# THE USE OF DYNAMIC SOFTWARE TO POTENTIALLY ENHANCE

# CONCEPTUAL UNDERSTANDING AND A PRODUCTIVE DISPOSITION IN THE VISUAL LEARNING OF ALGEBRA:

# AN INTERVENTIONIST CASE STUDY

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# ABSTRACT

Over the decades, the didactics and practice of teaching mathematics has offered many unique challenges and opportunities for exploration and understanding. The introduction and development of technology into mathematics is one of the occurrences which has also contributed to a new discourse in teaching mathematics – in this case the teaching of algebra.

Algebra is still seen as a gatekeeper and remains as one of the key reasons for a negative disposition amongst learners towards learning the subject. Across the globe psychologists, philosophers and educators continue to engage in debates and research projects in search of answers and solutions for the improvement of algebra teaching and an improvement in dispositions towards learning algebra. This thesis reports on a research project that focused on the use of dynamic software to enhance the conceptual understanding and productive dispositions of selected learners through the visual learning of abstract algebraic concepts. The research was executed as an interventionist case study.

A case study methodological strategy was adopted with two groups of 30 Grade 9 learners. One group was a Grade 9 mathematics class of a school in Windhoek who scored above average in algebra but showed a very low disposition score, while the other group was made up of learners from a community project who scored high on the disposition scale but achieved below average results in algebra.

The analytical framework of the case study is structured around a combination of complementary algebraic topics presented through visual learning, with GeoGebra as a medium of instruction. With the focus on visualisation and the use of technology the study investigated and attempted to understand how participants processed and internalised algebraic concepts to make sense of abstract algebraic concepts and eventually gain sustained conceptual understanding. The study was framed by the theoretical theories of constructivism and the Dual Coding Theory.

For the collection of data, a mixed methods approach was adopted following three cycles. Three algebraic topics were taught with GeoGebra applets yielding both qualitative and quantitative data, through the observation of participants, screen captures and reflective interviews, using instruments designed specifically for the study and collecting quantitative achievement test results.

The study, a journey that both the participants and the researcher embarked upon, revealed that the use of technology enhanced conceptual understanding for both groups and both groups showed a positive change in disposition towards learning algebra. The intervention with GeoGebra consistently and progressively improved in terms of conceptual understanding and dispositions towards learning algebra significant improvements in results were achieved.

The findings showed that this approach to teaching algebra yielded positive results and gave new insights into visual teaching with technology. New opportunities for further research were created.

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All the participants, who participated in the study. For being positive and faithfully attending classes. It was a journey over an extensive period.

My friends and colleagues at the school for their support.

To God all the glory.

# DEDICATION

I dedicate this thesis to my wife Erina. For sitting long hours with me while I worked – sometimes until very late at night. For never complaining when we had to cancel an event because I had to work on my thesis. You are the silent hero in this chapter of our lives.

# DECLARATION OF ORIGINALITY

I, Daniel Franscius Junius (Student number 11J5235) declare that this doctoral thesis entitled: The use of dynamic software to potentially enhance conceptual understanding and a productive disposition in the visual learning of algebra: an interventionist case study", is my own work written in my own words. Where I have drawn on the words or ideas of others, these have been fully acknowledged and referenced in the manner required by the Rhodes University Department of Education Guide to referencing.

Daniel Franscius Junius February 2023

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| | |
|---|---|
| AEI | Applet Evaluation Instrument |
| AR | Adaptive reasoning |
| BODMAS | Acronym for order of operations |
| CU | Conceptual understanding |
| DCT | Dual Coding Theory |
| DMS | Dynamic Mathematics Software |
| ETSIP | Education and Training Sector Improvement Programme |
| FOIL | Multiplying binomials acronym for First Outside, Inside Last |
| HE | Human Ethics |
| ICT | Information and Communications Technology |
| LCE | Learner-Centred Education |
| MTP | Mathematics Teaching Proficiency |
| PD | Productive Disposition |
| PF | Procedural Fluency |
| RUESC | Rhodes University Ethical Standards Committee |
| SC | Strategic competence |
| VIF | Visual Indicator Framework |

# CHAPTER 1

# INTRODUCTION

## 1.1 PURPOSE

In spite of many years teaching Grade 9 algebra I was still left with the questions on why learners find the conceptual understanding of basic algebra so difficult and why algebra is seen as a gatekeeper preventing learners from developing a positive disposition towards learning algebra and mathematics in general.

This first chapter introduces the background, context and intended purpose for doing this research project and places them in the perspective of the research questions. In this chapter the theoretical and methodological underpinnings of this research and the significance of the research within the Namibian and broader context will be addressed. The chosen method of reporting on the research project will be explained and established. The chapter will end with a chapter-by-chapter outline of the research process of the case study.

## 1.2 BACKGROUND AND CONTEXT OF THE STUDY

I teach mathematics at a private school in Namibia and initiated a community project – the Namibian Visualisation Project (NAMVISPRO) – where learners from different communities in Windhoek are invited to receive extra support to improve their marks in mathematics by receiving additional tutoring and having access to additional resources such as a well-equipped computer laboratory with high-speed internet access. Participants of the NAMVISPRO class can log into the many free lessons running on the *GeoGebra* platform, exposing them to the visual learning of mathematics.

I found that learners from the community project displayed a more positive disposition towards the learning of algebra, while learners from my classes at school often expressed their dismay at learning algebra or regarded algebra as being nonsensical without any use in real-life situations. Learners participating in the community project often expressed their enjoyment when doing mathematics on the *GeoGebra* platform. They persistently attended the classes despite the financial restraints of paying a taxi fare to come to the laboratory. Many of these learners attend under-resourced schools where they share textbooks and attend school in crowded classes, often with teachers not qualified to teach secondary mathematics.

On the other hand, learners from the well-resourced school appeared to dislike mathematics, expressed their negative disposition through a lack of motivation to acquire new mathematical skills and in general questioned the usefulness and need for being taught algebra. Despite their low dispositions, however, some of these learners achieved higher marks in mathematics.

Both groups of learners posed an excellent opportunity to compile a case for research to investigate and find answers to the research question that accumulated from teaching these different groups of learners.

Septian et al. (2019) are convinced that a positive disposition towards mathematics improves learners' problem solving abilities. They further state that an improvement in problem solving abilities can be achieved when a *GeoGebra*-assisted approach is preferred to conventional teaching approach (Septian, et al. 2019). Huda (2019) regarded the ability to integrate computer technology into the teaching of mathematics as an important pedagogical ability that all mathematics teachers should have. Applying pedagogical competency to use computers to support the teaching of mathematics motivates learners and improves their dispositions towards learning mathematics. Eventually this improves their achievement in mathematics (Huda, 2019).

Teaching within a Namibian context left me with similar questions, specifically when teaching algebra to junior secondary learners. My knowledge about visualisation as a learning tool – having had experience with integrating *GeoGebra* with algebra lessons and having well-equipped computer laboratories available – steered me towards the formulation of the main research question.

## 1.3 RESEARCH GOALS AND QUESTIONS

The aim of my research is three-fold:

1. To investigate the use of *GeoGebra* as a learning tool for teaching algebra to Grade 9 learners.

2. To examine the conceptual understanding participants can achieve when learning with dynamic visual software.
3. To analyse the changes in dispositions of learners when taught with *GeoGebra*.

The study aims to answer the main research question:

*How can the use of dynamic software potentially enhance conceptual understanding and a productive disposition through the visual learning of algebra?*

## 1.4 THEORETICAL FRAMEWORK OF THE RESEARCH

This section provides a brief overview of the study within a contextual, conceptual, and theoretical setting. Key concepts such as conceptual understanding of algebra and a productive disposition towards the visual learning of algebra, are placed in context according to relevant theories of learning. This study was undertaken within a Namibian context, therefore current trends and policies in Namibia are reviewed.

Mayer (2003) developed a cognitive theory of multimedia learning based on a constructivist epistemology of learning. Constructivism implies that learners actively create their own understanding and sense-making of the world, rather than having such understanding delivered to them (Thompson 1995; von Glasersfeld 1995). Learners become active participants in the learning process, rather than merely absorbing concepts presented to them. Thompson (1995) suggests that learners already have well developed cognitive structures, allowing them to select and transform the learning material into constructs, develop hypotheses and make calculated decisions. During the cognitive processing of mathematical concepts, learners spontaneously select information which is easiest to comprehend and manage mentally (Thompson 1995).

Ground-breaking ideas by Kilpatrick et al. (2001) define conceptual understanding as “an integrated and functional grasp of mathematical ideas” (Kilpatrick et al., 2001, p. 118). In the new Namibian secondary school syllabus these ideas are separated into different *proficiencies and skills* that a learner should acquire to progress towards another level (Namibian Ministry of Education, Arts and Culture, National Institute for

Educational Development (NIED), 2015). This is in line with Kilpatrick et al.'s claim that mathematics relies on many concepts or ideas that are mostly abstract and interwoven with each other.

For Kilpatrick et al. (2001), a learner acquires mathematical proficiency when several strands of mathematical learning are present. One of these intertwined strands refers to a learner's disposition towards the learning of mathematics. When the mathematics is conceptually understood, the learner will be able to acknowledge the necessity and practical possibilities of a concept (Kilpatrick et al. 2001). It implies an important symbiotic relationship between conceptual understanding and productive disposition – often ignored by teachers – and therefore the focus of this study.

## 1.5 METHODOLOGY

This case study is oriented within the interpretive paradigm, since it is my aim to understand how the use of learning with technology can enhance conceptual understanding of algebra and to track, if any, positive changes in disposition when doing so. Carefully selected applets from the *GeoGebra* platform were used to replace classroom instruction to teach participants the required algebraic skills. Several elements of action research methodology were also present during the research process. Creswell and Plano Clark (2004) refer to such mixed methods designs as a multi-phase design. The research process went through several phases and data collection took place during three repeating cycles, each time with a new algebra topic.

The case was compiled of two groups of learners doing mathematics at two different sites. The first group of 16 participants were selected from the school where I am a teacher while the second group of 14 participants were selected from the Namibian Visualisation Project (NAMVISPRO) group, I wanted to provide learners from our whole city an opportunity to use technology to improve their mathematical skills by attending free laboratory sessions. The case consisted of a cohort of mixed-gender and mixed-ability non-repeating Grade 9 learners.

The research was conducted over three cycles with each cycle spanning nine phases. For each cycle, a new algebra topic was selected, then taught by using very carefully selected *GeoGebra* applets. The same number of lesson periods were allowed for each topic as prescribed by the scheme of work we followed in classroom teaching. After each cycle, a refinement process followed, based upon the reflective interviews done during and after each cycle.

Both qualitative and quantitative sets of data were collected. A set of instruments was developed to evaluate applets and to measure and quantify each participant's disposition after each lesson and after each cycle. At the end of each cycle participants had to write a standardised achievement test, set and moderated by the mathematics department of my school. Constant monitoring of the research process and data collection was done by a senior Head of Department at my school.

Data consisted of several screen captures while participants were engaged with applets. The screen captures were used as referral material when conducting semi-structured interviews with individual participants. Responses were coded and analysed after each interview. After each lesson and after each cycle participants had the opportunity to complete a Disposition Instrument to reflect upon their experiences with applets. After each cycle participants had to write an achievement test and the results of the tests were quantified to provide quantitative data. Each participant was tracked and observed over the entire research period.

For validation purposes a final reflective session was held by using ten questions available to participants before the reflection session. All participants were present during the final reflection.

Two major sets of data were collected and analysed qualitatively and quantitatively. Firstly, from a horizontal perspective during each cycle and secondly, from a longitudinal perspective over the whole research period. All sets of data were collected and analysed anonymously and filed electronically to be available to my supervisor. An agreement was made with participants for them to have access to the results of the research project.

## 1.6 REPORTING THE RESEARCH PROJECT

I remained a professional mathematics teacher during the whole research period, but the close interaction between me as researcher and the participants for the purpose of data collection allowed me to adapt a narrative approach of reporting on the research process. Constant communication (reflection) between me as researcher and the participants became a mathematical journey we embarked upon. The intervention required that algebra be taught through technology in the place of normal classroom instruction. It was required that learning took place and I had no second opportunity to re-teach any topic. Not only did it create a very special opportunity for data collection, but also necessitated the monitoring of each participant very closely. The facilities included in *GeoGebra* lessons provided a clear storyline of each participant that made a narrative reporting approach a natural choice.

The chosen method of reporting allowed participants to feel safe and comfortable when reflecting upon their experiences during the lessons with the applets. It provided me with honest and valid sets of data, valuable for the results of the research process. I was provided with valuable sets of individual and collective data.

Beyers and Elliot (2005) regard a narrative approach as suitable for collecting both qualitative and quantitative data, especially when a mixed methods approach is adopted. According to McAlpine (2016) a narrative approach allows participants of a case study to provide the researcher with a collective coherent story from all individuals, while the researcher becomes an active agent during the research.

## 1.7 SIGNIFICANCE OF THE STUDY

Unexpectantly the COVID-19 pandemic and the accompanying lockdown period necessitated the teaching of mathematics with the use of technology. However, in Southern Africa and in Namibia there is widespread agreement that the learning of mathematics did not benefit by the sudden switch to learning with technology. An ongoing debate is still happening with many arguments for and against the use of technology to teach mathematics.

Not only does this study provide clear-cut answers to the major research questions about learning algebra with technology but it will also make a valuable contribution towards the current discourse about the use of technology to teach mathematics to junior learners. One of the unexpected but significant findings lies within the role that a qualified mathematics teacher can play in the driving of learning through technology.

The research also contributes towards the ongoing debate about visualisation as an epistemological tool in mathematics teaching. Specific focus is placed upon the role of visualisation in the teaching of algebra which is usually regarded as being without visualisation possibilities. Presmeg (2014) argues that several questions regarding the use of visualisation in mathematics education calls for further research and answers. This study aims to answer some of those questions, but also provides an opportunity for further research and investigations.

## 1.8 STRUCTURE OF THE THESIS

### 1.8.1 Chapter 2: Literature review

Chapter 2 provides a theoretical underpinning for this study. Literature pertinent for the formulation of founded research questions are reviewed and unpacked. I interrogate algebra as a gatekeeper and discuss the use of technology– especially *GeoGebra* – as an instruction medium within this context as a research instrument. Recent literature places visualisation and visual learning of algebra with dynamic software in context.

Secondly, ideas regarding conceptual understanding and a productive disposition towards mathematics learning, specifically algebra, are reviewed according to relevant authors and researchers. Opportunities and challenges about learning algebra visually are also investigated.

Thirdly, because the study was undertaken within a Namibian context, I discuss the current policies and developments within Namibia and use them as guidelines for the research. Finally, I investigate the chosen mixed methods research approach and bring it into context with the case study research project. I then explain the reasons why the research was placed within an interpretive paradigm.

### 1.8.2 Chapter 3: Methodology

In this chapter I investigate and discuss the research paradigm and different research methodological approaches. Finally I justify the chosen research methodology is with reference to the theoretical framework and the different practicalities that I needed to consider when doing the research. Important ethical considerations, especially because research was done with children and because it was an interventionist study are discussed. Finally, reference is made to validity, reliability and positionality.

### 1.8.3 Chapter 4: Applet selection process

Part of the validity of the research project relied upon the careful selection of relevant algebra topics that can be taught through visual learning with the aid of dynamic software. *GeoGebra* was the chosen platform for the project and great care had to be taken to select applets that met the requirements of being pedagogically correct and in line with the prescribed topics of the Namibian school syllabus.

An instrument was designed to filter applets to fulfil the requirements set out in the research design. This chapter discusses the steps taken to ensure that the applets would meet the requirements of the instrument and that the relevant part of the syllabus was fully incorporated to the correct level for Grade 9.

### 1.8.4 Chapters 5, 6 and 7: Cyclic execution of the research process

Chapters 5 to 7 provide a detailed discussion of the research process followed to collect rich and quantitative data. Examples of the work by participants are extracted and discussed. The research went through three tedious cycles with several lessons and laboratory sessions to fully cover the three chosen algebraic topics.

Within each cycle one topic is taught through *GeoGebra* lessons. The same time is spent per sub-topic to cover the prescribed work within the number of lessons as advised by the scheme of work we use in my school. Each cycle is concluded with an achievement test set according to the detailed prescriptions of the Namibian mathematics syllabus. Reflective interviews and discussions were held according to screen captures collected during laboratory sessions. The results of the achievement tests are discussed and analysed and finally trends regarding the participants' progress on the previously designed Disposition Instruments are captured.

### 1.8.5 Chapter 8: Horizontal data analyses of the three research cycles

Chapter 8 discuss and analyses all the data collected during the first three cycles from a horizontal perspective. Clear conclusions and findings are made by placing the data collected during the previous cycles horizontally next to each other. All data is analysed qualitative and quantitatively to find conclusive evidence for answering the posed research question.

Findings of the study in relation to the research goals and methodological approach are discussed and summarised to substantiate the findings and conclusions made through the first three cycles. A final reflective interview with all participants is discussed to support previous evidence and findings.

### 1.8.6 Chapter 9: Conclusion

Chapter 9 I reflect on the research findings that surfaced over a long period of time and discuss them to consolidate the significant findings of the study.

The limitations and the significance of the study are highlighted. The possibilities of contributions towards the learning of algebra with technology is placed within the context of the research design. The chapter concludes with personal reflections and important recommendations for further research.

# CHAPTER 2

# CONTEXTUAL OVERVIEW AND LITERATURE REVIEW

## 2.1 INTRODUCTION

The aim of this chapter is to place the study within a contextual, conceptual and theoretical setting. Here I unpack key concepts such as conceptual understanding and productive disposition towards mathematics learning, specifically algebra, according to relevant authors and researchers. Pertinent issues relating to learning algebra visually are also discussed. Visualisation and visual learning with dynamic software are explained within the context of recent literature. This study was undertaken within a Namibian context, therefore current trends and policies in Namibia were taken into consideration. Gaps in knowledge are identified in the light of current debates, schools of thought and hypotheses. In conclusion the chapter will end with a brief introduction to my research rationale and questions.

## 2.2 TOWARDS CONCEPTUAL UNDERSTANDING AND DEVELOPING A PRODUCTIVE DISPOSITION

### 2.2.1 Mathematical representation

Inherent to conceptually understanding of algebra, lies not only the ability to discover, develop, understand and design patterns, but also the ability to explain and impart the newly discovered patterns to other learners (Goldin, 2002). Algebraic understanding, according to Goldin and Shteingold (2001, p. 2), has two 'systems of representation', described as 'internal' and an 'external' component. External systems of representation include all the conventional algebraic notations made up of symbols, letters, the Cartesian coordinate notation and very complex manipulatives. Goldin and Shteingold (2001) argue that the internal system of representation is embedded in learners' personal ability to construct a form of visual symbolisation and give meaning to mathematical concepts. This is done by relying on their own vocabulary abilities, visual imagery and spatial concepts, in addition to attempting various problem solving strategies achieved within their own heuristics and personal dispositions (affective emotions towards mathematics learning) (Goldin & Shteingold, 2001).

For one learner, the internal representation of the numeral '$-3$' could merely be the minus sign symbol followed by the symbol 3 without forming a meaningful related concept, while another could have a visual image of '$-3$' as a number less than zero on a number line (Goldin & Shteingold, 2001, p. 5). Kosslyn (1987) argues that a system is needed to characterise the complex internal representations learners might have of mathematical concepts. Except for the concrete visual images learners have of objects – like an apple – there is also "high-level visual processing involved in visual recognition, navigation, tracking, and mental imagery". He refers to this as "visual perceptions and visual imagery" or "perceptual units that correspond to objects and parts thereof whenever a mathematical problem is solved." (Kosslyn 1987, p. 148).

Being able to communicate algebraic imagery forms one part of the cognitive understanding of algebra, but Mason et al. (2010) take it further to include specialisation and generalisation as a basis for the conceptual understanding of algebra. They define specialisation as the use of adequate examples or the use of specific cases in the operational process of teaching mathematical concepts (Mason et al., 2010, p. 209). Carefully selected examples should be used to illustrate an algebraic concept. Examples should be used to learn more about the main goal of a concept. Learners can change the focus of a problem by 'specialising' the circumstances in order to progress with the problem. Specialisation can also mean modifying a problem in such a way that the learner can recall the visual 'pathways' they have created for solving the problem. Specialising as a process can involve both the teacher and the learner to unfold a concept for the learner (Mason et al., 2010). For Mason et al. (2010), generalisation is the next logical stage after specialisation. When a learner is taught new concepts, conjecturing and formulation happens in their mind so that generalisation can evolve. For Mason et al. (2010) conjecturing means relying on any statement that could be true, even if the mathematical truth of the statement has not yet been established. This often involves integrating previous knowledge with newly related concepts. Becker and Rivera (2008) describe generalisation as being 'constructive' and 'deconstructive' (Becker & Rivera, 2008. p. 70), where *constructive* implies that a learner cognitively creates a figure from the given clues. *Deconstructive* implies that a learner sees overlapping figures and deduces a pattern from this. Examples of constructive generalisation could include arriving at a general term for a mathematical sequence, while deconstructive generalisation could include the use of a series of squares with shared sides to deduce a formula for a sequence. Both processes of generalisation form an integral part of algebraic thinking and reasoning. Kaput (1999, p. 2) defines generalisation as "deliberately extending the range of reasoning and communication beyond the concept considered". Kaput (1999, p. 2) is of the opinion that "learning algebra has for too long been the *simplifying* of algebraic expressions, *solving* equations and *learning* the rules for manipulating symbols". Algebra syllabi in schools generally fail in the construction of relationships or applying newly acquired knowledge. Learners do not have the opportunity to articulate their newly acquired knowledge to others. Success in algebra is based on "producing the right strings of symbols" (Kaput, 1999, p. 2). Figure 2.1 below shows the relationship between generalisation and specialisation.

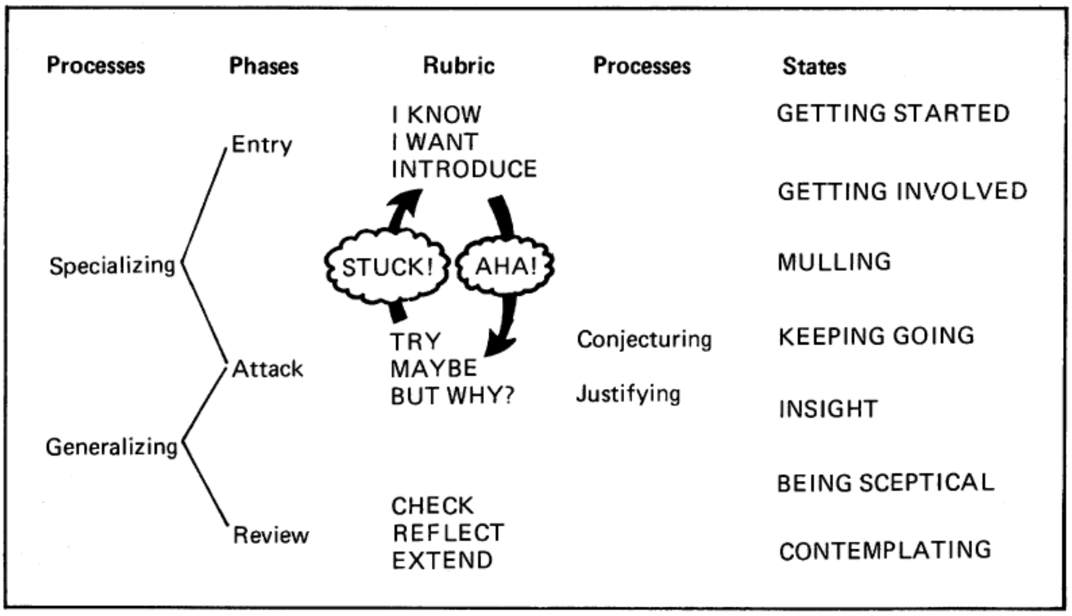



*Figure 2.1: Relationship between generalisation and specialisation*

(Mason et al., 2010, p. 219).

Mason et al. (2010), use the diagram in Figure 2.1 to point out the complex and interlinked relationship between generalisation and specialisation. For them, the need to solve a problem necessitates the learner to create several visual structures in their mind. They called the process 'mulling' (Mason et al., 2010, p. 219). It is a process of moving back and forth between specialising and generalising while conjecturing and justifying take place during each step.

Watson & Mason (2005) and Zaskis and Leikin (2007) turned examples of generalisation produced by learners into teaching tools or learning opportunities for other learners. According to Watson and Mason a collection of learner generated examples enhance the process of mathematisation at different levels in learners. Learner generated examples can also serve as pedagogical tools for teachers to teach generalisation. Zazkis and Leikin (2007, p. 15) remarked that: “Watson and Mason focus on learner generated examples, a teaching strategy of asking learners to construct their own examples of mathematical objects under given constraints,” (Zazkis & Leikin, 2007, p. 15). Watson & Mason (2005, p. 76) categorise these examples into what they call different “example spaces”. By creating a set of *example spaces* learners create their own framework of representation to conceptually understand a topic.

### 2.2.2 Visualisation

Yilmaz and Argun (2018) built upon earlier definitions and defined visualisation as complex processes of changing and re-constructing mental imagery. It should lead to the creation of a relationship between what is already known, the previously unknown and new ideas. This is a gradually developing process. It includes a reflection upon new interpretations of and the creation of visual structures in our minds and on paper, or computer images that can be used as tools for describing and explaining (Bishop, 2003; Zimmermann & Cunningham, 1991; Yilmaz & Argun, 2018).

I argue that defining visualisation is an ongoing evolutionary process, with many authors, philosophers and researchers still making valuable contributions to visualisation as a pedagogical tool for educators, and as a learning instrument for learners. Earlier definitions for visualisation evolved from the visual nature of geometry (Zimmermann & Cunningham, 1991). With the development of information technology and the possibilities offered using mathematical software packages like *GeoGebra*, there has however been a shift in focus amongst mathematics education practitioners to include visualisation in the understanding of algebra. Many software applications are available for visualisation of algebraic concepts but more research about visualisation and the mechanism for cognitive understanding of algebra is needed (Varankina et al., 2019).

Yilmaz and Argun (2018) believe that visualisation occurs as a continuum of processes which is present from the developing of mathematical thoughts and the discovering of new concepts, to the creation of complex relations between mathematical objects. Visualisation can assist to untangle the complexity of large volumes of mathematical constructs (Yilmaz & Argun, 2018). Arcavi (2003) and Stylianou and Silver (2004) believe that visualisation has limitations. Some learners experience difficulties in applying visualisation when engaging in new mathematical tasks. The limitations are rooted in the high concentration and complexity of the information often embedded in a visual diagram or graph. This is evident in the reluctance of some learners to untangle cognitively difficult visual structures. Stylianou and Silver (2004) claimed that learners often revert to algebraic solutions, rather than using visual methods. An example would be the situation where some learners easily solve two simultaneous linear equations by reading from a graph, while others rather attempt finding the algebraic solution of two simultaneous equations. Contrarily, Presmeg (2006) claims that learners more often switch to a visual register when struggling with a mathematical problem. Presmeg (2006) calls for further qualitative research to substantiate any of these claims.

For Presmeg (1997, p. 205), visualisation incorporates all 'inscriptions', (representations) of a spatial nature and the creation of new mental images. The term 'inscriptions' (Presmeg, 2006, p. 205) is defined as: "Graphical representations …which have come to be known as *inscriptions… in* scientific practice" (Presmeg 2006, p. 205). Presmeg (1985) regards visualisation as supporting understanding and as a tool towards generating a solution when solving mathematical problems. Visualisation is an effective application used in solving a mathematical problem and can also be used to create a visual image of the problem, which enables understanding the problem in terms of visual imagery. The process of solving the problem includes different images in the mind of the learner, sometimes supported by a diagram, as a necessary part of the solution method. Presmeg refers to it as a mental scheme, in which visual or spatial information is conceived. The mental schemata emerge as a product of the visualisation process (Presmeg, 1985).

Presmeg (1986) broadened the definition by Clements (1982, p. 36) from a "picture in the mind" to also include "verbal, numerical or mathematical symbols … arranged spatially to form a kind of imagery" Presmeg (1986, p. 42). This imagery relates to the images that Paivio (1971, p. 482) called 'number forms'. Presmeg (1986, p. 43) categorised visual images into five different types:

- concrete imagery (a holistic picture, when everyday objects come together and are constructed in the mind);
- pattern imagery (simple relationships, which are described in a visual-spatial scheme);
- memory imagery of formulae (formation of formulae, which we generally see in our minds, and which are written as a note in our memory);
- kinaesthetic imagery (images requiring muscle strength activities); and
- dynamic imagery (active images). With the aid of technology animations and the rotations of two- or three-dimensional objects are examples of dynamic imagery.

Bishop (1989) defines mathematics as the abstraction and objective representation of real-life situations. Visual imagery and spatial inscriptions are always present in mathematical representations and add meaning to evolving mathematical thoughts (Bishop, 1989). From this perspective visualisation guides the analytical process towards finding a solution for a problem (Presmeg, 1997). According to Wheatley (1991), visualisation is a vehicle to move from the concrete to the abstract. Yilmaz & Argun (2018) stated that visualisation promotes and develops multidimensional thinking in individuals. (Yilamaz & Argun, 2018, p. 41).

When any form of spatial arrangement happens in the mind of a learner it is always underpinned by some form of visual image to guide this formation. For Presmeg (1997), mathematical "visualisation includes all inscriptions of a spatial nature, as well as the construction and transformation processes of visual-mental images" (Presmeg, 1997, p. 43). Mathematics is a process of abstracting real situations. Visualisation guides these representations and adds meaning to knowledge (Bishop, 1989).

Guzman (2002) is of the opinion that visualisation requires optical processes and often a more superficial experience – an abstract cognitive type of vision. For Guzman (2002, p. 3) mathematics contains a "great richness of visual relationships that are intuitively representable in a variety of ways". Using visualisation to solve these mathematical problems requires not only optical processes, but also a more "superficial experience, a psychological sense of vision" (Guzman, 2002, p. 3).

### 2.2.3 Towards conceptual understanding

Conceptual understanding is often defined as a process of developing an interconnected network of abstract cognitive structures (Hiebert & Carpenter, 1992). Schoenfeld (1992) states that when learners can construct meaning for new mathematical concepts and connect them to existing (visual) structures, mathematics is easier and makes more sense to them.

For my research, the above-mentioned ideas align well with Piaget's (Beth & Piaget, 1966, pp. 203-208) idea of "reflective abstraction" consisting of processes of assimilation and accommodation fundamental to developing conceptual understanding. Piaget introduced four categories of reflective abstraction, namely interiorisation, coordination, encapsulation and generalisation. Dubinsky (1991) expanded Piaget's ideas on reversibility into a fifth category of reflective abstraction called reversal.

The concept of reflective abstraction introduced by Piaget (Beth & Piaget, 1966, pp. 203–208) claims that learners create knowledge through a process of reflective abstraction consisting of two fundamental components called "assimilation and accommodation". Assimilation is an active process of constructing new cognitive structures and accommodation is an active process of re-defining those structures to align them coherently with existing structures (Dubinsky 1991, pp. 8-10)). In describing Piaget's work, Dubinsky (1991) believes that active construction develops from a base structure into a structure of assimilation, then through a process of transformation and finally the creation of the construction. Active construction is continuously revised through a process of accommodation. Piaget (1985) believed that modern mathematics is the result of reflective abstraction.

Von Glasersfeld (1991) defines reflection as a process of "thinking about" where the mind "stands still for a moment", and then comparing and connecting the presented units (von Glasersfeld, 1991, p. 2). For von Glasersfeld (1991) abstraction is to apply new constructs to images in the mind. This may imply the substituting of certain parts of the image with placeholders or variables, resulting in a new cognitive structure that he refers to as a "concept" (von Glasersfeld, 1991, p. 9). A concept may differ from the original image, but still have specific characteristics. One can imagine different colours of an image, but the specific characteristics will still be that of the original image (von Glasersfeld, 1991).

Where Piaget's concept of reflective abstraction mainly focuses on early childhood, the mathematics community expanded his ideas to include advanced mathematics (Dubinsky 1991). Cappetta and Zollman (2013) studied student performance using reflective abstraction as a methodological tool which they called "agents of change" (Capetta & Zollman, 2013, p. 345) during an investigation using limits, from the college calculus curriculum. For their study, they defined the constructs of reflective abstraction as:

- interiorisation: follow the steps in a procedure, reflect on the procedure and a concept starts to emerge;
- coordination: Two different processes are analysed and then integrated into one coordinated process to analyse and grasp a mathematical concept;
- encapsulation: apply to and construct meaning for a concept and to personify understanding of a concept so that it becomes meaningful to the individual;
- generalisation: extend the encapsulated meaning to a broader spectrum of mathematical problems;

reversal: construct a new mathematical notion by stepping backwards from the initial steps (adapted from Beth & Piaget, 1966; von Glasersfeld, 1991; Capetta & Zollman, 2014).

For my research I built on the definitions above.

Dubinsky (2002) believes that when abstraction and the process of understanding conceptually is defined in terms of being processes of interiorisation, coordination, encapsulation, generalisation and reversal then the visualisations that happen as part of these processes become measurable entities. Dubinsky (2002) refers to the earlier example by Piaget (1966) of the commutativity of addition. For instance, the learner performs several addition operations with the series of numbers $a + b + c$ to grasp a concept (interiorisation). As soon as the learner sees that addition of a different sequence $b + a + c$ yields the same result, coordination happens. For the learner, the process is meaningful (encapsulation), therefore through generalisation the learner is able to extend the acquired knowledge to longer series of numbers or other addition examples. Reversal happens when the learner can reverse the steps of the original process. The visualisations a learner have are measurable when the learner is able to recognise similar problems during assessment and can visualise a solution to the problem (Dubinsky, 2002).

The Spanish mathematician Miquel De Guzman (2002) is of the opinion that one can determine exactly which visual imageries are present while the abstraction process takes place. He labels visualisation types as isomorphic, homoeomorphic, analogical and diagrammatic, as elaborated on below.

Guzman (2002) is convinced that visualisation in mathematics is not one single linear process with defined steps. He rather proposes several different types of visualisations based on the relationship between the mathematical object and its visualisation. Depending on the interpretation the learner has of the problem, different forms of visualisation could be applied. Figure 2.2 illustrates two different visualisations of Pythagoras' theorem. For Guzman there is a correlation between the mathematical situation and the concrete way of representation. Every type of visualisation can be natural, symbolic or even personal for an individual, and sometimes almost impossible to communicate or explain to others.

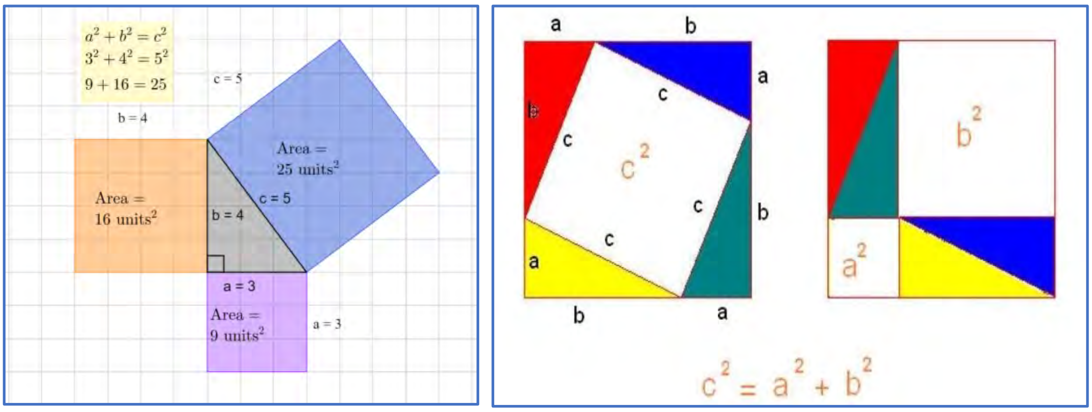



*Figure 2.2: Different forms of visualisations of Pythagoras' theorem.*

(Extracted from different *GeoGebra* representations of the theorem).

Guzman (2002) argues that the classification of visualisation into different types of visualisations is necessary, although he also states that despite the classification, the borders between the different types of visualisations are not always absolutely defined (Guzman, 2002).

*Isomorphic visualisation*: “The objects may have an *exac*t correspondence with the representations we make of them” (Guzman, 2002 p. 5). Kulcsár (2018) explains this as being the visualisation that happens when there is a strong relation between the visualisation and the mathematical problem. Figure 2.3 below illustrates isomorphic visualisation.

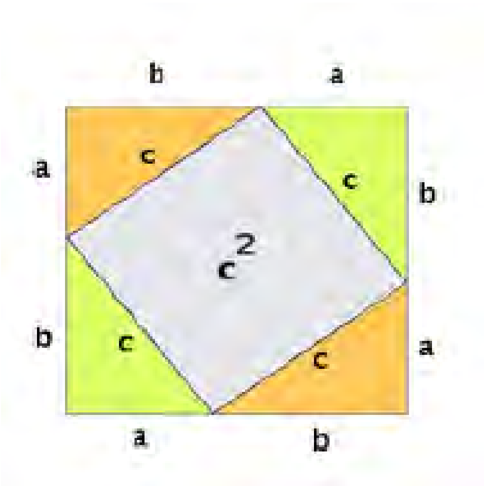



*Figure 2.3: An example of isomorphic visualisation, visually illustrating Pythagoras’ Theorem.*

Source: M.D. Guzman (2002).

According to Kulscsár (2018, p. 56) this is the most ‘common’ type of visualisation. For Kulscsár all graphs of functions, areas under curves and tangent lines are examples of isomorphic visualisations.

*Homoeomorphic visualisation*: “Some of the elements have certain mutual relations that imitate sufficiently well the relationships between the abstract objects to provide us with support, … to guide our imagination in the mathematical processes of conjecturing” Guzman (2002, p. 6). Kulscsár (2018, p. 57) elaborates on this and points out that “two structures are homoeomorphic if some elements of the two structures are the same, but the visualisation has a different form, often with a subjective element and not easy to communicate.” Figure 2.4 below illustrated homeomorphic visualisation.

| LEARNER | PIANO | FLUTE | GUITAR |
|---|---|---|---|
| | | | |
| Anne | X | X | X |
| Betty | X | | X |
| Charl | | X | |
| Damian | X | | |
| Daniela | X | X | |
| Esther | X | | |
| Franck | | | |
| Georgia | X | | X |
| Gisela | | | |
| Ian | | | X |
| Ida | X | X | X |
| James | | | X |
| 12 | 7 | 4 | 7 |

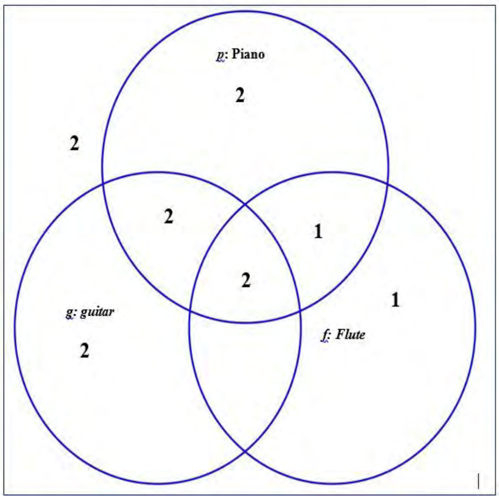



*Figure 2.4: An example of homoeomorphic visualisation, showing a situation where different elements have a mutual relationship.*

*Analogical visualisation*: “When we mentally substitute the object, we are working with, with another that relate between themselves in an analogous way that is perhaps easier to work with.” (Guzman, 2002 p. 7). Archimedes relied on this type of visualisation to postulate his formula to calculate the volume of a sphere (Guzman, 2002). Figure 2.5 below shows how an object is substituted by or replaced with another image that the learner finds easier to grasp conceptually.

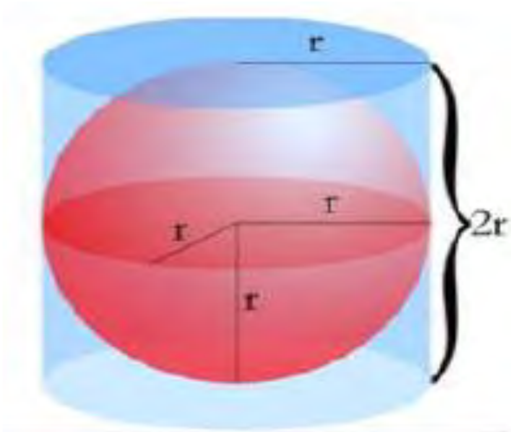
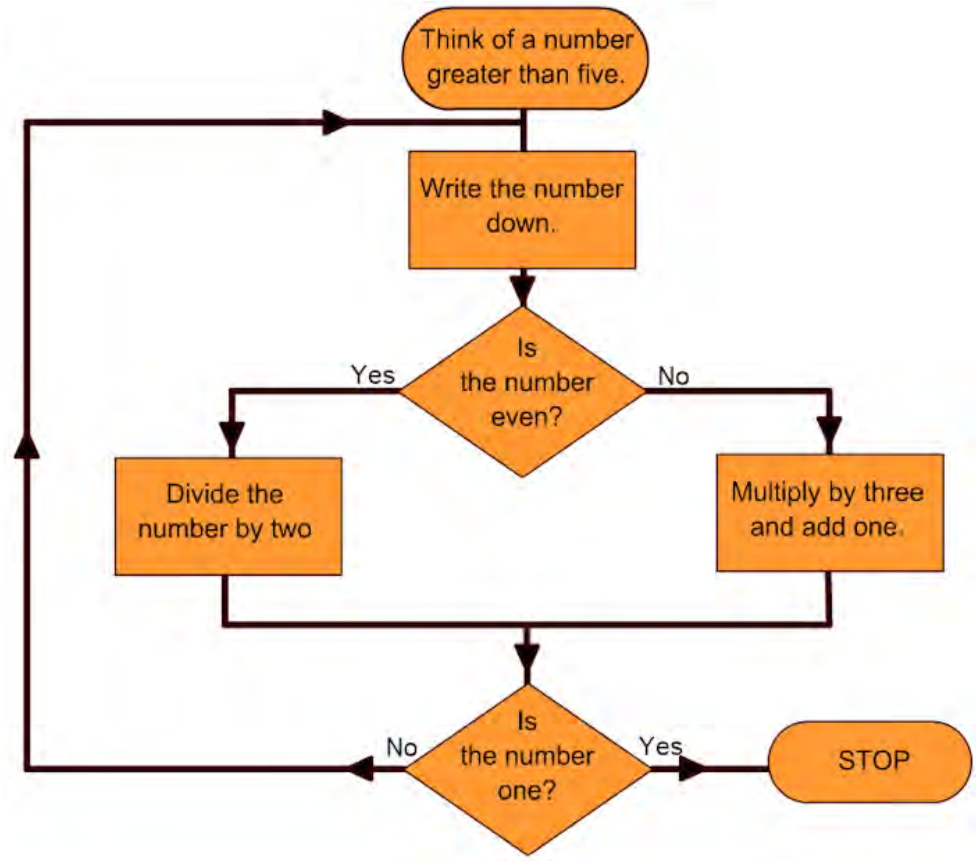



*Figure 2.5: An example of analogical visualisation.*

Extracted from Kulcsár (2018, p. 58)

*Diagrammatic visualisation*: "Our mental objects and their mutual relationships concerning the aspects which are of interest for us are merely represented by diagrams that constitute a useful help in our thinking processes. One could say that in many cases such diagrams are like mnemotechnic rules" (Guzman, 2002, p. 8). Good examples are the use of flowcharts or personalised structures to make concepts understandable for the learner (Kulcsár, 2019). According to Kulcsár and Szakos (2019, p. 674) diagrammatic visualisation could include Venn diagrams of functions and all structures associated with the understanding of algebraic concepts. These structures could include any visual aid applied by the learner to unpack a new concept and enhance long-term memory of the concept.

Figure 2.6 below shows an example of how a function or process is substituted into a diagram.




Downloaded from:
https://www.transum.org/Software/SW/Starter_of_the_day/starter_October13.ASP

*Figure 2.6: Example of diagrammatic visualisation.*

For this research project, I built on these definitions of visualisation to analyse the collected data on the use of visualisation during the learning process of school algebra in selected participants. It formed the basis of identifying suitable topics from the school syllabus to present using the visual abilities of *GeoGebra* computer software.

Guzman (2002) advocates that although a great deal of visualisation can be achieved using our imaginations and resources such as pen, paper and diagrams, the visualisation process can be enhanced using the inherent visual abilities of computer software. Kulscsár and Szakos (2019) state that visual aids help learners to better comprehend, and one should not underestimate its potential as a learning aid.

In defining visualisation, Guzman, (2002, p. 2) also noted that:

> Through the mathematical activity man tries to explore many different structures of reality that are apt to be handled by the process we call mathematization in the following way. Initially we have the perception of certain similarities in the real objects that guide us to the abstraction from these perceptions of what is common and to submit it to a peculiar rational and symbolic elaboration that allow us to efficiently handle the structures which lie behind such perceptions.
>
> Algebra, in a second order abstraction process, explores the structures lying behind numbers and operations related to them. It deals with a sort of symbol of symbol.

For Guzman (2002) visualisation is a natural process and its associated mathematisation is useful to improve understanding and manipulating the common structures of many real situations. Human perceptions are generally visual – therefore we continuously rely on visual processes to complete many of the tasks related to mathematisation, not only in geometry (that is inherently visual), but also in other domains, like algebra, which is often abstract and obscure.

### 2.2.4 Constructing conceptual understanding using visual models

Mayer (2003) developed a cognitive theory of multimedia learning based on a constructivist epistemology of learning. Constructivism implies that learners actively create their own understanding and sense-making of the world, rather than having such understanding delivered to them (Thompson 1995; von Glasersfeld 1995). Learners become active participants in the learning process, rather than merely absorbing all the information presented to them. Thompson (1995) suggests that, because learners already have an existing cognitive structure, they can select and transform the information, construct hypotheses and make decisions. During the cognitive processing of information, learners naturally select information which is easiest to comprehend and manage mentally (Thompson, 1995). Newly constructed information is then stored in the long-term memory as mental imagery or cognitive structures that serve as a framework for one’s knowledge. When required, learners can access these mental mappings to be able to reproduce and communicate these mental images. Instruments to achieve this may include drawings on paper, generating visual models on a computer screen, manipulating visual models with software tools or manipulating a physical visual model (Thompson, 1995).

Mnguni (2014) argues that emanating from the constructivist epistemology of learning and Mayer’s (2003) cognitive theory of multimedia learning, the visualisation process consists of at least three stages or levels. He names them:

(1) internalisation (Mnguni, 2014, p. 3), a low level of processing visual objects;
(2) conceptualisation of visual objects (Mnguni, 2014, p.4), relating to the comprehension of visual information and processing of this information in cognitive structures; and
(3) externalisation (Mnguni, 2014, p. 6) of information as visual models – a process to graphically represent the visual models or to communicate the visual model to others (Mnguni, 2014).

The three levels of visualisation overlap without clear boundaries between the three levels. For Mnguni (2014), visualisation is a process to select and effectively use different cognitive skills for recognising, processing and creating visual models. These deductions form the basis of the proposed theoretical cognitive process of visualisation of algebraic concepts for this study.

Mnguni (2014) aligns his ideas with the constructivist learning model proposed by Mayer (2002) where knowledge is seen by the learner, who then cognitively gives attention to the material, processes the material into visual-mental imagery and finally forms a coherent memory representation for long-term remembering (Mnguni, 2014).

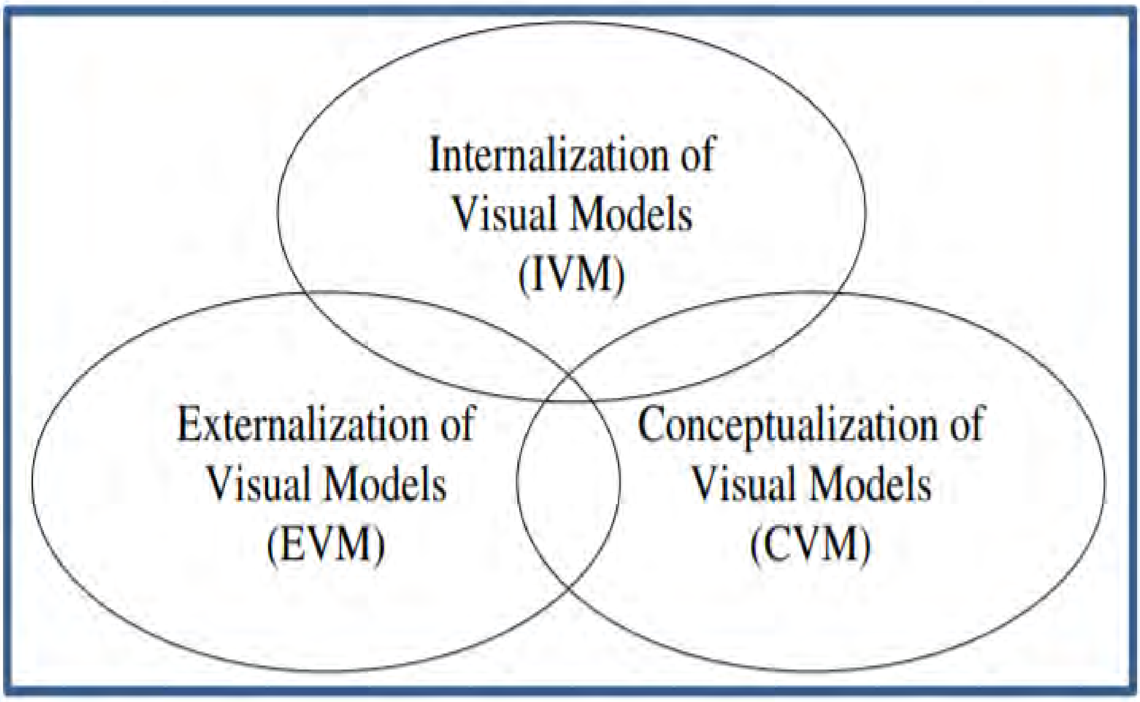



*Figure 2.7: The overlapping stages of the cognitive process of visualisation.*

Source: Mnguni (2014, p. 3)

## 2.3 ALGEBRA AS GATEKEEPER

Muchoko et al. (2018) argued that basic algebra is the foundation of all further courses in algebra in secondary school and beyond. As such, it acts as a gatekeeper for more abstract algebra. The gate needs to be kept open so that learners can develop strategies in solving algebraic expressions, learn to think out of the box and apply algebraic "mathematisation in converting real-life problems into mathematical tasks" (Muchoko et al. 2018, p. 1).

Muchoko et al. (2018) believe this is because learners are accustomed to mechanical manipulations to solve problems that they do not conceptually understand the abstract acuities of algebra. One approach to keep the gate open is to use visualisation strategies in teaching algebra. Visualisation in algebra is then defined as "seeing mathematics in the algebraic form and using algebraic concepts in solving tasks easily and correctly". (Muchoko et al., 2018, p. 2).

Laughbaum (2017) argues that there are many issues in the mathematics education of algebra. He believes that the current approach used in the algebra curriculum is a gatekeeper at two levels. The first level is in high school – as confirmed by the large number of learners progressing to university entrance who must then repeat sections of high school algebra before continuing their studies, because their conceptual understanding framework of algebra is too weak to continue with university mathematics. Algebra is taught and the syllabi align well with those of universities, but it still prevents learners from continuing with mathematics at university level. The second level is apparent by the large number of learners who receive remedial classes in algebra. Learners can achieve excellent results in geometry and other topics of mathematics but keep on failing in algebra. The core problem is not the teachers, but the tools they apply to teach algebra that cause "[a] fragile memory and understanding issues" (Laughbaum, 2017, p. 1).

Laughbaum (2017) is critical of textbooks and many approaches used by teachers to teach algebra. Unfortunately, many teachers argue that understanding algebra comes from repetitive practice through exercises given as homework assignments.

Laughbaum (2017) listed several mainstream approaches that diminish understanding and minimise long-term memory. These practices include:

a) atopic based curriculum, where the prescribed syllabus is broken up into separate chunks, each one covering one aspect of the learning material. This is done without presenting the syllabus as a continuum of mathematical material;
b) the improper use of visualisations. Little effort is made to visually represent the learning material in a manner that allows the learner to form their own cognitive structures;
c) a non-student-centered approach. The teacher remains the main role player in the presenting of the mathematical learning material;
d) a failure to address non-interested learners and non-participating learners are simply left behind; and
e) a reliance on verbal explanations. When a teacher-centered approach is followed, learners merely absorb the learning material, without forming any visual pictures of the concepts presented.

Laughbaum and Crocker (2004) propose a function-based approach in algebra versus the topic based approach. A function-based approach places the learning material in relation to the world the learners are living in, usually with the aid of information technology. By relating the learning material to the reality of the learner, especially with the incorporation of technology, Laughbuam believes that permanent connections are then be made as "neural associations" in the minds of learners (Laughbaum, 2018, p. 3). These visual images enhance long-term algebraic knowledge. A topic based approach, where the algebra syllabus is presented as different sections is merely practising of the underlying concepts. "Real understanding" can come only after students have practised *and* mastered the concept (Delvin, 2010, p. 174). For Delvin (2010). understanding means building a pattern and making connections through appropriately used visualisations, based on previously learnt algebra (Delvin, 2010). Laughbaum (2017) supports the idea of having connecting themes when presenting algebra.

Laughbaum (2017) suggests that visualisations should form the introductory part of any new mathematical concept, before lecturing the concept to learners. He suggests that visualisations should be done in a way that involve the learners. Laughbaum (2017) is of the opinion that many teachers only use visualisations to confirm an algebraic concept already taught, instead of using visualisations as introductions to concepts where so that they have more impact on long-term memory and understanding.

Dehaene et al. (2014) argue that learner understanding of algebra comes from repetition through exercises assigned as homework. The argument is that through repetition the neuronal circuits used to solve problems become more myelinated, meaning the forming of permanent tracks in the mind of the learner with repetitive practice of a concept. This process can happen unconsciously to create paths in the mind of a learner and can be recalled when problems must be solved. Advancements in technology to register and measure brain rhythms is used as a foundation for this argument. For Laughbaum (2017) this is a non-student oriented approach dating back to the forgetting curve that states that long-term memory is enhanced by repetition. For Laughbaum, long-term retention in algebra is based on cognitive understanding, dynamic visualisations, interleaving, dynamic engagement and the use of abstract symbols. (Laughbaum, 2017, p. 11).

Laughbaum's (2017) conviction is that an approach based on repetition is a major reason for learners losing interest in algebra. There are many classroom situations where learners pay attention without being engaged with the algebra. "Engagement has little to do with learning algebra unless it is the focus of the engagement" (Laughbaum, 2017, p. 5).

In the 1990's Demana and Waits introduced graphing calculators, which enabled one to draw on-screen graphs and apply visualisation as a method of teaching algebraic functions and calculus to learners. They stated that "… every classroom could be turned into a computer lab, and every student could own his or her own inexpensive personal computer with built-in mathematics software" (Demana & Waits 1992, p. 3). After their initial success with visualisation, Demana and Waits (1992) encouraged teachers to apply visualisation strategies with the use of technology in their classrooms. Demana and Waits (2000) were convinced that the inherent visual properties of computer software would improve the learning of mathematics. They envisaged that the rapid development of computer technology would drive a constant need for research about technology and its ability to improve mathematics teaching.

Several authors such as Guzman (2002), Arcavi (2003), and Presmeg (2006), agree that visualisation addresses the issue of conceptual understanding of algebra and long-term retention of skills. The above shows a wider consensus that more research about the epistemology of visualisation as a teaching instrument is needed, especially with the rapid development of technology with its inherent visual impact. More research needs to be done in the area of mathematical content (ie algebra) as well as on the visual impact of computer technology (Presmeg, 2014, p. 31).

This is underlined by Roschelle et al. (1998) when they state that current technology research does not result in a visually integrated mathematics classroom, but rather produces different fragmented collections of incompatible software applications (Roschelle et al., 1998).

## 2.4 CONCEPTUAL UNDERSTANDING

According to Kilpatrick et al. (2001) “Conceptual understanding refers to an integrated and functional grasp of mathematical ideas” (Kilpatrick et al., 2001, p. 118). In the new Namibian secondary school syllabus these ideas are separated into different proficiencies and skills that a learner should acquire to progress towards another level (MoEAC, NIED., 2015). This is in line with Kilpatrick et al.’s claim that mathematics relies on many concepts or ideas that are mostly abstract and interwoven with each other. To solve a typical mathematics problem a learner can either apply the steps of an algorithm memorised by the learner, or identify the underlying mathematical concept embedded in the problem and then solve the problem with understanding and insight. By grasping the concepts, learners develop new ideas of their own by connecting their ideas to what they already know. Kilpatrick et al. (2001) found that understanding conceptually improves a learner’s retention. Kilpatrick et al. (2001) stated that once a concept is grasped and applied it becomes part of a learner’s *knowledge base* (Kilpatrick et al., 2001) but when a learner only memorises a rigid method to apply to a specific problem, the method is easily forgotten. This means that conceptual understanding implies learning which enables the learner to make sense of the problem and apply his/her knowledge effectively and appropriately to other knowledge the learner already has. The learning can make new connections and relate this knowledge to other topics previously learnt. This is also in line with von Glasersfeld's (1995) idea of radical constructivism where a learner can construct knowledge when they understand. When a learner has conceptual understanding, the mathematical concepts are assimilated and integrated so that the learner can use the knowledge to reason adaptively and strategically (Kilpatrick et al., 2001).

Another argument for teaching for conceptual understanding is highlighted by Rittle-Johnson et al. (1998, p.181) who posit that "conceptual understanding frequently results in students having less [procedures] to learn because they can see the deeper similarities between superficially unrelated situations". Mathematics is based on logical rules, and conceptual understanding enables a learner to apply them in a universal and connected sense across the mathematical spectrum. Then the subject becomes an adventure and learners might develop a more positive disposition towards learning the subject matter.

To teach conceptually, teachers need to have conceptual understanding of the subject matter themselves, and be flexible in creating suitable learning strategies, relating mathematical concepts to one another and being aware of and rectifying possible misconceptions. Teachers need to visualise how mathematical ideas connect across fields and search for connections with real-life situations. For Shulman (1998) this is the foundation for pedagogical content knowledge which allows teachers to make conceptual ideas accessible to others (Shulman, 1987). Adding to these ideas, Sullivan (2011) advises that teachers should use research-informed strategies to teach mathematics better (Sullivan, 2011). This, I argue, includes using technology and visualisation appropriately and effectively.

The Kilpatrick et al. (2001) framework of mathematics teaching proficiency (MTP) builds on Shulman's (1986) pedagogical model of teaching competence. This model is useful, because it was based on the notion of mathematical proficiency – a theoretical concept that can be observed and measured. Kilpatrick et al.'s (2001, p. 380) conceptual and analytical framework comprises five interwoven and interdependent strands of MTP. These are:

- conceptual understanding (CU) of core knowledge that encourages comprehension of concepts, operations and relations as required in the practice of teaching;
- procedural fluency (PF) in carrying out basic instructional routines;
- strategic competence (SC) in planning effective instruction and solving problems that arise during instruction;
- adaptive reasoning (AR) in justifying and explaining one's instructional practices and in reflecting on those practices to improve them; and
- a productive disposition (PD) towards mathematics, the teaching, the learning, and the improvement of practice.

(Extracted from Kilpatrick et al. 2001, p. 380)

Although the five strands are intertwined and interconnected, it is the first and last strands of the Kilpatrick's framework of teaching for MTP particularly, that guide this study and my thinking, regarding conceptual understanding of, and a positive disposition towards learning algebra. I argue that the strands of conceptual understanding and productive disposition are inextricably linked as the one cannot exist without the other. A productive disposition with its associated strengthening of confidence and attitude leads to the strengthening of conceptual understanding and vice-versa (Kilpatrick et al., 2001). It implies a symbiotic relationship between conceptual understanding and productive disposition – often ignored by teachers and therefore the focus of this study.

## 2.5 PRODUCTIVE DISPOSITION

Kilpatrick et al. (2001) characterised a productive disposition by stating that a "*productive disposition* refers to a habitual inclination to see mathematics as sensible, useful and worthwhile, coupled with a belief in diligence and one's own efficacy" (Kilpatrick et al., 2001, p. 116). The learner and the teacher recognise opportunities to apply mathematics in real-life situations enthusiastically.

Kilpatrick et al. (2001) also note that developing a productive disposition requires that learners should often be reminded that mathematics make sense, has benefits and that perseverance has rewards. They argue that a productive disposition develops as a result of the development of the other strands. Learners must believe that mathematics is “understandable” and not merely “arbitrary” figures and numbers (Kilpatrick et al., 2001, p. 131).

Regarding a productive disposition, Carr and Claxton (2002) focused on three important learning dispositions, namely playfulness, resilience, and reciprocity. These are defined as:

- Resilience: The inclination to take on (at least some) learning challenges where the outcome is uncertain, to persist with learning despite temporary confusion or frustration and to recover from setbacks or failures and rededicate oneself to the learning task. (Carr & Claxton, 2002, p. 14)
- Playfulness: To be ready, willing and able to perceive or construct variations on learning situations and thus to be more creative in interpreting and reacting to problems. In this context three different types of playfulness are identified. These are mindfulness, imagination, and experimentation. (Carr & Claxton, 2002, p. 14)
- Reciprocity: The most valuable learning resources are other people. Those who lack the awareness to articulate their own learning processes and problems, the ability to communicate these to others or the inclination or the courage to do so, are inevitably handicapped as learners. (Carr & Claxton, 2002, p. 15)

Even before the study by the National Research Council headed by Kilpatrick et al. (2001) different researchers recognised that learners are often guided by negative perspectives about mathematics. McLeod (1992) acknowledged that many learners believe that mathematics is difficult consisting purely of the manipulation of rules and having little to do with real-world applications. Frank (1988) found that learners regarded mathematics as being only computations, that all mathematical knowledge should be transmitted from the teacher, that any mathematical problem should be solved within a few steps and that the answer is all that is required from any mathematical problem.

Responding to Kilpatrick et al. ’s (2001) productive disposition strands, Groves (2012) suggests that the above key learning dispositions are often absent in real classroom situations. From early primary years many learners have negative mathematical experiences which are expressed as helplessness and hopelessness in accessing mathematics (Graven & Buytenhuys, 2011). According to Groves (2012) this however provides the researcher with excellent opportunities for repairing negative relationships with mathematics and to contribute towards a shifting disposition. This research gap resonates strongly with my proposed research project. Critical indicators, as drawn from the Kilpatrick et al. (2001) definition of a productive disposition and Carr and Claxton's (2002) three key learning dispositions, thus includes increasing:

(1) improved sense-making;

(2) constant effort,

(3) self-efficacy,

(4) daring experimentation/playfulness,

(5) resilience,

(6) reciprocity (willingness to engage), and finally love and passion for mathematics. In my analytical framework I will rely on the above-mentioned dispositions to analyse and identify a change in disposition towards learning algebra.

During a study on dispositions of primary teachers towards mathematics learning in primary learners Siegfried (2012) identified some constructs embedded in the concept of a positive disposition. My research will be guided by these definitions:

- affect: a person's feelings and attitudes that shape the way one looks at the world;
- beliefs: psychological understandings about how one perceives the world;
- goals: the status that human beings desire to obtain;
- identity: qualities people recognise in themselves or that are recognised by others (a person's type or kind);
- mathematical Integrity: knowing what one knows, knowing what one does not know, and being honest about these assessments;
- motivation: the inclination people have to do certain things and avoid doing others;
- risk taking: willingness to ask questions or share ideas that may expose one's misconceptions or weaknesses; and

- self-efficacy: a person's own belief in his or her ability to act on a particular problematic situation.

## 2.6 LEARNING ALGEBRA WITH CONCEPTUAL UNDERSTANDING

### 2.6.1 Conceptual understanding of algebra with reference to the Dual Coding Theory

My focus is on visualisation as a learning tool. A clear understanding and definition of visualisation is thus important. Paivio (1971) attempted to define visualisation as part of his Dual Coding Theory. Paivio (1971) proposed the Dual Coding Theory (DCT) of memory to explain the powerful mnemonic effects of imagery that he and others were interrogating. The DCT is firmly grounded on the use of mental images (called imagery) to support memory (Paivio & Clark, 2006). Paivio emphasises that the DCT developmental theory relies on the early development of nonverbal systems for the later development of cognitive skills. In general, cognitive growth relies on nonverbal experiences and images. Clark and Campbell (1991) departed from dual coding mechanisms to develop a coding theory for number processing by learners. To them number concepts are concrete and associations and imagery build on that principle when numerical operations are executed. John Paivio (2006) refers to a programme that involved systematic concretisation and visual representations of mathematical concepts and operations, many of which are familiar to teachers (eg pie charts and box diagrams). Paivio (2006), concluded that many learners classified as slow learners excelled in mathematics following a DCT approach during an intervention.

The DCT is regarded as "one of the most influential theories of cognition of the 20th century" (Sadoski & Paivio, 2001, p. 433). For them verbally conveyed problems and mathematical symbols are arranged spatially to create 'number forms' visually in the mind of the learner. The same happens when a written problem is presented to a learner of mathematics. I intend to build on similar ideas of a spatial and visual approach to learning mathematics – specifically algebra with its abstract characteristics – with a special focus on the use of Information and Communications Technology (ICT) applications with *GeoGebra*.

Ramlatchan (2019) refers to the expansion of the DCT to a multimedia learning theory, which in this age of technology, colour, sound, movement and screen imagery can contribute simultaneously towards the conceptual understanding of mathematics.

Instructional design should incorporate forms of multimedia to limit the reliance on the cognitive resources of the learner as suggested by the DCT. The dynamic use of text with additional multimedia resources might decrease the dependency learners have on their own cognitive skills in order to code the learning material for themselves.

My research project investigated the use of *GeoGebra* as a form of multimedia instruction to visually teach abstract algebraic concepts to learners. Carefully selected topics from the Namibian algebra syllabus were taught through *GeoGebra* lessons. Attention was given to diverse types of visualisations as identified by Guzman (2002). For each selected topic, the analytical framework made provision to identify the presence of any of these types of visualisations.

My research further tried to identify how learners *process* each type of visualisation using the constructivist epistemology of learning and Mayer's (2003) cognitive theory of multimedia learning as points of departure. During the visualisation process, I identified the three stages, namely the comprehension of visual information, the processing of this information in cognitive structures and the externalisation of learning (Mnguni, 2014, pp. 3-6).

My study argued that the process of visualisation using technology relies on the strongly inherent visual and intriguing capacity of that technology. Costley (2014) asserts that when they are allowed to engage with their own learning processes, learners retain more information and mathematical concepts than if they simply listen to the teacher. Costley (2014, p. 3) argues that:

> … the rapid arrival of new technologies globally, makes it relevant to the students. Technology provides meaningful learning experiences. Technology also provides hands-on learning opportunities that can be integrated into all school curricular areas, including mathematics.

The process of visualisation in teaching mathematics according to Presmeg (1986) is "the extent to which that person prefers to use visual methods when attempting mathematical problems which may be solved by both visual and nonvisual methods." (Presmeg, 1986, p. 42). This is corroborated by Arcavi (2003, p. 56) who stated that

> Visualization is the ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understandings. (p. 59)

In this research I interrogated the use of visualisation processes in conjunction with *GeoGebra* to advance the conceptual understanding of algebraic concepts, which arguably then leads to a positive disposition towards learning algebra.

Mayer (2018) is of the opinion that fundamental research should be done into the inclusion of different forms of multimedia during instructional design. The current trends with tablets and e-learning in schools calls for a thorough investigation into the effectiveness of the expansion of the DCT to a multimedia theory of learning to ensure that conceptual understanding happens without creating a negative disposition amongst learners. (Mayer, 2018). For Mayer (2018) it is fundamental that any change in instruction should be professionally researched.

### 2.6.2 Constructivism and visualisation when learning algebra

The study also rests on the theoretical principles of social constructivism (Piaget, 1966) and radical constructivism as defined by von Glasersfeld (1995). Traditional teaching methods often over-emphasise the execution of algorithms and the procedural execution of the symbolic aspects of mathematical concepts. Learners thus construct only partial mathematical knowledge consisting mainly of algorithms that makes them manipulate symbols routinely, without giving significance to the basic concepts of mathematics. Duval (1999) pointed out that there exists an inconsistency between the different semiotic registers that prevents learners to conceptualise and comprehend the underlying concepts. Caligaris et al. (2014), suggest that visualisation could facilitate and enhance mathematical conceptualisation, particularly when using technology in mathematics (in this case, algebra). They define visualisation according to the dictionary of the Spanish Real Academy as “the action to make in the mind a visual image of an abstract concept” (Caligaris et al., 2014).

Malabar and Pountney (2002), stated that learners find computer environments or intriguing and stimulating. The computer environment keeps learners involved for longer in their own learning and conceptualisation process (Malabar & Pountney, 2002). Computers provide a means to highlight the different semiotic registers of a concept and allow learners to visually experience mathematics (Rodríguez et al., 2014).

Robinson et al., (2008) explain that teachers supporting a constructivist approach “facilitate learning and place[s] the emphasis on the learners and their interests and abilities (or disabilities)" (Robinson et al., 2008, p. 17). The teacher supports the learners by identifying their learning problems and working towards developing their internal mental processes. Constructivism views teachers and learners as partners in learning, resulting in the joint motivation of both teacher and learner. (Robinson et al., 2008). Driscoll (2005) highlighted "ownership of learning and self-awareness (reflection)". A key element of constructivism is to link new knowledge to existing knowledge and as a result enhance retention and communication of acquired knowledge (Robinson et al., 2008).

Constructivism and visualisation are well linked when Molenda and Robinson (2008) propose that

> educational technology focusses on the processes of creating instructional materials and learning environments and on relations with learners during the use of those materials and environments. (p. 245)

A constructivist teaching approach emphasises that each learner is unique and we "must acknowledge the diversity of the individual" (Robinson et al., 2008, p. 42). Modern computing and social software offer new learning opportunities, allowing for individual needs and varied abilities. Atwell (2007) believes the computer keeps the learner at the centre. In constructivist learning, students construct their own knowledge and teaching strategies should be designed around this. Previous knowledge and expected knowledge can easily be linked. While the learner is engaged with the computer the process of constructing knowledge is allowed to take its own course. For Atwell (2008), both the learner and the teacher are empowered when technology is integrated into the teaching and learning process. Teachers become facilitators of constructivism, while learners are actively thinking knowledge constructors.

Amarin and Ghishan (2013) believe that learning with technology addresses each learner's individual demand to grasp underlying concepts. It also aligns well with the constructivist philosophy of learning that instruction should adjust to keep up with the technological era.

## 2.7 A PRODUCTIVE DISPOSITION TOWARDS LEARNING ALGEBRA

As with MacGregor (2004), learners' attitudes to learning algebra are central to my research because early failure in algebra is likely to prevent learners from further study in this area, or even result in a dislike of learning mathematics. Proficiency in algebra yields positive results for the learning of other mathematical domains (MacGregor, 2004), therefore the development of positive attitudes or productive dispositions to algebra is essential. I argue that a productive disposition also increases learner enrolments in advanced mathematics subjects.

Khoon & Ainley (2005) remark that learners come to the learning process with their own perceptions and expectations and value their own contributions towards learning tasks. If allowed to, they keep on participating and contributing towards the learning process. According to Norton and Irwin (2007), learners lose interest in algebra and further studies for two reasons:

1. algebra is perceived as uninteresting and based only on symbolic manipulation with limited meaning and little relevance to everyday life; and
2. algebra is perceived as difficult.

Norton & Irvin (2007) argue that over-reliance on textbooks, an inevitable bias towards a procedural approach, too few challenges and a teacher-oriented approach prevent learners from developing a productive disposition towards learning mathematics (Norton & Irvin, 2007).

## 2.8 RELATIONSHIP BETWEEN CONCEPTUAL UNDERSTANDING AND DISPOSITION

Gabriel et al. (2018) state that cognitive conceptual understanding of mathematical matter and the psychological disposition towards mathematics are not directly related. Their own research affirmed the complex relationship between conceptual understanding and dispositions in mathematical proficiency. Demographics, self-efficacy, self-concept, mathematics anxiety and socioeconomic status were found to be contributing factors towards a negative disposition in learning mathematics. To be good at mathematics does not mean a learner will have a positive disposition towards learning mathematics. Although motivation contributes towards changes in disposition, it does it not seem to be enough to allow learners to embark on a life-long process of learning and doing mathematics (Gabriel et al. 2018).

Beyers (2011), regards disposition to be a cognitive function where a learner takes a cognitive decision to engage with the mental processes associated with doing and understanding mathematics. In doing so, the learner is able to recognise structures, reason, be judgemental and apply mental functions. All these actions contribute towards a conceptual grasp of the learning material.

Kusmaryono et al. (2019) regard disposition as a powerful mental function comprised of cognitive, affective and conative abilities. Teachers are challenged to transform dispositions from negative ones where learners see mathematics as a hated subject to a productive disposition where learners develop a trust in their mathematical abilities, discover their mathematical power and begin to believe in their own mathematical abilities. Learners start to attempt to solve problems and gain better conceptual understanding of a concept. What happens in the mind of a learner can be realised as actions with a positive outcome. When learners start to act, a future vision develops, giving insight into possible career opportunities. Disposition and better conceptual understanding are closely linked: the challenge is for the teacher to take advantage of every teaching opportunity (Kusmaryono et al, 2019).

## 2.9 CONCEPTUAL UNDERSTANDING WITH VISUALISATION AS A BRIDGE TO A PRODUCTIVE DISPOSITION

Waits & Demana (1998), after using calculators with graphing capabilities, were convinced that understanding a scientific or mathematical concept means a visualisation of the concept. They firmly believed in the ‘power of visualisations’ and advocated that those dynamic visualisations should be integral to any mathematics curriculum, because visualisation promotes understanding.

Cognitive scientist Pinker (1997) looks at it the same way, concluding that “Thanks to graphs, we primates grasp mathematics with our eyes and our mind’s eye. … vision was co-opted for mathematical thinking, which helps us see [understand] the world” (Pinker, 1997, p. 360). Pinker believes that mathematics is processed primarily by the visual system.

Laughbaum (2017) also recognised the value of visualisations but cautioned that it should not be used to confirm an algebraic concept already taught. Laughbaum (2017) states that visualisation used to introduce a topic will have more impact on memory than when used to underline an already taught concept. Further, not using any visualisation may impact negatively on learners understanding of an algebraic concept or skill.

Laughbaum (2017) further claims that the ability to visualise algebraic relations is not just incidental; it is the true understanding of algebra. Numbers, for example, are not just abstract values; they have ordered relations and spatial relations and can be placed precisely on a number line. Their position on the number line is visually determinable where we can see the relative positions of numbers. Operations on numbers such as addition, subtraction, division and multiplication are therefore spatial operations on spatial structures.

According to Laughbaum (2017) the normal human brain finds abstract symbols to be of low value, especially the brains of learners who have no interest in algebra. This indicates that an algebra lesson should not start with symbolic work. Laughbaum (2017) argues that the proper use of visualisations adds attention, especially when using technology. Laughbaum (2017) further suggests that visualisations should be placed first and they should be dynamic – as found on a graphing calculator. Paivio (1971) demonstrated that learners remember concrete words more easily than abstract words. Laughbaum (2017) believes that when learning algebra, learners have the option of visualising abstract symbols (e.g. +, −, Σ, $\sqrt{a^2 + b^2}$). Moving from real-world examples to abstract symbols gives the abstract symbols a concrete meaning, which further assists understanding (e.g. when 2 ml is diluted with 10 cc water it becomes a visual algebraic equation: 2 + 10 = 12).

Laughbaum (2017) refers to static and dynamic visualisations, where static visualisations refer to images like tables or mnemonics in mathematics, while dynamic visualisations have to do with movement, usually with the aid of computer software. *GeoGebra* provides excellent examples of dynamic visualisation. When the function $y = 2x^2 + 3x + 1$ is typed in, the learner can immediately visually see the drawing of the graph on the screen. Visualisations that are *connected* to the algebra being taught helps learners to remember for longer, especially when applied in the early moments of memory creation. This is supported by Carrol (2016) who states that visual images allow the brain to create permanent neural synapses at the beginning of memory processing. Human brains remember images far better, as described by Lynch et al.

> Professor Leo Standing, a Canadian psychologist, [asked psychology students to view 100 pictures for 5 seconds each] … He brought them back in a week, and showed them the pictures again, mixed with 100 new pictures, … The students correctly recognised more than 90% of the pictures, having seen them

> only once, for just five seconds The same 90% results were returned for 1,000 pictures and then for 10 000 pictures. (Lynch et al., 2008, p. 158)

For Lauchbaum (2003) visualisations are significant for the memory process as well as for understanding the abstractness of algebra. He believes static visualisations (geometric figures) are easily remembered, while dynamic visualisations (graphs of functions) increase understanding of algebra. For Laughbaum (2003) it appears that the normal brain takes notice of visualisations in a lesson. Often *only* the visual parts are remembered. Learners often select specific sections to remember.

Lauchbaum (2003) suggests three moments for the use of visualisations during any algebra lesson:

- at the beginning of the lesson: resulting in attention, understanding, and memory;
- at the end of the lesson: resulting in revised understanding; and
- airing the lesson: to assist learners struggling to pay attention with a potential loss of understanding.

Lauchbaum (2003) emphasised that it is important to apply visualisation as recommended by him to enhance conceptual understanding. Tammet (2009) supports the idea and states that

> We rely heavily on our eyes to provide much of the information we obtain about the world around us, and it is for this reason that a significant portion of the human brain is devoted entirely to visual processing. (Tammet, 2009, p. 2).

The new Namibian syllabus (MoEAC, 2016) prescribes a new and different approach towards the teaching of mathematics. It requires that learners conceptualise different algebraic concepts and a productive disposition should be an inherent part of the outcome. To capitalise on the ‘new approach’ visualisation can form an inherent part of teaching mathematics. This aligns very well with the strands of learning mathematics as proposed by Kilpatrick et al. (2001).

Groves (2012) confirms that for many learners, mathematics has no connection with their lives outside school. They do not see any applications of mathematics around them. Often, they ask, *“Where will I ever use this mathematics?”* This is more a plea of being ‘lost’ in a problem, rather than an academic question. Schoenfeld (2007, p. 69) discussed the results of the well-known bus problem presented to 45 000 students:

| |
|---|
| *An army bus holds 36 soldiers. If 1128 soldiers are being bussed to their training site, how many buses are needed?* |

Schoenfeld found that many did not care to do the correct calculation and others could not translate the fraction of a person into the correct whole number. He attributes this to the belief that mathematics is meaningless, disconnected from practical situations and only of academic importance to learners.

Wilson (1993) suggests that learners with a productive disposition are those who believe mathematical activities will have benefits for themselves and therefore they attempt it with more confidence. A productive disposition changes attitudes from *“I cannot do it,” to “I think I can solve this problem. Let me see how…”*

There is agreement amongst researchers that visualisation is an effective tool in enriching the teaching of mathematics (Arcavi, 2003; Presmeg 2020). Nevertheless there are many questions about visualisation as an epistemological tool in mathematics education. Arcavi (2003) rightfully asks whether visualisation is about seeing the unseen or about foreseeing the unseen, and do all people visualise in the same way (do you see what I see?). In the conceptual understanding of mathematics, especially when visualisation is used to create knowledge, it is important that both the teacher and learner have the same visual imagery (Arcavi, 2003) of the concept being taught and learnt. Technology and associated mathematical applets might contribute to address this inconsistency and uncertainty about visual imagary by creating clear unambiguous visual pictures or concepts of abstract concepts (Arcavi, 2003)

According to Alsina & Nelson (2006, p. 121) technology with its interactive nature can provide excellent opportunities for the construction of knowledge of mathematical concepts and improve visual thinking. They are convinced that visualisation improves problem solving skills in mathematics and in general.

Rösken and Rolka (2006) regard visualisation as a powerful tool in providing meaning for mathematical concepts, reducing complexity and enabling the handling of large volumes of information. The question that my research study seeks to answer is how visualisation as a learning tool can enhance the development of mathematical knowledge, particularly the learning and understanding of the more abstract concepts in algebra. The study takes cognisance of three broader questions, namely:

(1) Are there kinds, qualities and/or hierarchies of visualisation and visual skills?
(2) How do learners of varying abilities employ visualisation in learning mathematics with the aid of technology?
(3) What visualisation strategies do learners employ to enable them to construct conceptual content?

My research question thus aligns neatly with these. It asks: How *can the use of ICT and its inherent visualisation opportunities be harnessed to potentially enhance conceptual understanding and a productive disposition in the learning of algebra?*

### 2.10 MY RESEARCH WITHIN THE NAMIBIAN CONTEXT

The Namibian government embarked on vigorous and urgent educational reform processes soon after independence in 1990 (Ministry of Education and Culture, 1993). All children gained access to schools and mathematics and science were made compulsory from Grade 1 to 10 (Tjikuua, 2000). Learner-centred teaching was foregrounded as a new approach (MoEAC, 2006).

Learner-centred education (LCE) is regarded as democratic, where the learner can participate as an equal partner in the teaching process. The needs of different abilities and the uniqueness and socio-background of each learner are taken as points of departure (MoEAC, 1999).

- the new syllabus suggests that teachers who follow a learner-centred approach will: use technology and teaching aids to enhance learner activities;make the learning experience relevant;
- start with the concrete and move to the abstract;
- teach general concepts and not unrelated facts. Allow learners to understand; and

- allow peer teaching. (Syllabus policy MoEAC, 2015).

Every year the Ministry of Education undertakes a complete re-assessment of the available resources at schools. The 2019 statistics painted a dim picture with many schools in Namibia still having limited resources, a lack of textbooks and big class groups, up to 54 learners per teacher. Many teachers are enthusiastic but underqualified, particularly in mathematics and sciences (MoEAC, 2019). Currently the fallout rate of learners is high with less than 10% of learners who start Grade 1 reaching Grade 12. Despite the challenges there are also many opportunities to engage learners with e-learning, using the many facilities put in place by the Ministry of Education.

According to the 2019 education statistics there are reportedly 517 computer laboratories available for learners in junior secondary and senior secondary schools. This means that about 86% of junior secondary and secondary schools have computer laboratories (MoEAC, 2019). High-speed internet access is available to 98% of all schools (Mobile Telecomunications Corporation, 2019) All this sets the scene for opportunities to use the plethora of freely available ICT resources fruitfully and meaningfully in Namibian classrooms. One such resource is the globally acclaimed *GeoGebra* software (http//www.*GeoGebra*.org). *GeoGebra* is freely available to all schools and individuals the world over. It offers readymade applications covering the Namibian mathematics syllabus. The *GeoGebra* platform also makes provision for the development and uploading of topic specific applets from www.*GeoGebra*.com, developed by any user of the software.

Since 2017 the new Namibian mathematics syllabus has been progressively phased in from Grade 8 through to Grade 12. As mathematics is compulsory, and in order to accommodate all learners, the subject is taught on one level only. The curriculum states that:

> In the mathematical area of learning, learners understand and master a variety of mathematical skills, knowledge, concepts, and processes, in order to investigate and interpret numerical and spatial relationships and patterns that exist in the world. Mathematics helps learners to develop accuracy as well as logical and analytical thinking." (MoE, NIED, 2015, p. 1).

The syllabus is topic based, with specific competencies a learner needs to acquire. Although all the classical topics (algebra, geometry etc.) are present, teachers should relate each topic to Namibian society in a practical manner. Emphasis is placed on learner-centred teaching, which I argue could be read as 'conceptual teaching' (MoE, 2015). This aligns well with the theoretical concepts of teaching for conceptual understanding and changing the dispositions of learners towards learning mathematics (Kilpatrick et al., 2001). The objective with the syllabus is to enhance conceptual understanding and attitudes to provide society with skilled citizens. The teacher remains key in choosing methods in relation to the learning objectives and competencies to be achieved. (MoE, NIED, 2015, pp. 4 – 5). This allows every teacher to use technology in the classroom and to capitalise on its inherent visual advantage when using associated applets in the classroom.

## 2.11 AN INTERVENTION WITH *GeoGebra*

*GeoGebra* is an educational technology tool used in mathematics instruction. This software was developed by Markus Hohenwarter at the University of Salzburg in 2001 as part of his Masters thesis. It is a Dynamic Mathematics Software (DMS) for visual mathematics instruction. It provides interfaces that include algebra, calculus and geometry. *GeoGebra* may be freely downloaded and implemented for educational purposes since it is open-source software. Both teachers and learners can download it from www.geogebra.org (Majerek, 2014). Hohenwarter (2002) refers to it as a dynamic platform for visual interaction in teaching and learning mathematics (Hohenwarter & Fuchs, 2004). *GeoGebra* motivates learners to approach mathematics from an experimental perspective (Hohenwarter & Fuchs, 2004).

Kutluca (2013) is of the opinion that *GeoGebra* instruction improves van Hiele geometry thinking levels more than classical class teaching. Kutluca (2013) attributes it to the learners being participants in a constructivist way to create knowledge for themselves in an interactive investigative environment (Kutluca, 2013).

Bhagat & Chang (2014) described a study among Indian learners and found significant improvement in the achievement scores of learners taught with *GeoGebra* and that of learners taught through class instruction. Both cognitive and visualisation skills of the study group improved significantly. Bhagat & Chang (2014) did their study on learning geometry with *GeoGebra* and pointed out that the reasoning, visualisation skills and cognitive understanding of mathematical concepts improved.

## 2.12 MY NAMIBIAN STUDY

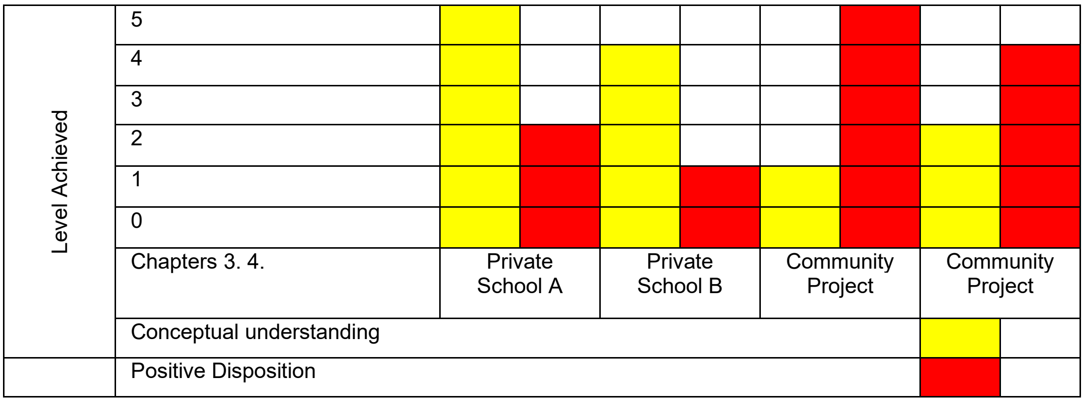
I teach at a very well-resourced private school where all learners have access to well-equipped laboratories and are supplied with their own tablet computers. Programmes were designed and training provided to use and integrate technology for teaching. As part of our community outreach programme, we support learners in the greater Windhoek area who do not have access to these resources, who are interested in technology and mathematics to improve their skills and who wish to work in well-equipped laboratories.

Integral to this programme is a community laboratory where parents can bring their children to improve their mathematical skills in the laboratory. This community laboratory is in one of the suburbs of Windhoek and I coordinate and manage this facility. Mathematics classes based on the *GeoGebra* applications are offered in this laboratory. The entire Namibian mathematics syllabus is covered. The programme is constantly monitored by the senior head of the mathematics department of the school. When *GeoGebra* applets for a specific topic or task are not available, I develop them myself using the *GeoGebra* script. This laboratory forms the empirical field of my study. I am particularly interested in the learning implications of the applets we use, about their inherent visualisation capacities to promote conceptual understanding and their influence on a positive disposition towards learning algebra.

As participation in the laboratory is voluntary, the learners are eager to use the technology to improve their mathematical skills. It is the aim of the laboratory to assist them to improve their mathematics to a level that will allow them entry into tertiary institutions. Although the learners from outside my own school who make use of the community laboratory are motivated and eager to learn, they often struggle with the required conceptual understanding of many mathematical concepts. This is in contrast with the learners from my own school attending laboratory sessions, who show a relatively high conceptual understanding of mathematics but a low productive disposition. The evidence for this observation is illustrated in Table 2.1 below.

It is standard procedure in both the school and the community computer laboratory that the participating learners write a standard achievement test after the completion of a syllabus topic to test their level of conceptual understanding of the topic. A random sample of both groups are also evaluated in terms of their productive disposition towards learning mathematics. The average results of four achievement tests conducted with 120 learners are represented in Table 2.1 below. The designed tool proposed for quantifying positive disposition were used, in addition to the results from standardised achievement algebra topic tests.

*Table 2.1: Conceptual understanding and productive disposition of learners from the school and the community programme.*




Fundamentally, my research project is about whether a carefully planned and implemented *GeoGebra* intervention programme in the two laboratories can lead to positive results in conceptual understanding and productive disposition for both groups of learners.

## 2.13 RESEARCH ON ALGEBRA USING VISUALISATION AND TECHNOLOGY

Presmeg (2014) identified 13 'big' research questions that still need to be addressed by researchers on visualisation. This study falls within the questions selected from the range of questions highlighted by Presmeg (2014, p. 31):

Pressing research questions according to Presmeg are:

3. What aspects of the use of different types of imagery and visualisation are effective in mathematical problem solving at various levels?
9. How may the use of imagery and visual inscriptions facilitate or hinder the reification of processes as mathematical objects?
10. How may visualisation be harnessed to promote mathematical abstraction and generalisation?
11. How may the affect generated by personal imagery be harnessed by teachers to increase the enjoyment of learning and doing mathematics?

12. How do visual aspects of computer technology change the dynamics of the learning of mathematics?

Presmeg (2014) identified the need for research using "qualitative methodologies that include clinical interviewing and classroom observation, which are powerful in yielding the opportunity for depth of insight" (Presmeg, 2014, p. 32). My research approach is partially qualitative, with in-depth structured interviews with selected participants. Presmeg raises as one of her 13 questions, the question about the inherent visual effects of computer technology and how it can influence the teaching of mathematics. My research addresses one aspect of that question by interrogating the visual aspects of abstract components of mathematics learning through the medium of *GeoGebra*.

Stylianou (2001) believes that 'visual imagery' in mathematical problem solving is still a pressing matter and needs to be researched. This is supported by Presmeg (2014) stating that the need for visualisation in learning and teaching mathematics at all levels remains strong.

Fundamentally, this research project will try to establish how a carefully planned and implemented *GeoGebra* intervention programme executed in two laboratories can lead to positive results in achieving conceptual understanding and productive disposition for two different groups of learners, by implementing visual teaching strategies based on *GeoGebra.*

## 2.14 ARRIVING AT RESEARCH QUESTIONS

The aim of the research is to investigate and analyse how both the conceptual understanding of algebra and a productive disposition towards it can be enhanced in selected learners using visualisation as an epistemological tool through the medium of *GeoGebra* because of participating in an intervention programme. In pursuance of this goal ICT technology was used extensively as a learning instrument.

By pursuing these goals, the following research questions needed to be answered.

How can the use of visualisation in the learning process of algebra:

(a) enhance conceptual understanding of algebra, and

(b) foster a productive disposition towards algebra, when mediated through *GeoGebra*?

## 2.15 CONCLUSION

Based on recent literature the need for more research on visualisation and the relationship with technology exists. For Presmeg (2020), research on visual thinking in learning mathematics is still relatively new and can still grow in volume and depth. Yilmaz & Argun (2018) call for research about visualisation in learning mathematics especially in the abstraction process vital to learning algebra. To align with noteworthy authors my research project relies on the definitions for conceptual understanding and positive disposition as constructed by Kilpatrick et al. (2001), the views of visualisation as suggested by Presmeg (2006), and the different types of visualisations as identified by Guzman (2002). These constructs I applied to the Namibian situation where the research is situated.

# CHAPTER THREE

# METHODOLOGY

## 3.1 INTRODUCTION

This chapter provides an overview of the design, techniques, strategies and procedures that were employed to execute the research plan. A fixed framework guided the writing of this chapter. The sections in this chapter are listed below:

- 3.2 Research goal
- 3.3 Orientation of the study: the chosen paradigm for the study
- 3.4 A case study and action research: units of analyses
- 3.5 Underlying methodology: mixed methods approach
- 3.6 Research design and data collection
- 3.7 Research population and site
- 3.8 Data analysis
- 3.9 Narrative approach
- 3.10 Validity and ethical issues
- 3.11 Limitations and challenges
- 3.12Conclusion

## 3.2 RESEARCH GOAL

The research methodology that a researcher adopts depends on their goals and should thus be aligned to these goals. A goal is anything that includes motives, desires and a purpose that can lead to an outcome. It includes anything that can induce someone to ask questions, do a study or to engage in research, to get answers or to implement changes (Patton, 1990).

With this research I wished to explore several issues that would assist me in understanding the visual properties of technology in teaching algebra to junior secondary learners and to explore the possibilities of changing their disposition towards learning algebra. In particular, the goal of this study was to analyse and understand visualisation as a teaching approach characterised by using technology to enhance conceptual understanding; and to observe any positive changes in Grade 9 learners' dispositions towards learning algebra.

## 3.3 ORIENTATION OF THE STUDY

This study is situated within an interpretive paradigm. According to Bertram and Christiansen (2014) such a study will seek "to develop a greater understanding of how people make sense of contexts in which they live and work" (Bertram & Christiansen, 2014, p. 26). With reference to my study, I investigated possible ways – with the aid of ICT and visualisation – of how learners make sense in their construction of knowledge of algebraic concepts. Against that backdrop, I wanted to discover how they develop a positive disposition towards doing algebra.

Cohen and Manion (2011) state that the interpretive paradigm tries to understand the subjective world of human experience through their "actions" when they "construct their social world," and should be studied in "its natural state" (Cohen et al., 2011, p. 17). Furthermore, an interpretive paradigm allows for a mixed methods approach in data collection (Cohen et al., 2011).

An interpretive paradigm is reliant on observation and interpretation. To observe is to collect information about events, while to interpret is to make meaning of that information by drawing inferences or by identifying the match between the information and a set of observable indicators. It attempts to gain understanding of complex phenomena guided by the meanings people assign to them, but simultaneously the interpretive paradigm can be dynamic and acts as a foundation for further quantitative research (Schwarz-Shea & Yanow, 2013).

Reeves and Hedberg (2003, p. 32) note that the interpretivist paradigm relies on a perspective to understand within a specific context and "to improve a program under development." It gains understanding of the world through the subjective experiences of individuals. In this paradigm researchers use methodologies, such as interviewing and participant observation. These methods rely on a subjective relationship between the researcher and subjects. This might complement a quantitative paradigm where the descriptive parameters alone do not sketch a clear picture. Interpretive research excludes dependent and independent variables but looks rather at a complexity of sense-making through the lens defined by a specific situation (Kaplan & Maxwell, 1994).

My research went through several cycles, each time with a new algebra topic. An interpretive paradigm also requires that the researcher remains observant and constantly interprets collected results to refine the research process as the research progresses (Schwarz-Shea & Yanow, 2013). During my research and as knowledge was generated, applets were evaluated and refined according to the needs of the participating learners for conceptual understanding. Careful selection of applets had to take place, based not only upon covering the prescribed syllabus, but also on being visually interesting and their ability to provoke curiosity in the participants – to eventually improve their conceptual understanding and enhance a positive disposition. The cycling nature of the research allowed for refinement in the types of applets used in presenting algebra topics to the participants.

### 3.4 METHOD: A CASE STUDY AND ACTION RESEARCH

Mixed methods research is characterised by the integration of both quantitative and qualitative methods within a study, to capitalise on the advantages of multiple sources of data (Creswell & Plano Clark, 2003). Both types of data are often required when verifying and generating new knowledge (Teddlie & Tashakkori, 2009). When comprehensive solutions are required for research issues, multiple sources of evidence yield reliable results.

My research project was a multi-sited case study, consisting of two groups of participants located at two different sites (Yin, 2000). The group of 16 participants was selected from Grade 9 learners from mathematics classes in the private school where I am a teacher. The second group of 14 participants was gathered from Grade 9 learners who participated in the NAMVISPRO (Namibia Visualisation Programme) project at the community mathematics laboratory. As mentioned above, many of the school learners revealed a relatively high level of conceptual understanding, but a very low productive disposition, while exactly the opposite was true for the learners from the outreach group. In total the case had 30 participants of both genders and similar ages.

The unit of analysis was the changes in conceptual understanding and disposition of all participants when algebra was taught using visualisation through the medium of ICT technology. The investigation took place while teaching algebra using the *same carefully selected GeoGebra* applets for all participants, as substitution for normal class teaching. I ensured that participants at both locations were similar, in so far that

(1) all participants were Grade 9 learners,
(2) the same intervention was implemented at both locations,
(3) both genders were represented, and
(4) all participants were of the same age group.
(5) differentl levels of cognitive abilities were included.

My approach aligned well with the ideas of McMillan and Schumacher (2001), that a case may also be a programme, an event, or an activity bounded in time and place. According to McMillan and Schumacher (2001), a case study examines situations with people in detail, over a period, identifying sources of data found in the setting.

Guided by the interpretive position of this research and the nature of the research questions, I found that the case study methodology was the most appropriate approach to adopt. It entails a systematic way to collect data, analyses information, and reports the results, to better understand a specific problem or situation in great depth.

More specifically, it:

(i) leaves room for various participant perspectives.
(ii) uses multiple data collection techniques.
(iii) interrogates the integration of e-learning within a technology rich.
(iv) environment.
(v) allows for a mixed method approach, where quantitative and qualitative data.
(vi) can be integrated successfully with each other.

(Adapted from Creswell, 2017).

Case study research does not utilise any specific methods of data collection or data analysis (Merriam, 1998). A combination of data collection methods was selected in this study in anticipation of providing a comprehensive picture of the situation and for answering both ‘how’ and ‘why’ questions (Cohen et al., 2011, p. 111) thus allowing for the adoption of both qualitative and quantitative data collection approaches. (Cohen et al., 2011).

As the case study unfolded, elements of action research were present as participants went through cycles of reflection, planning and implementing. Action research or “practitioner-based research” (McNiff, 2002, p. 6) was found to be a powerful tool to change and improve at a local level, especially where disadvantages, prejudice or short comings were identified. “The word itself explains the nature of the research, *action* is taken as it is *researched* and *improvements* [my italics] of the practice can be implemented” (Cohen et al., 2011, p. 344).

Action research has a wide scope and is suitable for a setting where a situation involves people, or different engagements or procedures, with the aim of improving the situation and producing a desirable outcome. It is ideal when a practice needs better understanding, interpretation and improvement (Cohen et al., 2011). According to Cohen et al. (2011), action research is an ideal research methodology when there is a call for:

- evaluating and changing learning strategies, and
- changing attitudes and values.

Within the case study methodology, I also employed elements of an action research strategy. Different authors have set out the stages for action research, but the work of McNiff (2002, p. 71), resonates well with my planned ideas of executing the research.

1. The current practice under review was how participants perceived the traditional ways of teaching algebra to learners.
2. Aspect(s) that could be improved and changed were identified.
3. A plan and a visual way forward were investigated and designed.
4. The imagined plan was tested with participants.
5. Sense-making, understanding of and reflecting on situations was involved.
6. Modifications and improvements of the outcomes were made in the light of findings.
7. The modified actions were evaluated.
8. This was repeated process until I was satisfied with the outcome and final product.

(adapted from Cohen et al., 2007, pp. 352-353).

The action research process allowed for collection of data, reflection and the development and improvement of my *GeoGebra* applets that could be used in classrooms by learners who wish to further improve their mathematical skills. It is envisaged that the visualisation characteristics of the applets will enhance conceptual understanding and foster a change in dispositions towards learning algebra.

## 3.5 RESEARCH APPROACH: MIXED METHODS APPROACH

According to Creswell (2003) mixed methods research design has come of age. To only use a qualitative or a quantitative approach for research may fall short of the requirements of approaches being used in research today (Creswell, 2003). For Cohen et al. (2011), certain research questions can *only* be answered by adapting a mixed methods approach, especially when educational research is done.

### 3.5.1 Qualitative research

According to Cohen et al. (2011), the educational world is full of contradictions, richness, complexity, connectedness, the making of new connections and the breaking of such combinations. Due to the multi-layered complexity of the research questions most of my data collection took place from a qualitative perspective. Rich data was collected during the engagement with both groups of learners involved in the research. Data collection occurred through structured interviews, short questionnaires and reflections after the completion of each topic cycle.

### 3.5.2 Quantitative research

Quantitative research allows for the measuring of variables. In my case it included measuring changes in dispositions using specifically designed instruments, and the statistical analysis of test results from standardised achievement tests to measure changes in conceptual understanding. (Hittleman & Simon, 1997).

## 3.6 RESEARCH DESIGN AND DATA COLLECTION

### 3.6.1 Phases and cycles

Creswell and Plano Clark (2011) refer to mixed methods designs as sometimes comprising of multi-phased designs. Teddlie and Tashakkori (2009) observed that sequential mixed methods designs are often iterative in nature and thus cyclical.

Based on the above, my research was conducted in four cycles (one cycle per selected algebra topic), with each cycle consisting of six phases. During each phase data was collected by using selected screen captures, progress reports of standardised conceptual understanding achievement tests and the completed disposition questionnaires. Each phase was followed by a session of reflection to review outcomes and to improve on the tools used. Finally, outcomes were critically evaluated in order to be certain that the acquired outcomes had been achieved (McNiff, 2002, p. 71).

The research was conducted during daily laboratory sessions, Mondays to Fridays. Guided by the research design and for fairness and validity of results, the research process and collection of data had to run simultaneously with the daily lessons prescribed by the scheme of work we use for completion of the Grade 9 mathematics syllabus. The same algebra topics were covered with the research participants as with non-participating learners receiving class instruction. The same number of lesson periods were allocated to the participants to engage with the applets that other learners received for normal class instruction. Focus interviews were done after completion of each lesson and after completion of a chosen topic. The scheme of work made provision for specific sub-topics to span over several lesson periods. For instance, for factorisation the factoring of trinomials forms a sub-topic of the topic factorisation. The scheme of work allocated six lesson periods for the completion of the sub-topic: factoring trinomials. No extra time could be allocated to laboratory sessions as participants and non-participants had to write the achievement tests simultaneously. The research process spanned a period of nine school weeks. For each achievement test one lesson period was allowed.

The dispositions instruments were designed using emojis that allowed participants to complete it within a few seconds. Selected screen captures were used to direct reflective interviews. The *GeoGebra* platform allows the facilitator to follow all participants simultaneously on a central computer or to zoom in on one specific lesson. This allowed for screens to be captured without interrupting the learning process of any participants.

Three different algebra topics were identified and selected from the ten prescribed topics in the new Namibian mathematics syllabus for Grade 9. The selection of the three topics was dependent on what had to be covered during formal lessons at school. Each selected topic was introduced and then taught using *GeoGebra* applets. After the completion of each topic a diagnostic achievement test was written covering the topic. Tests were set and standardised by the senior subject head and written under examination conditions, allowing the use of results as quantifiable data.

Participants from my school attended laboratory sessions during their mathematics periods and did the reflective interviews during break or just after school, while the NAMVISPRO participants came to the laboratory in the afternoons when they normally attended the NAMVISPRO lessons.

### 3.6.2 Unfolding in phases

The research process unfolded in six phases:

**Phase 1**

During Phase 1, one algebra topic from the prescribed Namibian Junior Secondary Syllabus (MoEAC, 2017) was selected. An important criterion for the selection of a particular topic was its suitability to be interrogated visually. Then *GeoGebra* applets for the teaching of this topic were sought, re-designed or adapted to fulfil the requirements for being visual (see Appendix A).

**Phase 2**

During Phase 2 the selected topic with sub-topics was taught and practised, using the selected *GeoGebra* applet. A process of exploring and experimenting with the applet to interrogate the algebraic concept followed. Several screen captures were recorded focusing on the interactions between each participant, the applet and its instructions on the computer screen.

**Phase 3**

After the completion of each topic a standardised achievement test was written by all participants. As a control, the results for the learners not participating in the research were collected to compare their results those of non-participating learners. Quantitative data was collected to measure the level of conceptual understanding of the two case study groups and the remaining Grade 9 learners. Each participant's test results were tabulated as the results went through the different cycles. The tests coincided with the achievement tests of all the other Grade 9 learners in the school. Quantitative data was extracted by selecting the marks from questions covering the selected algebra topics. A comparison of marks over the wider spectrum (participants and non-participants taught without *GeoGebra)*, allowed for triangulation and ensured validity.

**Phase 4**

After the completion of a specific topic or sub-topic in Phase 2 participants were asked to complete the Positive Disposition Instrument by merely choosing an emoji to express their disposition about the lessons covering a sub-topic. After the completion of a topic participants had the opportunity to place themselves on the podium instrument to express their disposition in terms of the covered material. – refer to Appendix B and Appendix C: Positive Disposition Instruments 1 and 2. Comparing the two instruments allowed me to track any changes in their dispositions towards the learning of algebra through the *GeoGebra* applets. Structured stimulus-recall interviews were held where the captured screen shots formed the basis of the interviews. A system of coding answers that is related to conceptual understanding and changes in disposition were implemented. Rich qualitative data could be collected with the interviews.

**Phase 5**

A mixed methods study has at least one "point of interface" (Morse & Niehaus, 2009, p. 25), where the qualitative and the quantitative data are brought together. During this phase, the various data sets were brought together into a coherent set of results by imposing the analytical tools on the data. As the research went through the different cycles, the different sets of data were compared to validate the results against a previous set of results.

**Phase 6**

This phase consisted of reflecting on the previous phases, drawing tentative conclusions, identifying conflicting results and preparing for the next cycle. This is also the phase where the next cycle of the research was refined by me before going through the phases again until all three intended cycles are completed.

## 3.7 RESEARCH DESIGN AND DATA COLLECTION

### 3.7.1 Research Instruments and data collection

Several instruments were developed to ensure the generation of appropriate data for my study.

The following instruments formed the basis of the data collection proses:

1. **Applet evaluation framework**. This instrument was used to either select, improve or develop appropriate applets from the *GeoGebra* platform to provide for genuine visualisation opportunities and processes for the learning of algebra. The framework draws from the concepts of different types of visualisations as proposed by Laughbaum (2002) in Chapter 2 (see Appendix B).

2. **Observations and screen captures** of participants' engagement with the applets were conducted. The screen captures provided rich qualitative data but also formed the basis for structured reflective interviews with participants. These captures were selected during observations while participants engaged with the chosen applets.

3. **Standardised achievement tests** written after each cycle by all research participants. These tests were set by experienced external examiners from my school. Tests were set on a rotational basis by staff members appointed by the mathematics Head of Department. All tests were set to (a) ensure that the syllabus was covered, (b) the correct levels of difficulty were maintained and (c) to measure conceptual understanding of the taught topic according to norms and standards prescribed by the mathematics syllabus (MoE, 2017).

4. A **positive disposition instruments 1 and 2** user-friendly questionnaire using emojis, was administered to the participants after each sub-topic and the podium instrument was complete by participants after the completion of specific topics. These questionnaires provided data regarding the participant's dispositions on a five-point scale. This was a repetitive progression. As progress was made through the topics, participants placed themselves on a Disposition Podium to reflect on their feelings towards the selected approach for each topic.

5. Also, after each few lessons on topic a stimulus-recall **interview** was conducted with each participant. The interviews took place in conjunction with the captured screen analyses and in classroom observations.
6. At the end of each cycle a **standardised conceptual understanding test** was written by all the participants covering the work done during the cycle. That test allowed for evaluating the level of conceptual understanding of a theme.

### 3.7.2 Research participants and site

Essentially two cohorts of learners constituted my case. I purposefully and conveniently selected 14 participants from the NAMVISPRO project and 16 learners from my school to make up the case. According to Cohen et al. (2011), purposive sampling allows the researcher to achieve representation, especially when comparisons must be made. The selected participants volunteered to participate in the research project and written approval was obtained from their parents or guardians. The criteria for selection and participation were done strictly according to the ethical requirements of Rhodes University. All research involving human participants, especially children, has to be done in accordance with the guidelines of the National Health Act 61 of 2003. Research at Rhodes University is overseen by the Rhodes University Ethical Standards Committee (RUESC). (See Appendix D). Permission for the research project was granted by the Ministry of Education and Culture in Namibia as well as from the head of the school where I am teaching. (See Appendix E and Appendix F). The whole research process was overseen by a senior Head of Department where I am teaching. She attended laboratory lessons and provided written confirmation that the research was done didactically correctly and in a professional manner. Eventually the case was constituted of participants from four different schools in the Khomas Region of Namibia.

The research site consisted of two mathematics computer laboratories, of which one is a community computer laboratory where learners from the wider community voluntarily receive mathematical instruction to improve their marks in mathematics. This laboratory is operated as a community project to help improve the mathematical skills of Namibian learners. The other one is situated at a private school where mostly learners from the school can do online classes, but the laboratory is also available for learners from the community project to attend lessons. Both laboratories had fast internet access and excellent equipment.

Convenient sampling was done to select the participants for the research project. The case consisted of a cohort of 30 mixed-gender and mixed-ability first-time Grade 9 learners. I opted for first-time Grade 9 learners to allow for participants of similar age. All participants were eloquent in English, even when it was not their mother tongue.

Convenience sampling implies that the researcher choses the sample to allow participants easy access to the research site and to allow the research process to be conducted conveniently, easily accessible for attending participants. Sampling was also done conveniently to allow for different genders and different abilities to be included in the case (Cohen et al. (2011)).

For Rule and John (2011) convenience sampling of research participants implies that participants are selected because they are the most suitable for the research to be undertaken.

## 3.8 DATA ANALYSIS

After extracting data, over a period of nine weeks (by adhering to the cyclic nature of the research) the next task was to extract meaning from the data and to interpret and extract information from the quantitative results (Newby, 2010). I used a colour coding system to

(a) look for patterns and measurable changes and
(b) identify wider implications of the findings.

All data for each participant was filed and analysed individually, then all sets of data were combined to qualitatively identify patterns and quantitatively draw statistical conclusions. Finally, I had to judge whether the intervention has answers to the research question(s) and to suggest modifications to processes where learners use *GeoGebra* when learning algebra (Newby, 2014).

An analytical tool/framework was applied for each of the above data gathering instruments:

1. **Applet evaluation framework**. *GeoGebra* applets covering the pre-defined algebra topics were selected, tested and then sorted as being appropriate or not.

Selected applets with *clear visual* properties went through a final selection process applying the previously developed visual indicator framework (VIF) (see Appendix A). Even visually suitable applets had to be rejected when they did not apply to pedagogical requirements, were too difficult or did not comply with approved mathematical principles. A score of 0, 1 or 2 was awarded to the selected sections on the instrument. Only applets that scored a mark of 8 or higher on the selection framework were included in the research project.

2. **Achievement test analysis.** Three achievement tests were written over the research period. The tests were set by experienced senior subject teachers appointed by the Head of Department for mathematics. All tests were set to:

- ensure that the syllabus topics were covered,
- the correct level of procedural fluency was reached, and that
- conceptual understanding of a topic was achieved by the participants (MoE, 2017).

Applicable questions were selected to extract data from the results. The quantitative data provided were analysed to find any significant changes in the achievement of conceptual understanding.

These were analysed quantitatively. The results of the tests contributed to independently triangulate changes in the conceptual understanding of participants when taught with *GeoGebra* applets. Extracted data statistically quantified the changes for each participant over the research period and collectively to statistically verify any changes. The statistical ability of Excel allowed me to draw graphs for each participant over the research period and to combine each set of data to draw conclusions collectively to find significant changes in results over the entire research period. Changes in average scores were tabled to identify positive improvements for each individual participant.

The results of the achievement test allowed for the comparison of modal scores of non-participating classes in addition to the changes of the mode and modal scores within the case over the research period. Final distribution curves as advised by Cohen et al. (2011) were used to identify any significant changes in the achievement and thus conceptual understanding of the algebraic concepts over the three cycles of the research.

3. **Completion of the positive disposition questionnaires**. The research project occurred over a period of nine weeks in which changes in disposition were tracked and identified. Components of a longitudinal study had to be incorporated into the tracking over time. Two instruments were used to track and quantify changes in disposition over the research period (Appendices B and C). These questionnaires could be completed very quickly and provided quantitative data regarding a participant's disposition on a five-point scale. After the completion of the lessons on sub-topics participants were requested to complete a questionnaire with emojis to express their dispositions after the lesson. At selected intervals (usually at the end of a cycle) participants placed themselves with a second Disposition Instrument on a podium to measure changes in their disposition towards the learning of algebra using technological instruments.

4. **Analysis of captured screen shots and structured stimulus-recall interviews**. Observations while participants were engaged with the applets were written down for referral purposes during the structured stimulus-recall interviews. Observations were done purposefully and included notes on remarks, gestures, body language, pencil notes made by participants and learners' ability to engage with the applets for an extended period. When working on the computers short screen captures were made of specific moments when participants engaged with a concept posed by the applets. Screen captures of different attempts by participants were printed and later discussed with individual participants.

The structured stimulus interviews were conducted adhering to the pre-set interview instrument. (See Appendix E)**.** Both the observations and structured interviews provided the following data:

(1) the time the applet kept the participant interested and engaged;
(2) the visual concepts the participant applied when solving problems with the applets; and
(3) the advancement towards more complex problems with the applets to provide data on the participants' conceptual understanding of the topic under review.

Observations and experiences of the participants were discussed during interviews. This was done using a fixed set of questions. Answers were written down, coded, and used to extract rich quantitative data.

## 3.9 VALIDITY

Copies of all the data material collected, such as transcripts of interviews and screen captures of lessons, were made available to the participants for the interpretive validation of the research project (Maxwell, 1992). The fact that I am a mathematics teacher for some of the participants, all the participants had to be convinced that no form of bias or judgement existed in the collection of the data. The analytical frameworks are based on a multilevel mixed methods design approach, which makes provision for validation and cross validation of data (Ponce & Pagán-Maldonado, 2015). According to Ponce and Pagán-Maldonado (2015) a researcher could, with careful planning, use parallel phases that rely on quantitative and qualitative approaches to study "in depth the same aspects of the research problem" (Ponce & Pagán-Maldonado, 2015, p. 122).

The credibility of research data, objectivity, reliability, and validity were upheld by the collection of both quantitative and qualitative data that were based on standardised instruments developed according to the framework set out by Merriam (1998). Creswell (2006, p. 247) suggests that

> the trustworthiness of qualitative research can be established by using four strategies: *credibility, transferability, dependability and confirmability*, constructed parallel to the analogous quantitative criteria of *reflexibility, triangulation* and *dense descriptions"*. [italics added]

For Creswell (2006), validity is based on establishing both quantitative validity (eg construct) and qualitative validity (e.g. triangulation) for each set of data. To ensure that there were no threats to the validity of the data I followed Creswell's recommendations for equal (large enough to be significant) sample sizes and used a limited number of variables for the quantitative data. This ensured that quantitative and qualitative data were comparable (Creswell, 2006). To address validity for the qualitative data I ensured honesty, depth, richness, and a wide scope when collecting data. This was achieved through the objectivity of the researcher.

### 3.9.1 Credibility transferability, dependability, and confirmability.

All data were collected under the supervision of the senior Head of Department as required by the school. Quantitative data were collected through standardised tests moderated by the Head of Department or subject head, who is an established external examiner for national examinations. According to Rule and John (2011) triangulation refers to the use of multiple sources and methods to support the findings of a case study. The repetitive (cyclic) nature of data collection during and after covering each topic in the algebra syllabus and the use of multiple instruments for data collection enhanced the credibility of each set of data and allowed for triangulation of the data.

Creswell (2006) advised that to avoid credibility concerns there should be *transferability,* meaning one should be able to apply one's findings to the wider mathematics community in Namibia. This was established by selecting adequate sample sizes purposely from several schools and by confirming that the qualitative and quantitative results 'speak' to each other. To prevent compromising the accuracy of any set of data, I identified different explanations that might need further investigation and understanding (Creswell, 2006).

Cohen et al. (2011) define d*ependability* as the stability of data over time and over conditions. This is provided for by the constant evaluation of the quality of the integrated processes of data collection, data analysis and theory generation. During my research, dependability was established by covering several algebra topics. After each topic the research design allowed for an evaluation and improvement process (Creswell, 2006; Cohen et al., 2011).

For the results to pass the test of trustworthiness, they were shaped by confirmability. Confirmability requires that the results should also be confirmed by other researchers in the same field of study (Baxter & Evles, 1997). To establish confirmability of an inquiry a researcher should derive findings along well-defined lines of interpretation. Such findings should be corroborated by other researchers (Tobin & Begley, 2004). I constantly ensured that the results were shaped by the participants and not by the researcher. This was established with the triangulation of results and by using several well-designed and tested instruments.

To ensure the reliability (internal consistency) of the data collected I regarded it as important to do some piloting before the actual data collection process started. Piloting has several functions, principally to increase reliability, validity and practicability

(Cohen et al., 2011). The large number of available algebra topics allowed for piloting some topics. Piloting furthermore allowed for the refining of the research instruments, leaving enough scope for sufficient data collection opportunities during the actual research process.

## 3.10 ETHICS

This case study involved Grade 9 learners from a well-established private school as well as learners from the wider community in the Khomas Region. I had to strictly adhere to the ethical guidelines as per the Rhodes University's Ethical Code. All research was done according to the stipulations and undertakings of the approved ethical proposal submitted to Rhodes University (see Appendix D)

Participants were informed regarding all the ethical implications of the research. The parents or legal guardians of all the participants were well briefed about the whole research process and were asked to sign a consent form, before any data collection took place.

### 3.10.1 Respect and dignity

After an initial interview with the parents and guardians of the selected participants, I acquired written permission from the parents or legal guardians of all learners who participated in this study (Bertram & Christiansen, 2014, p.66). All the permission letters and consent forms are kept on file. Participants were informed that their children's participation in the study was voluntarily and that they had the right to withdraw from it at any time. No names of any participants were disclosed anywhere in the report, instead pseudonyms were used. Raw data collected through interviews and observations are securely stored electronically and it will only be accessible to me and the participants and if required, to the supervisor of my study.

### 3.10.2 Transparency and honesty

In the consent form, the purpose of the research study and the process of data collection was clearly outlined to the participants. They were informed of the benefits of participating in this study, especially for future algebra courses (Bertram & Christiansen, 2014, p. 67) and how any possible risks (e.g. breach of anonymity) would be handled. Participants were given an opportunity to partake in member checking of interview transcriptions to ensure that they were correctly heard and understood. The latter action supported the proper interpretation of the results which will be to benefit of whoever is going to use the report. The participants were informed that the final report would be posted on Rhodes University's website. This means that the report can be viewed globally.

### 3.10.3 Accountability and responsibility

As a researcher and as a teacher of the participants I protected the participants from any possible harm, unnecessary risks or mental/emotional and physical discomfort that might be inherent in the research procedure. Every participant had an equal opportunity to participate in all the activities of the research study. I avoided the potential for bias that could jeopardise the validity of the results of the research. This was done by allowing the technology to be the sole instructor of the lessons presented. During the intervention programme, each participant benefitted from the activities and those who were slower were allowed to carry out their activities at their own pace.

### 3.10.4 Integrity, academic professionalism, and researcher positionality

The instruments used to collect data were cleared by my supervisor for this study to avoid potential bias. Interviews were conducted without assumptions and perceptions which could impact negatively on my participants or on the research itself. The interview questions were selected to not to cause negative emotions in the participants. All research was done under the supervision of the senior Head of Department who constantly monitored the process.

The conclusion reached in the study can be traced through the summaries made from the original observation and interview data. This would be done through an adequate audit trail and easily accessible records of the observation and interview data. The latter actions were instituted to ensure that the findings are factual and accurately reported.

Given that I worked under the supervision of my senior Head of Department both in school and at the outreach centre where the research study was conducted, I was always supervised and externally observed. The normal lesson approach remained available to the participants, whenever they felt uncertain about any topic. I ensured that my position did not influence the participants' responses in the research study. An objective instrument was designed for selecting applets to be used during the research process. Evaluating applets by awarding a measurable score in terms of the different visual categories prevented any form of bias by me as researcher.

### 3.10.5 Criteria for selecting the participants

Academic results did not play a role in the selection of participants. Only volunteers of the same grade, both at school and at the outreach centre, were selected. Participants were selected after written consent was obtained from their parents or guardians.

### 3.10.6 Confidentiality and anonymity

Information from this research was used solely for the purpose of this study. The final report of the study uses pseudonyms to maintain anonymity of the participants. The data collected will only be accessible to the researcher and the supervisor of the study.

The transcriptions of interviews and any other information that may lead to the identification of the participants, are kept in a locked file with a password in the personal possession of the researcher. The data will be kept for a period of about two years until it will no longer be necessary for the research and then be destroyed by the researcher.

### 3.10.7 Selection of the members of the case

Members of the case entered the research process with a relatively low disposition while others entered the research process with a relatively high disposition towards learning algebra. Participants covered the spectrum of underachievers through average learners to high scoring candidates. The specific selection and number of participants allowed for validity in answering the research questions. Although the case study had 30 participants, to ensure that the data remains manageable, it was representative enough to ensure validity of the results.

### 3.10.8 Positionality

I, as researcher, am also a mathematics teacher for all the participants. That could have been problematic on four levels: Firstly, as a person of power to them, secondly as an adult person, thirdly as a PhD student and fourthly as a teacher from a highly functional school. Rule and John (2011) recommend total transparency towards the participants of a case study. It is advised that the researcher should constantly be aware of his/her positionality to avoid influencing the outcome of the study.

Fortunately, the engagement with the technology and the monitoring features offered by the *GeoGebra* platform allowed me to be less involved with the data collection processes. Participants saw the intervention as a new adventure with technology that they were accustomed to and comfortable to use. The research process became a journey that both I as researcher and the participants embarked upon.

## 3.11 A NARRATIVE APPROACH TOWARDS RESEARCH

Due to the extended period of data collection and the many reflective interviews with each participant to verify and extract data, the responses of participants became a story of each participant's journey through the intervention period. Ultimately, a narrative approach characterised the research reporting. It highlighted challenges and opportunities during the research process.

For Bell (2003) it is a natural process when data collection happens over a long period of time with many personal interviews conducted with participants almost daily. Afful (2008) believes that thesis writing that includes narratives of participants can highlight human expression to help them construct meaning for themselves. It often happens when the researcher is situated firmly within the research process. The researcher should always be reminded about objectivity (Afful, 2008). Great care was therefore taken to not guide or influence participants with vocabulary about visualisation or disposition. In their efforts for the Grade 9 participants to convey their message to me as researcher it became a process of narrating their own experiences.

## 3.12 CONCLUSION

We live in a highly visual digital era. Information Technology and connectivity is part of the existence of every young Namibian. The COVID-19 pandemic made the ability to learn with technology an even more important matter. The investigation aimed at determining whether a better conceptual understanding could be achieved by using visualisation through technology and in doing so, how the disposition that learners have about learning algebra could change to become more positive. A well designed research process and methodology could contribute towards answering important research questions. It was important to ensure that all data would be statistically meaningful.

Chapter 4 describes the tedious steps followed during the applet selection process to ensure credibility transferability, dependability, and confirmability of the collected data. The research process was dependant on the visual teaching embedded in every applet used for data collection.

# CHAPTER FOUR

# APPLET SELECTION FOR DATA COLLECTION

## 4.1 INTRODUCTION

The aim of the applet evaluation process was to carefully select and find applets that would serve the purpose of my research. I was careful not to be judgemental about the contributions of any of the many applet contributors on the *GeoGebra* platform. Many brilliant applets were evaluated but could not be included in the research process simply because they did not meet one or more of the critical criteria of my Applet Evaluation Instrument.

This chapter interrogates the tedious process followed to pre-select and test *GeoGebra* applets for the three cycles of data collection envisaged in the research design. For selection, applets had to fulfil several requirements: firstly, suitability for teaching algebra visually; secondly, appropriate in-depth coverage of all the nuances of the Namibian syllabus for Grade 9 algebra to fully comply with all the prescriptions required by the syllabus; and thirdly, to be mathematically correct (MoEAC, 2017). The research design with its research questions required a deviation from traditional classroom teaching to teaching with technology. Three abstract algebra topics had to be identified for teaching using technology only, to a level of conceptual understanding that would allow participants to successfully complete a standardised test at the end of each topic. The research design called for a cyclical process of data collection aligned and in sync with the normal teaching of the school, with the same number of lesson periods allowed by the scheme of work for all Grade 9 mathematics learners. Provision had to be made to collect two different sets of quantitative and qualitative data to answer my research questions.

While the applets were selected to explore several issues in unpacking the visual properties of teaching algebra with technology to junior secondary learners, the applets also had to fulfil the intended need to track participants' changes in their dispositions towards learning algebra.

Streeb et al. (2019) argue that there is a large pool of evidence from researchers and practitioners that visualisation can improve conceptualisation and the cognitive processing of mathematics. They suggest the building of a visualisation theory and encourage empirical research into machine-assisted visualisation – in this case for the learning of the more abstract and less obvious visual properties of algebra (Streeb et al., 2019).

The research was done in conjunction with the ordinary and ongoing teaching and learning process of the participants, within the limitations of the COVID-19 pandemic which engulfed the entire globe during this research project. Extreme care had to be taken with the selection of the applets to ensure that quality teaching was not compromised for the participants.

Because the research coincided with the ordinary scheme of work that the school followed, the selected *GeoGebra* applets thus aligned with the algebra content for the normal class teaching.

## 4.2 CHALLENGES AND CONSIDERATIONS DURING THE APPLET SELECTION

The selection of applets was done with careful and purposeful consideration to enhance conceptual understanding of algebra. The selection of applets was an integral component of my research, and it was clear that the selection of the correct applets was a determining factor of the success of the whole research process. The cyclical nature of the action research implied that the applet selection would be an evolutionary and iterative process to eventually introduce participants to applets of appropriate quality that would satisfy the aim of quality teaching.

Morphett et al. (2015) suggested a framework of criteria for applets to have conceptual impact, enhance learning and promote the engagement of users of the applets. Figure 4.1 is a diagrammatic representation of the evolving framework that Morphet et al. (2015) suggested.

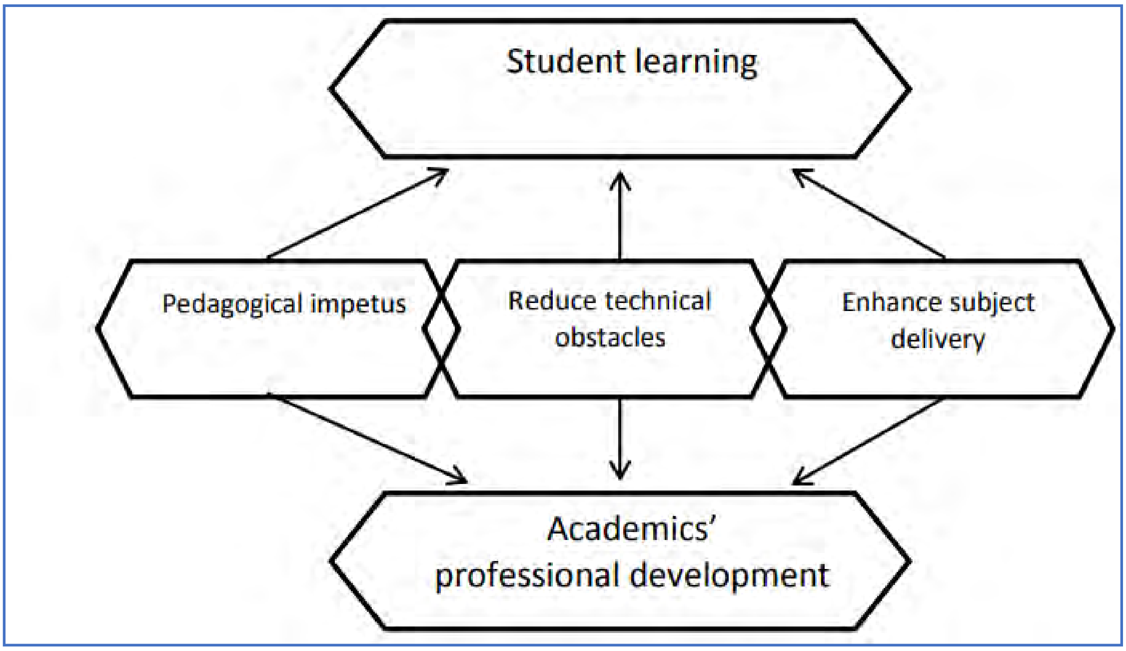



*Figure 4.1: Framework used for designing the applet selection instrument to ensure academic impact.*

Source: Morphett et al., 2015, p. 8

To satisfy the required criteria for my research, I adapted the suggested framework to meet the requirements of my research. An instrument was developed based upon the adopted requirements. After each cycle of the research the instrument was refined to ensure that all applets fully adhered to the following set of criteria:

1. The applets should replace the normal class teaching with interactive lessons with the same or better outcomes than the normal teacher-learner lessons.
2. Each set of selected applets should be done coherently with the prescribed topics as defined in the scheme of work and within the allocated number of prescribed lessons.
3. Applets selected should cover the same detail of every aspect as prescribed by the syllabus.
4. Lesson periods should be of the same length as that of non-participating classes.
5. Applets should build upon the established knowledge and conceptual understanding of the participants.

6. All applets should comply with acceptable mathematical practice, logic and accepted laws of mathematics.
7. Applets should pass the visualisation test of the Applet Selection Instrument that makes provision for several different types of visualisations within a specific applet.
8. All applets should have the potential to enhance a positive disposition within participants.
9. Applets should have the potential to engage participants for the entire lesson period, without them getting bored. Applets based upon rote learning and repetition of the same skills are immediately disqualified.
10. Specific attention should be given to potentially develop any of the previously identified different types of visualisations, to contribute towards the development of visual conceptual understanding of the topic under review.
11. Applets should provide participants with immediate non-judgemental feedback and reinforce any newly acquired visual conceptual understanding skill.

(Adapted from Morphett et al., 2015)

## 4.3 COGNITIVE CONSIDERATIONS OF THE APPLET SELECTION PROCESS

Data collection during normal class instruction is challenging. To keep data collection objective, it was decided not to make video-recordings of the participants, but to rather rely on screen captures of specific moments of the participants' engagement with the applets and to use these as reference material during reflective interviews after the observed lessons. *GeoGebra* Classroom makes provision for monitoring and direct communication with participants from one central console. This facility provided an excellent opportunity for the collection of several screen captures without interrupting participants when involved with an applet. To be able to extract data from the screen captures, the applets should provide moments where the participants' cognitive processes are displayed on the screens as they navigate through the lessons. Most applets tested for selection expected only an answer from participants, without typing any cognitive progression towards reaching the expected answer. However, no data can be extracted from answers only, and the research design required that the adaptive reasoning and visual-symbolic visualisations of the participants were visible and thus observable. One way of overcoming this challenge was to select applets at a level where participants were unable to guess or visualise solutions without manoeuvring step-by-step through the applet and thus displaying each cognitive thought on the screen.

## 4.4 COMPONENTS OF THE INSTRUMENT

With visualisation as the focus of the research, any applet based on rote learning or enhancing conceptual understanding through repetitive exercises of a previously explained skill could not be considered for selection. While undertaking the selection phase I gained valuable insight into the possibilities embedded in visual applets to enlighten the sometimes-hidden conceptual structures needed to master basic school algebra. However, most applets that I reviewed had to be rejected for not providing any opportunity for any form of visualisation that could enhance conceptual understanding, nor prospects to develop affinity for learning algebra. Right from the start of the research project it was interesting to observe that the participants themselves rejected those applets that required repetitive skills and actions. As soon as a similar type of applet was presented to them, albeit with only small variations, the participants simply randomly ticked answers without attempting to solve the problem. Often, they showed their disapproval by navigating away from those particular applets under investigation and started to play games or ceased working on the applet. Figure 4.2 is a typical example of an applet that randomly posed the same type of problem to the participants. By ticking boxes, the participants could guess the correct answers without having to conceptually solve the problem.

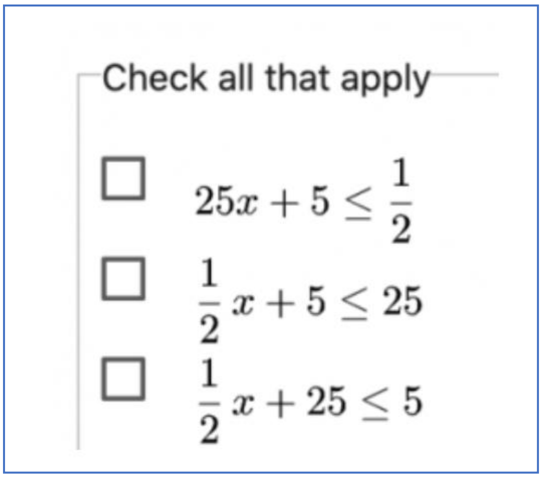



*Figure 4.2: Example of an applet without an opportunity for visual conceptual understanding*

Very few of the algebra applets complied with the forms of visualisation I intended to investigate. Although *GeoGebra* relies on visualisation, many contributors of applets simply reinforced rote learning by producing applets that required repetitive actions as a method to enhance understanding of concepts as illustrated below in Figure 4.2.

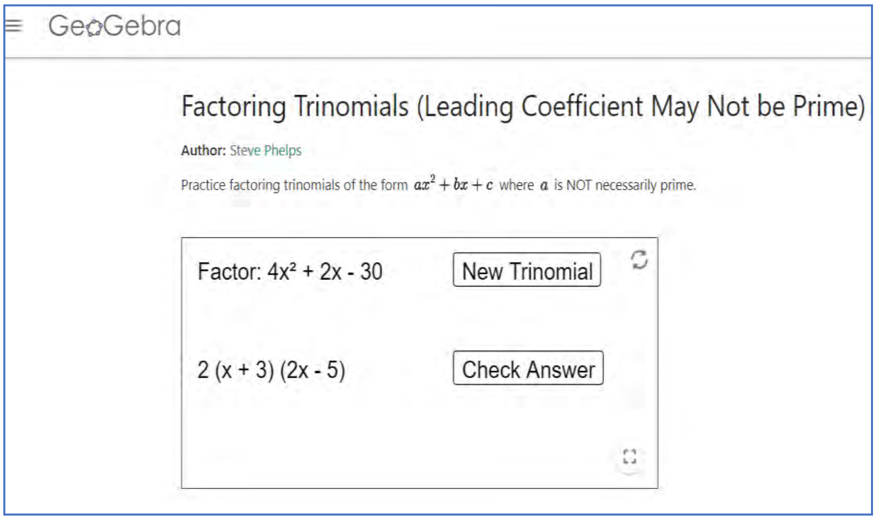



*Figure 4.3 Example of an applet with no visualisation opportunity*

Figure 4.3 is an example of an applet on factorisation. It expected the participant to ‘see’ the common factor and then factorise the trinomial using any method previously taught. During trial sessions the participants either guessed answers or grabbed a pencil and a sheet of paper and applied a method of factoring introduced by better applets. The applet randomly generated trinomials to be factorised by participants. Within the first five trinomials posed to them, participants started to click on the ‘Check Answer’ box merely to complete the exercise as soon as possible.

A far more suitable applet would be one as illustrated in Figure 4.3 below, where the applet covers the factorisation of trinomials using the completion of the square as one solution option, followed by a practice session where participants can factorise posed trinomials applying the proposed method by using a slider to *see* how to factorise the expression.

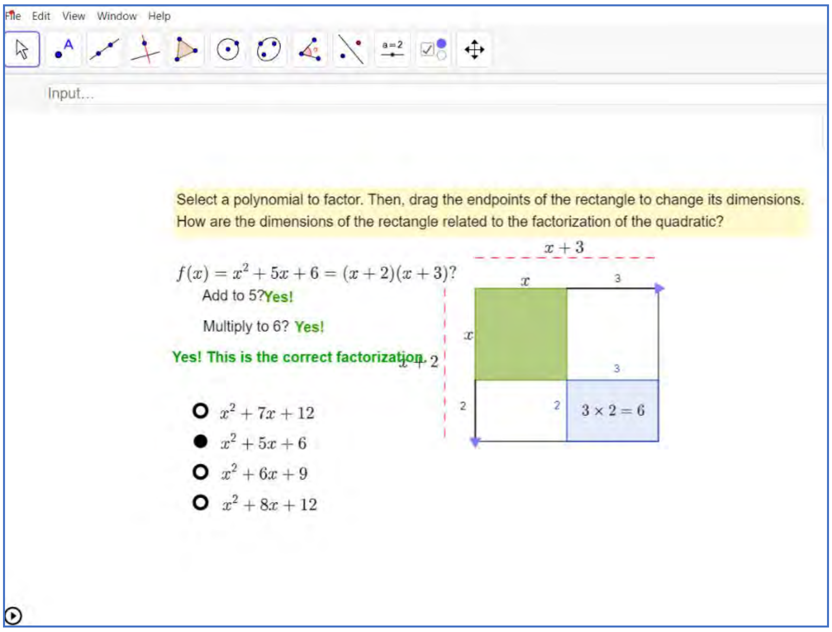



*Figure 4.4: Example of an applet with a slider and visualisation opportunities*

Preference was given to applets that were non-judgmental and patient with participants while they attempted to conceptually grasp concepts or tried to solve a problem, by allowing them enough time to find solutions.

Only applets with a high score on the Applet Evaluation Instrument, containing clear visual characteristics, could engage participants for any extensive period. During our reflections, participants expressed their satisfaction with those applets and progressively gave higher scores on the two Disposition Instruments. ( Appendix A and Appendix B)

Three different algebra topics were identified and selected from the ten prescribed algebra topics in the new Namibian mathematics syllabus for Grade 9 (MoEAC, 2016). The selection of the three topics had to be guided by the currently covered topics in the formal lessons at school.

More than 150 different applets and *GeoGebra* lessons were measured against the Applet Selection Instrument above to ensure that they complied with the requirements set out in the instrument. Eventually 70 applets were selected to be included as research material. Previous experiences with the ongoing NAMVISPRO project revealed that many applets were so unclear in instructing participants in how to use the applet that their inclusion would be a waste of laboratory time. Other applets lacked variation in the type of problem, or in my assessment, progressed too slowly towards developing conceptual understanding and generalisation. This is supported by Laughbaum (2017) who believes that for most learners conceptual understanding and generalisation occurs within the first three repetitions of similar examples. It must be said however that many *interesting and creative* applets had to be rejected because they did not serve my interest in learning through visualisation. Participants showed a preference for applets with variations in levels of difficulty and with a distinct progress from interiorisation towards coordination or encapsulation. Applets which posed only problems followed by the solutions without any cognitive interaction from the participant, were immediately rejected. The instrument envisaged that active engagement by the participants would eventually instil visual structures in their minds and assist them to cognitively grasp the intended concepts and create a visual form of conceptual understanding. I assumed that a form of visual-symbolic and visual-spatial conceptual understanding could be developed through the engagement with the applets. The algebra teaching programme aimed to not only enhance participants' conceptual understanding, but to improve the participants' disposition towards learning algebra.

## 4.5 APPLYING THE APPLET SELECTION INSTRUMENT TO THE APPLETS

Selecting applets using the Applet Selection Instrument was an integral part of the research process. Wrongly selected applets would not yield data and could potentially undermine the outcome of the research process and even cause long-term confusion amongst participants regarding the learning of algebra.

Initially the research design intended that the Applet Selection Instrument should make provision for applets to be evaluated on a scale of 1 to 5, but when the selection process started it was clear that some of the requirements identified in Section 4.2 are critical and should be answered by either a 'yes' (1) or a 'no' (0) response. A single ‘no response’ to any one of the vital questions led to the elimination of the applet under review. Some critical criteria included: Is the Namibian syllabus at Grade 9 level covered to the correct depth while adhering to mathematical rules and laws?

Muchoko et al. (2018) expressed concern that learners have difficulty identifying and recognising algebraic structures when they lack context as is often the case with test and examination problems. It is common practice to evaluate an expression as part of an achievement test or examination, without stating a required procedure or outcome. For example, learners are expected to be able to identify an expression as a trinomial in the form $ax^2 + bx + c$. Often manipulation of an expression is required before an attempt to solve the problem can be made. With trinomials, rearranging the order of the expression and removing a common factor should be done before factorising it. A typical question could present the expression $10x + 4x^2 + 6$, stating that the expression should be ‘evaluated’ by the learner. Presmeg (2017) refers to this ability as visio-symbolic visualisation. The ability to *recognise* a problem stated out of context and to ‘*see*’ or *visualise* that the problem requires some manipulation before solving the problem, lies at the heart of visio-symbolic visualisation. During the applet evaluation, specific attention was given to the inclusion of various examples to practice a skill to allow participants to be aware and confident when confronted with problems requiring additional manipulation. The same applies to other algebraic structures containing many hidden pitfalls for unsuspecting leaners. The evaluation of applets was done not merely by ticking boxes, but required a detailed scrutiny of each applet, applying stringent requirements and rigorous testing to ensure that I could implement it during the data collection phase.

The visual learning approach offered by using technology was chosen to replace topic based classroom teaching. Applets presented a concept followed by examples and exer based on the concept. Visio-symbolic visualisation skills allowed participants to place posed questions within context and to apply a solving strategy to the problem. I envisaged that over time, carefully selected *GeoGebra* applets would improve and strengthen the visualisation skills of participants when solving out of context problems.

The applet in Figure 4.4 above is based on the method of completion of the square for factoring trinomials. The dynamic square in assorted colours and with the slider provided participants with immediate visualisation opportunities of the process of factorising the trinomial.

Immediate visual feedback when sliding over the correct solution boosts the confidence of any hesitant participant and encourages them to attempt more advanced trinomials without hesitation. The applet above varied in terms of the numerical coefficients of the trinomial, but never used higher numbers for factorising. Regarding a given trinomial with higher numerical coefficients as being more advanced is senseless. The same principles apply to solving $45x^2 - 8x - 77$ as solving the trinomial $2x^2 - x - 3$. Many applets evaluated on the topic progressed towards larger numerical coefficients instead of varying between positive and negative coefficients, and eventually towards the expected first step of removing a highest common factor first.

Another consideration in the selection of appropriate applets was their appropriateness for a screen capture methodology to observe the participants' interaction with them. The *GeoGebra* classroom provided an opportunity to observe and capture the progress of any participant joining a lesson, even remotely, without interference.

A further consideration in the selection of applets was the efficiency of internet connectivity. Some applets required a high-speed and capacity internet connection which is not always available in Namibia. Thus, some applets had to be eliminated. Fortunately, many alternative applets are available.

Figure 4.5 below is another illustration of an unacceptable situation when selecting applets for inclusion in the data collection process. From an early age, learners are taught that division by zero is undefined and should never be done. Although the applet functioned very well, it used a zero as a sort of placeholder while a participant had to find the lowest common denominator. This tends to create the idea that division by zero occurs until the correct lowest common denominator has been found. By simply replacing the zero with a little □ the issue could have been resolved.

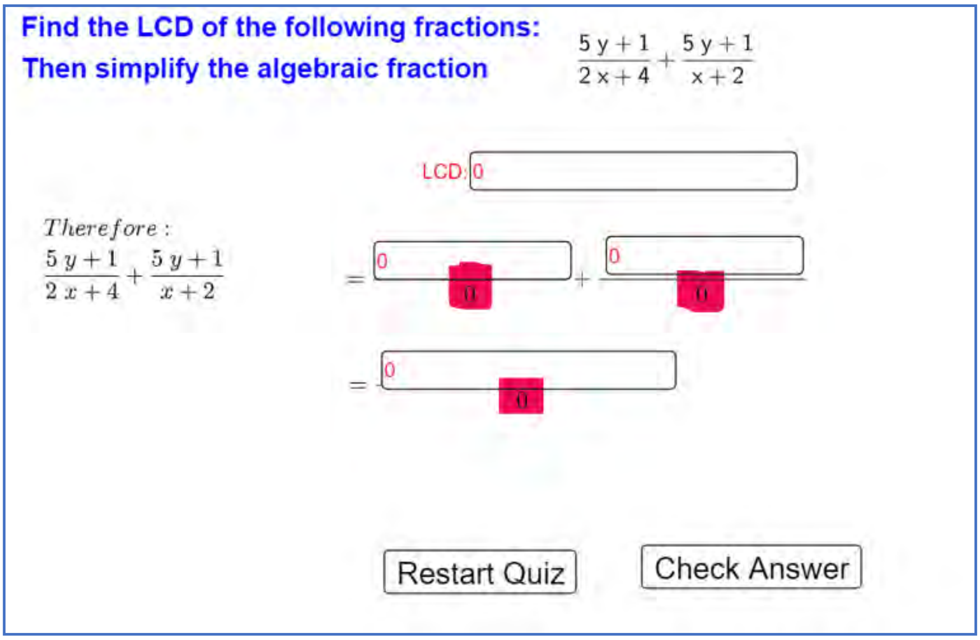



*Figure 4.5: Illustrating mathematically incorrect situations encountered with applets*

Already during the first cycle of the research during the applet selection process for data collection purposes, the importance of a well-qualified mathematics teacher to manage and facilitate the process of teaching with dynamic software was very clear. A mathematics teacher cannot be replaced by technology. This realisation was an unintended revelation in my research. As the research progressed, I continuously found confirmation that an able mathematics teacher is irreplaceable in the learning of mathematics through dynamic software. My research project confirmed that despite the visual learning opportunities offered by technology, the guidance of a mathematics teacher is critical to the learning process. A teacher's solid overall subject knowledge is vital for the optimum use of *GeoGebra* as a teaching and learning tool.

Figure 4.6 below is another illustration of an excellent visual applet that did not meet the requirements set out in the Applet Evaluation Instrument.

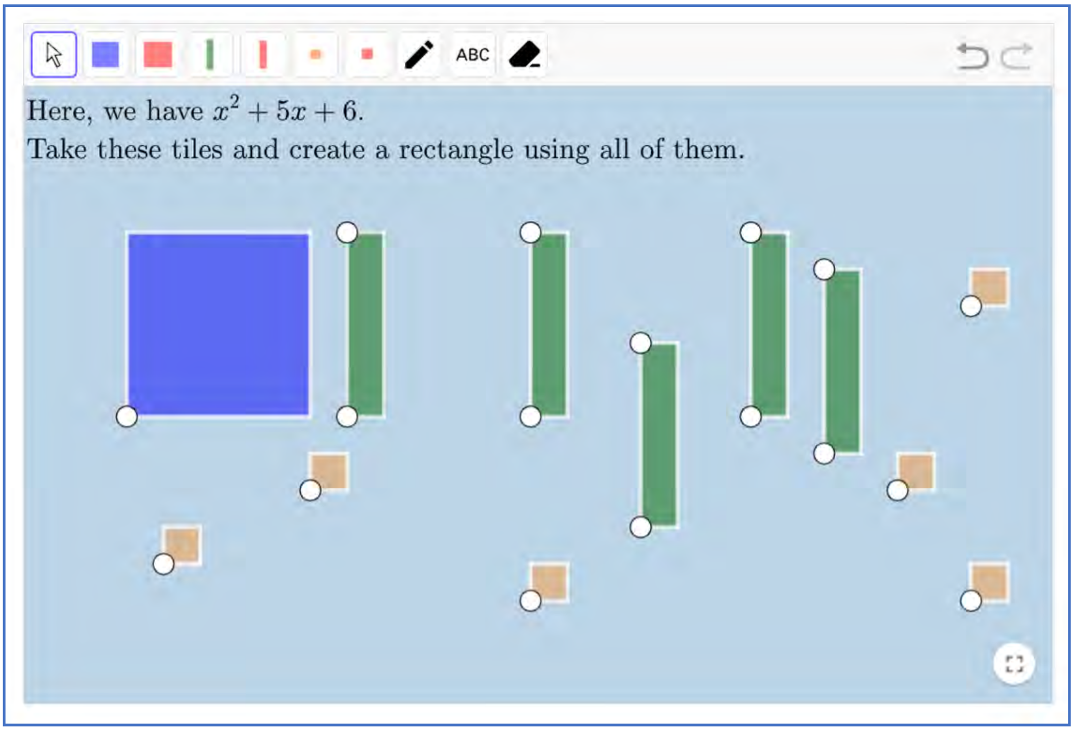



*Figure 4.6: A visual applet on factorisation trinomials*

This applet is based on the completion of a rectangle to visually illustrate the factorisation of trinomials. Initially the applet appeared to follow an exceptional approach, but on further investigation it was found that the applet did not provide the user with enough information to successfully complete the task. Firstly, no relationship between the sizes of the little tiles and the given expression were given. Secondly the completion of the rectangle was a very time consuming process to conclude successfully and thirdly, the applet failed to draw a relationship between the visual elements and the theoretical factorising of the trinomial.

Some applets such as Figure 4.7 below, were so overwhelming in terms of the quantity of information they tried to provide on one screen that they had to be rejected as learning material for the research process.

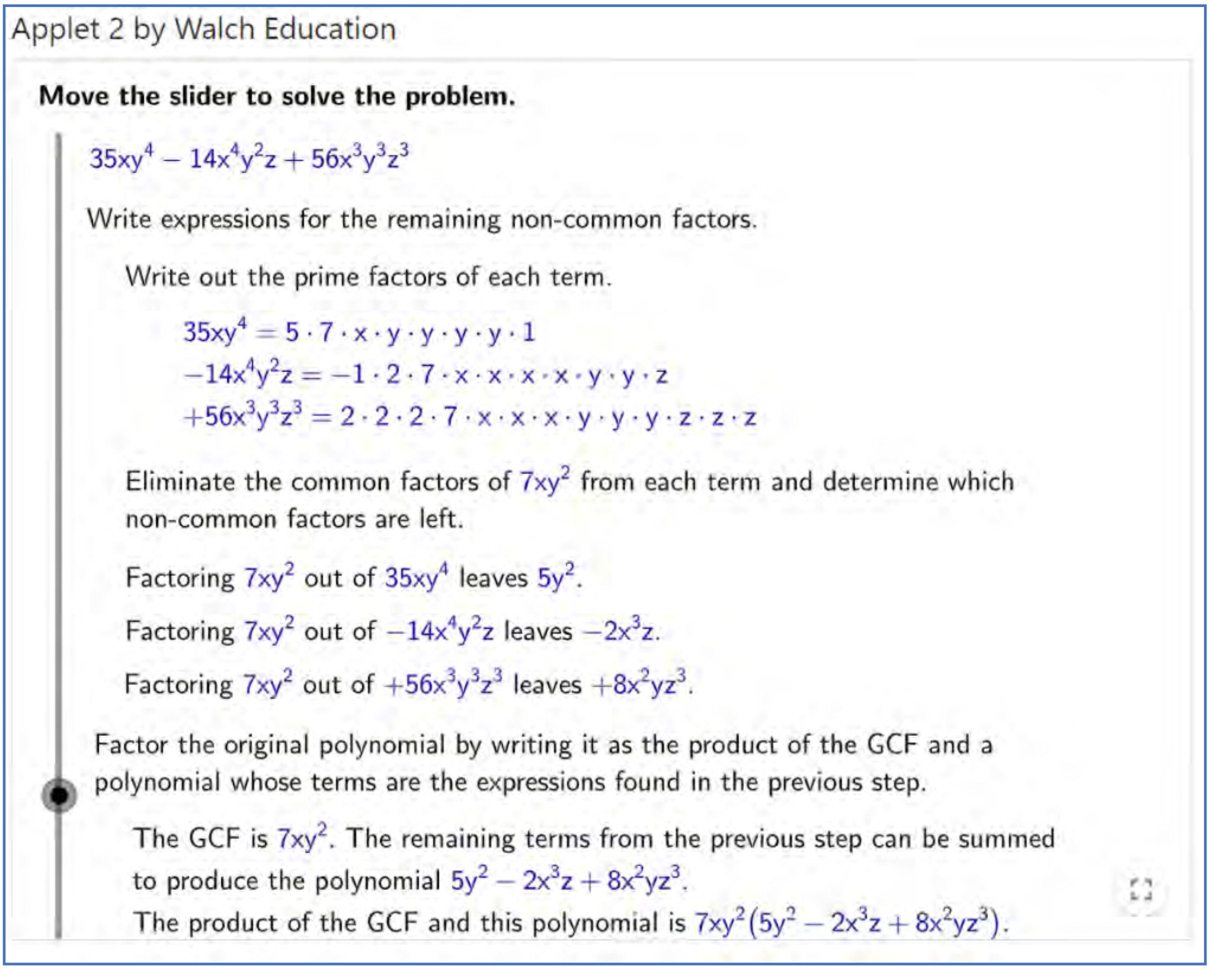



*Figure 4.7: Example of an applet flooded with too much information*

The small size of the computer screen and particularly phone screens was another factor that had to be considered when selecting applets. Despite the use of colour, the vast quantity of information that some applets, such as the one illustrated in Figure 4.7, was not clearly visible on the screen.

## 4.6 CONCLUSION

The selection of applets for inclusion in visual learning is a dynamic process that can never be concluded. Newer and better applets appear constantly on the *GeoGebra* platform. These will enable the teacher to constantly renew and improve his/her repertoire or arsenal of learning material. A dynamic teacher will have an ever evolving set of lessons on the *GeoGebra* platform to draw from for interactive visual lessons.

Full marks on the Applet Selection Instrument would provide an applet with a score of 60. A zero score on any of the first five categories of the instrument disqualified an applet from inclusion in the research project. Of all the applets evaluated for my project, the highest score awarded to any applet was 54 out of 60. It was decided that applets obtaining a mark of at least 80% or 16 out 20 would be included in the research material (Appendix A).

This cycle of the research added a dynamic component to the research process and had to be done repeatedly to ensure that only the best applets were used for data collection.

# CHAPTER FIVE

# DATA COLLECTION, ANALYSIS AND DISCUSSION OF TOPIC 1

## 5.1 INTRODUCTION

This chapter provides an in-depth narrative about the data collected in the first cycle of data collection and the interpretation and analyses thereof, within an interpretive paradigm. In this chapter the first algebraic topic is unpacked and analysed. A fixed framework guided the data collection process as it unfolded in pre-planned steps during this first cycle. Two different sets of quantitative and qualitative data were generated and had to be brought together to provide answers to the two main research questions.

This research explored several issues in unpacking the visual properties of technology in teaching algebra to junior secondary learners. The focus was to interrogate whether enhancement of the participants' conceptual understanding of algebra was evident, and to investigate any changes in their disposition towards learning algebra.

The research plan was to use *GeoGebra* to analyse and understand visualisation as a teaching approach to enhance conceptual understanding and to observe whether there were any positive changes in Grade 9 learners' dispositions towards learning the abstract concepts of algebra. Data was collected in the search for evidence that visualisation, when applied to learning algebra, could possibly enhance doing algebra conceptually and thus change learners' disposition towards doing algebra.

Streeb et al. (2019) argue that there is a large pool of evidence from researchers and practitioners that visualisation can indeed improve conceptualisation and the cognitive processing of mathematics. They point out the lack of a firm visualisation theory and the need for empirical research into machine-assisted visualisation (Streeb et al. 2019).

I built upon the notions of Cohen et al. (2011), who argued that a mixed methods approach is appropriate, especially when educational research is done. Participants went through several cycles of reflection, re-planning and implementing adapted teaching strategies. Stenhouse (1975) suggests that teachers should research their own practice with the support of professional researchers to change and improve their teaching practice. My research was done in a context of intermittent face-to-face and online learning and teaching due to the COVID-19 pandemic. The *GeoGebra* platform inadvertently provided me with new opportunities amidst the many challenges of the pandemic. The introduction of the *GeoGebra* classroom and the ability to create online group lessons allowed participants and non-participants to continue with their education. I was still able to collect data and communicate during interviews with my participants who were either actively involved in lessons, or online via either Google classroom or Teams.

## 5.2 GEOGEBRA LESSONS AND ACTIVITIES

It immediately became clear that many participants had had very little exposure to computer technology therefore I had to allocate time to allow all participants to log into the system and to master the loading of the *GeoGebra* platform. All participants had to be taught how to join a specific lesson or activity by typing in the lesson code on the platform or by joining with a shared link.

Earlier versions of *GeoGebra* lacked the facility that allows learners to participate individually or collectively in a classroom activity, but the recent inclusion of the 'create a lesson' option gave a totally new perspective to the use of the platform. Participants could now be invited to participate in a lesson and the selected practice activities by sending them a link. Clicking on the link would open the selected lesson and everybody could start working at their own pace. I, as the owner and creator of the lesson could see which participants had joined the lesson. Activities and lessons were either created by me or selected from a vast selection of pre-defined lessons and activities placed on the GeoGebra website (www.geogebra.org).

Features built into the *GeoGebra* lesson programme allow the facilitator to:

- see immediately when a learner started to work on a lesson;
- monitor the progress of the learner in real time via thumbnails;
- view answers of learners;
- pause a lesson, hide the names of the participants, share an individuals' screen with the whole class;
- edit and make changes to an individual's work; and
- give step-by-step instruction by using the overview function.
  (downloaded from https://www.geogebra.org/m/hncrgruu April 2021)

## 5.3 DESCRIPTION OF FIRST RESEARCH TOPIC: LAWS OF INDICES

I decided to be guided and limited by the prescribed topics and time frames of the new Namibian Grade 9 mathematics syllabus (MoEAC, 2015). For fairness and consistency, I collected all data within the time frames allowed by the official schemes of work of the school where I am a teacher. The schemes of work had to comply with the period allocations stated in the new Namibian syllabus for mathematics, but makes provision for extra lesson periods because our school has decided that learners will not write examinations at the end of the first quarter. We could continue with classes for about two weeks longer which allowed us to have about 12 extra lesson periods (MoEAC, 2015).

According to the new Namibian mathematics syllabus for Grades 8 and 9 (MoEAC, 2015) the topic 'laws of indices' should be introduced after a detailed study of number systems. The notation and terminology of powers and roots should be understood and applied in conversions of powers, with both positive and negative indices. The cases of fractional and zero indices should be introduced and learners should be able to apply the correct order of operations to numbers with powers and roots. A calculator can be used extensively to find powers and square and cube roots. As a cross curriculum topic learners are also expected to write numbers in scientific notation, convert them back to decimal form and be able to do the basic operations with numbers in scientific notation (MoEAC, NIED 2015 p.18). As scientific notation was the last section of the topic, I used the concept of writing numbers in scientific notation as an introductory lesson to index notation and the laws of indices. This was intended to enable a smooth transition from classroom instruction where learners were able to use their calculators to solve problems about scientific notation, to instruction using computers and *GeoGebra*.

The laws of indices are introduced to Grade 9 Namibian learners under the heading 'Algebraic manipulation' (MoEAC, 2015, p. 19). According to the syllabus (and our scheme of work) the laws of indices are seen as the introduction to formal algebra in Grade 9 and the general objective should be that learners should conceptually understand that "the transformation of algebraic expressions obeys and generalises the rules of arithmetic" (MoEAC, 2015. p. 19). The specific objective of the syllabus merely states that the laws of indices should be applied for expressions with a positive, negative and zero index.

## 5.4 PRE-DATA COLLECTION INDUCTION LESSON

In the first induction lesson using *GeoGebra*, the participants were requested to convert numbers into scientific notation. All participants were able to do most conversions correctly, by mechanically counting the decimal places either to the left or to the right of the decimal point. Once participants grasped the action required to convert to or from scientific notation, they could complete the lesson with ease. Most learners were able to score full marks with the applet. During the exercise I had short reflective conversations with individual participants and immediately found that learners were unable to visualise a simple number like $0.1 \times 10^2$ in any form other than what is prescribed by the scientific notation. Michael saw $0.1 \times 10^2$ as a *wrong* number that could not be converted into scientific notation. For him there was no relationship between the index of 10 and the 0.1 decimal number. Michael was then requested to write $0.1 \times 10^2$ as a decimal number, which he could do easily and after doing so, he was able to rewrite 10 into scientific notation.

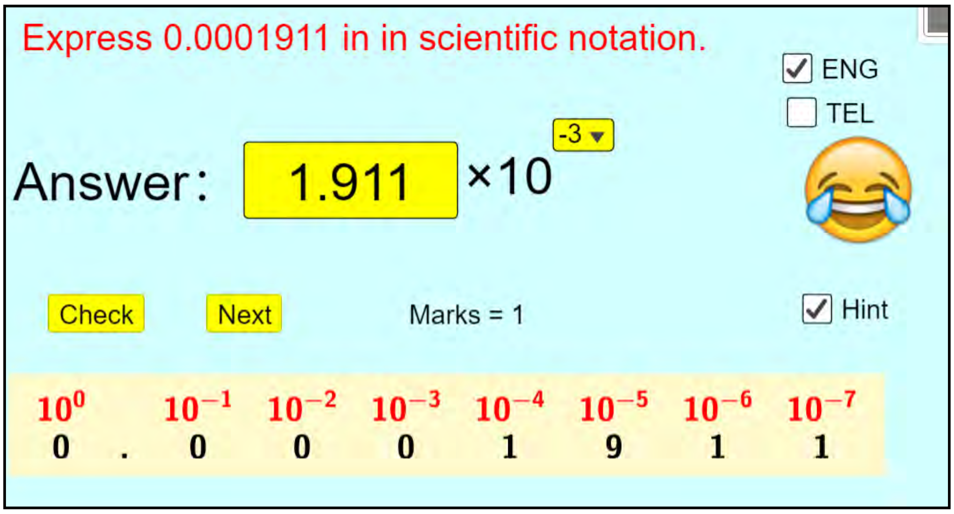



*Figure 5.1: Screen capture of a participant's incorrect conversion to scientific notation*

Source: www.geogebra.org

Figure 5.1 shows the work of Surya. While using the applet she did fairly well with the conversions, but in her visualisation, she had to move the decimal point three places from the right to the left to get to the answer of $1.922 \times 10^{-3}$. She could also not visualise the relationship between the current position of the decimal point and the actual number she was busy manipulating. She knew she had to move the decimal point to get 1.911 but lacked the visual insight that the position of the decimal point is 'connected' to the rest of the number.

In total, 13 participants from both the NAMVISPRO and school groups indicated that to them, $1.922 \times 10^{-3}$ represents two different numbers, namely, $1.922$ and $10^{-3}$. When asked to decide, by raising their hands, whether $1.922 \times 10^{-3}$ represents a large or a small number almost half of both groups said that it must be a large number. They argued that when a number is *multiplied* by ten, it must be a large number. Although they all were aware of indices, some were not able to visualise the role of the negative index.

From the introductory session I recognised the value of each participant working individually with the dynamic software. By working individually with the *GeoGebra* applets everyone *had* to respond and be involved with the posed problem in front of them before being allowed to continue with the applet. This is a major change from the normal teacher to class teaching situation where the teacher usually asks questions of the learners and instructs them on the next steps. In these circumstances, often only a few learners are actively involved in the lesson, and everyone is reliant on the teacher to instruct them step by step through the new concept. It is impossible for a teacher to know which individuals are really conceptually understanding the lesson. By allowing learners to work individually on applets, faster learners are also provided with an opportunity to progress on their own without being held back by slower learners.

While working with the *GeoGebra* applet, every participant had to interact with the applet to make progress. The patience of the applets and the demand to respond forced participants to make sense of the learning material and to cognitively respond to the applets. It was intriguing and insightful to see how different participants constructed meaning and made connections by themselves from the symbolic images on the screens in front of them. I found the process of constructing meaning for themselves to be in line with the definition of conceptual understanding by Capetta and Zollmann (2013) who asserted that learners apply and create their own meaning to understand a set of mathematical symbols, a process they called encapsulation (Capetta & Zollmann, 2014).

Already at this stage I saw the need to further investigate the process of forming a visual understanding of a set of mathematical symbols and expressions.

## 5.5 FIRST DYNAMIC SOFTWARE LESSON ONE: LAWS OF INDICES

### 5.5.1 Pre-lesson disposition questionnaire

In Grade 8 all Namibian learners are introduced to the algebraic concepts of linear equations, variables, algebraic manipulatives, index notation of numbers and unknowns. After the first introductory session in the laboratory all participants were requested to complete the Disposition Instrument. A summary of the results is presented in Figure 5.2.

| **Positive Disposition Questionnaire: Summary** | Present: 17(S) of School and 13(N) from NAMVISPRO | | | | |
|---|---|---|---|---|---|
| Let the emoji's help you to answer the following questions. | | | | Session: 1 | |
| | 1 | 2 | 3 | 4 | 5 |
| |  | | | | |
| For every question choose one emoji that will best describe your feelings. Mark your choice with an X below the emoji of your choice. | I detest it. Make me scared. Intimidating. | Boring. Not interesting. I do not understand it. | Makes me nervous. I don't care. Takes time to understand. | I like it. I will do more. I understand it. | I love it. I find it easy. I can use it in future. |
| (1) How do you feel about Mathematics in general? | 10 + 5 | 2 + 3 | 5 + 3 | 0 + 1 | 0 + 1 |
| (2) How do you feel about learning Algebra? | 12 + 5 | 5 + 5 | 0 + 3 | 0 + 0 | 0 + 0 |
| TOTAL QUESTION 1 | 15 | 5 | 8 | 1 | 1 |
| TOTAL QUESTION 2 | 17 | 10 | 3 | 0 | 0 |
| Mean per Question | 16 | 7.5 | 6.5 | 0.5 | 0.5 |
| MEAN SCORE FOR THE CASE: | 1.7 | | | | |
| MEDIAN SCORE: | 1.0 | | | | |

*Figure 5.2: Summary of scores on the pre-lessons with the GeoGebra disposition instrument*

Of the two groups which formed my case, 30 participants attended the first laboratory session – 17 participants from my school and 13 participants from the NAMVISPRO project. The NAMVISPRO community project participants gave themselves a slightly higher score on the Disposition Instrument, but for the whole case under investigation the mean score was only 1.7 out of 5, while the modal score was a mere 1 out of 5. This implied that on average the disposition towards learning mathematics and specifically algebra was concerningly less than 2 out of 5 for the whole case.

After the first session, participants were also requested to express their current feelings about mathematics by using the second instrument. They had to place themselves on a little podium with five places by drawing a little person (representing themselves) on the podium as illustrated in Figure 5.3.

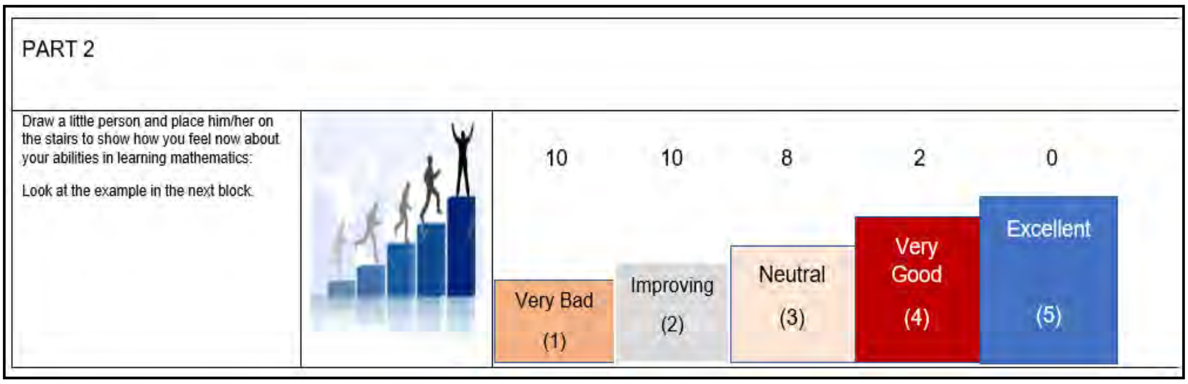



*Figure 5.3: Summary of scores of participants' feelings about mathematics*

Again, the participants did not value themselves as winners or even positive about learning mathematics. During the reflective interviews I interrogated the situation further to clarify the low placings. Figure 5.3 shows the combined results from all the participants. No participants placed themselves in the top position, and 20 out of the 30 participants placed themselves in the lowest two podiums. The results correlated well with the previous scores participants gave themselves with the Disposition Instrument.

### 5.5.2 Lesson 1: Execution and screen captures

The official scheme of work recommends that six lessons of 40 minutes should be spent on the topic 'laws of indices', including one lesson period for evaluation. Doing lessons in the laboratory required that participants log on and load the selected lessons before they could start working on the applet. I envisaged that it would be time consuming, but both groups of participants were eager to start working on the applets as they rushed to the laboratory and had no trouble logging onto the *GeoGebra* platform and opening the appropriate links to the chosen lessons. My early lesson (mentioned above) on how to begin navigating the computer clearly paid off. The

*GeoGebra* platform also allows participants to remotely join a lesson and complete work on their own.

In contrast to the accustomed class instruction where the teacher introduces the topic of laws of indices with appropriate examples on the whiteboard or in a data presentation to learners, here participants were taken to the computer laboratory and given a link to the selected *GeoGebra* introductory lesson on that topic. The first selected applet explained the first law of indices, namely $a^b \times a^c = a^{b+c}$. Participants could randomly move sliders to change the index of the randomly created base number. At the same time the participants could visually see *why,* for the same base, the indices are simply added. The Namibian syllabus for mathematics does not require learners to prove any laws of indices at this grade.

Figure 5.4 is a screen capture from Lesson One explaining the first law of indices visually to participants. Firstly, the applet uses a slider with numbers to visually connect the first number to the next. Participants had to anticipate the next step in solving the posed example. By ticking the appropriate box, the correct next steps are revealed and finally the answer is revealed. I observed that most participants took out pencils and papers to physically calculate and scribble the outcome of the posed problem. All participants were fully engaged with the applet and no participants simply ticked the boxes.

As soon as an individual participant felt ready to move from concrete numbers to the more abstract letters or variables, they had to tick a box and the applet would replace the numbers with letters. I observed that some participants navigated very quickly through the steps while others spent more time experimenting and pondering their responses. Eventually all participants showed their satisfaction with the learning process and indicated that they were ready to progress to the next applet. I did not intervene at any stage and allowed all the participants to engage with the applets for as long as they wanted to.

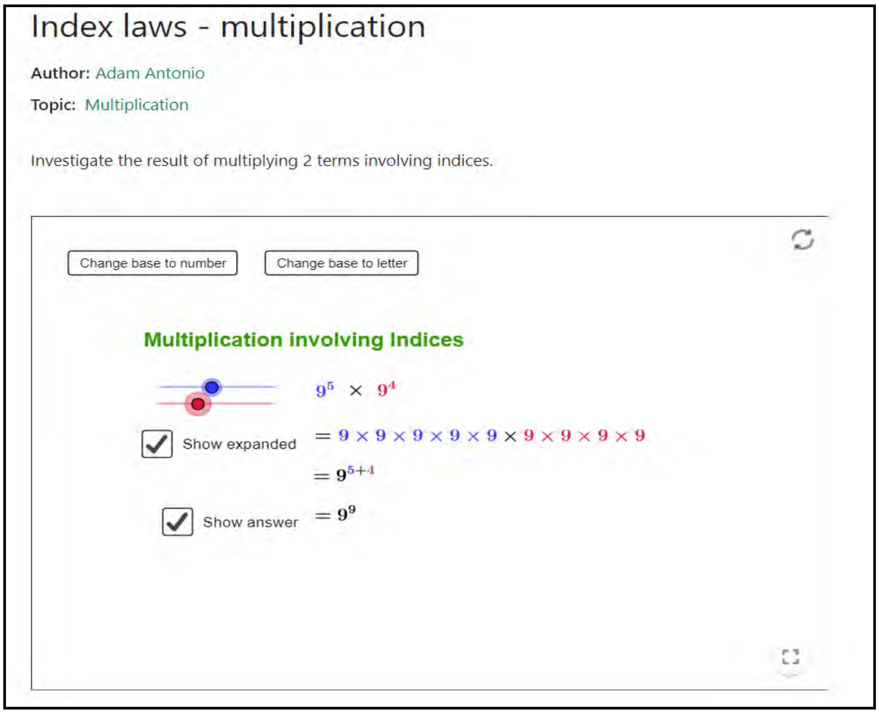



*Figure 5.4: First lesson on laws of indices, with sliders to explain the law*

Source: [www.geogebra.org/classroom](http://www.geogebra.org/classroom)

After the first short session, the participants had the opportunity to change the base to a random letter, while the same principle was explained visually, by using letters instead of numbers. Figure 5.5 shows a screen capture that illustrates how the applet allowed the learners to make the transition from numbers to the generalised expression of the first law by using variables as base and index. In both cases, participants were allowed to *visualise* the results for themselves, before ticking a box to show the expanded step of the multiplication law. Finally, the participants could see the results by ticking a second box. It was observed that *all* participants were able to apply the law after interacting just once or twice with the slider.

I noted that on their own, the participants started to move the sliders to zero to investigate the results of $a^0 \times a^0$. Again, they were allowed to explore interactively with letters as bases. Most participants quickly moved the slider to provide a zero index. Some visualised the answer as $d^{0+0} = d^0$, but could not fully visualise the relationship with a number with a zero index. That would have enabled them to get the answer $d^0 = 1$. During a discussion amongst themselves, some suggested that $d^0$

should be equal to 1. I could confirm their intuitive finding that $d^0 = 1$. An opportunity to introduce the next law was provided, because $\frac{x^a}{x^a} = 1 = x^{a-a} = x^0$. The visuality of the sliders used in the applet provided an opportunity for the participants to discover an important algebraic concept themselves.

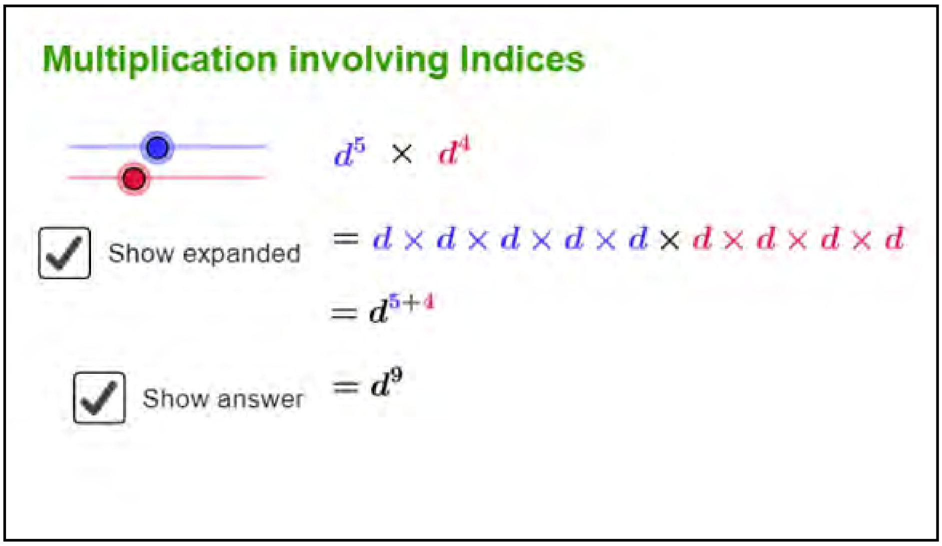



*Figure 5.5 First lesson on laws of indices with a letter base*

Source: [www.geogebra.org/classroom](www.geogebra.org/classroom)

As seen in Figure 5.5 the well-designed applet prevented participants from moving the sliders to negative integers, but some were intrigued as to what could happen with a negative index. This provided an opportunity to introduce a next law to them.

The participants progressed so well that the second law of indices, namely the division law could also be introduced during the first lesson. I was eager to let the participants do some practical exercises with the applets and decided to also introduce the second law of indices before allowing participants to do applet exercises.

The second law of indices, namely $\frac{a^p}{a^q} = a^{p-q}$, was introduced to the participants by again using the same approach. Initially the participants would be instructed to visualise the outcome of a random example using numbers, followed by an intermediate step where the concept was explained by showing the cancelling process. The last step was the displayed answer and the meaning of a negative was revealed. Finally, the applet moved from concrete numbers to abstract variables to demonstrate the law. It was found that the approach of using sliders to demonstrate the law helped participants to conceptually understand the law within a few minutes.

I anticipated that the process with the interactive applets would be time consuming, but soon concluded that engaging with the correct and appropriate applets is less time consuming then class teaching. As soon as participants felt they had fully grasped the concept, they sat back and waited for further instructions.

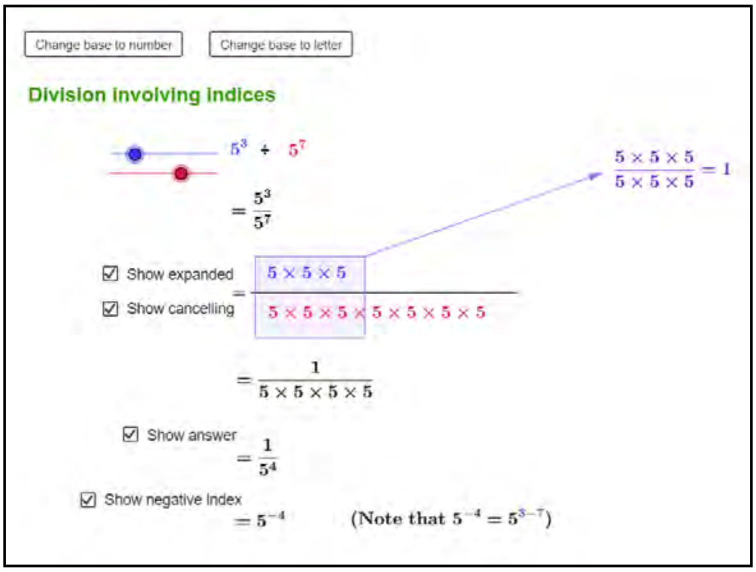



*Figure 5.6: Second law of indices with sliders and visualised steps*

Source: www.geogebra.org/classroom

Figure 5.6 illustrates the principles of the second law, using numbers. By clicking on the 'change base to letter', the participants could progress to the generalisation of the law. Again, I observed that some participants switch between numbers and letters. It confirmed the steps of conceptual understanding identified by Capetta and Zolmann (2014) – coordination, encapsulation and generalisation.

### 5.5.3 Lesson 2: Revision exercises and third law

The research was executed within the limitations of a real learning environment, including complying with the school timetable. I was not able to conclude the first lesson by doing revision exercises about the first two laws within one session because of time constraints The next scheduled session therefore started with a few interactive exercises to allow the participants to apply the acquired knowledge.

I chose different applets with three levels of difficulty – from elementary, intermediate to advanced for the participants to practise the acquired skills and to do some self-evaluation of their conceptual understanding of the laws.

Figure 5.7 illustrates their engagement with the first evaluating applet. The practical session took place one day after the introductory lesson on the first two laws of indices. I observed that of the two groups very few participants went back to the previous lesson while solving the problems with the applet.

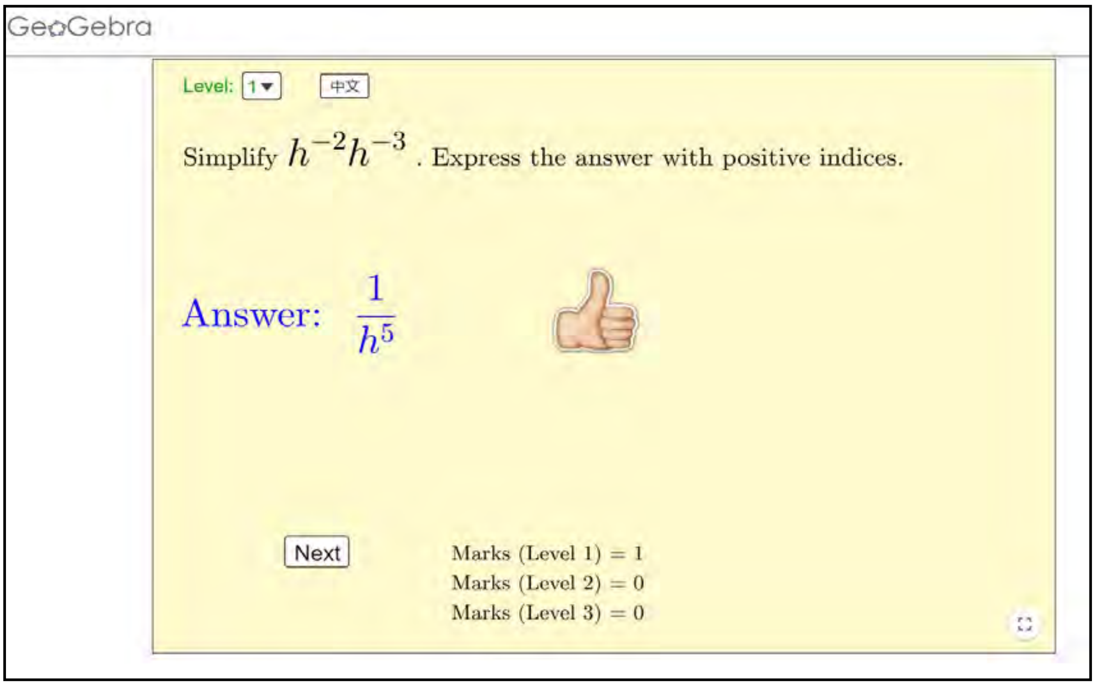



*Figure 5.7: Elementary exercise with GeoGebra posing questions on three levels*

Source: www.geogebra.org/classroom

In the chosen applet participants were rewarded with a little emoji to show when they had answered the question correctly. As participants progressed through the applet their score was displayed at the bottom of the screen. As facilitator I was able to track each one's progress from my desktop in the *GeoGebra* classroom. As illustrated in Figure 5.7, the applet does not allow for the participants to show any workings or intermediate steps when solving problems. Again, I observed that all participants used an extra piece of paper and pen to write and jot down the intermediate steps when solving the problems. These were kept for use during the reflective interviews. As the research progressed more and more participants wrote the laws of indices onto a piece of paper before attempting the applet. While solving problems with the applets, participants either solve the problem on paper before providing the final answer to the posed question or they used paper to give structure to their thoughts and efforts to solve a problem. When questioned why they divert to paper most of them refer to the necessity *to see* the steps towards the solution of the problem *"I have to see, how I get to the solution of the problem*." It provided me with evidence that the participants relied on visual imagery to complete their problems. It was also evident that the jottings provided the participants with a visual basis from where they could take different avenues to find a solution for a problem or return to if their attempts with a particular avenue reached a dead end.

As participants progressed to Level 3 questions, we recognised that these questions all required the third law of indices, namely $(a^b)^c = a^{b+c}$. I was pleasantly surprised that many participants intuitively started to rewrite the expression $(m^2)^3$ as $m^2 \times m^2 \times m^2 = m^{2+2+2} = m^6$, applying their previously acquired skills. In both groups some discussions among the participants took place about *seeing* the relationship between the first two laws and what is implied by the third law. Some participants started to peer teach the principle to others. Megan from the NAMVISPRO group mentioned that working with the applets provided her with an ability to *see* relationships between one law and the next one. I started to call it a 'relationship visualisation' as more and participants had this type of *"aha"* or *"I can see it"* moments as the research progressed. I noticed that the concrete structured images on the screens and the relationships with their underlying principles allowed participants to visually expand their knowledge to the next level or next concept. The visual imagery of laws 1 and 2 allowed Megan to visualise what would be implied by law 3. This finding was confirmed more than once during the interviews with participants.

Although screen captures were taken from individual participants it appeared that a certain trend was taking place among all participants of both groups of the case study. They showed an ability to visualise either in their minds or on-screen to integrate previously taught concepts into the next level of conceptual understanding. I called this process *relational-visualisation*. I believe it can be attributed to the tidy structured visual imagery that the participants constantly had in front of them on their screens. This allowed participants to draw clear visual links from one concept to another. During the work with the applets, the participants often made remarks pointing towards some form of recognition of a problem based upon a previous example or finding a relationship between the explained work and a problem at hand. I could often hear: "*This is the same as the example*" or, "*I recognise this problem from the second example.*" Another remark often heard was: "*I see the example in this problem.*" To me it supported evidence that the participants drew from some form of visual imagery created during the lessons with the revision applets.

When reflecting on lessons afterwards similar remarks were made. Evidence was emerging that the ability to think deductively and to see relationships when solving problems were based upon the re-appearance of images in some format in their minds. During the reflective interviews participants seldom said: "*I remember the law*," but often rather used phrases like "*The laws came back into my mind. I still see them on the screen*" (referring to images created by the computer.)

This is further supported and demonstrated by the next example from Figure 5.9.

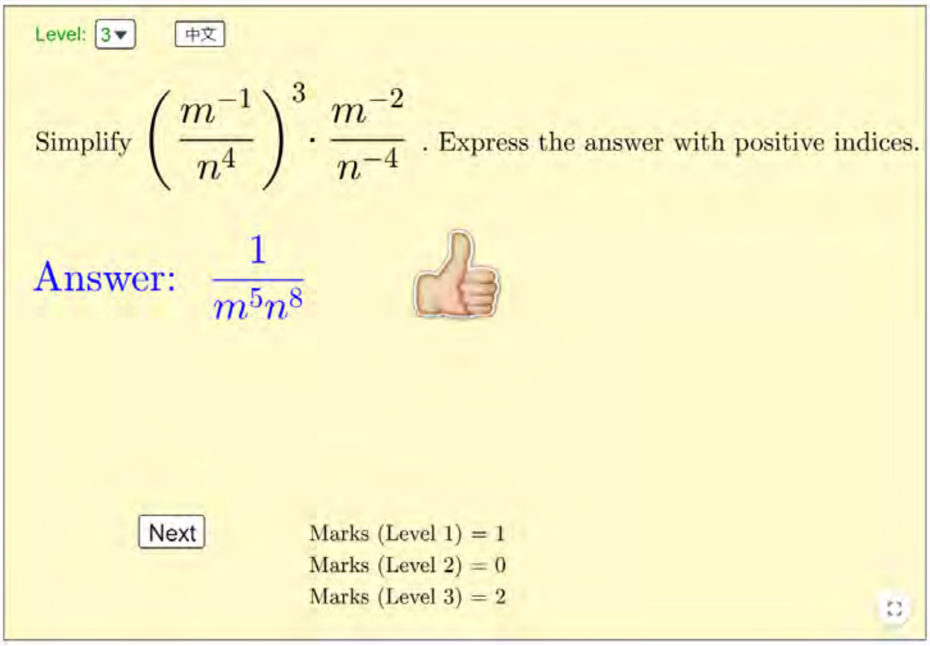



*Figure 5.8: Delvin's unsuccessful attempts at Level 2*

Source: www.geogebra.org/classroom

Figure 5.8 is a screen capture of the work of Delvin, one of the NAMVISPRO participants as I observed his progress with the revision applet. Initially he attempted only one Level 1 question, found it too easy and moved onto Level 2 questions. Delvin was unable to correctly answer any of the Level 2 questions but decided to attempt

Level 3 questions. Suddenly he was successful with all the Level 3 questions even though they were significantly more difficult that the Level 2 ones. I decided to sit with him to gain insight into what was happening. As he continued, he explained, "*Sir, now I can see what the relationship between each law and the questions is. Level 1 is just the law arranged differently. On Level 2, I just guessed, but here on Level 3 I can see there is a direct link between the law and the question. As soon as I was able to see the link in each question, I can apply the different laws to every part of the question"*.

Eventually Delvin completed 12 Level 3 questions and made only one calculation error. I tried to monitor him during further lessons. He gained confidence and constantly tried to visualise the relationships between the theoretical laws and their applications, by putting his hands over his eyes before attempting the next step of a problem.

At this stage not all participants had acquired the ability to visualise relationships between theoretical concepts and practical problems or evolving new theoretical concepts. I was cautious to find out if the relational visualisations would develop as we progressed with the research. I eagerly wanted to apply and investigate this suspected type of visualisation in further lessons.

Participants were never informed that the research was about visualisation or about their dispositions towards learning algebra. I listened to the vocabulary used by the participants when they spoke to each other about their experiences within the visual environment offered by the technology we used. While assisting another participant, I often heard the following: "*Try to see the relationship between this law and the problem.*" "*Can you <u>see</u> that this problem has more than one law in it that you should apply*" "*Keep a little <u>picture</u> of the laws in your mind while you try to solve this problem*." "*When you <u>see</u> the problem, you must also have a <u>picture</u> of the law in your mind.*"

Further lessons on the laws of indices had to follow and the question remained whether participants could construct a form of visual imagery of all the laws together. I also wondered whether they would be able to visually identify the laws of indices when they re-appeared in an unrelated topic like algebraic fractions.

### 5.5.4 Lessons 1 and 2: Reflective interviews

In my research design I made provision for reflective interviews to be done after every lesson. My study had elements of a longitudinal study over an extensive period, requiring that I collect data not only after each session, but monitor changes over a long period. All interviews were done while bearing in mind my own positionality, and ethical considerations towards participants. This was discussed in the methodology chapter.

Staying within the framework of a normal school day with timetables and learner programmes to be adhered to, I had to make separate appointments with participants for interviews to allow for data collection. The literature suggests that it is desirable to allow time for participants to re-think their actions before being interviewed (Bolton, 2010).

During the first reflective interviews I found that the participants were eager to share their experiences with me. I believe that this confirmed that a relationship of trust between me and the participants had been established. I was confident that I could work through a list of semi-structured questions with each interviewee. For ethical purposes I told the participants that notes would be taken, but that anonymity was guaranteed.

When analysing the interviews, I found very similar responses to certain questions. I experienced honesty and openness from participants. When I analysed responses to questions there was a developing trend by participants to repetitively use words and phrases that I could link to underlying visualisation abilities they showed or started to develop. Examples like "*I have a picture in mind how to apply the law for this problem*" or "*When I see the problem, I can also see the law*." These questions remained my focus during further interviews.

### 5.5.5 Responses to some interview questions discussed

To my question *"Did you enjoy working with the applets?"* A 100% "*yes*" response was received. *"Why do you say yes?"* Most answers centered around the fact that learners felt "exposed" within the classroom situation, especially when they were asked questions that they could not answer immediately. Several responses from the respondents suggested that they enjoyed working with technology, mainly because the computers offered them the opportunity to revisit and re-do theories or problems that they did not understand immediately. Almost 50% of the participants' responses indicated that they preferred to work on their own and enjoyed the privacy and anonymity of the computer environment. Darren said: ''*When I am alone on the computer, I can be more adventurous, without anybody laughing at me."* More than 50% of the respondents said that "*the computer helped [them] to immediately see* [*visualise] the relationship between the rule [theoretical laws] and the given problems."* I interrogated these responses further and it appeared that participants were able to create mental links back to the theoretical law the moment they saw a problem. Gabby told me: "*For the first time when I see a problem, I ask myself: Now what can I do here to solve the problem? Then I can see how the different laws can be applied to each section of the problem."* She was supported by the remarks of Requel: *"I can <u>see</u> how the laws are hiding inside the problems. This never happened to me before, because we are never allowed time to think on our own."* I could not establish whether was perhaps due to the large screens close to them, which participants could focus on more easily in comparison with class teaching where the theory was presented on a board further away. Usually, during a class lesson the theory is presented step-by-step with the verbal explanations by the teacher. Participants preferred the visually exclusive presentation of the computer. The lessons on *GeoGebra* are *not* narrated lessons, but solely reliant on the visual imagery and written explanations on-screen. I could not help but notice how silent it was while participants were working on the computers.

Another question that got a 100% response was the question *"Would you like to have all your classes on the computer?"* All participants responded with "*No*" and reasons varied from "*We also like it when you teach us*," and "*You know exactly what we don't know"* and "*You can explain very well*," to "*I think we like the personal atmosphere and interaction in our classroom.*" A general feeling amongst many participants was summarised by one respondent as: "*It is still mathematics that we are doing on the computer, and I do not like mathematics*."

Lastly, participants were asked what made learning with the computer different from class teaching. I expected that their answers would focus on their enjoyment of working on a computer, but to my surprise I received answers that I could relate to visualisation. Examples of words and phrases they used during conversations with each other or during interviews with me varied from "*I can **see** the connections.*" "*Different **pictures** formed in my mind,*" "*I have **an image** in my brain of the laws*", to "*There are links between things"*. I concluded that the visual impact the first lessons had on the participants was surfacing through their answers, although it was clear to me that they were unaware of the role the visuality of the applets played.

I assumed that Grade 9 learners cannot always express in words the abstract process occurring between the images in their minds and the complex processing of those images, to conceptually apply it to a problem they must solve. I therefore refrained from making any suggestions or trying to lead them towards answers during interviews.

### 5.5.6 Diagnostic outcome of one specific reflective interview

The interview with Mike needs mentioning. Mike has always been hard working but showed little academic progress. He never answered any questions in class and preferred to sit apart from other learners. He stayed on during breaks in my class and preferred to spend time on his tablet rather than play with his classmates. He was invited to the reflective interviews because I observed that he progressed rapidly through the applets and he achieved excellent results with the revision applets. When the interview started, he indicated that he would prefer to write down his responses instead of verbally answering them. Knowing his almost unreadable handwriting, I requested that he take his time and to rather type his responses to me. In his first response to the question of enjoying working with the applets he typed that he cannot operate amongst other people and that he has a total block when it comes to communicating verbally. He responded further that the computer environment removed him from acting socially with others. The computer provided him with a situation where he could *hide* behind the screen. When he is alone and isolated by the computer his ability to concentrate improves, therefore he can be more productive. This anti-social behaviour concerned me and left me with a dilemma of whether I should talk to his parents or not. From an ethical research perspective, I was not sure whether I would compromise the trust that I had established with Mike, but from a professional perspective, I felt that it was my responsibility to alert his parents. After much thought, I decided to communicate with his parents and make them aware of his anti-social tendencies. After an evaluation by a psychologist Mike was diagnosed with mild autism. He decided to remain part of the research project and improved his marks in mathematics from passing the previous year with 40%, to 52% in the final Grade 9 examination. Eventually his parents decided to withdraw him from the mainstream school and to enroll him in specialised home-schooling. I am convinced that teaching with dynamic software in all his subjects would have been to his benefit and staying in a mainstream school could have been to his advantage. To my knowledge are there no schools in Namibia for learners with autism.

### 5.5.7 Post-Lesson 1 and 2: Results on the Disposition Instrument

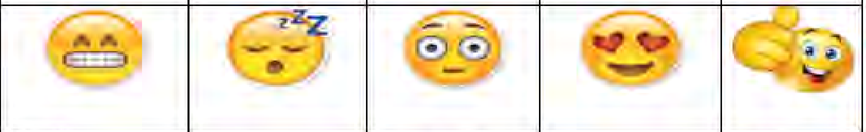
| Positive Disposition Questionnaire: Summary | Present: 17(S) of School and 13(N) from NAMVISPRO | | | | |
|---|---|---|---|---|---|
| Let the emoji's help you to answer the following questions. | | | | Session: 2 & 3 | |
| | 1 | 2 | 3 | 4 | 5 |
| |  | | | | |
| For every question choose one emoji that will best describe your feelings. Mark your choice with an X below the emoji of your choice. | I detest it.<br>Make me scared.<br>Intimidating. | Boring.<br>Not interesting.<br>I do not understand it. | Makes me nervous.<br>I don't care.<br>Takes time to understand. | I like it.<br>I will do more.<br>I understand it. | I love it.<br>I find it easy.<br>I can use it in future. |
| (1) How do you feel about Mathematics in general? | 5 + 0 | 7 + 4 | 4 + 4 | 1 + 4 | 0 + 1 |
| (2) How do you feel about learning Algebra? | 10 + 3 | 5 + 1 | 1 + 4 | 1 + 5 | 0 + 0 |
| TOTAL QUESTION 1 | 5 | 11 | 8 | 5 | 1 |
| TOTAL QUESTION 2 | 13 | 5 | 5 | 6 | 0 |
| Mean for 2 Questions | 9 | 8 | 6.5 | 5.5 | 0.5 |
| **MEAN SCORE FOR THE CASE:** | | **2.3** | | | |
| **MEDIAN SCORE FOR THE CASE:** | | **2** | | | |

*Figure 5.9: Summary post Lessons 1& 2 on the disposition scale*

Figure 5.9 is a summary of the results of the post-lesson dispositions of the participants after completion of the first three laws of indices and a revision exercise on *GeoGebra*. I was confident that there would have been a relatively big shift towards a positive disposition, due to the excitement that was displayed regarding working in the computer laboratory without any class instruction. I was however a little surprised at the relatively small shift towards a more positive disposition. The mean score shifted from 1.7 to 2.3 and the average from 1 to 2 for the whole case, despite the observations and my assumptions that that the whole research group was very excited to learn mathematics using technology. It was clear that the participants had not overcome initial fears, uncertainties and feelings of incompetence when doing mathematics in the context of our lessons. Up to this stage the research participants had not yet written any benchmark achievement tests to measure themselves in terms of their own conceptual understanding of the concepts taught through technology.

It is compulsory for all learners to have a homework manuscript book in which written review exercises are done and class tests are written. I had been advised by my Head of Department to revert back to the written exercise books from time to time and that participants should also do written exercises covering the prescribed syllabus topics. The school expected me to have hard copies of the participants' work and the participants would still have to write tests and examinations of all the work covered. I was however, hesitant to interrupt the flow of my lessons and on-screen exercises, therefore a decision was made to compile electronic worksheets covering the textbook exercises and to request participants to type their answers onto the worksheets. The worksheets were then printed by all participants and glued into their workbooks for marking by me. It appeared that although they found typing mathematics time consuming, they enjoyed doing mathematics homework on the computer. Unfortunately, I had to adhere to the instructions from my superiors and could not do the required achievement test online. I also could not find any ministerial restriction on doing online tests and decided that I would try it during a future cycle.

## 5.6 SECOND AND THIRD LESSONS: LAWS OF INDICES

### 5.6.1 Pre-lesson disposition scale

The initial research design made provision for participants to complete a pre- and post-lesson Disposition Instrument for each lesson. As the research progressed, however, I found that it would be time consuming to re-do the exercise for every lesson so soon after completing the instrument just after the previous lesson. I decided therefore to give the participants the disposition questionnaire after each completed lesson that covered a completed sub-section and after homework about the specific sub-section was done. This allowed me more time for reflective interviews. Participants had to attend at least one or two lessons per day as the time allocation prescribed by the

Namibian syllabus and by the scheme of work I had to adhere to (MoEAC, NIED, 2015).

Each session started with a few minutes of reflection and oral revision of the previous work, but time did not allow for any revision exercises using applets. The main purpose of the short reflection and questioning sessions was to re-establish the relationship of trust between me and the participants and to constantly reiterate the importance of each lesson. The research replaced class teaching and we had no opportunity to re-do any of the topics. Dates for achievement test were set by the scheme of work and had to be written simultaneously with the rest of the Grade 9 learners not participating in the research.

### 5.6.2 Screen captures of the second and third lessons

I was convinced that the participants were now ready to progress faster through all the remaining laws of indices, even though they were still unaware of symbolic visualisation (Presmeg, 2016) and relational-visualisation skills. In the first few lessons I made them more aware of the approach followed by the lessons and everyone I observed consciously searched for the underlying principles that the lessons tried to put across. I believed that they had acquired the ability to firstly see the relationships between the different laws and secondly, that all the laws have a logical structure. The next selected applet sub-divided the main laws into further laws. I found them to be useful and introduced them to the participants in a similar way, using sliders and abstracting from numbers to symbols.

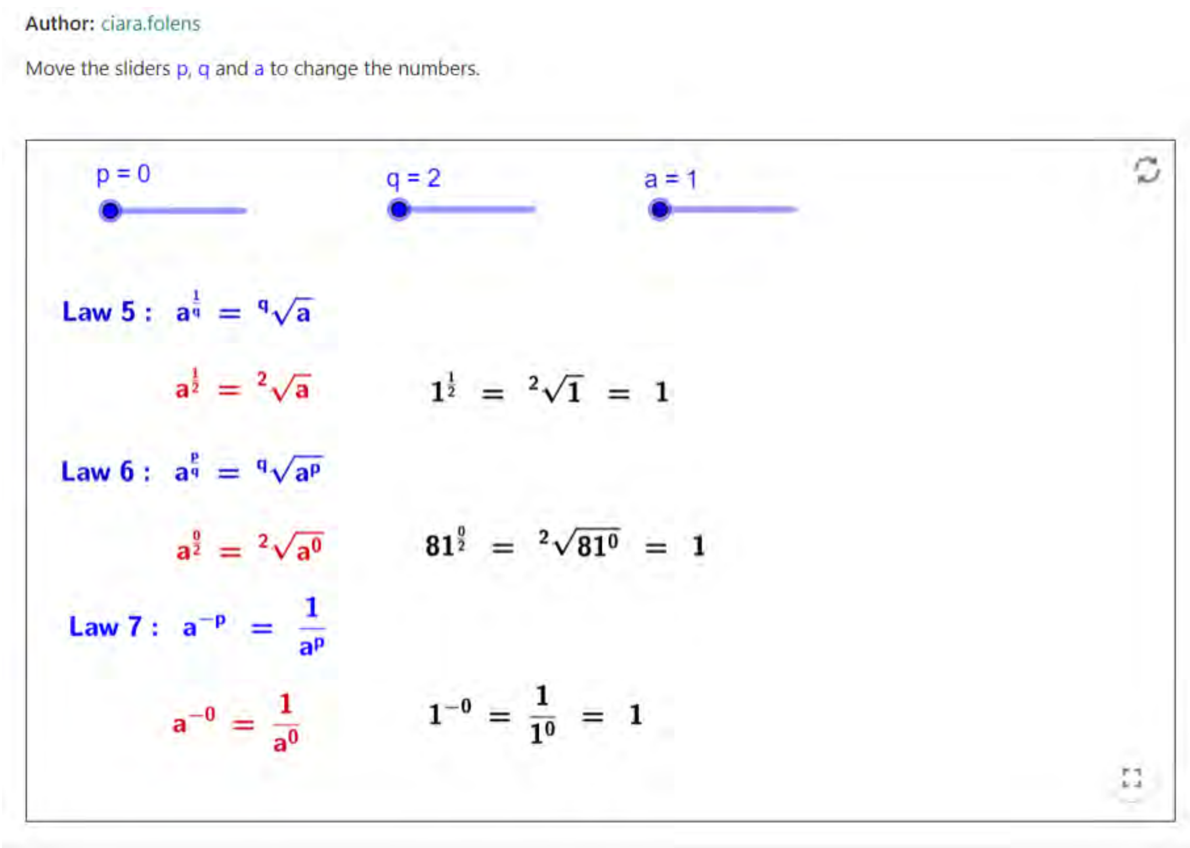



*Figure 5.10: The next set of laws of indices introduced with sliders and numbers*

Source: [www.geogebra.org/classroom](www.geogebra.org/classroom)

During the introductory session on the sliders and abstraction from numbers to variables as illustrated in Figure 5.10, it appeared that the participants were confident with the laws. They were already comfortable with roots but were unable to draw the relationship between the root notation and index notation. During a little diagnostic exercise with a revision applet, I found that participants were confused about the relationship between roots and indices.

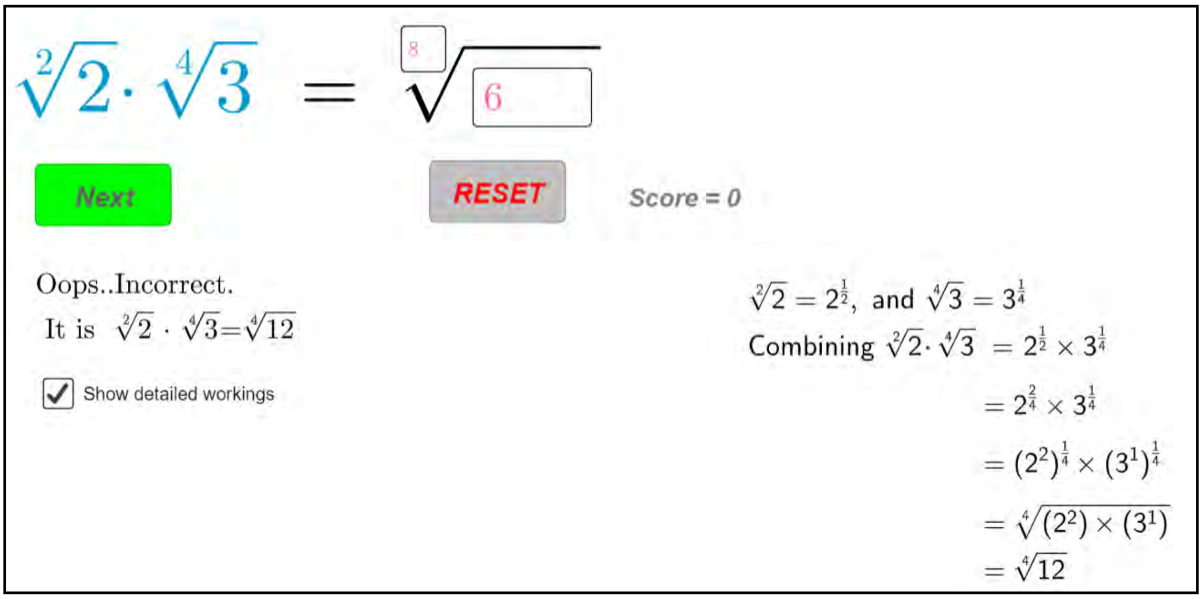



*Figure 5.11: Screen capture of participants struggling with roots and indices*

Source: www.geogebra.org/classroom

As illustrated in Figure 5.11 participants were struggling and the session had to be interrupted so that they could be reminded that during their study of number systems, they could easily calculate the square root and cube roots of large numbers by using products of prime factors. Nobody was able to visualise the relationship between $\sqrt[2]{81} = \sqrt[2]{3 \times 3 \times 3 \times 3} = \sqrt[2]{3^4} = 3^{\frac{4}{2}} = 3^2$ (which they could do very well a few lessons ago) and $\sqrt[a]{b^c} = b^{\frac{c}{a}}$. I had to point out the similarities between what they did in class and what was displayed on the screen. Using prime factors to find roots was done before the research within a classroom teaching situation and I had to conclude that it was difficult for participants to visualise relationships between visual imagery obtained on the whiteboard with the imagery obtained on the interactive computer screen.

I had the privilege of demonstrating *GeoGebra* applets on the large interactive television in the laboratory. The interactive television is a complete computerised system with a real time internet connection and full processing capabilities, directly connected to the network in the laboratory. After a very short illustrative session on the large interactive television screen, participants continued with the laws on their own. They engaged with the applets again and I asked them to spend more time with each law to ensure that they grasped the illustrated concepts.

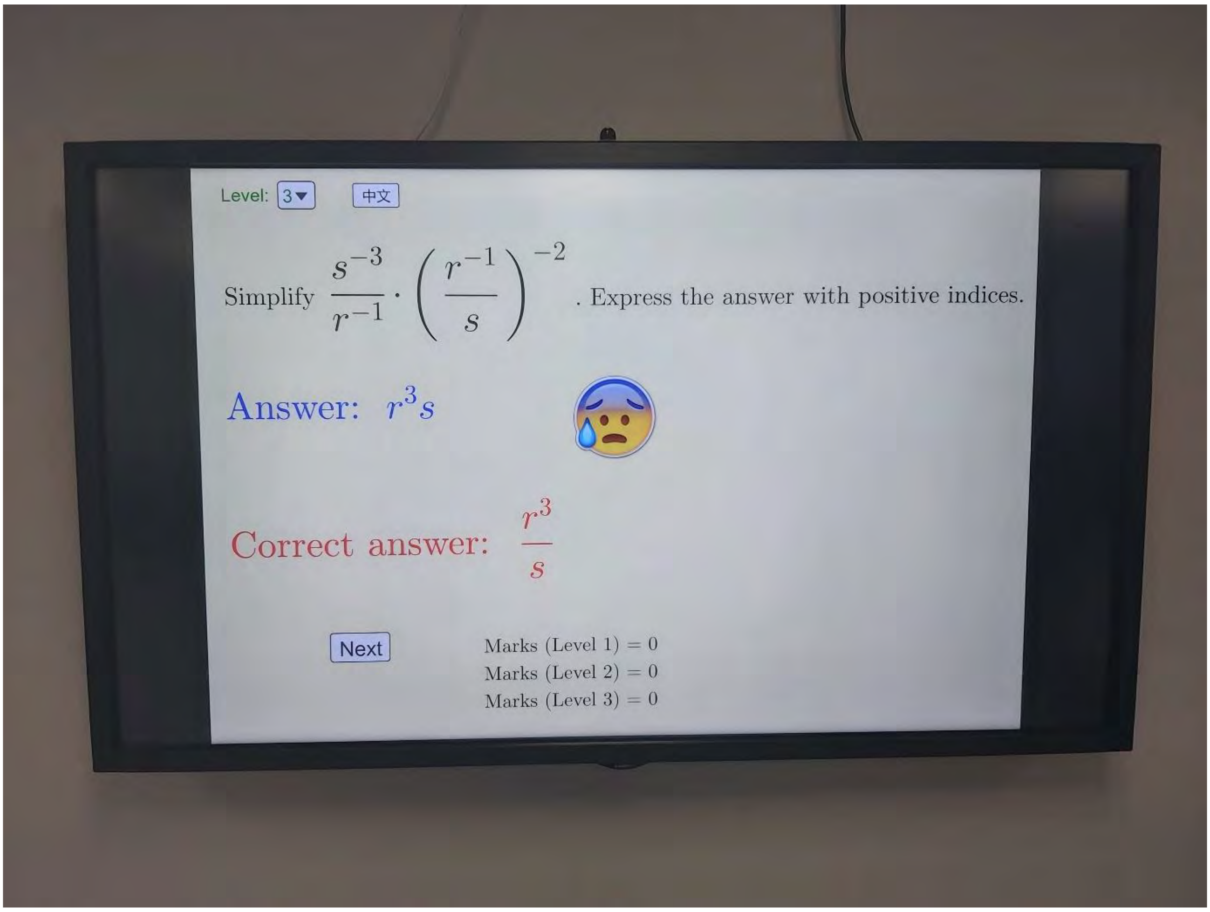



*Figure 5.12: How applets are introduced to participants on the interactive TV*

I decided to split the lesson into two separate lessons and rather deal with the last few laws during another lesson. All participants put in a great effort to conceptually understand the laws, especially where the slider moved to the zero position as illustrated by law 7 in Figure 5.10. I observed that the communicated with each other and that they relied on peer teaching to explain concepts to each other.

I also noted that the applet covered examples that would normally not be included in a classroom instruction situation where only the whiteboard is used. The exceptional situation of law 7 in the Figure 5.10 where $a^{-p} = \frac{1}{a^p}$ and $p = 0$ is one example. We concluded the lesson by doing a practical revision exercise using one of the revision applets. As participants progressed with the exercises, they remarked that the *tidiness* *(structured layout) of the algebra on the screen, the close proximity of the examples to their faces* and "*the fact that we can communicate with the computer*," (interactive reasoning opportunities) all contributed to their active participation and their conviction that they understood the concepts. It was also apparent that after their initial struggle to grasp the last few laws, they enthusiastically approached the revision applet and concluded it with an acceptable level of success.

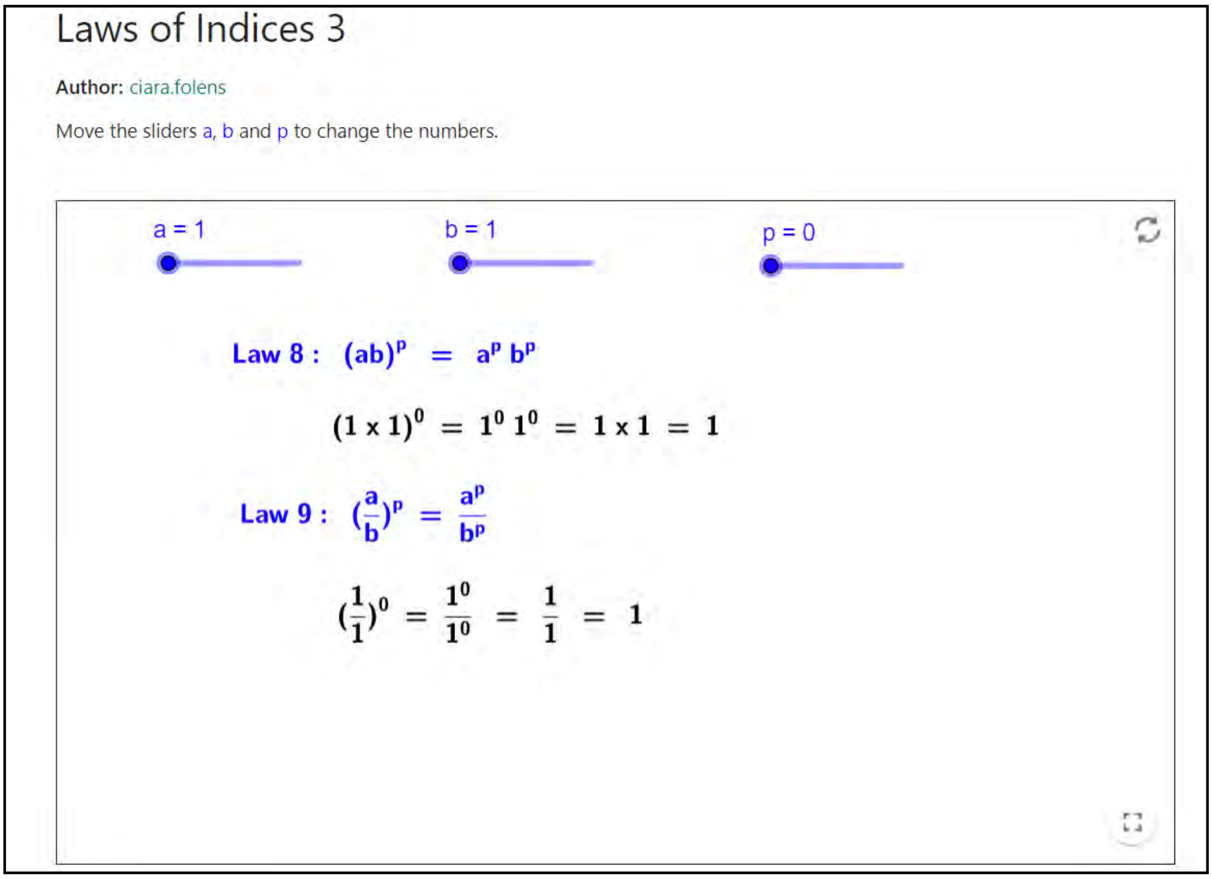



*Figure 5.13: Screen capture of the last lesson on indices*

Source: [www.geogebra.org/classroom](www.geogebra.org/classroom)

The last formal lesson on the laws of indices took participants only a few minutes to complete. We could thus spend time on doing revision applets about all the laws of indices. This was also the time that participants had to prepare for the standardised achievement test on the first researched topic.

### 5.6.3 Revision applets on all the laws of indices

With class teaching, learners must receive written homework tasks that they have to do at home and in their homework manuscripts. Due to the nature of the research, I planned that at least a section of the revision homework be done in class and on the computers in order to capitalise on the graphical nature of computers. During the applet selection period I included applets that served that purpose. Figure 5.14below is an example of a revision applet that covered all the laws of indices. The applet randomly posed questions to the participants. When answered incorrectly, a similar question is then given to the participant. Participants had the opportunity to continue until they were certain that they had reached a satisfactory level of achievement with the applet.

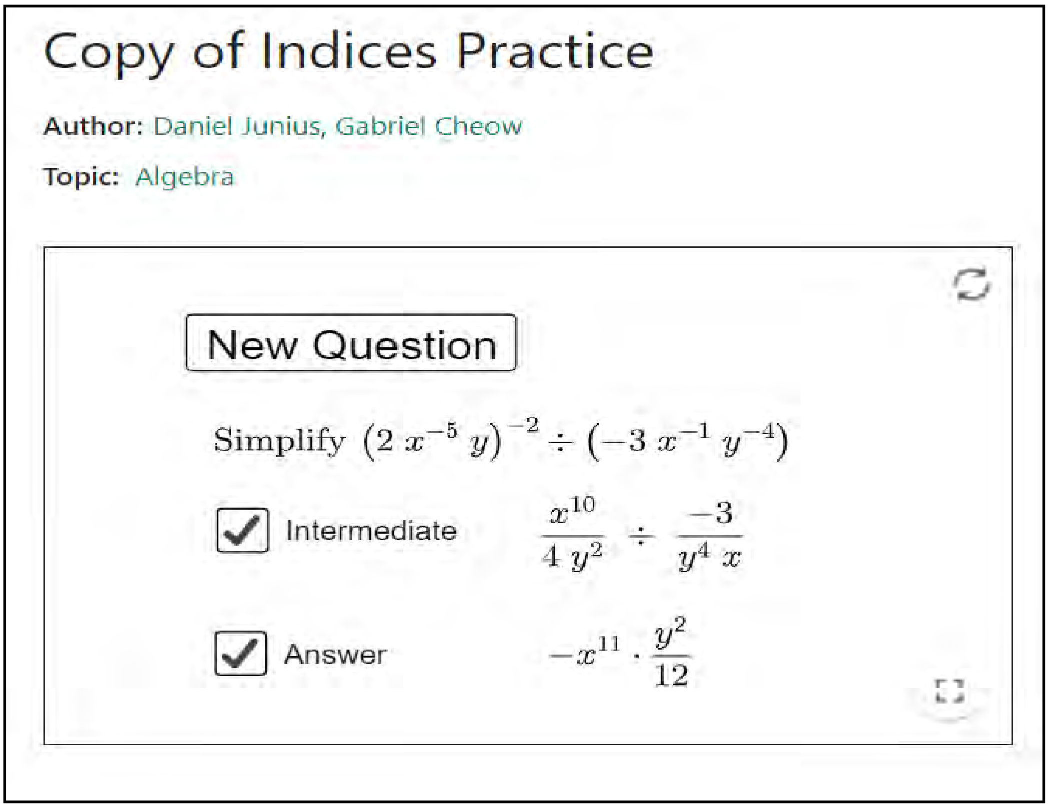



*Figure 5.14: Screen capture of mixed examples revision applet*

Source: www.geogebra.org/classroom)

Participants had the opportunity to receive questions in various formats, such as $\frac{a}{b}$ or in $a \div b$. This gave them the opportunity to prepare for an achievement test that was set by another person who could potentially deviate from the format of questioning that they were used to. I was satisfied that the posed questions were of a higher standard that what was normally expected from Grade 9 learners.

**5.6.4 Post-lesson 3 and 4: Results on the Disposition Instrument**

After the conclusion of the formal lessons and the revision applets I requested that participants complete the Disposition Instrument again.

| **Positive Disposition Questionnaire: Summary C1 Post- L3** Present: 14(S) of School and 13(N) from NAMVISPRO | | | | | |
|---|---|---|---|---|---|
| Let the emoji's help you to answer the following questions. | | | | Session:C1 L3/4 | |
| | 1 | 2 | 3 | 4 | 5 |
| |  |  |  |  |  |
| For every question choose one emoji that will best describe your feelings. Mark your choice with an X below the emoji of your choice. | I detest it.<br>Make me scared.<br>Intimidating. | Boring<br>Not interesting.<br>I do not understand it. | Makes me nervous<br>I don't care.<br>Takes time to understand. | I like it.<br>I will do more<br>I understand it. | I love it.<br>I find it easy.<br>I can use it in future. |
| (1) How do you feel about Mathematics in general? | 6 + 3 | 4 + 7 | 3 + 1 | 1 + 1 | 0 + 1 |
| (2) How do you feel about learning Algebra? | 7 + 3 | 3 + 5 | 3 + 3 | 1 + 1 | 0 + 1 |
| TOTAL QUESTION 1 | 9 | 11 | 4 | 2 | 1 |
| TOTAL QUESTION 2 | 10 | 8 | 6 | 2 | 1 |
| Mean per Question | 9.5 | 9.5 | 5 | 2 | 1 |
| MEAN SCORE FOR THE CASE: | | | 1.8 | | |
| MEDIAN SCORE | | | 1.0 | | |

*Figure 5.15: Post Lessons 3 and 4 Results of Disposition Instrument*

I was expecting that after the completion of the first topic a major shift might happen, but this did not occur. I did take notice of the shift at the bottom of the scale as it appeared that fewer participants detested algebra and moved to the second last position on the scale, indicating that they were still not interested or still had trouble understanding the work. The mean score for the case improved slightly to 1.8 while the median score remained on 1. I found it disappointing. At least I could see that participants were honest in expressing their feelings.

## 5.7 POST-TOPIC DISPOSITION PODIUM

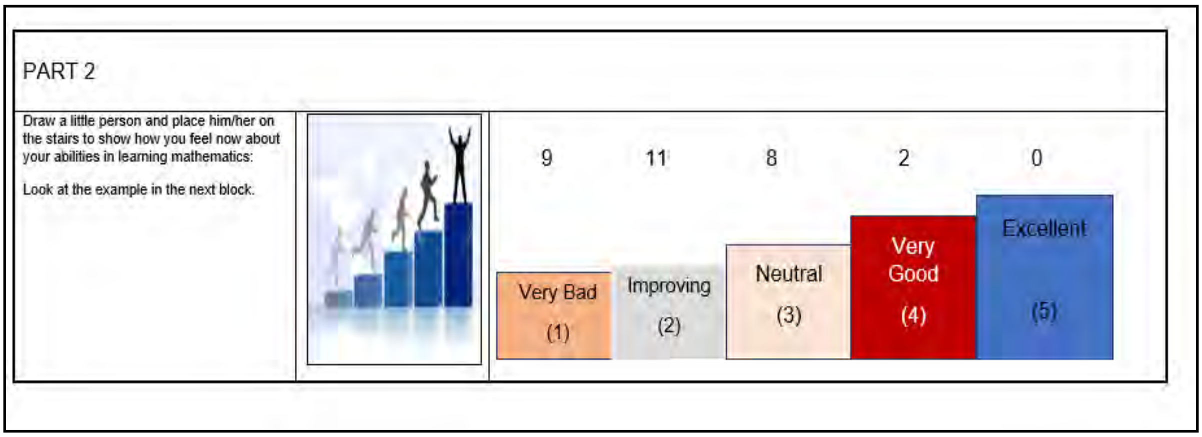



*Figure 5.16: Post-topic placement on Disposition Podium*

None of the participants felt confident enough to place themselves at the top position of the Disposition Podium. Two participants felt very good about their achievements with the applets and placed themselves on the second highest position. Unfortunately, of the 30 participants, 20 were still convinced that they could only position themselves on the lower two positions. Eight participants were neutral in terms of their experience with learning algebra with technology. This provided me with an opportunity to swing their sentiments towards a more positive experience. I concluded that any changes in their dispositions would only happen over a period and not within one cycle.

## 5.8 RESULTS OF THE ACHIEVEMENT TEST

All the participants wrote the achievement test. Due to the COVID-19 pandemic many parents were hesitant to send their children to tests and examinations due to the lack of social distancing. However, the participants expressed their wishes to participate in the achievement test and special arrangements were made so that they could write in two different venues, which accommodated 15 participants in each.

I was disappointed with an average score of 52.6%. I had only the grade score to benchmark the results against and it turned out that the participants scored well above the grade average. Participant Christine, who has always been a top scorer, got 100% for the test. There were no failures, although ten participants remained at the lower end of the score distribution (a score below 45%). I decided to hand out the tests individually to allow me some reflection opportunity with each participant on a one-on-

one basis. After the individual reflective interviews, I was able to come to some clear conclusions.

## 5.9 CONCLUSIONS ON TOPIC 1

I spent some time going through each participant's test with them, listening carefully to why they repetitively made specific mistakes. It led me to conclude the following:

- Many participants are not able to do basic arithmetic. Simple addition and multiplication with positive and negative integers are incorrectly done. In time, one mistake transforms a simple problem into something too complex for them to handle.
- They have not yet mastered basic calculator skills. Many participants did not retrace their steps to confirm that the initial answer was correct.
- An over-reliance on the calculators confused many. The calculator is useful to simplify $(\left(\frac{2}{3}\right)^2 .3)$, but as soon as it is changed to $(\left(\frac{2a}{3b}\right)^2 .3ab)$, they fail to solve the problem with their calculators.
- Some participants lost more than 80 teaching days due to the COVID-19 pandemic. Serious remedial work needed to be done. Fortunately, their acquired skills with *GeoGebra* and the assistance of a teacher will offer a catch-up opportunity.
- All participants confirmed that they started to gain some confidence in doing mathematics again.

Contrary to my initial expectations that the change from classroom teaching to teaching with technology would yield immediate results, one after another participant related how they were slowly getting used to the new approach and that it takes some time to adapt to an environment where understanding can only be achieved through their own interactions with the applets. Initially I was very hesitant to interrupt any activity during a session, but I was requested by several participants to *confirm* that they are on the right track while engaged with an applet. I have the advantage of the interactive TV to anonymously display the work of any participant and discuss it with all the other participants.

I reached the conclusion that the individual approach of the applets assisted many participants to progress at their own pace – an ideal situation for any classroom where learners’ mathematical skills lie over a wide spectrum.

In chapter 6, the next topic is unpacked, illustrating the cyclic nature of the research process. More data regarding conceptual understanding is collected based upon another topic and further tracking of possible changes in dispositions is done.

# CHAPTER 6

# DATA COLLECTION, ANALYSIS AND DISCUSSION OF TOPIC 2

## 6.1 INTRODUCTION

After the somewhat disappointing results obtained during the first cycle of the research, this chapter provides a narrative about the in-depth reflection I had with the participants about the outcome of the first cycle, the remedial steps taken, the data collected during the second cycle and the interpretation and analyses thereof. According to my research plan reflection is important before a subsequent topic is unpacked and analysed. Therefore, after analysing the recommendations of the participants, the framework guiding the data collection process was slightly adapted for this second cycle so that two different sets of quantitative and qualitative data could be generated. The focus remained on the two main research questions, namely whether enhancement of the participants' conceptual understanding of algebra was evident, and to instill positive changes in their disposition towards learning algebra.

## 6.2 REFLECTION WITH BOTH GROUPS OF THE CASE STUDY

Braa and Vidgen (2000) emphasise that reflection does not only contribute towards the research content but can also improve the process of inquiry. According to Coghlan and Brannick (2005) reflection should be seen as a learning process about the research and as a valuable data source of the research itself. They identified three types of critical reflections:

- **content reflection:** interrogating underlying difficulties and reviewing what is happening;
- **process reflection:** critically thinking about the methodology and possible improvements or changes to yield better results; and
- **premise reflection:** taking a critical look at assumptions or perceptions that could influence the course of the research.

The cyclic process the research followed was in line with the thoughts of Coghlin and Brannick (2005), as it allowed me to do reflections at the end of every lesson as well as at the end of each cycle. It provided me with the opportunity to refine each subsequent cycle in order to yield specific answers to the research questions. The reflection session discussed below was planned to reflect on the results of the achievement test after the first cycle, but participants were eager to share their experiences with me within the larger group of participants. Valuable, rich data was collected during the reflection session so I decided to make some changes to the research process before we continued with the second cycle.

I scheduled a combined reflection session with both groups of participants. All participants except for one, were able to attend the session held during the afternoon to allow the community participants to attend. Besides reflecting, I also wanted the two groups to interact so that all the participants would continue to see themselves as part of one case; secondly, I wanted to do an introductory lesson before the data collection lessons of the topic started, in order to familiarise myself with the required pre-knowledge of the participants regarding the topic. The laboratory has 28 computers, and 27 participants of the case could attend the introductory session, allowing each one to work on a computer.

I started the reflection session by expressing my disappointment with the average results of 52% the 30 participants scored for the achievement test and mentioned that the adapted approach might not yield positive results right from the start. To my surprise the overall response from the participants was that everyone experienced their results as being an improvement on their previous results. I focused on the average results for the case, but each participant reviewed their own individual achievements. Christine never wanted to participate in any reflective discussion. She was always focused on her work and seldom participated in any class discussions, but nonetheless decided to air her view. She mentioned that *although she always scored above 90% in tests, for the first time she achieved 100% in a test*. She also explained that she has a specific study method of *making summaries and writing down selected examples while studying mathematics. From her summaries she then creates memory pictures that she can recall when answering test questions. She found that the applets assisted her to create the type of memory pictures she relies on when writing tests.*

During further interrogation it became clear to me that she is a visual learner, who recalls from the visual structures she has created for herself when preparing for a test. During the conversation it surfaced that the visual and organised nature of the applets helped her to create the memory "*pictures*" that she usually relies on when learning mathematics. I decided to further interrogate her views after the introductory lesson and to give other participants the opportunity to express their views and experiences regarding their own learning and understanding of algebra.

The laws of indices are experienced as a difficult topic to master. My initial perspectives were wrong as the participants all agreed that the approach we followed unlocked the topic so well that they were less anxious about writing the test. Several participants agreed that: "*for the first time I had some confidence to go write a mathematics test.*" This view was supported by the whole group and there was consensus about the following:

- None of the questions in the test were *impossible* to the participants.
- Several participants attributed their mistakes either to calculation errors or to mistakes with basic algebraic rules. For example, several of them wrote $(-2a)(-3a)$ as equal to $-6a^2$ or even $-5a$.
- All participants agreed that they were able to '*visualise*' and see the correct law within the posed problems: "Sir*, I was able to recognise every law within a question. I could* ***see*** *what law I should use to solve the question, but I made careless mistakes".*
- Alzonia, who earlier reported that "*all mathematics is just a maze of nonsensical symbols and figures"* reported that for the first time she could "*visually recognise the sums"*. *"For the first time in my life, I passed a mathematics achievement test."*
- Some participants expressed their concern that they might have a *gap* in their basic mathematical knowledge and understanding. Some attributed it to the extended period we were in lockdown and had to do online classes. Other attributed it to the methods of teaching mathematics to them in earlier grades. We agreed that they would be allowed to visit the laboratory in their own time to work on revision applets or to re-do lessons on their own.

The next prescribed topic according to the scheme of work was multi-term products. I did not select this topic to be taught visually using *GeoGebra* applets, simply because it did not score high on the Applet Evaluation Instrument. Initially I believed it would be a good exercise to go back to face-to-face class teaching for a while so the participants were informed that the topic would be taught face-to-face and not be part of the research. However, they requested that, even though the topic would not form part of the research, they would prefer to continue with the topic using *GeoGebra*. I asked them: "*Motivate why you would prefer to do this topic with GeoGebra, when you can just sit and listen to my lessons in class?*" Various responses were given:

"*I am forced to give attention with GeoGebra.*"

"*My focus is better when I do the topic on the computer*."

"*I enjoy mathematics more, when I do it on the computer*."

"*The program gives structures that help me to understand better.*"

"*I take longer to understand, GeoGebra allows me to work on my own.*"

"*I start to like mathematics more when we do it with GeoGebra.*"

I therefore decided to teach the topic using previously considered applets, although the topics had not passed the selection process on the Applet Evaluation Instrument (AEI). I made this decision, not only at the request of the participants but also because I was convinced that I was observing slight changes in their disposition towards the visual learning of algebra. I did not want to miss the opportunity to capitalise on the process. We were allowed four lessons with the topic. We did not do any reflections, neither did we complete the Disposition Instruments. This provided me with the opportunity to take a step back to observe the participants while they worked with the applets. The participants knew that this was not part of the research project, and the exercise could be observed objectively as they did not have to try to impress me or feel that they were being observed.

## 6.3 DECSRIPTION OF SELECTED TOPIC 2 AND RESEARCH CYCLE 2

The second topic I selected for research purposes was factorisation. According to the requirements of the syllabus Grade 9 learners should be able to do the following regarding factorisation:

- find the highest common factor, numerical and non-numerical from any multi-term expression;
- do grouping and remove a set of common brackets. (This can include expression of the form $a(x-y)+b(y-x)+b^2(x-y)-(y-x)$);
- identify and factorise the difference between two squares, including cases where a common factor must be removed to expose the difference between two squares. (eg: factorise completely: $-16a^2b^3+4a^2b$ solved to $-4a^2b(4b^2-1) = -4a^2b(2b-1)(2b+1)$); and
- factorise trinomials of the form $\pm ax^2 \pm bx \pm c$. It is not expected that Grade 9 learners be able to rewrite trinomials of the order $ax^2+bx+c$, but learners should be able to first remove common factors before attempting to factorise a trinomial.

We were allowed to spend ten lesson periods on the topic of factorisation, but neither the syllabus nor the scheme of work allocated a specific number of lesson periods to any of the sub-divisions of the overall topic. I decided to be guided by the pace of the participants.

## 6.4 SECOND CYCLE: SECOND DYNAMIC TOPIC: FACTORISATION

### 6.4.1 Sub-topic: Common factors

#### *6.4.1.1 Pre-lesson Disposition Questionnaire*

In Grade 8, all Namibian learners are introduced to the concept of the highest common factor (HCF) where they are expected to apply prime factors to find the highest common factor of two numbers, by writing each number as the product of its prime factors. I capitalised on that concept to introduce the topic vocabulary, namely ‘factors’ and ‘factorisation’ to the participants. This was also an opportunity to draw a parallel between products and factorisation. The participants did a simple revision exercise with *GeoGebra* where they had to find the HCF of two numbers by writing them as products of their prime factors. I found that not all participants used the same method for this exercise. Some used long division to write the numbers as the products of their prime factors while others used a method of placing all the prime factors as elements within a Venn diagram with the common factors within the intersection.

All the participants showed appreciation for each other's methods. I seized the opportunity to ask them which method they would describe as being **visually** the easiest to understand. After a lengthy discussion and the different groups explaining their methods to the others, **most** participants agreed that the method using Venn diagrams explained the concept the best to them. I concluded by instructing the participants to apply the chosen method to find the HCF of $12a^2b^3$ and $18a^3b^2$. This was the first time that they had to construct a method for finding the HCF of a term containing not only numbers.
Figure 6.1 shows the successful attempt by Kyle, who was one of the lower scorers of the first achievement test. He used the visual Venn diagram as a starting point and then created a conceptual framework to solve the problem.

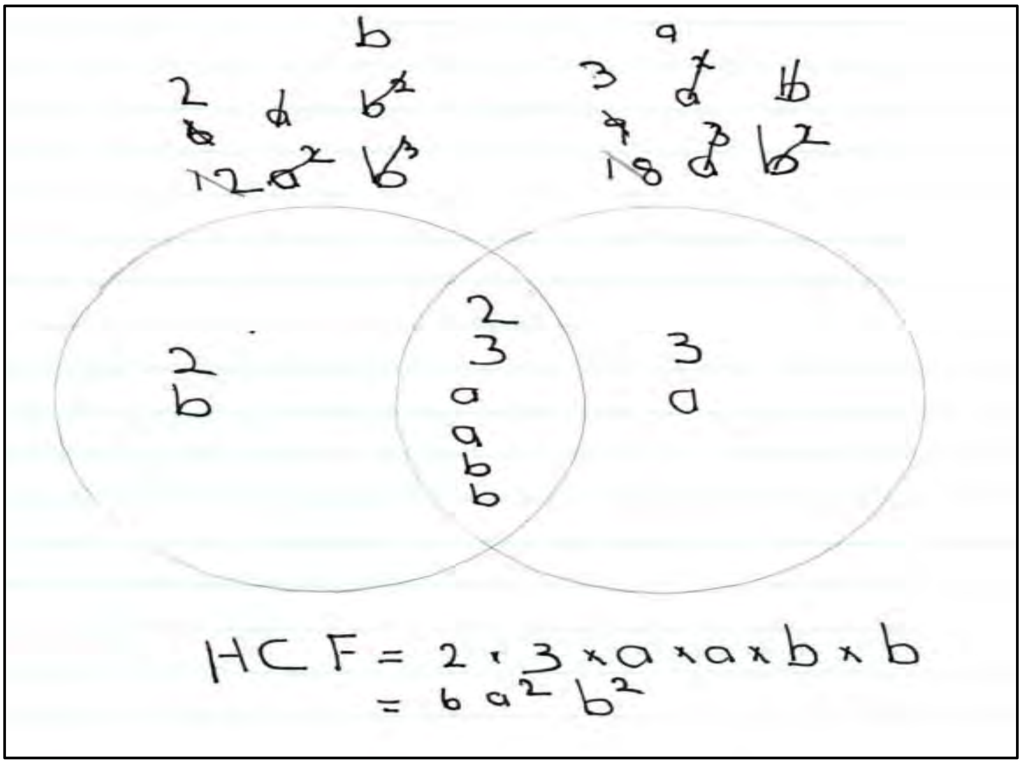



*Figure 6.1: Kyle's visual finding of the HCF using a Venn diagram*

I was pleasantly surprised that all participants could construct a mathematical notion to solve the problem by taking a step backwards and using previous knowledge to solve the given problem. Some participants used the Venn diagram and others wrote the two terms as strings of prime and letter factors. Figure 6.2 is another example of a participant's efforts. Univi typed a solution for herself, deviating somewhat from the two known methods but could successfully construct a method that *worked* for her to arrive at the correct answer. I allowed all the participants to be creative in designing a method that would consistently solve a problem correctly.

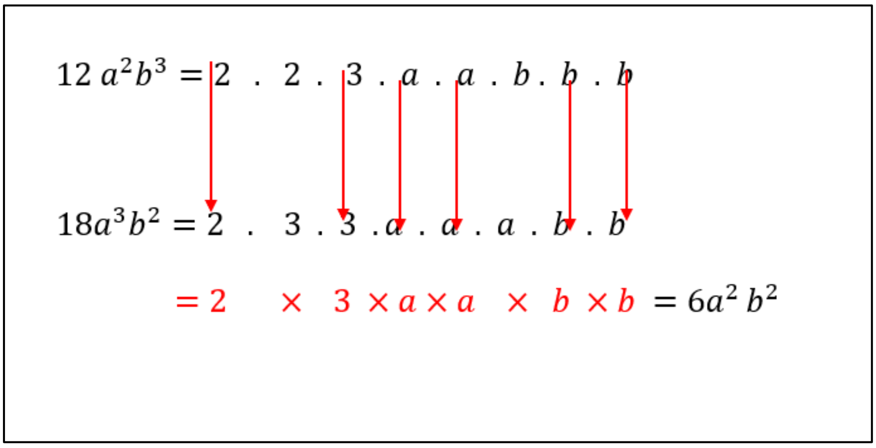



*Figure 6.2: Univi's solution to find the HCF*

In Figure 6.2 Univi used a visual structure based upon her previous knowledge of writing a number as the product of its prime factors to construct a visual method to find the HCF of the two given terms. Both Kyle and Univi showed and applied the elements of conceptual understanding as proposed by Capetta and Zollmann (2014). Both methods in my view, contain elements of visual understanding of a concept. Although sets as such had not yet been taught to the participants, Kyle could use elements of the concept of sets to solve the problem for himself.

While the participants were trying to find a solution for the simple problem it became clear that they were all trying to integrate their pre-knowledge about prime factors and their knowledge about indices to construct a method of solving the posed problem. They either reverted to either pen-and-paper as shown in Figure 6.1, or preferred to use the computer to create logical structures for the solving of the simple problem so that it contained not only numerical values. A few participants tried to disprove their own methods by creating random examples for themselves. Christine tried to find the HCF of $17a^3$ and $18a^{-3}$. Figure 6.3 illustrates how she struggled to reach a conclusive answer for her own problem.

Highest common factor: $17a^3$ and $18a^{-3}$ is???

$17a^3 = 1 \times 17\ a \times a \times a$

$18a^{-3} = 2 \times 3 \times 3\ \frac{1}{a} \times \frac{1}{a} \times \frac{1}{a}$

There are NO HCF.

$2^3 = 2 \times 2 \times 2$ AND $2^{-3} = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$

The HCF can only be 0 or 1??? BUT 0 cannot be a factor.

The answer is 1.

*Figure 6.3: Christine's efforts to find the HCF for her own problem*

Christine tried to integrate the previously learned law of indices and her knowledge about prime factors to solve the problem for herself. It was clear to me that Christine had to use algebraic structures visually to reach conclusions for herself. She spent some time integrating her knowledge into the solution for the problem. While she was working on the computer, she mimed some operations with her hands, with her eyes closed. I could clearly observe how she moved the letters/symbols in her mind before she typed anything on the computer. During a conversation after the exercise, she confirmed that she had to create *some sort of algebraic structure in her mind* for herself to be able to *understand* the mathematics. "*Once I have a picture in my mind, the work becomes easy for me.*" I suspected that Christine's algebraic proficiency relies on a type of symbolic visualisation as proposed by Bruner (1966) and endorsed by Presmeg (2017). I made a note to further investigate the strong visual concepts she relies on to conceptually understand algebra so well.

After the introductory lesson and the unlocking of terminology for the participants, they completed the Disposition Instrument. Figure 6.4 presents a summary of the results.

**Positive Disposition Questionnaire: Summary C2 Pre- L1** Present: 15(S) of School and 12(N) from NAMVISPRO

Let the emoji's help you to answer the following questions. Session:C2 L1

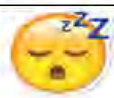
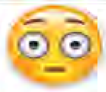
| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| |  |  |  |  |  |
| For every question choose one emoji that will best describe your feelings. Mark your choice with an X below the emoji of your choice. | I detest it.<br>Make me scared,<br>Intimidating. | Boring<br>Not interesting.<br>I do not understand it. | Makes me nervous<br>I don't care.<br>Takes time to understand. | I like it.<br>I will do more<br>I understand it. | I love it.<br>I find it easy.<br>I can use it in future. |
| (1) How do you feel about Mathematics in general? | 4 + 4 | 4 + 3 | 5 + 3 | 1 + 1 | 1 + 1 |
| (2) How do you feel about learning Algebra? | 5 + 2 | 5 + 4 | 2 + 4 | 2 + 1 | 1 + 1 |
| TOTAL QUESTION 1 | 8 | 7 | 8 | 2 | 2 |
| TOTAL QUESTION 2 | 7 | 9 | 6 | 3 | 2 |
| Mean per Question | 7.5 | 8 | 7 | 2.5 | 2 |
| MEAN SCORE FOR THE CASE: | | 2.4 | | | |
| MEDIAN SCORE: | | 2 | | | |

*Figure 6.4: Scores for pre-Lesson 1, Cycle 2 on the Disposition scale*

After the introductory lesson on factorisation, I was concerned that the participants might underestimate the level and scope the of factorisation topic and that it would lead to a false score on the disposition scale. But contrary to my expectations there was not a significant increase in the mean and median scores on the disposition scale. However, it was encouraging to see that for the first time both groups gave a score of 5 for both the algebra and mathematics in general questions. I suspected that the higher scores were from participants who also did well in the previous achievement test, although it was significant that the number of participants who detest doing algebra decreased from 13 to 7. I believed that if this change in disposition was permanent it would be revealed by the answers on the post-lesson Disposition Instrument after the next topic with *GeoGebra*.

#### 6.4.1.2 *Screen captures of the first factorisation topic: Highest common factor*

The time frame for completion of the factorisation topic was ten lesson periods. As mentioned earlier, I decided to be guided by the pace of the participants. The introductory session that focused on their pre-knowledge and required terminology inspired many participants to immediately commence with more advanced examples on the chosen applet. The general feeling amongst the participants was that they had *a picture of what to do, right from the start* with the applet.

Figure 6.5 shows the introductory screen of the applet with step-by-step instructions on finding the HCF of a multi-term expression. Most participants were confident enough to not spend any time on this screen but to immediately progress to the next screen as illustrated in Figure 6.6. The example section of the applet allowed the participants to use a slider to step-by-step engage in the solving of the problem under discussion. The participants reacted in two ways: some moved the slider one third down to disclose the first section of the problem, while others cautiously moved step-by-step through the whole applet. I appreciated the individualisation and freedom offered by the applet as well as the visuality of the explanation provided. The applet did not follow the usual whiteboard approach where an example is done line by line; instead numerical and non-numerical coefficients were investigated separately and finally the HCF was identified.

HERE'S HOW ...

1. Determine the greatest common factor of the numerical coefficients for each term.

2. Determine the smallest power of each variable common to all terms, and multiply this by the common factor from step 1 to complete the GCF.

3. Write expressions for the remaining non-common factors.

4. Verify the result using the Distributive Property.

The applet below illustrates the process.

*Figure 6.5: Introductory summary of the chosen applet for common factors*

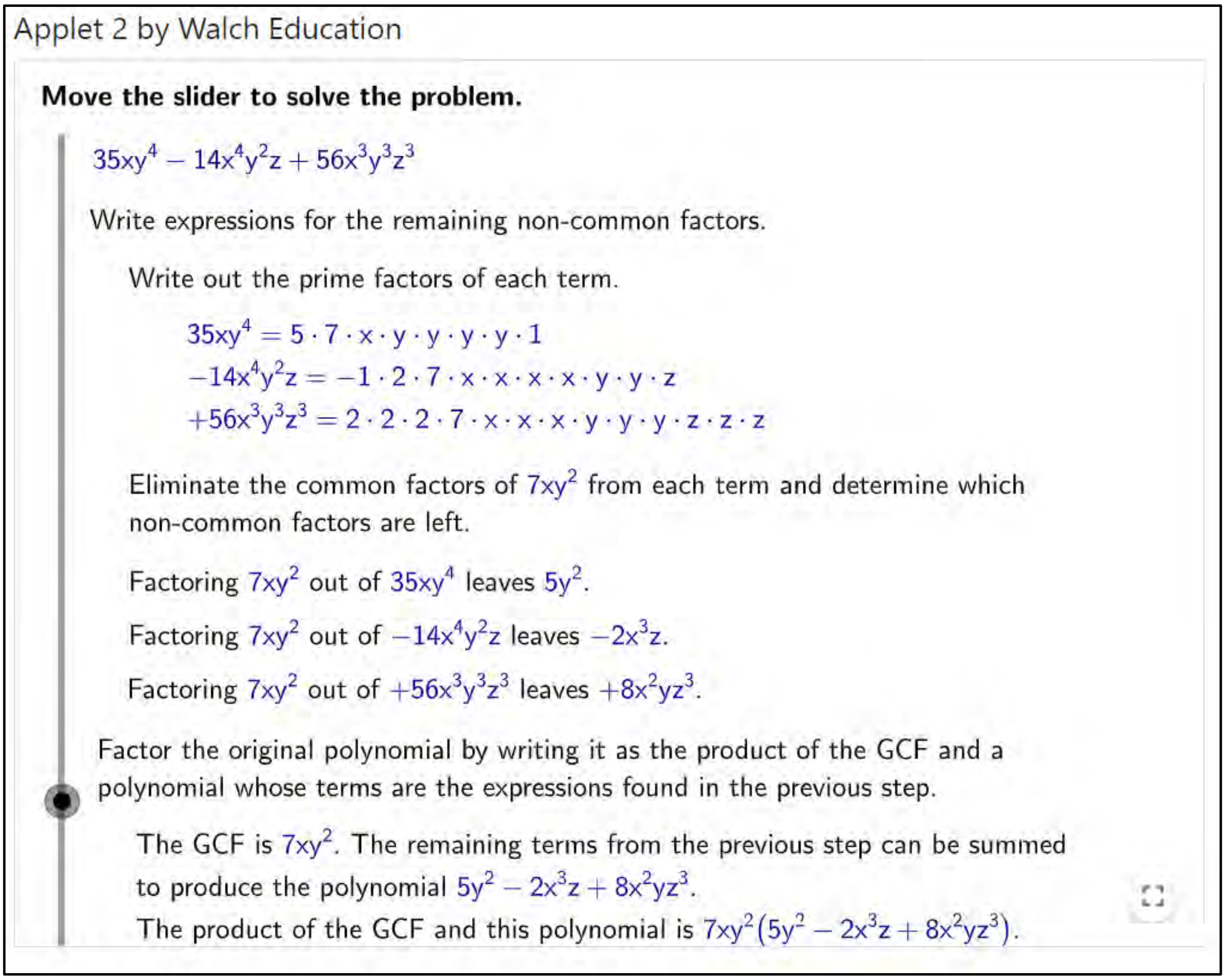



*Figure 6.6: The approach followed by the applet to use the slider to reveal sections of the applet. Each term is inspected and factorised individually*

Source: www.GeoGebra.org

Verify the result using the Distributive Property.

$$
\begin{aligned}
&7xy^2(5y^2 - 2x^3z + 8x^2yz^3) \\
&= 7xy^2 \cdot (5y^2) + 7xy^2 \cdot (-2x^3z) + 7xy^2 \cdot (+8x^2yz^3) \\
&= 35xy^4 - 14x^4y^2z + 56x^3y^3z^3
\end{aligned}
$$

The result of applying the Distributive Property is the original expression, $35xy^4 - 14x^4y^2z + 56x^3y^3z^3$. Thus, the polynomial is factored correctly and $35xy^4 - 14x^4y^2z + 56x^3y^3z^3 = 7xy^2(5y^2 - 2x^3z + 8x^2yz^3)$.

*Figure 6.7: The opportunity to verify results*

As seen in Figure 6.7, the applet included a feature where the participants could verify the results by applying the distributive property to get the original multinomial. Three more features formed part of the applet: firstly a feature where the participant could go back to the initial first example; secondly a feature to try more advanced examples; and thirdly when a participant felt ready, they could complete a quiz with multiple choice answers to which the correct answer would instantly be revealed as soon as the question was answered.

I noticed that in both groups of the case there were participants progressing rapidly through the lesson, but that more than half of each group of participants either progressed very slowly or stopped working when they reached applet 2 as illustrated in Figure 6.6. This was not what I expected as initially I was convinced that all the participants would fly through the lesson and be able to complete the quiz with ease. A decision was therefore taken not to complete the Disposition Instrument but to rather have an in-depth reflection session with the participants who progressed very slowly and those who gave up on the applet. Amongst them were participants who struggled academically and others who excelled in the test.

### 6.4.2 Lesson 1: Highest common factor reflective interviews

I had to capitalise on the feelings and experiences the specific participants had during the lesson session. I was also concerned that I might lose the momentum of the changes I intended to bring about. I suggested that we should have an immediate reflection session together so that they did not fall behind with their work and to ensure that they were on par with the others. The lesson period was at the end of the school day and school participants agreed to stay behind a little longer.

I mentioned to the participants that it appeared to me that they did not have an enjoyable experience with the lesson. I suspected that the session with the NAMVISPRO participants later the afternoon might follow the same pattern and I believed that I might get similar answers from both the school participants and the NAMVISPRO participants. Haley summarised the feelings of the group: "*It is not that we struggle with the work, but the screen is too full of stuff. I lost track of what I was supposed to do.*" Others agreed: ''*They make simple things complex with all the explanations in between."* Univi said: "*I had a picture in my mind of how to do this work, but the program messed with my plan to solve the problem, especially when I reached the second screen. Sir I already had a solution in my mind, and I know I understand the work because I skipped the explanations and just did the quiz with no wrong answers.*" Another one added: "*I know exactly what to do and all these things in between just waste time.*" There was mutual agreement amongst the participants that the applet *took away the enjoyment of learning with it.*

After the lesson with the NAMVISPRO participants (where even more participants did not complete the full cycle intended for the day) we did the same reflection exercise. Responses received from the NAMVISPRO group were very similar to the responses from the school participants. However, these participants are determined to make a success of every lesson opportunity and although they struggled to follow the sequence of screens with each one a little more advanced than the previous one, none of them simply gave up on the applet.

These are some of their responses: *"The teacher wants to do too much work at once. It is like a teacher that talks too much."* Paulo made an important remark and I decided to do another reflective interview with him after a few more lessons. *"Sir I just started to learn a new process of understanding the work better. The computer allows me to see in my imagination the different types of problems we are doing but today I did not have the opportunity to form my own picture of the work."* The response from Helena was also important: "*It is almost as if I can almost hear the teacher feeding me with all the information. My brain cannot process so many things."* She was supported by others who worded their experience as: "*There were just too many thingies [sic] on the screen. So many numbers and letters confuse me." "Why do they always have to use big numbers to explain something simple?"*

### 6.4.3 A lesson in applet selection for enhancing conceptual understanding

In my view the applet tried to explain the work systematically to the participants but failed by going into too much detail (with many elements on the screen). I recognised that it is important for any applet to progress systematically from an elementary level to a more advanced level and that factorising $5a^2b^3 - 10a^3b$ illustrates the same principle as factorising $55a^{12}b^9 - 110a^{14}b$. The participants underlined the fact that they are determined to grasp a concept and did not want to be caught up in an situation where they would have to deviate by using a calculator or other methods to create the important conceptual mind map for themselves. One positive point made was: "*I decided to just follow the steps in blue, then the whole process became clear to me. Now it is in my mind, and I can, well I believe, I can do similar problems now."* That remark eased my concern that I might lose some participants along the way. It was also re-assuring that we could end the reflections with the participants agreeing that in spite of this, they enjoyed learning with the applets.

Based upon my observations in the classes and the repeated confirmation during the reflective sessions I concluded that simplicity and very well-defined symbolic structures are important for the conceptual understanding and learning of algebra. I was searching for more evidence to support my suspicion that the participants had started to develop visual structures for themselves which they could rely on when confronted with a problem out of context – as would be the case in an examination. The fact was that the visuality of the screen prevented some of them from gaining understanding of the concept that they were already familiar with. I decided to revisit my applet selection process and to re-design the applet evaluation framework with a simpler rubric that would take the simplicity of layout on the screen and the linear progress of the applet into consideration. (See Appendix A).

## 6.5 SECOND AND THIRD SUB-TOPICS: GROUPING AND DIFFERENCE OF TWO SQUARES

The next pre-lesson disposition exercise was important to me as it would reveal any regression in terms of the positive change in disposition I was hoping to bring about. The results are summarised in Figure 6.8.

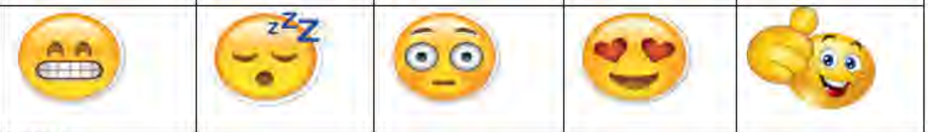
| Positive Disposition Questionnaire: Summary C2 Pre- L1 Present: 16(S) of School and 14(N) from NAMVISPRO | | | | | |
|---|---|---|---|---|---|
| Let the emoji's help you to answer the following questions. | | | | Session:C2 L2 & 3 | |
| | 1 | 2 | 3 | 4 | 5 |
| |  | | | | |
| For every question choose one emoji that will best describe your feelings. Mark your choice with an X below the emoji of your choice. | I detest it. Make me scared. Intimidating. | Boring Not interesting. I do not understand it. | Makes me nervous I don't care. Takes time to understand. | I like it. I will do more I understand it. | I love it. I find it easy. I can use it in future. |
| (1) How do you feel about Mathematics in general? | 2 + 3 | 6 + 7 | 6 + 3 | 1 + 1 | 1 + 1 |
| (2) How do you feel about learning Algebra? | 3 + 2 | 6 + 5 | 4 + 5 | 2 + 1 | 1 + 1 |
| TOTAL QUESTION 1 | 5 | 13 | 9 | 2 | 2 |
| TOTAL QUESTION 2 | 5 | 11 | 9 | 3 | 2 |
| Mean per Question | 5 | 12 | 9 | 2.5 | 2 |
| MEAN SCORE FOR THE CASE: | | | 2.5 | | |
| MEDIAN SCORE | | | 2 | | |

*Figure 6.8: Pre-lesson disposition questionnaire results before second and third sub-topics*

The results did not show any major shifts on the Disposition Instrument, however the number of participants on the lower end of the scale decreased, despite the less positive experience most of them had had with the last applet. The median score remained on 2, but the mean score improved from 2.4 to 2.5 since the previous completion of the instrument. The answers during the last reflection session suggested that they still *preferred to work with the applets* because: "*the computer does not judge me when I struggle.*" That might suggest why the mean score improved slightly. I also concluded that the changes in disposition might be permanent in nature and would likely not fluctuate too much after a single less positive experience. During the reflection interviews with the participants, they were very opiniated and honest when answering my questions. Contrary to my initial concerns that they would try to impress me, their answers confirmed that their focus was to master mathematics better and to improve their results.

### 6.5.1 Lesson 2: Grouping execution and screen captures

A decision was taken to allow the participants to work progressively at their own pace and to continue with a subsequent topic when they felt ready. The basis for the decision was the difference in pace among the participants. Some concluded an applet in a few minutes while others spent more time and preferred to revisit previous sections of an applet. Putting pressure on slower learners would defy the purpose of the research and holding back faster participants would annoy them and not contribute towards the purpose of the research project. We had the freedom of individualisation offered by the *GeoGebra* applets.

Then the cloud of a lockdown due to COVID-19 hung over us. We had permission to continue with our research, although in smaller groups. The participants were allowed the option of contact sessions or doing the work via remote learning. I decided to offer the participants the freedom of choice to either work online from home or to attend contact sessions in the laboratory. The links to the different lessons were forwarded to the participants but they all preferred to attend the contact classes.

The laboratory allowed for social distancing and participants could focus on their applets. *GeoGebra* allows me to follow participants when they work online, but I suspect that working in the laboratory provided the participants with a form of security during uncertain situations. The participants confirmed this when they mentioned that attending the sessions added to: "*a feeling that we continue as normally.*" Others noted that: "*We just started to make progress and begin to understand mathematics. We must not stop now.*"

The first set of applets explained factoring polynomials through grouping. The participants had been given a firm foundation of finding the HCF in any expression. The chosen applets focused on the grouping inside an expression to eventually remove a common set of brackets to complete the factoring process. Participants showed confidence with the process and grasped the underlying principles very quickly. They showed some confidence when working with the applet and investigated the results when they re-arranged the terms of the polynomial to ensure that it is possible to arrive at the same result. I also observed them showing the insight of removing 1 as a common factor to get the same results. This is illustrated in Figure 6.9.

| $2x^2 + 6x^3 - 3y - 9xy$ | $2x^2 - 3y + 6x^3 - 9xy$ |
|---|---|
| $= 2x^2 + 6x^3 - 3y - 9xy$ | $= 2x^2 - 3y + 6x^3 - 9xy$ |
| $=2x^2(1 + 3x) - 3y(1 + 3x)$ | $= 1(2x^2 - 3y) + 3x(2x^2 - 3y)$ |
| $=(1 + 3x)(2x^2 - 3y)$ | $= (1 + 3x)(2x^2 - 3y)$ |
| | |

*Figure 6.9: Participants' efforts with grouping of polynomials*

The syllabus requires that learners should be able to recognise prime polynomials (that is polynomials of the type $ax + by + cm + dn$, where no grouping can be done), re-arrange terms before grouping and be able to remove an HCF from an expression before grouping. It is also required that learners should be able to complete the factorising of situations where the intermediate steps may lead to an expression of the form $2x(a - b) + 3y(b - a)$. The role of the teacher remains important as none of the applets suitable for the exercise covered any of the mentioned requirements. I changed the revision section of the applet to cover all the syllabus requirements. Participants were allowed to do it as an online challenge. Figure 6.9 also shows that most participants not only illustrated an excellent understanding of the underlying concepts, but also enhanced their thoughts by applying different colours in line with the applet's illustration, to explain the process of factoring.

We did not complete the Disposition Instrument between the two lessons, neither did we spend time on reflective interviews. As the participants progressed, I made some screen captures and used the opportunity to ask them about their experiences, especially why some preferred to use colours when working online. The overall response suggested that they acted with more *confidence when they worked with the applets and that the use of the colours helped them to apply the* ***visual pictures they had in their minds,*** to the problem to be solved. I also got confirmation that the *structured presentation* of the applets assisted the participants with their cognitive understanding of the work. In Figure 6.9 it is also clear how they applied the same visual structures used in the applet to their own work. They took time getting all the equal signs below each other and highlighting the $(1 + 3x)$ in the answer. When I enquired about it the answers were always: *I apply what I remember from the applet to the problem.* To me it served as concrete evidence that participants drew from some form of '*mind picture'.*

### 6.5.2 Lesson 2: Difference of two squares

Many participants were confident enough to start with the next set of applets covering the difference between two squares. I deliberately selected three sets of applets to explain this simple concept to the participants. The first set included the words geometric visualisation as seen in Figure 6.10. The applet started with a set of questions that could be answered by applying a movement on a slider to illustrate the principle.

Geometric Visualization of Factoring : The Difference of Two Squares

Express the area of the figure algebraically in two different ways. Used the guide questions below to help you determine these expressions and its relationship to each other.

1. What is the area of the polygon ABDJ with side "a"?
2. What is the area of the polygon FEJG with side "b"?
3. What is the area then of the polygon ABDEFG?
4. Try moving point M to the end of the slider, what new plane figure is formed?
5. What are the dimensions of the newly formed figure in terms of a and b?
6. What is the area of the newly formed figure?
7. What is the relationship between these two areas?
8. Make an algebraic sentence that would express the relationship between the areas of polygon ABDEFG and the newly formed figure?

*Figure 6.10: Questions for the Geometric Visualisation: Difference of two squares*

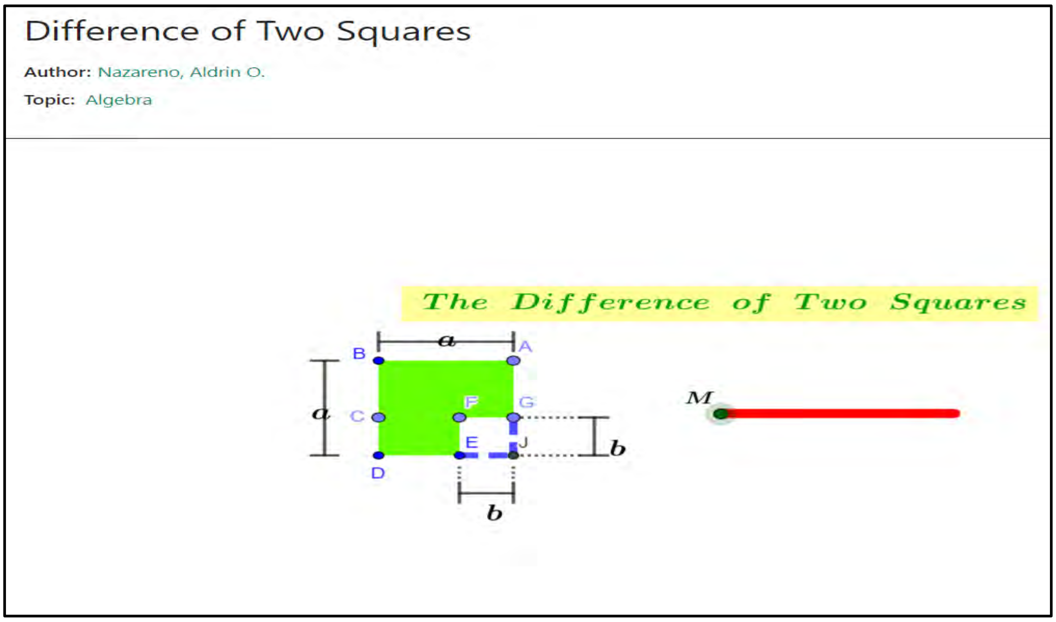



*Figure 6.11: Second screen. Geometric visualisation: Difference of two squares*

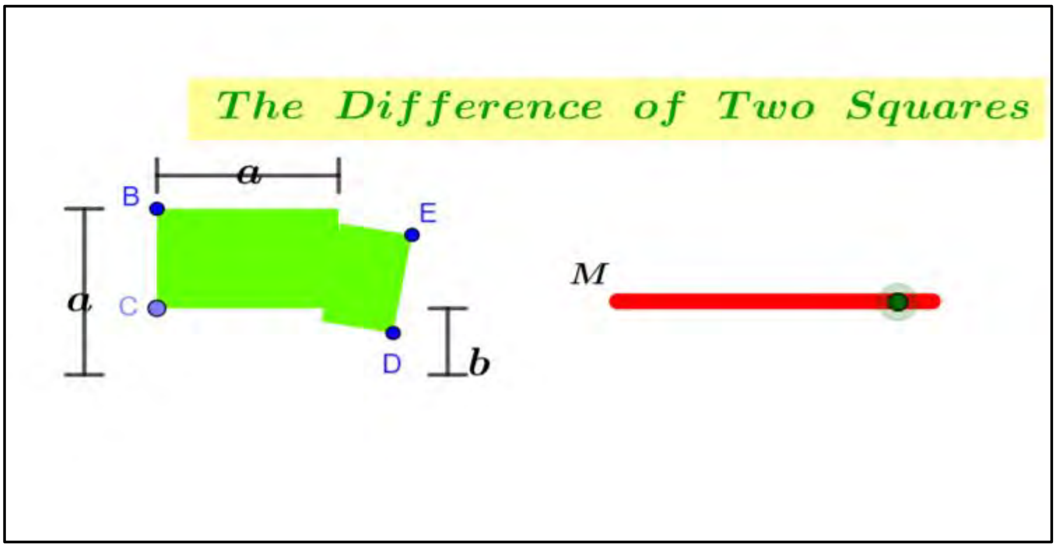



*Figure 6.12: Third screen capture animation of difference of two squares*

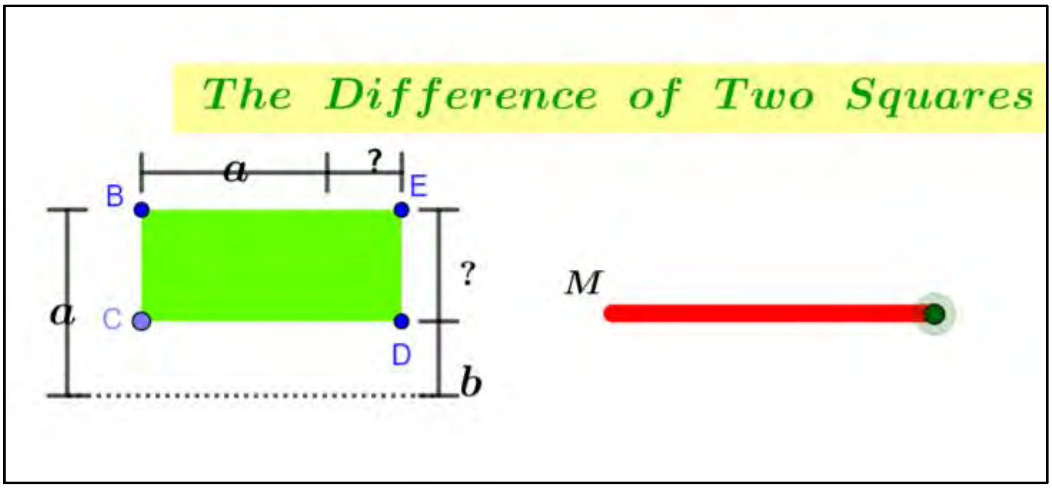



*Figure 6.13: Animation of difference of two squares concluded*

It started to occur to me that the participants differed in terms of the way that they achieve conceptual understanding of a topic. To investigate my suspicion further, the participants had a list of different applets available to assist them during the unfolding of the topic and the process of achieving conceptual understanding. Figures 6.10 to 6.13 are screen captures of the unfolding of the first applet available to participants. The applet referred as a geometrical visualisation of the difference between two squares is based on animation to explain the process of how to find the difference between two squares. All the participants were asked to spend time will all three applets and to decide for themselves which one they found the most appropriate to reveal the underlying concepts to them. I observed that the time spent with this applet varied from only a few seconds for some participants, to others spending quite some time repeating the exercise of moving the slider from left to right.

The second chosen applet as seen in Figure 6.14 was similar to the first one and based on a geometrical representation of the difference of two squares, but was more descriptive in explaining the principle of $a^2 - b^2$ = (a – b)$(a + b)$.

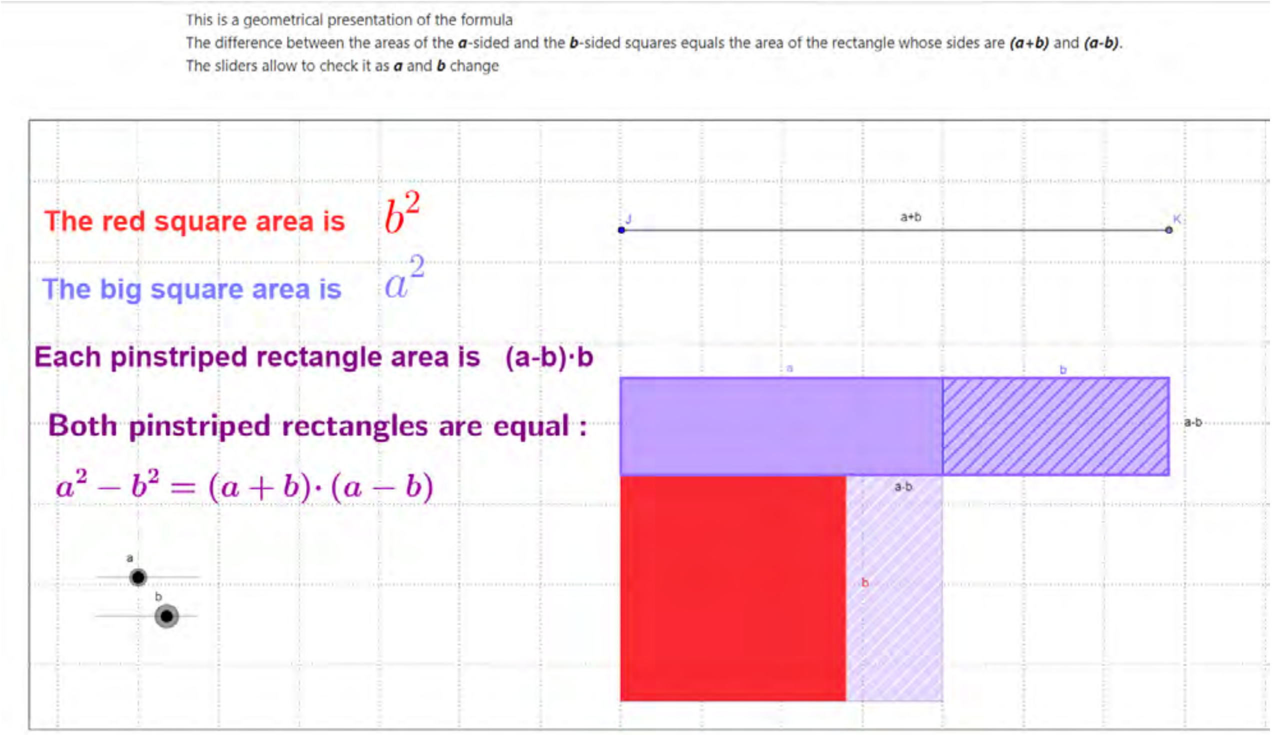



*Figure 6.14: Second applet: Difference of two squares*

The applet in Figure 6.14 also contained a slider to illustrate the principle but included some algebraic elements as illustration of the concept. I observed that all the participants spent a little more time exploring the applet. I noticed that some were more focused on the algebraic concepts on the left than on the geometrical figures that could be manipulated with the two sliders. Some participants randomly moved the different sliders and only when they focused on the <u>very small printed</u> $a - b$ and $a + b$ on the picture on screen, their faces lit up and they turned to their neighbours to draw their attention to the representation. I suspected that the aim of the exercise was clear to the participants but that very few really understood conceptually. As an intermediate step towards the final applet as illustrated in Figure 6.15, I assumed that both applets mentioned above did contribute towards conceptual understanding but the geometrical visuality of both applets failed to unlock the principle to the participants.

It was only after the participants engaged with the applet in Figure 6.15 that I could observe the "aha" moments on their faces. They were also more vocal after they completed the applet in Figure 6.15. This applet contained several visual illustrations of a symbolic nature and revealed the underlying principles by starting with numerical values and ending with variables. The applet further illustrated that factorising is an inverse operation from finding products.

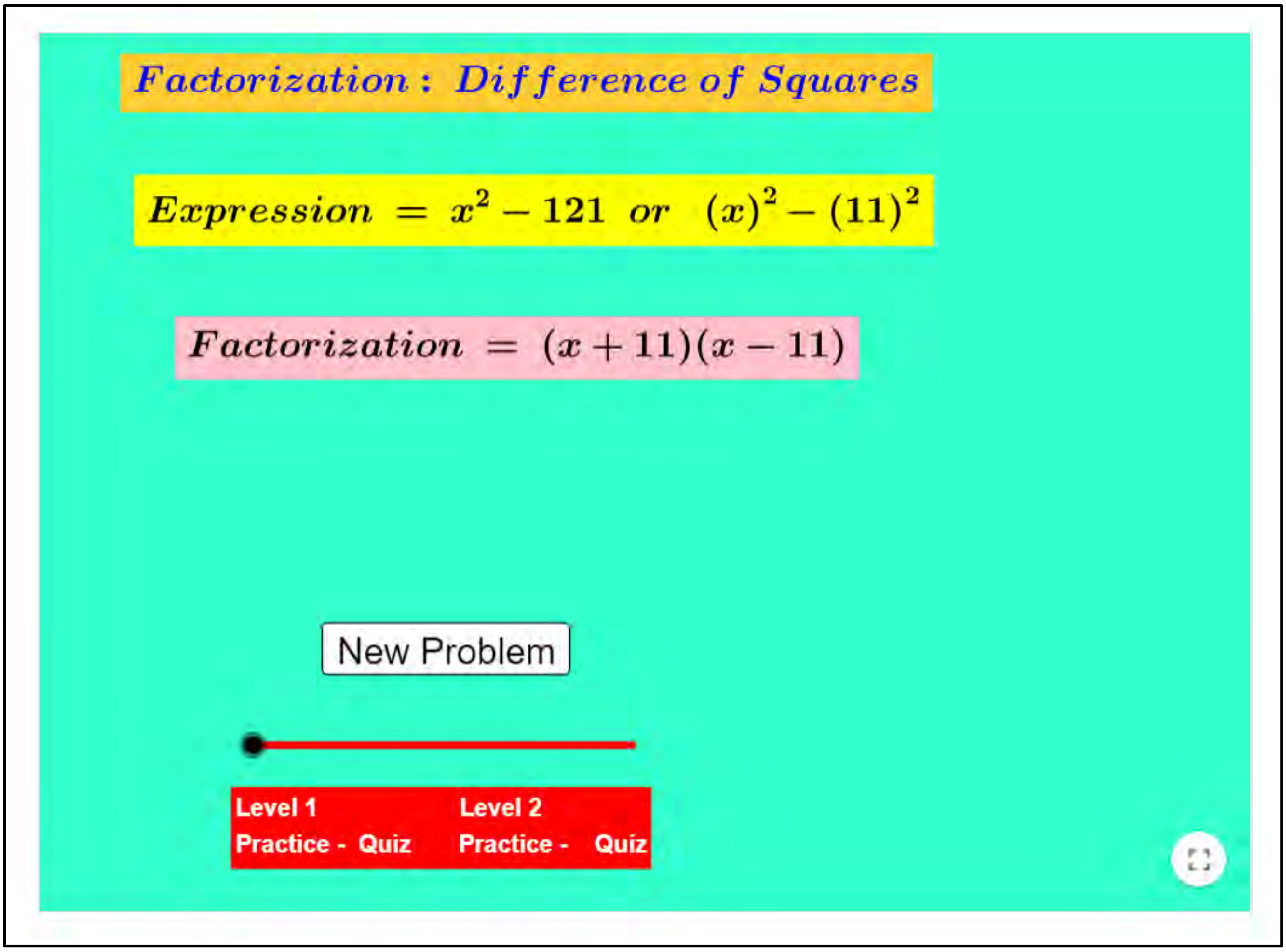



*Figure 6.15: Third applet to illustrate the difference of two squares*

All the participants spent more time with the third applet. I observed that some spent only a few seconds with applets 1 and 2 but longer with applet 3. The participants that moved quickly through applets 1 and 2 were also the ones who had previously disclosed that they found some *visuality in the applets relying on algebraic or symbolic structures*. To them conceptual understanding is embedded *within the algebra symbols and workings.* I have no doubt that they visualise by *mind mapping* (visualising) the symbolic structures of a specific algebraic concept. A little exercise later confirmed my beliefs.

The participants were instructed to complete the online worksheet that was part of the last applet. Figure 6.16 contains a screen capture of the online exercise. This type of worksheet serves as confirmation to participants that they have mastered the principle being taught. The problems given in the exercise cover most of the type of questions learners might expect to see in a test, but I added a few more that could pose a problem to some learners in an examination. Firstly, often learners do not see 1 as a square number; secondly at Grade 9 level learners should be able to factorise fractions and roots as part of this topic; and thirdly learners must be able to remove a common fraction to reveal the difference between two squares. My additions are shown in Figure 6.17.

1. -Apply the formula to write directly the result, without multiplying term to term :

a) (x+2)·(x-2)=

b) (3x-1)·(3x+1)=

c) $(2x^2 - 5x) \cdot (2x^2 + 5x) =$

d) $(3x^3y - 4y^2) \cdot (3x^3y - 4y^2) =$

2.- Identify the squares and factor the expression

e) $x^2 - 9 =$

f) $9x^2 - 1 =$

g) $4x^6 - 9y^2 =$

h) $25 - 4y^8 =$

*Figure 6.16: Applet worksheet based upon the difference of two squares*

3. -Factorise completely: Show your working.

i) $1 - 81b^4$

j) $\frac{1}{16} - b^6$

k) $45a^2b - 75b^3$

l) $\sqrt{16x^4} - \sqrt{81}$

*Figure 6.17: Additions to the applet worksheet to cover more possibilities*

The importance of the role of the teacher is not only to facilitate visual learning, but also to ensure that the learning material fulfils the requirements of the syllabus. This is clearly highlighted by the shortcomings of some applets. If participants can recognise the above examples in a test and examination the topic has been covered to the extent not only expected by the syllabus but also to the required level of displaying conceptual understanding of the topic.

### 6.5.3 Lessons 1 to 3: Reflective interviews

After the sections about common factors, grouping and difference of two squares, I decided that we should halt for a moment and reflect on the sessions we had covered. I noticed that the participants were more at ease during reflections and that they felt secure enough to air their views.

To maintain the structure and without being repetitive in the questions asked, I tried to rephrase the questions I needed answers to. Firstly, we reflected on the the participants' experiences when working on the applets. The overall response was that: "*We find it easier to work with the applets*". Paulo's response was important to me and I decided to do an individual reflection with him later *"Somehow I can continue when I understand and when I still struggle, I can go back*, or do another example" "*I am not forced to listen over and over when Sir explain the work to others that do not* *understand again. Also, that I understand differently from other people and Sir allows me to make sense in my own way".* I asked the participants to always tell me why they gave a particular answer. The motivation for their answers varied: "*I find it difficult to get a picture in my mind when I have to listen to a teacher and try to follow on a board far away from me". Another one added: "Did Sir notice how quiet it is when we work on the computers? At least now we may think on our own."* Paulo made an important comment: *"**<u>When I see I understand</u>**. To me it is as if the work is easier when I see it explained on the screen."* Alzonia explained that she was classified as "*an auditive learner and advised to always sit on the left-hand side more to the front of the class. It did not really help and I still struggle with mathematics.*" Now she has developed a method where she looks at the examples on-screen and narrates the steps to herself. "*At least I can recall the examples later on when I have to do homework."*

### 6.5.4 Responses to some reflective interview questions discussed

It was difficult to stand back and allow the participants to respond to my questions without leading them to give the responses I would like to hear. I had to be careful when I asked a follow up question after a response not to guide the respondent in any specific direction. It started to surface that some changes were taking place within the participants in terms of the way they understand algebra. Many participants remarked several times that they used to try to conceptually understand the algebra by either doing the examples over and over until they reach what they referred to as: "*the ability to can [sic] do the sums.*" But that they now started to: "*have pictures in their mind*s *that come back when they must solve problems".* I read it as a form of procedural fluency as identified by Kilpatrick et al. (2001), an emerging form of visual mind mapping, allowing them *to immediately recognise a problem.* This was confirmed by Alzonia who earlier referred to algebra as "*just a lot of symbols and letters.*" During the last reflective interview, she mentioned: "*I start to see what type of problem you give us to do at least I can try to solve the problems.*" Further confirmation came from Helena when she said: "*When I see the problems, I am able to recognise each one of them for what they are."*

I noticed that the participants fell into different categories in terms of their preferences for certain applets. Some participants preferred to work with geometrical applets (as in Figure 6.14), spending more time with the geometrical applets and enjoying gaining insight by moving sliders and animated explanations. A second distinct group preferred applets based on colours, lines and arrows. They remarked that the applets of choice for them were the ones which explained products (First Outside, Inside Last, acronym for the FOIL method) with arrows and explaining the difference of two squares such as in Figure 6.15 A third group preferred algebraic structures in black and white, with not too much information on the screen. During the reflective interviews they were clear about their preferences: if *there is not too many typing on the screen.* I decided to interrogate this phenomenon further during individual reflections from the different groups. Within the same sub-topic of algebra different learners could be visualising differently to create knowledge and conceptual understanding for themselves. This observation lies firmly within the theory of constructivism.

**6.5.5 Diagnostic outcome of one specific reflective interview**

Paulo is the son of refugee parents. He is articulate, older and taller than the other participants. Initially when he joined the NAMVISPRO group he did not attend school for a period of about one year. His command of the English language is excellent and he often acted a spokesperson for the other participants. Paulo achieved average marks during tests, and I attributed it to the fact that he had missed formal schooling for one year. In the first achievement test we conducted during the research project his marks improved dramatically to 76%. This was another reason why it was important to have an individual reflective interview with him.

Danie: *Paulo why do you enjoy working with the applets? Is it mainly to work on the computers or has it to do with the program we do?*

Paulo: *Sir I always loved working with computers, but for the first time I understand algebra faster and even better. It is not that the computer explains better than you, but the programs force me to concentrate and to save the work in my brain. While I look at the screen, I must untangle the examples and save it in my brain. Those images of the examples sort of get burnt into my memory."*

Danie: *What type of applets do you prefer?*

Paulo: *I like the single-coloured ones without images that change or have drawings around them. I found that the ones with one problem on the screen remains in my brain as if a picture was taken with a camera. I* *spend some time to imagine step-by-step how to solve the problem and then store them in this hard disk drive.* (pointing to his head).

Danie: *How did you find the other applets with the sliders and animations?*

Paulo: *I never liked mathematics with drawings or graphs. That is why I do not enjoy geometry so much. You helped me to like the structures of algebra.*

Danie: *How did I help you to like algebra?*

Paulo: *The programs you let us use. I create pictures that stay in my head after we work on the computers. When I must do homework, I go through a brain slideshow until I find a slide that reminds me how to solve a problem. Sometimes I recognise a problem without even knowing the name of the type of problem, but I am immediately able to visualise the steps to solve the problem.*

Danie: *Your marks improved a lot, but you have not achieved above 80% yet. Explain.*

Paulo: *Don't worry it will come*. *I am in such a hurry when I see that I can do a problem that I make many careless mistakes. When we mark the work, we received for homework most of the mistakes I made happened when I rewrite from one step to another.*"

Danie: *How do you study the algebra we are doing, and how do you know that you are ready for a test or examination?*

Paulo: *Wow! I re-do the examples of the applets or from the worksheets. After attempting some examples, the methods for answering the problems just flow from my brain. That is also when I know I am prepared. Sometimes I forget about some exceptions, like to first look for a common factor in those square problems. You mean difference of two squares? That one.*

Danie: *Last question. What changes did you experience about learning algebra?*

Paulo: *I always enjoyed math, maybe now even more. I think I learn now faster and the steps to solve problems just pour from my brain. I really want to improve my math more.*

After analysing some of Paulo's work I found that he mostly made mistakes with the correct application of the order of algebraic operations, namely brackets first then Of, followed by Division and Multiplication and finally Addition and Subtraction, (acronym

BODMAS). When he rewrote the problem from one line to a next, he often did not rewrite a minus sign, producing an incorrect final answer. I noted that Paulo was the first participant to use the word 'visualising'. When he referred to the mind images that he created when working with the applets it was evident that he relied on logical symbolic structures to gain insight into the algebra. These images formed the basis of the *flow from his brain to the paper* when he solved problems. This put me in mind of the previously mentioned symbolic visualisation of Bruner (1994) and a concept that I called communication visualisation where the image in the mind of the participant directed the solution of a problem at hand.

### 6.5.6 Post-lesson Disposition Instrument

As intended by the research design I requested that participants complete the Disposition Instrument shown below in Figure 6.18.

| **Positive Disposition Questionnaire: Summary C2 Post- L2&3** | | Present: 15(S) of School and 13(N) from NAMVISPRO | | | |
|---|---|---|---|---|---|
| Let the emoji's help you to answer the following questions. | | | | Session:C3 L2 & 3 | |
| | 1 | 2 | 3 | 4 | 5 |
| |  | | | | |
| For every question choose one emoji that will best describe your feelings. Mark your choice with an X below the emoji of your choice. | I detest it.<br>Make me scared.<br>Intimidating. | Boring<br>Not interesting.<br>I do not understand it. | Makes me nervous<br>I don't care.<br>Takes time to understand. | I like it.<br>I will do more<br>I understand it. | I love it.<br>I find it easy.<br>I can use it in future. |
| (1) How do you feel about Mathematics in general? | 1 + 1 | 6 + 4 | 6 + 6 | 1 + 1 | 1 + 1 |
| (2) How do you feel about learning Algebra? | 3 + 1 | 4 + 4 | 5 + 6 | 2 + 1 | 1 + 1 |
| TOTAL QUESTION 1 | 2 | 10 | 12 | 2 | 2 |
| TOTAL QUESTION 2 | 4 | 8 | 11 | 3 | 2 |
| Mean per Question | 3 | 9 | 10.5 | 2.5 | 2 |
| MEAN SCORE FOR THE CASE: | | 2.6 | | | |
| MEDIAN SCORE | | 3 | | | |

*Figure 6.18: Post-Lesson 2 and 3 results of Disposition Instruments*

One of the questions the research project tried to answer was to investigate the change in the dispositions of learners towards algebra when taught visually using dynamic software. Halfway through the interventionist study it was clear that the switched approach and use of technology did not bring immediate changes in the disposition of the participants. However, as illustrated in Figure 6.18, the number of participants on the lower end of the instrument showed a steady decrease. According to the summary in Figure 6.18 the median score, for the first time was 3. It appeared that the participants might be developing new skills. It is also important to notice that the instrument never recorded a decline in terms of the dispositions of the participants.

## 6.6 FOURTH SUB-TOPIC: FACTORISING TRINOMIALS

An abundance of applets was found for factorising trinomials. Several applets with a similar approach were selected to ensure that the participants had a choice when they engaged on their own with the applets. I uploaded several links to lessons about the factorising of trinomials. The syllabus for Grade 9 prescribes trinomials of the form $+ax^2 \pm bx \ \pm c$, with $a$ being a positive integer and the expression in standard form. Problems may include the removal of a common factor. Participants were instructed to spend time with each uploaded applet before choosing one to focus on.

### 6.6.1 Pre-lesson disposition scale

Instead of asking the participants to complete the Disposition Instrument I decided to reflect with them about their dispositions regarding learning mathematics in general and algebra specifically. They used the opportunity to air their views. I selected the participants randomly from all over the city so that a wide variety of different personalities was represented. I commended them on their positive attitudes towards the project and the unity among all of them. Greg, an outspoken participant responded: "*As we grow closer to each other we also grow closer to mathematics."* Other participants rejoined: "*If I can learn on my own and just call teacher when I need you, I kind of like mathematics." "When I am successful, I like the subject." "In the past I just gave up, because I struggled so much". "GeoGebra is a lifeline when I am lost. I found that I can access the lessons from home. Sometimes just a little reminder is needed to help me continue." "I have nobody to ask when I struggle, now it changed with the lessons that we can use again".*

We concluded the disposition conversation with all in agreement that it is not *about the working on the computers* that they enjoy the work but *definitely about the way the content is brought to them.* One remarked: "*This class is giving me hope again."*

6.6.2 Screen captures of fourth sub-topic: Factorising trinomials. The first chosen applet was very visual. It expected from participants to complete a square figure by moving little blocks around. By correctly completing the square, participants are able to read the factors of the trinomial, directly from the completed figure. Initially the participants enjoyed moving the factors around to complete the square. Figures 6.19 to 6.21 illustrate the mechanical geometrical steps to factorise the trinomial by completing the square.

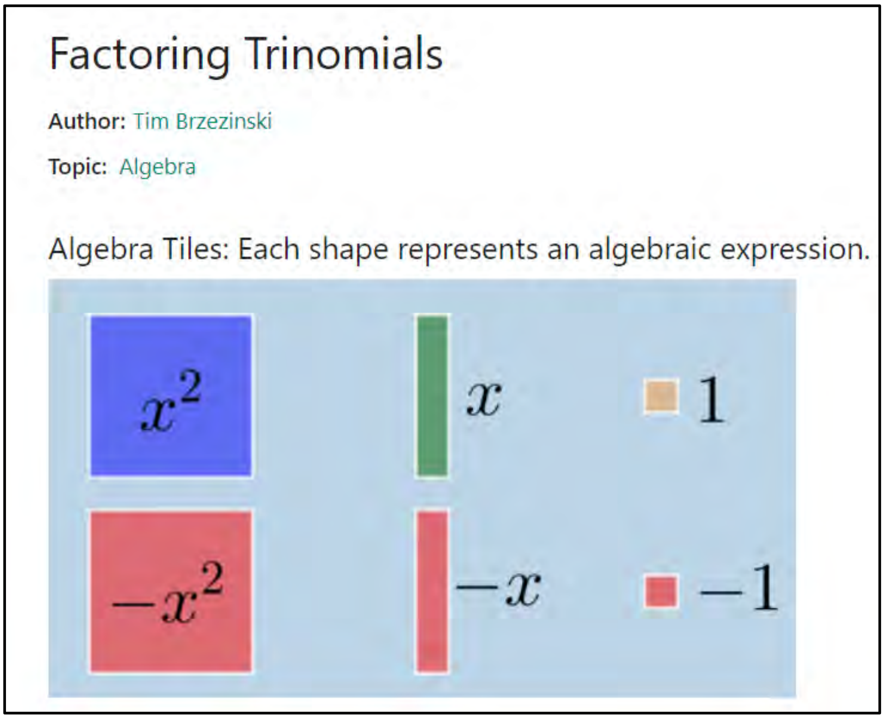



*Figure 6.19: Introductory screen: Factorising trinomials*

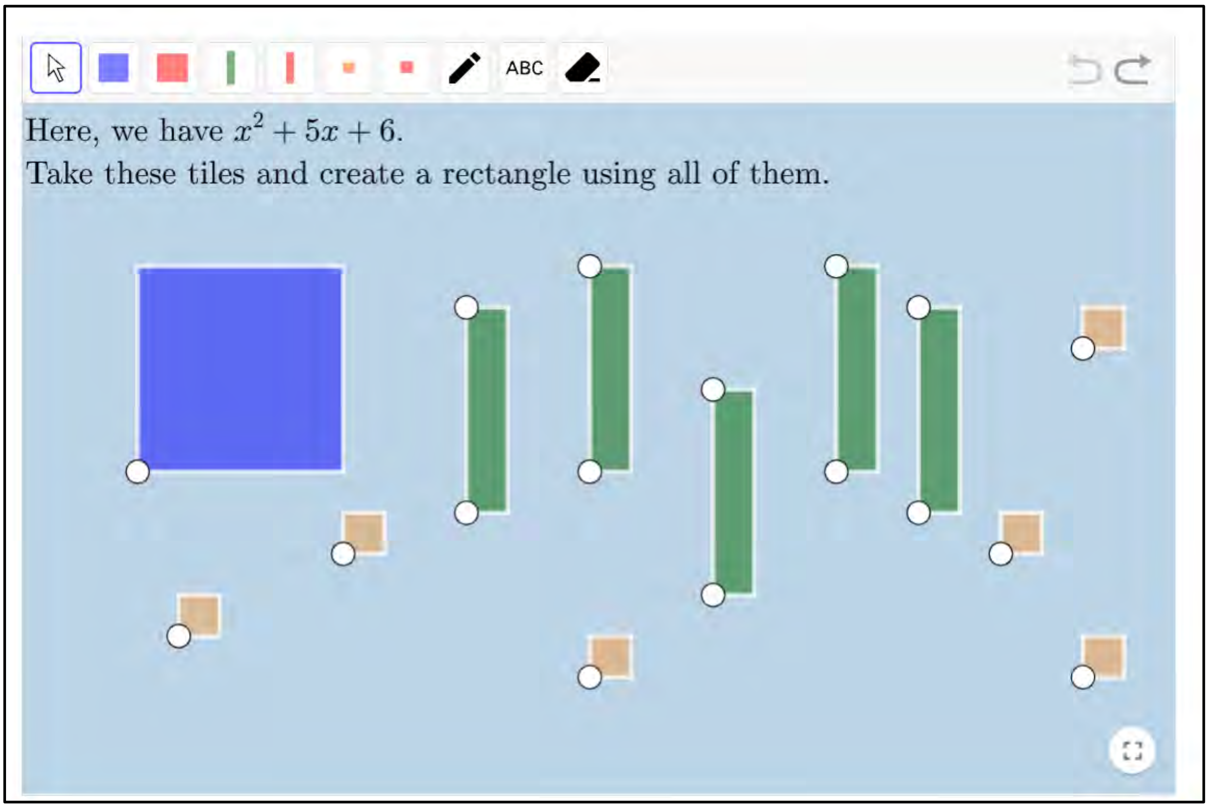



*Figure 6.20: Limited Instructions to visually illustrate factorising*

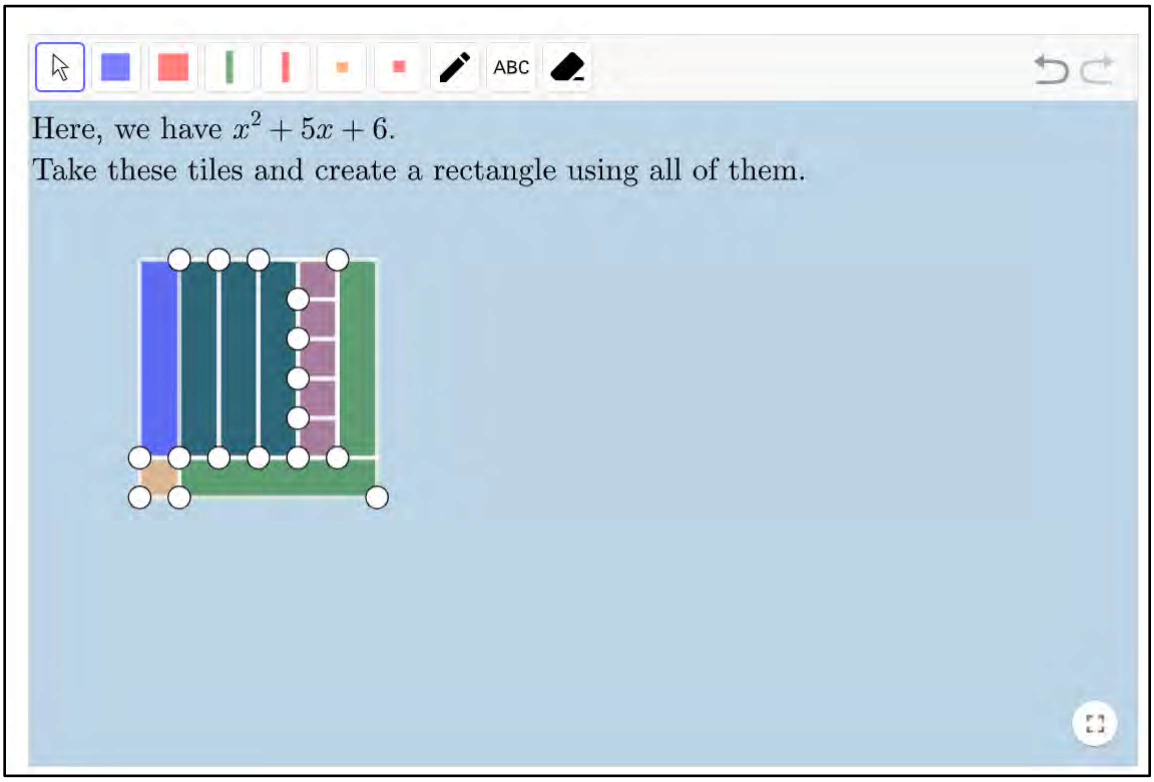



*Figure 6.21: One participant's effort to complete the rectangle*

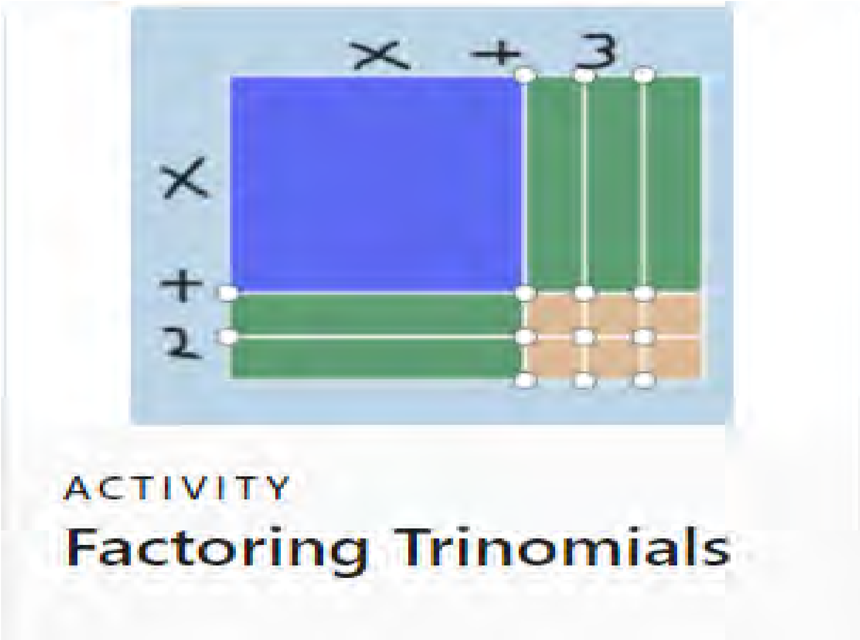



*Figure 6.22: The expected answer when moving algebraic tiles*

The applet included a link to a video explaining the expected steps to conclude that factorising $x^2 + 5x + 6 = (x + 2)(x + 3)$. Initially participants enjoyed playing with the applet, but very few were successful in completing the rectangle. Nobody was able to progress further than the second problem before giving up. Despite the visuality of the applet it was clear that *no conceptual understanding* took place. Participants found it was a game in tiling and the link between the visual demonstration and factorisation never happened.

Fortunately, more applets were selected for instruction purposes. Without intervention from my side most participants progressed to the next applet which was more instructional than geometrical. Although links were uploaded to different applets, it appeared that one specific applet was preferred by the participants. It was not initially a *GeoGebra* applet but a linked animation presentation that fulfilled the requirements of the Applet Selection Instrument. The applet focused on the factors that made up the coefficient of the middle term by applying a simple set of rules based upon the signs of the middle and last terms' numerical coefficients. This was presented as being an inverse process from finding the initial products of two linear expressions.

The applet started by revising the topic ‘products’, to illustrate that products and factorisation have an inverse relationship. Figure 6.23 is a screen capture of the introductory screen of the applet. The products between the two brackets are illustrated and as the participants progress, attention is drawn to the signs of the terms within each set of brackets and how they determine the terms of the trinomial – particularly when the trinomial is factorised.

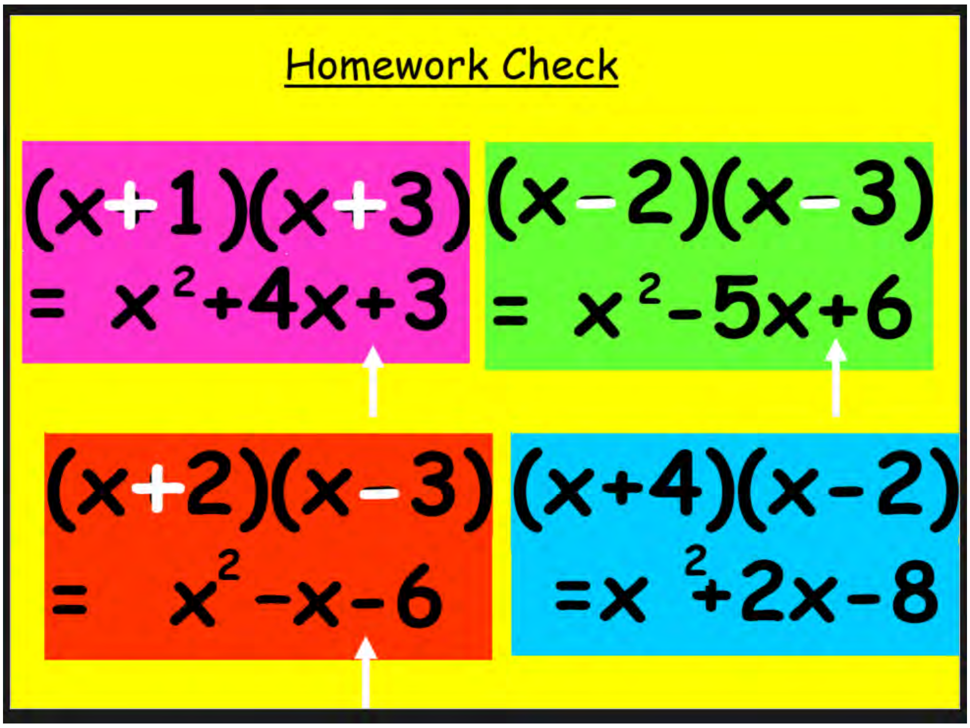



*Figure 6.23: The relationship between a trinomial and its factors*

The animation progressed step-by-step and attention was drawn to the important considerations when factorising a trinomial. In this applet, few words were used, and each new step was well illustrated. Participants had the opportunity to follow explanations for seven examples, each one covering a different scenario. The screen capture shown below in Figure 6.24 is one example. Based upon the factors of the middle term and applying the sign rule the trinomial is re-written as a polynomial and then finally factorised into two brackets.

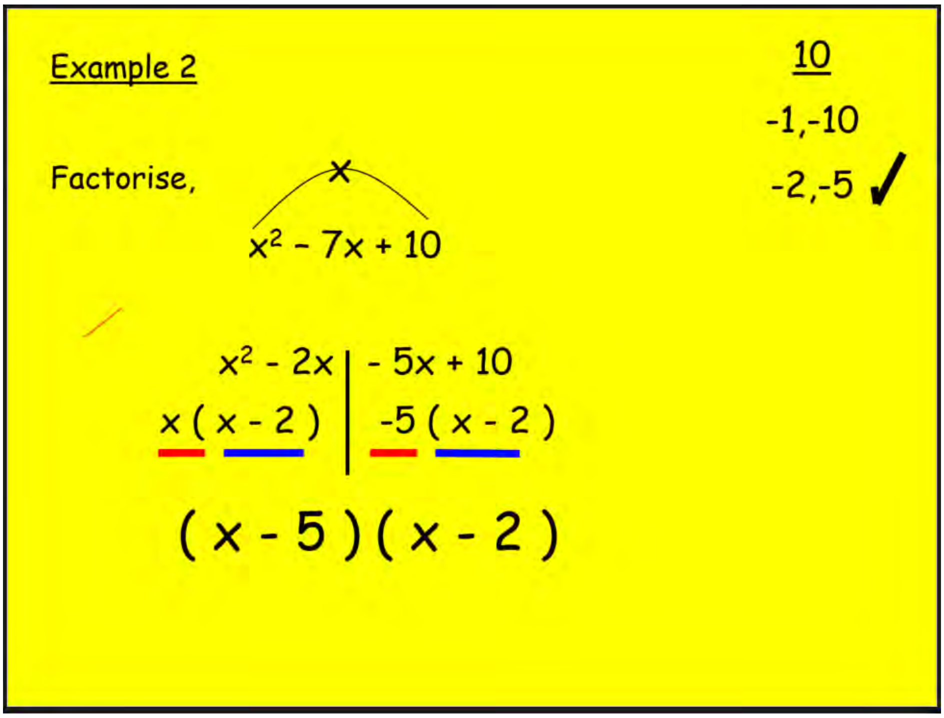



*Figure 6.24: Intermediate steps in factorising a trinomial*

The animation was un-narrated and although I initially intended to add some narration, I decided against it as the interaction between the participants and the screen contributed towards the visual learning process. All the participants remained fully occupied while they followed the steps on the screen, and nobody asked questions. It was deadly silent in the laboratory while the participants were engaged with the application. They could not complete the animation in one session, and we agreed that we would only do a worksheet after all the examples were completed. I noticed how the situation in the laboratory switched from being noisy and uneasy to absolute engagement. The same happened during the session with the NAMVISPRO participants.

### 6.6.3 Post-lesson revision worksheet

After the completion of the work on factorisation of trinomials a worksheet covering factorisation was handed out to the participants. They had the choice of submitting the worksheet as either a handwritten exercise or electronically as a typed document. Initially most participants started to type their work, but because they found typing mathematical equations challenging and time consuming, they eventually submitted written worksheets.

As mentioned, the worksheet consisted of all forms of factorisation in random order. Many participants handed in worksheets with answers scratched out and then replaced with the correct answer. Figure 6.25 is an example of Grace's work. I made a point to ask her about it and her answer was: "*Suddenly the problems were mixed, and I could not immediately identify the type of factorisation I have to do*." Other participants agreed: *"Initially I struggled to identify what factorisation I should do. When I attempted to solve the problem the image of the type of problem came back into my mind."*

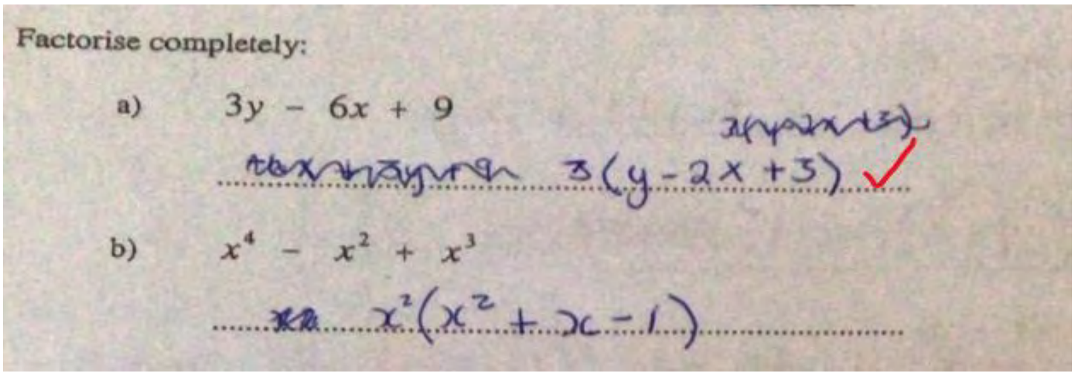



*Figure 6.25: Example of the work by participants to factorise random problems*

Figure 6.25 also illustrates that once the participant was familiar with a certain type of factorisation, they were able to recognise it when it appeared as part of another problem. Figure 6.26 is another extract from the work of the same participant. In problem number 4 on the worksheet, she was able to visualise the steps to change $(2 - x)$ to $-1(x - 2)$ and consequently ***deleted*** the sign between the two brackets to give a final correct answer. Fortunately, she realised her mistake, removing the $-1$ between the two brackets and wrote the answer in the correct format.

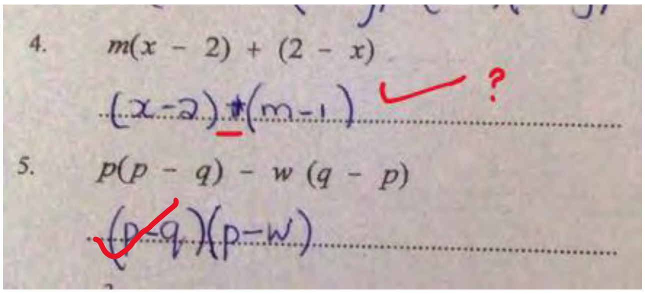


*Figure 6.26: How Grace applied the imagery from her mind to a problem.*

The illustrated examples were picked up in several of the worksheets when I marked them. I discussed my observations regarding the way they had answered the worksheet with individual participants. The overall response was that they *relied on examples in their minds to recognise the problem.* For some it happened immediately while others took longer to ***see** the examples we did during lessons.* Once the participants *recognised* the problem as being: "*the same as the example in my head,"* they were able to apply a method to solve it. Many lacked the vocabulary to communicate the processes in their minds, but one participant said: "*I had to integrate two different sums we did to solve that problem.*" To me it was evident that the participants were drawing from images created in their minds when they used the applets. I could relate their remarks to the definition of visualisation as an epistemological tool by Presmeg (2006a). Their experiences with the applets also related to the early classification by Skemp (1971) of visual symbolism that applies to more abstract concepts as found in algebra. The reflective processes revealed that the participants drew from some form of visual *symbolism* and as Skemp (1971) mentioned, it is difficult to communicate, but realistic and visual. No evidence could be found that any participants are total non-visualisers (Presmeg, 2006). During the individual reflections, they all constantly mentioned *images in my mind* or *seeing the examples in my imagination.*

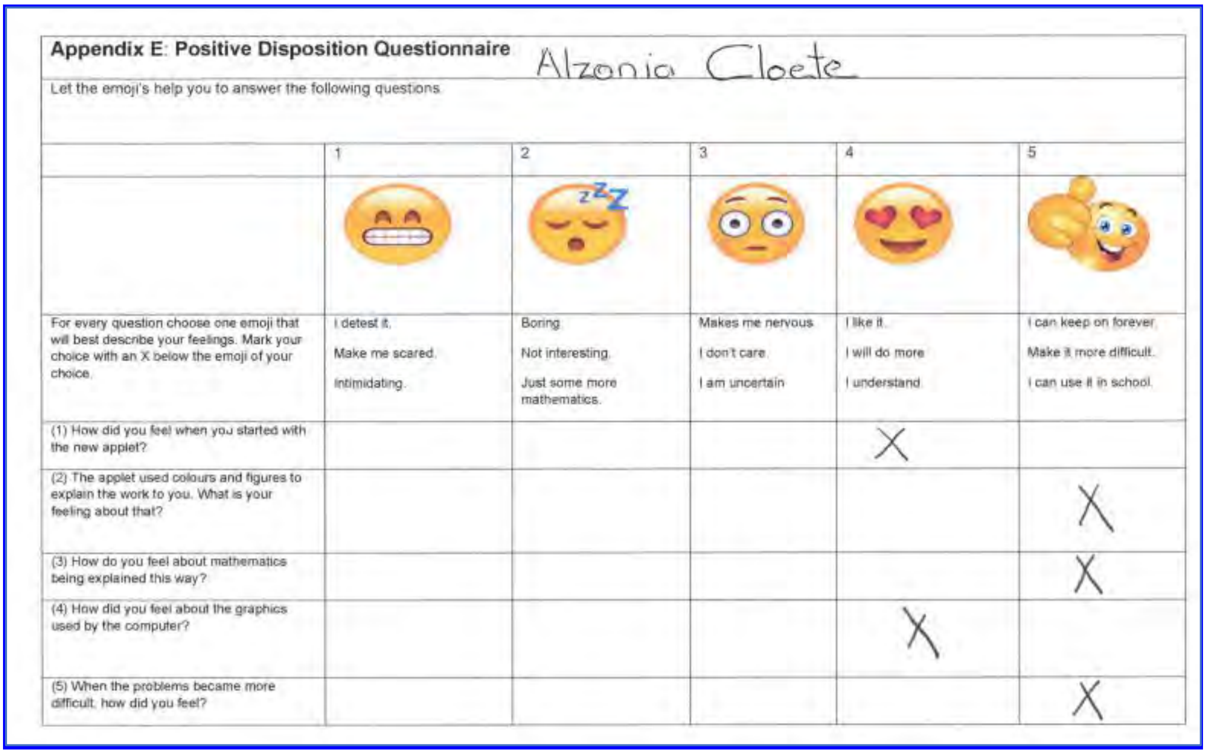
### 6.6.4 Post-lesson: Results of the Disposition Instrument

| | | | | | |
|---|---|---|---|---|---|
| **Positive Disposition Questionnaire: Summary C2 Post- L4** | Present: 14(S) of School and 12(N) from NAMVISPRO | | | | |
| Let the emoji's help you to answer the following questions. | | | | Session:C2 L4 | |
| | 1 | 2 | 3 | 4 | 5 |
| |  |  |  |  |  |
| For every question choose one emoji that will best describe your feelings. Mark your choice with an X below the emoji of your choice. | I detest it.<br>Make me scared.<br>Intimidating. | Boring<br>Not interesting.<br>I do not understand it. | Makes me nervous<br>I don't care.<br>Takes time to understand. | I like it.<br>I will do more<br>I understand it. | I love it.<br>I find it easy.<br>I can use it in future. |
| (1) How do you feel about Mathematics in general? | 1 + 0 | 5 + 7 | 6 + 2 | 1 + 2 | 1 + 1 |
| (2) How do you feel about learning Algebra? | 0 + 0 | 6 + 4 | 4 + 5 | 3 + 2 | 1 + 1 |
| TOTAL QUESTION 1 | 1 | 12 | 8 | 3 | 2 |
| TOTAL QUESTION 2 | 0 | 10 | 9 | 5 | 2 |
| Mean per Question | 0.5 | 11 | 8.5 | 4 | 2 |
| MEAN SCORE FOR THE CASE: | | 2.8 | | | |
| MEDIAN SCORE | | 2.5 | | | |

*Figure 6:27: Results of the post-topic Disposition Instrument*

I was delighted to see the results of the positive disposition questionnaire. For the first time no one recorded that they detest learning algebra. This included the one outspoken participant who referred constantly to algebra as *nonsensical numbers and letters, without any real-life application.* We were in the final phases of the research project, but for the first time the median score reached the mid-point of 2.5. The upper scorers remained stable on 2 for both groups. The mean score also showed some improvement to 2.8.

One specific participant caught my attention, and the following is an excerpt from a stimulus-recall interview with Alzonia from the NAMVISPRO group. Initially she gave herself a very low score of only 1 for each question on the Positive Disposition Instrument. Figure 6.28 shows a dramatic change in her disposition after the third topic was covered with *GeoGebra*. This score even improved more during the last cycle.



**Appendix E: Positive Disposition Questionnaire** Alzonia Cloete

Let the emoji's help you to answer the following questions

| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| For every question choose one emoji that will best describe your feelings. Mark your choice with an X below the emoji of your choice. | I detest it.<br>Make me scared.<br>Intimidating. | Boring.<br>Not interesting.<br>Just some more mathematics. | Makes me nervous.<br>I don't care.<br>I am uncertain. | I like it.<br>I will do more.<br>I understand. | I can keep on forever.<br>Make it more difficult.<br>I can use it in school. |
| (1) How did you feel when you started with the new applet? | | | | X | |
| (2) The applet used colours and figures to explain the work to you. What is your feeling about that? | | | | | X |
| (3) How do you feel about mathematics being explained this way? | | | | | X |
| (4) How did you feel about the graphics used by the computer? | | | | X | |
| (5) When the problems became more difficult, how did you feel? | | | | | X |

*Figure 6.28: Sudden change in score for participant Alzonia Cloete*

Danie: *Alzonia did you enjoy learning with GeoGebra? I see you gave yourself a score of almost 5.*

Alzonia: *Yes Sir, very much. I was never afraid to try to solve the problems. I could see what the solution would be, and the mover helped me to test my answers.*

Danie: *How did you 'see' the answers?*

Alzonia: *I cannot explain it, but I have a picture in my head that connects the problem with the square* [referring to the rectangle]. *I feel I have two methods to answer the same question.*

Danie: *What would you prefer, to work on the computers or that I explain the work to the class with examples?*

Alzonia: *I was a bit afraid of the computers. This is the first time I work on a computer, but now I prefer to work on the computers. I have time to try until I understand the work. I make my own methods to solve problems. Nobody sees when I struggle and that I take longer to understand what to do.*

Danie: *Will you be able to solve similar problems in the examination?*

Alzonia: *Yes. When I see the problem, I have my own thing in my head to solve the problem.*

## 6.7 POST-TOPIC ACHIEVEMENT TEST RESULTS

I was instructed to set the achievement test covering the topic under review. In our school, we make a distinction between topic tests and achievement tests: topic tests are short worksheet-type tests to confirm that learners have reached a satisfactory level of understanding and fluency regarding a specific topic or sub-topic, while achievement tests are drafted on three levels of difficulty, taking into consideration the whole spectrum of achievers. Achievement tests normally cover more than one syllabus topic and are often written during a test period allocated on the term plan of the school. All achievement tests are moderated by the subject head responsible for mathematics to ensure that we maintain a high level of questioning. Eight class groups participated in the test, plus the NAMVISPRO participants who wrote it during their interactive class period.

To me, the average results of the participants were disappointing, with many learners not working accurately. The research participants achieved an average of 57.5%. When compared with the average of the other Grade 9 classes, the participants exceeded the grade average of 49.0% by 8.5% so from this perspective their results should be seen as a significant improvement. Again, there were no failures and the lowest mark exceeded the pass mark by 5%. The highest scorer achieved 96% and for the first time the top position was shared by two participants. Paulo increased his results from 64% to 96%.

One of the larger pitfalls remains factorisation involving the number 1: firstly, within the difference of two squares and secondly, when $-1$ must be removed as a common factor. Somehow many learners and participants could not factorise: $1 - \frac{x^4}{16}$ and the classic example $(2x - 3) - (3 - 2x)b$, where very few were able to factorise the expression to $(2x - 3)(1 + b)$. Their answers varied between: *Cannot be factorised,* or: $(2x - 3)(b)$ or even: $2x - 3 - 3b - 2bx$.

The research participants did very well in factorising the trinomials.

After handing out the test results and having a short discussion about recurring mistakes we completed the topic Disposition Podium.

## 6.8 POST-TOPIC DISPOSITION PODIUM

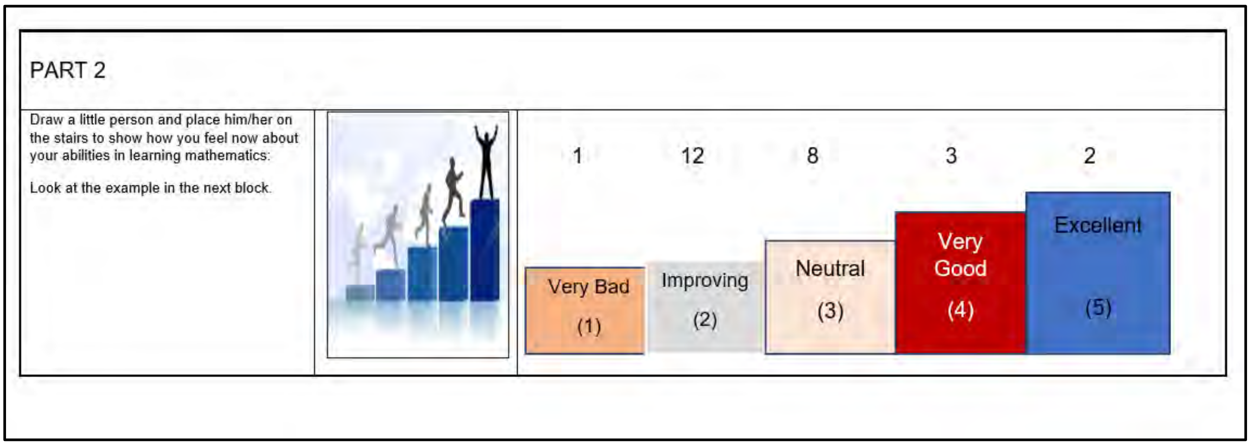



*Figure 6.29: Summary of placements on the Disposition Podium*

The participants confirmed the results of the previous exercise with the Disposition Instrument. It appeared that they were gaining confidence in doing algebra and that fewer participants placed themselves on the lower end of the podium. Time was made to have an individual reflective interview with the participant who placed herself on the lowest position. This proved to be a fruitful exercise for her to reflect with me on her own learning of mathematics. Helena remarked: "*I find the work that we are doing on the computer easier to understand, but nobody ever taught me how to understand and remember mathematics. I am afraid when I must do mathematics, and when I try to do homework, I have no confidence. In my mind I believe that there are many gaps of knowledge and that will always prevent me from doing better in the subject. We are told in our school that mathematics is like building a wall. If the foundation is weak, you will never be successful in learning mathematics*. *I am now learning that one can.*"

I was somehow optimistic that a further swing towards the higher end might still be possible, but during the next cycle we would have to bring together all previously covered topics and add new algebraic skills and knowledge.

## 6.9 CONCLUSIONS ON TOPIC 2

Individual participants received their tests from me personally and after spending some time reviewing their mistakes in general, I discussed each one's mistakes with them. It provided each participant with the opportunity to revisit their own work and to verbally explain why they made specific mistakes. It led me to conclude the following:

- Basic arithmetic remains a challenge for participants. Many participants are so dependent on their calculators that they do not have the confidence to do simple calculations of the order $-1 \times -2$ without the assistance of their calculators. So much time was spent doing basic arithmetic on the calculators that some participants could not complete the last section of the test or had to rush through the last section, causing them to make mistakes.
- Constant evidence was found that many participants relied upon the examples that they had practised on *GeoGebra*. They tried to apply the memory imagery they had of specific examples directly to the questions in the test. Some were able to grasp when a little manipulation was required before the *template* or *memory pictures* could be applied. During the reflective interviews everyone who failed to generalise or relate the examples (they had practised) to the questions in the test remarked with disappointment: "*Yes, now I see what I should have done.*" Another remark made by participants was: "*With a little more practice I will be able to see how to apply the work we did correctly in the test.*" I found clear evidence of better conceptual understanding of the basic concepts, but some participants did not achieve the ability to progress from interiorisation to coordination, encapsulation, generalisation and reversal as identified by Capetta and Zollmann (2014). Many remarked that their abilities *are improving* and for the next test they would be aware of the *pitfalls.*
- The participants were hesitant to attempt problems containing fractions. Many could not complete the factorising of $1 - \frac{x^4}{16}$ and although they wrote $(1 - )( 1 + )$, they did not complete the factorising. Many correctly wrote down the square root of the fraction $\frac{x^4}{16}$ as $\frac{x^2}{4}$, but failed to bring it from their rough work into the brackets. It appeared that they had conceptual understanding of

the required steps but could not visualise the last step to complete the expression.

- Again, participants confirmed that they started to believe in their abilities again and that the images of the questions recalled images created in their minds during the *GeoGebra* lessons.

I was convinced that the work with the applets was transforming the participants' learning of algebra from doing repetitive examples to the creation of visual-symbolic figures of specific algebraic concepts in their brains. They started to rely on their recognition skills to visualise relationships between the imagery they had in their mind and the finding of solutions to problems. This conclusion is supported by their mentioning of: "*I had a picture in my mind, that helped me to recognise the problem.*" Or their constant referring to: "*I could immediately see how to solve the problem.*" It convinced me that with more exposure to the *GeoGebra* applets, their visual skills could develop further, and they would attempt problems with greater confidence. Participants ceased to use phrases like: "*I had no idea what to do,*" or "*I could not recognise the problem".*

After the completion of the second topic and cycle the research plan made provision for a third cycle of data collection. Despite the cyclic nature of the research, is the data collection a continuum process to track changes in the conceptual understanding of algebraic concepts and monitoring participants for positive changes in disposition. Every new cycle introduces a new topic slightly more difficult than the previous one. Chapter 7 focus on the last cycle, covering topic three.

# CHAPTER SEVEN

# DATA COLLECTION & DATA ANALYSIS OF TOPIC 3

## 7.1 INTRODUCTION

This chapter provides an in-depth discussion about the data collected and the analyses of the data within an interpretive paradigm, of the third cycle of data collection according to the research plan. Not only will this chapter conclude the cyclic nature of the research by unpacking and analysing the third algebra topic but will bring it all together by building upon the algebraic content of the first two cycles. Previously acquired knowledge and skills are integrated into the newly acquired knowledge of the third topic. This completes a full circle of learning algebra with technology creating conceptual understanding, achieved through visual learning within the technological arena. At the end of the previous cycle, I found evidence that participants had some consensus about acquiring new relational and communication structures to solve problems based upon the recollection of visual imagery created during the *GeoGebra* lessons. The data collection process was based upon the same instruments that guided data collection during the previous cycles. This chapter will conclude the process of vertical data analyses.

Data collection took place as part of the ongoing teaching and learning process of the prescribed syllabus content of the Ministry of Education in Namibia. I did not deviate from the visual learning with *GeoGebra* as the chosen method of teaching the syllabus topic to the participants. As with the previous cycles, the work had to be completed within the prescribed time frames of the scheme of work. The syllabus referred to specific skills that need to be covered. Examples of what is expected from learners include the ability to:

(1) recognise $1 - x^2$ or $y^2 - 1$ as the difference of two squares;

(2) recognise and re-arrange $c + ax^2 + bx$ to form a trinomial that can be factored within any algebraic fraction: and

(3) do addition, subtraction, multiplication and division of a combination of algebraic fractions that include different types of factorising.

To conclude the cycle, participants had to write a standardised achievement test drawn up by an independent examiner. All the topics covered during previous cycles had to be included in the test.

## 7.2 *GEOGEBRA* LESSONS AND ACTIVITIES

All the lessons were presented using the scope of the applets available on the *GeoGebra* platform. To fully cover the requirements of the Namibian syllabus, I had to create a few more lessons. A decision was taken to use the *GeoGebra* platform for revision purposes by using *GeoGebra* worksheets and interactive quizzes. This proved to be fruitful as all the participants stayed engaged with the revision applets and commented positively on their structured approach. *GeoGebra* offered the participants an individualised experience and they could repeat sub-topics with different examples until they reached a satisfactory level of understanding. During reflective interviews, the participants appreciated the ability of the applets to adjust to their phase of learning and the opportunity provided for re-doing sub-sections to achieve conceptual understanding.

Permission was granted, allowing the participants to complete some homework tasks and most of the prescribed worksheets on the computer and to submit them electronically for marking.

## 7.3 DESCRIPTION OF THIRD RESEACH TOPIC: ALGEBRAIC FRACTIONS

According to the new Namibian Grade 9 mathematics syllabus (MoEAC, 2015) learners should be able to:

1. simplify algebraic fractions by factorising the numerators and the denominators,
2. multiply and divide algebraic fractions after factorising,
3. do addition and subtraction of algebraic fractions, and
4. solve simple equations with fractions through cross-multiplication.

According to the new Namibian mathematics syllabus for Grades 8 and 9 (MoEAC, 2015) the topic ‘Algebraic fractions’ should build upon the foundation of simplifying fractions done in Grade 8, then include a detailed study of simplifying simple algebraic fractions including multiplying and dividing to the point where learners are able to factorise and solve an equation with algebraic fractions. Finally, the laws of indices should be integrated into an algebraic fraction. For my research, the integration of all previously covered topics into one problem provided the final opportunity for data collection. This was to provide evidence that the cyclic process of visual learning accumulated in the creation of a firm foundation for the conceptual understanding of all the topics covered. This would also hopefully confirm that a positive disposition was developed amongst the participants.

Again, I decided to collect all data within the time frames allowed by the schemes of work of the school. Both participants and non-participants had to write the prescribed and independently set achievement test.

## 7.4 PRE-DATA COLLECTION INDUCTION LESSONS

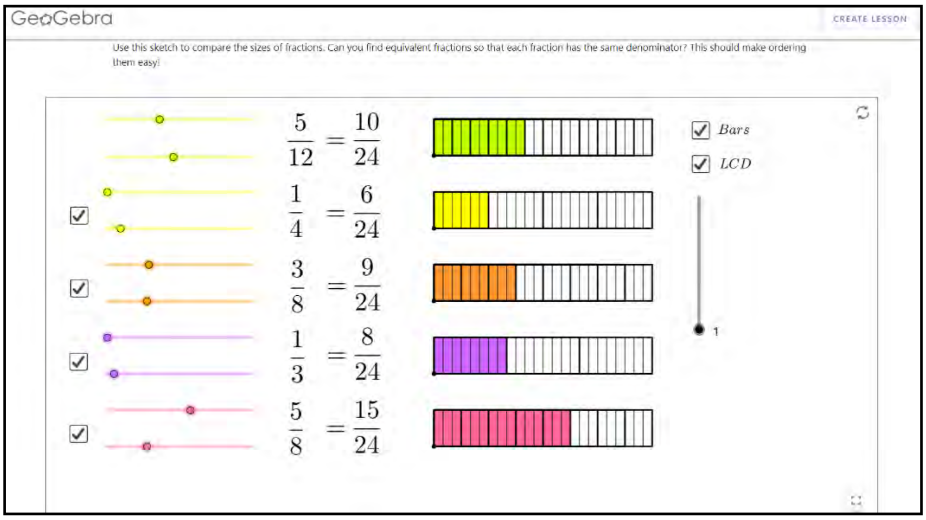



*Figure 7.1: An introductory lesson about fractions and finding the lowest common denominator (LCD)*

**Source:** [www.geogebra.org/classroom](www.geogebra.org/classroom)

The reflective interviews after the second cycle convinced me that I had to make a few changes to the scheduled plan for executing Cycle 3. Initially I intended to start immediately with the applets introducing algebraic fractions. However, during the reflective interviews after Cycle 2, the participants referred to several uncertainties that they experienced which prevented them from performing optimally. Fractions was mentioned specifically as one area.

Didactically, fractions are seen as challenging for learners – therefore teachers should lay a firm foundation when teaching fractions. Research spanning three continents concluded that learners can only improve their numerical skills when they fully understand fractions in terms of magnitude, representation and their relationship with whole numbers and manipulations. It is important that learners bridge the gap between fraction understanding and numerical fluency. The understanding of fractions is essential for the conceptual understanding of mathematics (Torbeyns et al., 2015).

To evaluate and improve their numerical skills with fractions participants had the opportunity to engage with selected fraction applets. Concepts like the LCD, operations with fractions and the ability to see fractions as part of a whole were dealt with. During several interviews with participants, it surfaced that they found a tidy layout of the screen with just a few important concepts highlighted or pointed out assisted them best to conceptually grasp underlying algebraic principles. The use of many different colours or elaborate explanations on-screen were found to be confusing and did not contribute towards conceptual understanding.

Almost every participant mentioned how they created memory templates of principles explained on-screen. Both Alzonia and AJ who *detested* algebra and called algebra, "*just confusing letters and symbols*" attributed their improved performance to the *clear and minimalistic amount of information displayed on-screen*. Both mentioned that they were developing "*pictures and structures in their minds*" that they can recall to identify and solve problems. AJ mentioned that the "*computer lessons helped me to order my mind. I still struggle because many previous topics are still not clear in my mind, but the more I practise the better I can remember. I have overcome many fears*." Precious referred to "*a noise in my mind whenever I saw the mixed letters and numbers, but now they appear to be organised*."

During several interviews it surfaced that a simple, tidy structure plays a particularly important role in the creation of the memory templates for the participants. Other participants confirmed that the applets helped them to regain confidence when working with fractions. While some applets were presented in entertaining ways, all the participants were focused on acquiring the required skills that apply to fractions. Figure 7.2 is an example of an applet with a colourful although simple logical layout. No participants mentioned that colour played a role in achieving conceptual understanding of the concepts explained.

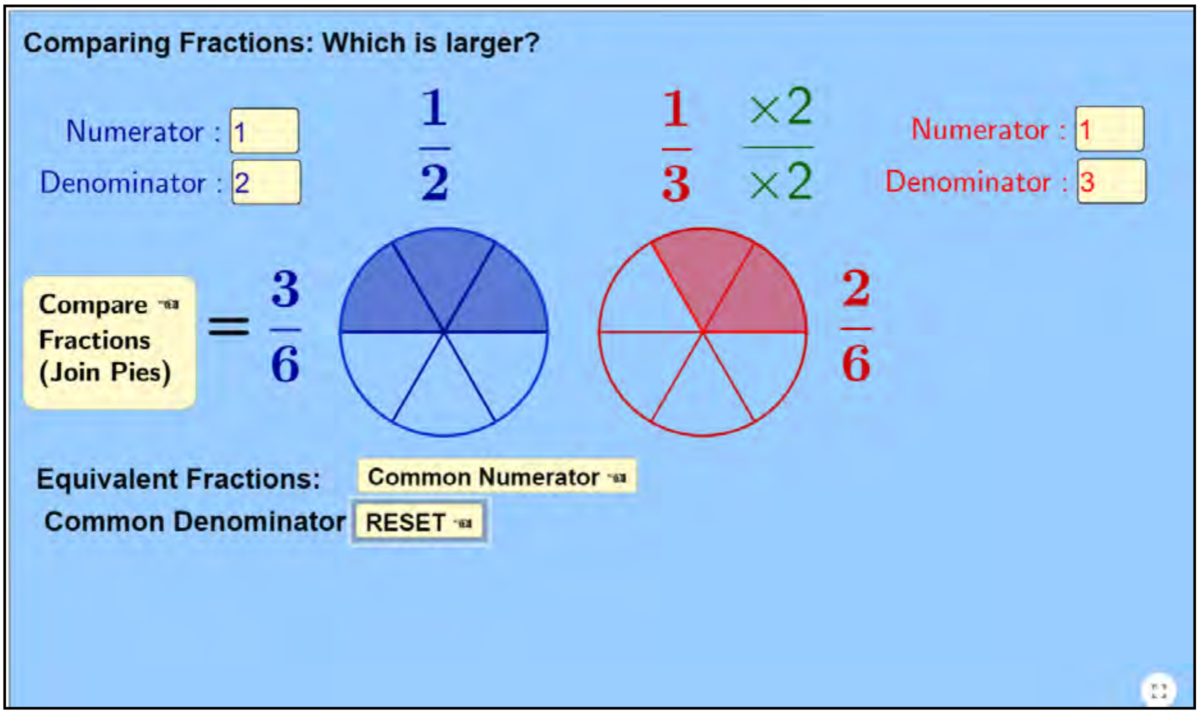



*Figure 7.2: Revision applet using colours to demonstrate the concept*

With the applet in Figure 7.2, participants could revise the use of the LCD to compare fractions. Again, the participants related how they focused rather on the lesson content about the nominator and denominator than on the colourful layout of the applet. When asked if the colours would play a role in helping them to recapture the skill, I received a 100% *no* response. The only contribution that the colours made was to draw attention to important aspects required for understanding. For everyone, the layout of the pie figures and the eventual numerical comparison of $\frac{3}{6}$ and $\frac{2}{6}$ was the most important. Participants were already aware of multiplication with the identity element of multiplication, (a special form of multiplication by 1, for example: $\frac{2}{2}$ $or$ $\frac{x}{x}$ ) and the term LCD. It re-confirmed to me that although graphical figures can be used, the layout rather than the use of colours is more important for the participants to understand concepts. This conclusion was supported by a remark from Paulo: "*We all have pictures in our minds when we think about things, but for these sums I just try to recall the examples we did. I think I see black typed sums in my mind.*"

## 7.5 FIRST DYNAMIC SOFTWARE LESSON ONE LAWS OF INDICES

### 7.5.1 Pre-lesson disposition questionnaire

To ensure consistency of the results, the participants were requested to complete the Disposition Instrument at regular intervals, even if no new materials were introduced or the previous laboratory session was before a weekend. As illustrated in Figure 7.3 the pre-lesson disposition stayed in line with their post-lesson dispositions after the previous cycle. The only visible trend was that participants at the lower end of the disposition scale started to move away from the lowest scores. For the first time the median score was 3. It was encouraging to observe the progress made in the change in the dispositions of participants in learning algebra. Fewer participants scored algebra lower than mathematics in general. I found it to be consistent with the recommendations of Ojaleye and Awofala, (2018) who recommend a blended instructional approach that includes learning with computer technology and a problem based strategy for teaching algebra to secondary school learners (Ojaleye & Awofala, 2018).

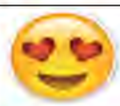
| **Positive Disposition Questionnaire: Summary C3 Pre- L1** Present: 14(S) of School and 13(N) from NAMVISPRO | | | | | |
|---|---|---|---|---|---|
| Let the emoji's help you to answer the following questions. | | | | Session:C3 L1 | |
| | 1 | 2 | 3 | 4 | 5 |
| |  |  |  |  |  |
| For every question choose one emoji that will best describe your feelings. Mark your choice with an X below the emoji of your choice. | I detest it. Make me scared. Intimidating. | Boring Not interesting. I do not understand it. | Makes me nervous I don't care. Takes time to understand. | I like it. I will do more I understand it. | I love it. I find it easy. I can use it in future. |
| (1) How do you feel about Mathematics in general? | 1 + 0 | 5 + 7 | 6 + 3 | 1 + 2 | 1 + 1 |
| (2) How do you feel about learning Algebra? | 0 + 0 | 5 + 4 | 4 + 4 | 4 + 1 | 1 + 2 |
| TOTAL QUESTION 1 | 1 | 12 | 9 | 3 | 2 |
| TOTAL QUESTION 2 | 0 | 9 | 8 | 5 | 3 |
| Mean per Question | 0.5 | 10.5 | 8.5 | 4 | 2.5 |
| MEAN SCORE FOR THE CASE: | | | 2.8 | | |
| MEDIAN SCORE | | | 3.0 | | |

*Figure 7.3: Summary of pre-lesson scores on the Disposition Instrument*

Only one of the school participants still felt that they found mathematics frightening and that detest doing mathematics. None of the participants placed algebra on the lowest score. This was significant because the whole research project focused only on algebra.

### 7.5.2 Pre-lesson 1 Impromptu Reflection by participants

For the first lesson on algebraic fractions, I scheduled a combined session with all participants of the case. Before we could log-in and start working on the applet, Paulo mentioned that the group requested a discussion with me about the research project to clarify some uncertainties they had and to reflect on the process so far. I agreed on condition that we record the conversation as their reflections could provide me with data for the research. Again, I outlined my own objectivity and the importance of not giving any direction regarding the outcome of the research to the participants.

Paulo acted as spokesperson for the participants during the interview: "*Sir, may we ask what you try to find with your research? We really enjoy these classes and the way we learn with the computers. Was that what you wanted to research?"* The relationship between me as researcher and the participants was consistently particularly good, and I felt that they trusted me regarding the new approach, my honesty and my transparency with what we were doing. I did not want to compromise it in any way. I tried to explain the research process but avoided using the words visualisation and (conceptual) understanding. I had to steer clear from mentioning any changes in disposition to them.

> *I try to learn from you how you understand algebra and how you recall what you have learnt. I rely on your help to find answers for a few questions about learning algebra in Grade 9. Can GeoGebra help us to learn algebra? I would like to know what is going through your mind when you answer algebra questions. In other words: When you say WOW! I know how to solve this new problem, then I want to know what it is that makes it possible for you to solve the problem correctly. We all have one common aim: to learn algebra better and to enjoy what we do.*

More participants joined the conversation and shared their experiences. Unintentionally, important remarks were made:

> We find the *algebra easier to understand from the computer screen. The algebra lessons are like a movie that I can follow. It just looks easier than in my books or when you do it on the board.*

*I can see on the screen what I should understand. When I study, I remember the work we did better.*

*We work now without fear, Sir. I always was afraid to bring wrong homework to school.*

*I am afraid to ask teacher to go back and explain something again, but now I can go back myself and look at specific things I struggle to understand.*

*I feel I concentrate longer when I work on the computer and the examples stay in my mind. All the work makes more sense to me now. It helps me to do my homework faster.*

Marco concluded the interview by stating:

*Sir we all feel that we are improving, but we do not know if we are helping you with anything. We wanted to tell you that all of us look forward to coming to class every day. We hope we can continue having our lessons like this.*

The remarks by Marco provided me with the opportunity to re-assure all participants that their contributions are important and in line with the expectations of the research process.

### 7.5.3 Lesson 1: Algebraic Fractions: Execution and Screen captures

The selected applet used to introduce algebraic fractions to the participants progressed slowly from simplifying very simple algebraic fractions step-by-step to more advanced algebraic fractions where they had to find a common denominator and factorise before simplifying the fraction. I observed that all the participants were engaged with the applet and everyone completed all the posed questions. From the *GeoGebra* classroom screen I could also see that some participants took quite a while before attempting to solve a problem. There were few mistakes happened, but the applet design allowed for a similar problem when a wrong answer was given. Progression started from basic algebraic fractions such as $\frac{2a}{6a}$ to $\frac{3b^2}{b}$ to fractions requiring the removal of a common factor such as $\frac{2a+4b}{a+2b}$. Figure 7.4 is a screen capture of a problem where a participant has to first find the LCD containing variables before doing the addition. The logical linear progress followed by the applet, where participants could engage in many examples covering a vast quantity of pre-knowledge, allowed the participants to gradually establish their own mind maps as they tried to create knowledge and understanding. A classroom situation where the teacher does some selected examples step-by-step on the white bord cannot fulfil this function. Firstly, it is time consuming to write every step on the board and to explain every step to the learners; and secondly, not all learners can stay focused on the board for the entire process taking place in front of them. For the first time I experienced how disconnected learners are in a teaching with a whiteboard situation.

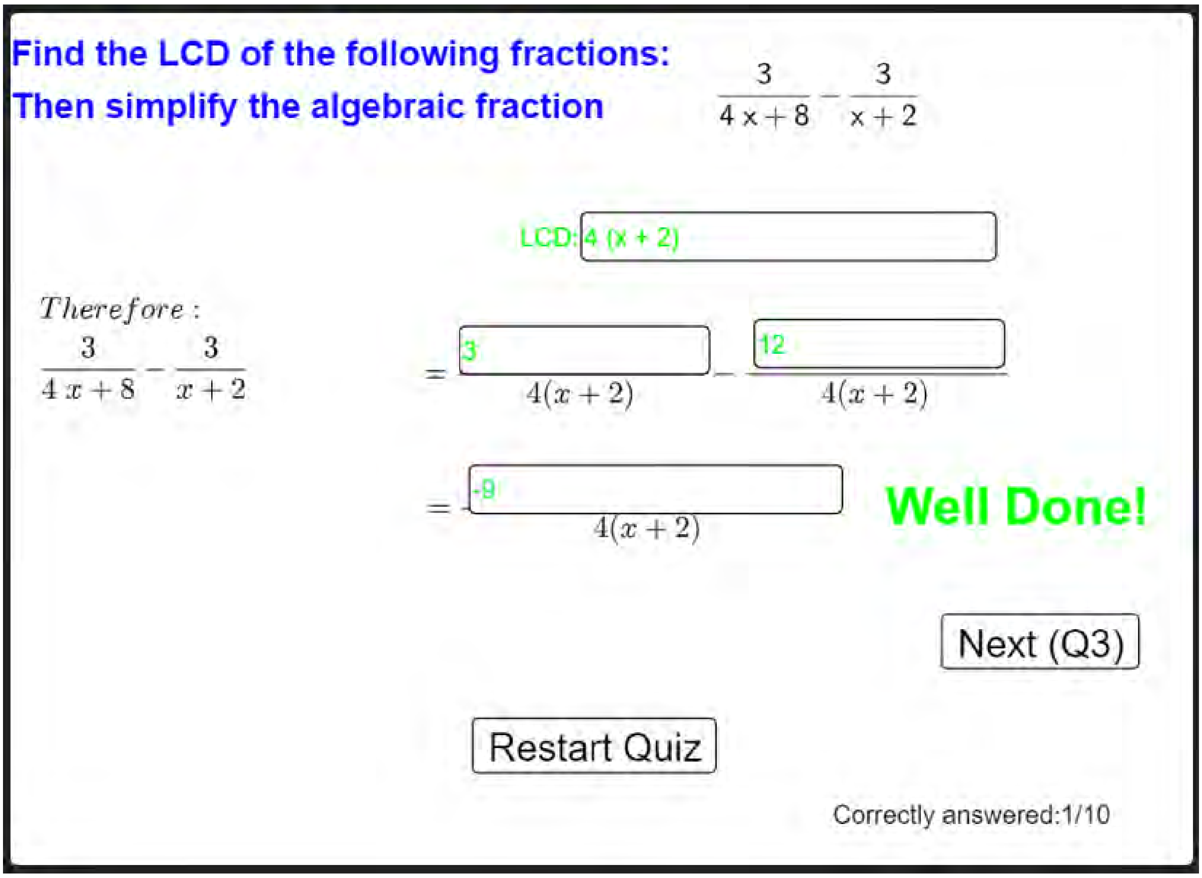



*Figure 7.4: Example of a fraction where participants must find the LCD first*

As shown in Figure 7.4, the participant fully engaged and was in control of the example in front of them. They could control the speed of progress through the visual explanation and decide to skip similar examples or re-do any of the ten examples, if needed.

During monitoring the participants while they were engaged with the applet, it was clear that they all progressed at an increased speed and were able to use their previously required knowledge to successfully solve problems. Fewer questions were asked, and no participants sat back and withdrew from interacting with the applet at any stage. I concluded that the participants were able to apply a process of reversal by relying on their previously acquired knowledge of numerical fractions and visualise new mathematical notions applicable to algebraic fractions. They had no problems with the added variables that required different manipulations.

Again, the lesson concluded with a multiple choice quiz. I observed that many participants first rewrote the question on a piece of paper, circled sections, then solved the problem before selecting the given options. Their actions and the way they closed their eyes to recall previous knowledge, convinced me that they drew from some imagery in their minds to assist them in solving the problems.

### 7.5.4 Post-Lesson 1 placement on the Disposition Podium

After the first session the participants were requested to express their current feelings about mathematics by using the second instrument. Again, they had to place themselves on a little podium with five places by drawing a little person on the podium as illustrated in Figure 7.5.

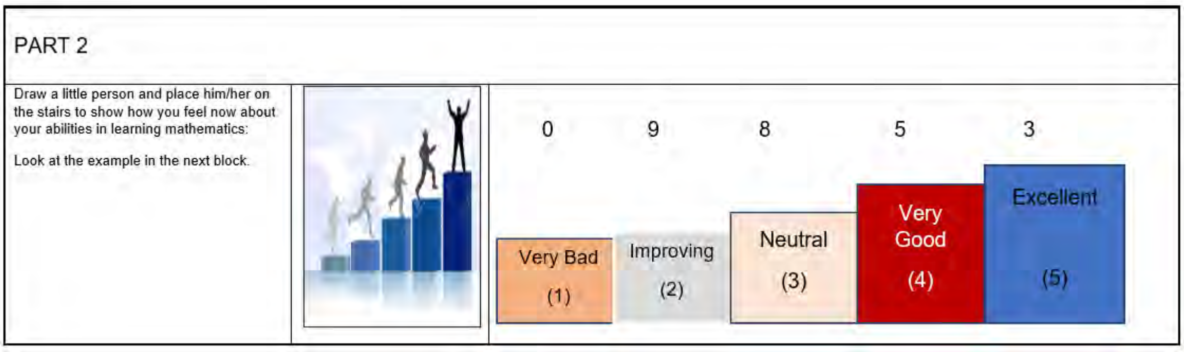



*Figure 7.5: Summary of scores about participants' feelings about mathematics*

The results of the podium instrument aligned 100% with the results obtained for question 2 of the Disposition Instrument. Again, nobody placed themselves on the lowest position. The figures drawn on position 2 (improving) showed people walking towards the higher scores. For the first time participants wrote comments below the podium, especially those who chose the improving podium, commented. "*I am improving but can improve more.*" "*I struggle but understand better now.*" "*I find algebra very difficult, but the work we did is ok*" One comment caught my attention: "*Nobody ever taught me how to learn algebra. Thank you.*"

Figure 7.6 shows an example from a participant who progressed in terms of his achievements academically, but also started to enjoy the visual approach of the research process. The drawing of the figure running towards the higher steps of the podium is symbolic of the journey he has embarked upon. He underlines his feelings by adding a comment: "*I feel I'm improving in mathematics and starting to like it*."

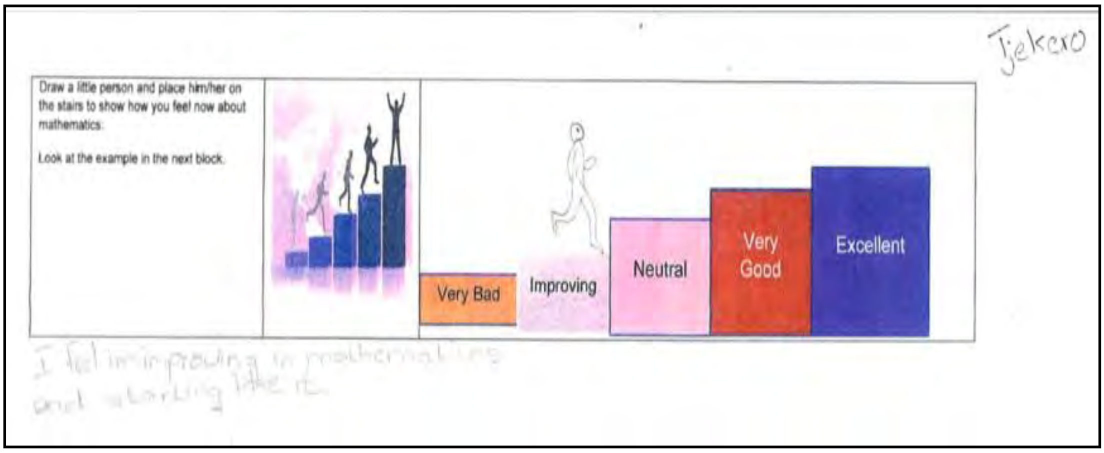



*Figure 7.6: Example of participant adding comments to express his feelings*

Applying the two instruments at different intervals provided triangulation of the data. The consistency of the data collected a few days later assured me that the instruments served the purpose they were designed for. A 100% correlation was found between the first Disposition Instrument and the Disposition Podium instrument completed a few days later. I was delighted by the honesty and consistency of the participants and the seriousness with which they approached the journey we had embarked upon.

### 7.5.5 Lessons 2 & 3: Execution and screen: Complex algebraic fractions

Lessons 2 and 3 spanned over several periods as recommended by the scheme of work. One lesson period had to be reserved for the writing of an achievement test.

According to the scheme of work learners should be able to apply the previously acquired algebraic skills to simplify algebraic fractions or equations containing algebraic fractions. The integrated topic of algebraic fractions required several sessions in order to cover:

- simple algebraic fractions with single variables;
- fractions with multiple variables with positive and negative indices;
- simplifying fractions with common factors;
- fractions requiring factorisation including difference of two squares, trinomials, and grouping;
- equations with fractions; and

- complex algebraic expressions with addition, subtraction, products and division of combined algebraic fractions.

The topic provided an excellent opportunity to conclude the cyclic processes of the research. In the final cycle participants had to apply all the skills acquired during the previous cycles, integrated into solving challenging problems about algebraic fractions. Again, the applets provided the opportunity for participants to progress through the learning curve at their own phase. Some participants had to revisit previous lessons to refresh their minds about the procedural steps required to solve a problem, while others easily recalled the required pre-knowledge to solve integrated problems contained in earlier topics. Participants re-visiting previous applets usually remarked: *"We have done this work, just must go back to refresh my mind. I can still do it.*" For some participants, the *brain-mapping* was so well done that they were able to apply the foundational knowledge continuously to each new problem.

A progressive process was followed, advancing step-by-step from very simple problems to more complex and integrated problems. All the participants differed in terms of the time spent with a specific problem, and the types of problem they had to spend more time with, but I could observe an increase in the average time that participants spent on the different applets. They attributed it to an acquired confidence to engage with new applets and an established ability to recall work previously done with the *GeoGebra* applets.

Once the participants had grasped the concept of simplifying algebraic fractions, they were able to attempt and successfully solve more complex problems with confidence. Figures 7.7 to 7.9 illustrate the progression followed by the applets towards instilling the required conceptual abilities. While observing the participants, I noticed how involved they were with the applets and that all the participants tirelessly attempted to solve the problems step-by-step. They continuously switched from the applets on the screen to working on paper. Their dedication and seriousness were clearly visible. The selection of applets for this cycle offered participants the opportunity to follow different approaches to solve the posed examples. They did not show a preference for a specific applet, but it was obvious that they attempted the problems with confidence and were comfortable to attempt a variety of problems on the topic.

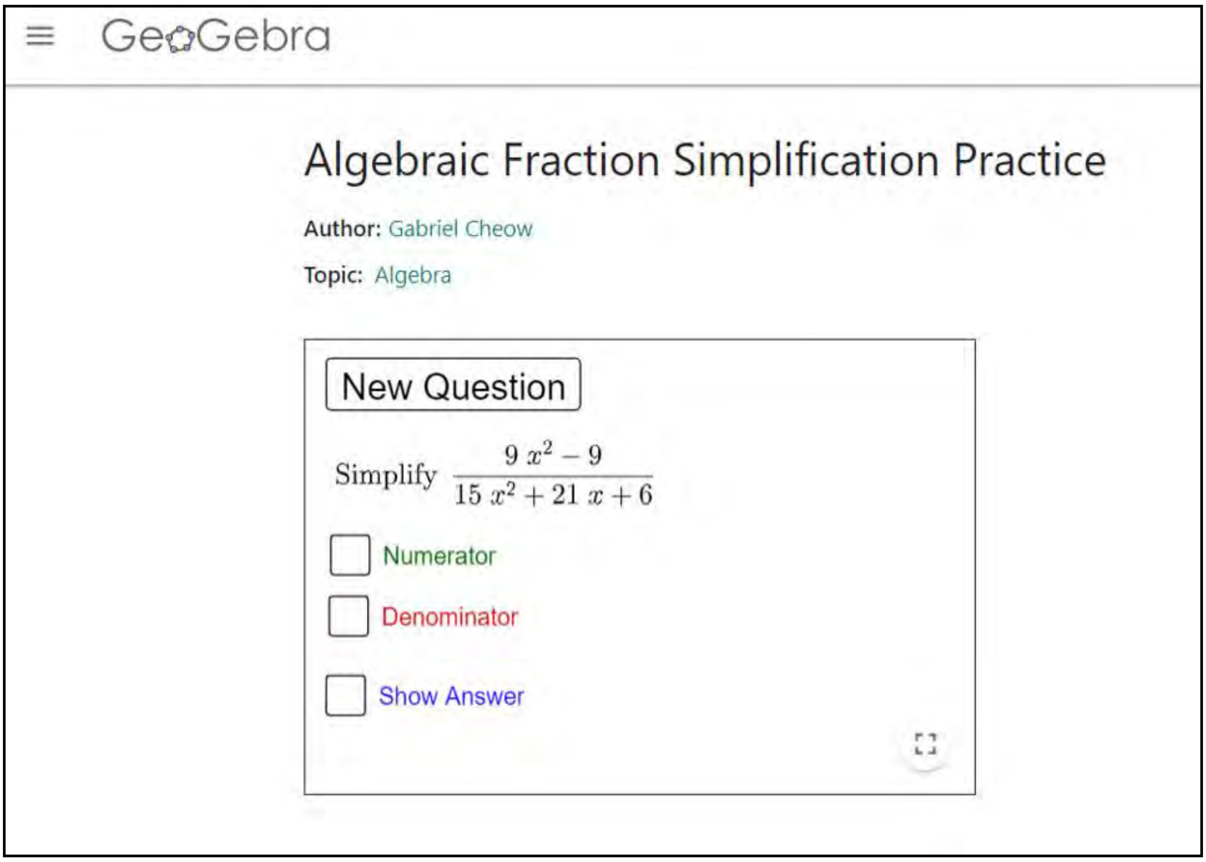



*Figure 7.7: Example question from one applet*

The first chosen applet required that participants dissect a posed problem by firstly simplifying the numerator, then the denominator and lastly, simplifying it by cancelling the common factors. During each step participants could request some guidance. By ticking a box, the applet would reveal how to factorise the numerator first and then the denominator. The step-by-step guided approach assisted participants to gain confidence when solving more complex algebraic fractions. For most participants it was an enjoyable experience because it "*helped me to break the problem down into smaller pieces. I am not deterred by one big problem.*"

I observed that all the participants kept a piece of paper and pencil next to them. After writing the problem onto the piece of paper, all of them divided the problem into different segments. They revealed different methods of guiding and structuring their thoughts to solve the problem. Some circled segments of the problem which they then would identify as being a trinomial or the difference of two squares. Others used highlighter pens to identify the different forms of factorising required during the initial steps. Often learners tend to see the removal of a common factor as a first step. It appeared that, by inspecting a problem on-screen and by re-writing it onto the paper, they could easily spot the option of removing a common factor first. This behaviour was observed in both groups of participants. Although the applet only required final answers, all the participants wrote down all the intermediate steps before typing their final answers on the computer. This was was encouraging as learners often merely wrote down the answer during tests and examinations, losing the opportunity to score marks for intermediate steps done correctly.

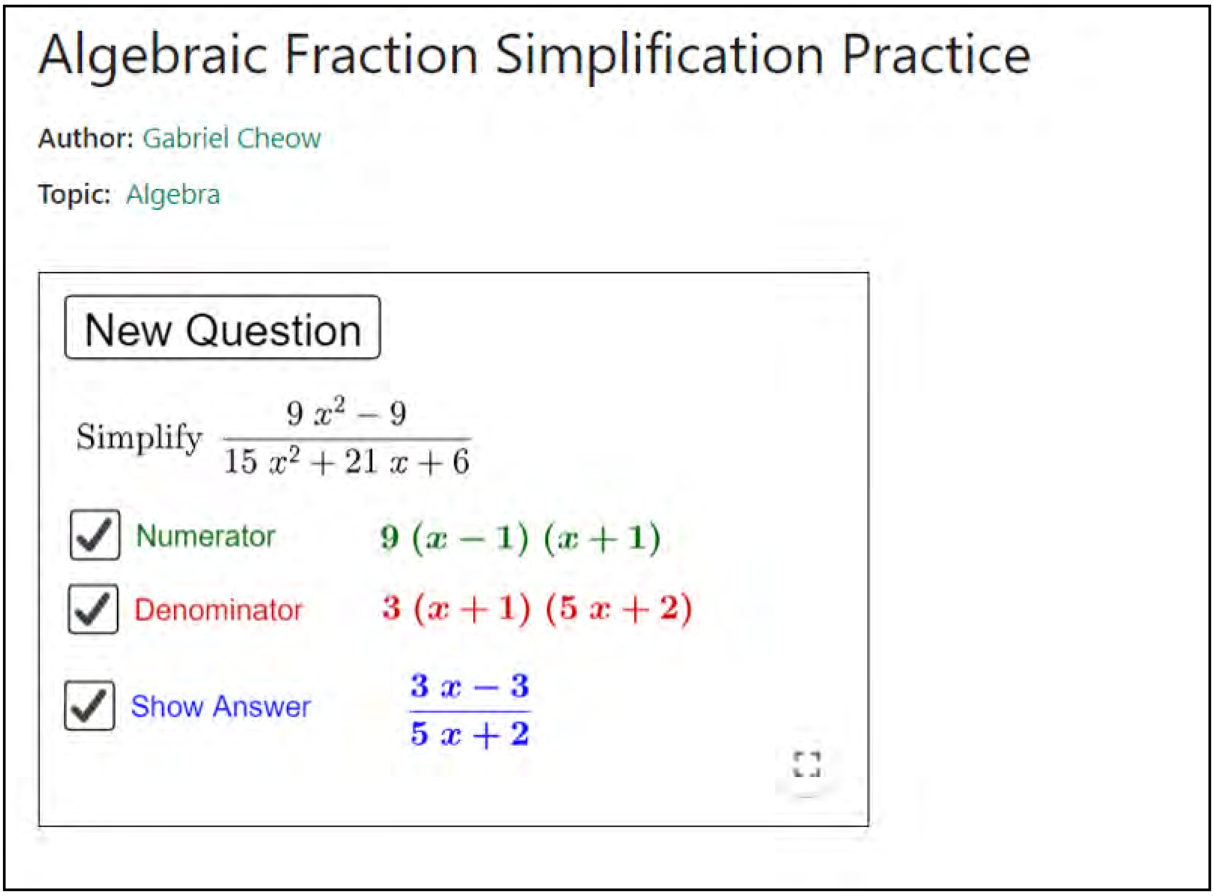



*Figure 7.8: Step-by-step guidance provided by the applet*

As illustrated by Figure 7.8 the participants could receive hints to guide them to solve the problem. These intermediate steps were mostly used to confirm that they had solved the problem correctly. With the illustrated problem, the participants were confident enough to critique the final answer of $\frac{3x-3}{5x+2}$ by stating that: *the applet did not give the answer in its simplest form which should be:* $\frac{3(x-1)}{5x+2}$. During reflections the participants were often asked to motivate their answers or to give a reason for a specific answer. Again, they motivated their remarks by adding: "*It looks incomplete when we just leave* $3x-3$ *in the answer while one can still remove a common factor*"

Constantly, evidence emerged that they relied on some form of visual imagery to motivate their answers or to construct a solution to a problem.

## 7.6 Post-Lesson Disposition Instruments

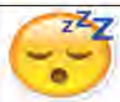
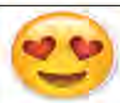
| **Positive Disposition Questionnaire: Summary C3 Post- L3** Present: 14(S) of School and 13(N) from NAMVISPRO | | | | | |
|---|---|---|---|---|---|
| Let the emoji's help you to answer the following questions. | | | | Session:C3 L3 | |
| | 1 | 2 | 3 | 4 | 5 |
| |  |  |  |  |  |
| For every question choose one emoji that will best describe your feelings. Mark your choice with an X below the emoji of your choice. | I detest it. Make me scared. Intimidating. | Boring Not interesting. I do not understand it. | Makes me nervous I don't care. Takes time to understand. | I like it. I will do more I understand it. | I love it. I find it easy. I can use it in future. |
| (1) How do you feel about Mathematics in general? | 1 + 0 | 4 + 5 | 7 + 4 | 1 + 3 | 1 + 1 |
| (2) How do you feel about learning Algebra? | 0 + 0 | 3 + 3 | 6 + 5 | 3 + 4 | 2 + 2 |
| TOTAL QUESTION 1 | 1 | 9 | 11 | 4 | 2 |
| TOTAL QUESTION 2 | 0 | 6 | 11 | 7 | 4 |
| Mean per Question | 0.5 | 7.5 | 11 | 5.5 | 3 |
| MEAN SCORE FOR THE CASE: | | | 3.2 | | |
| MEDIAN SCORE | | | 3.0 | | |

*Figure 7.9: Post-topic Disposition Instrument results*

Figure 7.9 contains the results of the Disposition Instrument after all the lessons were concluded by the participants. As mentioned before, the topic of the third cycle built upon the work covered in the previous cycles. All the algebra covered in the previous cycles was integrated into the required skills to successfully complete the topic. Participants had the opportunity to complete the instrument after completion of all the chosen applets, and when they felt that they had mastered the required topic.

During a subject meeting with the Grade 9 mathematics teachers, the head of the mathematics department pointed out that learners find algebraic fractions based upon the manipulation of algebraic fractions containing expressions that need to be factorised or including elements of indices, as the most difficult topic of the Grade 9 algebra syllabus. She advised teachers to ensure conceptual understanding of algebraic fractions in Grade 9 to prevent problems with algebra in later grades. I can testify from my own experience of teaching mathematics from junior to senior level that this is true.

## 7.7 ACHIEVEMENT TEST

I was responsible for setting the standardised achievement test for the Grade 9 classes. It is normal practice to differentiate between three different levels, with 30% as level one questions. 40% intermediate questions and 30% on an advanced level. Figures 7.10 and 7.11 contain inserts from the memorandum showing two advanced questions to illustrate the level of questions asked in the achievement test. The questions illustrated were also the ones that most learners had trouble answering.

7. Simplify the following algebraic fractions.

$$\frac{2x-8x^3}{2x-1}$$

$$=\frac{2x(1-4x^2)}{2x-1}\ \surd$$

$$=\frac{2x(1-2x)(1+2x)}{-1(1-2x)\surd}\ \surd\ (\text{M}_1 \text{ for correct factorisation})$$

$$=\frac{2x(1+2x)}{-1}\ or -2x(1+2x)\surd \qquad (4)$$

*Figure 7.10: Advanced level question that participants struggled with*

During the marking of the participants' achievement tests, I observed that many participants constantly used a highlighter or circled different sections of a problem when they recognised it as being a trinomial or the difference of two squares. Some added small letters when they recognised that an HCF must be removed. This was consistent with the methods they applied during the *GeoGebra* lessons.

During the marking of question 7, two aspects came to light: firstly, some participants were unable to recognise that $2x$ should be removed as a common factor from $2x - 8x^2$ to allow $1 - 4x^2$ to be factored further as a difference between two squares.

Secondly, some participants failed to comprehend that the removal of $-1$ as common factor from $(2x-1)$ would transform it to $-1(1-2x)$ to allow cancellation with $(1-2x)$ above the line. However some attempted some form of cancelling, but no marks could be awarded.

10. Evaluate

$$\frac{4x^2+4x-3}{3(2x-1)} \div \frac{2ax+4bx+3a+6b}{3a+6b}$$

$$= \frac{(2x-1)(2x+3)\surd}{3(2x-1)} \times \frac{3(a+2b)\surd}{(2x+3)(a+2b)\surd}$$

$$= \frac{3}{3}$$ *(ft) follow through if factorised incorrectly.*

$$= 1\surd$$ (4)

*Figure 7.11: Last advanced level question of the achievement test*

Theoretically the last question is one that the least learners would be able to answer. Despite this, the research participants did fairly well. All the participants could identify each section of the question and applied their own methods to mark out each type of factorisation or made notes next to each section to assist them in solving the problem. Unfortunately, marks were lost due to a lack of algebraic and numerical fluency. All research participants attempted to solve the two advanced level questions whereas many of the non-participating Grade 9 learners decided to not answer either of the two questions. I found it to be evident of a visual foundation to rely on that made them confident enough to engage with the more difficult questions.

### 7.7.1 Discussion of test results

I had hoped to conclude the research process with higher results, but in comparison with the rest of the grade, the research participants scored an average of 58.5%, a significant 8% higher than the grade average. Two participants achieved a 100% score and none of the participants failed the test. One participant increased his marks by 34% in comparison with the previous test.

Figure 7.12 shows the distribution of the test results. Participants for the research project were randomly selected, including several participants who constantly failed mathematics. Some top achievers formed part of the project, but none of them had beed able to achieve full marks for any previous achievement tests. For a standardised achievement test and for the sample size, one would normally expect the distribution of marks to form a normal curve. The results of the final achievement test showed a distribution of 25% of the participants in the first quadrille (75% to 100%), 67% in the second quadrille (50% to 74%) and 7% in the third quadrille (25% to 49%) with 0% in the last quadrille.

| SCORE | A 80 – 100% | B 70 – 79% | C 60 – 69% | D 50 – 59% | E 40 – 49% | F 35 – 39% | FF 30 - 34% | G 20 – 29% | H 0% 19% |
|---|---|---|---|---|---|---|---|---|---|
| CASE | 3 | 4 | 8 | 10 | 2 | 0 | 0 | 0 | 0 |
| TOTAL | 3 | 4 | 8 | 10 | 2 | 0 | 0 | 0 | 0 |

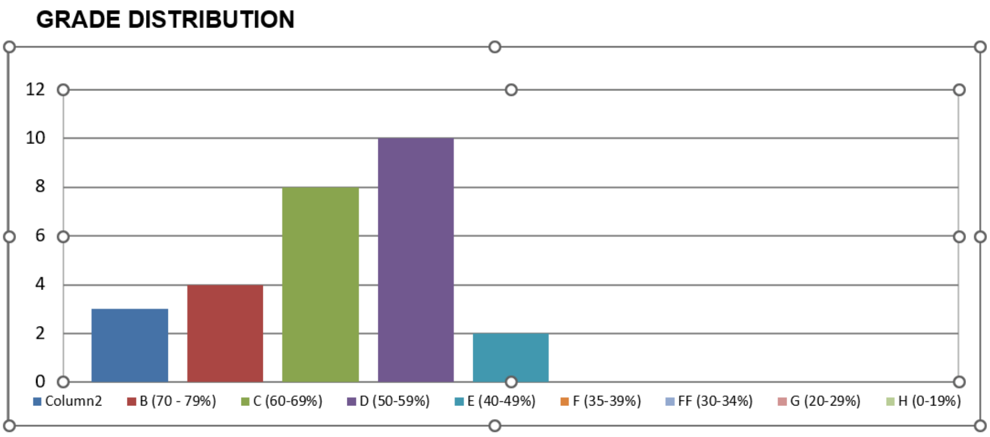



*Figure 7.12: Distribution of marks for final achievement test*

Taking into consideration that the final achievement test included all the topics covered during the three cycles of the research process, the test results could be interpreted as providing conclusive evidence that teaching algebra with the aid of dynamic software will enhance conceptual understanding.

Further evidence that participants relied on visualisation during the establishment of their conceptual understanding is provided by their application of relational visualisation during the test. All of them first analysed each question to identify the type of factorising or operation required. Some used codes, some used highlighters and others circled the sub-sections. I am convinced that they drew on the visual imagery established during learning with the applets. This is clearly illustrated with the segments from two tests in Figure 7.13.

## 7.8 POST-TOPIC REFLECTIVE INTERVIEWS

During the post-topic interviews two major points were made by the participants. All remarked that they were more *relaxed* when they attended the mathematics classes and that they felt *at ease* when they worked on the computers. They were *not subjected to any pressure*, neither from the teacher or from their peers. They gained *confidence* and were convinced that all the above-mentioned aspects, plus the change in approach, had a *positive influence on how they feel about learning mathematics, especially algebra.* The participants expressed a need to have individual interviews with me and it was agreed that a set of questions would be handed out to each one to help them prepare for a final individual interview with me.

None of the participants referred to visualisation or visual imagery when learning and doing algebra, but a few referred to the "*pictures created in their minds*" as being *helpful instruments when learning mathematics*. Paulo, whose marks increased the most during the research period, referred to a *skill that he never had, namely the ability to recognise images from the examples done with the applets when he had to do a problem they apply to.*

## 7.9 POST-TOPIC PODIUM INSTRUMENT

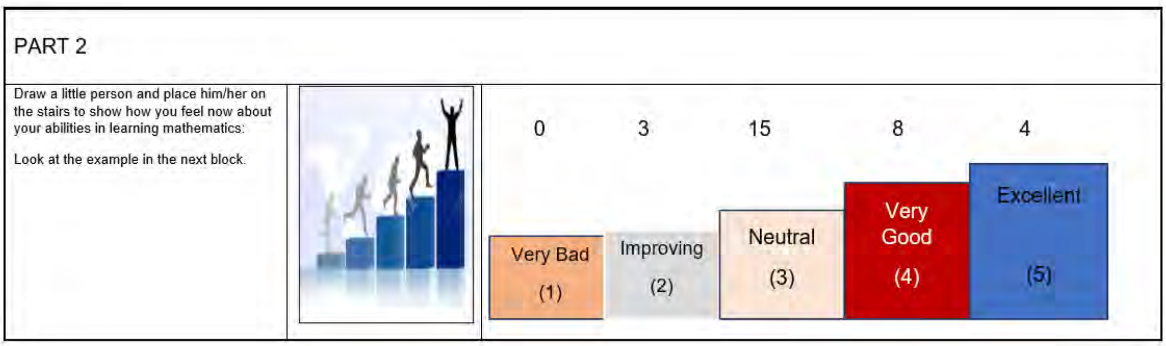



*Figure 7.13: Placement of participants on Disposition Podium*

The participants had to complete the Post-topic Podium Instrument after the achievement test was handed back and discussed individually with each participant. The instrument correlated well with the results of the achievement test. Both the NAMVISPRO participants and the school participants showed a positive change in terms of their dispositions. Both groups were consistent with each other in terms of their placements on the podium. In Figure 7.13 the placements of the participants on the Disposition Podium are shown. Not only did the test results improve over the research period, but a positive change in disposition was also visible.

## 7.10 CONCLUSION

This chapter reviewed the final cycle of the research process. In this chapter the methodology implemented to collect data for this case study and the methods employed to analyse the data proved to be successful. The measures to uphold the validity and reliability of the research findings were found to be successful. The ethical measures to uphold the dignity and rights of the participants could be implemented successfully while the test results showed that successful learning took place.

The completion of all three cycles and the constant completion of the disposition instruments provided me with the opportunity to place all the collection data within a horizontal perspective to support the findings of each cycle and to make final conclusions. This is done in chapter 8.

# CHAPTER EIGHT

# HORIZONTAL DATA ANALYSIS OF THREE CYCLES

## 8.1 INTRODUCTION

This chapter analyses the data that was collected in three distinct cycles from a horizontal perspective. Both research questions, the acquiring of conceptual understanding with technology and a change in positive disposition will be reviewed. Doing so will provide another perspective on the collected data and could provide more supporting evidence of the final findings. This chapter will also act as triangulation of the collected data that took place over the nine weeks of research and data collection. Data collected from the case will be analysed within a qualitative and quantitative paradigm by applying the analytical tools developed during the drafting of the research plan and presented in Chapter 3

By analysing the results of the different achievement tests over the entire research process, conclusive evidence of the change in conceptual understanding and the participants' application of visual tools and reasoning when learning algebra through dynamic software will be presented. Section 8.3 of this chapter focuses on the research participants' changes in disposition over the entire research process and their reflections on the visual learning of algebra with *GeoGebra* applets.

It was decided to not only consider the data of the whole case, but also to track specific individual participants horizontally as they provided valuable insight and data for the final conclusions.

## 8.2 HORIZONTAL DATA ANALYSIS PHASE 1

Participants were randomly selected without taking their academic backgrounds into consideration. Eventually the case was made up of 16 participants from the school where I am a teacher at and 14 participants from the NAMVISPRO project. The NAMVISPRO project focused on visualisation and technology to assist learners from different local schools to improve their marks in mathematics, with the aid of the visual properties provided using technology.

After the compilation of the case, it appeared that the participants from the NAMVISPRO project scored higher on the Disposition Instrument than the participants from the school, while the participants from school started the research project with slightly higher marks in mathematics, but lower scores on the Disposition Instrument.

As soon as the research project started it was found that the different schools that the NAMVISPRO participants came from, had all covered different topics of the Grade 9 syllabus, preventing the writing of a standardised pre-test to benchmark all members of the case study. Marks received from the schools that participants came from, were not taken into consideration for the research project. The research intended to investigate if the use of visualisation with the aid of dynamic software, – specifically the *GeoGebra* interactive software program – would improve results in the learning of algebra while changing the participants' disposition positively. None of the Grade 9 algebra topics had been done by any of the schools that made up the case, meaning that nobody entered the research programme with any advantage or had any pre-knowledge of the topics under investigation. It was not stipulated as a requirement for participants to be computer literate to become a member of the research group, but fortunately it turned out that all the participants had excellent computer skills. No time had to be spent to bring everyone to the same level of computer literacy.

### 8.2.1 Analyses of results of three achievement tests

In Table 8.1 the results of the three achievement tests written by participants after each cycle are summarised and analysed. The three data collection cycles happened within the prescribed number of lessons as stipulated by the Grade 9 syllabus. Each cycle covered one complete algebra topic from the Namibian Grade 9 syllabus. An achievement test was written after each cycle covering the work done during the cycle. All achievement tests were written under examination conditions and set to match the prescribed requirements of the Namibian mathematics syllabus.

During the COVID-19 pandemic the prescribed pass requirements were lowered from 40% to 35% by the Ministry of Education. This was made applicable to all school subjects as an interim measure to counter the negative effects the pandemic had on the marks of learners. I decided to keep the pass requirements for the research participants at 40% in line with the policy of my school.

*Table 8.1: Horizontal analyses of achievement tests over three cycles*

| ANALYSES OF ACHIEVEMENT TESTS OVER 3 CYCLES | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| Number | NAME | CODE | TEST 1 | SYMBOL | %< 45% | TEST 2 | SYMBOL | TEST 3 | AVERAGE | CHANGE |
| | | | | | | | | | | |
| 1 | Alzonia | S1 | 40 | E | 5 | 45 | E | 48 | E | +8 |
| 2 | Artem | S2 | 44 | E | 1 | 48 | E | 50 | D | +6 |
| 3 | Beatrice | N1 | 45 | E | | 52 | D | 54 | D | +9 |
| 4 | Benna | N2 | 40 | E | 5 | 48 | E | 50 | D | +10 |
| 5 | Christine | S3 | 100 | A | | 96 | A | 100 | A | 0 |
| 6 | Darren | S4 | 50 | D | | 55 | D | 58 | D | +8 |
| 7 | Delvin | S5 | 58 | D | | 56 | D | 58 | D | 0 |
| 8 | Ernah | N3 | 44 | E | 1 | 48 | E | 50 | D | +6 |
| 9 | Ethan | N4 | 56 | D | | 60 | C | 62 | C | +6 |
| 10 | Gabby | S6 | 40 | E | 5 | 45 | E | 48 | E | +8 |
| 11 | Gladys | N5 | 56 | D | | 58 | D | 60 | C | +4 |
| 12 | Greg | S7 | 64 | C | | 64 | C | 70 | B | +6 |
| 13 | Haley | S8 | 72 | B | | 80 | A | 82 | A | +10 |
| 14 | Helena | S9 | 44 | E | 1 | 45 | E | 52 | D | +8 |
| 15 | Jenna | S10 | 52 | D | | 55 | D | 56 | D | +4 |
| 16 | Johson | N6 | 48 | E | | 52 | D | 52 | D | +4 |
| 17 | Lydia | N7 | 44 | E | 1 | 48 | E | 52 | D | +8 |
| 18 | Magareth | N8 | 48 | E | | 52 | D | 56 | D | +8 |
| 19 | Michael | S11 | 64 | C | | 84 | A | 86 | A | +26 |
| 20 | Mike | S12 | 52 | D | | 55 | D | 58 | D | +6 |
| 21 | Nathan | N9 | 52 | D | | 50 | D | 84 | A | +32 |
| 22 | Paula | S13 | 52 | D | | 58 | D | 58 | D | +6 |
| 23 | Paulo | N10 | 64 | C | | 96 | A | 100 | A | +36 |
| 24 | Penda | S14 | 58 | D | | 62 | C | 70 | B | +12 |
| 25 | Requel | S15 | 68 | C | | 58 | D | 68 | C | 0 |
| 26 | Rocher | S16 | 48 | E | | 60 | C | 62 | C | +14 |
| 27 | Rowan | N11 | 40 | E | 5 | 46 | E | 50 | D | +10 |
| 28 | Surya | N12 | 44 | E | 1 | 48 | E | 52 | D | +8 |
| 29 | Tjekere | N13 | 52 | D | | 55 | D | 60 | C | +8 |
| 30 | Univi | N14 | 40 | E | 5 | 45 | E | 48 | E | +8 |
| **TOTALS** | | | **1579** | | | **1724** | | **1854** | | **277** |
| **AVERAGE** | | | **52.6** | | | **57.5** | | **61.8** | | **9.2** |

### 8.2.2 Horizontal analyses for individual achievement test results

For the first achievement test participants were requested to set a target mark for themselves of at least 45%. The purpose was to set a benchmark for the participants to track their own progress during the whole research process. All participants achieved the required pass mark of 40%, but ten participants failed to achieve the goal mark of 45%, with five who missed it by 5% and another five participants who missed it by only 1%. The results for the ten participants are displayed in Table 8.1.

It was also found that seven of the lowest scorers persistently placed themselves on the bottom end of the Disposition Instrument. The reflective interviews after the first few lessons revealed that the same participants constistently referred to algebra as: *"a pile of letters and numbers that they are not able to make any sense of."* During the reflective interviews after the first achievement test it surfaced that five of the participants had a history of constantly failing mathematics tests. Their marks for the achievement test was their first experience of success in learning mathematics, specifically algebra. Passing the achievement test after the first cycle is significant for the further horizontal analyses of the data, because it indicated that changes in their conceptual understanding of algebra were already evident with only one cycle completed.

One participant who drew attention during the horizontal review of the results was Christine who scored full marks for the achievement test at the end of Cycle 1. During the reflective interview, Christine related how she always did very well in mathematics but was never able to achieve her dream mark of 100%. After the completion of the first cycle, she achieved her personal goal of full marks. Tracking her results for the entire research period was regarded as important for the horizontal analyses of the data, because she said during the reflective interview at the end of the cycle: "*The applets helped me to remember better the finer tricks of the different sums. During the test all the examples we did came back into my mind. It gave me confidence and I will keep on learning mathematics with this new method. I remember the work much better."*

Paulo, enrolled with the NAMVISPRO project with below average marks and although he always showed good insight into any work done, achieved only 64% for the first test. He articulated his results as being: "*Not a true reflection of [my] abilities*." Over the duration of the research project, he was able to improve his marks by 36% to achieve full marks during the final test. Tracking Paulo on the Disposition Instrument showed that he scored high on the Disposition Instrument right from the start of the research project. During the initial interviews he related:

> *I enjoy doing algebra and am convinced that I understand the work very well, but when I have to solve a problem during a test, I am unable to picture a solution for the question based upon the work we have done.*"

After the completion of the research project, he was asked what value the project had for him, and he explained:

> *I enjoyed the way you taught the algebra to us. I learnt to keep on working with the program until I can see all the different types in my mind. In the last test I could clearly see a picture of the examples in front of me. I will keep on doing this and do not think I will ever struggle with mathematics again.*

Another participant who drew attention during the horizontal analyses of the data was Nathan, a participant from my school. He constantly scored himself very low on the disposition scale. Only during the last two opportunities he increased his self-evaluation by two points. During the final reflective interview, he revealed that mathematics, and in particular algebra, is not seen as being important for any future plans he might have, but he also added:

> … *in later years, things might change for me, and I actually started to enjoy the algebra more. Now that I can remember the examples better, I am more confident about my own abilities. I did not study more for the last test, I just found it easier to do the questions. Things just came back into to my mind; therefore, my marks were so much better. At least I will try improving even more.*

For the first achievement test Nathan scored 52%, then his marks decreased to 50% for the second test and in the final test he scored 84% – a pleasing increase of 32% from the first test.

Nathan's experience relates very well to Michael's, who increased his score by 26% from the first test. Both participants referred to their abilities to better "*recall*" from the lessons done and both referred to a newly acquired skill that developed within themselves during the research project, namely, to draw from "*mind images/pictures,*" created during the work with the applets. Both participants mentioned that they "*learnt to rely upon and apply what they remember of the pictures in their minds, to solve different problems.*"

Except for Christine, who constantly averaged very high marks in the first achievement test to the final one, two other participants did not show any increase in their average marks from the first to the third achievement tests. Delvin achieved 58% for the first test, 56% for the second test and 58% for the final achievement test. He attributed his inability to show a significant increase in his marks to his difficulty in attending all the classes. He lives far from the laboratory and often experienced transport problems to get to the research classes. Participants were able and allowed to join *GeoGebra* lessons online (using the unique lesson codes provided) but did he not have any computer facilities or internet access at home.

For the first test, Raquel achieved a mark of 68% but developed health issues during the second cycle of the research. For the second test her marks decreased by 10% to 58%. For the third test she was able to recover to equal her initial mark of 68%. Although she did not increase her average score from the first cycle to the third cycle, she recovered from dropping from 68% to 58%, to achieving 68% again. She explained that:

> *After I recovered*, *I was still able to log into the lessons that I missed while I was ill and could manage to cover the lost lessons on my own. Without the ability to join the classes I would not be able to do well in the test.*

It is significant that she stated during the reflection after the final test that:

> *GeoGebra showed me the examples in a way that I needed less time to understand the algebra. I was also able to recognise the questions in the last test from the examples I did with GeoGebra, and I could see how the examples were similar to the test questions.*

It is evident that although she worked on her own, she succeeded in creating the visio-spatial imagery in her mind to recall the examples during the test and to apply some relational-visualisation between the examples and the test questions.

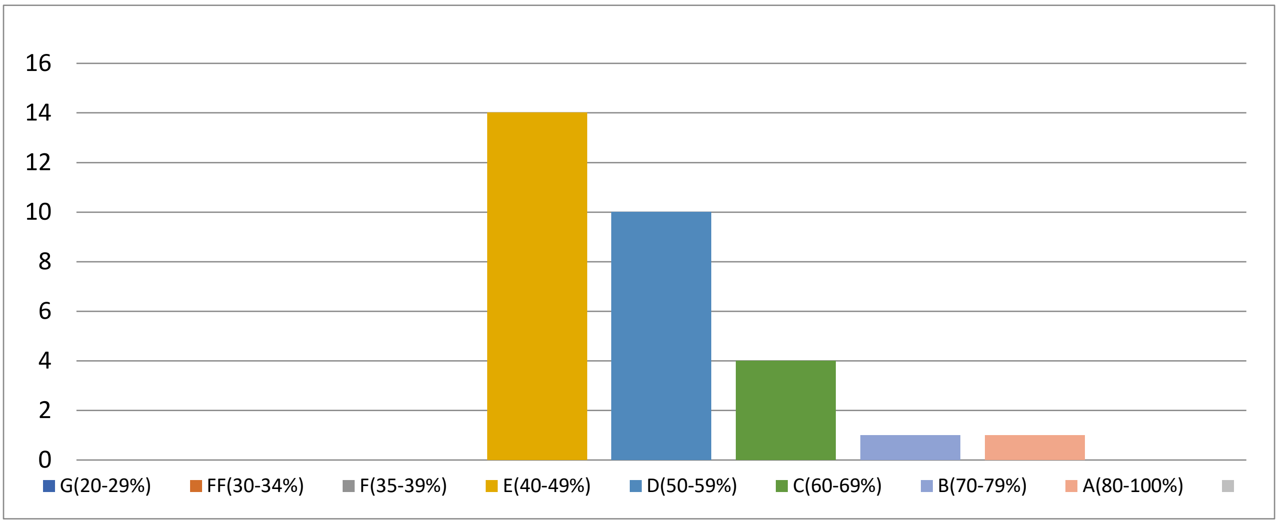
During the horizontal analyses of the data, several participants referred to spatial images they relied upon during tests and often mentioned: "*being able to immediately see the relationship between examples done and the problem in front of them.*" It is clear to me that the participants relied upon forms of visio-spatial visualisation as well as relational-visualisation when they solved problems. I am convinced that the *GeoGebra* applets made participants aware of this ability or that they developed it during the nine weeks research period.

### 8.2.3 Mark distribution curves for first and third tests

In Figure 8.1 the marks distribution for the first and the third achievement test are placed below each other to illustrate the shift in the results from the first test to the third achievement test. When analysed, the largest shifts took place from the 40% to 49% range to the 50% to 59% range. For a normal distribution it is expected that fewer participants would score in the 80% to 100% range, but a significant increase occurred here. That range is consisted of participants who normally scored above average. These participants have developed the ability to conceptually understand the algebra more proficiently and were able to improve on their abilities to recall from examples embedded in their minds. They clearly illustrated an improvement of their generalisation and reversal skills when answering the third achievement test. This conclusion was confirmed by them during the interviews held in retrospect when handing out the third achievement test.

> *I could clearly remember the examples we practised, and I found it very easy to see how they can help in answering the questions.*
>
> *Previously I relied on my memory to remember what we did in class and then it was still difficult to see the relationship between the examples you did and the questions in the test. Now they just re-appear when I see a question.*

**GRADE DISTRIBUTION FOR FIRST ACHIEVEMENT TEST**




**GRADE DISTRIBUTION FOR THIRD ACHIEVEMENT TEST**

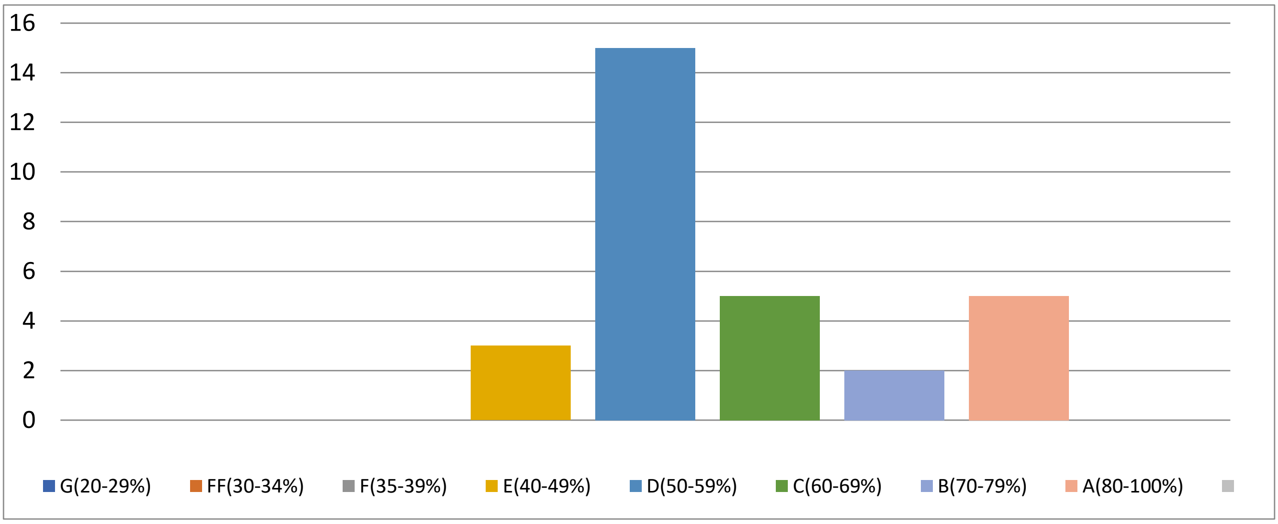



*Figure 8.1: Comparison of the grade distribution for Test 1 and Test 3*

The remarks by the higher scoring participants confirmed that visual imagery played a role in their ability to learn and recall algebraic structures. Reflective remarks from participants in the midrange who also improved their results also confirmed their reliance on visual imagery when questions had to be answered:

> *The examples we did on the computer were burnt into my mind. I could recall them to answer most of the questions in the test. The only time I struggled was when you changed the questions a bit and I could not recognise the examples immediately.*

Rocher, who improved by 14% from the first to the third test was critical of himself:

*I remembered the work very well and could clearly see what I have to do with each question, but I either made simple mistakes or when I had to hand in the correct solution for tricky questions jumped into my mind. In the next test I will not be caught out again.*

### 8.2.4 Correlations between results of the three achievement tests

The results of the three achievement tests were analysed to find the correlation between the results of the three different tests. As displayed in Figures 8.2, Figure 8.3 and Figure 8.4, a clear linear correlation was found between the results of the first, second and third tests.

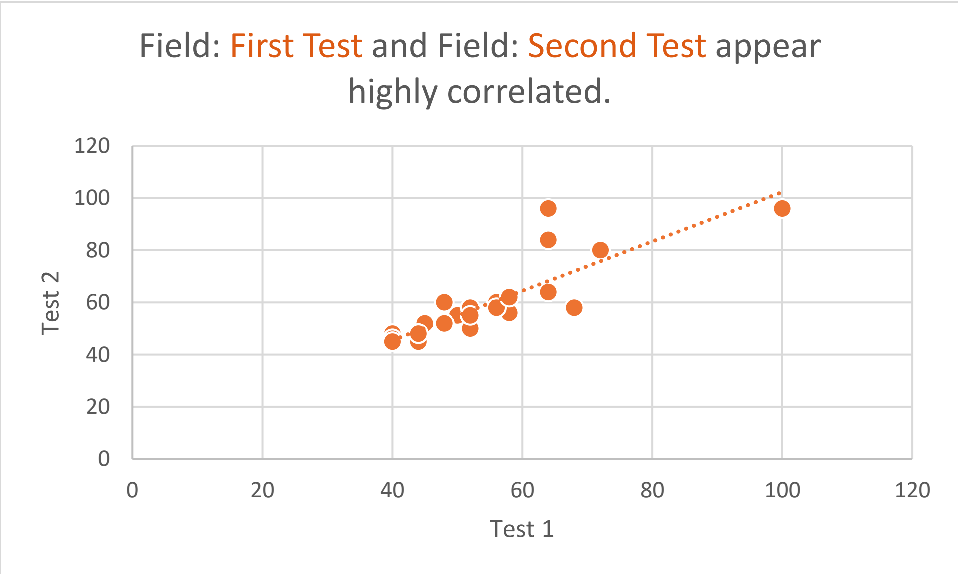



*Figure 8.2: Correlation between results of first and second tests*

From Test 1 to Test 2 the results of participants who scored between 60% and 70% showed the largest increase, causing the results to appear scattered, but for the whole case a clear linear correlation was found.

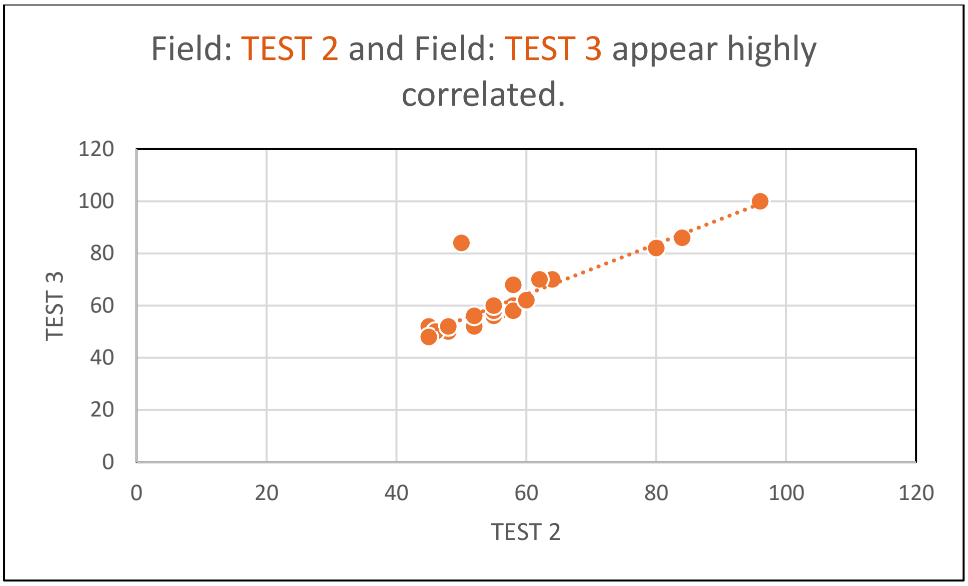



*Figure 8.3: Correlation between results of second and third tests*

Figure 8.3 clearly illustrates how the results for Test 2 and Test 3 moved in linear association with each other. It was clear that all the participants began to consistently improve their ability to conceptually understanding algebra.

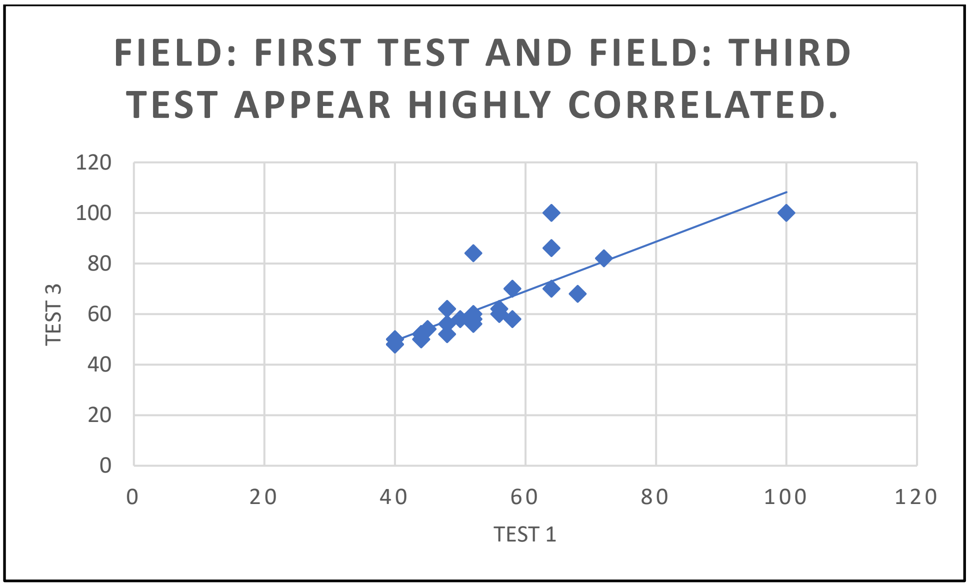



*Figure 8.4: Correlation between results of first and third tests*

Even when the results of the first and third tests are analysed a clear linear correlation is visible. From the onset of learning algebra with the use of *GeoGebra* until the final achievement test was written, participants improved their conceptual understanding of algebra at a steady rate.

### 8.3 Horizontal Data Analyses of Disposition Instruments

During the horizontal analysing of the data, it was found that although the case showed a significant positive change in their dispositions towards learning algebra with technology, most participants fluctuated when expressing their feelings towards learning algebra. Ultimately, all participants showed a positive change towards learning algebra, but there was not a linear progression from the first cycle towards the last cycle. Figure 8.4 illustrates the progression of two participants who showed significant changes in their dispositions towards learning algebra. Other participants displayed similar trends during the research period. The emotional well-being of participants on a specific day could play a role, but when they struggled to conceptually understand a specific topic it was clear that they scored themselves lower on the Disposition Instrument.

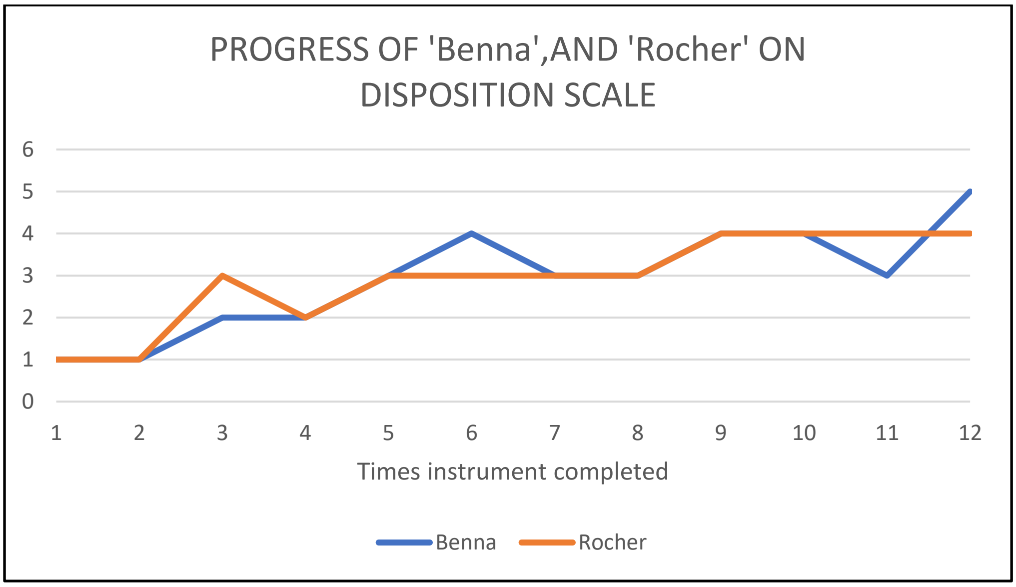



*Figure 8.5: Progression of two participants who showed a significant increase on the disposition scale over the research period*

During reflective interviews, *many* participants revealed how they experienced *re-occurrences of the fears and uncertainties* that initially drove their negative disposition towards learning algebra. *The re-occurrence of previous experiences of failure* when they initially struggled to conceptually understand the work presented by the applets was another reason many participants had moments of moving down on the disposition scale. When they eventually *understood the applet topics successfully it* always led to a feeling of self-actualisation amongst the participants and that was then the primary driving force to place themselves higher on the Disposition Instrument. Both Hayley and Helena explained:

> I*n the school classroom my biggest fear is to be exposed as being stupid. With GeoGebra only the computer knows when I answer something wrong.”*

Helena continued:

> *In the classroom I was so afraid that I would have to stand up and answer questions that I might not be able to do. It unnerved me so much that I could never focus on understanding or even trying to remember and recall any of the examples done on the bord. It is only now that I slowly begin to believe in myself again.*

## 8.4 Final Reflective Interviews to Review the Research Project

To provide me with a horizontal perspective over the whole research period, I decided to do a final reflective interview with all the participants. It provided the opportunity to participants to reflect on the several weeks of the research project and would also provide them with the opportunity to share any experiences they still wanted to share with me.

I decided to provide all the participants with a list of ten questions that would guide the individual interviews. The rationale to follow this approach was to ensure that they were at ease during the interviews, had the opportunity to provide me with well-considered answers and to ensure that participants would use the opportunity to be well prepared and to express themselves as well as possible. It was a fair approach and provided me with the opportunity to code answers to identify any similar trends in their answers. The participants made it clear that they would appreciate the opportunity to prepare themselves as they often lack the vocabulary to express their feelings and to accurately word the processes in their minds. Participants were allowed enough time to prepare for the interviews and could make appointments when they were fully prepared.

The research project was based upon mutual trust between the researcher and the participants. I experienced the participants as honest when reflecting upon lessons and not trying to give the answers they suspected I wanted to hear.

Table 8.2 contains the coded overview of the interviews, showing the frequency of similar answers. It was agreed with participants that their answers would be recorded and then transcribed later to provide me with an overall view of the answers provided. Answers were colour coded in search of trends and clues that might answer the two main research questions. The frequency of answers served as support for the specific conclusions I reached.

*Table 8.2: Concise summary of the final structured reflective interviews with participants*

**Final Reflective Interview Questions**

| Question number | Questions | Most relevant responses with some motivations | Frequency for 30 participants | Remarks |
|---|---|---|---|---|
| 1 | **Did you enjoy the new approach to teaching algebra we had?** | 1. Yes, very much.<br>2. *The work was easier to understand.*<br>3. Not in the beginning, but later I enjoyed it.<br>4. *I was a bit nervous when we started.* | 22<br>8 | **All participants concluded:** *I started to look forward to the algebra classes.* |
| 2 | **What stood out for you during the research period?** | 1. *I learned how to understand algebra.*<br>2. *Computers make algebra easier.*<br>3. *I was never under pressure. Nobody looked over my shoulder. I worked on my own / at my own pace.*<br>4. *There was no teacher.* | 9<br>7<br>10<br>2 | |
| 3 | **What changed in the way that you understand and learn algebra?** | 1. *The programs taught me how to study algebra/mathematics.*<br>2. *The more I practise the better I remember the problems.*<br>3. *I recall the examples now in my mind. I see the sums in my head.* | 9<br>5<br>11 | **Constant referring to the *images (pictures or systems*) in their minds** |
| 4 | **Did your experience of learning algebra change during our research?** | 1. *Yes. All the numbers and symbols changed to something that I could recognise.*<br>2. *Yes. I started to enjoy learning algebra.* | 12<br>8 | |
| 5 | **When you study for an algebra test, how do you remember the work?** | 1. *I see the examples we did in my mind.*<br>2. *I can immediately recognise a problem when I see it.* | 10<br>8 | |
| 6 | **What do you see in your mind when you have to solve an algebra problem?** | 1. *The examples we did. On the computers.*<br>2. *I can break a problem up into smaller sections.*<br>3. *The method how to solve the problem.*<br>4. *I just know how to do the sum.*<br>5. *I try to recognise what I have to do.* | 6<br>6<br>7<br>7<br>3 | |
| 7 | **How did you feel about algebra before we started with the project?** | 1. *I liked it.*<br>2. *I did not like it at all.*<br>3. *I hated it.* | 7<br>18<br>5 | **Several referred to seeing algebra as useless with no future value. Others referred to the complex structures that did** |

| Question number | Questions | Most relevant responses with some motivations | Frequency for 30 participants | Remarks |
|---|---|---|---|---|
| | | | | not make sense to them. |
| 8 | How do you feel about learning algebra now? | 1. *I enjoy it more now.*<br>2. *I am less afraid of algebra*<br>3. *I am still afraid when it gets difficult.*<br>4. *I have more confidence now.* | 14<br>6<br>3<br>4 | |
| 9 | Which approach would you recommend for the future learning of algebra? Teaching algebra with ***GeoGebra*** and computers or teaching algebra by a teacher only? | 1. *Teaching on the computer will be great.*<br>2. *A combination of class teaching and learning with computers.*<br>3. *I am fine with any method.* | 15<br>13<br>2 | |
| 10 | Are you convinced that technology could help with the teaching of algebra? | 1. *Yes. It helped me. It changed my mind.* | 30 | All participants referred to some changes about the way they either recall algebra or use some form of symbolic or structural system to understand and recall work done. "I learnt with my eyes, there was no teacher to listen to." |
| | | | | |

After the final interviews it was noticeably clear that none of the participants were "*unchanged*" after the conclusion of the research process.

Conclusions made after the interviews were:

1. Participants entered the research process with "*muddled concepts"* about algebraic structures in their minds.
2. The research transformed the visual structures participants had about algebraic structures, from: being *"just confusing letters and numbers"* to "*becoming orderly structures that can be simplified."*
3. Although the word 'visualisation' was never used by any participant, they confirmed that the research instilled in their minds a reliance on visio-spatial and visio-symbolic systems to recall and apply their conceptual understanding of algebra when solving problems.
4. It was unintended but all participants referred to the research as a process that taught them "*how to learn algebra.*"

5. No correlation between success and disposition could be found, but a significant positive change in disposition was recorded. Participants attributed that to the visual approach followed with the applets, especially the structured visually pleasing layout of the on-screen examples.
6. Half of the participants preferred a combination between teaching with technology and teaching with the involvement of the teacher. The role of the teacher cannot be underestimated, from the selection of the applets to the overseeing of the learning process.

## 8.5 CONCLUSION

Both the qualitative and quantitative sets of data provided enough evidence to conclude that the use of dynamic software enhances conceptual understanding and a productive disposition) in the visual learning of algebra. No conflicts were found between the different sets of data.

From the collected data, qualitative and quantitative I was able to make final conclusions and place the study within a Namibian context. This is done in chapter 9.

# CHAPTER 9

# CONCLUSIONS AND SIGNIFICANCE OF THE STUDY

## 9.1 INTRODUCTION

With this concluding chapter I want to consolidate the results of this case study in terms of the research questions and contextualise them in my theoretical and methodological frameworks. The significance of the study within the limitations experienced in terms of the case, the period and other external factors are interrogated. Recommendations for further research on visualisation in the learning of algebra that surfaced during the research process are made. Valuable experiences are shared to inspire and guide other teachers and researchers interested in the use of technology as a learning tool in algebra. Finally, I reflect on the development of a positive disposition as a driving force in the learning of abstract concepts in junior algebra to motivate other teachers in similar situations.

## 9.2 THE RESEARCH GOALS AND QUESTIONS REVISITED

### 9.2.1 The Research goals

This study had a three-fold goal:

1. To investigate the use of dynamic software and technology as medium of instruction for the visual teaching of algebra to Grade 9 mathematics learners.
2. To analyse changes in conceptual understanding of algebra amongst the participants when technology replaces traditional classroom teaching.
3. To examine the changes in productive disposition when conceptual understanding is achieved through visual learning.

To achieve these goals, the study was an interventionist case study consisting of 30 Grade 9 learners – 14 of whom came from a community project focusing on the improvement of mathematical skills among learners from different schools in

Windhoek, and 16 participants from the private school where I am teaching mathematics.
The study was guided by one main research question.

### 9.2.2 The research question

How can the use of dynamic software potentially enhance conceptual understanding and a productive disposition through the visual learning of algebra?

## 9.3 THE RESEARCH FINDINGS

The key research findings from the case study are presented as a result of analysing several sets of qualitative and quantitative data collected from a horizontal as well as a vertical perspective, during three recurring cycles of the research process.

### 9.3.1 Findings about visualisation with technology

During the research process algebra was presented visually to the participants by using *GeoGebra* software. It was found that the visuality of the algebra was not based upon pictures, diagrams, graphical structures or any physical construction, but mostly on the visio-symbolic structures embedded within the algebraic compilation of numbers, letters and symbols. All the lessons were without any verbal narration and the visual instructions guided participants towards conceptual understanding of the computer-based lessons. Lessons relied purely upon the visio-symbolic structures of algebra, while others combined the algebraic structures with sliders and/or interactive diagrams next to the algebraic expressions to visually explain concepts.

All 30 participants repeatedly reflected that colour, arrows, lines or highlighted text did not contribute towards the conceptual understanding of algebraic concepts. Conclusive evidence was found that participants relied on various aspects while learning with technology, to establish a complete understanding of the algebra. Answers during reflective interviews were colour coded to group together similar responses – regarded as pointers – to a specific concept. Doing so revealed and confirmed that:

1. Being situated close to the computer screen with only a carefully selected section of the learning material in front of them, guided the participants' focus

on the matter at hand. It excluded distracting factors and kept them focused. Lessons were presented on-screen only, and all participants experienced each lesson as being a personal lesson targeted at him or her as an individual. It demanded concentration, and they could relate to the images on-screen as nobody else was involved during the interactive individual learning process.

2. Each participant had total control of the pace of analysing, assimilating and accommodating the learning material. It was never a shared experience with other classmates and allowed them to take ownership of their own learning.
3. Applets with too much information on one screen confused participants. Previous experiences with the NAMVISPRO project guided the AEI to exclude applets with vast quantities of information on a single screen. Participants backed away from screens with an overload of information to try to enlighten a principle. Participants confirmed their preference of focused and minimalistic quantity of information presented to them during lessons. Conclusive evidence was found that the simplicity in the layout of the symbolic structures on the screen, the size of the algebraic symbols presented and a structured line-by-line explanation to solve a problem were key to the creation of visual structures in their minds, assisting them in unlocking concepts and reaching conceptual understanding.
4. There was repeated confirmation that a simplistic approach in presenting learning material created lasting visual-symbolic structures in the minds of participants.
5. Newly constructed visual representations changed the views participants had of algebra, from being random senseless symbols and numbers to well-structured imagery that could be *read as lines* of logical argumentation and *not so difficult to remember or recall.*

Despite a still-developing vocabulary and difficulty in communicating in detail the visual imagery in their minds when retrieving algebraic processes, **all** participants witnessed and experienced a progression towards a conceptual understanding of algebraic concepts linked to the concrete forms of imagery created in their minds. Specific evidence pointed towards the recalling of *pictures of previously done examples in their minds,* allowing them to solve problems easily. I found that participants who entered the research process who already scored higher marks, were aware of visual imagery in their minds as a source they could use when problems had to be solved. Others eventually referred to a learning process that enabled them to *look back into their minds* to create solutions for problems. A third group saw it as a process of *ordering* and re-structuring the existing imagery in their minds into useful concepts that could be applied in problem solving situations. Even participants previously classified as non-visualisers by the school psychologist were among the first ones to express their preference towards the adopted visual approach.

Conclusive evidence was found that the whole case changed their vocabulary from "*I remember*" or "*It comes into my mind*" towards referring to "*images in my mind* or *I have pictures of the examples in my mind that help me to solve problems".* Articulate participants even referred to the visual imagery that they developed during the research period as: *"picture images that I can call back*" when solving similar problems. All participants confirmed a learned process of "*seeing images somewhere behind their eyes*, *which directed or guided them when attempting to solve problems.*" Others even referred to "*videos in their mind that replayed the work on the computer screen when doing a test.*"

The research focused on the visual representation of the subject content to the participants, with better conceptual understanding as outcome. Ultimately, the visual representation contributed towards the creation of a form of *"visual conceptual understanding"* within participants. Visualisation and visual imagery in the minds of the participants became the foundation of their conceptual understanding.

The prospect of teaching participants to think visually when doing algebra was not the initial intention or outcome of my study. It emerged as the study unfolded, and contrary to other research findings classifying learners as being either visualiser or non-visualisers, this research showed that all participants acquired some form of visualising skills during the research process. Non-visualisors like Alzonia and Arthem who initially described algebra as "*a pile of nonsensical letters and numbers without structure and without any recognisable form,"* attributed their improvement in doing algebra to "*the ability to create understandable pictures in their minds*, *that helped them to recognise problems*" All the participants confirmed that they ***learned** to rely on visual images in their minds, video images or pictures of the examples when they have to solve posed problems.* Conclusive empirical results proved that participants improved their marks after learning to recall from the visual imagery in their minds, created through the visual teaching of algebra.

Another unexpected but important outcome of the research was that all participants repetitively started to say that nobody ever taught them how to learn and remember algebra. *"The greatest skill I have learnt, is how to remember algebra better."* Better conceptual understanding was the initial focus of my research but during the cycles of the research I discovered that visualisation that leads to better conceptual understanding of algebra can become a learned skill. Through visual learning and careful didactical guidance by a mathematics teacher, learners can discover and develop their own visualisation abilities as an important vehicle towards growing conceptual understanding. Skilful selection and professional representation of the learning material can lead to the formation of the ability to visually gain conceptual understanding of algebra. I believe a need exists for more research to unravel all the elements required to teach learners to rely upon visual imagery as basis of their own conceptual understanding of algebra.

### 9.3.2 Findings about conceptual understanding

My research project relied upon specific definitions of conceptual understanding of mathematics. The ground breaking work by Kilpatrick et al. (2001) on the five strands of mathematical fluency and their definition of conceptual understanding as "an integrated and functional grasp of mathematical ideas" (Kilpatrick et al. 2001, p. 118), served as a point of departure for my research project. This was integrated with the constructivist ideas of visually understanding mathematics by Mayer (2003) and Mnguni (2014). Besides this, the criticism of Laughbaum (2017) on learning algebra was considered when planning the research model. Evidence to support the ideas of the mentioned authors and some evidence to expand and develop previously constructed definitions were found. Lastly, the DCT on visual understanding and learning algebra by relying on visualisation supported the results of the research (Sadoski & Paivio, 2001).

The first finding was the formation of what could be called **reconstructive or communicative visualisation.** This is the visual ability that assisted participants to reconstruct a previously practised example in so much detail that they could apply the reconstructed knowledge to solve a problem. Participants attributed their newly acquired skill of being able to recall clear images of previously done examples to the adopted teaching approach. As the research progressed, different participants mentioned that they started to develop better memory images of examples in vivid detail. The tight schedule of the teaching and the completion of the research project, however, did not allow for any repetition of examples. The visio-symbolic diagram presented to the participants on screens became a communicative diagram that guided them from within their own minds towards a solution to a problem. These communicative diagrams were not static figures but dynamic instruments that participants could apply when solving other problems. I argue that this is the main reason why so many participants applauded the simplicity and clarity of on-screen lessons.

A second finding was expressed in the words of the articulate Paulo who excelled, from being average to acquiring full marks at the end of the project. *"I learnt to see the relationship between the computer examples and the test questions. Previously they had nothing to do with each other."* When he explained his newly acquired ability during a group reflection session, many participants agreed that the focused instructions of *GeoGebra* developed abilities within them to draw lines between practised examples and problems to be solved. "*Up to now I was unable to see how any question asked had to do with the work covered in class. For me the examples done had nothing to do with questions asked.*" Christine summarised the general feeling of participants: *"Problems in tests were like foreign people to me. I did not recognise them, so I did not even know where to start to solve them. Now I can see the link between questions and the work we did in class."* I recognised the creation of **relational-visualisation** as the visual ability to draw lines between a problem at hand and the previously created mental imagery. Relational-visualisation allowed participants to apply acquired skills to a problem to find a solution for the problem. A problem on paper in front of participants triggers the recalling of the practised example as an image in the mind. That relationship aided the participants to deduce a solution of the asked question being asked.

Sets of both qualitative and quantitative data were collected in trying to answer the research question: whether the visual learning of algebra with technology would enhance conceptual understanding. The focus was on visual learning with the expected outcome of improved marks for the participants. The research project became a journey with the participants revealing the visuality contained within the conceptual understanding of algebra.

During the first cycle of the research the qualitative data extracted during reflection interviews revealed references to changes in the way that participants understand algebra. As the research progressed most responses referred to *learning a new way to understand algebra* and to establishing visual imagery within their minds that participants called the *seeing of the examples in their minds.* When the participants reflected upon their conceptual understanding of an algebraic topic, they ceased to refer to *being able to do the problem* but referred to *seeing how to solve the problem.* Others mentioned *images behind their eyes* that enabled them to solve problems successfully. The reliance on visuality as part of their conceptual understanding of algebra was an unexpected result of the research journey with the participants. The frequent remark: "*I learned how to better understand algebra,*" provided proof and confirmed that a visual learning process can teach learners to better understand algebra conceptually.

The quantitative data extracted from the three achievement tests written at the end of each cycle conclusively revealed an average increase in the results of the participants of 9.2%. Already after the completion of the first cycle, none of the ten weaker participants who normally failed their tests, were unsuccessful in the achievement test. All of them achieved the minimum score of 40%. Five of them missed the target mark of 45% by 5% and five missed it by only 1%. For the first test, results ranged between 40% and 100% with a mean score of 52.6%.

For the final test, after completion of the third cycle, the lowest mark scored was 48% and two participants scored full marks. The third test contained elements of all three topics and covered a larger quantity of learning material than the first or second test. For the final test, the results ranged between 48% and 100% with a mean score of 61.8%

Both the quantitative and qualitative sets of data yielded no contradictions, and both sets of data conclusively pointed at a significant improvement in conceptual understanding among the participants.

### 9.3.3 Findings about a productive disposition

Regular reflective interviews were conducted, and two instruments were constantly applied to track any changes in the dispositions participants had towards the learning of algebra. At the end of the research cycles a final structured interview with ten questions was conducted with all participants together. All ten questions of the final structured interview were made available to the participants beforehand.

All sets of collected data confirmed a gradual change in positive disposition amongst all participants. No participants showed a regression in terms of disposition by the end of the research process.

Between each cycle I found that several participants showed marked changes towards a more positive disposition on both instruments. To ensure that the instruments were not completed randomly, I had individual reflection sessions with each participant who displayed sudden positive changes, especially when moving from a very low disposition score towards scores above three. Participants provided reasons why they entered the research process with an exceptionally low score of 0 or 1 on the disposition scale and what brought about the positive change.

### 9.3.4 Specific findings about changes in terms of disposition

1. ***Difficulty to follow explanations by the teacher on whiteboard or on PowerPoint.*** The participants interviewed stated this as the main reason why they detested being in a mathematics class. Many participants explained that they lacked the courage to interrupt a lesson when they got lost. A positive change was brought about by the ability of the applets to adapt to everyone's unique learning pace and the ability to create imagery in their minds that they can use as a resource when solving problems.
2. ***Feeling exposed in front of others*.** Often teachers randomly ask questions as they progress through a lesson. Many teenage learners are, however, very sensitive to being placed 'on the spot'. When teaching with technology that exposure is minimised as everyone can interact on a one-on-one basis with the computer. Teachers are often impatient when learners take longer to unravel a problem.
3. ***The perception that mathematics, specifically algebra, is only for clever learners***. The participants found the computer to be impartial about their abilities or previous failures. Every effort by the participants is evaluated as being either successful or else they are given the option to try again. They felt protected against any derogative remarks by insensitive teachers.

4. ***Most participants felt unsafe and experienced anxiety in mathematics classes*.** Anxiety often occurs when a large quantity of abstract information is displayed. Learners have to assimilate large volumes of information, often written illegibly. The applets provided all the participants with the opportunity to revisit the presented material again at any convenient time and thus removed the anxiety that participants had before.
5. **The ingenuity and simplistic approach by well selected applets placed participants at ease and allowed them to form structures of conceptual understanding in their minds.** All the re-interviewed participants referred to the easy-to-remember explanations on-screen.

Tracking the two Disposition Instruments over the entire research period and analysing all the reflective interviews, conclusively showed a noticeable shift towards a positive disposition in the participants in the learning of algebra with the assistance of dynamic software. The formation of visual structures as a basis for the conceptual understanding of algebra proved to be a major reason for a positive change in the disposition participants had towards learning algebra.

### 9.3.5 Findings about the role of the teacher

A computer cannot replace a well-qualified mathematics teacher. My research confirmed that a skilful mathematics teacher remains essential in steering learners towards conceptual understanding of algebra. Computers are excellent in lightening the burden of a large class group and to assist with the didactical plan to unlock the essence of algebraic concepts for learners. The selection of the learning material, time management, choosing applicable software, defining the level of teaching and monitoring the progress of the learners should be done by a mathematics teacher with thorough subject content knowledge and pedagogical content knowledge. Learners are not able to complete a course on their own without the intervention and guidance of the mathematics teacher.

Computer software can diagnose shortcomings in the learning process of individuals and is even able to suggest remedial processes, but eventually the in-depth knowledge the teacher has of each learner's specific needs, will determine the success of computer-aided learning. One single sentence from the teacher can bring a learner back on track into learning with dynamic software.

I advise teachers to use computers and dynamic software as partners in the teaching of mathematics, but not as a replacement. The participants of my research project requested a balance between classroom instruction, individual explanations and the use of dynamic software.

## 9.4 SIGNIFICANCE OF THE RESEARCH

My research project highlighted the successful integration of *GeoGebra* as an instruction tool to not only enhance conceptual understanding and bring about a positive disposition in the learning of algebra, but also reveal that carefully selected applets systematically instil structures to develop a visual form of conceptual understanding of algebra.

The research underlined the importance of the relationship between the teacher and the learners. Teaching should not always be a one way instruction process initiated by the teacher. Two-way communication happens when learners can communicate individually with the computer, without feeling exposed. The ability of the selected software to monitor everyone's progress can lead to a higher level of conceptual understanding, improving the confidence of learners to take on problems and help to avoid feelings of anxiety. Changes in their disposition towards learning algebra is possible and should ultimately lead to a further positive change in conceptual understanding.

## 9.5 OPPORTUNITY FOR FURTHER RESEARCH

My own research only focused on the use of *GeoGebra* as a teaching and learning tool. There is a need however to explore other software that is available to teach algebra and mathematics visually. Further research to investigate the integration of other senses with visual learning is possible. My research took place in total silence with only whispering among participants, whereas allowing participants headsets and adding audio explanations to the lessons might bring about a different outcome for each lesson.

A need exists for research about the didactical approaches followed by dynamic software to unlock conceptual understanding for participants. Would a different didactical plan or other ingenuities lead to an even better conceptual understanding and visual memory of the learning material?

All participants displayed a positive change in their dispositions towards learning algebra. Immediately after the research, the participants started the geometry of quadrilaterals and circles. The lasting effect of the acquired positive dispositions towards other disciplines would be interesting to research.

*GeoGebra* provides unlimited opportunities for teachers and being an open-source platform, teachers can constantly contribute towards improved and didactically better applets to enhance conceptual understanding, change dispositions and support learners with limited access to quality teachers.

## 9.6 PERSONAL REFLECTION

The adopted mixed methods approach contained elements of an interventionist approach and action research where the practice of teaching algebra to Grade 9 learners was researched. From the study it emerged that a blended approach with the use of technology and classical classroom teaching was preferred by all participants. I became aware of the important role the teacher must play, to not only be a facilitator of learning algebra to learners, but to strategically create visio-symbolic structures in the minds of the learners, to the point where they can say*" I understand. I have it in my mind*." The mathematics teacher can facilitate conceptual understanding by guiding learners to construct well-defined and properly organised visio-symbolic structures in their minds. Once these structures have been created in their minds, the learners can state with confidence that the work is understood and that they have the confidence to dig into their own *visual* database of knowledge and to apply it to any posed problem.

I found that teaching algebra to young high school learners not only unlocks the logical sequences of algebraic problems to learners by simply explaining examples to them, but also develops an ability to use their visual minds as instruments to recall concepts and construct their own meaning for a concept. It cannot happen by just showing them examples on the whiteboard and hoping that they will understand conceptually. Visio-conceptual understanding requires the implementation of a long-term strategic plan to teach learners to understand each different concept that makes up algebra. It is a step-by-step journey with everyone involved establishing conceptual understanding for themselves by learning how to create visio-symbolic structures in their own minds. In doing so the visual imagery becomes a source to draw from when needed to solve any problem that requires conceptual insight. It is the weaving and constructing of a complex network of visio-symbolic concepts in their minds *that makes sense* to the individual. A teacher can assist with mapping constructs visually in the minds of learners and build a database of well-defined constructs to draw on during problem solving activities. This will enable them to apply conceptual skills to systematically deconstruct and solve problems. Learners can develop relational-visualisation and visual-communicative skills to solve a problem, and to communicate the solution either verbally or in writing. The research confirmed that the use of technology empowers the teacher with additional tools to achieve this goal.

Participants remarked that they enjoyed being taught with technology as confirmed in Table 9.2, but during the reflective interviews they reasoned that the *technology only approach* could not be the ideal method of teaching algebra to them. All the participants asked for a blended approach, involving both the teacher and technology. Eventually none of the participants wished to be taught by either technology only or by a teacher only.

## 9.7 CONCLUSION

This research project was a journey in gaining valuable insight into the learning of algebra. Participants progressively transformed until learning algebra became an enjoyable process boosting their confidence.

None of the thirty participants withdrew from the research project and their commitment, honesty, and loyalty during the lockdown period is commendable. I applaud each one of them, and am thankful for their sustained participation, and for the professional development that happened within myself.

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# Appendix A: Analytical Instrument for Applet evaluation: Visualisation for Conceptual understanding and Positive Disposition

| No | Procedure | Requirements | Categories of Observable indicators | Remarks | Marks: 0 – Not at all 1 – partially 2 - Fully |
|---|---|---|---|---|---|
| 1 | Is the applet suitable to replace formal class teaching? | ✓ Appropriate applications covering the Namibian algebra syllabus. | Covers the Namibian syllabus and of adequate difficulty. | Algebra applets for use during laboratory teaching sessions. | |
| | | ✓ Level of applet | Covers syllabus but too difficult or too easy for Grade 9. | Apps cover the topic, but not as prescribed by the syllabus | |
| | | ✓ Focus on teaching the syllabus | Applications relevant to the syllabus, but too limited or too wide | Topic does not emerge directly from the applet | |
| 2 | Does the applet apply the correct mathematical principles? | ✓ Nothing mathematically wrong.<br>✓ No shortcuts.<br>✓ Clear in terms of logic. | A clear progression to conceptual understanding should be followed.<br>Mathematical logical thinking should be followed. | Tricks to apply to reach solutions are not acceptable. Any mathematical errors like division by zero must disqualify applet. | |
| 3 | Will the applet enhance visualisation? | This implies the presence of the visual indicators previously identified. | No or only one or two visual indicators present. No visual value at all. | Does the visual elements contribute towards the conceptual understanding of the underlying algebraic concepts? | |
| | | | Visual elements present but not contributing towards the process of visualisaisaisaisation | | |
| | | | Too many visual indicators present. Hindering the learning process. | | |
| | | | Two or more relevant visual indicators present in the applet | | |
| | | | Several visual indicators present. Relevant for the understanding of underlying algebraic concepts. | | |
| 4 | Can the applet be done in one lesson period? | ✓ Is the applet too long / short?<br>✓ Is the applet error free?<br>✓ Can the applet be split into more than one lesson period?<br>✓ | Participants should potentially be engaged for the full time allocated for a lesson. | | |
| 5 | Will the applet foster a positive disposition? | ✓ Interesting approach<br>✓ Variation<br>✓ Appearance on-screen | Clear instruction.<br>Applet is focused on topic.<br>Explanation of concept.<br>Interesting approach. | According to Kilpatrick (2001, p. 118), learners understand a topic conceptually, when they have an "integrated and functional grasp of mathematical ideas." | |

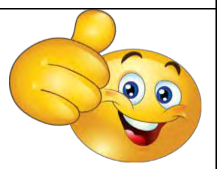
| | | | | | |
|---|---|---|---|---|---|
| **No** | **Procedure** | ✓ **Requirements** | **Observable Indicators** | **Remarks** | **Marks** |
| 6 | Does the applet have the potential to enhance conceptually understanding of the topic and the underlying principles? | ✓ The level of conceptual understanding of the topic.<br>✓ Identify special cases that can contribute towards a better conceptual understanding of the topic.<br>✓ An interesting approach. | Participant indicates no conceptual understanding of the topic. | Participant completed the topic but shows no understanding of the underlying topic being taught through the applet | |
| | | ✓ | Conceptual understanding is shown, but only to solve problems very similar to those in the applet. | | |
| | | ✓ | Conceptually understands the concept, able to solve a variety of different problems on the topic. | | |
| | | ✓ | A good conceptual understanding of the algebraic concept. Confident to attempt even more advanced problems on the topic. | | |
| | | ✓ | High level of conceptual understanding of the topic. Confident with any problem and able to create new problems about the topic. Able to work independent. | | |
| 7 | Does the applet provide the participant with immediate feedback? | ✓ Feedback regarding correctly answered question?<br>✓ Comments regarding applied skills to solve a problem.<br>✓ Positive encouragement when challenging problems are solved. | | Right/wrong feedback is not sufficient.<br>Additional explanatory information to help the development of conceptual understanding. | |
| 8 | Does the applet make provision for revision questions to prepare participants for prescribed standardised achievement test on the topic? | Exercises that can potentially improve performance. | Mainly repetition of previously covered exercises.<br>Progression in terms of difficulty.<br>Confirmation that a specific skill has been mastered.<br>Exceptions that should be known and practised by participants. | It is standard practice to write a standardised test after the completion of every topic. These tests are set by an external examiner.<br>Test scores will provide empirical data that could be compared with the results of the non-participating learners as control. | |

| No | Procedure | Requirements | Observable Indicators | Remarks | Marks |
|---|---|---|---|---|---|
| 9 | Applet has an internal verification system | ✓ To check progress towards challenging problems.<br>✓ To acknowledge creative attempts to solve problems by participants.<br>✓ To measure the period, they stayed engaged with the applet before losing interest. | Participant randomly attempts to solve the problems but loses very quickly interest.<br>Participant stays involved for a short period of time and shows some ingenuity in attempting to solve problems.<br>Participant is actively involved with the problem posed, try several different attempts to solve the problem.<br>Participant complete task enthusiastically, show a high level of ingenuity in solving problems.<br>Some ground breaking new attempts on a higher level. | An internal analytical tool will contribute towards a very high level of visual teaching by the applet. | |
| 10 | A dual method approach to explain concepts. | ✓ The use of other than visual-symbolic structures to support conceptual understanding.<br>✓ A minimalistic screen layout without an overload of information. | A focus on the concept to be transferred.<br>The use of colour or symbols like arrows or diagrams.<br>Hints provided to keep participants engaged and in control of their own progress. | Carefully selected structures can potentially support the process of conceptual understanding and changing a participant's disposition. | |
| | | | | **TOTAL SCORE** | |

# Appendix B: Positive Disposition Instrument 1

Appendix B: Positive Disposition Instrument 1

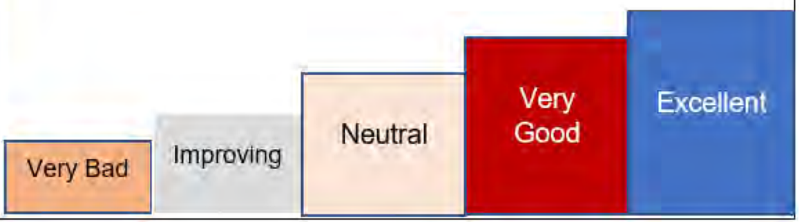
Let the emoji's help you to answer the following questions.
Just mark your choice with an X.

| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| |  |  |  |  |  |
| For every question choose one emoji that will best describe your feelings. Mark your choice with an X below the emoji of your choice. | I detest it.<br>Make me scared.<br>Intimidating. | Boring<br>Not interesting.<br>Just some more mathematics. | Makes me nervous<br>I don't care.<br>I am uncertain. | I like it.<br>I will do more<br>I understand. | I can keep on forever.<br>Make it more difficult.<br>I can use it in school. |
| (1) How did you feel when you started with the new applet? | | | | | |
| (2) The applet used colours and figures to explain the work to you. What is you're feeling about that? | | | | | |
| (3) How do you feel about mathematics being explained this way? | | | | | |
| (4) How did you feel about the graphics used by the computer? | | | | | |
| (5) When the problems became more difficult, how did you feel? | | | | | |

# Appendix C: Positive Disposition Instrument 2

**ADDENDUM C**

Positive Disposition Instrument 2.

| | | |
|---|---|---|
| Draw a little person and place him/her on the stairs to show how you feel now about mathematics. <br><br> Look at the example in the next block. |  |   |

# Appendix D: Permission from Rhodes University Ethics Committee

RHODES UNIVERSITY
Where leaders learn

Human Ethics subcommittee
Rhodes University Ethical Standards Committee
PO Box 94, Grahamstown, 6140, South Africa
t: +27 (0) 46 603 8055
f: +27 (0) 46 603 8822
e: ethics-committee@ru.ac.za

www.ru.ac.za/research/research/ethics
NHREC Registration no. REC-241114-045

13 February 2020

Prof Marc Schäfer

Review Reference: 2020-0990-3199

Email: m.schafer@ru.ac.za

Dear Prof Marc Schäfer

Re: To potentially enhance conceptual understanding and a productive disposition in the visual learning of algebra: an interventionist analyzing case study.,

Principal Investigator: Prof Marc Schäfer

Collaborators: Mr. Daniel Junius

This letter confirms that the above research proposal has been reviewed by the Rhodes University Ethical Standards Committee (RUESC) – Human Ethics (HE) sub-committee and **PROVISIONALLY APPROVED PENDING GATEKEEPER PERMISSION**.

Gatekeeper permission is required from:

a) Windhoek Gymnasium Private School

b) Directorate of Education, Namibia

Once the Gatekeeper permission letter/s have been received please forward it to the Ethics Coordinator, (s.manqele@ru.ac.za) in order to finalize your ethics approval.

Sincerely

**Prof Joanna Dames**

**Chair: Human Ethics sub-committee, RUESC- HE**

## Appendix E: Permission from the school

RHODES UNIVERSITY
*Where leaders learn*

**Permission Letter to Windhoek Gymnasium to participate in research project.**

D. F Junius

P O Box 30065

Pionierspark

WINDHOEK

Windhoek Gymnasium CURRO 2020 -04- 17 OFFICIAL STAMP
PRIMARY SCHOOL Tel: 061-381 463 Fax: 061-381 466
SECONDARY SCHOOL Tel: 061-381 459 Fax: 061-381 460

Mr A Myburgh

The Rector

Windhoek Gymnasium Private School

Windhoek

Dear Mr Myburgh

**REQUEST FOR PERMISSION TO CONDUCT RESEARCH AT WINDHOEK GYMNASIUM**

I am a PhD student at Rhodes University in Grahamstown, South Africa. My study requires me to conduct research about the learning process of Algebra with the aid of ICT. In principle I will investigate the question: *How can the use of ICT and its inherent visualisation opportunities be harnessed (implemented) to potentially enhance conceptual understanding and a productive disposition in the learning of Algebra*. I intend to do a case study of the learning process of Algebra by grade 8 learners with the assistance of technology. Only one grade 8 class per school will be involved in the research. Only learners willing to participate in the research and whose parents signed the letters of consent will participate in the research project. The research will provide participants with the opportunity to incorporate technology formally in the learning process – which is in line with the mission of your institution to encourage the use of technology in the classroom.

This letter therefore seeks formal consent from your office to allow Grade 8 mathematics learners to participate in the research process. Formal permission will also be sought from the parents/guardians of the selected learners. The study suggests that the use of ICT and the GeoGebra software will improve learners' understanding of Algebra and change their attitude towards the learning of Algebra. The program will run in sync with the planned scheme of work for the Grade 8 Mathematics syllabus. This means that the learners will be involved in the research project during the same time that Algebra is covered according to the planning schedule of the grade 8 syllabus. I envisage that learners will enjoy participating in the project, as we know from experience that they enjoy using computers in class room.

The research is done under the strict ethical guidelines of Rhodes University that requires confidentiality, trust, safety and learning without treat or fear for the participants. All research will be done under the supervision of Ms Maretha du Plessis, senior HOD and councillor at my school. All learners' identity will be protected by the use of pseudonyms and the name of the school will never be revealed. Data collected will only be accessible to me and my research supervisor. Data will be collected by observing the learners when working on the computers doing Algebra, structured interviews and the completion of questionnaires about their experiences. Results of the achievement tests will be analysed anonymously and only used to track changes.

Participation is entirely voluntary and if any participant feel uncomfortable, or do not want to continue for any reasons, that participant will be free to withdraw from the study. Withdrawal will not bear any consequences for the participant. The proposed research period is from August 2020 until end of September 2020, when the last of the Algebra topics of the Grade 8 syllabus will be covered.

Upon completion of my study, I undertake to provide your office with access to the research findings. Positive findings that will improve our practise will be made available to colleagues. It is envisaged that the use of technology could improve results in Algebra.

Yours sincerely

Daniel F. Junius

Student: 11J5235 Rhodes University

I, ........J. A. MYBURGH................................ have read the above information / confirm that the above information has been explained to me in a language that I understand and I am aware of this document's contents. I have asked all questions that I wished to ask and these have been answered to my satisfaction. I fully understand what the research project entails.

I have not been pressurised in any way and I voluntarily agree that the project may proceed.

...................................... ...17/04/2020.

**Signature** **Date**

Rhodes University, Research Office, Ethics
Ethics Coordinator: ethics-commitee@ru.ac.za
t: +27 (0) 46 603 7727 f: +27 (0) 86 616 7707
Room 220, Main Admin Building, Drostdy Road, Grahamstown, 6139

## Appendix F: Permission to conduct research

REPUBLIC OF NAMIBIA

UNITY LIBERTY JUSTICE

# MINISTRY OF EDUCATION, ARTS AND CULTURE

*Enquiries: Mr. Gibson Munene*
*Tel: +264 61 2933202*
*Fax: +264 61 2933922*
*Email: Gibson.Munene@moe.gov.na*
File no. 13/2/9

*Luther Street, Govt. Office Park*
*Private Bag 13186*
*Windhoek*
*Namibia*

Ms Daniel Franscius Junius
P. O. Box 30065
Pioneerspark
Windhoek

Dear Ms. Junius,

**SUBJECT: PERMISSION TO CONDUCT RESEARCH IN KHOMAS REGION**

Kindly be informed that permission to conduct your academic research for your PhD studies on "*How to Potentially Enhance Conceptual Understanding and a Productive Disposition in the Visual Learning of Algebra: An Interventionist Analyzing Case study*," is here with granted. You are requested to present this letter of approval to the Khomas Regional Director of Education, Arts and Culture to ensure that research ethics are adhered to and disruption of curriculum delivery is avoided.

Furthermore, you are requested to share your research findings with the Ministry. You may contact Mr. G. Munene at the Directorate of Programmes and Quality Assurance (PQA) for submission of a summary of your research findings.

I wish you the best in conducting your research and I look forward to hearing from you upon completion of your study.

Sincerely Yours,

24.4.2020.

Ministry of Education, Arts and Culture
Office of the Executive Director
2020 -04- 24
Tel: 061-2933523/4
Fax: 061-253671
Private Bag 13186, Windhoek, Namibia

Sanet L. Steenkamp
**EXECUTIVE DIRECTOR**

*All official correspondence must be addressed to the Executive Director.*

# Appendix G: Permission Letter to parents requesting learners to participate in research project.

P O Box 30065

Pionierspark

WINDHOEK

The Guardian/Parent

………………………………

………………………………

Dear …………………………………………

**REQUEST FOR PERMISSION TO PARTICIPATE IN RESEARCH AT WINDHOEK GYMNASIUM**

I am a PhD student at Rhodes University in Grahamstown, South Africa. My study requires me to conduct research about the learning process of Algebra with the aid of ICT. In principle I will investigate the question: *How can the use of ICT and its inherent visualisation opportunities be harnessed (implemented) to potentially enhance conceptual understanding and a productive disposition in the learning of Algebra*. I intend to do a case study of the learning process of Algebra by Grade 9 learners with the assistance of technology. Only one Grade 9 class at the school and one Grade 9 group of the outreach group will be involved in the research. Only learners willing to participate in the research and whose parents signed the letters of consent will participate in the research project. The research will provide participants with the opportunity to incorporate technology formally in the learning process – which is in line with the mission of your institution to encourage the use of technology in the classroom.

This letter therefore seeks formal consent from you to allow your child ……………………. to participate in the research process. The study suggests that the use of ICT and the *GeoGebra* software will improve learner' understanding of Algebra and change their attitude towards the learning of Algebra. The program will run in sync with the planned scheme of work for the Grade 9 mathematics syllabus. This means that the learners will be involved in the research project during the same time that Algebra is covered according to the planning schedule of the Grade 9 syllabus. Beginning March 2020 until 20 April 2020. I envisage that learners will enjoy participating in the project, as we know from experience that they enjoy using computers in classroom.

The research is done under the strict ethical guidelines of Rhodes University that requires confidentiality, trust, safety and learning without treat or fear for the participants. All learners' identity will be protected using pseudonyms and the name of the school will never be revealed.

Data collected will only be accessible to me and my research supervisor. Data will be collected by observing the learners when working on the computers in their Algebra class, structured interviews, and the completion of questionnaires about their experiences. Results of the achievement tests will be analysed anonymously and only used to track changes.

Participation is entirely voluntary and if any participant feels uncomfortable, or do not want to continue for any reasons, that participant will be free to withdraw from the study. Withdrawal will not bear any consequences for the participant. The proposed research period is from February 2020 until end of April 2020, when the last of the Algebra topics of the Grade 9 syllabus will be covered.

Upon completion of my study, I undertake to provide you with access to the research findings. Positive findings that will improve our practise will be made available to colleagues.

Yours sincerely

Daniel F. Junius

Student: Rhodes University

I, ………………………………………………………………………….. have read the above information / confirm that the above information has been explained to me in a language that I understand, and I am aware of this document's contents. I have asked all questions that I wished to ask, and these have been answered to my satisfaction. I fully understand what the research project entails.

I have not been pressurised in any way and I voluntarily agree that the project may proceed.

| ………………………………….. | ………………………… | ………………………….. |
|---|---|---|
| **Signature** | **Witness** | **Date** |

# Appendix H: Flowchart of cycles and phases of research design

**ADDENDUM H** **Flowchart of Cycles and Phases of Research Design**