Delta f^{\leftarrow}(G_2)\,(\alpha 1_x).
\end{aligned}
$$

(iv) Let $\mu \in L^X$, then

$$
\begin{aligned}
f^{\leftarrow}(G_1) \circ f^{\leftarrow}(G_2)(\mu) &= f^{\leftarrow}(G_1(f^{\rightarrow}[f^{\leftarrow}(G_2(f^{\rightarrow}(\mu)))])) \\
&\leq f^{\leftarrow}(G_1(G_2(f^{\rightarrow}(\mu)))) \\
&= f^{\leftarrow}(G_1 \circ G_2)(\mu).
\end{aligned}
$$

(v) It will suffice to show that $[f^{\leftarrow}(G)]^{-1} \leq f^{\leftarrow}(G^{-1})$. Indeed if we replace G by G^{-1} in the above inequality we will have that $[f^{\leftarrow}(G^{-1})]^{-1} \leq f^{\leftarrow}((G^{-1})^{-1})$. Then since $(G^{-1})^{-1} = G$ and by (I.2.9(iv)) we have that

$$f^{\leftarrow}(G^{-1}) \leq [f^{\leftarrow}(G)]^{-1}.$$

To prove that $[f^{\leftarrow}(G)]^{-1} \leq f^{\leftarrow}(G^{-1})$, we show that $[f^{\leftarrow}(G)]^{-1}(\mu) \leq f^{\leftarrow}(G^{-1})(\mu)$ for each $\mu \in L^X$. By (I.2.9(i)) this is equivalent to showing that

$$f^{\leftarrow}(G)\,([f^{\leftarrow}(G^{-1})(\mu)]')\leq \mu'.$$

Now

$$\begin{aligned} f^{\leftarrow}(G^{-1})(\mu) &= f^{\leftarrow}(\wedge\{\nu \in L^Y : G(\nu') \leq (f^{\rightarrow}(\mu))'\}) \\ &= \wedge\{f^{\leftarrow}(\nu) : \nu \in L^Y \text{ and } G(\nu') \leq (f^{\rightarrow}(\mu))'\}. \end{aligned}$$

We have that $f^{\rightarrow}(f^{\leftarrow}(\nu)') \leq \nu'$ for $\nu \in L^Y$ and $f^{\leftarrow}(f^{\rightarrow}(\mu)') \leq \mu'$. Thus

$$\begin{aligned} & f^{\leftarrow}(G)([f^{\leftarrow}(G^{-1})(\mu)]') \\ &= f^{\leftarrow}(G)\,(\vee\{f^{\leftarrow}(\nu)' : \nu \in L^Y \text{ and } G(\nu') \leq (f^{\rightarrow}(\mu))'\}) \\ &= f^{\leftarrow}(G\,(\vee\{f^{\rightarrow}(f^{\leftarrow}(\nu)') : \nu \in L^Y \text{ and } G(\nu') \leq (f^{\rightarrow}(\mu))'\}) \\ &\leq f^{\leftarrow}(G\,(\vee\{\nu' : \nu \in L^Y \text{ and } G(\nu') \leq (f^{\rightarrow}(\mu))'\}) \\ &= f^{\leftarrow}(\vee\{G(\nu') : \nu \in L^Y \text{ and } G(\nu') \leq (f^{\rightarrow}(\mu))'\}) \\ &\leq f^{\leftarrow}(f^{\rightarrow}(\mu)') \\ &\leq \mu'. \end{aligned}$$

■

We now show how each $F \in \mathscr{H}_2(X)$ can be extended to an element of $\mathscr{H}_L(X)$. We begin with a simple proposition.

I.2.14 PROPOSITION

For each μ in L^X let C_μ be the subset of X defined by

$$C_\mu = \cap\{B \in 2^X : \mu \leq 1_B\}.$$

Then

(i) $C_{\underline{0}} = \emptyset$.

(ii) $\mu \leq 1_{C_\mu}$.

(iii) If μ is of the form $\mu = \bigvee_{j \in J} \mu_j$, then

$$C_\mu = \bigcup_{j \in J} C_{\mu_j}.$$

PROOF

We can write C_μ as $C_\mu = \{x \in X : 0 < \mu(x)\}$. Parts (i) and (ii) then follow easily. To prove part (iii) let μ be of the form $\mu = \bigvee_{j \in J} \mu_j$. Now for $x \in X$, $\mu(x) > 0$ if and only if for some $j \in J$ $\mu_j(x) > 0$. Thus $C_\mu = \bigcup_{j \in J} C_{\mu_j}$.

■

I.2.15 DEFINITION

For $F \in \mathscr{H}_2(X)$ let $G_F : L^X \longrightarrow L^X$ be defined by

$$G_F(\mu) = 1_{F(C_\mu)} \text{ for each } \mu \in L^X.$$

■

I.2.16 PROPOSITION

Let $F, F_1, F_2 \in \mathscr{H}_2(X)$. Then

(i) $G_F \in \mathscr{H}_L(X)$,

(ii) $G_{F_1 \Delta F_2} = G_{F_1} \Delta G_{F_2}$,

(iii) $G_{F_1 \circ F_2} = G_{F_1} \circ G_{F_2}$,

(iv) $[G_F]^{-1} = G_{F^{-1}}$.

PROOF

(i) We check that G_F satisfies (LH1), (LH2) and (LH3).

(LH1) $G_F(\underline{0}) = 1_{F(C_{\underline{0}})} = 1_{F(\emptyset)} = 1_\emptyset = \underline{0}$.

(LH2) $\mu \leq 1_{C_\mu} \leq 1_{F(C_\mu)} = G_F(\mu)$ for each $\mu \in L^X$.

(LH3) Let $\mu = \bigvee_{j \in J} \mu_j$, then

$$G_F(\mu) = 1_{F(C_\mu)} = 1_{F(\bigcup_j C_{\mu_j})} = 1_{\bigcup_j F(C_{\mu_j})} = \bigvee_j 1_{F(C_{\mu_j})}$$

$$= \bigvee_j G_F(\mu_j).$$

(ii) By (I.2.11) it is enough to show that

$G_{F_1 \Delta F_2}(\alpha 1_x) = G_{F_1}(\alpha 1_x) \wedge G_{F_2}(\alpha 1_x)$ for each $\alpha \in COP(L)$ and $x \in X$.

If $\alpha = 0$ then the above clearly holds for each $x \in X$. Therefore let $\alpha \neq 0$ and $x \in X$. Then $C_{\alpha 1_x} = \{x\}$. Furthermore $F_1 \Delta F_2(\{x\}) = F_1(\{x\}) \cap F_2(\{x\})$.

Hence

$$\begin{aligned} G_{F_1 \Delta F_2}(\alpha 1_x) &= 1_{F_1(\{x\}) \cap F_2(\{x\})} \\ &= 1_{F_1(\{x\})} \wedge 1_{F_2(\{x\})} \\ &= G_{F_1}(\alpha 1_x) \wedge G_{F_2}(\alpha 1_x). \end{aligned}$$

(iii) Let μ in L^X then

$$\begin{aligned} G_{F_1} \circ G_{F_2}(\mu) &= G_{F_1}(1_{F_2(C_\mu)}) \\ &= 1_{F_1(F_2(C_\mu))} \\ &= 1_{F_1 \circ F_2(C_\mu)} \\ &= G_{F_1 \circ F_2}(\mu). \end{aligned}$$

(iv) Again it will suffice to show that $[G_F]^{-1} \leq G_{F^{-1}}$. If we replace F by F^{-1} and use (I.2.9(iv)) we get that $G_{F^{-1}} \leq [G_F]^{-1}$. As in the proof of (I.2.13(v)) to prove that

$[G_F]^{-1} \leq G_{F^{-1}}$, we show that $G_F((G_{F^{-1}}(\mu))') \leq \mu'$. We have that

$(G_{F^{-1}}(\mu))' = 1_{X/F^{-1}(C_\mu)}$. Therefore $G_F((G_{F^{-1}}(\mu))') = 1_{F(X/F^{-1}(C_\mu))}$.

By (I.1.2(vii)) we have $F(X/F^{-1}(C_\mu)) \subseteq X/C_\mu$ and since $\mu \leq 1_{C_\mu}$ we also have that $1_{X/C_\mu} \leq \mu'$. This all implies that $G_F((G_{F^{-1}}(\mu))') = 1_{F(X/F^{-1}(C_\mu))} \leq \mu'$.

■

I.3 : L–UNIFORMITIES AND L–UNIFORM SPACES.

Using the appropriate results in Section 1 as motivation we define the concepts of L–uniformity and L–uniformly continuous. We also defined the allied concepts of L–uniform base and L–uniform subbase. Next we assign to each L–uniformity on a set an L–topology via an interior operation. In this assignment L–uniformly continuous functions become L–continuous. We conclude this section by assigning an L–uniformity to each uniformity on a set. We study in some detail the L–uniformity assigned to the uniformity which induces the natural topology on the unit interval.

All material in this chapter with the exception of I.3.5, I.3.9 – I.3.11 is that of Hutton ([6]).

I.3.1 DEFINITION

An L–uniformity on X is a subfamily $\mathscr{D} \subseteq \mathscr{H}_L(X)$ which satisfies the following conditions.

(LU1) If $D \in \mathscr{D}$ and $D \leq E \in \mathscr{H}_L(X)$ then $E \in \mathscr{D}$.

(LU2) If $D_1, D_2 \in \mathscr{D}$, then $D_1 \Delta D_2 \in \mathscr{D}$.

(LU3) For every $D \in \mathscr{D}$ there exists $E \in \mathscr{D}$ such that $E \circ E \leq D$.

(LU4) For every $D \in \mathscr{D}$ there exists $E \in \mathscr{D}$ such that $E^{-1} \leq D$.

A family $\mathscr{B} \subseteq \mathscr{D}$ is called a base for the L–uniformity $\mathscr{D}$ if for every $D \in \mathscr{D}$ there exists $E \in \mathscr{B}$ such that $E \leq D$. A collection $\mathscr{B} \subseteq \mathscr{H}_L(X)$ is a base for some L–uniformity on X if and only if

(LBU1) For every $D_1, D_2 \in \mathscr{B}$ there exists $E \in \mathscr{B}$ such that $E \leq D_1$ and $E \leq D_2$.

(LBU2) For every $D \in \mathscr{B}$ there exists $E \in \mathscr{B}$ such that $E \circ E \leq D$.

(LBU3) For every $D \in \mathscr{B}$ there exists $E \in \mathscr{B}$ such that $E^{-1} \leq D$.

A collection $\mathscr{S} \subseteq \mathscr{H}_L(X)$ is a subbase for some L–uniformity on X if and only if

$$\mathscr{B}(\mathscr{S}) = \{ \mathop{\Delta}_{i=1}^{n} D_i : n \in \mathbb{N} \text{ and } D_i \in \mathscr{S}\} \text{ is an L–uniform base.}$$

Conditions (LU1) and (LU2) can be combined and stated as

(LU1′) $\mathscr{D}$ is a filter on $(\mathscr{H}_L(X), \leq)$.

Similary condition (LBU1) can be stated as

(LBU1′) $\mathscr{B}$ is a filter base on $(\mathscr{H}_L(X), \leq)$.

An L–uniform space is a pair $(X, \mathscr{D})$ consisting of a non–empty set X and an L–uniformity $\mathscr{D}$ on X.

■

I.3.2 PROPOSITION

Let $(X, \mathscr{D})$ be an L–uniform space. Then $\text{int}_{\mathscr{D}} : L^X \longrightarrow L^X$ where

$$\text{int}_{\mathscr{D}}(\mu) = \vee \{\nu \in L^X : \text{for some } D \in \mathscr{D},\ D(\nu) \leq \mu\}$$

is an interior operation on X. Hence

$$\tau_{\mathscr{D}} = \{\mu \in L^X : \mu = \text{int}_{\mathscr{D}}(\mu)\}$$

is an L–topology on X and shall be referred to as the L–topology on X induced by $\mathscr{D}$.

PROOF

We show that $\text{int}_{\mathscr{D}} : L^X \longrightarrow L^X$ satisfies the required properties of an interior operation (0.2.8).

(IO1) : For $\mu \in L^X$ and $D \in \mathscr{D}$ we have that $\mu \leq D(\mu)$. Thus $D(\underline{1}) = \underline{1}$. This implies that $\text{int}_{\mathscr{D}}(\underline{1}) = \underline{1}$.

(IO2) : Let μ, $\nu \in L^X$ be such that for some $D \in \mathscr{D}$ we have that $D(\nu) \leq \mu$. Then $\nu \leq \mu$. Therefore $\text{int}_{\mathscr{D}}(\mu) \leq \mu$.

(IO3) : Let μ, $\nu \in L^X$. From the definition of $\text{int}_{\mathscr{D}}$, if $\mu \leq \nu$ then $\text{int}_{\mathscr{D}}(\mu) \leq \text{int}_{\mathscr{D}}(\nu)$. Thus $\text{int}_{\mathscr{D}}(\mu \wedge \nu) \leq \text{int}_{\mathscr{D}}(\mu) \wedge \text{int}_{\mathscr{D}}(\nu)$. To reverse this equality suppose that μ_1, $\nu_1 \in L^X$ such that there exists $D_1, D_2 \in \mathscr{D}$ satisfying $D_1(\mu_1) \leq \mu$ and $D_2(\nu_1) \leq \nu$. Then

$$\begin{aligned} D_1 \Delta D_2(\mu_1 \wedge \nu_1) &\leq D_2(\mu_1 \wedge \nu_1) \wedge D_2(\mu_1 \wedge \nu_1) \\ &\leq D_1(\mu_1) \wedge D_2(\nu_1) \\ &\leq \mu \wedge \nu. \end{aligned}$$

Thus $\mu_1 \wedge \nu_1 \leq \text{int}_{\mathscr{D}}(\mu \wedge \nu)$. Since L is completely distributive we may write

$$\text{int}_{\mathscr{D}}(\mu) \wedge \text{int}_{\mathscr{D}}\, \nu) = \vee \{\mu_1 \wedge \nu_1 : \text{for some } D_1, D_2 \in \mathscr{D}, D_1(\mu_1) \leq \mu, D_2(\mu_2) \leq \nu\}.$$

By what we have previously shown this implies that $\text{int}_{\mathscr{D}}(\mu) \wedge \text{int}_{\mathscr{D}}(\nu) \leq \text{int}_{\mathscr{D}}(\mu \wedge \nu)$. This proves that $\text{int}_{\mathscr{D}}(\mu) \wedge \text{int}_{\mathscr{D}}(\nu) = \text{int}_{\mathscr{D}}(\mu \wedge \nu)$.

(IO4) : Let $\mu \in L^X$. By (IO2) we have that $\text{int}_{\mathscr{D}}(\text{int}_{\mathscr{D}}(\mu)) \leq \text{int}_{\mathscr{D}}(\mu)$. To reverse the inequality let $\nu \in L^X$ and $D \in \mathscr{D}$ such that $D(\nu) \leq \mu$. Since $D \in \mathscr{D}$ there exists $E \in \mathscr{D}$ such that $E \circ E \leq D$. Thus $\nu \leq E(\nu) \leq E(E(\nu)) \leq D(\nu) \leq \mu$. This implies that $\nu \leq E(\nu) \leq \text{int}_{\mathscr{D}}(\mu)$. Therefore $\nu \leq \text{int}_{\mathscr{D}}(\text{int}_{\mathscr{D}}(\mu))$. Hence $\text{int}_{\mathscr{D}}(\mu) \leq \text{int}_{\mathscr{D}}(\text{int}_{\mathscr{D}}(\mu))$.

■

I.3.3 COROLLARY

Let $\mathscr{B}$ be a base of $\mathscr{D}$. Then for each $\mu \in L^X$

$$\begin{aligned}\text{int}_{\mathscr{D}}(\mu) &= \vee \{\nu \in L^X : \text{for some } E \in \mathscr{B}, E(\nu) \leq \mu\} \\ &= \vee \{E(\nu) : \nu \in L^X \text{ and } E \circ E(\nu) \leq \mu \text{ for some } E \in \mathscr{B}\}.\end{aligned}$$

PROOF

Let $\mu \in L^X$ and let $A_\mu = \{\nu \in L^X : \text{for some } D \in \mathscr{D}, D(\nu) \leq \mu\}$. Thus $\text{int}_{\mathscr{D}}(\mu) = \vee A_\mu$ and for each $\nu \in A_\mu$ there exists $D_\nu \in \mathscr{D}$ such that $D_\nu(\nu) \leq \mu$. Since $\mathscr{B}$ is a base there exists $E_\nu \in \mathscr{B}$ such that $E_\nu \circ E_\nu \leq D_\nu$.

This implies that

$$\nu \leq E_\nu(\nu) \leq E_\nu(E_\nu(\nu)) \leq D_\nu(\nu) \leq \mu.$$

Thus $A_\mu \subseteq \{\nu \in L^X : \text{for some } E \in \mathscr{B}, E(\nu) \leq \mu\}$ and

$$\{E_\nu(\nu) : \nu \in A_\mu\} \subseteq \{E(\nu) : \nu \in L^X \text{ and } E \circ E(\nu) \leq \mu \text{ for some } E \in \mathscr{B}\}.$$

We therefore have that

$\text{int}_{\mathscr{D}}(\mu) \leq \vee \{\nu \in L^X : \text{for some } E \in \mathscr{B},\ E(\nu) \leq \mu\} \leq \text{int}_{\mathscr{D}}(\mu)$

and

$\text{int}_{\mathscr{D}}(\mu) \leq \vee \{E(\nu) : \nu \in L^X \text{ and } E \circ E(\nu) \leq \mu \text{ for some } E \in \mathscr{B}\} \leq \text{int}_{\mathscr{D}}(\mu).$

This completes the proof.

■

I.3.4 DEFINITION

An L–space (X,τ) is said to be L–uniformizable if there exists an L–uniformity $\mathscr{D}$ on X such that $\tau_{\mathscr{D}} = \tau$.

■

The following proposition will prove useful when we consider weak L–uniformities generated by collections of functions in Section 4.

I.3.5 PROPOSITION

Let $(X, \mathscr{D})$ be an L–uniform space. Then $\mathscr{D}$ has a base $\mathscr{B} \subseteq \mathscr{D}$ such that for each $E \in \mathscr{B}$, $E^{\rightarrow}(L^X) \subseteq \tau_{\mathscr{D}}$.

PROOF

For each $D \in \mathscr{D}$ we define the mapping $\text{int}_{\mathscr{D}}(D) : L^X \to L^X$ given by

$\text{int}_{\mathscr{D}}(D)(\mu) = \text{int}_{\mathscr{D}}(D(\mu)).$

It is easy to see that $\text{int}_{\mathscr{D}}(D)$ satisfies requirements (i), (ii), (iii) of (I.2.4). That is $\text{int}_{\mathscr{D}}(D)(\underline{0}) = \underline{0}$, $\mu \leq \text{int}_{\mathscr{D}}(D)(\mu)$ for $\mu \in L^X$ and $\text{int}_{\mathscr{D}}(D)(\mu) \leq \text{int}_{\mathscr{D}}(D)(\nu)$ for $\mu, \nu \in L^X$, $\mu \leq \nu$.

Therefore by (I.2.4)

$$(\text{int}_{\mathscr{D}}(D))^* : L^X \to L^X,$$

given by $(\text{int}_{\mathscr{D}}(D))^*(\mu) = \bigvee_{\nu \in B(\mu)} \text{int}_{\mathscr{D}}(D)(\nu)$ belongs to $\mathscr{H}_L(X)$. Since $\text{int}_{\mathscr{D}}(D)(\nu) \in \tau$ for each $\nu \in B(\mu)$ we have immediately that $(\text{int}_{\mathscr{D}}(D))^*(\mu) \in \tau_{\mathscr{D}}$.

Furthermore $(\text{int}_{\mathscr{D}}(D))^* \leq D$. Let $\mathscr{B} = \{(\text{int}_{\mathscr{D}}(D))^* : D \in \mathscr{D}\}$. If we can show that $\mathscr{B} \subseteq \mathscr{D}$, then by the previous inequality we will have that $\mathscr{B}$ is a base for $\mathscr{D}$. To this end let $D \in \mathscr{D}$. There exists $E \in \mathscr{D}$ such that $E \circ E \leq D$. We then have that

$$E(\mu) \leq \text{int}_{\mathscr{D}}(D(\mu)) = \text{int}_{\mathscr{D}}(D)(\mu) \leq D(\mu)$$

By (I.2.4) it follows that

$$E \leq (\text{int}_{\mathscr{D}}(D))^* \leq D.$$

This implies that $(\text{int}_{\mathscr{D}}(D))^* \in \mathscr{D}$, whence $\mathscr{B} \subseteq \mathscr{D}$.

■

I.3.6 DEFINITION

Let $(X, \mathscr{D})$ and $(Y, \mathscr{E})$ be L–uniform spaces. Then $f : X \longrightarrow Y$ is L–uniformly continuous if and only if for each $E \in \mathscr{E}$ there exists $D \in \mathscr{D}$ such that $D \leq f^{\leftarrow}(E)$. If f is a bijection, then we say that f is an L–uniform isomorphism if and only if f, f^{-1} are L–uniformly continuous.

■

We observe that as in the uniform case for $f : X \longrightarrow Y$ to be L–uniformly continuous it is enough to show that if $\mathscr{B} \subseteq \mathscr{E}$ is a base for $\mathscr{E}$, then for each $E \in \mathscr{B}$ there exists $D \in \mathscr{D}$ such that $D \leq f^{\leftarrow}(E)$.

I.3.7 PROPOSITION

Let $(X, \mathscr{D})$ and $(Y, \mathscr{E})$ be L–uniform spaces. Let $\mathscr{S}$ be a subbase for $\mathscr{E}$. Then $f : X \longrightarrow Y$ is L–uniformly continuous if and only if for each $E \in \mathscr{S}$ there exists $D \in \mathscr{D}$ such that $D \leq f^{\leftarrow}(E)$.

PROOF

The forward implication follows directly from the definition since $\mathscr{S} \subseteq \mathscr{E}$. We now consider the reverse implication. We are given that

$\mathscr{B}(\mathscr{S}) = \{ \overset{n}{\underset{i=1}{\Delta}} E_i : n \in \mathbb{N} \text{ and } E_i \in \mathscr{S} \}$ is a base for $\mathscr{D}$. By the observation preceeding the proposition to show that $f : X \longrightarrow Y$ is L–uniformly continuous it is sufficient to show that for each $E \in \mathscr{B}(\mathscr{S})$ there exists $D \in \mathscr{D}$ such that $D \leq f^{\leftarrow}(E)$. Let $E \in \mathscr{B}(\mathscr{S})$. Then E is of the form $E = \overset{n}{\underset{i=1}{\Delta}} E_i$. For each E_i there exists $D_i \in \mathscr{D}$ such that $D_i \leq f^{\leftarrow}(E_i)$. By (I.2.13(iii)) we have that $\overset{n}{\underset{i=1}{\Delta}} f^{\leftarrow}(E_i) = f^{\leftarrow}(\overset{n}{\underset{i=1}{\Delta}} E_i)$. This implies that $\overset{n}{\underset{i=1}{\Delta}} D_i \leq f^{\leftarrow}(\overset{n}{\underset{i=1}{\Delta}} E_i)$. But $\overset{n}{\underset{i=1}{\Delta}} D_i \in \mathscr{D}$. Thus for each $E \in \mathscr{B}(\mathscr{S})$ there exists $D \in \mathscr{D}$ satisfying $D \leq f^{\leftarrow}(E)$.

■

I.3.8 PROPOSITION

Let $f : (X, \mathscr{D}) \longrightarrow (Y, \mathscr{E})$ be L–uniformly continuous. Then $f : (X.\tau_{\mathscr{D}}) \longrightarrow (Y,\tau_{\mathscr{E}})$ is L–continuous.

PROOF

Let $\mu \in \tau_{\mathscr{E}}$ and $A_\mu = \{\nu \in L^Y : E_\nu(\nu) \leq \mu$ for some $E_\nu \in \mathscr{E}\}$. Since $\mu \in \tau_{\mathscr{E}}$ we have that $\mu = \text{int}_{\mathscr{E}}(\mu) = \vee A_\mu$. For each $\nu \in A_\mu$, $f^{\leftarrow}(E_\nu) \in \mathscr{D}$ since f is L–uniformly continuous and

$$f^{\leftarrow}(E_\nu)(f^{\leftarrow}(\nu)) \leq f^{\leftarrow}(E_\nu(\nu)) \leq f^{\leftarrow}(\mu).$$

This implies that $f^{\leftarrow}(\nu) \leq \text{int}_{\mathscr{D}}(f^{\leftarrow}(\mu))$ for each $\nu \in A_\mu$. Thus

$$f^{\leftarrow}(\mu) = \bigvee_{\nu \in A_\mu} f^{\leftarrow}(\nu) \leq \text{int}_{\mathscr{D}}(f^{\leftarrow}(\mu)) \leq f^{\leftarrow}(\mu).$$

This means $f^{\leftarrow}(\mu) \in \tau_{\mathscr{D}}$, which proves that f is L–continuous.

■

We now assign to each uniformity on a set a related L–uniformity.

I.3.9 PROPOSITION

Let $(X,\mathbb{D})$ be a uniform space.

(i) The collection

$$\mathscr{B}_{\mathbb{D}} = \{G_D : D \in \mathscr{D}(\mathbb{D})\}$$

where $\mathscr{D}(\mathbb{D}) = \Phi(\mathbb{D})$ and G_D is given in (I.2.15), is an L–uniform base on X.

(ii) If B is a base for $\mathbb{D}$ then

$$\mathscr{B}_B = \{G_E : E \in \mathscr{B}(B) = \Phi(B)\}$$

is a base for the L–uniformity generated by $\mathscr{B}_{\mathbb{D}}$.

(iii) The L–topology $\tau_{\mathscr{B}_{\mathbb{D}}}$ induced by $\mathscr{B}_{\mathbb{D}}$ is precisely the L–topology

$$\chi_{\tau_{\mathbb{D}}} = \{1_A : A \in \tau_{\mathbb{D}}\}.$$

PROOF

(i) The collection $\mathscr{D}(\mathbb{D})$ is a filter on $\mathscr{H}_2(X)$ and by (I.2.16(ii)) $G_{D_1} \Delta G_{D_2} = G_{D_1 \Delta D_2}$. Hence $\mathscr{B}_{\mathbb{D}}$ is a filter base on $\mathscr{H}_L(X)$. For $D \in \mathscr{D}(\mathbb{D})$ there exists $E_1, E_2 \in \mathscr{D}(\mathbb{D})$ such that $E_1 \circ E_1 \leq D$ and $E_2^{-1} \leq D$. Therefore by (I.2.16(iii) – (v)) we have that $G_{E_1} \circ G_{E_1} \leq G_D$ and $(G_{E_2})^{-1} \leq G_D$. Consequently $\mathscr{B}_{\mathbb{D}}$ is an L–uniform base on X.

(ii) We have that $\mathscr{B}_B \subseteq \mathscr{B}_{\mathbb{D}}$ and for each $G_D \in \mathscr{B}_{\mathbb{D}}$ there exists $G_E \in \mathscr{B}_B$ such that $G_E \leq G_D$. Thus $\mathscr{B}_B$ is a base for the L–uniformity generated by $\mathscr{B}_{\mathbb{D}}$.

(iii) Applying (I.3.5) to the special case of L = {0,1} = 2, we have that there exists a base $\mathscr{B} \subseteq \mathscr{D}(\mathbb{D})$ of $\mathscr{D}(\mathbb{D})$ such that for each $E \in \mathscr{B}$, $E^{\rightarrow}(2^X) \subseteq \tau_{\mathbb{D}}$. Then by (ii) and (I.3.3), we have for $\mu \in L^X$,

$$\text{int}_{\mathscr{B}_{\mathbb{D}}}(\mu) = \vee \{G_E(\nu) : \nu \in L^X \text{ and } G_E \circ G_E(\nu) \leq \mu \text{ for some } E \in \mathscr{B}\}.$$

But $G_E(\nu) = 1_{E(C_\nu)}$ where $C_\nu = \cap \{B \in 2^X : \nu \leq 1_B\}$. Hence $G_E(\nu) \in \chi_{\tau_{\mathbb{D}}}$. Therefore $\text{int}_{\mathscr{B}_{\mathbb{D}}}(\mu) \in \chi_{\tau_{\mathbb{D}}}$, proving that $\tau_{\mathscr{B}_{\mathbb{D}}} \subseteq \chi_{\tau_{\mathbb{D}}}$. To reverse the inequality, let $A \in \tau_{\mathbb{D}}$. Then by (I.1.15)

$$A = \cup \{B \in 2^X : \text{there exists } D \in \mathscr{D}(\mathbb{D}) \text{ such that } D(A) \subseteq B\}.$$

Therefore

$$\begin{aligned} 1_A &= \vee \{1_B : \text{there exists } G_D \in \mathscr{B}_{\mathbb{D}} \text{ such that } G_D(1_B) \leq 1_A\} \\ &\leq \text{int}_{\mathscr{B}_{\mathbb{D}}}(1_A) \\ &\leq 1_A. \end{aligned}$$

This implies that $1_A \in \tau_{\mathscr{B}_{\mathbb{D}}}$. Hence $\chi_{\tau_{\mathbb{D}}} \subseteq \tau_{\mathscr{B}_{\mathbb{D}}}$.

■

A base for the unique uniformity $\mathbb{D}$ on the interval I is given by $B = \{V_\epsilon : \epsilon > 0\}$ where $V_\epsilon = \{(x,y) \in I \times I : |x-y| < \epsilon\}$. If we let $W_\epsilon = \{(x,y) \in I \times I : x - \epsilon < y\}$, then $V_\epsilon = W_\epsilon \cap W_\epsilon^{-1}$. Let $D_\epsilon = \Phi(V_\epsilon)$ and $E_\epsilon = \Phi(W_\epsilon)$.

Concerning D_ϵ, E_ϵ we have the following proposition.

I.3.10 PROPOSITION

(i) $G_{D_\epsilon} = G_{E_\epsilon \Delta E_\epsilon^{-1}} = G_{E_\epsilon} \Delta G_{E_\epsilon^{-1}} = G_{E_\epsilon} \Delta (G_{E_\epsilon})^{-1}$.

(ii) For $\mu \in L^X$ and $\mu \neq \underline{0}$.

(a) $G_{E_\epsilon}(\mu) = 1_{(r_\mu - \epsilon, 1]}$ where $r_\mu = \sup \{r \in I : \mu \leq 1_{[r,1]}\}$ and if $r_\mu - \epsilon < 0$ then $(r_\mu - \epsilon, 1]$ is taken to be I.

(b) $(G_{E_\epsilon})^{-1}(\mu) = 1_{[0, \ell_\mu + \epsilon)}$ where $\ell_\mu = \inf\{\ell \in I : \mu \leq 1_{[0,\ell]}\}$ and if $1 < \ell_\mu + \epsilon$ then $[0, \ell_\mu + \epsilon)$ is taken to be I.

PROOF

Part (i) is a consequence of (I.2.16). We now prove (ii).

(ii) (a) By definition $G_{E_\epsilon}(\mu) = 1_{E_\epsilon(C_\mu)}$ where $C_\mu = \cap \{B \in 2^X : \mu \leq 1_B\}$.

We have that
$$E_\epsilon(C_\mu) = \bigcup_{x \in C_\mu} E_\epsilon(x) = \bigcup_{x \in C_\mu} (x - \epsilon, 1] = (\inf C_\mu - \epsilon, 1].$$

If $\mu \leq 1_{[r,1]}$, then $C_\mu \subseteq [r,1]$ which implies that $r \leq \inf C_\mu$. But $\inf C_\mu \in \{r \in I : \mu \leq 1_{[r,1]}\}$ since $\mu \leq 1_{C_\mu} \leq 1_{[\inf C_\mu, 1]}$. Thus $r_\mu = \inf C_\mu$. This proves that $G_{E_\epsilon}(\mu) = 1_{(r_\mu - \epsilon, 1]}$.

(b) $(G_{E_\epsilon})^{-1}(\mu) = G_{E_\epsilon^{-1}}(\mu) = 1_{E_\epsilon^{-1}(C_\mu)}$

where
$$E_\epsilon^{-1}(C_\mu) = \bigcup_{x \in C_\mu} E_\epsilon^{-1}(x) = \bigcup_{x \in C_\mu} [0, x + \epsilon) = [0, \sup C_\mu + \epsilon).$$

Arguing along the same lines as in (a) we conclude that $\ell_\mu = \sup C_\mu$. Hence $(G_{E_\epsilon})^{-1}(\mu) = 1_{[0,\, \ell_\mu + \epsilon)}$.

■

Summarizing what we have done so far, we have

I.3.11 PROPOSITION

The L–uniformity on I generated by $\mathscr{B}_{\mathbb{D}}$ has as a base

$$\mathscr{B}_B = \{G_{E_\epsilon} \Delta (G_{E_\epsilon})^{-1} : \epsilon > 0\}$$

where the action of G_{E_ϵ}, $(G_{E_\epsilon})^{-1}$ on elements of L^I is given in (I.3.10).

■

I.4 : OPERATIONS ON L–UNIFORMITIES.

In this section three operations on L–uniform spaces are examined. We first consider subspaces of L–uniform spaces. Next we consider weak L–uniformities generated by collections of functions and products of L–uniform spaces.

Unlike the uniform case the definition of a subspace L–uniformity is a non–trival matter. Most of what is done here has appeared in Rodabaugh's paper ([17]). In this paper he claims without proof that his "subspace L–uniformity" may in fact not be an L–uniformity. He is incorrect and we give a proof to this effect. We proceed as follows.

Let Y be a subset of a set X. For each L–set $\mu \in L^Y$, we define an extension μ_X of μ to X :

$$\mu_X : X \to L,\ \mu_X(x) = \begin{cases} \mu(x) & \text{if } x \in Y \\ 0 & \text{otherwise.} \end{cases}$$

Conversely given $\lambda \in L^X$, we define λ_Y to be the restriction of λ to Y, that is $\lambda_Y = \lambda_{|Y}$.

If $\mu \in L^Y$, we define the complement μ^c of μ relative to Y to be the L–set :

$$\mu^c : Y \to L\ ,\ \mu^c(y) = (\mu(y))' \quad \text{for } y \in Y.$$

We list some obvious properties :

I.4.1 PROPOSITION

Let $\mu, \nu \in L^Y$; $\eta, \lambda \in L^X$; $A \subseteq L^Y$; $B \subseteq L^X$.

(i) $(\bigvee_{\mu \in A} \mu)_X = \bigvee_{\mu \in A} \mu_X$, $(\bigvee_{\lambda \in B} \lambda)_Y = \bigvee_{\lambda \in B} \lambda_Y$,

$(\bigwedge_{\mu \in A} \mu)_X = \bigwedge_{\mu \in A} \mu_X$, $(\bigwedge_{\lambda \in B} \lambda)_Y = \bigwedge_{\lambda \in B} \lambda_Y$.

(ii) $(\mu_X)_Y = \mu$, $\mu^c = (\mu_X)'_Y$,$[((\mu_X)')_Y]^c = \mu$, $[((\mu^c)_X)']_Y = \mu$.

(iii) $(\mu^c)_X \leq (\mu_X)'$, $(\lambda_Y)_X \leq \lambda$.

(iv) $\nu \leq \mu$ implies $\nu_X \leq \mu_X$, $\eta \leq \lambda$ implies $\eta_Y \leq \lambda_Y$.

■

I.4.2 DEFINITION

For each $F \in \mathscr{H}_L(X)$ define the function F_Y :

$$F_Y : L^Y \rightarrow L^Y , F_Y(\mu) = F(\mu_X)_Y \text{ for each } \mu \in L^Y .$$

■

I.4.3 PROPOSITION

Let $E, F \in \mathscr{H}_L(X)$.

Then :

(i) $\{F_Y : F \in \mathscr{H}_L(X)\} \subseteq \mathscr{H}_L(Y)$.

(ii) If $E \leq F$ then $E_Y \leq F_Y$.

(iii) $E_Y \circ E_Y \leq (E \circ E)_Y$.

(iv) $(F^{-1})_Y = (F_Y)^{-1}$.

PROOF :

Parts (i) and (ii) are straightforward. We shall prove parts (iii) and (iv).

(iii) Let $\mu \in L^Y$. Then $E_Y \circ E_Y(\mu) = E_Y(E_Y(\mu)) = E(E_Y(\mu)_X)_Y$,but

$$E_Y(\mu)_X = (E(\mu_X)_Y)_X \leq E(\mu_X).$$

We therefore have

$$E_Y \circ E_Y(\mu) \le E(E(\mu_X))_Y = (E \circ E(\mu_X))_Y = (E \circ E)_Y(\mu) .$$

Thus $E_Y \circ E_Y \le (E \circ E)_Y$.

(iv) Note that it is sufficient to show that $(F^{-1})_Y \le (F_Y)^{-1}$. Since replacing F by F^{-1}, we get $F_Y \le ((F^{-1})_Y)^{-1}$. This implies $(F_Y)^{-1} \le (F^{-1})_Y$. If $\mu \in L^Y$, then $(F_Y)^{-1}(\mu) = \wedge \{\nu \in L^Y : F_Y(\nu^c) \le \mu^c\}$.

Let $\nu \in L^Y$ such that $F_Y(\nu^c) \le \mu^c$, then $F_Y(\nu^c)_X \le (\mu^c)_X \le (\mu_X)'$. But $F_Y(\nu^c)_X = F((\nu^c)_X) \wedge 1_Y$ and $(\mu_X)'(x) = 1$ for all $x \in X/Y$.

This implies that $F((\nu^c)_X) \le (\mu_X)'$.

We have that

$F^{-1}(\mu_X) = \wedge \{\lambda \in L^X : F(\lambda') \le (\mu_X)'\}$.

Hence $F^{-1}(\mu_X) \le [(\nu^c)_X]'$. It follows that

$(F^{-1})_Y(\mu) = F^{-1}(\mu_X)_Y \le [((\nu^c)_X)']_Y = \nu$.

We can now conclude that $(F^{-1})_Y(\mu) \le (F_Y)^{-1}(\mu)$, and therefore the inequality $(F^{-1})_Y \le (F_Y)^{-1}$ follows.

■

The reason for choosing our particular extension μ_X of $\mu \in L^Y$, is now apparent. The inequality $(\mu^c)_X \le (\mu_X)'$ is crucial to (I.4.3.(iv)). This inequality and other crucial inequalities of the form $(\mu_Y)_X \le \mu$ needed later on would not be available with another form of μ_X .

I.4.4 PROPOSITION

Let $(X, \mathscr{D})$ be an L–uniform space and $Y \subseteq X$. Then $\mathscr{B}_Y = \{D_Y : D \in \mathscr{D}\}$ is an L–uniform base on Y, and the L–topology $\tau_{\mathscr{B}_Y}$ generated by $\mathscr{B}_Y$ coincides with the subspace L–topology $\tau_{\mathscr{D}|Y}$.

PROOF :

By (II.1.3(i)) $\mathscr{B}_Y \subseteq \mathscr{H}_L(Y)$. To show that $\mathscr{B}_Y$ is an L–uniform base it remains to show that :

(a) $\mathscr{B}_Y$ is a filter base,

(b) For each $G \in \mathscr{B}_Y$ there exists

(i) $H \in \mathscr{B}_Y$ such that $H \circ H \leq G$,

(ii) $H \in \mathscr{B}_Y$ such that $H^{-1} \leq G$.

(a) Let $G_1, G_2 \in \mathscr{B}_Y$, then there exists $D_1, D_2 \in \mathscr{D}$ such that

$G_i = (D_i)_Y$ for $i = 1,2$.

Since $\mathscr{D}$ is an L–uniformity $D_1 \Delta D_2 \in \mathscr{D}$ and further $D_1 \Delta D_2 \leq D_i$ for $i = 1,2$. Then $(D_1 \Delta D_2)_Y \in \mathscr{B}_Y$ and by (I.4.3(ii)),

$(D_1 \Delta D_2)_Y \leq (D_i)_Y = G_i$ for $i = 1,2$.

(b)(i) If $G \in \mathscr{B}_Y$, then there exists $D \in \mathscr{D}$ such that $D_Y = G$. Since $\mathscr{D}$ is an L–uniformity there exists $E \in \mathscr{D}$ such that $E \circ E \leq D$. Now $E_Y \in \mathscr{B}_Y$ and by (I.4.3(iii)) $E_Y \circ E_Y \leq (E \circ E)_Y$. But $(E \circ E)_Y \leq D_Y$ by (I.4.3(ii)). Thus $E_Y \circ E_Y \leq D_Y$.

(ii) Let $G \in \mathscr{B}_Y$, then G is of the form $G = D_Y$, where $D \in \mathscr{D}$. Since $\mathscr{D}$ is an L–uniformity there exists $E \in \mathscr{D}$ such that $E^{-1} \leq D$. Then by (I.4.3(ii), (iv)), we have $(E_Y)^{-1} \leq D_Y = G$.

First observe that if $D \in \mathscr{D}$ such that $D^{\to}(L^X) \subseteq \tau_{\mathscr{D}}$, then $D_Y^{\to}(L^Y) \subseteq \tau_{\mathscr{D}}|Y$. Therefore since we may write each $\mu \in \tau_{\mathscr{B}_Y}$ as

$$\mu = \vee \{D_Y(\nu) : \nu \in L^Y, D_Y \circ D_Y(\nu) \leq \mu \text{ and } D \in \mathscr{B}\}$$

where $\mathscr{B}$ is a base of $\mathscr{D}$ for which $D^{\to}(L^X) \subseteq \tau_{\mathscr{D}}$ for $D \in \mathscr{B}$, (I.3.5), we have immediately that $\tau_{\mathscr{B}_Y} \subseteq \tau_{\mathscr{D}}|Y$.

We conclude by showing the reverse inclusion $\tau_{\mathscr{D}}|Y \subseteq \tau_{\mathscr{B}_Y}$. To show this let $\mu \in \tau_{\mathscr{D}}|Y$. Then μ is of the form $\lambda_{|Y}$ where $\lambda \in \tau_{\mathscr{D}}$.

Let $A_\lambda = \{\eta \in L^X : \text{there exists } D \in \mathscr{D} \text{ such that } D(\eta) \leq \lambda\}$.

Then since $\lambda \in \tau_{\mathscr{D}}$, $\lambda = \bigvee_{\eta \in A_\lambda} \eta$. But for each $\eta \in A_\lambda$, $(\eta_Y)_X \leq \eta$ and for some $D \in \mathscr{D}$, $D(\eta) \leq \lambda$. Thus $D_Y(\eta_Y) \leq \lambda_Y$.

Consequently $\bigvee_{\eta \in A_\lambda} \eta_Y \leq \text{int}_{\mathscr{B}_Y}(\lambda_Y) \leq \lambda_Y$. But $\lambda_Y = \bigvee_{\eta \in A_\lambda} \eta_Y$.

Hence $\lambda_Y = \text{int}_{\mathscr{B}_Y}(\lambda_Y)$. Thus $\tau_{\mathscr{D}|Y} \subseteq \tau_{\mathscr{B}_Y}$.

■

We can now formally define a subspace L–uniformity.

I.4.5 DEFINITION

Let $(X, \mathscr{D})$ be an L–uniform space and $Y \subseteq X$. The induced subspace L–uniformity, denoted by $\mathscr{D}_Y$, on Y is that L–uniformity on Y having as a base

$$\mathscr{B}_Y = \{D_Y : D \in \mathscr{D}\}.$$

■

We now turn to the problem of defining an L–uniformity on a set X induced by a collection of functions from X to a corresponding collection of L–uniform spaces.

I.4.6 PROPOSITION

Let $\mathscr{A}$ be a collection of functions, where each function f is a function from X to an L–uniform space $(Y_f, \mathscr{D}_f)$.

(i) Then

$$\mathscr{B}(X) = \{ \mathop{\Delta}_{i=1}^{n} f_i^{\leftarrow}(D_i) : f_i \in \mathscr{A}, D_i \in \mathscr{D}_{f_i}, n \in \mathbb{N}\}$$

is an L–uniform base on X.

(ii) The induced L–topology $\tau_{\mathscr{B}(X)}$ is given by the weak L–topology $\tau_{w(\mathscr{A})}$ generated by $\mathscr{A}$.

PROOF

(i) The set $\mathscr{B}(X)$ is clearly a filter base on $\mathscr{H}_L(X)$. Let $\overset{n}{\underset{i=1}{\Delta}} f_i^{\leftarrow}(D_i) \in \mathscr{B}(X)$.

For each i there exists $E_i^1, E_i^2 \in \mathscr{D}_{f_i}$ such that

(a) $E_i^1 \circ E_i^1 \leq D_i$

(b) $(E_i^2)^{-1} \leq D_i$

By (I.2.13) we have

(a′) $f_i^{\leftarrow}(E_i^1) \circ f^{\leftarrow}(E_i^1) \leq f_i^{\leftarrow}(D_i)$

(b′) $(f_i^{\leftarrow}(E_i^2))^{-1} \leq f_i^{\leftarrow}(D_i)$.

Relations (a′), (b′) together with (I.2.7(iii)) and (I.2.10) imply that

$$(\overset{n}{\underset{i=1}{\Delta}} f_i^{\leftarrow}(E_i^1)) \circ (\overset{n}{\underset{i=1}{\Delta}} f_i^{\leftarrow}(E_i^1)) \leq \overset{n}{\underset{i=1}{\Delta}} f_i^{\leftarrow}(D_i)$$

and

$$(\overset{n}{\underset{i=1}{\Delta}} f_i^{\leftarrow}(E_i^2))^{-1} \leq \overset{n}{\underset{i=1}{\Delta}} f_i^{\leftarrow}(D_i).$$

Hence $\mathscr{B}(X)$ satisfies the requirements of an L–uniform base.

(ii) For each $f \in \mathscr{A}$ let $\mathscr{B}_f \subseteq \mathscr{D}_f$ be a base of $\mathscr{D}_f$ satisfying the conditions of (I.3.5). That is for each $D \in \mathscr{B}_f$, $D^{\rightarrow}(L^{Y_f}) \subseteq \tau_{\mathscr{D}_f}$. Then

$$\tilde{\mathscr{B}}(X) = \{ \overset{n}{\underset{i=1}{\Delta}} f_i^{\leftarrow}(D_i) : f_i \in \mathscr{A}, D_i \in \mathscr{B}_{f_i}, n \in \mathbb{N}\}.$$

is a base for the L–uniformity generated by $\mathscr{B}(X)$. By (I.3.3) for $\mu \in L^X$

$$\text{int}_{\mathscr{B}(X)}(\mu) = \vee \{E(\nu) : \nu \in L^X, E \in \tilde{\mathscr{B}}(X) \text{ and } E \circ E(\nu) \leq \mu\}.$$

Let $E \in \tilde{\mathscr{B}}(X)$ and $\nu \in L^X$. Then E is of the form $\overset{n}{\underset{i=1}{\Delta}} f_i^{\leftarrow}(D_i)$ where

$D_i \in \mathscr{B}_{f_i}$. Then by (I.2.4)

$$E(\nu) = \bigvee_{\lambda \in B(\nu)} f_1^{\leftarrow}(D_1)(\lambda) \wedge \cdots \wedge f_n^{\leftarrow}(D_n)(\lambda).$$

For each i and $\lambda \in B(\nu)$,

$$f_i^{\leftarrow}(D_i)(\lambda) = f_i^{\leftarrow}(D_i(f_i^{\rightarrow}(\lambda)))$$

where $D_i(f_i^{\leftarrow}(\lambda)) \in \tau_{\mathscr{D}_{f_i}}$. Therefore $f_i^{\leftarrow}(D_i)(\lambda) \in \tau_{w(\mathscr{A})}$.

This implies that $E(\nu) \in \tau_{w(\mathscr{A})}$. Hence $\text{int}_{\mathscr{B}(X)}(\mu) \in \tau_{w(\mathscr{A})}$. Thus $\tau_{\mathscr{B}}(X) \subseteq \tau_{w(\mathscr{A})}$. Conversely since each $f \in \mathscr{A}$ is L–uniformly continuous with respect to $\mathscr{B}(X)$, we have $\tau_{w(\mathscr{A})} \subseteq \tau_{\mathscr{B}(X)}$.

■

An immediate corollary is that products of L–uniform spaces are L–uniformizable.

I.4.7 COROLLARY

Let $\{(X_j, \mathscr{D}_j) : j \in J\}$ be a collection of L–uniform spaces. Then

$$\mathscr{B}(\prod_j X_j) = \{ \bigtriangleup_{i=1}^{n} \pi_{j_i}^{\leftarrow}(D_i) : j_i \in J,\ D_i \in \mathscr{D}_{j_i},\ n \in \mathbb{N}\},$$

where $\pi_j : \prod_j X_j \longrightarrow X_j$ is the j'th projection mapping, is an L–uniform base on $\prod_j X_j$ which generates the product L–topology.

■

I.5 : THE L–UNIT INTERVAL I(L)

Hutton ([6],[7]) invented the L–unit interval. This section is a thorough treatment of his construction. All definitions and results are essentially his. We begin immediately with a definition.

I.5.1 DEFINITION

A function $f : \mathbb{R} \longrightarrow L$ is said to be monotonically decreasing if

$$f(t) \leq f(s) \quad \text{for all } s < t.$$

We define

$$f(t+) = \bigvee_{t < s} f(s),\ f(t-) = \bigwedge_{s < t} f(s) \quad \text{for } t \in \mathbb{R}$$

These generalised right–hand and left–hand limits induce the monotonically decreasing functions

$$f^{+} : \mathbb{R} \longrightarrow L,\ f^{-} : \mathbb{R} \longrightarrow L$$

where $f^{+}(t) = f(t+)$, $f^{-}(t) = f(t-)$ for $t \in \mathbb{R}$.

■

I.5.2 PROPOSITION ([11])

Let f,g : $\mathbb{R} \longrightarrow L$ be monotonically decreasing.

(i) For s, t ∈ $\mathbb{R}$ and $s < t$ we have that

$$f(t+) \leq f(t) \leq f(t-) \leq f(s+) \leq f(s) \leq f(s-).$$

(ii) For t ∈ $\mathbb{R}$,

$$f(t+) = \bigvee_{t < s} f(s+) = \bigvee_{t < s} f(s-)$$

$$f(t-) = \bigwedge_{s < t} f(s-) = \bigwedge_{s < t} f(s+).$$

(iii) $(f^+)^+ = f^+$, $(f^-)^- = f^-$.

(iv) $f^+ \leq g^+$ if and only if $f^- \leq g^-$ if and only if $f^+ \leq g^-$.

(v) $f^+ = g^+$ if and only if $f^- = g^-$.

(vi) $(f \vee g)^+ = f^+ \vee g^+$, $(f \wedge g)^+ = f^+ \wedge g^+$.

(vii) $(f \vee g)^- = f^- \vee g^-$, $(f \wedge g)^- = f^- \wedge g^-$.

PROOF

Part (i) follows directly from the fact that f is monotonically decreasing and the definition of f(t+), f(t–) for t ∈ $\mathbb{R}$. Using (i) the proof of (ii) is straightforward. Parts (iii) and (iv) are consequences of (ii), and (v) is a special case of (iv). Part (vii) follows from (vi) by duality. To prove (vi) let t ∈ $\mathbb{R}$. Then

$$(f \vee g)^+(t) = \bigvee_{t < s} f(s) \vee g(s),$$

$$(f^+ \vee g^+)(t) = \Big(\bigvee_{t < s_1} f(s_1)\Big) \vee \Big(\bigvee_{t < s_2} f(s_2)\Big)$$

$$= \bigvee_{t < s_1, s_2} (f(s_1) \vee g(s_2)).$$

Suppose that $t < s_1$, $t < s_2$. Let $s = \min\{s_1, s_2\}$. Then $t < s \leq s_1$ and

$t < s \leq s_2$. This implies that

$$f(s_1) \vee g(s_2) \leq f(s) \vee g(s).$$

Therefore

$$\bigvee_{t < s_1, s_2} (f(s_1) \vee g(s_2)) \leq \bigvee_{t < s} f(s) \vee g(s)$$

$$\leq \bigvee_{t < s_1, s_2} f(s_1) \vee g(s_2).$$

Hence $(f \vee g)^+(t) = (f^+ \vee g^+)(t)$.

In a similar manner we have that

$$(f \wedge g)^+(t) = \bigvee_{t < s} f(s) \wedge g(s)$$

$$= \bigvee_{t < s_1, s_2} f(s_1) \wedge g(s_2).$$

Since L is completely distributive we then have

$$\bigvee_{t < s_1, s_2} f(s_1) \wedge g(s_2) = (\bigvee_{t < s} f(s)) \wedge (\bigvee_{t < s} g(s))$$

$$= f(t+) \wedge g(t+)$$

$$= (f^+ \wedge g^+)(t)$$

This proves that $(f \wedge g)^+(t) = (f^+ \wedge g^+)(t)$.

■

I.5.3 DEFINITION

Let I_L denote the collection

$$\{f : \mathbb{R} \longrightarrow L : f \text{ is monotonically decreasing, } f(0-) = 1 \text{ and } f(1+) = 0\}.$$

We define an equivalence relation on I_L. For $f, g \in I_L$:

$$f \sim g \text{ if and only if } f^- = g^-.$$

The L–unit interval I(L), is then defined to be the set of equivalence classes of the equivalence relation $\sim$ on I_L. We usually write f in place of [f]. Consistency then requires that we re–define our notion of equality in I_L so that $f = g$ if and only if $f^- = g^-$.

■

We note that by (I.5.2(v)) that for $f, g \in I_L$, $f \sim g$ if and only if $f^+ = g^+$. Next we define an injective mapping of I into I(L).

I.5.4 PROPOSITION

The mapping

$$e : I \longrightarrow I(L)$$

given by $e(t) = 1_{(-\infty,t]}$, is injective and in the special case of $L = \{0,1\} = 2$, e is bijective.

PROOF

First note that for $t \in I$, $(1_{(-\infty,t]})^{-} = 1_{(-\infty,t]}$. Let t_1,t_2 be two distinct points in I. Without loss of generality we may assume that $t_1 < t_2$. Therefore $1_{(-\infty,t_1]}(t_2) = 0$ and $1_{(-\infty,t_2]}(t_2) = 1$. Hence $e(t_1) \neq e(t_2)$. This proves that e is an injective mapping. To prove that for $L = \{0,1\}$ e is a bijection, it remains to show that e is surjective. Let $f \in I(\{0,1\})$ and put $t_f = \sup f^{-1}(1)$. Then $f^{-} = 1_{(-\infty,t_f]} = e(t_f)$. Therefore $e(t_f) = f$.

■

When defining an L–topology on I(L) we shall obviously require that in the special case of $L = \{0,1\}$ the mapping

$$e : (I, \chi(\Delta)) \longrightarrow I(L)$$

is an $\{0,1\}$–homeomorphism, where Δ is the standard interval topology.

I.5.5 DEFINITION

For $t \in \mathbb{R}$ define the L–sets

(i) $R_t : I(L) \longrightarrow L$, $R_t(f) = f(t+)$,

(ii) $L_t : I(L) \longrightarrow L$, $L_t(f) = (f(t-))'$.

■

Observe that for $t < 0$, $R_t = \underline{1}$ and for $t \geq 1$, $R_t = \underline{0}$. Also note that for $t \leq 0$, $L_t = \underline{0}$ and for $t > 1$, $L_t = \underline{1}$.

I.5.6 PROPOSITION

(i) For $0 \leq t < 1$, $e^{\leftarrow}(R_t) = 1_{(t,1]}$.

(ii) For $0 < t \leq 1$, $e^{\leftarrow}(L_t) = 1_{[0,t)}$.

(iii) Let τ be some L–topology on I(L). If the L–sets $R_{t|e(I)}$, $L_{t|e(I)}$ for $t \in I$ are open in the relative L–topology $\tau_{|e(I)}$ on e(I) then $e^{-1} : (e(I), \tau_{|e(I)}) \longrightarrow (I, \chi(\Delta))$ is L–continuous.

PROOF

(i) Let $0 \leq t < 1$. Then for $s \in I$,

$$\begin{aligned} e^{\leftarrow}(R_t)(s) &= R_t(1_{(-\infty,s]}) \\ &= (1_{(-\infty,s]})^{+}(t) \\ &= \begin{cases} 1 & \text{for } t < s \\ 0 & \text{for } s \leq t \end{cases} \\ &= 1_{(-\infty,s)}(t) \\ &= 1_{(t,1]}(s) \end{aligned}$$

Thus $e^{\leftarrow}(R_t) = 1_{(t,1]}$.

(ii) Let $0 < t \leq 1$. Then for $s \in I$

$$\begin{aligned} e^{\leftarrow}(L_t)(s) &= L_t(1_{(-\infty,s]}) \\ &= (1_{(-\infty,s]}(t))' \\ &= \begin{cases} 0 & \text{for } t \leq s \\ 1 & \text{for } s < t \end{cases} \\ &= 1_{[0,t)}(s). \end{aligned}$$

Therefore $e^{\leftarrow}(L_t) = 1_{[0,t)}$.

(iii) The collection $\{1_{[0,s)}, 1_{(t,1]} : 0 < s \leq 1 \text{ and } 0 \leq t < 1\}$ forms a subbase of $\chi(\Delta)$. Therefore to show that $e^{-1} : e(I) \longrightarrow I$ is L–continuous it suffices to show that $(e^{-1})^{\leftarrow}(1_{[0,s)})$, $(e^{-1})^{\leftarrow}(1_{(t,1]})$ are open in the relative L–topology $\tau_{|e(I)}$ for $0 < s \leq 1$ and $0 \leq t < 1$.

By (i) and (ii) we have

$$(e^{-1})^{\leftarrow}(1_{(t,1]}) = (e^{-1})^{\leftarrow}(e^{\leftarrow}(R_t)) = R_{t|\,e(I)},$$

$$(e^{-1})^{\leftarrow}(1_{[0,s)}) = (e^{-1})^{\leftarrow}(e^{\leftarrow}(L_s)) = L_{s|\,e(I)}.$$

Thus for e^{-1} to be L–continuous we require that $R_{t|\,e(I)}$, $L_{t|\,e(I)}$ be open in $\tau_{|\,e(I)}$.

■

Our next proposition is essentially a restatement of (I.5.2) in terms of R_t, L_t and therefore can be easily verified.

I.5.7 PROPOSITION

Let s, t ∈ ℝ, A ⊆ ℝ.

(i) $\bigvee_{t \in A} R_t = R_{\inf A}$ if A is bounded from below, otherwise $\bigvee_{t \in A} R_t = \underline{1}$.

(ii) $\bigvee_{t \in A} L_t = L_{\sup A}$ if A is bounded from above, otherwise $\bigvee_{t \in A} L_t = \underline{1}$

(iii) If $s < t$ then $R_s \wedge R_t = R_t$ and $L_s \wedge L_t = L_s$.

■

I.5.8 DEFINITION

(i) The L–topology

$$\tau_r = \{R_t : t \in \mathbb{R}\}$$

is called the right–hand interval L–topology on I(L).

(ii) The L–topology

$$\tau_\ell = \{L_t : t \in \mathbb{R}\}$$

is called the left–hand interval L–topology on I(L).

(iii) The common refinement $\tau_r \vee \tau_\ell$ is called the interval L–topology on I(L) and is denoted by $\tau_{I(L)}$.

■

From the definition of $\tau_{I(L)}$ and (I.5.6) we have immediately

I.5.9 PROPOSITION

The mapping

$$e : (I, \chi(\Delta)) \rightarrow (I(L), \tau_{I(L)})$$

is an L–embedding. In the special case of L = {0,1}, e is an {0,1}–homeomorphism.

■

We now define an L–uniform base on I(L) which generates the L–topology $\tau_{I(L)}$ on I(L). As we can see from previous results (I, $\chi(\Delta)$) can be treated as a subspace of I(L). The L–sets R_t, L_t can be thought of as extensions of the L–sets $1_{(t,1]}$, $1_{[0,t)}$ from I to I(L). Keeping within this framework it would seem natural to try to define an L–uniform base on I(L) in which each element can be treated as an extension of some $G_{E_\epsilon} \Delta (G_{E_\epsilon})^{-1} \in \mathscr{B}_B$. With this in mind we are motivated by the analysis of $\mathscr{B}_B$, (I.3.9), to make the next definition.

I.5.10 DEFINITION

For $\mu \in L^{I(L)}$ let $\mathscr{R}_\mu$, $\mathscr{L}_\mu \subseteq \mathbb{R}$ be defined as

$$\mathscr{R}_\mu = \{t \in \mathbb{R} : \mu \leq L'_t\},\ \mathscr{L}_\mu = \{s \in \mathbb{R} : \mu \leq R'_s\}.$$

■

Observe that if $\mu = \underline{0}$ then $\mathscr{R}_\mu$ is unbounded from above and $\mathscr{L}_\mu$ is unbounded from below.

I.5.11 PROPOSITION

Let $\mu \in L^{I(L)}$ and $\mu \neq \underline{0}$.

(i) $\mathscr{R}_\mu$ is bound above by 1 and $t_\mu = \sup \mathscr{R}_\mu \in \mathscr{R}_\mu$, $0 \leq t_\mu$.

(ii) $\mathscr{L}_\mu$ is bounded below by 0 and $s_\mu = \inf \mathscr{L}_\mu \in \mathscr{L}_\mu$, $s_\mu \leq 1$.

(iii) $\mu \leq \nu$ implies $\mathscr{R}_\mu \subseteq \mathscr{R}_\nu$ and $\mathscr{L}_\mu \subseteq \mathscr{L}_\nu$. Therefore $t_\mu \leq t_\nu$ and $s_\nu \leq s_\mu$.

(iv) If μ is of the form $\mu = \bigvee_{j \in J} \mu_j$, $\mu_j \neq \underline{0}$, then $t_\mu = \inf_{j \in J} t_{\mu_j}$.

(v) For $s < 1$, $s \leq t_{R_s}$ with equality for $0 \leq s < 1$, and for $s \leq 1$, $s \leq t_{L'_s}$ with equality for $0 \leq s \leq 1$.

(vi) For $0 < t$, $s_{L_t} \leq t$ with equality for $0 < t \leq 1$ and for $0 \leq t$, $s_{R'_t} \leq t$ with equality for $0 \leq t \leq 1$.

PROOF

Parts (iii), (v), (vi) are self evident and part (ii) follows from (i) by duality. To prove (i) let $\mu \neq \underline{0}$. Since $L'_t = \underline{0}$ for $1 < t$ and $L'_0 = \underline{1}$ we have that $\mathscr{R}_\mu$ is bounded by 1 and $0 \leq \sup \mathscr{R}_\mu$. Furthermore since $\mu \leq L'_t$ for each $t \in \mathscr{R}_\mu$ we have that

$$\begin{aligned} \mu &\leq \bigwedge_{t \in \mathscr{R}_\mu} L'_t \\ &= L'_{\sup \mathscr{R}_\mu}. \end{aligned}$$

Therefore $t_\mu = \sup \mathscr{R}_\mu$.

To establish (iv) let $\mu = \bigvee_{j \in J} \mu_j$, $\mu_j \neq \underline{0}$. For each j we have by (i) that $\mu_j \leq L'_{t_{\mu_j}}$.

Therefore

$$\begin{aligned} \bigvee_j \mu_j &\leq \bigvee_j L'_{t_{\mu_j}} \\ &\leq L'_{\inf_j t_{\mu_j}}. \end{aligned}$$

Thus $\inf_j t_{\mu_j} \in \mathscr{R}_\mu$. If $t \in \mathscr{R}_\mu$ then $t \in \bigcap_j \mathscr{R}_{\mu_j}$.

This implies that $t_\mu \leq \inf_j t_{\mu_j}$. Hence since $\inf_j t_{\mu_j} \in \mathscr{R}_\mu$ we have that $t_\mu = \inf_j t_{\mu_j}$.

■

I.5.12 DEFINITION

For $\epsilon > 0$, define

$$B_\epsilon : L^{I(L)} \longrightarrow L^{I(L)}$$

by $B_\epsilon(\mu) = R_{t_\mu - \epsilon}$ for $\mu \neq \underline{0}$ and $B_\epsilon(\underline{0}) = \underline{0}$.

■

I.5.13 PROPOSITION

Let ϵ, ϵ_1, $\epsilon_2 > 0$.

(i) $B_\epsilon \in \mathscr{H}_L(I(L))$.

(ii) $\epsilon_1 < \epsilon_2$ implies that $B_{\epsilon_1} \leq B_{\epsilon_2}$.

(iii) $B_{\epsilon_1} \circ B_{\epsilon_2} = B_{\epsilon_1 + \epsilon_2}$.

(iv) For $\mu \in L^{I(L)}$ and $\mu \neq \underline{0}$ we have that

$$B_\epsilon(\mu) = B_\epsilon(L'_{t_\mu}).$$

(v) The inverse B_ϵ^{-1} of B_ϵ is given by

$$B_\epsilon^{-1}(\mu) = L_{s_\mu + \epsilon} \text{ for } \mu \neq \underline{0} \text{ and } B_\epsilon^{-1}(\underline{0}) = \underline{0}.$$

(vi) For $s < 1$, $B_\epsilon(R_s) = R_{s-\epsilon}$ and for $0 < t$, $B_\epsilon^{-1}(L_t) = L_{t+\epsilon}$.

(vii) $(B_\epsilon \,\Delta\, B_\epsilon^{-1})^{\rightarrow}(L^{I(L)}) \subseteq \tau_{I(L)}$.

PROOF

(i) By definition $B_\epsilon(\underline{0}) = \underline{0}$. For $\mu \neq \underline{0}$ we have that

$$\mu \leq R'_{t_\mu - \epsilon}.$$

This implies that $\mu \leq B_\epsilon(\mu)$. Thus B_ϵ satisfies requirements (LH1), (LH2).

Let $\mu \neq \underline{0}$ be of the form $\mu = \bigvee_{j \in J} \mu_j$. We wish to show that

$B_\epsilon(\mu) = \bigvee_{j \in J} B_\epsilon(\mu_j)$. Since $B_\epsilon(\underline{0}) = \underline{0}$ we may assume that each $\mu_j \neq \underline{0}$.

By (I.5.11(v)) we have that $t_\mu = \inf_{j \in J} t_{\mu_j}$.

Therefore

$$\begin{aligned} B_\epsilon(\mu) &= R_{\inf\limits_{j \in J} t_{\mu_j} - \epsilon} \\ &= \bigvee_{j \in J} R_{t_{\mu_j} - \epsilon} \\ &= \bigvee_{j \in J} B_\epsilon(\mu_j). \end{aligned}$$

This proves that B_ϵ satisfies (LH3). Hence $B_\epsilon \in \mathscr{H}_L(I(L))$.

(ii) The inequality $B_{\epsilon_1}(\mu) \leq B_{\epsilon_2}(\mu)$ holds trivially for $\mu = \underline{0}$. Suppose that $\mu \neq \underline{0}$. Since $\epsilon_1 < \epsilon_2$ we have that $t_\mu - \epsilon_2 < t_\mu - \epsilon_1$. This implies that

$$B_{\epsilon_1}(\mu) = R_{t_\mu - \epsilon_1} \leq R_{t_\mu - \epsilon_2} = B_{\epsilon_2}(\mu).$$

(iii) For $\mu \neq \underline{0}$ we have

$$\begin{aligned} B_{\epsilon_1} \circ B_{\epsilon_2}(\mu) &= B_{\epsilon_1}(B_{\epsilon_2}(\mu)) \\ &= B_{\epsilon_1}(R_{t_\mu - \epsilon_2}) \\ &= R_{t_\mu - (\epsilon_1 + \epsilon_2)} \\ &= B_{\epsilon_1 + \epsilon_2}(\mu). \end{aligned}$$

Again $B_{\epsilon_1} \circ B_{\epsilon_2}(\mu) = B_{\epsilon_1 + \epsilon_2}(\mu)$ holds trivially for $\mu = \underline{0}$.

(iv) If $\mu \neq \underline{0}$, then $t_\mu \leq 1$ by (I.5.11(i)). Then by (I.5.11(v)) we have that $t_{L'_{t_\mu}} = t_\mu$. This implies that $B_\epsilon(L'_{t_\mu}) = R_{t_\mu - \epsilon}$. But $B_\epsilon(\mu) = R_{t_\mu - \epsilon}$. Hence $B_\epsilon(\mu) = B_\epsilon(L'_{t_\mu})$.

(v) Since $B_\epsilon \in \mathscr{H}_L(I(L))$ we have that $B_\epsilon^{-1} \in \mathscr{H}_L(I(L))$. Therefore $B_\epsilon^{-1}(\underline{0}) = \underline{0}$. Suppose that $\mu \neq \underline{0}$. By definition

$$B_\epsilon^{-1}(\mu) = \wedge \{\nu \in L^{I(L)} : B_\epsilon(\nu') \leq \mu'\}.$$

Suppose that the only $\nu \in L^{I(L)}$ satisfying $B_\epsilon(\nu') \leq \mu'$ is $\nu = \underline{1}$. This situation will occur for large ϵ. If we can show that $1 < s_\mu + \epsilon$ we will have that $L_{s_\mu+\epsilon} = \underline{1}$ and consequently $B_\epsilon^{-1}(\mu)$ will be correctly given by $L_{s_\mu+\epsilon}$. Suppose that $s_\mu+\epsilon \leq 1$. Then $L_{s_\mu+\epsilon} \neq \underline{1}$ and $B_\epsilon(L'_{s_\mu+\epsilon}) = R_{s_\mu}$. But $\mu \leq R'_{s_\mu}$. This implies that $B_\epsilon(L'_{s_\mu+\epsilon})) \leq \mu'$. This contradiction proves that $1 < s_\mu+\epsilon$.

Suppose that there exists $\nu \neq \underline{1}$ such that $B_\epsilon(\nu') \leq \mu'$. We may then write

$$B_\epsilon^{-1}(\mu) = \wedge \{\nu \in L^{I(L)}/\{\underline{1}\} : B_\epsilon^{-1}(\nu') \leq \mu'\}$$

Let $\nu \in L^{I(L)}/\{\underline{1}\}$ such that $B_\epsilon(\nu') \leq \mu'$. Since $\nu \neq \underline{1}$, we have that $\nu' \neq \underline{0}$. Therefore $B_\epsilon(\nu') = B_\epsilon(L'_{t_{\nu'}})$ and $\nu' \leq L'_{t_{\nu'}}$.

This inequality implies that $L_{t_{\nu'}} \leq \nu$. Hence

$$\begin{aligned} B_\epsilon^{-1}(\mu) &= \wedge \{L_s : B_\epsilon(L'_s) \leq \mu'\} \\ &= \wedge \{L_s : R_{s-\epsilon} \leq \mu'\} \\ &= \wedge \{L_{s+\epsilon} : \mu \leq R'_s\} \\ &= \wedge \{L_{s+\epsilon} : s \in \mathscr{L}_\mu\}. \\ &= L_{s_\mu+\epsilon}. \end{aligned}$$

(vii) Let $\mu \in L^{I(L)}$. Then by (I.2.4)

$$B_\epsilon \,\Delta\, B_\epsilon^{-1}(\mu) = \bigvee_{\lambda \in B(\mu)} B_\epsilon(\lambda) \wedge B_\epsilon^{-1}(\lambda).$$

For each $\lambda \in B(\mu)$ we have that $B_\epsilon(\lambda) \wedge B_\epsilon^{-1}(\lambda) = R_s \wedge L_t$ for some $s,t \in \mathbb{R}$. Hence $B_\epsilon(\lambda) \wedge B_\epsilon^{-1}(\lambda) \in \tau_{I(L)}$. This implies that $B_\epsilon \,\Delta\, B_\epsilon^{-1}(\mu) \in \tau_{I(L)}$.

■

I.5.14 PROPOSITION

The collection

$$\mathscr{B}(I(L)) = \{B_\epsilon \Delta B_\epsilon^{-1} : \epsilon > 0\}$$

is an L–uniform base on I(L). Let $\mathscr{D}(I(L))$ denote the L–uniformity generated by $\mathscr{B}(I(L))$. The L–topology $\tau_{\mathscr{D}(I(L))}$ induced by $\mathscr{D}(I(L))$ is precisely $\tau_{I(L)}$.

PROOF

By (I.5.13) $\mathscr{B}(I(L))$ is a chain of symmetric elements of $\mathscr{H}_L(I(L))$. Therefore $\mathscr{B}(I(L))$ satisfies (LBU1), (LBU3). By (I.5.13(iii)) $B_{\epsilon/2} \circ B_{\epsilon/2} \leq B_\epsilon$. This implies that $B^{-1}_{\epsilon/2} \circ B^{-1}_{\epsilon/2} \leq B^{-1}_{\epsilon/2}$. Then using (I.2.7(iii)) we have that

$$B_{\epsilon/2} \Delta B^{-1}_{\epsilon/2} \circ (B_{\epsilon/2} \Delta B^{-1}_{\epsilon/2}) \leq B_\epsilon \Delta B_\epsilon^{-1}.$$

Thus $\mathscr{B}(I(L))$ satisfies (LBU2). Hence $\mathscr{B}(I(L))$ is an L–uniform base on I(L). By (I.3.3) we have that for $\mu \in L^{I(L)}$,

$\text{int}_{\mathscr{D}(I(L))}(\mu) = \vee \{B_\epsilon \Delta B_\epsilon^{-1}(\nu) : \nu \in L^{I(L)}$ and $(B_\epsilon \Delta B_\epsilon^{-1}) \circ (B_\epsilon \Delta B_\epsilon^{-1}(\nu)) \leq \mu$ for some $\epsilon > 0\}$

By (I.5.13(vii)), $(B_\epsilon \Delta B_\epsilon^{-1})^{\rightarrow}(L^{I(L)}) \subseteq \tau_{I(L)}$. Thus $\text{int}_{\mathscr{D}(I(L))}(\mu) \in \tau_{I(L)}$.
This implies that $\tau_{\mathscr{D}((I(L))} \subseteq \tau_{I(L)}$. For $t \in \mathbb{R}$,

$$\begin{aligned} R_t &= \bigvee_{\epsilon > 0} R_{t+\epsilon} \\ &\leq \bigvee_{\epsilon > 0} B_\epsilon(R_{t+\epsilon}) \\ &\leq R_t \end{aligned}$$

and

$$\begin{aligned} L_t &= \bigvee_{\epsilon > 0} L_{t-\epsilon} \\ &\leq \bigvee_{\epsilon > 0} B_\epsilon^{-1}(L_{t-\epsilon}) \\ &\leq L_t. \end{aligned}$$

Therefore by the definition of the interior operation $\text{int}_{\mathscr{D}(I(L))}$ we have that $\text{int}_{\mathscr{D}(I(L))}(R_t) = R_t$ and $\text{int}_{\mathscr{D}(I(L))}(L_t) = L_t$.

Hence $\tau_{I(L)} \subseteq \tau_{\mathscr{D}(I(L))}$.

Therefore $\tau_{I(L)} = \tau_{\mathscr{D}(I(L))}$.

■

We show how G_{E_ϵ} and B_ϵ are related.

I.5.15 PROPOSITION

Let $\epsilon > 0$ and let $e : I \longrightarrow I(L)$ be the mapping defined in (I.5.4). Then

(i) $e^{\leftarrow}(B_\epsilon) = G_{E_\epsilon}$.

(ii) $(e^{-1})^{\leftarrow}(G_{E_\epsilon}) = (B_\epsilon)_{e(I)}$.

PROOF

(i) Let $\mu \in L^I$ and $\mu \neq \underline{0}$. First note that for $t \in I$,

$$\mu \leq 1_{[t,1]} \text{ if and only if } e^{\rightarrow}(\mu) \leq L'_t.$$

Therefore $t_{e^{\rightarrow}(\mu)} = \sup\{t \in I : \mu \leq 1_{[t,1]}\}$. Hence $B_\epsilon(e^{\rightarrow}(\mu)) = R_{t_{e^{\rightarrow}(\mu)}-\epsilon}$ and $G_{E_\epsilon}(\mu) = 1_{(t_{e^{\rightarrow}(\mu)}-\epsilon,1]}$.

Therefore

$$\begin{aligned} e^{\leftarrow}(B_\epsilon)(\mu) &= e^{\leftarrow}(B_\epsilon(e^{\rightarrow}(\mu))) \\ &= e^{\leftarrow}(R_{t_{e^{\rightarrow}(\mu)}-\epsilon}) \\ &= 1_{(t_{e^{\rightarrow}(\mu)}-\epsilon,1]} \\ &= G_{E_\epsilon}(\mu). \end{aligned}$$

Thus $e^{\leftarrow}(B_\epsilon) = G_{E_\epsilon}$

(ii) From (i) we have for $\mu \in L^{e(I)}$

$$\begin{aligned} (e^{-1})^{\leftarrow}(G_{E_\epsilon})(\mu) &= (e^{-1})^{\leftarrow}(G_{E_\epsilon}[(e^{-1})^{\rightarrow}(\mu)]) \\ &= (e^{-1})^{\leftarrow}((e^{\leftarrow}(B_\epsilon))[(e^{-1})^{\rightarrow}(\mu)]). \end{aligned}$$

Then

$$e^{\leftarrow}(B_\epsilon)[(e^{-1})^{\rightarrow}(\mu)] \qquad = e^{\leftarrow}(B_\epsilon(e^{\rightarrow}((e^{-1})^{\rightarrow}(\mu))))$$
$$= e^{\leftarrow}(B_\epsilon(\mu_{I(L)})).$$

Hence

$$(e^{-1})^{\leftarrow}(G_{E_\epsilon})(\mu) \qquad = (e^{-1})^{\leftarrow}[e^{\leftarrow}(B_\epsilon(\mu_{I(L)}))]$$
$$= B_\epsilon(\mu_{I(L)})_{e(I)}$$
$$= (B_\epsilon)_{e(I)}(\mu).$$

Therefore $(e^{-1})^{\leftarrow}(G_{E_\epsilon}) = (B_\epsilon)_{e(I)}$.

■

Next we show that not only is e an L–embedding, it is also an L–uniform embedding, in the sense that e, e^{-1} are both L–uniformly continuous.

I.5.16 PROPOSITION

The mapping
$$e : (I, \mathscr{D}_{\mathbb{D}}) \longrightarrow (I(L), \mathscr{D}(I(L)))$$
is an L–uniform embedding and in the case of L = {0,1}, e is {0,1}–uniform isomorphism.

PROOF

For each $\epsilon > 0$ we have by (I.5.15) and (I.2.12) that
$$e^{\leftarrow}(B_\epsilon^{-1}) = (G_{E_\epsilon})^{-1},$$
$$(e^{-1})^{\leftarrow}((G_{E_\epsilon})^{-1}) = ((B_\epsilon)_{e(I)})^{-1}.$$
Therefore since $\{B_\epsilon, B_\epsilon^{-1} : \epsilon > 0\}$ and $\{G_{E_\epsilon}, (G_{E_\epsilon})^{-1} : \epsilon > 0\}$ form subbases for the L–uniformities $\mathscr{D}(I(L))$ and $\mathscr{D}_{\mathbb{D}}$, we have by (I.3.7) that e and e^{-1} are L–uniformly continuous. Hence e is an L–uniform embedding. In the case of L = {0,1}, e(I) = I({0,1}). Thus $e : I \longrightarrow I(\{0,1\})$ is an {0,1}–uniform isomorphism.

■

We close this section with a result concerning I(L)–valued L–continuous functions which will be needed in Chapter III.

I.5.17 PROPOSITION

Let (X,τ) be an L–space and $f,g : X \longrightarrow I(L)$ be L–continuous functions.

(i) $f \wedge g : X \longrightarrow I(L)$ is L–continuous.

(ii) $f \vee g : X \longrightarrow I(L)$ is L–continuous.

PROOF

(i) Since $\{R_t, L_t : t \in \mathbb{R}\}$ from a subbase for $\tau_{I(L)}$, to show that $f \wedge g$, $f \vee g$ are L–continuous it is sufficient to show that $(f \wedge g)^{\leftarrow}(R_t)$, $(f \vee g)^{\leftarrow}(R_t) \in \tau$ and $(f \wedge g)^{\leftarrow}(L_t)$, $(f \vee g)^{\leftarrow}(L_t) \in \tau$ for $t \in \mathbb{R}$. We will only show that $(f \wedge g)^{\leftarrow}(R_t)$, $(f \vee g)^{\leftarrow}(L_t) \in \tau$.

The corresponding results for $f \vee g$ can be proved in exactly the same way.

Let $t \in \mathbb{R}$, then for $x \in X$

$$(f \wedge g)^{\leftarrow}(R_t)(x) = R_t(f(x) \wedge g(x)) = (f^{\leftarrow}(R_t) \wedge g^{\leftarrow}(R_t))(x)$$

and

$$\begin{aligned}(f \wedge g)^{\leftarrow}(L_t)(x) &= L_t(f(x) \wedge g(x)) \\ &= (f(x)(t-) \wedge g(x)(t-))' \\ &= (f(x)(t-))' \vee (f(x)(t-))' \\ &= (f^{\leftarrow}(L_t) \vee g^{\leftarrow}(L_t))(x).\end{aligned}$$

Therefore

$$(f \wedge g)^{\leftarrow}(R_t) = f^{\leftarrow}(R_t) \wedge f^{\leftarrow}(R_t)$$

and

$$(f \wedge g)^{\leftarrow}(L_t) = f^{\leftarrow}(L_t) \vee g^{\leftarrow}(L_t).$$

This proves that $f \wedge g$ is L–continuous.

∎

CHAPTER II

EXTENDING I(L)–VALUED L–UNIFORMLY CONTINUOUS FUNCTIONS.

INTRODUCTION

The aim of this chapter is to show that an L–uniformly continuous function from a subspace of an L–uniform space to I(L) can be extended to the whole space.

Using a series of lemmas on binary relations Katetov ([8], [9]), showed how a bounded real–valued uniformly continuous function defined on a subspace of a uniform space can be extended to the entire space. His first lemma deals with a special type of binary relation on the collection of subsets of a set, and can easily be extended to L–sets. By defining a relation on an L–uniform space satisfying the conditions of an L–version of Katetov's first lemma, we are able to use his second lemma, an insertion type lemma, to obtain a parallel L–version of his result.

In section 1, we give an L–version of Katetov's first lemma, and state and prove his second lemma.

The final section, section 2, is devoted to showing how an L–uniformly continuous I(L)–valued function from a subspace of an L–uniform space can be extended to the whole space. This section is our own work.

Thoughout this chapter L will be assumed to be a completely distributive lattice, unless otherwise stated.

II.1 KATETOV'S LEMMAS

We begin with a definition.

II.1.1 DEFINITION ([8],[9])

Let X,Y be two sets and let ρ , ξ be two binary relations defined on X and Y respectively.

(i) The induced binary relation ρ^{ξ} is defined as follows :
For f, g $\in X^Y$:
$f \rho^{\xi} g$ if and only if $f(y_1) \rho g(y_2)$ for each $y_1, y_2 \in Y$ satisfying $y_1 \xi y_2$.

(ii) If A, B $\subseteq$ X such that $a \rho b$ whenever $a \in A$, $b \in B$ we write $A \rho B$.

(iii) We say that ρ satisfies the Interpolation Property (IntP) if given finite subsets A, B $\subseteq$ X such that $A \rho B$, then there exists $c \in X$ such that $A \rho c$, $c \rho B$.

■

II.1.2 DEFINITION

Let $(X, \mathscr{D})$ be an L–uniform space. Define a binary relation $\rho_{\mathscr{D}}$ on L^X in the following way :

For $\mu, \nu \in L^X$: $\mu \rho_{\mathscr{D}} \nu$ if and only if there exists $D \in \mathscr{D}$ such that $D(\mu) \leq \nu$.

■

II.1.3 PROPOSITION

Observe that

(i) $\mu \rho_{\mathscr{D}} \nu$ implies $\mu \leq \text{int}_{\mathscr{D}}(\nu)$.

(ii) $\mu \leq \mu_1 \rho_{\mathscr{D}} \nu_1 \leq \nu$ implies $\mu \rho_{\mathscr{D}} \nu$ for $\mu, \mu_1, \nu, \nu_1 \in L^X$.

■

II.1.4 PROPOSITION

$\rho_{\mathscr{D}}$ satisfies (IntP).

PROOF

Let $\{\mu_1 ; \cdots ; \mu_m\}$, $\{\nu_1 ; \cdots ; \nu_n\}$ be finite subsets of L^X such that

$$\mu_i \rho_{\mathscr{D}} \nu_j \quad \text{for } i \in [1 \cdots m], j \in [1 \cdots n].$$

Thus for each i, j there exists $D_{ij} \in \mathscr{D}$ such that $D_{ij}(\mu_i) \leq \nu_j$.

Since $\mathscr{D}$ is an L–uniformity $E = \mathop{\Delta}\limits_{j=1}^{n} \left(\mathop{\Delta}\limits_{i=1}^{m} D_{ij}\right) \in \mathscr{D}$ and therefore there exists $E_1 \in \mathscr{D}$ such that $E_1 \circ E_1 \leq E$. Set

$$\lambda = \bigvee_{i=1}^{m} E_1(\mu_i).$$

Then for each μ_i, $\mu_i \leq E_1(\mu_i) \leq \lambda$. Hence

$$\mu_i \, \rho_{\mathscr{D}} \, \lambda \ \text{ for } i = 1,\cdots,m.$$

We also have that

$$\begin{aligned} E_1(\lambda) &= E_1\Big(\bigvee_{i=1}^{m} E_1(\mu_i)\Big) \\ &= \bigvee_{i=1}^{m} (E_1 \circ E_1)(\mu_i) \\ &\leq \bigvee_{i=1}^{m} E(\mu_i). \end{aligned}$$

But $E(\mu_i) = \mathop{\Delta}\limits_{j=1}^{n} \left(\mathop{\Delta}\limits_{i=1}^{m} D_{ij}\right)(\mu_i) \leq D_{ij}(\mu_i) \leq \nu_j$. Hence $E_1(\lambda) \leq \nu_j$, and we therefore have that

$$\lambda \, \rho_{\mathscr{D}} \, \nu_j \ \text{ for } j = 1,\cdots,m.$$

We have now shown that for any two finite sets A, $B \subseteq L^X$ such that $A \, \rho_{\mathscr{D}} \, B$, then there exists $\lambda \in L^X$ such that

$$A \, \rho_{\mathscr{D}} \, \lambda, \ \lambda \, \rho_{\mathscr{D}} \, B.$$

This proves that $\rho_{\mathscr{D}}$ satisfies (IntP).

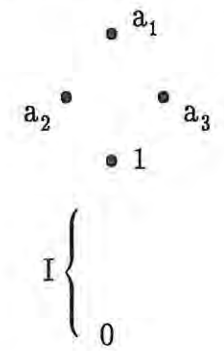
■

For a relation to satisfy (IntP) is a strong requirement. In the unit interval the strictly less than relation <, satisfies (IntP). In the class of completely distributive lattices as the following example will show, the way below relation <<, does not always satisfy (IntP).

II.1.5 EXAMPLE

Add the three points $\{a_1, a_2, a_3\}$ to the unit interval I to obtain the lattice L described in the diagram below

L :



The lattice L is complete with top element a_1 and bottom element 0. The following condition in a complete lattice (0.1.17(ii)) is equivalent to complete distributivity :

> the lattice is continuous and every element can be written as the supremum of a collection of co–prime elements. (1)

Let $D \subseteq L$ be a directed subset such that $a_i \leq \vee D$. Then it can readily be seen that either $a_1 \in D$ or $a_i \in D$.

It therefore follows if we let $<<_L$ denote the way below relation on L, that $a_i <<_L a_i$ for i = 1,2,3. It can also be easily seen that $<<_{L|I} = <<_I$, where $<<_I$ denotes the way below relation on I. Combining these two results it follows that L is a continuous lattice.

All the elements of I remain coprime in L, and a_2,a_3 are also coprime. To see the latter, consider for example a_2. If $\alpha, \beta \in L$ such that $a_2 \leq \alpha \vee \beta$, then $\alpha \vee \beta = a_2$ or $\alpha \vee \beta = a_1$. Hence

$$(\alpha,\beta) \in \{(a_2,a_2); (a_2,a_3); (a_3,a_2)\} \cup \{(a_1,\delta); (\delta,a_1): \delta \in L\} \cup \{(a_2,t); (t,a_2): t \in I\}$$

Thus either $a_2 \leq \alpha$ or $a_2 \leq \beta$. Since $a_1 = a_2 \vee a_3$, we have that every element in L can be written as the supremum of coprimes less than or equal to itself. According to (1) L is a completely distributive lattice.

Now $1 <<_L a_i$ for i = 2,3. Suppose that $<<_L$ satisfies (IntP). Then there exists $\alpha \in L$ such that $1 <<_L \alpha <<_L a_i$ for i = 1,2. This implies that $1 \leq \alpha \leq a_2 \wedge a_3$, thus $\alpha = 1$. The set [0,1) is directed in L and $\vee[0,1) = 1$, but for each $t \in [0,1)$, $t < 1$. We can therefore not have that $1 <<_L 1$. This contradiction proves that $<<_L$ does not satisfy (IntP).

■

Binary relations satisfying (IntP) and induced binary relations of the type described in (II.1.1(i)) occur in some well known areas of mathematics.

II.1.6 EXAMPLE

For any topological space (X,Δ) define ρ on $\mathscr{P}(X)$ by

$$A \,\rho\, B \text{ if and only if } \bar{A} \subset B.$$

Then ρ satisfies (IntP). To see this let $\{A_1 ; \cdots ; A_m\}$ and $\{B_1 ; \cdots ; A_n\}$ be two finite subsets of $\mathscr{P}(X)$ such that

$$A_i \,\rho\, B_j \text{ for } i \in [1 \cdots m], j \in [1 \cdots n]$$

Then $A_i \subseteq \bar{A}_i \subseteq \bigcup_{i=1}^{m} \bar{A}_i \subseteq B_j$. Let $C = \bigcup_{i=1}^{m} \bar{A}_i$, then since C is closed, we have $A_i \,\rho\, C$ and $C \,\rho\, B_j$ for all i,j.

If (X,Δ) is a normal space, then for any pair of disjoint closed subsets A and B, a family $\{U_d : d \in Q_D\}$ of open subsets indexed by Q_D, the set of dyadic rationals, can be constructed satisfying :

(i) $U_{d_1} \subseteq \bar{U}_{d_1} \subseteq U_{d_2}$ for $d_1 < d_2$.

(ii) $A \subseteq U_d$ and $\bar{U}_d \cap B = \emptyset$.

Thus $d_1 < d_2$ implies $U_{d_1} \,\rho\, U_{d_2}$. Let $<$ be the strictly less than relation on Q_D and let $f : Q_D \rightarrow \mathscr{P}(X)$ be defined as $f(d) = U_d$. Then $f \underset{\rho}{\overset{<}{\ }} f$.

■

We need some additional definitions :

II.1.7 DEFINITION ([8],[9])

Let X be a set and ρ a binary relation on X. The induced binary relation $\bar{\rho}$ is defined by :

For x, y $\in$ X, x $\bar{\rho}$ y if and only if y ρ z implies x ρ z and v ρ x implies v ρ y for z, v $\in$ X.

■

The induced relation ρ is transitive :
Let x, y, z $\in$ X such that x $\bar{\rho}$ y, y $\bar{\rho}$ z. Suppose u $\in$ X such that u ρ x. This implies u ρ y (since x $\bar{\rho}$ y), which in turn implies u ρ z (since y $\bar{\rho}$ z). If v $\in$ X such that z ρ v, we have y ρ v, which implies x ρ v. Therefore x $\bar{\rho}$ z.

II.1.8 EXAMPLE

Let L be a continuous lattice, then the induced relation $\overline{<<}$, is precisely $\leq$. To show this let β_1, $\beta_2 \in L_1$ such that $\beta_1 \overline{<<} \beta_2$. Since L is continuous,

$\beta_1 = \vee\{\alpha \in L : \alpha << \beta_1\}$. But for each $\alpha \in L$ satisfying $\alpha << \beta_1$, we have $\alpha << \beta_2$. Hence since $\alpha << \beta_1$ implies $\alpha \leq \beta_2$, we have $\beta_1 = \vee\{\alpha \in L : \alpha << \beta_1\} \leq \beta_2$. Conversely, if $\beta_1,\beta_2 \in L$ such that $\beta_1 \leq \beta_2$, then if $\alpha \in L$ such that $\alpha << \beta_1$ we have immediately that $\alpha << \beta_2$ since $\beta_1 \leq \beta_2$. Similarly if $\beta_2 << \alpha$, then $\beta_1 << \alpha$.

II.1.9 PROPOSITION

The binary relation $\bar{\rho}_{\mathscr{D}}$ on L^X where $(X, \mathscr{D})$ is an L–uniform space has the following property :

$$\mu \leq \nu \text{ implies } \mu \,\bar{\rho}_{\mathscr{D}}\, \nu \text{ for } \mu, \nu \in L^X.$$

■

This follows directly from (II.1.3(ii)).

II.1.10 DEFINITION ([8],[9])

Let ρ be a binary relation on a set X. Then the relation ρ is said to have

(i) the property (I) if the following holds :

if M,N $\subseteq$ X are countable, and there exists x,y $\in$ X such that

M $\bar{\rho}$ x, x ρ N, M ρ y, y $\bar{\rho}$ N,

then there exists z $\in$ X such that

M ρ z, z ρ N.

(ii) the property (L) if for any finite $A \subseteq X$ there exists $x, y \in X$ such that

(1) $A \bar{\rho} x$ and $x \rho z$ whenever $A \rho z$.

(2) $y \bar{\rho} A$ and $z \rho y$ whenever $z \rho A$

for $z \in X$.

■

We have the following L–version of Katetov's first lemma :

II.1.11 LEMMA

Let X be a set and ρ a binary relation on L^X such that

(i) ρ satisfies (IntP),

(ii) for $\mu, \nu \in L^X$: $\mu \leq \nu$ implies $\mu \bar{\rho} \nu$,

(iii) for $\mu, \nu \in L^X$: $\mu \rho \nu$ implies $\mu \leq \nu$.

Then ρ satisfies (I) and (L).

■

Before we prove this lemma we note that the relation $\rho_{\mathscr{D}}$ on L^X, where $(X, \mathscr{D})$ is an L–uniform space, satisfies (i), (ii) and (iii) of (II.1.11). By (II.1.4) $\rho_{\mathscr{D}}$ satisfies (IntP), requirements (ii) and (iii) follow from (II.1.9) and (II.1.3(i)) respectively.

PROOF OF LEMMA

PROPERTY (I)

Let $M = \{\mu_i : i \in \mathbb{N}\}$, $N = \{\nu_j : j \in \mathbb{N}\}$ be countable subsets of L^X, and suppose that there exists $\mu, \nu \in L^X$ such that

$$\mu_i \bar{\rho} \mu, \ \mu \rho \nu_j, \ \mu_i \rho \nu, \ \nu \bar{\rho} \nu_j \quad \text{for } i,j \in \mathbb{N}. \tag{1}$$

Since $\mu_1 \rho \nu$ and ρ satisfies (IntP), there exists $\lambda_1 \in L^X$ such that

$$\mu_1 \rho \lambda_1 , \ \lambda_1 \rho \nu \tag{2}$$

By (1) we have $\nu \bar{\rho} \nu_1$, which implies that $\lambda_1 \rho \nu_1$. Consider the sets $\{\mu,\lambda_1\}$, $\{\nu_1\}$. From (1) we have that $\mu \rho \nu_1$ and we have already shown that $\lambda_1 \rho \nu_1$. Hence $\{\mu, \lambda_1\} \rho \{\nu_1\}$.

Applying (IntP) we have that there exists $\eta_1 \in L^X$ such that

$$\{\mu, \lambda_1\} \ \rho \ \eta_1, \ \eta_1 \ \rho \ \nu_1 \tag{3}$$

Together with (2) we now have :

$$\mu_1 \ \rho \ \lambda_1 \ , \ \eta_1 \ \rho \ \nu_1 \ , \ \mu \ \rho \ \eta_1 \ , \ \lambda_1 \ \rho \ \nu \ , \ \lambda_1 \ \rho \ \eta_1. \tag{4}$$

Suppose now that λ_k , $\eta_k \in L^X$, $k = 1,2, \cdots, n$ have been found such that

$$(S_n) \qquad \mu_i \ \rho \ \lambda_i \ , \ \eta_i \ \rho \ \nu_i \ , \ \mu \ \rho \ \eta_i, \ \lambda_i \ \rho \ \nu, \ \lambda_i \ \rho \ \eta_j$$
$$\text{for } i,j \in [1 \cdot\cdot n].$$

By (4) this can be done for $n = 1$.

We have by (1) that $\mu_{n+1} \ \bar{\rho} \ \mu$, and by (S_n) $\mu \ \rho \ \eta_j$ for $j = 1,\cdots, n$.
Therefore by definition of $\bar{\rho}$, $\mu_{n+1} \ \rho \ \eta_j$ for $j = 1,\cdots,n$. Also by (1) we have that $\mu_{n+1} \ \rho \ \nu$. Thus $\mu_{n+1} \ \rho \ \{\eta_1;\cdots;\eta_n, \nu\}$. Applying (IntP) there exists $\lambda_{n+1} \in L$ such that

$$\mu_{n+1} \ \rho \ \lambda_{n+1}, \ \lambda_{n+1} \ \rho \ \nu \ \text{ and } \ \lambda_{n+1} \ \rho \ \eta_i \ \text{ for } i = 1,\cdots,n. \tag{5}$$

By (S_n) we have $\lambda_i \ \rho \ \nu$ for $i = 1,\cdots,n$ and by (1) $\nu \ \bar{\rho} \ \nu_{n+1}$. This together with (5) (in particular the fact that $\lambda_{n+1} \ \rho \ \nu$) and the definition of $\bar{\rho}$ implies $\lambda_i \ \rho \ \nu_{n+1}$ for $i = 1,\cdots,n+1$. By (1) $\mu \ \rho \ \nu_{n+1}$. Therefore $\{\mu, \lambda_1;\cdots;\lambda_{n+1}\} \ \rho \ \nu_{n+1}$, and applying (IntP) there exists $\eta_{n+1} \in L^X$ such that :

$$\mu \ \rho \ \eta_{n+1}, \ \lambda_i \ \rho \ \eta_{n+1} \text{ and } \ \eta_{n+1} \ \rho \ \nu_{n+1} \ \text{ for } i = 1,\cdots,n+1 \tag{6}$$

We now check that (S_{n+1}) holds,

$$(S_{n+1}) \qquad \mu_i \ \rho \ \lambda_i, \ \eta_i \ \rho \ \nu_i, \ \mu \ \rho \ \eta_i, \ \lambda_i \ \rho \ \nu, \ \lambda_i \ \rho \ \eta_j$$
$$\text{for } i,j \in [1 \cdot\cdot n+1].$$

By (S_n) and (5) $\mu_i \ \rho \ \lambda_i$, $\lambda_i \ \rho \ \nu$ holds for $i = 1,\cdots,n+1$. Similarly by (S_n) and (6) $\eta_i \ \rho \ \nu_i$, $\mu \ \rho \ \eta_i$ holds for $i = 1,\cdots,n+1$. Finally (S_n) together with (5) and (6) shows that $\lambda_i \ \rho \ \eta_j$ holds for $i,j \in [1 \cdot\cdot n+1]$.

We have shown that S_1 holds and if S_n holds for $n \in \mathbb{N}$, then S_{n+1} holds. Therefore by induction S_n holds for all $n \in \mathbb{N}$.

Let $\lambda = \bigvee_{i \in \mathbb{N}} \lambda_i$. Then for each $i \in \mathbb{N}$, $\lambda_i \leq \lambda$. By (ii) this implies that $\lambda_i \,\bar{\rho}\, \lambda$. Hence since $\mu_i \,\rho\, \lambda_i$ it follows that

$$\mu_i \,\rho\, \lambda \quad \text{for } i \in \mathbb{N}. \tag{7}$$

Let $j \in \mathbb{N}$. Then for each $i \in \mathbb{N}$, $\lambda_i \,\rho\, \eta_j$. By (iii) this implies that $\lambda_i \leq \eta_j$. Hence $\lambda = \bigvee_i \lambda_i \leq \eta_j$. By (ii) we have $\lambda \,\bar{\rho}\, \eta_j$. But $\eta_j \,\rho\, \nu_j$. Thus by definition of $\bar{\rho}$,

$$\lambda \,\rho\, \nu_j \quad \text{for all } j \in \mathbb{N}. \tag{8}$$

Using (7) and (8) we see that there exists $\lambda \in L^X$ such that

$$M \,\rho\, \lambda,\ \lambda \,\rho\, N.$$

This proves that ρ satisfies (I).

PROPERTY (L)

Let $M = \{\mu_1;\cdots;\mu_n\}$ be a finite subset of L^X. Let

$$\mu = \bigwedge_{i=1}^{n} \mu_i\,,\ \nu = \bigvee_{i=1}^{n} \mu_i$$

By (ii) we have $\mu \,\bar{\rho}\, M$, $M \,\bar{\rho}\, \nu$.

Suppose that $\lambda \in L^X$ and $\lambda \,\rho\, M$. By (IntP) there exists $\eta \in L^X$ such that $\lambda \,\rho\, \eta$ and $\eta \,\rho\, M$. By (iii) since $\eta \,\rho\, \mu_i$, it follows that $\eta \leq \mu_i$ for $i = 1,\cdots,n$. Hence $\eta \leq \bigwedge_{i=1}^{n} \mu_i = \mu$, and by (ii) this implies $\eta \,\bar{\rho}\, \mu$. But $\lambda \,\rho\, \eta$. Thus by definition of $\bar{\rho}$, $\lambda \,\rho\, \mu$. We have therefore shown that

$$\mu \,\bar{\rho}\, M, \text{ and } \lambda \,\rho\, \mu \text{ whenever } \lambda \,\rho\, M \text{ for } \lambda \in L^X.$$

The same sequence of steps appropriately relabelled establishes

$$M \,\bar{\rho}\, \nu \text{ and } \nu \,\rho\, \lambda \text{ whenever } M \,\rho\, \lambda \text{ for } \lambda \in L^X.$$

■

We now state and prove Katetov's second lemma.

II.1.12 LEMMA

Let ρ be a binary relation on X possessing properties (I) and (L). Let Y be a countable set and let ξ be a transitive, irreflexive relation on Y. Further let ρ^ξ be the induced relation described in (II.1.1(i)). Let f, g $\in X^Y$ such that

(i) $f \rho^\xi g$

(ii) If $y_o \in Y$ and $U = \{f(y) : y \xi y_o\}$, $V = \{g(y)\ y_o \xi y\}$, then
$U \bar{\rho} f(y_o)$ and $g(y_o) \bar{\rho} V$.

Then there exists $h \in X^Y$ such that

$$f \rho^\xi h,\ h \rho^\xi h,\ h \rho^\xi g.$$

PROOF :

The set Y is countable, so we can arrange the elements of Y in a sequence $\{y_n : n \in \mathbb{N}\}$. Let $M = \{f(y) : y \xi y_1\}$, $N = \{g(y) : y_1 \xi y\}$. Then by (i) and (ii)

$$M \bar{\rho} f(y_1),\ f(y_1) \rho N,\ M \rho g(y_1),\ g(y_1) \bar{\rho} N. \tag{1}$$

An application of (I) will yield $x_1 \in X$ such that

$$M \rho x_1,\ x_1 \rho N. \tag{2}$$

Set $h(y_1) = x_1$. We then have

$$y \xi y_1 \text{ implies } f(y) \rho h(y_1), \tag{3}$$

$$y_1 \xi y \text{ implies } h(y_1) \rho g(y),$$

for $y \in Y$.

Suppose that for $k = 1,\cdots,n$, $h(y_k)$ has been defined such that

$$(T_n)\quad \begin{array}{ll} y \xi y_k,\ k = 1,\cdots,n & \text{implies } f(y) \rho h(y_k) \\ y_k \xi y,\ b = 1,\cdots,n & \text{implies } h(y_k) \rho g(y), \\ y_i \xi y_j,\ i,j \in [1\cdot\cdot n] & \text{implies } h(y_i) \rho h(y_j), \end{array}$$

for $y \in Y$.

The relation ξ is irreflexive. Thus $y_1 \not\xi\, y_1$. Therefore by the latter and (3), (T_n) holds for $n = 1$.

Consider the following sets

$$M_{n+1} = \{h(y_k) : k = 1,\cdots,n \text{ and } y_k \xi y_{n+1}\} \cup \{f(y) : y \xi y_{n+1}\}$$
$$N_{n+1} = \{h(y_k) : k = 1,\cdots,n \text{ and } y_{n+1} \xi y_k\} \cup \{g(y) : y_{n+1} \xi y\}$$
$$A_{n+1} = \{h(y_k) : k = 1,\cdots,n \text{ and } y_k \xi y_{n+1}\} \cup \{f(y_{n+1})\}$$
$$B_{n+1} = \{h(y_k) : k = 1,\cdots,n \text{ and } y_{n+1} \xi y_k\} \cup \{g(y_{n+1})\}.$$

Using the fact that ξ is transitive and (T_n), (i) we have

$$M_{n+1}\, \rho\, B_{n+1},\ A_{n+1}\, \rho\, N_{n+1} \tag{4}$$

Applying (L) to the finite sets A_{n+1}, B_{n+1} there exists $x, y \in X$ such that

$$A_{n+1}\, \bar{\rho}\, x \text{ and } x\, \rho\, z \text{ whenever } A_{n+1}\, \rho\, z, \tag{5}$$
$$y\, \bar{\rho}\, B_{n+1} \text{ and } z\, \rho\, y \text{ whenever } z\, \rho\, B_{n+1},$$

for $z \in X$.

Using part (ii) of the hypothesis of the lemma we have

$$\{f(y) : y \xi y_{n+1}\}\, \bar{\rho}\, f(y_{n+1}) \text{ and } g(y_{n+1})\, \bar{\rho}\, \{g(y) : y_{n+1} \xi y\}.$$

Now $f(y_{n+1}) \in A_{n+1}$ and $A_{n+1}\, \bar{\rho}\, x$. Thus by the transitivity of $\bar{\rho}$ we have $\{f(y) : y \xi y_{n+1}\}\, \bar{\rho}\, x$, since $f(y_{n+1})\, \bar{\rho}\, x$. Since $M_{n+1} \subseteq A_{n+1} \cup \{f(y) : y \xi y_{n+1}\}$ we have $M_{n+1}\, \bar{\rho}\, x$.

Similarly since $\bar{\rho}$ is transitive, $g(y_{n+1}) \in B_{n+1}$ and $y\, \bar{\rho}\, B_{n+1}$ we have that $y\, \bar{\rho}\, \{g(y) : y_{n+1} \xi y\}$. We also have $N_{n+1} \subseteq B_{n+1} \cup \{g(y) : y_{n+1} \xi y\}$. Hence $y\, \bar{\rho}\, N_{n+1}$. Combining (4) and (5), $M_{n+1}\, \bar{\rho}\, x$ and $y\, \bar{\rho}\, N_{n+1}$ we have

$$M_{n+1}\, \bar{\rho}\, x,\ x\, \rho\, N_{n+1},\ M_{n+1}\, \rho\, y,\ y\, \bar{\rho}\, N_{n+1}. \tag{6}$$

Applying (I) to (6) there exists $x_{n+1} \in X$ such that

$$M_{n+1}\, \rho\, x_{n+1},\ x_{n+1}\, \rho\, N_{n+1}. \tag{7}$$

Setting $h(y_{n+1}) = x_{n+1}$, we see that (T_{n+1}) holds. We have shown that (T_1) is true and that if (T_n) holds for $n \in \mathbb{N}$, then (T_{n+1}) holds. Therefore by induction (T_n) holds for all $n \in \mathbb{N}$. The function

$$h : Y \longrightarrow X \, , \, h(y_n) = x_n.$$

so constructed satisfies

$$f \, \rho^{\xi} \, h, \; h \, \rho^{\xi} \, h, \; h \, \rho^{\xi} \, g.$$

■

II.2 EXTENDING I(L)–VALUED L–UNIFORMLY CONTINUOUS FUNCTIONS.

Let $(X, \mathscr{D})$ be an L–uniform space and $Y \subseteq X$. Let

$$f : (Y, \mathscr{D}_Y) \longrightarrow I(L) \tag{2.1}$$

be an L–uniformly continuous function. In this section we shall use the results of (II.1) to construct an L–uniformly continuous function

$$F : (X, \mathscr{D}) \longrightarrow I(L)$$

such that $F_{|Y} = f$.

We first prove a lemma which will simplify matters greatly.

For uniform spaces we have the following :

Let $(Z, \mathbb{D})$ be a uniform space and $g: Z \longrightarrow I$. Then g is uniformly continuous if and only if for each $\epsilon > 0$ there exists $V_\epsilon \in \mathbb{D}$ such that

$$V_\epsilon(g^{-1}([t,1])) \subseteq g^{-1}((t-\epsilon,1]) \quad \text{for each } t \in I.$$

This translates into L–uniform spaces as

II.2.2 LEMMA

Let $(V, \mathscr{V})$ be an L–uniform space. Then $g : (V, \mathscr{V}) \longrightarrow I(L)$ is L–uniformly continuous if and only if for each $\epsilon > 0$, there exists $D_\epsilon \in \mathscr{V}$ such that

$$D_\epsilon(g^{\leftarrow}(L'_t)) \leq g^{\leftarrow}(R_{t-\epsilon}) \quad \text{for all } t \in \mathbb{R}.$$

PROOF

Since $\{B_\epsilon, B_\epsilon^{-1} : \epsilon > 0\}$ forms a subbase for the natural L–uniformity on I(L) and $(g^{\leftarrow}(B_\epsilon))^{-1} = g^{\leftarrow}(B_\epsilon^{-1})$ for $\epsilon > 0$, we have by (I.3.7) :

> $g : (V, \mathscr{V}) \rightarrow I(L)$ is L–uniformly continuous if and only if for each $\epsilon > 0$ there exists $D_\epsilon \in \mathscr{V}$ such that $D_\epsilon \leq g^{\leftarrow}(B_\epsilon)$.

($\Rightarrow$) : Let $\epsilon > 0$, then there exists $D_\epsilon \in \mathscr{V}$ such that $D_\epsilon \leq g^{\leftarrow}(B_\epsilon)$.

Consider the L–set $g^{\leftarrow}(L'_t)$, $t \in \mathbb{R}$. We will show that

$$D_\epsilon(g^{\leftarrow}(L'_t)) \leq g^{\leftarrow}(R_{t-\epsilon}) \text{ for all } t \in \mathbb{R}. \tag{1}$$

$$\begin{aligned} \text{Now } g^{\leftarrow}(B_\epsilon)(g^{\leftarrow}(L'_t)) &= g^{\leftarrow}(B_\epsilon(g^{\rightarrow}(g^{\leftarrow}(L'_t)))) \\ &\leq g^{\leftarrow}(B_\epsilon(L'_t)) \\ &\leq g^{\leftarrow}(R_{t-\epsilon}). \end{aligned}$$

The last inequality cannot be replaced by equality since if $t > 1$ then $B_\epsilon(L'_t) = \underline{0}$ and there may exists $v \in V$ that

$$g^{\leftarrow}(R_{t-\epsilon})(v) = g(v)((t-\epsilon)+) \neq 0.$$

Thus $D_\epsilon(g^{\leftarrow}(L'_t)) \leq g^{\leftarrow}(B_\epsilon)(g^{\leftarrow}(L'_t)) \leq g^{\leftarrow}(R_{t-\epsilon})$. Hence (1) follows.

($\Leftarrow$) : Let $\epsilon > 0$ then there exists $D_\epsilon \in \mathscr{V}$ such that $D_\epsilon(g^{\leftarrow}(L'_t)) \leq g^{\leftarrow}(R_{t-\epsilon})$ for all $t \in \mathbb{R}$. We now claim that

$$D_\epsilon(\mu) \leq g^{\leftarrow}(B_\epsilon)(\mu) \text{ for all } \mu \in L^V. \tag{2}$$

Let $\mu \in L^V$. If $\mu = \underline{0}$ then $D_\epsilon(\mu) = D_\epsilon(\underline{0}) = \underline{0}$ and (2) holds trivially. Suppose therefore that $\mu \neq \underline{0}$. Let

$$A_\mu = \{r \in \mathbb{R} : g^{\rightarrow}(\mu) \leq L'_r\}. \tag{3}$$

Since $\mu \neq \underline{0}$ we have $\sup A_\mu \in \mathbb{R}$. In fact for $\mu \neq \underline{0}$, $g^{\rightarrow}(\mu) \leq L'_r$ fails for $r > 1$. We also have that

$$g^{\rightarrow}(\mu) \leq \bigwedge_{r \in A_\mu} L'_r = L'_{\sup A_\mu}. \tag{4}$$

Thus $\sup A_\mu \in A_\mu$ and $\mu \leq g^{\leftarrow}(L'_{\sup A_\mu})$. Now

$$\begin{aligned} D_\epsilon(\mu) &\leq D_\epsilon(g^{\leftarrow}(L'_{\sup A_\mu})) \\ &\leq g^{\leftarrow}(R_{\sup A_\mu - \epsilon}) \\ &= g^{\leftarrow}(B_\epsilon(g^{\rightarrow}(\mu))) \\ &= g^{\leftarrow}(B_\epsilon)(\mu) \end{aligned} \tag{5}$$

We have from (5) that (2) holds for all $\epsilon > 0$. This proves the L–uniform continuity of g.

■

We now return to (2.1). Since $f : (Y, \mathscr{D}_Y) \longrightarrow I(L)$ is L–uniformly continuous, we have by Lemma II.2.2 that for each $\epsilon > 0$ there exists $E_\epsilon \in \mathscr{D}_Y$ such that $E_\epsilon(f^{\leftarrow}(L'_t)) \leq f^{\leftarrow}(R_{t-_\epsilon})$ for all $t \in \mathbb{R}$. Recall from (I.4.5) that $\mathscr{B}_Y = \{D_Y : D \in \mathscr{D}\}$ forms a base for $\mathscr{D}_Y$. Thus for each $E \in \mathscr{D}_Y$ there exists $D \in \mathscr{D}$ such that $D_Y \leq E$. We therefore have that for each $\epsilon > 0$ there exists $D_\epsilon \in \mathscr{D}$ such that $(D_\epsilon)_Y (f^{\leftarrow}(L'_t)) \leq f^{\leftarrow}(R_{t-_\epsilon})$ for all $t \in \mathbb{R}$. In addition we always have $f^{\leftarrow}(R_t) \leq f^{\leftarrow}(L'_t)$. Hence $(D_\epsilon)_Y(f^{\leftarrow}(R_t)) \leq f^{\leftarrow}(R_{t-_\epsilon})$ for $t \in \mathbb{R}$.

Collecting the results obtained in the above paragraph we have :
For each $\epsilon > 0$ there exists $D_\epsilon \in \mathscr{D}$ such that

$$(D_\epsilon)_Y (f^{\leftarrow}(R_t)) \leq f^{\leftarrow}(R_{t-_\epsilon}) \text{ for all } t \in \mathbb{R}. \tag{2.3}$$

Now $(D_\epsilon)_Y (f^{\leftarrow}(R_t)) = D_\epsilon(f^{\leftarrow}(R_t)_X)_Y$. If $\mu \in L^X$ and $\nu \in L^Y$ such that $\mu_Y \leq \nu$, then $\mu \leq \nu_X \vee 1_{X/Y}$. Thus $D_\epsilon (f^{\leftarrow}(R_t)_X) \leq f^{\leftarrow}(R_{t-_\epsilon})_X \vee 1_{X/Y}$ for all $\epsilon > 0$ and $t \in \mathbb{R}$. If we let $\nu_t = f^{\leftarrow}(R_t)_X \vee 1_{X/Y}$ and $\mu_t = f^{\leftarrow}(R_t)$ we have

$$D_\epsilon(\mu_t) \leq \nu_{t-_\epsilon} \text{ for all } \epsilon > 0 \text{ and } t \in \mathbb{R} \tag{2.4}$$

Writing this in terms of the relation $\rho_{\mathscr{D}}$ defined on L^X, (II.1.2),

$$\mu_t \, \rho_{\mathscr{D}} \, \nu_{t-_\epsilon} \text{ for all } \epsilon > 0 \text{ and } t \in \mathbb{R}. \tag{2.5}$$

Let Q_D be the set of dyadic rationals (each rational of the form $d = k/2^n$, $k = 0,1,\cdots,2^n$, $n \in \mathbb{N}$). Then for $d_1,d_2 \in Q_D$ such that $d_1 > d_2$, it follows that $\mu_{d_1} \rho_{\mathscr{D}} \nu_{d_2}$.

Define the functions Φ, $\Psi : Q_D \longrightarrow L^X$ by

$$\Phi(d) = \mu_d, \Psi(d) = \nu_d \tag{2.6}$$

II.2.7 PROPOSITION

For $d_1,d_2 \in Q_D$ such that $d_1 > d_2$,

(i) $\Phi(d_1) \rho_{\mathscr{D}} \Psi(d_2)$,

(ii) $\Phi(d_1) \leq \Phi(d_2)$, $\Psi(d_1) \leq \Psi(d_2)$.

PROOF

(i) $\Phi(d_1) = \mu_{d_1} \rho_{\mathscr{D}} \nu_{d_2} = \Psi(d_2)$.

(ii) Since $d_1 > d_2$ implies $R_{d_1} \leq R_{d_2}$, we have $f^{\leftarrow}(R_{d_1}) \leq f^{\leftarrow}(R_{d_2})$. Therefore $f^{\leftarrow}(R_{d_1})_X \leq f^{\leftarrow}(R_{d_2})_X$. It now follows that $\Phi(d_1) \leq \Phi(d_2)$ and $\Psi(d_1) \leq \Psi(d_2)$.

■

We now wish to show that Φ, Ψ satisfy the requirements of Katetov's insertion lemma (II.1.12). By the remarks preceding the proof of (II.1.11) we know that $\rho_{\mathscr{D}}$ satisfies (I) and (L). The set Q_D is countable and the relation $>$ on Q_D is irreflexive and transitive. It therefore remains to show that :

II.2.8 PROPOSITION

(i) $\Phi \overset{>}{\rho}_{\mathscr{D}} \Psi$.

(ii) If $d_o \in Q_D$ and $U = \{\Phi(d) : d > d_o\}$, $V = \{\Psi(d) : d_o > d\}$ then $U \bar{\rho}_{\mathscr{D}} \Phi(d_o)$ and $\Psi(d_o) \bar{\rho}_{\mathscr{D}} V$.

PROOF

(i) This we have already proved in (II.2.7(i)).

(ii) By (II.2.7(ii)) we have that for $d > d_o$, $\Phi(d) \leq \Phi(d_o)$ and for $d_o > d$, $\Psi(d_o) \leq \Psi(d)$. Since $\mu \leq \nu$ implies $\mu\ \bar{\rho}_{\mathscr{D}}\ \nu$, it follows that $U\ \bar{\rho}_{\mathscr{D}}\ \Phi(d_o)$ and $\Psi(d_o)\ \bar{\rho}_{\mathscr{D}}\ V$.

■

We can now apply Katetov's insertion lemma. This lemma yields $\Xi : Q_D \longrightarrow L^X$ such that :

$$\Phi\ \bar{\rho}_{\mathscr{D}}\ \Xi,\ \Xi\ \bar{\rho}_{\mathscr{D}}\ \Xi,\ \Xi\ \bar{\rho}_{\mathscr{D}}\ \Psi \tag{2.9}$$

Define $F : X \longrightarrow I(L)$ by

$$F(x)(t) = \begin{cases} 1 & t < 0 \\ \bigvee\limits_{\substack{t \leq d \\ d \in Q_D}} \Xi(d)(x) & 0 \leq t \leq 1 \\ 0 & 1 < t \end{cases} \tag{2.10}$$

for $t \in \mathbb{R}$.

II.2.11 PROPOSITION

$F_{|Y} = f$.

PROOF

It suffices to show that for each $y \in Y$, $F(y)(t+) = f(y)(t+)$ for all $t \in \mathbb{R}$. Since $f(y)(t+) = F(y)(t+) = 1$ for all $t < 0$, $f(y)(t+) = F(y)(t+) = 0$ for all $t \geq 1$, for each $y \in Y$ we need only consider $t \in [0,1)$.

Let $y \in Y$ and $t \in [0,1)$. Then

$$F(y)(t+) = \bigvee_{s > t} F(y)(s)$$

$$= \bigvee_{s > t} \bigvee_{\substack{d \geq s \\ d \in Q_D}} \Xi(d)(y) \tag{1}$$

$$= \bigvee_{\substack{d > t \\ d \in Q_D}} \Xi(d)(y)$$

Let $d \in Q_D$ such that $d > t$. Then since Q_D is dense in $[0,1]$ there exists $d_1, d_2 \in Q_D$ such that $d > d_1 > d_2 > t$. Since $\Phi \mathrel{\overset{>}{\rho}}_{\mathscr{D}} \Xi$, $\Xi \mathrel{\overset{>}{\rho}}_{\mathscr{D}} \Psi$ we have $\Phi(d) \, \rho_{\mathscr{D}} \, \Xi(d_1)$, $\Xi(d_1) \, \rho_{\mathscr{D}} \, \Psi(d_2)$. This implies that $\Phi(d) \leq \Xi(d_1) \leq \Psi(d_2)$.

Therefore :

$$\bigvee_{d > t} \Phi(d) \leq \bigvee_{d > t} \Xi(d) \leq \bigvee_{d > t} \Psi(d) \tag{2}$$

Recalling that $\Phi(d) = f^{\leftarrow}(R_d)_X$ and $\Psi(d) = f^{\leftarrow}(R_d) \vee 1_{X/Y}$, we see that, for $z \in Y$,

$$\Phi(d)(z) = f(z)(d+) = \Psi(d)(z) \text{ for all } d \in Q_D. \tag{3}$$

Then we have that

$$f(y)(t+) = \bigvee_{\substack{d > t \\ d \in Q_D}} f(y)(d+) = \bigvee_{\substack{d > t \\ d \in Q_D}} \Phi(d)(y) = \bigvee_{\substack{d > t \\ d \in Q_D}} \Psi(d)(y) \tag{4}$$

Thus combining (1), (2), (3) and (4) we have

$$f(y)(t+) = \bigvee_{\substack{d > t \\ d \in Q_D}} \Xi(d)(y) = F(y)(t+)$$

∎

II.2.12 PROPOSITION

$F : (X, \mathscr{D}) \longrightarrow I(L)$ is L–uniformly continuous.

PROOF

By Lemma II.2.2, it suffices to show that for each $\epsilon > 0$, there exists $D_\epsilon \in \mathscr{D}$ such that $D_\epsilon(F^{\leftarrow}(L'_t)) \leq F^{\leftarrow}(R_{t-_\epsilon})$ for all $t \in \mathbb{R}$. To show this it suffices to show that for each $n \in \mathbb{N}$ there exists $D_{1/2^{n+1}} \in \mathscr{D}$ such that

$$D_{1/2^{n+1}}(F^{\leftarrow}(L'_t)) \leq F^{\leftarrow}(R_{t-1/2^{n-1}}) \text{ for all } t \in \mathbb{R}. \quad (1)$$

Let $n \geq 1$. Since $\Xi \gtrsim_{\rho\, \mathscr{D}} \Xi$, for $k = 0,1,\cdots,2^{n+1}-1$ there exists $D_k \in \mathscr{D}$ such that

$$D_k(\Xi(k+1/2^{n+1})) \leq \Xi(k/2^{n+1}) \quad (2)$$

Let $D_{1/2^{n+1}} = \mathop{\Delta}\limits_{k=0}^{2^{n+1}-1} D_k$. We have $D_{1/2^{n+1}} \in \mathscr{D}$. We now establish (1).

If $t > 1$, then $F^{\leftarrow}(L'_t) = \underline{0}$ and since $D_{1/2^{n+1}}(\underline{0}) = \underline{0}$, (1) holds for $t > 1$.
If $t < 1/2^{n-1}$, then $F^{\leftarrow}(R_{t-1/2^{n-1}}) = \underline{1}$, and (1) holds trivially. We need therefore only consider $t \in [1/2^{n-1}, 1]$.
Let $t \in [1/2^{n-1}, 1]$ and $x \in X$. Then $F^{\leftarrow}(L'_t)(x) = F(x)(t-) = \bigwedge_{s>t} F(x)(s)$

But Q_D is dense in $[0,1]$. Thus

$$F(x)(t-) = \bigwedge_{s>t} F(x)(s) = \bigwedge_{\substack{d>t \\ d \in Q_D}} F(x)(d). \quad (2)$$

By definition $F(x)(d) = \bigvee_{\substack{d' \geq d \\ d' \in Q_D}} \Xi(d')(x)$. And since $\Xi \gtrsim_{\rho\, \mathscr{D}} \Xi$, we have $\Xi(d') \leq \Xi(d)$ for all $d' \in Q_D$ satisfying $d' \geq d$.

Therefore $F(x)(d) = \Xi(d)(x)$. Hence $\bigwedge_{s>t} F(x)(s) = \bigwedge_{\substack{d>t \\ d \in Q_D}} \Xi(d)(x)$.

All this implies that

$$F^{\leftarrow}(L'_t) = \bigwedge_{\substack{d>t \\ d \in Q_D}} \Xi(d). \quad (3)$$

Let $k_o = \max\{k : k \in [1 \cdot\cdot 2^{n+1} - 1] \text{ and } {}^{k}/_{2^{n+1}} < t\}$, since $t \in [\,{}^{1}/_{2^{n-1}}, 1]$

we have $k_o \geq 3$ and $t - {}^{1}/_{2^{n-1}} < {}^{k_o - 1}/_{2^{n+1}}$. The latter follows since

$t - {}^{k_o}/_{2^{n+1}} \leq {}^{1}/_{2^{n+1}}$, , which implies $t - {}^{1}/_{2^{n-1}} \leq {}^{k_o-3}/_{2^{n+1}} < {}^{k_o-1}/_{2^{n+1}}$.

Then $F^{\leftarrow}(L'_t) = \bigwedge_{d > t} \Xi(d) \leq \Xi(\,{}^{k_o}/_{2^{n+1}})$. We have by construction of $D_{1/_{2^{n+1}}}$:

$$D_{1/_{2^{n+1}}}(F^{\leftarrow}(L'_t)) \leq D_{k_o}(\,\Xi(\,{}^{k_o}/_{2^{n+1}})) \leq \Xi(\,{}^{k_o^{-1}}/_{2^{n+1}}). \tag{4}$$

Since $t - {}^{1}/_{2^{n-1}} < {}^{k_o-1}/_{2^{n+1}}$, we also have

$$\Xi(\,{}^{k_o-1}/_{2^{n+1}}) \leq \bigvee_{\substack{d > t - {}^{1}/_{2^{n-1}} \\ d \in Q_D}} \Xi(d)$$

$$= F^{\leftarrow}(R_{t - {}^{1}/_{2^{n-1}}}). \tag{5}$$

Combining (4) and (5) we have :

$$D_{1/_{2^{n+1}}}(F^{\leftarrow}(L'_t)) \leq F^{\leftarrow}(R_{t - {}^{1}/_{2^{n-1}}}). \tag{6}$$

We have now established (1) and this concludes the proof.

■

We can now state the results of this section in the following theorem.

II.3.13 THEOREM

Let $(X, \mathscr{D})$ be an L–uniform space and $Y \subseteq X$. Then if $f : (Y, \mathscr{D}_Y) \longrightarrow I(L)$ is L–uniformly continuous there exists an L–uniformly continuous function

$$F : (X, \mathscr{D}) \longrightarrow I(L)$$

such that $$F_{|X} = f.$$

■

CHAPTER III

L–UNIFORMIZABILITY AND L–COMPLETE REGULARITY

INTRODUCTION

In the opening section, we first consider a lemma of Hutton's ([6]) which no doubt motivated his definition of L–complete regularity. In addition we show that the class of L–completely regular spaces is, as in the topological case, the class of L–spaces having enough I(L)–valued L–continuous functions to determine their L–topologies completely. All material in this section is that of Hutton.

The second section deals mainly with products of L–completely regular spaces. Our proof of the fact that products of L–completely regular spaces are L–completely regular is based on that of Liu ([15]). In this chapter the lattice L will be assumed to be completely distributive and X a non–empty set.

III.1 : A DEFINITION OF L–COMPLETE REGULARITY AND A CHARACTERISATION OF L–UNIFORMIZABILITY.

We begin immediately with Hutton's motivation of his definition of L–complete regularity.

III.1.1 LEMMA

Let $(X, \mathscr{D})$ be an L–uniform space and $D \in \mathscr{D}$. Suppose $D(\mu) \leq \nu$ for $\mu, \nu \in L^X$. Then there exists an L–uniformly continuous function $f\colon X \longrightarrow I(L)$ such that

$$\mu \leq f^{\leftarrow}(L_1') \leq f^{\leftarrow}(R_0) \leq \nu.$$

PROOF

Before we begin, we establish some notation. For $D_n \in \mathscr{D}$ indexed by $n \in \mathbb{N}$, we take $\lambda_1 <_n \lambda_2$ for $\lambda_1, \lambda_2 \in L^X$ to mean $D_n(\lambda_1) \leq \lambda_2$.

Let $\mu_0 = \nu$, $\mu_1 = \mu$ and $D_0 = D$.

Since $\mathscr{D}$ is an L–uniformity there exists $D_1 \in \mathscr{D}$ such that $D_1 \circ D_1 \leq D_0$. Set $\mu_{1/2} = D_1(\mu_1)$, then

$$\mu_1 <_1 \mu_{1/2} <_1 \mu_0.$$

Continuing, there exists $D_2 \in \mathscr{D}$ such that $D_2 \circ D_2 \leq D_1$. Setting $\mu_{1/4} = D_2(\mu_{1/2})$, $\mu_{3/4} = D_2(\mu_1)$, we have

$$\mu_1 <_2 \mu_{3/4} <_2 \mu_{1/2} <_2 \mu_{1/4} <_2 \mu_0.$$

Therefore by induction, for $n \in \mathbb{N}$:

there exists $D_n \in \mathscr{D}$ such that $D_n \circ D_n \leq D_{n-1}$ and a collection $\{\mu_{p/2^n} : p = 0,1,\cdots,2^n\}$ satisfying

$$\mu_1 <_n \mu_{2^n-1/2^n} <_n \cdots <_n \mu_{1/2^n} <_n \mu_0.$$

By replacing D_n by $D_{1/2^n}$ and setting $\mu_{-1/2^n} = \underline{1}$ for each $n \in \mathbb{N}$, we have for $p = 0,1,\cdots,2^n$

$$D_{1/2^n}(\mu_{p/2^n}) \leq \mu_{p-1/2^n}.$$

Denoting the set of dyadic rationals by Q_D we define the function $f : X \longrightarrow I(L)$ by

$$f(x)(t) = \begin{cases} 1 & t < 0 \\ \bigvee_{\substack{t \leq d \\ d \in Q_D}} \mu_d(x) & 0 \leq t \leq 1 \\ 0 & 1 < t \end{cases}$$

From the definition of f

$$\mu(x) = \mu_1(x) \leq f(x)(t) \leq \mu_0(x) = \nu(x) \quad \text{for } 0 \leq t \leq 1,\ x \in X.$$

Therefore

$$\mu \leq f^{\leftarrow}(L'_1) \leq f^{\leftarrow}(R_0) \leq \nu.$$

To show that f is L–uniformly continuous, we use Lemma II.2.2. According to this lemma we need only show that for each $d \in Q_D / \{0\}$ there exists $D \in \mathscr{D}$ such that

$$D(f^{\leftarrow}(L'_t)) \leq f^{\leftarrow}(R_{t-d}) \text{ for each } t \in \mathbb{R}.$$

Each $d \in Q_D / \{0\}$ can be written in the form $p/2^n$ where $p \geq 3$, $n \geq 2$.

With the latter condition on p and n, we claim that

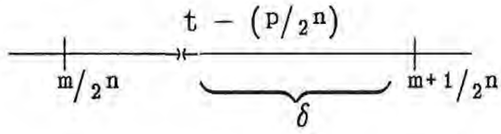
$$D_{1/2^n}(f^{\leftarrow}(L'_t)) \leq f^{\leftarrow}(R_{t-(p/2^n)}) \text{ for } t \in \mathbb{R}.$$

To prove our claim we must consider four cases.

CASE 1 : $p/2^n \leq t < 1$

Choose m such that $m/2^n \leq t - (p/2^n) < m+1/2^n$. Then $m+p/2^n \leq t$. But since $p \geq 3$ and $t < 1$, we have $m+3/2^n \leq m+p/2^n \leq t < 1$.

Consider the following sketch :



From the sketch we see that $0 < \delta \leq 1/2^n$ and $t - (p/2^n) + \delta + 1/2^n = m+2/2^n$.
Since $m+2/2^n < m+p/2^n \leq t$, we have $t-(p/2^n) + \delta + 1/2^n < t$.

By the construction of $\{D_{1/2^n} : n \in \mathbb{N}\}$ and $\{\mu_d : d \in Q_D\}$,

$$D_{1/2^n}(\mu_{t-(p/2^n)+\delta+1/2^n}) \leq \mu_{t-(p/2^n)+\delta}$$

But

$$\mu_{t-(p/2^n)+\delta} \leq \bigvee_{d > t-(p/2^n)} \mu_d = f^{\leftarrow}(R_{t-(p/2^n)})$$

and

$$f^{\leftarrow}(L'_t) = \bigwedge_{d < t} \mu_d \leq \mu_{t-(p/2^n)+\delta+1/2^n}.$$

Hence

$$D_{1/2^n}(f^{\leftarrow}(L'_t)) \leq D_{1/2^n}(\mu_{t-(p/2^n)+\delta+1/2^n}) \leq \mu_{t-(p/2^n)+\delta} \leq f^{\leftarrow}(R_{t-(p/2^n)})$$

CASE 2 : $t < p/2^n$.

If $t < p/2^n$, then $t-(p/2^n) < 0$. But $f^{\leftarrow}(R_s) = \underline{1}$ for $s < 0$. Hence

$D_{1/2^n}(f^{\leftarrow}(L'_t)) \leq f^{\leftarrow}(R_{t-(p/2^n)})$ holds trivially.

CASE 3 : $t = 1$

By imposing the condition $n \geq 2$, we have ensured that $0 < 1 - 3/2^n$.
Since

$$f^{\leftarrow}(L'_1) = \bigwedge_{d < 1} \mu_d$$

and

$$\bigvee_{1-(3/2^n) < d} \mu_d = f^{\leftarrow}(R_{1-(3/2^n)}) \leq f^{\leftarrow}(R_{1-(p/2^n)}) ,$$

we have

$$D_{1/2^n}(f^{\leftarrow}(L'_1)) \leq D_{1/2^n}(\mu_{2^n-1/2^n}) \leq \mu_{2^n-2/2^n} \leq f^{\leftarrow}(R_{1-(3/2^n)}) \leq f^{\leftarrow}(R_{1-(p/2^n)}).$$

CASE 4 : t > 1.

When $t > 1$, then $f^{\leftarrow}(L'_t) = \underline{0}$. Therefore $D_{1/2^n}(f^{\leftarrow}(L'_t)) = \underline{0}$ and the inequality $D_{1/2^n}(f^{\leftarrow}(L'_t)) \leq f^{\leftarrow}(R_{t-(p/2^n)})$ holds trivially.

■

We can now prove

III.1.2 PROPOSITION

Let $(X, \mathscr{D})$ be an L–uniform space and μ be an open L–set in the L–topology $\tau_{\mathscr{D}}$, generated by $\mathscr{D}$. Then there exists a collection $\{\mu_j : j \in J\}$ of L–sets and a corresponding collection $\{f_j : X \longrightarrow I(L) : j \in J\}$ of L–continuous functions which satisfy

(a) $\quad \vee \mu_j = \mu$,

(b) $\quad \mu_j \leq f_j^{\leftarrow}(L'_1) \leq f_j^{\leftarrow}(R_0) \leq \mu$.

PROOF

Let $A_\mu = \{\nu \in L^X : \text{there exists } D \in \mathscr{D} \text{ such that } D(\nu) \leq \mu\}$. Since $\mu \in \tau_{\mathscr{D}}$ we have $\mu = \bigvee_{\nu \in A_\mu} \nu$. By Lemma III.1.1, for each $\nu \in A_\mu$, there exists an L–continuous function $f_\nu : X \longrightarrow I(L)$ satisfying

$$\nu \leq f_\nu^{\leftarrow}(L'_1) \leq f_\nu^{\leftarrow}(R_0) \leq \mu.$$

■

Hutton then went on to define L–complete regularity.

III.1.3 DEFINITION

Let (X,τ) be an L–space. Then X is L–completely regular if for each $\mu \in \tau$, there exists a collection $\{\nu_j : j \in J\}$ of L–sets and a corresponding collection $\{f_j : X \longrightarrow I(L) : j \in J\}$ of I(L)–valued L–continuous functions which satisfy

(a) $\bigvee_j \nu_j = \mu$,

(b) $\nu_j \leq f_j^{\leftarrow}(L_1') \leq f_j^{\leftarrow}(R_0) \leq \mu$ for $j \in J$.

■

We note that when $L = \{0,1\}$, then Definition III.1.3 is equivalent to the usual complete regularity. Let (X,Δ) be a topological space, that is a $\{0,1\}$–topology, which is $\{0,1\}$–completely regular. For any closed subset $A \subseteq X$, there exists a collection of subsets $\{B_j : j \in J\}$ and continuous functions $\{f_j : X \longrightarrow I : j \in J\}$ such that

(a) $\bigcup_j B_j = X/A$,

(b) $B_j \subseteq f_j^{-1}(\{1\}) \subseteq f_j^{-1}((0,1]) \subseteq X/A$.

Therefore for each $x \notin A$ there exists a continuous function $f : X \longrightarrow I$ such that $f(A) = \{0\}$ and $f(x) = 1$.

Conversely suppose that (X,Δ) is completely regular. Let G be an open subset. For each $x \notin X/G$ there exists a continuous function $f_x : X \longrightarrow I$ such that $f_x(x) = 1$ and $f_x(X/G) = 0$. Then $x \in f_x^{-1}(\{1\}) \subseteq f^{-1}((0,1]) \subseteq G$. Conditions (a) and (b) of $\{0,1\}$–complete regularity are then satisfied by $\{\{x\} : x \in G\}$ and $\{f_x : X \longrightarrow I : x \in G\}$.

III.1.4 DEFINITION

Let (X,τ) be an L–space. Then the collection $\{f : (X,\tau) \longrightarrow I(L) : f \text{ is L–continuous}\}$ will be denoted by $C(X,I(L))$. The weak L–topology generated by $C(X,I(L))$ will be denoted by $\tau_{C(X,I(L))}$.

■

From the definition of L–complete regularity, it follows that if (X,τ) is L–completely regular then $\tau_{C(X,I(L))} = \tau$. The converse is true and will be proved later.

From (I.4.6) together with the fact that $\tau_{C(X,I(L))} \subseteq \tau$ we have

III.1.5 PROPOSITION

The subset $\mathscr{B}(X) \subseteq \mathscr{H}_L(X)$, where

$$\mathscr{B}(X) = \{ \mathop{\Delta}_{i=1}^{n} f_i^{\leftarrow}(B_{\epsilon_i} \Delta B_{\epsilon_i}^{-1}) : \epsilon_i > 0, n \in \mathbb{N} \text{ and } f_i \in C(X,I(L))\}$$

is an L–uniform base on X and $\tau_{C(X,I(L))} = \tau_{\mathscr{B}(X)} \subseteq \tau$.

■

We have now prepared the way to prove a characterisation of L–uniformizability. In general topology two criteria for uniformizability of a topology are that the topology be completely regular or be given by the weak topology generated by the collection of real valued, bounded continuous functions. By replacing the unit interval by the L–unit interval we have exactly the same criteria for the L–uniformizability of L–topologies.

III.1.6 PROPOSITION

Let (X,τ) be an L–space. Then the following are equivalent :

(i) (X,τ) is L–uniformizable,

(ii) (X,τ) is L–completely regular,

(iii) (X,τ) has the weak L–topology generated by C(X,I(L)).

PROOF

We have already (i) ⟹ (ii) ⟹ (iii). Now if $\tau_{C(X,I(L))} = \tau$, then by (III.1.5) $\tau = \tau_{\mathscr{B}(X)}$. Thus (X,τ) is L–uniformizable. This proves that (iii) ⟹ (i).

■

III.2 : SUBSPACES AND PRODUCTS OF L–COMPLETELY REGULAR SPACES

We now look at some basic questions about subspaces and products of L–completely regular spaces.

III.2.1 PROPOSITION

Let (X,τ) be an L–completely regular space and $A \subseteq X$. Then the induced L–space (A,τ_A) is L–completely regular.

■

In (II.4.4) we showed that a subspace of an L–uniformizable space is uniformizable. Thus the above result follows as a consequence of (III.1.6). We shall nevertheless give an explicit proof below.

PROOF OF (III.2.1)

Let $\lambda \in \tau_A$, then λ is of the form $\lambda = \mu_{|A}$, $\mu \in \tau$. Since (X,τ) is L–completely regular there exists a collection $\{\nu_j \in L^X : j \in J\}$ of L–sets and a corresponding collection $\{f_j : X \longrightarrow I(L) : j \in J\}$ of L–continuous functions such that

(a) $\quad \bigvee_{j \in J} \nu_j = \mu$,

(b) $\quad \nu_j \le f_j^{\leftarrow}(L_1') \le f_j^{\leftarrow}(R_0) \le \mu$ for $j \in J$.

Now $\bigvee_j \nu_{j|A} = \lambda$, and for each $j \in J$

$\nu_{j|A} \le (f_{j|A})^{\leftarrow}(L_1') \le (f_{j|A})^{\leftarrow}(R_0) \le \lambda$

and $f_{j|A} \in C(A,I(L))$. This proves that (A,τ_A) is L–completely regular.

■

III.2.2 PROPOSITION

Let $\{(X_j,\tau_j) : j \in J\}$ be a collection of L–completely regular spaces indexed by the set J. Then $(\prod_j X_j,\tau)$, where τ denotes the product L–topology, is an L–completely regular space.

■

Again the above result follows as a consequence of (III.1.6). In (I.4.7) we show that the product of a collection of L–uniformizable spaces is L–uniformizable. Hence the above result follows from (III.1.6). We give an explicit proof below.

PROOF OF (III.2.2)

Since L is a completely distributive lattice, the set

$$\{ \bigwedge_{i=1}^{n} \pi_{j_i}^{\leftarrow}(\mu_{j_i}) : n \in \mathbb{N}, j_i \in J \text{ and } \mu_{j_i} \in \tau_{j_i}\}$$

of L–sets forms a base for the product L–topology τ on $\prod_j X_j$. Let

$\mu = \bigwedge_{i=1}^{n} \pi_{j_i}^{\leftarrow}(\mu_{j_i})$ be a basic open L–set. For each j_i there exists sets

$\{\nu_{j_ik} : k \in K_i\} \subseteq L^{X_{j_i}}$ and $\{f_{j_ik} : k \in K_i\} \subseteq C(X_{j_i}, I(L))$ satisfying

(a) $\bigvee_k \nu_{j_ik} = \mu_{j_i}$,

(b) $\nu_{j_ik} \le (f_{j_ik}^{\leftarrow}(L_1')) \le (f_{j_ik}^{\leftarrow}(R_0)) \le \mu_{j_i}$ for $k \in K_i$.

By the complete distributive law

$$\begin{aligned}\mu &= \bigwedge_{i=1}^{n} \pi_{j_i}^{\leftarrow}\Big(\bigvee_{k \in K_i} \nu_{j_ik}\Big) \\ &= \bigwedge_{i=1}^{n} \bigvee_{k \in K_i} \pi_{j_i}^{\leftarrow}(\nu_{j_ik}) \\ &= \bigvee_{\varphi \in \prod_{i=1}^{n} K_i} \bigwedge_{i=1}^{n} \pi_{j_i}^{\leftarrow}(\nu_{j_{i\varphi(i)}})\end{aligned}$$

We now show that for each $\varphi \in \prod_{i=1}^{n} K_i$ there exists an L–continuous function

$f : \prod_j X_j \longrightarrow I(L)$ such that

$$\bigwedge_{i=1}^{n} \pi_{j_i}^{\leftarrow}(\nu_{j_{i\varphi(i)}}) \le f^{\leftarrow}(L_1') \le f^{\leftarrow}(R_0) \le \mu$$

This will complete the proof of the L–complete regularity of $(\prod_j X_j, \tau)$.

Let $\varphi \in \prod_{i=1}^{n} K_i$. For each $\varphi_{(i)}$ the function

$$f_{j_{i\varphi(i)}} : X_{j_i} \longrightarrow I(L)$$

satisfies

$$\nu_{j_{i\varphi(i)}} \le f_{j_{i\varphi(i)}}^{\leftarrow}(L_1') \le f_{j_{i\varphi(i)}}^{\leftarrow}(R_0) \le \mu_{j_i}.$$

Set $f_{\varphi(i)} = f_{j_{i\varphi(i)}} \circ \pi_{j_i}$. From (I.5.17) we have that the infima of a finite collection of L–continuous I(L)–valued functions is L–continuous. Thus since $f_{\varphi(i)}$ is the composition of two L–continuous functions, we have that $\bigwedge_{i=1}^{n} f_{\varphi(i)} \in C(\prod_j X_j, I(L))$. Furthermore we have that

$$(\bigwedge_{i=1}^{n} f_{\varphi(i)})^{\leftarrow}(L'_1) = \bigwedge_{i=1}^{n} f_{\varphi(i)}^{\leftarrow}(L'_1),$$

and

$$(\bigwedge_{i=1}^{n} f_{\varphi(i)})^{\leftarrow}(R_0) = \bigwedge_{i=1}^{n} f_{\varphi(i)}^{\leftarrow}(R_0).$$

Now

$$\begin{aligned}
\pi_{j_i}^{\leftarrow}(\nu_{j_{i\varphi(i)}}) &\leq \pi_{j_i}^{\leftarrow}(f_{j_{i\varphi(i)}}^{\leftarrow}(L'_1)) \\
&= f_{\varphi(i)}^{\leftarrow}(L'_1) \\
&\leq \pi_{j_i}^{\leftarrow}(f_{j_{i\varphi(i)}}^{\leftarrow}(R_0)) \\
&= f_{\varphi(i)}^{\leftarrow}(R_0)
\end{aligned}$$

for $i = 1,\cdots,n$.

Hence

$$\begin{aligned}
\bigwedge_{i=1}^{n} \pi_{j_i}^{\leftarrow}(\nu_{j_i}) &\leq \bigwedge_{i=1}^{n} (f_{\varphi(i)}^{\leftarrow}(L'_1)) \\
&= (\bigwedge_{i=1}^{n} f_{\varphi(i)})^{\leftarrow}(L'_1) \\
&\leq \bigwedge_{i=1}^{n} (f_{\varphi(i)}^{\leftarrow}(R_0)) \\
&= (\bigwedge_{i=1}^{n} f_{\varphi(i)})^{\leftarrow}(R_0) \leq \bigwedge_{i=1}^{n} \pi_{j_i}^{\leftarrow}(\mu_{j_i}) = \mu.
\end{aligned}$$

Thus $f = \bigwedge_{i=1}^{n} f_{\varphi(i)}$ is the required function.

■

CHAPTER IV

THE TOPOLOGICAL MODIFICATION OF L–COMPLETELY REGULAR L–SPACES AND THE TOPOLOGICALLY GENERATED L–TOPOLOGIES OF COMPLETELY REGULAR SPACES

INTRODUCTION

Lowen ([16]) observed that for any topological space (X,Δ) the collection of lower semicontinuous functions $f : X \longrightarrow I$, is closed under arbitary suprema and finite infima. Therefore this collection constitutes an I–topology on X. For an I–topological space (X,τ) Lowen also considered a weak topology generated by the collection τ of I–sets. The resulting topological space is called the topological modification of (X,τ). With varying success Kubiak ([11]) extended these concepts to more general types of lattices. These extended concepts are dealt with in Sections 1,2. Section 8 contains a counterexample which we have constructed, which shows that completeness is not a sufficient condition for many of the propositions in Section 2. In Section 3, using our extension of Katsaras's ([10]) construction of I–uniformities from uniformities to completely distributive lattices, we show that the topologically generated L–topology of a completely regular space is L–completely regular. Section 4 contains Kubiak's ([11]) result that the topological modification of the L–unit interval I(L), is a compact, Hausdorff topology. Using our own embedding lemma, developed in Section 5, we are able to reprove Liu's ([15]) result that L–Tychonoff spaces can be L–embedded in a cube of I(L). This is done is Section 6. This result allows us to prove that the topological modification of L–Tychonoff spaces are Tychonoff. Having extended a T_0–modification technique to L–spaces, we are able in Section 7 to prove that the topological modification of an L–completely regular space is completely regular.

IV.1 THE UPPER INTERVAL TOPOLOGY

Let (X,Δ) be a topological space. The collection

$$\text{lsc}(X,I) = \{f : X \longrightarrow I : f \text{ is lower semicontinuous}\}$$

of functions is closed under arbitary suprema and finite infima. Hence lsc(X,I) defines an I–topology on X.

To extend the concept of lower semicontinuity to an arbitary complete lattice L, we first need to define a topology on L which reduces to the upper interval topology when L = I. We shall also need to define topologies on complete lattices analogous to the lower interval topology and the interval topology on I. We make this the subject of our next definition.

IV.1.1 DEFINITION ([2] : II.1.8, VII.1.11)

Let L be a complete lattice

(i) The upper interval topology u(L) is the smallest topology for which all sets of the form $L/{\downarrow}\alpha$, $\alpha \in L$, are open.

(ii) The lower interval topology is defined to be $u(L^{op})$. That is the smallest topology for which all sets of the form $L/{\downarrow}^{op}\alpha = L/{\uparrow}\alpha$, $\alpha \in L$, are open.

(iii) The common refinement $u(L) \vee u(L^{op})$ of the upper and lower interval topologies is called the interval topology.

■

All these topologies occur naturally on complete lattices and reduce to the familiar topologies on the unit interval I when L = I. We make note of the following easily verified facts.

(a) The interval topologies on L and L^{op} coincide.
(b) The interval topology on L can be equivalently defined as the smallest topology for which all sets of the form

$$[\alpha,\beta] = \{\gamma \in L : \alpha \leq \gamma \leq \beta\},$$

are closed.

We are now able to extend the concept of lower semicontinuous I–valued functions from a topological space to a complete lattice.

IV.1.2 DEFINITION

Let (X,Δ) be a topological space. The collection

$$lsc(X,L) = \{f : X \to L : f \text{ is } (\Delta,u(L))\text{–continuous}\}$$

of functions is referred to as the collection of lower semicontinuous L–valued functions on X.

■

IV.1.3 DEFINITION ([1], II:1.10, III:1.5)

Let L be a continuous lattice

(i) The topology which has as a base

$$\mathscr{B}(L) = \{\Uparrow \alpha : \sigma \in L\}$$

is called the Scott topology and is denoted by $\sigma(L)$.

(ii) The common refinement $\sigma(L) \vee u(L^{op})$ of the Scott topology and the lower interval topology is called the Lawson topology and is denoted by $\lambda(L)$.

■

In fact the Scott topology is defined for any complete lattice L, and then it can be shown that when L is a continuous lattice $\mathscr{B}(L)$ forms a base for the Scott topology on L.

We shall now show that when L is completely distributive $u(L) = \sigma(L)$. This fact then provides us with $\mathscr{B}(L)$ as a base for $u(L)$.

In this section and in section 3 we shall need to know some of the topological properties of the interval topology on a complete lattice. To determine these properties we use the characterisation of ultrafilter convergence.

IV.1.4 LEMMA ([2]: VII.1.11)

Let L be a complete lattice and $\mathscr{F}$ a filter on the underlying set of L. Then

$$\bigvee_{F \in \mathscr{F}} (\wedge F) \leq \bigwedge_{F \in \mathscr{F}} (\vee F).$$

If L is completely distributive, equality is achieved when $\mathscr{F}$ is an ultrafilter.

PROOF

Let $\{\alpha_{st} : t \in T_s \text{ and } s \in S\}$ be a subset of L. Then we always have

$\bigvee_{\varphi \in \prod_S T_s} \bigwedge_{s \in S} \alpha_{s\varphi(s)} \leq \bigwedge_{s \in S} \bigvee_{t \in T_s} \alpha_{st}$, with equality being achieved when L is completely distributive.

Applying this to our situation we get

$$\bigvee_{\varphi \in \prod_{F \in \mathscr{F}} F} \left(\bigwedge_{F \in \mathscr{F}} \varphi(F)\right) \leq \bigwedge_{F \in \mathscr{F}} \left(\bigvee_{\alpha \in F} \alpha\right) \qquad (1).$$

Let $G \in \mathcal{F}$. Then for each $F \in \mathcal{F}$, $G \cap F \neq \emptyset$. Thus $\prod_{F \in \mathcal{F}} G \cap F \neq \emptyset$ by the Axiom of Choice. For each $\varphi \in \prod_{F \in \mathcal{F}} G \cap F$, we have $\wedge G \leq \bigwedge_{F \in \mathcal{F}} \varphi(F)$. But $\prod_{F \in \mathcal{F}} G \cap F \subseteq \prod_{F \in \mathcal{F}} F$. Hence for each $F \in \mathcal{F}$ there exists $\varphi \in \prod_{F \in \mathcal{F}} F$ such that $\wedge F \leq \bigwedge_{F \in \mathcal{F}} \varphi(F)$. Thus together with (1) we have :

$$\bigvee_{F \in \mathcal{F}} (\wedge F) \leq \bigvee_{\varphi \in \prod_{F \in \mathcal{F}} F} \left(\bigwedge_{F \in \mathcal{F}} \varphi(F) \right) \leq \bigwedge_{F \in \mathcal{F}} (\vee F). \tag{2}$$

Suppose now that $\mathcal{F}$ is an ultrafilter. Then for each $\varphi \in \prod_{F \in \mathcal{F}} F$, the set $G_\varphi = \{\varphi(F) : F \in \mathcal{F}\}$ meets every $F \in \mathcal{F}$. Therefore $G_\varphi \in \mathcal{F}$. From (2) we then have $\bigvee_{F \in \mathcal{F}} (\wedge F) = \bigvee_{\varphi \in \prod_{F \in \mathcal{F}} F} \left(\bigwedge_{F \in \mathcal{F}} \varphi(F) \right)$. When L is a completely distributive lattice we have equality in (1) and the required result, $\bigvee_{F \in \mathcal{F}} (\wedge F) = \bigwedge_{F \in \mathcal{F}} (\vee F)$, follows.

■

IV.1.5 PROPOSITION ([2] : VII.1.11)

Let L be a complete lattice and $\mathcal{F}$ an ultrafilter on L. Then $\mathcal{F}$ converges to $\alpha \in L$ in the interval topology if and only if $\bigvee_{F \in \mathcal{F}} \wedge F \leq \alpha \leq \bigwedge_{F \in \mathcal{F}} \vee F$.

PROOF

A basic closed set in the interval topology is of the form $\bigcup_{i=1}^{n} [\alpha_i, \beta_i]$.

Let $\bigcup_{i=1}^{n} [\alpha_i, \beta_i] \in \mathcal{F}$. Since $\mathcal{F}$ is an ultrafilter, for at least one $i \in [1 \cdot\cdot n]$, $[\alpha_i, \beta_i] \in \mathcal{F}$. Together with Lemma IV.1.4 we have $\alpha_i \leq \bigvee_{F \in \mathcal{F}} (\wedge F) \leq \bigwedge_{F \in \mathcal{F}} (\vee F) \leq \beta_i$. Every closed set belonging to $\mathcal{F}$ is the intersection of a collection of basic closed sets.

Thus every closed set of $\mathcal{F}$ contains $[\bigvee_{F \in \mathcal{F}} (\wedge F), \bigwedge_{F \in \mathcal{F}} (\vee F)]$.

If $\alpha \in [\bigvee_{F \in \mathcal{F}} (\wedge F), \bigwedge_{F \in \mathcal{F}} (\vee F)]$, then $\alpha \in \cap \{\bar{F} : F \in \mathcal{F}\}$. But an ultrafilter converges to each of its adherence points. Thus $\mathcal{F}$ converges to α.

Conversely, suppose $\mathscr{F}$ converges to α. If there exists $F \in \mathscr{F}$ such that $\wedge F \nleq \alpha$, then $L/\uparrow(\wedge F)$ is a nbd of α which does not meet F. But this contradicts the fact that $\mathscr{F}$ converges to α. Hence $\bigvee_{F \in \mathscr{F}} (\wedge F) \leq \alpha$. The proof that $\alpha \leq \bigwedge_{F \in \mathscr{F}} (\vee F)$ follows similarly.

■

IV.1.6 COROLLARY ([2]: VII.1.11)

Let L be a complete lattice.

(i) The interval topology is compact.

(ii) If L is completely distributive, the interval topology is Hausdorff.

PROOF

(i) By Lemma IV.1.4 for an ultrafilter $\mathscr{F}$, the set $[\bigvee_{F \in \mathscr{F}} (\wedge F), \bigwedge_{F \in \mathscr{F}} (\vee F)]$ is non–empty. Therefore by the previous proposition every ultrafilter is convergent. Hence the interval topology is compact.

(ii) A necessary and sufficient condition for the interval topology to be Hausdorff is that each convergent ultrafilter have a unique limit. By Lemma (IV.1.4) for each ultrafilter $\mathscr{F}$ on L, $\bigwedge_{F \in \mathscr{F}} (\vee F) = \bigvee_{F \in \mathscr{F}} (\wedge F)$. Thus by the previous proposition the limit of each ultrafilter is unique.

■

We also need to consider various properties of the Lawson and Scott topologies on continuous lattices.

IV.1.7 PROPOSITION ([1],III:1.10)

The Lawson topology $\lambda(L)$ on a continuous lattice L is compact and Hausdorff.

PROOF

(a) To prove the compactness of $\lambda(L)$ we use the Alexander subbase theorem :

X is compact if and only if there is a subbase $\mathscr{S}$ for X such that each cover of X by elements of $\mathscr{S}$ has a finite subcover.

The collection of sets $\mathscr{S} = \{ \Uparrow \alpha : \alpha \in L\} \cup \{L / \uparrow \alpha : \alpha \in L\}$ forms a subbase for $\lambda(L)$. Let $\{ \Uparrow \alpha_j : j \in J\} \cup \{L/\uparrow \alpha_k : k \in K\}$ be a cover of L by elements of $\mathscr{S}$. Then

$$\bigcup_{k \in K} L/\uparrow \alpha_k = L/ \bigcup_{k \in K} \uparrow \alpha_k = L/\uparrow \alpha,$$

where $\alpha = \bigvee_{k \in K} \alpha_k$. Now $\alpha \notin L/\uparrow \alpha$, thus for some $j \in J$, $\alpha \in \Uparrow \alpha_j$.

Since L is continuous there exists $\beta \in L$ such that $\alpha_j << \beta << \alpha$. Furthermore since $\beta << \alpha = \bigvee_{k \in K} \alpha_k$ there exists a finite subset $\{\alpha_{k_i} : i \in [1 \cdot\cdot n]\} \subseteq \{ \alpha_k : k \in K\}$ such that $\beta \leq \alpha_{k_1} \vee \cdots \vee \alpha_{k_n}$. We now claim that

$$L = \Uparrow \alpha_j \cup (\bigcup_{i=1}^{n} L/\uparrow \alpha_{k_i}).$$

Let $\beta \in L$ such that $\beta \notin \Uparrow \alpha_j$, then $\alpha_{k_1} \vee \cdots \vee \alpha_{k_n} \nleq \beta$. This means that for some $i \in [1 \cdot\cdot n]$, $\alpha_{k_i} \nleq \beta$. Thus $\beta \in L/\uparrow \alpha_{k_i}$.

(b) To show that $\lambda(L)$ is Hausdorff let α,β be two distinct elements of L. Without loss of generality we may assume that $\alpha \nleq \beta$. Since L is continuous there exists $\alpha_0 \in L$ such that $\alpha_0 << \alpha$ and $\alpha_0 \nleq \beta$. Then the sets $\Uparrow \alpha_0$, $L/\uparrow \alpha_0$ are disjoint neighbourhoods of α and β respectively.

■

IV.1.8 PROPOSITION

If L is a continuous lattice, then $u(L) \subseteq \sigma(L)$.

PROOF

A typical subbasic open set in u(L) is of the form $L/\downarrow \alpha$, $\alpha \in L$. Let $\beta \in L/\downarrow \alpha$. Then $\beta \nleq \alpha$ and since L is continuous there exists $\beta_0 \in L$ such that $\beta_0 << \beta$ and $\beta_0 \nleq \alpha$. Now $\Uparrow \beta_0 \subseteq L/\downarrow \alpha$ and $\beta \in \Uparrow \beta_0$. Thus $L/\downarrow \alpha$ is open in the Scott topology $\sigma(L)$ on L. We therefore have that every subbasic open set in u(L) is open in $\sigma(L)$. Hence $u(L) \subseteq \sigma(L)$.

■

We now have at our disposal sufficient properties of the interval, Lawson and Scott topologies to prove our main result.

IV.1.9 PROPOSITION

Let L be a completely distributive lattice. Then

$$\sigma(L) = u(L).$$

■

Before we proceed with the proof of this proposition, we shall need the following lemma.

IV.1.10 LEMMA

Let L be a continuous lattice whose interval topology is Hausdorff. Then every upper closed set is closed in the lower interval topology, $u(L^{op})$.

PROOF

By (IV.1.8), $u(L) \subseteq \sigma(L)$. Thus the Lawson topology $\lambda(L)$ is always stronger than the interval topology. But by (IV.1.7) $\lambda(L)$ is a compact topology, which implies that the interval topology on L is compact. Therefore since compact, Hausdorff topologies are maximal we have that the Lawson and interval topologies are identical.

Let U be an upper closed set in L. If $\alpha \notin U$, then for each $\beta \in U$, $\beta \nleq \alpha$ (U is an upper set). But L is continuous. This means that for each $\beta \in U$, there exists $\alpha_\beta \in L$ such that $\alpha_\beta << \beta$ and $\alpha_\beta \nleq \alpha$. The collection $\{\Uparrow \alpha_\beta : \beta \in U\}$ of open sets forms an open cover of U, a closed subset of a compact space. Therefore since U is compact there exists a finite subcollection $\{\Uparrow \alpha_{\beta_i} : i \in [1 \cdot\cdot n]\}$ which will cover U. Now $\alpha \notin \bigcup_{i=1}^{n} \uparrow \alpha_{\beta_i}$ and $U \subseteq \bigcup_{i=1}^{n} \Uparrow \alpha_{\beta_i} \subseteq \bigcup_{i=1}^{n} \uparrow \alpha_{\beta_i}$.

Let $U_\alpha = \bigcup_{i=1}^{n} \uparrow \alpha_{\beta_i}$. We can then write U as

$$U = \bigcap_{\alpha \notin U} U_\alpha ,$$

where each U_α is a basic closed set in $u(L^{op})$. Therefore U is a closed set in the lower interval topology, $u(L^{op})$.

■

PROOF OF PROPOSITION

Completely distributive lattices are continuous (0.1.16). If L is completely distributive then L^{op} is completely distributive (0.1.14). Hence L^{op} is continuous. By (IV.1.6) the interval topology on L^{op} is compact and Hausdorff, we may therefore apply Lemma IV.1.10 to L^{op}. Since $\{\twoheaduparrow \alpha : \alpha \in L\}$ is a base for $\sigma(L)$ to show that $\sigma(L) \subseteq u(L)$ it will suffice to show that each of the sets $L/\twoheaduparrow \alpha$, $\alpha \in L$, are closed in the upper interval topology, u(L).

Let $\alpha \in L$. As noted in the proof of the above lemma, $\twoheaduparrow \alpha$ is an open set in the interval topology topology on L. Thus $L/\twoheaduparrow \alpha$ is a closed set. Furthermore since $L/\twoheaduparrow \alpha$ is a lower set in L and the interval topologies on L and L^{op} coincide, we have that $L/\twoheaduparrow \alpha$ is a closed upper set in the interval topology on L^{op}. By Lemma IV.1.10 $L/\twoheaduparrow \alpha$ is closed in $u((L^{op})^{op})$. But $(L^{op})^{op} = L$. Thus $L/\twoheaduparrow \alpha$ is closed in the upper interval topology, u(L).

■

The following appears as an exercise ([1]: III.3.22) :

If L is a continuous lattice, then $u(L) = \sigma(L)$ if and only if the Lawson topology $\lambda(L)$ and interval topologies coincide.

This appears to hinge on a result quoted without proof in ([2]: VII.1.13). This result states that in a complete lattice L for which the interval topology is Hausdorff, every upper closed set is closed in the lower interval topology, $u(L^{op})$.

The dual of this result can then be used to prove the result appearing in the exercise.

As Lemma IV.1.10 indicates we can only prove this result for continuous lattices. The lattice L^{op} of a continuous lattice L is not generally continuous. However if L is completely distributive then L^{op} is continuous. Using this fact, we are able to prove (IV.1.9) for completely distributive lattices.

We close this section with the following proposition which will be needed in section 2.

IV.1.11 PROPOSITION ([11])

If L is completely distributive, then

$$\wedge : (L,u(L)) \times (L,u(L) \longrightarrow (L,u(L))$$

is continuous.

PROOF

Recall (0.1.18) that for any $\alpha_0 \in L$, $\alpha_0 = \wedge\, (\uparrow \alpha_0 \cap \text{PRIME}(L))$. Thus the inverse image of any subbasic closed set $\downarrow \alpha_0$ is given by

$$\begin{aligned}
\wedge^{-1}(\downarrow \alpha_0) &= \{(\alpha,\beta) \in L\times L : \alpha \wedge \beta \leq \alpha_0\} \\
&= \{(\alpha,\beta) \in L\times L : \alpha \wedge \beta \leq \wedge\, (\uparrow \alpha_0 \cap \text{PRIME}(L))\} \\
&= \{(\alpha,\beta) \in L\times L : \alpha \wedge \beta \leq p \text{ for all } p \geq \alpha_0 \text{ and } p \in \text{PRIME}(L)\} \\
&= \bigcap_{\substack{p \geq \alpha_0 \\ p \in \text{PRIME}(L)}} \{(\alpha,\beta) \in L\times L : \alpha \leq p \text{ or } \beta \leq p\} \\
&= \bigcap_{\substack{p \geq \alpha_0 \\ p \in \text{PRIME}(L)}} (\downarrow p\times L) \cup (L\times \downarrow p)
\end{aligned}$$

which is closed in $u(L) \times u(L)$. Thus $\wedge$ is continuous.

■

IV.2 TOPOLOGICALLY GENERATED L–TOPOLOGIES AND TOPOLOGICAL MODIFICATIONS OF L–TOPOLOGIES.

We can now extend Lowen's natural way of associating an I–topological space with a given topological space and vice–versa, to a much larger class of lattices. In this section L will be assumed to be a complete lattice. All definitions and propositions in this section are those of Kubiak ([11]).

IV.2.1 DEFINITION

(a) Let (X,Δ) be a topological space. The topologically generated L–topology, denoted by $\omega(\Delta)$, is the L–topology generated by $lsc(X,L)$.

(b) Let (X,τ) be an L–topological space. The topological modification of τ, denoted by $\iota(\tau)$, is the weakest topology on X with respect to which the collection τ of L–sets are continuous from X to $(L,u(L))$.

■

From this definition we make the following observations.

(a) If $\Delta_1 \subseteq \Delta_2$ then $\omega(\Delta_1) \subseteq \omega(\Delta_2)$, and if $\tau_1 \subseteq \tau_2$ then $\iota(\tau_1) \subseteq \iota(\tau_2)$.

(b) If Δ is any topology on X such that each $\mu \in \tau$ is $(\Delta, u(L))$–continuous then $\iota(\tau) \subseteq \Delta$.

IV.2.2 PROPOSITION

If L is completely distributive, then $\omega(\Delta) = \mathrm{lsc}(X,L)$.

PROOF

By (IV.1.11), the operation $\wedge : (L,u(L)) \times (L,u(L)) \longrightarrow (L,u(L))$ is continuous. Let $f,g \in \mathrm{lsc}(X,L)$ and define

$$F : X \longrightarrow L \times L,\ F(x) = (f(x),g(x)).$$

Then F is continuous. Thus the composition $\wedge \circ F = f \wedge g$ is continuous. Therefore $\mathrm{lsc}(X,L)$ is closed under finite infima. Secondly, let $\{f_k : k \in K\} \subseteq \mathrm{lsc}(X,L)$. To show that $\bigvee_{k \in K} f_k \in \mathrm{lsc}(X,L)$, it will suffice to show that $(\bigvee_k f_k)^{-1}(\downarrow \alpha)$ is closed in (X,Δ) for each $\alpha \in L$. The continuity of $\bigvee_k f_k$ will then follow since $\{\downarrow \alpha : \alpha \in L\}$ forms a subbase for the closed sets of $(L,u(L))$.

$$\begin{aligned}(\bigvee_k f_k)^{-1}(\downarrow \alpha) &= \{x \in X : \bigvee_k f_k(x) \leq \alpha\} \\ &= \{x : f_k(x) \leq \alpha \text{ for all } k \in K\} \\ &= \bigcap_k f_k^{-1}(\downarrow \alpha)\end{aligned}$$

But for each $k \in K$, $f_k^{-1}(\downarrow \alpha)$ is closed and thus $(\bigvee_k f_k)^{-1}(\downarrow \alpha)$ is closed. We now see that $\mathrm{lsc}(X,L)$ is closed under arbitary suprema and finite infima. Hence $\mathrm{lsc}(X,L)$ is an L–topology.

■

IV.2.3 PROPOSITION

Let (X,Δ) be a topological space and (Y,τ) be an L–topological space.

(a) $\Delta \subseteq \iota(\omega(\Delta))$ and when L is completely distributive, then $\iota(\omega(\Delta)) = \Delta$.

(b) $\tau \subseteq \omega(\iota(\tau))$.

PROOF

(a) Let $U \in \Delta$. Consider the characteristic function 1_U of the set U. The collection $T_U = \{G \subseteq L : (1_U)^{-1}(G) \in \Delta\}$ is a topology on L which contains the defining subbase of u(L). Indeed let $\alpha \in L$, then

$$(1_U)^{-1}(L/\downarrow \alpha) = \begin{cases} U & \text{if } \alpha < 1 \\ \emptyset & \text{if } \alpha = 1 \end{cases}$$

Thus $u(L) \subseteq T_U$. Hence 1_U is $(\Delta, u(L))$–continuous and therefore $1_U \in \omega(\Delta)$. Now any topology making 1_U continuous with respect to (L,u(L)) for each $U \in \Delta$, must contain Δ. Hence $\Delta \subseteq \iota(\omega(\Delta))$. When L is a completely distributive lattice $\omega(\Delta) = \text{lsc}(X,L)$. The topology $\iota(\omega(\Delta))$ is defined to be the weakest topology on X making each $\mu \in \omega(\Delta) = \text{lsc}(X,L)$ continuous with respect (L,u(L)). Therefore $\iota(\omega(\Delta)) \subseteq \Delta$.

(b) Let $\mu \in \tau$, by definition of $\iota(\tau)$, $\mu \in \text{lsc}((Y,\iota(\tau)),L)$. But $\text{lsc}((Y,\iota(\tau)),L) \subseteq \omega(\iota(\tau))$. Hence $\tau \subseteq \omega(\iota(\tau))$.

■

The inclusion in part(b) of (IV.2.3) cannot be generally reversed. The reason being that topologically generated L–topologies will always contain the constant L–sets. For example : let X be a non–empty set, then the discrete L–topology, $\tau = \{\underline{0},\underline{1}\}$, is an L–topology which does not contain all the constant L–sets.

IV.2.4 PROPOSITION

(a) Let $f : (X,\tau_1) \longrightarrow (Y,\tau_2)$. If f is L–continuous, then f is $(\iota(\tau_1),\iota(\tau_2))$–continuous.

(b) Let $f : (X,\Delta_1) \longrightarrow (Y,\Delta_2)$. If f is continuous, then f is $(\omega(\Delta_1),\omega(\Delta_2))$ – L–continuous, and the converse holds when L is completely distributive.

PROOF

(a) A subbase for the open sets of $\iota(\tau_2)$ is given by the collection $S = \{\mu^{-1}(L/\downarrow\alpha) : \mu \in \tau_2, \alpha \in L\}$. Now

$$f^{-1}(\mu^{-1}(L/\downarrow\alpha)) = (f^{\leftarrow}(\mu))^{-1}(L/\downarrow\alpha)$$

for each $\mu^{-1}(L/\downarrow\alpha) \in S$. But since f is L–continuous, $f^{\leftarrow}(\mu) \in \tau_1$. Hence $(f^{\leftarrow}(\mu))^{-1}(L/\downarrow\alpha) \in \iota(\tau_1)$. This proves that f is $(\iota(\tau_1),\iota(\tau_2))$–continuous.

(b) Let $\mu \in \mathrm{lsc}(Y,L)$, then $f^{\leftarrow}(\mu) = \mu \circ f \in \mathrm{lsc}(X,L)$. Now $\{\mu \in L^Y : f^{\leftarrow}(\mu) \in \omega(\Delta_1)\}$ contains lsc(Y,L) and is an L–topology on Y. Thus it must contain $\omega(\Delta_2)$. Hence whenever $\mu \in \omega(\Delta_2)$, $f^{\leftarrow}(\mu) \in \omega(\Delta_1)$. This proves that f is $(\omega(\Delta_1),\omega(\Delta_2))$–L–continuous. When L is completely distributive $\iota(\omega(\Delta_i)) = \Delta_i$ for $i = 1,2$. Then applying part (a), if f is $(\omega(\Delta_1),\omega(\Delta_2))$–L–continuous, then f is $(\iota(\omega(\Delta_1)),\iota(\omega(\Delta_2)))$–continuous. Thus f is (Δ_1,Δ_2)–continuous.

■

In (III.1.6) we showed that when L is completely distributive, L–completely regular spaces have the weak L–topology generated by their collections C(X,I(L)) of I(L)–valued, L–continuous functions. To study the topological modifications of such L–spaces we must consider the topological modification of weak L–topologies generated by collections of functions.

In Propositions IV.2.5 – IV.2.7, L will be assumed to be completely distributive.

IV.2.5 PROPOSITION

Let $\{\tau_k : k \in K\}$ be a collection of L–topologies on X. Then

$$\iota(\bigvee_k \tau_k) = \bigvee_k \iota(\tau_k).$$

PROOF

By observation (a) following Definition IV.2.1, we have for each $k \in K$, $\iota(\tau_k) \subseteq \iota(\bigvee_k \tau_k)$. Thus $\bigvee_k \iota(\tau_k) \subseteq \iota(\bigvee_k \tau_k)$.

Since L is completely distributive, each $\mu \in \bigvee_k \tau_k$ is of the form $\mu = \bigvee_{j \in J} \mu_j$ where each μ_j is of the form $\mu_j = \lambda_1 \wedge \cdots \wedge \lambda_n$, where $\{\lambda_i : i \in [1 \cdots n]\} \subseteq \bigcup_k \tau_k$. If we can show that each $\mu \in \bigvee_k \tau_k$ is $(\bigvee_k \iota(\tau_k), u(L))$–continuous, we will then have that $\iota(\bigvee_k \tau_k) \subseteq \bigvee_k \iota(\tau_k)$ and this shall complete the proof.

To this end let μ be as described above. Let $j \in J$ and consider $\mu_j = \lambda_1 \wedge \cdots \wedge \lambda_n$. Each λ_i is $(\bigvee_k \iota(\tau_k), u(L))$–continuous, then since $\wedge : L \times L \longrightarrow L$ is continuous we can argue as in (IV.2.2) to show that $\mu_j = \lambda_1 \wedge \cdots \wedge \lambda_n$ is $(\bigvee_k \iota(\tau_k), u(L))$–continuous. We also proved in (IV.2.2) that supremum of a collection L–valued, u(L)–continuous functions is continuous. Thus $\mu = \bigvee_j \mu_j$ is $(\bigvee_k \iota(\tau_k), u(L))$–continuous.

■

IV.2.6 PROPOSITION

Let $f : X \longrightarrow (Y, \tau)$. Then

$$\iota(f^{\leftarrow}(\tau)) = f^{-1}(\iota(\tau)),$$

where $f^{-1}(\iota(\tau))$ is the weak topology given by $f : X \longrightarrow (Y, \iota(\tau))$.

PROOF

By (IV.2.4(a)), we immediately have that $f : (X, \iota(f^{\leftarrow}(\tau))) \longrightarrow (Y, \iota(\tau))$ is continuous, thus $f^{-1}(\iota(\tau)) \subseteq \iota(f^{\leftarrow}(\tau))$. But by (IV.2.4(b)), f is $(\omega(f^{-1}(\iota(\tau))), \omega(\iota(\tau)))$– L–continuous. Therefore $f^{\leftarrow}(\tau) \subseteq \omega(f^{-1}(\iota(\tau)))$. Hence $\iota(f^{\leftarrow}(\tau)) \subseteq \iota(\omega(f^{-1}(\iota(\tau)))) = f^{-1}(\iota(\tau))$.

■

IV.2.7 PROPOSITION

If $\{(X_k, \tau_k) : k \in K\}$ is a collection of L–topological spaces and $\{f_k : X \longrightarrow X_k : k \in K\}$ a collection of functions. Then

$$\iota(\bigvee_k f^{\leftarrow}(\tau_k)) = \bigvee_k f^{-1}(\iota(\tau_k)).$$

PROOF

Applying (IV.2.5) to $\bigvee_k f_k^{\leftarrow}(\tau_k)$, we get $\iota(\bigvee_k f_k^{\leftarrow}(\tau_k)) = \bigvee_k \iota(f_k^{\leftarrow}(\tau_k))$. Then applying (IV.2.6) to each $\iota(f_k^{\leftarrow}(\tau_k))$, we arrive at the required result.

■

In the closing section, section 8, we will give a counterexample of a lattice L which is not completely distributive and for which some of the propositions in this section requiring complete distributivity, do not hold.

IV.3 TOPOLOGICALLY GENERATED L–UNIFORMITIES.

Working with the unit interval I, it was shown by Katsaras ([10]) that to every uniformity $\mathbb{D}$ on a set X, there corresponds an I–uniformity. The I–topology generated by this I–uniformity is then precisely $\omega(\Delta)$ where Δ is the topology generated by $\mathbb{D}$. Using the way below relation <<, we can extend this correspondence to completely distributive lattices.

In this section L will be assumed to be a completely distributive lattice.

Let X be a non–empty set. Further let
$L_p(X) = \{\alpha 1_x : \alpha \in L, x \in X\}$. For each $W \in DIAG(X)$, define

$$K_W : L_p(X) \to L^X, K_W(\alpha 1_x) = \alpha 1_{W(x)} \quad \textbf{(3.1)}$$

We extend K_W to L^X by defining

$$K_W(\mu) = \bigvee_{x \in X} K_W(\mu(x)1_x) \text{ for } \mu \in L^X \quad \textbf{(3.2)}$$

IV.3.3 PROPOSITION

(a) $\{K_W : W \in DIAG(X)\} \subseteq \mathscr{H}_L(X)$.

(b) For U, $W \in DIAG(X)$.

  (i) If $U \subseteq W$ then $K_U \leq K_W$.

  (ii) $K_U \circ K_U \leq K_{U \circ U}$.

  (iii) $(K_W)^{-1} = K_{W^{-1}}$.

PROOF

Parts (a) and (b)(i) follow directly from (IV.3.1) and (IV.3.2) and their proofs shall not be given here.

b(ii) : For each $\alpha 1_x \in L_p(X)$

$$K_U \circ K_U(\alpha 1_x) = K_U(\alpha 1_{U(x)}) = \bigvee_{y \in U(x)} \alpha 1_{U(y)}.$$

Now $y \in U(x)$ implies $U(y) \subseteq U \circ U(x)$. Therefore

$$K_U \circ K_U(\alpha 1_x) = \bigvee_{y \in U(x)} \alpha 1_{U(y)} \leq \alpha 1_{U \circ U(x)} = K_{U \circ U}(\alpha 1_x).$$

By (IV.3.2) it now follows that $K_U \circ K_U(\mu) \leq K_{U \circ U}(\mu)$ for each $\mu \in L^X$.

b(iii) : To show that $(K_W)^{-1} = K_{W^{-1}}$, it will again suffice to show that $(K_W)^{-1}(\alpha 1_x) = K_{W^{-1}}(\alpha 1_x)$ for each $\alpha 1_x \in L_p(X)$. By definition

$$\begin{aligned}(K_W)^{-1}(\alpha 1_x) &= \wedge\{\nu \in L^X : K_W(\nu') \leq (\alpha 1_x)'\} \\ &= \wedge\{\nu \in L^X : K_W(\nu')(x) \leq \alpha'\}.\end{aligned}$$

Let A denote the set $\{\nu \in L^X : K_W(\nu')(x) \leq \alpha'\}$.

Now $K_W(\nu')(x) \leq \alpha'$

if and only if $\bigvee_{y \in X} \nu'(y)\, 1_{W(y)}(x) \leq \alpha'$

if and only if for each $y \in X : \nu'(y)\, 1_{W(y)}(x) \leq \alpha'$

if and only if for each $y \in X : (y,x) \in W$ implies $\nu'(y) \leq \alpha'$

if and only if for each $y \in W^{-1}(x) : \nu'(y) \leq \alpha'$

if and only if for each $y \in W^{-1}(x) : \alpha \leq \nu(y)$.

Thus $\alpha 1_{W^{-1}(x)} \in A$ and for each $\nu \in A$, $\alpha 1_{W^{-1}(x)} \leq \nu$.

Hence $(K_W)^{-1}(\alpha 1_x) = \alpha 1_{W^{-1}(x)} = K_{W^{-1}}(\alpha 1_x)$.

■

IV.3.4 PROPOSITION

Let $\mathbb{D}$ be a uniformity on X. Then $\mathscr{B}(\mathbb{D}) = \{K_W : W \in \mathbb{D}\}$ is an L–uniform base.

PROOF

Let $\mathscr{B}$ denote $\mathscr{B}(\mathbb{D})$. To show that $\mathscr{B}$ is an L–uniform base, we must show that

(a) $\mathscr{B}$ is a filter base on $\mathscr{H}_L(X)$,

(b) for each $K_W \in \mathscr{B}$, there exists

  (i) $K_V \in \mathscr{B}$ such that $K_V \circ K_V \leq K_W$.

  (ii) $K_U \in \mathscr{B}$ such that $(K_U)^{-1} \leq K_W$.

(a) Let $K_{W_1}, K_{W_2} \in \mathscr{B}$. Then $W_1 \cap W_2 \in \mathbb{D}$. But $W_1 \cap W_2 \subseteq W_i$ for $i = 1,2$.

Thus by (IV.3.3(b)(i))
$K_{W_1 \cap W_2} \leq K_{W_i}$ for $i = 1,2$.

(b)(i) : If $W \in \mathbb{D}$, there exists $V \in \mathbb{D}$ such that $V \circ V \subseteq W$. By (IV.3.3(b)(ii)), $K_V \circ K_V \leq K_{V \circ V}$ and then by (IV.3.3(b)(i)) $K_{V \circ V} \leq K_W$.

(b)(ii): For each $W \in \mathbb{D}$, $W^{-1} \in \mathbb{D}$. By (IV.3.3(b)(iii)), $(K_W)^{-1} = K_{W^{-1}}$. Therefore there exists $K_U \in \mathscr{B}$ such that $(K_U)^{-1} \leq K_W$.

■

IV.3.5 DEFINITION

Let $(X,\mathbb{D})$ be a uniform space. Then the topologically generated L–uniform space, denoted by $(X,\omega(\mathbb{D}))$, is that L–uniformity having $\mathscr{B}(\mathbb{D})$ as an L–uniform base.

■

This correspondence preserves uniform mappings.

IV.3.6 PROPOSITION

Let $(X,\mathbb{D}_1)$, $(Y,\mathbb{D}_2)$ be two uniform spaces. Then $f : (X,\mathbb{D}_1) \to (Y,\mathbb{D}_2)$ is uniformly continuous if and only if $f : (X, \omega(\mathbb{D}_1)) \to (Y, \omega(\mathbb{D}_2))$ is L–uniformly continuous.

PROOF

Suppose that f is uniformly continuous. Then for each $V \in \mathbb{D}_2$, there exists $W \in \mathbb{D}_1$ such that $W \subseteq (f \times f)^{-1}(V)$. We now show that $K_W \leq f^{\leftarrow}(K_V)$.

Let $\alpha\, 1_x \in L_p(X)$, then

$$\begin{aligned} f^{\leftarrow}(K_V)(\alpha\, 1_x) &= f^{\leftarrow}(K_V(f^{\rightarrow}(\alpha\, 1_x))) \\ &= f^{\leftarrow}(K_V(\alpha\, 1_{f(x)})) \\ &= f^{\leftarrow}(\alpha\, 1_{V(f(x))}) \\ &= \alpha\, 1_{f^{-1}(V(f(x)))} \\ &= \alpha\, 1_{(f\times f)^{-1}(V)(x)} \\ &\geq \alpha\, 1_{W(x)} \\ &= K_W(\alpha\, 1_x). \end{aligned}$$

Thus $K_W \leq f^{\leftarrow}(K_V)$. Hence f is L–uniformly continuous. Conversely suppose that f is L–uniformly continuous. Let $V \in \mathbb{D}_2$, then there exists $W \in \mathbb{D}_1$ such that $K_W \leq f^{\leftarrow}(K_V)$. Thus $K_W(1_x) \leq f^{\leftarrow}(K_V)(1_x)$ for each $x \in X$. But

$$K_W(1_x) = 1_{W(x)} \leq f^{\leftarrow}(K_V)(1_x) = 1_{(f\times f)^{-1}(V)(x)}.$$

Therefore $W \subseteq (f\times f)^{-1}(V)$. This proves that f is uniformly continuous.

■

IV.3.7 PROPOSITION

Let (X,Δ) be a topological space and $\mathbb{D}$ a uniformity on X compatible with the topology Δ. Then the L–topology $\tau_{\omega(\mathbb{D})}$, generated by $\omega(\mathbb{D})$, is precisely $\omega(\Delta)$.

PROOF

Firstly we will show that $\tau_{\omega(\mathbb{D})} \subseteq \omega(\Delta)$. Since L is completely distributive $\omega(\Delta) = \mathrm{lsc}(X,L)$. It will therefore suffice to show that each $\mu \in \tau_{\omega(\mathbb{D})}$ is $(\Delta, u(L))$–continuous.

Let $\mu \in \tau_{\omega(\mathbb{D})}$ and $\alpha \in L$. By (I.3.3), there exists a collection $\{(D_j, \mu_j) : j \in J\} \subseteq \mathscr{B}(\mathbb{D}) \times L^X$ such that $\mu = \bigvee_j D_j(\mu_j)$. If $x \in \mu^{-1}(L/\!\downarrow \alpha)$ then there exists $j \in J$ such that $\mu_j(x) \nleq \alpha$. Now D_j is of the form K_W, where $W \in \mathbb{D}$.

The set $W(x)$ is a nbd of x and if $y \in W(x)$:

$$\mu_j(x) = \mu_j(x) 1_{W(x)}(y) \leq K_W(\mu_j(x)1_x)(y) \leq K_W(\mu_j)(y) \leq \mu(y).$$

Thus $\mu(y) \nleq \alpha$. Hence $W(x) \subseteq \mu^{-1}(L/\downarrow \alpha)$. The L–set μ is therefore $(\Delta,u(L))$–continuous.

Conversely, let $\mu \in \omega(\Delta)$. To show that $\omega(\Delta) \subseteq \tau_{\omega(\mathbb{D})}$, we show that $\text{int}_{\omega(\mathbb{D})}(\mu) = \mu$. The lattice L is continuous. Therefore for $x_0 \in X$, $\mu(x_0) = \vee\{\alpha \in L : \alpha << \mu(x_0)\}$. Suppose that $\alpha \in L$ such that $\alpha << \mu(x_0)$.

By (0.1.12) there exists $\beta \in L$ such that $\alpha << \beta << \mu(x_0)$. The collection $\{\Uparrow \alpha : \alpha \in L\}$ forms a base for the open sets in u(L) by (IV.1.9). Thus $\mu^{-1}(\Uparrow \beta)$ is an open set in X containing x_0, hence for some $W \in \mathbb{D}$, $W(x_0) \subseteq \mu^{-1}(\Uparrow \beta)$.

Let $\nu \in L$ be defined by

$$\nu(x) = \wedge\{\mu(y) : y \in W(x)\}.$$

Then

$$\bigvee_{y \in X} \nu(y)\, 1_{W(y)} = K_W(\nu) \leq \mu$$

and

$$\alpha << \beta \leq \bigwedge_{y \in W(x_0)} \mu(y) = \nu(x_0)$$

Thus $\alpha << \nu(x_0) \leq \text{int}_{\omega(\mathbb{D})}(\mu)(x_0) \leq \mu(x_0)$. It now follows that for all $x_0 \in X$, $\alpha << \text{int}_{\omega(\mathbb{D})}(\mu)(x_0) \leq \mu(x_0)$ for $\alpha \in \{\alpha \in L : \alpha << \mu(x_0)\}$. This implies that $\mu = \text{int}_{\omega(\mathbb{D})}(\mu)$.

■

IV.3.8 COROLLARY

If (X,Δ) is a completely regular topological space then $(X,\omega(\Delta))$ is an L–completely regular space.

PROOF

We have that a topological (resp L–topological) space is completely regular (resp L–completely regular) if and only if it is uniformizable (resp L–uniformizable). Let $\mathbb{D}$ be a uniformity on X compatible with the topology Δ. Then by (IV.3.7) the topologically generated L–topology $\omega(\Delta)$ and the L–topology induced by $\omega(\mathbb{D})$ coincide. Thus $(X,\omega(\Delta))$ is L–completely regular.

■

The converse of the above corollary is true and will be proved as a corollary to a more general result appearing in (IV.7).

IV.4 THE TOPOLOGICAL MODIFICATION OF THE L–UNIT INTERVAL I(L).

In due course we shall show that for a class of L–completely regular spaces satisfying a certain separation condition, we can L–embed each member in a cube of the L–unit interval I(L). In the topological modification, this L–embedding translates into a topological embedding. Therefore a thorough study of the topological modification of the L–unit interval I(L), is necessary.

The definitions and propositions of this section are those of Kubiak ([11],[13]). We shall again assume that L is a completely distributive lattice.

First we need to recall the definition of Helly's space. The subspace X of I^I consisting of all monotonically increasing functions with the subspace topology is called Helly's space. Henceforth L^I will denote I copies of $(L, u(L) \vee u(L^{op}))$. By using the generalised concepts of left and right hand limits, and extending the construction of Helly's space in I^I to L^I, we construct an L–space L–homeomorphic to I(L). The topological modification of this L–space is then homeomorphic to the topological modification of I(L). Consequently a detailed investigation of this L–space will reveal much about the topological modification of I(L).

IV.4.1 DEFINITION

Let H_L be the collection

$$\{f \in L^I : f \text{ is monotonically decreasing}\}$$

of functions. Then $(H_L, \mathscr{H}_L)$ is called Helly's space with the subspace topology, $\mathscr{H}_L$.

■

To agree with the definition of I(L), we consider instead the monotonically decreasing functions of L^I. However recalling that L and L^{op} are homeomorphic with respect to the interval topology on L and L^{op}, we have that H_L and $H_{L^{op}}$, the subspace of monotonically increasing functions of L^I, are homeomorphic. To see this let

$$\varphi : L \longrightarrow L^{op}, \ \varphi(\alpha) = \alpha.$$

Then for $\alpha \leq \beta$, $\varphi(\beta) \leq^{op} \varphi(\alpha)$. The mapping φ is then a homeomorphism between L and L^{op}. Then

$$\Phi : H_L \rightarrow H_{L^{op}},\ \Phi(f) = \varphi \circ f,$$

is the required homeomorphism between H_L and $H_{L^{op}}$.

IV.4.2 PROPOSITION

$(H_L, \mathscr{H}_L)$ is a compact, Hausdorff space.

PROOF

By (IV.1.6), (L, $u(L) \vee u(L^{op})$) is compact and Hausdorff. Thus by Tychonoff's product theorem and the fact that products of Hausdorff spaces are Hausdorff, L^I is a compact, Hausdorff space. To prove that H_L is compact, we need only show that H_L is closed. If $f \in L^I/H_L$, then there exists s, t $\in$ I such that $s < t$ and $f(t) \nleq f(s)$. Since L is continuous, there exists $\alpha \in$ L such that $\alpha << f(t)$ and $\alpha \nleq f(s)$. The open set

$$V = \pi_t^{-1}(\twoheaduparrow \alpha) \cap \pi_s^{-1}(L/\uparrow \alpha),$$

contains f.

We shall now show that $H_L \cap V = \emptyset$. Suppose $h \in H_L \cap V$, then $\alpha << h(t)$ and $\alpha \nleq h(s)$. But $\alpha << h(t)$ implies $\alpha \leq h(t)$. This implies that $\alpha \leq h(s)$ since $h(t) \leq h(s)$. Therefore we have a contradiction. Hence $H_L \cap V = \emptyset$. This proves that H_L is closed. Furthermore subspaces of Hausdorff spaces are Hausdorff. Therefore $(H_L, \mathscr{H}_L)$ is a compact, Hausdorff space.

■

IV.4.3 DEFINITION

(a) For each $f \in H_L$ and $t \in$ I, let

$$f(t+) = \vee\{f(s) : t < s\},\ f(t-) = \wedge\{f(s) : s < t\}$$

(since $\vee\emptyset = 0$, $\wedge\emptyset = 1$ we have $f(1+) = 0$, $f(0-) = 1$)

(b) Define the relation $\sim$ on H_L as follows. For f, g $\in H_L$:

$$f \sim g \text{ if and only if } f(t+) = g(t+) \text{ for each } t \in I.$$

■

The relation $\sim$ is obviously an equivalence relation. We define $(H_{L|\sim}, \Delta)$ to be the quotient space whose elements are the equivalence classes for $\sim$.

IV.4.4 DEFINITION

The natural L–topology δ, on $H_{L|\sim}$ is the L–topology generated by

$$\{\bar{R}_t, \bar{L}_t \in L^{H_{L|\sim}} : t \in I\},$$

where $\bar{R}_t$, $\bar{L}_t$ are defined by $\bar{R}_t([f]) = f(t+)$ and $\bar{L}_t([f]) = f(t-)'$ for each $[f] \in H_{L|\sim}$.

■

IV.4.5 PROPOSITION

$(H_{L|\sim}, \delta)$ is L–homeomorphic to $(I(L), \tau_{I(L)})$.

PROOF

Let $\varphi : I(L) \longrightarrow H_{L|\sim}$, $\varphi([f]) = [f|_I]$. Recalling the construction of I(L), (I.5), it is clear that φ is a bijection. Furthermore $\varphi^{\leftarrow}(\bar{R}_t) = R_t$, $\varphi^{\leftarrow}(\bar{L}_t) = L_t$ for all $t \in I$. Thus φ is L–continuous. But

$$(\varphi^{-1})^{\leftarrow}(R_t) = \begin{cases} \underline{1} & \text{for } t < 0 \\ \bar{R}_t & \text{for } t \in I \\ \underline{0} & \text{for } 1 < t \end{cases}$$

$$(\varphi^{-1})^{\leftarrow}(L_t) = \begin{cases} \underline{0} & \text{for } t < 0 \\ \bar{L}_t & \text{for } t \in I \\ \underline{1} & \text{for } 1 < t \end{cases}$$

This proves that φ^{-1} is L–continuous. Hence φ is an L–homeomorphism.

■

IV.4.6 PROPOSITION

$\delta \subseteq \omega(\Delta)$.

PROOF

To show the inclusion we show that for each $t \in I$, $\bar{L}_t$ and $\bar{R}_t$ belong in $\omega(\Delta) = \mathrm{lsc}(H_{L|\sim}, L)$. Then since δ is the L–topology generated by $\{\bar{R}_t, \bar{L}_t \in L^{H_{L|\sim}} : t \in I\}$ and $\omega(\Delta)$ is an L–topology, we have then that $\delta \subseteq \omega(\Delta)$.

Let $\rho : H_L \to H_{L|\sim}$ denote the quotient map of H_L onto $H_{L|\sim}$.
From standard quotient theory a sufficient condition for $\bar{L}_t, \bar{R}_t$ to be $(\Delta, u(L))$–continuous is that $\bar{L}_t \circ \rho$, $\bar{R}_t \circ \rho$ be $(\mathscr{H}_L, u(L))$–continuous. To show this it will suffice to show that the sets $(\bar{L}_t \circ \rho)^{-1}(\downarrow \alpha)$, $(\bar{R}_t \circ \rho)^{-1}(\downarrow \alpha)$ are closed for each $\alpha \in L$. We have that

$$
\begin{aligned}
(\bar{L}_t \circ \rho)^{-1}(\downarrow \alpha) &= \{f \in H_L : f(t-)' \leq \alpha\} \\
&= \{f \in H_L : (\bigwedge_{s<t} f(s))' \leq \alpha\} \\
&= \{f \in H_L : \bigvee_{s<t} f(s)' \leq \alpha\} \\
&= \{f \in H_L : \bigvee_{s<t} \pi_s(f)' \leq \alpha\} \\
&= \{f \in H_L : \alpha' \leq \pi_s(f) \quad \text{for } s < t\} \\
&= (\bigcap_{s<t} \pi_s^{-1}(\uparrow \alpha')) \cap H_L,
\end{aligned}
$$

and similarly $(R_t \circ \rho)^{-1}(\downarrow \alpha) = (\bigcap_{t<s} \pi_s^{-1}(\downarrow \alpha)) \cap H_L$. Now $\downarrow \alpha$ and $\uparrow \alpha'$ are closed subsets in the interval topology on L and each projection mapping π_r, $r \in I$, is continuous with respect to the interval topology. Thus $(\bar{L}_t \circ \rho)^{-1}(\downarrow \alpha)$ and $(\bar{R}_t \circ \rho)^{-1}(\downarrow \alpha)$ are closed subsets of $(H_L, \mathscr{H}_L)$.

■

IV.4.7 COROLLARY

$\iota(\delta) \subseteq \Delta$.

PROOF

By (IV.4.6), $\delta \subseteq \omega(\Delta)$. Hence $\iota(\delta) \subseteq \iota(\omega(\Delta)) = \Delta$.

∎

Our main result is :

IV.4.8 PROPOSITION

$(H_{L|\sim}, \iota(\delta))$ is a compact, Hausdorff space.

PROOF

A quotient space of a compact space is always compact. Therefore $(H_{L|\sim}, \Delta)$ is compact. Compactness is preserved under weakening of topologies so by (IV.4.7), $(H_{L|\sim}, \iota(\delta))$ is compact.

Before proving that $(H_{L|\sim}, \iota(\delta))$ is a Hausdorff space, recall from (I.5.2) that if $[f], [g] \in I(L)$, then $f(t+) \leq g(t+)$ for each $t \in \mathbb{R}$ if and only if $f(t+) \leq g(t-)$ for each $t \in \mathbb{R}$.

Since $H_{L|\sim}$ is the collection $\{[f|_I] : [f] \in I(L)\}$, we have that this result holds in $H_{L|\sim}$ with the restriction $t \in I$.

Let $[f],[g]$ be distinct in $H_{L|\sim}$. By the above, without loss of generality, we may assume that there exists $t \in I$, such that $f(t+) \nleq g(t-)$. Once again since L is continuous, there exists $\alpha \in L$ such that $\alpha << f(t+)$ and $\alpha \nleq g(t-)$. Then $g(t-)' \nleq \alpha'$. The sets

$$U = (\bar{R}_t)^{-1}(\uparrow \alpha), V = (\bar{L}_t)^{-1}(L/\downarrow \alpha')$$

are open neighbourhoods of $[f]$ and $[g]$ respectively. The sets U, V are disjoint as well. Indeed if $U \cap V \neq \emptyset$, there exists $[h] \in H_{L|\sim}$ such that $\alpha << h(t+)$ and $h(t-)' \nleq \alpha'$, which implies $\alpha \nleq h(t-)$. But $\alpha << h(t+)$ implies $\alpha \leq h(t+)$, and $h(t+) \leq h(t-)$. This contradiction proves that $U \cap V = \emptyset$. We have shown that any two distinct points in $H_{L|\sim}$ can be separated by two disjoint open sets. Therefore $(H_{L|\sim}, \iota(\delta))$ is Hausdorff.

∎

IV.4.9 PROPOSITION

$\Delta = \iota(\delta)$.

PROOF

The identity mapping $i : (H_{L|\sim}, \Delta) \longrightarrow (H_{L|\sim}, \iota(\delta))$ is a continuous one to one map from the compact space $(H_{L|\sim}, \Delta)$ onto the Hausdorff space $(H_{L|\sim}, \iota(\delta))$. Hence i is a homeomorphism.

∎

IV.4.10 COROLLARY

$(I(L), \iota(\tau_{I(L)}))$ is a compact, Hausdorff space.

PROOF

The mapping $\varphi : (I(L), \iota(\tau_{I(L)})) \longrightarrow (H_{L|\sim}, \delta)$ described in (IV.4.5) translates into a homeomorphism in the topological modification of I(L) and $H_{L|\sim}$. Since the homeomorphic image of a compact, Hausdorff space is compact and Hausdorff, it follows that $(I(L), \iota(\tau_{I(L)}))$ is compact and Hausdorff.

∎

IV.4.11 COROLLARY

Let (Γ, τ_Γ) denote the cube $\prod_{j \in J} I(L)$ given the product L–topology. Then $(\Gamma, \iota(\tau_\Gamma))$ is a compact, Hausdorff space.

PROOF

The product L–topology is the weak L–topology given by the projections $\{\pi_j : j \in J\}$. That is $\tau_\Gamma = \bigvee_j \pi_j^{\leftarrow}(\tau_{I(L)})$. By (IV.2.7), $\iota(\tau_\Gamma) = \bigvee_j \pi_j^{-1}(\iota(\tau_{I(L)}))$. The required result then follows by Tychonoff's product theorem and the fact that products of Hausdorff spaces are Hausdorff.

∎

As mentioned earlier we shall define a class of L–completely regular spaces for which each member can be L–embedded in a cube of I(L).

Then by a corollary in the next section the topological modification of each of these L–completely regular spaces is embedded in a cube of the topological modification of I(L), which is a compact, Hausdorff space.

IV.5 AN L–EMBEDDING LEMMA

Let L be completely distributive. As in the topological case, with a simple extra condition on the generating collection of maps, any L–space with a weak L–topology can be L–embedded as a subspace of the product of the range L–spaces.

IV.5.1 DEFINITION

If, for each $k \in K$, $f_k : X \longrightarrow X_k$, then the evaluation map $e : X \longrightarrow \prod_k X_k$ induced by the collection $\{f_k : k \in K\}$ is defined as follows :

For each $x \in X$, $k \in K$,

$$[e(x)]_k = f_k(x)$$

A collection $\{f_k : k \in K\}$ of functions on X will be said to separate points in X if and only if whenever $x \neq y$ in X, then for some $k \in K$, $f_k(x) \neq f_k(y)$.

■

IV.5.2 PROPOSITION (L–EMBEDDING LEMMA)

Let (X,τ) be an L–space. For each $k \in K$, let $f_k : X \longrightarrow (X_k,\tau_k)$. Then the evaluation map $e: X \longrightarrow \prod_k X_k$ is an L–embedding if and only if X has the weak L–topology given by the functions f_k and the collection $\{f_k : k \in K\}$ separates points in X.

PROOF

Our proof depends essentially on the way below relation $<<$ and the observation that, for each $k \in K$, $\pi_k \circ e = f_k$.

($\Longrightarrow$): First suppose e is an L–embedding. If x,y are distinct in X, then $e(x) \neq e(y)$. This implies that for some $k \in K$, $[e(x)]_k \neq [e(y)]_k$. Therefore $f_k(x) \neq f_k(y)$.

Thus the collection $\{f_k : k \in K\}$ separates points in X. We will first show that $\bigvee_k f_k^{\leftarrow}(\tau_k) \subseteq \tau$ and then that $\tau \subseteq \bigvee_k f_k^{\leftarrow}(\tau_k)$.

Let $k \in K$, then $f_k = \pi_k \circ e$. Thus f_k is (τ,τ_k)–L–continuous. Hence $\bigvee_k f_k^{\leftarrow}(\tau_k) \subseteq \tau$.

Conversely let $\mu \in \tau$. For each $x \in X$, $\mu(x) = \vee\{\alpha \in L : \alpha << \mu(x)\}$. Let $x \in X$ and $\alpha \in L$ satisfying $\alpha << \mu(x)$. Since e is an L–embedding, $e^{\rightarrow}(\mu)$ is an open L–set in the relative L–topology on e(X).

Since L is completely distributive $e^{\rightarrow}(\mu)$ is of the form $\bigvee_{j \in J} \nu_j$, where each ν_j is itself of the form

$$\bigwedge_{l=1}^{n_j} (\pi_{k_l}|\, e(X))^{\leftarrow}(\mu_{k_l}), \text{ where } \mu_{k_l} \in \tau_{k_l}.$$

Now

$$\alpha << e^{\leftarrow}(e^{\rightarrow}(\mu))(x) = \bigvee_j \nu_j(e(x)) = \mu(x).$$

Put $S = \{\bigvee_{j \in A} \nu_j(e(x)) : A \in 2^{(J)}\}$. The set S is then a directed subset of L and $\vee S = \mu(x)$. Since $\alpha << \mu(x)$ and $\mu(x) = \vee S$, there exists $A \in 2^{(J)}$ such that $\alpha \leq \bigvee_{j \in A} \nu_j(e(x))$.

Then

$$\begin{aligned} e^{\leftarrow}\Big(\bigvee_{j \in A} \nu_j\Big) &= \bigvee_{j \in A} e^{\leftarrow}(\nu_j) \\ &= \bigvee_{j \in A} e^{\leftarrow}\Big(\bigwedge_{l=1}^{n_j} (\pi_{k_l}|\, e(X))^{\leftarrow}(\mu_{k_l})\Big) \\ &= \bigvee_{j \in A} \bigwedge_{l=1}^{n_j} f_{k_l}^{\leftarrow}(\mu_{k_l}) \\ &\leq \mu \end{aligned}$$

and

$$\alpha \leq \bigvee_{j \in A} \nu_j(e(x)) = e^{\leftarrow}\Big(\bigvee_{j \in A} \nu_j\Big) = \bigvee_{j \in A} \bigwedge_{l=1}^{n_j} f_{k_l}^{\leftarrow}(\mu_{k_l})(x) \leq \mu(x).$$

Therefore μ can be written as the supremum of elements of $\bigvee_k f^{\leftarrow}(\tau_k)$. Thus $\mu \in \bigvee_k f^{\leftarrow}(\tau_k)$. This proves that $\tau \subseteq \bigvee_k f^{\leftarrow}(\tau_k)$.

($\Longleftarrow$): This half of the proof is a duplication of the usual topological proof.

If $x \neq y$ in X, then for some $k \in K$, $f_k(x) \neq f_k(y)$. Hence $[e(x)]_k \neq [e(y)]_k$. Thus e is one to one.

Since X has the weak L–topology given by $\{f_k : k \in K\}$, we have for each $k \in K$, $\pi_k \circ e = f_k$ is an L–continuous function. This proves that e is L–continuous. Finally we must show that for each $\mu \in \tau$, $e^{\rightarrow}(\mu)$ is open in the relative L–topology.

Since L is completely distributive and e is one to one it will suffice to show for each $f_k^{\leftarrow}(\mu_k)$, $k \in K$, $\mu_k \in \tau_k$ that $e^{\rightarrow}(f_k^{\leftarrow}(\mu_k))$ is an open L–set in the relative L–topology on e(X). But

$$e^{\rightarrow}(f_k^{\leftarrow}(\mu_k)) = (\pi_k|_{e(X)})^{\leftarrow}(\mu_k),$$

is an open L–set in the relative L–topology.

■

IV.5.3 COROLLARY

If the evaluation map $e : X \rightarrow \prod_k X_k$ is an L–embedding then in the topological modification e is an embedding.

PROOF

If e is an L–embedding, then by (IV.5.2) the L–topology τ on X is given by $\tau = \bigvee_k f_k^{\leftarrow}(\tau_k)$. By (IV.2.7)

$$\iota(\tau) = \bigvee_k f^{-1}(\iota(\tau_k))$$

Furthermore, $\{f_k : k \in K\}$ separates points in X. Thus X has the weak topology given by $\{f_k : X \rightarrow (X_k, \iota(\tau_k)) : k \in K\}$ and this collection of functions also separates points in X. By the standard topological embedding lemma, the evaluation mapping e is an embedding.

■

IV.6 L–T_o L–SPACES AND L–TYCHONOFF SPACES.

A topological space (X,Δ) is a T_o–space if and only if whenever x,y are distinct in X, there is an open set containing one and not the other. Liu ([15]) translated this separation axiom to L–spaces. We use a well known equivalent definition ([18]). Throughout this section we will assume L to be completely distributive.

IV.6.1 DEFINITION

An L–space (X,τ) is an L–T_o space if and only if whenever x and y are distinct in X, then there is an open L–set μ such that $\mu(x) \neq \mu(y)$.

■

Part (a) of the following proposition appears in ([15]).

IV.6.2 PROPOSITION

(a) The L–unit interval I(L) with the right hand topology τ_r, is L–T_o.

(b) $(I(L)), \iota(\tau_r))$ is a T_o–space.

PROOF

Suppose [f], [g] are distinct in I(L). Then for some $t \in \mathbb{R}$, $f(t+) \neq g(t+)$ and without loss of generality we may assume that $f(t+) \nleq g(t+)$. Thus $R_t([f]) \neq R_t([g])$, proving that $(I(L),\tau_r)$ is L–T_o. Further let $\alpha = g(t+)$. Then $R_t^{-1}(L/\!\downarrow \alpha)$ is an open neighbourhood of [f] in $\iota(\tau_r)$ not containing [g]. Therefore $(I(L),\iota(\tau_r))$ is T_o.

■

Let (X,Δ) be a topological space. Since the characteristic function of a subset A of X is lower semicontinuous if and only if A is open, we have that X has the weak topology given by lsc(X,I). From this observation we can conclude that X is T_o if and only if lsc(X,I) separates points in X. By replacing I with I(L) we have similar results in L–topological spaces.

Let (X,τ) be an L–topological space. For each $\mu \in \tau$ define

$$f_\mu : X \longrightarrow I(L)\ ,\ f(x) = [\lambda_{\mu(x)}]$$

where $$\lambda_{\mu(x)}(r) = \begin{cases} 1 & r < 0 \\ \mu(x) & 0 \le r \le 1 \\ 0 & 1 < r \end{cases}$$

Then f_μ is (τ,τ_r) – L–continuous since

$$f_\mu^{\leftarrow}(R_t) = \begin{cases} \underline{1} & t < 0 \\ \mu & 0 \le t \le 1 \\ \underline{0} & 1 < t \end{cases}$$

Let LSC(X,I(L)) denote the collection

$$\{f : X \longrightarrow I(L) : f \text{ is } (\tau,\tau_r) - \text{L–continuous}\}.$$

Thus X has the weak L–topology given by LSC(X,I(L)).

An easy consequence of the above is :

IV.6.3 PROPOSITION

(X,τ) is L–T_o if and only if X has the weak L–topology given by LSC(X,I(L)) and the collection LSC(X,I(L)) separates points in X.

PROOF

As shown above every L–topological space (Y,δ) has the weak L–topology given by LSC(Y,I(L)). We need therefore only show that (X,τ) is L–T_o if and only if LSC(X,I(L)) separates points in X. Suppose (X,τ) is L–T_o, then $\{f_\mu : \mu \in \tau\} \subseteq$ LSC(X,I(L)) separates points in X. Conversely let x,y be distinct in X. Then there exists $f \in$ LSC(X,I(L)) such that $f(x) \neq f(y)$. Therefore for some $t \in \mathbb{R}$

$$f(x)(t+) = f^{\leftarrow}(R_t)(x) \neq f^{\leftarrow}(R_t)(y) = f(y)(t+).$$

But $f^{\leftarrow}(R_t) \in \tau$. This proves that (X,τ) is L–T_o.

■

Concerning products and subspaces we have the following easily verified propositions of Liu ([15]).

IV.6.4 PROPOSITION

(a) Subspaces of L–T_o topological spaces are L–T_o.

(b) Products of L–T_o L–topological spaces are L–T_o.

■

Applying our L–embedding lemma to the L–T_o L–topological spaces we have :

IV.6.5 PROPOSITION

(X,τ) is L–T_o if and only if X can be L–embedded in a cube of $(I(L),\tau_r)$.

PROOF

($\Longrightarrow$): Suppose (X,τ) is L–T_o then by (IV.6.3) X has the weak L–topology given by LSC(X,I(L)) and this collection separates points in X. Applying (IV.5.2), the evaluation mapping

$$e : X \longrightarrow \prod_{f \in LSC(X,I(L))} I(L)$$

is an L–embedding.

($\Longleftarrow$): Subspaces of L–T_o L–topological spaces are L–T_o.

■

IV.6.6 PROPOSITION

If (X,τ) is L–T_o then $(X,\iota(\tau))$ is T_o.

PROOF

In the proof of the preceeding proposition we saw that $e : X \longrightarrow \prod_{f \in LSC(X,I(L))} I(L)$ is an L–embedding. By (IV.5.3) e is an embedding in the topological modification. Thus $(X,\iota(\tau))$ can be embedded in a cube of a T_o–space $(I(L),\iota(\tau_r))$. This proves that $(X,\iota(\tau))$ is T_o.

■

Alternatively if (X,τ) is L–T_o, then for two distinct points x,y there exists $\mu \in \tau$ such that $\mu(x) \neq \mu(y)$. Let $\alpha = \mu(x)$, and without loss of generality we may assume that $\mu(x) = \alpha \nleq \mu(y)$. Then the open set $\mu^{-1}(L/\downarrow \alpha)$ is a neighbourhood of y not containing x. Thus $(X,\iota(\tau))$ is T_o.

IV.6.7 DEFINITION ([15])

An L–completely regular L–topological space (X,τ) which is L–T_o shall be called an L–Tychonoff space.

■

IV.6.8 PROPOSITION

If (X,τ) is L–Tychonoff, then C(X,I(L)) separates points in X.

PROOF

Let x,y be distinct in X. Then there exists $\mu \in \tau$ such that $\mu(x) \neq \mu(y)$. But since X is L–completely regular, there exists a collection $\{f_k : k \in K\} \subseteq C(X,I(L))$ such that $\bigvee_k f_k^{\leftarrow}(R_o) = \mu$. Thus for some $k \in K$, $f_k(x)(0+) \neq f_k(y)(0+)$ which implies that $f_k(x) \neq f_k(y)$.

■

In general topology Tychonoff spaces are precisely those which can be embedded in a cube of the unit interval. Replacing the unit interval by the L–unit interval, we have :

IV.6.9 PROPOSITION

(X,τ) is L–Tychonoff if and only if X can be L–embedded in a cube of $(I(L),\tau_{I(L)})$.

PROOF

($\Leftarrow$): Products of L–completely regular spaces are L–completely regular (III.2.2), and products of L–T_o spaces are L–T_o. Therefore a cube of $(I,(L),\tau_{I(L)})$ is L–Tychonoff, (the L–space $(I,(L),\tau_{I(L)})$ is L–T_o since $\tau_r \subset \tau_{I(L)}$). Hence (X,τ) is L–Tychonoff.

(⟹): We know from (III.1.6) that (X,τ) has the weak L–Topology given by $C(X,I(L))$ and from (IV.6.8) $C(X,I(L))$ separates points in X. Applying the L–embedding lemma (IV.5.2) we have that the evaluation mapping

$$e : X \longrightarrow \prod_{f \in C(X,I(L))} I(L)$$

is an L–embedding.

∎

IV.6.10 PROPOSITION

If (X,τ) is L–Tychonoff, then $(X,\iota(\tau))$ is Tychonoff.

PROOF

In the proof of (IV.6.9) we saw that $e : X \longrightarrow \prod_{f \in C(X,I(L))} I(L)$ is an L–embedding. By (IV.5.3) e is an embedding in the topological modification. Further by (IV.4.9), $\prod_{f \in C(X,I(L))} I(L)$ with the product topology $\bigvee_{f \in C(X,I(L))} \pi_f^{-1}(\iota(\tau_{I(L)}))$, is a compact, Hausdorff space. Thus $(X,\iota(\tau))$ is Tychonoff.

∎

Using a different type of L–embedding lemma, Liu proved Proposition IV.6.9. He showed that if (X,τ) is an L space and $\{f_k : X \longrightarrow (X_k,\tau_k) : k \in K\}$ distinguishes L–points and closed L–sets in the following sense :

If for each $x \in X$, $\alpha \in L$ and μ a closed L–set satisfying $\alpha \nleq \mu(x)$ there exists $k \in K$ such that

$\alpha \nleq \overline{f_k^{\leftarrow}(\mu)}(x)$, then the mapping

$$e : X \longrightarrow \prod_k X_k$$

is an L–embedding if $\{f_k : k \in K\}$ separates points. He then went on to show that if (X,τ) is L–Tychonoff, then $C(X,I(L))$ satisfies the conditions of his lemma.

IV.7 L–T_o IDENTIFICATION OF L–SPACES

It is well known that each topological space can be modified into a T_o space by identifying points with identical neighbourhood systems. This simple technique is very useful since a large number of topological properties are preserved and inherited from the modified space. Using (III.6.3) we extend this technique to L–spaces. In this section we are mainly concerned with showing that the L–T_o modification of an L–completely regular space is an L–Tychonoff space, and that if the topological modification of the L–T_o modification of an L–space is completely regular then the topological modification of the original L–space is completely regular. The main result of this chapter stating that the topological modification of an L–completely regular space is completely regular then follows easily.

Let L be a completely distributive lattice.

IV.7.1 DEFINITION

Let (X,τ) be an L–space. Let $\sim$ be the relation on X defined as

$$x \sim y \text{ if and only if for each } f \in LSC(X,I(L)) : f(x) = f(y).$$

The relation $\sim$ is an equivalence relation, and $(X_{|\sim}, \tau_{|\sim})$ shall denote the quotient L–space given the L–topology $\tau_{|\sim}$, induced by the quotient map $\rho : X \longrightarrow X_{|\sim}$.

■

The following lemma will help prove that $(X_{|\sim}, \tau_{|\sim})$ is an L–T_o space.

IV.7.2 LEMMA

The quotient L–topology $\tau_{|\sim}$ is the weak L–topology given by

$$S = \{\hat{f} : X_{|\sim} \longrightarrow I(L) : f \in LSC(X,I(L))\}$$

where $\hat{f}([x]) = f(x)$ for each $[x] \in X_{|\sim}$.

PROOF

Since each $f \in LSC(X,I(L))$ is constant on each of the equivalence classes $\{[x] : x \in X\}$, each $\hat{f}$ is well defined.

First we show that $S \subseteq LSC(X_{|\sim}, I(L))$. Let $\hat{f} \in S$, to show that $\hat{f}$ is $(\tau_{|\sim}, \tau_R) -$ L–continuous it will by (0.2.9(v)) suffice to show that for each $t \in \mathbb{R}$ $\rho^{\leftarrow}(\hat{f}^{\leftarrow}(R_t))$ is an open L–set in (X,τ). But

$$\rho^{\leftarrow}(\hat{f}^{\leftarrow}(R_t)) = R_t \circ \hat{f} \circ \rho = R_t \circ f = f^{\leftarrow}(R_t),$$

is an open L–set in τ, since $f \in LSC(X,I(L))$. Hence the weak L–topology given by S is contained in $\tau_{|\sim}$.

Conversely, let $\mu \in \tau_{|\sim}$, then $\rho^{\leftarrow}(\mu) \in \tau$. Define $f_{\rho^{\leftarrow}(\mu)} : X \longrightarrow I(L)$ as in the paragraph preceeding (IV.6.3). Then $\mu = (\hat{f}_{\rho^{\leftarrow}(\mu)})^{\leftarrow}(R_t)$ for $0 \leq t \leq 1$, and $\hat{f}_{\rho^{\leftarrow}(\mu)} \in S$. This implies that $\tau_{|\sim}$ is contained in the weak L–topology given by S. Therefore $\tau_{|\sim}$ is precisely the weak L–topology given by S.

■

IV.7.3 PROPOSITION

$(X_{|\sim}, \tau_{|\sim})$ is an L–T_o L–space.

PROOF

Suppose $[x] \neq [y]$, then for some $f \in LSC(X,I(L))$, $f(x) \neq f(y)$. Thus there exists $t \in \mathbb{R}$ such that $f(x)(t+) \neq f(y)(t+)$. This implies that $\hat{f}^{\leftarrow}(R_t)([x]) \neq \hat{f}^{\leftarrow}(R_t)([y])$. But by (IV.7.2) $\hat{f}^{\leftarrow}(R_t)$ is an open L–set in $\tau_{|\sim}$. Hence by the definition of L–T_o L–spaces $(X_{|\sim}, \tau_{|\sim})$ is an L–T_o L–space.

■

We shall now refer to $(X|\sim, \tau|\sim)$ as the L–T_o modification of (X,τ).

Using the L–T_o modification technique we show

(a) if (X,τ) is L–completely regular then $(X_{|\sim}, \tau_{|\sim})$ is L–completely regular,

(b) if $(X_{|\sim}, \iota(\tau_{|\sim}))$ is completely regular then $(X,\iota(\tau))$ is completely regular.

IV.7.4 PROPOSITION

If (X,τ) is L–completely regular then $(X_{|\sim}, \tau_{|\sim})$ is L–Tychonoff.

PROOF

Since L–Tychonoff spaces are L–T_o, L–completely regular spaces we need only, by (IV.7.3), show that $(X_{|\sim}, \tau_{|\sim})$ is L–completely regular.

Suppose $\mu \in \tau_{|\sim}$, then $\rho^{\leftarrow}(\mu) \in \tau$. Since (X,τ) is L–completely regular, there are collections $\{\nu_k : k \in K\}$ of L–sets and $\{f_k : X \longrightarrow I(L) : k \in K\}$ of L–continuous functions such that :

$$\bigvee_k \nu_k = \rho^{\leftarrow}(\mu) \text{ and for each } k \in K$$
$$\nu_k \leq f_k^{\leftarrow}(L_1') \leq f_k^{\leftarrow}(R_o) \leq \rho^{\leftarrow}(\mu).$$

For each f_k, $k \in K$ let $\hat{f}_k$ be defined as in Proposition IV.7.2 (since $C(X,I(L)) \subseteq LSC(X,I(L))$, $\hat{f}_k$ is well defined).

For each $t \in \mathbb{R}$

$$\rho^{\leftarrow}(\hat{f}_k^{\leftarrow}(R_t)) = f^{\leftarrow}(R_t), \ \rho^{\leftarrow}(\hat{f}_k^{\leftarrow}(L_t)) = f_k^{\leftarrow}(L_t).$$

Hence $\hat{f}_k$ is $(\tau_{|\sim}, \tau_{I(L)})$–L– continuous.

Let $k \in K$ and $x \in X$. Then

(i) $\hat{f}_k^{\leftarrow}(L_1')([x]) = f_k^{\leftarrow}(L_1')(x) \leq f_k^{\leftarrow}(R_o)(x) = \hat{f}_k^{\leftarrow}(R_o)([x])$
and $f_k^{\leftarrow}(R_o)(x) \leq \rho^{\leftarrow}(\mu)(x) = \mu([x])$,

(ii) $\rho^{\rightarrow}(\nu_k)([x]) = \bigvee_{y \in [x]} \nu_k(y) \leq \bigvee_{y \in [x]} f_k^{\leftarrow}(L_1')(y)$. But $f_k \in C(X,I(L)) \subseteq LSC(X,I(L))$. Thus f_k is constant on the set $[x]$. Therefore

$$\rho^{\rightarrow}(\nu_k)([x]) \leq f_k^{\leftarrow}(L_1')(x) = \hat{f}_k^{\leftarrow}(L_1')([x]).$$

(iii) Since ρ is onto we have

$$\bigvee_k \rho^{\rightarrow}(\nu_k) = \rho^{\rightarrow}(\bigvee_k \nu_k) = \rho^{\rightarrow}(\rho^{\leftarrow}(\mu)) = \mu.$$

Combining (i), (ii) and (iii) we have a collection $\{\rho^{\leftarrow}(\nu_k) : k \in K\}$ of L–sets and a collection $\{\hat{f}_k : X_{|\sim} \longrightarrow I(L) : k \in K\}$ of $(\tau_{|\sim}, \tau_{I(L)})$– L–continuous functions such that :

$$\vee \rho^{\rightarrow}(\nu_k) = \mu$$

and for each $k \in K$

$$\rho^{\rightarrow}(\nu_k) \leq \hat{f}_k^{\leftarrow}(L_1') \leq \hat{f}_k^{\leftarrow}(R_o') \leq \mu.$$

This proves that $(X_{|\sim}, \tau_{|\sim})$ is L–completely regular.

∎

IV.7.5 PROPOSITION

Let (X,τ) be an L–space. If $(X_{|\sim}, \iota(\tau_{|\sim}))$ is completely regular, then $(X,\iota(\tau))$ is completely regular.

PROOF

We must show that whenever A is a closed set in $(X,\iota(\tau))$ and $x \notin A$, there is a continuous function $f : X \longrightarrow I$ such that $f(x) = 1$ and $f(A) = 0$.

By (IV.6.3) we have that, $\tau = \bigvee_{f \in LSC(X,I(L))} f^{\leftarrow}(\tau_r)$. Thus by (IV.2.7)

$$\iota(\tau) = \bigvee_{f \in LSC(X,I(L))} f^{-1}(\iota(\tau_r)).$$

The topology $\iota(\tau_r)$ has as a subbase the collection

$$\{R_t^{-1}(L/\downarrow \alpha) : t \in \mathbb{R} \text{ and } \alpha \in L\}.$$

Therefore

$$\{f^{-1}(R_t^{-1}(L/\downarrow \alpha)) : f \in LSC(X,I(L)), t \in \mathbb{R}, \alpha \in L\}$$

forms a subbase for $\iota(\tau)$.

Let A be closed in $(X,\iota(\tau))$ and $x \notin A$. Since A is closed there must exist a basic open set such that $x \in U$ and $U \cap A = \emptyset$.

The set U is of the form

$$U = \bigcap_{i=1}^{n} f_i^{-1}(R_{t_i}^{-1}(L/\downarrow \alpha_i))$$

where $\alpha_i \in L$ and $f_i \in LSC(X,I(L))$.

The set $\hat{U} = \bigcap_{i=1}^{n} \hat{f}_i^{-1}(R_{t_i}^{-1}(L/\downarrow \alpha_i)$ is an open set in $(X_{|\sim}, \iota(\tau_{|\sim}))$. We claim that $\hat{U} \cap \rho(A) = \emptyset$. Suppose otherwise. Then since $\rho(U) = \hat{U}$, there exists $u \in U$ and $a \in A$ such that $[u] = [a]$. Thus $f_i(u) = f_i(a)$ for $i = 1,\cdots,n$. This implies that $a \in U$, a contradiction since $U \cap A = \emptyset$. We have now that $[x] \in \hat{U}$ and $\hat{U} \cap \rho(A) = \emptyset$. Thus $[x] \notin \overline{\rho(A)}$. Since $(X_{|\sim}, \iota(\tau_{|\sim}))$ is completely regular there is a continuous function

$$h : X_{|\sim} \longrightarrow I$$

such that $h([x]) = 1$ and $h(\overline{\rho(A)}) = 0$.

By (IV.2.4(a)) $\rho : X \longrightarrow X_{|\sim}$ is $(\iota(\tau), \iota(\tau_{|\sim}))$–continuous. Define the function

$$f : X \longrightarrow I$$

by $f = h \circ \rho$. Then f is continuous and $f(x) = h([x]) = 1$ and $f(A) = f(\rho(A)) = 0$.

■

Finally our main result :

IV.7.6 PROPOSITION

Let (X,τ) be L–completely regular, then $(X,\iota(\tau))$ is completely regular.

PROOF

By (IV.7.4) $(X_{|\sim}, \tau_{|\sim})$ is L–Tychonoff. Then by (IV.6.10), $(X_{|\sim}, \iota(\tau_{|\sim}))$ is Tychonoff. Applying (IV.7.5) we have $(X, \iota(\tau))$ is completely regular.

■

IV.7.7 COROLLARY

(X,Δ) is completely regular if and only if $(X,\omega(\Delta))$ is completely regular.

PROOF

($\Longrightarrow$): Proved in (IV.3.8).

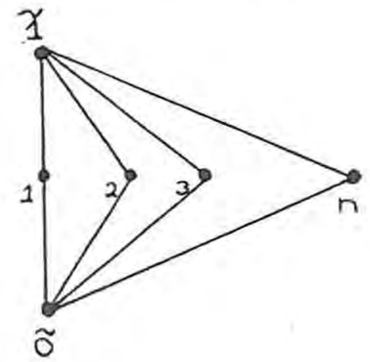
($\Longleftarrow$): If $(X,\omega(\Delta))$ is L–completely regular, then by (IV.7.6) $(X,\iota(\omega(\Delta)))$ is completely regular. But since we have assumed that L is completely distributive, $\iota(\omega(\Delta)) = \Delta$.

■

IV.8 A COUNTEREXAMPLE

We construct a continuous, non–distributive lattice L with an order reversing involution, and a topological space (X,Δ) for which $lsc(X,L) \subsetneq \omega(\Delta)$. This then shows that continuity and therefore also completeness are insufficient conditions on a lattice for $lsc(X,L)$ and $\omega(\Delta)$ to coincide.

Let L be the lattice given by the diagram

L : 

, $n \in \mathbb{N}$

The operation $'$: L $\longrightarrow$ L defined by

$$\tilde{0}' = \tilde{1},\ \tilde{1}' = \tilde{0} \text{ and } n' = n \text{ for } n \in \mathbb{N}$$

is an order reversing involution. It is easily seen that L is a complete lattice with an order reversing involution.

IV.8.1 PROPOSITION

(a) L is not distributive.

(b) L is a continuous lattice.

PROOF

(a) Let p,q,r be distinct in $\mathbb{N}$. Then $p \wedge (q \vee r) = p$, but $(p \wedge q) \vee (p \wedge r) = \tilde{0} \vee \tilde{0} = \tilde{0}$. Thus $p \wedge (q \vee r) \neq (p \wedge q) \vee (p \wedge r)$, proving that L is not distributive.

(b) By example (c) in (0.1) it is sufficient to note that L contains no strictly increasing infinite chains.

■

The upper topology u(L) on L is the topology is generated by

$$\mathscr{S} = \{L/\downarrow n : n \in \mathbb{N}\} \cup \{\mathbb{N} \cup \{\tilde{1}\}\}.$$

The collection of all finite intersections of elements of $\mathscr{S}$ forms a base for u(L). Thus

$$\mathscr{B}(u(L)) = \{L\} \cup \{\{\tilde{1}\} \cup (\mathbb{N}/A) : A \in 2^{(\mathbb{N})}\}$$

is a base for u(L). We can now observe the following about u(L) :

(i) Each non–empty open set U of u(L) will contain a tail of $\mathbb{N}$. That is there exists $n \in \mathbb{N}$ such that $\{n+k : k \in \mathbb{N}\} \subseteq U$.

(ii) No open set other than L itself will contain $\tilde{0}$ and each non–empty open set contains $\tilde{1}$.

The product topology on $L \times L$ has as a base

$$\mathscr{B}(u(L) \times u(L)) = \{B_1 \times B_2 : B_1, B_2 \in \mathscr{B}(u(L))\}$$

Thus for each non–empty $W \in u(L) \times u(L)$ there exists $(m,n) \in \mathbb{N} \times \mathbb{N}$ such that $\{(n+k, n+l) : k,l \in \mathbb{N}\} \subseteq W$.

IV.8.2 PROPOSITION

The operation

$$\wedge : L \times L \longrightarrow L,$$

is not continuous.

PROOF

Let U be a subbasic open set. If $\wedge^{-1}(U)$ where open in $L \times L$, then $\wedge^{-1}(U)$ would contain a tail of $\mathbb{N} \times \mathbb{N}$.

Thus there exists $(m,n) \in \mathbb{N}$ such that
$\{(m+k, n+l) : k, l \in \mathbb{N}\} \subseteq \wedge^{-1}(U)$. Therefore $\wedge^{-1}(U)$ contains two distinct elements of $\mathbb{N}$, say p,q. Then $p \wedge q = \tilde{0} \in U$, a contradiction.

■

Knowing now that $\wedge : L \times L \longrightarrow L$ is not continuous, we construct a topological space (X,Δ) such that $lx(X,L) \subsetneqq \omega(\Delta)$.

Define f, g : $\mathbb{N} \longrightarrow L$ by $f(1) = g(1) = \tilde{1}$ and $f(n) = n$, $g(n) = n+1$ for $n \in \mathbb{N}/\{1\}$.
Let Δ be the weak topology on $\mathbb{N}$ generated by f,g with respect to the upper interval topology u(L) on L. A base for Δ is given by taking all finite intersections of elements of $\mathscr{S}(\Delta) = \{f^{-1}(U), g^{-1}(U) : U \in u(L)\}$. Each non–empty element of $\mathscr{S}(\Delta)$ will contain 1 and a tail of $\mathbb{N}$. We have that

$$f \wedge g(n) = \begin{cases} \tilde{1} & n = 1 \\ \tilde{0} & n \geq 2 \end{cases}$$

Let U be a non–empty open set of (L,u(L)) other than L. Then $(f \wedge g)^{-1}(U) = 1$ since $\tilde{0} \notin U$ and $\tilde{1} \in U$. Since each open subset of $(\mathbb{N},\Delta)$ contains a tail of $\mathbb{N}$, $(f \wedge g)^{-1}(U)$ is not open. Thus $f \wedge g$ is not continuous. But $f, g \in lsc(\mathbb{N},L) \subseteq \omega(\Delta)$, which implies since $\omega(\Delta)$ is an L–topology that $f \wedge g \in \omega(\Delta)$. We therefore have

$$lsc(\mathbb{N},L) \subsetneqq \omega(\Delta).$$

The topology $\iota(\omega(\Delta))$ on $\mathbb{N}$ will make $f \wedge g$ continuous with respect to (L,u(L)), thus

$$\Delta \subsetneqq \iota(\omega(\Delta)).$$

Therefore continuity of our lattice is not a sufficient condition for Proposition IV.2.2 to hold for all topological spaces.

We can also use this example to show that continuity of our lattice L, is insufficient to have $\iota(\bigvee_k \tau_k) = \bigvee_k \iota(\tau_k)$ for a collection of L–topologies $\{\tau_k : k \in K\}$ on a set X, (Proposition IV.2.5).

Let $\tau_f = \{\underline{\tilde{1}} \,;\, \underline{\tilde{0}} \,;\, f\}$ and $\tau_g = \{\underline{\tilde{1}} \,;\, \underline{\tilde{0}} \,;\, g\}$. Then τ_f, τ_g are L–topologies on $\mathbb{N}$. The topology Δ can be equivalently defined to be $\iota(\tau_f) \vee \iota(\tau_g)$. But $f \wedge g$ is $(\iota(\tau_f \vee \tau_g)$, u(L))–continuous. Hence $\iota(\tau_f) \vee \iota(\tau_g) \subsetneqq \iota(\tau_f \vee \tau_g)$.

REFERENCES

MONOGRAPHS AND BOOKS

[1] G. Gierz, K.H. Hofmann, K. Keimel, J.D. Lawson, M. Mislove and D.S. Scott, "*A Compendium on Continuous Lattices*", Springer–Verlag, Berlin, Heidelberg, New York, 1980.

[2] P.T. Johnstone, "*Stone Spaces*", Camb. Univ. Press, Cambridge, 1982.

[3] S. Willard, "*General Topology*", Addison Wesley, Reading, Massachusetts, 1970.

ARTICLES

[4] T.E. Gantner, R.C. Steinlage, R.H. Warren, "*Compactness in Fuzzy Topological Spaces*", J. Math. Anal, Appl. **62**(1978) 547–562.

[5] J.A. Goguen, "*L–fuzzy Sets*", J. Math. Anal. Appl. **18**(1967), 145–174.

[6] B.W. Hutton, "*Uniformities on Fuzzy Topological Spaces*", J. Math. Anal. Appl. **58**(1977) 559–571.

[7] B.W. Hutton, "*Normality in Fuzzy Topological Spaces*", J. Math. Anal. Appl. **50**(1975) 74–79.

[8] M. Katetov, "*On Real–Valued Functions in Topological Spaces*", Fund. Math. **38**(1951) 85–91.

[9] M. Katetov, Correction to "*On Real–Valued Functions in Topological Spaces*", Fund. Math. **40**(1953) 203–205.

[10] A.K. Katsaras, "*On Fuzzy Uniform Spaces*", J. Math. Anal. Appl. **101**(1984) 97–113.

[11] T. Kubiak, "*The topological modification of the L–fuzzy unit interval*", In : Applications of Category Theory to Fuzzy Subsets (Rodabaugh, S.E., Klement, E.P., and Höhle, U., Eds.) Kluwer Academic Rubl., Dordrecht, 1991. (To appear).

[12] T. Kubiak, "*L–Fuzzy Normal Spaces and Tietze Extension Theorem*", J. Math. Anal. Appl. **125**(1987) 141–153.

[13] T. Kubiak, "*A Quotient of the Helly Space which is AR(Normal)*", Math. Japonica **36**, No.4(1991) 633–637.

[14] Y.–M. Liu, "*Intersection Operation on Union–Preserving Mappings in Completely Distributive Lattices*", J. Math. Anal. Appl **84**(1981) 249–255.

[15] Y.–M. Liu, "*Pointwise Characterization of Complete Regularity and Imbedding Theorem in Fuzzy Topological Spaces*", Scientia Sinica (Series A) *b*26(1983) 138–147.

[16] R. Lowen, "*Initial and Final Fuzzy Topologies and the Fuzzy Tychonoff Theorem*", J. Math. Anal. Appl. **58**(1977) 11–21.

[17] S.E. Rodabaugh, "*A Theory of Fuzzy Uniformities with Application to the Fuzzy Real Lines*", J. Math. Anal. Appl. **129**(1988) 37–70.

[18] S.E. Rodabaugh, "*The Hausdorff Separation Axiom for Fuzzy Topological Spaces*", Topology Appl. **11**(1980) 319–334.

[19] L.A. Zadeh, "*Fuzzy Sets*", Inform. Contr. **8**(1965) 338–353.