A COMPUTER ANALYSIS OF SOME OF THE HARRISON METRICS.

Thesis

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INTRODUCTION.

In his paper B.K.Harrison concludes with the observation that his"solutions....... are presented as raw material for further research in General Relativity". In the same spirit, the present work started out as an attempt to process that raw material in a production-line powered by a computer. Harrison's solutions would be fed in at one end, and the finished product, as yet undecided, would appear at the other.

In the event, however, the project became more like an exercise in quality control, to continue the analogy. A search was made for algebraic criteria which would distinguish between those solutions which were acceptable for further analysis with particular regard to gravitational radiation, and those which were not. Regrettably, no criteria could be found which characterised radiative solutions unequivocally, and, at the same time, lent themselves to a computer approach.

The result is that the discussion of radiative solutions has had to be relegated to an appendix (Appendix 1), while the main body of the work is concerned with the determination of those quantities (the Newman-Penrose scalars) which would seem to be the foundation of any future computer-based analysis of gravitational radiation.

Chapter 1 is an account of the underlying mathematical formulation, defining the terms, concepts and processes involved. In Chapter 2 the transformation of some of the ideas of Chapter 1 into computer software is presented. Chapter 3 is concerned with the specific metrics (the Harrison metrics) and the extent to which they have been processed.

The project has leaned heavily on papers by Harrison(1), for the "raw material", by D'Inverno and Russell-Clark(1), who pioneered the techniques and classified the Harrison metrics, and by Sachs (1), for the treatment of gravitational radiation. However, the analysis of diagonal metrics, the special tetrad of Chapter 2 and the results in Appendix 2 are new.

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CONTENTS.

CHAPTER 1

MATHEMATICS

1. The Mathematical Framework

Fundamental to the study of gravitation in this context is the assertion that the geometry of space-time is Riemannian. The components of the metric tensor, g_{ij}, in a coordinate neighbourhood of the manifold, are related to the fundamental form

$$ds^2 = g_{ij}dx^i dx^j \ , \tag{1.1}$$

of the space. The signature of the metric tensor is taken as +2 and the summation convention is assumed. A metric tensor for which

$$g_{ij} = 0 \ , \qquad \text{for all} \quad i \neq j \ , \tag{1.2}$$

is known as a "diagonal" metric. Non-zero components are functions of class C^2 of the coordinates.

The Christoffel symbols are defined by

$$[ij,k] = \tfrac{1}{2}(g_{ik,j} + g_{jk,i} - g_{ij,k}) \ , \tag{1.3}$$

where the comma denotes partial differentiation, and

$$\left\{ {i \atop j\ k} \right\} = g^{ir}[jk,r] \ , \tag{1.4}$$

where the g^{ij} represent the contravariant components of the metric tensor. The Riemann (or curvature) tensor [Synge (1)]

$$R_{ijkm} = g^{is}\left[\left\{ {s \atop j\ m} \right\}_{,k} - \left\{ {s \atop j\ k} \right\}_{,m} + \left\{ {s \atop r\ k} \right\}\left\{ {r \atop j\ m} \right\} - \left\{ {s \atop r\ m} \right\}\left\{ {r \atop j\ k} \right\} \right] , \tag{1.5}$$

the Ricci tensor

$$R_{ij} = R^{k}{}_{ijk}, \tag{1.6}$$

and the curvature invariant

$$R = R^{i}{}_{i} , \tag{1.7}$$

are all geometrical constructs which are determined by the metric tensor. While the metric itself contains the geometrical, and therefore physical information about the particular space-time in question, these constructs often have to be employed to develop significant insight into the space-time geometry.

Einstein's equations

$$R_{ij} - \tfrac{1}{2} g_{ij} R = k\, T_{ij} , \tag{1.8}$$

where T_{ij} is the energy-momentum tensor, provide the fundamental link between the "geometry" and the "physics" of the space-time. These may also be written, with a little manipulation [Lawden (1)]

$$R_{ij} = k(\tfrac{1}{2} g_{ij} T - T_{ij}) , \tag{1.9}$$

where $T = T^{i}{}_{i}$. If the space is a vacuum space - i.e. devoid of all energy in the form of discrete masses and electromagnetic radiation - then the energy-momentum tensor must be zero, and consequently

$$R_{ij} = 0 . \tag{1.10}$$

Conversely, if $R_{ij} = 0$, then the metric describes a vacuum space, or is, loosely, a "vacuum metric".

The problem of gravitation would appear to be solved! The "geometry" can be deduced from the "physics" using (1.9), or the "physics" can be deduced from the "geometry" using (1.8). Unfortunately, Einstein's equations form a very formidable set of non-linear partial differential equations [Synge (1) Ch. 4], and once solved, it is not always easy to extract the physical information from the metric. It becomes pertinent therefore to ask just how much information can be obtained from an exact solution to the equations (1.8).

As outlined above, it is a fairly simple matter to establish whether or not a metric describes a vacuum space, using only the techniques of classical differential geometry. Before discussing other useful techniques, it is appropriate to dwell a little further on the properties of a vacuum metric.

There are a number of symmetries inherent in the curvature tensor [Synge (1)] which reduce the number of independent components of that tensor to 20. The symmetry relations are

$$R_{ijkm} = -R_{jikm} = -R_{ijmk} = R_{kmij} , \tag{1.11}$$

$$R_{ijkm} + R_{ikmj} + R_{imjk} = 0 , \tag{1.12}$$

while a convenient set of independent components is given by

$$\begin{array}{ccccc} R_{1212} & R_{1213} & R_{1214} & R_{1223} & R_{1224} \\ R_{1234} & R_{1313} & R_{1314} & R_{1323} & R_{1334} \\ R_{1414} & R_{1423} & R_{1424} & R_{1434} & R_{2323} \\ R_{2324} & R_{2334} & R_{2424} & R_{2434} & R_{3434} \end{array} \tag{1.13}$$

If the metric is a vacuum metric, then

$$R_{ij} = R^{k}{}_{ijk} = g^{km}R_{mijk} = 0 , \tag{1.14}$$

which are 10 independent linear equations in 20 unknowns. Consequently, a vacuum curvature tensor has only 10 independent components. For the diagonal case, of particular interest in this work, this figure is reduced to 8 independent components. This is shown below explicitly, since certain identities which arise out of the calculations will be required later on.

The first thing to notice is that, if the metric is diagonal then $k = m$ always, and therefore every component which appears in (1.14) must have at least one repeated numerical index. However, two of the set (1.13) (i.e. R_{1234}, R_{1423}) do not have this property, which leads to the suspicion that these may be zero. Indeed, from (1.5),

$$R_{1234} = g^{11}\left[\left\{ {1 \atop 2\ 4} \right\}_{,3} - \left\{ {1 \atop 2\ 3} \right\}_{,4} + \left\{ {1 \atop r\ 3} \right\}\left\{ {r \atop 2\ 4} \right\} - \left\{ {1 \atop r\ 4} \right\}\left\{ {r \atop 2\ 3} \right\}\right] , \tag{1.15}$$

but if the metric is diagonal, no Christoffel symbol with three different numerical indices can be non-zero, and therefore, by inspection, $R_{1234} = 0$. A similar argument holds in the case of R_{1423}.

Now, writing out (1.14) explicitly, the components become

$$R_{11} = -g^{22}R_{1212} - g^{33}R_{1313} - g^{44}R_{1414} = 0 \ , \qquad (1.16)$$

$$R_{12} = -g^{33}R_{1323} - g^{44}R_{1424} = 0 \ , \qquad (1.17)$$

$$R_{13} = g^{22}R_{1223} - g^{44}R_{1434} = 0 \ , \qquad (1.18)$$

$$R_{14} = g^{22}R_{1224} + g^{33}R_{1334} = 0 \ , \qquad (1.19)$$

$$R_{22} = g^{11}R_{1212} - g^{33}R_{2323} - g^{44}R_{2424} = 0 \ , \qquad (1.20)$$

$$R_{23} = g^{11}R_{1213} + g^{44}R_{2434} = 0 \ , \qquad (1.21)$$

$$R_{24} = g^{11}R_{1214} + g^{33}R_{2334} = 0 \ , \qquad (1.22)$$

$$R_{33} = g^{11}R_{1313} - g^{22}R_{2323} - g^{44}R_{3434} = 0 \ , \qquad (1.23)$$

$$R_{34} = g^{11}R_{1314} - g^{22}R_{2324} = 0 \ , \qquad (1.24)$$

$$R_{44} = g^{11}R_{1414} - g^{22}R_{2424} - g^{33}R_{3434} = 0 \ , \qquad (1.25)$$

where the symmetry conditions (1.11) have been used.

Eliminate R_{1212} between (1.16) and (1.20) giving, with some manipulation

$$g_{22}g_{44}R_{1313} + g_{22}g_{33}R_{1414} + g_{11}g_{44}R_{2323} + g_{11}g_{33}R_{2424} = 0 \ , \qquad (1.26)$$

where, because the metric is diagonal, $g_{11} = (g^{11})^{-1}$, etc. Similarly, eliminate R_{3434} between (1.23) and (1.25) to give

$$g_{22}g_{44}R_{1313} - g_{11}g_{44}R_{2323} - g_{33}g_{22}R_{1414} + g_{11}g_{33}R_{2424} = 0 \ . \qquad (1.27)$$

Adding (1.26) and (1.27)

$$g_{22}g_{44}R_{1313} = -g_{33}g_{11}R_{2424} \ , \qquad (1.28)$$

and subtracting

$$g_{22}g_{33}R_{1414} = -g_{11}g_{44}R_{2323} \ , \qquad (1.29)$$

which show the dependence on R_{2424} and R_{2323} .

R_{1212}, R_{3434} can be obtained by substitution. A useful identity arises out of (1.28) and (1.29), namely that

$$\frac{g_{22}}{g_{11}}\left(\frac{R_{1313}}{g_{33}} - \frac{R_{1414}}{g_{44}}\right) = \left(\frac{R_{2323}}{g_{33}} - \frac{R_{2424}}{g_{44}}\right) . \qquad (1.30)$$

(a) Decomposition of the Curvature Tensor. A set of four linearly independent, complex unit vectors, ℓ^i, m^i, n^i, p^i, is called a null tetrad if the conditions

$$\begin{aligned} \ell_i \ell^i &= m_i m^i = n_i n^i \\ &= p_i p^i = 0 , \end{aligned} \qquad (1.31)$$

hold. Furthermore, if $p^i = \bar{m}^i$ is the complex conjegate of m^i, and ℓ^i, n^i are real, and if the orthogonality conditions

$$\ell_i n^i = - m_i \bar{m}^i = -1 , \qquad (1.32)$$

are imposed, then the null tetrad becomes a Newman-Penrose null tetrad [Newman and Penrose (1)] .

It is convenient in the tetrad notation to employ a Greek suffix to denote a particular tetrad, viz.

$$z_{\alpha i} = (\ell_i , n_i , m_i , \bar{m}_i) , \qquad (1.33)$$

and these indices (i.e. the Greek indices) can be raised or lowered with the flat-space metric

$$\eta^{\alpha\beta} = \eta_{\alpha\beta} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} . \qquad (1.34)$$

The tetrad components of the metric tensor are defined by

$$g_{ij} = \eta^{\alpha\beta} a_{\alpha i} a_{\beta j} , \tag{1.35}$$

whence

$$g_{ij} = - \ell_i n_j - \ell_j n_i + m_i \bar{m}_j + m_j \bar{m}_i . \tag{1.36}$$

Transformations of the tetrad which preserve the orthogonality conditions (equation (1.32)) are limited to a six-parameter restricted Lorentz group [Janis and Newman (1)] as follows:

A) A two-parameter group of null rotations about ℓ_i

$$\begin{aligned} \tilde{\ell}_i &= \ell_i , \\ \tilde{m}_i &= m_i + a \ell_i , \\ \tilde{n}_i &= n_i + a\bar{m}_i + \bar{a}m_i + a\bar{a} \ell_i . \end{aligned} \tag{1.37}$$

B) An ordinary Lorentz transformation in the $\ell_i n_i$ plane, and a spatial rotation in the $m_i \bar{m}_i$ plane.

$$\begin{aligned} \tilde{\ell}_i &= \lambda \ell_i , \\ \tilde{n}_i &= \lambda^{-1} n_i , \\ \tilde{m}_i &= e^{i\phi} m_i . \end{aligned} \tag{1.38}$$

where λ and ϕ are real functions.

C) A two-parameter group of null rotations about n_i

$$\begin{aligned} \tilde{\ell}_i &= \ell_i + b\bar{m}_i + \bar{b}m_i + b\bar{b}n_i , \\ \tilde{m}_i &= m_i + bn_i , \\ \tilde{n}_i &= n_i . \end{aligned} \tag{1.39}$$

The curvature tensor of a vacuum metric can be resolved into five complex tetrad components [Newman and Penrose (1)], known as the Newman-Penrose scalars,

$$\Psi_0 = - R_{ijkm} \ell^i m^j \ell^k m^m , \qquad (1.40)$$

$$\Psi_1 = - R_{ijkm} \ell^i n^j \ell^j m^m , \qquad (1.41)$$

$$\Psi_2 = - \tfrac{1}{2} R_{ijkm} (\ell^i n^j \ell^k n^m - \ell^i n^j m^k \bar{m}^m) , \qquad (1.42)$$

$$\Psi_3 = - R_{ijkm} n^i \ell^j n^k \bar{m}^m , \qquad (1.43)$$

$$\Psi_4 = - R_{ijkm} n^i \bar{m}^j n^k \bar{m}^m . \qquad (1.44)$$

This resolution of the curvature tensor leads to 5 complex (or 10 real) components which are independent, and therefore, in view of the previous result (page 4), contain all the information about the curvature.

(b) <u>The Spinor Formalism</u>. A full discussion of spinor algebra appears in Pirani (2), and the reader is referred to this for more details.

Spinors are the elements of a two-dimensional complex "spin"-space. Transformations in this space form a double-valued representation of the homogeneous Lorentz group. With every Lorentz transformation in Minkowski space

$$m^i \rightarrow L^i_{\ j} m^j , \qquad (1.45)$$

where the small Roman indices (i.e. the tensor indices) i, j have the range 1 to 4, there is associated a pair of spin transformations

$$\eta^A \rightarrow L^A_{\ B}\, \eta^B , \tag{1.46}$$

$$\bar{\eta}^A \rightarrow \bar{L}^A_{\ B}\, \bar{\eta}^B , \tag{1.47}$$

where the second representation is merely the conjugate representation (since spin-space is complex) and the capital Roman indices (the spin indices) have the range 1 to 2. The above transformations apply to "contravariant" spinors. If $\ell^B_{\ A}$ is the reciprocal of $L^B_{\ A}$, then a "covariant" spinor transforms according to

$$\mu_A \rightarrow \ell^B_{\ A}\, \mu_B , \tag{1.48}$$

$$\text{or} \quad \bar{\mu}_A \rightarrow \bar{\ell}^B_{\ A}\, \bar{\mu}_B . \tag{1.49}$$

To avoid confusion between these representations, the indices of spinors which transform according to the conjugate representation will be primed, and will be drawn from the end of the alphabet. Spinors of order greater than one may contain any number of contravariant or covariant indices of either representation. Thus $T^{AX'}_{\ \ \ B}$ transforms

$$T^{AX'}_{\ \ \ B} = L^A_{\ C}\, \bar{L}^{X'}_{\ Y'}\, \ell^D_{\ B}\, T^{CY'}_{\ \ \ D} . \tag{1.50}$$

The properties of spinors are sought by analogy with those of tensors and in particular, inner and outer products, symmetrization and anti-symmetrization are preserved by spin transformations.

The transition from tensors to spinors may be accomplished by means of Infeld-van der Waerden connecting quantities, represented by $\sigma^{AX'}_{\ \ \ i}$ for a contravariant index, and by $\sigma^{\ \ i}_{AX'}$ for a covariant index. A typical mixed second-rank tensor thus becomes

$$T^{AX'}_{BY'} = \sigma^{AX'}_{\ \ i} \, \sigma^{\ j}_{BY'} \, T^{i}_{\ j} \ . \tag{1.51}$$

The connecting quantities are defined in a local Cartesian coordinate frame. Usually they form a set of Pauli spin-matrices although often a coordinate- or spin-transformation is required to bring them into such a form.

For a tensor of rank four

$$B_{ijkm} \rightarrow B_{AW'BX'CY'DZ'} \ , \tag{1.52}$$

and in particular, if $B_{ijkm} = R_{ijkm}$, the curvature tensor of a vacuum metric, it can be shown [Pirani (2)] that $R_{AW'BX'CY'DZ'}$ can be written in terms of a symmetric four-spinor, viz.

$$R_{AW'BX'CY'DZ'} = \Psi_{(ABCD)} \epsilon_{W'X'} \epsilon_{Y'Z'} + \epsilon_{AB} \epsilon_{CD} \Psi_{(W'X'Y'Z')} \ , \tag{1.53}$$

where ϵ_{AB} is the two-dimensional Levi-Civita alternating symbol. The independent components of such a symmetric four-spinor are Ψ_{1111}, Ψ_{1112}, Ψ_{1122}, Ψ_{1222}, Ψ_{2222} - i.e. 5 complex, or 10 real components, in accordance with the previous representations of the vacuum curvature tensor.

2. The Mathematical Structure

The Significance of the Curvature Tensor. Geometry provides the conceptual key to the problem of gravitation, and the curvature tensor is the mathematical key to geometry [Synge (1) Intro. p.viii]. The reason for this lies in the demands made by the postulates of relativity, namely, that in general different gravitational fields may only be distinguished by the variations in the field quantities (the principle of equivalence)[1], and that these variations should manifest themselves as tensors (the principle of covariance) [Pirani (2)]. These demands are most simply met by the curvature tensor, and it is natural that any search for a method of distinguishing between different space-times should begin with a study of the structure of this tensor [Pirani (1)].

A convenient decomposition of the curvature tensor splits it into two parts, the trace (or Ricci tensor), and the trace free part (or Weyl tensor) [Plebanski and Stachel (1)]. The information contained in the curvature tensor is spread out over these two entities, and in order to characterize the curvature tensor, it is necessary to develop a classification scheme for both.

The classification of the Weyl tensor is called the "Petrov Classification" after Petrov who first developed it. Work on a classification of the Ricci tensor has been done by Petrov (1) and Plebanski and Stachel (1). This work is concerned solely with exact solutions of the vacuum equations and therefore only the Weyl tensor will be discussed.

The Weyl tensor is defined by [Pirani (1)]

$$W_{ijkm} = R_{ijkm} - g_{i[k}R_{m]j} + g_{j[k}R_{m]i} - \frac{1}{3} g_{i[m}g_{k]j} R , \qquad (1.54)$$

---

[1] The Principle of Equivalence: The effects of a gravitational field are indistinguishable from those of inertial forces by means of purely <u>local</u> experiments [Trautman (1)]. Non-local experiments imply a knowledge of the spatial variations in the field.

and the following algebraic properties

$$W_{ijkm} = -W_{jikm} = -W_{ijmk} = W_{kmij} , \qquad (1.55)$$

$$W_{ijkm} + W_{ikmj} + W_{imjk} = 0 , \qquad (1.56)$$

$$W^{i}_{jki} = 0 . \qquad (1.57)$$

For a vacuum metric, $R = R_{ij} = 0$, so the Weyl tensor is identical to the curvature tensor and the Petrov Classification provides a complete characterization of the space-time.

The Petrov Classification. There are four main techniques for performing the Petrov Classification. These are:

(i) The Matrix Method
(ii) The Tensor Method
(iii) The Method of the Newman-Penrose Scalars
(iv) The Spinor Method.

Each of these four methods will be briefly reviewed with the ultimate aim of discussing their relative merits in the context of a computer system.

(i) The Matrix Method was the original method used by Petrov (1), and also discussed by Géhéniau (1), Debever (1) and Synge (2). A survey of this method, together with methods (ii) and (iv) appears in Pirani (1). The approach used here follows that of Synge (2), who shows that, if

$$g_{ijkm} \overset{D}{=} g_{ik}\, g_{jm} - g_{im}\, g_{jk} , \qquad (1.58)$$

then the eigenvalues of the 6 X 6 determinantal equation

$$R_{ijkm} F^{km} = \lambda g_{ijkm} F^{km} , \tag{1.59}$$

where F^{km} is any skew-symmetric tensor, will be similar for curvature tensors of the same class. The word "similar" is used here in the sense that there will be the same number of distinct, and the same number of coincident eigenvalues in each case. The classification is based on the relative numbers of distinct and coincident eigenvalues, together with the resulting (distinct) eigenvectors [Pirani (1)].

First however, the whole problem is mapped into a complex Euclidean 3-space where several simplifications result. This is achieved by correlating the pairs of numerical indices of the independent components of the curvature tensor with single numbers as follows:

$$(23) \rightarrow 1, \quad (31) \rightarrow 2, \quad (12) \rightarrow 3, \quad (14) \rightarrow 4, \quad (24) \rightarrow 5, \quad (34) \rightarrow 6. \tag{1.60}$$

[for example $R_{1213} \rightarrow R_{32}$]

The resulting quantities become the elements of a symmetric 6 X 6 matrix W. If these are expressed in coordinates which give $g_{ij} = \delta_{ij}$ at a point in space-time, and for which the time coordinate (x_4, by convention) is imaginary, then every single appearance of the matrix suffixes 4,5,6 denotes an imaginary element. Whence the matrix W can be written,

$$W = \begin{pmatrix} M & N \\ N & M \end{pmatrix} \tag{1.61}$$

where M and N are symmetric 3 X 3 trace-free matrices with M real and N imaginary. Furthermore, the skew-symmetric tensor F^{ij} becomes a column matrix (using (1.60)),

$$F = \begin{pmatrix} G \\ H \end{pmatrix} , \tag{1.62}$$

where G is real and H imaginary. Equation (1.59) thus becomes

$$\begin{pmatrix} M & N \\ N & M \end{pmatrix}\begin{pmatrix} G \\ H \end{pmatrix} = I \begin{pmatrix} G \\ H \end{pmatrix} , \tag{1.63}$$

where I is a 6 dimensional identity matrix, or,

$$\begin{aligned} MG + NH &= \lambda G, \\ NG + MH &= \lambda H. \end{aligned} \tag{1.64}$$

Adding,

$$\begin{aligned} (M+N)(G+H) &= \lambda(G+H) , \\ \text{or} \quad \det (K - \lambda I) &= 0 , \end{aligned} \tag{1.65}$$

where $K = M+N$ is a symmetric, complex 3×3 matrix. The number of distinct eigenvalues and eigenvectors of K indicates the Petrov Class.

Petrov Type	I	II	D	III	N
Eigenvalues	all distinct	2 equal	distinct	zero	zero
Distinct Eigenvectors	3	3	2	1	2

If the eigenvectors are mapped back into space-time, it is discovered that there is a correspondence between the Petrov types and the configurations of the resulting four null vectors [Ludwig (1)] . This correspondence may be illustrated thus:

Petrov Type	Distinct Null Vectors	Coincident Null Vectors	Null Directions
I	4	–	4
II	2	2	3
D	–	2 pairs	2
III	1	3	2
N	–	4	1

The last column above reflects the number of null directions which are defined by the four null vectors produced by the mapping.

(ii) The Tensor Method, due to Sachs (1), originated with a statement to the effect that "In every empty space-time, there exist(s) at least one and at most four direction(s) $k^i \neq 0$, which obey

$$k_{[i}R_{j]km[p}k_{q]}k^k k^m = 0 \,, \tag{1.66}$$

where
$$k_i k^i = 0 \text{."} \tag{1.67}$$

These "directions" are known variously as "Debever vectors" [Pirani (1)] or principle null directions [Sachs (2)]. Israel (1, p.50) has shown that these principle null directions correspond directly to the null directions encountered in the Matrix method above.

The classification follows that of (i), but in fact, curvature tensors of classes other than Class 1 satisfy certain supplementary equations given by Sachs (1). These are as follows:

$$\text{for} \quad \text{Type II;} \quad R_{ijk[m}k_{n]}k^j k^k = 0 , \tag{1.68}$$

$$\text{Type D ;} \quad R_{ijk[m}m_{n]}m^j m^k = 0 , \tag{1.69}$$

in addition to (1.68),

$$\text{Type III;} \quad R_{ijk[m}k^k k_{n]} = 0 , \tag{1.70}$$

$$\text{Type N ;} \quad R_{ijkm}k^m = 0 . \tag{1.71}$$

A curvature tensor which satisfies any particular equation of the set (1.66), (1.68), (1.69), (1.70) and (1.71) will satisfy all the preceding equations of the set, so that, moving down the set a degree of specialization becomes apparent. In fact, all classes but Class I are referred to collectively as "algebraically special". This concept is often represented in the form of a "Penrose Diagram" in which the arrows indicate increasing specialization.

$$\begin{tikzcd}
 & & \mathrm{I} \arrow[dl] \arrow[dr] & & \\
 & \mathrm{II} \arrow[rr] \arrow[dl] & & \mathrm{D} \arrow[dr] & \\
\mathrm{III} \arrow[rr] & & \mathrm{N} \arrow[rr] & & 0
\end{tikzcd}$$

Class 0 here refers to the flat-space case

$$R_{ijkm} = 0 . \tag{1.72}$$

(iii) The Method of the Newman-Penrose Scalars owes its existence to the work of Newman and Penrose (1), Janis and Newman (1) and Ludwig (1). The Newman-Penrose scalars (equations (1.40) to (1.44)) may be subjected to the transformations (1.37), (1.38) and (1.39) with the following results [Janis and Newman (1)] :

for (1.37);
$$\begin{aligned}
\widetilde{\Psi}_0 &= \Psi_0 \ , \\
\widetilde{\Psi}_1 &= \Psi_1 + \bar{a}\,\Psi_0 \ , \\
\widetilde{\Psi}_2 &= \Psi_2 + 2\bar{a}\,\Psi_1 + \bar{a}^2\Psi_0 \ , \\
\widetilde{\Psi}_3 &= \Psi_3 + 3\bar{a}\,\Psi_2 + 3\bar{a}^2\Psi_1 + \bar{a}^3\Psi_0 \ , \\
\widetilde{\Psi}_4 &= \Psi_4 + 4\bar{a}\,\Psi_3 + 6\bar{a}^2\Psi_2 + 4\bar{a}^3\Psi_3 + \bar{a}^4\Psi_4 \ .
\end{aligned} \tag{1.73}$$

for (1.38);
$$\begin{aligned}
\widetilde{\Psi}_0 &= \lambda^2\, e^{2i\phi}\,\Psi_0 \ , \\
\widetilde{\Psi}_1 &= \lambda\, e^{i\phi}\,\Psi_1 \ , \\
\widetilde{\Psi}_2 &= \Psi_2 \ , \\
\widetilde{\Psi}_3 &= \lambda^{-1}\, e^{-i\phi}\,\Psi_3 \ , \\
\widetilde{\Psi}_4 &= \lambda^{-2}\, e^{-2i\phi}\,\Psi_4 \ .
\end{aligned} \tag{1.74}$$

for (1.39);
$$\begin{aligned}
\widetilde{\Psi}_0 &= \Psi_0 + 4b\,\Psi_1 + 6b^2\Psi_2 + 4b^3\Psi_3 + b^4\Psi_4 \ , \\
\widetilde{\Psi}_1 &= \Psi_1 + 3b\,\Psi_2 + 3b^2\Psi_3 + b^3\Psi_4 \ , \\
\widetilde{\Psi}_2 &= \Psi_2 + 2b\,\Psi_3 + b^2\Psi_4 \ , \\
\widetilde{\Psi}_3 &= \Psi_3 + b\,\Psi_4 \ , \\
\widetilde{\Psi}_4 &= \Psi_4 \ .
\end{aligned} \tag{1.75}$$

In the first of equations (1.75), $\widetilde{\Psi}_0$ is a quartic function of b. Thus, there are at most four values of b for which

$$\widetilde{\Psi}_0 = \Psi_0 + 4b\,\Psi_1 + 6b^2\Psi_2 + 4b^3\Psi_3 + b^4\Psi_4 = 0 \ . \tag{1.76}$$

b however, is a transformation parameter for, in particular, the null vector $\widetilde{\ell}_i$, and according to Newman and Penrose (1), if $\widetilde{\Psi}_0 = 0$, then $\tilde{\ell}_{i}$ is a principle null direction. Therefore, if equation (1.76) has four distinct roots, then there are four principle null directions and the curvature tensor is of Type I. Alternatively, by comparison with the table on page 15,

	2 distinct and 2 coincident roots	$\Rightarrow$	Type II,
	2 pairs of coincident roots	$\Rightarrow$	Type D,
	1 distinct and 3 coincident roots	$\Rightarrow$	Type III,
and	4 coincident roots	$\Rightarrow$	Type N.

(iv) The Spinor Method was developed by Penrose (1), following Witten (1). Arising from the form of the curvature spinor as expressed in equation (1.53), an analysis is made of the symmetric spinor $\Psi_{(ABCD)}$. An invariant may be formed as follows,

$$\Psi_{ABCD}\,\xi^A \xi^B \xi^C \xi^D \, , \tag{1.77}$$

where ξ^A is arbitrary. This invariant is in fact a quartic equation as may be seen by considering

$$\xi^A = (\xi^1, \xi^2) = (1, \xi) \, . \tag{1.78}$$

Then if the total symmetry inherent in Ψ_{ABCD} is

$$\begin{aligned} \phi(\xi) &= \Psi_{ABCD}\,\xi^A \xi^B \xi^C \xi^D \\ &= \Psi_{1111} + 4\xi\Psi_{1112} + 6\xi^2\Psi_{1122} + 4\xi^3\Psi_{1222} + \xi^4\Psi_{2222} \, , \end{aligned} \tag{1.79}$$

and the quartic form emerges. Any quartic can be written as a product of linear factors

$$\phi(\xi) = (\alpha_1 + \xi\alpha_2)(\beta_1 + \xi\beta_2)(\gamma_1 + \xi\gamma_2)(\delta_1 + \xi\delta_2)$$

$$= \alpha_A \xi^A \beta_B \xi^B \gamma_C \xi^C \delta_D \xi^D , \tag{1.80}$$

or

$$(\Psi_{ABCD} - \alpha_{(A}\beta_B \gamma_C \delta_{D)}) \xi^A \xi^B \xi^C \xi^D = 0 . \tag{1.81}$$

If ξ^A is arbitrary, then

$$\Psi_{ABCD} = \alpha_{(A}\beta_B \gamma_C \delta_{D)} . \tag{1.82}$$

These 1-spinors, α_A etc., correspond to the Debever vectors of (ii) [Pirani (1)] , and consequently the classification emerges from their multiplicity.

Petrov Type	R_{ABCD} =
I	$\alpha_{(A}\,\beta_{B}\,\gamma_{C}\,\delta_{D)}$
II	$\alpha_{(A}\,\alpha_{B}\,\beta_{C}\,\gamma_{D)}$
D	$\alpha_{(A}\,\alpha_{B}\,\beta_{C}\,\beta_{D)}$
III	$\alpha_{(A}\,\alpha_{B}\,\alpha_{C}\,\beta_{D)}$
N	$\alpha_{A}\,\alpha_{B}\,\alpha_{C}\,\alpha_{D}$

CHAPTER II

COMPUTATION.

1. It is natural to look to the computer as a means of eliminating much of the tedium and some of the human error inherent in the considerable manipulative demands made by the analyses described in the previous chapter. The brief exposition of the four main techniques for the Petrov classification has been given for the purpose of making a comparative study with an eye to effecting a computer-based classification scheme.

In the context of computer programming, it is often the simplest approach that is the most efficient in the long run. The reason behind this is that an intricate algorithm can invite time-consuming slips in the logic of the programme or in the manipulation of the language. A further consideration is that, in a large computer system, efficiency is often linked to the demands made by the programme on the system. A small, fast programme receives a higher priority than a larger,time-consuming programme, and can consequently be run more often. Lastly, if a large number of data-sets are to be handled by the same programme, the fewer the elements to be generated by the data for each run, the better. Thus it is desirable to try to free specific steps in the process from the particular properties of any given data-set. In comparing the four classification schemes, these criteria of "simplicity","brevity" and "generality" respectively will be invoked.

The first question to ask with regard to setting up an algorithm is, What sort of problem is it? The answers are; for the matrix method, it is an eigenvalue problem in complex Euclidean 3-space (equation (1.65)). - i.e. the solution of a cubic; for the tensor method, it is the simultaneous solution of thirty-six fourth order equations :

(equation (1.66)); and for both the Spinor and the Newman-Penrose scalars methods, it is the solution of a quartic equation (equations (1.80) and (1.76) respectively). Application of the "simplicity" criterion indicates that the tensor method should be rejected in favour of the other three. On the same basis, the matrix method seems to be the most promising.

Synge (2) has given what is, in effect, an algorithm for the matrix method. There are many steps involved in setting up the eigenvalue problem, including the reduction of the metric (locally) to Minkowski form and the inversion of the resulting transformation matrix (see Synge(2) for details). Another disadvantage is that the eigenvectors must be transformed back into space-time, should the principal null directions be required. In contrast, the methods involving the solution of a quartic require fewer steps (see 2.3) and thus, on the grounds of "brevity", the matrix method may be rejected.

The requirements of the spinor method are that a suitable set of van der Waerden connecting quantities, dependent on the metric tensor (input data), be found and that (as with the matrix method), should the principal null directions be required, a transformation back into space-time is necessary. The Newman-Penrose scalars method requires that a suitable null-tetrad, also dependent on the metric, but less intricately so, (see 2.2) be set up. Thus, on the grounds of "generality", the method of the Newman-Penrose scalars may be selected in preference to the Spinor method.

It must be remembered, however, that the process is alegebraic while computers are numeric machines, and therefore a special compiler, or language, is required to deal with the "algebraic manipulation". Computer systems adapted for such work have been in existence for some time (Tobey, Bobrow& Zilles (1)), and some have reached a high degree of sophistication (Barton, Bourne & Horton (1)), but a detailed knowledge of the operation of the computer is not, in general, required for such systems. The techniques involved differ sufficiently from the conventional programming techniques to impose a slight reorientation of attitude on the programmer, and equally, the programmes written differ sufficiently from the conventional analytic modes of mathematical operation (vide Chapter I), to require some preliminary adaption which is the subject of this chapter.

2. The Decomposition of the Curvature Tensor.

It is usual, when decomposing the curvature tensor into the Newman-Penrose scalars, to select a tetrad which has special properties with regard to the space-time under consideration (Newman & Penrose (1)). However, for a computer programme which is designed to process a number of different metrics, this approach is inappropriate. Instead it is preferable to find some more general tetrad in terms of which all the curvature tensors can be decomposed. Although this may cause problems in an attempt to interpret the metrics physically [1], in a straightforward algebraic task like the Petrov Classification no difficulties arise.

Taking into account the simplifications already mentioned (that only diagonal vacuum metrics are to be considered), it is possible to find a tetrad of the Newman-Penrose type which reduces the total number of computations required for the analysis. It is simplest to treat the wholly real and the complex elements of this tetrad separately. The complex elements are dealt with first. Newman & Unti(1) state that, if a_j and b_j are two orthogonal space-like unit vectors, then m_j (in the notation of Chapter 1) can be

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1. See Newman &Penrose(1) or Sachs(1), for instance, where a "special" tetrad requires an affine parameter.

calculated as follows,

$$m_j = \frac{1}{\sqrt{2}} (a_j + i\, b_j), \tag{2.1}$$

where $i = \sqrt{-1}$.

By a suitable choice of a_j & b_j, namely

$$a_j = (0, 0, \sqrt{g_{33}}, 0), \tag{2.2}$$

$$b_j = (0, 0, 0, \sqrt{g_{44}}), \tag{2.3}$$

which is permissible since the metric is diagonal,

m_j becomes

$$m_j = \left(0, 0, \sqrt{g_{33}/2}, i\sqrt{g_{44}/2}\right), \tag{2.4}$$

$$\text{and} \quad \bar{m}_j = \left(0, 0, \sqrt{g_{33}/2}, -i\sqrt{g_{44}/2}\right). \tag{2.5}$$

Since the whole tetrad must obey (see equation(1.36)),

$$g_{ij} = -l_i n_j - l_j n_i + m_i \bar{m}_j + \bar{m}_i m_j, \tag{2.6}$$

the choice of m_j and $\bar{m}_j$ above implies that the last two components of l_j and n_j can be set equal to zero. Thus (2.6) provides three equations in the four unknowns l_1, l_2, n_1, n_2.

If n_2 is arbitrarily chosen as 1 and subtituted into (2.6), then

$$l_j = \left(\frac{g_{11}}{2}\sqrt{\frac{g_{22}}{g_{11}}}, -g_{22}/2, 0, 0\right), \tag{2.7}$$

and

$$n_j = \left(\sqrt{g_{11}/g_{22}}, 1, 0, 0\right). \tag{2.8}$$

It is easily verified that the tetrad defined by (2.7), (2.8), (2.4) and (2.5) obeys the conditions (1.31) and (1.32).

The Newman-Penrose scalars, when written out explicitly in terms of this tetrad, take the form

$$\Psi_0 = -\tfrac{1}{8}\Big[\frac{g_{22}}{g_{11}}\Big(\frac{R_{1313}}{g_{33}} - \frac{R_{1414}}{g_{44}}\Big) + \Big(\frac{R_{2323}}{g_{33}} - \frac{R_{2424}}{g_{44}}\Big) + 2\sqrt{\frac{g_{22}}{g_{11}}}\Big(\frac{R_{1323}}{g_{33}} - \frac{R_{1424}}{g_{44}}\Big) + \frac{2i}{\sqrt{g_{33}g_{44}}}\Big\{\Big(\frac{g_{22}}{g_{11}}R_{1314} + R_{2324}\Big) + \sqrt{\frac{g_{22}}{g_{11}}}\big(R_{1324} + R_{1423}\big)\Big\}\Big], \tag{2.9}$$

$$\Psi_1 = \frac{-1}{2\sqrt{g_{11}g_{22}}}\Big[\sqrt{\frac{g_{22}}{g_{11}}}\Big(\frac{R_{1213}}{\sqrt{g_{33}}} + i\frac{R_{1214}}{\sqrt{g_{44}}}\Big) + \Big(\frac{R_{1223}}{\sqrt{g_{33}}} + i\frac{R_{1224}}{\sqrt{g_{44}}}\Big)\Big], \tag{2.10}$$

$$\Psi_2 = -\tfrac{1}{2}\Big[\frac{R_{1212}}{g_{11}g_{22}} - \frac{i\,R_{1234}}{\sqrt{g_{11}g_{22}g_{33}g_{44}}}\Big], \tag{2.11}$$

$$\Psi_3 = \frac{1}{\sqrt{2g_{11}g_{22}^3}}\Big[\sqrt{\frac{g_{22}}{g_{11}}}\Big(\frac{R_{1213}}{\sqrt{g_{33}}} - i\frac{R_{1214}}{\sqrt{g_{44}}}\Big) - \Big(\frac{R_{1223}}{\sqrt{g_{33}}} - i\frac{R_{1224}}{\sqrt{g_{44}}}\Big)\Big], \tag{2.12}$$

$$\Psi_4 = -\frac{1}{g_{22}^2}\Big[\frac{g_{22}}{g_{11}}\Big(\frac{R_{1313}}{g_{33}} - \frac{R_{1414}}{g_{44}}\Big) + \Big(\frac{R_{2323}}{g_{33}} - \frac{R_{2424}}{g_{44}}\Big) - 2\sqrt{\frac{g_{22}}{g_{11}}}\Big(\frac{R_{1323}}{g_{33}} - \frac{R_{1424}}{g_{44}}\Big) - \frac{2i}{\sqrt{g_{33}g_{44}}}\Big\{\Big(\frac{g_{22}}{g_{11}}R_{1314} + R_{2324}\Big) + \sqrt{\frac{g_{22}}{g_{11}}}\big(R_{1324} + R_{1423}\big)\Big\}\Big]. \tag{2.13}$$

By invoking equations (1.17), (1.24), (1.30) and the argument immediately after equation(1.15), Ψ_0 , Ψ_2 and Ψ_4 can be rendered in terms of the known independent components. The equivalence of certain terms in some of the expressions can be exploited to yield

$$\Psi_0 = -\tfrac{1}{4}\,(A + D), \tag{2.14}$$

$$\Psi_1 = -\tfrac{\sqrt{2}}{4}\,Q\,(B + C), \tag{2.15}$$

$$\Psi_2 = \frac{1}{2g_{11}}\Big(\frac{R_{1313}}{g_{33}} + \frac{R_{1414}}{g_{44}}\Big), \tag{2.16}$$

$$\psi_3 = \frac{\sqrt{2}}{2} \cdot \frac{Q}{g_{22}} \left(\bar{B} - \bar{C} \right), \tag{2.17}$$

and

$$\psi_4 = -\frac{1}{g_{22}^2} \left(A - D \right), \tag{2.18}$$

where

$$Q = \frac{1}{\sqrt{g_{11}\, g_{22}}}, \tag{2.19}$$

$$A = \frac{g_{22}}{g_{11}} \left(\frac{R_{1313}}{g_{33}} - \frac{R_{1414}}{g_{44}} \right), \tag{2.20}$$

$$D = 2\sqrt{\frac{g_{22}}{g_{11}}} \left(\frac{R_{1323}}{g_{33}} + i \sqrt{\frac{g_{22}}{g_{11}\, g_{33}\, g_{44}}}\; R_{1314} \right), \tag{2.21}$$

$$B = \sqrt{\frac{g_{22}}{g_{11}}} \left(\frac{R_{1213}}{\sqrt{g_{33}}} + i \frac{R_{1214}}{\sqrt{g_{44}}} \right), \tag{2.22}$$

$$C = \left(\frac{R_{1223}}{\sqrt{g_{33}}} + i \frac{R_{1224}}{\sqrt{g_{44}}} \right). \tag{2.23}$$

3. <u>The Petrov Classification.</u>

Because of the theory of the solution of quartic polynomials is so well-developed (Archbold(1)), the Petrov classification by means of the Newman-Penrose scalars can be performed without finding the explicit roots of the quartic or the corresponding principal null directions. It is possible, from the coefficients of the quartic (1.76), to define a set of four quantities as follows,

$$I = \psi_0 \psi_4 - 4 \psi_1 \psi_3 + 3 \psi_2^2, \tag{2.24}$$

$$J = \begin{vmatrix} \psi_0 & \psi_1 & \psi_2 \\ \psi_1 & \psi_2 & \psi_3 \\ \psi_2 & \psi_3 & \psi_4 \end{vmatrix}, \tag{2.25}$$

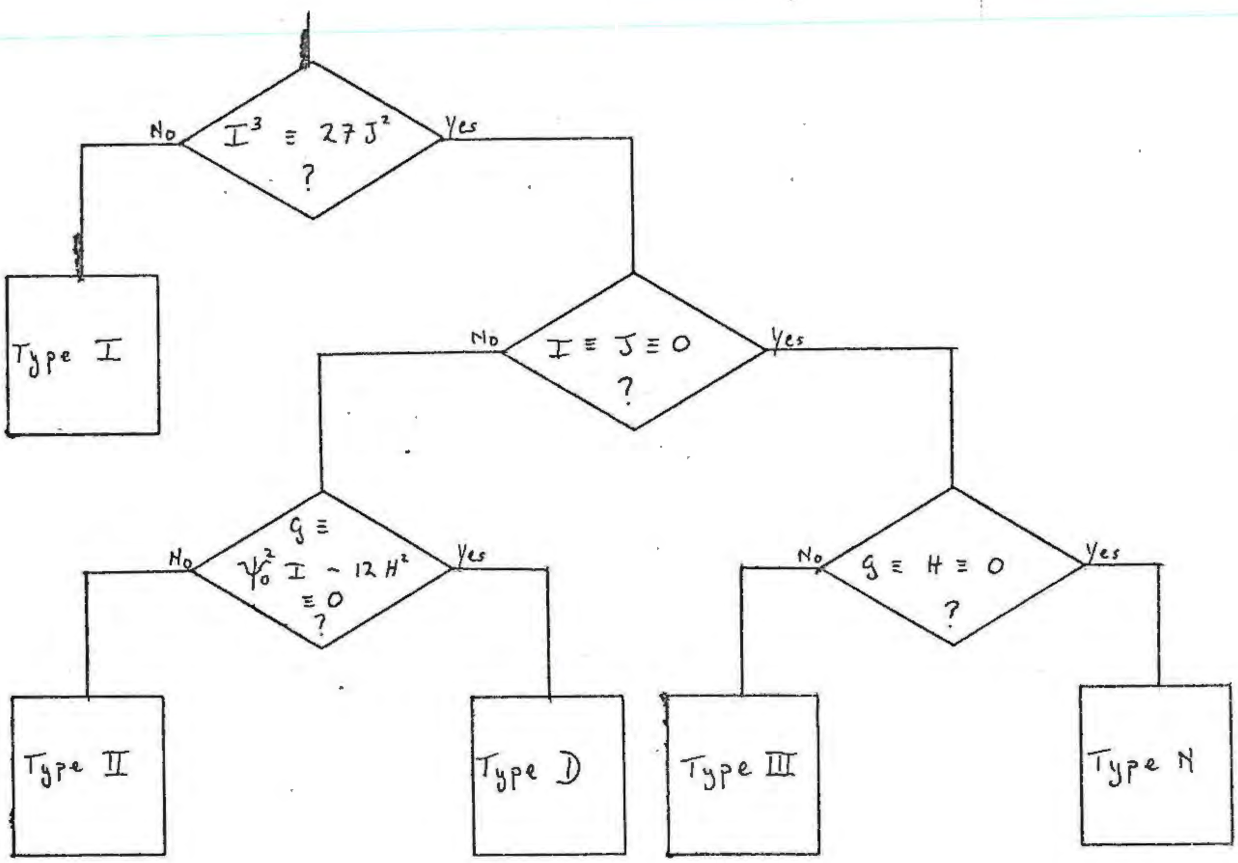
$$G = \psi_0^2 \psi_3 - 3 \psi_0 \psi_1 \psi_2 + 2 \psi_1^3, \tag{2.26}$$

$$H = \Psi_0 \Psi_2 - \Psi_1^2 \tag{2.27}$$

Conditions for the quartic to have any number of distinct or coincident roots depend on various identities involving I, J, G and H. The details are to be found in Archbold(1), but the precise relationship between the identities and the number of distinct and coincident roots (and therefore the Petrov types- see Chapter 1) is reflected in the table below.

Identity	Number of Equal Roots	Petrov Type
$I^3 \not\equiv 27 J^2$	0	I
$I^3 \equiv 27 J^2$ $\Psi_0^2 I - 12 H^2 \not\equiv 0$ or $G \not\equiv 0$	2 (and 2 distinct)	II
$I^3 \equiv 27 J^2$ $G \equiv \Psi_0^2 I - 12 H^2 \equiv 0$	2 pairs	D
$I \equiv J \equiv 0$ $G \not\equiv 0$ or $H \not\equiv 0$	3	III
$I \equiv J \equiv G \equiv H \equiv 0$	4	N

D'Inverno and Russell-Clark (1) have created an algorithm by means of which a space-time may be classified once its Newman-Penrose scalars are known. Essentially, the algorithm enables the computer to place the metric into its position in the table. Schematically, the algorithm may be represented as a flow-diagram (D'Inverno & Russell-Clark (1)). The diamond-shaped blocks represent decisions to be made by the computer, the path of action depending on the results of the decision.



This algorithm is easily programmed, and, in theory, the classification is quite straightforward, but in practice several problems remain. All computer systems have limited storage space and available machine-time, and algebraic manipulation compilers are capable of stretching both of these to the limit. This is due to an effect known as "expression swell", where expressions held in store in the course of manipulation become unmanagably large. The reason for this is that the operation of simplifying an algebraic expression is essentially a human one. Often the simplifier has some idea of what his expression will look like, and this "intuition" cannot be replaced by a machine. Consequently, the unsimplified expressions become very large and their manipulation uses prodigious amounts of machine time.

There are several ways of dealing with this problem. One is to make use of "automatic simplification" facilities which most systems have to a greater or lesser degree (Moses(1)). These facilities perform basic simplification tasks, (e.g. setting the product of anything multiplied by zero equal to zero), and thus they serve to reduce expression size somewhat. An additional, and more direct approach, is to break the analysis up into segments and attempt to simplify the intermediate expressions "by hand", after which the resulting reduced expressions can be input into the next step of the process. This is clearly a compromise between a totally manual (pencil-and-paper) process and its totally machine-based equivalent, but in the circumstances it is the only available course of action.

This approach was adopted in section 2.2 where, before calculating ψ_0, ψ_1, etc., the "intermediate" quantities A, B etc. were obtained. In the same way, it is possible to insert intermediate quantities into the present analysis.

Let K, L and M be given by

$$K = \psi_0 \psi_3 - \psi_1 \psi_2 , \tag{2.28}$$

$$L = \psi_1 \psi_3 - \psi_2^2 , \tag{2.29}$$

$$M = \psi_0 \psi_4 - \psi_1 \psi_3 , \tag{2.30}$$

whence

$$I = M - 3L , \tag{2.31}$$

$$J = \Psi_4 H - \Psi_3 K + \Psi_2 L , \tag{2.32}$$

and

$$G = \Psi_0 K - 2\Psi_1 H , \tag{2.33}$$

where, as before, $H = \Psi_0 \Psi_2 - \Psi_1^2$. (2.34)

This device truncates the analysis so that a simplification step can be interposed between the calculation of K , L , M and H , and that of I , J and G .

It may happen that such attempts do not simplify the expressions sufficiently to make their manipulation practicable, and further limiting measures must be resorted to. D'Inverno &Russell-Clark (1), for instance, were forced to investigate the roots of the quartic for some metrics at a single point in space-time. Since the condition for algebraic speciality is an identity (see table above), if it is violated anywhere, then the metric cannot be algebraically special.

4. Conclusion.

In the present study the Harrison metrics (Harrison(1)) have motivated the investigation of classification techniques and, in particular, the processes whereby these may be made programmable[1]. The deciding factor in favour of the Newman-Penrose scalars method was the fact that the classification could be performed without actually solving the quartic which it entails.

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1. The Harrison metrics have already been classified by D'Inverno and Russell-Clark (1) using the Newman-Penrose scalars method. (see Chapter III)

If the roots of the quartic can be derived, then it is possible to open up a much richer vein of investigation, since each distinct root yields a principal null direction explicitly (via equation 1.39). The principal null directions of the Weyl tensor form the basis of a preliminary investigation into the gravitational radiation characteristics of a metric (Pirani (2)). The theoretical groundwork of this investigation was laid down in Sachs (1), and many results (most of which appear in Unti & Torrence (1)) have followed. A brief summary of this theory is given in Appendix 1.

Unfortunately, it would appear that the solution of a quartic with algebraic coefficients of the complexity encountered in the Harrison metrics is not practicable, and therefore the principal null vectors are unattainable using this approach. Attempts to limit the investigation to a single point in space-time (thereby yielding numerical coefficients for the quartic) have proved unsatisfactory owing to the difficulties involved in drawing generalized conclusions from particular results. No key identity such as that used by D'Inverno & Russell-Clark (1) in their work (see section 2.3) could be established in this connection.

However, it is thought that the results both of D'Inverno & Russell-Clark (1) and the present study, discussed in the next chapter and presented in Appendix 2 are worth recording in themselves, although the possibility of further analysis should not be precluded.

CHAPTER III

THE HARRISON METRICS.

1. Harrison(1) has found a number of solutions to the field equations. These metrics constitute a class of solutions with the following properties.

1) All are independent of one space-like coordinate taken to be x^3.

2) All are invariant under a transformation $x^{3'} = -x^3$. Consequently, the g_{i3} are all zero, and a simple coordinate transformation yields $g_{12} = g_{14} = g_{24} = 0$. Therefore the metrics are diagonal.

3. All have a "linked-pair" form,

$$g_{ij} = \delta_{ij} \; e_i \; A_i^2 \, (x^1, x^2) \; B_i^2 \, (x^1, x^4), \tag{3.1}$$

where $e_1 = -1$, $e_2 = e_3 = e_4 = 0$ to designate the signature, and x^2, x^4 are space-like coordinates. The summation convention has been suspended.

Harrison (1) suggests that this functional relationship (the "linked-pairs" form) may imply radiative solutions[1].

Property (2) dictates that Einstein's equations in vacuo (equation 1.10) should reduce to seven differential~~s~~ equations, since R_{13}, R_{23} and R_{43} (equations 1.18,1.21 and.1.24 respectively) vanish automatically, by the same sort of reasoning as that immediately after equation (1.15). The linked-pairs restriction implies that these equations take the form,

$$F_i \; (x^1, x^2) \; \mathcal{G}_i \; (x^1, x^4) = 0 \, , \tag{3.2}$$

1. See Appendix 1 for a treatment of null-vectors (which could arise from the linked pairs) tangent to null geodesics, and their role as propagation vectors for gravitational radiation. While the idea is attractive, the precise consequences of this assertion are not yet clear.

where F_i and G_i are formed from A_i, B_i and their derivatives, i goes from 1 to 3, and the summation convention again applies.

Harrison(1) succeeded in finding the whole class of these solutions by the separation of the variables. The time-like coordinate is separated out, reducing F_i, G_i to functions dependent on one of the space-like variables only. Further separation yields eight mutually exclusive solutions to the equations. The time dependence is then restored to these solutions.

This process need only be applied to a single member of equations (1.10) ($R_{24}=0$), the resulting six values of A_i, B_i being substituted back into the other six equations to establish their precise algebraic dependence on the coordinates. Any solutions showing a two-variable ("degenerate" in Harrison terminology) rather than a three-variable dependence, together with certain flat-space solutions which arose, were discarded on the grounds that two-variable solutions are comparatively well-understood and flat-space metrics are unrealistic.

The remaining sixteen solutions, together with fourteen generated therefrom by complex coordinate transformations, and a further ten solutions obtained by substituting $\cosh x^i$ and e^{x^i} for $\sinh x^i$ everywhere it appears, were presented in Harrison(1). These solutions were classified as follows.

Type I[1] - each non-zero component of the metric is the product of three factors, with sub-classification into:

I - A - 1 and 2, involving components in which all factors are dependent on a single variable.

and I - B - 1 to 4, with components in which at least one factor is dependent on a single variable, and at least one dependent on a linear combination of two variables.

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1. This does not refer to the Petrov type.

Type II - each component is a product of two factors, one dependent on a single variable, the other on a function of two variables, at least one of them quadratic.

II - A - 1to 7, in which the single-variable factor is dependent on x^4, the other factor being dependent on x^1 and x^2.

II - B - 1 to 3, in which x^1 is the variable pertaining to the single-variable factor, and x^2 and x^4 the variables in the second factor.

II - C - 1 to 4, in which x^4 is again the single variable but x^2 and x^3 are the variables of the second factor. It is not clear how this departure from the stated independence of the metric on x^3 comes about, nor how the time-dependence vanishes. Nevertheless, these are still three-variable metrics.

Type III - 1 to 10. These solutions are degenerate in the sense that they are solutions which were obtained using the technique outlined above but were subsequently discovered, by means of coordinate transformations, to be dependent on only two variables.

Two groups, D'Inverno and Russell-Clark(1) and Barton and Fitch(1) have done subsequent work on the Harrison metrics. Both attempted a computer-based Petrov classification of the metrics. D'Inverno and Russell-Clark(1), following the approach outlined in Section 2.3 succeeded in performing the classification, having first checked that the solutions were indeed vacuum solutions. They discovered that four of the metrics(I - B- 1(b)[1], I - B - 1(c),

1. The labels (a),(b) and (c) refer to solutions containing $\sinh x^i$, $\cosh x^i$ and e^{x^i} respectively, as mentioned previously. This notation is due to D'Inverno and Russell-Clark (1).

I - B - 2 and III -6) were, in fact, non-vacuum solutions. It appears that their inclusion is the result of computational errors, and not because the method fails (Matravers (2)).

Solutions I - A - 1 and 2 were found to be degenerate[1] and hence re-labelled III - 11 and III - 12 respectively. In all, twenty - one of the vacuum solutions, both degenerate and non-degenerate were established as Petrov Type I, fourteen as Type D and one as Type N. All these solutions are listed in Appendix 2.

Finally, D'Inverno and Russell-Clark placed the Type D metrics into their Kinnersley (1) classes (see D'Inverno and Russell-Clark (1) for details). As far as is known, this represents all the published work on the Harrison metrics to date.

2. Further Work

Additional computational work on the Harrison metrics was undertaken as a result of a two-month studentship at the National Research Institute for Mathematical Sciences, Council for Scientific and Industrial Research, Pretoria. The installation included an I.B.M. 360/60 computer equipped with a "hands-on" card-reader and line-printer, and 256 K (bytes) immediate access core-store with an additional 100 K back-up store. This machine was equipped with a FORMAC compiler, based on PL/1, which was used for the processing of some of the Harrison metrics.

The FORMAC language is one of the most powerful symbolic manipulators generally available (Tobey et al(1)). It is not as efficient as the more-specialised "relativity" manipulators (Barton and Fitch (1)),

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1. This point is also made in Harrison(1) (Footnote P 1289).

but is more versatile. It exploits the list-processing properties of some of the PL/1 functions, and itself possesses functions with the following essential capabilities.

(i) It allows for the storage of mathematical expressions, both as "atoms" (unknown algebraic quantities) or as "variables" (strings of atoms linked by the conventional algebraic operators). Variables can be output either in edited form, for ease of reading, or unedited form, coded for use in some other FORMAC programme.

(ii) It can perform the elementary algebraic operations of addition, multiplication, subtraction or division of variables or atoms.

(iii) It allows for the substitution of variables into one another, or into atoms, of atoms into one another, or indeed of actual numerical values into atoms.

(iv) It contains certain automatic simplification capabilities such as the elimination from a variable of two atoms of opposite sign. $A = B + C - B \rightarrow A = C.$

(v) It contains certain user-controlled simplification capabilities, allowing, for instance, for the expansion of expressions with numerical exponents,

$$(A + B)^2 \rightarrow A^2 + 2AB + B^2,$$

and the collection of like terms in an expression

$$AB + AC \rightarrow A(B + C).$$

(vi) It allows for the differentiation of expressions with respect to variables or atoms within those expressions, recognising the functional nature of certain variable-names such as $SIN(X)$, $EXP(X)$ etc.

These specifications, combined with the standard programming techniques contained in PL/1 (eg looping and branching) provide a powerful means whereby general information (non-zero components of the Riemann tensor,or indeed, the Petrov class (see Chapter 2)) about metrics can be processed with precision and speed. Unfortunately, the programmes do take some time to devise, and, in the two months available, it was not possible to accomplish the entire programme of work, providing a confirmation of the results given in Section 3.1. Instead of arriving at a complete Petrov classification of all the Harrison metrics, it was possible only to obtain the Newman-Penrose Scalars (see Chapter 1) for a selected set of the solutions.

The basis of selection was as follows:

A) Those metrics listed as non-vacuum by D'Inverno and Russell -Clark(1) were rejected on the grounds that the analysis, as presented in Chapter 2 (and therefore, as programmed in FORMAC) did not apply to non-vacuum metrics. Metrics so eliminated were I - B - 1(b)&(c), I - B - 2, III - 6.

B) Those metrics containing unknown analytical forms (see Harrison (1)) were eliminated on the grounds that it would be impossible to assess the significance of these metrics until more was known about the analytical forms involved. The metrics rejected in this way were II -A -4, II - A - 5, II - A - 6, II - A - 7, II - B - 2, II - B - 3, II - C - 3, and II - C - 4.

C) III - 2 was rejected on the grounds that it was not listed in diagonal form.

D) The (b) and (c) versions (corresponding to $\cosh x^i$ and e^{x^i}) of III - 4, III - 7, III - 9 and III - 12 were eliminated to save time.

The remaining metrics were processed as far as the Newman-Penrose scalars, and are presented in Appendix 2. The processing was achieved with four programmes connected by unedited FORMAC input/

output data-links in the form of punched cards.

Programme 1. (HARIM) separated out the non-zero metric components from the form given in Harrison(1), and expressed then as FORMAC variables.

Programme 2. (WEYL) carried out the analysis as far as the relevant components of the Riemann tensor (equation 1.5). This involved the first and second derivatives of the metric components. All the symmetries inherent in the Christoffel symbols (equations (1.3),(1.4) including those brought about by the diagonal character of the metric, were exploited, reducing the machine-time, at the cost of some intricate programming. The symmetries inherent in the Riemann tensor were not used as only the eight well-specified components were required (see Chapter 2), and these were computed directly from the Christoffel symbols.

Programme 3. was not given a general name because this stage in the process was one in which an individual programme was written for each metric, in an attempt to cut down on expression- size[1]. In general, expressions were completely expanded and inspected, subsequent programming being tailored to fit the requirements of each expression in turn. On average, these programmes succeeded in reducing the size of the expressions by a factor of about two-thirds.

Programme 4. (QUARTIC) was a FORMAC version of equations (2.14) to (2.23). Unfortunately, an error in the programming, discovered subsequently, has meant that

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1. A technical report (unpublished) made to N.R.I.M.S. by the author at the time this work was done lists the largest expressions output by WEYL as occupying about 30 lines of print-out.

the output of QUARTIC has been abandoned, and the results appearing in Appendix 2 have been calculated by hand.

3. Conclusion.

Research in General Relativity can follow one of two lines. Either new ideas and techniques can be tried out on solutions which are already established and well-understood, or, new solutions can be sought. The Harrison metrics are new in the sense that their properties are comparatively unknown and there are many methods of approach which could be adopted in the hopes that one or more of these solutions will prove physically fruitful.

It is unfortunate that, in the present work, no decisive results were forthcoming, enabling a selection of the more interesting metrics to be made. However, the Newman-Penrose scalars, which appear in Appendix 2, may prove to be a useful foundation for future work. It should be remembered that they are expressed in terms of a particular, and coordinate-dependent tetrad (equations (2.4),(2.5),(2.7) and (2.8), the properties of which are not fully understood. An investigation of the implications of this tetrad, whose main advantage is the ease with which it can be set up in a computational context, could be a profitable future line of approach.

The attractions of symbolic manipulation to the student of General Relativity should become apparent from an inspection of Appendix 2. The sheer size of the expressions is daunting enough, and these are fairly simple metrics. It is thought that the versatility of a language like FORMAC does not outweigh the obvious advantages of the more specialised relativity languages (Barton &Fitch(1)),

although neither approach can overcome entirely the formidable simplification problems. New algorithms like that of D'Inverno and Russell-Clark(1) (see Chapter 2) are required. These would exploit other modes of analysis and add significantly to the scope and sophistication of these languages.

ABSTRACT.

Several alternative mathematical formulations of the Petrov classification of metrics in General Relativity are examined. One, involving the Newman-Penrose scalars, is selected, together with a tetrad with special properties pertaining to diagonal metrics, for a computer-based analysis. Using a computer, the Newman-Penrose scalars are calculated for certain of the Harrison metrics, but the analysis has not been carried through to the Petrov classification.

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APPENDIX I

Gravitational Radiation.

The study of gravitational radiation and of radiative solutions forms an important part of current research in gravitation. The reasons for this lie in the continued attempts to unite relativity, through a study of "gravitons", with the quantum theory(Sachs(2)), and with the investigation of the claims of Weber(1) to have detected gravitational waves.

The concept of radiation arises out of electromagnetic theory, in which a wave may be viewed as a surface of discontinuity in the (electromagnetic) field, which propagates with the fundamental velocity (Pirani(3)). By direct analogy, a gravitational wave is considered to be present if, across a null hypersurface, there exist discontinuities in the curvature tensor. Mathematically, this is just a characteristic initial value problem, since a solution is specified on a characteristic surface tangent to the local light-cone - i.e. a null hypersurface (Sachs(2)). In the theory of partial differential equations, the Cauchy data for equations of the type considered here are propagated along curves lying in the characteristic surface, called bicharacteristics(Pirani(2)). Physically, these bicharacteristics are equivalent to "rays" along which the wave is propagated. For this reason, the analysis which follows below is referred to as the "Ray Optics" of a gravitational field (Pirani(2)).

A null hypersurface is defined by the equation

$$g^{ij}\, u_{,i}\, u_{,j} = 0, \qquad \text{(A1.1)}$$

where

$$u(x^i) = 0 \qquad \text{(A1.2)}$$

is the equation of a hypersurface in space-time(Newman &Unti(1)). If l^i is defined as

$$l^i = g^{ij}\, u_{,j}\,, \qquad \text{(A1.3)}$$

then l^i is the tangent field to a congruence of null geodesics which are bicharacteristics of the null hypersurface u (Synge(1)). This can be verified by observing that (A1.1) is just the Hamilton-Jacobi equation for null-geodesics, whence the Euler-Lagrange equations may be written (Matravers(1))

$$l^i{}_{;j}\, l^j = 0, \qquad \text{(A1.4)}$$

where it assumed that the congruence may be affinely parametrized (Pirani(2)).

Now that the null hypersurface and its bicharacteristics have been defined in the space-time, it is necessary to find out if the curvature tensor suffers a discontinuity across the hypersurface. The means to this end lies in a statement, due to Sachs(1), which may be paraphrased thus

> "A vacuum metric is said to have "Geodesic Rays" if one of the principal null directions of the curvature tensor to geodesics -i.e. obeys (A1.4)." ?

This is reinforced by two theorems:

(i) " All algebraically special vacuum metrics have geodesic rays, but, there exist algebraically general vacuum fields with geodesic rays." (Sachs(1)).

(ii) " For algebraically special metrics, which are degenerate in the sense that at least one of the principal null directions corresponds to the repeated roots of the quartic (1.76), it is this degenerate direction which is tangent to the geodesics" (Adapted from Pirani(2)).

For the purposes of the following discussion, the existence of geodesics rays for a metric of type I[1] will be taken as a criterion

1. It seems that exact, physically realistic radiation fields can only be of type I (Pirani(2), P364).

for the existence of radiation[1].

In order to unravel more of the structure pertaining to radiative metrics, it is useful to consider the following "experiment", (Sachs(1)).

Consider a small object in a null geodesic congruence illuminated by some source of light. Suppose that the shadow cast by the object is observed on a screen and compared with a parallelly displaced projection of the object, at the screen. Then, depending on the particular (radiation) properties of the space-time, the shadow can differ from the projection in size, shape and orientation.

Mathematically, these differences are expressed by three "optical" parameters, expansion, shear and twist, respectively. These parameters are obtained from the congruence of null geodesics , the tangents being given by l^i , as follows:

expansion
$$\theta = \tfrac{1}{2} \, l^i{}_{;i} \quad , \tag{A1.5}$$

twist
$$\omega = \left[\tfrac{1}{2} \, l_{[i;j]} \, l^{i;j} \right]^{\frac{1}{2}} , \tag{A1.6}$$

shear
$$|\sigma| = \left[\tfrac{1}{2} \, l_{(i;j)} \, l^{i;j} - \theta^2 \right]^{\frac{1}{2}} . \tag{A1.7}$$

In particular, the definitions (A1.5), (A1.6) and (A1.7) hold for geodesic rays (Sachs(1)).

It is now proposed that metrics which admit geodesic rays be classified, following Unti & Torrence(1), according to the restrictions which may or may not be imposed on the optical parameters. Much work has been done in this direction. In particular, the Goldberg-Sachs Theorem (Newman & Tamburino(1)) asserts that, for all algebraically special vacuum metrics, $\sigma = 0$. Newman &Tamburino(1) have worked with metrics which admit a

1. In fact, a restriction has been introduced here, since this criterion properly belongs to "outgoing" radiation - see Sachs(1).

"hypersurface orthogonal" congruence of geodesic rays. Pirani(2) has shown that this is equivalent to requiring that the congruence, has zero twist. The accompanying table outlines the classification together with references to the literature pertinent to each class, and a brief discussion.

Class	Parameters	Discussion	References
1.	$\Theta = 0, \omega = 0, \sigma = 0$	Algebraically special, hypersurface orthogonal rays. Plane symmetry.	Kundt(1)
2.	$\Theta = 0, \omega = 0, \sigma \neq 0$	Not permitted by the field equations.	Unti&Torrence(1)
3.	$\Theta = 0, \omega \neq 0, \sigma = 0$	Not permitted by the field equations.	Unti&Torrence(1)
4.	$\Theta = 0, \omega \neq 0, \sigma \neq 0$	Rays satisfy $\omega^2 = \sigma^{-2}$ (cylindrical symmetry).	Unti&Torrence(1)
5.	$\Theta \neq 0, \omega = 0, \sigma = 0$	Algebraically special, hypersurface orthogonal rays, all of which develop line singularities.	Robinson and Trautman(1)
6.	$\Theta \neq 0, \omega = 0, \sigma \neq 0$	a)Hypersurface orthogonal rays which satisfy $\Theta^2 \neq \sigma^{-2}$ (spherical symmetry) and which do not depend on arbitrary functions. b) Hypersurface orthogonal rays which satisfy $\Theta^2 = \sigma^{-2}$ (cylindrical symmetry) and which do not depend on arbitrary functions.	Newman and Tamburino (1)
7.	$\Theta \neq 0, \omega \neq 0, \sigma = 0$	Algebraically special solutions	Newman, Unti and Tamburino(1) Kerr(1)
8.	$\Theta \neq 0, \omega \neq 0, \sigma \neq 0$	Solutions satisfy $\Theta^2 + \omega^2 = \sigma^{-2}$ (cylindrical symmetry)	Unti and Torrence (1).

The conclusions which can be drawn from this table are disappointing. A realistic radiation metric ought to be algebraically general, spherically symmetric and depend on arbitrary functions (Unti and Torrence(1)). In the table,

classes 1, 5 and 7 are algebraically special
classes 4, 6b and 8 are cylindrically symmetric
classes 2, 3 and 5 are algebraically unacceptable and
class 6a does not depend on arbitrary functions.

Consequently, the criterion that radiative metrics must be those type 1 metrics which admit geodesic rays leads to an interesting set of solutions, none of which are physically realizable. Either the concept of gravitational radiation must be rejected totally, or the tenets of the theory must be re-examined. Apart from the fact that Weber's(1) results must somehow be explained, the former course is contrary to very widely held views and strikes at the roots of field theories in general. A return must, therefore, be made to the original assumptions on which the theory is based.

The first of these assumptions is that a graviational wave can be considered as a surface in space-time, across which the curvature tensor is discontinuous. It is not very easy to modify this view without drastically affecting the whole theory of relativity, since it is based on the fundamental postulates (Pirani(2)).

The second assumption is that the discontinuity should be propagated with the fundamental velocity. Again, this assumption is worth retaining since it is difficult to accept the idea that it could be propagated faster, and there is no reason to suppose that it should travel more slowly.

The third assumption requires that the wave be propagated along a geodesic ray. While this appears reasonable enough by analogy with electromagnetic theory, it should be remembered that the field equations (unlike the electromagnetic case) are non-linear, and perhaps a re-examination is called for. Interpreted physically, the non-linearity of the field equations means that a gravitational field generates its own field. It is reasonable, therefore, to expect that a disturbance in the field should initiate other disturbances (Sachs(1)). This phenomenon, known as "back-scatter" (Penrose(2)), seems to be inevitable, and implies that a wavefront is not propagated entirely along a geodesic ray (Bardeen & Press(1)).

The final assumption upon which the theory hangs involves the restriction to outgoing radiation. Clearly, if backscattering is present the radiation will be mixed and thus the outgoing condition is no longer valid.

In searching for a new characterization of radiation, it should be remembered that the foregoing analysis is not completely valueless since it holds in the linearized (Arnowitt, Deser & Misner(1)) and asymptotic approximations (Newman & Unti (1)), and also, the optical parameters have clear physical interpretations. An interesting extension to the theory is found in Sachs(1), where metrics with "asymptotically geodesic rays" are suggested as possible radiative candidates. The method of testing a metric for asymptotically geodesic rays (comparable to A1.4) is more complicated than for the exact exposition given above. In particular, an affine parameter, whose existence was assumed above, must be found explicitly.

Another technique which has received attention recently involves the analysis of perturbations (test-fields) imposed on a given background metric. It is usual to select the Schwarzschild metric (Kundt & Newman(1)) for which the Newman-Penrose scalars can be expanded in terms of spherical harmonics, exploiting the known spherical symmetry of the solution (Bardeen& Press(1)).

APPENDIX 2

Information about each solution is listed as follows.

1. Label
2. Non-zero components of metric tensor [Harrison (1)]
3. Petrov type [D'Inverno + Russell-Clark (1)]
4. 8 components of the curvature tensor. For those metrics independent of x^3, $R_{1314} = R_{1213} = R_{1223} = 0$, while those independent of x^1 have $R_{1323} = R_{1223} = R_{1224} = 0$.
5. The Newman-Penrose scalars.

ϵ, ϵ_1, ϵ_2 and ϵ_3 take values ± 1.

1. $\underline{\text{I - B - 1}}$

2.
$$g_{11} = \sinh(2x_4)\,(x_1 - x_2)^{1-\epsilon\sqrt{2}}\,(x_1 + x_2)^{1+\epsilon\sqrt{2}}$$
$$g_{22} = \sinh^{-1}(2x_4)\,(x_1 - x_2)^{1-\epsilon\sqrt{2}}\,(x_1 + x_2)^{1+\epsilon\sqrt{2}}$$
$$g_{33} = (x_1 - x_2)^{\epsilon\sqrt{2}}\,(x_1 + x_2)^{-\epsilon\sqrt{2}}$$
$$g_{44} = \sinh^{-1}(2x_4)\,(x_1 - x_2)^{2-\epsilon\sqrt{2}}\,(x_1 + x_2)^{2+\epsilon\sqrt{2}}$$

3. Petrov Type I

4.
$$R_{1313} = \epsilon\sqrt{2}\,(x_1 + x_2)^{-\epsilon\sqrt{2}-2}\,(x_1 - x_2)^{\epsilon\sqrt{2}-2}\Big[-\epsilon\sqrt{2}\big(x_1^2\sinh^2(2x_4) + 2x_2^2\big) + x_1 x_2\big(3 + \sinh^2(2x_4)\big)\Big]$$
$$R_{1214} = \epsilon\sqrt{2}\,x_1\,(x_1 + x_2)^{\epsilon\sqrt{2}}\,(x_1 - x_2)^{-\epsilon\sqrt{2}}\cosh(2x_4)$$
$$R_{1224} = \epsilon\sqrt{2}\,x_2\,(x_1 + x_2)^{\epsilon\sqrt{2}}\,(x_1 - x_2)^{-\epsilon\sqrt{2}}\cosh(2x_4)\,\sinh^{-2}(2x_4)$$
$$\begin{aligned} R_{1414} = {} & \sinh^{-3}(2x_4)\,(x_1 - x_2)^{-1+\epsilon\sqrt{2}}\,(x_1 + x_2)^{-1-\epsilon\sqrt{2}}\Big(\Big(4 + \frac{\epsilon\sqrt{2}}{8}\Big) \\ & - (3+\epsilon)\cosh^2(2x_4)\Big) - \tfrac{1}{4}\sinh^{-1}(2x_4)\,(x_1 - x_2)^{-2+\epsilon\sqrt{2}}(x_1+x_2)^{\epsilon\sqrt{2}} \\ & \Big(3(2-\epsilon\sqrt{2})\sinh^{-2}(2x_4) - (2 - 2\epsilon - \epsilon\sqrt{2})\Big) \\ & - \tfrac{1}{4}\sinh^{-1}(2x_4)\,(x_1 - x_2)^{\epsilon\sqrt{2}}\,(x_1 + x_2)^{-2-\epsilon\sqrt{2}}\Big((2 - 4\epsilon - 3\epsilon\sqrt{2})\sinh^{-1}(2x_4) \\ & - (2 - 2\epsilon + \epsilon\sqrt{2})\Big) \end{aligned}$$
$$R_{1323} = \sinh^2(2x_4)\,(x_1 - x_2)^{-2+3\epsilon\sqrt{2}}\,(x_1 + x_2)^{-2-3\epsilon\sqrt{2}}\Big[(2 + \epsilon\sqrt{2})(x_1 + x_2)^{-2} - \tfrac{3}{2}(x_1 - x_2)^{-2}\Big]$$

5.
$$\begin{aligned} \Psi_0 = {} & -\tfrac{1}{4}(x_1 - x_2)^{-2}(x_1 + x_2)^{-2}\sinh^{-2}(2x_4)\Big[\epsilon\sqrt{2}\Big\{(x_1 x_2 - \epsilon\sqrt{2}\,x_1^2)\sinh^2(2x_4) \\ & + 3x_1 x_2 - 2\epsilon\sqrt{2}\,x_2^2\Big\} \\ & + (x_1 - x_2)^{2\epsilon\sqrt{2}}(x_1 + x_2)^{-2\epsilon\sqrt{2}}\Big\{\tfrac{1}{4}\sinh^{-2}(2x_4)\Big(3(2 - \epsilon\sqrt{2})(x_1 - x_2)^{-2}(x_1 + x_2)^{2\epsilon\sqrt{2}} \\ & + \big(2 - \epsilon(4 + 3\sqrt{2})\big)(x_1 + x_2)^{-2}\Big) \\ & - \sinh^{-2}(2x_4)(x_1 - x_2)^{-1}(x_1 + x_2)^{-1}\Big(4 + \frac{\epsilon\sqrt{2}}{8} - (3+\epsilon)\cosh^2(2x_4)\Big) \\ & - \tfrac{1}{4}\Big(\big(2 - \epsilon(2 + \sqrt{2})\big)(x_1 - x_2)^{-2}(x_1 + x_2)^{2\epsilon\sqrt{2}} + \big(2 - \epsilon(2 - \sqrt{2})\big)(x_1 + x_2)^{-2}\Big) \\ & + 2\sinh^3(2x_4)\Big((2 + \epsilon\sqrt{2})(x_1 + x_2)^{-2} - \tfrac{3}{2}(x_1 - x_2)^{-2}\Big)\Big\}\Big] \end{aligned}$$

$$\Psi_1 = -i\,\frac{\epsilon}{2}\,(x_1 - x_2)^{-2+\epsilon\frac{\sqrt{2}}{2}}\,(x_1 + x_2)^{-2-\epsilon\frac{\sqrt{2}}{2}}\,\sinh^{-\frac{1}{2}}(2x_4)\cosh(2x_4)$$
$$\left[x_1 + x_2 \sinh^{-2}(2x_4)\right]$$

$$\Psi_2 = -(x_1 - x_2)^{-1+\epsilon\sqrt{2}}\,(x_1 + x_2)^{-3-\epsilon\sqrt{2}}\Big[x_2\,(\epsilon\sqrt{2}\,x_1 - x_2)\sinh(2x_4)$$
$$- x_1\,(\epsilon\sqrt{2}\,x_2 - x_1)\sinh^{-1}(2x_4)\Big]$$

$$\Psi_3 = -i\,\epsilon\,(x_1 - x_2)^{-3+3\epsilon\frac{\sqrt{2}}{2}}\,(x_1 + x_2)^{-3-3\epsilon\frac{\sqrt{2}}{2}}\,\sinh^{\frac{1}{2}}(2x_4)\cosh(2x_4)$$
$$\left[x_1 - x_2 \sinh^{-2}(2x_4)\right]$$

$$\Psi_4 = -(x_1 - x_2)^{-4+2\epsilon\sqrt{2}}\,(x_1 + x_2)^{-4-2\epsilon\sqrt{2}}\Big[\epsilon\sqrt{2}\,\Big\{(x_1x_2 - \epsilon\sqrt{2}\,x_1^2)\sinh^2(2x_4)$$
$$+ 3x_1x_2 - 2\epsilon\sqrt{2}\,x_2^2\Big\}$$
$$+ (x_1 - x_2)^{2\epsilon\sqrt{2}}\,(x_1 + x_2)^{-2\epsilon\sqrt{2}}\Big\{\frac{1}{4}\sinh^{-2}(2x_4)\Big(3(2 - \epsilon\sqrt{2})(x_1 - x_2)^{-2}(x_1 + x_2)^{2\epsilon\sqrt{2}}$$
$$+ (x_1 + x_2)^{-2}\big(2 - \epsilon(4 + 3\sqrt{2})\big)\Big)$$
$$- (x_1 - x_2)^{-1}(x_1 + x_2)^{-1}\sinh^{-2}(2x_4)\Big(4 + \frac{\epsilon\sqrt{2}}{8} - (3 + \epsilon)\cosh^2(2x_4)\Big)$$
$$- \frac{1}{4}\Big(\big(2 - \epsilon(2 + \sqrt{2})\big)(x_1 - x_2)^{-2}(x_1 + x_2)^{2\epsilon\sqrt{2}} + \big(2 - \epsilon(2 - \sqrt{2})\big)(x_1 + x_2)^{-2}\Big)$$
$$- 2\sinh^3(2x_4)\Big((2 + \epsilon\sqrt{2})(x_1 + x_2)^{-2} - \frac{3}{2}(x_1 - x_2)^{-2}\Big)\Big\}\Big]$$

1. <u>I - B - 2</u>

2.

$$g_{11} = +(x_2 - x_1)^{1+\epsilon\sqrt{2}}(x_2 + x_1)^{1-\epsilon\sqrt{2}} \sin^{-1}(x_4)$$
$$g_{22} = (x_2 - x_1)^{1+\epsilon\sqrt{2}}(x_2 + x_1)^{1-\epsilon\sqrt{2}} \sin(x_4)$$
$$g_{33} = (x_2 - x_1)^{-\epsilon\sqrt{2}}(x_2 + x_1)^{\epsilon\sqrt{2}}$$
$$g_{44} = (x_2 - x_1)^{2+\epsilon\sqrt{2}}(x_2 + x_1)^{2-\epsilon\sqrt{2}} \sin^{-1}(x_4)$$

3. Petrov Type I

No further analysis since this is not a vacuum metric.

1. I - B - 3

2.
$$g_{11} = (x_1 - x_2)^{1+\sqrt{2}\epsilon_1} x_1^{2\left(\frac{5}{2} + \epsilon_2 - \sqrt{2}\epsilon_1\left(\frac{1}{2} + 2\epsilon_2\right)\right)} x_4^2$$
$$g_{22} = \lambda^2 (x_1 - x_2)^{1+\sqrt{2}\epsilon_1} x_1^{2\left(\frac{1}{2} + \epsilon_2 - \frac{1}{2}\sqrt{2}\epsilon_1\right)} x_4^{\frac{1}{2}\sqrt{2}\epsilon_1(1+\epsilon_2)}$$
$$g_{33} = (x_1 - x_2)^{-\sqrt{2}\epsilon_1} x_1^{2\left(-\epsilon_2 + \frac{1}{2}\sqrt{2}\epsilon_1\right)} x_4^{2\left(\frac{1}{4}(1-\epsilon_2) + \frac{1}{8}\sqrt{2}\epsilon_1(-3+\epsilon_2)\right)}$$
$$g_{44} = (x_1 - x_2)^{2+\sqrt{2}\epsilon_1} x_1^{2\left(3 + \epsilon_2 - \sqrt{2}\epsilon_1\left(\frac{1}{2} + 2\epsilon_2\right)\right)} \left(\frac{1}{4}(1-\epsilon_2) + \frac{1}{8}\sqrt{2}\epsilon_1(3\epsilon_2 - 1)\right)^2$$

3. Petrov Type I

The analysis continued with $\epsilon_1 = \epsilon_2 = \lambda^2 = 1$

4.
$$R_{1313} = \tfrac{1}{2} x_4^{-\frac{\sqrt{2}}{2}} x_1^{-2+\sqrt{2}} (x_1 - x_2)^{3\sqrt{2}} \Big[x_1^{-2}\left(17 + \tfrac{25}{2}\sqrt{2}\right) + (x_1 - x_2)^{-1} x_1^{-1}\left(-19 + \tfrac{37}{2}\sqrt{2}\right) + (x_1 - x_2)^{-2}(2 + \sqrt{2}) \Big] + \tfrac{\sqrt{2}}{4}(1 + \sqrt{2})(x_1 - x_2)^{-2-\sqrt{2}} x_1^{2-3\sqrt{2}} x_4^{2-\frac{3}{2}\sqrt{2}}$$
$$R_{1214} = (x_1 - x_2)^{\sqrt{2}} x_1^{7-5\sqrt{2}} x_4 \left(-4 - \tfrac{7}{2}\sqrt{2}\right)$$
$$R_{1224} = (x_1 - x_2)^{1+\sqrt{2}} x_1^{3-\sqrt{2}} x_4^{-1+\sqrt{2}} \left\{\left(\tfrac{1}{2} + \tfrac{3\sqrt{2}}{4}\right)\left((x_1 - x_2)^{-1} - x_1^{-1}\right)\right\}$$
$$R_{1414} = -\tfrac{1}{32}\Big\{ (x_1 - x_2)^{-1-\sqrt{2}} x_1^{-7+5\sqrt{2}} x_4^{-4} \Big[10(11 + 5\sqrt{2})(x_1 - x_2)^{-1} x_1 + (26 + 9\sqrt{2})(x_1 - x_2)^{-1} x_2 + 35(10 + 7\sqrt{2}) x_2 x_1^{-1} + 2(15 + 7\sqrt{2}) \Big] + (x_1 - x_2)^{-2-\sqrt{2}} x_1^{-2+9\sqrt{2}} x_4^{-2-\sqrt{2}} \Big\}$$
$$R_{1323} = (x_1 - x_2)^{-3-3\sqrt{2}} x_1^{-16+11\sqrt{2}} x_4^{-\frac{\sqrt{2}}{2}-4} \Big[x_1^{-1}\left(1 - \tfrac{3\sqrt{2}}{2}\right) - (x_1 - x_2)^{-1}\left(\tfrac{3}{2} + \sqrt{2}\right) \Big]$$

5.
$$\psi_0 = -\tfrac{1}{4}(x_1 - x_2)^{-3-2\sqrt{2}} x_1^{-14+4\sqrt{2}} x_4^{\sqrt{2}-2} \Big[\tfrac{1}{2}(x_1 - x_2)^{3+4\sqrt{2}} x_1^{14-4\sqrt{2}} x_4^{2-\sqrt{2}} \Big\{ x_1^{-2}\left(17 + \tfrac{25}{2}\sqrt{2}\right) + (x_1 - x_2)^{-1} x_1^{-1}\left(-19 + \tfrac{37}{2}\sqrt{2}\right) + (x_1 - x_2)^{-2}(2 + \sqrt{2}) \Big\}$$
$$+ \tfrac{1}{4}(2 + \sqrt{2})(x_1 - x_2)^{-2-4\sqrt{2}} x_1^{4-4\sqrt{2}} x_4^{2-2\sqrt{2}}$$
$$+ \tfrac{1}{4}\Big\{ x_1^{-5} x_4^{-2} \Big[10(11 + 5\sqrt{2})(x_1 - x_2)^{-1} x_1 + (26 + 9\sqrt{2})(x_1 - x_2)^{-1} x_2 + 35(10 + 7\sqrt{2}) x_2 x_1^{-1} + 2(15 + 7\sqrt{2}) \Big] + (x_1 - x_2)^{-1} x_1^{4\sqrt{2}} x_4^{-\sqrt{2}} \Big\} x_4^{-\sqrt{2}}$$
$$+ 2\Big\{ x_1^{-1}\left(1 - \tfrac{3\sqrt{2}}{2}\right) - (x_1 - x_2)^{-1}\left(\tfrac{3}{2} + \sqrt{2}\right) \Big\} x_1^{-2+3\sqrt{2}} x_4^{-\frac{\sqrt{2}}{2}-3} \Big]$$

$$
\begin{aligned}
\psi_1 &= -i\,(x_1-x_2)^{-1}\,x_1^{-1+\frac{3\sqrt{2}}{2}}\,x_4^{-1}\left\{\tfrac{1}{8}\,(x_1-x_2)^{-1-\frac{\sqrt{2}}{2}}\left(-4-\tfrac{7}{2}\sqrt{2}\right)\right.\\
&\quad\left.+\,x_1^{-2+2\sqrt{2}}\,x_4^{-1+\sqrt{2}/2}\left(\tfrac{1}{2}+\tfrac{3\sqrt{2}}{2}\right)\left((x_1-x_2)^{-1}-x_1^{-1}\right)\right\}\\
\psi_2 &= \tfrac{1}{2}\,(x_1-x_2)^{-1-\sqrt{2}}\,x_1^{-1-3\sqrt{2}}\,x_4^{-2}\left\{x_1^{-1}\,(x_1-x_2)^{-1}\left(1-4\sqrt{2}\right)\right.\\
&\quad\left.+\,\tfrac{1}{2}\,x_1^{-2}\left(-13+9\sqrt{2}\right)+\tfrac{1}{2}\,(x_1-x_2)^{-2}\left(1+\sqrt{2}\right)\right\}\\
&\quad+\tfrac{1}{4}\,(x_1-x_2)^{-3-\sqrt{2}}\,x_1^{3-\sqrt{2}}\,x_4^{-\sqrt{2}}\left(-1+3\sqrt{2}\right)\\
\psi_3 &= -\tfrac{i}{4}\,(x_1-x_2)^{-3-3\frac{\sqrt{2}}{2}}\,x_1^{-4-5\frac{\sqrt{2}}{2}}\,x_4^{-1-\sqrt{2}}\left(-4-\tfrac{7}{2}\sqrt{2}\right)\\
&\quad+2i\,(x_1-x_2)^{-2-3\frac{\sqrt{2}}{2}}\,x_1^{-6+\frac{9}{2}\sqrt{2}}\,x_4^{-2-\frac{\sqrt{2}}{2}}\left(\left(\tfrac{1}{2}+\tfrac{3\sqrt{2}}{4}\right)\left((x_1-x_2)^{-1}\right.\right.\\
&\quad\left.\left.-\,x_1^{-1}\right)\right)\\
\psi_4 &= -(x_1-x_2)^{-5}\,x_1^{-20+2\sqrt{2}}\,x_4^{-2-\sqrt{2}}\left[\tfrac{1}{2}\,(x_1-x_2)^{3+4\sqrt{2}}\,x_1^{14-4\sqrt{2}}\,x_4^{2-\sqrt{2}}\right.\\
&\quad\left\{x_1^{-2}\left(17+\tfrac{25}{2}\sqrt{2}\right)+(x_1-x_2)^{-1}x_1^{-1}\left(-19+\tfrac{37}{2}\sqrt{2}\right)+(x_1-x_2)^{-2}\left(2+\sqrt{2}\right)\right\}\\
&\quad+\tfrac{1}{4}\left(2+\sqrt{2}\right)(x_1-x_2)^{-2-4\sqrt{2}}\,x_1^{4-4\sqrt{2}}\,x_4^{2-2\sqrt{2}}\\
&\quad+\tfrac{1}{4}\,x_4^{-\sqrt{2}}\left\{x_1^{-5}x_4^{-2}\left[10\left(11+5\sqrt{2}\right)(x_1-x_2)^{-1}x_1+\left(26+9\sqrt{2}\right)(x_1-x_2)^{-1}x_2\right.\right.\\
&\quad\left.\left.+\,35\left(10+7\sqrt{2}\right)x_2\,x_1^{-1}+2\left(15+7\sqrt{2}\right)\right]+(x_1-x_2)^{-1}x_1^{4\sqrt{2}}\,x_4^{-\sqrt{2}}\right\}\\
&\quad\left.-\,2\left(x_1^{-1}\left(1-\tfrac{3\sqrt{2}}{2}\right)-(x_1-x_2)^{-1}\left(\tfrac{3}{2}+\sqrt{2}\right)\right)x_1^{-2+3\sqrt{2}}\,x_4^{-\frac{\sqrt{2}}{2}-3}\right]
\end{aligned}
$$

1. I - B - 4

2. $$
\begin{aligned}
g_{11} &= \lambda^2 (x_1 - x_2)^{1+\sqrt{2}\epsilon_1} x_2^{1+2\epsilon_2-\sqrt{2}\epsilon_1} x_4^{\frac{\sqrt{2}}{2}\epsilon_1(1+\epsilon_2)} \\
g_{22} &= (x_1 - x_2)^{1+\sqrt{2}\epsilon_1} x_2^{5+2\epsilon_2-2\sqrt{2}\epsilon_1(\frac{1}{2}+2\epsilon_2)} \\
g_{33} &= (x_1 - x_2)^{-\sqrt{2}\epsilon_1} x_2^{-2\epsilon_2+\sqrt{2}\epsilon_1} x_4^{\frac{1}{2}(1-\epsilon_1)+\frac{1}{4}\sqrt{2}\epsilon_1(-3+\epsilon_2)} \\
g_{44} &= \left(\tfrac{1}{4}(1-\epsilon_2) + \tfrac{1}{8}\sqrt{2}\,\epsilon_1(3\epsilon_2 - 1)\right)^2 (x_1 - x_2)^{2+\sqrt{2}\epsilon_1} x_2^{6+2\epsilon_2-2\sqrt{2}\epsilon_1(\frac{1}{2}+2\epsilon_2)}
\end{aligned}
$$

3. Petrov Type I

Analysis continued with $\epsilon_1 = 1$, $\epsilon_2 = -1$

4. $$
\begin{aligned}
R_{1313} &= \lambda^2 (x_1 - x_2)^{-2-\sqrt{2}} x_4^{1-\sqrt{2}} \Big[\tfrac{1}{\lambda^2}\left(1 + \tfrac{3\sqrt{2}}{4}\right) x_2^{2+\sqrt{2}} \\
&\quad + \tfrac{1}{4}(1+\sqrt{2})\, x_2^{-4+3\sqrt{2}} \left\{ 2(x_1 - x_2)x_2 + \sqrt{2}\,x_1^2 \right\} \\
&\quad - \tfrac{(1-\sqrt{2})}{(3-2\sqrt{2})} (x_1 - x_2)\, x_2^{-3+3\sqrt{2}} x_4^{-1} \Big] \\
R_{1224} &= \tfrac{1}{4}(2+\sqrt{2})(x_1 - x_2)^{\sqrt{2}} x_2^{2-3\sqrt{2}} \\
R_{1214} &= \tfrac{1}{4}\lambda^2 (x_1 - x_2)^{1+\sqrt{2}} x_2^{-1-\sqrt{2}} \left((4-3\sqrt{2})x_2^{-1} - (2+\sqrt{2})(x_1 - x_2)^{-1} \right) \\
R_{1414} &= \tfrac{1}{16}\lambda^{-2}(1-\sqrt{2})(x_1 - x_2)^{-\sqrt{2}} x_2^{2+\sqrt{2}} \Big\{ (2+\sqrt{2})(x_1 - x_2)^{-2} \\
&\quad - (6-2\sqrt{2})(x_1 - x_2)^{-1} x_2^{-1} - (4-3\sqrt{2})x_2^{-2} \Big\} \\
&\quad - \tfrac{1}{16}\lambda^{-4}(2-\sqrt{2})(x_1 - x_2)^{-2-\sqrt{2}} x_2^{5-\sqrt{2}} x_4^{-1} \\
R_{1323} &= \tfrac{1}{4}\lambda^{-4}(x_1 - x_2)^{-3-3\sqrt{2}} x_2^{-3-3\sqrt{2}} x_4^{-\sqrt{2}} \Big\{ (2+\sqrt{2})(x_1 - x_2)^{-1} x_2 \\
&\quad - (6+5\sqrt{2}) + \left(\sqrt{2}(x_1 - x_2)^{-1} + (2+\sqrt{2})x_2^{-1}\right)\left(\sqrt{2} - (1+\sqrt{2})x_2^{1+6\sqrt{2}}\right) \Big\}
\end{aligned}
$$

5. $$
\begin{aligned}
\Psi_0 &= -\tfrac{1}{4}(x_1 - x_2)^{-2-2\sqrt{2}} x_2^2 \Big\{ (x_1 - x_2)^{2\sqrt{2}} x_2^{2-3\sqrt{2}} x_4^{1-2\sqrt{2}} \Big[\lambda^{-2}\left(1 + \tfrac{3\sqrt{2}}{4}\right) x_2^{2+\sqrt{2}} \\
&\quad + \tfrac{1}{4}(1+\sqrt{2})\, x_2^{-4+3\sqrt{2}} \left\{ 2(x_1 - x_2)x_2 + \sqrt{2}\,x_1^2 \right\} \\
&\quad - \tfrac{(1-\sqrt{2})}{(3-2\sqrt{2})} (x_1 - x_2)\, x_2^{-3+3\sqrt{2}} x_4^{-1} \Big] \\
&\quad - \tfrac{1}{16}\lambda^{-4}(1-\sqrt{2})\left(\tfrac{1}{2} - 2\sqrt{2}\right)^{-2} x_2^{2+2\sqrt{2}} x_4 \Big\{ (2+\sqrt{2})(x_1 - x_2)^{-2} \\
&\quad - (6-2\sqrt{2})(x_1 - x_2)^{-1} x_2^{-1} - (4-3\sqrt{2})x_2^{-2} \Big\} \\
&\quad + \tfrac{1}{16}\lambda^{-6}(2-\sqrt{2})\left(\tfrac{1}{2} - 2\sqrt{2}\right)^{-2}(x_1 - x_2)^2 x_2^5 \\
&\quad + \tfrac{1}{2}\lambda^{-5}(x_1 - x_2)^{-1} x_2^{3+\sqrt{2}} x_4^{-\frac{1}{2}-2\sqrt{2}} \Big\{ (2+\sqrt{2})(x_1 - x_2)^{-1} x_2 - (6+5\sqrt{2}) + \Big[\sqrt{2}(x_1 - x_2)^{-1} \\
&\quad + (2+\sqrt{2})x_2^{-1} \Big]\Big[\sqrt{2} - (1+\sqrt{2})x_2^{1+6\sqrt{2}} \Big] \Big\} \Big\}
\end{aligned}
$$

$$\Psi_1 = -\frac{i}{4}(\sqrt{2}-2)^{-1}\lambda^{-1}(x_1-x_2)^{-1-\frac{\sqrt{2}}{2}}x_2^{-1-5\frac{\sqrt{2}}{2}}x_4^{-1}\Big[x_2^{-1}\big\{(4-3\sqrt{2})x_2^{-1}$$
$$-(2+\sqrt{2})(x_1-x_2)^{-1}\big\}+(2+\sqrt{2})(x_1-x_2)^{-1}x_2^{-\sqrt{2}/2}\Big]$$

$$\Psi_2 = -\frac{1}{2}(x_1-x_2)^{3-\sqrt{2}}x_2^{-3-5\sqrt{2}}x_4^{-1}\Big[\frac{1}{2}(1+\sqrt{2})\left(1-\lambda^{-2}x_2^{4+4\sqrt{2}}\right)(x_1-x_2)^{-2}$$
$$-\left(2(3-2\sqrt{2})^{-1}x_4^{-1}+\frac{\sqrt{2}}{2}\right)x_2^{-1}(x_1-x_2)^{-1}-\frac{1}{4}(5+4\sqrt{2})x_2^{-2}\Big]$$

$$\Psi_3 = -i\,\frac{\lambda^{-1}}{\sqrt{2}}(x_1-x_2)^{-2-3\frac{\sqrt{2}}{2}}x_2^{-4+\sqrt{2}/2}x_4^{-2}\,\frac{1}{2}(\sqrt{2}-2)\Big[x_2^{-1}\big\{(4-3\sqrt{2})x_2^{-1}$$
$$-(2+\sqrt{2})(x_1-x_2)^{-1}\big\}-(2+\sqrt{2})(x_1-x_2)^{-1}x_2^{-\sqrt{2}/2}\Big]$$

$$\Psi_4 = -(x_1-x_2)^{-4-4\sqrt{2}}x_2^{-4+6\sqrt{2}}\Big[(x_1-x_2)^{2\sqrt{2}}x_2^{2-3\sqrt{2}}x_4^{1-2\sqrt{2}}$$
$$\Big\{\lambda^{-2}\left(1+\frac{3\sqrt{2}}{4}\right)x_2^{2+\sqrt{2}}+\frac{1}{4}(1+\sqrt{2})x_2^{-4+3\sqrt{2}}\left(2(x_1-x_2)x_2+\sqrt{2}x_1^2\right)$$
$$-\frac{(1-\sqrt{2})}{(3-2\sqrt{2})}(x_1-x_2)x_2^{-3+3\sqrt{2}}x_4^{-1}\Big\}$$
$$-\frac{1}{16}\lambda^{-4}(1-\sqrt{2})\left(\frac{1}{2}-2\sqrt{2}\right)^{-2}x_2^{2+2\sqrt{2}}x_4\Big\{(2+\sqrt{2})(x_1-x_2)^{-2}$$
$$-(6-2\sqrt{2})(x_1-x_2)^{-1}x_2^{-1}-(4-3\sqrt{2})x_2^{-2}\Big\}$$
$$+\frac{1}{16}\lambda^{-6}(2-\sqrt{2})\left(\frac{1}{2}-2\sqrt{2}\right)^{-2}(x_1-x_2)^2x_2^5$$
$$-\frac{1}{2}\lambda^{-5}(x_1-x_2)^{-1}x_2^{3+\sqrt{2}}x_4^{-\frac{1}{2}-2\sqrt{2}}\Big(\left((2+\sqrt{2})(x_1-x_2)^{-1}x_2-(6+5\sqrt{2})\right)$$
$$+\left(\sqrt{2}(x_1-x_2)^{-1}+(2+\sqrt{2})x_2^{-1}\right)\left(\sqrt{2}-(1+\sqrt{2})x_2^{1+6\sqrt{2}}\right)\Big)\Big]$$

1. $\underline{\mathrm{II} - A - 1}$

2.
$$
\begin{aligned}
g_{11} &= (x_1^2 + x_2)^{1+\sqrt{2}\epsilon}\, x_4^2 \\
g_{22} &= \lambda^2 (x_1^2 + x_2)^{1+\sqrt{2}\epsilon}\, x_4^{\frac{2}{7}(1+2\sqrt{2}\epsilon)} \\
g_{33} &= (x_1^2 + x_2)^{-\sqrt{2}\epsilon}\, x_4^{\frac{2}{7}(2-3\sqrt{2}\epsilon)} \\
g_{44} &= \left(\tfrac{1}{7}(3 - \sqrt{2}\epsilon)\right)^2 (x_1^2 + x_2)^{2+\sqrt{2}\epsilon}
\end{aligned}
$$

3. Petrov Type I

Analysis continued with $\epsilon = 1$

4.
$$
\begin{aligned}
R_{1313} &= (x_1^2 + x_2)^{-1-\sqrt{2}}\, x_4^{\frac{2}{7}(2-3\sqrt{2})} \Big[\Big\{ \big(-\sqrt{2} - \tfrac{1}{2}\big(\tfrac{1}{7}(3-\sqrt{2})\big)^{-1} \\
&\quad + x_1^2 (x_1^2 + x_2)^{-1} (4 + 3\sqrt{2}) \Big\} \\
&\quad + \tfrac{1}{2}\sqrt{2}\,\lambda^{-2} (1+\sqrt{2})(x_1^2 + x_2)^{-1}\, x_4^{\frac{2}{7}(6-2\sqrt{2})} \Big] \\
R_{1214} &= \tfrac{3}{14} (x_1^2 + x_2)^{\sqrt{2}}\, x_4 (4 + \sqrt{2}) \\
R_{1224} &= -\frac{3\lambda^2}{14} (x_1^2 + x_2)^{\sqrt{2}}\, x_1\, x_4^{\frac{2}{7}(1+2\sqrt{2}) - 1} \{6 + \sqrt{2}\} \\
R_{1414} &= \left(\tfrac{1}{7}(3-\sqrt{2})\right)^2 (x_1^2 + x_2)^{-1-\sqrt{2}}\, x_4^{-2} \Big[\big(1 - (x_1^2 + x_2)^{-1} x_1^2\big)(2 + \sqrt{2})\, x_4^{-2} \\
&\quad - \tfrac{1}{4}\lambda^{-2} (4 + 3\sqrt{2})(x_1^2 + x_2)^{-1}\, x_4^{-\frac{2}{7}(1+2\sqrt{2})} \Big] \\
R_{1323} &= (3 + 2\sqrt{2})\, x_1 (x_1^2 + x_2)^{-4-3\sqrt{2}}\, x_4^{\frac{3}{7}(2-3\sqrt{2})-4}
\end{aligned}
$$

5.
$$
\begin{aligned}
\Psi_0 &= -\tfrac{1}{4}\, x_4^{-\frac{2}{7}(6+2\sqrt{2})} \Big[\lambda^2 \Big((x_1^2 + x_2)^{-1} \Big[\Big\{ \big(-\sqrt{2} - \tfrac{1}{2}\big(\tfrac{1}{7}(3-\sqrt{2})\big)^{-1}\big) \\
&\quad + x_1^2 (x_1^2 + x_2)^{-1} (4 + 3\sqrt{2}) \Big\} + \tfrac{1}{2}\sqrt{2}\,\lambda^{-2} (1+\sqrt{2})(x_1^2 + x_2)^{-1}\, x_4^{\frac{2}{7}(6-2\sqrt{2})} \Big] \Big) \\
&\quad - (x_1^2 + x_2)^{-3-2\sqrt{2}}\, x_4^{-2} \Big[\big(1 - (x_1^2 + x_2)^{-1} x_1^2\big)(2 + \sqrt{2})\, x_4^{-2} \\
&\quad - \tfrac{1}{4}\lambda^{-2} (4 + 3\sqrt{2})(x_1^2 + x_2)^{-1}\, x_4^{-\frac{2}{7}(1+2\sqrt{2})} \Big] \\
&\quad + 2\lambda (3 + 2\sqrt{2})\, x_1 (x_1^2 + x_2)^{-4-3\sqrt{2}}\, x_4^{-\frac{2}{7}(9+2\sqrt{2})} \Big] \\
\Psi_1 &= \frac{3}{4\sqrt{2}}\, i\, (3-\sqrt{2})^{-1} (x_1^2 + x_2)^{-2-\frac{\sqrt{2}}{2}} \Big[(4+\sqrt{2})\, x_4^{-1} + \lambda^{-1}(6+\sqrt{2})\, x_1\, x_4^{\frac{2}{7}(-3+\sqrt{2})-1} \Big] \\
\Psi_2 &= -\tfrac{1}{2} (x_1^2 + x_2)^{-2-\sqrt{2}}\, x_4^{-2} \Big\{ (1+\sqrt{2}) - \tfrac{2}{7}(1+2\sqrt{2})(3-\sqrt{2})^2 \\
&\quad - (2 + 2\sqrt{2})\, x_1^2 (x_1^2 + x_2)^{-1} \Big\} - \tfrac{1}{8}\lambda^2 (1+\sqrt{2})(x_1^2 + x_2)^{-3-\sqrt{2}} \\
&\quad x_4^{-\frac{2}{7}(1+2\sqrt{2})} \Big\{ (1-\sqrt{2}) + (1+\sqrt{2})\, x_1 \Big\}
\end{aligned}
$$

$$\Psi_3 = -\frac{3}{2\sqrt{2}}\, i\, \lambda^{-3}(3-\sqrt{2})^{-1}\,(x_1^2+x_2)^{-3-\frac{3\sqrt{2}}{2}}\, x_4^{-\frac{2}{7}(5+3\sqrt{2})}\Big[\lambda(4+\sqrt{2})\,x_4^{\frac{2}{7}(-3+\sqrt{2})+1}$$
$$-(6+\sqrt{2})x_1\, x_4^{\frac{2}{7}(1+2\sqrt{2})-1}\Big]$$

$$\Psi_4 = -\lambda^{-3}(x_1^2+x_2)^{-2-2\sqrt{2}}\, x_4^{-\frac{3}{7}(8+6\sqrt{2})}\Big[\lambda(x_1^2+x_2)^{-1}\Big[\Big\{\Big(-\sqrt{2}$$
$$-\frac{1}{2}\Big(\frac{1}{7}(3-\sqrt{2})\Big)^{-1}\Big) + x_1^2(x_1^2+x_2)^{-1}(4+3\sqrt{2})\Big\}$$
$$+\frac{1}{2}\sqrt{2}\,\lambda^{-2}(1+\sqrt{2})(x_1^2+x_2)^{-1}\, x_4^{\frac{3}{7}(6-2\sqrt{2})}\Big]$$
$$-(x_1^2+x_2)^{-3-2\sqrt{2}}\, x_4^{-2}\Big[\Big(1-(x_1^2+x_2)^{-1}x_1^2\Big)(2+\sqrt{2})\,x_4^{-2}$$
$$-\frac{1}{4}\lambda^{-2}(4+3\sqrt{2})(x_1^2+x_2)^{-1}\, x_4^{-\frac{3}{7}(1+2\sqrt{2})}\Big]$$
$$-2\,(3+2\sqrt{2})x_1\,(x_1^2+x_2)^{-4-3\sqrt{2}}\, x_4^{-\frac{2}{7}(9+2\sqrt{2})}\Big]$$

1. $\underline{\text{II - A - 2}}$

2.
$$
\begin{aligned}
g_{11} &= \left(x_2 - \tfrac{x_1^2}{16}\right)^{1+\sqrt{2}\epsilon} (x_4 - \epsilon_1)^{2-\sqrt{2}\epsilon} (x_4 + \epsilon_1)^{4+3\sqrt{2}\epsilon} \left(\epsilon_2 x_4 + \tfrac{\sqrt{2}}{2}\epsilon_3\right)^{-2(3+\sqrt{2}\epsilon)} \\
g_{22} &= \lambda^2 \left(x_2 - \tfrac{x_1^2}{16}\right)^{1+\sqrt{2}\epsilon} (x_4 - \epsilon_1)^{\sqrt{2}\epsilon} (x_4 + \epsilon_1)^{2+\sqrt{2}\epsilon} \left(\epsilon_2 x_4 + \tfrac{\sqrt{2}}{2}\epsilon_3\right)^{-2(1+\sqrt{2}\epsilon)} \\
g_{33} &= \left(x_2 - \tfrac{x_1^2}{16}\right)^{-\sqrt{2}\epsilon} (x_4 - \epsilon_1)^{1-\sqrt{2}\epsilon} (x_4 + \epsilon_1)^{-1-\sqrt{2}\epsilon} \left(\epsilon_2 x_4 + \tfrac{\sqrt{2}}{2}\epsilon_3\right)^{2\sqrt{2}\epsilon} \\
g_{44} &= 8\left(x_2 - \tfrac{x_1^2}{16}\right)^{2+\sqrt{2}\epsilon} (x_4 - \epsilon_1)^{1-\sqrt{2}\epsilon} (x_4 + \epsilon_1)^{3+3\sqrt{2}\epsilon} \left(\epsilon_2 x_4 + \tfrac{\sqrt{2}}{2}\epsilon_3\right)^{-2(4+\sqrt{2}\epsilon)}
\end{aligned}
$$

3. Petrov Type I

Analysis continued with $\epsilon = \epsilon_1 = \epsilon_2 = \epsilon_3 = 1$

4.
$$
\begin{aligned}
R_{1313} &= \left(x_2 - \tfrac{x_1^2}{16}\right)^{-1-\sqrt{2}} (x_4 - 1)^{-\sqrt{2}} (x_4 + 1)^{-1-\sqrt{2}} \left(x_4 + \tfrac{\sqrt{2}}{2}\right)^{2\sqrt{2}} \Big[\left(x_2 - \tfrac{x_1^2}{16}\right)^{-1} \\
&\quad (x_4 - 1) \left(\tfrac{x_1}{16}\right)^2 (4 + 3\sqrt{2}) + \left(x_2 - \tfrac{x_1^2}{16}\right)^{-1} (x_4 - 1)^3 (x_4 + 1)^{2+2\sqrt{2}} \big(2 \\
&\quad + \sqrt{2}\big) \left(x_4 + \tfrac{\sqrt{2}}{2}\right)^{-4} \tfrac{\lambda^{-2}}{4} \\
&\quad - \tfrac{1}{8} \left\{ x_4^2 (-2 - 3\sqrt{2}) + x_4 (7 + 2\sqrt{2}) + (-1 - 2\sqrt{2}) \right\} (x_4 + 1)^{-1} \left(x_4 + \tfrac{\sqrt{2}}{2}\right)^2 \\
&\quad - \tfrac{1}{32} \left\{ x_4^2 (4 + 12\sqrt{2}) + x_4 (-8 - 2\sqrt{2}) - 2(1 + 2\sqrt{2}) \right\} \Big] \\
R_{1214} &= \tfrac{1}{4} \left(x_2 - \tfrac{x_1^2}{16}\right)^{\sqrt{2}} (x_4 - 1)^{2-\sqrt{2}} (x_4 + 1)^{4+3\sqrt{2}} \left(x_4 + \tfrac{\sqrt{2}}{2}\right)^{-6-2\sqrt{2}} \Big[4 (x_4 - 1)^{-1} \\
&\quad + (20 - 6\sqrt{2})(x_4 + 1)^{-1} + (12 - 5\sqrt{2}) \left(x_4 + \tfrac{\sqrt{2}}{2}\right)^{-1} \Big] \\
R_{1224} &= \tfrac{1}{16} \lambda^2 x_1 \left(x_2 - \tfrac{x_1^2}{16}\right)^{\sqrt{2}} (x_4 - 1)^{\sqrt{2}} (x_4 + 1)^{2+\sqrt{2}} \left(x_4 + \tfrac{\sqrt{2}}{2}\right)^{-2-2\sqrt{2}} \Big[-\sqrt{2} (x_4 - 1)^{-1} \\
&\quad + \tfrac{1}{2} (14 - 11\sqrt{2}) \left(x_4 + \tfrac{\sqrt{2}}{2}\right)^{-1} - 2(3 + 2\sqrt{2})(x_4 + 1)^{-1} \Big] \\
R_{1414} &= \left(x_2 - \tfrac{x_1^2}{16}\right)^{-1-\sqrt{2}} (x_4 - 1)^{-2+\sqrt{2}} (x_4 + 1)^{4-3\sqrt{2}} \left(x_4 + \tfrac{\sqrt{2}}{2}\right)^{6+2\sqrt{2}} \Big[- \Big(\tfrac{x_1^2}{32} \left(x_2 - \tfrac{x_1^2}{16}\right)^{-1} \\
&\quad + \tfrac{1}{2} \Big) (2 + \sqrt{2}) (x_4 - 1)^{-1} (x_4 + 1)^{-1} \left(x_4 + \tfrac{\sqrt{2}}{2}\right)^{-2} \\
&\quad - 2\lambda^{-2} (4 + 3\sqrt{2}) (x_4 - 1)^{1-2\sqrt{2}} (x_4 + 1)^{1+2\sqrt{2}} \left(x_4 + \tfrac{\sqrt{2}}{2}\right)^{-6} \\
&\quad + \tfrac{1}{4} (x_4 - 1)^{-2} (2 - \sqrt{2}) + \tfrac{1}{4} (4 + 3\sqrt{2})(x_4 + 1)^{-2} + \tfrac{1}{4} (6 + \sqrt{2}) \left(x_4 + \tfrac{\sqrt{2}}{2}\right)^{-2} \\
&\quad - \tfrac{1}{2} (3 + \sqrt{2})(x_4 - 1)^{-1} (x_4 + 1)^{-1} + \left(-11 + \tfrac{53}{4}\sqrt{2}\right)(x_4 - 1)^{-1} \left(x_4 + \tfrac{\sqrt{2}}{2}\right)^{-1} \\
&\quad - 2 (3 + 5\sqrt{2})(x_4 + 1)^{-1} \left(x_4 + \tfrac{\sqrt{2}}{2}\right)^{-1} \Big] \\
R_{1323} &= -\tfrac{x_1}{16} (3 + 2\sqrt{2}) \left(x_2 - \tfrac{x_1^2}{16}\right)^{-4-3\sqrt{2}} (x_4 - 1)^{-3+\sqrt{2}} (x_4 + 1)^{-9-7\sqrt{2}} \\
&\qquad \left(x_4 + \tfrac{\sqrt{2}}{2}\right)^{6(2+\sqrt{2})}
\end{aligned}
$$

5.

$$
\begin{aligned}
\Psi_0 = & -\tfrac{1}{4}\lambda^2\left(x_2-\tfrac{x_1^2}{16}\right)^{-1}(x_4-1)^{-2+2\sqrt{2}}(x_4+1)^{-2-2\sqrt{2}}\left(x_4+\tfrac{\sqrt{2}}{2}\right)^2 \Big\{ \left(x_2-\tfrac{x_1^2}{16}\right)^{-1}\left(\tfrac{x_1}{16}\right)^2(4+3\sqrt{2}) \\
& + \left(x_2-\tfrac{x_1^2}{16}\right)^{-1}(x_4-1)^2(x_4+1)^{2+2\sqrt{2}}\left(x_4+\tfrac{\sqrt{2}}{2}\right)^{-4}\Big[\tfrac{1}{4}\lambda^{-2}(2+\sqrt{2})\Big] \\
& -\tfrac{1}{8}(x_4-1)^{-1}(x_4+1)^{-1}\left(x_4+\tfrac{\sqrt{2}}{2}\right)^2\Big[x_4^2(-2-3\sqrt{2})+x_4(7+2\sqrt{2}) \\
& -(1+2\sqrt{2})\Big] + \tfrac{1}{32}\Big[x_4^2(4+12\sqrt{2})+x_4(-8-2\sqrt{2})-2(1+2\sqrt{2})\Big] \\
& -\tfrac{1}{8}\left(x_2-\tfrac{x_1^2}{16}\right)^{-2-2\sqrt{2}}(x_4-1)^{-3+2\sqrt{2}}(x_4+1)^{-7-6\sqrt{2}}\left(x_4+\tfrac{\sqrt{2}}{2}\right)^{14+4\sqrt{2}}\Big[-\Big(\tfrac{x_1^2}{32}\left(x_2-\tfrac{x_1^2}{16}\right)^{-1} \\
& +\tfrac{1}{2}\Big)(2+\sqrt{2})(x_4-1)^{-1}(x_4+1)^{-1}\left(x_4+\tfrac{\sqrt{2}}{2}\right)^{-2} - 2\lambda^{-2}(4+3\sqrt{2})(x_4-1)^{1-2\sqrt{2}} \\
& (x_4+1)^{1+2\sqrt{2}}\left(x_4+\tfrac{\sqrt{2}}{2}\right)^{-6} + \tfrac{1}{4}(x_4-1)^{-2}(2-\sqrt{2}) + \tfrac{1}{4}(4+3\sqrt{2})(x_4+1)^{-2} \\
& + \tfrac{1}{4}(6+\sqrt{2})\left(x_4+\tfrac{\sqrt{2}}{2}\right)^{-2} - \tfrac{1}{2}(3+\sqrt{2})(x_4-1)^{-1}(x_4+1)^{-1} + \left(-11+5\tfrac{3}{4}\sqrt{2}\right) \\
& (x_4-1)^{-1}\left(x_4+\tfrac{\sqrt{2}}{2}\right)^{-1} - 2(3+5\sqrt{2})(x_4+1)^{-1}\left(x_4+\tfrac{\sqrt{2}}{2}\right)^{-1}\Big] \\
& -(3+2\sqrt{2})\left(\tfrac{x_1}{8}\right)\lambda^{-1}\left(x_2-\tfrac{x_1^2}{16}\right)^{-3-2\sqrt{2}}(x_4-1)^{-3+\sqrt{2}}(x_4+1)^{-7-5\sqrt{2}} \\
& \left(x_4+\tfrac{\sqrt{2}}{2}\right)^{11+4\sqrt{2}}\Big\}
\end{aligned}
$$

$$
\begin{aligned}
\Psi_1 = & -\tfrac{1}{32}\, i\left(x_2-\tfrac{x_1^2}{16}\right)^{-2-\frac{\sqrt{2}}{2}}(x_4-1)^{-\frac{1}{2}+\frac{\sqrt{2}}{2}}(x_4+1)^{-\frac{1}{2}-\frac{\sqrt{2}}{2}}\left(x_4+\tfrac{\sqrt{2}}{2}\right)^{4+\sqrt{2}} \\
& \Big\{(x_4+1)^{-1-\sqrt{2}}\Big(4(x_4-1)^{-1}+(20-6\sqrt{2})(x_4-1)^{-1}+(12-5\sqrt{2})\left(x_4+\tfrac{\sqrt{2}}{2}\right)^{-1}\Big) \\
& + \tfrac{\lambda}{4}x_1(x_4-1)^{-1+\sqrt{2}}\left(x_4+\tfrac{\sqrt{2}}{2}\right)^2\Big(-\sqrt{2}(x_4-1)^{-1}+(-6-4\sqrt{2})(x_4+1)^{-1} \\
& + \tfrac{1}{2}(14-11\sqrt{2})\left(x_4+\tfrac{\sqrt{2}}{2}\right)^{-1}\Big)\Big\}
\end{aligned}
$$

$$
\begin{aligned}
\Psi_2 = & -\tfrac{1}{2}\lambda^{-2}\left(x_2-\tfrac{x_1^2}{16}\right)^{-2-\sqrt{2}}(x_4-1)^{-2+\sqrt{2}}(x_4+1)^{-4-3\sqrt{2}}\left(x_4+\tfrac{\sqrt{2}}{2}\right)^{6+2\sqrt{2}}\Big[\tfrac{x_1^2}{32}\Big(x_2 \\
& -\tfrac{x_1^2}{16}\Big)(-3+\sqrt{2}) + x_4^2(-1+7\sqrt{2}) + x_4(1+\sqrt{2}) - (26+19\sqrt{2})\Big] \\
& + \tfrac{1}{4}\lambda^{-2}\left(x_2-\tfrac{x_1^2}{16}\right)^{-2-\sqrt{2}}(x_4-1)^{-3+\sqrt{2}}(x_4+1)^{-5-3\sqrt{2}}\left(x_4+\tfrac{\sqrt{2}}{2}\right)^{8+2\sqrt{2}}\Big[x_4^2(20 \\
& + 15\sqrt{2}) + x_4(-32-18\sqrt{2}) + (4+7\sqrt{2})\Big] \\
& -\tfrac{1}{4}\left(x_2-\tfrac{x_1^2}{16}\right)^{-3-\sqrt{2}}(x_4-1)^{-2\sqrt{2}}(x_4+1)^{-2-\sqrt{2}}\left(x_4+\tfrac{\sqrt{2}}{2}\right)^{2+2\sqrt{2}}(1+\sqrt{2})
\end{aligned}
$$

$$
\begin{aligned}
\Psi_3 = & -\tfrac{1}{8}\, i\,\lambda^{-2}\left(x_2-\tfrac{x_1^2}{16}\right)^{-3-\frac{3\sqrt{2}}{2}}(x_4-1)^{-\frac{1}{2}-\frac{\sqrt{2}}{2}}(x_4+1)^{-\frac{7}{2}-\frac{5\sqrt{2}}{2}}\left(x_4+\tfrac{\sqrt{2}}{2}\right)^{6+3\sqrt{2}} \\
& \Big\{\Big(4(x_4-1)^{-1}+(20-6\sqrt{2})(x_4+1)^{-1}+(12-5\sqrt{2})\left(x_4+\tfrac{\sqrt{2}}{2}\right)^{-1}\Big) \\
& -\tfrac{1}{8}\lambda x_1(x_4-1)^{-1-\sqrt{2}}(x_4+1)^{-1-\sqrt{2}}\left(x_4+\tfrac{\sqrt{2}}{2}\right)^2\Big(-\sqrt{2}(x_4-1)^{-1} \\
& + (-6-4\sqrt{2})(x_4+1)^{-1} + \tfrac{1}{2}(14-11\sqrt{2})\left(x_4+\tfrac{\sqrt{2}}{2}\right)^{-1}\Big)\Big\}
\end{aligned}
$$

$$
\begin{aligned}
\Psi_4 = & -\lambda^{-2}\left(x_2-\tfrac{x_1^2}{16}\right)^{-3-\sqrt{2}}(x_4-1)^{-2}(x_4+1)^{-6-4\sqrt{2}}\left(x_4+\tfrac{\sqrt{2}}{2}\right)^{6+2\sqrt{2}}\Big\{\left(\tfrac{x_1}{16}\right)^2\left(x_2-\tfrac{x_1^2}{16}\right)^{-1} \\
& (4+3\sqrt{2}) + \left(x_2-\tfrac{x_1^2}{16}\right)^{-1}(x_4-1)^2(x_4+1)^{2+2\sqrt{2}}\left(x_4+\tfrac{\sqrt{2}}{2}\right)^{-4}\Big[\tfrac{1}{4}\lambda^2(2+\sqrt{2}) \Big] \\
& -\tfrac{1}{8}(x_4-1)^{-1}(x_4+1)^{-1}\left(x_4+\tfrac{\sqrt{2}}{2}\right)^2\Big[x_4^2(-2-3\sqrt{2})+x_4(7+2\sqrt{2})-(1+2\sqrt{2})\Big] \\
& +\tfrac{1}{32}\Big[x_4^2(4+12\sqrt{2})+x_4(-8-2\sqrt{2})-2(1+2\sqrt{2})\Big] \\
& -\tfrac{1}{8}\left(x_2-\tfrac{x_1^2}{16}\right)^{-2-2\sqrt{2}}(x_4-1)^{-3+2\sqrt{2}}(x_4+1)^{-7-6\sqrt{2}}\left(x_4+\tfrac{\sqrt{2}}{2}\right)^{14+4\sqrt{2}}\Big[-\Big(\tfrac{x_1^2}{32} \\
& \left(x_2-\tfrac{x_1^2}{16}\right)^{-1}+\tfrac{1}{2}\Big)(2+\sqrt{2})(x_4-1)^{-1}(x_4+1)^{-1}\left(x_4+\tfrac{\sqrt{2}}{2}\right)^{-2} \\
& -2\lambda^{-2}(4+3\sqrt{2})(x_4-1)^{1-2\sqrt{2}}(x_4+1)^{1+2\sqrt{2}}\left(x_4+\tfrac{\sqrt{2}}{2}\right)^{-2} \\
& -\tfrac{1}{2}(3+\sqrt{2})(x_4-1)^{-1}(x_4+1)^{-1}+\left(-11+\tfrac{53}{4}\sqrt{2}\right)(x_4-1)^{-1}\left(x_4+\tfrac{\sqrt{2}}{2}\right)^{-1} \\
& -2(3+5\sqrt{2})(x_4+1)^{-1}\left(x_4+\tfrac{\sqrt{2}}{2}\right)^{-1}\Big] \\
& +\tfrac{1}{8}x_1\lambda^{-1}(3+2\sqrt{2})\left(x_2-\tfrac{x_1^2}{16}\right)^{-3-2\sqrt{2}}(x_4-1)^{-3+\sqrt{2}}(x_4+1)^{-7-5\sqrt{2}} \\
& \left(x_4+\tfrac{\sqrt{2}}{2}\right)^{11+4\sqrt{2}}\Big\}
\end{aligned}
$$

1. $\underline{\text{II} - A - 3}$

2.
$$g_{11} = \lambda^2 \left(x_1 + \frac{x_2^2}{16}\right)^{1+\sqrt{2}\epsilon} (x_4 - \epsilon_1)^{2+\sqrt{2}\epsilon} (x_4 + \epsilon_1)^{\sqrt{2}\epsilon} \left(\epsilon_2 x_4 + \tfrac{1}{2}\sqrt{2}\,\epsilon_3\right)^{-2-2\sqrt{2}\epsilon}$$
$$g_{22} = \left(x_1 + x_2^2/16\right)^{1+\sqrt{2}\epsilon} (x_4 - \epsilon_1)^{4+3\sqrt{2}\epsilon} (x_4 + \epsilon_1)^{2-\sqrt{2}\epsilon} \left(\epsilon_2 x_4 + \tfrac{1}{2}\sqrt{2}\,\epsilon_3\right)^{-6-2\sqrt{2}\epsilon}$$
$$g_{33} = \left(x_1 + x_2^2/16\right)^{-\sqrt{2}\epsilon} (x_4 - \epsilon_1)^{-1-\sqrt{2}\epsilon} (x_4 + \epsilon_1)^{1-\sqrt{2}\epsilon} \left(\epsilon_2 x_4 + \tfrac{1}{2}\sqrt{2}\,\epsilon_3\right)^{2\sqrt{2}\epsilon}$$
$$g_{44} = 8\left(x_1 + x_2^2/16\right)^{2+\sqrt{2}\epsilon} (x_4 - \epsilon_1)^{3+3\sqrt{2}\epsilon} (x_4 + \epsilon_1)^{1-\sqrt{2}\epsilon} \left(\epsilon_2 x_4 + \tfrac{1}{2}\sqrt{2}\,\epsilon_3\right)^{-8-2\sqrt{2}}$$

3. Petrov Type I

Analysis continued with $\epsilon = \epsilon_1 = \epsilon_2 = \epsilon_3 = 1$

4.
$$\begin{aligned}
R_{1313} &= \left(x_1 + x_2^2/16\right)^{-2-\sqrt{2}} (x_4 - 1)^{-1-\sqrt{2}} (x_4+1)^{-2-\sqrt{2}} \left(x_4 + \tfrac{\sqrt{2}}{2}\right)^{2\sqrt{2}} \\
&\quad \Big\{ -(4+3\sqrt{2})(x_4+1)^3 - \tfrac{1}{16}\lambda^2 (x_4-1)^{-3-2\sqrt{2}} \left(x_4 + \tfrac{\sqrt{2}}{2}\right)^4 \\
&\quad \Big[\tfrac{x_2^2}{16} (2+\sqrt{2})(x_4^2 - 1) + \tfrac{1}{2}\left(x_1 + \tfrac{x_2^2}{16}\right) \big\{ x_4^2 (16 + 4\sqrt{2}) \\
&\quad + x_4 (14 + 24\sqrt{2}) + (18 + 6\sqrt{2}) \big\} \Big] \Big\} \\
R_{1224} &= \left(x_1 + x_2^2/16\right)^{\sqrt{2}} (x_4-1)^{4+3\sqrt{2}} (x_4+1)^{2-\sqrt{2}} \left(x_4 + \tfrac{\sqrt{2}}{2}\right)^{-6-2\sqrt{2}} \Big[6(2+\sqrt{2})\left(x_4 + \tfrac{\sqrt{2}}{2}\right)^{-1} \\
&\quad - (x_4+1)^{-1} \Big] \\
R_{1214} &= \tfrac{1}{32} \lambda^2 x_2 \left(x_1 + x_2^2/16\right)^{\sqrt{2}} (x_4-1)^{2+\sqrt{2}} (x_4+1)^{\sqrt{2}} \left(x_4 + \tfrac{\sqrt{2}}{2}\right)^{-2-2\sqrt{2}} \Big\{ (12 + \\
&\quad + 10\sqrt{2})\left(x_4 + \tfrac{\sqrt{2}}{4}\right)^{-1} - 2\sqrt{2}(x_4+1)^{-1} - 4(3+2\sqrt{2})(x_4-1)^{-1} \Big\} \\
R_{1414} &= \lambda^{-2} \left(x_1 + x_2^2/16\right)^{-1-\sqrt{2}} (x_4-1)^{-2-\sqrt{2}} (x_4+1)^{-\sqrt{2}} \left(x_4 + \tfrac{\sqrt{2}}{2}\right)^{2+2\sqrt{2}} \Big[\tfrac{1}{4} (10 \\
&\quad + 7\sqrt{2})(x_4-1)^{-2} - \tfrac{1}{4}(4+3\sqrt{2})(x_4+1)^{-2} + 2(1+\sqrt{2})\left(x_4 + \tfrac{\sqrt{2}}{2}\right)^{-2} \\
&\quad + \tfrac{1}{2}(1-\sqrt{2})(x_4-1)^{-1}(x_4+1)^{-1} - (11 + 6\sqrt{2})(x_4-1)^{-1}\left(x_4 + \tfrac{\sqrt{2}}{2}\right)^{-1} \\
&\quad + \tfrac{1}{2}(3 - 2\sqrt{2})(x_4+1)^{-1}\left(x_4 + \tfrac{\sqrt{2}}{2}\right)^{-1} \Big] - 2\lambda^{-4}(4 + 3\sqrt{2})\left(x_1 + x_2^2/16\right)^{-2-\sqrt{2}} \\
&\quad (x_4-1)^{-1+\sqrt{2}} (x_4+1)^{1-3\sqrt{2}} \left(x_4 + \tfrac{\sqrt{2}}{2}\right)^{-4+2\sqrt{2}} - \tfrac{1}{32}\lambda^{-2} x_2^2 (4 + 3\sqrt{2}) \\
&\quad \left(x_1 + x_2^2/16\right)^{-2-\sqrt{2}} (x_4-1)^{-3-\sqrt{2}} (x_4+1)^{-1-\sqrt{2}} \left(x_4 + \tfrac{\sqrt{2}}{2}\right)^{1+2\sqrt{2}} \\
R_{1323} &= \tfrac{x_2}{16} \lambda^{-4} (3 + 2\sqrt{2})\left(x_1 + x_2^2/16\right)^{-4-3\sqrt{2}} (x_4-1)^{-5-3\sqrt{2}} (x_4+1)^{1-3\sqrt{2}} \left(x_4 + \tfrac{\sqrt{2}}{2}\right)^{4+6\sqrt{2}}
\end{aligned}$$

5.

$$
\begin{aligned}
\Psi_0 = & -\tfrac{1}{4}\left(x_1 + \tfrac{x_2^2}{16}\right)^{-4-2\sqrt{2}} (x_4-1)^{-4-2\sqrt{2}} (x_4+1)^{1-2\sqrt{2}} \left(x_4 + \tfrac{\sqrt{2}}{2}\right)^{4+4\sqrt{2}} \Big\{ \lambda^{-2} \\
& \left(x_1 + \tfrac{x_2^2}{16}\right)^{2+2\sqrt{2}} (x_4-1)^{6+4\sqrt{2}} (x_4+1)^{-8-4\sqrt{2}} \Big(-(4+3\sqrt{2})(x_4+1)^3 \\
& -\tfrac{1}{16}\lambda^2 (x_4-1)^{-3-2\sqrt{2}} (x_4+1)^{2\sqrt{2}} \left(x_4+\tfrac{\sqrt{2}}{2}\right)^4 \Big[\tfrac{x_2^2}{16}(2+\sqrt{2})(x_4^2-1) \\
& + \tfrac{1}{2}\left(x_1 + \tfrac{x_2^2}{16}\right)\big\{ x_4^2(16+4\sqrt{2}) + x_4(14+24\sqrt{2}) + (18+6\sqrt{2})\big\}\Big]\Big) \\
& -\tfrac{1}{8}\lambda^{-4}\left(x_1 + \tfrac{x_2^2}{16}\right)(x_4 - x_1)\left(x_4 + \tfrac{\sqrt{2}}{2}\right)^2 \Big[\tfrac{1}{4}(10+7\sqrt{2})(x_4-1)^{-2} \\
& -\tfrac{1}{4}(4+3\sqrt{2})(x_4+1)^{-2} + 2(1+\sqrt{2})\left(x_4+\tfrac{\sqrt{2}}{2}\right)^{-2} + \tfrac{1}{2}(1-\sqrt{2})(x_4-1)^{-1}(x_4+1)^{-1} \\
& -(11+6\sqrt{2})(x_4-1)^{-1}\left(x_4+\tfrac{\sqrt{2}}{2}\right)^{-1} + \tfrac{1}{2}(3-\sqrt{2})(x_4+1)^{-1}\left(x_4+\tfrac{\sqrt{2}}{2}\right)^{-1}\Big] \\
& + 2\lambda^{-6}(4+3\sqrt{2})(x_4-1)^{2+2\sqrt{2}}(x_4+1)^{1-2\sqrt{2}}\left(x_4+\tfrac{\sqrt{2}}{2}\right)^{-4} \\
& + \tfrac{1}{32}x_2^2\lambda^{-4}(4+3\sqrt{2})\left(x_1 + \tfrac{x_2^2}{16}\right)^{-2\sqrt{2}}(x_4+1)^{-1}\left(x_4+\tfrac{\sqrt{2}}{2}\right) \\
& + 2\lambda^{-5}(x_4-1)^{1+\sqrt{2}}(x_4+1)^{-\sqrt{2}}\left(x_4+\tfrac{\sqrt{2}}{2}\right)^{-2}\tfrac{x_2}{16}(3+2\sqrt{2})\Big\}
\end{aligned}
$$

$$
\begin{aligned}
\Psi_1 = & -\tfrac{1}{8} i \lambda^{-1}\left(x_1 + \tfrac{x_2^2}{16}\right)^{-2-\sqrt{2}/2}(x_4-1)^{-3/2 - 3\sqrt{2}/2}(x_4+1)^{-1/2+\sqrt{2}/2} \\
& \left(x_4+\tfrac{\sqrt{2}}{2}\right)^{4+\sqrt{2}}\Big\{\tfrac{1}{32}\lambda x_2\Big[(12+10\sqrt{2})\left(x_4+\tfrac{\sqrt{2}}{2}\right)^{-1} - 2\sqrt{2}(x_4+1)^{-1} \\
& - 4(3+2\sqrt{2})(x_4-1)^{-1}\Big] + (x_4-1)^{1+\sqrt{2}}(x_4+1)^{1-\sqrt{2}}\left(x_4+\tfrac{\sqrt{2}}{2}\right)^{-2} \\
& \big\{6(2+\sqrt{2})\left(x_4+\tfrac{\sqrt{2}}{2}\right)^{-1} - (x_4+1)^{-1}\big\}\Big]
\end{aligned}
$$

$$
\begin{aligned}
\Psi_2 = & -\tfrac{1}{2}\lambda^{-2}\left(x_1 + \tfrac{x_2^2}{16}\right)^{-2-\sqrt{2}}(x_4-1)^{-5-3\sqrt{2}}(x_4+1)^{-2+\sqrt{2}}\left(x_4+\tfrac{\sqrt{2}}{2}\right)^{6+2\sqrt{2}} \\
& \Big[(1+\sqrt{2})(x_4-1)\Big\{\tfrac{1}{2}\left(x_1+\tfrac{x_2^2}{16}\right)^{-1}(x_4-1)^{2+2\sqrt{2}}(x_4+1)^{2-2\sqrt{2}}\left(x_4+\tfrac{\sqrt{2}}{2}\right)^{-4} \\
& - \lambda^2\left(\tfrac{1}{128}x_2^2\left(x_1+\tfrac{x_2^2}{16}\right)^{-1} - \tfrac{1}{16}\right)\Big\} \\
& - \tfrac{1}{32}\lambda^2(x_4+1)^{-1}\left(x_4+\tfrac{\sqrt{2}}{2}\right)^{-4\sqrt{2}}\big\{x_4^2(-30-32\sqrt{2}) + 48x_4(-2-\sqrt{2}) \\
& - (38+36\sqrt{2})\big\}\Big]
\end{aligned}
$$

$$
\begin{aligned}
\Psi_3 = & -\tfrac{1}{4} i \lambda^{-1}\left(x_1 + \tfrac{x_2^2}{16}\right)^{-3-\frac{3}{2}\sqrt{2}}(x_4-1)^{-11/2 - 9\sqrt{2}/2}(x_4+1)^{-5/2+3\sqrt{2}/2} \\
& \left(x_4+\tfrac{\sqrt{2}}{2}\right)^{10+3\sqrt{2}}\Big\{\tfrac{1}{32}\lambda x_2\Big[(12+10\sqrt{2})\left(x_4+\tfrac{\sqrt{2}}{2}\right)^{-1} - 2\sqrt{2}(x_4+1)^{-1} \\
& - 4(3+2\sqrt{2})(x_4-1)^{-1}\Big] - (x_4-1)^{1+\sqrt{2}}(x_4+1)^{1-\sqrt{2}}\left(x_4+\tfrac{\sqrt{2}}{2}\right)^{-2}\big\{6(2+\sqrt{2}) \\
& \left(x_4+\tfrac{\sqrt{2}}{2}\right)^{-1} - (x_4+1)^{-1}\big\}\Big\}
\end{aligned}
$$

$$
\begin{aligned}
\Psi_4 = & -\left(x_1 + x_2^2/16\right)^{-6-4\sqrt{2}} (x_4-1)^{-12-8\sqrt{2}} (x_4+1)^{-3} \left(x_4 + \tfrac{\sqrt{2}}{2}\right)^{16+8\sqrt{2}} \\
& \Big\{ \left(x_1 + x_2^2/16\right)^{2+2\sqrt{2}} (x_4-1)^{6+4\sqrt{2}} (x_4+1)^{-8-4\sqrt{2}} \lambda^{-2} \Big\{ -(4+3\sqrt{2})(x_4+1)^3 \\
& - \tfrac{1}{16} \lambda^2 (x_4-1)^{-3-2\sqrt{2}} (x_4+1)^{2\sqrt{2}} \left(x_4 + \tfrac{\sqrt{2}}{2}\right)^4 \Big[\tfrac{x_2^2}{16} (2+\sqrt{2})(x_4^2-1) \\
& + \tfrac{1}{2}\left(x_1 + x_2^2/16\right) \left\{ x_4^2 (16+4\sqrt{2}) + x_4 (14+24\sqrt{2}) + (18+6\sqrt{2}) \right\} \Big] \Big\} \\
& - \tfrac{1}{8} \lambda^{-4} \left(x_1 + x_2^2/16\right)(x_4-1)\left(x_4 + \tfrac{\sqrt{2}}{2}\right)^2 \Big[\tfrac{1}{4}(10+7\sqrt{2})(x_4-1)^2 \\
& - \tfrac{1}{4}(4+3\sqrt{2})(x_4+1)^{-2} + 2(1+\sqrt{2})\left(x_4 + \tfrac{\sqrt{2}}{2}\right)^{-2} + \tfrac{1}{2}(3-2\sqrt{2})(x_4+1)^{-1}\left(x_4 + \tfrac{\sqrt{2}}{2}\right)^{-1} \\
& + \tfrac{1}{2}(1-\sqrt{2})(x_4-1)^{-1}(x_4+1)^{-1} - (11+6\sqrt{2})(x_4-1)^{-1}\left(x_4 + \tfrac{\sqrt{2}}{2}\right)^{-1} \Big] \\
& + 2\lambda^{-6}(4+3\sqrt{2})(x_4-1)^{2+2\sqrt{2}}(x_4+1)^{1-2\sqrt{2}}\left(x_4 + \tfrac{\sqrt{2}}{2}\right)^{-4} \\
& + \tfrac{1}{32} x_2^2 \lambda^{-4} (4+3\sqrt{2})\left(x_1 + x_2^2/16\right)^{-2\sqrt{2}} (x_4+1)^{-1}\left(x_4 + \sqrt{2}/2\right) \\
& - 2\lambda^{-5}(x_4-1)^{1+\sqrt{2}}(x_4+1)^{-\sqrt{2}}\left(x_4 + \sqrt{2}/2\right)^{-2} \tfrac{x_2}{16}(3+2\sqrt{2}) \Big\}
\end{aligned}
$$

1. II - A - 4

2.
$$g_{11} = \left(\tfrac{1}{16}(\epsilon_1 x_1^2 + x_2^2) + \lambda\right)^{1+\sqrt{2}\epsilon} \exp\left(\tfrac{1}{2}(1+\sqrt{2}\epsilon)\int \tfrac{u}{x_4}\, dx_4\right) (\epsilon_1 x_4)^{-\frac{1}{2}}$$
$$g_{22} = \left(\tfrac{1}{16}(\epsilon_1 x_1^2 + x_2^2) + \lambda\right)^{1+\sqrt{2}\epsilon} \exp\left(\tfrac{1}{2}(1+\sqrt{2}\epsilon)\int \tfrac{u}{x_4}\, dx_4\right) (\epsilon_1 x_4)^{\frac{1}{2}}$$
$$g_{33} = \left(\tfrac{1}{16}(\epsilon_1 x_1^2 + x_2^2) + \lambda\right)^{-\sqrt{2}\epsilon} \exp\left(-\tfrac{1}{2}\int \tfrac{u}{x_4}\, dx_4\right)$$
$$g_{44} = \left(\tfrac{1}{16}(\epsilon_1 x_1^2 + x_2^2) + \lambda\right)^{2+\sqrt{2}\epsilon} \exp\left(\tfrac{1}{2}(1+\sqrt{2}\epsilon)\int \tfrac{u}{x_4}\, dx_4\right)(\epsilon_1 x_4)^{-\frac{3}{2}} \left(\frac{u^2-1}{x_4-1}\right)$$

where u is a solution of
$$\frac{du}{dx_4} = \left(\frac{u^2-1}{4x_4}\right)\left[u\left(\frac{x_4+1}{x_4-1}\right) + \sqrt{2}\,\epsilon\right]$$

3. Petrov Type I

No further analysis owing to unknown analytical forms.

1. II - A - 5

2.

$$
\begin{aligned}
g_{11} &= \left(\lambda - \tfrac{1}{16}\left(x_1^2 + \epsilon_1 x_2^2\right)\right)^{1+\sqrt{2}\epsilon} \exp\left(\tfrac{1}{2}(1+\sqrt{2}\epsilon)\int \tfrac{u}{x_4}\, dx_4\right)(\epsilon_1 x_4)^{\frac{1}{2}} \\
g_{22} &= \left(\lambda - \tfrac{1}{16}\left(x_1^2 + \epsilon_1 x_2^2\right)\right)^{1+\sqrt{2}\epsilon} \exp\left(\tfrac{1}{2}(1+\sqrt{2}\epsilon)\int \tfrac{u}{x_4}\, dx_4\right)(\epsilon_1 x_4)^{-\frac{1}{2}} \\
g_{33} &= \left(\lambda - \tfrac{1}{16}\left(x_1^2 + \epsilon_1 x_2^2\right)\right)^{-\sqrt{2}\epsilon} \exp\left(-\tfrac{1}{2}\int u/x_4\, dx_4\right) \\
g_{44} &= \left(\lambda - \tfrac{1}{16}\left(x_1^2 + \epsilon_1 x_2^2\right)\right)^{2+\sqrt{2}\epsilon} \exp\left(\tfrac{1}{2}(1+\sqrt{2}\epsilon)\int \tfrac{u}{x_4}\, dx\right)(\epsilon_1 x_4)^{-\frac{3}{2}} \left(\frac{u^2-1}{x_4-1}\right)
\end{aligned}
$$

where u is a solution of

$$
\frac{du}{dx_4} = \left(\frac{u^2-1}{4x_4}\right)\left[u\left(\frac{x_4+1}{x_4-1}\right) + \sqrt{2}\,\epsilon\right]
$$

3. Petrov Type I

No further analysis owing to unknown analytical form.

1. II - A - 6

2.

$$g_{11} = \left(\tfrac{\sqrt{3}}{3} x_2 - \tfrac{x_1^2}{12}\right)^{2+\sqrt{3}\epsilon} \exp\left(\tfrac{1}{3}(3+2\sqrt{3}\epsilon)\int \tfrac{v}{x_4}\, dx_4\right)(-x_4)^{-\sqrt{3}\epsilon/3}$$

$$g_{22} = \left(\tfrac{\sqrt{3}}{3} x_2 - \tfrac{x_1^2}{12}\right)^{1+\sqrt{3}\epsilon} \exp\left(\tfrac{1}{3}(3+2\sqrt{3}\epsilon)\int \tfrac{v}{x_4}\, dx_4\right)(-x_4)^{-1-\sqrt{3}\epsilon/3}$$

$$g_{33} = \left(\tfrac{\sqrt{3}}{3} x_2 - \tfrac{x_1^2}{12}\right)^{-\sqrt{3}\epsilon} \exp\left(-\int \tfrac{v}{x_4}\, dx_4\right)(-x_4)^{\sqrt{3}\epsilon/3}$$

$$g_{44} = \left(\tfrac{\sqrt{3}}{3} x_2 - \tfrac{x_1^2}{12}\right)^{3+\sqrt{3}\epsilon} \exp\left(\tfrac{1}{3}(3+2\sqrt{3}\epsilon)\int \tfrac{v}{x_4}\, dx_4\right)(-x_4)^{-2-\sqrt{3}\epsilon/3}\left(\frac{v^2-1}{x_4-1}\right)$$

where v is a solution of

$$\frac{dv}{dx_4} = \left(\frac{v^2-1}{4x_4}\right)\left[\frac{2v}{(x_4-1)} + \frac{4\sqrt{3}}{3}\epsilon\right]$$

3. Petrov Type I

No further analysis owing to unknown analytical form.

1. $\underline{\text{II - A - 7}}$

2.
$$g_{11} = \left(\frac{\sqrt{3}}{3}x_1 + \frac{x_2^2}{12}\right)^{1+\sqrt{3}\epsilon} \exp\left(\frac{1}{3}(3 + 2\sqrt{3}\epsilon)\int \frac{v}{x_4}\,dx_4\right) x_4^{-1-\sqrt{3}\epsilon/3}$$
$$g_{22} = \left(\frac{\sqrt{3}}{3}x_1 + \frac{x_2^2}{12}\right)^{2+\sqrt{3}\epsilon} \exp\left(\frac{1}{3}(3 + 2\sqrt{3}\epsilon)\int \frac{v}{x_4}\,dx_4\right) x_4^{-\sqrt{3}\epsilon/3}$$
$$g_{33} = \left(\frac{\sqrt{3}}{3}x_1 + \frac{x_2^2}{12}\right)^{-\sqrt{3}\epsilon} \exp\left(-\int \frac{v}{x_4}\,dx_4\right) x_4^{\sqrt{3}\epsilon/3}$$
$$g_{44} = \left(\frac{\sqrt{3}}{3}x_1 + \frac{x_2^2}{12}\right)^{3+\sqrt{3}\epsilon} \exp\left(\frac{1}{3}(3 + 2\sqrt{3}\epsilon)\int \frac{v}{x_4}\,dx_4\right) x_4^{-2-\sqrt{3}\epsilon/3} \left(\frac{v^2-1}{x_4-1}\right)$$

where v is a solution of

$$\frac{dv}{dx_4} = \left(\frac{v^2-1}{4x_4}\right)\left[\frac{2v}{(x_4-1)} + \frac{4\sqrt{3}}{3}\epsilon\right]$$

3. Petrov Type I

No further analysis owing to unknown analytical form.

1. $\underline{\text{II} - \text{B} - 1}$

2.
$$g_{11} = 8\left(x_4 - \tfrac{x_2^2}{16}\right)^{2+\sqrt{2}\epsilon}(x_1 - \epsilon_1)^{3+3\sqrt{2}\epsilon}(x_1 + \epsilon_1)^{1-\sqrt{2}\epsilon}\left(\epsilon_2 x_1 + \tfrac{\sqrt{2}}{2}\epsilon_3\right)^{-8-2\sqrt{2}\epsilon}$$
$$g_{22} = \left(x_4 - \tfrac{x_2^2}{16}\right)^{1+\sqrt{2}\epsilon}(x_1 - \epsilon_1)^{4+3\sqrt{2}\epsilon}(x_1 + \epsilon_1)^{2-\sqrt{2}\epsilon}\left(\epsilon_2 x_1 + \tfrac{\sqrt{2}}{2}\epsilon_3\right)^{-6-2\sqrt{2}\epsilon}$$
$$g_{33} = \left(x_4 - \tfrac{x_2^2}{16}\right)^{-\sqrt{2}\epsilon}(x_1 - \epsilon_1)^{-1-\sqrt{2}\epsilon}(x_1 + \epsilon_1)^{1-\sqrt{2}\epsilon}\left(\epsilon_2 x_1 + \tfrac{\sqrt{2}}{2}\epsilon_3\right)^{2\sqrt{2}\epsilon}$$
$$g_{44} = \lambda^2\left(x_4 - \tfrac{x_2^2}{16}\right)^{1+\sqrt{2}\epsilon}(x_1 - \epsilon_1)^{2+\sqrt{2}\epsilon}(x_1 + \epsilon_1)^{\sqrt{2}\epsilon}\left(\epsilon_2 x_1 + \tfrac{\sqrt{2}}{2}\epsilon_3\right)^{-2-2\sqrt{2}\epsilon}$$

3. Petrov Type I

Analysis continued with $\epsilon = \epsilon_1 = \epsilon_2 = \epsilon_3 = 1$.

4.
$$R_{1313} = \tfrac{1}{4}\left(x_4 - \tfrac{x_2^2}{16}\right)^{-\sqrt{2}}(x_1-1)^{-3-\sqrt{2}}(x_1+1)^{-1-\sqrt{2}}\left(x_1 + \tfrac{\sqrt{2}}{2}\right)^{-2+2\sqrt{2}}\Big\{(-8+24\sqrt{2})x_1^3$$
$$+ (72+21\sqrt{2})x_1^2 + (58+26\sqrt{2})x_1 + 35\sqrt{2}\Big\}$$
$$+ \tfrac{1}{16}x_2^2(1+\sqrt{2})\left(x_4 - \tfrac{x_2^2}{16}\right)^{1-\sqrt{2}}(x_1-1)^{-2-\sqrt{2}}(x_1+1)^{-\sqrt{2}}\left(x_1 + \tfrac{\sqrt{2}}{2}\right)^{-2+2\sqrt{2}}$$
$$R_{1224} = \tfrac{1}{2}\left(x_4 - \tfrac{x_2^2}{16}\right)^{\sqrt{2}}(x_1-1)^{4+3\sqrt{2}}(x_1+1)^{2-\sqrt{2}}\left(x_1 + \tfrac{\sqrt{2}}{2}\right)^{-6-2\sqrt{2}}\Big\{(x_1-1)^{-1}(-6-4\sqrt{2})$$
$$+ (x_1+1)^{-1}(2-\sqrt{2}) + \left(x_1 + \tfrac{\sqrt{2}}{2}\right)^{-1}(4+5\sqrt{2})\Big\}$$
$$R_{1214} = 2\left(x_4 - \tfrac{x_2^2}{16}\right)^{\sqrt{2}}(x_1-1)^{3+3\sqrt{2}}(x_1+1)^{1-\sqrt{2}}\left(x_1 + \tfrac{\sqrt{2}}{2}\right)^{-8-2\sqrt{2}}x_2(1+\sqrt{2})$$
$$R_{1414} = \tfrac{1}{8}\left(x_4 - \tfrac{x_2^2}{16}\right)^{-2-\sqrt{2}}(x_1-1)^{-3-3\sqrt{2}}(x_1+1)^{-1+\sqrt{2}}\left(x_1 + \tfrac{\sqrt{2}}{2}\right)^{8+2\sqrt{2}}\Big[\tfrac{\lambda^2}{16}\left(x_4 - \tfrac{x_2^2}{16}\right)^{-1}$$
$$(x_1-1)^{-1-2\sqrt{2}}(x_1+1)^{-1+2\sqrt{2}}\left(x_1 + \tfrac{\sqrt{2}}{2}\right)^{6}\Big\{-(4+3\sqrt{2})(x_1-1)^{-2}$$
$$+ \left(2 - \tfrac{3\sqrt{2}}{2}\right)(x_1+1)^{-2} - 8(1+\sqrt{2})\left(x_1 + \tfrac{\sqrt{2}}{2}\right)^{-2} - \left(1 - \tfrac{3\sqrt{2}}{2}\right)(x_1-1)^{-1}(x_1+1)^{-1}$$
$$+ 5(2+\sqrt{2})(x_1-1)^{-1}\left(x_1 + \tfrac{\sqrt{2}}{2}\right)^{-1} - (3-2\sqrt{2})(x_1+1)^{-1}\left(x_1 + \tfrac{\sqrt{2}}{2}\right)^{-1}\Big\}$$
$$+ \tfrac{1}{4}(2+\sqrt{2})\left(x_4 - \tfrac{x_2^2}{16}\right)^{-2}$$
$$- \tfrac{1}{16^2}x_2^2\lambda^2(4+3\sqrt{2})\left(x_4 - \tfrac{x_2^2}{16}\right)^{-2}(x_1-1)^{-2-2\sqrt{2}}(x_1+1)^{-2+2\sqrt{2}}\left(x_1 + \tfrac{\sqrt{2}}{2}\right)^{4}\Big]$$
$$R_{1323} = -\tfrac{1}{8}\left(\tfrac{x_2}{16^2}\right)\left(x_4 - \tfrac{x_2^2}{16}\right)^{-5-3\sqrt{2}}(x_1-1)^{-7-5\sqrt{2}}(x_1+1)^{-1+\sqrt{2}}\left(x_1 + \tfrac{\sqrt{2}}{2}\right)^{16+6\sqrt{2}}$$
$$\Big\{(10+6\sqrt{2})(x_1-1)^{-1} + 2(x_1+1)^{-1} - (8+6\sqrt{2})\left(x_1 + \tfrac{\sqrt{2}}{2}\right)^{-1}\Big\}$$

5.

$$\psi_0 = -\tfrac{1}{4}\Big[\tfrac{1}{8}\left(x_4-\tfrac{x_2^2}{16}\right)^{-1}\Big(\tfrac{1}{4}(x_1-1)^{-1}(x_1+1)^{-1}\Big\{(-8+24\sqrt{2})x_1^3$$
$$+(58+26\sqrt{2})x_1+(72+21\sqrt{2})x_1^2+35\sqrt{2}\Big\}+\tfrac{x_2^2}{16}(1+\sqrt{2})\left(x_4-\tfrac{x_2^2}{16}\right)\Big)$$
$$-\tfrac{1}{8}\lambda^{-2}\left(x_4-\tfrac{x_2^2}{16}\right)^{-3-2\sqrt{2}}(x_1-1)^{-4-4\sqrt{2}}\left(x_1+\tfrac{\sqrt{2}}{2}\right)^{12+4\sqrt{2}}\Big[\tfrac{1}{16}\lambda^2\left(x_4-\tfrac{x_2^2}{16}\right)^{-1}$$
$$(x_1-1)^{-1-2\sqrt{2}}(x_1+1)^{-1+2\sqrt{2}}\left(x_1+\tfrac{\sqrt{2}}{2}\right)^6\Big\{-(4+3\sqrt{2})(x_1-1)^{-2}$$
$$+\left(2-\tfrac{3}{2}\sqrt{2}\right)(x_1+1)^{-2}-8(1+\sqrt{2})\left(x_1+\tfrac{\sqrt{2}}{2}\right)^{-2}-\left(1-\tfrac{3\sqrt{2}}{2}\right)(x_1-1)^{-1}(x_1+1)^{-1}$$
$$+5(2+\sqrt{2})(x_1-1)^{-1}\left(x_1+\tfrac{\sqrt{2}}{2}\right)^{-1}-(3-\sqrt{2})(x_1+1)^{-1}\left(x_1+\tfrac{\sqrt{2}}{2}\right)^{-1}\Big\}$$
$$+\tfrac{1}{4}(2+\sqrt{2})\left(x_4-\tfrac{x_2^2}{16}\right)^{-2}$$
$$+\left(\tfrac{x_2\lambda}{16}\right)^2(4+3\sqrt{2})\left(x_4+\tfrac{x_2^2}{16}\right)^{-2}(x_1-1)^{-2-2\sqrt{2}}(x_1+1)^{-2+2\sqrt{2}}\left(x_1+\tfrac{\sqrt{2}}{2}\right)^4\Big]$$
$$-2\left(x_4-\tfrac{x_2^2}{16}\right)^{-\frac{11}{2}-2\sqrt{2}}(x_1+1)^{-\frac{3}{2}+2\sqrt{2}}\left(x_1+\tfrac{\sqrt{2}}{2}\right)^{17+4\sqrt{2}}\tfrac{x_2}{8\times 16^2}\Big]$$

$$\psi_1 = -\tfrac{1}{8}i\lambda^{-1}\left(x_4-\tfrac{x_2^2}{16}\right)^{-2-\sqrt{2}/2}(x_1-1)^{-1-\sqrt{2}/2}(x_1+1)^{-\sqrt{2}/2}\left(x_1+\tfrac{\sqrt{2}}{2}\right)^{1+\sqrt{2}}$$
$$\Big\{\tfrac{1}{\sqrt{2}}\left(x_4-\tfrac{x_2^2}{16}\right)^{-\frac{1}{2}}+\tfrac{1}{2}(x_1-1)^{\frac{1}{2}}(x_1+1)^{\frac{1}{2}}\left(x_1+\tfrac{\sqrt{2}}{2}\right)\Big[(x_1-1)^{-1}(-6-4\sqrt{2})$$
$$+(x_1+1)^{-1}(2-\sqrt{2})+\left(x_1+\tfrac{\sqrt{2}}{2}\right)^{-1}(4+5\sqrt{2})\Big]\Big\}$$

$$\psi_2 = -\tfrac{1}{16}\left(x_4-\tfrac{x_2^2}{16}\right)^{-3-\sqrt{2}}(x_1-1)^{-5-3\sqrt{2}}(x_1+1)^{-3+\sqrt{2}}\left(x_1+\tfrac{\sqrt{2}}{2}\right)^{6+2\sqrt{2}}$$
$$\Big\{\tfrac{1}{4}\left(x_4-\tfrac{x_2^2}{16}\right)\Big[-2x_1^2(10+3\sqrt{2})-8x_1(2+3\sqrt{2})-2(7+2\sqrt{2})\Big]$$
$$+\tfrac{x_4}{2}(2+\sqrt{2})(x_1^2-1)+2\sqrt{2}\lambda^{-2}(x_1-1)^{2+2\sqrt{2}}(x_1+1)^{3-\sqrt{2}}\left(x_1+\tfrac{\sqrt{2}}{2}\right)^{-4}\Big\}$$

$$\psi_3 = -\tfrac{1}{4}i\lambda^{-1}\left(x_4-\tfrac{x_2^2}{16}\right)^{-3-3\sqrt{2}/2}(x_1-1)^{-5-7\sqrt{2}/2}(x_4+1)^{2-\sqrt{2}/2}\left(x_4+\tfrac{\sqrt{2}}{2}\right)^{7+3\sqrt{2}}$$
$$\Big\{\tfrac{1}{\sqrt{2}}\left(x_4-\tfrac{x_2^2}{16}\right)^{-\frac{1}{2}}-\tfrac{1}{2}(x_1-1)^{\frac{1}{2}}(x_1+1)^{\frac{1}{2}}\left(x_1+\tfrac{\sqrt{2}}{2}\right)\Big[(x_1-1)^{-1}(-6-4\sqrt{2})$$
$$+(x_1+1)^{-1}(2-\sqrt{2})+\left(x_1+\tfrac{\sqrt{2}}{2}\right)^{-1}(4+5\sqrt{2})\Big]\Big\}$$

$$\psi_4 = -\left(x_4-\tfrac{x_2^2}{16}\right)^{-2-2\sqrt{2}}(x_1-1)^{-8-6\sqrt{2}}(x_1+1)^{-4+2\sqrt{2}}\left(x_1+\tfrac{\sqrt{2}}{2}\right)^{12+4\sqrt{2}}\Big[\tfrac{1}{8}\left(x_4-\tfrac{x_2^2}{16}\right)^{-1}$$
$$\Big(\tfrac{1}{4}(x_1-1)^{-1}(x_1+1)^{-1}\Big\{(-8+24\sqrt{2})x_1^3+(58+26\sqrt{2})x_1+(72+21\sqrt{2})x_1^2$$
$$+35\sqrt{2}\Big\}+\tfrac{1}{16}x_2^2(1+\sqrt{2})\left(x_4-\tfrac{x_2^2}{16}\right)\Big)-\tfrac{1}{8}\lambda^{-2}\left(x_4-\tfrac{x_2^2}{16}\right)^{-3-2\sqrt{2}}(x_1-1)^{-4-4\sqrt{2}}$$
$$\left(x_1+\tfrac{\sqrt{2}}{2}\right)^{12+4\sqrt{2}}\Big[\tfrac{1}{16}\lambda^2\left(x_4-\tfrac{x_2^2}{16}\right)^{-1}(x_1-1)^{-1-2\sqrt{2}}(x_1+1)^{-1+2\sqrt{2}}\left(x_1+\tfrac{\sqrt{2}}{2}\right)^6$$
$$\Big\{-(4+3\sqrt{2})(x_1-1)^{-2}+\left(2-\tfrac{3}{2}\sqrt{2}\right)(x_1+1)^{-2}-8(1+\sqrt{2})\left(x_1+\tfrac{\sqrt{2}}{2}\right)^{-2}$$
$$-\left(1-\tfrac{3}{2}\sqrt{2}\right)(x_1-1)^{-1}(x_1+1)^{-1}+5(2+\sqrt{2})(x_1-1)^{-1}\left(x_1+\tfrac{\sqrt{2}}{2}\right)^{-1}$$
$$-(3-\sqrt{2})(x_1+1)^{-1}\left(x_1+\tfrac{\sqrt{2}}{2}\right)^{-1}\Big\}+\tfrac{1}{4}(2+\sqrt{2})\left(x_4-\tfrac{x_2^2}{16}\right)^{-2}$$
$$-\left(\tfrac{x_2\lambda}{16}\right)^2(4+3\sqrt{2})\left(x_4+\tfrac{x_2^2}{16}\right)^{-2}(x_1-1)^{-2-2\sqrt{2}}(x_1+1)^{-2+2\sqrt{2}}\left(x_1+\tfrac{\sqrt{2}}{2}\right)^4\Big]$$
$$+2\left(x_4-\tfrac{x_2^2}{16}\right)^{-\frac{4}{2}-2\sqrt{2}}(x_1-1)^{-\frac{11}{2}-4\sqrt{2}}(x_4+1)^{-\frac{3}{2}+2\sqrt{2}}\left(x_1+\tfrac{\sqrt{2}}{2}\right)^{17+4\sqrt{2}}$$
$$\times\tfrac{x_2}{8\times 16^2}\Big]$$

1. II - B - 2

2. $$g_{11} = \left(\tfrac{1}{16}\left(\epsilon_1 x_4^2 - x_2^2\right) + \lambda\right)^{2+\sqrt{2}\epsilon} \exp\left(\tfrac{1}{2}\left(1+\sqrt{2}\epsilon\right)\int \tfrac{u}{x_1}\,dx_1\right)\left(\epsilon_1 x_1\right)^{-\frac{3}{2}}\left(\frac{u^2-1}{x_1-1}\right)$$
$$g_{22} = \left(\tfrac{1}{16}\left(\epsilon_1 x_4^2 - x_2^2\right) + \lambda\right)^{1+\sqrt{2}\epsilon} \exp\left(\tfrac{1}{2}\left(1+\sqrt{2}\epsilon\right)\int \tfrac{u}{x_1}\,dx_1\right)\left(\epsilon_1 x_1\right)^{\frac{1}{2}}$$
$$g_{33} = \left(\tfrac{1}{16}\left(\epsilon_1 x_4^2 - x_2^2\right) + \lambda\right)^{-\sqrt{2}\epsilon} \exp\left(-\tfrac{1}{2}\int \tfrac{u}{x_1}\,dx_1\right)$$
$$g_{44} = \left(\tfrac{1}{16}\left(\epsilon_1 x_4^2 - x_2^2\right) + \lambda\right)^{1+\sqrt{2}\epsilon} \exp\left(\tfrac{1}{2}\left(1+\sqrt{2}\epsilon\right)\int \tfrac{u}{x_1}\,dx_1\right)\left(\epsilon_1 x_1\right)^{-\frac{1}{2}}$$

where u is a solution of

$$\frac{du}{dx_1} = \left(\frac{u^2-1}{4x_1}\right)\left[u\left(\frac{x_1+1}{x_1-1}\right) + \sqrt{2}\epsilon\right]$$

3. Petrov Type I

No further analysis owing to unknown analytical form.

1. $\underline{\text{II} - \text{B} - 3}$

2.
$$g_{11} = \left(\frac{\sqrt{3}}{3}\, x_4 - \frac{x_2^2}{12}\right)^{3+\sqrt{3}\epsilon} \exp\left(\tfrac{1}{3}(3+2\sqrt{3}\epsilon)\int \frac{v}{x_1}\, dx_1\right) x_1^{-2-\sqrt{3}\epsilon/3} \left(\frac{v^2-1}{x_1-1}\right)$$
$$g_{22} = \left(\frac{\sqrt{3}}{3}\, x_4 - \frac{x_2^2}{12}\right)^{2+\sqrt{3}\epsilon} \exp\left(\tfrac{1}{3}(3+2\sqrt{3}\epsilon)\int \frac{v}{x_1}\, dx_1\right) x_1^{-\frac{\sqrt{3}}{3}\epsilon}$$
$$g_{33} = \left(\frac{\sqrt{3}}{3}\, x_4 - \frac{x_2^2}{12}\right)^{-\sqrt{3}\epsilon} \exp\left(-\int \frac{v}{x_1}\, dx_1\right) x_1^{\sqrt{3}\epsilon/3}$$
$$g_{44} = \left(\frac{\sqrt{3}}{3}\, x_4 - \frac{x_2^2}{12}\right)^{1+\sqrt{3}\epsilon} \exp\left(\tfrac{1}{3}(3+2\sqrt{3}\epsilon)\int \frac{v}{x_1}\, dx_1\right) x_1^{-1-\sqrt{3}\epsilon/3}$$

where v is a solution of

$$\frac{dv}{dx_1} = \left(\frac{v^2-1}{4x_1}\right)\left[\frac{2v}{(x_1-1)} + \frac{4\sqrt{3}\,\epsilon}{3}\right]$$

3. Petrov Type I

No further analysis owing to unknown analytical form.

1. $\underline{\text{II} - \text{C} - 1}$

2.
$$
\begin{aligned}
g_{11} &= (x_2 - x_3^2)^{-\sqrt{2}\,\epsilon}\; x_4^{\frac{2}{7}(2 - 3\sqrt{2}\,\epsilon)} \\
g_{22} &= \lambda^2 (x_2 - x_3^2)^{1+\sqrt{2}\,\epsilon}\; x_4^{\frac{2}{7}(1 + 2\sqrt{2}\,\epsilon)} \\
g_{33} &= (x_2 - x_3^2)^{1+\sqrt{2}\,\epsilon}\; x_4^2 \\
g_{44} &= \left(\tfrac{1}{7}(3 - \sqrt{2}\,\epsilon)\right)^2 (x_2 - x_3^2)^{2+\sqrt{2}\,\epsilon}
\end{aligned}
$$

3. Petrov Type I

Analysis continued with $\epsilon = 1$

4.
$$
\begin{aligned}
R_{1313} &= (x_2 - x_3^2)^{-1-\sqrt{2}}\, x_4^{\frac{2}{7}(2-3\sqrt{2})} \Big\{ \sqrt{2} + \tfrac{1}{4}(2 - 3\sqrt{2})\left(\tfrac{1}{7}(3-\sqrt{2})\right)^2 \\
&\quad + (x_2 - x_3^2)^{-1} x_4^{\frac{2}{7}(6-2\sqrt{2})} \sqrt{2}\left(3x_3^2 - \tfrac{1}{4}\lambda^{-2}(1+\sqrt{2})\right)\Big\} \\
R_{1314} &= \tfrac{1}{14}(8 - 13\sqrt{2})(x_2 - x_3^2)^{-1-\sqrt{2}}\, x_3\, x_4^{\frac{2}{7}(2-3\sqrt{2})-1} \\
R_{1213} &= -(x_2 - x_3^2)^{-2-\sqrt{2}}\, x_3\, x_4^{\frac{2}{7}(2-3\sqrt{2})}(1 + 2\sqrt{2}) \\
R_{1214} &= \tfrac{1}{14}(x_2 - x_3^2)^{-1-\sqrt{2}}\, x_4^{\frac{2}{7}(2-3\sqrt{2})-1}(-4 + 7\sqrt{2}) \\
R_{1414} &= -\tfrac{1}{7}(3-\sqrt{2})(x_2 - x_3^2)^{\sqrt{2}}\, x_4^{-\frac{2}{7}(2-3\sqrt{2})}\Big[\left(1 + \tfrac{1}{7}(3-\sqrt{2})\right)x_4^{-1} \\
&\quad + \sqrt{2}(2+\sqrt{2})\left(\tfrac{1}{7}(3-\sqrt{2})\right)(x_2 - x_3^2)^{-1}\Big\{\tfrac{1}{4}\lambda^{-2} x_4^{-\frac{2}{7}(1+2\sqrt{2})} + x_3^2 x_4^{-2}\Big\}\Big]
\end{aligned}
$$

5.
$$
\begin{aligned}
\Psi_0 &= -\tfrac{1}{4}\lambda^2 (x_2 - x_3^2)^{-1+2\sqrt{2}}\, x_4^{\frac{2}{7}(1+2\sqrt{2})}\Big\{(x_2 - x_3^2)^{-2\sqrt{2}} x_4^{-2}\Big(\sqrt{2} \\
&\quad + \tfrac{1}{4}(2-3\sqrt{2})\left(\tfrac{1}{7}(3-\sqrt{2})\right)^2 + (x_2 - x_3^2)^{-1} x_4^{\frac{2}{7}(6-2\sqrt{2})}\sqrt{2}\,\Big(3x_3^2 \\
&\quad - \tfrac{1}{4}\lambda^{-2}(1+\sqrt{2})\Big)\Big) + \left(\tfrac{1}{7}(3-\sqrt{2})\right)^{-1} x_4^{-\frac{4}{7}(2-3\sqrt{2})}\Big[\left(1 + \tfrac{1}{7}(3-\sqrt{2})\right)x_4^{-2} \\
&\quad + \sqrt{2}(2+\sqrt{2})\left(\tfrac{1}{7}(3-\sqrt{2})\right)(x_2 - x_3^2)^{-1}\left(\tfrac{1}{4}\lambda^{-2} x_4^{-\frac{2}{7}(1+2\sqrt{2})} + x_3^2 x_4^{-2}\right)\Big]\Big\} \\
&\quad + \tfrac{i}{28}\lambda^2(8 - 13\sqrt{2})\left(\tfrac{1}{7}(3-\sqrt{2})\right)^{-1}(x_2 - x_3^2)^{\frac{3}{2}}\, x_3\, x_4^{\frac{2}{7}(1+2\sqrt{2})-2} \\
\Psi_1 &= -\tfrac{1}{2\sqrt{2}}(x_2 - x_3^2)^{-\frac{5}{2}-\frac{3\sqrt{2}}{2}}\, x_4^{\frac{1}{7}(-7+3\sqrt{2})}\Big[x_3(-1 - 2\sqrt{2}) \\
&\quad + \tfrac{1}{4}i(-4 + 7\sqrt{2})(3-\sqrt{2})^{-1}(x_2 - x_3^2)^{\frac{1}{2}}\Big] \\
\Psi_2 &= -\tfrac{1}{2}\lambda^{-2}(x_2 - x_3^2)^{-2-\sqrt{2}}\, x_4^2\Big[\sqrt{2}(x_2 - x_3^2)^{-1}\left(\tfrac{3}{4} - \lambda^2(1+\sqrt{2})x_3^2\right) \\
&\quad + 2\lambda^2(-10 + \sqrt{2})(3-\sqrt{2})^{-2}\Big]
\end{aligned}
$$

$$\psi_3 = \frac{1}{\sqrt{2}} \lambda^{-2} (x_2 - x_3^2)^{-\frac{7}{2} - \frac{5\sqrt{2}}{2}} x_4^{-\frac{1}{7}(9+\sqrt{2})} \Big[x_3 (-1 - 2\sqrt{2})$$
$$- \frac{1}{4} i (-4 + 7\sqrt{2}) (3 - \sqrt{2})^{-1} (x_2 - x_3^2)^{\frac{1}{2}} \Big]$$

$$\psi_4 = -\lambda^{-2} (x_2 - x_3^2)^{-3} x_4^{\frac{2}{7}(-1-2\sqrt{2})} \Big\{ (x_2 - x_3^2)^{-2\sqrt{2}} x_4^{-2} \Big(\sqrt{2} + \frac{1}{7}(2 - 3\sqrt{2})\big(\tfrac{1}{7}(3-\sqrt{2})\big)^{2}$$
$$+ (x_2 - x_3^2)^{-1} x_4^{\frac{2}{7}(6-2\sqrt{2})} \sqrt{2} \big(3x_3^2 - \tfrac{1}{4} \lambda^{-2} (1 + \sqrt{2}) \big)\Big) + \big(\tfrac{1}{7}(3-\sqrt{2})\big)^{-1}$$
$$x_4^{-\frac{4}{7}(2-3\sqrt{2})} \Big[\big(1 + \tfrac{1}{7}(3-\sqrt{2})\big) x_4^{-2} + \sqrt{2} (2+\sqrt{2}) \big(\tfrac{1}{7}(3-\sqrt{2})\big) (x_2 - x_3^2)^{-1}$$
$$\Big(\tfrac{1}{4} \lambda^{-2} x_4^{-\frac{3}{7}(1+2\sqrt{2})} + x_3^2 x_4^{-2} \Big)\Big]\Big\}$$
$$- \frac{i}{7} \lambda^{-2} (8 - 13\sqrt{2}) \big(\tfrac{1}{7}(3-\sqrt{2})\big)^{-1} (x_2 - x_3^2)^{-\frac{1}{2} - 2\sqrt{2}} x_3 \, x_4^{-\frac{2}{7}(1+2\sqrt{2}) - 2} .$$

1. II - C - 2

2.
$$g_{11} = \left(x_2 + \tfrac{x_3^2}{16}\right)^{-\sqrt{2}\epsilon} (x_4 - \epsilon_1)^{1-\sqrt{2}\epsilon} (x_4 + \epsilon_1)^{-1-\sqrt{2}\epsilon} \left(\epsilon_2 x_4 + \tfrac{\sqrt{2}}{2}\epsilon_2\right)^{2\sqrt{2}\epsilon}$$
$$g_{22} = \lambda^2 \left(x_2 + \tfrac{x_3^2}{16}\right)^{1+\sqrt{2}\epsilon} (x_4 - \epsilon_1)^{\sqrt{2}\epsilon} (x_4 + \epsilon_1)^{2+\sqrt{2}\epsilon} \left(\epsilon_2 x_4 + \tfrac{\sqrt{2}}{2}\epsilon_2\right)^{-2-2\sqrt{2}\epsilon}$$
$$g_{33} = \left(x_2 + \tfrac{x_3^2}{16}\right)^{1+\sqrt{2}\epsilon} (x_4 - \epsilon_1)^{2-\sqrt{2}\epsilon} (x_4 + \epsilon_1)^{4+3\sqrt{2}\epsilon} \left(\epsilon_2 x_4 + \tfrac{\sqrt{2}}{2}\epsilon_2\right)^{-6-2\sqrt{2}\epsilon}$$
$$g_{44} = 8\left(x_2 + \tfrac{x_3^2}{16}\right)^{2+\sqrt{2}\epsilon} (x_4 - \epsilon_1)^{1-\sqrt{2}\epsilon} (x_4 + \epsilon_1)^{3+3\sqrt{2}\epsilon} \left(\epsilon_2 x_4 + \tfrac{\sqrt{2}}{2}\epsilon_2\right)^{-8-2\sqrt{2}\epsilon}$$

3. Petrov Type I

Analysis continued with $\epsilon = \epsilon_1 = \epsilon_2 = 1$

4.
$$\begin{aligned}
R_{1313} = {} & \left(x_2 + \tfrac{x_3^2}{16}\right)^{-2-\sqrt{2}} (x_4 - 1)^{1-\sqrt{2}} (x_4 + 1)^{-1-\sqrt{2}} \left(x_4 + \tfrac{\sqrt{2}}{2}\right)^{2\sqrt{2}} \Big[\Big\{ \left(\tfrac{x_3}{16}\right)^2 (4 + \sqrt{2}) \\
& + \tfrac{\sqrt{2}}{4} \lambda^{-2} (1 + \sqrt{2}) (x_4 - 1)^{2-2\sqrt{2}} (x_4 + 1)^{2+2\sqrt{2}} \left(x_4 + \tfrac{\sqrt{2}}{2}\right)^{-4} \Big\} \\
& + \tfrac{1}{16} \left(x_2 + \tfrac{x_3^2}{16}\right) \Big\{ \sqrt{2} - (x_4 - 1)^{-1-2\sqrt{2}} (x_4 + 1)^{-1} \big(-2x_4^3 + \sqrt{2}\, x_4^2 \\
& + (5 + 2\sqrt{2}) x_4 + 1\big) \Big\} \Big]
\end{aligned}$$
$$R_{1213} = -(3 + 2\sqrt{2}) \left(x_2 + \tfrac{x_3^2}{16}\right)^{-2-\sqrt{2}} (x_4 - 1)^{1-\sqrt{2}} (x_4 + 1)^{-1-\sqrt{2}} \left(x_4 + \tfrac{\sqrt{2}}{2}\right)^{2\sqrt{2}}$$
$$\begin{aligned}
R_{1214} = {} & \tfrac{1}{4} \left(x_2 + \tfrac{x_3^2}{16}\right)^{-1-\sqrt{2}} (x_4 - 1)^{1-\sqrt{2}} (x_4 + 1)^{-1-\sqrt{2}} \left(x_4 + \tfrac{\sqrt{2}}{2}\right)^{2\sqrt{2}} \Big[4(1 + \sqrt{2})(x_4 - 1)^{-1} \\
& - 2(4 + 3\sqrt{2})(x_4 + 1)^{-1} + 2(-2 + \sqrt{2})\left(x_4 + \tfrac{\sqrt{2}}{2}\right)^{-1} \Big]
\end{aligned}$$
$$\begin{aligned}
R_{1314} = {} & \tfrac{1}{32} x_3 \left(x_2 + \tfrac{x_3^2}{16}\right)^{-1-\sqrt{2}} (x_4 - 1)^{1-\sqrt{2}} (x_4 + 1)^{-1-\sqrt{2}} \left(x_4 + \tfrac{\sqrt{2}}{2}\right)^{2\sqrt{2}} \\
& \Big[(6 + \sqrt{2})(x_4 - 1)^{-1} - (10 + 7\sqrt{2})(x_4 + 1)^{-1} + 2(4 + 5\sqrt{2})\left(x_4 + \tfrac{\sqrt{2}}{2}\right)^{-1} \Big]
\end{aligned}$$
$$\begin{aligned}
R_{1414} = {} & \left(x_2 + \tfrac{x_3^2}{16}\right)^{\sqrt{2}} (x_4 - 1)^{-1+\sqrt{2}} (x_4 + 1)^{1+\sqrt{2}} \left(x_4 + \tfrac{\sqrt{2}}{2}\right)^{-2\sqrt{2}} \Big[2\sqrt{2}\lambda^{-2} (2 + \sqrt{2}) \\
& \left(x_2 + \tfrac{x_3^2}{16}\right)^{-1} (x_4 - 1)^{1-2\sqrt{2}} (x_4 + 1)^{1+2\sqrt{2}} \left(x_4 + \tfrac{\sqrt{2}}{2}\right)^{-6} \\
& + \tfrac{1}{32} \sqrt{2}\, x_3^2 (2 + \sqrt{2}) \left(x_2 + \tfrac{x_3^2}{16}\right)^{-1} (x_4 - 1)^{-1} (x_4 + 1)^{-1} \left(x_4 + \tfrac{\sqrt{2}}{2}\right)^{-1} \\
& - \tfrac{1}{4} (3 + \sqrt{2})(x_4 - 1)^{-2} - \tfrac{1}{2} (8 + 3\sqrt{2})(x_4 + 1)^{-2} - \tfrac{1}{2} (8 + 7\sqrt{2})\left(x_4 + \tfrac{\sqrt{2}}{2}\right)^{-2} \\
& + \tfrac{1}{2} (8 + 5\sqrt{2})(x_4 - 1)^{-1}(x_4 + 1)^{-1} - \tfrac{1}{2} (14 + \sqrt{2})(x_4 - 1)^{-1}\left(x_4 + \tfrac{\sqrt{2}}{2}\right)^{-1} \\
& + (8 + 5\sqrt{2})(x_4 + 1)^{-1}\left(x_4 + \tfrac{\sqrt{2}}{2}\right)^{-1} \Big]
\end{aligned}$$

5.

$$\psi_0 = -\tfrac{1}{4}\lambda^2 (x_2 + \tfrac{x_3^2}{16})^{-1+\sqrt{2}} (x_4-1)^{-2+2\sqrt{2}} (x_4+1)(x_4 + \tfrac{\sqrt{2}}{2})^{4-4\sqrt{2}} \Big\{ (x_2 + \tfrac{x_3^2}{16})^{-1-2\sqrt{2}}$$

$$(x_4+1)^{-3-2\sqrt{2}} (x_4 + \tfrac{\sqrt{2}}{2})^{4\sqrt{2}} \Big[\Big\{ (\tfrac{x_3}{16})^2 (4+\sqrt{2}) + \tfrac{1}{4}\sqrt{2}\lambda^{-2}(1+\sqrt{2})(x_4-1)^{2-2\sqrt{2}}$$

$$(x_4+1)^{2+2\sqrt{2}} (x_4 + \tfrac{\sqrt{2}}{2})^{-4} \Big\}$$

$$+ \tfrac{1}{16}(x_2 + \tfrac{x_3^2}{16})(\sqrt{2} - (x_4-1)^{-1-2\sqrt{2}}(x_4+1)^{-1} \{ -2x_4^3 + \sqrt{2}\,x_4^2 + (5+2\sqrt{2})x_4$$

$$+1\}) \Big] - \tfrac{1}{8}(x_4-1)^{-1+2\sqrt{2}} (x_4 + \tfrac{\sqrt{2}}{2})^2 \Big[2\sqrt{2}\lambda^{-2}(2+\sqrt{2})(x_2 + \tfrac{x_3^2}{16})^{-1}(x_4-1)^{1-2\sqrt{2}}$$

$$(x_4+1)^{1+2\sqrt{2}} (x_4 + \tfrac{\sqrt{2}}{2})^{-6} + \tfrac{1}{32}\sqrt{2}\,x_3^2 (2+\sqrt{2})(x_2 + \tfrac{x_3^2}{16})^{-1}(x_4-1)^{-1}(x_4+1)^{-1}$$

$$(x_4 + \tfrac{\sqrt{2}}{2})^{-1} - \tfrac{1}{4}(3+\sqrt{2})(x_4-1)^{-2} - \tfrac{1}{2}(8+3\sqrt{2})(x_4+1)^{-2}$$

$$- \tfrac{1}{2}(8+7\sqrt{2})(x_4 + \tfrac{\sqrt{2}}{2})^{-2} + \tfrac{1}{2}(8+5\sqrt{2})(x_4-1)^{-1}(x_4+1)^{-1}$$

$$- \tfrac{1}{2}(14+\sqrt{2})(x_4-1)^{-1}(x_4 + \tfrac{\sqrt{2}}{2})^{-1} + (8+5\sqrt{2})(x_4+1)^{-1}(x_4 + \tfrac{\sqrt{2}}{2})^{-1} \Big] \Big\}$$

$$+ \tfrac{i}{128\sqrt{2}} \lambda^2 x_3 (x_2 + \tfrac{x_3^2}{16})^{-3/2} (x_4-1)^{-3/2+2\sqrt{2}} (x_4+1)^{-3/2-2\sqrt{2}} (x_4 + \tfrac{\sqrt{2}}{2})^5$$

$$\{ (6+\sqrt{2})(x_4-1)^{-1} + (-10-7\sqrt{2})(x_4+1)^{-1} + (8+10\sqrt{2})(x_4 + \tfrac{\sqrt{2}}{2})^{-1} \}$$

$$\psi_1 = -\tfrac{1}{2\sqrt{2}} (x_2 + \tfrac{x_3^2}{16})^{-2-\sqrt{2}/2} (x_4-1)^{-1+\sqrt{2}/2} (x_4+1)^{-3/2-3\sqrt{2}/2} (x_4 + \tfrac{\sqrt{2}}{2})^{3+\sqrt{2}}$$

$$\Big\{ -(3+2\sqrt{2})(x_2 + \tfrac{x_3^2}{16})^{-1/2}(x_4+1)^{-1/2} + \tfrac{i}{8\sqrt{2}}(x_4-1)^{-1/2}(x_4 + \tfrac{\sqrt{2}}{2})$$

$$\Big[4(1+\sqrt{2})(x_4-1)^{-1} - 2(4+3\sqrt{2})(x_4+1)^{-1} - 2(2-\sqrt{2})(x_4 + \tfrac{\sqrt{2}}{2})^{-1} \Big] \Big\}$$

$$\psi_2 = -\tfrac{1}{2}\lambda^{-2} (x_2 + \tfrac{x_3^2}{16})^{-3-\sqrt{2}} (x_4-1)^{-\sqrt{2}} (x_4+1)^{-2-\sqrt{2}} (x_4 + \tfrac{\sqrt{2}}{2})^{2+2\sqrt{2}}$$

$$\Big[-1 + \tfrac{\sqrt{2}}{4} + (x_4-1)^{-2}(x_4+1)^{-2-2\sqrt{2}}(x_4 + \tfrac{\sqrt{2}}{2})^4 \Big\{ \tfrac{1}{16^2} x_3^2 \lambda^2 \sqrt{2}(1+\sqrt{2})$$

$$(x_4-1)^{2\sqrt{2}} - \tfrac{1}{32}\lambda^2 (x_2 + \tfrac{x_3^2}{16})(x_4^2-1)\big[4x_4^3 + (8+6\sqrt{2})x_4^2$$

$$+ (6+8\sqrt{2})x_4 + 2 \big] \Big\} \Big]$$

$$\psi_3 = +\tfrac{1}{\sqrt{2}} \lambda^{-2} (x_2 + \tfrac{x_3^2}{16})^{-3-3\sqrt{2}/2} (x_4-1)^{-1-\sqrt{2}/2} (x_4+1)^{-7/2-5\sqrt{2}/2} (x_4 + \tfrac{\sqrt{2}}{2})^{5+3\sqrt{2}}$$

$$\Big\{ -(3+2\sqrt{2})(x_2 + \tfrac{x_3^2}{16})^{-1/2}(x_4+1)^{-1/2} - \tfrac{i}{8\sqrt{2}}(x_4-1)^{-1/2}(x_4 + \tfrac{\sqrt{2}}{2})$$

$$\Big[(4+4\sqrt{2})(x_4-1)^{-1} - 2(4+3\sqrt{2})(x_4+1)^{-1} + (-4+2\sqrt{2})(x_4 + \tfrac{\sqrt{2}}{2})^{-1} \Big] \Big\}$$

$$
\begin{aligned}
\psi_4 = & -\lambda^{-2}\left(x_2 + \tfrac{x_3^2}{16}\right)^{-3-\sqrt{2}}(x_4-1)^{-2}(x_4+1)^{-3-2\sqrt{2}}\left(x_4+\tfrac{\sqrt{2}}{2}\right)^{8}\Big\{\left(x_2+\tfrac{x_3^2}{16}\right)^{-1-2\sqrt{2}} \\
& (x_4+1)^{-3-2\sqrt{2}}\left(x_4+\tfrac{\sqrt{2}}{2}\right)^{4\sqrt{2}}\Big[\Big\{\left(\tfrac{x_3}{16}\right)^2(4+\sqrt{2}) + \tfrac{1}{4}\sqrt{2}\,\lambda^{-2}(1+\sqrt{2})(x_4-1)^{2-2\sqrt{2}} \\
& (x_4+1)^{2+2\sqrt{2}}\left(x_4+\tfrac{\sqrt{2}}{2}\right)^{-4}\Big\} \\
& + \tfrac{1}{16}\left(x_2+\tfrac{x_3^2}{16}\right)\Big(\sqrt{2} - (x_4-1)^{-1-2\sqrt{2}}(x_4+1)^{-1}\big\{-2x_4^3+\sqrt{2}\,x_4^2 \\
& + (5+2\sqrt{2})x_4 + 1\big\}\Big)\Big] \\
& - \tfrac{1}{8}(x_4-1)^{-1+2\sqrt{2}}\left(x_4+\tfrac{\sqrt{2}}{2}\right)^{2}\Big[2\sqrt{2}\,\lambda^{-2}(2+\sqrt{2})\left(x_2+\tfrac{x_3^2}{16}\right)^{-1}(x_4-1)^{1-2\sqrt{2}} \\
& (x_4+1)^{1+2\sqrt{2}}\left(x_4+\tfrac{\sqrt{2}}{2}\right)^{-6} \\
& + \tfrac{1}{32}\sqrt{2}\,x_3^2(2+\sqrt{2})\left(x_2+\tfrac{x_3^2}{16}\right)^{-1}(x_4-1)^{-1}(x_4+1)^{-1}\left(x_4+\tfrac{\sqrt{2}}{2}\right)^{-1} \\
& - \tfrac{1}{4}(3+\sqrt{2})(x_4-1)^{-2} - \tfrac{1}{2}(8+3\sqrt{2})(x_4+1)^{-2} - \tfrac{1}{2}(8+7\sqrt{2})\left(x_4+\tfrac{\sqrt{2}}{2}\right)^{-2} \\
& + \tfrac{1}{2}(8+5\sqrt{2})(x_4-1)^{-1}(x_4+1)^{-1} - \tfrac{1}{2}(14+\sqrt{2})(x_4-1)^{-1}\left(x_4+\tfrac{\sqrt{2}}{2}\right)^{-1} \\
& + (8+5\sqrt{2})(x_4+1)^{-1}\left(x_4+\tfrac{\sqrt{2}}{2}\right)^{-1}\Big]\Big\} \\
& - \tfrac{i}{32\sqrt{2}}\lambda^{-2}x_3\left(x_2+\tfrac{x_3^2}{16}\right)^{-\frac{7}{2}-2\sqrt{2}}(x_4-1)^{-3/2}(x_4+1)^{-\frac{11}{2}-4\sqrt{2}} \\
& \left(x_4+\tfrac{\sqrt{2}}{2}\right)^{9+4\sqrt{2}}\Big\{(6+\sqrt{2})(x_4-1)^{-1} - (10+7\sqrt{2})(x_4+1)^{-1} \\
& + 2(4+5\sqrt{2})\left(x_4+\tfrac{\sqrt{2}}{2}\right)^{-1}\Big\}
\end{aligned}
$$

1. $\underline{\text{II} - C - 3}$

2.
$$g_{11} = \left(\tfrac{1}{16} (x_2^2 - \epsilon_1 x_3^2) + \lambda \right)^{-\sqrt{2}\epsilon} \exp\left(-\tfrac{1}{2} \int \tfrac{u}{x_4}\, dx_4 \right)$$
$$g_{22} = \left(\tfrac{1}{16} (x_2^2 - \epsilon_1 x_3^2) + \lambda \right)^{1+\sqrt{2}\epsilon} \exp\left(\tfrac{1}{2}(1+\sqrt{2}\epsilon) \int \tfrac{u}{x_4}\, dx_4 \right) (\epsilon_1 x_4)^{\frac{1}{2}}$$
$$g_{33} = \left(\tfrac{1}{16} (x_2^2 - \epsilon_1 x_3^2) + \lambda \right)^{1+\sqrt{2}\epsilon} \exp\left(\tfrac{1}{2}(1+\sqrt{2}\epsilon) \int \tfrac{u}{x_4}\, dx_4 \right) (\epsilon_1 x_4)^{-\frac{1}{2}}$$
$$g_{44} = \left(\tfrac{1}{16} (x_2^2 - \epsilon_1 x_3^2) + \lambda \right)^{2+\sqrt{2}\epsilon} \exp\left(\tfrac{1}{2}(1+\sqrt{2}\epsilon) \int \tfrac{u}{x_4}\, dx_4 \right) (\epsilon_1 x_4)^{-\frac{3}{2}} \left(\frac{u^2-1}{x_4-1} \right)$$

where u is a solution of
$$\frac{du}{dx_4} = \left(\frac{u^2-1}{4x_4} \right) \left[u \left(\frac{x_4+1}{x_4-1} \right) + \sqrt{2}\,\epsilon \right]$$

3. Petrov Type I

No further analysis owing to unknown analytical form.

1. <u>II - C - 4</u>

2.
$$g_{11} = \left(\tfrac{\sqrt{3}}{3} x_2 + x_3^2/12\right)^{-\sqrt{3}\epsilon} \exp\left(-\int \tfrac{v}{x_4}\, dx_4\right) (-x_4)^{\sqrt{3}\epsilon/3}$$
$$g_{22} = \left(\tfrac{\sqrt{3}}{3} x_2 + x_3^2/12\right)^{1+\sqrt{3}\epsilon} \exp\left(\tfrac{1}{3}(3+2\sqrt{3}\epsilon)\int \tfrac{v}{x_4}\, dx_4\right) (-x_4)^{-1-\sqrt{3}\epsilon/3}$$
$$g_{33} = \left(\tfrac{\sqrt{3}}{3} x_2 + x_3^2/12\right)^{2+\sqrt{3}\epsilon} \exp\left(\tfrac{1}{3}(3+2\sqrt{3}\epsilon)\int \tfrac{v}{x_4}\, dx_4\right) (-x_4)^{-\sqrt{3}\epsilon/3}$$
$$g_{44} = \left(\tfrac{\sqrt{3}}{3} x_2 + x_3^2/12\right)^{3+\sqrt{3}\epsilon} \exp\left(\tfrac{1}{3}(3+2\sqrt{3}\epsilon)\int \tfrac{v}{x_4}\, dx_4\right) (-x_4)^{-2-\sqrt{3}\epsilon/3} \left(\frac{v^2-1}{x_4-1}\right)$$

where v is a solution of

$$\frac{dv}{dx} = \left(\frac{v^2-1}{4x_4}\right)\left[\frac{2v}{(x_4-1)} + \frac{4\sqrt{3}\,\epsilon}{3}\right]$$

3. Petrov Type I

No further analysis owing to unknown analytical form.

1. $\underline{\text{III} - 1}$

2. $g_{11} = x_1$

   $g_{22} = x_1^2$

   $g_{33} = x_1^{-1}$

   $g_{44} = x_1^2$

3. Petrov Type D

4. $R_{1313} = -x_1^{-3}$

   $R_{1414} = -\frac{1}{2}x_1^{-2}$

5. $\Psi_0 = \frac{1}{4}x_1^{-2}\left(1 - \frac{1}{2}x_1^{-1}\right)$

   $\Psi_1 = 0$

   $\Psi_2 = -\frac{1}{4}x_1^{-3}$

   $\Psi_3 = 0$

   $\Psi_4 = x_1^{-6}\left(1 - \frac{1}{2}x_1^{-1}\right)$

1. III - 2

2. $$g_{12} = -\exp\left(2x_3 + \tfrac{5}{2}x_1\right)$$
$$g_{33} = \lambda^2 \exp\{3(x_3 + x_1)\}$$
$$g_{44} = \exp(-x_3 - x_1)$$

3. Petrov Type D

No further analysis as this is not in diagonal form.

1. III - 3

2.
$$g_{11} = + x_2 \sin(x_4)$$
$$g_{22} = x_2^{-\frac{1}{2}} \sin^{-\frac{1}{2}}(x_4)$$
$$g_{33} = x_2 \sin(x_4)$$
$$g_{44} = x_2^{\frac{3}{2}} \sin^{-\frac{1}{2}}(x_4)$$

3. Petrov Type D

4.
$$R_{1313} = -\tfrac{1}{2}\, x_2^{\frac{1}{2}} \sin^{\frac{1}{2}}(x_4)$$
$$R_{1414} = \tfrac{1}{8}\, x_2^{-1} \sin^{-3} x_4 \left(1 + \cos^2(x_4)\right)$$
$$R_{1323} = -\tfrac{1}{4}\, x_2^{-\frac{3}{2}} \sin^{\frac{1}{2}}(x_4)$$

5.
$$\Psi_0 = -\tfrac{1}{4}\, x_2^{-4} \sin^{-\frac{5}{4}}(x_4) \Big\{ \sin^{\frac{3}{4}}(x_4)\left(-\tfrac{1}{2} - \tfrac{1}{8}\, x_2^{-2} \sin^{-2}(x_4)(1+\cos^2(x_4))\right) - \tfrac{1}{2}\, x_2^{-\frac{1}{4}} \Big\}$$
$$\Psi_1 = 0$$
$$\Psi_2 = -\tfrac{1}{16}\, x_2^{-\frac{3}{2}} \sin^{-\frac{3}{2}}(x_4)$$
$$\Psi_3 = 0$$
$$\Psi_4 = -\, x_2^{-3} \sin^{-\frac{1}{4}}(x_4) \Big\{ \sin^{\frac{3}{4}}(x_4)\left(-\tfrac{1}{2} + \tfrac{1}{8}\, x_2^{-2} \sin^{-2}(x_4)(1+\cos^2(x_4))\right) - \tfrac{1}{2}\, x_2^{-\frac{1}{4}} \Big\}$$

1. $\underline{\text{III} - 4}$

2. $$g_{11} = x_1^{-\frac{1}{2}} \sinh^{-\frac{1}{2}}(x_4)$$
$$g_{22} = x_1 \sinh(x_4)$$
$$g_{33} = x_1 \sinh(x_4)$$
$$g_{44} = x_1^{\frac{3}{2}} \sinh^{-\frac{1}{2}}(x_4)$$

3. Petrov Type D

4. $$R_{1313} = -\tfrac{1}{8}\, x_1^{-1} \sinh^{-1}(x_4)$$
$$R_{1414} = \tfrac{1}{4}\, x_1^{\frac{1}{2}} \sinh^{-\frac{3}{2}}(x_4)$$

5. $$\Psi_0 = \tfrac{1}{32}\, x_1^{-\frac{1}{2}} \sinh^{-\frac{1}{2}}(x_4) \left(1 + x_1^{-\frac{1}{2}} \sinh^{-\frac{1}{2}}(x_4)\right)$$
$$\Psi_1 = 0$$
$$\Psi_2 = \tfrac{1}{16}\, x_1^{-\frac{3}{2}} \sinh^{-\frac{3}{2}}(x_4)$$
$$\Psi_3 = 0$$
$$\Psi_4 = \tfrac{1}{8}\, x_1^{-\frac{5}{2}} \sinh^{-\frac{5}{2}}(x_4) \left(1 + x_1^{-\frac{1}{2}} \sinh^{-\frac{1}{2}}(x_4)\right)$$

1. III - 5

2.
$$g_{11} = \lambda^2 (x_1+x_2)^{1+\sqrt{2}\epsilon_1} x_4^{\frac{2}{7}(3+\epsilon_2+\sqrt{2}\epsilon_1(1-2\epsilon_2))}$$
$$g_{22} = (x_1+x_2)^{1+\sqrt{2}\epsilon_1} x_4^{\frac{2}{7}(3-\epsilon_2+\sqrt{2}\epsilon_1(1+2\epsilon_2))}$$
$$g_{33} = (x_1+x_2)^{-\sqrt{2}\epsilon_1} x_4^{\frac{2}{7}(1-2\sqrt{2}\epsilon_1)}$$
$$g_{44} = (x_1+x_2)^{2+\sqrt{2}\epsilon_1}$$

3. Petrov Type I

Analysis continued with $\epsilon_1 = 1$, $\epsilon_2 = -1$

4.
$$R_{1313} = -\sqrt{2}\,(x_1+x_2)^{-2-\sqrt{2}} x_4^{\frac{2}{7}(-4-2\sqrt{2})}\Big[\tfrac{1}{2}\left(-\tfrac{3}{2}-\sqrt{2}\right)x_4^{10/7} - \tfrac{1}{4}(1+\sqrt{2})\,x_4^{\frac{3}{7}(3+4\sqrt{2})} - \tfrac{1}{49}\lambda^2 (x_1+x_2)^{-1+\sqrt{2}}(-10-\sqrt{2})\,x_4^{6\sqrt{2}/7}\Big]$$
$$R_{1214} = \tfrac{1}{4}\lambda^2 (4+6\sqrt{2})(x_1+x_2)^{\sqrt{2}} x_4^{\frac{2}{7}(2+3\sqrt{2})-1}$$
$$R_{1224} = \tfrac{1}{7}\cdot(3+\sqrt{2})(x_1+x_2)^{\sqrt{2}} x_4^{\frac{2}{7}(2-4\sqrt{2})-1}\left(1 - x_4^{\frac{3}{7}(2+3\sqrt{2})}\right)$$
$$R_{1414} = \lambda^{-2}(x_1+x_2)^{-1-\sqrt{2}} x_4^{-\frac{2}{7}(2+3\sqrt{2})}\Big[\tfrac{1}{49}(-2-\sqrt{2})\,x_4^{-2} - \tfrac{1}{4}\lambda^{-2}(2+\sqrt{2})(x_1+x_2)^{-1} x_4^{-\frac{2}{7}(2+3\sqrt{2})} - \tfrac{1}{4}(4+3\sqrt{2})(x_1+x_2)^{-1} x_4^{-\frac{3}{7}(4-\sqrt{2})}\Big]$$
$$R_{1323} = \lambda^{-4}(x_1+x_2)^{-4-3\sqrt{2}} x_4^{-\frac{3}{7}(1-2\sqrt{2}}\left(\tfrac{3}{2}+\sqrt{2}\right)$$

5.
$$\Psi_0 = -\tfrac{1}{4}\lambda^{-2}(x_1+x_2)^2 x_4^{4\sqrt{2}/7}\Big\{-\sqrt{2}\,x_4^{-6/7}\Big[\tfrac{1}{2}\left(-\tfrac{3}{2}-\sqrt{2}\right)x_4^{10/7} - \tfrac{1}{4}(1+\sqrt{2})\,x_4^{\frac{2}{7}(3+4\sqrt{2})} - \tfrac{1}{49}\lambda^2(x_1+x_2)^{-1+\sqrt{2}}(-10-\sqrt{2})\,x_4^{6\sqrt{2}/7}\Big]$$
$$- (x_1+x_2)^{-1-2\sqrt{2}}\lambda^{-2} x_4^{-6\sqrt{2}/7}\Big[\tfrac{1}{49}(-2-\sqrt{2})\,x_4^{-2} - \tfrac{1}{4}\lambda^{-2}(2+\sqrt{2})(x_1+x_2)^{-1} x_4^{-\frac{2}{7}(2+3\sqrt{2})} - \tfrac{1}{4}(4+3\sqrt{2})(x_1+x_2)^{-1} x_4^{-\frac{2}{7}(4-\sqrt{2})}\Big]$$
$$+ 2\lambda^{-3}\left(\tfrac{3}{2}+\sqrt{2}\right)(x_1+x_2)^{-2-\cancel{2}\sqrt{2}} x_4^{\frac{2}{7}(-1+4\sqrt{2})}\Big\}$$
$$\Psi_1 = -\tfrac{1}{14\sqrt{2}}\, i\,\lambda^{-1}(x_1+x_2)^{-2-\sqrt{2}/2} x_4^{-\frac{2}{7}(2+2\sqrt{2})}\Big\{\tfrac{\lambda}{2}(4+6\sqrt{2})\,x_4^{12\sqrt{2}/7} + (3+\sqrt{2})\,x_4^{2/7}\left(x_4^{-6\sqrt{2}/7}-1\right)\Big\}$$
$$\Psi_2 = -\tfrac{1}{2}\lambda^2(x_1+x_2)^{-2-\sqrt{2}} x_4^{-\frac{3}{7}(7+3\sqrt{2})}\Big[-\tfrac{1}{2}(1+\sqrt{2})(x_1+x_2)^{-1} x_4^{10/7} + \tfrac{1}{2}\lambda^2(1+\sqrt{2})(x_1+x_2)^{-1} x_4^{\frac{3}{7}(3+4\sqrt{2})} - \tfrac{1}{49}\lambda^2(-2-10\sqrt{2})\,x_4^{6\sqrt{2}/7}\Big]$$

$$
\begin{aligned}
\Psi_3 = & -\tfrac{1}{7\sqrt{2}}\, i\lambda^{-1} (x_1+x_2)^{-3-3\sqrt{2}/2}\, x_4^{\frac{2}{7}(6-\sqrt{2})-1} \Big\{ \tfrac{1}{2}\lambda (4+6\sqrt{2})\, x_4^{12\sqrt{2}/7} \\
& - (3+\sqrt{2})\, x_4^{2/7} \left(x_4^{-6\sqrt{2}/7} - 1 \right) \Big\} \\
\Psi_4 = & -\lambda^{-2} (x_1+x_2)^{-4-2\sqrt{2}}\, x_4^{-16/7} \Big\{ -\sqrt{2}\, x_4^{-6/7} \Big[\tfrac{1}{2} \left(-\tfrac{3}{2}-\sqrt{2}\right) x_4^{10/7} \\
& - \tfrac{1}{4} (1+\sqrt{2})\, x_4^{\frac{2}{7}(2+4\sqrt{2})} - \tfrac{\lambda^2}{49} (x_1+x_2)^{-1+\sqrt{2}} (-10-\sqrt{2})\, x_4^{6\sqrt{2}/7} \Big] \\
& - (x_1+x_2)^{-1-2\sqrt{2}} \lambda^{-2}\, x_4^{-6\sqrt{2}/7} \Big[\tfrac{1}{49} (-2-\sqrt{2})\, x_4^{-2} - \tfrac{1}{4}\lambda^{-2} (2+\sqrt{2}) (x_1+x_2)^{-1} \\
& x_4^{-\frac{3}{7}(2+3\sqrt{2})} - \tfrac{1}{4} (4+3\sqrt{2}) (x_1+x_2)^{-1/1}\, x_4^{-\frac{3}{7}(4-\sqrt{2})} \Big] \\
& - 2\lambda^{-3} \left(\tfrac{3}{2}+\sqrt{2} \right) (x_1+x_2)^{-2-2\sqrt{2}}\, x_4^{\frac{3}{7}(-1+4\sqrt{2})} \Big\}
\end{aligned}
$$

1. III − 6

2. $$g_{11} = \lambda^2 \left(x_1 + \tfrac{\sqrt{3}}{3} x_2\right)^{2+\sqrt{3}\epsilon} (x_4 - \epsilon_2)^{\frac{1}{2}(1+\sqrt{3}\epsilon)} (x_4 + \epsilon_2)^{\frac{1}{2}(3+\sqrt{3}\epsilon)} x_4^{-2-\sqrt{3}\epsilon}$$
$$g_{22} = \left(x_1 + \tfrac{\sqrt{3}}{3} x_2\right)^{1+\sqrt{3}\epsilon} (x_4 - \epsilon_2)^{\frac{1}{2}} (x_4 + \epsilon_2)^{\frac{1}{2}(3+2\sqrt{3}\epsilon)} x_4^{-2-\sqrt{3}\epsilon}$$
$$g_{33} = \left(x_1 + \tfrac{\sqrt{3}}{3} x_2\right)^{-\sqrt{3}\epsilon} (x_4 - \epsilon_2)^{\frac{1}{2}(1-\sqrt{3}\epsilon)} (x_4 + \epsilon_2)^{-\frac{1}{2}(1+\sqrt{3}\epsilon)} x_4^{\sqrt{3}\epsilon}$$
$$g_{44} = 3\left(x_1 + \tfrac{\sqrt{3}}{3} x_2\right)^{3+\sqrt{3}\epsilon} (x_4 - \epsilon_2)^{-\frac{1}{2}} (x_4 + \epsilon_2)^{\frac{1}{2}(1+2\sqrt{3}\epsilon)} x_4^{-\frac{1}{2}(4+\sqrt{3}\epsilon)}$$

3. Petrov Type I

No further analysis since the metric is not a vacuum metric.

1. $\underline{\text{III} - 7}$

2. $$g_{11} = (1 - x_4^2)^{-2}$$
$$g_{22} = \sinh^2(2x_1)\,(1 - x_4^2)^{-2}$$
$$g_{33} = x_4^2$$
$$g_{44} = (1 - x_4^2)^{-4}$$

3. Petrov Type D

4. $$R_{1313} = 2x_4^2\,(1 - x_4^2)$$
$$R_{1414} = -2\,(1 - x_4^2)$$

5. $$\Psi_0 = -\tfrac{1}{2}(1 - x_4^2)\sinh^2(2x_1)\left(1 + (1 - x_4^2)^2\right)$$
$$\Psi_1 = 0$$
$$\Psi_2 = 2\,(1 - x_4^2)^3$$
$$\Psi_3 = 0$$
$$\Psi_4 = -2\,(1 - x_4^2)^5\,\sinh^{-2}(2x_1)\left(1 + (1 - x_4^2)^2\right)$$

1. $\underline{\text{III} - 8}$

2. $$g_{11} = (1 + x_4^2)^{-2}$$
$$g_{22} = \sin^2(2x_1)\,(1 + x_4^2)^{-2}$$
$$g_{33} = x_4^2$$
$$g_{44} = (1 + x_4^2)^{-4}$$

3. Petrov Type D

4. $$R_{1313} = -2x_4^2(1 + x_4^2)$$
$$R_{1414} = 2(1 + x_4^2)$$

5. $$\Psi_0 = \tfrac{1}{2}\sin^2(2x_1)(1 + x_4^2)\left(1 + (1 + x_4^2)^4\right)$$
$$\Psi_1 = 0$$
$$\Psi_2 = -2(1 + x_4^2)^3$$
$$\Psi_3 = 0$$
$$\Psi_4 = 2\sin^{-2}(2x_1)(1 + x_4^2)^5\left(1 + (1 + x_4^2)^4\right)$$

1. $\underline{\text{III} - 9}$

2. $g_{11} = x_4^2$

$g_{22} = \sinh^2(2x_3)\,(1+x_4^2)^{-2}$

$g_{33} = (1+x_4^2)^{-2}$

$g_{44} = (1+x_4^2)^{-4}$

3. Petrov Type D

4. $R_{1313} = -2x_4^2(1+x_4^2)$

$R_{1414} = -4\,x_4^{-2}\,(1+x_4^2)^{-1}$

5. $\Psi_0 = \frac{1}{2}\sinh^2(2x_3)\;x_4^4\,(1+x_4^2)(1-2x_4^{-2})$

$\Psi_1 = 0$

$\Psi_2 = (1+x_4^2)^3$

$\Psi_3 = 0$

$\Psi_4 = 2\sinh^{-2}(2x_3)\,x_4^4\,(1+x_4^2)^3\,(1-2x_4^{-2})$

1. $\underline{\text{III}} - 10$

2. $$g_{11} = x_4^2$$
$$g_{22} = \sin^2(2x_3)\,(1-x_4^2)^{-2}$$
$$g_{33} = (1-x_4^2)^{-2}$$
$$g_{44} = (1-x_4^2)^{-4}$$

3. Petrov Type D

4. $$R_{1313} = 2x_4^2\,(1-x_4^2)^{-1}$$
$$R_{1414} = 2\,x_4^{-2}(1-x_4^2)^{-3}$$

5. $$\Psi_0 = -\tfrac{1}{2}\,\sin^2(2x_3)\,(1-x_4^2)^{-1}\,x_4^4\,(1-x_4^{-4})$$
$$\Psi_1 = 0$$
$$\Psi_2 = -(1-x_4^2)$$
$$\Psi_3 = 0$$
$$\Psi_4 = -2\,\sin^{-2}(2x_3)\,(1-x_4^2)^3\,x_4^4\,(1-x_4^{-4})$$

1. $\underline{\text{III} - 11}$ $(\text{I} - A - 1)$

2. $$g_{11} = \lambda^2 \sin^3(x_4)\, x_1 x_2^3$$
$$g_{22} = \sin^{3/2}(x_4)\, x_1^2\, x_2^{3/2}$$
$$g_{33} = x_1^{-1} x_2^{-1} \sin^{-1}(x_4)$$
$$g_{44} = \sin^{3/2}(x_4)\, x_1^2\, x_2^{7/2}$$

3. Petrov Type I

4. $$R_{1313} = x_1^{-2} x_2^{-1} \sin^{-3/2}(x_4) \left(- x_1 \sin^{1/2}(x_4) - \tfrac{3}{4}\lambda^2 x_2^{-1/2}\right)$$
$$R_{1214} = -\tfrac{3}{2}\lambda^2 x_1 x_2^2 \cos(x_4) \sin^2(x_4)$$
$$R_{1224} = -\tfrac{3}{2} x_1 x_2^{3/2} \cos(x_4) \sin^{1/2}(x_4)$$
$$R_{1414} = \lambda^{-2} \sin^{-3}(x_4)\, x_1^{-1} x_2^{-3} \left[\tfrac{3}{2}\cos^2(x_4) - \tfrac{9}{8} - \tfrac{1}{2}\lambda^{-2} x_1^{-1} x_2^{1/2} \sin^{1/2}(x_4)\right]$$
$$R_{1323} = -\tfrac{1}{2}\, 5 \lambda^{-4} x_1^{-4} x_2^{-8} \sin^{-7}(x_4)$$

5. $$\Psi_0 = \tfrac{1}{4}\lambda^{-2} \sin^{-2}(x_4)\, x_2^{-3/2}\Big[\left(x_1 \sin^{1/2}(x_4) + \tfrac{3}{4}\lambda^2 x_2^{-1/2}\right)$$
$$+ \sin^4(x_4)\, x_1^{-2} x_2^{-9/2}\left(\tfrac{3}{2}\cos^2(x_4) - \tfrac{9}{8} - \tfrac{1}{2}\lambda^{-2} x_1^{-1} x_2^{1/2} \sin^{1/2}(x_4)\right)$$
$$+ 5\lambda^{-3} \sin^{-19/4}\, x_1^{-5/2}\, x_2^{-25/4}\Big]$$
$$\Psi_1 = \tfrac{3}{4\sqrt{2}}\, i \lambda^{-1} \sin^{-2}(x_4) \cos(x_4)\, x_1^{-3/2} x_2^{-11/4} \left\{\lambda \sin^{3/4}(x_4)\, x_1^{1/2} + x_1^{1/4}\right\}$$
$$\Psi_2 = -\tfrac{1}{4}\lambda^{-2} \sin^{-7/2}(x_4)\, x_1^{-3} x_2^{-7/2}\Big[\tfrac{1}{8}\lambda^2 x_1 \big\{(-3\sin^2(x_4))$$
$$+ 9\cos^2(x_4)\big\} + x_2^{1/2} \sin^{1/2}(x_4)\Big]$$
$$\Psi_3 = \tfrac{3}{2\sqrt{2}}\, i \lambda^{-1} \sin^{-4}(x_4) \cos(x_4)\, x_1^{-7/2} x_2^{-17/2}\left[\lambda \sin^{3/4}(x_4)\, x_1^{1/2} - x_2^{1/4}\right]$$
$$\Psi_4 = \lambda^{-2} \sin^{-5}(x_4)\, x_1^{-2} x_2^{-9/2}\Big[\left(x_1 \sin^{1/2}(x_4) + \tfrac{3}{4}\lambda^2 x_2^{-1/2}\right)$$
$$+ \sin^4(x_4)\, x_1^{-2} x_2^{-9/2}\left(\tfrac{3}{2}\cos^2(x_4) - \tfrac{9}{8} - \tfrac{1}{2}\lambda^{-2} x_1^{-1} x_2^{1/2} \sin^{1/2}(x_4)\right)$$
$$- 5\lambda^{-3} \sin^{-19/4}(x_4)\, x_1^{-5/2}\, x_2^{-25/4}\Big]$$

1. $\underline{\text{III} - 12}$ $\quad (\text{I} - A - 2)$

2.
$$g_{11} = \sinh^{3/2}(x_4)\, x_2^2\, x_1^{3/2}$$
$$g_{22} = \lambda^2 \sinh^3(x_4)\, x_2\, x_1^3$$
$$g_{33} = \sinh^{-1}(x_4)\, x_2^{-1} x_1^{-1}$$
$$g_{44} = \sinh^{3/2}(x_4)\, x_2^2\, x_1^{7/2}$$

3. <u>Petrov Type I</u>

4.
$$R_{1313} = -\tfrac{1}{8} \sinh^{-3}(x_4)\, x_1^{-3} x_2^{-1} \Big(3\cosh^2(x_4) + 9\sinh^2(x_4) + 4\lambda^2 \sinh^{1/2}(x_4)\, x_2^{-1}\, x_1^{1/2} \Big)$$
$$R_{1214} = -\tfrac{3}{2} \cosh(x_4) \sinh^{1/2}(x_4)\, x_2\, x_1^{3/2}$$
$$R_{1224} = -\tfrac{3}{2} \lambda^2 \cosh(x_4) \sinh^2(x_4)\, x_2 x_1^2$$
$$R_{1414} = x_1^{-3/2}\, x_2^{-2} \sinh^{-3/2}(x_4) \Big[\tfrac{3}{4} \operatorname{cosech}^2(x_4) - \lambda^{-2}\, x_1^{1/2}\, x_2^{-1} \sinh^{-3/2}(x_4) \Big]$$
$$R_{1323} = \tfrac{3}{2}\, x_1^{-5}\, x_2^{-6} \sinh^{-4}(x_4)$$

5.
$$\Psi_0 = \tfrac{1}{4} \lambda^2 \sinh^{-1/2}(x_4)\, x_2^{-3}\, x_1^{-3/2} \Big[\tfrac{1}{8} \Big(3\cosh^2(x_4) + 9\sinh^2(x_4) + 4\lambda^2 \sinh^{1/2}(x_4)\, x_2^{-1} x_1^{1/2} \Big) + \sinh^{-1}(x_4)\, x_2^{-2}\, x_1^{-5/2} \Big(\tfrac{3}{4} \operatorname{cosech}^2(x_4) - \lambda^{-2} x_1^{1/2} x_2^{-1} \sinh^{-3/2}(x_4) - \tfrac{3}{2} \lambda^{-1} \sinh^{-7/4}(x_4)\, x_2^{-5/2}\, x_1^{-13/4} \Big]$$
$$\Psi_1 = \tfrac{3}{4\sqrt{2}}\, i \cosh(x_4) \sinh^{-7/4}(x_4)\, x_2^{-2}\, x_1^{-2}$$
$$\Psi_2 = -\tfrac{1}{16} \lambda^{-2} \sinh^{-7/2}(x_4)\, x_2^{-3}\, x_1^{-7/2} \Big[3\lambda^2 x_2 \big(\sinh^2(x_4) + \cosh^2(x_4) \big) - 4\sinh^{1/2}(x_4)\, x_1^{1/2} \Big]$$
$$\Psi_3 = \tfrac{3}{2\sqrt{2}}\, i \lambda^{-2} \cosh(x_4) \sinh^{-19/4}\, x_2^{-3} x_1^{-5} \Big(x_1^{1/4} - \lambda x_1^{1/2} \sinh^{3/4}(x_4) \Big)$$
$$\Psi_4 = \lambda^{-2} \sinh^{-11/2}(x_4)\, x_2^{-5}\, x_1^{-15/2} \Big[\tfrac{1}{8} \Big(3\cosh^2(x_4) + 9\sinh^2(x_4) + 4\lambda^2 \sinh^{1/2}(x_4)\, x_2^{-1} x_1^{1/2} \Big) + \sinh^{-1}(x_4)\, x_2^{-2}\, x_1^{-5/2} \Big(\tfrac{3}{4} \operatorname{cosech}^2(x_4) - \lambda^2 x_1^{1/2} x_2^{-1} \sinh^{-3/2}(x_4) + \tfrac{3}{2} \lambda^{-1} \sinh^{-7/4}(x_4)\, x_2^{-5/2}\, x_1^{-13/14} \Big]$$