A STUDY OF FOUR-DIMENSIONAL OSCILLATOR GROUPS AND THE ASSOCIATED LEFT-INVARIANT CONTROL AFFINE SYSTEMS

A thesis submitted in fulfilment of the

requirements for the degree of

MASTERS IN SCIENCE

of

RHODES UNIVERSITY

by

RORY BIGGS

December 2011

Abstract

In this thesis we consider four-dimensional oscillator Lie groups. We start by classifying all such connected Lie groups (with isomorphic Lie algebras) and show that there are only four types. Various properties of these groups and their algebras are investigated. Thereafter, we classify all left-invariant control affine systems evolving on these groups, under (local and global) detached feedback equivalence. This is accomplished by reducing this classification problem to classifying affine subspaces of the Lie algebra involved, under an appropriate equivalence relation. Controllability criteria are then produced for a subclass of these systems. Finally, we investigate a general optimal control problem (with fixed terminal time and quadratic cost) associated to the homogeneous two-input systems. This investigation mainly involves a qualitative investigation of the reduced normal extremals and finding explicit expressions for a subclass of these extremals.

Key words and phrases. (Four-dimensional) oscillator Lie group, (detached) feedback equivalence, left-invariant control affine system, optimal control.

Acknowledgements

I would like to thank my supervisor Dr. C.C. Remsing, and my colleagues Ross Adams and Helen Henninger, for the many hours we shared investigating and discussing a multitude of fascinating topics. In particular, a special word of thanks is extended to Dr. C.C. Remsing, without whom these last two years would not have been nearly as exciting and fun as they were. Finally, I would also like to thank the National Research Foundation and Rhodes University for financial support.

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Introduction

In recent decades much interest has been shown in invariant control affine systems evolving on Lie groups, as well as the optimal control problems associated to such systems. In this thesis we will study left-invariant control affine systems on four-dimensional oscillator Lie groups. This will involve a study of the four-dimensional oscillator Lie groups themselves, a classification of said systems, and the investigation of an optimal control problem associated to one of these systems.

A four-dimensional oscillator Lie group was introduced by Streater [36], and so named as its Lie algebra can be identified to that generated by the differential operators associated to the harmonic oscillator problem. The oscillator Lie groups have many interesting geometric features, as well as being of much interest from the viewpoint of physics (see, e.g., [9], [10], [13], [23], [27]). In particular, the four-dimensional simply connected oscillator Lie group can be described as the only simply connected four-dimensional non-abelian solvable Lie group which admits a bi-invariant Lorentzian metric (see [25], [27]).

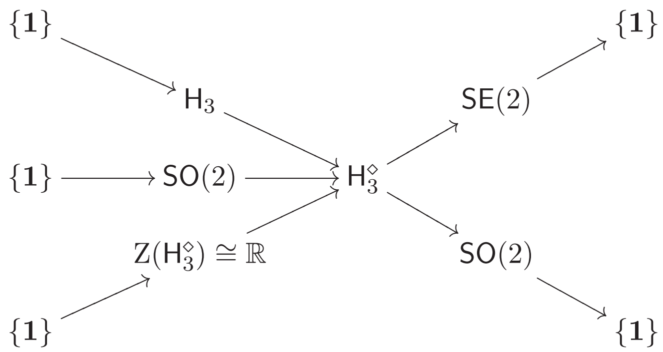
In this thesis, we commence our study of four-dimensional oscillator Lie groups by realising such a connected Lie group as a linear Lie group $\mathsf{H}_3^\diamond$, decomposing as a semi-direct product of the Heisenberg group H_3 and $\mathsf{SO}(2)$, with Lie algebra having commutator relations as given in table 1. Thereafter we find the universal covering Lie group (with an isomorphic Lie algebra)

$[\downarrow, \rightarrow]$	E_1	E_2	E_3	E_4
E_1	0	E_3	0	E_2
E_2	$-E_3$	0	0	$-E_1$
E_3	0	0	0	0
E_4	$-E_2$	E_1	0	0

Table 1: Commutator table for the oscillator Lie algebra

and classify all connected Lie groups (with isomorphic Lie algebras). We then proceed to investigate the adjoint and coadjoint orbits and show that they are “naturally” equivalent. Consequently, we find an invariant scalar product on the oscillator Lie algebra and show that this algebra may be realised as a double extension of the abelian Lie algebra $\mathbb{R}^2$ by $\mathbb{R}$.

Regarding the structure of the oscillator Lie group $\mathsf{H}_3^\diamond$, it is of interest to note that it has subgroups $\mathsf{SO}(2)$, $(\mathbb{R}, +)$, H_3 and quotient groups $\mathsf{SO}(2)$, $\mathsf{SE}(2)$. In particular, we have the following diagram of exact sequences of Lie group homomorphisms (i.e., the image of each arrow is the kernel of the next arrow in the same line)



Having concluded our study of the (four-dimensional) oscillator Lie groups themselves, we then move on to considering left-invariant control affine systems evolving on these oscillator Lie groups. Such a system can be described as a pair $\Sigma = (\mathsf{G}, \Xi)$, where the state space G is an oscillator Lie group and $\Xi : \mathsf{G} \times \mathbb{R}^\ell \to T\mathsf{G}$ is some left-invariant dynamics of the form

$$\Xi(g, u) = g\,\Xi(\mathbf{1}, u) = g\left(A + \sum_{i=1}^{\ell} u_i B_i\right)$$

where the set $\{B_i\}_{i=\overline{1,\ell}}$ is linearly independent. We then classify all such systems (locally and globally) under detached feedback equivalence (see [6], [7]). Briefly, two systems are (locally) detached feedback equivalent if there exists a (local) Lie group isomorphism mapping trajectories to trajectories, and an affine linear isomorphism mapping the corresponding controls. Detached feedback equivalence may be described as the most general feedback equivalence for which the state component (i.e., the diffeomorphism between state spaces involved) preserves left-invariant vector fields by the push forward (cf. [6]).

In order to classify locally (and globally) the above mentioned class of systems, we reduce the problem to classifying affine subspaces of the oscillator Lie algebra $\mathfrak{h}_3^\diamond$ (by an appropriate equivalence relation). This approach is briefly described in [7] and uses the algebraic characterisations of local and global detached feedback equivalence presented in [7]. For the sake of brevity, we will however restrict ourselves to only considering linear oscillator Lie groups when making a global classification. Within this class of systems we then also fully investigate controllability.

In the last chapter of this thesis we investigate a general optimal control problem (with fixed terminal time and quadratic cost) associated to the homogeneous two-input systems (i.e., $\Xi(\mathbf{1}, u) = u_1 B_1 + u_2 B_2$) which is lifted, via the Pontryagin Maximum Principle, to a Hamiltonian on the dual of the Lie algebra. Specifically, the system we consider is an equivalence representative of the whole class of two-input homogeneous systems evolving on the simply connected oscillator Lie group. This investigation will entail the study of the normal and abnormal extremals. In the normal case, we make a detailed qualitative analysis (including the study of the stability nature of all equilibrium states), and explicitly integrate a subclass of the extremal equations.

Original Contributions

To the best of our knowledge, the following results of this thesis are original.

Chapter 1. The classification of all connected four-dimensional oscillator Lie groups; theorem 1.3.7 and its corollary. An investigation as to which of the aforementioned Lie groups admit faithful linear representations; proposition 1.3.10. An investigation of which four-dimensional oscillator Lie groups cover which; propositions 1.3.12 through 1.3.15. Explicit calculation of the adjoint orbits of the four-dimensional oscillator Lie algebra; proposition 1.4.5. A complete list of ideals of the four-dimensional oscillator Lie algebra; proposition 1.5.3.

Chapters 2 & 3. A result for obtaining a preclassification of homogeneous systems from a classification of inhomogeneous systems; proposition 2.1.8 and its corollary (locally), proposition 3.1.4 and its corollary (globally). Results pertaining to invariants of (the traces of) left-invariant control affine systems on the four-dimensional oscillator Lie groups; propositions 2.2.3, 2.4.4 and 2.5.2. A complete classification of locally detached feedback equivalent left-invariant control affine systems of full rank evolving on four-dimensional oscillator Lie groups; propositions 2.2.4, 2.3.1, 2.4.5, 2.5.3, 2.6.1, theorem 2.7.1 and corollary 2.7.2. A useful necessary condition for a Lie algebra automorphism to be the tangent map (at identity) of a Lie group automorphism; proposition 3.1.7. A useful characterisation of Lie group automorphisms (specifically, existence of an automorphism given its tangent map); theorem 3.1.8 and corollary 3.1.9. A calculation of the group of automorphisms of (the n-fold covers of) the oscillator Lie group $\mathsf{H}_3^\diamond$; theorem 3.3.2. A complete classification of globally detached feedback equivalent left-invariant control affine systems of full rank evolving on four-dimensional linear oscillator Lie groups; corollary 3.2.1, propositions 3.3.3, 3.3.4, 3.3.5, 3.3.6, 3.3.7 theorem 3.3.8 and corollary 3.3.9. Controllability criteria for the aforementioned classes of systems; theorem 3.2.2, corollary 3.2.3 and theorem 3.3.10.

Chapter 4. An investigation of the abnormal extremals of an optimal control problem associated to a homogeneous two-input system; proposition 4.1.2. Results pertaining to the normal extremals of the aforementioned problem; propositions 4.1.3 and 4.1.5. An investigation of the equilibriums states and their stability nature, for the reduced normal extremals; propositions 4.2.1, 4.2.2 and 4.2.3. A qualitative investigation and classification of integral curves of a family of Hamiltonian vector fields; propositions 4.2.5, 4.2.7, 4.2.10, 4.2.11, 4.2.12, corollary 4.2.8 and theorem 4.2.16. An involution preserving the aforementioned integral curves; proposition 4.2.17. Explicit expressions for a subclass of the aforementioned integral curves; propositions 4.3.1, 4.3.3, 4.3.5, 4.3.8 and corollary 4.3.10.

Notation

Lie groups, as a rule, will be denoted with upper case Sans Serif letters (with the exception of $\mathcal{H}_{2n+1}$) and their corresponding Lie algebras with the corresponding small case Fraktur letters.

In additions we will use the following notation:

$\mathbf{1}$ – the identity element of a Lie group;

Aut G – the (Lie) group of Lie group automorphisms $\phi : \mathsf{G} \to \mathsf{G}$;

Aut $\mathfrak{g}$ – the (Lie) group of Lie algebra automorphisms $\psi : \mathfrak{g} \to \mathfrak{g}$;

dAut G – the subgroup of Aut $\mathfrak{g}$ consisting of tangent maps at identity of elements of Aut G;

$\rtimes$ – the semi-direct product of groups (normal subgroup on the left);

$\leq$ – subgroup, subalgebra;

$\trianglelefteq$ – normal subgroup, ideal of an algebra;

$\mathbb{T}$, $\mathbb{S}$ – the one-dimensional compact Lie group represented as the 1-torus or circle;

$\mathsf{GL}(n, \mathbb{R})$ – the general linear group of $\mathbb{R}^n$, represented as $n \times n$ invertible matrices;

$\mathsf{GL}(V)$ – the general linear group of a (real) vector space V;

$\mathsf{SO}(n)$ – The special orthogonal linear Lie group $\{A \in \mathsf{GL}(n, \mathbb{R}) \mid A^\top A = I, \det A = 1\}$;

δ_{ij} – Kronecker delta ($\delta_{ij} = 1$ if $i = j$, else $\delta_{ij} = 0$);

$|\cdot|$ – determinant (of a matrix) or absolute value (of a real number);

$\mathrm{diag}(r_1, r_2, \ldots, r_n)$ – a diagonal $n \times n$ matrix with diagonal entries r_1, r_2,$\ldots$ r_n;

$\wedge, \vee, \Rightarrow, \Leftrightarrow$ – logical conjunction (and), disjunction (or), implication and equivalence, respectively.

Chapter 1

The Oscillator Lie Groups

In this chapter we introduce the (four-dimensional) oscillator Lie groups and investigate some of their properties. Note that the term "oscillator Lie group" is used in the literature primarily only for the simply connected oscillator Lie groups. We start by briefly reviewing the definition and basic properties of the (three-dimensional) Heisenberg group H_3. We then proceed to introduce a oscillator group $\mathsf{H}_3^\diamond$ as a semi-direct product of H_3 and $\mathsf{SO}(2)$. Next we establish this Lie group as a closed (or embedded) linear Lie group and study its properties, mainly locally on its Lie algebra $\mathfrak{h}_3^\diamond$. We then proceed to find a universal covering $\widetilde{\mathsf{H}}_3^\diamond$ of $\mathsf{H}_3^\diamond$ which we use to classify all connected Lie groups with the same Lie algebra as $\mathsf{H}_3^\diamond$. (We refer to these Lie groups collectively as the four-dimensional oscillator Lie groups.)

At that stage we turn our attention to the adjoint and coadjoint orbits. We show that they are "equivalent" in the sense that there exists a linear isomorphism mapping the adjoint and coadjoint orbits bijectively. We then observe that this in turn is equivalent to a non-degenerate invariant scalar product existing on $\mathfrak{h}_3^\diamond$ and show that $\mathfrak{h}_3^\diamond$ together with this scalar product is a double extension of the Lie algebra $\mathbb{R}^2$, endowed with the dot product, by $\mathbb{R}$.

1.1 The Heisenberg Group

Representations

The **Heisenberg group**, denoted $\mathcal{H}_{2n+1}$, is defined ([12]) as the set $\mathbb{R}^{2n+1} = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ with group law

$$(x, y, z)(x', y', z') = (x + x', y + y', z + z' + \tfrac{1}{2}(x \bullet y' - y \bullet x')).$$

We define the **polarized Heisenberg group**, H_{2n+1}, as

$$\mathsf{H}_{2n+1} = \{h_n(x, y, z) \in \mathsf{GL}(n + 2, \mathbb{R}) \mid x, y \in \mathbb{R}^n, z \in \mathbb{R}\}$$

where $h_n : \mathbb{R}^3 \to \mathsf{H}_{2n+1}$ is (convenient notation) defined by

$$h_n(x,y,z) = \begin{bmatrix} 1 & x_1 & x_2 & \dots & x_n & z \\ 0 & 1 & 0 & \dots & 0 & y_1 \\ 0 & 0 & 1 & \dots & 0 & y_2 \\ \vdots & & & \ddots & & \vdots \\ & & & & & y_n \\ 0 & \dots & & & 0 & 1 \end{bmatrix}.$$

1.1.1 Remark. Note that $h_n : \mathbb{R}^{2n+1} \to \mathsf{H}_{2n+1}$ is a diffeomorphism and thus H_{2n+1} is a closed linear Lie group.

1.1.2 Proposition. *The Heisenberg Lie group $\mathcal{H}_{2n+1}$ is isomorphic to the polarised Heisenberg Lie group H_{2n+1} (for $n \in \mathbb{N}$).*

Proof. Define a mapping $\phi : \mathcal{H}_{2n+1} \to \mathsf{H}_{2n+1}$, $(x,y,z) \mapsto h_n(x,y,z+\frac{1}{2}(x \bullet y))$. We get that ϕ is injective and well defined as

$$\begin{aligned} & h_n(x,y,z+\tfrac{1}{2}(x \bullet y)) = h_n(x',y',z'+\tfrac{1}{2}(x' \bullet y')) \\ \Leftrightarrow\quad & \begin{cases} x = x' \\ y = y' \\ z + \frac{1}{2}(x \bullet y) = z' + \frac{1}{2}(x' \bullet y') \end{cases} \\ \Leftrightarrow\quad & (x,y,z) = (x',y',z'). \end{aligned}$$

Next ϕ is surjective as $\forall h_n(x,y,z) \in \mathsf{H}_{2n+1}$, $(x,y,z-\frac{1}{2}(x \bullet y)) \in \mathcal{H}_{2n+1}$ and $\phi(x,y,z-\frac{1}{2}(x \bullet y)) = h_n(x,y,z)$. Now ϕ is a group homomorphism as

$$\begin{aligned} \phi((x,y,z)\,(x',y',z')) &= \phi(x+x', y+y', z+z'+\tfrac{1}{2}(x \bullet y' - y \bullet x')) \\ &= h_n(x+x', y+y', z+z'+\tfrac{1}{2}(x \bullet y' - y \bullet x') + \tfrac{1}{2}(x+x') \bullet (y+y')) \\ &= h_n(x+x', y+y', z+z'+x \bullet y' + \tfrac{1}{2}(x \bullet y + x' \bullet y')) \\ &= h_n(x,y,z+\tfrac{1}{2}x \bullet y)\, h_n(x',y',z'+\tfrac{1}{2}x' \bullet y') \\ &= \phi(x,y,z)\,\phi(x',y',z'). \end{aligned}$$

Finally as $h_n : \mathbb{R}^{2n+1} \to \mathsf{H}_{2n+1}$ is a diffeomorphism we have that ϕ is a diffeomorphism and hence a Lie group isomorphism. □

Topological properties

As H_{2n+1} is diffeomorphic to $\mathbb{R}^{2n+1}$ the topological properties of H_{2n+1} are just those of $\mathbb{R}^{2n+1}$. We therefore note that H_{2n+1} is not compact, but is connected and simply connected.

Algebraic properties

We list (and prove) some algebraic properties of H_{2n+1} in the form of propositions.

1.1.3 Proposition. *The centre of* H_{2n+1} *is given by* $\mathsf{Z}(\mathsf{H}_{2n+1}) = \{h_n(0,0,z) \mid z \in \mathbb{R}\}$.

Proof. Indeed we have that

$$\begin{aligned}
& h_n(x,y,z) \in \mathsf{Z}(\mathsf{H}_{2n+1}) \\
\Leftrightarrow\ & h_n(x,y,z)\, h_n(x',y',z') = h_n(x',y',z')\, h_n(x,y,z) && \text{for } \forall h_n(x',y',z') \in \mathsf{H}_{2n+1} \\
\Leftrightarrow\ & x \bullet y' = y \bullet x' && \text{for } \forall (x',y',z') \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \\
\Leftrightarrow\ & x = 0 \text{ and } y = 0.
\end{aligned}$$

□

We introduce the mapping $H_n : \mathbb{R}^3 \to \mathfrak{h}_3$, $H_n(x,y,z) = h_n(x,y,z) - I_{n+2}$ as convenient notation, i.e.,

$$H_n(x,y,z) = \begin{bmatrix} 0 & x_1 & x_2 & \dots & x_n & z \\ 0 & 0 & 0 & \dots & 0 & y_1 \\ 0 & 0 & 0 & \dots & 0 & y_2 \\ \vdots & & & \ddots & & \vdots \\ & & & & & y_n \\ 0 & \dots & & & 0 & 0 \end{bmatrix}.$$

1.1.4 Proposition. *The Lie algebra of* H_{2n+1} *is given by* $\mathfrak{h}_{2n+1} = \{H_n(x,y,z) \mid x,y \in \mathbb{R}^n,\ z \in \mathbb{R}\}$.

Proof. Let $x,y \in \mathbb{R}^n$, $z \in \mathbb{R}$, then we define a curve $\gamma : \mathbb{R} \to \mathsf{H}_{2n+1}$, $t \mapsto t\, H_n(x,y,z) + I_{n+2} \in \mathsf{H}_{2n+1}$ and note that $\gamma(0) = \mathbf{1}$ and $\dot\gamma(0) = H_n(x,y,z)$. Thus $\{H_n(x,y,z) \mid x,y \in \mathbb{R}^n,\ z \in \mathbb{R}\} \subseteq \mathfrak{h}_{2n+1}$. Now consider an arbitrary curve $\gamma(\cdot) : [-\epsilon,\epsilon] \to \mathsf{H}_{2n+1}$ such that $\gamma(0) = \mathbf{1}$. Then $\gamma(t) = h_n(x(t),y(t),z(t))$ for some curves $x(\cdot),\ y(\cdot) : [-\epsilon,\epsilon] \to \mathbb{R}^n$ and $z(\cdot) : [-\epsilon,\epsilon] \to \mathbb{R}$. Differentiating and evaluating at 0 then yields $\dot\gamma(0) = H_n(\dot x(0), \dot y(0), \dot z(0))$. Thus $\mathfrak{h}_{2n+1} \subseteq \{H_n(x,y,z) \mid x,y \in \mathbb{R}^n,\ z \in \mathbb{R}\}$. □

We use the notation $\{e_i\}_{i=\overline{1,n}}$ for the standard basis of $\mathbb{R}^n$. That is, e_i is the vector with i'th entry having value 1 and all other entries zero.

1.1.5 Proposition. *Let* $X_i = H_n(e_i,0,0)$, $Y_i = H_n(0,e_i,0)$ *and* $Z = H_n(0,0,1)$. *Then the set* $\{X_i, Y_i, Z\}_{i\in\overline{1,n}}$ *is a basis for* $\mathfrak{h}_{2n+1}$ *and has commutator relations* $[X_i, Y_j] = \delta_{ij} Z$ *with the rest zero.*

1.1.6 Proposition. *The centre of* $\mathfrak{h}_{2n+1}$ *is given by* $\mathsf{Z}(\mathfrak{h}_{2n+1}) = \{H_n(0,0,z) \mid z \in \mathbb{R}\}$.

Proof. Indeed we have that

$$\begin{aligned}
& H_n(x,y,z) \in \mathsf{Z}(\mathfrak{h}_{2n+1}) \\
\Leftrightarrow\ & H_n(x,y,z)\, H_n(x',y',z') = H_n(x',y',z')\, H_n(x,y,z) && \text{for } \forall H_n(x',y',z') \in \mathfrak{h}_{2n+1} \\
\Leftrightarrow\ & x \bullet y' = x' \bullet y && \text{for } \forall (x',y',z') \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \\
\Leftrightarrow\ & x = 0 \text{ and } y = 0.
\end{aligned}$$

□

1.1.7 PROPOSITION. *The Lie algebra $\mathfrak{h}_{2n+1}$ is nilpotent (of class 2) and hence triangular, exponential and solvable.*

PROOF. We have that

$$\begin{aligned}
\mathfrak{g}_0 &= \mathfrak{h}_{2n+1} \\
\mathfrak{g}_1 &= [\mathfrak{h}_{2n+1}, \mathfrak{h}_{2n+1}] \\
&= \left\{ \left[H_n(x,y,z), H_n(x',y',z')\right] \mid x,y,x',y' \in \mathbb{R}^n, z,z' \in \mathbb{R} \right\} \\
&= \left\{ H_n(0,0,x \bullet y' - x' \bullet y) \mid x,y,x',y' \in \mathbb{R}^n \right\} \\
&= \{ H_n(0,0,z) \mid z \in \mathbb{R} \} \\
&= \mathrm{Z}\,(\mathfrak{h}_{2n+1}) \\
\mathfrak{g}_2 &= [\mathfrak{h}_{2n+1}, \mathrm{Z}\,(\mathfrak{h}_{2n+1})] = \{0\}, \text{ the trivial algebra.}
\end{aligned}$$

Thus $\mathfrak{h}_{2n+1}$ is nilpotent. Applying theorem A.1.18 then yields the required results. □

1.1.8 PROPOSITION. *The exponential map* $\exp : \mathfrak{h}_{2n+1} \to \mathsf{H}_{2n+1}$ *is given by* $\exp(H_n(x,y,z)) = h_n(x,y,z+\frac{1}{2}x \bullet y)$ *and is a diffeomorphism.*

PROOF. Notice that $H_n(x,y,z)^2 = H_n(0,0,x \bullet y)$ and $H_n(x,y,z)^3 = 0$. Thus $\exp(H_n(x,y,z)) = I_{n+2} + H_n(x,y,z) + \frac{1}{2}H_n(x,y,z)^2 = h_n(x,y,z+\frac{1}{2}x \bullet y)$. As H_3 is simply connected and $\mathfrak{h}_3$ is exponential (proposition 1.1.7), it follows that exp is a diffeomorphism. □

1.2 The Oscillator Lie Group $\mathsf{H}_3^\diamond$

1.2.1 As a semi-direct product of H_3 and $\mathsf{SO}(2)$

We define the **oscillator Lie group $\mathsf{H}_3^\diamond$** as $\mathsf{H}_3^\diamond = \{m(x,y,z,\theta) \mid x,y,z,\theta \in \mathbb{R}\}$, where $m : \mathbb{R}^4 \to \mathsf{GL}(4,\mathbb{R})$ is a continuous mapping (used as convenient notation) given by

$$m(x,y,z,\theta) = \begin{bmatrix} 1 & -x\cos\theta + y\sin\theta & x\sin\theta + y\cos\theta & -2z \\ 0 & \cos\theta & -\sin\theta & y \\ 0 & \sin\theta & \cos\theta & x \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

1.2.1 REMARK. This definition (made in retrospect) is the result of trying to construct a closed linear Lie group that is isomorphic to a semi-direct product of H_3 and $\mathsf{SO}(2)$. Additionally we required that its Lie algebra has table 1 as commutator table (in some ordered basis). As a starting point for this construction we considered the linear representation (for the Lie algebra) suggested in [36]. (The notation $\mathsf{H}_3^\diamond$ for this Lie group is motivated by the fact that H_3 forms a subgroup and that $\mathfrak{h}_3^\diamond$ is one of the diamond Lie algebras.)

We will show that $\mathsf{H}_3^\diamond$ as a closed linear Lie group and decomposes as a semi-direct product of subgroups H_3 and $\mathsf{SO}(2)$. We start by making some basic calculations (performed in Mathematica, see section C.1) for $\mathsf{H}_3^\diamond$. We have that $m(0,0,0,0) = I_4$ and that

$$m(x,y,z,\theta)^{-1} = m\left(-x\cos\theta + y\sin\theta, -x\sin\theta - y\cos\theta, -z, -\theta\right).$$

Also note that $m(x,y,z,\theta) = m(x,y,z,0)\,m(0,0,0,\theta)$. Finally note that the equation

$$m(x^*,y^*,z^*,\theta^*) = m(x,y,z,\theta)\,m(x',y',z',\theta')$$

always has a solution and that one such solution is given by

$$x^* = x + y'\sin\theta + x'\cos\theta \qquad y^* = y + y'\cos\theta - x'\sin\theta$$
$$z^* = z + z' + \tfrac{1}{2}(xy' - yx')\cos\theta - \tfrac{1}{2}(xx' + yy')\sin\theta \qquad \theta^* = \theta + \theta'.$$

1.2.2 REMARK. The above solution may be written more compactly using complex numbers as

$$x^* + iy^* = x + iy + \mathrm{e}^{-i\theta}(x' + iy'), \quad z^* = z + z' + \tfrac{1}{2}\mathrm{Im}\,(\mathrm{e}^{-i\theta}(x - iy)(x' + iy')), \quad \theta^* = \theta + \theta'.$$

1.2.3 PROPOSITION. *$\mathsf{H}_3^\diamond$ is a closed linear Lie group.*

PROOF. By the preceding calculations we have that $\mathsf{H}_3^\diamond$ is an abstract group and hence a subgroup of $\mathsf{GL}(4,\mathbb{R})$. We are left to show that $\mathsf{H}_3^\diamond$ is closed in $\mathsf{GL}(4,\mathbb{R})$. We leave this to the next subsection where we show it in proposition 1.2.9. □

1.2.4 PROPOSITION. *The subsets H_3 (redefined here) and $\mathsf{SO}(2)$ of $\mathsf{H}_3^\diamond$ given by*

$$\mathsf{H}_3 = \{m(x,y,z,0) \mid x,y,z \in \mathbb{R}\} \qquad \mathsf{SO}(2) = \{m(0,0,0,\theta) \mid \theta \in \mathbb{R}\}$$

are closed Lie subgroups of $\mathsf{H}_3^\diamond$, isomorphic to the (three-dimensional) Heisenberg group and the one-dimensional special orthogonal group, respectively. Moreover the mapping $\phi : \mathcal{H}_3 \to \mathsf{H}_3$, $(x,y,z) \mapsto m(x,y,z,0)$ is a (Lie group) isomorphism.

PROOF. As

$$m(x,y,z,0) = \begin{bmatrix} 1 & -x & y & -2z \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & x \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

we get that ϕ is a diffeomorphism. Then we have that

$$\begin{aligned} \phi\,(x,y,z)\;\phi\left(x',y',z'\right) &= m(x,y,z,0)\,m(x',y',z',0) \\ &= m(x+x', y+y', z+z'+\tfrac{1}{2}(xy'-yx'), 0) \\ &= \phi\left(x+x', y+y', z+z'+\tfrac{1}{2}(xy'-yx')\right) \\ &= \phi\left((x,y,z)\,(x',y',z')\right), \end{aligned}$$

thus ϕ is a Lie group isomorphism. Now as $\mathbb{R}^3$ is complete, so is H_3 and hence H_3 is closed in $\mathsf{H}_3^\diamond$. Thus H_3 is indeed a closed Lie subgroup of $\mathsf{H}_3^\diamond$. Note that

$$m(0,0,0,\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

making it clear that $\mathsf{SO}(2)$ here is just the special orthogonal group $\{g \in \mathsf{GL}(2,\mathbb{R}) \mid g^\top g = I_2,\ \det g = 1\}$ embedded in $\mathsf{GL}(4,\mathbb{R})$. □

1.2.5 LEMMA. *$rhr^{-1} \in \mathsf{H}_3$ for $h \in \mathsf{H}_3$ and $r \in \mathsf{SO}(2)$*

PROOF. Indeed (for some $x, y, z, \theta \in \mathbb{R}$) we have that

$$\begin{aligned} rhr^{-1} &= m(0,0,0,\theta)\, m(x,y,z,0)\, m(0,0,0,-\theta) \\ &= \begin{bmatrix} 1 & -x\cos\theta - y\sin\theta & y\cos\theta - x\sin\theta & -2z \\ 0 & 1 & 0 & y\cos\theta - x\sin\theta \\ 0 & 0 & 1 & y\sin\theta + x\cos\theta \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= m(y\sin\theta + x\cos\theta, y\cos\theta - x\sin\theta, z, 0) \\ &\in \mathsf{H}_3. \end{aligned}$$

□

With this information at hand we are now in a position to decompose $\mathsf{H}_3^\diamond$ as a semi-direct product.

1.2.6 PROPOSITION. *$\mathsf{H}_3^\diamond$ decomposes as a semi-direct product $\mathsf{H}_3^\diamond = \mathsf{H}_3 \rtimes \mathsf{SO}(2)$, i.e.,*

1. $\mathsf{H}_3 \trianglelefteq \mathsf{H}_3^\diamond$; *2.* $\mathsf{H}_3 \cap \mathsf{SO}(2) = \{\mathbf{1}\}$; *3.* $\mathsf{H}_3\, \mathsf{SO}(2) = \mathsf{H}_3^\diamond$.

PROOF. Item 2 is trivial. Recall that for any element $m(x,y,z,\theta) \in \mathsf{H}_3^\diamond$ we have that

$$m(x,y,z,\theta) = m(x,y,z,0)\, m(0,0,0,\theta),$$

proving item 3. Let $g \in \mathsf{H}_3^\diamond$ and $h \in \mathsf{H}_3$. Then $g = h'r$ for some $h' \in \mathsf{H}_3$ and $r \in \mathsf{SO}(2)$ by item 3. Thus $ghg^{-1} = h'rhr^{-1}(h')^{-1}$. But by the preceding lemma $rhr^{-1} \in \mathsf{H}_3$ and thus $h'rhr^{-1}(h')^{-1} \in \mathsf{H}_3$ (as H_3 is a subgroup) yielding item 1. □

1.2.7 REMARK. Explicitly we have that $\mathsf{H}_3^\diamond$ is Lie group isomorphic to $\mathsf{H}_3 \rtimes_\mu \mathsf{SO}(2)$, where

$$\mu : \mathsf{SO}(2) \to \mathrm{Aut}(\mathsf{H}_3),\ r \mapsto \mu(r) \qquad \mu(r)(h) = rhr^{-1}.$$

This isomorphism is given by

$$\begin{aligned} \phi : \quad & \mathsf{H}_3 \rtimes_\mu \mathsf{SO}(2) \to \mathsf{H}_3^\diamond \\ & (m(x,y,z,0), m(0,0,0,\theta)) \mapsto m(x,y,z,0)\, m(0,0,0,\theta) = m(x,y,z,\theta) \end{aligned}$$

where the homomorphism property follows as

$$\phi\left((h_1,r_1)\,(h_2,r_2)\right) = \phi\left((h_1r_1h_2r_1^{-1}, r_1r_2)\right) = h_1r_1h_2r_2 = \phi(h_1,r_1)\,\phi(h_2,r_2).$$

1.2.2 Topological properties

We briefly investigate the topological properties of $\mathsf{H}_3^\diamond$ and in particular show that $\mathsf{H}_3^\diamond$ is diffeomorphic to $\mathbb{R}^3 \times \mathbb{T}$.

1.2.8 LEMMA. *Let $\phi : \mathbb{R} \to \mathbb{R}^2$, $\theta \mapsto (\cos\theta, \sin\theta)$. Then for $\forall p \in \mathbb{R}$ there exists an open neighbourhood W of $p \in \mathbb{R}$ such that $\phi|_W : W \to \phi(W)$ has smooth inverse $\phi|_W^{-1}$.*

PROOF. $D\phi = [-\sin\theta, \cos\theta]$ is of full rank for all $\theta \in \mathbb{R}$. Thus, by the inverse mapping theorem ([22]), for $\forall p \in \mathbb{R}$ there exists an open neighbourhood W of $p \in \mathbb{R}$ such that $\phi|_W : W \to \phi(W)$ is a diffeomorphism. Hence $\phi|_W$ has smooth inverse $\phi|_W^{-1}$. □

1.2.9 PROPOSITION. $\mathsf{H}_3^\diamond$ *is topologically closed in* $\mathsf{GL}(4,\mathbb{R})$.

PROOF. Let $m(x_n, y_n, z_n, \theta_n)$ be a sequence in $\mathsf{H}_3^\diamond$, $m(x_n, y_n, z_n, \theta_n) \to B \in \mathsf{GL}(4,\mathbb{R})$. First we recall that

$$m(x,y,z,\theta) = \begin{bmatrix} 1 & -x\cos\theta + y\sin\theta & x\sin\theta + y\cos\theta & -2z \\ 0 & \cos\theta & -\sin\theta & y \\ 0 & \sin\theta & \cos\theta & x \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

So, as the projection maps (to entries) are continuous, we have: $x_n \to B_{3,4}$, $y_n \to B_{2,4}$, $z_n \to \frac{B_{1,4}}{-2}$ (multiplication by $\frac{1}{-2}$ is continuous) and $\theta_n^* = \phi|_W^{-1}(\cos\theta_n, \sin\theta_n) \to \phi|_W^{-1}(B_{2,2}, B_{3,2})$ (as in lemma 1.2.8 for some neighbourhood W). Thus as m is continuous and $m(x_n, y_n, z_n, \theta_n) = m(x_n, y_n, z_n, \theta_n^*)$ we have $m(x_n, y_n, z_n, \theta_n) \to m(B_{3,4}, B_{2,4}, \frac{B_{1,4}}{-2}, \phi|_W^{-1}(B_{2,2}, B_{3,2}))$. Finally as limits are unique in $\mathsf{GL}(4,\mathbb{R})$ (Hausdorff space), $B = m(B_{3,4}, B_{2,4}, \frac{B_{1,4}}{-2}, \phi|_W^{-1}(B_{2,2}, B_{3,2})) \in \mathsf{H}_3^\diamond$. Note that if $B \notin \mathsf{GL}(4,\mathbb{R})$ then there is nothing to show. □

1.2.10 PROPOSITION. $\mathsf{H}_3^\diamond$ *is diffeomorphic to* $\mathbb{R}^3 \times \mathbb{T}$.

PROOF. We show that the mapping

$$\phi : \mathbb{R}^3 \times \mathbb{T} \to \mathsf{H}_3^\diamond, \quad \big((x,y,z), (\cos\theta, \sin\theta)\big) \mapsto m(x,y,z,\theta)$$

is a diffeomorphism. We note that here we represent the 1-torus (or circle) as a submanifold of $\mathbb{R}^2$ with two charts $f_i : (\cos\theta, \sin\theta) \mapsto \theta$, with $\theta \in (0, \frac{2}{3}\pi)$ for $i = 1$ and $\theta \in (\frac{1}{2}\pi, 2\pi)$ for $i = 2$. Then we have that ϕ is injective and well defined as

$$\begin{aligned} & \big((x,y,z), (\cos\theta, \sin\theta)\big) = \big((x',y',z'), (\cos\theta', \sin\theta')\big) \\ \Leftrightarrow\ & (x = x') \wedge (y = y') \wedge (z = z') \wedge (\cos\theta = \cos\theta') \wedge (\sin\theta = \sin\theta') \\ \Leftrightarrow\ & m(x,y,z,\theta) = m(x',y',z',\theta'). \end{aligned}$$

As $m : \mathbb{R}^4 \to \mathsf{H}_3^\diamond$ is surjective and smooth, we get that ϕ is surjective and smooth. Next we have that (in a suitable local chart)

$$\frac{\partial\phi}{\partial x} = \begin{bmatrix} 0 & -\cos\theta & \sin\theta & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \frac{\partial\phi}{\partial y} = \begin{bmatrix} 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\frac{\partial\phi}{\partial z} = \begin{bmatrix} 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \frac{\partial\phi}{\partial \theta} = \begin{bmatrix} 0 & y\cos\theta + x\sin\theta & x\cos\theta - y\sin\theta & 0 \\ 0 & -\sin\theta & -\cos\theta & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus we have that $T_p\phi : \mathbb{R}^4 \to T_{\phi(p)}\mathsf{H}_3^\diamond$ is a linear isomorphism for any $p \in \mathbb{R}^3 \times \mathbb{T}$. Thus, by the inverse mapping theorem ([22]), as ϕ is a full rank bijective smooth mapping, it is a diffeomorphism. □

1.2.11 COROLLARY. $\mathsf{H}_3^\diamond$ *is not compact.*

1.2.12 COROLLARY. $\mathsf{H}_3^\diamond$ *is connected, but not simply connected. Indeed we have that the fundamental group of* $\mathsf{H}_3^\diamond$ *is given by*

$$\pi_1(\mathsf{H}_3^\diamond) \cong \pi_1(\mathbb{R}^3 \times \mathbb{T}) \cong \pi_1(\mathbb{R}^3) \times \pi_1(\mathbb{T}) \cong \{\mathbf{1}\} \times \pi_1(\mathbb{S}^1) \cong \mathbb{Z}.$$

1.2.3 Algebraic properties

We list a number of algebraic properties of $\mathsf{H}_3^\diamond$. In particular: we determine its centre, its quotient groups and its Lie algebra; we show that its Lie algebra is a unimodular and solvable but not exponential; we find the automorphism group of its Lie algebra.

1.2.13 PROPOSITION. *The centre of* $\mathsf{H}_3^\diamond$ *is given by* $\mathsf{Z}(\mathsf{H}_3^\diamond) = \{m(0,0,z,0) \mid z \in \mathbb{R}\}$.

PROOF. We have the following sequence of implications

$$\begin{aligned}
& m(x,y,z,\theta) \in \mathsf{Z}(\mathsf{H}_3^\diamond) \\
\Rightarrow\ & m(x,y,z,\theta)\, m(x',y',z',\theta') - m(x',y',z',\theta')\, m(x,y,z,\theta) = 0 \qquad \text{for } \forall x',y',z',\theta' \in \mathbb{R} \\
\Rightarrow\ & \text{(from row 3, column 4 we get)} \\
& y'\sin\theta + x'\cos\theta - y\sin\theta' - x\cos\theta' + x - x' = 0 \qquad \text{for } \forall x',y',z',\theta' \in \mathbb{R} \\
\Rightarrow\ & \begin{cases} (x'=0 \wedge y'=0 \wedge \theta'=\frac{\pi}{2}) & \Rightarrow\ x-y=0 \\ (x'=0 \wedge y'=0 \wedge \theta'=\frac{3\pi}{2}) & \Rightarrow\ x+y=0, \qquad \text{thus } (x=0 \wedge y=0) \\ (x'=1 \wedge y'=0 \wedge x=0 \wedge y=0) & \Rightarrow\ \cos(\theta)=1 \\ (x'=0 \wedge y'=1 \wedge x=0 \wedge y=0) & \Rightarrow\ \sin(\theta)=0, \qquad \text{thus } \theta = 2k\pi,\ k\in\mathbb{Z} \end{cases} \\
\Rightarrow\ & \mathsf{Z}(\mathsf{H}_3^\diamond) \subseteq \{m(0,0,z,0) \mid z \in \mathbb{R}\}.
\end{aligned}$$

Conversely we have that

$$\begin{aligned}
& m(0,0,z,0) \in \{m(0,0,z,0) \mid z \in \mathbb{R}\} \\
\Rightarrow\ & m(0,0,z,0)\, m(x',y',z',\theta') - m(x',y',z',\theta')\, m(0,0,z,0) = 0 \\
\Rightarrow\ & \mathsf{Z}(\mathsf{H}_3^\diamond) \supseteq \{m(0,0,z,0) \mid z \in \mathbb{R}\}.
\end{aligned}$$

□

1.2.14 PROPOSITION. *The quotient group* $\mathsf{H}_3^\diamond/\mathrm{Z}(\mathsf{H}_3^\diamond)$ *is isomorphic to the Euclidean group* $\mathsf{SE}(2)$*, where*

$$\mathsf{SE}(2) = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ x & \cos\theta & -\sin\theta \\ y & \sin\theta & \cos\theta \end{bmatrix} \;\middle|\; x,y,\theta \in \mathbb{R} \right\}.$$

PROOF. As $\mathsf{Z}(\mathsf{H}_3^\diamond)$ is a closed normal subgroup, the quotient is well defined (see section A.1.2). Define a map $\phi : \mathsf{H}_3^\diamond(n) \to \mathsf{SE}(2)$ by

$$\phi : m(x, y, z, \theta) \mapsto \begin{bmatrix} 1 & 0 & 0 \\ x & \cos\theta & \sin\theta \\ y & -\sin\theta & \cos\theta \end{bmatrix}.$$

Using the diffeomorphism from the proof of proposition 1.2.10, we get that ϕ is smooth and surjective. Using Mathematica (see section C.1) we verify that ϕ is a homomorphism (specifically, we show that $\phi(m(x, y, z, \theta)\, m(x', y', z', \theta')) - \phi(m(x, y, z, \theta))\, \phi(m(x', y', z', \theta')) = 0$). Hence, by the epimorphism theorem (i.e., theorem A.1.5), we get that $\mathsf{H}_3^\diamond / \ker\phi \cong \mathsf{SE}(2)$. But $\ker\phi = \mathsf{Z}(\mathsf{H}_3^\diamond)$, proving result. □

1.2.15 PROPOSITION. *The quotient group* $\mathsf{H}_3^\diamond/\mathsf{H}_3$ *is isomorphic to the subgroup* $\mathsf{SO}(2)$.

PROOF. As H_3 is a closed normal subgroup (propositions 1.2.4 and 1.2.6), the quotient is well defined. Define a map $\phi : \mathsf{H}_3^\diamond(n) \to \mathsf{SO}(2)$, $m(x, y, z, \theta) \mapsto m(0, 0, 0, \theta)$. Using the diffeomorphism from the proof of proposition 1.2.10, we get that ϕ is smooth and surjective. Then we have that

$$\begin{aligned} \phi(m(x, y, z, \theta)\, m(x', y', z', \theta')) &= \phi(m(\cdot, \cdot, \cdot, \theta + \theta')) \\ &= m(0, 0, 0, \theta + \theta') = m(0, 0, 0, \theta)\, m(0, 0, 0, \theta') \\ &= \phi(m(x, y, z, \theta))\, \phi(m(x', y', z', \theta')). \end{aligned}$$

Thus ϕ is a Lie group epimorphism. Hence, by the epimorphism theorem (i.e., theorem A.1.5), we get that $\mathsf{H}_3^\diamond / \ker\phi \cong \mathsf{SO}(2)$. But $\ker\phi = \mathsf{H}_3$, proving result. □

1.2.16 REMARK. The above two quotient groups are the only two quotient groups of $\mathsf{H}_3^\diamond$. This follows from the fact that the Lie algebras of H_3 and $\mathsf{Z}(\mathsf{H}_3^\diamond)$ are the only two ideals of $\mathfrak{h}_3^\diamond$ (which will be shown in section 1.5).

1.2.17 PROPOSITION. *The Lie algebra* $\mathfrak{h}_3^\diamond$ *of* $\mathsf{H}_3^\diamond$ *is given by*

$$\mathfrak{h}_3^\diamond = \{M(x, y, z, \theta) \mid x, y, z, \theta \in \mathbb{R}\}$$

where $M : \mathbb{R}^4 \to \mathfrak{h}_3^\diamond$ *is (convenient notation) given by*

$$M(x, y, z, \theta) = \begin{bmatrix} 0 & -x & y & -2z \\ 0 & 0 & -\theta & y \\ 0 & \theta & 0 & x \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

PROOF. Consider curve $\gamma(t) = m(x(t), y(t), z(t), \theta(t))$ in $\mathsf{H}_3^\diamond$ such that $\gamma(0) = I_4$ (i.e., $x(0) = 0$, $y(0) = 0$, $z(0) = 0$ and $\theta(0) = 0 + 2k\pi$, $k \in \mathbb{Z}$). Then $\dot\gamma(0) = M(\dot x(0), \dot y(0), \dot z(0), \dot\theta(0))$. Thus $\mathfrak{h}_3^\diamond \subseteq \{M(x, y, z, \theta) \mid x, y, z, \theta \in \mathbb{R}\}$. For the converse if $M(x, y, z, \theta) \in \{M(x, y, z, \theta) \mid x, y, z, \theta \in \mathbb{R}\}$ then $\gamma(t) = m(tx, ty, tz, t\theta)$ is a curve in $\mathsf{H}_3^\diamond$, $\gamma(0) = m(0, 0, 0, 0) = I_4$ and $\dot\gamma(0) = M(x, y, z, \theta)$. Thus we have that $\mathfrak{h}_3^\diamond \supseteq \{M(x, y, z, \theta) \mid x, y, z, \theta \in \mathbb{R}\}$. □

1.2.18 Proposition. *Let* $E_1 = M(1,0,0,0) = M(e_1)$, $E_2 = M(e_2)$, $E_3 = M(e_3)$ *and* $E_4 = M(e_4)$. *Then* $\{E_i\}_{i=\overline{1,4}}$ *is a (ordered) basis for* $\mathfrak{h}_3^\diamond$ *and has commutator relations:* $[E_1, E_2] = E_3$, $[E_1, E_4] = E_2$, $[E_2, E_4] = -E_1$ *with the rest zero.*

1.2.19 Proposition. *The Lie algebra* $\mathfrak{h}_3^\diamond$ *is not nilpotent.*

Proof. For two arbitrary elements of $\mathfrak{h}_3^\diamond$ we have that

$$[M(x,y,z,\theta), M(x',y',z',\theta')] = M(\theta y' - y\theta', x\theta' - \theta x', xy' - yx', 0).$$

In particular, note that $[E_4, E_2] = E_1$, $[E_1, E_4] = E_2$ and $[E_2, E_1] = E_3$. Thus $[\mathfrak{h}_3^\diamond, \mathfrak{h}_3^\diamond] = \operatorname{span}\{[A,B] \mid A, B \in \mathfrak{h}_3^\diamond\} = \{M(a,b,c,0) \mid a,b,c \in \mathbb{R}\} = \mathfrak{h}_3$ (i.e., the Lie algebra of H_3). Now $[M(x,y,z,0), M(x',y',z',\theta')] = M(-y\theta', x\theta', xy' - yx', 0)$, so $[\mathfrak{h}_3, \mathfrak{h}_3^\diamond] = \mathfrak{h}_3$. That is to say the lower central series of $\mathfrak{h}_3^\diamond$ does not terminate and thus $\mathfrak{h}_3^\diamond$ is not nilpotent. (Alternatively this follows from proposition 1.2.22 and theorem A.1.18.) □

1.2.20 Proposition. *The Lie algebra* $\mathfrak{h}_3^\diamond$ *is solvable.*

Proof. We have already shown in proposition 1.2.19 that $(\mathfrak{h}_3^\diamond)^{(1)} = [\mathfrak{h}_3^\diamond, \mathfrak{h}_3^\diamond] = \mathfrak{h}_3$. Thus we have that $(\mathfrak{h}_3^\diamond)^{(2)} = [\mathfrak{h}_3, \mathfrak{h}_3] = \mathrm{Z}(\mathfrak{h}_3)$. So it follows that $(\mathfrak{h}_3^\diamond)^{(3)} = [\mathrm{Z}(\mathfrak{h}_3), \mathrm{Z}(\mathfrak{h}_3)] = 0$. □

1.2.21 Proposition. *For* $g = m(x,y,z,\theta) \in \mathsf{H}_3^\diamond$ *and* $A = M(x,y,z,\theta) \in \mathfrak{h}_3^\diamond$ *we have (w.r.t. the ordered basis* $\{E_i\}_{i=\overline{1,4}}$*) that*

$$\operatorname{Ad} g = \begin{bmatrix} \cos\theta & \sin\theta & 0 & -y \\ -\sin\theta & \cos\theta & 0 & x \\ -y\cos\theta - x\sin\theta & x\cos\theta - y\sin\theta & 1 & \frac{1}{2}\left(x^2+y^2\right) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\operatorname{ad} A = \begin{bmatrix} 0 & \theta & 0 & -y \\ -\theta & 0 & 0 & x \\ -y & x & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Proof. We use Mathematica for calculations in this proof, see section C.2. We calculate vectors $V_i = \operatorname{Ad} g \cdot E_i = m(x,y,z,\theta)\, E_i\, m(x,y,z,\theta)^{-1}$ for $i = \overline{1,4}$. Then, as $\operatorname{Ad} g$ is a linear map, we have that $\operatorname{Ad} g = \begin{bmatrix} V_1 & V_2 & V_3 & V_4 \end{bmatrix}^\top$. Next we calculate vectors $V_i = \operatorname{ad} A \cdot E_i = [M(x,y,z,\theta), E_i]$ for $i = \overline{1,4}$. Then, as $\operatorname{ad} A$ is a linear map, we have that $\operatorname{ad} A = \begin{bmatrix} V_1 & V_2 & V_3 & V_4 \end{bmatrix}^\top$. (Alternatively $\operatorname{ad} A$ can be calculated by differentiating a curve $g(\cdot)$ such that $\dot g(0) = A$, i.e., $\operatorname{ad} A = \frac{d}{dt} \operatorname{Ad} g(t)\big|_{t=0}$.) □

1.2.22 Proposition. *The Lie algebra* $\mathfrak{h}_3^\diamond$ *is not exponential and hence not triangular.*

Proof. The characteristic polynomial of $\operatorname{ad} A$ (as represented in preceding proposition) is given by $\det(\operatorname{ad} A - \lambda I_4) = \lambda^2\left(\theta^2 + \lambda^2\right)$, thus yielding eigenvalues of 0, $-i\theta$ and $i\theta$. Thus there exists $A \in \mathfrak{h}_3^\diamond$ such that $\operatorname{ad} A$ has purely imaginary eigenvalues. By theorem A.1.18, it then follows that $\mathfrak{h}_3^\diamond$ is not exponential. Using theorem A.1.17, we then get that $\mathfrak{h}_3^\diamond$ is not triangular (or nilpotent). □

1.2.23 PROPOSITION. *Any connected Lie group (including* $\mathsf{H}_3^\diamond$*) with Lie algebra (isomorphic to)* $\mathfrak{h}_3^\diamond$ *is unimodular.*

PROOF. This follows immediately from proposition A.1.20 as the trace of $\operatorname{ad} A$ (as described in proposition 1.2.21) is zero for $A \in \mathfrak{h}_3^\diamond$. □

1.2.24 PROPOSITION. *The exponential map,* $\exp : \mathfrak{h}_3^\diamond \to \mathsf{H}_3^\diamond$*, is given by*

$$\exp\left(M(x,y,z,\theta)\right) = \begin{bmatrix} 1 & \frac{y(1-\cos\theta)-x\sin\theta}{\theta} & \frac{x(1-\cos\theta)+y\sin\theta}{\theta} & \frac{(x^2+y^2)(\theta-\sin\theta)-2z\theta^2}{\theta^2} \\ 0 & \cos\theta & -\sin\theta & \frac{-x(1-\cos\theta)+y\sin\theta}{\theta} \\ 0 & \sin\theta & \cos\theta & \frac{y(1-\cos\theta)+x\sin\theta}{\theta} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$= m\left(\frac{y(1-\cos\theta)+x\sin\theta}{\theta}, \frac{-x(1-\cos\theta)+y\sin\theta}{\theta}, \frac{(x^2+y^2)(\theta-\sin\theta)-2z\theta^2}{-2\theta^2}, \theta\right).$$

Furthermore $\exp$ *is onto and for*

$$x^* = \frac{\frac{1}{2}\theta x(\cos\theta+1)}{\sin\theta} - \frac{1}{2}y\theta \qquad y^* = \frac{\frac{1}{2}\theta y(\cos\theta+1)}{\sin\theta} + \frac{1}{2}x\theta$$
$$z^* = \frac{4z(\cos\theta-1)+(x^2+y^2)(\sin\theta-\theta)}{4(\cos\theta-1)} \qquad \theta^* = \theta \in [-\pi,\pi]$$

we get that $\exp(M(x^*,y^*,z^*,\theta^*)) = m(x,y,z,\theta)$.

PROOF. We use Mathematica (see section C.1) to calculate the exponential of an element $M(x,y,z,\theta) \in \mathfrak{h}_3^\diamond$. We also use Mathematica to verify that $\exp(M(x^*,y^*,z^*,\theta^*)) = m(x,y,z,\theta)$ (and by extension through limit is valid for $\theta \in \{-\pi, 0, \pi\}$). Note that the exponential map is not injective, as $\exp(M(0,0,0,0)) = \exp(M(0,0,0,2\pi))$. □

1.2.25 REMARK. Note that, as

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1 \qquad \lim_{\theta\to 0}\frac{\cos\theta-1}{\theta} = 0 \qquad \lim_{\theta\to 0}\frac{\sin\theta-\theta}{\theta^2} = 0,$$

the above formula for $\exp\left(M(x,y,z,\theta)\right)$ is valid everywhere (by extension through limit).

1.2.26 PROPOSITION. *The centre of* $\mathfrak{h}_3^\diamond$ *is given by* $\mathrm{Z}\left(\mathfrak{h}_3^\diamond\right) = \{M(0,0,z,0) \mid z \in \mathbb{R}\}$.

PROOF. Indeed we have that

$$\begin{aligned} & m(x,y,z,\theta) \in \mathsf{Z}\left(\mathsf{H}_3^\diamond\right) \\ \Leftrightarrow\ & [M(x,y,z,\theta), M(x',y',z',\theta')] = 0 \text{ for } \forall x',y',z',\theta' \in \mathbb{R} \\ \Leftrightarrow\ & M(\theta y' - y\theta', x\theta' - \theta x', xy' - yx', 0) = 0 \text{ for } \forall x',y',z',\theta' \in \mathbb{R} \\ \Leftrightarrow\ & \left\{\begin{array}{rcl} \theta y' &=& y\theta' \\ x\theta' &=& \theta x' \\ xy' &=& yx' \end{array}\right. \text{ for } \forall x',y',z',\theta' \in \mathbb{R} \quad \Leftrightarrow \quad \left\{\begin{array}{rcl} x &=& 0 \\ y &=& 0 \\ \theta &=& 0. \end{array}\right. \end{aligned}$$
□

1.2.27 PROPOSITION. *The (linear Lie) group of automorphism of* $\mathfrak{h}_3^\diamond$ *is given by*

$$\operatorname{Aut}\mathfrak{h}_3^\diamond = \left\{ \begin{bmatrix} x & y & 0 & u \\ -ky & kx & 0 & v \\ kux - vy & kuy + xv & k(x^2+y^2) & w \\ 0 & 0 & 0 & k \end{bmatrix} \;\middle|\; x, y, u, v, w \in \mathbb{R},\ k \in \{-1, 1\},\ x^2 + y^2 \neq 0 \right\}$$

We note that E_3 is an eigenvector of every automorphism.

PROOF. Let $\psi : \mathfrak{h}_3^\diamond \to \mathfrak{h}_3^\diamond$ be a linear map. Then we can represent ψ as

$$\psi = \begin{bmatrix} x & y & a_3 & u \\ b_1 & b_2 & b_3 & v \\ c_1 & c_2 & c_3 & w \\ d_1 & d_2 & d_3 & d_4 \end{bmatrix}$$

with respect to the ordered basis $\{E_i\}_{i=\overline{1,4}}$. Then requiring ψ to be an automorphism of $\mathfrak{h}_3^\diamond$ implies the following equation (produced in Mathematica, see section C.1)

$$\begin{bmatrix} x & y & a_3 & u \\ b_1 & b_2 & b_3 & v \\ c_1 & c_2 & c_3 & w \\ d_1 & d_2 & d_3 & d_4 \end{bmatrix} = \begin{bmatrix} b_2 d_4 & -b_1 d_4 & 0 & u \\ -y d_4 & x d_4 & 0 & v \\ -yv + u b_2 & xv - u b_1 & x b_2 - y b_1 & w \\ 0 & 0 & 0 & d_4 \end{bmatrix}$$

(from requirement that $\psi \cdot [E_i, E_j] = [\psi \cdot E_i, \psi \cdot E_j]$ for $i, j = \overline{1,4}$) along with the requirement that $\det \psi \neq 0$. Substituting the values for b_1 and b_2 back into our equation we then have that

$$\begin{bmatrix} x & y & a_3 & u \\ b_1 & b_2 & b_3 & v \\ c_1 & c_2 & c_3 & w \\ d_1 & d_2 & d_3 & d_4 \end{bmatrix} = \begin{bmatrix} x d_4^2 & y d_4^2 & 0 & u \\ -y d_4 & x d_4 & 0 & v \\ -yv + ux d_4 & xv + uy d_4 & d_4(x^2+y^2) & w \\ 0 & 0 & 0 & d_4 \end{bmatrix}$$

Then we have that $\det \psi = d_4^5 (x^2 + y^2)^2$ and, as $\det \psi \neq 0$, we require that $d_4 \neq 0$ and either $x \neq 0$ or $y \neq 0$. Hence, considering the equalities from the first two entries of the first row, i.e., $x = x d_4^2$ and $y = y d_4^2$, we get that $d_4 = \pm 1$. Substituting this value for d_4 back we then have that

$$\begin{bmatrix} x & y & a_3 & u \\ b_1 & b_2 & b_3 & v \\ c_1 & c_2 & c_3 & w \\ d_1 & d_2 & d_3 & d_4 \end{bmatrix} = \begin{bmatrix} x & y & 0 & u \\ \mp y & \pm x & 0 & v \\ -yv \pm ux & xv \pm uy & \pm(x^2+y^2) & w \\ 0 & 0 & 0 & \pm 1 \end{bmatrix}$$

as required. Again using Mathematica, we verify that any ψ of the given form is an automorphism. $\square$

1.3 Connected Lie Groups with Lie Algebra $\mathfrak{h}_3^\diamond$

In this section we find the universal covering Lie group of $\mathsf{H}_3^\diamond$ and then proceed to find all connected Lie groups with Lie algebra (isomorphic to) $\mathfrak{h}_3^\diamond$.

1.3.1 The universal covering Lie group

We define mappings $\widetilde{m} : \mathbb{R}^4 \to \mathsf{GL}(5, \mathbb{R})$ and $\widetilde{M} : \mathbb{R}^4 \to \mathbb{R}^{5\times 5}$ (as convenient notation) by

$$\widetilde{m}(x, y, z, \theta) = \begin{bmatrix} m(x, y, z, \theta) & 0 \\ 0 & \mathrm{e}^\theta \end{bmatrix} \qquad \widetilde{M}(x, y, z, \theta) = \begin{bmatrix} M(x, y, z, \theta) & 0 \\ 0 & \theta \end{bmatrix}.$$

Observe that (for all $x, y, z, \theta, x', y', z', \theta' \in \mathbb{R}$)

$$\begin{aligned} \widetilde{m}(0, 0, 0, 0) &= I_5 \\ \widetilde{m}(x, y, z, \theta)^{-1} &= \widetilde{m}\left(-x\cos\theta + y\sin\theta, -x\sin\theta - y\cos\theta, -z, -\theta\right) \\ \widetilde{m}(x, y, z, \theta) &= \widetilde{m}(x, y, z, 0)\, \widetilde{m}(0, 0, 0, \theta) \end{aligned}$$

and that $\widetilde{m}(x, y, z, \theta)\, \widetilde{m}(x', y', z', \theta') \in \{\widetilde{m}(x^*, y^*, z^*, \theta^*) \mid x^*, y^*, z^*, \theta^* \in \mathbb{R}\}$. This is simply a slight generalisation of the calculations made at the beginning of section 1.2.1.

1.3.1 Proposition. *The set $\widetilde{\mathsf{H}}_3^\diamond = \{\widetilde{m}(x, y, z, \theta) \mid x, y, z, \theta \in \mathbb{R}\}$ is a simply connected closed linear Lie group with Lie algebra $\widetilde{\mathfrak{h}}_3^\diamond = \{\widetilde{M}(x, y, z, \theta) \mid x, y, z, \theta \in \mathbb{R}\}$ isomorphic to $\mathfrak{h}_3^\diamond$.*

Proof. In the same way as we proved $\mathsf{H}_3^\diamond$ was topologically closed in $\mathsf{GL}(4, \mathbb{R})$ (proposition 1.2.9), we can show that $\widetilde{\mathsf{H}}_3^\diamond$ is topologically closed in $\mathsf{GL}(5, \mathbb{R})$. The crucial difference is that we use of the universal covering $\widetilde{\mathsf{SO}}(2)$ (diffeomorphic to $\mathbb{R}$) of $\mathsf{SO}(2)$ here, where

$$\widetilde{\mathsf{SO}}(2) \cong \left\{ \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & \mathrm{e}^\theta \end{bmatrix} \;\middle|\; \theta \in \mathbb{R} \right\}.$$

Thus, as $\widetilde{\mathsf{H}}_3^\diamond$ is an abstract group, we have that $\widetilde{\mathsf{H}}_3^\diamond$ is a closed linear Lie group. Similarly to proposition 1.2.10, we have that $\widetilde{m} : \mathbb{R}^4 \to \widetilde{\mathsf{H}}_3^\diamond$ is a diffeomorphism. Hence, as $\mathbb{R}^4$ is simply connected, we have that $\widetilde{\mathsf{H}}_3^\diamond$ is simply connected.

Alternatively, consider the mapping $\phi : \widetilde{\mathsf{H}}_3^\diamond \to \mathbb{R}^4$, $\phi = \mathrm{pr}_{3,4} \times \mathrm{pr}_{2,4} \times -\frac{1}{2}\mathrm{pr}_{1,4} \times \ln \circ\, \mathrm{pr}_{5,5}$ (where $\mathrm{pr}_{i,j}$ denotes the projection to the entry in the i'th row and j'th column). Now this is a bijective mapping and as projections are continuous open mappings, $\ln(\cdot)$ and multiplication by a non-zero constant are homeomorphisms, we have that ϕ is a homeomorphism. Thus as $\mathbb{R}^4$ is connected and simply connected, $\widetilde{\mathsf{H}}_3^\diamond$ is connected and simply connected.

We may show that the Lie algebra of $\widetilde{\mathsf{H}}_3^\diamond$ is given by $\widetilde{\mathfrak{h}}_3^\diamond = \{\widetilde{M}(x, y, z, \theta) \mid x, y, z, \theta \in \mathbb{R}\}$ similarly to proposition 1.2.17. Then one may easily verify that $\psi : \widetilde{\mathfrak{h}}_3^\diamond \to \mathfrak{h}_3^\diamond$, $\widetilde{M}(x, y, z, \theta) \mapsto M(x, y, z, \theta)$ is a Lie algebra isomorphism. □

Let $\widetilde{E}_1 = \widetilde{M}(1,0,0,0) = \widetilde{M}(e_1)$, $\widetilde{E}_2 = \widetilde{M}(e_2)$, $\widetilde{E}_3 = \widetilde{M}(e_3)$ and $\widetilde{E}_4 = \widetilde{M}(e_4)$. Then $\{\widetilde{E}_i\}_{i=\overline{1,4}}$ is a (ordered) basis for $\mathfrak{h}_3^\diamond$ and has commutator relations: $[\widetilde{E}_1, \widetilde{E}_2] = \widetilde{E}_3$, $[\widetilde{E}_1, \widetilde{E}_4] = \widetilde{E}_2$, $[\widetilde{E}_2, \widetilde{E}_4] = -\widetilde{E}_1$ with the rest zero. We note that the linear isomorphism $\psi : \widetilde{\mathfrak{h}}_3^\diamond \to \mathfrak{h}_3^\diamond$, $\widetilde{M}(x,y,z,\theta) \mapsto M(x,y,z,\theta)$ represented as a matrix (w.r.t. the ordered bases $\{\widetilde{E}_i\}_{i=\overline{1,4}}$ and $\{E_i\}_{i=\overline{1,4}}$) is simply I_4. That is $\psi : \widetilde{E}_i \mapsto E_i$. This being the case we have that the group $\operatorname{Aut}\widetilde{\mathfrak{h}}_3^\diamond$ represented as matrices (w.r.t. the ordered bases $\{\widetilde{E}_i\}_{i=\overline{1,4}}$) is identical to the group $\operatorname{Aut}\mathfrak{h}_3^\diamond$ represented as matrices (w.r.t. the ordered bases $\{E_i\}_{i=\overline{1,4}}$). (This follows as $\operatorname{Aut}\widetilde{\mathfrak{h}}_3^\diamond = \psi^{-1} \circ \operatorname{Aut}\mathfrak{h}_3^\diamond \circ \psi$.)

1.3.2 REMARK. The "oscillator group" Osc may alternatively be defined ([11]) as the semi-direct product of $\mathbb{R}$ and the Heisenberg group $\mathsf{H} = \mathbb{R} \times \mathbb{C}$, where $\mathbb{R}$ acts by rotation on $\mathbb{C}$. Specifically we have that $\mathsf{Osc} = \mathbb{R} \times \mathbb{C} \times \mathbb{R}$ with group multiplication given by,

$$(z,c,r)\,(z',c',r') = (z + z' + \tfrac{1}{2}\mathrm{Im}\,(\mathrm{e}^{i(r+r')}\bar{c}c'), \mathrm{e}^{ir}c' + \mathrm{e}^{-ir}c, r + r').$$

We note that Osc is isomorphic to $\widetilde{\mathsf{H}}_3^\diamond$. Indeed we claim that $\phi : \widetilde{\mathsf{H}}_3^\diamond \to \mathbb{R}\times\mathbb{C}\times\mathbb{R}$, $\widetilde{m}(x,y,z,\theta) \mapsto (z, \mathrm{e}^{i\frac{\theta}{2}}(x+iy), -\frac{\theta}{2})$ is the required Lie group isomorphism. We have that ϕ is injective and well defined as

$$(z, \mathrm{e}^{i\frac{\theta}{2}}(x+iy), -\tfrac{\theta}{2}) = (z', \mathrm{e}^{i\frac{\theta'}{2}}(x'+iy'), -\tfrac{\theta'}{2}) \;\Leftrightarrow\; \begin{cases} z = z' \\ \theta = \theta' \\ x + iy = x' + iy'. \end{cases}$$

Furthermore ϕ is surjective as, for $(z, x+iy, \theta) \in \mathbb{R}\times\mathbb{C}\times\mathbb{R}$, $\widetilde{m}(x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta, z, \theta)$ is an element in $\widetilde{\mathsf{H}}_3^\diamond$ that maps to it. Next we show that ϕ is a homomorphism:

$$\begin{aligned}
&\phi(\widetilde{m}(x,y,z,\theta))\,\phi(\widetilde{m}(x',y',z',\theta')) \\
&= (z, \mathrm{e}^{i\frac{\theta}{2}}(x+iy), -\tfrac{\theta}{2})\,(z', \mathrm{e}^{i\frac{\theta'}{2}}(x'+iy'), -\tfrac{\theta'}{2}) \\
&= (z + z' + \tfrac{1}{2}\mathrm{Im}\,(\mathrm{e}^{-i\frac{1}{2}(\theta+\theta')}\mathrm{e}^{-i\frac{\theta}{2}}(x-iy)\mathrm{e}^{i\frac{\theta'}{2}}(x'+iy')), \\
&\qquad \mathrm{e}^{-i\frac{\theta}{2}}\mathrm{e}^{i\frac{\theta'}{2}}(x'+iy') + \mathrm{e}^{i\frac{\theta'}{2}}\mathrm{e}^{i\frac{\theta}{2}}(x+iy), -\tfrac{1}{2}(\theta+\theta')) \\
&= (z + z' + \tfrac{1}{2}\mathrm{Im}\,(\mathrm{e}^{-i\frac{\theta}{2}}(x-iy)(x'+iy')), \\
&\qquad \mathrm{e}^{i\frac{1}{2}(\theta'+\theta)}(\mathrm{e}^{-i\theta}(x'+iy') + x + iy, -\tfrac{1}{2}(\theta+\theta')) \\
&= (z^*, \mathrm{e}^{i\frac{1}{2}(\theta'+\theta)}(x^* + iy^*), -\tfrac{1}{2}(\theta+\theta')) \qquad \text{(as in remark 1.2.2)} \\
&= \phi(\widetilde{m}(x^*, y^*, z^*, \theta+\theta')) \\
&= \phi(\widetilde{m}(x,y,z,\theta)\,\widetilde{m}(x',y',z',\theta')).
\end{aligned}$$

Finally we note that as $\widetilde{m} : \mathbb{R}^4 \to \widetilde{\mathsf{H}}_3^\diamond$ and $\mathbb{R}^4 \to \mathbb{R}\times\mathbb{C}\times\mathbb{R}$, $(x,y,z,\theta) \mapsto (z, \mathrm{e}^{i\frac{\theta}{2}}(x+iy), -\frac{\theta}{2})$ are diffeomorphisms, so is ϕ.

1.3.3 PROPOSITION. *The mapping $\phi : \widetilde{\mathsf{H}}_3^\diamond \to \mathsf{H}_3^\diamond$, $\widetilde{m}(x,y,z,\theta) \mapsto m(x,y,z,\theta)$ is a Lie group covering homomorphism. Specifically,* $\ker\phi = \widetilde{m}(0,0,0,2\pi\mathbb{Z})$.

PROOF. First recall that

$$\widetilde{m}(x, y, z, \theta) = \begin{bmatrix} m(x, y, z, \theta) & 0 \\ 0 & \mathrm{e}^{\theta} \end{bmatrix}.$$

So ϕ is simply a projection and may easily be shown to be a homomorphism. The projection of the helix $\mathbb{R}$ onto the circle $\mathbb{S}$, $p : \mathbb{R} \to \mathbb{R}^2$, $\theta \mapsto (\cos\theta, \sin\theta)$ has $\ker p = 2\pi\mathbb{Z}$. It follows that $\ker\phi = \widetilde{m}(0, 0, 0, 2\pi\mathbb{Z})$. □

1.3.4 PROPOSITION. *The centre of* $\widetilde{\mathsf{H}}_3^\diamond$ *is given by* $\mathrm{Z}\,(\widetilde{\mathsf{H}}_3^\diamond) = \{\widetilde{m}(0, 0, z, \theta) \mid z \in \mathbb{R}, \theta \in 2\pi\mathbb{Z}\}$.

PROOF. We have that $\widetilde{m}(x, y, z, \theta) \in \mathrm{Z}\,(\widetilde{\mathsf{H}}_3^\diamond)$ if and only if for all $x', y', z', \theta' \in \mathbb{R}$

$$\widetilde{m}(x', y', z', \theta')\, \widetilde{m}(x, y, z, \theta)\, \widetilde{m}(x', y', z', \theta')^{-1}\, \widetilde{m}(x, y, z, \theta)^{-1} = I_5.$$

We use Mathematica to facilitate calculations. Setting $\theta' = 0$ and equating entries (of row 3, column 4) we get the equation $x' - x'\cos\theta - y'\sin\theta = 0$ implying that $\theta \in 2\pi\mathbb{Z}$. Then assuming $z' = 0$ and substituting value for θ we get (row 1, column 4) the equation $2(xy' - yx') = 0$, yielding $x = y = 0$. Finally we have that $\widetilde{m}(x', y', z', \theta')\, \widetilde{m}(0, 0, z, \theta)\, \widetilde{m}(x', y', z', \theta')^{-1}\, \widetilde{m}(0, 0, z, \theta)^{-1} = I_5$ for $z \in \mathbb{R}$, $\theta \in 2\pi\mathbb{Z}$ yielding result. □

1.3.5 PROPOSITION. *The exponential map,* $\exp : \widetilde{\mathfrak{h}}_3^\diamond \to \widetilde{\mathsf{H}}_3^\diamond$, *is given by*

$$\exp(\widetilde{M}(x, y, z, \theta)) = \begin{bmatrix} 1 & \frac{y(1-\cos\theta) - x\sin\theta}{\theta} & \frac{x(1-\cos\theta) + y\sin\theta}{\theta} & \frac{\left(x^2+y^2\right)(\theta - \sin\theta) - 2z\theta^2}{\theta^2} & 0 \\ 0 & \cos\theta & -\sin\theta & \frac{-x(1-\cos\theta) + y\sin\theta}{\theta} & 0 \\ 0 & \sin\theta & \cos\theta & \frac{y(1-\cos\theta) + x\sin\theta}{\theta} & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{e}^{\theta} \end{bmatrix},$$

which is valid everywhere by extension through limit. Furthermore exp *is neither injective nor surjective.*

PROOF. This is very similar to the result in proposition 1.2.24. Again we can use Mathematica to calculate the exponential of an element $\widetilde{M}(x, y, z, \theta) \in \widetilde{\mathfrak{h}}_3^\diamond$. Notice that

$$\exp(\widetilde{M}(x, y, z, 2n\pi)) = \widetilde{m}\left(0, 0, -\tfrac{x^2+y^2-4n\pi z}{4n\pi}, 2n\pi\right)$$

for $n \in \mathbb{N}$, $n \neq 0$. Thus we have that $\widetilde{m}(x, y, 0, 2n\pi) \notin \operatorname{im}\exp$ for $x, y \in \mathbb{R}\backslash\{0\}$, i.e., the exponential map is not surjective. Also $\exp(\widetilde{M}(x, y, \frac{x^2+y^2}{4n\pi}, 2n\pi)) = \widetilde{m}\,(0, 0, 0, 2n\pi)$ for $x, y \in \mathbb{R}$, proving that the exponential map is not injective. □

1.3.2 Classification of connected Lie groups with Lie algebra $\mathfrak{h}_3^\diamond$

We classify the discrete subgroups of $\widetilde{\mathsf{H}}_3^\diamond$ and then use this classification to classify all connected Lie groups with Lie Algebra $\mathfrak{h}_3^\diamond$. We conclude the section by investigating which of these Lie groups have faithful linear representations and which ones cover which. However, before we proceed to the classification, we prove a supporting lemma.

1.3.6 Lemma. *Suppose $z_1, z_2 \in \mathbb{R}$, $\theta_1, \theta_2 \in \mathbb{Z}$, $\theta_1 \neq 0$ and $\{(z_1, 2\pi\theta_1), (z_2, 2\pi\theta_2)\} \subset \mathbb{R}^2$ is a linearly independent set. Then there exists $x, w \in \mathbb{R}$, $x \neq 0$ such that*

$$\begin{bmatrix} x^2 & w \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} z_1 \\ 2\pi\theta_1 \end{bmatrix} \mathbb{Z} + \begin{bmatrix} z_2 \\ 2\pi\theta_2 \end{bmatrix} \mathbb{Z} \right) = \begin{bmatrix} 0 \\ 2\pi \gcd(\theta_1, \theta_2) \end{bmatrix} \mathbb{Z} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mathbb{Z}.$$

Proof. First suppose that $\theta_2 = 0$. Then $\gcd(\theta_1, \theta_2) = |\theta_1|$. Let $x^2 = \frac{1}{|z_2|}$ and $w = -\frac{z_1}{2\pi|z_2|\theta_1}$. (We have that $z_2 \neq 0$ by linear independence.) Then

$$\begin{bmatrix} \frac{1}{|z_2|} & -\frac{z_1}{2\pi(|z_2|\theta_1)} \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} z_1 \\ 2\pi\theta_1 \end{bmatrix} \mathbb{Z} + \begin{bmatrix} z_2 \\ 0 \end{bmatrix} \mathbb{Z} \right) = \begin{bmatrix} 0 \\ 2\pi|\theta_1| \end{bmatrix} \mathbb{Z} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mathbb{Z}.$$

On the other hand suppose $\theta_2 \neq 0$. Then, as θ_1 and θ_2 are non-zero integers, there exists integers $a, b \in \mathbb{Z}$ such that $a\theta_1 + b\theta_2 = \gcd(\theta_1, \theta_2) > 0$ by Bézout's identity. Also, as $\{(z_1, 2\pi\theta_1), (z_2, 2\pi\theta_2)\}$ is linearly independent, we have that $z_1\theta_2 - z_2\theta_1 \neq 0$. Let $x^2 = \frac{a\theta_1 + b\theta_2}{|z_1\theta_2 - z_2\theta_1|}$ and $w = -\frac{az_1 + bz_2}{2\pi(|z_1\theta_2 - z_2\theta_1|)}$. We claim that this choice of x^2 and w satisfies the lemma's requirements. (This choice of values for x^2 and w are the result of trying to construct the required linear map; first by reducing $(z_1, 2\pi\theta_1)$ and $(z_2, 2\pi\theta_2)$ to $(0, 2\pi\theta_1)$ and $(1, 2\pi\theta_2)$, respectively, and then attempting to solve for values of x^2 and w such that the two discrete subgroups are equal.) We now prove this claim. We get that

$$\begin{aligned}
&\begin{bmatrix} \frac{a\theta_1 + b\theta_2}{|z_1\theta_2 - z_2\theta_1|} & -\frac{az_1 + bz_2}{2\pi|z_1\theta_2 - z_2\theta_1|} \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} z_1 \\ 2\pi\theta_1 \end{bmatrix} \mathbb{Z} + \begin{bmatrix} z_2 \\ 2\pi\theta_2 \end{bmatrix} \mathbb{Z} \right) \\
&= \begin{bmatrix} \frac{(a\theta_1 + b\theta_2)z_1}{|z_1\theta_2 - z_2\theta_1|} - \frac{(az_1 + bz_2)\theta_1}{|z_1\theta_2 - z_2\theta_1|} \\ 2\pi\theta_1 \end{bmatrix} \mathbb{Z} + \begin{bmatrix} \frac{(a\theta_1 + b\theta_2)z_2}{|z_1\theta_2 - z_2\theta_1|} - \frac{(az_1 + bz_2)\theta_2}{|z_1\theta_2 - z_2\theta_1|} \\ 2\pi\theta_2 \end{bmatrix} \mathbb{Z} \\
&= \begin{bmatrix} b\sigma \\ 2\pi\theta_1 \end{bmatrix} \mathbb{Z} + \begin{bmatrix} -a\sigma \\ 2\pi\theta_2 \end{bmatrix} \mathbb{Z}
\end{aligned}$$

where $\sigma = \operatorname{sgn}(z_1\theta_2 - z_2\theta_1)$.

Now, in order to show that the discrete subgroups $\mathsf{N} = (b\sigma, 2\pi\theta_1)\mathbb{Z} + (-a\sigma, 2\pi\theta_2)\mathbb{Z}$ and $\mathsf{N}' = (0, \gcd(\theta_1, \theta_2))\mathbb{Z} + (1, 0)\mathbb{Z}$ of $(\mathbb{R}^2, +)$ are identical, we show that each one contains the other. This is done by showing that each contains the generators of the other. Note that

$$\begin{aligned}
\begin{bmatrix} b\sigma \\ 2\pi\theta_1 \end{bmatrix} &= \begin{bmatrix} 0 \\ 2\pi \gcd(\theta_1, \theta_2) \end{bmatrix} \frac{\theta_1}{\gcd(\theta_1, \theta_2)} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} b\sigma \\
\begin{bmatrix} -a\sigma \\ 2\pi\theta_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 2\pi \gcd(\theta_1, \theta_2) \end{bmatrix} \frac{\theta_2}{\gcd(\theta_1, \theta_2)} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} (-a\sigma).
\end{aligned}$$

Therefore we get that $\mathsf{N} \subseteq \mathsf{N}'$. On the other hand note that

$$\begin{aligned}
\begin{bmatrix} 0 \\ 2\pi \gcd(\theta_1, \theta_2) \end{bmatrix} &= \begin{bmatrix} b\sigma \\ 2\pi\theta_1 \end{bmatrix} a + \begin{bmatrix} -a\sigma \\ 2\pi\theta_2 \end{bmatrix} b \\
\begin{bmatrix} 1 \\ 0 \end{bmatrix} &= \begin{bmatrix} b\sigma \\ 2\pi\theta_1 \end{bmatrix} \frac{\theta_2\sigma}{\gcd(\theta_1, \theta_2)} + \begin{bmatrix} -a\sigma \\ 2\pi\theta_2 \end{bmatrix} \frac{-\theta_1\sigma}{\gcd(\theta_1, \theta_2)}.
\end{aligned}$$

Thus $\mathsf{N}' \subseteq \mathsf{N}$ and so $\mathsf{N} = \mathsf{N}'$. □

1.3.7 Theorem. *Any non-trivial normal discrete subgroup* N *of* $\widetilde{\mathsf{H}}_3^\diamond$ *is mapped (or related), by an element of* $\operatorname{Aut}\widetilde{\mathsf{H}}_3^\diamond$*, to exactly one of the following discrete subgroups*

$$\begin{aligned} \mathsf{N}_{1,n} &= \{\widetilde{m}(0,0,0,2n\pi\theta) \mid \theta \in \mathbb{Z}\}\,, \quad n \in \mathbb{N} \\ \mathsf{N}_2 &= \{\widetilde{m}(0,0,z,0) \mid z \in \mathbb{Z}\} \\ \mathsf{N}_{1,n} \oplus \mathsf{N}_2 &= \{\widetilde{m}(0,0,z,2n\pi\theta) \mid \theta, z \in \mathbb{Z}\}\,, \quad n \in \mathbb{N}. \end{aligned}$$

Proof. Note that any normal discrete subgroup of a connected Lie group is contained in its centre (proposition A.1.9). Now notice that $\mathsf{Z} = \{\widetilde{m}(0,0,z,\theta) \mid z,\theta \in \mathbb{R}\}$ is an abelian subgroup of $\widetilde{\mathsf{H}}_3^\diamond$. (Indeed, $\widetilde{m}(0,0,z,\theta)\,\widetilde{m}(0,0,z',\theta') = \widetilde{m}(0,0,z+z',\theta+\theta')$.) Moreover we have that $\Phi : \mathsf{Z} \to (\mathbb{R}^2,+)$, $\widetilde{m}(0,0,z,\theta) \to (z,\theta)$ is a Lie group isomorphism. Now as $\mathsf{Z}(\widetilde{\mathsf{H}}_3^\diamond) \subset \mathsf{Z}$, we have that any discrete subgroup N of $\mathsf{Z}(\widetilde{\mathsf{H}}_3^\diamond)$, is a discrete subgroup of Z and hence, by theorem A.1.11, is of the form $\Phi(\mathsf{N}) = (z_1, 2\pi\theta_1)\mathbb{Z} + (z_2, 2\pi\theta_2)\mathbb{Z}$ for some $z_1, z_2 \in \mathbb{R}$ and $\theta_1, \theta_2 \in \mathbb{Z}$. To simplify notation we will write $r\mathsf{N}$ for $\Phi^{-1}(r\Phi(\mathsf{N})) = \Phi^{-1}\big((z_1, 2\theta_1\pi)r\mathbb{Z} + (z_2, 2\theta_2\pi)r\mathbb{Z}\big)$. Notice that in this notation $\mathsf{N}_{1,n} = n\mathsf{N}_{1,1}$.

Now observe that $\mathsf{N}_{1,n}$, N_2 and $\mathsf{N}_{1,n} \oplus \mathsf{N}_2$ are discrete central subgroups. We additionally define the following family of discrete central subgroups

$$\mathsf{N}_{3,n} = \{\widetilde{m}(0,0,z,2n\pi z) \mid z \in \mathbb{Z}\}\,, \quad n \in \mathbb{Z}\backslash\{0\}.$$

We note that $d\operatorname{Aut}\widetilde{\mathsf{H}}_3^\diamond = \operatorname{Aut}\mathfrak{h}_3^\diamond$ (as $\widetilde{\mathsf{H}}_3^\diamond$ is simply connected) and that the restriction

$$\begin{aligned} \exp : \{\widetilde{M}(0,0,z,\theta) \mid z,\theta \in \mathbb{R}\} &\to \{\widetilde{m}(0,0,z,\theta) \mid z,\theta \in \mathbb{R}\} \\ \widetilde{M}(0,0,z,\theta) &\mapsto \widetilde{m}(0,0,z,\theta) \end{aligned}$$

of the exponential map is bijective. Thus an automorphism $\phi \in \operatorname{Aut}\widetilde{\mathsf{H}}_3^\diamond$, acting on an element in $\{\widetilde{m}(0,0,z,\theta) \mid z,\theta \in \mathbb{R}\}$, satisfies

$$\phi(\widetilde{m}(0,0,z,\theta)) = \exp(T_{\mathbf{1}}\phi \cdot \widetilde{M}(0,0,z,\theta)).$$

We recall (from proposition 1.2.27 and preceding section) that the general form of an automorphism $\psi \in \operatorname{Aut}\widetilde{\mathfrak{h}}_3^\diamond$, w.r.t. the ordered basis $\{\widetilde{E}_i\}_{i=\overline{1,4}}$, is

$$\psi = \begin{bmatrix} x & y & 0 & u \\ -ky & kx & 0 & v \\ kux - vy & kuy + xv & k(x^2+y^2) & w \\ 0 & 0 & 0 & k \end{bmatrix}$$

where $x, y, u, v, w \in \mathbb{R}$, $k \in \{-1, 1\}$ and $x^2 + y^2 \neq 0$.

First we show that $r\mathsf{N}_2 = \Phi^{-1}(r\,\Phi(\mathsf{N}_2)) = \{\widetilde{m}(0,0,rz,0) \mid z \in \mathbb{Z}\}$ and N_2 are related by an element of $\operatorname{Aut}\widetilde{\mathsf{H}}_3^\diamond$ for $r \in \mathbb{R}\backslash\{0\}$. Consider the automorphism ϕ with tangent map

$$T_{\mathbf{1}}\phi = \operatorname{diag}\left(\sqrt{\operatorname{sgn}(r)r}, \operatorname{sgn}(r)\sqrt{\operatorname{sgn}(r)r}, r, \operatorname{sgn}(r)\right).$$

Then, for $z \in \mathbb{Z}$, we get that $\phi(\widetilde{m}(0,0,z,0)) = \widetilde{m}(0,0,rz,0) \in r\mathsf{N}_2$ and so $\phi(\mathsf{N}_2) \subseteq r\mathsf{N}_2$. Similarly $\phi^{-1}(r\mathsf{N}_2) \subseteq \mathsf{N}_2$. Hence we get that $\phi(\mathsf{N}_2) = r\mathsf{N}_2$.

Next, for $n \in \mathbb{N}$ and $k \in \{-1, 1\}$, we claim that $\mathsf{N}_{1,n}$ and $\mathsf{N}_{3,kn}$ are related by an element of $\operatorname{Aut} \widetilde{\mathsf{H}}_3^\diamond$. Consider an automorphism ϕ defined by

$$T_{\mathbf{1}}\phi = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & k & 0 & 0 \\ 0 & 0 & k & \frac{1}{2n\pi} \\ 0 & 0 & 0 & k \end{bmatrix}.$$

Then, for $\widetilde{m}(0,0,0,2n\pi z) \in \mathsf{N}_{1,n}$, we get that

$$\phi(\widetilde{m}(0,0,0,2n\pi z)) = \widetilde{m}(0,0,z,2kn\pi z) \in \mathsf{N}_{3,kn}.$$

Hence $\phi(\mathsf{N}_{1,n}) \subseteq \mathsf{N}_{3,kn}$. Similarly $\phi^{-1}(\mathsf{N}_{3,kn}) \subseteq \mathsf{N}_{1,n}$. Thus $\phi(\mathsf{N}_{1,n}) = \mathsf{N}_{3,kn}$

Now let N be any non-trivial central discrete subgroup of $\widetilde{\mathsf{H}}_3^\diamond$. Recall that we have subgroups $\mathsf{H}_3 = \{\widetilde{m}(x,y,z,0) \mid x,y,z \in \mathbb{R}\}$ and $\widetilde{\mathsf{SO}}(2) = \{\widetilde{m}(0,0,0,\theta) \mid \theta \in \mathbb{R}\}$ of $\widetilde{\mathsf{H}}_3^\diamond$. We use these subgroups to distinguish between various cases.

Case: $\mathsf{N} \cap \mathsf{H}_3 = \{\mathbf{1}\}$

In this case, as $\mathrm{Z}(\widetilde{\mathsf{H}}_3^\diamond) = \{\widetilde{m}(0,0,z,\theta) \mid z \in \mathbb{R}, \theta \in 2\pi\mathbb{Z}\}$, we have that $\mathsf{N} \subseteq \mathsf{N}_{1,1}$, or more generally $\mathsf{N} \subseteq \{\widetilde{m}(0,0,0,\theta), \theta \in \mathbb{R}\}$. Thus we get that $\Phi(\mathsf{N}) = (0,a)2\pi\mathbb{Z}$, for some $a \in \mathbb{R}$, $a \neq 0$ (otherwise N is trivial). That is to say $\mathsf{N} = a\mathsf{N}_{1,1}$. But as $\mathsf{N} \subseteq \mathsf{N}_{1,1}$ we require that $a \in \mathbb{Z}$. Therefore we have that $\mathsf{N} = \mathsf{N}_{1,n}$, where $n = |a|$.

Case: $\mathsf{N} \cap \mathsf{H}_3 \neq \{\mathbf{1}\}$ **and** $\mathsf{N} \cap \widetilde{\mathsf{SO}}(2) = \{\mathbf{1}\}$

As $\mathrm{Z}(\widetilde{\mathsf{H}}_3^\diamond) = \{\widetilde{m}(0,0,z,\theta) \mid z \in \mathbb{R}, \theta \in 2\pi\mathbb{Z}\}$, we have that $\mathsf{N} \subseteq \{\widetilde{m}(0,0,z,0), z \in \mathbb{R}\}$. Thus we get that $\Phi(\mathsf{N}) = (a,0)\mathbb{Z}$, for some $a \in \mathbb{R}$ (with $a \neq 0$ as $\mathsf{N} \cap \mathsf{H}_3 \neq \{\mathbf{1}\}$). That is to say $\mathsf{N} = a\mathsf{N}_2$. Thus (as $r\mathsf{N}_2 \sim \mathsf{N}_2$ for $r \in \mathbb{R}\backslash\{0\}$) we have that N is related to N_2.

Case: $\mathsf{N} \cap \mathsf{H}_3 \neq \{\mathbf{1}\}$ **and** $\mathsf{N} \cap \widetilde{\mathsf{SO}}(2) \neq \{\mathbf{1}\}$

We have that $\mathsf{N} \cap \mathsf{H}_3$ is a non-trivial discrete subgroup of $\{\widetilde{m}(0,0,z,0), z \in \mathbb{R}\}$ and thus (like above) must be of the form $\mathsf{N} \cap \mathsf{H}_3 = r\mathsf{N}_2$, for some $r \in \mathbb{R}$, $r \neq 0$. But, as explained before, there exists $\phi \in \operatorname{Aut} \widetilde{\mathsf{H}}_3^\diamond$ with tangent map

$$T_{\mathbf{1}}\phi = \operatorname{diag}\left(\sqrt{\tfrac{\operatorname{sgn}(r)}{r}}, \operatorname{sgn}(r)\sqrt{\tfrac{\operatorname{sgn}(r)}{r}}, \tfrac{1}{r}, \operatorname{sgn}(r)\right),$$

such that $\phi(r\mathsf{N}_2) = \mathsf{N}_2$. Moreover, note that $\phi(\mathsf{H}_3) = \mathsf{H}_3$ (as H_3 is connected and $T_{\mathbf{1}}\phi$ preserves its Lie algebra $\mathfrak{h}_3$). Hence we get that $\phi(\mathsf{N}) \cap \mathsf{H}_3 = \mathsf{N}_2$. Next, as ϕ preserves $\mathrm{Z}(\widetilde{\mathsf{H}}_3^\diamond)$ and $\phi(\mathsf{N}) \cap \widetilde{\mathsf{SO}}(2) \neq \{\mathbf{1}\}$, we have that $\phi(\mathsf{N}) \cap \widetilde{\mathsf{SO}}(2) = \mathsf{N}_{1,n}$ for some $n \in \mathbb{N}$ as discussed in the first case. Now note that $1 \leq \dim \operatorname{span} \Phi(\phi(\mathsf{N})) \leq 2$.

We consider the case $\dim \operatorname{span} \Phi(\phi(\mathsf{N})) = 1$ first. As $\phi(\mathsf{N}) \cap \mathsf{H}_3 = \mathsf{N}_2$, there exists a non-zero integer θ such that $\widetilde{m}(0,0,1,2\pi\theta) \in \phi(\mathsf{N})$ and $\operatorname{span} \Phi(\phi(\mathsf{N})) = \operatorname{span} \Phi(\widetilde{m}(0,0,1,2\pi\theta))$. We claim that $\Phi(\phi(\mathsf{N})) = (1, 2\pi\theta)\mathbb{Z}$. As $\widetilde{m}(0,0,1,2\pi\theta) \in \phi(\mathsf{N})$ we have (by using "additive" inverse and product in $Z(\widetilde{\mathsf{H}}_3^\diamond)$ repeatedly) that $(1, 2\pi\theta)\mathbb{Z} \subseteq \Phi(\phi(\mathsf{N}))$. Conversely if $g \in \phi(\mathsf{N})$, then $\Phi(g) = (a, 2\pi b)$ for some $a, b \in \mathbb{Z}$. Then as $\operatorname{span}\Phi(\phi(\mathsf{N})) = \operatorname{span}\Phi(\widetilde{m}(0,0,1,2\pi\theta)) = \operatorname{span}(1, 2\pi\theta)$, there exists an $r \in \mathbb{R}$ such that $(a, 2\pi b) = r(1, 2\pi\theta)$ implying $r = a \in \mathbb{Z}$ and thus that $(a, 2\pi b) \in (1, 2\pi\theta)\mathbb{Z}$. That is to say $\Phi(\phi(\mathsf{N})) \subseteq (1, 2\pi\theta)\mathbb{Z}$. Thus we have that $\Phi(\phi(\mathsf{N})) = (1, 2\pi\theta)\mathbb{Z}$. Next, as $\phi(\mathsf{N}) \cap \widetilde{\mathsf{SO}}(2) = \mathsf{N}_{1,n}$ we get that

$$\Phi\big(\phi(\mathsf{N}) \cap \widetilde{\mathsf{SO}}(2)\big) = (0, 2\pi\theta)\mathbb{Z} = (0, 2n\pi)\mathbb{Z} = \Phi(\mathsf{N}_{1,n}).$$

Therefore $k\theta = n$ for some $k \in \{-1, 1\}$. So we have that $\phi(\mathsf{N}) = \Phi^{-1}\big((1, 2\pi kn)\mathbb{Z}\big) = \mathsf{N}_{3,kn}$. Hence we conclude that N is mapped to $\mathsf{N}_{1,n}$ by an element of $\operatorname{Aut}\widetilde{\mathsf{H}}_3^\diamond$.

We now consider the case $\dim\operatorname{span}\Phi(\phi(\mathsf{N})) = 2$. We have that $\phi(\mathsf{N})$ is a discrete subgroup of $Z(\widetilde{\mathsf{H}}_3^\diamond)$. Thus we have that

$$\Phi(\phi(\mathsf{N})) = (z_1, 2\pi\theta_1)\mathbb{Z} + (z_2, 2\pi\theta_2)\mathbb{Z}$$

for some $z_1, z_2 \in \mathbb{R}$ and $\theta_1, \theta_2 \in \mathbb{Z}$. Now not both θ_1 and θ_2 can be zero (otherwise it would contradict the assumption $\dim\operatorname{span}\Phi(\phi(\mathsf{N})) = 2$). Hence w.l.o.g. we assume that $\theta_1 \neq 0$. Now consider an automorphism ϕ' with tangent map

$$T_{\mathbf{1}}\phi' = \begin{bmatrix} x & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & x^2 & w \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Next observe that

$$\Phi\big((\phi' \circ \phi)(\mathsf{N})\big) = \begin{bmatrix} x^2 & w \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} z_1 \\ 2\pi\theta_1 \end{bmatrix}\mathbb{Z} + \begin{bmatrix} z_2 \\ 2\pi\theta_2 \end{bmatrix}\mathbb{Z} \right).$$

Hence, by lemma 1.3.6, we may choose x and w such that

$$\Phi\big((\phi' \circ \phi)(\mathsf{N})\big) = (0, \gcd(\theta_1, \theta_2))\mathbb{Z} + (1, 0)\mathbb{Z} = \Phi(\mathsf{N}_{1,\gcd(\theta_1,\theta_2)} \oplus \mathsf{N}_2).$$

That is to say $(\phi' \circ \phi)(\mathsf{N}) = \mathsf{N}_{1,n} \oplus \mathsf{N}_2$, where $n = \gcd(\theta_1, \theta_2)$.

Thus we have shown that any discrete central subgroup is related to one of $\mathsf{N}_{1,n}$, N_2, $\mathsf{N}_{1,n} \oplus \mathsf{N}_2$ for some $n \in \mathbb{N}$. We are left to show that none of the listed discrete central subgroups are related to one another by an element of $\operatorname{Aut}\widetilde{\mathsf{H}}_3^\diamond$.

$(n \neq m \wedge n, m \in \mathbb{N}) \Rightarrow \mathsf{N}_{1,n} \nsim \mathsf{N}_{1,m}$: Suppose there exists an automorphism $\phi \in \operatorname{Aut}\widetilde{\mathsf{H}}_3^\diamond$ such that $\phi(\mathsf{N}_{1,n}) = \mathsf{N}_{1,m}$. Then we have the following sequence of implications

$$\begin{aligned} \phi(\mathsf{N}_{1,n}) = \mathsf{N}_{1,m} \quad &\Rightarrow \quad \exp(T_{\mathbf{1}}\phi \cdot \widetilde{M}(0,0,0,2n\pi\mathbb{Z}))) = \mathsf{N}_{1,m} \\ &\Rightarrow \quad T_{\mathbf{1}}\phi \cdot \widetilde{M}(0,0,0,2n\pi\mathbb{Z}) = \widetilde{M}(0,0,0,2m\pi\mathbb{Z}) \\ &\Rightarrow \quad 2n\pi k\mathbb{Z} = 2m\pi\mathbb{Z}, \quad k \in \{-1, 1\} \\ &\Rightarrow \quad n\mathbb{Z} = m\mathbb{Z} \\ &\Rightarrow \quad n = m \end{aligned}$$

providing a contradiction.

$\mathsf{N}_{1,n} \not\sim \mathsf{N}_2$, $n \in \mathbb{N}$: Suppose there exists an automorphism ϕ such that $\phi(\mathsf{N}_{1,n}) = \mathsf{N}_2$. Then we get that

$$\begin{aligned}\phi(\mathsf{N}_{1,n}) = \mathsf{N}_2 \quad &\Rightarrow \quad \exp(T_{\mathbf{1}}\phi \cdot \widetilde{M}(0,0,0,2n\pi\mathbb{Z}))) = \mathsf{N}_2 \\ &\Rightarrow \quad T_{\mathbf{1}}\phi \cdot \widetilde{M}(0,0,0,2n\pi\mathbb{Z}) = \widetilde{M}(0,0,\mathbb{Z},0) \\ &\Rightarrow \quad 2n\pi k\mathbb{Z} = 0, \quad k \in \{-1,1\}\end{aligned}$$

providing a contradiction.

$\mathsf{N}_{1,n} \not\sim \mathsf{N}_{1,m} \oplus \mathsf{N}_2$, $n,m \in \mathbb{N}$: Suppose there exists an automorphism ϕ such that $\phi(\mathsf{N}_{1,n}) = \mathsf{N}_{1,m} \oplus \mathsf{N}_2$. Then

$$\begin{aligned}\phi(\mathsf{N}_{1,n}) = \mathsf{N}_{1,m} \oplus \mathsf{N}_2 \quad &\Rightarrow \quad \exp(T_{\mathbf{1}}\phi \cdot \widetilde{M}(0,0,0,2n\pi\mathbb{Z}))) = \mathsf{N}_{1,m} \oplus \mathsf{N}_2 \\ &\Rightarrow \quad T_{\mathbf{1}}\phi \cdot \widetilde{M}(0,0,0,2n\pi\mathbb{Z}) = \widetilde{M}(0,0,\mathbb{Z},2m\pi\mathbb{Z}).\end{aligned}$$

But $\dim \operatorname{span} \Phi(T_{\mathbf{1}}\phi \cdot \widetilde{M}(0,0,0,2n\pi\mathbb{Z})) = 1$ and $\dim \operatorname{span} \Phi(\widetilde{M}(0,0,\mathbb{Z},2m\pi\mathbb{Z})) = 2$, a contradiction.

$\mathsf{N}_2 \not\sim \mathsf{N}_{1,n} \oplus \mathsf{N}_2$, $n \in \mathbb{N}$: Suppose there exists an automorphism ϕ such that $\phi(\mathsf{N}_2) = \mathsf{N}_{1,n} \oplus \mathsf{N}_2$. Then

$$\begin{aligned}\phi(\mathsf{N}_2) = \mathsf{N}_{1,n} \oplus \mathsf{N}_2 \quad &\Rightarrow \quad \exp(T_{\mathbf{1}}\phi \cdot \widetilde{M}(0,0,\mathbb{Z},0))) = \mathsf{N}_{1,n} \oplus \mathsf{N}_2 \\ &\Rightarrow \quad T_{\mathbf{1}}\phi \cdot \widetilde{M}(0,0,\mathbb{Z},0) = \widetilde{M}(0,0,\mathbb{Z},2n\pi\mathbb{Z}).\end{aligned}$$

Now note that $\dim \operatorname{span} \Phi(T_{\mathbf{1}}\phi \cdot \widetilde{M}(0,0,\mathbb{Z},0)) = 1$ and $\dim \operatorname{span} \Phi(\widetilde{M}(0,0,\mathbb{Z},2n\pi\mathbb{Z})) = 2$, yielding a contradiction.

$(n \neq m \wedge n,m \in \mathbb{N}) \;\Rightarrow\; \mathsf{N}_{1,n} \oplus \mathsf{N}_2 \not\sim \mathsf{N}_{1,m} \oplus \mathsf{N}_2$: Suppose there exists an automorphism ϕ such that $\phi(\mathsf{N}_{1,n} \oplus \mathsf{N}_2) = \mathsf{N}_{1,m} \oplus \mathsf{N}_2$. Then

$$\begin{aligned}\phi(\mathsf{N}_{1,n} \oplus \mathsf{N}_2) = \mathsf{N}_{1,m} \oplus \mathsf{N}_2 \quad &\Rightarrow \quad \exp(T_{\mathbf{1}}\phi \cdot \widetilde{M}(0,0,\mathbb{Z},2n\pi\mathbb{Z}))) = \mathsf{N}_{1,m} \oplus \mathsf{N}_2 \\ &\Rightarrow \quad T_{\mathbf{1}}\phi \cdot \widetilde{M}(0,0,\mathbb{Z},2n\pi\mathbb{Z}) = \widetilde{M}(0,0,\mathbb{Z},2m\pi\mathbb{Z}) \\ &\Rightarrow \quad 2n\pi k\mathbb{Z} = 2m\pi\mathbb{Z}, \quad k \in \{-1,1\} \\ &\Rightarrow \quad n\mathbb{Z} = m\mathbb{Z} \\ &\Rightarrow \quad n = m\end{aligned}$$

providing a contradiction. □

1.3.8 COROLLARY. *There are four types of connected Lie groups (up to isomorphism) with Lie algebra (isomorphic to) $\mathfrak{h}_3^\diamond$, namely the universal covering group $\widetilde{\mathsf{H}}_3^\diamond$, the n-fold coverings $\mathsf{H}_3^\diamond(n)$ of $\mathsf{H}_3^\diamond = \widetilde{\mathsf{H}}_3^\diamond/\mathsf{N}_{1,1}$, $\widetilde{\mathsf{H}}_3^\diamond/\mathsf{N}_2$ and the n-fold coverings of $\widetilde{\mathsf{H}}_3^\diamond/(\mathsf{N}_{1,1} \oplus \mathsf{N}_2)$. That is, we have the following list of groups (where $n \in \mathbb{N}$ parametrises families of Lie groups)*

$$\widetilde{\mathsf{H}}_3^\diamond, \qquad \mathsf{H}_3^\diamond(n) \cong \widetilde{\mathsf{H}}_3^\diamond/\mathsf{N}_{1,n}, \qquad \widetilde{\mathsf{H}}_3^\diamond/\mathsf{N}_2, \qquad \widetilde{\mathsf{H}}_3^\diamond/(\mathsf{N}_{1,n} \oplus \mathsf{N}_2).$$

The above corollary simply follows from theorem A.1.8 and proposition A.1.10. Having classified all connected Lie groups with Lie algebra (isomorphic to) $\mathfrak{h}_3^\diamond$, we now move on to considering which of them have faithful linear representation.

1.3.9 LEMMA. *The commutator subgroup of* $\widetilde{\mathsf{H}}_3^\diamond$ *(and its Lie algebra) is given by*

$$\left(\widetilde{\mathsf{H}}_3^\diamond, \widetilde{\mathsf{H}}_3^\diamond\right) = \{\widetilde{m}(x,y,z,0) \mid x,y,z \in \mathbb{R}\}, \qquad \left[\widetilde{\mathfrak{h}}_3^\diamond, \widetilde{\mathfrak{h}}_3^\diamond\right] = \{\widetilde{M}(x,y,z,0) \mid x,y,z \in \mathbb{R}\}.$$

PROOF. In the proof of proposition 1.2.19 we showed that $[\mathfrak{h}_3^\diamond, \mathfrak{h}_3^\diamond] = \{M(x,y,z,0) \mid x,y,z \in \mathbb{R}\} = \mathfrak{h}_3$. Thus, using the covering homomorphism $q : \widetilde{\mathsf{H}}_3^\diamond \to \mathsf{H}_3^\diamond$, $\widetilde{m}(x,y,z,\theta) \mapsto m(x,y,z,\theta)$ (proposition 1.3.3) and noting that $T_{\mathbf{1}}q$ is a Lie algebra isomorphism, we may conclude that $[\widetilde{\mathfrak{h}}_3^\diamond, \widetilde{\mathfrak{h}}_3^\diamond] = \{\widetilde{M}(x,y,z,0) \mid x,y,z \in \mathbb{R}\}$. Now, by theorem A.1.29, the commutator subgroup $(\widetilde{\mathsf{H}}_3^\diamond, \widetilde{\mathsf{H}}_3^\diamond)$ is the unique connected Lie subgroup with Lie algebra $[\widetilde{\mathfrak{h}}_3^\diamond, \widetilde{\mathfrak{h}}_3^\diamond]$. Hence $(\widetilde{\mathsf{H}}_3^\diamond, \widetilde{\mathsf{H}}_3^\diamond) = \{\widetilde{m}(x,y,z,0) \mid x,y,z \in \mathbb{R}\} \cong \mathsf{H}_3$. □

1.3.10 PROPOSITION. *The Lie groups* $\widetilde{\mathsf{H}}_3^\diamond$ *and* $\widetilde{\mathsf{H}}_3^\diamond/\mathsf{N}_{1,n}$ *admit faithful (finite-dimensional) linear representations, but* $\widetilde{\mathsf{H}}_3^\diamond/\mathsf{N}_2$ *and* $\widetilde{\mathsf{H}}_3^\diamond/(\mathsf{N}_{1,n} \oplus \mathsf{N}_2)$, *do not (admit faithful linear representations).*

PROOF. We apply the result of theorem A.1.31. As $(\widetilde{\mathsf{H}}_3^\diamond, \widetilde{\mathsf{H}}_3^\diamond) = \{\widetilde{m}(x,y,z,0) \mid x,y,z \in \mathbb{R}\} \cong \mathsf{H}_3$ is simply connected, $\widetilde{\mathsf{H}}_3^\diamond$ admits a faithful linear representation. Suppose N is a discrete central subgroup and $q : \widetilde{\mathsf{H}}_3^\diamond \to \widetilde{\mathsf{H}}_3^\diamond/\mathsf{N}$ is the corresponding covering homomorphism. Let $\widetilde{\mathsf{H}}_3 = (\widetilde{\mathsf{H}}_3^\diamond, \widetilde{\mathsf{H}}_3^\diamond)$. Then we have that

$$\begin{aligned}
\left(\widetilde{\mathsf{H}}_3^\diamond/\mathsf{N}, \widetilde{\mathsf{H}}_3^\diamond/\mathsf{N}\right) &= \{(q(\widetilde{m}(x_1,y_1,z_1,\theta_1)), q(\widetilde{m}(x_2,y_2,z_2,\theta_2))) \mid x_i,y_i,z_i,\theta_i \in \mathbb{R}, i = \overline{1,2}\} \\
&= \{q((\widetilde{m}(x_1,y_1,z_1,\theta_1)), \widetilde{m}(x_2,y_2,z_2,\theta_2)) \mid x_i,y_i,z_i,\theta_i \in \mathbb{R}, i = \overline{1,2}\} \\
&= q\left(\widetilde{\mathsf{H}}_3^\diamond, \widetilde{\mathsf{H}}_3^\diamond\right) \\
&= q(\widetilde{\mathsf{H}}_3) \\
&\cong \widetilde{\mathsf{H}}_3/(\widetilde{\mathsf{H}}_3 \cap \mathsf{N}).
\end{aligned}$$

Now, as $\widetilde{\mathsf{H}}_3$ is simply connected, the fundamental group of $\widetilde{\mathsf{H}}_3/(\widetilde{\mathsf{H}}_3 \cap \mathsf{N})$ is isomorphic to $\widetilde{\mathsf{H}}_3 \cap \mathsf{N}$ (see theorem A.1.12). Thus $(\widetilde{\mathsf{H}}_3^\diamond/\mathsf{N}, \widetilde{\mathsf{H}}_3^\diamond/\mathsf{N})$ is simply connected if and only if $\widetilde{\mathsf{H}}_3 \cap \mathsf{N} = \{\mathbf{1}\}$. The result then follows, as

$$\widetilde{\mathsf{H}}_3 \cap \mathsf{N}_{1,n} = \{\mathbf{1}\} \qquad \widetilde{\mathsf{H}}_3 \cap \mathsf{N}_2 = \mathsf{N}_2 \qquad \widetilde{\mathsf{H}}_3 \cap (\mathsf{N}_{1,n} \oplus \mathsf{N}_2) = \mathsf{N}_2.$$

□

We have the following linear representation for the **n-fold covers $\mathsf{H}_3^\diamond(n)$** $\cong \widetilde{\mathsf{H}}_3^\diamond/\mathsf{N}_{1,n}$ of $\mathsf{H}_3^\diamond \cong \widetilde{\mathsf{H}}_3^\diamond/\mathsf{N}_{1,1}$ and their Lie algebras (which we will denote $\mathfrak{h}_3^\diamond(n)$):

$$\mathsf{H}_3^\diamond(n) = \left\{ \begin{bmatrix} 1 & -x\cos\theta + y\sin\theta & x\sin\theta + y\cos\theta & -2z & 0 \\ 0 & \cos\theta & -\sin\theta & y & 0 \\ 0 & \sin\theta & \cos\theta & x & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{e}^{\frac{i\theta}{n}} \end{bmatrix} = m_n(x,y,z,\theta) \;\middle|\; x,y,z,\theta \in \mathbb{R} \right\}$$

$$\mathfrak{h}_3^\diamond(n) = \left\{ \begin{bmatrix} 1 & -x & y & -2z & 0 \\ 0 & 0 & -\theta & y & 0 \\ 0 & \theta & 0 & x & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{i\theta}{n} \end{bmatrix} = M_n(x,y,z,\theta) \;\middle|\; x,y,z,\theta \in \mathbb{R} \right\}.$$

Here $m_n : \mathbb{R}^3 \to \mathsf{H}_3^\diamond(n)$ and $M_n : \mathbb{R}^3 \to \mathfrak{h}_3^\diamond(n)$ are parametrisations used to simplify notation. Proving that $\mathsf{H}_3^\diamond(n)$, as given above, is a linear Lie group (for $n \in \mathbb{N}$) and has specified Lie algebra is almost identical to what we did for $\mathsf{H}_3^\diamond \cong \mathsf{H}_3^\diamond(1)$. When working with $\mathfrak{h}_3^\diamond(n)$, we will use $E_1 = M_n(1,0,0,0)$, $E_2 = M_n(0,1,0,0)$, $E_3 = M_n(0,0,1,0)$ and $E_4 = M_n(0,0,0,1)$ as our ordered basis.

1.3.11 REMARK. As we have restricted ourselves to subgroups of $\mathsf{GL}(n,\mathbb{R})$, we consider $\mathrm{e}^{\frac{i\theta}{n}}$ simply as convenient notation for $\begin{bmatrix} \cos\frac{\theta}{n} & -\sin\frac{\theta}{n} \\ \sin\frac{\theta}{n} & \cos\frac{\theta}{n} \end{bmatrix}$. That is, an arbitrary element $m_n(x,y,z,\theta) \in \mathsf{H}_3^\diamond(n)$ ought to be represented as

$$m_n(x,y,z,\theta) = \begin{bmatrix} m(x,y,z,\theta) & 0 & 0 \\ 0 & \cos\frac{\theta}{n} & -\sin\frac{\theta}{n} \\ 0 & \sin\frac{\theta}{n} & \cos\frac{\theta}{n} \end{bmatrix}.$$

We have that $q : \widetilde{\mathsf{H}}_3^\diamond \to \mathsf{H}_3^\diamond(n)$, $\widetilde{m}(x,y,z,\theta) \mapsto m_n(x,y,z,\theta)$ is a covering homomorphism (the same argument as in proposition 1.3.3 can be made). Note that $\ker q = \mathsf{N}_{1,n}$. Thus (by theorem A.1.5) we have that $\widetilde{\mathsf{H}}_3^\diamond/\mathsf{N}_{1,n} \cong \mathsf{H}_3^\diamond(n)$. Furthermore note that $T_{\mathbf{1}}q \cdot \widetilde{M}(x,y,z,\theta) = M_n(x,y,z,\theta)$. We also note that (w.r.t. the respective bases) the covering homomorphism q defined above and the covering homomorphism in proposition 1.3.12, is simply represented as I_4. The exponential map is given by

$$\exp : \mathfrak{h}_3^\diamond(n) \to \mathsf{H}_3^\diamond(n), \qquad M_n(x,y,z,\theta) \mapsto q(\exp(\widetilde{M}(x,y,z,\theta))).$$

Also, note (by use of proposition A.1.13) that

$$\mathsf{Z}(\mathsf{H}_3^\diamond(n)) = q(\mathsf{Z}(\widetilde{\mathsf{H}}_3^\diamond)) = \{m_n(0,0,z,\theta) \mid z \in \mathbb{R}, \theta \in 2\pi\mathbb{Z}\}.$$

Next we investigate which of the groups listed in corollary 1.3.8 cover which.

1.3.12 PROPOSITION. *$\mathsf{H}_3^\diamond(n)$ is an n-fold cover of $\mathsf{H}_3^\diamond \cong \mathsf{H}_3^\diamond(1)$. Specifically, we have that $q : \mathsf{H}_3^\diamond(n) \to \mathsf{H}_3^\diamond$, $m_n(x,y,z,\theta) \mapsto m(x,y,z,\theta)$ is a Lie group covering homomorphism such that $\ker q = m_n(0,0,0,\frac{2\pi}{n}\mathbb{Z})$, $|\ker q| = n$.*

PROOF. We can interpret q as the projection to the top left 4×4 block of elements. Thus we have that it is smooth and can easily verify that it is an epimorphism. Now note that $q(m_n(x,y,z,\theta)) = \mathbf{1} \Leftrightarrow (x = 0 \wedge y = 0 \wedge z = 0 \wedge \theta \in 2\pi\mathbb{Z})$. Thus $\ker q = \operatorname{diag}(1,1,1,1,\mathrm{e}^{\frac{i2\pi\mathbb{Z}}{n}})$. Finally note that $\mathrm{e}^{\frac{i2\pi\mathbb{Z}}{n}} = \{1, \mathrm{e}^{\frac{i}{n}}, \mathrm{e}^{\frac{2i}{n}}, \ldots, \mathrm{e}^{\frac{(n-1)i}{n}}\}$, and so $|\ker q| = n$. $\square$

1.3.13 PROPOSITION. $\mathsf{H}_3^\diamond(n) \cong \widetilde{\mathsf{H}}_3^\diamond/\mathsf{N}_{1,n}$ *covers* $\widetilde{\mathsf{H}}_3^\diamond/(\mathsf{N}_{1,n} \oplus \mathsf{N}_2)$ *for each* $n \in \mathbb{N}$.

PROOF. We have a covering homomorphism $q : \widetilde{\mathsf{H}}_3^\diamond \to \widetilde{\mathsf{H}}_3^\diamond/\mathsf{N}_{1,n} \cong \mathsf{H}_3^\diamond(n)$. Now note that $q|_{\widetilde{\mathsf{H}}_3} : (\widetilde{\mathsf{H}}_3^\diamond, \widetilde{\mathsf{H}}_3^\diamond) \to (\widetilde{\mathsf{H}}_3^\diamond/\mathsf{N}_{1,n}, \widetilde{\mathsf{H}}_3^\diamond/\mathsf{N}_{1,n})$, $\widetilde{m}(x,y,z,0) \to m_n(x,y,z,0)$ is a diffeomorphism. Thus, as $\mathsf{N}_2 \subset \widetilde{\mathsf{H}}_3$, $q(\mathsf{N}_2)$ is a discrete central subgroup of $\widetilde{\mathsf{H}}_3^\diamond/\mathsf{N}_{1,n}$. Hence there exists a covering homomorphism $q' : \widetilde{\mathsf{H}}_3^\diamond/\mathsf{N}_{1,n} \to (\widetilde{\mathsf{H}}_3^\diamond/\mathsf{N}_{1,n})/q(\mathsf{N}_2)$. Then $q' \circ q : \widetilde{\mathsf{H}}_3^\diamond \to (\widetilde{\mathsf{H}}_3^\diamond/\mathsf{N}_{1,n})/q(\mathsf{N}_2)$ is a covering morphism and

$$\ker(q' \circ q) = q^{-1}(\ker q') = q^{-1}(q(\mathsf{N}_2)) = \mathsf{N}_{1,n}\mathsf{N}_2 = \mathsf{N}_{1,n} \oplus \mathsf{N}_2.$$

Thus we have (by the epimorphism theorem) that $(\widetilde{\mathsf{H}}_3^\diamond/\mathsf{N}_{1,n})/q(\mathsf{N}_2) \cong \widetilde{\mathsf{H}}_3^\diamond/(\mathsf{N}_{1,n} \oplus \mathsf{N}_2)$, proving $\widetilde{\mathsf{H}}_3^\diamond/\mathsf{N}_{1,n}$ indeed covers $\widetilde{\mathsf{H}}_3^\diamond/(\mathsf{N}_{1,n} \oplus \mathsf{N}_2)$ for $n \in \mathbb{N}$. $\square$

1.3.14 PROPOSITION. $\widetilde{\mathsf{H}}_3^\diamond/(\mathsf{N}_{1,n} \oplus \mathsf{N}_2)$ *is a n-fold cover of* $\widetilde{\mathsf{H}}_3^\diamond/(\mathsf{N}_{1,1} \oplus \mathsf{N}_2)$ *for each* $n \in \mathbb{N}$.

PROOF. We have a covering homomorphism $q : \widetilde{\mathsf{H}}_3^\diamond \to \widetilde{\mathsf{H}}_3^\diamond/(\mathsf{N}_{1,n} \oplus \mathsf{N}_2)$. Now note that $|q(\mathsf{N}_{1,1})| = |\mathsf{N}_{1,1}/\mathsf{N}_{1,n}| = |\mathbb{Z}/n\mathbb{Z}| = n$. Thus $q(\mathsf{N}_{1,1})$ is finite and hence a discrete central subgroup of $\widetilde{\mathsf{H}}_3^\diamond/(\mathsf{N}_{1,n} \oplus \mathsf{N}_2)$. So then (the canonical mapping) $q' : \widetilde{\mathsf{H}}_3^\diamond/(\mathsf{N}_{1,n} \oplus \mathsf{N}_2) \to (\widetilde{\mathsf{H}}_3^\diamond/(\mathsf{N}_{1,n} \oplus \mathsf{N}_2))/q(\mathsf{N}_{1,1})$ is a covering homomorphism. Thus $q' \circ q : \widetilde{\mathsf{H}}_3^\diamond \to (\widetilde{\mathsf{H}}_3^\diamond/(\mathsf{N}_{1,n} \oplus \mathsf{N}_2))/q(\mathsf{N}_{1,1})$ is a covering morphism and

$$\ker(q' \circ q) = q^{-1}(\ker q') = q^{-1}(q(\mathsf{N}_{1,1})) = (\mathsf{N}_{1,n} \oplus \mathsf{N}_2)\mathsf{N}_{1,1} = \mathsf{N}_{1,n}\mathsf{N}_2\mathsf{N}_{1,1} = \mathsf{N}_{1,n} \oplus \mathsf{N}_2.$$

Hence we get (by the epimorphism theorem) that $(\widetilde{\mathsf{H}}_3^\diamond/(\mathsf{N}_{1,n} \oplus \mathsf{N}_2))/q(\mathsf{N}_{1,1}) \cong \widetilde{\mathsf{H}}_3^\diamond/(\mathsf{N}_{1,1} \oplus \mathsf{N}_2)$, proving $\widetilde{\mathsf{H}}_3^\diamond/(\mathsf{N}_{1,n} \oplus \mathsf{N}_2)$ indeed covers $\widetilde{\mathsf{H}}_3^\diamond/(\mathsf{N}_{1,1} \oplus \mathsf{N}_2)$. Moreover it is an n-fold covering as

$$\left|\ker q'\right| = |q(\mathsf{N}_{1,1})| = n.$$

$\square$

1.3.15 PROPOSITION. $\widetilde{\mathsf{H}}_3^\diamond/\mathsf{N}_2$ *covers* $\widetilde{\mathsf{H}}_3^\diamond/(\mathsf{N}_{1,n} \oplus \mathsf{N}_2)$ *(for each* $n \in \mathbb{N}$*).*

PROOF. We have a covering homomorphism $q : \widetilde{\mathsf{H}}_3^\diamond \to \widetilde{\mathsf{H}}_3^\diamond/\mathsf{N}_2$. Now note that $q|_{\widetilde{\mathsf{SO}(2)}} : \{\widetilde{m}(0,0,0,\theta) \,|\, \theta \in \mathbb{R}\} \to \{q(\widetilde{m}(0,0,0,\theta)) \,|\, \theta \in \mathbb{R}\}$ is a diffeomorphism. Thus, as $\mathsf{N}_{1,n} \subset \widetilde{\mathsf{SO}}(2)$, $q(\mathsf{N}_{1,n})$ is a discrete central subgroup of $\widetilde{\mathsf{H}}_3^\diamond/\mathsf{N}_2$. So then (the canonical mapping) $q' : \widetilde{\mathsf{H}}_3^\diamond/\mathsf{N}_2 \to (\widetilde{\mathsf{H}}_3^\diamond/\mathsf{N}_2)/q(\mathsf{N}_{1,n})$ is a covering homomorphism. Thus $q' \circ q : \widetilde{\mathsf{H}}_3^\diamond \to (\widetilde{\mathsf{H}}_3^\diamond/\mathsf{N}_2)/q(\mathsf{N}_{1,n})$ is a covering morphism and

$$\ker(q' \circ q) = q^{-1}(\ker q') = q^{-1}(q(\mathsf{N}_{1,n})) = \mathsf{N}_2\mathsf{N}_{1,n} = \mathsf{N}_{1,n} \oplus \mathsf{N}_2.$$

Hence we get (by the epimorphism theorem) that $(\widetilde{\mathsf{H}}_3^\diamond/\mathsf{N}_2)/q(\mathsf{N}_{1,n}) \cong \widetilde{\mathsf{H}}_3^\diamond/(\mathsf{N}_{1,n} \oplus \mathsf{N}_2)$, proving $\widetilde{\mathsf{H}}_3^\diamond/\mathsf{N}_2$ indeed covers $\widetilde{\mathsf{H}}_3^\diamond/(\mathsf{N}_{1,n} \oplus \mathsf{N}_2)$. $\square$

1.3.16 Remark. We can summarise the structure of the connected Lie groups with Lie algebra (isomorphic to) $\mathfrak{h}_3^\diamond$ with the following diagram

$$\begin{array}{ccccc}
\widetilde{\mathsf{H}}_3^\diamond & \longrightarrow & \mathsf{H}_3^\diamond(n) \cong \widetilde{\mathsf{H}}_3^\diamond/\mathsf{N}_{1,n} & \longrightarrow & \mathsf{H}_3^\diamond \cong \widetilde{\mathsf{H}}_3^\diamond/\mathsf{N}_{1,1} \\
\downarrow & & \downarrow & & \downarrow \\
\widetilde{\mathsf{H}}_3^\diamond/\mathsf{N}_2 & \longrightarrow & \widetilde{\mathsf{H}}_3^\diamond/(\mathsf{N}_{1,n} \oplus \mathsf{N}_2) & \longrightarrow & \widetilde{\mathsf{H}}_3^\diamond/(\mathsf{N}_{1,1} \oplus \mathsf{N}_2)
\end{array}$$

in which the arrows represent Lie group covering homomorphisms (see propositions 1.3.12 through 1.3.15 for details).

1.4 Orbits and Scalar Products

In this section we find the adjoint and coadjoint orbits of the oscillator Lie algebra $\mathfrak{h}_3^\diamond$. We will show that these two sets of orbits are "equivalent" in the sense that they are mapped bijectively by a linear isomorphism. Interpreting these results, we then show that there exists an invariant scalar product on $\mathfrak{h}_3^\diamond$.

1.4.1 Preliminaries

We will view the adjoint and coadjoint actions (denoted by Ad and Ad* respectively) as linear representations of a Lie group G in its Lie algebra $\mathfrak{g}$ and the dual of its Lie algebra $\mathfrak{g}^*$ respectively (see section A.1.4 for details). We will use the following convention for these linear representations

$$\begin{array}{rccl}
\mathrm{Ad}: & \mathsf{G} & \to & \mathsf{GL}(\mathfrak{g}) \\
 & g & \mapsto & \mathrm{Ad}\, g \\
\mathrm{Ad}\, g: & \mathfrak{g} & \to & \mathfrak{g} \\
 & A & \mapsto & g\, A\, g^{-1}
\end{array}
\qquad
\begin{array}{rccl}
\mathrm{Ad}^*: & \mathsf{G} & \to & \mathsf{GL}(\mathfrak{g}^*) \\
 & g & \mapsto & \mathrm{Ad}^*\, g \\
\mathrm{Ad}^*\, g: & \mathfrak{g}^* & \to & \mathfrak{g}^* \\
 & \mu(\cdot) & \mapsto & \mu(\mathrm{Ad}\, g^{-1}(\cdot)).
\end{array}$$

We denote their tangent maps at identity by ad and ad* and note that they are given by

$$\begin{array}{rccl}
\mathrm{ad}: & \mathfrak{g} & \to & \mathrm{Der}\,\mathfrak{g} \\
 & A & \mapsto & \mathrm{ad}\, A \\
\mathrm{ad}\, A: & \mathfrak{g} & \to & \mathfrak{g} \\
 & B & \mapsto & [A, B]
\end{array}
\qquad
\begin{array}{rccl}
\mathrm{ad}^*: & \mathfrak{g} & \to & \mathrm{Der}\,\mathfrak{g}^* \\
 & A & \mapsto & \mathrm{Ad}^* A \\
\mathrm{ad}^*\, A: & \mathfrak{g}^* & \to & \mathfrak{g}^* \\
 & \mu(\cdot) & \mapsto & -\mu(\mathrm{ad}\, A(\cdot)).
\end{array}$$

The **adjoint orbit** through a point $A \in \mathfrak{g}$ is simply the orbit of the adjoint action through that point, i.e., $\mathrm{Ad}\,\mathsf{G} \cdot A$. The **coadjoint orbit** through a point $A^* \in \mathfrak{g}^*$ is similarly given by $\mathrm{Ad}^*\,\mathsf{G} \cdot A^*$. Note that these orbits are independent of the connected Lie group chosen (i.e., they are solely properties of the Lie algebra). As such we will conveniently use $\mathsf{H}_3^\diamond$ for calculating the adjoint and coadjoint orbits of $\mathfrak{h}_3^\diamond$. For the remainder of these preliminaries, we investigate when adjoint and coadjoint orbits are "equivalent". In particular, we characterise such an equivalence by the existence of an invariant scalar product on the Lie algebra.

Let G be a (finite-dimensional) Lie group with Lie algebra $\mathfrak{g}$. A bilinear form $\langle\cdot,\cdot\rangle : \mathfrak{g}\times\mathfrak{g} \to \mathbb{R}$, $(A, B) \mapsto \langle A, B\rangle$ on $\mathfrak{g}$ is a non-degenerate if $\forall A \in \mathfrak{g}$, $\langle A, B\rangle = 0$ implies $B = 0$. We will say that $\langle\cdot,\cdot\rangle$ is invariant if

$$\forall A, B, C \in \mathfrak{g}, \quad \langle [A, B], C\rangle = \langle A, [B, C]\rangle \,.$$

We will refer to such an non-degenerate invariant bilinear form as an **invariant scalar product** on $\mathfrak{g}$. A vector space isomorphism $\psi : \mathfrak{g} \to \mathfrak{g}^*$ (and dually its inverse $\psi^{-1} : \mathfrak{g}^* \to \mathfrak{g}$) will be termed **adjoint invariant** (shortly Ad-invariant) if it satisfies $\psi \circ \operatorname{Ad} g = \operatorname{Ad}^* g \circ \psi$ for any $g \in \mathsf{G}$. A (non-degenerate) scalar product $\langle\cdot,\cdot\rangle : \mathfrak{g}\times\mathfrak{g} \to \mathbb{R}$ is called **adjoint invariant** (shortly Ad-invariant) if it satisfies $\langle A, B\rangle = \langle \operatorname{Ad} g \cdot A, \operatorname{Ad} g \cdot B\rangle$ for any $g \in \mathsf{G}$ and $A, B \in \mathfrak{g}$.

1.4.1 Remark. Suppose that G is a Lie group with Lie algebra $\mathfrak{g}$ and $\psi : \mathfrak{g} \to \mathfrak{g}^*$ is an adjoint invariant linear isomorphism. Then, for A $\in \mathfrak{g}$, we get that $\psi(\operatorname{Ad}\mathsf{G} \cdot A) = \operatorname{Ad}^* \mathsf{G} \cdot \psi(A)$. That is to say that ψ maps the adjoint and coadjoint orbits bijectively.

1.4.2 Proposition. *Let G be a finite-dimensional connected Lie group with Lie algebra $\mathfrak{g}$. Then the bilinear form $\langle\cdot,\cdot\rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ is an* Ad*-invariant scalar product if and only if it is invariant scalar product.*

Proof. Assume that the scalar product $\langle\cdot,\cdot\rangle$ is Ad-invariant. Let $A, B, C \in \mathfrak{g}$, then (for $t \in \mathbb{R}$) we get that $\langle A, C\rangle = \left\langle \operatorname{Ad} \mathrm{e}^{tB} \cdot A, \operatorname{Ad} \mathrm{e}^{tB} \cdot C\right\rangle$. Differentiating this equation w.r.t. t and evaluating at $t = 0$ then yields

$$\begin{aligned} 0 &= \langle \operatorname{ad} B \cdot A, C\rangle + \langle A, \operatorname{ad} B \cdot C\rangle \\ \langle [A, B], C\rangle &= \langle A, [B, C]\rangle \,. \end{aligned}$$

For the converse we assume our scalar product is invariant. Let $A, B, C \in \mathfrak{g}$, then

$$\begin{aligned} \frac{d}{dt}\left\langle \operatorname{Ad}\mathrm{e}^{tA} \cdot B, \operatorname{Ad}\mathrm{e}^{tA} \cdot C\right\rangle\Big|_{t=0} &= \langle \operatorname{ad} A \cdot B, C\rangle + \langle B, \operatorname{ad} A \cdot C\rangle \\ &= -\langle [B, A], C\rangle + \langle B, [A, C]\rangle \\ &= 0 \end{aligned}$$

That is to say $\left\langle \operatorname{Ad}\mathrm{e}^{tA} \cdot B, \operatorname{Ad}\mathrm{e}^{tA} \cdot C\right\rangle$ is constant as a function of t. But at $t = 0$ we have that it evaluates as $\langle B, C\rangle$. Hence $\left\langle \operatorname{Ad}\mathrm{e}^{tA} \cdot B, \operatorname{Ad}\mathrm{e}^{tA} \cdot C\right\rangle = \langle B, C\rangle$ for $\forall A, B, C \in \mathfrak{g}$ and $t \in \mathbb{R}$. Now let $g \in \mathsf{G}$, then as G is connected there exists $n \in \mathbb{N}$ and $\{A_i\}_{i\in\overline{1,n}} \subseteq \mathfrak{g}$ such that $g = \prod_{i=1}^{n} \mathrm{e}^{A_i}$. Thus (for $A, B \in \mathfrak{g}$) we have that

$$\begin{aligned} \langle \operatorname{Ad} g \cdot A, \operatorname{Ad} g \cdot B\rangle &= \left\langle \operatorname{Ad}\left(\prod_{i=1}^{n} \mathrm{e}^{A_i}\right) \cdot A, \operatorname{Ad}\left(\prod_{i=1}^{n} \mathrm{e}^{A_i}\right) \cdot B\right\rangle \\ &= \left\langle \operatorname{Ad}\mathrm{e}^{A_1} \cdot \left(\operatorname{Ad}\left(\prod_{i=2}^{n} \mathrm{e}^{A_i}\right) \cdot A\right), \operatorname{Ad}\mathrm{e}^{A_1} \cdot \left(\operatorname{Ad}\left(\prod_{i=2}^{n} \mathrm{e}^{A_i}\right) \cdot B\right)\right\rangle \\ &= \left\langle \operatorname{Ad}\left(\prod_{i=2}^{n} \mathrm{e}^{A_i}\right) \cdot A, \operatorname{Ad}\left(\prod_{i=2}^{n} \mathrm{e}^{A_i}\right) \cdot B\right\rangle \\ &= \langle A, B\rangle \,. \end{aligned}$$

□

1.4.3 Theorem. *Let G be a (finite-dimensional) connected Lie group with Lie algebra $\mathfrak{g}$. Then there exists an invariant scalar product on $\mathfrak{g}$ if and only if there exists an adjoint invariant vector space isomorphism $\psi : \mathfrak{g} \to \mathfrak{g}^*$ (or dually $\psi^{-1} : \mathfrak{g}^* \to \mathfrak{g}$). Moreover, either one of these can be constructed from the other by means of the identification $A \mapsto \psi(A) \ \leftrightarrow \ (A, B) \mapsto \psi(A) \cdot B$.*

Proof. Assume we have an Ad-invariant vector space isomorphism $\psi : \mathfrak{g} \to \mathfrak{g}^*$. Define a mapping $\langle \cdot, \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ by $\langle A, B \rangle = \psi(A) \cdot B$. We show that $\langle \cdot, \cdot \rangle$ is an invariant scalar product. For $A, B, C, D \in \mathfrak{g}$ and $a, c \in \mathbb{R}$ we have that

$$\begin{aligned}\langle aA + B, cC + D \rangle &= \psi(aA + B) \cdot (cC + D) \\ &= a\psi(A) \cdot (cC + D) + \psi(B) \cdot (cC + D) \\ &= ac\psi(A) \cdot C + a\psi(A) \cdot D + c\psi(B) \cdot C + \psi(B) \cdot D \\ &= ac \langle A, C \rangle + a \langle A, D \rangle + c \langle B, C \rangle + \langle B, D \rangle .\end{aligned}$$

Thus $\langle \cdot, \cdot \rangle$ is a bilinear form. Next we show that $\langle \cdot, \cdot \rangle$ is non-degenerate. Let $\{E_i\}_{i \in \overline{1,n}}$ be a basis for $\mathfrak{g}$ and $\{E_i^*\}_{i \in \overline{1,n}}$ be the associated dual basis for $\mathfrak{g}^*$. Furthermore let $A = \sum_{i=1}^n a_i E_i$ be an arbitrary element in $\mathfrak{g}$. Then as ψ is a vector space isomorphism we have that $\forall i = \overline{1,n}\ \exists B_i$ such that $\psi(B_i) = a_i E_i^*$. So then $\langle B_i, A \rangle = \psi(B_i) \cdot A = a_i E_i^* \sum_{j=1}^n a_j E_j = a_i^2$. Thus if $\langle B_i, A \rangle = 0$ for all $B_i \in \mathfrak{g}$ we have that for $\forall i \in \overline{1,n},\ a_i = 0$ and consequently $A = 0$. Thus we have that $\langle \cdot, \cdot \rangle$ is non-degenerate. Now, for $A, B, C \in \mathfrak{g}$, we have that

$$\begin{aligned}\langle \operatorname{Ad} g \cdot A, \operatorname{Ad} g \cdot B \rangle &= \psi(\operatorname{Ad} g \cdot A) \cdot (\operatorname{Ad} g \cdot B) \\ &= (\operatorname{Ad}^* g \cdot \psi(A)) \cdot (\operatorname{Ad} g \cdot B) \\ &= \psi(A) \cdot (\operatorname{Ad} g^{-1} \operatorname{Ad} g \cdot B) \\ &= \langle A, B \rangle\end{aligned}$$

Thus $\langle \cdot, \cdot \rangle$ is Ad-invariant and hence invariant by proposition 1.4.2.

For the converse we assume we have an invariant scalar product $\langle \cdot, \cdot \rangle$. By proposition 1.4.2 we have that $\langle \cdot, \cdot \rangle$ is Ad-invariant. We define a mapping $\psi : \mathfrak{g} \to \mathfrak{g}^*$ as $A \mapsto \langle A, \cdot \rangle$ where $\langle A, \cdot \rangle : \mathfrak{g} \to \mathbb{R},\ B \mapsto \langle A, B \rangle$. First we show that $\psi(A) \in \mathfrak{g}^*$ for all $A \in \mathfrak{g}$. Let $B, C \in \mathfrak{g},\ b \in \mathbb{R}$ then

$$\psi(A) \cdot (bB + C) = \langle A, bB + D \rangle = b \langle A, B \rangle + \langle A, D \rangle = b\psi(A) \cdot B + \psi(A) \cdot D.$$

Next, for $A, B \in \mathfrak{g}$, we have that

$$\psi(A) = \psi(B) \quad \Leftrightarrow \quad \forall C \in \mathfrak{g},\ \langle A, C \rangle = \langle B, C \rangle \quad \Leftrightarrow \quad \forall C \in \mathfrak{g},\ \langle A - B, C \rangle = 0 \quad \Leftrightarrow \quad A = B,$$

where the last equivalence follows from non-degeneracy. Thus we have that ψ is injective and well defined. Next we observe that ψ is linear as (for $A, B, C \in \mathfrak{g},\ a \in \mathbb{R}$)

$$\psi(aA + B)(C) = \langle aA + B, C \rangle = a \langle A, C \rangle + \langle B, C \rangle = a\psi(A) \cdot C + \psi(B) \cdot C,$$

and thus $\psi(aA + B) = a\psi(A) + \psi(B)$. Note that as ψ is an injective linear mapping between vector spaces of the same dimension it is surjective. Moreover $\{\psi(E_i)\}_{i \in \overline{1,n}}$ is a basis for $\mathfrak{g}^*$. Lastly we show that ψ is Ad-invariant. For $A, B \in \mathfrak{g},\ g \in \mathsf{G}$ we have that

$$\psi(\operatorname{Ad} g \cdot A) \cdot B = \langle \operatorname{Ad} g \cdot A, B \rangle = \left\langle A, \operatorname{Ad} g^{-1} B \right\rangle = \psi(A) \cdot (\operatorname{Ad} g^{-1} \cdot B) = (\operatorname{Ad}^* g \cdot \psi(A)) \cdot B,$$

and thus $\psi(\operatorname{Ad} g \cdot A) = \operatorname{Ad}^* g \cdot \psi(A)$, yielding $\psi \circ \operatorname{Ad} g = \operatorname{Ad}^* g \circ \psi$. □

1.4.4 REMARK. The scalar product $\langle \cdot, \cdot \rangle$ as in theorem 1.4.3 is symmetric (respectively antisymmetric) if and only if the vector space isomorphism φ is symmetric (respectively antisymmetric). We say φ is symmetric if for $\forall A, B \in \mathfrak{g}$, $\varphi(A) \cdot B = \varphi(B) \cdot A$ (i.e., ${}_{\mathfrak{g}}[\varphi]_{\mathfrak{g}^*}$ is symmetric) and antisymmetric if for $\forall A, B \in \mathfrak{g}$, $\varphi(A) \cdot B = -\varphi(B) \cdot A$ (i.e., ${}_{\mathfrak{g}}[\varphi]_{\mathfrak{g}^*}$ is antisymmetric).

1.4.2 The adjoint orbits

We recall (from proposition 1.2.21) that for $g = m(x_g, y_g, z_g, \theta_g) \in \mathsf{H}_3^\diamond$ we have (w.r.t. the ordered basis $\{E_i\}_{i=\overline{1,4}}$) that

$$\operatorname{Ad} g = \begin{bmatrix} \cos\theta_g & \sin\theta_g & 0 & -y_g \\ -\sin\theta_g & \cos\theta_g & 0 & x_g \\ -y_g\cos\theta_g - x_g\sin\theta_g & x_g\cos\theta_g - y_g\sin\theta_g & 1 & \frac{1}{2}\left(x_g^2 + y_g^2\right) \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The action of g on an arbitrary element $A = M(x_a, y_a, z_a, \theta_a) \in \mathfrak{h}_3^\diamond$ is then given by $\operatorname{Ad} g \cdot A = M(x_*, y_*, z_*, \theta_*)$, where

$$\begin{aligned} x_* &= x_a\cos\theta_g + y_a\sin\theta_g - \theta_a y_g \\ y_* &= y_a\cos\theta_g - x_a\sin\theta_g + \theta_a x_g \\ z_* &= z_a + (y_a x_g - x_a y_g)\cos\theta_g - (y_a y_g + x_a x_g)\sin\theta_g + \tfrac{1}{2}\theta_a(x_g^2 + y_g^2) \\ &= x_g y_* - y_g x_* - \tfrac{1}{2}\theta_a(x_g^2 + y_g^2) + z_a \\ &= x_g(y_* - \theta_a x_g) - y_g(x_* + \theta_a y_g) + \tfrac{1}{2}\theta_a(x_g^2 + y_g^2) + z_a \\ \theta_* &= \theta_a. \end{aligned}$$

Notice that

$$\begin{aligned} &(x_a\cos\theta_g + y_a\sin\theta_g)^2 + (y_a\cos\theta_g - x_a\sin\theta_g)^2 \\ &= x_a^2(\cos^2\theta_g + \sin^2\theta_g) + y_a^2(\sin^2\theta_g + \cos^2\theta_g) + 2x_a y_a(\cos\theta_g\sin\theta_g - \cos\theta_g\sin\theta_g) \\ &= x_a^2 + y_a^2. \end{aligned}$$

Thus we can parametrise a circle

$$x_a\cos\theta_g + y_a\sin\theta_g = r\cos\vartheta \qquad -(y_a\cos\theta_g - x_a\sin\theta_g) = r\sin\vartheta$$

where,

$$r = \sqrt{x_a^2 + y_a^2} \qquad \vartheta = \arctan\left(x_a\cos\theta_g + y_a\sin\theta_g, -y_a\cos\theta_g + x_a\sin\theta_g\right) \in [-\pi, \pi).$$

Here $\arctan(x, y)$ is the four-quadrant inverse tangent of the point $(x, y) \in \mathbb{R}^2$. (That is, in polar coordinates with $r > 0$, $\theta \in [-\pi, \pi)$, we have that $\arctan(r\cos\theta, r\sin\theta) = \theta$.) Note that as θ_g ranges through $\mathbb{R}$ for some fixed $x_a, y_a \in \mathbb{R}$, ϑ takes all values in $[-\pi, \pi)$. The above expressions for x_*, y_*, z_* and θ_* then simplify to

$$\begin{aligned} x_* &= r\cos\vartheta - \theta_a y_g & y_* &= -r\sin\vartheta + \theta_a x_g \\ z_* &= z_a + \tfrac{1}{2}\theta_a(x_g^2 + y_g^2) - r(x_g\sin\vartheta + y_g\cos\vartheta) & \theta_* &= \theta_a. \end{aligned}$$

In the case that $\theta_a \neq 0$ we may simplify further by solving for y_g and x_g. We get that

$$y_g = \frac{x_* - r\cos\vartheta}{-\theta_a} \qquad x_g = \frac{y_* + r\sin\vartheta}{\theta_a}.$$

Hence

$$\begin{aligned} x_g \sin\vartheta + y_g\cos\vartheta &= \frac{1}{\theta_a}(y_*\sin\vartheta - x_*\cos\vartheta + r) \\ x_g^2 &= \frac{1}{\theta_a^2}(y_*^2 + 2y_* r\sin\vartheta + r^2\sin^2\vartheta) \\ y_g^2 &= \frac{1}{\theta_a^2}(x_*^2 - 2x_* r\cos\vartheta + r^2\cos^2\vartheta) \\ x_g^2 + y_g^2 &= \frac{1}{\theta_a^2}(x_*^2 + y_*^2 + 2r(y_*\sin\vartheta - x_*\cos\vartheta) + r^2). \end{aligned}$$

Using these identities we can further simplify z_* to find

$$z_* = \frac{1}{2\theta_a}(x_*^2 + y_*^2) + z_a - \frac{r^2}{2\theta_a}.$$

Table 1.1 gives a summary of our calculations. With these at hand we are now ready to classify

$\operatorname{Ad} g \cdot A = M(x_*, y_*, z_*, \theta_*)$	
θ_*	θ_a
x_*	$r\cos\vartheta - \theta_a y_g$
y_*	$-r\sin\vartheta + \theta_a x_g$
z_* $(\theta_a = 0)$	$z_a + x_g y_* - y_g x_*$
z_* $(\theta_a \neq 0)$	$\frac{1}{2\theta_a}(x_*^2 + y_*^2) + z_a - \frac{r^2}{2\theta_a}$
r	$\sqrt{x_a^2 + y_a^2}$
ϑ	$\arctan(x_a\cos\theta_g + y_a\sin\theta_g, -y_a\cos\theta_g + x_a\sin\theta_g)$
g	$m(x_g, y_g, z_g, \theta_g)$
A	$M(x_a, y_a, z_a, \theta_a)$

Table 1.1: Formulae for adjoint action in $\mathfrak{h}_3^\diamond$

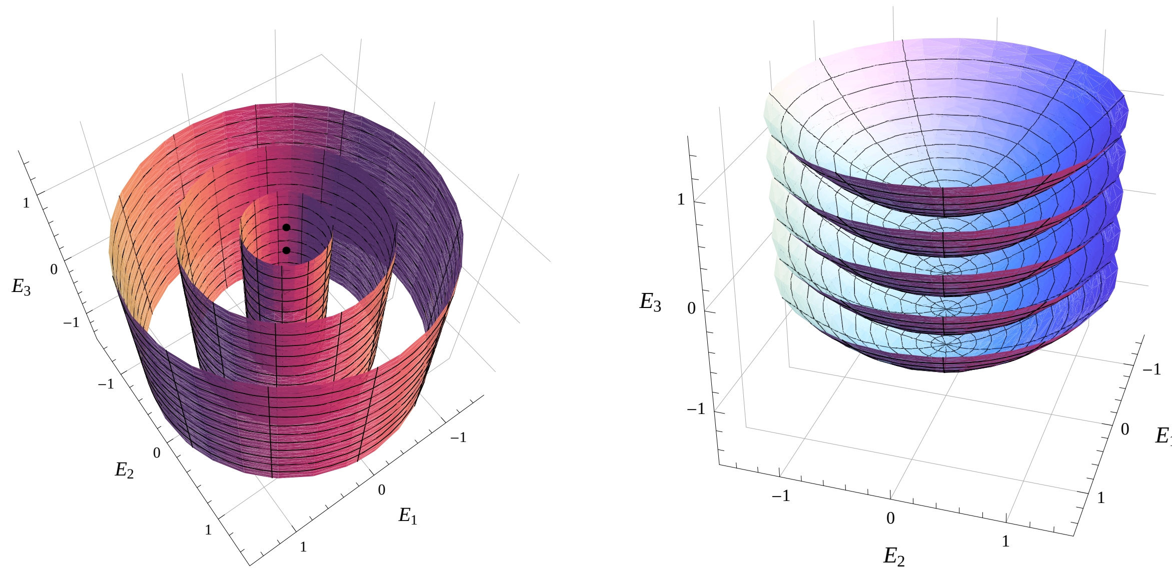
the adjoint orbits

$$\operatorname{Ad} \mathsf{H}_3^\diamond \cdot A = \{x_* E_1 + y_* E_2 + z_* E_3 + \theta_* E_4 \mid x_g, y_g, z_g, \theta_g \in \mathbb{R}\}.$$

1.4.5 Proposition. *The adjoint orbit through a point* $A = M(x_a, y_a, z_a, \theta_a) \in \mathfrak{h}_3^\diamond$ *is one of three types listed below.*

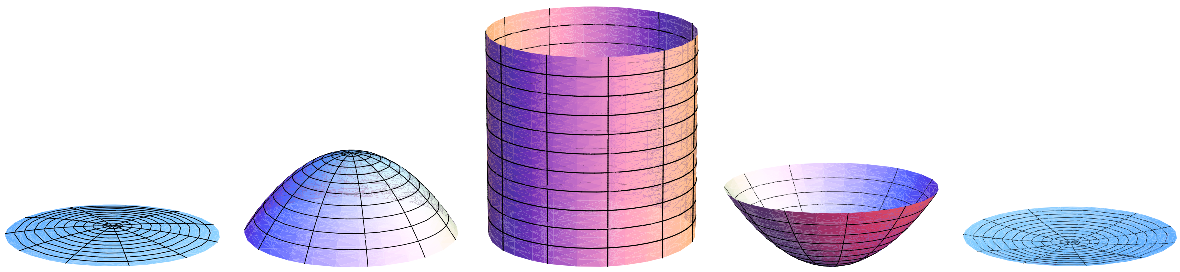
1. *Case* $\theta_a = x_a = y_a = 0$. *Then* $\operatorname{Ad} \mathsf{H}_3^\diamond \cdot A = \{z_a E_3\}$, *a single point on the* E_3 *axis. Fig 1.1a.*
2. *Case* $\theta_a = 0$, $x_a^2 + y_a^2 \neq 0$. *Then* $\operatorname{Ad} \mathsf{H}_3^\diamond \cdot A = \{r\cos\vartheta E_1 - r\sin\vartheta E_2 + z_* E_3 \mid z_*, \vartheta \in \mathbb{R}\}$, *cylinders about the* E_3 *axis of radius* $r = \sqrt{x_a^2 + y_a^2}$, *contained in the hyperplane through* $\theta_a E_4$, *with normal vector parallel to* E_4 *axis. Fig 1.1a.*

3. *Case* $\theta_a \neq 0$. *Then* $\operatorname{Ad} \mathsf{H}_3^\diamond \cdot A = \{x_* E_1 + y_* E_2 + (\frac{1}{2\theta_a}(x_*^2 + y_*^2) + z_a - \frac{x_a^2 + y_a^2}{2\theta_a})E_3 + \theta_a E_4 \mid x_*, y_* \in \mathbb{R}\}$, *paraboloids about* E_3 *axis contained in the hyperplane through* $\theta_a E_4$ *with normal vector parallel to* E_4 *axis, cupping up or down for* θ_a *greater or less than* 0 *respectively, stacked along* E_3 *axis. Fig 1.1b.*



(a) $\theta_a = 0$ (b) $\theta_a > 0$

Figure 1.1: The adjoint orbits of $\mathfrak{h}_3^\diamond$



(a) $\theta_a << 0$ (b) $\theta_a < 0$ (c) $\theta_a = 0$ (d) $\theta_a > 0$ (e) $\theta_a >> 0$

Figure 1.2: Sliding window of typical adjoint orbits of $\mathfrak{h}_3^\diamond$

Proof. In each case we suppose $A = M(x_a, y_a, z_a, \theta_a)$ is fixed. Case 1 follows trivially. For case 2 we have that ϑ takes all values in $[-\pi, \pi)$ and $r = \sqrt{x_a^2 + y_a^2}$ is fixed. Next note that (as $\theta_a = 0$) that $x_* = r\cos\vartheta$, $y_* = -r\sin\vartheta$, $z_* = z_a + x_g y_* - y_g x_*$ and $\theta_* = 0$. As x_* and y_* are not simultaneously zero we have that as x_g and y_g range through $\mathbb{R}$, z_* takes all values in $\mathbb{R}$. Note that ϑ does not depend on x_g and y_g, thus z_* takes all values in $\mathbb{R}$ for every value of $\vartheta \in [-\pi, \pi)$. Finally for case 3 we have that (as $\theta_a \neq 0$) that x_* takes all values in $\mathbb{R}$ (as y_g

ranges through $\mathbb{R}$) and y_* takes all values in $\mathbb{R}$ (as x_g ranges through $\mathbb{R}$). Finally note that $\theta_* = \theta_a$ is fixed and $z_* = \frac{1}{2\theta_a}(x_*^2 + y_*^2) + z_a - \frac{r^2}{2\theta_a}$ depends only on x_*, y_* and fixed quantities.□

We observe that our orbits are "flat" with respect to the E_4 axis, that is to say any orbit's projection onto the E_4 axis yields a single point or that any orbit is contained in a hyperplane with surface normal parallel to E_4. This fact allows one to faithfully represent the orbits as three-dimensional slices; this is done in figure 1.1. Figure 1.2 shows a typical slice for a range of values along the E_4 axis. The code used to produce these figures is supplied in section C.3.

1.4.3 The coadjoint orbits

We now proceed to calculate the coadjoint orbits of $\mathfrak{h}_3^\diamond$. Let $\{E_1^*, E_2^*, E_3^*, E_4^*\}$ be the dual basis for $(\mathfrak{h}_3^\diamond)^*$. In a similar fashion to proposition 1.2.21 we represent $\mathrm{Ad}^* g \in \mathsf{GL}(\mathfrak{h}_3^\diamond)^*$ as a matrix w.r.t. this basis. Then note that $\mathrm{Ad}^* g = (E_j^* \cdot (\mathrm{Ad}\, g^{-1} \cdot E_i))_{ij}$. Thus, for $g = m(x_g, y_g, z_g, \theta_g)$ we get (again using Mathematica, see section C.4) that

$$\mathrm{Ad}^* g = \begin{bmatrix} \cos\theta_g & \sin\theta_g & y_g & 0 \\ -\sin\theta_g & \cos\theta_g & -x_g & 0 \\ 0 & 0 & 1 & 0 \\ \sin\theta_g x_g + \cos\theta_g y_g & -\cos\theta_g x_g + \sin\theta_g y_g & \frac{1}{2}\left(x_g^2 + y_g^2\right) & 1 \end{bmatrix}.$$

Thus, for $A^* = x_c E_1^* + y_c E_2^* + z_c E_3 * + \theta_c E_4^* \in (\mathfrak{h}_3^\diamond)^*$, we get that

$$\begin{aligned} \mathrm{Ad}^* g \cdot A^* &= (x_c \cos\theta_g + y_c \sin\theta_g + z_c y_g) E_1^* + (y_c \cos\theta_g - x_c \sin\theta_g - z_c x_g) E_2^* \\ &+ z_c E_3^* + (\theta_c + (x_c y_g - y_c x_g)\cos\theta_g + (y_c y_g + x_c x_g)\sin\theta_g + \tfrac{1}{2} z_c (x_g^2 + y_g^2)) E_4^*. \end{aligned}$$

We notice that this looks similar to the result for the adjoint orbits (see remark 1.4.1), which leads to the next proposition.

1.4.6 Proposition. *Let $\psi : \mathfrak{h}_3^\diamond \to (\mathfrak{h}_3^\diamond)^*$ be the vector space isomorphism (represented as a matrix w.r.t. the respective bases) given by*

$$\psi = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}.$$

Then ψ is an adjoint invariant vector space isomorphism. Moreover ψ is symmetric.

Proof. We have that ψ is symmetric as the matrix representing it is symmetric. We show that ψ is Ad-invariant. Let $g = m(x_g, y_g, z_g, \theta_g)$, then utilizing previous results (in particular proposition 1.2.21) and making some calculations (in matrix form) we find that

$$\begin{aligned}
\psi \circ \operatorname{Ad} g &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta_g & \sin\theta_g & 0 & -y_g \\ -\sin\theta_g & \cos\theta_g & 0 & x_g \\ -x_g\sin\theta_g - y_g\cos\theta_g & x_g\cos\theta_g - y_g\sin\theta_g & 1 & \frac{1}{2}\left(x_g^2+y_g^2\right) \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \cos\theta_g & \sin\theta_g & 0 & -y_g \\ -\sin\theta_g & \cos\theta_g & 0 & x_g \\ 0 & 0 & 0 & -1 \\ x_g\sin\theta_g + y_g\cos\theta_g & -x_g\cos\theta_g + y_g\sin\theta_g & -1 & \frac{1}{2}\left(-x_g^2-y_g^2\right) \end{bmatrix} \\
&= \begin{bmatrix} \cos\theta_g & \sin\theta_g & y_g & 0 \\ -\sin\theta_g & \cos\theta_g & -x_g & 0 \\ 0 & 0 & 1 & 0 \\ x_g\sin\theta_g + y_g\cos\theta_g & -x_g\cos\theta_g + y_g\sin\theta_g & \frac{1}{2}\left(x_g^2+y_g^2\right) & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \\
&= \operatorname{Ad}^* g \circ \psi.
\end{aligned}$$

In other words for $A \in \mathfrak{h}_3^\diamond$ and $g \in \mathsf{H}_3^\diamond$, we have that $\psi(\operatorname{Ad} g \cdot A) = \operatorname{Ad}^* g \cdot \psi(A)$. □

We may thus conclude that the coadjoint orbits of $\mathfrak{h}_3^\diamond$ are "naturally" equivalent to the adjoint orbits (as we have an adjoint invariant linear isomorphism mapping them bijectively). Furthermore, as expected, the coadjoint orbits are even-dimensional.

As discussed in theorem 1.4.3, we may construct a symmetric invariant scalar product on $\mathfrak{h}_3^\diamond$ from the above linear isomorphism. We will now continue on to explicitly construct such a scalar product.

1.4.4 An invariant scalar product on $\mathfrak{h}_3^\diamond$

1.4.7 Proposition. *The oscillator Lie algebra $\mathfrak{h}_3^\diamond$ admits a symmetric invariant scalar product*

$$\begin{aligned}
&\langle \cdot, \cdot \rangle : \mathfrak{h}_3^\diamond \times \mathfrak{h}_3^\diamond \to \mathbb{R} \\
&(M(x,y,z,\theta), M(x',y',z',\theta')) \mapsto \begin{bmatrix} x \\ y \\ z \\ \theta \end{bmatrix}^\top \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \\ \theta' \end{bmatrix}.
\end{aligned}$$

Proof. We construct this scalar product, as described in theorem 1.4.3, from the Ad-invariant linear isomorphism ψ introduced in proposition 1.4.6. Indeed we have that

$$\begin{aligned}
\psi(M(x,y,z,\theta)) \cdot M(x',y',z',\theta') &= (xE_1^* + yE_2^* - \theta E_3^* - zE_4^*) \cdot (x'E_1 + y'E_2 + z'E_3 + \theta' E_4) \\
&= xx' + yy' - \theta z' - z\theta'.
\end{aligned}$$

□

1.4.8 COROLLARY. *The quadratic form $\mathcal{C} : (\mathfrak{h}_3^\diamond)^* \to \mathbb{R}$,*

$$xE_1^* + yE_2^* + zE_3^* + \theta E_4^* \mapsto \begin{bmatrix} x \\ y \\ z \\ \theta \end{bmatrix}^\top \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ \theta \end{bmatrix} = x^2 + y^2 - 2z\theta$$

is adjoint invariant (i.e., for $g \in \mathsf{H}_3^\diamond$, $\mathcal{C} = \mathcal{C} \circ \mathrm{Ad}^* g$) and hence a Casimir function (on $(\mathfrak{h}_3^\diamond)^*$ with Lie-Poisson structure as described in section A.3.5).

PROOF. Let $g \in \mathsf{H}_3^\diamond$, $A \in \mathfrak{g}_3^\diamond$, $A^* \in (\mathfrak{h}_3^\diamond)^*$ and $\psi : \mathfrak{h}_3^\diamond \to (\mathfrak{h}_3^\diamond)^*$ be the Ad-invariant linear isomorphism as introduced in proposition 1.4.6. Then notice that $\mathcal{C}(A^*) = A^* \cdot \psi^{-1}(A^*)$. Also note that, as $\psi(\mathrm{Ad}\, g \cdot A) = \mathrm{Ad}^* g \cdot \psi(A)$, we have that $\psi^{-1}(\mathrm{Ad}^*\, g \cdot A^*) = \mathrm{Ad}\, g \cdot \psi^{-1}(A^*)$. Hence we get that

$$\mathcal{C}(\mathrm{Ad}^*\, g \cdot A^*) = (\mathrm{Ad}^*\, g \cdot A^*) \cdot \psi^{-1}(\mathrm{Ad}^*\, g \cdot A^*) = A^*(\mathrm{Ad}\, g^{-1} \cdot \mathrm{Ad}\, g \cdot \psi^{-1}(A^*)) = \mathcal{C}(A^*).$$

Thus $\mathcal{C}$ is adjoint invariant and thus a Casimir function by proposition A.3.18. □

1.5 The Ideals of $\mathfrak{h}_3^\diamond$

In order to study the decomposition of $\mathfrak{h}_3^\diamond$ and its invariant scalar product (specifically to show that they form an indecomposable pair, as will be explained in the next section) we need to find all the ideals of $\mathfrak{h}_3^\diamond$. We do so here.

1.5.1 Preliminaries

We start by giving a characterisation for a subspace $\mathfrak{a}$ to be an ideal, which we will then use to calculate all the ideals.

1.5.1 PROPOSITION. *Let $\mathfrak{a}$ be a m-dimensional vector subspace of a n-dimensional real Lie algebra $\mathfrak{g}$. Let $\{A_i\}_{i\in\overline{1,m}}$ be a basis for $\mathfrak{a}$ and $\{E_i\}_{i\in\overline{1,n}}$ be a basis for $\mathfrak{g}$. Then $\mathfrak{a}$ is an ideal of $\mathfrak{g}$ if and only if there exists functions $f_i : \mathbb{R}^{n+m} \to \mathbb{R}$, $i \in \overline{1,m}$ such that, for all $i \in \overline{1,m}$, $j \in \overline{1,n}$, $a_i, e_j \in \mathbb{R}$,*

$$\left[\sum_{i=1}^m a_i A_i, \sum_{j=1}^n e_j E_j\right] - \sum_{i=1}^m f_i(e_1, e_2, \ldots, e_n, a_1, a_2, \ldots, a_m)\, A_i = 0.$$

PROOF. The subspace $\mathfrak{a}$ is an ideal if and only if $[\mathfrak{a}, \mathfrak{g}] \leq \mathfrak{a}$. In particular, this means for any element $\sum_{i=1}^m a_i A_i \in \mathfrak{a}$ and any element $\sum_{j=1}^n e_j E_j \in \mathfrak{g}$ we have that $\left[\sum_{i=1}^m a_i A_i, \sum_{j=1}^n e_j E_j\right]$ is in $\mathfrak{a}$. Hence $\left[\sum_{i=1}^m a_i A_i, \sum_{j=1}^n e_j E_j\right] = \sum_{i=1}^m b_i A_i$ for some scalars $b_i \in \mathbb{R}$. These b_i depend solely on the a_i and e_i. Moreover this defines functions $f_i(e_1, e_2, \ldots, e_n, a_1, a_2, \ldots, a_m) = b_i$ as the b_i are uniquely determined given a_i and e_i. Conversely if such functions exist the subspace $\mathfrak{a}$ is an ideal. □

We now proceed to calculating the ideals of $\mathfrak{h}_3^\diamond$.

1.5.2 One-dimensional

Recall that the centre of $\mathfrak{h}_3^\diamond$ is given by $\mathrm{Z}(\mathfrak{h}_3^\diamond) = \{M(0,0,z,0) \mid z \in \mathbb{R}\}$. Hence, $\mathrm{Z}(\mathfrak{h}_3^\diamond)$ is a one-dimensional ideal. We show that it is the only one-dimensional ideal.

Let $\mathfrak{a}$ be a one-dimensional subspace of $\mathfrak{h}_3^\diamond$ with basis $\{A_1\}$, where $A_1 = M(x_1, y_1, z_1, \theta_1)$ for some $x_1, y_1, z_1, \theta_1 \in \mathbb{R}$ not all zero. Then $\mathfrak{a}$ is an ideal if and only if there exists a function $f_1 : \mathbb{R}^5 \to \mathbb{R},\ (x, y, z, \theta, a) \mapsto f_1(x, y, z, \theta, a)$ such that for $\forall x, y, z, \theta, a \in \mathbb{R}$ we have

$$[aA_1, B] - f_1(x, y, z, \theta, a)\, A_1 = 0 \tag{1.5.1}$$

where $B = M(x, y, z, \theta) \in \mathfrak{h}_3^\diamond$. Suppressing the input arguments for f_1, the above equation (1.5.1) in matrices is equivalent the following set of four equations in four variables and one function:

$$a\theta_1 y - ay_1\theta - f_1 x_1 = 0 \tag{1.5.2}$$
$$ax_1\theta - a\theta_1 x - f_1 y_1 = 0 \tag{1.5.3}$$
$$-ax_1 y + ay_1 x + f_1 z_1 = 0 \tag{1.5.4}$$
$$f_1\theta_1 = 0. \tag{1.5.5}$$

Assume $\theta_1 \neq 0$, then from (1.5.5) we have $f_1 = 0$. So then from (1.5.2) we have that $ay = a\theta\frac{y_1}{\theta_1}$. But we require this to be true for $\forall y, \theta, a \in \mathbb{R}$ which is untenable. Thus we conclude $\theta_1 = 0$. Thus our conditions become

$$-ay_1\theta = f_1 x_1 \tag{1.5.6}$$
$$ax_1\theta = f_1 y_1 \tag{1.5.7}$$
$$a(-x_1 y + y_1 x) = -f_1 z_1 \tag{1.5.8}$$
$$\theta_1 = 0. \tag{1.5.9}$$

We now assume that $x_1 \neq 0$. Then from (1.5.6) it follows that $f_1 = -a\theta\frac{y_1}{x_1}$. Substituting this into (1.5.7) we have that $a\theta x_1^2 = -a\theta y_1^2$. But this is not true for $\forall \theta, a \in \mathbb{R}^2$. Thus $x_1 = 0$. Substituting this into (1.5.6) we have $y_1 = 0$ (as it must be true for $\forall \theta, a \in \mathbb{R}$).

Thus we have $x_1 = y_1 = \theta_1 = 0$, thus $z_1 \neq 0$ as $\mathfrak{a}$ is one-dimensional. Then from (1.5.8) it follows that $f_1 = 0$. We note that these conditions ($x_1 = y_1 = \theta_1 = 0,\ z_1 \neq 0,\ f_1 = 0$) are also sufficient for (1.5.1). Thus $\mathfrak{a}$ is an ideal if and only if

$$\mathfrak{a} = \{M(0, 0, z_1, 0) \mid z_1 \in \mathbb{R}\} = \mathrm{Z}(\mathfrak{h}_3^\diamond).$$

That is to say, the only one-dimensional ideal of $\mathfrak{h}_3^\diamond$ is its centre.

1.5.3 Two-dimensional

Let $\mathfrak{a}$ be a two-dimensional subspace of $\mathfrak{h}_3^\diamond$ with basis $\{A_1, A_2\}$ where $A_1 = M(x_1, y_1, z_1, \theta_1)$, $A_2 = M(x_2, y_2, z_2, \theta_2)$ and $[x_1, y_1, z_1, \theta_1],\ [x_2, y_2, z_2, \theta_2] \in \mathbb{R}^4$ are two linearly independent vectors. That is, $\mathfrak{a} = \{aA_1 + bA_2 \mid a, b \in \mathbb{R}\}$. Then $\mathfrak{a}$ is an ideal if only if there exists functions

$f_1, f_2 : \mathbb{R}^6 \to \mathbb{R}$, $(x, y, z, \theta, a, b) \mapsto f_i(x, y, z, \theta, a, b)$, $i = \overline{1, 2}$, such that for $\forall x, y, z, \theta, a, b \in \mathbb{R}$ we have

$$[aA_1 + bA_2, B] - f_1 A_1 - f_2 A_2 = 0 \tag{1.5.10}$$

where $B = M(x, y, z, \theta)$. The above equation (1.5.10) in matrices is equivalent to the following set of four equations in eight variables and two functions:

$$(a\theta_1 + b\theta_2)y - (ay_1 + by_2)\theta - f_1 x_1 - f_2 x_2 = 0 \tag{1.5.11}$$
$$(ax_1 + bx_2)\theta - (a\theta_1 + b\theta_2)x - f_1 y_1 - f_2 y_2 = 0 \tag{1.5.12}$$
$$(ay_1 + by_2)x - (ax_1 + bx_2)y + f_1 z_1 + f_2 z_2 = 0 \tag{1.5.13}$$
$$f_1\theta_1 + f_2\theta_2 = 0. \tag{1.5.14}$$

Assume $\theta_1 \neq 0$, then from (1.5.14) it follows that $f_1 = -\frac{\theta_2}{\theta_1} f_2$. Substituting this into (1.5.11), (1.5.12) and (1.5.13) we get

$$(a\theta_1 + b\theta_2)y - (ay_1 + by_2)\theta + \left(\frac{\theta_2}{\theta_1}x_1 - x_2\right) f_2 = 0 \tag{1.5.15}$$
$$(ax_1 + bx_2)\theta - (a\theta_1 + b\theta_2)x + \left(\frac{\theta_2}{\theta_1}y_1 - y_2\right) f_2 = 0 \tag{1.5.16}$$
$$(ay_1 + by_2)x - (ax_1 + bx_2)y - \left(\frac{\theta_2}{\theta_1}z_1 - z_2\right) f_2 = 0. \tag{1.5.17}$$

Now we note that one of $\left(\frac{\theta_2}{\theta_1}x_1 - x_2\right)$, $\left(\frac{\theta_2}{\theta_1}y_1 - y_2\right)$ or $\left(\frac{\theta_2}{\theta_1}y_1 - y_2\right)$ is non-zero. If not then

$$\frac{\theta_2}{\theta_1}x_1 = x_2 \qquad \frac{\theta_2}{\theta_1}y_1 = y_2 \qquad \frac{\theta_2}{\theta_1}y_1 = y_2 \qquad \frac{\theta_2}{\theta_1}\theta_1 = \theta_2$$

contradicting linear independence. So we have that one of the following three equations are true

$$\left(\frac{\theta_2}{\theta_1}x_1 - x_2\right) \neq 0 \qquad \text{i.e., } f_2 = -\frac{(a\theta_1 + b\theta_2)y - (ay_1 + by_2)\theta}{\frac{\theta_2}{\theta_1}x_1 - x_2} \tag{1.5.18}$$
$$\left(\frac{\theta_2}{\theta_1}y_1 - y_2\right) \neq 0 \qquad \text{i.e., } f_2 = -\frac{(ax_1 + bx_2)\theta - (a\theta_1 + b\theta_2)x}{\frac{\theta_2}{\theta_1}y_1 - y_2} \tag{1.5.19}$$
$$\left(\frac{\theta_2}{\theta_1}z_1 - z_2\right) \neq 0 \qquad \text{i.e., } f_2 = \frac{(ay_1 + by_2)x - (ax_1 + bx_2)y}{\frac{\theta_2}{\theta_1}z_1 - z_2}. \tag{1.5.20}$$

We show that each of the three equations above lead to a contradiction.

1. Assuming (1.5.18) is true we substitute it into (1.5.16) to find

$$((ax_1 + bx_2)\theta - (a\theta_1 + b\theta_2)x)\left(\tfrac{\theta_2}{\theta_1}x_1 - x_2\right) = ((a\theta_1 + b\theta_2)y - (ay_1 + by_2)\theta)\left(\tfrac{\theta_2}{\theta_1}y_1 - y_2\right).$$

We require this to be true for $\forall x, y, z, \theta, a, b \in \mathbb{R}$. As $\theta_1 \neq 0$ we can choose $a \neq 0$ and $b = 0$ such that $(a\theta_1 + b\theta_2) \neq 0$. Next we may choose $y = \theta = 0$ and $x \neq 0$, then as $\left(\frac{\theta_2}{\theta_1}x_1 - x_2\right) \neq 0$ this leads to a contradiction as

$$\underbrace{\left(\underbrace{(ax_1 + bx_2)\theta}_{=0} - \underbrace{(a\theta_1 + b\theta_2)x}_{\neq 0}\right)}_{\neq 0}\underbrace{\left(\frac{\theta_2}{\theta_1}x_1 - x_2\right)}_{\neq 0} = \underbrace{((a\theta_1 + b\theta_2)y - (ay_1 + by_2)\theta)}_{=0}\left(\frac{\theta_2}{\theta_1}y_1 - y_2\right).$$

2. Assuming (1.5.19) is true we substitute it into (1.5.15) to find

$$((a\theta_1 + b\theta_2)y - (ay_1 + by_2)\theta)\left(\frac{\theta_2}{\theta_1}y_1 - y_2\right) = ((ax_1 + bx_2)\theta - (a\theta_1 + b\theta_2)x)\left(\frac{\theta_2}{\theta_1}x_1 - x_2\right).$$

We require this to be true for $\forall x, y, z, \theta, a, b \in \mathbb{R}$. As $\theta_1 \neq 0$ we can choose $a \neq 0$ and $b = 0$ such that $(a\theta_1 + b\theta_2) \neq 0$. Next we may choose $\theta = x = 0$ and $y \neq 0$, then as $\left(\frac{\theta_2}{\theta_1}y_1 - y_2\right) \neq 0$ this leads to a contradiction as

$$\underbrace{\left(\underbrace{(a\theta_1 + b\theta_2)y}_{\neq 0} - \underbrace{(ay_1 + by_2)\theta}_{=0}\right)}_{\neq 0}\underbrace{\left(\frac{\theta_2}{\theta_1}y_1 - y_2\right)}_{\neq 0} = \underbrace{((ax_1 + bx_2)\theta - (a\theta_1 + b\theta_2)x)}_{=0}\left(\frac{\theta_2}{\theta_1}x_1 - x_2\right).$$

3. Assuming (1.5.20) is true we substitute it into (1.5.15) to find

$$\begin{aligned}&((a\theta_1 + b\theta_2)y - (ay_1 + by_2)\theta)\left(\frac{\theta_2}{\theta_1}z_1 - z_2\right)\\ &= -((ay_1 + by_2)x - (ax_1 + bx_2)y)\left(\frac{\theta_2}{\theta_1}x_1 - x_2\right)\end{aligned} \tag{1.5.21}$$

and substituting it into (1.5.16) we have

$$\begin{aligned}&((ax_1 + bx_2)\theta - (a\theta_1 + b\theta_2)x)\left(\frac{\theta_2}{\theta_1}z_1 - z_2\right)\\ &= -((ay_1 + by_2)x - (ax_1 + bx_2)y)\left(\frac{\theta_2}{\theta_1}y_1 - y_2\right).\end{aligned} \tag{1.5.22}$$

We require this to be true for $\forall x, y, z, \theta, a, b \in \mathbb{R}$. We separate into two cases.

(a) There exists a and b such that $ay_1 + by_2 \neq 0$. We consider (1.5.21). Choose $x = y = 0$ and $\theta \neq 0$ then as $\left(\frac{\theta_2}{\theta_1}z_1 - z_2\right) \neq 0$ this leads to a contradiction as

$$\begin{aligned}&\underbrace{\left(\underbrace{(a\theta_1 + b\theta_2)y}_{=0} - \underbrace{(ay_1 + by_2)\theta}_{\neq 0}\right)}_{\neq 0}\underbrace{\left(\frac{\theta_2}{\theta_1}z_1 - z_2\right)}_{\neq 0}\\ &= -\underbrace{((ay_1 + by_2)x - (ax_1 + bx_2)y)}_{=0}\left(\frac{\theta_2}{\theta_1}x_1 - x_2\right).\end{aligned}$$

(b) For all a and b we have $ay_1 + by_2 = 0$, i.e., $y_1 = 0$ and $y_2 = 0$. We consider (1.5.22). As $\theta_1 \neq 0$ we may choose a and b such that $a\theta_1 + b\theta_2 \neq 0$. We choose $y = \theta = 0$ and $x \neq 0$ then as $\left(\frac{\theta_2}{\theta_1} z_1 - z_2\right) \neq 0$ this leads to a contradiction as

$$\underbrace{\left(\underbrace{(ax_1 + bx_2)\theta}_{=0} - \underbrace{(a\theta_1 + b\theta_2)x}_{\neq 0}\right)}_{\neq 0} \underbrace{\left(\frac{\theta_2}{\theta_1} z_1 - z_2\right)}_{\neq 0}$$
$$= -\underbrace{\left(\underbrace{(ay_1 + by_2)}_{=0} x - \underbrace{(ax_1 + bx_2)y}_{=0}\right)}_{=0} \left(\frac{\theta_2}{\theta_1} y_1 - y_2\right).$$

Hence we conclude that $\theta_1 = 0$.

As we did not specify A_1 and A_2 in proving this, and our equations are formally the same if we swap the subscripts (1) and (2) we may conclude $\theta_2 = 0$. Our equations (from (1.5.10)) then simplify to:

$$(ay_1 + by_2)\theta + f_1 x_1 + f_2 x_2 = 0 \tag{1.5.23}$$
$$(ax_1 + bx_2)\theta - f_1 y_1 - f_2 y_2 = 0 \tag{1.5.24}$$
$$(ay_1 + by_2)x - (ax_1 + bx_2)y + f_1 z_1 + f_2 z_2 = 0. \tag{1.5.25}$$

Now suppose $x_1 \neq 0$ then from (1.5.23) we have that $f_1 = \frac{1}{x_1}(-(ay_1 + by_2)\theta - f_2 x_2)$. Substituting this into (1.5.24) and (1.5.25) get

$$(ax_1 + bx_2)\theta + \frac{y_1}{x_1}(ay_1 + by_2)\theta + \left(\frac{x_2}{x_1} y_1 - y_2\right) f_2 = 0 \tag{1.5.26}$$
$$(ay_1 + by_2)(x - \frac{z_1}{x_1}\theta) - (ax_1 + bx_2)y - \left(\frac{x_2}{x_1} z_1 - z_2\right) f_2 = 0. \tag{1.5.27}$$

Now we note that $\left(\frac{x_2}{x_1} y_1 - y_2\right)$ or $\left(\frac{x_2}{x_1} z_1 - z_2\right)$ is non zero. If both were zero we would have

$$\frac{x_2}{x_1} x_1 = x_2 \qquad \frac{x_2}{x_1} y_1 = y_2 \qquad \frac{x_2}{x_1} z_1 = z_2 \qquad \frac{x_2}{x_1} \theta_1 = \theta_2$$

contradicting our requirement of linear independence. So we have that one of the following two equations are true

$$\left(\frac{x_2}{x_1} y_1 - y_2\right) \neq 0 \qquad \text{i.e., } f_2 = -\frac{(ax_1 + bx_2)\theta + \frac{y_1}{x_1}(ay_1 + by_2)\theta}{\frac{x_2}{x_1} y_1 - y_2} \tag{1.5.28}$$
$$\left(\frac{x_2}{x_1} z_1 - z_2\right) \neq 0 \qquad \text{i.e., } f_2 = \frac{(ay_1 + by_2)(x - \frac{z_1}{x_1}\theta) - (ax_1 + bx_2)y}{\frac{x_2}{x_1} z_1 - z_2}. \tag{1.5.29}$$

We show that both cases lead to a contradiction.

1. Assuming (1.5.28) is true we substitute it into (1.5.27) to get

$$\left((ay_1+by_2)(x-\tfrac{z_1}{x_1}\theta)-(ax_1+bx_2)y\right)\left(y_2-\tfrac{x_2}{x_1}y_1\right)$$
$$=((ax_1+bx_2)\theta+\tfrac{y_1}{x_1}(ay_1+by_2)\theta)\left(\tfrac{x_2}{x_1}z_1-z_2\right).$$

We require this to be true for $\forall x,y,z,\theta,a,b\in\mathbb{R}$. As $x_1\neq 0$ we can choose $a\neq 0$ and $b=0$ so that $(ax_1+bx_2)\neq 0$. Next we may choose $\theta=x=0$ and $y\neq 0$. Then as $\left(y_2-\frac{x_2}{x_1}y_1\right)\neq 0$ we have a contradiction as

$$\underbrace{\left(\underbrace{(ay_1+by_2)(x-\tfrac{z_1}{x_1}\theta)}_{=0}-\underbrace{(ax_1+bx_2)y}_{\neq 0}\right)}_{\neq 0}\underbrace{\left(y_2-\tfrac{x_2}{x_1}y_1\right)}_{\neq 0}$$
$$=\underbrace{((ax_1+bx_2)\theta+\tfrac{y_1}{x_1}(ay_1+by_2)\theta)}_{=0}\left(\tfrac{x_2}{x_1}z_1-z_2\right).$$

2. Assuming (1.5.29) is true we substitute it into (1.5.26) to get

$$\left((ax_1+bx_2)\theta+\tfrac{y_1}{x_1}(ay_1+by_2)\theta\right)\left(z_2-\tfrac{x_2}{x_1}z_1\right)$$
$$=\left((ay_1+by_2)(x-\tfrac{z_1}{x_1}\theta)-(ax_1+bx_2)y\right)\left(\tfrac{x_2}{x_1}y_1-y_2\right). \tag{1.5.30}$$

We require this to be true for $\forall x,y,z,\theta,a,b\in\mathbb{R}$. As $x_1\neq 0$ we can choose $a\neq 0$ and $b=0$ so that $(ax_1+bx_2)\neq 0$. Next we may choose $\theta=1$, $y=0$, and $x=\frac{z_1}{x_1}$. Then as $\left(z_2-\frac{x_2}{x_1}z_1\right)\neq 0$ and as $\left(x_1+\frac{y_1^2}{x_1}\right)\neq 0$ we have a contradiction as

$$\left(\underbrace{a\theta(x_1+\tfrac{y_1^2}{x_1})}_{\neq 0}+\underbrace{b\theta(x_2+\tfrac{y_1y_2}{x_1})}_{=0}\right)\underbrace{\left(z_2-\tfrac{x_2}{x_1}z_1\right)}_{\neq 0}$$
$$=\underbrace{\left((ay_1+by_2)(x-\tfrac{z_1}{x_1}\theta)-(ax_1+bx_2)y\right)}_{=0}\left(\tfrac{x_2}{x_1}y_1-y_2\right). \tag{1.5.31}$$

Hence we conclude that $x_1=0$.

As we did not specify A_1 and A_2 in proving this, and our equations are "symmetric" in the subscripts, we may conclude $x_2=0$. Our equations (from (1.5.10)) then simplify to:

$$(ay_1+by_2)\theta=0 \tag{1.5.32}$$
$$-f_1y_1-f_2y_2=0 \tag{1.5.33}$$
$$(ay_1+by_2)x+f_1z_1+f_2z_2=0 \tag{1.5.34}$$

We require this to be true for $\forall x,y,z,\theta,a,b\in\mathbb{R}$. Then (1.5.32) yields that $y_1=y_2=0$.

Thus in conclusion we have z_1 and z_2 free and the others $(x_1,x_2,y_1,y_2,\theta_1,\theta_2)$ all zero (i.e., we find the ideal $\mathrm{Z}\left(\mathfrak{h}_3^{\diamond}\right)$ again). But this contradicts $\mathfrak{a}$ being of dimension two. Thus there are *no* ideals of dimension two.

1.5.4 Three-dimensional

If we were to proceed here as we did for the two-dimensional case we would need to solve four equations in twelve variables and three functions, which may become *very* "technical" and cumbersome. Fortunately another avenue of investigation is open to us in this case.

Recall that $\mathsf{H}_3^\diamond$ decomposes as a semi-direct product $\mathsf{H}_3 \rtimes \mathsf{SO}_2$. This in particular means that H_3 is a normal (Lie) subgroup. Therefore, by proposition A.1.28, it follows that the Lie algebra $\mathfrak{h}_3$ of H_3 is an ideal of $\mathfrak{h}_3^\diamond$. We recall that $\mathfrak{h}_3 = \{M(x,y,z,0) \mid x,y,z \in \mathbb{R}\}$. So we already have one three-dimensional ideal of $\mathfrak{h}_3^\diamond$. We now show that this is the *only* three-dimensional ideal. First we have a small result.

1.5.2 Lemma. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be two vector subspaces of some n-dimensional vector space. If* $\dim\mathfrak{a} = \dim\mathfrak{b} = \dim\mathfrak{a}\cap\mathfrak{b}$*, then* $\mathfrak{a} = \mathfrak{b}$.

Proof. As $\mathfrak{a}$ and $\mathfrak{b}$ are vector subspaces so is $\mathfrak{a}\cap\mathfrak{b}$. Now let $\dim\mathfrak{a} = \dim\mathfrak{b} = \dim\mathfrak{a}\cap\mathfrak{b} = m$. Then there exists a basis $\{B_i\}_{i\in\overline{1,m}}$ such that $\mathfrak{a}\cap\mathfrak{b} = \operatorname{span}\{B_i\}_{i\in\overline{1,m}}$. But $\mathfrak{a}\cap\mathfrak{b} \subseteq \mathfrak{a}$ and $\mathfrak{a}\cap\mathfrak{b} \subseteq \mathfrak{b}$, thus $\{B_i\}_{i\in\overline{1,m}}$ is a basis for $\mathfrak{a}$ and for $\mathfrak{b}$. Therefore $\mathfrak{a} = \operatorname{span}\{B_i\}_{i\in\overline{1,m}} = \mathfrak{b}$. □

Now suppose $\mathfrak{a}_3$ is another (other than $\mathfrak{h}_3$) three-dimensional ideal. Then for $\mathfrak{a}_3$ and $\mathfrak{h}_3$, as subspaces of the vector space $\mathfrak{h}_3^\diamond$, we have that

$$2 = \dim\mathfrak{h}_3 + \dim\mathfrak{a}_3 - \dim\mathfrak{h}_3^\diamond \leq \dim\mathfrak{h}_3\cap\mathfrak{a}_3 \leq \min(\dim\mathfrak{h}_3, \dim\mathfrak{a}_3) = 3.$$

Thus there are two possibilities

1. $\dim\mathfrak{h}_3\cap\mathfrak{a}_3 = 3$, in which case $\mathfrak{a}_3 = \mathfrak{h}_3$ (lemma 1.5.2).
2. $\dim\mathfrak{h}_3\cap\mathfrak{a}_3 = 2$. Now by proposition 1.5.1 we have that $\mathfrak{h}_3\cap\mathfrak{a}_3$ is an ideal of $\mathfrak{h}_3^\diamond$. But this contradicts the fact the $\mathfrak{h}_3^\diamond$ has no two-dimensional ideals as shown in subsection 1.5.3.

So we conclude that there is only one ideal of dimension three in $\mathfrak{h}_3^\diamond$, namely $\mathfrak{h}_3$.

1.5.5 Summary

1.5.3 Proposition. *The Lie algebra $\mathfrak{h}_3^\diamond$ has exactly two proper ideals namely*

1. $\mathrm{Z}(\mathfrak{h}_3^\diamond) = \{M(0,0,z,0) \mid z \in \mathbb{R}\}$;
2. $\mathfrak{h}_3 = \{M(x,y,z,0) \mid x,y,z \in \mathbb{R}\}$.

1.6 Decomposition of $\mathfrak{h}_3^\diamond$ and its Invariant Scalar Product

The main purpose of this section is to show that $\mathfrak{h}_3^\diamond$, together with the invariant scalar product on it (proposition 1.4.7), is a double extension of the Lie algebra $\mathbb{R}^2$ endowed with the dot product by $\mathbb{R}$. We start by reviewing some of the structure of the class invariant scalar products on real Lie algebras ([26]). We then proceed to show that $\mathfrak{h}_3^\diamond$, together with the its invariant scalar product, is an indecomposable element of this class. Hence, we conclude the section by showing that $\mathfrak{h}_3^\diamond$ is a double extension.

1.6.1 Preliminaries

We define $\mathscr{C}$ as the **class of couples** $(\mathfrak{g}, \varphi)$, where $\mathfrak{g}$ is a finite-dimensional real Lie algebra and $\varphi : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ is a invariant scalar product. We will denote φ by $\langle \cdot, \cdot \rangle$ when convenient. An object $(\mathfrak{g}, \varphi)$ of $\mathscr{C}$ will be termed **indecomposable** if the restriction of φ to any proper ideal of $\mathfrak{g}$ is degenerate (and **decomposable** if not). If $(\mathfrak{g}_i, \varphi_i)$, $i \in \overline{1,2}$ are objects in $\mathscr{C}$, then their **orthogonal direct sum** $(\mathfrak{g}_1 \oplus \mathfrak{g}_2, \varphi_1 \oplus \varphi_2)$ is an object of $\mathscr{C}$. (We note that $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are ideals of $\mathfrak{g}_1 \oplus \mathfrak{g}_2$, to which the restriction of $\varphi_1 \oplus \varphi_2$ will be non-degenerate.) The algebra of dimension one and the simple algebras are indecomposable elements (the non-degenerate invariant scalar products being given by the Killing form). We say that elements $(\mathfrak{g}, \varphi)$ and $(\mathfrak{g}', \varphi')$ of $\mathscr{C}$ are **isomorphic** if there exists a Lie algebra isomorphism $\psi : \mathfrak{g} \to \mathfrak{g}'$ such that $\varphi = \varphi' \circ (\psi \times \psi)$.

For an element $(\mathfrak{g}, \varphi) \in \mathscr{C}$ we define the **right orthogonal complement** of an ideal $\mathfrak{a}$ by $\mathfrak{a}^\perp = \{B \in \mathfrak{g} \mid \varphi(A, B) = 0, \ \forall A \in \mathfrak{a}\}$. We include the following proposition to form a more complete perspective.

1.6.1 Proposition. *Let $(\mathfrak{g}, \varphi) \in \mathscr{C}$ be a decomposable object and $\mathfrak{a}$ be a proper ideal to which the restriction of φ is non-degenerate. Then $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^\perp$ and $(\mathfrak{a}, \varphi|_{\mathfrak{a}}), (\mathfrak{a}^\perp, \varphi|_{\mathfrak{a}^\perp}) \in \mathscr{C}$.*

Proof. We start by showing that $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^\perp$ as vector spaces. Clearly $0 \in \mathfrak{a} \cap \mathfrak{a}^\perp$. For $A \in \mathfrak{a} \cap \mathfrak{a}^\perp$ we have $\forall B \in \mathfrak{a}, \ \langle B, A \rangle = 0$ but as $\mathfrak{a}$ is non-degenerate this implies $A = 0$. Thus we have that $\mathfrak{a} \cap \mathfrak{a}^\perp = \{0\}$. We are left to show that $\mathfrak{a} + \mathfrak{a}^\perp = \mathfrak{g}$. Let $\{A_i\}_{i \in \overline{1,m}}$ be a basis for $\mathfrak{a}$. Extend this to a basis for $\mathfrak{g}$, i.e., $\{A_i\}_{i \in \overline{1,n}}$ is a basis for $\mathfrak{g}$ (with $n > m$). Now with respect to this basis we may realise φ by a matrix M. That is, for $v = \sum_{i=1}^n v_i A_i, \ w = \sum_{i=1}^n w_i A_i$ in $\mathfrak{g}$ we have $\varphi(v, w) = v^\top M w$. Define an induced mapping $\varphi_{\mathfrak{a}} : v \mapsto [I_m \ 0] M v \in \mathbb{R}^n$. Then the null space of $\varphi_{\mathfrak{a}}$ is $\mathfrak{a}^\perp$ (in coordinates). Now as φ is non-degenerate, M is of full rank, and hence $[I_m \ 0]M$ is of rank m. Thus by the rank nullity theorem $[I_m \ 0]M$ has nullspace of dimension $n - m$. Thus as $\mathfrak{a} \cap \mathfrak{a}^\perp = \{0\}$ we have $\mathfrak{a} + \mathfrak{a}^\perp = \mathfrak{g}$. Hence $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^\perp$ as vector spaces.

Next let $A \in \mathfrak{a}$, $B \in \mathfrak{a}^\perp$ and $C \in \mathfrak{g}$. Then $\langle A, [C, B] \rangle = \langle [A, C], B \rangle = 0$ as $[A, C] \in \mathfrak{a}$ and $B \in \mathfrak{a}^\perp$. Thus $[C, B] \in \mathfrak{a}^\perp$ and so $\mathfrak{a}^\perp$ is an ideal of $\mathfrak{g}$. Now let $B \in \mathfrak{a}^\perp$ and suppose that $\forall A \in \mathfrak{a}^\perp, \ \langle A, B \rangle = 0$. Then we get that

$$\begin{aligned} & \forall A \in \mathfrak{a}^\perp, \ \forall C \in \mathfrak{a}, \ \langle A + C, B \rangle = 0 \\ \Rightarrow \ & \forall C \in \mathfrak{g}, \ \langle C, B \rangle = 0 \qquad (\text{as } \mathfrak{g} = \mathfrak{a} + \mathfrak{a}^\perp) \\ \Rightarrow \ & B = 0 \quad (\varphi \text{ is non-degenerate}). \end{aligned}$$

Thus $\varphi|_{\mathfrak{a}^\perp}$ is non-degenerate. So we have $(\mathfrak{a}, \varphi|_{\mathfrak{a}}), (\mathfrak{a}^\perp, \varphi|_{\mathfrak{a}^\perp}) \in \mathscr{C}$ (and thus $(\mathfrak{a} \oplus \mathfrak{a}^\perp, \varphi|_{\mathfrak{a}} \perp \varphi|_{\mathfrak{a}^\perp}) \in \mathscr{C}$). We already have that $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^\perp$ as vector spaces. We are left to show that $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^\perp$ as Lie algebras. That is, we are left to show that the Lie bracket on $\mathfrak{g}$ is the direct sum of Lie brackets on $\mathfrak{a}$ and $\mathfrak{a}^\perp$. We first show that for $A \in \mathfrak{a}$ and $B \in \mathfrak{a}^\perp$ we have that $[A, B] = 0$. Indeed

$$\forall C \in \mathfrak{g}, \ \langle [A, B], C \rangle = \langle \underbrace{A}_{\in \mathfrak{a}}, \underbrace{[B, C]}_{\in \mathfrak{a}^\perp} \rangle = 0$$

and thus as φ is non-degenerate $[A, B] = 0$. We denote projections to $\mathfrak{a}$ and $\mathfrak{a}^\perp$ by $\mathrm{pr}_{\mathfrak{a}}$ and $\mathrm{pr}_{\mathfrak{a}^\perp}$

respectively. So then for $A, B \in \mathfrak{g}$ we have that

$$\begin{aligned}
[A, B] &= [\mathrm{pr}_{\mathfrak{a}} A + \mathrm{pr}_{\mathfrak{a}^\perp} A, \mathrm{pr}_{\mathfrak{a}} B + \mathrm{pr}_{\mathfrak{a}^\perp} B] \\
&= [\mathrm{pr}_{\mathfrak{a}} A, \mathrm{pr}_{\mathfrak{a}} B] + [\mathrm{pr}_{\mathfrak{a}} A, \mathrm{pr}_{\mathfrak{a}^\perp} B] + [\mathrm{pr}_{\mathfrak{a}^\perp} A, \mathrm{pr}_{\mathfrak{a}} B] + [\mathrm{pr}_{\mathfrak{a}^\perp} A, \mathrm{pr}_{\mathfrak{a}^\perp} B] \\
&= [\mathrm{pr}_{\mathfrak{a}} A, \mathrm{pr}_{\mathfrak{a}} B] + 0 + 0 + [\mathrm{pr}_{\mathfrak{a}^\perp} A, \mathrm{pr}_{\mathfrak{a}^\perp} B] \\
&= [\mathrm{pr}_{\mathfrak{a}} A, \mathrm{pr}_{\mathfrak{a}} B]_{\mathfrak{a}} + [\mathrm{pr}_{\mathfrak{a}^\perp} A, \mathrm{pr}_{\mathfrak{a}^\perp} B]_{\mathfrak{a}^\perp}. \qquad \square
\end{aligned}$$

Thus any Lie algebra admitting an invariant scalar product and forming a decomposable pair (together with this scalar product) is the direct sum of Lie algebras admitting invariant scalar products. We now consider another type of decomposition, namely double extensions.

Let $(\mathfrak{a}, \varphi)$ be an object of $\mathscr{C}$, $\mathfrak{b}$ be a Lie algebra and $d : \mathfrak{b} \to \mathrm{Der}\, \mathfrak{a}$, a homomorphism (of Lie algebras) from $\mathfrak{b}$ to the algebra of antisymmetric derivations of $\mathfrak{a}$. Here, by an antisymmetric derivation we mean antisymmetric with respect to φ, i.e.,

$$\forall B \in \mathfrak{b},\ A_1, A_2 \in \mathfrak{a},\ \varphi(d(B) \cdot A_1, A_2) + \varphi(A_1, d(B) \cdot A_2) = 0.$$

We define a mapping

$$\omega : \mathfrak{a} \times \mathfrak{a} \to \mathfrak{b}^*, \qquad (A, A') \mapsto \varphi(d(\cdot)\ A, A').$$

Then the vector space $\mathfrak{g} = \mathfrak{b}^* \oplus \mathfrak{a} \oplus \mathfrak{b}$ is a Lie algebra with Lie bracket given by

$$\begin{aligned}
&[(p, A, B), (p', A', B')] \\
&= \Big(\mathrm{ad}^* B' \cdot p - \mathrm{ad}^* B \cdot p' + \omega(A, A'), [A, A']_{\mathfrak{a}} + d(B) \cdot A' - d(B') \cdot A,\ [B, B']_{\mathfrak{b}}\Big). \qquad (1.6.1)
\end{aligned}$$

The algebra $\mathfrak{g}$ is called the **double extension** of $(\mathfrak{a}, \varphi)$ by $\mathfrak{b}$, by means of d. Moreover if ϕ is a invariant bilinear form on $\mathfrak{b}$, then the form $\varphi' : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ given by

$$\varphi'((p, A, B), (p', A', B')) = \varphi(A, A') + \phi(B, B') + p(B') + p'(B) \qquad (1.6.2)$$

is a non-degenerate invariant scalar product on $\mathfrak{g}$. In this case we say the pair $(\mathfrak{g}, \varphi')$ is a **double extension** of $(\mathfrak{a}, \varphi)$ by $\mathfrak{b}$, by means of d and ϕ.

We are now in a position to quote some results regarding the structure of $\mathscr{C}$ as presented in [26]. The main result is the following.

1.6.2 Theorem. ([26]) *Every indecomposable object $(\mathfrak{g}, \varphi)$ of $\mathscr{C}$, where $\mathfrak{g}$ is not simple, is (isomorphic to) a double extension of a couple $(\mathfrak{a}, \psi)$ by a simple or one-dimensional algebra.*

Among the consequence of this result we have the following (included only to form a more complete perspective).

1.6.3 Theorem. ([26]) *The class $\mathscr{C}$ of Lie algebras admitting invariant scalar products is the smallest class of Lie algebras which contains the simple and abelian Lie algebras and are stable w.r.t. direct sum and double extension.*

1.6.4 Theorem. ([26]) *The class $\mathscr{C}'$ of solvable Lie algebras admitting invariant scalar products is the smallest class of Lie algebras which contains the abelian Lie algebras and is stable w.r.t. direct sum and double extensions by algebras of dimension one.*

1.6.2 The Lie algebra $\mathfrak{h}_3^\diamond$ as a double extension

Having found all the ideals of $\mathfrak{h}_3^\diamond$ (in section 1.5) we can show that the invariant scalar product (as defined in proposition 1.4.7) on $\mathfrak{h}_3^\diamond$ restricted to any proper ideal of $\mathfrak{h}_3^\diamond$ is degenerate.

1.6.5 PROPOSITION. *The invariant scalar product φ (introduced in proposition 1.4.7)*

$$\varphi : \mathfrak{h}_3^\diamond \times \mathfrak{h}_3^\diamond \to \mathbb{R}$$
$$(M(x,y,z,\theta), M(x',y',z',\theta')) \mapsto xx' + yy' - z\theta' - \theta z'$$

restricted to any ideal of $\mathfrak{h}_3^\diamond$ is degenerate.

PROOF. We show φ is degenerate when restricted to each of the only two ideals of $\mathfrak{h}_3^\diamond$ as listed in proposition 1.5.3.

1. When restricted to $\mathrm{Z}(\mathfrak{h}_3^\diamond) = \{M(0,0,z,0) \mid z \in \mathbb{R}\}$ our scalar product becomes:

$$\varphi|_{\mathrm{Z}(\mathfrak{h}_3^\diamond)\times \mathrm{Z}(\mathfrak{h}_3^\diamond)} : (M(0,0,z,0), M(0,0,z',0)) \mapsto 0$$

   which is clearly degenerate.

2. When restricted to $\mathfrak{h}_3 = \{M(x,y,z,0) \mid x,y,z \in \mathbb{R}\}$ our scalar product becomes:

$$\varphi|_{\mathfrak{h}_3\times\mathfrak{h}_3} : (M(x,y,z,0), M(x',y',z',0)) \mapsto xx' + yy'$$

   which is degenerate as for $\forall z \in \mathbb{R},\ A \in \mathfrak{h}_3,\ \langle M(0,0,z,0), A\rangle = 0$. □

1.6.6 COROLLARY. *The pair $(\mathfrak{h}_3^\diamond, \varphi)$, with φ as described above, is an indecomposable element of $\mathscr{C}$.*

Theorem 1.6.2 states that any indecomposable element of $\mathscr{C}$ may be realised as a double extension. We will now proceed to show that $\mathfrak{h}_3^\diamond$ is a double extension of $\mathbb{R}^2$ by $\mathbb{R}$.

In order to keep our calculations relatively simple and to expose the structure better we will represent $\mathbb{R}^2$ and $\mathbb{R}$ as subspaces of $\mathfrak{h}_3^\diamond$. We represent $\mathbb{R}^2$ as

$$\mathbb{R}^2 = \{M(x,y,0,0) = xE_1 + yE_2 \mid x,y \in \mathbb{R}\}$$

If we endow $\mathbb{R}^2$ with the trivial Lie bracket we have that $\mathbb{R}^2$ is a Lie algebra. The dot product on $\mathbb{R}^2$, given (in our representation) by

$$M(x,y,0,0) \bullet M(x',y',0,0) = xx' + yy'$$

is then a non-degenerate invariant scalar product on $\mathbb{R}^2$. Moreover we note that the dot product is the restriction of our scalar product on $\mathfrak{h}_3^\diamond$ to the subspace $\mathbb{R}^2$ as represented here. We represent $\mathbb{R}$ as

$$\mathbb{R} = \{M(0,0,0,\theta) = \theta E_4 \mid \theta \in \mathbb{R}\}$$

and endow it with the trivial lie bracket. We also note that for each $a \in \mathbb{R}$,

$$\phi_a : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \quad (\theta E_4, \theta' E_4) \mapsto a\theta\theta'$$

is a invariant bilinear form.

The required homomorphism then turns out to be given by $d = \mathrm{ad}$, i.e.,

$$\begin{aligned} \mathrm{ad} : \mathbb{R} &\to \mathrm{Der}(\mathbb{R}^2) \\ M(0,0,0,\theta) &\mapsto \mathrm{ad}\, M(0,0,0,\theta) \\ \mathrm{ad}\, M(0,0,0,\theta) \cdot M(x,y,0,0) &= M(\theta y, -\theta x, 0, 0). \end{aligned}$$

We note that as the dot product on $\mathbb{R}^2$ is a restriction of our invariant scalar product we get (for $B \in \mathbb{R}$ and $A_1, A_2 \in \mathbb{R}^2$ in our convention) that

$$\langle \mathrm{ad}\, B \cdot A_1, A_2 \rangle + \langle A_1, \mathrm{ad}\, B \cdot A_2 \rangle = \langle [B, A_1], A_2 \rangle + \langle A_1, [B, A_2] \rangle = 0.$$

Thus ad indeed maps to antisymmetric derivations. The mapping $\omega : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^*$ is then given by

$$\omega(xE_1 + yE_2, x'E_1 + y'E_2) = (yx' - xy')E_4^*$$

Finally note that $\mathrm{ad}^*(\theta' E_4) \cdot zE_4^* = \mathrm{ad}^*(\theta E_4) \cdot z'E_4^* = 0$.

We now have all the ingredients required to state our claim.

1.6.7 Proposition. *The indecomposable element* $(\mathfrak{h}_3^\diamond, \varphi)$ *of* $\mathscr{C}$ *(as in corollary 1.6.6) is isomorphic to the double extension of* $(\mathbb{R}^2, \bullet)$ *by* $\mathbb{R}$ *by means of* ad *and the trivial bilinear product* ϕ_0 *on* $\mathbb{R}$*.*

Proof. We claim that $\psi : \mathbb{R}^* \oplus \mathbb{R}^2 \oplus \mathbb{R} \to \mathfrak{h}_3^\diamond$, $(zE_4^*, xE_1 + yE_2, \theta E_4) \mapsto xE_1 + yE_2 - zE_3 + \theta E_4$ is the required Lie algebra isomorphism. Now ψ is clearly a vector space isomorphism so we need only show that it preserves the Lie bracket to prove that it is a Lie algebra isomorphism. First note that the Lie bracket on $\mathbb{R}^* \oplus \mathbb{R}^2 \oplus \mathbb{R}$ (as in equation (1.6.1)) is given by

$$\begin{aligned} &[(zE_4^*, xE_1 + yE_2, \theta E_4), (z'E_4^*, x'E_1 + y'E_2, \theta' E_4)] \\ &= \Big((yx' - xy')E_4^*, \mathrm{ad}(\theta E_4) \cdot (x'E_1 + y'E_2) - \mathrm{ad}(\theta' E_4) \cdot (xE_1 + yE_2),\ 0 \Big) \\ &= \Big((yx' - xy')E_4^*, (\theta y' - y\theta')E_1 + (x\theta' - \theta x')E_2,\ 0 \Big). \end{aligned}$$

Then we have that

$$\begin{aligned} &[\psi(zE_4^*, xE_1 + yE_2, \theta E_4), \psi(z'E_4^*, x'E_1 + y'E_2, \theta' E_4)] \\ &= [xE_1 + yE_2 - zE_3 + \theta E_4, x'E_1 + y'E_2 - z'E_3 + \theta' E_4] \\ &= (\theta y' - y\theta')E_1 + (x\theta' - \theta x')E_2 + (xy - yx')E_4 \\ &= \psi((yx' - xy')E_4^*, (\theta y' - y\theta')E_1 + (x\theta' - \theta x')E_2, 0) \\ &= \psi\left(\left[(zE_4^*, xE_1 + yE_2, \theta E_4), (z'E_4^*, x'E_1 + y'E_2, \theta' E_4) \right] \right). \end{aligned}$$

We are left to show that the invariant scalar product φ' on $\mathbb{R}^* \oplus \mathbb{R}^2 \oplus \mathbb{R}$, as introduced in equation (1.6.2), satisfies $\varphi' = \varphi \circ (\psi \times \psi)$. Indeed we have that

$$\begin{aligned}
&\varphi\Big(\psi(zE_4^*, xE_1 + yE_2, \theta E_4), \psi(z'E_4^*, x'E_1 + y'E_2, \theta' E_4)\Big) \\
&= \varphi\Big(xE_1 + yE_2 - zE_3 + \theta E_4, x'E_1 + y'E_2 - z'E_3 + \theta' E_4\Big) \\
&= xx' + yy' + z\theta' + \theta z' \\
&= \big\langle xE_1 + yE_2, x'E_1 + y'E_2\big\rangle + \phi_0(\theta E_4, \theta' E_4) + zE_4^* \cdot \theta' E_4 + z'E_4^* \cdot \theta E_4 \\
&= \varphi'\Big((zE_4^*, xE_1 + yE_2, \theta E_4), (z'E_4^*, x'E_1 + y'E_2, \theta' E_4)\Big).
\end{aligned}$$

□

1.6.8 COROLLARY. *We have a family $\{\varphi_a \mid a \in \mathbb{R}\}$ of invariant scalar products on $\mathfrak{h}_3^\diamond$ given by*

$$\varphi_a : \mathfrak{h}_3^\diamond \times \mathfrak{h}_3^\diamond \to \mathbb{R}$$

$$(M(x,y,z,\theta), M(x',y',z',\theta')) \mapsto \begin{bmatrix} x \\ y \\ z \\ \theta \end{bmatrix}^\top \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & a \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \\ \theta' \end{bmatrix}.$$

PROOF. We have that φ'_a defines a family of invariant scalar products on $\mathbb{R}^* \oplus \mathbb{R}^2 \oplus \mathbb{R}$ as defined in general in equation (1.6.2) (depending on the family of invariant bilinear forms $\{\phi_a \mid a \in \mathbb{R}\}$ on $\mathbb{R}$). Let $\psi : \mathbb{R}^* \oplus \mathbb{R}^2 \oplus \mathbb{R} \to \mathfrak{h}_3^\diamond$ be the Lie algebra isomorphism introduced in the proof of the preceding proposition. Then define $\varphi_a = \varphi'_a \circ (\psi^{-1} \times \psi^{-1})$ to get a family of invariant scalar products. Invariance follows as (for $A, B, C \in \mathfrak{h}_3^\diamond$)

$$\varphi_a([A,B],C) = \varphi'_a([\psi^{-1}(A), \psi^{-1}(B)], C) = \varphi'_a(\psi^{-1}(A), \psi^{-1}([B,C])) = \varphi_a(A,[B,C]).$$

Nondegeneracy simply follows as ψ is a linear isomorphism. □

1.6.9 REMARK. In a similar fashion as in the proof of corollary 1.4.8 we get that $xE_1^* + yE_2^* + zE_3^* + \theta E_4^* \mapsto x^2 + y^2 + az^2 - 2z\theta$ is a Casimir function for $a \in \mathbb{R}$. But this Casimir function is simply a linear combination of our Casimir function $\mathcal{C}$ (given in corollary 1.4.8, corresponding to $a = 0$ here) and the Casimir function $xE_1^* + yE_2^* + zE_3^* + \theta E_4^* \mapsto z^2$ (introduced in chapter 4 as a consequence of the result $\mathrm{Z}(\mathfrak{h}_3^\diamond) = \{M(0,0,z,0) \mid z \in \mathbb{R}\}$).

Chapter 2

Local Classification of Full Rank Systems on Oscillator Lie Groups

In this chapter we classify all full rank left invariant control affine systems, with Lie algebra isomorphic to $\mathfrak{h}_3^\diamond$, under local detached feedback equivalence. This is achieved as follows. First, we reduce the problem (cf. [7]) of classifying under DF_{loc}-equivalence to classifying affine subspaces of $\mathfrak{h}_3^\diamond$ under an appropriate equivalence relation. Then, we postulate a classification and explicitly construct relations mapping any given affine subspace to one of a list of equivalence representatives. Finally, we verify that all of the said equivalence representatives represent distinct equivalence classes.

We briefly recall some concepts as introduced in section A.2. A left invariant control affine system is a pair (G, Ξ), where G is a Lie Group and $\Xi : \mathsf{G} \times \mathbb{R}^\ell \to T\mathsf{G}$ is a smooth embedding such that

$$\Xi(g,u) = g\,\Xi(\mathbf{1},u) = g\left(A + \sum_{i=1}^{\ell} u_i B_i\right)$$

where the set $\{B_i\}_{i=\overline{1,\ell}}$ is linearly independent. These systems can be organised into a category (see [6] and section A.2) which we denote **LiCAS**. The **trace** Γ of the system is defined as $\Gamma = A + \Gamma^0$, where $\Gamma^0 = \operatorname{span}\{B_i\}_{i=\overline{1,\ell}}$. A system Σ is said to be of full rank if $\operatorname{Lie}\Gamma = \mathfrak{g}$ (where $\operatorname{Lie}\Gamma$ is the smallest Lie subalgebra containing Γ).

Let $\Sigma = (\mathsf{G}, \Xi)$ and $\Sigma' = (\mathsf{G}', \Xi')$ be two systems. Then Σ and Σ' are called **locally detached feedback equivalent** (shortly DF_{loc}-equivalent) at points $a \in \mathsf{G}$ and $a' \in \mathsf{G}'$ if there exist open neighbourhoods N and N' of a and a', respectively, and a (local) diffeomorphism $\Phi : N \times \mathbb{R}^\ell \to N' \times \mathbb{R}^{\ell'}$, $(g,u) \mapsto (\phi(g), \bar{\varphi}(u))$ such that $\phi(a) = a'$ and $T_g\phi \cdot \Xi(g,u) = \Xi'(\phi(g), \bar{\varphi}(u))$ for $g \in N$ and $u \in \mathbb{R}^\ell$. Note that any DF_{loc}-equivalence between two system can be reduced to a local equivalence between neighbourhoods of identity (see proposition A.2.13). We base our work on the following characterisation of DF_{loc}-equivalence.

2.0.1 Theorem. ([7]) *Two systems Σ and Σ' of full rank are DF_{loc}-equivalent (at identity) if and only if there exists a Lie algebra isomorphism $\psi : \mathfrak{g} \to \mathfrak{g}'$ such that $\psi \cdot \Gamma = \Gamma'$.*

2.1 Preliminaries

We use the following conventions. For a subset S of a linear space we will use the notation $\langle S \rangle$ to mean the linear span of S. Furthermore, for the span of a finite set of elements $\{A_i\}_{i=\overline{1,n}}$ we will simply write $\langle A_1, A_2, \ldots, A_n \rangle$. We will call an affine subspace Γ **homogeneous** if $0 \in \Gamma$ (i.e. Γ is subspace of $\mathfrak{g}$), and **inhomogeneous** if not. Let S be any subset of a Lie algebra $\mathfrak{g}$. Then by $\mathrm{Lie}\, S$ we mean the smallest Lie sub algebra of $\mathfrak{g}$ containing S. An affine subspace Γ of $\mathfrak{g}$ is said to be of **full rank** if $\mathrm{Lie}\,\Gamma = \mathfrak{g}$ or equivalently $\dim(\mathrm{Lie}\,\Gamma) = \dim \mathfrak{g}$.

Two affine subspaces Γ_1 and Γ_2 of some Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$, respectively, are said to be **$\mathfrak{L}$-related** (shortly $\Gamma_1 \sim \Gamma_2$) if there exists a Lie algebra isomorphism $\psi : \mathfrak{g}_1 \to \mathfrak{g}_2$ such that $\psi \cdot \Gamma_1 = \Gamma_2$. Hence, by theorem 2.0.1, systems Σ_1 and Σ_2 are DF_{loc}-equivalent if and only if $\Gamma_1 \sim \Gamma_2$.)

2.1.1 Lemma. *The above relation is an equivalence relation.*

Proof. Reflexivity. The identity mapping is always an automorphism, hence $id \cdot \Gamma = \Gamma$, i.e., $\Gamma \sim \Gamma$. Symmetry. if $\psi : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a Lie algebra isomorphism such that $\psi \cdot \Gamma_1 = \Gamma_2$, then $\psi^{-1} : \mathfrak{g}_2 \to \mathfrak{g}_1$ is also a Lie algebra isomorphism and $\psi^{-1} \cdot \Gamma_2 = \Gamma_1$. Transitivity. Suppose $\Gamma_1 \sim \Gamma_2$ and $\Gamma_2 \sim \Gamma_3$. Then we have Lie algebra isomorphisms $\psi_1 : \mathfrak{g}_1 \to \mathfrak{g}_2$ and $\psi_2 : \mathfrak{g}_2 \to \mathfrak{g}_3$ such that $\psi_1 \cdot \Gamma_1 = \Gamma_2$ and $\psi_2 \cdot \Gamma_2 = \Gamma_3$. Hence, $\psi_2 \cdot \psi_1 : \mathfrak{g}_1 \to \mathfrak{g}_3$ is a Lie algebra isomorphism such that $\psi_2 \cdot \psi_1 \cdot \Gamma_1 = \Gamma_3$. Thus $\Gamma_1 \sim \Gamma_3$. □

2.1.2 Lemma. *Let $\psi : \mathfrak{g}_1 \to \mathfrak{g}_2$ be a Lie algebra isomorphism, S be any subset of $\mathfrak{g}_1$. Then $\psi \cdot \mathrm{Lie}\, S = \mathrm{Lie}\,(\psi \cdot S)$. Hence, if $\Gamma_1 \sim \Gamma_2$, then Γ_1 is of full rank if and only if Γ_2 is of full rank.*

Proof. As ψ is a Lie algebra isomorphism, it maps subalgebras to subalgebras. So, as $\mathrm{Lie}\, S$ is a subalgebra containing S, we have that $\psi \cdot \mathrm{Lie}\, S$ is a subalgebra containing $\psi \cdot S$. As $\mathrm{Lie}\,(\psi \cdot S)$ is the smallest subalgebra containing $\psi \cdot S$, it follows that $\mathrm{Lie}\,(\psi \cdot S) \subseteq \psi \cdot \mathrm{Lie}\, S$. Similarly (as ψ^{-1} is also a Lie algebra isomorphism) we have that $\mathrm{Lie}\,(\psi^{-1}(\psi \cdot S)) \subseteq \psi^{-1} \cdot \mathrm{Lie}\,(\psi \cdot S)$, i.e., $\mathrm{Lie}\, S \subseteq \psi^{-1} \cdot \mathrm{Lie}\,(\psi \cdot S)$ and so $\psi \cdot \mathrm{Lie}\, S \subseteq \mathrm{Lie}\,(\psi \cdot S)$. Thus $\psi \cdot \mathrm{Lie}\, S = \mathrm{Lie}\,(\psi \cdot S)$. Now suppose $\Gamma_1 \sim \Gamma_2$ and Γ_1 is of full rank. Then we have a Lie algebra isomorphism $\psi : \mathfrak{g}_1 \to \mathfrak{g}_2$ such that $\psi \cdot \Gamma_1 = \Gamma_2$. Hence $\dim \mathrm{Lie}\,\Gamma_2 = \dim \mathrm{Lie}\,(\psi \cdot \Gamma_1) = \dim(\psi \cdot \mathrm{Lie}\,\Gamma_1) = \dim \mathrm{Lie}\,\Gamma_1 = \dim \mathfrak{g}_1 = \dim(\psi \cdot \mathfrak{g}_1) = \dim \mathfrak{g}_2$. The converse holds by the same argument. □

2.1.3 Lemma. *Two affine subspaces $\Gamma_1 = A_1 + \Gamma_1^0$ and $\Gamma_2 = A_2 + \Gamma_2^0$ (of some Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$, respectively) are $\mathfrak{L}$-related if and only if there exists a Lie algebra isomorphism $\psi : \mathfrak{g}_1 \to \mathfrak{g}_2$ such that $\psi \cdot A_1 \in \Gamma_2$ and $\psi \cdot \Gamma_1^0 = \Gamma_2^0$.*

Proof. Suppose $\Gamma_1 \sim \Gamma_2$, then there exists a Lie algebra isomorphism $\psi : \mathfrak{g}_1 \to \mathfrak{g}_2$ such that $\psi \cdot \Gamma_1 = \Gamma_2$. Now as $A_1 \in \Gamma_1$ we have that $\psi \cdot A_1 \in \Gamma_2$, that is to say $\psi \cdot A_1 = A_2 + B_2$ for some $B_2 \in \Gamma_2^0$. Then, as $\psi \cdot \Gamma_1 = \Gamma_2$, we have that $A_2 + B_2 + \psi \cdot \Gamma_1^0 = A_2 + \Gamma_2^0$. Hence, we get that $\psi \cdot \Gamma_1^0 = \Gamma_2^0$. For the converse assume that we have a Lie algebra automorphism ψ satisfying our conditions. Then, as $\psi \cdot A_1 \in \Gamma_2$, there exists a $B_2 \in \Gamma_2^0$ such that $\psi \cdot A_1 = A_2 + B_2$. Thus $\psi \cdot \Gamma_1 = \psi \cdot A_1 + \psi \cdot \Gamma_1^0 = A_2 + B_2 + \Gamma_2^0 = \Gamma_2$. □

At this stage we have the following necessary conditions for systems Σ_1 and Σ_2 to be DF_{loc}-equivalent.

2.1.4 LEMMA. *If* Σ_1 *and* Σ_2 *are* DF_{loc}*-equivalent, then:* $\mathfrak{g}_1 \cong \mathfrak{g}_2$*;* $\dim \Gamma_1^0 = \dim \Gamma_2^0$*;* Γ_1 *and* Γ_2 *are both homogeneous or both inhomogeneous.*

PROOF. The first two items follow from theorem 2.0.1. If Σ_1 and Σ_2 are DF_{loc}-equivalent, then there exists a Lie algebra isomorphism $\psi : \mathfrak{g}_1 \to \mathfrak{g}_2$ such that $\psi \cdot \Gamma_1 = \Gamma_2$ (and $\Gamma_1 = \psi^{-1} \cdot \Gamma_2$). Thus if $0 \in \Gamma_1$, then $0 \in \Gamma_2$ (and conversely). □

Thus we can separate our systems by the associated Lie algebra (up to isomorphism), dimension of trace and homogeneity. This motivates us to define the following family of classes of full rank systems:

$$\mathscr{C}_{(\mathfrak{c},\ell,\varepsilon)} = \left\{\Sigma \in \mathrm{Ob}\,\mathbf{LiCAS} \mid \Sigma \text{ is of full rank}, \ \mathfrak{g} \cong \mathfrak{c}, \ \ \dim \Gamma^0 = \ell, \mathrm{Hom}\,\Gamma = \varepsilon\right\}$$

where $\mathfrak{c}$ is a (finite-dimensional) Lie algebra, $\ell \in \mathbb{N}, \ell \leq \dim(\mathfrak{c})$, $\varepsilon \in \{0,1\}$ and

$$\mathrm{Hom}\,\Gamma = \begin{cases} 0 & \Gamma \text{ is homogeneous, i.e., } \Gamma = \Gamma^0 \\ 1 & \Gamma \text{ is inhomogeneous.} \end{cases}$$

Having fixed $\mathfrak{c}$ we will refer to an affine subspace of dimension ℓ and homogeneity ε as a **(ℓ, ε)-affine subspace**. Note that if two systems are DF_{loc} equivalent they necessarily belong to the same class (by preceding lemma). Furthermore note that if $\mathfrak{g}_1 \cong \mathfrak{g}_2$ then $\mathscr{C}_{(\mathfrak{g}_1,\ell,\varepsilon)} = \mathscr{C}_{(\mathfrak{g}_2,\ell,\varepsilon)}$.

Finding ourselves in one of these classes the question of classification then reduces to finding $\mathfrak{L}$-related affine subspaces. Now given two systems Σ_1 and Σ_2 in some class $\mathscr{C}_{(\mathfrak{c},\ell,\varepsilon)}$ we have that both $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are isomorphic to (the common representation) $\mathfrak{c}$. Then we have Lie algebra isomorphisms $\psi_1 : \mathfrak{g}_1 \to \mathfrak{c}$ and $\psi_2 : \mathfrak{g}_2 \to \mathfrak{c}$. Thus we may transfer the affine subspaces of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ to $\mathfrak{c}$ by means of these isomorphisms. Then the two systems are DF_{loc}-equivalent if and only if these transferred affine subspaces are $\mathfrak{L}$-related by a Lie algebra automorphism. We now make a concrete statement.

2.1.5 PROPOSITION. *Let* Σ_1 *and* Σ_2 *be elements of a class* $\mathscr{C}_{(\mathfrak{c},\ell,\varepsilon)}$ *and suppose* $\psi_1 : \mathfrak{g}_1 \to \mathfrak{c}$ *and* $\psi_2 : \mathfrak{g}_2 \to \mathfrak{c}$ *are Lie algebra isomorphisms. Then* Σ_1 *and* Σ_2 *are* DF_{loc}*-equivalent if and only if there exist a Lie algebra automorphism* $\psi_{\mathfrak{c}} \in \mathrm{Aut}\,\mathfrak{c}$ *such that* $\psi_{\mathfrak{c}} \cdot (\psi_1 \cdot \Gamma_1) = (\psi_2 \cdot \Gamma_2)$*, i.e.,* $(\psi_1 \cdot \Gamma_1) \sim (\psi_2 \cdot \Gamma_2)$.

PROOF. Suppose that Σ_1 and Σ_2 are DF_{loc}-equivalent. Then there exists a Lie algebra isomorphism $\psi : \mathfrak{g}_1 \to \mathfrak{g}_2$ such that $\psi \cdot \Gamma_1 = \Gamma_2$. Now consider the mapping $\psi_{\mathfrak{c}} = \psi_2 \circ \psi \circ \psi_1^{-1} : \mathfrak{c} \to \mathfrak{c}$. As a composition of Lie algebra isomorphisms it is a Lie algebra isomorphism and hence a Lie algebra automorphism. Finally we have that

$$\psi_{\mathfrak{c}} \cdot (\psi_1 \cdot \Gamma_1) = \psi_2 \cdot \psi \cdot \psi_1^{-1} \cdot \psi_1 \cdot \Gamma_1 = \psi_2 \cdot \psi \cdot \Gamma_1 = (\psi_2 \cdot \Gamma_2).$$

Conversely, suppose we have a Lie algebra automorphism $\psi_{\mathfrak{c}} : \mathfrak{c} \to \mathfrak{c}$ such that $\psi_{\mathfrak{c}} \cdot (\psi_1 \cdot \Gamma_1) = (\psi_2 \cdot \Gamma_2)$. Define a mapping $\psi = \psi_2^{-1} \circ \psi_{\mathfrak{c}} \circ \psi_1 : \mathfrak{g}_1 \to \mathfrak{g}_2$. Then as a composition of Lie algebra isomorphisms it is a Lie algebra isomorphism and we have that

$$\psi \cdot \Gamma_1 = \psi_2^{-1} \cdot \psi_{\mathfrak{c}} \cdot (\psi_1 \cdot \Gamma_1) = \psi_2^{-1} \cdot (\psi_2 \cdot \Gamma_2) = \Gamma_2$$

Thus Σ_1 and Σ_2 are DF_{loc}-equivalent. □

Hence classification within a class $\mathscr{C}_{(\mathfrak{c},\ell,\varepsilon)}$ may be accomplished by choosing a specific representation of the Lie algebra and then classifying full rank $\mathfrak{L}$-related affine subspaces. Our problem is thus mainly one of classifying affine subspaces $\mathfrak{L}$-related by Lie algebra automorphisms for a given Lie algebra. We may then "decode" our classification as follows. Suppose we have a complete list $\{\Gamma_i \mid i \in I\}$ of equivalence representatives of full rank $\mathfrak{L}$-related affine subspaces of a Lie algebra $\mathfrak{g}$. Let G be a Lie group with Lie algebra (isomorphic to) $\mathfrak{g}$ (for example the simply connected or universal covering Lie group). Then we need only to choose affine parametrisations $\Xi_i(\mathbf{1}, \cdot)$ of each Γ_i to get a complete list of equivalence representatives of DF_{loc} equivalent systems with Lie algebra isomorphic to $\mathfrak{g}$.

2.1.6 Remark. There is only one affine subspace whose dimension coincides with that of the Lie algebra $\mathfrak{g}$, namely the space itself (and it is necessarily homogeneous). Thus from the standpoint of classification there is nothing to be done here. Also note that there are no $(0,0)$, $(0,1)$ or $(1,0)$-affine subspaces of full rank for $\mathfrak{h}_3^\diamond$. As such we may safely ignore these cases.

So in order to classify DF_{loc}-equivalent full rank systems with Lie algebra isomorphic to $\mathfrak{h}_3^\diamond$, we need only to classify $\mathfrak{L}$-related full rank affine subspaces of $\mathfrak{h}_3^\diamond$. Note that the restriction to full rank systems is natural as being of full rank is a necessary condition for controllability. *For the remainder of this chapter we assume that all affine subspaces mentioned are of full rank.* Our approach is as follows.

We fix dimension and homogeneity (i.e., we fix ℓ and ε) and then postulate and prove a classification. This is accomplished as follows. We create a classifying table (list of mutually exclusive exhaustive conditions) that given a (ℓ,ε)-affine subspace Γ, asserts that Γ is $\mathfrak{L}$-related to some specific equivalence representative. Then given an arbitrary (ℓ,ε)-affine subspace we construct an automorphism that $\mathfrak{L}$-relates Γ and the stated equivalence representative. Finally we show that the different equivalence representatives are all distinct (i.e., no two are $\mathfrak{L}$-related). Note that, when convenient, we use a parameter (usually α) to parametrise an infinite family of (distinct) equivalence representatives.

At this stage we recall that an automorphism $\psi \in \operatorname{Aut}\mathfrak{h}_3^\diamond$, written w.r.t. the ordered basis $\{E_i\}_{i=\overline{1,4}}$, is of the form (see proposition 1.2.27)

$$\psi = \begin{bmatrix} x & y & 0 & u \\ -ky & kx & 0 & v \\ kux - vy & kuy + xv & k(x^2+y^2) & w \\ 0 & 0 & 0 & k \end{bmatrix}$$

for some $u, v, w, x, y \in \mathbb{R}$ and $k \in \{-1, 1\}$ such that $x^2 + y^2 \neq 0$. *For the remainder of this chapter, when an arbitrary automorphism $\psi \in \operatorname{Aut}\mathfrak{h}_3^\diamond$ is called for, we assume it is of this form.*

In order to separate cases (and create a classifying table), we will make use of the dual basis $\{E_i^*\}_{i=\overline{1,4}}$ of $\mathfrak{h}_3^\diamond$. To support our view of the dual basis as **linear projections** in this chapter (and the next), we will use the notation $\boldsymbol{\pi_i} = E_i^* : \mathfrak{h}_3^\diamond \to \mathbb{R}$, $E_j \mapsto \delta_{ij}$ for $i = \overline{1,4}$. Further separation of cases is accomplished by the introduction of some **invariants**. An invariant $\mathfrak{A}$ (represented by an upper case Fraktur letter) will associate to an (ℓ,ε)-affine subspace Γ a scalar value $\mathfrak{A}(\Gamma)$. Invariance here means that for any $\psi \in \operatorname{Aut}\mathfrak{h}_3^\diamond$ we have that $\mathfrak{A}(\psi \cdot \Gamma) = \mathfrak{A}(\Gamma)$.

A summary of invariants and classifying tables is given in appendix B. Before proceeding to the classification of the affine subspaces of $\mathfrak{h}_3^\diamond$, we make some general observations.

Suppose $\Gamma = A + \Gamma^0$ is an inhomogeneous affine subspace. Then one of the ways in which we separate cases (in the classifying table), is by a condition of the form $\pi_i(\Gamma^0) = \{0\}$, $\pi_i(A) = \alpha$. We show that such a condition is well defined (i.e., does not depend on the parametrisation chosen for Γ).

2.1.7 Lemma. *Let $\Gamma = A + \Gamma^0$ be an affine subspace of $\mathfrak{h}_3^\diamond$ and suppose that $\pi_i(\Gamma^0) = \{0\}$ for some $i \in \{1, 2, \ldots, 4\}$. Then the property $\pi_i(A) = \alpha$ is independent of parametrisation of Γ (i.e., the choice of A).*

Proof. Suppose $\Gamma = A + \Gamma^0$ such that $\pi_i(\Gamma^0) = \{0\}$ for some $i \in \{1, 2, \ldots, 4\}$ and $\pi_i(A) = \alpha$. Now parametrise Γ as $\Gamma = \widetilde{A} + \Gamma^0$. That is we have that $\widetilde{A} = A + B$ for some $B \in \Gamma^0$. Then as $\pi_i(\Gamma^0) = \{0\}$ we have that $\pi_i(B) = 0$ and hence that $\pi_i(\widetilde{A}) = \pi_i(A + B) = \pi_i(A) + \pi_i(B) = \alpha$.□

Next we show how a preclassification of $\mathfrak{L}$-related $(\ell+1, 0)$-affine subspaces may be produced from a classification of $\mathfrak{L}$-related $(\ell, 1)$-affine subspaces. (This result is captured in the corollary to the next proposition.)

2.1.8 Proposition. *If $\Gamma_1 = \langle A_1, A_2, \ldots, A_\ell \rangle$ is a $(\ell, 0)$-affine subspace (of full rank) of a Lie algebra $\mathfrak{g}$, then $\overline{\Gamma}_1 = A_1 + \langle A_2, \ldots, A_\ell \rangle$ is a $(\ell - 1, 1)$-affine subspace (of full rank) of $\mathfrak{g}$. Moreover if $\overline{\Gamma}_1$ is $\mathfrak{L}$-related to another $(\ell - 1, 1)$-affine subspace $\overline{\Gamma}_2$, then Γ_1 is $\mathfrak{L}$-related to $\langle \overline{\Gamma}_2 \rangle$.*

Proof. Suppose $\Gamma_1 = \langle A_1, A_2, \ldots, A_\ell \rangle$ is a $(\ell, 0)$-affine subspace (of full rank). Then $\{A_i\}_{i=\overline{1,\ell}}$ is linearly independent and hence $\overline{\Gamma}_1 = A_1 + \langle A_2, \ldots, A_\ell \rangle$ is a $(\ell - 1, 1)$-affine subspace. We have that $\dim \mathrm{Lie}\, \overline{\Gamma}_1 = \dim \langle \mathrm{Lie}\, \overline{\Gamma}_1 \rangle = \dim \mathrm{Lie} \langle \overline{\Gamma}_1 \rangle = \dim \mathrm{Lie}\, \Gamma_1 = \dim \mathfrak{g}$. That is, $\overline{\Gamma}_1$ is of full rank. Now suppose $\overline{\Gamma}_1$ and $\overline{\Gamma}_2$ are $\mathfrak{L}$-related. Then there exists $\psi \in \mathrm{Aut}\, \mathfrak{g}$ such that $\psi \cdot \overline{\Gamma}_1 = \overline{\Gamma}_2$. Hence $\langle \psi \cdot \overline{\Gamma}_1 \rangle = \langle \overline{\Gamma}_2 \rangle$ and thus $\psi \cdot \Gamma_1 = \langle \overline{\Gamma}_2 \rangle$. That is, Γ_1 and $\langle \overline{\Gamma}_2 \rangle$ are $\mathfrak{L}$-related. □

2.1.9 Corollary. *Let Γ be a $(\ell, 0)$-affine subspace (of full rank) of a Lie algebra $\mathfrak{g}$. Suppose $\{\overline{\Gamma}_i \mid i \in I\}$ is a complete list of equivalence representatives of $\mathfrak{L}$-related $(\ell - 1, 1)$-affine subspaces (of full rank) of $\mathfrak{g}$. Then Γ is $\mathfrak{L}$-related to at least one of $\{\langle \overline{\Gamma}_i \rangle \mid i \in I\}$.*

Proof. As Γ is a $(\ell, 0)$-affine subspace we can write Γ as $\Gamma = \langle A_1, A_2, \ldots, A_\ell \rangle$ for some $A_i \in \mathfrak{g}$, $i = \overline{1, \ell}$. Then $\overline{\Gamma} = A_1 + \langle A_2, \ldots, A_\ell \rangle$ is a $(\ell - 1, 1)$-affine subspace and consequently $\mathfrak{L}$-related to one of $\{\overline{\Gamma}_i \mid i \in I\}$. Hence Γ is $\mathfrak{L}$-related to one of $\{\langle \overline{\Gamma}_i \rangle \mid i \in I\}$. □

2.2 Dimension One, Inhomogeneous

Before postulating a classification of the $(1, 1)$-affine subspaces, we motivate a separating condition and introducing an invariant $\mathfrak{P}$. Let $A = \sum_{i=1}^{4} a_i E_i \in \mathfrak{h}_3^\diamond$, $\psi \in \mathrm{Aut}\, \mathfrak{h}_3^\diamond$ and $\Gamma^0 = \langle A \rangle$. Then notice that $\pi_4(A) = a_4$ and $\pi_4(\psi \cdot A) = k a_4$ for some $k \in \{-1, 1\}$. Hence we have that $\pi_4(\Gamma^0) = \{0\}$ if and only if $\pi_4(\psi \cdot \Gamma^0) = \{0\}$. Thus any $(1, 1)$-affine subspace (of $\mathfrak{h}_3^\diamond$) for which $\pi_4(\Gamma^0) = \{0\}$, cannot be $\mathfrak{L}$-related to one for which $\pi_4(\Gamma^0) \neq \{0\}$. In due course we will show that any $(1, 1)$-affine subspace for which $\pi_4(\Gamma^0) \neq \{0\}$ is $\mathfrak{L}$-related to $E_1 + \alpha E_3 + \langle E_4 \rangle$ for some $\alpha \geq 0$.

2.2.1 Lemma. *Let Γ be a $(1,1)$-affine subspace with parametrisation $\Gamma = \sum_{i=1}^{4} a_i E_i + \left\langle \sum_{i=1}^{4} b_i E_i \right\rangle$, for some constants $a_i, b_i \in \mathbb{R}$, $i = \overline{1,4}$ such that $\pi_4(\Gamma^0) \neq \{0\}$. Then there exists $\psi \in \operatorname{Aut} \mathfrak{h}_3^\diamond$ such that $\psi \cdot \Gamma = E_1 + c(x^2+y^2)E_3 + \langle E_4 \rangle$, where*

$$c(x^2+y^2) = \frac{b_4(-a_1b_1 - a_2b_2 + a_3b_4) + a_4\left(b_1^2 + b_2^2 - b_3b_4\right)}{a_4^2\left(b_1^2+b_2^2\right) - 2a_4(a_1b_1+a_2b_2)b_4 + \left(a_1^2+a_2^2\right)b_4^2}.$$

Proof. Now as $\pi_4(\Gamma^0) \neq \{0\}$ we have that $b_4 \neq 0$ and hence get that $\Gamma = \sum_{i=1}^{3}(a_i - \frac{a_4 b_i}{b_4})E_i + \left\langle \sum_{i=1}^{3}\left(\frac{b_i}{b_4}E_i\right) + E_4 \right\rangle$. Now we have that

$$\psi_1 = \begin{bmatrix} 1 & 0 & 0 & -\frac{b_1}{b_4} \\ 0 & 1 & 0 & -\frac{b_2}{b_4} \\ -\frac{b_1}{b_4} & -\frac{b_2}{b_4} & 1 & -\frac{b_3}{b_4} + \frac{b_1^2+b_2^2}{b_4^2} \\ 0 & 0 & 0 & 1 \end{bmatrix} \in \operatorname{Aut} \mathfrak{h}_3^\diamond$$

and so get that

$$\begin{aligned} \psi_1 \cdot \Gamma &= \sum_{i=1}^{2}\left((a_i - \tfrac{a_4 b_i}{b_4})E_i\right) + \left(\sum_{i=1}^{2}\left(-\tfrac{b_i}{b_4}(a_i - \tfrac{a_4b_i}{b_4})\right) + (a_3 - \tfrac{a_4b_3}{b_4})\right)E_3 \\ &\quad + \left\langle \tfrac{b_1}{b_4}E_1 + \tfrac{b_2}{b_4}E_2 + \left(-\tfrac{b_1^2}{b_4^2} - \tfrac{b_2^2}{b_4^2} + \tfrac{b_3}{b_4}\right)E_3 + \left(-\tfrac{b_1}{b_4}E_1 - \tfrac{b_2}{b_4}E_2 + \left(-\tfrac{b_3}{b_4} + \tfrac{b_1^2+b_2^2}{b_4^2}\right)E_3 + E_4\right)\right\rangle \\ &= \sum_{i=1}^{2}\left((a_i - \tfrac{a_4b_i}{b_4})E_i\right) + cE_3 + \langle E_4 \rangle, \end{aligned}$$

where $c = \sum_{i=1}^{2}\left(-\frac{b_i}{b_4}(a_i - \frac{a_4b_i}{b_4})\right) + (a_3 - \frac{a_4b_3}{b_4})$. With the "target" of $E_1 + \alpha E_3 + \langle E_4 \rangle$ in mind we try and find an automorphism taking $\sum_{i=1}^{2}\left((a_i - \frac{a_4b_i}{b_4})E_i\right)$ to E_1 but not changing E_4. Consider the following equation (motivated by applying an automorphism to $\psi_1 \cdot \Gamma$)

$$\left(\left(a_1 - \tfrac{a_4b_1}{b_4}\right)x + \left(a_2 - \tfrac{a_4b_2}{b_4}\right)y\right)E_1 + \left(-\left(a_1 - \tfrac{a_4b_1}{b_4}\right)y + \left(a_2 - \tfrac{a_4b_2}{b_4}\right)x\right)E_2 = E_1,$$

or equivalently

$$\begin{bmatrix} a_1 - \frac{a_4b_1}{b_4} & a_2 - \frac{a_4b_2}{b_4} \\ a_2 - \frac{a_4b_2}{b_4} & -a_1 + \frac{a_4b_1}{b_4} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{2.2.1}$$

We claim that if Γ is of full rank then either $a_1 - \frac{a_4b_1}{b_4} \neq 0$ or $a_2 - \frac{a_4b_2}{b_4} \neq 0$. Suppose that $a_i - \frac{a_4b_i}{b_4} = 0$, $i = \overline{1,2}$. Then $\langle \Gamma \rangle = \left\langle (a_3 - \frac{a_4b_3}{b_4})E_3, \sum_{i=1}^{3}\left(\frac{b_i}{b_4}E_i\right) + E_4 \right\rangle$. But we have that $[E_3, E_i] = 0$, $i = \overline{1,4}$ and hence that $\dim \operatorname{Lie}\Gamma = \dim \langle \Gamma \rangle = 2$. Thus Γ is not of full rank, providing the needed contradiction. Therefore the square matrix in equation (2.2.1) has non-zero determinant and hence we get that

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_1 - \frac{a_4b_1}{b_4} & a_2 - \frac{a_4b_2}{b_4} \\ a_2 - \frac{a_4b_2}{b_4} & -a_1 + \frac{a_4b_1}{b_4} \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad x^2 + y^2 = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 - \frac{a_4b_1}{b_4} & a_2 - \frac{a_4b_2}{b_4} \\ a_2 - \frac{a_4b_2}{b_4} & -a_1 + \frac{a_4b_1}{b_4} \end{bmatrix}^{-2} \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Then we have that

$$\psi_2 = \begin{bmatrix} x & y & 0 & 0 \\ -y & x & 0 & 0 \\ 0 & 0 & x^2+y^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is an automorphism such that

$$\begin{aligned}\psi_2 \cdot \psi_1 \cdot \Gamma &= \left(x(a_1 - \tfrac{a_4 b_1}{b_4}) + y(a_2 - \tfrac{a_4 b_2}{b_4})\right) E_1 + \left(-y(a_2 - \tfrac{a_4 b_2}{b_4}) + x(a_1 - \tfrac{a_4 b_1}{b_4})\right) E_2 \\ &\qquad + c(x^2+y^2)E_3 + \langle E_4 \rangle \\ &= E_1 + c(x^2+y^2)E_3 + \langle E_4 \rangle .\end{aligned}$$

Furthermore we get that (simplification done in Mathematica)

$$\begin{aligned} c(x^2+y^2) &= \left(\sum_{i=1}^{2}\left(-\tfrac{b_i}{b_4}(a_i - \tfrac{a_4 b_i}{b_4})\right) + a_3 - \tfrac{a_4 b_3}{b_4}\right)\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix} a_1 - \frac{a_4 b_1}{b_4} & a_2 - \frac{a_4 b_2}{b_4} \\ a_2 - \frac{a_4 b_2}{b_4} & -a_1 + \frac{a_4 b_1}{b_4}\end{bmatrix}^{-2}\begin{bmatrix}1\\0\end{bmatrix} \\ &= \frac{b_4(-a_1 b_1 - a_2 b_2 + a_3 b_4) + a_4\left(b_1^2 + b_2^2 - b_3 b_4\right)}{a_4^2\left(b_1^2+b_2^2\right) - 2a_4(a_1 b_1 + a_2 b_2)b_4 + \left(a_1^2+a_2^2\right)b_4^2}.\end{aligned}$$

Notice that, as the matrix in equation (2.2.1) has non-zero determinant, the denominator of this expression is non-zero. □

The above lemma motivates the introduction of **the invariant $\mathfrak{P}$**. Let Γ be a $(1,1)$-affine subspaces of $\mathfrak{h}_3^\diamond$ with parametrisation $P : x \mapsto \sum_{i=1}^{4}(a_i E_i) + x\sum_{i=1}^{4} b_i E_i$ such that $\pi_4(\Gamma^0) \neq \{0\}$. Then we define $\mathfrak{P}(\Gamma)$ with respect to the parametrisation P as

$$\mathfrak{P}_P(\Gamma) = \left|\frac{b_4(-a_1 b_1 - a_2 b_2 + a_3 b_4) + a_4\left(b_1^2 + b_2^2 - b_3 b_4\right)}{a_4^2\left(b_1^2+b_2^2\right) - 2a_4(a_1 b_1 + a_2 b_2)b_4 + \left(a_1^2+a_2^2\right)b_4^2}\right|.$$

From the preceding work we have that it is indeed defined for all such Γ. We now proceed to show that $\mathfrak{P}$ is independent of the parametrisation chosen.

2.2.2 Lemma. *Let $P_1 : x \mapsto A + xB$ and $P_2 : x \mapsto \widetilde{A} + x\widetilde{B}$ be two parametrisations of a $(1,1)$-affine subspace Γ (with the property $\pi_4(\Gamma^0) \neq \{0\}$). Then $\mathfrak{P}_{P_1}(\Gamma) = \mathfrak{P}_{P_2}(\Gamma)$.*

Proof. Let $A = \sum_{i=1}^{4} a_i E_i$, $B = \sum_{i=1}^{4} b_i E_i$, $\widetilde{A} = \sum_{i=1}^{4}\widetilde{a}_i E_i$, and $\widetilde{B} = \sum_{i=1}^{4}\widetilde{b}_i E_i$. Then as P_1 and P_2 parametrise the same $(1,1)$-affine subspace we have that $A = \widetilde{A} + \nu\widetilde{B}$ and $B = \mu\widetilde{B}$ for some constants $\nu, \mu \in \mathbb{R}$, $\mu \neq 0$. Substituting these identities into the expression for $\mathfrak{P}_{P_1}(\Gamma)$ and simplifying (using Mathematica, see section C.5) yields $\mathfrak{P}_{P_1}(\Gamma) = \mathfrak{P}_{P_2}(\Gamma)$. □

Thus the value $\mathfrak{P}(\Gamma)$ is only dependent on the affine subspace in question and not the parametrisation of it. That is to say $\mathfrak{P}(\Gamma)$ is well defined for a $(1,1)$-affine subspace (such that $\pi_4(\Gamma^0) \neq \{0\}$). We now show that automorphisms leave $\mathfrak{P}$ invariant.

2.2.3 Proposition. *If* Γ *is a* $(1,1)$*-affine subspace such that* $\pi_4(\Gamma^0) \neq \{0\}$*, then* $\mathfrak{P}(\psi \cdot \Gamma) = \mathfrak{P}(\Gamma)$ *for* $\psi \in \operatorname{Aut}\mathfrak{h}_3^\diamond$.

Proof. We use Mathematica (see section C.5) to verify that $\mathfrak{P}(\psi\cdot\Gamma) = \mathfrak{P}(\Gamma)$ for $\psi \in \operatorname{Aut}\mathfrak{h}_3^\diamond$.□

Having introduced the invariant $\mathfrak{P}$ we now classify the $(1,1)$-affine subspaces.

2.2.4 Proposition. *Any* $(1,1)$*-affine subspace* $\Gamma = A + \Gamma^0 \subset \mathfrak{h}_3^\diamond$ *(of full rank) is* $\mathfrak{L}$*-related to exactly one of the following affine subspaces:*

Case: $\pi_4(\Gamma^0) \neq \{0\}$**,** $\mathfrak{P}(\Gamma) = \alpha$**,** $\alpha \geq 0$

$$\Gamma^{(1,1)}_{1,\alpha} = E_1 + \alpha E_3 + \langle E_4 \rangle$$

Case: $\pi_4(\Gamma^0) = \{0\}$**,** $\pi_4(A) = \pm\alpha$**,** $\alpha > 0$

$$\Gamma^{(1,1)}_{2,\alpha} = \alpha E_4 + \langle E_1 \rangle.$$

Proof. (We omit the superscript $(1,1)$ in this proof.) Suppose Γ is an $(1,1)$-affine subspace such that $\pi_4(\Gamma^0) \neq \{0\}$ and $\mathfrak{P}(\Gamma) = \alpha$ for some $\alpha \geq 0$. We wish to show that $\Gamma \sim \Gamma_{1,\alpha} = E_1 + \alpha E_3 + \langle E_4 \rangle$. We may write Γ as $\Gamma = \sum_{i=1}^4 a_i E_i + \left\langle \sum_{i=1}^4 b_i E_i \right\rangle$, for some constants $a_i, b_i \in \mathbb{R}$, $i = \overline{1,4}$. Then, by lemma 2.2.1, there exists a $\psi_1 \in \operatorname{Aut}\mathfrak{h}_3^\diamond$ such that $\psi_1 \cdot \Gamma = E_1 + k\mathfrak{P}(\Gamma)\, E_3 + \langle E_4 \rangle$ for some $k \in \{-1, 1\}$. Hence, by assumption, $\psi_1 \cdot \Gamma = E_1 + k\alpha E_3 + \langle E_4 \rangle$. Then $\psi_2 = \operatorname{diag}(1, k, k, k) \in \operatorname{Aut}\mathfrak{h}_3^\diamond$ and $\psi_2 \cdot \psi_1 \cdot \Gamma = E_1 + k^2\alpha E_3 + \langle kE_4 \rangle = E_1 + \alpha E_3 + \langle E_4 \rangle$. Thus Γ is $\mathfrak{L}$-related to $\Gamma_{1,\alpha}$.

We now proceed to the second case. Suppose Γ is an $(1,1)$-affine subspace such that $\pi_4(\Gamma^0) = \{0\}$ and $\pi_4(A) = \pm\alpha$ for some $\alpha > 0$. (Note that if $\alpha = 0$ then Γ would not be of full rank, hence we can ignore this case.) We wish to show that $\Gamma \sim \Gamma_{2,\alpha} = \alpha E_4 + \langle E_1 \rangle$. Now as $\pi_4(\Gamma^0) = \{0\}$ we have constants $a_i, a_4, b_i \in \mathbb{R}$, $i = \overline{1,3}$ such that $\Gamma = \sum_{i=1}^4 a_i E_i + \left\langle \sum_{i=1}^3 b_i E_i \right\rangle$. Note that as $\pi_4(A) = \pm\alpha$ we have that $a_4 = k\alpha$ (and thus $a_4 \neq 0$) for some $k \in \{-1, 1\}$. Hence we get that

$$\psi_1 = \begin{bmatrix} 1 & 0 & 0 & -\frac{a_1}{a_4} \\ 0 & 1 & 0 & -\frac{a_2}{a_4} \\ -\frac{a_1}{a_4} & -\frac{a_2}{a_4} & 1 & -\frac{a_3}{a_4} + \frac{a_1^2 + a_2^2}{a_4^2} \\ 0 & 0 & 0 & 1 \end{bmatrix} \in \operatorname{Aut}\mathfrak{h}_3^\diamond \qquad \psi_2 = \operatorname{diag}(1, k, k, k) \in \operatorname{Aut}\mathfrak{h}_3^\diamond$$

and that

$$\psi_2 \cdot \psi_1 \cdot \Gamma = \psi_2 \Bigg(a_1(E_1 - \tfrac{a_1}{a_4} E_3) + a_2(E_2 - \tfrac{a_2}{a_4} E_3) + a_3 E_3 + a_4 \left(-\tfrac{a_1}{a_4} E_1 - \tfrac{a_2}{a_4} E_2 + (-\tfrac{a_3}{a_4} + \tfrac{a_1^2 + a_2^2}{a_4^2}) E_3 + E_4 \right) + \left\langle \psi_1 \cdot \sum_{i=1}^3 b_i E_i \right\rangle \Bigg).$$

Hence, $\psi_2 \cdot \psi_1 \cdot \Gamma = k a_4 E_4 + \left\langle \psi_2 \cdot \psi_1 \cdot \sum_{i=1}^3 b_i E_i \right\rangle = \alpha E_4 + \left\langle \sum_{i=1}^3 b_i' E_i \right\rangle$, for some new constants $b_i' \in \mathbb{R}$, $i = \overline{1,3}$. Next we claim that either $b_1' \neq 0$ or $b_2' \neq 0$. Supposing $b_1' = b_2' = 0$ we have that $\langle \psi_2 \cdot \psi_1 \cdot \Gamma \rangle = \langle \alpha E_4, b_3' E_3 \rangle$. But as $[E_4, E_3] = 0$, we then have that $\dim \operatorname{Lie}\Gamma = \dim \operatorname{Lie}(\psi_2 \cdot \psi_1 \cdot \Gamma) = \dim \langle \alpha E_4, b_3' E_3 \rangle = 2 \neq 4$, providing a contradiction.

Now, keeping the target $\Gamma_{2,\alpha} = \alpha E_4 + \langle E_1 \rangle$ in mind, we consider the equation (motivated by applying an automorphism to $\psi_2 \cdot \psi_1 \cdot \Gamma$)

$$\begin{bmatrix} b_1' & b_2' \\ b_2' & -b_1' \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Then, as either $b_1' \neq 0$ or $b_2' \neq 0$ (hence the above matrix is invertible), the equation has a solution (for x and y such that $(x, y) \neq (0, 0)$) from which we may define an automorphism

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1' & b_2' \\ b_2' & -b_1' \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \psi_3 = \begin{bmatrix} x & y & 0 & 0 \\ -y & x & 0 & 0 \\ 0 & 0 & x^2 + y^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in \operatorname{Aut} \mathfrak{h}_3^{\diamond}.$$

Hence,

$$\begin{aligned} \psi_3 \cdot \psi_2 \cdot \psi_1 \cdot \Gamma &= \alpha E_4 + \left\langle \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} b_1' \\ b_2' \end{bmatrix} E_1 + \begin{bmatrix} -y & x \end{bmatrix} \begin{bmatrix} b_1' \\ b_2' \end{bmatrix} E_2 + (x^2 + y^2) b_3' E_3 \right\rangle \\ &= \alpha E_4 + \left\langle E_1 + (x^2 + y^2) b_3' E_3 \right\rangle. \end{aligned}$$

Finally we have that

$$\psi_4 = \begin{bmatrix} 1 & 0 & 0 & -(x^2 + y^2) b_3' \\ 0 & 1 & 0 & 0 \\ -(x^2 + y^2) b_3' & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in \operatorname{Aut} \mathfrak{h}_3^{\diamond}$$

and that

$$\begin{aligned} &\psi_4 \cdot \psi_3 \cdot \psi_2 \cdot \psi_1 \cdot \Gamma \\ &\quad = \alpha(E_4 - (x^2 + y^2) b_3' E_1) + \left\langle (E_1 - (x^2 + y^2) b_3' E_3) + (x^2 + y^2) b_3' E_3 \right\rangle \\ &\quad = \alpha E_4 - \alpha (x^2 + y^2) b_3' E_1 + \langle E_1 \rangle \\ &\quad = \alpha E_4 + \langle E_1 \rangle. \end{aligned}$$

Hence Γ is $\mathfrak{L}$-related (by $\psi_4 \cdot \psi_3 \cdot \psi_2 \cdot \psi_1$) to $\Gamma_{2,\alpha}$.

Now as our list is exhaustive we are left to show that none of the affine subspaces

$$\Gamma_{1,\alpha} = E_1 + \alpha E_3 + \langle E_4 \rangle, \ \alpha \geq 0 \qquad \Gamma_{2,\alpha} = \alpha E_4 + \langle E_1 \rangle, \ \alpha > 0$$

are $\mathfrak{L}$-related. This is done by assuming that they are, giving us an automorphism ψ taking one to the other, and producing a contradiction from this.

$(\alpha \neq \beta, \ \alpha, \beta \geq 0) \Rightarrow \Gamma_{1,\alpha} \nsim \Gamma_{1,\beta}$: Assume there exists $\psi \in \operatorname{Aut} \mathfrak{h}_3^{\diamond}$ such that $\psi \cdot \Gamma_{1,\alpha} = \Gamma_{2,\alpha}$. Now we have that $\beta = \mathfrak{P}(\Gamma_{1,\beta}) = \mathfrak{P}(\psi \cdot \Gamma_{1,\alpha}) = \mathfrak{P}(\Gamma_{1,\alpha}) = \alpha$, yielding a contradiction.

$\Gamma_{1,\alpha} \nsim \Gamma_{2,\beta}$: Assume there exists $\psi \in \operatorname{Aut} \mathfrak{h}_3^{\diamond}$ such that $\psi \cdot \Gamma_{1,\alpha}^0 = \Gamma_{2,\beta}^0$. Now $E_4 \in \Gamma_{1,\alpha}^0$ hence $\psi \cdot E_4 \in \Gamma_{2,\beta}^0$. But we have that $\pi_4(\psi \cdot E_4) = k = \pm 1$ and $\pi_4(\Gamma_{2,\beta}^0) = \{0\}$, a contradiction.

$(\alpha \neq \beta, \ \alpha, \beta > 0) \Rightarrow \Gamma_{2,\alpha} \nsim \Gamma_{2,\beta}$: Assume there exists $\psi \in \operatorname{Aut} \mathfrak{h}_3^{\diamond}$ such that $\psi \cdot \Gamma_{2,\alpha} = \Gamma_{2,\beta}$. Now $\alpha E_4 \in \Gamma_{2,\alpha}$, thus $\psi \cdot \alpha E_4 \in \Gamma_{2,\beta}$. Therefore $\pm\alpha = \beta$, but as $\alpha, \beta > 0$ this means $\alpha = \beta$. □

2.3 Dimension Two, Homogeneous

2.3.1 Proposition. *Any* $(2,0)$*-affine subspace* $\Gamma = \Gamma^0$ *(of full rank) is* $\mathfrak{L}$*-related to* $\Gamma_1^{(2,0)} = \langle E_1, E_4\rangle$.

Proof. By proposition 2.2.4 and corollary 2.1.9, we get that Γ is $\mathfrak{L}$-related to either $\left\langle \Gamma_{1,\alpha}^{(1,1)} \right\rangle = \langle E_1 + \alpha E_3, E_4\rangle$ for some $\alpha \geq 0$, or $\left\langle \Gamma_{2,\beta}^{(1,1)} \right\rangle = \langle \beta E_4, E_1\rangle$ for some $\beta > 0$. Note that $\langle \beta E_4, E_1\rangle = \langle E_1, E_4\rangle$, thus in the second case there is nothing more to show. Assume Γ is $\mathfrak{L}$-related to $\langle E_1 + \alpha E_3, E_4\rangle$. Then we have that

$$\psi = \begin{bmatrix} 1 & 0 & 0 & -\alpha \\ 0 & 1 & 0 & 0 \\ -\alpha & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in \operatorname{Aut} \mathfrak{h}_3^\diamond$$

and get that $\psi \cdot \langle E_1 + \alpha E_3, E_4\rangle = \langle (E_1 - \alpha E_3) + \alpha E_3, -\alpha E_1 + E_4\rangle = \langle E_1, E_4\rangle$. Thus in either case Γ is $\mathfrak{L}$-related to $\langle E_1, E_4\rangle$. □

2.4 Dimension Two, Inhomogeneous

Before postulating and proving a classification of the $(2,1)$-affine subspaces we introduce two **invariants** $\mathfrak{T}$ and $\mathfrak{S}$. These invariants are motivated by calculations made in attempting to classify these affine subspaces, similarly to how we motivated the invariant $\mathfrak{P}$ in section 2.2. However, for the sake of brevity, we will not show all the calculations motivating these invariants as we did for $\mathfrak{P}$, but simply present the final result.

Let $\Gamma = A + \Gamma^0 \subset \mathfrak{h}_3^\diamond$ be a $(2,1)$-affine subspace (of full rank). We define $\mathfrak{T}(\Gamma)$, w.r.t. parametrisation $P : (x,y) \mapsto \Gamma = \sum_{i=1}^4 a_i E_i + x\left(\sum_{i=1}^4 b_i E_i\right) + y\left(\sum_{i=1}^4 c_i E_i\right)$ as

$$\mathfrak{T}_P(\Gamma) = \left| \operatorname{sgn}\left(\begin{vmatrix} a_1 & a_2 & a_4 \\ b_1 & b_2 & b_4 \\ c_1 & c_2 & c_4 \end{vmatrix} \right) \right|.$$

If $\pi_4(\Gamma^0) \neq \{0\}$ and $\mathfrak{T}(\Gamma) \neq 0$, then Γ has a parametrisation of the form $P : (x,y) \mapsto \sum_{i=1}^4 a_i E_i + x\left(\sum_{i=1}^3 b_i E_i\right) + y\left(\sum_{i=1}^4 c_i E_i\right)$, with $c_4 \neq 0$. We define $\mathfrak{S}(\Gamma)$ w.r.t. such a parametrisation P as

$$\mathfrak{S}(\Gamma) = \left| \frac{\begin{matrix} c_4\left(-\left(a_2 b_1 - a_1 b_2\right)\left(-b_2 c_1 + b_1 c_2\right) + \left(a_3\left(b_1^2 + b_2^2\right) - \left(a_1 b_1 + a_2 b_2\right) b_3\right) c_4\right) \\ +a_4\left(\left(b_2 c_1 - b_1 c_2\right)^2 + \left(b_3\left(b_1 c_1 + b_2 c_2\right) - \left(b_1^2 + b_2^2\right) c_3\right) c_4\right) \end{matrix}}{\begin{vmatrix} a_1 & a_2 & a_4 \\ b_1 & b_2 & 0 \\ c_1 & c_2 & c_4 \end{vmatrix}^2} \right|.$$

In particular if $\Gamma = a_2 E_2 + a_3 E_3 + \langle E_1, E_4\rangle$ we have that $\mathfrak{S}(\Gamma) = \left|\frac{a_3}{a_2^2}\right|$.

2.4.1 REMARK. We specialise the parametrisations P (as done above) with respect to which we define $\mathfrak{S}(\Gamma)$ in order to simplify the expression of $\mathfrak{S}_P(\Gamma)$. Note that if $\pi_4(\Gamma^0) \neq \{0\}$ then we can always find $A, B, C \in \mathfrak{h}_3^\diamond$ such that $\pi_4(B) = 0$, $\pi_4(C) \neq 0$ and $\Gamma = A + \langle B, C \rangle$. Thus any affine subspace for which $\pi_4(\Gamma^0) \neq \{0\}$ has a parametrisation of this specialised type. Therefore, if $\mathfrak{S}(\Gamma)$ does not depend on this specialised type of parametrisation, then $\mathfrak{S}(\Gamma)$ is well defined (irrespective of parametrisation).

In the next two lemmas we show that $\mathfrak{T}(\Gamma)$ and $\mathfrak{S}(\Gamma)$ do not depend on the parametrisation chosen.

2.4.2 LEMMA. *Let $P_1 : (x,y) \mapsto A + xB + yC$ and $P_2 : x \mapsto \widetilde{A} + x\widetilde{B} + y\widetilde{C}$ be two parametrisations of a $(2,1)$-affine subspace Γ. Then $\mathfrak{T}_{P_1}(\Gamma) = \mathfrak{T}_{P_2}(\Gamma)$.*

PROOF. Let $A = \sum_{i=1}^4 a_i E_i$, $\widetilde{A} = \sum_{i=1}^4 \widetilde{a}_i E_i$ and so on for B, $\widetilde{B}$, C and $\widetilde{C}$. Then, as P_1 and P_2 parametrise the same $(2,1)$-affine subspace, we have that

$$A = \widetilde{A} + \eta_1 \widetilde{B} + \eta_2 \widetilde{C} \qquad B = \nu_1 \widetilde{B} + \nu_2 \widetilde{C} \qquad C = \mu_1 \widetilde{B} + \mu_2 \widetilde{C}$$

for some constants $\eta_i, \nu_i, \mu_i \in \mathbb{R}$, $i = \overline{1,2}$ such that $\begin{vmatrix} \nu_1 & \nu_2 \\ \mu_1 & \mu_2 \end{vmatrix} \neq 0$ (from linear independence). Using these identities we get that

$$\mathfrak{T}_{P_1}(\Gamma) = \left| \operatorname{sgn} \left(\begin{vmatrix} a_1 & a_2 & a_4 \\ b_1 & b_2 & b_4 \\ c_1 & c_2 & c_4 \end{vmatrix} \right) \right| = \left| \operatorname{sgn} \left(\begin{vmatrix} 1 & \eta_1 & \eta_2 \\ 0 & \nu_1 & \nu_2 \\ 0 & \mu_1 & \mu_2 \end{vmatrix} \begin{vmatrix} \widetilde{a}_1 & \widetilde{a}_2 & \widetilde{a}_4 \\ \widetilde{b}_1 & \widetilde{b}_2 & \widetilde{b}_4 \\ \widetilde{c}_1 & \widetilde{c}_2 & \widetilde{c}_4 \end{vmatrix} \right) \right| = \mathfrak{T}_{P_2}(\Gamma). \qquad \square$$

2.4.3 LEMMA. *Let Γ be a $(2,1)$-affine subspace such that $\pi_4(\Gamma^0) \neq \{0\}$ and $\mathfrak{T}(\Gamma) \neq 0$. Let $P_1 : (x,y) \mapsto A + xB + yC$ and $P_2 : x \mapsto \widetilde{A} + x\widetilde{B} + y\widetilde{C}$ be two parametrisations of Γ such that $\pi_4(B) = \pi_4(\widetilde{B}) = 0$, $\pi_4(C) \neq 0$ and $\pi_4(\widetilde{C}) \neq 0$. Then $\mathfrak{S}_{P_1}(\Gamma) = \mathfrak{S}_{P_2}(\Gamma)$.*

PROOF. As P_1 and P_2 both parametrise Γ we have constants $\eta_1, \eta_2, \nu_1, \mu_1, \mu_2 \in \mathbb{R}$, $\nu_1 \neq 0$, $\mu_2 \neq 0$ such that

$$A = \widetilde{A} + \eta_1 \widetilde{B} + \eta_2 \widetilde{C} \qquad B = \nu_1 \widetilde{B} \qquad C = \mu_1 \widetilde{B} + \mu_2 \widetilde{C}$$

Notice that the condition $\mathfrak{T}(\Gamma) \neq 0$ implies that the denominator in our expression for $\mathfrak{S}(\Gamma)$ is non-zero, i.e., $\mathfrak{S}(\Gamma)$ is defined with respect to any (specialised) parametrisation. Using these identities we get (using Mathematica for simplification, see section C.5) that $\mathfrak{S}_{P_1}(\Gamma) = \frac{\mu_2^2 \nu_1^2}{\mu_2^2 \nu_1^2} \mathfrak{S}_{P_2}(\Gamma) = \mathfrak{S}_{P_2}(\Gamma)$. $\square$

Next we show that automorphisms leave $\mathfrak{T}(\Gamma)$ and $\mathfrak{S}(\Gamma)$ invariant.

2.4.4 PROPOSITION. *If Γ is a $(2,1)$-affine subspace, then $\mathfrak{P}(\psi \cdot \Gamma) = \mathfrak{P}(\Gamma)$ for $\psi \in \operatorname{Aut} \mathfrak{h}_3^\diamond$. If in addition $\pi_4(\Gamma^0) \neq \{0\}$ and $\mathfrak{T}(\Gamma) \neq 0$, then $\mathfrak{S}(\psi \cdot \Gamma) = \mathfrak{S}(\Gamma)$ for $\psi \in \operatorname{Aut} \mathfrak{h}_3^\diamond$.*

Proof. Let $\psi \in \operatorname{Aut}\mathfrak{h}_3^\diamond$ be an arbitrary automorphism. Let $\Gamma = A + \langle B, C\rangle = \sum_{i=1}^4 a_i E_i + \left\langle \sum_{i=1}^4 b_i E_i, \sum_{i=1}^4 c_i E_i \right\rangle$. Then we get that

$$\mathfrak{T}(\psi\cdot\Gamma) = \left|\operatorname{sgn}\left(\begin{vmatrix} x & y & u \\ -ky & kx & v \\ 0 & 0 & k\end{vmatrix}\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_4 & b_4 & c_4\end{vmatrix}\right)\right| = \left|\operatorname{sgn}\left(k^2(x^2+y^2)\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_4 & b_4 & c_4\end{vmatrix}\right)\right| = \mathfrak{T}(\Gamma).$$

Now suppose that $\mathfrak{T}(\Gamma) \neq 0$, $c_4 \neq 0$ and $b_4 = 0$ (w.l.o.g., see remark 2.4.1). Note that $\pi_4(\psi\cdot C) \neq 0$, $\pi_4(\psi\cdot B) = 0$ and $\mathfrak{T}(\psi\cdot\Gamma) = \mathfrak{T}(\Gamma) \neq 0$. Thus $\mathfrak{S}(\psi\cdot\Gamma)$ is defined. Then we get (using Mathematica for simplification, see section C.5) that $\mathfrak{S}(\psi\cdot\Gamma) = |k|\,\mathfrak{S}(\Gamma) = \mathfrak{S}(\Gamma)$. □

Now, having set up our invariants, we proceed to classifying the affine subspaces.

2.4.5 Proposition. *Any $(2,1)$-affine subspace $\Gamma = A + \Gamma^0 \subset \mathfrak{h}_3^\diamond$ (of full rank) is $\mathfrak{L}$-related to exactly one of the following affine subspaces:*

Case: $\pi_4(\Gamma^0) \neq \{0\}$**,** $\dim\operatorname{Lie}\Gamma^0 = 4$

Case: $\mathfrak{T}(\Gamma) = 1$**,** $\mathfrak{S}(\Gamma) = \alpha$**,** $\alpha \geq 0$

$\Gamma_{1,\alpha}^{(2,1)} = E_2 + \alpha E_3 + \langle E_1, E_4\rangle$

Case: $\mathfrak{T}(\Gamma) = 0$

$\Gamma_{2}^{(2,1)} = E_3 + \langle E_1, E_4\rangle$

Case: $\pi_4(\Gamma^0) \neq \{0\}$**,** $\dim\operatorname{Lie}\Gamma^0 \neq 4$

$\Gamma_{3}^{(2,1)} = E_1 + \langle E_3, E_4\rangle$

Case: $\pi_4(\Gamma^0) = \{0\}$**,** $\pi_4(A) = \pm\alpha$**,** $\alpha > 0$

Case: $\mathfrak{T}(\Gamma) = 1$

$\Gamma_{4,\alpha}^{(2,1)} = \alpha E_4 + \langle E_1, E_2\rangle$

Case: $\mathfrak{T}(\Gamma) = 0$

$\Gamma_{5,\alpha}^{(2,1)} = \alpha E_4 + \langle E_1, E_3\rangle.$

Proof. (We omit the superscript $(2,1)$ in this proof.) Let $\Gamma = A + \Gamma^0$ be a $(2,1)$-affine subspace (of full rank) of $\mathfrak{h}_3^\diamond$.

Case: $\pi_4(\Gamma^0) \neq \{0\}$**,** $\dim\operatorname{Lie}\Gamma^0 = 4$

As $\dim\operatorname{Lie}\Gamma^0 = 4$, we have that Γ^0 is a $(2,0)$-affine subspace (of full rank). Thus, by proposition 2.3.1, Γ^0 is $\mathfrak{L}$-related to $\langle E_1, E_4\rangle$. That is, we have an automorphism ψ_1 such that $\psi_1\cdot\Gamma^0 = \langle E_1, E_4\rangle$. Thus we have constants $a_i \in \mathbb{R}$, $i = \overline{1,4}$ such that $\psi_1\cdot\Gamma = \sum_{i=1}^4 a_i E_i + \langle E_1, E_4\rangle = a_2E_2 + a_3E_3 + \langle E_1, E_4\rangle$.

Case: $\mathfrak{T}(\Gamma) = 1$**,** $\mathfrak{S}(\Gamma) = \alpha$**,** $\alpha \geq 0$

We wish to show that $\Gamma \sim \Gamma_{1,\alpha} = E_2 + \alpha E_3 + \langle E_1, E_4\rangle$. Now as $\mathfrak{T}(\Gamma) = 1$ we have that $\mathfrak{T}(\psi_1\cdot\Gamma) = 1$ and hence that $\operatorname{sgn}|a_2| = 1$ meaning $a_2 \neq 0$. Consequently note that $\mathfrak{S}(\psi_1\cdot\Gamma)$ is defined. Then we get that $\mathfrak{S}(\psi_1\cdot\Gamma) = \mathfrak{S}(\Gamma) = \alpha$, which is to say $\left|\frac{a_3}{a_2^2}\right| = \alpha$. If $\alpha = 0$ (i.e., $a_3 = 0$) we have that $\psi_2 = \operatorname{diag}(\frac{1}{a_2}, \frac{1}{a_2}, \frac{1}{a_2^2}, 1) \in \operatorname{Aut}\mathfrak{h}_3^\diamond$ and hence that $\psi_2\cdot\psi_1\cdot\Gamma = E_2 + \langle E_1, E_4\rangle = \Gamma_{1,0}$.

If however $\alpha \neq 0$ (i.e., $a_3 \neq 0$), we get that

$$\psi_3 = \operatorname{diag}\left(\tfrac{\operatorname{sgn}(a_3)}{a_2}, \tfrac{1}{a_2}, \tfrac{\operatorname{sgn}(a_3)}{a_2^2}, \operatorname{sgn}(a_3)\right) \in \operatorname{Aut}\mathfrak{h}_3^\diamond$$

and hence that

$$\psi_3 \cdot \psi_1 \cdot \Gamma = E_2 + \tfrac{\operatorname{sgn}(a_3)a_3}{a_2^2} E_3 + \left\langle \tfrac{\operatorname{sgn}(a_3)}{a_2} E_1, \operatorname{sgn}(a_3)E_4 \right\rangle = E_2 + \alpha E_3 + \langle E_1, E_4\rangle .$$

That is either $\psi_2 \cdot \psi_1$ or $\psi_3 \cdot \psi_1$ is an automorphism taking Γ to $\Gamma_{1,\alpha}$ as required.

Case: $\mathfrak{T}(\Gamma) = 0$

We wish to show that $\Gamma \sim \Gamma_2 = E_3 + \langle E_1, E_4\rangle$. Recall that we already have that $\psi_1 \cdot \Gamma = a_2E_2 + a_3E_3 + \langle E_1, E_4\rangle$ for some constants $a_2, a_3 \in \mathbb{R}$. Now as $\mathfrak{T}(\Gamma) = 0$ we have that $\operatorname{sgn}|a_2| = 0$ and hence that $a_2 = 0$. We then require that $a_3 \neq 0$ as Γ is inhomogeneous. Thus we get that

$$\psi_2 = \operatorname{diag}\left(\frac{1}{\sqrt{\operatorname{sgn}(a_3)a_3}}, \frac{\operatorname{sgn}(a_3)}{\sqrt{\operatorname{sgn}(a_3)a_3}}, \frac{1}{a_3}, \operatorname{sgn}(a_3)\right) \in \operatorname{Aut}\mathfrak{h}_3^\diamond$$

and that $\psi_2 \cdot \psi_1 \cdot \Gamma = E_2 + \langle E_1, E_4\rangle$.

Case: $\pi_4(\Gamma^0) \neq \{0\}$**,** $\dim \operatorname{Lie}\Gamma^0 \neq 4$

We wish to show that $\Gamma \sim \Gamma_3 = E_1 + \langle E_3, E_4\rangle$. We have constants $a_i, b_i, c_i \in \mathbb{R}$, $i = \overline{1,4}$ such that $\Gamma = \sum_{i=1}^4 a_iE_i + \left\langle \sum_{i=1}^4 b_iE_i, \sum_{i=1}^4 c_iE_i \right\rangle$. Now as $\pi_4(\Gamma^0) \neq \{0\}$ we have that either $c_4 \neq 0$ or $b_4 \neq 0$. W.l.o.g. we thus assume $c_4 \neq 0$. Then $\Gamma = \sum_{i=1}^3 (a_i - \frac{a_4c_i}{c_4})E_i + \left\langle \sum_{i=1}^3 (b_i - \frac{b_4c_i}{c_4})E_i, \sum_{i=1}^3 \frac{c_i}{c_4}E_i + E_4 \right\rangle$. Hence we get that

$$\psi_1 = \begin{bmatrix} 1 & 0 & 0 & -\frac{c_1}{c_4} \\ 0 & 1 & 0 & -\frac{c_2}{c_4} \\ -\frac{c_1}{c_4} & -\frac{c_2}{c_4} & 1 & -\frac{c_3}{c_4} + \frac{c_1^2+c_2^2}{c_4^2} \\ 0 & 0 & 0 & 1 \end{bmatrix} \in \operatorname{Aut}\mathfrak{h}_3^\diamond$$

and that

$$\begin{aligned} \psi_1 \cdot \Gamma = \psi_1 \cdot \sum_{i=1}^3 (a_i - \tfrac{a_4c_i}{c_4})E_i + \Bigg\langle \psi_1 \cdot \sum_{i=1}^3 (b_i - \tfrac{b_4c_i}{c_4})E_i, \tfrac{c_1}{c_4}\left(E_1 - \tfrac{c_1}{c_4}E_3\right) \\ + \tfrac{c_2}{c_4}\left(E_2 - \tfrac{c_2}{c_4}E_3\right) + \tfrac{c_3}{c_4}E_3 + \left(-\tfrac{c_1}{c_4}E_1 - \tfrac{c_2}{c_4}E_2 + \left(-\tfrac{c_3}{c_4} + \tfrac{c_1^2+c_2^2}{c_4^2}\right)E_3 + E_4\right) \Bigg\rangle \\ = \sum_{i=1}^3 a_i'E_i + \left\langle \sum_{i=1}^3 b_i'E_i, E_4 \right\rangle \end{aligned}$$

for some corresponding new constants $a_i', b_i' \in \mathbb{R}$, $i = \overline{1,3}$. Next we claim that $b_1' = b_2' = 0$. To this end, let $B = \sum_{i=1}^3 b_i'E_i$. Now we have that $[B, E_4] = b_1'E_2 - b_2'E_1$ and

$[B,[B,E_4]] = ((b_1')^2 + (b_2')^2)E_3$. Thus if $b_1' \neq 0$ or $b_2' \neq 0$, then

$$\begin{vmatrix} 0 & 0 & 0 & 1 \\ b_1' & b_2' & b_3' & 0 \\ -b_2' & b_1' & 0 & 0 \\ 0 & 0 & (b_1')^2 + (b_2')^2 & 0 \end{vmatrix} = -\left((b_1')^2 + (b_2')^2\right)^2 \neq 0,$$

which is to say that $\{E_4, B, [B,E_4], [B,[B,E_4]]\}$ is linearly independent. Hence we get that $\dim \operatorname{Lie}\Gamma^0 = 4$, a contradiction. Hence $b_1' = b_2' = 0$ and so $b_3' \neq 0$ (as $\dim \Gamma^0 = 2$). Consequently $\psi_1 \cdot \Gamma = a_1'E_1 + a_2'E_2 + \langle E_3, E_4\rangle$. Now note that the condition that Γ is inhomogeneous (and so $\psi_1 \cdot \Gamma$ is inhomogeneous) requires that either $a_1' \neq 0$ or $a_2' \neq 0$. Thus the equation

$$\begin{bmatrix} a_1' & a_2' \\ a_2' & -a_1' \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

has a solution for x and y (such that $(x,y) \neq (0,0)$), from which we may define an automorphism

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_1' & a_2' \\ a_2' & -a_1' \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \psi_2 = \begin{bmatrix} x & y & 0 & 0 \\ -y & x & 0 & 0 \\ 0 & 0 & x^2+y^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in \operatorname{Aut} \mathfrak{h}_3^\diamond.$$

Hence we get that

$$\psi_2 \cdot \psi_1 \cdot \Gamma = a_1'(xE_1 - yE_2) + a_2'(yE_1 + xE_2) + \langle E_3, E_4\rangle = E_1 + \langle E_3, E_4\rangle .$$

Case: $\pi_4(\Gamma^0) = \{0\}$**,** $\pi_4(A) = \pm\alpha,\ \alpha > 0$

We have that $\Gamma = \sum_{i=1}^4 a_iE_i + \left\langle \sum_{i=1}^4 b_iE_i, \sum_{i=1}^4 c_iE_i \right\rangle$ for some constants $a_i, b_i, c_i \in \mathbb{R}$, $i = \overline{1,4}$. Now as $\pi_4(\Gamma^0) = \{0\}$ we have that $b_4 = c_4 = 0$. Next we claim $a_4 \neq 0$. If $a_4 = 0$ we have that $\langle \Gamma \rangle = \left\langle \sum_{i=1}^3 a_iE_i, \sum_{i=1}^3 b_iE_i, \sum_{i=1}^3 c_iE_i \right\rangle$ and as $\pi_4([E_i,E_j]) = 0$, $i,j = \overline{1,4}$ we then have that Γ is not of full rank, a contradiction. Thus $a_4 \neq 0$ and hence there is no case for which $\alpha = 0$. Then as $\pi_4(A) = \pm\alpha$, $\alpha > 0$ we get that $\psi_1 = \operatorname{diag}(1, \operatorname{sgn}(a_4), \operatorname{sgn}(a_4), \operatorname{sgn}(a_4)) \in \operatorname{Aut}\mathfrak{h}_3^\diamond$ and that $\psi_1 \cdot \Gamma = \sum_{i=1}^3 a_i'E_i + \alpha E_4 + \left\langle \sum_{i=1}^3 b_i'E_i, \sum_{i=1}^3 c_i'E_i \right\rangle$ for some new constants $a_i', b_i', c_i' \in \mathbb{R}$, $i = \overline{1,3}$. Next we have that

$$\psi_2 = \begin{bmatrix} 1 & 0 & 0 & -\frac{a_1'}{\alpha} \\ 0 & 1 & 0 & -\frac{a_2'}{\alpha} \\ -\frac{a_1'}{\alpha} & -\frac{a_2'}{\alpha} & 1 & -\frac{a_3'}{\alpha} + \frac{(a_1')^2+(a_2')^2}{\alpha^2} \\ 0 & 0 & 0 & 1 \end{bmatrix} \in \operatorname{Aut}\mathfrak{h}_3^\diamond$$

and hence get that

$$\psi_2 \cdot \psi_1 \cdot \Gamma = \sum_{i=1}^{2} a_i'(E_i - \tfrac{a_i'}{\alpha}E_3) + a_3'E_3 + \alpha\left(-\tfrac{a_1'}{\alpha}E_1 - \tfrac{a_2'}{\alpha}E_2 + (-\tfrac{a_3'}{\alpha} + \tfrac{(a_1')^2+(a_2')^2}{\alpha^2})E_3\right)$$
$$+ \alpha E_4 + \left\langle \psi_2 \cdot \sum_{i=1}^{3} b_i'E_i, \psi_2 \cdot \sum_{i=1}^{3} c_i'E_i \right\rangle$$
$$= \alpha E_4 + \left\langle \sum_{i=1}^{3} b_i''E_i, \sum_{i=1}^{3} c_i''E_i \right\rangle$$

for some new constants $b_i'', c_i'' \in \mathbb{R}$, $i = \overline{1,3}$. Now we claim that at least one of b_1'', b_2'', c_1'' and c_2'' is non-zero, i.e., $(b_1'' \neq 0 \vee b_2'' \neq 0) \vee (c_1'' \neq 0 \vee c_2'' \neq 0)$. Suppose not, i.e., $b_1'' = b_2'' = c_1'' = c_2'' = 0$. Then $\dim \Gamma^0 = \dim \langle b_3''E_3, c_3''E_3\rangle < 2$, a contradiction. Thus w.l.o.g. we may assume $(b_1'' \neq 0 \vee b_2'' \neq 0)$. Then the equation

$$\begin{bmatrix} b_1'' & b_2'' \\ b_2'' & -b_1'' \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

has a solution for x and y (such that $(x, y) \neq (0, 0)$), from which we may define an automorphism

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1'' & b_2'' \\ b_2'' & -b_1'' \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \psi_3 = \begin{bmatrix} x & y & 0 & 0 \\ -y & x & 0 & 0 \\ 0 & 0 & x^2+y^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in \operatorname{Aut}\mathfrak{h}_3^\diamond.$$

Let $\psi = \psi_3 \cdot \psi_2 \cdot \psi_1$. Hence we get that

$$\psi \cdot \Gamma = \alpha E_4 + \left\langle b_1''(xE_1 - yE_2) + b_2''(yE_1 + xE_2) + b_3''\left(x^2+y^2\right)E_3, \psi_3 \cdot \sum_{i=1}^{3} c_i''E_i \right\rangle$$
$$= \alpha E_4 + \left\langle E_1 + b_3'''E_3, c_2'''E_2 + c_3'''E_3 \right\rangle$$

for some new constants $b_3''', c_i''' \in \mathbb{R}$, $i = \overline{1,3}$.

Case: $\mathfrak{T}(\Gamma) = 1$

We wish to show that $\Gamma \sim \Gamma_{4,\alpha} = \alpha E_4 + \langle E_1, E_2\rangle$. Now by our preliminary discussion we have that there is an automorphism ψ and constants b_3, c_2, c_3 such that $\psi \cdot \Gamma = \alpha E_4 + \langle E_1 + b_3E_3, c_2E_2 + c_3E_3\rangle$. Then, as $\mathfrak{T}(\Gamma) = 1$, we have that $|\operatorname{sgn}(c_2)| = 1$, i.e., $c_2 \neq 0$. Thus $\psi \cdot \Gamma = \alpha E_4 + \left\langle E_1 + b_3E_3, E_2 + \frac{c_3}{c_2}E_3\right\rangle$. Now we have that

$$\psi_4 = \begin{bmatrix} 1 & 0 & 0 & -b_3 \\ 0 & 1 & 0 & -\frac{c_3}{c_2} \\ -b_3 & -\frac{c_3}{c_2} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in \operatorname{Aut}\mathfrak{h}_3^\diamond$$

and hence get that

$$\begin{aligned}\psi_4 \cdot \psi \cdot \Gamma &= \alpha(E_4 - b_3E_1 - \tfrac{c_3}{c_2}E_2) + \left\langle (E_1 - b_3E_3) + b_3E_3, (E_2 - \tfrac{c_3}{c_2}E_3) + \tfrac{c_3}{c_2}E_3 \right\rangle \\ &= \alpha E_4 - \alpha b_3 E_1 - \alpha\tfrac{c_3}{c_2}E_2 + \langle E_1, E_2\rangle \\ &= \alpha E_4 + \langle E_1, E_2\rangle\,.\end{aligned}$$

Case: $\mathfrak{T}(\Gamma) = 0$

We wish to show that $\Gamma \sim \Gamma_{5,\alpha} = \alpha E_4 + \langle E_1, E_3\rangle$. Now by our preliminary discussion we have that there is an automorphism ψ and constants b_3, c_2, c_3 such that $\psi \cdot \Gamma = \alpha E_4 + \langle E_1 + b_3E_3, c_2E_2 + c_3E_3\rangle$. Now, as $\mathfrak{T}(\Gamma) = 0$, we have that $|\text{sgn}(c_2)| = 0$, i.e., $c_2 = 0$. Thus $\psi \cdot \Gamma = \alpha E_4 + \langle E_1 + b_3E_3, c_3E_3\rangle$. Then, as $\dim \Gamma = 2$, we have that $\dim \psi \cdot \Gamma = 2$ and so $c_3 \neq 0$. Hence we get that $\psi \cdot \Gamma = \alpha E_4 + \langle E_1 + b_3E_3, E_3\rangle = \alpha E_4 + \langle E_1, E_3\rangle$.

Now as our list is exhaustive we are left to show that none of the affine subspaces

$$\begin{aligned}&\Gamma_{1,\alpha} = E_2 + \alpha E_3 + \langle E_1, E_4\rangle\,,\ \alpha \geq 0 && \Gamma_2 = E_3 + \langle E_1, E_4\rangle \\ &\Gamma_3 = E_1 + \langle E_3, E_4\rangle && \Gamma_{4,\alpha} = \alpha E_4 + \langle E_1, E_2\rangle\,,\ \alpha > 0 \\ &\Gamma_{5,\alpha} = \alpha E_4 + \langle E_1, E_3\rangle\,,\ \alpha > 0\end{aligned}$$

are $\mathfrak{L}$-related. In general this is done by assuming that they are, giving us an automorphism ψ taking one to the other, and producing a contradiction from this.

$(\alpha \neq \beta \wedge \alpha, \beta \geq 0) \Rightarrow (\Gamma_{1,\alpha} \nsim \Gamma_{1,\beta})$**:** Suppose there exists $\psi \in \text{Aut}\,\mathfrak{h}_3^\diamond$ such that $\psi \cdot \Gamma_{1,\alpha} = \Gamma_{1,\beta}$. Then we get that $\alpha = \mathfrak{S}(\Gamma_{1,\alpha}) = \mathfrak{S}(\psi \cdot \Gamma_{1,\alpha}) = \mathfrak{S}(\Gamma_{1,\beta}) = \beta$, a contradiction.

$\Gamma_{1,\alpha} \nsim \Gamma_2$**:** Suppose there exists $\psi \in \text{Aut}\,\mathfrak{h}_3^\diamond$ such that $\psi \cdot \Gamma_{1,\alpha} = \Gamma_2$. Then we get that $1 = \mathfrak{T}(\Gamma_{1,\alpha}) = \mathfrak{T}(\psi \cdot \Gamma_{1,\alpha}) = \mathfrak{T}(\Gamma_2) = 0$, a contradiction.

$\Gamma_{1,\alpha}, \Gamma_2 \nsim \Gamma_3$**:** For the first two $\dim \text{Lie}\,\Gamma^0 = 4$, but $\dim \text{Lie}\,\Gamma_3^0 = 2$. As $\dim \text{Lie}\,\Gamma^0$ is preserved by any automorphism (i.e., $\dim \text{Lie}\,(\psi \cdot \Gamma^0) = \dim \text{Lie}\,\Gamma^0$) we have that neither $\Gamma_{1,\alpha}$ nor Γ_2 can be $\mathfrak{L}$-related to Γ_3.

$\Gamma_{1,\alpha}, \Gamma_2, \Gamma_3 \nsim \Gamma_{4,\alpha}, \Gamma_{5,\alpha}$**:** For the first three $\pi_4(\Gamma^0) \neq \{0\}$ and for the last two $\pi_4(\Gamma^0) = \{0\}$. Suppose ψ is an automorphism taking one of $\{\Gamma_{1,\alpha}, \Gamma_2, \Gamma_3\}$ to one of $\{\Gamma_{4,\alpha}, \Gamma_{5,\alpha}\}$. Then in particular it must take one of $\{\Gamma^0_{1,\alpha}, \Gamma^0_2, \Gamma^0_3\}$ to one of $\{\Gamma^0_{4,\alpha}, \Gamma^0_{5,\alpha}\}$. But for each of $\{\Gamma^0_{1,\alpha}, \Gamma^0_2, \Gamma^0_3\}$, $E_4 \in \Gamma^0$ we have that $\pi_4(\psi \cdot E_4) = k = \pm 1$. But this contradicts the fact that $\pi_4(\Gamma^0) = \{0\}$ for $\Gamma^0 \in \{\Gamma^0_{4,\alpha}, \Gamma^0_{5,\alpha}\}$.

$(\alpha \neq \beta \wedge \alpha, \beta > 0) \Rightarrow (\Gamma_{4,\alpha} \nsim \Gamma_{4,\beta} \wedge \Gamma_{5,\alpha} \nsim \Gamma_{5,\beta})$**:** Suppose we have an automorphism ψ such that $\psi \cdot \Gamma_{4,\alpha} = \Gamma_{4,\beta}$. Then as $\alpha E_4 \in \Gamma_{4,\alpha}$ we have that $\psi \cdot \alpha E_4 \in \Gamma_{4,\beta}$. Hence we get that $\pi_4(\psi \cdot \alpha E_4) = k\alpha = \pm\alpha$, but for any element $A \in \Gamma_{4,\beta}$ we have that $\pi_4(A) = \beta$, a contradiction. The exact same argument can be made to show $\Gamma_{5,\alpha} \nsim \Gamma_{5,\beta}$.

$\Gamma_{4,\alpha} \nsim \Gamma_{5,\beta}$**:** Suppose there exists $\psi \in \text{Aut}\,\mathfrak{h}_3^\diamond$ such that $\psi \cdot \Gamma_{4,\alpha} = \Gamma_{5,\beta}$. Then we get that $1 = \mathfrak{T}(\Gamma_{4,\alpha}) = \mathfrak{T}(\psi \cdot \Gamma_{4,\alpha}) = \mathfrak{T}(\Gamma_{5,\beta}) = 0$, a contradiction. □

2.5 Dimension Three, Homogeneous

We start be defining an **invariant $\mathfrak{R}$** similar to the invariant $\mathfrak{T}$ defined for $(2,1)$-affine subspaces. Let Γ be a $(3,0)$-affine subspace and let $P : (x,y,z) \mapsto x\left(\sum_{i=1}^4 a_i E_i\right) + y\left(\sum_{i=1}^4 b_i E_i\right) + z\left(\sum_{i=1}^4 c_i E_i\right)$ be a parametrisation of Γ. Then we define $\mathfrak{R}(\Gamma)$, w.r.t. parametrisation P, as

$$\mathfrak{R}_P(\Gamma) = \left|\operatorname{sgn}\left(\begin{vmatrix} a_1 & a_2 & a_4 \\ b_1 & b_2 & b_4 \\ c_1 & c_2 & c_4 \end{vmatrix}\right)\right|.$$

We show that $\mathfrak{R}$ doesn't depend on the parametrisation chosen and that automorphisms leave $\mathfrak{R}$ invariant.

2.5.1 Lemma. *Let $P_1 : (x,y,z) \mapsto xA + yB + zC$ and $P_2 : (x,y,z) \mapsto x\widetilde{A} + y\widetilde{B} + z\widetilde{C}$ be two parametrisations of a $(3,0)$-affine subspace Γ. Then $\mathfrak{R}_{P_1}(\Gamma) = \mathfrak{R}_{P_2}(\Gamma)$.*

Proof. As P_1 and P_2 parametrise the same space we have that we have that

$$A = \eta_1\widetilde{A} + \eta_2\widetilde{B} + \eta_3\widetilde{C} \qquad B = \nu_1\widetilde{A} + \nu_2\widetilde{B} + \nu_3\widetilde{C} \qquad C = \mu_1\widetilde{A} + \mu_2\widetilde{B} + \mu_3\widetilde{C}$$

for some constants $\eta_i, \nu_i \mu_i \in \mathbb{R}$, $i = \overline{1,3}$, satisfying (as $\{A, B, C\}$ is linearly independent)

$$\begin{vmatrix} \eta_1 & \eta_2 & \eta_3 \\ \nu_1 & \nu_2 & \nu_3 \\ \mu_1 & \mu_2 & \mu_3 \end{vmatrix} \neq 0.$$

Using these identities for simplification we get that

$$\mathfrak{R}_{P_1}(\Gamma) = \left|\operatorname{sgn}\left(\begin{vmatrix} a_1 & a_2 & a_4 \\ b_1 & b_2 & b_4 \\ c_1 & c_2 & c_4 \end{vmatrix}\right)\right| = \left|\operatorname{sgn}\left(\begin{vmatrix} \eta_1 & \eta_2 & \eta_3 \\ \nu_1 & \nu_2 & \nu_3 \\ \mu_1 & \mu_2 & \mu_3 \end{vmatrix}\begin{vmatrix} \widetilde{a}_1 & \widetilde{a}_2 & \widetilde{a}_4 \\ \widetilde{b}_1 & \widetilde{b}_2 & \widetilde{b}_4 \\ \widetilde{c}_1 & \widetilde{c}_2 & \widetilde{c}_4 \end{vmatrix}\right)\right| = \mathfrak{R}_{P_2}(\Gamma). \qquad \square$$

2.5.2 Proposition. *If Γ is a $(3,0)$-affine subspace, then $\mathfrak{R}(\psi \cdot \Gamma) = \mathfrak{R}(\Gamma)$ for $\psi \in \operatorname{Aut}\mathfrak{h}_3^\diamond$.*

Proof. Let $\psi \in \operatorname{Aut}\mathfrak{h}_3^\diamond$ be an arbitrary automorphism. Let $\Gamma = A + \langle B, C\rangle = \sum_{i=1}^4 a_i E_i + \left\langle \sum_{i=1}^4 b_i E_i, \sum_{i=1}^4 c_i E_i \right\rangle$. Then we get that

$$\mathfrak{R}(\psi\cdot\Gamma) = \left|\operatorname{sgn}\left(\begin{vmatrix} x & y & u \\ -ky & kx & v \\ 0 & 0 & k \end{vmatrix}\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_4 & b_4 & c_4 \end{vmatrix}\right)\right| = \left|\operatorname{sgn}\left(k^2(x^2+y^2)\right)\right| \mathfrak{R}(\Gamma) = \mathfrak{R}(\Gamma). \qquad \square$$

Having set up our invariant in this dimension we proceed to classification.

2.5.3 Proposition. *Any $(3,0)$-affine subspace $\Gamma = A + \Gamma^0 \subset \mathfrak{h}_3^\diamond$ (of full rank) is $\mathfrak{L}$-related to exactly one of the following affine subspaces:*

Case: $\mathfrak{R}(\Gamma) = 1$, $\Gamma_1^{(3,0)} = \langle E_1, E_2, E_4\rangle$, **Case:** $\mathfrak{R}(\Gamma) = 0$, $\Gamma_2^{(3,0)} = \langle E_1, E_3, E_4\rangle$.

Proof. By proposition 2.4.5 and corollary 2.1.9, Γ is $\mathfrak{L}$-related to one of the following affine subspaces (with $\mathfrak{R}(\Gamma)$ given alongside)

$$\begin{aligned}
&\left\langle \Gamma^{(2,1)}_{1,\alpha} \right\rangle = \langle E_2 + \alpha E_3, E_1, E_4 \rangle\,, \ \alpha \geq 0 && \mathfrak{R}\big(\left\langle \Gamma^{(2,1)}_{1,\alpha} \right\rangle\big) = 1\\
&\left\langle \Gamma^{(2,1)}_{2} \right\rangle = \langle E_3, E_1, E_4 \rangle && \mathfrak{R}\big(\left\langle \Gamma^{(2,1)}_{2} \right\rangle\big) = 0\\
&\left\langle \Gamma^{(2,1)}_{3} \right\rangle = \langle E_1, E_3, E_4 \rangle && \mathfrak{R}\big(\left\langle \Gamma^{(2,1)}_{3} \right\rangle\big) = 0\\
&\left\langle \Gamma^{(2,1)}_{4,\alpha} \right\rangle = \langle \alpha E_4, E_1, E_2 \rangle\,, \ \alpha > 0 && \mathfrak{R}\big(\left\langle \Gamma^{(2,1)}_{4,\alpha} \right\rangle\big) = 1\\
&\left\langle \Gamma^{(2,1)}_{5,\alpha} \right\rangle = \langle \alpha E_4, E_1, E_3 \rangle\,, \ \alpha > 0 && \mathfrak{R}\big(\left\langle \Gamma^{(2,1)}_{5,\alpha} \right\rangle\big) = 0.
\end{aligned}$$

Assume $\mathfrak{R}(\Gamma) = 0$. Then as $\mathfrak{R}$ is invariant we have that Γ is $\mathfrak{R}$-related to $\left\langle \Gamma^{(2,1)}_{2} \right\rangle$, $\left\langle \Gamma^{(2,1)}_{3} \right\rangle$ or $\left\langle \Gamma^{(2,1)}_{5,\alpha} \right\rangle$. But $\left\langle \Gamma^{(2,1)}_{2} \right\rangle = \left\langle \Gamma^{(2,1)}_{3} \right\rangle = \left\langle \Gamma^{(2,1)}_{5,\alpha} \right\rangle = \langle E_1, E_3, E_4 \rangle$. Thus Γ is $\mathfrak{L}$-related to $\langle E_1, E_3, E_4 \rangle$. On the other hand assume $\mathfrak{R}(\Gamma) = 1$. Then Γ is $\mathfrak{L}$-related to either $\left\langle \Gamma^{(2,1)}_{1,\alpha} \right\rangle$ or $\left\langle \Gamma^{(2,1)}_{4,\alpha} \right\rangle = \langle E_1, E_2, E_4 \rangle$. If Γ is $\mathfrak{L}$-related to $\left\langle \Gamma^{(2,1)}_{4,\alpha} \right\rangle$ we are done. So assume Γ is $\mathfrak{L}$-related to $\left\langle \Gamma^{(2,1)}_{1,\alpha} \right\rangle$. Then we have that

$$\psi = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\alpha \\ 0 & -\alpha & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in \operatorname{Aut} \mathfrak{h}^{\diamond}_3$$

and get that

$$\psi \cdot \langle E_2 + \alpha E_3, E_1, E_4 \rangle = \langle (E_2 - \alpha E_3) + \alpha E_3, E_1, -\alpha E_2 + E_4 \rangle = \langle E_1, E_2, E_4 \rangle\,.$$

Finally note that Γ_1 is not $\mathfrak{L}$-related to Γ_2 as $\mathfrak{R}(\Gamma_1) = 1$ and $\mathfrak{R}(\Gamma_2) = 0$. (If $\psi \in \operatorname{Aut} \mathfrak{h}^{\diamond}_3$ and $\psi \cdot \Gamma_1 = \Gamma_2$, then $1 = \mathfrak{R}(\Gamma_1) = \mathfrak{R}(\psi \cdot \Gamma_1) = \mathfrak{R}(\Gamma_2) = 0$, a contradiction.) □

2.6 Dimension Three, Inhomogeneous

2.6.1 Proposition. *Any* $(3,1)$*-affine subspace* $\Gamma = A + \Gamma^0 \subset \mathfrak{h}^{\diamond}_3$ *(of full rank) is* $\mathfrak{L}$*-related to exactly one of the following affine subspaces:*

Case: $\pi_4(\Gamma^0) \neq \{0\}$

- **Case:** $\mathfrak{R}(\Gamma^0) = 0$
  - $\Gamma^{(3,1)}_{1} = E_2 + \langle E_1, E_3, E_4 \rangle$
- **Case:** $\mathfrak{R}(\Gamma^0) = 1$
  - $\Gamma^{(3,1)}_{2} = E_3 + \langle E_1, E_2, E_4 \rangle$

Case: $\pi_4(\Gamma^0) = \{0\}$**,** $\pi_4(A) = \pm\alpha$**,** $\alpha > 0$

- $\Gamma^{(3,1)}_{3,\alpha} = \alpha E_4 + \langle E_1, E_2, E_3 \rangle.$

Proof. (We omit the superscript $(3,1)$ in this proof.) Let $\Gamma = A + \Gamma^0$ be a $(3,1)$-affine subspace (of full rank) of $\mathfrak{h}^{\diamond}_3$.

Case: $\pi_4(\Gamma^0) \neq \{0\}$

We claim that Γ^0 is of full rank. We have constants $a_i, b_i, c_i \in \mathbb{R}$, $i = \overline{1,4}$ such that $\Gamma^0 = \left\langle \sum_{i=1}^4 a_i E_i, \sum_{i=1}^4 b_i E_i, \sum_{i=1}^4 c_i E_i \right\rangle$. Then as $\pi_4(\Gamma^0) \neq \{0\}$ we have that at least one of a_4, b_4 and c_4 is non-zero (i.e., $a_4 \neq 0 \vee b_4 \neq 0 \vee c_4 \neq 0$). So w.l.o.g. we assume $c_4 \neq 0$. Then we have that $\Gamma^0 = \left\langle \sum_{i=1}^3 a'_i E_i, \sum_{i=1}^3 b'_i E_i, \sum_{i=1}^3 c'_i E_i + E_4 \right\rangle$ for some new constants $a'_i, b'_i, c'_i \in \mathbb{R}$, $i = \overline{1,3}$. Let $A' = \sum_{i=1}^3 a'_i E_i$, $B' = \sum_{i=1}^3 b'_i E_i$ and $C' = \sum_{i=1}^3 c'_i E_i + E_4$. Now we claim that either $(a'_1)^2 + (a'_2)^2 \neq 0$ or $(b'_1)^2 + (b'_2)^2 \neq 0$. Suppose not, then $a'_1 = a'_2 = b'_1 = b'_2 = 0$ and hence $\dim \Gamma^0 = \dim \langle a'_3 E_3, b'_3 E_3, C \rangle < 3$, a contradiction. So w.l.o.g. we can assume that $(a'_1)^2 + (a'_2)^2 \neq 0$. Next we observe that

$$\begin{aligned} [A', C'] &= -a'_2 E_1 + a'_1 E_2 + (a'_1 c'_2 - a'_2 c'_1) E_3 \\ [A', [A', C']] &= \left((a'_1)^2 + (a'_2)^2 \right) E_3. \end{aligned}$$

Now if we consider the square matrix with rows A', $[A', C']$, $[A', [A', C']]$ and C' (w.r.t. the ordered basis) and take its determinant we get that

$$\begin{vmatrix} a'_1 & a'_2 & a'_3 & 0 \\ -a'_2 & a'_1 & (a'_1 c'_2 - a'_2 c'_1) & 0 \\ 0 & 0 & (a'_1)^2 + (a'_2)^2 & 0 \\ c'_1 & c'_2 & c'_3 & 1 \end{vmatrix} = \left((a'_1)^2 + (a'_2)^2 \right)^2 \neq 0.$$

Hence $\{A', [A', C'], [A', [A', C']], C'\}$ is linearly independent and thus Γ^0 is of full rank. Consequently Γ^0 is a $(3,0)$-affine subspace of full rank. Thus $\mathfrak{R}(\Gamma^0)$ is defined and we can apply the results of proposition 2.5.3.

Case: $\mathfrak{R}(\Gamma^0) = 0$

We wish to show that $\Gamma \sim \Gamma_1 = E_2 + \langle E_1, E_3, E_4 \rangle$. By proposition 2.5.3 (noting that $\mathfrak{R}(\Gamma^0) = 0$) we have that there exists an automorphism $\psi_1 \in \operatorname{Aut} \mathfrak{h}_3^\diamond$ such that $\psi_1 \cdot \Gamma^0 = \langle E_1, E_3, E_4 \rangle$. Thus (for some new constants $a_i \in \mathbb{R}$, $i = \overline{1,4}$) we get that $\psi_1 \cdot \Gamma = \sum_{i=1}^4 a_i E_i + \langle E_1, E_3, E_4 \rangle = a_2 E_2 + \langle E_1, E_3, E_4 \rangle$. Then (noting that $a_2 \neq 0$ as Γ is inhomogeneous) we have that $\psi_2 = \operatorname{diag}\left(\frac{1}{a_2}, \frac{1}{a_2}, \frac{1}{a_2^2}, 1 \right) \in \operatorname{Aut} \mathfrak{h}_3^\diamond$ and consequently get that $\psi_2 \cdot \psi_1 \cdot \Gamma = E_2 + \langle E_1, E_3, E_4 \rangle$.

Case: $\mathfrak{R}(\Gamma^0) = 1$

We wish to show that $\Gamma \sim \Gamma_2 = E_3 + \langle E_1, E_2, E_4 \rangle$. By proposition 2.5.3 (noting that $\mathfrak{R}(\Gamma^0) = 1$) we have that there exists an automorphism $\psi_1 \in \operatorname{Aut} \mathfrak{h}_3^\diamond$ such that $\psi_1 \cdot \Gamma^0 = \langle E_1, E_2, E_4 \rangle$. Thus (for some new constants $a_i \in \mathbb{R}$, $i = \overline{1,4}$) we get that $\psi_1 \cdot \Gamma = \sum_{i=1}^4 a_i E_i + \langle E_1, E_2, E_4 \rangle = a_3 E_2 + \langle E_1, E_2, E_4 \rangle$. Then (noting that $a_2 \neq 0$ as Γ is inhomogeneous) we have that

$$\psi_2 = \operatorname{diag}\left(\frac{1}{\sqrt{\operatorname{sgn}(a_3) a_3}}, \frac{\operatorname{sgn}(a_3)}{\sqrt{\operatorname{sgn}(a_3) a_3}}, \frac{1}{a_3}, \operatorname{sgn}(a_3) \right) \in \operatorname{Aut} \mathfrak{h}_3^\diamond$$

and consequently get that $\psi_2 \cdot \psi_1 \cdot \Gamma = E_3 + \langle E_1, E_2, E_4 \rangle$.

Case: $\pi_4(\Gamma^0) = \{0\}$**,** $\pi_4(A) = \pm\alpha,\ \alpha > 0$

We wish to show that $\Gamma \sim \Gamma_{3,\alpha} = \alpha E_4 + \langle E_1, E_2, E_3\rangle$. Now as $\pi_4(\Gamma^0) = \{0\}$ and $\Gamma^0 \subset \langle E_1, E_2, E_3, E_4\rangle$ we have that $\Gamma^0 = \langle E_1, E_2, E_3\rangle$. Now we have constants $a_i \in \mathbb{R}$, $i = \overline{1,4}$ such that $\Gamma = \sum_{i=1}^4 a_i E_i + \langle E_1, E_2, E_3\rangle = a_4 E_4 + \langle E_1, E_2, E_3\rangle$. We note that as $\pi_4(A) = \pm\alpha$, $\alpha > 0$ ($\alpha \neq 0$ required, otherwise Γ would be homogeneous), we have that $|a_4| = \alpha$. So then $\psi_1 = \operatorname{diag}(1, \operatorname{sgn}(a_4), \operatorname{sgn}(a_4), \operatorname{sgn}(a_4)) \in \operatorname{Aut}\mathfrak{h}_3^\diamond$ and hence we get that $\psi_1 \cdot \Gamma = |a_4| E_4 + \langle E_1, E_2, E_3\rangle = \alpha E_4 + \langle E_1, E_2, E_3\rangle$. (Note that the class containing $\Gamma_{3,\alpha}$ contains only two elements, namely $\Gamma_{3,\alpha}$ and $\Gamma_{3,-\alpha}$.)

Now as our list is exhaustive we are left to show that none of the affine subspaces

$$\Gamma_1 = E_2 + \langle E_1, E_3, E_4\rangle \qquad \Gamma_2 = E_3 + \langle E_1, E_2, E_4\rangle \qquad \Gamma_{3,\alpha} = \alpha E_4 + \langle E_1, E_2, E_3\rangle\,,\ \alpha > 0$$

are $\mathcal{L}$-related. This is done by assuming that they are, giving us an automorphism ψ taking one to the other, and producing a contradiction from this.

$\Gamma_1 \nsim \Gamma_2$**:** Assume there exists a $\psi \in \operatorname{Aut}\mathfrak{h}_3^\diamond$ such that $\psi \cdot \Gamma_1^0 = \Gamma_2^0$. Then $0 = \mathfrak{R}(\Gamma_1^0) = \mathfrak{R}(\psi \cdot \Gamma_1^0) = \mathfrak{R}(\Gamma_2^0) = 1$, a contradiction.

$\Gamma_1, \Gamma_2 \nsim \Gamma_{3,\alpha}$**:** Assume there exists $\psi_i \in \operatorname{Aut}\mathfrak{h}_3^\diamond$ such that $\psi_i \cdot \Gamma_i^0 = \Gamma_{3,\alpha}^0$, $i = \overline{1,2}$. Now we have that $E_4 \in \Gamma_i^0$, thus $\psi_i \cdot E_4 \in \Gamma_{3,\alpha}^0$. However we have that $\pi_4(\psi_i \cdot E_4) = k = \pm 1$, but $\pi_4(\Gamma_{3,\alpha}^0) = 0$, providing a contradiction.

$(\alpha \neq \beta \wedge \alpha, \beta > 0) \Rightarrow \Gamma_{3,\alpha} \nsim \Gamma_{3,\beta}$**:** Assume there exists a $\psi \in \operatorname{Aut}\mathfrak{h}_3^\diamond$ such that $\psi \cdot \Gamma_{3,\alpha} = \Gamma_{3,\beta}$. Then as $\alpha E_4 \in \Gamma_{3,\alpha}$ we have that $\psi \cdot \alpha E_4 \in \Gamma_{3,\beta}$. Thus we get that $\pi_4(\psi \cdot \alpha E_4) = k\alpha = \pm\alpha$. But $\pi_4(B) = \beta$ for $B \in \Gamma_{3,\beta}$. Hence $\alpha = \beta$, a contradiction. □

2.7 Summary

We collect our results in a theorem.

2.7.1 Theorem. *Any affine subspace Γ of $\mathfrak{h}_3^\diamond$ (of full rank) is $\mathfrak{L}$-related to exactly one of the following affine subspaces:*

$$\begin{aligned}
&\Gamma_{1,\alpha}^{(1,1)} = E_1 + \alpha E_3 + \langle E_4\rangle\,,\ \alpha \geq 0 && \Gamma_{2,\alpha}^{(1,1)} = \alpha E_4 + \langle E_1\rangle\,,\ \alpha > 0\\
&\Gamma_1^{(2,0)} = \langle E_1, E_4\rangle && \Gamma_{1,\alpha}^{(2,1)} = E_2 + \alpha E_3 + \langle E_1, E_4\rangle\,,\ \alpha \geq 0\\
&\Gamma_2^{(2,1)} = E_3 + \langle E_1, E_4\rangle && \Gamma_3^{(2,1)} = E_1 + \langle E_3, E_4\rangle\\
&\Gamma_{4,\alpha}^{(2,1)} = \alpha E_4 + \langle E_1, E_2\rangle\,,\ \alpha > 0 && \Gamma_{5,\alpha}^{(2,1)} = \alpha E_4 + \langle E_1, E_3\rangle\,,\ \alpha > 0\\
&\Gamma_1^{(3,0)} = \langle E_1, E_2, E_4\rangle && \Gamma_2^{(3,0)} = \langle E_1, E_3, E_4\rangle\\
&\Gamma_1^{(3,1)} = E_2 + \langle E_1, E_3, E_4\rangle && \Gamma_2^{(3,1)} = E_3 + \langle E_1, E_2, E_4\rangle\\
&\Gamma_{3,\alpha}^{(3,1)} = \alpha E_4 + \langle E_1, E_2, E_3\rangle\,,\ \alpha > 0 && \Gamma_1^{(4,0)} = \langle E_1, E_2, E_3, E_4\rangle\,.
\end{aligned}$$

2.7.2 COROLLARY. *Any full rank system* Σ *(specifically* $\Sigma \in \mathrm{Ob}\,\mathbf{LiCAS}$*) with Lie algebra isomorphic to* $\mathfrak{h}_3^\diamond$ *is* DF_{loc}*-equivalent to exactly one of the systems* $\Sigma_{i,\alpha}^{(\ell,\varepsilon)} = \left(\mathsf{H}_3^\diamond, \Xi_{i,\alpha}^{(\ell,\varepsilon)}\right)$, $\alpha \in \mathbb{R}$, *where*

$$\Xi_{1,\alpha}^{(1,1)}(\mathbf{1},u) = E_1 + \alpha E_3 + uE_4,\ \alpha \geq 0 \qquad \Xi_{2,\alpha}^{(1,1)}(\mathbf{1},u) = \alpha E_4 + uE_1,\ \alpha > 0$$
$$\Xi_{1}^{(2,0)}(\mathbf{1},u) = u_1E_1 + u_2E_4 \qquad \Xi_{1,\alpha}^{(2,1)}(\mathbf{1},u) = E_2 + \alpha E_3 + u_1E_1 + u_2E_4,\ \alpha \geq 0$$
$$\Xi_{2}^{(2,1)}(\mathbf{1},u) = E_3 + u_1E_1 + u_2E_4 \qquad \Xi_{3}^{(2,1)}(\mathbf{1},u) = E_1 + u_1E_3 + u_2E_4$$
$$\Xi_{4,\alpha}^{(2,1)}(\mathbf{1},u) = \alpha E_4 + u_1E_1 + u_2E_2,\ \alpha > 0 \qquad \Xi_{5,\alpha}^{(2,1)}(\mathbf{1},u) = \alpha E_4 + u_1E_1 + u_2E_3,\ \alpha > 0$$
$$\Xi_{1}^{(3,0)}(\mathbf{1},u) = u_1E_1 + u_2E_2 + u_3E_4 \qquad \Xi_{2}^{(3,0)}(\mathbf{1},u) = u_1E_1 + u_2E_3 + u_3E_4$$
$$\Xi_{1}^{(3,1)}(\mathbf{1},u) = E_2 + u_1E_1 + u_2E_3 + u_3E_4 \qquad \Xi_{2}^{(3,1)}(\mathbf{1},u) = E_3 + u_1E_1 + u_2E_2 + u_3E_4$$
$$\Xi_{3,\alpha}^{(3,1)}(\mathbf{1},u) = \alpha E_4 + u_1E_1 + u_2E_2 + u_3E_3,\ \alpha > 0 \qquad \Xi_{1}^{(4,0)}(\mathbf{1},u) = u_1E_1 + u_2E_2 + u_3E_3 + u_4E_4.$$

PROOF. A general method for proving precisely this was the motivation for making the classification of $\mathfrak{L}$-related affine subspaces, as described in the preliminaries. We now concisely apply this method to get this result. Suppose $\Sigma = (\mathsf{G}, \Xi)$ has Lie algebra isomorphic to $\mathfrak{h}_3^\diamond$. That is, we have a Lie algebra isomorphism $\psi : \mathfrak{g} \to \mathfrak{h}_3^\diamond$. By theorem 2.7.1 it then follows that $\psi \cdot \Gamma$ is $\mathfrak{L}$-related to $\Gamma_{i,\alpha}^{(\ell,\varepsilon)}$ for some ℓ, ε, i and α. Thus there exists a $\psi' \in \mathrm{Aut}\,\mathfrak{h}_3^\diamond$ such that $\psi' \cdot \psi \cdot \Gamma = \Gamma_{i,\alpha}^{(\ell,\varepsilon)}$. By theorem 2.0.1 it then follows that Σ is DF_{loc}-equivalent to $\Sigma_{i,\alpha}^{(\ell,\varepsilon)}$. □

2.7.3 REMARK. Note that the group $\mathsf{H}_3^\diamond$ in the above theorem may be replaced by any group G with Lie algebra isomorphic to $\mathfrak{h}_3^\diamond$. Let $\psi : \mathfrak{h}_3^\diamond \to \mathfrak{g}$ be such a Lie algebra isomorphism. Then the required ordered basis $\{E_i'\}_{i=\overline{1,4}}$ for $\mathfrak{g}$ (so as to define the parametrisation maps) is then given by $E_i' = \psi \cdot E_i$.

Chapter 3

Global Classification of Controllable Systems on Oscillator Lie Groups

The goal of this chapter is to classify controllable proper left-invariant control affine systems with Lie algebra isomorphic to $\mathfrak{h}_3^\diamond$ under (global) detached feedback equivalence. However such a (full) classification is quite extensive. As such, we only present results pertaining to systems with state space Lie group isomorphic to $\widetilde{\mathsf{H}}_3^\diamond$ or $\mathsf{H}_3^\diamond(n)$, $n \in \mathbb{N}$. In other words, we restrict ourselves to the Lie groups, with Lie algebra isomorphic to $\mathfrak{h}_3^\diamond$, which have faithful linear representation (see proposition 1.3.10 in regard to this).

We briefly recall some concepts as introduced in section A.2 (and at the start of chapter 2). A left-invariant control affine system is a pair (G, Ξ), where G is a Lie Group and $\Xi : \mathsf{G}\times\mathbb{R}^\ell \to T\mathsf{G}$ is a smooth embedding such that

$$\Xi(g, u) = g\,\Xi(\mathbf{1}, u) = g\left(A + \sum_{i=1}^{\ell} u_i B_i\right)$$

where the set $\{B_i\}_{i=\overline{1,\ell}}$ is linearly independent. These systems can be organised into a category (see [6] and section A.2) which we denote **LiCAS**. The **trace** Γ of the system is defined as $\Gamma = A + \Gamma^0$, where $\Gamma^0 = \operatorname{span}\{B_i\}_{i=\overline{1,\ell}}$. A system Σ is said to be **of full rank** if $\operatorname{Lie}\Gamma = \mathfrak{g}$ (where $\operatorname{Lie}\Gamma$ is the smallest Lie subalgebra containing Γ). We say that a system Σ is **connected** if its state space is connected. A system Σ is said to be **proper** if it is both connected and of full rank. Note that a restriction to proper systems is a natural as both conditions are necessary for controllability.

Let $\Sigma = (\mathsf{G}, \Xi)$ and $\Sigma' = (\mathsf{G}', \Xi')$ be two systems. Then Σ and Σ' are called **(globally) detached feedback equivalent** (shortly DF-equivalent) if there exists a diffeomorphism $\Phi : \mathsf{G}\times\mathbb{R}^\ell \to \mathsf{G}'\times\mathbb{R}^{\ell'}$, $(g, u) \mapsto (\phi(g), \bar{\varphi}(u))$ such that $T_g\phi\cdot\Xi\,(g, u) = \Xi'\,(\phi(g), \bar{\varphi}(u))$ for $g \in \mathsf{G}$ and $u \in \mathbb{R}^\ell$. We use the following characterisation of DF-equivalence as a starting point for our work.

3.0.1 Theorem. ([7]) *Two proper systems Σ and Σ' are DF-equivalent if and only if there exists a Lie group isomorphism $\phi : \mathsf{G} \to \mathsf{G}'$ such that $T_{\mathbf{1}}\phi\cdot\Gamma = \Gamma'$.*

3.1 Preliminaries

3.1.1 Approach to classification

We follow very much the same approach as in chapter 2. The main difference is that for a global classification we fix a Lie group G (rather that a Lie algebra $\mathfrak{g}$) and classify affine subspaces related by elements of $d\mathrm{Aut}\,\mathsf{G}$ (rather that $\mathrm{Aut}\,\mathfrak{g}$). We then also investigate which systems are controllable, mainly using theorem A.2.12 (a result adapted from [20]).

We again use the following conventions. For a subset S of a linear space we will use the notation $\langle S\rangle$ to mean the linear span of S. Furthermore for the span of a finite set of elements $\{A_i\}_{i=\overline{1,n}}$ we will simply write $\langle A_1, A_2, \ldots, A_n\rangle$. We will call an affine subspace Γ **homogeneous** if $0 \in \Gamma$ (i.e. Γ is subspace of $\mathfrak{g}$), and **inhomogeneous** if not. Let S be any subset of a Lie algebra $\mathfrak{g}$. Then by $\mathrm{Lie}\,S$ we mean the smallest Lie sub algebra of $\mathfrak{g}$ containing S. An affine subspace Γ of $\mathfrak{g}$ is said to be of **full rank** if $\mathrm{Lie}\,\Gamma = \mathfrak{g}$ or equivalently $\dim \mathrm{Lie}\,\Gamma = \dim \mathfrak{g}$.

For the purposes of this chapter we specialise out definition of a $\mathfrak{L}$-relation. Let G_1 and G_2 be Lie groups. Then two affine subspaces Γ_1 and Γ_2 of their respective Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are **$\mathfrak{L}$-related** (and we write $\Gamma_1 \sim \Gamma_2$) if there exists a Lie group isomorphism $\phi : \mathsf{G}_1 \to \mathsf{G}_2$ such that $T_{\mathbf{1}}\phi \cdot \Gamma_1 = \Gamma_2$. (Hence systems Σ_1 and Σ_2 are DF-equivalent if and only if $\Gamma_1 \sim \Gamma_2$.) Note that $\mathfrak{L}$-relations now implicitly involve specific Lie groups with specified Lie algebras. By the same reasoning as used in chapter 2 we have that this relation is an equivalence relation preserving the property of full rank. We also have the following adaptation of lemma 2.1.3.

3.1.1 Lemma. *Two affine subspaces $\Gamma_1 = A_1 + \Gamma_1^0$ and $\Gamma_2 = A_2 + \Gamma_2^0$ are $\mathfrak{L}$-related if and only if there exists a Lie group isomorphism $\phi : \mathsf{G}_1 \to \mathsf{G}_2$ such that $T_{\mathbf{1}}\phi \cdot A_1 \in \Gamma_2$ and $T_{\mathbf{1}}\phi \cdot \Gamma_1^0 = \Gamma_2^0$.*

Now, if two proper systems Σ_1 and Σ_2 are DF-equivalent, they must have Lie group isomorphic state spaces. Hence we have the following necessary conditions (similar to lemma 2.1.4).

3.1.2 Lemma. *If two proper systems Σ_1 and Σ_2 are DF-equivalent, then: $\mathsf{G}_1 \cong \mathsf{G}_2$; $\dim \Gamma_1^0 = \dim \Gamma_2^0$; Γ_1 and Γ_2 are both homogeneous or both inhomogeneous.*

Thus we may immediately separate our systems by the associated Lie group (up to isomorphism), dimension of trace and homogeneity. This motivates us to define the following family of classes of full rank systems:

$$\mathscr{C}_{(\mathsf{H},\ell,\varepsilon)} = \{\Sigma = (\mathsf{G}, \Xi) \in \mathrm{Ob}\,\mathbf{LiCAS} \mid \Sigma \text{ is proper},\ \mathsf{G} \cong \mathsf{H},\ \dim\Gamma = \ell, \mathrm{Hom}\,\Gamma = \varepsilon\}$$

where H is a (finite-dimensional) Lie Group, $\ell \in \mathbb{N}, \ell \leq \dim \mathsf{H}$, $\varepsilon \in \{0, 1\}$ and

$$\mathrm{Hom}\,\Gamma = \begin{cases} 0 & \Gamma \text{ is homogeneous, i.e., } \Gamma = \Gamma^0 \\ 1 & \Gamma \text{ is inhomogeneous.} \end{cases}$$

Having fixed H we will again refer to an affine subspace of $\mathfrak{h}$ of dimension ℓ and homogeneity ε as a **(ℓ, ε)-affine subspace**. Note that if two systems are DF equivalent they necessarily belong to the same class. Furthermore note that if $\mathsf{H}_1 \cong \mathsf{H}_2$ then $\mathscr{C}_{(\mathsf{H}_1,\ell,\varepsilon)} = \mathscr{C}_{(\mathsf{H}_2,\ell,\varepsilon)}$.

Finding ourselves in one of these classes the question of classification then reduces to finding $\mathfrak{L}$-related affine subspaces. Now given two systems Σ_1 and Σ_2 in some class $\mathscr{C}_{(\mathsf{H},\ell,\varepsilon)}$ we have that both G_1 and G_2 are isomorphic to (the common representation) H. Then we have Lie group isomorphisms $\phi_1 : \mathsf{G}_1 \to \mathsf{H}$ and $\phi_2 : \mathsf{G}_2 \to \mathsf{H}$. Thus we may transfer the dynamics (and thus the traces) of Σ_1 and Σ_2 to H (and to $\mathfrak{h}$, respectively) by means of these isomorphisms. We then have the that our systems are DF-equivalent if and only if these transferred traces are $\mathfrak{L}$-related (by an element of $d\mathrm{Aut}\,\mathsf{H}$). We now make a concrete statement.

3.1.3 Proposition. *Let Σ_1 and Σ_2 be two elements of a class $\mathscr{C}_{(\mathsf{H},\ell,\varepsilon)}$. In particular we have Lie Group isomorphisms $\phi_1 : \mathsf{G}_1 \to \mathsf{H}$ and $\phi_2 : \mathsf{G}_2 \to \mathsf{H}$. Then Σ_1 and Σ_2 are DF-equivalent if and only if there exist an element of $\psi \in d\mathrm{Aut}\,\mathsf{H}$ such that $\psi \cdot (T_{\mathbf{1}}\phi_1 \cdot \Gamma_1) = (T_{\mathbf{1}}\phi_2 \cdot \Gamma_2)$, i.e., $(T_{\mathbf{1}}\phi_1 \cdot \Gamma_1) \sim (T_{\mathbf{1}}\phi_2 \cdot \Gamma_2)$.*

Proof. Suppose that Σ_1 and Σ_2 are DF-equivalent. Then there exists a Lie group isomorphism $\phi : \mathsf{G}_1 \to \mathsf{G}_2$ such that $T_{\mathbf{1}}\phi \cdot \Gamma_1 = \Gamma_2$. Now consider the mapping $\phi_{\mathsf{H}} = \phi_2 \circ \phi \circ \phi_1^{-1} : \mathsf{H} \to \mathsf{H}$. As a composition of Lie group isomorphisms it is a Lie group isomorphism and hence a Lie group automorphism. Thus $\psi = T_{\mathbf{1}}\phi_{\mathsf{H}} \in d\mathrm{Aut}\,\mathsf{H}$ and we have that

$$\psi \cdot (T_{\mathbf{1}}\phi_1 \cdot \Gamma_1) = T_{\mathbf{1}}\phi_2 \cdot T_{\mathbf{1}}\phi \cdot T_{\mathbf{1}}\phi_1^{-1} \cdot T_{\mathbf{1}}\phi \cdot \Gamma_1 = T_{\mathbf{1}}\phi_2 \cdot T_{\mathbf{1}}\phi \cdot \Gamma_1 = (T_{\mathbf{1}}\phi_2 \cdot \Gamma_2).$$

Conversely, suppose we have an element $\psi \in d\mathrm{Aut}\,\mathsf{H}$ such that $\psi \cdot (T_{\mathbf{1}}\phi_1 \cdot \Gamma_1) = (T_{\mathbf{1}}\phi_2 \cdot \Gamma_2)$. Then there exists a Lie group automorphism $\phi_{\mathsf{H}} : H \to H$ such that $\psi = T_{\mathbf{1}}\phi_{\mathsf{H}}$. Now define a mapping $\phi = \phi_2^{-1} \circ \phi_{\mathsf{H}} \circ \phi_1 : \mathsf{G}_1 \to \mathsf{G}_2$. Then as a composition of Lie group isomorphisms it is a Lie group isomorphism and we get that

$$T_{\mathbf{1}}\phi \cdot \Gamma_1 = T_{\mathbf{1}}\phi_2^{-1} \cdot T_{\mathbf{1}}\phi_{\mathsf{H}} \cdot T_{\mathbf{1}}\phi_1 \cdot \Gamma_1 = T_{\mathbf{1}}\phi_2^{-1} \cdot \psi \cdot (T_{\mathbf{1}}\phi_1 \cdot \Gamma_1) = T_{\mathbf{1}}\phi_2^{-1} \cdot (T_{\mathbf{1}}\phi_2 \cdot \Gamma_2) = \Gamma_2 \quad \square$$

Thus classification within a class $\mathscr{C}_{(\mathsf{H},\ell,\varepsilon)}$ may be accomplished by choosing a specific representation of the Lie group and then classifying $\mathfrak{L}$-related affine subspaces of $\mathfrak{h}$ (i.e., related by elements of $d\mathrm{Aut}\,\mathsf{H}$). Our problem is thus mainly one of classifying $\mathfrak{L}$-related affine subspaces (cf. [7]). We may then “decode” our classification in a similar fashion as described in chapter 2. We note that remark 2.1.6 again applies.

So in order to classify controllable DF-equivalent proper systems with state space Lie group isomorphic to $\widetilde{\mathsf{H}}_3^\circ$ or $\mathsf{H}_3^\circ(n)$, we need to classify $\mathfrak{L}$-related full rank affine subspaces of $\widetilde{\mathfrak{h}}_3^\circ$ or $\mathfrak{h}_3^\circ(n)$. We then need to determine which of these full rank affine subspaces correspond to controllable systems. *For the remainder of this chapter we assume that all systems mentioned are proper and all affine subspaces mentioned are of full rank.* Our approach to classifying $\mathfrak{L}$-related affine subspaces is exactly the same as in chapter 2 (replacing $\mathrm{Aut}\mathfrak{g}$ with $d\mathrm{Aut}\,\mathsf{G}$). We then (mainly) use theorem A.2.12 to determine which of these affine subspaces correspond to controllable systems. A tabulation of results is given in appendix B.

At this stage we recall that a Lie algebra automorphism ψ of $\widetilde{\mathfrak{h}}_3^\circ$ or $\mathfrak{h}_3^\circ(n)$, written w.r.t. the respective ordered bases (see section 1.3), is of the form (see proposition 1.2.27)

$$\psi = \begin{bmatrix} x & y & 0 & u \\ -ky & kx & 0 & v \\ kux - vy & kuy + xv & k(x^2+y^2) & w \\ 0 & 0 & 0 & k \end{bmatrix}$$

for some $u, v, w, x, y \in \mathbb{R}$ and $k \in \{-1, 1\}$ such that $x^2 + y^2 \neq 0$. *For the remainder of this chapter, when an arbitrary Lie algebra automorphism is called for, we assume it is of this form.*

In order to separate cases (and create a classifying table), we will make use of the dual bases $\{\widetilde{E}_i^*\}_{i=\overline{1,4}}$ and $\{E_i^*\}_{i=\overline{1,4}}$ of $\widetilde{\mathfrak{h}}_3^\diamond$ and $\mathfrak{h}_3^\diamond(n)$, respectively. To support our view of the dual basis as **linear projections** in this chapter, we will use the notation $\boldsymbol{\pi_i} = \widetilde{E}_i^*, E_i^* : \widetilde{\mathfrak{h}}_3^\diamond, \mathfrak{h}_3^\diamond(n) \to \mathbb{R}$ (with domain and basis being inferred from context). We note that the invariants $\mathfrak{P}$, $\mathfrak{T}$, $\mathfrak{S}$ and $\mathfrak{R}$ introduced in chapter 2 are all invariants (defined in the same manner w.r.t. the respective ordered bases) for $\widetilde{\mathfrak{h}}_3^\diamond$ and $\mathfrak{h}_3^\diamond(n)$.

Before continuing, we recall and adapt the general observations made at the end of section 2.1. A condition of the form $\pi_i(\Gamma^0) = \{0\}$, $\pi_i(A) = \alpha$ (used to separate cases) does not depend on the parametrisation chosen for Γ. A preclassification of $\mathfrak{L}$-related $(\ell + 1, 0)$-affine subspaces may be produced from a classification of $\mathfrak{L}$-related $(\ell, 1)$-affine subspaces. (This result is captured in the corollary to the next proposition. This corollary may be proven in exactly the same way as corollary 2.1.9.)

3.1.4 PROPOSITION. *Let* G *be a Lie group with Lie algebra* $\mathfrak{g}$. *If* $\Gamma_1 = \langle A_1, A_2, \ldots, A_\ell \rangle$ *is a* $(\ell, 0)$-*affine subspace (of full rank) of* $\mathfrak{g}$, *then* $\overline{\Gamma}_1 = A_1 + \langle A_2, \ldots, A_\ell \rangle$ *is a* $(\ell - 1, 1)$-*affine subspace (of full rank) of* $\mathfrak{g}$. *Moreover if* $\overline{\Gamma}_1$ *is* $\mathfrak{L}$-*related to another* $(\ell - 1, 1)$-*affine subspace* $\overline{\Gamma}_2$, *then* Γ_1 *is* $\mathfrak{L}$-*related to* $\langle \overline{\Gamma}_2 \rangle$.

PROOF. We showed $\overline{\Gamma}_1$ is of full rank in proposition 2.1.5. Now suppose $\overline{\Gamma}_1$ and $\overline{\Gamma}_2$ are $\mathfrak{L}$-related. Then there exists $\psi \in d\mathrm{Aut}\,\mathsf{G}$ such that $\psi \cdot \overline{\Gamma}_1 = \overline{\Gamma}_2$. Hence $\langle \psi \cdot \overline{\Gamma}_1 \rangle = \langle \overline{\Gamma}_2 \rangle$ and thus $\psi \cdot \Gamma_1 = \langle \overline{\Gamma}_2 \rangle$. That is, Γ_1 and $\langle \overline{\Gamma}_2 \rangle$ are $\mathfrak{L}$-related. □

3.1.5 COROLLARY. *Let* Γ *be a* $(\ell, 0)$-*affine subspace (of full rank) of a Lie algebra* $\mathfrak{g}$. *Suppose* $\{\overline{\Gamma}_i \mid i \in I\}$ *is a complete list of equivalence representatives of* $\mathfrak{L}$-*related* $(\ell - 1, 1)$-*affine subspaces (of full rank) of* $\mathfrak{g}$. *Then* Γ *is* $\mathfrak{L}$-*related to at least one of* $\{\langle \overline{\Gamma}_i \rangle \mid i \in I\}$.

3.1.2 Regarding the calculation of $d\mathrm{Aut}\,\mathsf{G}$

We produce some general results pertaining to (the calculation of) the subgroup $d\mathrm{Aut}\,\mathsf{G}$ of $\mathrm{Aut}\,\mathfrak{g}$ (for a Lie group G with Lie algebra $\mathfrak{g}$).

3.1.6 LEMMA. *If* G *is a connected Lie group and* $\phi \in \mathrm{Aut}\,\mathsf{G}$, *then the mapping* $\prod_{i=1}^n \exp(A_i) \mapsto \prod_{i=1}^n \exp(T_{\mathbf{1}}\phi \cdot A_i)$, $A_j \in \mathfrak{g}$, $j = \overline{1,n}$, $n \in \mathbb{N}$, *is identical to* ϕ.

PROOF. We have that $\phi(\exp(A)) = \exp(T_{\mathbf{1}}\phi \cdot A)$ in general (see section A.1.7). Then as ϕ is an automorphism we have that $\phi\left(\prod_{i=1}^n \exp(A_i)\right) = \prod_{i=1}^n \phi\left(\exp(A_i)\right) = \prod_{i=1}^n \exp(T_{\mathbf{1}}\phi \cdot A_i)$. As G is connected, any neighbourhood of identity generates G, which is to say that $\mathsf{G} = \{\prod_{i=1}^n \exp(A_i) \mid A_j \in \mathfrak{g}, j = \overline{1,n}, n \in \mathbb{N}\}$. Thus the mapping $\prod_{i=1}^n \exp(A_i) \mapsto \prod_{i=1}^n \exp(T_{\mathbf{1}}\phi \cdot A_i)$, $A_j \in \mathfrak{g}$, $j = \overline{1,n}$, $n \in \mathbb{N}$, is specified for every element of G, and maps any such g to $\phi(g)$. Thus the specified mapping is equal to ϕ. □

3.1.7 PROPOSITION. *If* $\phi \in \mathrm{Aut}\,\mathsf{G}$, *then* $T_{\mathbf{1}}\phi \cdot \exp^{-1}(\mathbf{1}) = \exp^{-1}(\mathbf{1})$. *Here* $\exp^{-1}(\mathbf{1})$ *is the preimage of identity, i.e.,* $\exp^{-1}(\mathbf{1}) = \{A \in \mathfrak{g} \mid \exp(A) = \mathbf{1}\}$.

Proof. Let $A \in \exp^{-1}(\mathbf{1})$. Then $\phi(\exp(A)) = \mathbf{1} = \phi^{-1}(\exp(A))$ and so $\exp(T_{\mathbf{1}}\phi \cdot A) = \mathbf{1} = \exp(T_{\mathbf{1}}\phi^{-1} \cdot A)$. Hence $T_{\mathbf{1}}\phi \cdot A$ and $T_{\mathbf{1}}\phi^{-1} \cdot A$ are in $\exp^{-1}(\mathbf{1})$. Thus we have that $T_{\mathbf{1}}\phi \cdot \exp^{-1}(\mathbf{1}) \subseteq \exp^{-1}(\mathbf{1})$ and $T_{\mathbf{1}}\phi^{-1} \cdot \exp^{-1}(\mathbf{1}) \subseteq \exp^{-1}(\mathbf{1})$. Hence we conclude $T_{\mathbf{1}}\phi \cdot \exp^{-1}(\mathbf{1}) = \exp^{-1}(\mathbf{1})$. □

3.1.8 Theorem. *Let G be a connected Lie group with universal covering $q : \widetilde{\mathsf{G}} \to \mathsf{G}$. Further let $\psi \in \operatorname{Aut} \mathfrak{g}$ and $\widetilde{\phi} : \widetilde{\mathsf{G}} \to \widetilde{\mathsf{G}}$ be the unique automorphism such that $T_{\mathbf{1}}\widetilde{\phi} = (T_{\mathbf{1}}q)^{-1} \circ \psi \circ T_{\mathbf{1}}q$. Then the mapping $\phi : \mathsf{G} \to \mathsf{G}$, $q(g) \mapsto (q \circ \widetilde{\phi})(g)$ is a Lie group automorphism of G such that $T_{\mathbf{1}}\phi = \psi$ if and only if it is well defined and injective.*

For the sake of clarity, if such a $\phi \in \operatorname{Aut} \mathsf{G}$ exists, the following diagrams commute

$$\begin{array}{ccc} \widetilde{\mathsf{G}} & \xrightarrow{\;\widetilde{\phi}\;} & \widetilde{\mathsf{G}} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q} \\ \mathsf{G} & \xrightarrow[\;\phi\;]{} & \mathsf{G} \end{array} \qquad\qquad \begin{array}{ccc} \widetilde{\mathfrak{g}} & \xrightarrow{\;T_{\mathbf{1}}\widetilde{\phi}\;} & \widetilde{\mathfrak{g}} \\ {\scriptstyle T_{\mathbf{1}}q}\downarrow & & \downarrow{\scriptstyle T_{\mathbf{1}}q} \\ \mathfrak{g} & \xrightarrow[\;T_{\mathbf{1}}\phi=\psi\;]{} & \mathfrak{g} \end{array}$$

See section A.1.3 for universal coverings and section A.1.7 for existence of a unique Lie group automorphism (given a Lie algebra automorphism).

Proof. First note that, as q is surjective, every element $g \in \mathsf{G}$ may be written as $g = q(\widetilde{g})$ for some $\widetilde{g} \in \widetilde{\mathsf{G}}$ (i.e., ϕ is defined for all $g \in \mathsf{G}$). If ϕ is a Lie group automorphism it is necessarily well defined and injective (and takes the given form by construction). Conversely, suppose that ϕ is well defined and injective. Let $\widetilde{g}_1, \widetilde{g}_2 \in \widetilde{\mathsf{G}}$, then we get that

$$\phi(q(\widetilde{g}_1)\, q(\widetilde{g}_2)) = \phi(q(\widetilde{g}_1\, \widetilde{g}_2)) = (q \circ \widetilde{\phi})(\widetilde{g}_1\, \widetilde{g}_2) = (q \circ \widetilde{\phi})(\widetilde{g}_1)\, (q \circ \widetilde{\phi})(\widetilde{g}_2) = \phi(q(\widetilde{g}_1))\, \phi(q(\widetilde{g}_2)).$$

Thus ϕ is an abstract group homomorphism. Next we show that ϕ is surjective. Let $h \in \mathsf{G}$, then there exists a $g \in \widetilde{\mathsf{G}}$ such that $q(g) = h$. As $\widetilde{\phi}$ is an automorphism we have that $\widetilde{\phi}^{-1}(g) \in \widetilde{\mathsf{G}}$ and hence that $\phi(q(\widetilde{\phi}^{-1}(g))) = (q \circ \widetilde{\phi})(\widetilde{\phi}^{-1}(g)) = q(g) = h$. So at this stage we have that ϕ is an abstract group automorphism. We are left to prove that ϕ is a diffeomorphism. As q is a Lie group covering homomorphism, there exist neighbourhoods V_1 and W of $\mathbf{1}_{\widetilde{\mathsf{G}}}$ and $\mathbf{1}_{\mathsf{G}}$, respectively, such that $q|_{V_1} : V_1 \to W$ is a diffeomorphism. Then, as $\widetilde{\phi}$ is an automorphism, we have a neighbourhood V_2 of $\mathbf{1}_{\widetilde{\mathsf{G}}}$ such that $\widetilde{\phi}(V_2) \subseteq V_1$ and $\widetilde{\phi}\big|_{V_2} : V_2 \to \widetilde{\phi}(V_2)$ is a diffeomorphism. Now we show that $\phi\big|_{q(V_2)} : q(V_2) \to q(\widetilde{\phi}(V_2))$ is a diffeomorphism. We have that $\phi\big|_{q(V_2)} = q\big|_{\widetilde{\phi}(V_2)} \circ \widetilde{\phi}\big|_{V_2} \circ q\big|_{V_2}^{-1}$. Thus as a compositions of diffeomorphisms, $\phi\big|_{q(V_2)}$ is a diffeomorphism. That is to say, ϕ is a local diffeomorphism at identity and hence, by theorem A.1.1 we get that ϕ is a Lie group automorphism. Finally, for $A \in \mathfrak{g}$, we have that

$$\begin{aligned} \phi(\exp(tA)) &= \phi\Big(q\big(\exp((T_{\mathbf{1}}q)^{-1} \cdot tA)\big)\Big) = (q \circ \widetilde{\phi})\Big(\exp\big((T_{\mathbf{1}}q)^{-1} \cdot tA\big)\Big) \\ &= \exp\Big(T_{\mathbf{1}}q \cdot \big((T_{\mathbf{1}}q)^{-1} \cdot \psi \cdot T_{\mathbf{1}}q\big) \cdot (T_{\mathbf{1}}q)^{-1} \cdot tA\Big) = \exp(\psi \cdot tA) \end{aligned}$$

implying that $T_{\mathbf{1}}\phi = \psi$. □

3.1.9 Corollary. *Let* G *be a connected Lie group,* $\psi \in \mathrm{Aut}(\mathfrak{g})$*,* $r \in \mathbb{N}$ *and* $\{V_i \subseteq \mathfrak{g} \mid i = \overline{1,r}\}$ *be a collection of subsets such that* $\{\prod_{i=1}^r \exp(A_i) \mid A_i \in V_i \text{ for } i = \overline{1,r}\} = \mathsf{G}$*. Then the mapping*

$$\phi : \mathsf{G} \to \mathsf{G}, \qquad \textstyle\prod_{i=1}^r \exp(A_i) \mapsto \prod_{i=1}^r \exp(\psi \cdot A_i), \qquad A_i \in V_i \subseteq \mathfrak{g},\ i = \overline{1,r}$$

is a Lie group automorphism of G *such that* $T_{\mathbf{1}}\phi = \psi$ *if and only if it is well defined and injective.*

Proof. Necessity is trivial. Suppose ϕ is well defined and injective. Then, as G is connected, we have a universal covering $q : \widetilde{\mathsf{G}} \to \mathsf{G}$. Consequently, as $\psi \in \mathrm{Aut}\,\mathfrak{g}$, we have that $(T_{\mathbf{1}}q)^{-1} \circ \psi \circ T_{\mathbf{1}}q \in \mathrm{Aut}\,\widetilde{\mathfrak{g}}$. Hence there exists a unique $\widetilde{\phi} \in \mathrm{Aut}\,\widetilde{\mathsf{G}}$ such that $T_{\mathbf{1}}\widetilde{\phi} = (T_{\mathbf{1}}q)^{-1} \circ \psi \circ T_{\mathbf{1}}q$. Now define $\widetilde{A}_i = (T_{\mathbf{1}}q)^{-1} \cdot A_i$, $i = \overline{1,r}$. Then we have that $\prod_{i=1}^r \exp(\widetilde{A}_i) \in \widetilde{\mathsf{G}}$ and $q\left(\prod_{i=1}^r \exp(\widetilde{A}_i)\right) = \prod_{i=1}^r \exp(A_i)$. Furthermore we have that

$$\textstyle\prod_{i=1}^r \exp(\psi \cdot A_i) = \prod_{i=1}^r \exp(T_{\mathbf{1}}q \cdot (T_{\mathbf{1}}q)^{-1} \cdot \psi \cdot T_{\mathbf{1}}q \cdot \widetilde{A}_i) = (q \circ \widetilde{\phi})\left(\prod_{i=1}^r \exp(\widetilde{A}_i)\right).$$

That is to say we have that $\phi : q\left(\prod_{i=1}^r \exp(\widetilde{A}_i)\right) \mapsto (q \circ \widetilde{\phi})\left(\prod_{i=1}^r \exp(\widetilde{A}_i)\right)$ is a well defined and injective mapping. Thus, as ϕ satisfies the conditions of the foregoing theorem, we have that it is a Lie group automorphism such that $T_{\mathbf{1}}\phi = \psi$. □

3.2 Systems on the Universal Covering Lie Group $\widetilde{\mathsf{H}}_3^\diamond$

Our aim in this section is to classify all controllable DF-equivalent systems with state space isomorphic to $\widetilde{\mathsf{H}}_3^\diamond$. (Refer to 1.3.1 for the definition of $\widetilde{\mathsf{H}}_3^\diamond$.) Now, as $\widetilde{H}_3^\diamond$ is simply connected, we have that two systems Σ and Σ' (with state space isomorphic to $\widetilde{\mathsf{H}}_3^\diamond$) are DF-equivalent if and only if they are DF_{loc}-equivalent (by corollary A.2.16). As such, we may get as a corollary to theorem 2.7.1, a classification of DF-equivalent systems with state space isomorphic to $\widetilde{\mathsf{H}}_3^\diamond$. (With regard to this also see corollary 2.7.2 and remark 2.7.3.)

3.2.1 Corollary. *Any proper system* Σ *with state space isomorphic to* $\widetilde{\mathsf{H}}_3^\diamond$ *is* DF*-equivalent to exactly one of the systems* $\widetilde{\Sigma}_{i,\alpha}^{(\ell,\varepsilon)} = (\widetilde{\mathsf{H}}_3^\diamond, \widetilde{\Xi}_{i,\alpha}^{(\ell,\varepsilon)})$*,* $\alpha \in \mathbb{R}$*, where*

$\widetilde{\Xi}_{1,\alpha}^{(1,1)}(\mathbf{1},u) = \widetilde{E}_1 + \alpha\widetilde{E}_3 + u\widetilde{E}_4,\ \alpha \geq 0$

$\widetilde{\Xi}_{2,\alpha}^{(1,1)}(\mathbf{1},u) = \alpha\widetilde{E}_4 + u\widetilde{E}_1,\ \alpha > 0$

$\widetilde{\Xi}_{1}^{(2,0)}(\mathbf{1},u) = u_1\widetilde{E}_1 + u_2\widetilde{E}_4$

$\widetilde{\Xi}_{1,\alpha}^{(2,1)}(\mathbf{1},u) = \widetilde{E}_2 + \alpha\widetilde{E}_3 + u_1\widetilde{E}_1 + u_2\widetilde{E}_4,\ \alpha \geq 0$

$\widetilde{\Xi}_{2}^{(2,1)}(\mathbf{1},u) = \widetilde{E}_3 + u_1\widetilde{E}_1 + u_2\widetilde{E}_4$

$\widetilde{\Xi}_{3}^{(2,1)}(\mathbf{1},u) = \widetilde{E}_1 + u_1\widetilde{E}_3 + u_2\widetilde{E}_4$

$\widetilde{\Xi}_{4,\alpha}^{(2,1)}(\mathbf{1},u) = \alpha\widetilde{E}_4 + u_1\widetilde{E}_1 + u_2\widetilde{E}_2,\ \alpha > 0$

$\widetilde{\Xi}_{5,\alpha}^{(2,1)}(\mathbf{1},u) = \alpha\widetilde{E}_4 + u_1\widetilde{E}_1 + u_2\widetilde{E}_3,\ \alpha > 0$

$\widetilde{\Xi}_{1}^{(3,0)}(\mathbf{1},u) = u_1\widetilde{E}_1 + u_2\widetilde{E}_2 + u_3\widetilde{E}_4$

$\widetilde{\Xi}_{2}^{(3,0)}(\mathbf{1},u) = u_1\widetilde{E}_1 + u_2\widetilde{E}_3 + u_3\widetilde{E}_4$

$\widetilde{\Xi}_{1}^{(3,1)}(\mathbf{1},u) = \widetilde{E}_2 + u_1\widetilde{E}_1 + u_2\widetilde{E}_3 + u_3\widetilde{E}_4$

$\widetilde{\Xi}_{2}^{(3,1)}(\mathbf{1},u) = \widetilde{E}_3 + u_1\widetilde{E}_1 + u_2\widetilde{E}_2 + u_3\widetilde{E}_4$

$\widetilde{\Xi}_{3,\alpha}^{(3,1)}(\mathbf{1},u) = \alpha\widetilde{E}_4 + u_1\widetilde{E}_1 + u_2\widetilde{E}_2 + u_3\widetilde{E}_3,\ \alpha > 0$

$\widetilde{\Xi}_{1}^{(4,0)}(\mathbf{1},u) = u_1\widetilde{E}_1 + u_2\widetilde{E}_2 + u_3\widetilde{E}_3 + u_4\widetilde{E}_4.$

We now investigate which of these systems are controllable. We apply theorem A.2.12, item 3a in the following cases. For $\widetilde{\Sigma}_1^{(2,0)}$, $\widetilde{\Sigma}_{1,\alpha}^{(2,1)}$, $\widetilde{\Sigma}_2^{(2,1)}$, $\widetilde{\Sigma}_1^{(3,0)}$, $\widetilde{\Sigma}_2^{(3,0)}$, $\widetilde{\Sigma}_1^{(3,1)}$, $\widetilde{\Sigma}_2^{(3,1)}$ and $\widetilde{\Sigma}_1^{(4,0)}$ we have that $\operatorname{Lie}\Gamma^0 = \widetilde{\mathfrak{h}}_3^\diamond$. Hence they are controllable.

We apply theorem A.2.12, item 3c in the following cases. Calculations were made in Mathematica (see section C.6). For $\widetilde{\Sigma}_{1,\alpha}^{(1,1)}$ we have (assuming $\alpha > 0$) that $\exp(4\alpha\pi\Xi_{1,\alpha}^{(1,1)}(\mathbf{1}, \frac{1}{2\alpha})) = \exp(2\pi E_4) \in \mathcal{O}^0$ and hence get that $\widetilde{\Sigma}_{1,\alpha}^{(1,1)}$ is controllable for $\alpha > 0$. We will show that it is controllable for $\alpha = 0$ later on. For $\widetilde{\Sigma}_3^{(2,1)}$ we have that $\exp\big(\Xi_3^{(2,1)}(\mathbf{1}, (\frac{1}{4\pi}, 2\pi)\big) = \exp(2\pi E_4) \in \mathcal{O}^0$ and hence $\widetilde{\Sigma}_3^{(2,1)}$ is controllable.

Using the diffeomorphism $\widetilde{m} : \mathbb{R}^4 \to \widetilde{\mathsf{H}}_3^\diamond$ we may take the pull-back of a left-invariant vector field Ξ_v to get a system of "parametric" differential equations in $\mathbb{R}^4$; we will do so now. For a control system $\widetilde{\Sigma} = (\widetilde{\mathsf{H}}_3^\diamond, \Xi)$, $\Xi(\mathbf{1}, v(t)) = \sum_{i=1}^4 v_i(t)E_i$, we may parametrise a trajectory $g(\cdot)$, as $g(t) = \widetilde{m}(x(t), y(t), z(t), \theta(t))$. Then we get that $\frac{d}{dt}\widetilde{m}(x(t), y(t), z(t), \theta(t)) = \widetilde{m}(x(t), y(t), z(t), \theta(t))\,\Xi(\mathbf{1}, v(t))$. Specifically (using Mathematica, see section C.6) we get that

$$\begin{bmatrix} \cdot & \cdot & \cdot & -2\dot{z}(t) & 0 \\ \cdot & \cdot & \cdot & \dot{y}(t) & 0 \\ \cdot & \cdot & \cdot & \dot{x}(t) & 0 \\ \cdot & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & 0 & \dot{\theta}(t)\mathrm{e}^{\theta(t)} \end{bmatrix}$$

$$= \widetilde{m}(x(t), y(t), z(t), \theta(t))\, \widetilde{M}(v_1(t), v_2(t), v_3(t), v_4(t))$$

$$= \begin{bmatrix} \cdot & \cdot & \cdot & v_1(t)(x(t)\sin\theta(t) + y(t)\cos\theta(t)) + v_2(t)(-x(t)\cos\theta(t) + y(t)\sin\theta(t)) - 2v_3(t) & 0 \\ \cdot & \cdot & \cdot & -v_1(t)\sin\theta(t) + v_2(t)\cos\theta(t) & 0 \\ \cdot & \cdot & \cdot & v_1(t)\cos\theta(t) + v_2(t)\sin\theta(t) & 0 \\ \cdot & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & 0 & v_4(t)\mathrm{e}^{\theta(t)} \end{bmatrix}$$

yielding parametric equations (suppressing evaluation at t)

$$\begin{aligned} \dot{x} &= v_1\cos\theta + v_2\sin\theta \\ \dot{y} &= -v_1\sin\theta + v_2\cos\theta \\ \dot{z} &= -\tfrac{1}{2}v_1\big(x\sin\theta + y\cos\theta\big) + \frac{1}{2}v_2\big(x\cos\theta - y\sin\theta\big) + v_3 \\ \dot{\theta} &= v_4. \end{aligned}$$

These equations may be succinctly written as

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \qquad \dot{z} = \tfrac{1}{2}(x\dot{y} - y\dot{x}) + v_3 \qquad \dot{\theta} = v_4. \tag{3.2.1}$$

Requiring $g(0) = \mathbf{1}$, implies that $x(0) = 0$, $y(0) = 0$, $z(0) = 0$ and $\theta(0) = 0$. Now for each of $\widetilde{\Sigma}_{2,\alpha}^{(1,1)}$, $\widetilde{\Sigma}_{4,\alpha}^{(2,1)}$, $\widetilde{\Sigma}_{5,\alpha}^{(2,1)}$ and $\widetilde{\Sigma}_{3,\alpha}^{(3,1)}$ we have that the parametric equation of the θ-coordinate is given by $\dot{\theta}(\cdot) = \alpha$. Thus, as $\theta(0) = 0$, we have that $\operatorname{sgn}(\theta(t)) = \operatorname{sgn}(\alpha) = 1$ for all $t > 0$, showing that none of these systems are controllable.

We finally show that $\widetilde{\Sigma}_{1,0}^{(1,1)}$ is controllable. (Theorem A.2.12 was insufficient to prove controllability in this case.) We will show that $\widetilde{\Sigma}_{1,0}^{(1,1)}$ is locally controllable at the identity, i.e.,

$\mathbf{1} \in \operatorname{int} \mathcal{A}$. Then as $\mathcal{A}$ is a semi-group (proposition A.2.4) and any neighbourhood of identity generates a connected Lie group, it follows that $\mathcal{A} = \widetilde{\mathsf{H}}_3^\diamond$ and hence that $\widetilde{\Sigma}_{1,0}^{(1,1)}$ is controllable. (This is also known as the local controllability test.) We use Mathematica for the following calculations, see section C.6.

Consider the mapping $F : \mathbb{R}^4 \to \widetilde{\mathsf{H}}_3^\diamond$ given by

$$\begin{aligned} F(s_1, s_2, s_3, s_4) &= \mathrm{e}^{\pi \Xi_{1,0}^{(1,1)}(\mathbf{1}, s_1)}\, \mathrm{e}^{\pi \Xi_{1,0}^{(1,1)}(\mathbf{1}, s_2)}\, \mathrm{e}^{\frac{3\pi}{2} \Xi_{1,0}^{(1,1)}(\mathbf{1}, -s_3)}\, \mathrm{e}^{\frac{\pi}{2} \Xi_{1,0}^{(1,1)}(\mathbf{1}, -s_4)} \\ &= \mathrm{e}^{\pi(E_1 + s_1 E_4)}\, \mathrm{e}^{\pi(E_1 + s_2 E_4)}\, \mathrm{e}^{\frac{3\pi}{2}(E_1 - s_3 E_4)} \mathrm{e}^{\frac{\pi}{2}(E_1 - s_4 E_4)}. \end{aligned}$$

Notice that the image of F is contained in $\mathcal{A}$ (as $\mathcal{A}$ is a semi-group, proposition A.2.4, and $\exp(t\,\Xi(\mathbf{1}, u)) \in \mathcal{A}$ for $t > 0$). Let $p = (1, 1, 1, 1)$. Then calculation in Mathematica yields $F(p) = \mathbf{1}$. As the exponentiation map is smooth and products are smooth in a Lie group, we have that F is smooth. We claim that there exists some neighbourhood V of p such that $F\big|_V : V \to F(V)$ is a diffeomorphism. By use of the inverse function theorem (cf. [22]) we need only show that the set $\left\{ \frac{\partial F}{\partial s_1}\Big|_p, \frac{\partial F}{\partial s_2}\Big|_p, \frac{\partial F}{\partial s_3}\Big|_p, \frac{\partial F}{\partial s_4}\Big|_p \right\}$ of partial derivatives at p is linearly independent. But as $F(p) = \mathbf{1}$, we have that each of these partial derivatives are in $T_{\mathbf{1}}\widetilde{\mathsf{H}}_3^\diamond$ which we identify with $\widetilde{\mathfrak{h}}_3^\diamond$. Thus we can represent them as vectors (of length 4, with respect to the basis $\{\widetilde{E}_i\}_{i=\overline{1,4}}$). Let R be the (4×4) matrix with these vectors as its rows. Then we get (using Mathematica) that $\det R = -16\pi^2$. That is to say, the set of partial derivatives is independent at p. Thus we get that $F(V) \subseteq \mathcal{A}$ is an (open) neighbourhood of $F(p) = \mathbf{1}$. Thus $\mathbf{1} \in \operatorname{int} \mathcal{A}$ and so $\widetilde{\Sigma}_{1,0}^{(1,1)}$ is controllable as explained above.

In summary (noting that controllability is invariant under DF-equivalence, see [7]) we have the following.

3.2.2 Theorem. *Any system Σ with state space Lie group isomorphic to $\widetilde{\mathsf{H}}_3^\diamond$ is controllable if and only if it is DF-equivalent to (exactly) one of the systems $\widetilde{\Sigma}_{i,\alpha}^{(\ell,\varepsilon)} = (\widetilde{\mathsf{H}}_3^\diamond, \widetilde{\Xi}_{i,\alpha}^{(\ell,\varepsilon)})$, where*

$$\begin{aligned} &\widetilde{\Xi}_{1,\alpha}^{(1,1)}(\mathbf{1}, u) = E_1 + \alpha E_3 + u E_4,\ \alpha \geq 0 && \widetilde{\Xi}_{1}^{(2,0)}(\mathbf{1}, u) = u_1 E_1 + u_2 E_4 \\ &\widetilde{\Xi}_{1,\alpha}^{(2,1)}(\mathbf{1}, u) = E_2 + \alpha E_3 + u_1 E_1 + u_2 E_4,\ \alpha \geq 0 && \widetilde{\Xi}_{2}^{(2,1)}(\mathbf{1}, u) = E_3 + u_1 E_1 + u_2 E_4 \\ &\widetilde{\Xi}_{3}^{(2,1)}(\mathbf{1}, u) = E_1 + u_1 E_3 + u_2 E_4 && \widetilde{\Xi}_{1}^{(3,0)}(\mathbf{1}, u) = u_1 E_1 + u_2 E_2 + u_3 E_4 \\ &\widetilde{\Xi}_{2}^{(3,0)}(\mathbf{1}, u) = u_1 E_1 + u_2 E_3 + u_3 E_4 && \widetilde{\Xi}_{1}^{(3,1)}(\mathbf{1}, u) = E_2 + u_1 E_1 + u_2 E_3 + u_3 E_4 \\ &\widetilde{\Xi}_{2}^{(3,1)}(\mathbf{1}, u) = E_3 + u_1 E_1 + u_2 E_2 + u_3 E_4 && \widetilde{\Xi}_{1}^{(4,0)}(\mathbf{1}, u) = u_1 E_1 + u_2 E_2 + u_3 E_3 + u_4 E_4. \end{aligned}$$

3.2.3 Corollary. *A proper system $\Sigma = (\widetilde{\mathsf{H}}_3^\diamond, \Xi)$ is controllable if and only if $\pi_4(\Gamma^0) \neq \{0\}$.*

Proof. For any automorphism $\psi \in \operatorname{Aut} \widetilde{\mathfrak{h}}_3^\diamond$ we have that $\pi_4(\psi \cdot A) = k\pi_4(A) = \pm\pi_4(A)$ for $A \in \widetilde{\mathfrak{h}}_3^\diamond$. Thus we have that, $\pi_4(\Gamma^0) \neq 0 \Leftrightarrow \pi_4(\psi \cdot \Gamma^0) \neq 0$, for any $\psi \in \operatorname{Aut} \widetilde{\mathfrak{h}}_3^\diamond$. Now by corollary 3.2.1, there exists a $\psi \in \operatorname{Aut} \widetilde{\mathfrak{h}}_3^\diamond$ taking Γ^0 to ${}_{(\ell,\varepsilon)}\Gamma_{i,\alpha}^0 = \{A - B \mid A, B \in \Gamma_{i,\alpha}^{(\ell,\varepsilon)}\}$ for some ℓ, ε, i and α. Finally notice that the systems listed in corollary 3.2.1 that are controllable are exactly those for which $\pi_4({}_{(\ell,\varepsilon)}\Gamma_{i,\alpha}^0) \neq \{0\}$ (and those that are not controllable are exactly those for which $\pi_4({}_{(\ell,\varepsilon)}\Gamma_{i,\alpha}^0) = \{0\}$). ∎

3.2.4 REMARK. The necessity of the condition $\pi_4(\Gamma^0) \neq \{0\}$ is most apparent when looking at (the parametric) equation (3.2.1). We have that $\dot{\theta}(t) = v_4(t)$ for dynamics $\Xi(\mathbf{1}, v) = v_1 E_1 + v_2 E_2 + v_3 E_3 + v_4 E_4$. Thus if $\pi_4(\Gamma^0) = \{0\}$ then $v_4(\cdot)$ is constant and so $\theta(\cdot)$ is either strictly increasing or strictly decreasing (if $v_4(\cdot) = 0$, then the system is not of full rank). We can interpret this, using the decomposition $\widetilde{\mathsf{H}}_3^\diamond = \mathsf{H}_3 \rtimes \widetilde{\mathsf{SO}}(2)$, $\widetilde{m}(x, y, z, \theta) = \widetilde{m}(x, y, z, 0)\, \widetilde{m}(0, 0, 0, \theta)$, as either moving up or down the spiral $\{\widetilde{m}(0, 0, 0, \theta) \mid \theta \in \mathbb{R}\}$. We will see that for $\mathsf{H}_3^\diamond(n)$ this condition falls away, which corresponds to projecting the spiral $\mathbb{R} \cong \{\widetilde{m}(0, 0, 0, \theta) \mid \theta \in \mathbb{R}\}$ onto the circle $\mathbb{T} \cong \{m_n(0, 0, 0, \theta) \mid \theta \in \mathbb{R}\}$, making $m_n(0, 0, 0, \theta(\cdot))$ periodic.

3.3 Systems on $\mathsf{H}_3^\diamond$ and its n-Fold Covers

Our aim in this section is to classify all controllable DF-equivalent systems with state space isomorphic to a n-fold cover $\mathsf{H}_3^\diamond(n)$ of $\mathsf{H}_3^\diamond$, including $\mathsf{H}_3^\diamond(1) \cong \mathsf{H}_3^\diamond$. (Refer to section 1.3.2 for the definition of $\mathsf{H}_3^\diamond(n)$.) The map $d : \mathrm{Aut}\, \mathsf{H}_3^\diamond(n) \to \mathrm{Aut}\, \mathfrak{h}_3^\diamond(n)$ is not surjective and hence our local classification, as given in corollary 2.7.2, doesn't "lift" to a global classification (as was the case for the universal covering). So, in order to get the classification, we need to first calculate the subgroup $d\mathrm{Aut}\, \mathsf{H}_3^\diamond(n)$ of $\mathrm{Aut}\, \mathfrak{h}_3^\diamond(n)$, then classify $\mathfrak{L}$-related (full rank) affine subspaces of $\mathfrak{h}_3^\diamond(n)$, and finally investigate which them are controllable (it turns out all of them are). *Throughout we assume that all statements made are for any fixed positive integer n.* We use Mathematica to support many of the calculations made in this section; see section C.7.

3.3.1 Automorphisms of $\mathsf{H}_3^\diamond(n)$

Before we make a claim regarding $d\mathrm{Aut}\, \mathsf{H}_3^\diamond(n)$, we develop some background. We recall that $\mathfrak{h}_3^\diamond$ admits a invariant scalar product, which we introduced in proposition 1.4.7. Thus, as $\mathfrak{h}_3^\diamond(n)$ is isomorphic to $\mathfrak{h}_3^\diamond$, we have a symmetric invariant scalar product on $\mathfrak{h}_3^\diamond(n)$. Specifically, if $\psi : \mathfrak{h}_3^\diamond(n) \to \mathfrak{h}_3^\diamond$ is a Lie algebra isomorphism, such an invariant scalar product may be defined as $\mathfrak{h}_3^\diamond(n) \times \mathfrak{h}_3^\diamond(n) \to \mathbb{R}$, $(A, B) \mapsto \varphi(\psi \cdot A, \psi \cdot B)$, where φ is the invariant scalar product on $\mathfrak{h}_3^\diamond$. Hence (using the tangent map, at identity, of the covering homomorphism in proposition 1.3.12), we have an invariant scalar product $\omega : \mathfrak{h}_3^\diamond(n) \times \mathfrak{h}_3^\diamond(n) \to \mathbb{R}$ on $\mathfrak{h}_3^\diamond(n)$ given by

$$\begin{aligned}\omega : (M_n(x, y, z, \theta), M_n(x', y', z', \theta')) \mapsto & \begin{bmatrix} x \\ y \\ z \\ \theta \end{bmatrix}^\top \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \\ \theta' \end{bmatrix} \\ &= xx' + yy - z\theta' - \theta z'.\end{aligned}$$

(We use the notation ω here rather that $\langle \cdot, \cdot \rangle$ to avoid confusion with the linear span later on.)

3.3.1 LEMMA. *The preimage of identity under the exponential map is given by*

$$\begin{aligned}\exp^{-1}(\mathbf{1}) &= \{A \in \mathfrak{h}_3^\diamond(n) \mid \omega(A, A) = 0, \pi_4(A) \in 2n\pi\mathbb{Z}, \pi_4(A) = 0 \Rightarrow \pi_3(A) = 0\} \\ &= \{0\} \cup \{xE_1 + yE_2 + \tfrac{x^2+y^2}{2\theta} E_3 + \theta E_4 \mid x, y \in \mathbb{R}, \theta \in 2n\pi\mathbb{Z} \setminus \{0\}\}\end{aligned}$$

Proof. Let $A \in \exp^{-1}(\mathbf{1})$, $A = M_n(x,y,z,\theta)$ for some $x,y,z,\theta \in \mathbb{R}$. Now if $\theta = 0$, then $\exp(A) = M_n(x,y,z,0)$ and thus $x = y = z = 0$. Note that this case satisfies the conditions $\pi_4(A) \in 2n\pi\mathbb{Z}$, $\omega(A,A) = 0$ and $\pi_4(A) = 0 \Rightarrow \pi_3(A) = 0$. If on the other hand $\theta \neq 0$, we get (see proposition 1.3.5 and section 1.3.2) that

$$\exp(A) = \begin{bmatrix} 1 & \frac{y - y\cos\theta - x\sin\theta}{\theta} & \frac{x - x\cos\theta + y\sin\theta}{\theta} & \frac{\theta\left(x^2+y^2-2z\theta\right) - \left(x^2+y^2\right)\sin\theta}{\theta^2} & 0 \\ 0 & \cos\theta & -\sin\theta & \frac{x(-1+\cos\theta)+y\sin\theta}{\theta} & 0 \\ 0 & \sin\theta & \cos\theta & \frac{y - y\cos\theta + x\sin\theta}{\theta} & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{e}^{\frac{i\theta}{n}} \end{bmatrix}.$$

Consequently we have the following sequence of implications:

$$A \in \exp^{-1}(\mathbf{1}) \;\Rightarrow\; \begin{cases} \mathrm{e}^{\frac{i\theta}{n}} = 0 & (\text{hence } \cos\theta = 1 \text{ and } \sin\theta = 0) \\ \frac{\theta\left(x^2+y^2-2z\theta\right) - \left(x^2+y^2\right)\sin\theta}{\theta^2} = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \theta \in 2n\pi\mathbb{Z} \\ \frac{\theta\left(x^2+y^2-2z\theta\right)}{\theta^2} = 0 \end{cases} \;\Rightarrow\; \begin{cases} \theta \in 2n\pi\mathbb{Z} \\ x^2+y^2-2z\theta = 0 \end{cases} \;\Rightarrow\; \begin{cases} \theta \in 2n\pi\mathbb{Z} \\ \omega(A,A) = 0. \end{cases}$$

Thus we have shown that, if $A \in \exp^{-1}(\mathbf{1})$, then $\pi_4(A) \in 2n\pi\mathbb{Z}$, $\omega(A,A) = 0$ and $\pi_4(A) = 0 \Rightarrow \pi_3(A) = 0$. We now show that these conditions are also sufficient. Again let $A = M_n(x,y,z,\theta) \in \mathfrak{h}_3^\diamond(n)$ for some $x,y,z,\theta \in \mathbb{R}$. Then we have the following sequence of implications:

$$\begin{cases} \theta \in 2n\pi\mathbb{Z} \\ \omega(A,A) = 0 \\ \pi_4(A) = 0 \Rightarrow \pi_3(A) = 0 \end{cases} \;\Rightarrow\; \begin{cases} \theta \in 2n\pi\mathbb{Z} \\ x^2+y^2-2z\theta = 0 \\ \pi_4(A) = 0 \Rightarrow \pi_3(A) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \theta \in 2n\pi\mathbb{Z},\ \theta \neq 0 \\ \frac{\theta\left(x^2+y^2-2z\theta\right)}{\theta^2} = 0 \end{cases} \quad\text{or}\quad \begin{cases} \theta = 0 \\ x = y = z = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \mathrm{e}^{\frac{i\theta}{n}} = 0 \\ \cos\theta = 1 \text{ and } \sin\theta = 0 \\ \frac{\theta\left(x^2+y^2-2z\theta\right) - \left(x^2+y^2\right)\sin\theta}{\theta^2} = 0 \end{cases} \quad\text{or}\quad \begin{cases} \theta = 0 \\ x = y = z = 0 \end{cases}$$

$$\Rightarrow A \in \exp^{-1}(\mathbf{1}).$$

□

3.3.2 Theorem. *The subgroup* $d\mathrm{Aut}\,\mathsf{H}_3^\diamond(n)$ *of* $\mathrm{Aut}\,\mathfrak{h}_3^\diamond$ *is given by*

$$d\mathrm{Aut}\,\mathsf{H}_3^\diamond(n) = \left\{ \begin{bmatrix} a & b & 0 & u \\ -kb & ka & 0 & v \\ kua - vb & kub + av & k(a^2+b^2) & \frac{1}{2}k(u^2+v^2) \\ 0 & 0 & 0 & k \end{bmatrix} \;\middle|\; a,b,u,v \in \mathbb{R},\ a^2+b^2 \neq 0,\ k \in \{-1,1\} \right\}.$$

PROOF. Suppose $\phi \in \operatorname{Aut} \mathsf{H}_3^\diamond(n)$. Then, in particular $T_{\mathbf{1}}\phi \in \operatorname{Aut} \mathfrak{h}_3^\diamond$ and hence we get (by proposition 1.2.27 and section 1.3) that

$$T_{\mathbf{1}}\phi = \begin{bmatrix} a & b & 0 & u \\ -kb & ka & 0 & v \\ kua - vb & kub + av & k(a^2+b^2) & w \\ 0 & 0 & 0 & k \end{bmatrix}$$

for some $a, b, u, v, w \in \mathbb{R}$ and $k \in \{-1, 1\}$ such that $a^2 + b^2 \neq 0$. Let $A = M_n(x, y, z, \theta) \in \exp^{-1}(\mathbf{1})$. (So in particular $x^2 + y^2 - 2yz = 0$.) Then, by proposition 3.1.7 and the preceding lemma, we get that $\omega(T_{\mathbf{1}}\phi \cdot A, T_{\mathbf{1}}\phi \cdot A) = 0$. (The condition $\pi_4(T_{\mathbf{1}}\phi \cdot A) = 0 \Rightarrow \pi_3(T_{\mathbf{1}}\phi \cdot A) = 0$ may be shown to be satisfied by all automorphisms.) Consequently we get that

$$\begin{aligned} 0 &= \omega(T_{\mathbf{1}}\phi \cdot A, T_{\mathbf{1}}\phi \cdot A) \\ &= \begin{bmatrix} x & y & z & \theta \end{bmatrix} (T_{\mathbf{1}}\phi)^\top \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} (T_{\mathbf{1}}\phi) \begin{bmatrix} x \\ y \\ z \\ \theta \end{bmatrix} \\ &= \begin{bmatrix} x & y & z & \theta \end{bmatrix} \begin{bmatrix} a^2+b^2 & 0 & 0 & 0 \\ 0 & a^2+b^2 & 0 & 0 \\ 0 & 0 & 0 & -a^2-b^2 \\ 0 & 0 & -a^2-b^2 & u^2+v^2-2kw \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ \theta \end{bmatrix} \\ &= \left(a^2+b^2\right)\left(x^2+y^2-2z\theta\right) + \left(u^2+v^2-2kw\right)\theta^2 \\ &= \left(u^2+v^2-2kw\right)\theta^2. \end{aligned}$$

Now, as we require that $\omega(T_{\mathbf{1}}\phi \cdot A, T_{\mathbf{1}}\phi \cdot A) = 0$ for all $A \in \exp^{-1}(\mathbf{1})$, and thus in particular for $2n\pi E_4$, we get that $u^2 + v^2 - 2kw = 0$. Hence we have that $w = \frac{1}{2}k(u^2 + v^2)$.

We define a (linear Lie) group G as the subgroup of $\operatorname{Aut} \mathfrak{h}_3^\diamond$ satisfying the above (necessary) condition. That is,

$$\mathsf{G} = \left\{ \begin{bmatrix} a & b & 0 & u \\ -kb & ka & 0 & v \\ kua - vb & kub + av & k(a^2+b^2) & \frac{1}{2}k(u^2+v^2) \\ 0 & 0 & 0 & k \end{bmatrix} \;\middle|\; a, b, u, v \in \mathbb{R},\ a^2+b^2 \neq 0,\ k \in \{-1, 1\} \right\}.$$

We aim to show that $\mathsf{G} = d\operatorname{Aut} \mathsf{H}_3^\diamond(n)$. (Verification that G is a Lie group is standard, though unnecessary, as we will show that $\mathsf{G} = d\operatorname{Aut} \mathsf{H}_3^\diamond(n)$.)

At this stage we have that $d\operatorname{Aut} \mathsf{H}_3^\diamond(n) \subseteq \mathsf{G}$. Now let $\psi \in \mathsf{G}$, represented as above, for some $a, b, u, v \in \mathbb{R}$ and $k \in \{-1, 1\}$ such that $a^2 + b^2 \neq 0$. Then define a mapping

$$\phi : \exp(M_n(x, y, z, 0)) \exp(M_n(0, 0, 0, \theta)) \mapsto \exp(\psi \cdot M_n(x, y, z, 0)) \exp(\psi \cdot M_n(0, 0, 0, \theta)).$$

We note that any element $m(x,y,z,\theta)$ can be uniquely decomposed as

$$m_n(x,y,z,\theta) = m_n(x,y,z,0)\, m_n(0,0,0,z) = \exp(M_n(x,y,z,0))\, \exp(M_n(0,0,0,\theta))$$

from semi-direct product structure (see proposition 1.2.6). Thus ϕ is defined for all $g \in \mathsf{H}_3^\diamond(n)$. We claim that ϕ is a well defined and injective. Indeed we have that

$$\begin{aligned}
& \phi(m_n(x,y,z,\theta)) = \phi(m_n(x',y',z',\theta')) \\
\Leftrightarrow\ & \exp(\psi\cdot M(x,y,z,0))\,\exp(\psi\cdot M(0,0,0,\theta)) = \exp(\psi\cdot M(x',y',z',0)\,\exp(\psi\cdot M(0,0,0,\theta'))) \\
\Leftrightarrow\ & \begin{cases} \exp(\psi\cdot M_n(x,y,z,0)) = \exp(\psi\cdot M_n(x',y',z',0)) \\ \exp(\psi\cdot M_n(0,0,0,\theta)) = \exp(\psi\cdot M_n(0,0,0,\theta')) \end{cases} \\
\Leftrightarrow\ & \begin{cases}
\begin{bmatrix} 1 & -ax-by & k\,(-bx+ay) & -2\left((aku-bv)x+(bku+av)y+\left(a^2+b^2\right)kz\right) & 0 \\ 0 & 1 & 0 & k\,(-bx+ay) & 0 \\ 0 & 0 & 1 & ax+by & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \\
= \begin{bmatrix} 1 & -ax'-by' & k\,(-bx'+ay') & -2\left((aku-bv)x'+(bku+av)y'+\left(a^2+b^2\right)kz'\right) & 0 \\ 0 & 1 & 0 & k\,(-bx'+ay') & 0 \\ 0 & 0 & 1 & ax'+by' & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \\
\begin{bmatrix} 1 & \frac{v-v\cos(k\theta)-u\sin(k\theta)}{k} & \frac{u-u\cos(k\theta)+v\sin(k\theta)}{k} & -\left(u^2+v^2\right)\sin(k\theta) & 0 \\ 0 & \cos(k\theta) & -\sin(k\theta) & \frac{u\left(-1+\cos(k\theta)\right)+v\sin(k\theta)}{k} & 0 \\ 0 & \sin(k\theta) & \cos(k\theta) & \frac{v-v\cos(k\theta)+u\sin(k\theta)}{k} & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & e^{\frac{ik\theta}{n}} \end{bmatrix} \\
= \begin{bmatrix} 1 & \frac{v-v\cos(k\theta')-u\sin(k\theta')}{k} & \frac{u-u\cos(k\theta')+v\sin(k\theta')}{k} & -\left(u^2+v^2\right)\sin(k\theta') & 0 \\ 0 & \cos(k\theta') & -\sin(k\theta') & \frac{u\left(-1+\cos(k\theta')\right)+v\sin(k\theta')}{k} & 0 \\ 0 & \sin(k\theta') & \cos(k\theta') & \frac{v-v\cos(k\theta')+u\sin(k\theta')}{k} & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & e^{\frac{ik\theta'}{n}} \end{bmatrix}
\end{cases} \\
\Leftrightarrow\ & \begin{cases} \begin{bmatrix} -a & -b \\ -kb & ka \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -a & -b \\ -kb & ka \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} \\ k(a^2+b^2)z = k(a^2+b^2)z' \\ e^{\frac{ik\theta}{n}} = e^{\frac{ik\theta'}{n}} \end{cases}
\Leftrightarrow \begin{cases} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix} \\ z = z' \\ \theta + 2n\pi\mathbb{Z} = \theta' + 2n\pi\mathbb{Z} \end{cases} \\
\Leftrightarrow\ & \begin{cases} \exp(M_n(x,y,z,0)) = \exp(M_n(x',y',z',0)) \\ \exp(M_n(0,0,0,\theta)) = \exp(M_n(0,0,0,\theta')) \end{cases}
\Leftrightarrow \begin{cases} m_n(x,y,z,0) = m_n(x',y',z',0) \\ m_n(0,0,0,\theta) = m_n(0,0,0,\theta') \end{cases} \\
\Leftrightarrow\ & m_n(x,y,z,\theta) = m_n(x',y',z',\theta').
\end{aligned}$$

Thus we have that ϕ is a well defined injective mapping. Thus by corollary 3.1.9 we have that ϕ is a Lie group automorphism such that $T_{\mathbf{1}}\phi = \psi$. In other words we get that $\mathsf{G} \subseteq d\mathrm{Aut}\,\mathsf{H}_3^\diamond(n)$ and conclude that $d\mathrm{Aut}\,\mathsf{G} = \mathsf{H}_3^\diamond(n)$. □

3.3.2 Global classification of proper systems on $\mathsf{H}_3^\diamond(n)$

We now turn our attention to classifying $\mathfrak{L}$-related affine subspaces of $\mathfrak{h}_3^\diamond(n)$. As $d\mathrm{Aut}\,\mathsf{H}_3^\diamond(n)$ is a subgroup of $\mathrm{Aut}\,\mathfrak{h}_3^\diamond(n)$, this forms a sub-classification of the local classification produced in chapter 2. In this subsection we represent an arbitrary element $\psi \in d\mathrm{Aut}\,\mathsf{H}_3^\diamond(n)$ as

$$\psi = \begin{bmatrix} x & y & 0 & u \\ -ky & kx & 0 & v \\ kux - vy & kuy + xv & k(x^2+y^2) & \frac{1}{2}k(u^2+v^2) \\ 0 & 0 & 0 & k \end{bmatrix}$$

for some $x, y, u, v \in \mathbb{R}$ and $k \in \{-1, 1\}$ such that $x^2 + y^2 \neq 0$.

Dimension one, homogeneous

3.3.3 Proposition. *Any* $(1,1)$*-affine subspace* $\Gamma = A + \Gamma^0 \subset \mathfrak{h}_3^\diamond(n)$ *(of full rank) is* $\mathfrak{L}$*-related exactly one of the following affine subspaces:*

Case: $\pi_4(\Gamma^0) \neq \{0\}$**,** $\mathfrak{P}(\Gamma) = \alpha_1$**,** $\alpha_1 \geq 0$
$\Gamma_{1,\alpha}^{(1,1)} = E_1 + \alpha_1 E_3 + \langle \alpha_2 E_3 + E_4 \rangle$, $\alpha_2 \in \mathbb{R}$
Case: $\pi_4^{(1,1)}(\Gamma^0) = \{0\}$**,** $\pi_4(A) = \pm\alpha$**,** $\alpha > 0$
$\Gamma_{2,\alpha,\sigma}^{(1,1)} = \sigma E_3 + \alpha E_4 + \langle E_1 \rangle$, $\sigma \in \{-1, 0, 1\}$.

Proof. (We omit the superscript $(1,1)$ in this proof.) Suppose Γ is an $(1,1)$-affine subspace such that $\pi_4(\Gamma^0) \neq \{0\}$ and $\mathfrak{P}(\Gamma) = \alpha_1$ for some $\alpha_1 \geq 0$. We wish to show that $\Gamma \sim \Gamma_{1,\alpha} = E_1 + \alpha_1 E_3 + \langle \alpha_2 E_4 + E_4 \rangle$ for some $\alpha_2 \in \mathbb{R}$. We may write Γ as $\Gamma = \sum_{i=1}^4 a_i E_i + \left\langle \sum_{i=1}^4 b_i E_i \right\rangle$, for some constants $a_i, b_i \in \mathbb{R}$, $i = \overline{1,4}$. Now as $\pi_4(\Gamma^0) \neq \{0\}$, we have that $b_4 \neq 0$ and so $\Gamma = \sum_{i=1}^3 (a_i - \frac{a_4 b_i}{b_4}) E_i + \left\langle \sum_{i=1}^3 \left(\frac{b_i}{b_4} E_i \right) + E_4 \right\rangle$. Then

$$\psi_1 = \begin{bmatrix} 1 & 0 & 0 & -\frac{b_1}{b_4} \\ 0 & 1 & 0 & -\frac{b_2}{b_4} \\ -\frac{b_1}{b_4} & -\frac{b_2}{b_4} & 1 & \frac{b_1^2+b_2^2}{2b_4^2} \\ 0 & 0 & 0 & 1 \end{bmatrix} \in d\mathrm{Aut}\,\mathsf{H}_3^\diamond(n)$$

and thus

$$\begin{aligned} \psi_1 \cdot \Gamma &= \sum_{i=1}^2 \left((a_i - \tfrac{a_4 b_i}{b_4})(E_i - \tfrac{b_i}{b_4} E_3) \right) + (a_3 - \tfrac{a_4 b_3}{b_4}) E_3 \\ &\quad + \left\langle \tfrac{b_1}{b_4} E_1 + \tfrac{b_2}{b_4} E_2 + \left(-\tfrac{b_1^2}{b_4^2} - \tfrac{b_2^2}{b_4^2} + \tfrac{b_3}{b_4} \right) E_3 + \left(-\tfrac{b_1}{b_4} E_1 - \tfrac{b_2}{b_4} E_2 + \tfrac{b_1^2+b_2^2}{2b_4^2} E_3 + E_4 \right) \right\rangle \\ &= \sum_{i=1}^3 a_i' E_i + \langle b_3' E_3 + E_4 \rangle \end{aligned}$$

for some corresponding new constants $a_i', b_3' \in \mathbb{R}$, $i = \overline{1,3}$. With a target of $\Gamma_{1,\alpha} = E_1 + \alpha_1 E_3 + \langle \alpha_2 E_3 + E_4 \rangle$ in mind we try and find an element of $d\mathrm{Aut}\,\mathsf{H}_3^\diamond(n)$ taking $\sum_{i=1}^2 a_i' E_i$ to E_1, but not changing E_4. Toward this end we consider the following equation

$$\begin{bmatrix} a_1' & a_2' \\ a_2' & -a_1' \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

We claim that either $a_1' \neq 0$ or $a_2' \neq 0$. Suppose not, i.e., $a_1' = a_2' = 0$. Then $\langle\Gamma\rangle = \langle a_3'E_3, b_3'E_3 + E_4\rangle$, but we have that $[E_3, E_i] = 0$, $i = \overline{1,4}$ and hence that $\dim \operatorname{Lie}\Gamma = \dim\langle\Gamma\rangle = 2$, contradicting that Γ is of full rank. Thus the matrix in the above equation has non-zero determinant and hence the equation has a solution (for x and y such that $(x,y) \neq (0,0)$) from which we may define an automorphism

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_1' & a_2' \\ a_2' & -a_1' \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \psi_2 = \begin{bmatrix} x & y & 0 & 0 \\ -y & x & 0 & 0 \\ 0 & 0 & x^2+y^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in d\operatorname{Aut}\mathsf{H}_3^\diamond(n).$$

Hence we get that

$$\begin{aligned} \psi_2 \cdot \psi_1 \cdot \Gamma &= a_1'(xE_1 - yE_1) + a_2'(yE_1 + xE_2) + a_3'(x^2+y^2)E_3 + \left\langle b_3'(x^2+y^2)E_3 + E_4 \right\rangle \\ &= E_1 + a_3''E_3 + \left\langle b_3''E_3 + E_4 \right\rangle. \end{aligned}$$

for some corresponding new constants $a_3'', b_3'' \in \mathbb{R}$. Now we have that $\mathfrak{P}(\psi_2 \cdot \psi_1 \cdot \Gamma) = \mathfrak{P}(\Gamma) = \pm\alpha_1$. Thus $|a_3''| = \alpha_1$. If $a_3'' = 0$, then $\psi_2 \cdot \psi_1 \cdot \Gamma = E_1 + \langle \alpha_2 E_3 + E_4\rangle$ for some $\alpha_2 \in \mathbb{R}$ and we are done. On the other hand, if $a_3'' \neq 0$, we have that $\psi_3 = \operatorname{diag}(1, \operatorname{sgn}(a_3''), \operatorname{sgn}(a_3''), \operatorname{sgn}(a_3'')) \in d\operatorname{Aut}\mathsf{H}_3^\diamond(n)$ and so $\psi_3 \cdot \psi_2 \cdot \psi_1 \cdot \Gamma = E_1 + |a_3''|E_3 + \langle b_3''E_3 + E_4\rangle = E_1 + \alpha_1 E_3 + \langle \alpha_2 E_3 + E_4\rangle$ for some $\alpha_2 \in \mathbb{R}$. This concludes the first case.

Next we assume Γ is an $(1,1)$-affine subspace such that $\pi_4(\Gamma^0) = \{0\}$ and $\pi_4(A) = \pm\alpha$ for some $\alpha > 0$. (Note $\pi_4(A) = 0$ implies that Γ is not of full rank and may thus be ignored.) We wish to show that $\Gamma \sim \Gamma_{2,\alpha,\sigma} = \sigma E_3 + \alpha E_4 + \langle E_1\rangle$, for some $\sigma \in \{-1,0,1\}$. Now as $\pi_4(\Gamma^0) = \{0\}$ we have constants $a_i, a_4, b_i \in \mathbb{R}$, $i = \overline{1,3}$ such that $\Gamma = \sum_{i=1}^4 a_i E_i + \left\langle \sum_{i=1}^3 b_i E_i \right\rangle$. Then as $\pi_4(A) = \pm\beta$ we have that $a_4 = \pm\beta \neq 0$. Hence,

$$\psi_1 = \begin{bmatrix} 1 & 0 & 0 & -\frac{a_1}{a_4} \\ 0 & 1 & 0 & -\frac{a_2}{a_4} \\ -\frac{a_1}{a_4} & -\frac{a_2}{a_4} & 1 & \frac{a_1^2+a_2^2}{2a_4^2} \\ 0 & 0 & 0 & 1 \end{bmatrix} \in d\operatorname{Aut}\mathsf{H}_3^\diamond(n)$$

and so

$$\begin{aligned} \psi_1 \cdot \Gamma &= a_1(E_1 - \tfrac{a_1}{a_4}E_3) + a_2(E_2 - \tfrac{a_2}{a_4}E_3) + a_3E_3 \\ &\quad + a_4\left(-\tfrac{a_1}{a_4}E_1 - \tfrac{a_2}{a_4}E_2 + \tfrac{a_1^2+a_2^2}{2a_4^2}E_3 + E_4\right) + \left\langle \psi_1 \cdot \sum_{i=1}^3 b_i E_i \right\rangle \\ &= a_3'E_3 + a_4E_4 + \left\langle \psi_1 \cdot \sum_{i=1}^3 b_i E_i \right\rangle \end{aligned}$$

for some new constant $a_3' \in \mathbb{R}$. Then $\psi_2 = \operatorname{diag}(1, \operatorname{sgn}(a_4), \operatorname{sgn}(a_4), \operatorname{sgn}(a_4)) \in d\operatorname{Aut}\mathsf{H}_3^\diamond(n)$ and $\psi_2 \cdot \psi_1 \cdot \Gamma = a_3''E_3 + |a_4|E_4 + \left\langle \sum_{i=1}^3 b_i'E_i \right\rangle = a_3''E_3 + \alpha E_4 + \left\langle \sum_{i=1}^3 b_i'E_i \right\rangle$ for some new constants

$a_3'', b_i' \in \mathbb{R}$, $i = \overline{1,3}$. We claim that either $b_1' \neq 0$ or $b_2' \neq 0$. Supposing $b_1' = b_2' = 0$ we have that $\langle \psi_2 \cdot \psi_1 \cdot \Gamma \rangle = \langle a_3'' E_3 + \alpha E_4, b_3' E_3 \rangle$, contradicting that Γ is of full rank. Thus the equation

$$\begin{bmatrix} b_1' & b_2' \\ b_2' & -b_1' \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

has a solution (for x and y such that $(x, y) \neq (0, 0)$), as matrix has non-zero determinant, from which we may define an automorphism

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1' & b_2' \\ b_2' & -b_1' \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \psi_3 = \begin{bmatrix} x & y & 0 & 0 \\ -y & x & 0 & 0 \\ 0 & 0 & x^2+y^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in d\mathrm{Aut}\,\mathsf{H}_3^\diamond(n).$$

Consequently we get that $\psi_3 \cdot \psi_2 \cdot \psi_1 \cdot \Gamma = (x^2+y^2) a_3'' E_3 + \alpha E_4 + \left\langle E_1 + (x^2+y^2) b_3' E_3 \right\rangle$. Next we have that

$$\psi_4 = \begin{bmatrix} 1 & 0 & 0 & -(x^2+y^2)b_3' \\ 0 & 1 & 0 & 0 \\ -(x^2+y^2)b_3' & 0 & 1 & \frac{1}{2}((x^2+y^2)b_3')^2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and that

$$\begin{aligned} \psi_4 \cdot \psi_3 \cdot \psi_2 \cdot \psi_1 \cdot \Gamma \quad &= (x^2+y^2) a_3'' E_3 + \alpha (E_4 - (x^2+y^2) b_3' E_1 + \tfrac{1}{2}((x^2+y^2)b_3')^2 E_3) \\ &\qquad + \left\langle (E_1 - (x^2+y^2) b_3' E_3) + (x^2+y^2) b_3' E_3 \right\rangle \\ &= -\alpha (x^2+y^2) b_3' E_1 + ((x^2+y^2) a_3'' + \tfrac{1}{2}\alpha((x^2+y^2)b_3')^2) E_3 + \alpha E_4 + \langle E_1 \rangle \\ &= a_3''' E_3 + \alpha E_4 + \langle E_1 \rangle \end{aligned}$$

for some $a_3''' \in \mathbb{R}$. If $a_3''' = 0$ then $\Gamma \sim \alpha E_4 + \langle E_1 \rangle = \Gamma_{2,\alpha,0}$. On the other hand if $a_3''' \neq 0$, then

$$\psi_5 = \mathrm{diag}\left(\frac{1}{\sqrt{\mathrm{sgn}(a_3''') a_3'''}}, \frac{1}{\sqrt{\mathrm{sgn}(a_3''') a_3'''}}, \frac{1}{\mathrm{sgn}(a_3''') a_3'''}, 1 \right) \in d\mathrm{Aut}\,\mathsf{H}_3^\diamond(n)$$

and hence we get that $\psi_5 \cdot \psi_4 \cdots \psi_1 \cdot \Gamma = \mathrm{sgn}(a_3''') E_3 + \alpha E_4 + \langle E_1 \rangle = \sigma E_3 + \alpha E_4 + \langle E_1 \rangle = \Gamma_{2,\alpha,\sigma}$ for some $\sigma \in \{-1, 1\}$. This concludes the second case.

Now as our list is exhaustive we are left to show that none of the affine subspaces

$$\begin{aligned} \Gamma_{1,\alpha} &= E_1 + \alpha_1 E_3 + \langle \alpha_2 E_3 + E_4 \rangle & \quad \alpha_1 \geq 0, \alpha_2 \in \mathbb{R} \\ \Gamma_{2,\alpha,\sigma} &= \sigma E_3 + \alpha E_4 + \langle E_1 \rangle & \quad \alpha > 0,\ \sigma \in \{-1, 0, 1\} \end{aligned}$$

are $\mathfrak{L}$-related. This is done by assuming that they are, giving us an element ψ of $d\mathrm{Aut}\,\mathsf{H}_3^\diamond(n)$ taking one to the other, and producing a contradiction from this.

$(\alpha \neq \beta,\ \alpha_1, \beta_1 \geq 0) \Rightarrow \Gamma_{1,\alpha} \nsim \Gamma_{1,\beta}$**:** Assume there exists a $\psi \in d\mathrm{Aut}\,\mathsf{H}_3^\diamond(n)$ such that $\psi \cdot \Gamma_{1,\alpha} = \Gamma_{2,\beta}$. Then we get that $\alpha_1 = \mathfrak{P}(\Gamma_{1,\alpha}) = \mathfrak{P}(\psi \cdot \Gamma_{1,\alpha}) = \mathfrak{P}(\Gamma_{1,\beta}) = \beta_1$. Now as $E_1 + \alpha_1 E_3 \in$

$\Gamma_{1,\alpha}$ and $\alpha_2 E_3 + E_4 \in \Gamma^0_{1,\alpha}$ we require that $\psi\cdot(E_1+\alpha_1 E_3) \in \Gamma_{2,\beta}$ and $\psi\cdot(\alpha_2 E_3 + E_4) \in \Gamma^0_{2,\beta}$. That is to say there exists constants $r_1, r_2 \in \mathbb{R}$ such that

$$\begin{bmatrix} x \\ -ky \\ kux - vy + k\left(x^2+y^2\right)\alpha_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \beta_1 + r_1\beta_2 \\ r_1 \end{bmatrix}, \quad \begin{bmatrix} u \\ v \\ \frac{1}{2}k\left(u^2+v^2\right) + k\left(x^2+y^2\right)\alpha_2 \\ k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ r_2\beta_2 \\ r_2 \end{bmatrix}.$$

Hence we get that $x = 1$, $y = u = v = 0$. Then from the second equation we get that $k\alpha_2 = k\beta_2$, implying that $\alpha_2 = \beta_2$.

$\Gamma_{1,\alpha} \nsim \Gamma_{2,\beta,\sigma}$: This follows form proposition 2.2.4.

$(\alpha,\sigma) \neq (\beta,\varsigma),\ \alpha,\beta > 0) \Rightarrow \Gamma_{2,\alpha,\sigma} \nsim \Gamma_{2,\beta,\varsigma}$: Assume there exists a $\psi \in d\mathrm{Aut}\,\mathsf{H}_3^\diamond(n)$ such that $\psi \cdot \Gamma_{2,\alpha,\sigma} = \Gamma_{2,\beta,\varsigma}$. Then, as $\sigma E_3 + \alpha E_4 \in \Gamma_{2,\alpha,\sigma}$ and $E_1 \in \Gamma^0_{2,\beta,\varsigma}$, we require that $\psi\cdot(\sigma E_3 + \alpha E_4) \in \Gamma_{2,\beta,\varsigma}$ and $\psi \cdot E_1 \in \Gamma^0_{2,\beta,\varsigma}$. Hence, there exists constants $r_1, r_2 \in \mathbb{R}$ such that

$$\begin{bmatrix} u\alpha \\ v\alpha \\ \frac{1}{2}k\left(u^2+v^2\right)\alpha + k\left(x^2+y^2\right)\sigma \\ k\alpha \end{bmatrix} = \begin{bmatrix} r_1 \\ 0 \\ \varsigma \\ \beta \end{bmatrix}, \qquad \begin{bmatrix} x \\ -ky \\ kux - vy \\ 0 \end{bmatrix} = \begin{bmatrix} r_2 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Therefore $k = 1$ (as $\alpha, \beta > 0$) and so $\alpha = \beta$. Moreover we get that $v = 0$ and $y = 0$ (and so $x \neq 0$). Then $u = 0$ and so $x^2\sigma = \varsigma$, implying that $\sigma = \varsigma$. □

Dimension two, homogeneous

3.3.4 PROPOSITION. *Any* $(2,0)$*-affine subspace* $\Gamma = \Gamma^0 \subset \mathfrak{h}_3^\diamond(n)$ *(of full rank) is* $\mathfrak{L}$*-related to exactly one of the affine subspaces* $\Gamma^{(2,0)}_{1,\sigma} = \langle E_1, \sigma E_3 + E_4\rangle$*, where* $\sigma \in \{-1,0,1\}$.

PROOF. (We omit the superscript $(2,0)$ in this proof.) By proposition 3.3.3 and corollary 3.1.5 we have that Γ is $\mathfrak{L}$-related to $\left\langle\Gamma^{(1,1)}_{1,\alpha}\right\rangle = \langle E_1 + \alpha_1 E_3, \alpha_2 E_3 + E_4\rangle$, for some $\alpha_1 > 0$ and $\alpha_2 \in \mathbb{R}$, or $\left\langle\Gamma^{(1,1)}_{2,\beta,\sigma}\right\rangle = \langle \sigma E_3 + \beta E_4, E_1\rangle = \left\langle E_1, \frac{\sigma}{\alpha}E_3 + E_4\right\rangle$, for some $\beta > 0$ and $\sigma \in \{-1,0,1\}$. If $\Gamma \sim \left\langle\Gamma^{(1,1)}_{2,\beta,0}\right\rangle = \langle E_1, E_4\rangle$, then we are done. If $\Gamma \sim \left\langle\Gamma^{(1,1)}_{2,\beta,\sigma}\right\rangle$ and $\sigma \neq 0$, then $\Gamma \sim \langle E_1, aE_3 + E_4\rangle$ for some $a \in \mathbb{R}$, $a \neq 0$ (we will deal with this case presently). If $\Gamma \sim \left\langle\Gamma^{(1,1)}_{1,\alpha}\right\rangle$, then we have that

$$\psi_1 = \begin{bmatrix} 1 & 0 & 0 & -\alpha_1 \\ 0 & 1 & 0 & 0 \\ -\alpha_1 & 0 & 1 & \frac{1}{\alpha_1^2} \\ 0 & 0 & 0 & 1 \end{bmatrix} \in d\mathrm{Aut}\,\mathsf{H}_3^\diamond(n)$$

and get that $\psi_1 \cdot \left\langle\Gamma^{(1,1)}_{1,\alpha}\right\rangle = \left\langle E_1 - \alpha_1 E_3 + \alpha_1 E_3, \alpha_2 E_3 - \alpha_1 E_1 + \frac{1}{\alpha_1^2}E_3 + E_4\right\rangle = \langle E_1, aE_3 + E_4\rangle$ for some $a \in \mathbb{R}$. If $a = 0$, then $\Gamma \sim \langle E_1, E_4\rangle$ and we are done. So we are left to deal with the case $\Gamma \sim \langle E_1, aE_3 + E_4\rangle$ for some $a \neq 0$. In this case we have that

$$\psi_2 = \operatorname{diag}\left(\tfrac{1}{\sqrt{\operatorname{sgn}(a)a}}, \tfrac{1}{\sqrt{\operatorname{sgn}(a)a}}, \tfrac{1}{\operatorname{sgn}(a)a}, 1\right) \in d\mathrm{Aut}\,\mathsf{H}_3^{\diamond}(n)$$

and get that $\psi_2 \cdot \langle E_1, aE_3 + E_4\rangle = \langle E_1, \operatorname{sgn}(a)E_3 + E_4\rangle = \langle E_1, \sigma E_3 + E_4\rangle$ for some $\sigma \in \{-1, 1\}$.

We are left to show that no two of these three systems are $\mathfrak{L}$-related. We do this by showing that if $\Gamma_{1,\sigma} \sim \Gamma_{1,\varsigma}$, then $\sigma = \varsigma$. Assume there exists $\psi \in d\mathrm{Aut}\,\mathsf{H}_3^{\diamond}(n)$ such that $\psi \cdot \Gamma_{1,\sigma} = \Gamma_{1,\varsigma}$. Then, as $E_1, \sigma E_3 + E_4 \in \Gamma_{1,\sigma}$ we have that $\psi \cdot E_1, \psi \cdot (\sigma E_3 + E_4) \in \Gamma_{1,\varsigma}$. Hence, there exists constants $r_i \in \mathbb{R}$, $i = \overline{1,4}$ such that

$$\begin{bmatrix} x \\ -ky \\ kux - vy \\ 0 \end{bmatrix} = \begin{bmatrix} r_1 \\ 0 \\ \varsigma r_2 \\ r_2 \end{bmatrix}, \qquad \begin{bmatrix} u \\ v \\ \frac{1}{2}k\left(u^2 + v^2\right) + k\left(x^2 + y^2\right)\sigma \\ k \end{bmatrix} = \begin{bmatrix} r_3 \\ 0 \\ \varsigma r_4 \\ r_4 \end{bmatrix}.$$

Hence $y = 0$ (so $x \neq 0$), $r_2 = 0$, $v = 0$, $u = 0$ and $r_4 = k$. Thus $kx^2\sigma = k\varsigma$ implying that $\sigma = \varsigma$. □

Dimension two, inhomogeneous

3.3.5 Proposition. *Any $(2,1)$-affine subspace $\Gamma = A + \Gamma^0 \subset \mathfrak{h}_3^{\diamond}(n)$ (of full rank) is $\mathfrak{L}$-related to exactly one of the following affine subspaces:*

Case: $\pi_4(\Gamma^0) \neq \{0\}$

Case: $\dim \mathrm{Lie}\,\Gamma^0 = 4$

$\Gamma^{(2,1)}_{1,\alpha,\sigma} = \alpha_1 E_2 + \alpha_2 E_3 + \langle E_1, \sigma E_3 + E_4\rangle, \quad \alpha_1, \alpha_2 \geq 0,\ \alpha \neq 0,\ \sigma \in \{-1, 1\}.$

$\Gamma^{(2,1)}_{2,\alpha} = E_2 + \alpha E_3 + \langle E_1, E_4\rangle, \quad \alpha \geq 0$

$\Gamma^{(2,1)}_{3} = E_3 + \langle E_1, E_4\rangle.$

Case: $\dim \mathrm{Lie}\,\Gamma^0 \neq 4$

$\Gamma^{(2,1)}_{4} = E_1 + \langle E_3, E_4\rangle$

Case: $\pi_4(\Gamma^0) = \{0\}$, $\pi_4(A) = \pm\alpha$, $\alpha > 0$

Case: $\mathfrak{T}(\Gamma) = 1$

$\Gamma^{(2,1)}_{5,\alpha,\sigma} = \sigma E_3 + \alpha E_4 + \langle E_1, E_2\rangle, \quad \sigma \in \{-1, 0, 1\}$

Case: $\mathfrak{T}(\Gamma) = 0$

$\Gamma^{(2,1)}_{6,\alpha} = \alpha E_4 + \langle E_1, E_3\rangle.$

Proof. (We omit the superscript $(2,1)$ in this proof.)

Case: $\pi_4(\Gamma^0) \neq \{0\}$, $\dim \mathrm{Lie}\,\Gamma^0 = 4$

As $\dim \mathrm{Lie}\,\Gamma^0 = 4$, Γ^0 is a two-dimensional, homogeneous affine subspace of full rank. By application of proposition 3.3.4, we therefore get that Γ is $\mathfrak{L}$-related to $\langle E_1, \sigma E_3 + E_4\rangle$

for some $\sigma \in \{-1, 0, 1\}$. That is, there exists a $\psi_1 \in d\mathrm{Aut}\,\mathsf{H}_3^\diamond(n)$ such that $\psi_1 \cdot \Gamma^0 = \langle E_1, \sigma E_3 + E_4\rangle$. Thus we have constants $a_i \in \mathbb{R}$, $i = \overline{1,4}$, such that

$$\psi_1 \cdot \Gamma = \sum_{i=1}^{4} a_i E_i + \langle E_1, \sigma E_3 + E_4\rangle = a_2' E_2 + a_3' E_3 + \langle E_1, \sigma E_3 + E_4\rangle\,.$$

Note that as Γ is inhomogeneous we have that $(a_2', a_3') \neq (0,0)$. Now define $\mathrm{sgn}^* : \mathbb{R} \to \{-1, 1\}$ by: $\mathrm{sgn}^*(r) = \mathrm{sgn}(r)$ if $r \neq 0$ and $\mathrm{sgn}^*(0) = 1$. Then

$$\psi_2 = \mathrm{diag}\left(\mathrm{sgn}^*(a_2')\mathrm{sgn}^*(a_3'), \mathrm{sgn}^*(a_2'), \mathrm{sgn}^*(a_3'), \mathrm{sgn}^*(a_3')\right) \in d\mathrm{Aut}\,\mathsf{H}_3^\diamond(n)$$

and we get that $\psi_2 \cdot \psi_1 \cdot \Gamma = \mathrm{sgn}(a_2')a_2'E_2 + \mathrm{sgn}(a_3')a_3E_3 + \langle E_1, \mathrm{sgn}^*(a_3')(\sigma E_3 + E_4)\rangle$. Thus we have that $\psi_2 \cdot \psi_1 \cdot \Gamma = \alpha_1 E_2 + \alpha_2 E_3 + \langle E_1, \sigma E_3 + E_4\rangle$ for some $\alpha_1, \alpha_2 \geq 0$, $\alpha \neq 0$ and $\sigma \in \{-1, 0, 1\}$. If $\sigma \neq 0$, then $\psi_2 \cdot \psi_1 \cdot \Gamma = \Gamma_{1,\alpha,\sigma}$. Assume that $\sigma = 0$ and that $\alpha_1 \neq 0$. Then

$$\psi_3 = \mathrm{diag}\left(\tfrac{\mathrm{sgn}^*(\alpha_2)}{\alpha_1}, \tfrac{1}{\alpha_1}, \tfrac{\mathrm{sgn}^*(\alpha_2)}{\alpha_1^2}, \mathrm{sgn}^*(\alpha_2)\right) \in d\mathrm{Aut}\,\mathsf{H}_3^\diamond(n)$$

and hence we get that

$$\psi_3 \cdot \psi_2 \cdot \psi_1 \cdot \Gamma = E_2 + \tfrac{\mathrm{sgn}(\alpha_2)\alpha_2}{\alpha_1^2} E_3 + \langle E_1, E_4\rangle = E_2 + \beta E_3 + \langle E_1, E_4\rangle = \Gamma_{2,\beta},$$

for some $\beta \geq 0$. On the other hand assume that $\sigma = 0$ and that $\alpha_1 = 0$. Then as $\alpha \neq 0$ we have that $\alpha_2 \neq 0$. Consequently

$$\psi_4 = \mathrm{diag}\left(\tfrac{1}{\sqrt{\mathrm{sgn}(\alpha_2)\alpha_2}}, \tfrac{\mathrm{sgn}(\alpha_2)}{\sqrt{\mathrm{sgn}(\alpha_2)\alpha_2}}, \tfrac{1}{\alpha_2}, \mathrm{sgn}(\alpha_2)\right) \in d\mathrm{Aut}\,\mathsf{H}_3^\diamond(n)$$

and hence we get that $\psi_4 \cdot \psi_2 \cdot \psi_1 \cdot \Gamma = E_3 + \langle E_1, E_4\rangle = \Gamma_3$.

Case: $\pi_4(\Gamma^0) \neq \{0\}$**,** $\dim \mathrm{Lie}\,\Gamma^0 \neq 4$

We wish to show that $\Gamma \sim \Gamma_4 = E_1 + \langle E_3, E_4\rangle$. Now, we have constants $a_i, b_i, c_i \in \mathbb{R}$, $i = \overline{1,4}$ such that $\Gamma = \sum_{i=1}^4 a_i E_i + \left\langle \sum_{i=1}^4 b_i E_i, \sum_{i=1}^4 c_i E_i\right\rangle$. Thus, as $\pi_4(\Gamma^0) \neq \{0\}$, we have that either $c_4 \neq 0$ or $b_4 \neq 0$. W.l.o.g. we thus assume $c_4 \neq 0$. Then $\Gamma = \sum_{i=1}^3 (a_i - \frac{a_4 c_i}{c_4})E_i + \left\langle \sum_{i=1}^3 (b_i - \frac{b_4 c_i}{c_4})E_i, \sum_{i=1}^3 \frac{c_i}{c_4} E_i + E_4\right\rangle$. Now, we have that

$$\psi_1 = \begin{bmatrix} 1 & 0 & 0 & -\frac{c_1}{c_4} \\ 0 & 1 & 0 & -\frac{c_2}{c_4} \\ -\frac{c_1}{c_4} & -\frac{c_2}{c_4} & 1 & \frac{c_1^2 + c_2^2}{2c_4^2} \\ 0 & 0 & 0 & 1 \end{bmatrix} \in d\mathrm{Aut}\,\mathsf{H}_3^\diamond(n)$$

and hence get that

$$\psi_1 \cdot \Gamma = \psi_1 \cdot \sum_{i=1}^{3}(a_i - \tfrac{a_4 c_i}{c_4})E_i + \Bigg\langle \psi_1 \cdot \sum_{i=1}^{3}(b_i - \tfrac{b_4 c_i}{c_4})E_i,$$
$$\sum_{i=1}^{2} \tfrac{c_i}{c_4}(E_i - \tfrac{c_i}{c_4}E_3) + \tfrac{c_3}{c_4}E_3 + \Big(-\tfrac{c_1}{c_4}E_1 - \tfrac{c_2}{c_4}E_2 + \tfrac{c_1^2+c_2^2}{2c_4^2}E_3 + E_4\Big)\Bigg\rangle$$
$$= \sum_{i=1}^{3} a_i' E_i + \Bigg\langle \sum_{i=1}^{3} b_i' E_i, c_3' E_3 + E_4 \Bigg\rangle$$

for some corresponding new constants $a_i', b_i', c_3' \in \mathbb{R}$, $i = \overline{1,3}$. Next we claim that $b_1' = b_2' = 0$. Let $B' = \sum_{i=1}^{3} b_i' E_i$ and $C' = c_3' E_3 + E_4$. Then we have that $[B, C] = b_1' E_2 - b_2' E_1$ and $[B, [B, C]] = ((b_1')^2 + (b_2')^2)E_3$. So, if $b_1' \neq 0$ or $b_2' \neq 0$, then

$$\begin{vmatrix} 0 & 0 & c_3' & 1 \\ b_1' & b_2' & b_3' & 0 \\ -b_2' & b_1' & 0 & 0 \\ 0 & 0 & (b_1')^2 + (b_2')^2 & 0 \end{vmatrix} = -\left((b_1')^2 + (b_2')^2\right)^2 \neq 0,$$

which is to say that $\{C, B, [B, C], [B, [B, C]]\}$ is linearly independent. But then we have that $\dim \operatorname{Lie} \Gamma^0 = 4$, a contradiction. Hence $b_1' = b_2' = 0$, thus $b_3' \neq 0$ (as $\dim \Gamma^0 = 2$) and so $\psi_1 \cdot \Gamma = a_1' E_1 + a_2' E_2 + \langle E_3, c_3' E_3 + E_4 \rangle = a_1' E_1 + a_2' E_2 + \langle E_3, E_4 \rangle$. The condition that Γ is inhomogeneous implies that either $a_1' \neq 0$ or $a_2' \neq 0$. So the equation

$$\begin{bmatrix} a_1' & a_2' \\ a_2' & -a_1' \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

has a solution (for x and y such that $(x, y) \neq (0, 0)$), from which we define an automorphism,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_1' & a_2' \\ a_2' & -a_1' \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \psi_2 = \begin{bmatrix} x & y & 0 & 0 \\ -y & x & 0 & 0 \\ 0 & 0 & x^2 + y^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in d\mathrm{Aut}\, \mathsf{H}_3^\diamond(n).$$

Consequently we get that $\psi_2 \cdot \psi_1 \cdot \Gamma = a_1'(xE_1 - yE_2) + a_2'(yE_1 + xE_2) + \langle E_3, E_4 \rangle = E_1 + \langle E_3, E_4 \rangle = \Gamma_4$.

Case: $\pi_4(\Gamma^0) = \{0\}$**,** $\pi_4(A) = \pm\alpha$**,** $\alpha > 0$

We have that $\Gamma = \sum_{i=1}^{4} a_i E_i + \left\langle \sum_{i=1}^{4} b_i E_i, \sum_{i=1}^{4} c_i E_i \right\rangle$ for some constants $a_i, b_i, c_i \in \mathbb{R}$, $i = \overline{1,4}$. Now as $\pi_4(\Gamma^0) = \{0\}$, we have that $b_4 = c_4 = 0$. Next we claim $a_4 \neq 0$. If $a_4 = 0$ we have that $\langle \Gamma \rangle = \left\langle \sum_{i=1}^{3} a_i E_i, \sum_{i=1}^{3} b_i E_i, \sum_{i=1}^{3} c_i E_i \right\rangle$ and hence, as $\pi_4([E_i, E_j]) = 0$, $i, j = \overline{1,4}$, Γ is not of full rank, a contradiction. So $a_4 \neq 0$ (i.e., there is no case for which $\alpha = 0$). Then $\psi_1 = \operatorname{diag}(1, \operatorname{sgn}(a_4), \operatorname{sgn}(a_4), \operatorname{sgn}(a_4)) \in d\mathrm{Aut}\, \mathsf{H}_3^\diamond(n)$ and hence

$\psi_1 \cdot \Gamma = \sum_{i=1}^3 a_i' E_i + \alpha E_4 + \left\langle \sum_{i=1}^3 b_i' E_i, \sum_{i=1}^3 c_i' E_i \right\rangle$ for some new constants $a_i', b_i', c_i' \in \mathbb{R}$, $i = \overline{1,3}$. Next we have that

$$\psi_2 = \begin{bmatrix} 1 & 0 & 0 & -\frac{a_1'}{\alpha} \\ 0 & 1 & 0 & -\frac{a_2'}{\alpha} \\ -\frac{a_1'}{\alpha} & -\frac{a_2'}{\alpha} & 1 & \frac{(a_1')^2+(a_2')^2}{2\alpha^2} \\ 0 & 0 & 0 & 1 \end{bmatrix} \in d\mathrm{Aut}\,\mathsf{H}_3^\diamond(n)$$

and get that

$$\begin{aligned}\psi_2 \cdot \psi_1 \cdot \Gamma &= \sum_{i=1}^2 a_i'(E_i - \tfrac{a_i'}{\alpha} E_3) + a_3' E_3 + \alpha(-\tfrac{a_1'}{\alpha} E_1 - \tfrac{a_2'}{\alpha} E_2 + \tfrac{(a_1')^2+(a_2')^2}{2\alpha^2} E_3 + E_4) \\ &\quad + \left\langle \psi_2 \cdot \sum_{i=1}^3 b_i' E_i, \psi_2 \cdot \sum_{i=1}^3 c_i' E_i \right\rangle \\ &= a_3'' E_3 + \alpha E_4 + \left\langle \sum_{i=1}^3 b_i'' E_i, \sum_{i=1}^3 c_i'' E_i \right\rangle\end{aligned}$$

for some new constants $a_3'', b_i'', c_i'' \in \mathbb{R}$, $i = \overline{1,3}$. Now we claim that $(b_1'' \neq 0 \vee b_2'' \neq 0) \vee (c_1'' \neq 0 \vee c_2'' \neq 0)$. Suppose not, i.e., $b_1'' = b_2'' = c_1'' = c_2'' = 0$. Then $\langle \Gamma \rangle = \langle a_3'' E_3 + E_4, b_3'' E_3, c_3'' E_3 \rangle$ and (as $[E_3, E_4] = 0$) we then have that Γ is not of full rank, a contradiction. Thus w.l.o.g. we may assume $(b_1'' \neq 0 \vee b_2'' \neq 0)$. Then the equation

$$\begin{bmatrix} b_1'' & b_2'' \\ b_2'' & -b_1'' \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

has a solution (for x and y such that $(x, y) \neq (0, 0)$), from which we define an automorphism,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1'' & b_2'' \\ b_2'' & -b_1'' \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \psi_3 \qquad = \begin{bmatrix} x & y & 0 & 0 \\ -y & x & 0 & 0 \\ 0 & 0 & x^2+y^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in d\mathrm{Aut}\,\mathsf{H}_3^\diamond(n).$$

Therefore we get that $\psi_3 \cdot \sum_{1=1}^3 b_i'' E_i = b_1''(xE_1 - yE_2) + b_2''(yE_1 + xE_2) + b_3''(x^2+y^2)E_3 = E_1 + b_3''(x^2+y^2)E_3$. Let $\psi = \psi_3 \circ \psi_2 \circ \psi_1$ and hence get

$$\psi \cdot \Gamma = a_3''' E_3 + \alpha E_4 + \left\langle E_1 + b_3''' E_3, c_2''' E_2 + c_3''' E_3 \right\rangle$$

for some new constants $a_3''', b_3''', c_i''' \in \mathbb{R}$, $i = \overline{1,3}$.

Case: $\mathfrak{T}(\Gamma) = 1$

We wish to show that $\Gamma \sim \Gamma_{5,\alpha,\sigma} = \sigma E_3 + \alpha E_4 + \langle E_1, E_2 \rangle$, for some $\sigma \in \{-1, 0, 1\}$. Now, by the preliminary discussion we have that there exists a $\psi \in d\mathrm{Aut}\,\mathsf{H}_3^\diamond(n)$ and constants a_3, b_3, c_2, c_3 such that $\psi \cdot \Gamma = a_3 E_3 + \alpha E_4 + \langle E_1 + b_3 E_3, c_2 E_2 + c_3 E_3 \rangle$. Now as $\mathfrak{T}(\Gamma) = 1$ we have that $|\mathrm{sgn}(c_2)| = 1$, thus $c_2 \neq 0$. Hence, $\psi \cdot \Gamma = a_3E_3 + \alpha E_4 + \left\langle E_1 + b_3E_3, E_2 + \frac{c_3}{c_2}E_3 \right\rangle$. Then we have that

$$\psi_4 = \begin{bmatrix} 1 & 0 & 0 & -b_3 \\ 0 & 1 & 0 & -\frac{c_3}{c_2} \\ -b_3 & -\frac{c_3}{c_2} & 1 & \frac{1}{2}\left(b_3^2 + \frac{c_3^2}{c_2^2}\right) \\ 0 & 0 & 0 & 1 \end{bmatrix} \in d\mathrm{Aut}\,\mathsf{H}_3^\diamond(n)$$

and thus get that

$$\begin{aligned} \psi_4 \cdot \psi \cdot \Gamma &= a_3E_3 + \alpha(E_4 - b_3E_1 - \tfrac{c_3}{c_2}E_2 + \tfrac{1}{2}\left(b_3^2 + \tfrac{c_3^2}{c_2^2}\right)E_3) \\ &\qquad + \left\langle (E_1 - b_3E_3) + b_3E_3, (E_2 - \tfrac{c_3}{c_2}E_3) + \tfrac{c_3}{c_2}E_3 \right\rangle \\ &= a_1'E_1 + a_2'E_2 + a_3'E_3 + \alpha E_4 + \langle E_1, E_2\rangle \\ &= a_3'E_3 + \alpha E_4 + \langle E_1, E_2\rangle \end{aligned}$$

for some corresponding new constants $a_i' \in \mathbb{R}$, $i = \overline{1,3}$. Now, if $a_3' = 0$, we have that $\psi_4 \cdot \psi \cdot \Gamma = \alpha E_4 + \langle E_1, E_2\rangle = \Gamma_{5,\alpha,0}$. On the other hand, if $a_3' \neq 0$, then we have that

$$\psi_5 = \mathrm{diag}\left(\frac{1}{\sqrt{\mathrm{sgn}(a_3')a_3'}}, \frac{1}{\sqrt{\mathrm{sgn}(a_3')a_3'}}, \frac{1}{\mathrm{sgn}(a_3')a_3'}, 1\right) \in d\mathrm{Aut}\,\mathsf{H}_3^\diamond(n)$$

and get that $\psi_5 \cdot \psi_4 \cdot \psi \cdot \Gamma = \mathrm{sgn}(a_3')E_3 + \alpha E_4 + \langle E_1, E_2\rangle = \sigma E_3 + \alpha E_4 + \langle E_1, E_2\rangle = \Gamma_{5,\alpha,\sigma}$ for some $\sigma \in \{-1, 1\}$.

Case: $\mathfrak{T}(\Gamma) = 0$

We wish to show that $\Gamma \sim \Gamma_{6,\alpha} = \alpha E_4 + \langle E_1, E_3\rangle$. By the preliminary discussion we have that there exists a $\psi \in d\mathrm{Aut}\,\mathsf{H}_3^\diamond(n)$ and constants a_3, b_3, c_2, c_3 such that $\psi \cdot \Gamma = a_3E_3 + \alpha E_4 + \langle E_1 + b_3E_3, c_2E_2 + c_3E_3\rangle$. Now as $\mathfrak{T}(\Gamma) = 0$ we have that $|\mathrm{sgn}(c_2)| = 0$, i.e., $c_2 = 0$. Thus $\psi \cdot \Gamma = a_3E_3 + \alpha E_4 + \langle E_1 + b_3E_3, c_3E_3\rangle$. Then, as $\dim \Gamma^0 = 2$, we have that $c_3 \neq 0$. So we have that $\psi \cdot \Gamma = a_3E_3 + \alpha E_4 + \langle E_1 + b_3E_3, E_3\rangle = \alpha E_4 + \langle E_1, E_3\rangle = \Gamma_{6,\alpha}$.

Now as our list is exhaustive we are left to show that none of the affine subspaces

$$\begin{aligned} \Gamma_{1,\alpha,\sigma} &= \alpha_1E_2 + \alpha_2E_3 + \langle E_1, \sigma E_3 + E_4\rangle & \alpha_1, \alpha_2 \geq 0, \alpha \neq 0, \sigma \in \{-1, 1\} \\ \Gamma_{2,\alpha} &= E_2 + \alpha E_3 + \langle E_1, E_4\rangle, & \alpha \geq 0 \\ \Gamma_3 &= E_3 + \langle E_1, E_4\rangle & \\ \Gamma_4 &= E_1 + \langle E_3, E_4\rangle & \\ \Gamma_{5,\alpha,\sigma} &= \sigma E_3 + \alpha E_4 + \langle E_1, E_2\rangle & \alpha > 0, \sigma \in \{-1, 0, 1\} \\ \Gamma_{6,\alpha} &= \alpha E_4 + \langle E_1, E_3\rangle & \alpha > 0 \end{aligned}$$

are $\mathfrak{L}$-related. This is done by assuming that they are, giving us an element ψ of $d\mathrm{Aut}\,\mathsf{H}_3^\diamond(n)$ taking one to the other, and producing a contradiction from this.

$\Gamma_{1,\alpha,\sigma} \not\sim \Gamma_{1,\beta,\varsigma}$**:** Specifically we wish to show that

$$\Big(\big((\alpha,\sigma) \neq (\beta,\varsigma)\big) \wedge (\alpha_i, \beta_i \geq 0, i = \overline{1,2}) \wedge (\alpha, \beta \neq 0) \wedge (\sigma, \varsigma \in \{-1,1\})\Big) \Rightarrow \Gamma_{1,\alpha,\sigma} \not\sim \Gamma_{1,\beta,\varsigma}.$$

Assume there exists a $\psi \in d\mathrm{Aut}\,\mathsf{H}_3^\diamond(n)$ such that $\psi \cdot \Gamma_{1,\alpha,\sigma} = \Gamma_{1,\beta,\varsigma}$. Then, as $\alpha_1 E_2 + \alpha_2 E_3 \in \Gamma_{1,\alpha,\sigma}$ and $E_1, \sigma E_3 + E_4 \in \Gamma^0_{1,\alpha,\sigma}$, we get that $\psi \cdot (\alpha_1 E_2 + \alpha_2 E_3) \in \Gamma_{1,\beta,\varsigma}$ and $\psi \cdot E_1, \psi \cdot (\sigma E_3 + E_4) \in \Gamma^0_{1,\beta,\varsigma}$. That is to say there exists constants $r_i \in \mathbb{R}$, $i = \overline{1,6}$, such that

$$\begin{bmatrix} y\alpha_1 \\ kx\alpha_1 \\ (vx + kuy)\alpha_1 + k\left(x^2+y^2\right)\alpha_2 \\ 0 \end{bmatrix} = \begin{bmatrix} r_1 \\ \beta_1 \\ \varsigma r_2 + \beta_2 \\ r_2 \end{bmatrix}, \qquad \begin{bmatrix} x \\ -ky \\ kux - vy \\ 0 \end{bmatrix} = \begin{bmatrix} r_3 \\ 0 \\ \varsigma r_4 \\ r_4 \end{bmatrix},$$
$$\begin{bmatrix} u \\ v \\ \frac{1}{2}k\left(u^2+v^2\right) + k\left(x^2+y^2\right)\sigma \\ k \end{bmatrix} = \begin{bmatrix} r_5 \\ 0 \\ \varsigma r_6 \\ r_6 \end{bmatrix}.$$

From the second equation we get that $y = 0$ (and so $x \neq 0$), $r_4 = 0$ and thus $u = 0$. Then, from the third equation we get that $v = 0$, $r_6 = k$ and so $kx^2\sigma = k\varsigma$. Now as $\sigma, \varsigma \in \{-1,1\}$ we therefore have that $x = \pm 1$ and $\sigma = \varsigma$. Consequently, from the first equation we then get that $r_2 = 0$, $kx\alpha_1 = \beta_1$ and $k\alpha_2 = \beta_2$. But as $k, kx \in \{-1,1\}$ and $\alpha_i, \beta_i \geq 0$, we then get that $\alpha_i = \beta_i$ for $i = \overline{1,2}$.

$\Gamma_{1,\alpha,\sigma} \not\sim \Gamma_{2,\beta}, \Gamma_3$**:** Assume there exists a $\psi \in d\mathrm{Aut}\,\mathsf{H}_3^\diamond(n)$ such that $\psi \cdot \Gamma^0_{2,\alpha,\sigma} = \Gamma^0_{2,\beta} = \Gamma^0_3$. Then, as $E_1, \sigma E_3 + E_4 \in \Gamma^0_{1,\alpha,\sigma}$, we get that $\psi \cdot (\sigma E_3 + E_4) \in \Gamma^0_{2,\beta} = \Gamma^0_3$. That is to say there exists constants $r_i \in \mathbb{R}$, $i = \overline{1,4}$, such that

$$\begin{bmatrix} x \\ -ky \\ kux - vy \\ 0 \end{bmatrix} = \begin{bmatrix} r_1 \\ 0 \\ 0 \\ r_2 \end{bmatrix}, \qquad \begin{bmatrix} u \\ v \\ \frac{1}{2}k\left(u^2+v^2\right) + k\left(x^2+y^2\right)\sigma \\ k \end{bmatrix} = \begin{bmatrix} r_3 \\ 0 \\ 0 \\ r_4 \end{bmatrix}.$$

Thus $y = 0$ (and so $x \neq 0$), $u = 0$, $v = 0$ and thus $kx^2\sigma = 0$. Hence $\sigma = 0$, a contradiction.

$\Gamma_{1,\alpha,\sigma}, \Gamma_{2,\alpha}, \Gamma_3 \not\sim \Gamma_4$**:** For the first three $\dim \mathrm{Lie}\,\Gamma^0 = 4$, but $\dim \mathrm{Lie}\,\Gamma^0_4 = 2$. As $\dim \mathrm{Lie}\,\Gamma^0$ is preserved by automorphisms (i.e., $\dim \mathrm{Lie}\,(\psi \cdot \Gamma^0) = \dim \mathrm{Lie}\,\Gamma^0$), the result follows.

$\Gamma_{1,\alpha,\sigma}, \Gamma_{2,\alpha}, \Gamma_3, \Gamma_4 \not\sim \Gamma_{5,\beta,\varsigma}, \Gamma_{6,\beta}$**:** For the first four cases $\pi_4(\Gamma^0) \neq \{0\}$ and for the last two $\pi_4(\Gamma^0) = \{0\}$. The result then follows (as it did in proposition 2.4.5).

$\big((\alpha \neq \beta) \wedge (\alpha, \beta \geq 0)\big) \Rightarrow \Gamma_{2,\alpha} \not\sim \Gamma_{2,\beta}$**:** Assume there exists a $\psi \in d\mathrm{Aut}\,\mathsf{H}_3^\diamond(n)$ such that $\psi \cdot \Gamma_{2,\alpha} = \Gamma_{2,\beta}$. Then, as $E_2 + \alpha E_3 \in \Gamma_{2,\alpha}$ and $E_1, E_4 \in \Gamma^0_{2,\alpha}$, we get that $\psi \cdot (E_2 + \alpha E_3) \in \Gamma_{2,\beta}$ and $\psi \cdot E_1, \psi \cdot E_4 \in \Gamma^0_{2,\beta}$. That is to say there exists constants $r_i \in \mathbb{R}$, $i = \overline{1,6}$, such that

$$\begin{bmatrix} y \\ kx \\ vx + kuy + k\left(x^2+y^2\right)\alpha \\ 0 \end{bmatrix} = \begin{bmatrix} r_1 \\ 1 \\ \beta \\ r_2 \end{bmatrix}, \quad \begin{bmatrix} x \\ -ky \\ kux - vy \\ 0 \end{bmatrix} = \begin{bmatrix} r_3 \\ 0 \\ 0 \\ r_4 \end{bmatrix}, \quad \begin{bmatrix} u \\ v \\ \frac{1}{2}k\left(u^2+v^2\right) \\ k \end{bmatrix} = \begin{bmatrix} r_5 \\ 0 \\ 0 \\ r_6 \end{bmatrix}.$$

From the third equation we get that $u = v = 0$. Then from the second equation we get that $y = 0$. Therefore, from the first equation, we get that $x = \pm 1$ and hence that $k\alpha = \beta$. But as $\alpha, \beta \geq 0$, we get that $\alpha = \beta$.

$\Gamma_{2,\alpha} \nsim \Gamma_3$: Assume $\psi \in d\mathrm{Aut}\,\mathsf{H}_3^\diamond(n)$ and that $\psi \cdot \Gamma_3 = \Gamma_{2,\alpha}$. Now as $E_3 \in \Gamma_3$ we have that $\psi \cdot E_3 \in \Gamma_{2,\alpha}$. But, as E_3 is an eigenvector of any automorphism, we have that $\pi_2(\psi \cdot E_3) = 0$. This contradicts the fact that $\pi_2(A) = 1$ for any $A \in \Gamma_{2,\alpha}$.

$\Gamma_{5,\alpha,\sigma} \nsim \Gamma_{5,\beta,\varsigma}$: Specifically we wish to show that

$$\Big(\big((\alpha,\sigma) \neq (\beta,\varsigma)\big) \wedge (\alpha,\beta > 0) \wedge (\sigma,\varsigma \in \{-1,0,1\})\Big) \Rightarrow \Gamma_{5,\alpha,\sigma} \nsim \Gamma_{5,\beta,\varsigma}.$$

Assume there exists a $\psi \in d\mathrm{Aut}\,\mathsf{H}_3^\diamond(n)$ such that $\psi \cdot \Gamma_{5,\alpha,\sigma} = \Gamma_{5,\beta,\varsigma}$. Then, as $\sigma E_3 + \alpha E_4 \in \Gamma_{5,\alpha,\sigma}$ and $E_1, E_2 \in \Gamma^0_{5,\alpha,\sigma}$, we get that $\psi \cdot (\sigma E_3 + \alpha E_4) \in \Gamma_{5,\beta,\varsigma}$ and $\psi \cdot E_1, \psi \cdot E_2 \in \Gamma^0_{5,\beta,\varsigma}$. That is to say there exists constants $r_i \in \mathbb{R}$, $i = \overline{1,6}$, such that

$$\begin{bmatrix} u\alpha \\ v\alpha \\ \frac{1}{2}k\left(u^2+v^2\right)\alpha + k\left(x^2+y^2\right)\sigma \\ k\alpha \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \\ \varsigma \\ \beta \end{bmatrix}, \quad \begin{bmatrix} x \\ -ky \\ kux - vy \\ 0 \end{bmatrix} = \begin{bmatrix} r_3 \\ r_4 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} y \\ kx \\ vx + kuy \\ 0 \end{bmatrix} = \begin{bmatrix} r_5 \\ r_6 \\ 0 \\ 0 \end{bmatrix}.$$

From the second and third equations we get that

$$\begin{cases} kux - vy = 0 \\ vx + kuy = 0 \end{cases} \quad \Rightarrow \quad \begin{bmatrix} kx & -y \\ ky & x \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = 0 \quad \Rightarrow \quad u = v = 0.$$

Hence, returning to the first equation, we find that $k\alpha = \beta$. Thus (as $\alpha, \beta > 0$) we get that $k = 1$ and $\alpha = \beta$. Therefore (again from the first equation) we have that $(x^2+y^2)\sigma = \varsigma$. But, as $x^2 + y^2 > 0$ and $\sigma, \varsigma \in \{-1,0,1\}$, this then implies that $\sigma = \varsigma$.

$\Gamma_{5,\alpha,\sigma} \nsim \Gamma_{6,\beta}$: Suppose there exists a $\psi \in d\mathrm{Aut}\,\mathsf{H}_3^\diamond(n)$ such that $\psi \cdot \Gamma_{5,\alpha,\sigma} = \Gamma_{6,\beta}$. Then we get that $1 = \mathfrak{T}(\Gamma_{5,\alpha,\sigma}) = \mathfrak{T}(\psi \cdot \Gamma_{5,\alpha,\sigma}) = \mathfrak{T}(\Gamma_{6,\beta}) = 0$, a contradiction.

$\big((\alpha \neq \beta) \wedge (\alpha,\beta > 0)\big) \Rightarrow \Gamma_{6,\alpha} \nsim \Gamma_{6,\beta}$: Assume there exists a $\psi \in d\mathrm{Aut}\,\mathsf{H}_3^\diamond(n)$ such that $\psi \cdot \Gamma_{6,\alpha} = \Gamma_{6,\beta}$. Then, as $\alpha E_4 \in \Gamma_{6,\alpha}$, we get that $\psi \cdot \alpha E_4 \in \Gamma_{6,\beta}$. That is to say there exists constants $r_1, r_2 \in \mathbb{R}$, such that $k\alpha E_4 = \beta E_4 + r_1 E_1 + r_2 E_3$. Then, as $\alpha, \beta > 0$, equating the coefficients of E_4 yields $\alpha = \beta$. □

Dimension three, homogeneous

3.3.6 Proposition. *Any $(3,0)$-affine subspace $\Gamma = \Gamma^0 \subset \mathfrak{h}_3^\diamond(n)$ (of full rank) is $\mathfrak{L}$-related to exactly one of the following affine subspaces:*

Case: $\mathfrak{R}(\Gamma) = 1$: $\Gamma^{(3,0)}_{1,\sigma} = \langle E_1, E_2, \sigma E_3 + E_4 \rangle$, $\sigma \in \{-1,0,1\}$

Case: $\mathfrak{R}(\Gamma) = 0$: $\Gamma^{(3,0)}_{2} = \langle E_1, E_3, E_4 \rangle$.

Proof. (We omit the superscript $(3,0)$ in this proof.) By proposition 3.3.5 and corollary 3.1.5, Γ is $\mathfrak{L}$-related to one of the following affine subspaces (with $\mathfrak{R}(\Gamma)$ given alongside)

$$\left\langle\Gamma^{(2,1)}_{1,\alpha,\sigma}\right\rangle = \langle \alpha_1 E_2 + \alpha_2 E_3, E_1, \sigma E_3 + E_4\rangle, \quad \alpha_1,\alpha_2 \geq 0, \alpha \neq 0, \sigma \in \{-1,1\} \qquad \mathfrak{R}(\langle\Gamma^{(2,1)}_{1,\alpha,\sigma}\rangle) = |\mathrm{sgn}(\alpha_1)|$$

$$\left\langle\Gamma^{(2,1)}_{2,\alpha}\right\rangle = \langle E_2 + \alpha E_3, E_1, E_4\rangle, \ \alpha \geq 0 \qquad \mathfrak{R}(\langle\Gamma^{(2,1)}_{2,\alpha}\rangle) = 1$$

$$\left\langle\Gamma^{(2,1)}_{3}\right\rangle = \left\langle\Gamma^{(2,1)}_{4}\right\rangle = \langle E_1, E_3, E_4\rangle \qquad \mathfrak{R}(\langle\Gamma^{(2,1)}_{3}\rangle) = 0$$

$$\left\langle\Gamma^{(2,1)}_{5,\alpha,\sigma}\right\rangle = \langle \sigma E_3 + \alpha E_4, E_1, E_2\rangle, \ \alpha > 0, \sigma \in \{-1,0,1\} \qquad \mathfrak{R}(\langle\Gamma^{(2,1)}_{5,\alpha,\sigma}\rangle) = 1$$

$$\left\langle\Gamma^{(2,1)}_{6,\alpha}\right\rangle = \langle \alpha E_4, E_1, E_3\rangle, \ \alpha > 0 \qquad \mathfrak{R}(\langle\Gamma^{(2,1)}_{6,\alpha}\rangle) = 0.$$

Assume $\mathfrak{R}(\Gamma) = 0$. Then as $\mathfrak{R}$ is invariant we have that Γ is $\mathfrak{R}$-related to $\left\langle\Gamma^{(2,1)}_{1,(0,\alpha_2),\sigma}\right\rangle$ (with $\alpha_2 \neq 0$), $\left\langle\Gamma^{(2,1)}_{3}\right\rangle$ or $\left\langle\Gamma^{(2,1)}_{6,\alpha}\right\rangle$. But $\left\langle\Gamma^{(2,1)}_{1,(0,\alpha_2),\sigma}\right\rangle = \left\langle\Gamma^{(2,1)}_{3}\right\rangle = \left\langle\Gamma^{(2,1)}_{6,\alpha}\right\rangle = \langle E_1, E_3, E_4\rangle$. Thus Γ is $\mathfrak{L}$-related to $\langle E_1, E_3, E_4\rangle$. On the other hand assume $\mathfrak{R}(\Gamma) = 1$. Then Γ is $\mathfrak{L}$-related to $\left\langle\Gamma^{(2,1)}_{1,\alpha,\sigma}\right\rangle$ (with $\alpha_1 \neq 0$), $\left\langle\Gamma^{(2,1)}_{2,\alpha}\right\rangle$ or $\left\langle\Gamma^{(2,1)}_{5,\alpha,\sigma}\right\rangle$. Now notice that $\left\langle\Gamma^{(2,1)}_{1,\alpha,\sigma}\right\rangle = \left\langle E_1, E_2 + \frac{\alpha_2}{\alpha_1}E_3, E_4 + \sigma E_3\right\rangle$, $\left\langle\Gamma^{(2,1)}_{2,\alpha}\right\rangle = \langle E_1, E_2 + \alpha E_3, E_4\rangle$ and $\left\langle\Gamma^{(2,1)}_{5,\alpha,\sigma}\right\rangle = \left\langle E_1, E_2, E_4 + \frac{\sigma}{\alpha}E_3\right\rangle$. Therefore, there exists a $\psi_1 \in d\mathrm{Aut}\,\mathsf{H}_3^\diamond(n)$ such that $\psi_1 \cdot \Gamma = \langle E_1, E_2 + bE_3, cE_3 + E_4\rangle$ for some constants $b, c, \in \mathbb{R}$. Then

$$\psi_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -b \\ 0 & -b & 1 & \frac{1}{2}b^2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in d\mathrm{Aut}\,\mathsf{H}_3^\diamond(n)$$

and $\psi_2 \cdot \psi_1 \cdot \Gamma = \left\langle E_1, E_2 - bE_3 + bE_3, cE_3 + (-bE_2 + \frac{1}{2}b^2 E_3 + E_4)\right\rangle = \langle E_1, E_2, c'E_3 + E_4\rangle$ for some new constant $c' \in \mathbb{R}$. If $c' = 0$, then $\psi_2 \cdot \psi_1 \cdot \Gamma = \Gamma_{1,0}$ and we are done. On the other hand, if $c' \neq 0$, then

$$\psi_3 = \mathrm{diag}\left(\sqrt{\tfrac{\mathrm{sgn}(c')}{c'}}, \sqrt{\tfrac{\mathrm{sgn}(c')}{c'}}, \tfrac{\mathrm{sgn}(c')}{c'}, 1\right) \in d\mathrm{Aut}\,\mathsf{H}_3^\diamond(n)$$

and so $\psi_3 \cdot \psi_2 \cdot \psi_1 \cdot \Gamma = \langle E_1, E_2, \mathrm{sgn}(c_3)E_3 + E_4\rangle = \Gamma_{1,\sigma}$ for some $\sigma \in \{-1,1\}$.

We are left to show that $\Gamma_{1,-1}$, $\Gamma_{1,0}$, $\Gamma_{1,1}$ and Γ_2 are all distinct (i.e., no two are $\mathfrak{L}$-related).

$\big((\sigma \neq \varsigma) \wedge (\sigma, \varsigma \in \{-1,0,1\})\big) \Rightarrow \Gamma_{1,\sigma} \nsim \Gamma_{1,\varsigma}$**:** Suppose $\psi \in d\mathrm{Aut}\,\mathsf{H}_3^\diamond(n)$ and $\psi \cdot \Gamma_{1,\sigma} = \Gamma_{1,\varsigma}$. Then, as $E_1, E_2, \sigma E_3 + E_4 \in \Gamma_{1,\sigma}$, we have that $\psi \cdot E_1, \psi \cdot E_3, \psi \cdot (\sigma E_3 + E_4) \in \Gamma_{2,\varsigma}$. That is to say, there exist $r_i \in \mathbb{R}$, $i = \overline{1,9}$ such that

$$\begin{bmatrix} x \\ -ky \\ kux - vy \\ 0 \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \\ \varsigma r_3 \\ r_3 \end{bmatrix}, \quad \begin{bmatrix} y \\ kx \\ vx + kuy \\ 0 \end{bmatrix} = \begin{bmatrix} r_4 \\ r_5 \\ \varsigma r_6 \\ r_6 \end{bmatrix}, \quad \begin{bmatrix} u \\ v \\ \frac{1}{2}k\left(u^2+v^2\right) + k\left(x^2+y^2\right)\sigma \\ k \end{bmatrix} = \begin{bmatrix} r_7 \\ r_8 \\ \varsigma r_9 \\ r_9 \end{bmatrix}.$$

From the first and second equations we get (as $r_3 = 0$, $r_6 = 0$ and $x^2 + y^2 \neq 0$) that

$$\begin{cases} kux - vy = 0 \\ vx + kuy = 0 \end{cases} \Rightarrow \begin{bmatrix} kx & -y \\ ky & x \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = 0 \Rightarrow u = v = 0.$$

Hence from the third equation we get that $k(x^2+y^2)\sigma = k\sigma$ and so $(x^2+y^2)\sigma = \varsigma$. But, as $x^2+y^2>0$ and $\sigma, \varsigma \in \{-1,0,1\}$, this implies that $\sigma = \varsigma$.

$\Gamma_{1,s} \nsim \Gamma_2$: As $\mathfrak{R}$ is invariant under automorphisms, $\Gamma_{1,\sigma}$ and Γ_2 are distinct. (If $\psi \in \operatorname{Aut}\mathfrak{h}_3^\diamond$ and $\psi \cdot \Gamma_{1,\sigma} = \Gamma_2$, then $1 = \mathfrak{R}(\Gamma_{1,\sigma}) = \mathfrak{R}(\psi \cdot \Gamma_{1,\sigma}) = \mathfrak{R}(\Gamma_2) = 0$, a contradiction.) □

Dimension three, inhomogeneous

3.3.7 PROPOSITION. *Any $(3,1)$-affine subspace $\Gamma = A + \Gamma^0 \subset \mathfrak{h}_3^\diamond(n)$ (of full rank) is $\mathfrak{L}$-related to exactly one of the following affine subspaces:*

Case: $\pi_4(\Gamma^0) \neq \{0\}$

Case: $\mathfrak{R}(\Gamma^0) = 0$

$\Gamma_1^{(3,1)} = E_2 + \langle E_1, E_3, E_4\rangle$

Case: $\mathfrak{R}(\Gamma^0) = 1$

$\Gamma_{2,\alpha,\sigma}^{(3,1)} = \alpha E_3 + \langle E_1, E_2, \sigma E_3 + E_4\rangle, \quad \alpha > 0, \sigma \in \{-1,1\}$

$\Gamma_3^{(3,1)} = E_3 + \langle E_1, E_2, E_4\rangle$

Case: $\pi_4(\Gamma^0) = \{0\}$**,** $\pi_4(A) = \pm\alpha$**,** $\alpha > 0$

$\Gamma_{4,\alpha}^{(3,1)} = \alpha E_4 + \langle E_1, E_2, E_3\rangle.$

PROOF. (We omit the superscript $(3,1)$ in this proof.) By the same argument used in the proof of proposition 2.6.1, we get that: if $\pi_4(\Gamma_0) \neq 0$, then Γ^0 is a $(3,0)$-affine subspace of full rank. So, if $\pi_4(\Gamma^0) \neq \{0\}$, we have that $\mathfrak{R}(\Gamma^0)$ is defined and we may use the results of proposition 3.3.6.

Case: $\pi_4(\Gamma^0) \neq \{0\}$**,** $\mathfrak{R}(\Gamma^0) = \{0\}$

We wish to show that $\Gamma \sim \Gamma_1 = E_2 + \langle E_1, E_3, E_4\rangle$. Now as $\mathfrak{R}(\Gamma^0) = \{0\}$ and Γ^0 is of full rank, there exists an element $\psi_1 \in d\operatorname{Aut}\mathsf{H}_3^\diamond(n)$ such that $\psi_1 \cdot \Gamma^0 = \langle E_1, E_3, E_4\rangle$. Thus we have that $\psi_1 \cdot \Gamma = aE_2 + \langle E_1, E_3, E_4\rangle$ for some constant $a \in \mathbb{R}$. Then we have that $\psi_2 = \operatorname{diag}\left(\frac{1}{a}, \frac{1}{a}, \frac{1}{a^2}, 1\right) \in d\operatorname{Aut}\mathsf{H}_3^\diamond(n)$ and get that $\psi_2 \cdot \psi_1 \cdot \Gamma = E_2 + \langle E_1, E_3, E_4\rangle = \Gamma_1$.

Case: $\pi_4(\Gamma^0) \neq \{0\}$**,** $\mathfrak{R}(\Gamma^0) = 1$

We wish to show that Γ is $\mathfrak{L}$-related to either $\Gamma_{2,\alpha,\sigma}^{(3,1)} = \alpha E_3 + \langle E_1, E_2, \sigma E_3 + E_4\rangle$ for some $\alpha > 0$ and $\sigma \in \{-1,1\}$, or $\Gamma_3^{(3,1)} = E_3 + \langle E_1, E_2, E_4\rangle$. Now, as $\mathfrak{R}(\Gamma^0) = 1$ and Γ^0 is of full rank, there exists an element $\psi_1 \in d\operatorname{Aut}\mathsf{H}_3^\diamond(n)$ such that $\psi_1 \cdot \Gamma^0 = \langle E_1, E_2, \sigma E_3 + E_4\rangle$, for some $\sigma \in \{-1,0,1\}$. Thus we have that $\psi_1 \cdot \Gamma = aE_3 + \langle E_1, E_2, \sigma E_3 + E_4\rangle$ for some constant $a \in \mathbb{R}$. Now as Γ is inhomogeneous, we have that $a \neq 0$. Thus we have that $\psi_2 = \operatorname{diag}(1, \operatorname{sgn}(a), \operatorname{sgn}(a), \operatorname{sgn}(a)) \in d\operatorname{Aut}\mathsf{H}_3^\diamond(n)$ and get that $\psi_2 \cdot \psi_1 \cdot \Gamma = \operatorname{sgn}(a)aE_3 + \langle E_1, E_2, \operatorname{sgn}(a)(\sigma E_3 + E_4)\rangle = \alpha E_3 + \langle E_1, E_2, \sigma E_3 + E_4\rangle$ for some $\alpha > 0$. If $\sigma \neq 0$, then $\psi_2 \cdot \psi_1 \cdot \Gamma = \Gamma_{2,\alpha,\sigma}$. On the other hand, if $\sigma = 0$, then $\psi_2 \cdot \psi_1 \cdot \Gamma = \alpha E_3 + \langle E_1, E_2, E_4\rangle$. Hence

$$\psi_3 = \operatorname{diag}\left(\frac{\operatorname{sgn}(\alpha)}{\sqrt{\operatorname{sgn}(\alpha)\alpha}}, \frac{1}{\sqrt{\operatorname{sgn}(\alpha)\alpha}}, \frac{1}{\alpha}, \operatorname{sgn}(\alpha)\right) \in d\operatorname{Aut}\mathsf{H}_3^\diamond(n)$$

and so $\psi_3 \cdot \psi_2 \cdot \psi_1 \cdot \Gamma = E_3 + \langle E_1, E_2, E_3\rangle = \Gamma_3$.

Case: $\pi_4(\Gamma^0) = \{0\}$**,** $\pi_4(A) = \pm\alpha,\ \alpha > 0$

We wish to show that $\Gamma \sim \Gamma_{4,\alpha} = \alpha E_4 + \langle E_1, E_2, E_3\rangle$. Now, as $\pi_4(\Gamma^0) = \{0\}$ and $\Gamma^0 \subset \langle E_1, E_2, E_3, E_4\rangle$, we get that $\Gamma^0 = \langle E_1, E_2, E_3\rangle$. Hence $\Gamma = aE_4 + \langle E_1, E_2, E_3\rangle$ for some $a \in \mathbb{R}$. We note that as $\pi_4(A) = \pm\alpha$, $\alpha > 0$ ($\alpha \neq 0$ required else Γ would be homogeneous), we have that $|a_4| = \alpha$. So then $\psi_1 = \operatorname{diag}(1, \operatorname{sgn}(a), \operatorname{sgn}(a_4), \operatorname{sgn}(a_4)) \in d\mathrm{Aut}\,\mathsf{H}_3^\circ(n)$ and $\psi_1 \cdot \Gamma = |a_4|E_4 + \langle E_1, E_2, E_3\rangle = \alpha E_4 + \langle E_1, E_2, E_3\rangle = \Gamma_{4,\alpha}$.

Now as our list is exhaustive we are left to show that none of the affine subspaces

$$\begin{aligned}
\Gamma_1 &= E_2 + \langle E_1, E_3, E_4\rangle \\
\Gamma_{2,\alpha,\sigma} &= \alpha E_3 + \langle E_1, E_2, \sigma E_3 + E_4\rangle \qquad && \alpha > 0, \sigma \in \{-1, 1\} \\
\Gamma_3 &= E_3 + \langle E_1, E_2, E_4\rangle \\
\Gamma_{4,\alpha} &= \alpha E_4 + \langle E_1, E_2, E_3\rangle && \alpha > 0
\end{aligned}$$

are $\mathfrak{L}$-related. This is done by assuming that they are, giving us an element $\psi \in d\mathrm{Aut}\,\mathsf{H}_3^\circ(n)$ taking one to the other, and producing a contradiction from this.

$\Gamma_1 \nsim \Gamma_{2,\alpha,\sigma}, \Gamma_3$**:** This follows as $\mathfrak{R}(\Gamma_1^0) = 0$ and $\mathfrak{R}(\Gamma_{2,\alpha,\sigma}^0) = \mathfrak{R}(\Gamma_3^0) = 1$. (That is, the associated linear subspaces are not related, as in proposition 3.3.6.)

$\Gamma_1, \Gamma_{2,\alpha,\sigma}, \Gamma_3 \nsim \Gamma_4$**:** For the first three cases $\pi_4(\Gamma^0) \neq \{0\}$, but $\pi_4(\Gamma_4^0) = \{0\}$. The result then follows (as it did in proposition 2.5.3).

$\Gamma_{2,\alpha,\sigma} \nsim \Gamma_{2,\beta,\varsigma}$**:** Specifically we wish to show that

$$\Big(\big((\alpha, \sigma) \neq (\beta, \varsigma)\big) \wedge (\alpha, \beta > 0) \wedge (\sigma, \varsigma \in \{-1, 1\})\Big) \Rightarrow \Gamma_{2,\alpha,\sigma} \nsim \Gamma_{2,\beta,\varsigma}.$$

Assume there exists a $\psi \in d\mathrm{Aut}\,\mathsf{H}_3^\circ(n)$ such that $\psi \cdot \Gamma_{2,\alpha,\sigma} = \Gamma_{2,\beta,\varsigma}$. Then, as $\alpha E_3 \in \Gamma_{2,\alpha,\Sigma}$ and $E_1, E_2, \sigma E_3, E_4 \in \Gamma_{2,\alpha,\sigma}^0$, we get that $\psi \cdot \alpha E_3 \in \Gamma_{2,\beta,\varsigma}$ and $\psi \cdot E_1, \psi \cdot E_2, \psi \cdot (\sigma E_3 + E_4) \in \Gamma_{2,\beta,\varsigma}^0$. That is to say (by equating the coefficients of E_3 and E_4) there exists constants $r_i \in \mathbb{R}$, $i = \overline{1,4}$, such that

$$\begin{bmatrix} k\left(x^2 + y^2\right)\alpha \\ 0 \end{bmatrix} = \begin{bmatrix} \beta + \varsigma r_1 \\ r_1 \end{bmatrix}, \qquad \begin{bmatrix} kux - vy \\ 0 \end{bmatrix} = \begin{bmatrix} \varsigma r_2 \\ r_2 \end{bmatrix},$$

$$\begin{bmatrix} vx + kuy \\ 0 \end{bmatrix} = \begin{bmatrix} \varsigma r_3 \\ r_3 \end{bmatrix}, \qquad \begin{bmatrix} \frac{1}{2}k\left(u^2 + v^2\right) + k\left(x^2 + y^2\right)\sigma \\ k \end{bmatrix} = \begin{bmatrix} \varsigma r_4 \\ r_4 \end{bmatrix}.$$

From the second and third equation we get that $kux - vy = 0$ and $vx + kuy$, respectively. Hence we have that $\begin{bmatrix} kx & ky \\ -y & x \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = 0$, implying $u = v = 0$. Then from the fourth equation we get that $k\left(x^2 + y^2\right)\sigma = k\varsigma$ and so $\left(x^2 + y^2\right)\sigma = \varsigma$. Now as $x^2 + y^2 > 0$ and $\sigma, \varsigma \in \{-1, 1\}$, this implies that $\sigma = \varsigma$ and $x^2 + y^2 = 1$. Then from the first equation we get that $k\alpha = \beta$. But as $\alpha, \beta > 0$, this implies that $\alpha = \beta$.

$\Gamma_{2,\alpha,\sigma} \nsim \Gamma_3$**:** Notice that $\Gamma_{2,\alpha,\sigma}^0 = \Gamma_{1,\sigma}^{(3,0)}$ (for some $\sigma \in \{-1, 1\}$) and $\Gamma_3^0 = \Gamma_{1,0}^{(3,0)}$. But, by proposition 3.3.6, we have that $\Gamma_{1,\sigma}^{(3,0)} \nsim \Gamma_{1,0}^{(3,0)}$.

$(\alpha \neq \beta \wedge \alpha, \beta > 0) \Rightarrow \Gamma_{4,\alpha} \nsim \Gamma_{4,\beta}$**:** Assume there exists a $\psi \in d\mathrm{Aut}\,\mathsf{H}_3^\diamond(n)$ such that $\psi \cdot \Gamma_{4,\alpha} = \Gamma_{4,\beta}$. Then as $\alpha E_4 \in \Gamma_{4,\alpha}$ we have that $\psi \cdot \alpha E_4 \in \Gamma_{4,\beta}$. Thus we get that $\pi_4(\psi \cdot \alpha E_4) = k\alpha = \pm\alpha$. But $\pi_4(B) = \beta$ for $B \in \Gamma_{3,\beta}$. Hence $\alpha = \beta$. □

Summary

We collect our results in a theorem.

3.3.8 THEOREM. *Any affine subspace Γ of $\mathfrak{h}_3^\diamond(n)$ (of full rank) is $\mathfrak{L}$-related (w.r.t. $\mathsf{H}_3^\diamond(n)$) to exactly one of the following affine subspaces:*

$$\Gamma^{(1,1)}_{1,\alpha} = E_1 + \alpha_1 E_3 + \langle \alpha_2 E_3 + E_4 \rangle\,,\ \alpha_1 \geq 0, \alpha_2 \in \mathbb{R} \qquad \Gamma^{(1,1)}_{2,\alpha,\sigma} = \sigma E_3 + \alpha E_4 + \langle E_1 \rangle\,,\ \alpha > 0$$
$$\Gamma^{(2,0)}_{1,\sigma} = \langle E_1, \sigma E_3 + E_4 \rangle \qquad \Gamma^{(2,1)}_{1,\alpha,\varsigma} = \alpha_1 E_2 + \alpha_2 E_3 + \langle E_1, \varsigma E_3 + E_4 \rangle\,,\ \alpha_i \geq 0$$
$$\Gamma^{(2,1)}_{2,\alpha} = E_2 + \alpha E_3 + \langle E_1, E_4 \rangle\ \ \alpha \geq 0 \qquad \Gamma^{(2,1)}_{3} = E_3 + \langle E_1, E_4 \rangle$$
$$\Gamma^{(2,1)}_{4} = E_1 + \langle E_3, E_4 \rangle \qquad \Gamma^{(2,1)}_{5,\alpha,\sigma} = \sigma E_3 + \alpha E_4 + \langle E_1, E_2 \rangle\,,\ \alpha > 0$$
$$\Gamma^{(2,1)}_{6,\alpha} = \alpha E_4 + \langle E_1, E_3 \rangle\,,\ \alpha > 0 \qquad \Gamma^{(3,0)}_{1,\sigma} = \langle E_1, E_2, \sigma E_3 + E_4 \rangle$$
$$\Gamma^{(3,0)}_{2} = \langle E_1, E_3, E_4 \rangle \qquad \Gamma^{(3,1)}_{1} = E_2 + \langle E_1, E_3, E_4 \rangle$$
$$\Gamma^{(3,1)}_{2,\alpha,\varsigma} = \alpha E_3 + \langle E_1, E_2, \varsigma E_3 + E_4 \rangle\ \ \alpha > 0 \qquad \Gamma^{(3,1)}_{3} = E_3 + \langle E_1, E_2, E_4 \rangle$$
$$\Gamma^{(3,1)}_{4,\alpha} = \alpha E_4 + \langle E_1, E_2, E_3 \rangle\,,\ \alpha > 0 \qquad \Gamma^{(4,0)}_{1} = \langle E_1, E_2, E_3, E_4 \rangle\,,$$

where $\sigma \in \{-1, 0, 1\}$, $\varsigma \in \{-1, 1\}$ and for $\Gamma^{(2,1)}_{1,\alpha,\varsigma}$ we additionally require that $\alpha \neq 0$.

3.3.9 COROLLARY. *Any proper system Σ, with state space Lie group isomorphic to $\mathsf{H}_3^\diamond(n)$, is DF-equivalent to exactly one of the systems $\Sigma^{(\ell,\varepsilon)}_{i,\alpha,\sigma} = \left(\mathsf{H}_3^\diamond(n), \Xi^{(\ell,\varepsilon)}_{i,\alpha,\sigma}\right)$, where $\Xi^{(\ell,\varepsilon)}_{i,\alpha,\sigma}(\mathbf{1}, \cdot) : \mathbb{R}^\ell \to \mathfrak{h}_3^\diamond(n)$ is an affine parametrisation of the affine subspace $\Gamma^{(\ell,\varepsilon)}_{i,\alpha,\sigma}$ (as described in the above theorem).*

PROOF. A general method for proving precisely this was the motivation for making the classification of $\mathfrak{L}$-related affine subspaces, as described in the preliminaries. We now concisely apply this method to get this result. Suppose $\Sigma = (\mathsf{G}, \Xi)$ has state space Lie group isomorphic to $\mathsf{H}_3^\diamond(n)$. That is, we have a Lie group isomorphism $\phi : \mathsf{G} \to \mathsf{H}_3^\diamond(n)$. By theorem 3.3.8 it then follows that $T_{\mathbf{1}}\phi \cdot \Gamma$ is $\mathfrak{L}$-related to $\Gamma^{(\ell,\varepsilon)}_{i,\alpha,\sigma}$ for some ℓ, ε, i, α and σ. That is, there exists a $\phi' \in \mathrm{Aut}\,\mathsf{H}_3^\diamond(n)$ such that $T_{\mathbf{1}}\phi' \cdot T_{\mathbf{1}}\phi \cdot \Gamma = T_{\mathbf{1}}(\phi \circ \phi') \cdot \Gamma = \Gamma^{(\ell,\varepsilon)}_{i,\alpha,\sigma}$. By theorem 3.0.1, it then follows that Σ is DF-equivalent to $\Sigma^{(\ell,\varepsilon)}_{i,\alpha}$. □

3.3.3 Controllability of systems on $\mathsf{H}_3^\diamond(n)$

We now investigate which proper systems, as given in corollary 3.3.9, are controllable. We will consistently choose our parametrisation maps $\Xi^{(\ell,\varepsilon)}_{i,\alpha,\sigma}(\mathbf{1}, \cdot)$ as follows. Suppose Γ is an affine subspace written as $\Gamma = A + \langle B_1, B_2, \ldots, B_\ell \rangle$ for some $A, B_i \in \mathfrak{h}_3^\diamond(n)$, $i = \overline{1, \ell}$. Then we define an affine parametrisation $\Xi(\mathbf{1}, \cdot) : \mathbb{R}^\ell \to \mathfrak{h}_3^\diamond(n)$ of Γ as $\Xi(\mathbf{1}, u) = A + u_1 B_1 + u_2 B_2 + \cdots + u_\ell B_\ell$.

We apply theorem A.2.12, item 3a in the following cases. For $\Sigma_{1,\sigma}^{(2,0)}$, $\Sigma_{1,\alpha,\sigma}^{(2,1)}$, $\Sigma_{2,\alpha}^{(2,1)}$, $\Sigma_{3}^{(2,1)}$, $\Sigma_{1,\sigma}^{(3,0)}$, $\Sigma_{2}^{(3,0)}$, $\Sigma_{1}^{(3,1)}$, $\Sigma_{2,\alpha,\sigma}^{(3,1)}$, $\Sigma_{3}^{(3,1)}$ and $\Sigma_{1}^{(4,0)}$ we have that $\operatorname{Lie}\Gamma^0 = \mathfrak{h}_3^\diamond(n)$. Hence they are all controllable.

Next we consider $\Sigma_{1,\alpha}^{(1,1)}$. Recall, from section 1.3.2, that we have a universal covering homomorphism $q : \widetilde{\mathsf{H}}_3^\diamond \to \mathsf{H}_3^\diamond(n)$, $\widetilde{m}(x,y,z,\theta) \mapsto m_n(x,y,z,\theta)$. Then note that $T_{\mathbf{1}}q : \widetilde{M}(x,y,z,\theta) \mapsto M_n(x,y,z,\theta)$. Using this covering homomorphism we can construct a **LiCAS** covering $(\widetilde{\Sigma}, \Phi)$, as described in section A.2.2, as follows: $\widetilde{\Sigma} = (\widetilde{\mathsf{H}}_3^\diamond, \widetilde{\Xi})$, where $\widetilde{\Xi}(\mathbf{1}, u) = \widetilde{E}_1 + \alpha_1 \widetilde{E}_3 + u\big(\alpha_2 \widetilde{E}_3 + \widetilde{E}_4\big)$, and $\Phi = (q, \varphi)$ for some (uniquely defined) feedback component φ. Then we have that $\widetilde{\Gamma}$ is of full rank and hence, by corollary 3.2.3, that $\widetilde{\Sigma}$ is controllable. Therefore (as the state component q of Φ is surjective) it follows, by proposition A.2.2, that $\Sigma_{1,\alpha}^{(1,1)}$ is controllable. (Specifically, we get that the whole family, i.e., for all $\alpha \neq 0, \alpha_1 \geq 0, \alpha_2 \in \mathbb{R}$, is controllable as we didn't impose any of the conditions on α in showing that $\Sigma_{1,\alpha}^{(1,1)}$ is controllable.)

We show that $\Sigma_{2,\alpha,\sigma}^{(1,1)}$ is controllable (by use of theorem A.2.12, item 3c) for $\sigma \in \{0, 1\}$, but delay proving that it is controllable for $\sigma = -1$ till later. For $\sigma = 1$ we have that $\exp\left(\frac{2\pi n}{\alpha}\, \Xi_{2,\alpha,1}^{(1,1)}\left(\mathbf{1}, \sqrt{2\alpha}\right)\right) = \mathbf{1}$ and hence that $\Sigma_{2,\alpha,1}^{(1,1)}$ is controllable for $\alpha > 0$. For $\sigma = 0$ we have that $\exp\left(\frac{2\pi n}{\alpha}\, \Xi_{2,\alpha,0}^{(1,1)}\left(\mathbf{1}, 0\right)\right) = \mathbf{1}$ and hence that $\Sigma_{2,\alpha,0}^{(1,1)}$ is controllable for $\alpha > 0$.

We again apply theorem A.2.12, item 3c to prove controllability in the next few cases. For $\Sigma_{4}^{(2,1)}$ we have that $\exp\left(2\pi n\, \Xi_{4}^{(2,1)}\left(\mathbf{1}, \left(\tfrac{1}{2}, 1\right)\right)\right) = \mathbf{1}$ and hence it is controllable. Next we notice that $\alpha E_4 = \Xi_{6,\alpha}^{(2,1)}(\mathbf{1}, 0) = \Xi_{4,\alpha}^{(3,1)}(\mathbf{1}, 0)$ for some $\alpha > 0$. But $\exp\left(\frac{2n\pi}{\alpha}\, \alpha E_4\right) = \mathbf{1}$. Thus $\Sigma_{6,\alpha}^{(2,1)}$ and $\Sigma_{4,\alpha}^{(3,1)}$, are both controllable (for $\alpha > 0$).

Next we consider $\Sigma_{5,\alpha,\sigma}^{(2,1)}$. First notice that $\Gamma_{5,\alpha,\sigma}^{(2,1)} = \Gamma_{2,\alpha,\sigma}^{(1,1)} + \langle E_2 \rangle$. Hence the identity mapping $\phi : \mathsf{H}_3^\diamond(n) \to \mathsf{H}_3^\diamond(n), g \mapsto g$ is a Lie group automorphism such that $T_{\mathbf{1}}\phi \cdot \Gamma_{2,\alpha,\sigma}^{(1,1)} \subset \Gamma_{3,\alpha,\sigma}^{(2,1)}$. Thus, by proposition A.2.1, we have a **LiCAS**-morphism $\Phi : \Sigma_{2,\alpha,\sigma}^{(1,1)} \to \Sigma_{5,\alpha,\sigma}^{(2,1)}$ with state component ϕ. Hence controllability of $\Sigma_{2,\alpha,\sigma}^{(1,1)}$ implies controllability of $\Sigma_{3,\alpha,\sigma}^{(2,1)}$, by proposition A.2.2.

We now return to show that $\Sigma_{2,\alpha,-1}^{(1,1)}$ is controllable. Here our usual tools fail to prove this point and we are left to show directly that $\Sigma_{2,\alpha,-1}^{(1,1)}$ is locally controllable at the identity (i.e., $\mathbf{1} \in \operatorname{int}\mathcal{A}$). Then as $\mathcal{A}$ is a semi-group (proposition A.2.4) and any neighbourhood of identity generates a connected Lie group, it follows that $\mathcal{A} = \mathsf{H}_3^\diamond(n)$ and hence that $\Sigma_{2,\alpha,-1}^{(1,1)}$ is controllable. We use Mathematica for the following calculations, see section C.7.

First recall that $\Xi_{2,\alpha,-1}^{(1,1)}(\mathbf{1}, u) = -E_3 + \alpha E_4 + u E_1$, where $\alpha > 0$. Now consider the mapping $F : (0,\infty)^3 \times \mathbb{R} \to H_3^\diamond(n)$ given by

$$\begin{aligned} F(t_1,t_2,t_3,u) &= \mathrm{e}^{\frac{t_1\pi}{6\alpha}\, \Xi_{2,\alpha,-1}^{(1,1)}(\mathbf{1},u)}\, \mathrm{e}^{\frac{t_2\pi}{3\alpha}\, \Xi_{2,\alpha,-1}^{(1,1)}(\mathbf{1},-u)}\, \mathrm{e}^{\frac{t_3\pi}{3\alpha}\, \Xi_{2,\alpha,-1}^{(1,1)}(\mathbf{1},u)}\, \mathrm{e}^{\frac{\pi}{6\alpha}\, \Xi_{2,\alpha,-1}^{(1,1)}(\mathbf{1},-u)} \\ &= \mathrm{e}^{\frac{t_1\pi}{6\alpha}(-E_3+\alpha E_4+uE_1)}\, \mathrm{e}^{\frac{t_2\pi}{3\alpha}(-E_3+\alpha E_4-uE_1)}\, \mathrm{e}^{\frac{t_3\pi}{3\alpha}(-E_3+\alpha E_4+uE_1)}\, \mathrm{e}^{\frac{\pi}{6\alpha}(-E_3+\alpha E_4-uE_1)}. \end{aligned}$$

Notice that the image of F is contained in $\mathcal{A}$ (as $\mathcal{A}$ is a semi-group, proposition A.2.4, and $\exp(t\Xi(\mathbf{1}, u)) \in \mathcal{A}$ for $t > 0$). As the exponentiation map is smooth and products are smooth in a Lie group, we have that F is smooth. Let $p = \left(1, 1, 1, \sqrt{\frac{2\pi}{2\sqrt{3}-\pi}}\sqrt{\alpha}\right)$. Then we get (using

Mathematica) that

$$F(p) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & e^{\frac{i\pi}{n}} \end{bmatrix} \qquad (F(p))^{2n} = \mathbf{1}.$$

We claim that there exists some neighbourhood V of p such that $F\big|_V : V \to F(V)$ is a diffeomorphism. By use of the inverse function theorem (cf. [22]) we need only show that the set $\left\{ \frac{\partial F}{\partial t_1}\big|_p, \frac{\partial F}{\partial t_2}\big|_p, \frac{\partial F}{\partial t_3}\big|_p, \frac{\partial F}{\partial u}\big|_p \right\}$ of partial derivatives at p is linearly independent. Note that each of these partial derivatives are in $T_p\mathsf{H}_3^\diamond$. Thus we can use (the tangent map of) a left translation, $T_pL_{p^{-1}} : T_p\mathsf{H}_3^\diamond(n) \to T_1\mathsf{H}_3^\diamond(n))$, to map each of these partial derivatives to $T_1\mathsf{H}_3^\diamond$, which we identify with $\mathfrak{h}_3^\diamond(n)$. Furthermore, as $T_pL_{p^{-1}}$ is a linear isomorphism, linear independence is preserved. Note that, for a linear Lie group, we have that $T_pL_{p^{-1}} \cdot \xi = p^{-1}\xi \in \mathfrak{h}_3^\diamond(n)$ for $\xi \in T_p\mathsf{H}_3^\diamond(n)$ (see section A.1.1). Now represent each element of $\left\{ p^{-1}\frac{\partial F}{\partial t_1}\big|_p, p^{-1}\frac{\partial F}{\partial t_2}\big|_p, p^{-1}\frac{\partial F}{\partial t_3}\big|_p, p^{-1}\frac{\partial F}{\partial u}\big|_p \right\} \subset \mathfrak{h}_3^\diamond(n)$ as a vector (of length 4, with respect to the basis $\{E_i\}_{i=\overline{1,4}}$). Then let R be the (4×4) matrix with these vectors as its rows. Then we get (using Mathematica) that

$$\det R = -\frac{2\sqrt{2}\sqrt{\pi^9}}{9\sqrt{6\sqrt{3}-3\pi}\sqrt{\alpha^5}} \neq 0.$$

Thus the set of partial derivatives is independent at p. Therefore we have that $F(V) \subseteq \mathcal{A}$ is an (open) neighbourhood of $F(p)$. Thus $F(p) \in \operatorname{int}\mathcal{A}$. Then as $\operatorname{int}\mathcal{A}$ is a semi-group (proposition A.2.4), we get that $(F(p))^{2n} = \mathbf{1} \in \operatorname{int}\mathcal{A}$, and so $\widetilde{\Sigma}_{1,0}^{(1,1)}$ is controllable, as explained above.

In summary we have the following.

3.3.10 Theorem. *A system Σ with state space Lie group isomorphic to $\mathsf{H}_3^\diamond(n)$ is controllable if and only if it is of full rank.*

Proof. If Σ is controllable then it must be of full rank. On the other hand if Σ is of full rank (and has state space isomorphic to the connected Lie group $\mathsf{H}_3^\diamond(n)$), then it is DF-equivalent to one of systems listed in corollary 3.3.9, by this corollary. But we have shown that each of these systems is controllable. As controllability is invariant under DF-equivalence the result follows. □

That is to say, we have that all proper systems on $\mathsf{H}_3^\diamond(n)$ are controllable. Hence a classification of controllable systems, is just one of proper systems, as given in corollary 3.3.9. We have the following corollary to this theorem.

3.3.11 Corollary. *A system Σ with state space Lie group isomorphic to $\widetilde{\mathsf{H}}/(n\mathsf{N}_1 \oplus \mathsf{N}_2)$ (as described in theorem 1.3.7) is controllable if and only if it is a proper system.*

Proof. If Σ is controllable it is necessarily a proper system. Recall from proposition 1.3.13 that $\mathsf{H}_3^\diamond(n) \cong \widetilde{\mathsf{H}}_3^\diamond/n\mathsf{N}_1$ covers $\widetilde{\mathsf{H}}_3^\diamond/(n\mathsf{N}_1 \oplus \mathsf{N}_2)$, i.e., there exists a Lie group covering homomorphism

$q : \mathsf{H}_3^\diamond(n) \to \widetilde{\mathsf{H}}_3^\diamond/(n\mathsf{N}_1 \oplus \mathsf{N}_2)$. Thus given a proper system Σ (with state space Lie group isomorphic to $\widetilde{\mathsf{H}}/(n\mathsf{N}_1 \oplus \mathsf{N}_2)$) we can construct a **LiCAS**-covering (Σ', Φ), as described in section A.2.2, such that $\Sigma' = (\mathsf{H}_3^\diamond(n), \Xi')$ and q is the state component of Φ. Moreover as $T_1 q \cdot \Gamma' = \Gamma$ by construction, we get that Σ' is of full rank. Hence Σ' is controllable (by the above theorem) and so it follows (by proposition A.2.2) that Σ is controllable. $\square$

Chapter 4

An Optimal Control Problem on $\widetilde{\mathsf{H}}_3^\diamond$

We consider optimal control problems (with fixed terminal time and diagonal $\widetilde{\mathsf{H}}_3^\diamond$-invariant cost) associated to controllable left-invariant control affine systems on $\widetilde{\mathsf{H}}_3^\diamond$, as listed in theorem 3.2.2. That is, we consider problems of the form

$$\begin{aligned} \dot{g}(t) &= \widetilde{\Xi}_{i,\alpha}^{(\ell,\varepsilon)}(g(t), u(t)), \quad g(\cdot) : [0,T] \to \widetilde{\mathsf{H}}_3^\diamond, \quad u(\cdot) : [0,T] \to \mathbb{R}^\ell \\ g(0) &= g_0, \quad g(T) = g_1, \quad g_0, g_1 \in \widetilde{\mathsf{H}}_3^\diamond \text{ fixed}, \quad T > 0 \text{ fixed} \\ J(u(\cdot)) &= \int_0^T \chi(u(t))dt \to \min, \end{aligned}$$

(for some ℓ, ε, i and α) where the cost χ is of the form $\chi : \mathbb{R}^\ell \to \mathbb{R}$, $u \mapsto \sum_{i=1}^{\ell} \beta_i u_i^2$ for some $\beta_i > 0$, $i = \overline{1, \ell}$. Specifically we wish to investigate the (reduced) extremal curves (see section A.3.6). However such an investigation becomes fairly complicated and technical quite quickly. For this reason we will restrict ourselves to an investigation of the two-dimensional homogeneous system, i.e., $\widetilde{\Sigma}_1^{(2,0)}$. (A preliminary investigation suggested that $\widetilde{\Sigma}_1^{(2,0)}$ is one of the simpler cases to deal with. Note that even in this case, integrating the reduced extremal curves is quite complicated. As such we will only integrate a subclass of the reduced extremal curves.)

The choice of considering systems on $\widetilde{\mathsf{H}}_3^\diamond$ as opposed to on $\mathsf{H}_3^\diamond(n)$, is primarily motivated by being able to "cover" all system on connected Lie groups with Lie algebra isomorphic to $\mathfrak{h}_3^\diamond$, by systems on $\widetilde{\mathsf{H}}_3^\diamond$. (See section A.1.3 regarding Lie group coverings and section A.2.2 regarding **LiCAS**-coverings. At the time of writing this thesis we were engaged in investigating "cost extended systems" and, in particular, showing that extremal trajectories are projected by "cost extended coverings".)

Ultimately, this chapter will mainly be concerned with the study of the dynamics generated by a Hamiltonian

$$H : (\widetilde{\mathfrak{h}}_3^\diamond)^* \to \mathbb{R}, \quad p \mapsto \tfrac{1}{2}\left(\tfrac{1}{\beta_1}p_1^2 + \tfrac{1}{\beta_2}p_4^2\right), \qquad \beta_i > 0, \quad p_i = p(\widetilde{E}_i),$$

on $(\widetilde{\mathfrak{h}}_3^\diamond)^*$ equipped with the Lie-Poisson bracket.

4.1 Preliminaries

We will use the notation and conventions as laid out in section A.3. Specifically, we use the following trivialisation of the cotangent bundle

$$(g,p) \in \widetilde{\mathsf{H}}_3^\diamond \times (\widetilde{\mathfrak{h}}_3^\diamond)^* \qquad \longleftrightarrow \qquad (T_{\mathbf{1}}L_{g^{-1}})^* \cdot p \in T^*\widetilde{\mathsf{H}}_3^\diamond$$

and recall that the Poisson bracket (defined by the symplectic form $\omega = -d\theta$, $\theta = (T\pi)^*$, $\pi : T^*\widetilde{\mathsf{H}}_3^\diamond \to \widetilde{\mathsf{H}}_3^\diamond$, $(g,p) \mapsto g$) of two $\widetilde{\mathsf{H}}_3^\diamond$-invariant Hamiltonians H_1 and H_2 on $T^*\widetilde{\mathsf{H}}_3^\diamond$ is given by

$$\{H_1, H_2\}(g,p) = -p \cdot \left[\tfrac{\partial H_1}{\partial p}(g,p), \tfrac{\partial H_2}{\partial p}(g,p)\right].$$

The Lie-Poisson structure on $(\widetilde{\mathfrak{h}}_3^\diamond)$ is then defined as above, by considering $H_1, H_2 \in C^\infty((\widetilde{\mathfrak{h}}_3^\diamond)^*)$ as $\widetilde{\mathsf{H}}_3^\diamond$-invariant Hamiltonians on $T^*\widetilde{\mathsf{H}}_3^\diamond$. For the purposes of this chapter we will denote the ordered basis $\{\widetilde{E}_i\}_{i=\overline{1,4}}$ for $\widetilde{\mathfrak{h}}_3^\diamond$, as introduced in section 1.3.1, by $\{E_i\}_{i=\overline{1,4}}$. Furthermore we will denote the dual basis by $\{E_i^*\}_{i=\overline{1,4}}$ and the double dual basis by $\{P_i\}_{i=\overline{1,4}}$. That is $P_i(E_j^*) = E_j^*(E_i) = \delta_{ij}$ for $i,j = \overline{1,4}$. Then for $p \in (\mathfrak{h}_3^\diamond)^*$ we will write $p = (p_1,p_2,p_3,p_4) = \sum_{i=1}^4 p_i E_i^*$, i.e., we define the i'th component of p as $P_i(p)$. With regard to Casimir functions on $\big((\widetilde{\mathfrak{h}}_3^\diamond)^*, \{,\}\big)$, we have the following result (which in particular implies that $\dot{p}_3 = 0$ for all extremal curves).

4.1.1 PROPOSITION. *$P_3 : p \mapsto p_3$ and $\mathcal{C} : p \mapsto p_1^2 + p_2^2 - 2p_3p_4$ are Casimir functions on* $\big((\widetilde{\mathfrak{h}}_3^\diamond)^*, \{,\}\big)$

PROOF. By proposition 1.2.26, we have that $E_3 \in \mathrm{Z}\,(\widetilde{\mathfrak{h}}_3^\diamond)$ and hence (by proposition A.3.19) get that $P_3 = E_3^{**}$ is a Casimir function. We have that $\mathcal{C}$ is a Casimir function by corollary 1.4.8 (and noting that we have a Lie algebra isomorphism $\psi : \widetilde{\mathfrak{h}}_3^\diamond \to \mathfrak{h}_3^\diamond$, $\psi = I_4$ w.r.t. the respective ordered bases). □

Having dispensed with defining our notations and structures, we now turn our attention to the system $\widetilde{\Sigma}_1^{(2,0)}$. Recall that $\widetilde{\Sigma}_1^{(2,0)} = \Big(\widetilde{\mathsf{H}}_3^\diamond, \widetilde{\Xi}_1^{(2,0)}\Big)$, with arbitrary diagonal cost, is given by

$$\Big(\widetilde{\Sigma}_1^{(2,0)}, \chi_1^{(2,0)}\Big) : \begin{cases} \widetilde{\Xi}_1^{(2,0)}(\mathbf{1}, u) = u_1E_1 + u_2E_4 \\ \chi_1^{(2,0)}(u) = \beta_1u_1^2 + \beta_2u_2^2 \\ \beta_1, \beta_2 > 0. \end{cases}$$

Therefore, the cost extended Hamiltonian lift (on $T^*\widetilde{\mathsf{H}}_3^\diamond$, see section A.3.4) takes the form

$$H_u^\nu(g,p) = p_1u_1 + p_4u_2 + \nu\left(\beta_1u_1^2 + \beta_2u_2^2\right).$$

We wish to study the extremal triplets associated to H_u^ν. In other words, we seek $\nu \in \mathbb{R}$ and a triplet $t \mapsto (g(t), p(t), u(t))$ satisfying the necessary conditions of PMP (theorem A.3.20), i.e.,

$$\begin{aligned}
&g(\cdot) : [0,T] \to \widetilde{\mathsf{H}}_3^\diamond, \quad p(\cdot) : [0,T] \to \mathfrak{g}^*, \quad u(\cdot) : [0,T] \to \mathbb{R}^\ell, \\
&(\nu, p(\cdot)) \neq 0, \quad \nu \in \mathbb{R},\ \nu \leq 0, \\
&\tfrac{d}{dt}\big(g(t),p(t)\big) = \vec{H}_{u(t)}^\nu\big(g(t),p(t)\big), \\
&H_{u(t)}^\nu\big(g(t),p(t)\big) = \max_{u \in \mathbb{R}^\ell} H_u^\nu\big(g(t),p(t)\big) \quad \forall \text{ a.e. } t \in [0,T].
\end{aligned}$$

We consider the two cases for ν (either $\nu = 0$ or $\nu < 0$ and may be normalised) in the following subsections.

4.1.1 Abnormal extremals

We consider extremal triplets for which $\nu = 0$. In this case the condition $H^0_{u(t)}\big(g(t), p(t)\big) = \max_{u\in\mathbb{R}^\ell} H^0_u\big(g(t), p(t)\big)$ implies that

$$p_1(t)u_1(t) + p_4(t)u_2(t) = \max_{u\in\mathbb{R}^\ell} (p_1(t)u_1 + p_4(t)u_2) \quad \forall \text{ a.e. } t \in [0, T].$$

This in turn implies that $p_1(\cdot) = p_4(\cdot) = 0$. Then, as we assume $\frac{d}{dt}\big(g(t), p(t)\big) = \vec{H}^0_{u(t)}\big(g(t), p(t)\big)$, by applying propositions A.3.9 and A.3.17, we get that (for $t \in [0, T]$)

$$\begin{aligned} 0 = \dot{p}_1(t) &= -p(t)\cdot[E_1, u_1(t)E_1 + u_2(t)E_4] = -p(t)\cdot(u_2(t)E_4) = -p_2(t)u_2(t) \\ 0 = \dot{p}_4(t) &= -p(t)\cdot[E_4, u_1(t)E_1 + u_2(t)E_4] = -p(t)\cdot(-u_1(t)E_2) = p_2(t)u_1(t). \end{aligned}$$

Thus we have that either $u_1(\cdot) = u_2(\cdot) = 0$ or $p_2(\cdot) = 0$. In the first case we have trivial dynamics, i.e., $\widetilde{\Xi}_1^{(2,0)}(\mathbf{1}, u(t)) = 0$ and so $g(\cdot)$ is constant. So assume $p_2(\cdot) = 0$. Then, like above (noting $p_1(\cdot) = 0$), we get that

$$0 = \dot{p}_2(t) = -p(t)\cdot[E_2, u_1(t)E_1 + u_2(t)E_4] = -p(t)\cdot(-u_1(t)E_3 - u_2(t)E_1) = p_3(t)u_1(t).$$

Therefore, either $p_3(\cdot) = 0$, or $u_1(\cdot) = 0$. But $p_3(\cdot) = 0$ is untenable as $(\nu, p(\cdot)) \neq 0$. Thus $u_1(\cdot) = 0$. Finally notice, again by the same argument as above and as $E_3 \in \mathrm{Z}(\widetilde{\mathfrak{h}}_3^\diamond)$, that

$$\dot{p}_3(t) = -p(t)\cdot[E_3, u_1(t)E_1 + u_2(t)E_4] = 0$$

and so $p_3(\cdot)$ is constant. We are now ready to make a statement concerning all abnormal extremals.

4.1.2 PROPOSITION. *The abnormal extremal triplets $(g(\cdot), p(\cdot), u(\cdot))$, for which $g(\cdot)$ is not constant, are given by*

$$g(\cdot) : t \mapsto \widetilde{m}\Big(x, y, z, \textstyle\int_0^t u_2(s)ds + \theta\Big), \qquad p(\cdot) : t \mapsto (0, 0, p_3, 0), \qquad u(\cdot) : t \mapsto (0, u_2(t)),$$

where $u_2(\cdot) \neq 0$ is some admissible control and $p_3, x, y, z, \theta \in \mathbb{R}$ are constants such that $p_3 \neq 0$. (For the definition of $\widetilde{m} : \mathbb{R}^4 \to \widetilde{\mathsf{H}}_3^\diamond$, see section 1.3.1.)

PROOF. By the foregoing discussion we already have that $p(\cdot)$ and $u(\cdot)$ must be of the specified forms (the condition $u_2(\cdot) \neq 0$ is required for $g(\cdot)$ not to be constant).

By proposition A.3.9 (and the assumption $\frac{d}{dt}\big(g(t), p(t)\big) = \vec{H}^0_{u(t)}\big(g(t), p(t)\big)$), we get that $\dot{g}(t) = \widetilde{\Xi}_1^{(2,0)}(g(t), u(t)) = g(t)\, u_2(t)E_4$. Recall that in chapter 3 (equation (3.2.1)) we showed that we may parametrise an integral curve $g(\cdot)$ of a (left-invariant) vector field $\Xi : g \mapsto g\sum_{i=1}^4 v_i(t)E_i$ as

$$g(t) = \widetilde{m}(x(t), y(t), z(t), \theta(t)).$$

Accordingly we got a system of "parametric" differential equations

$$\begin{bmatrix}\dot{x}(t)\\ \dot{y}(t)\end{bmatrix} = \begin{bmatrix}\cos\theta(t) & \sin\theta(t)\\ -\sin\theta(t) & \cos\theta(t)\end{bmatrix}\begin{bmatrix}v_1(t)\\ v_2(t)\end{bmatrix} \quad \dot{z}(t) = \tfrac{1}{2}\big(x(t)\dot{y}(t) - y(t)\dot{x}(t)\big) + v_3(t) \quad \dot{\theta}(t) = v_4(t).$$

So in this case we have that $v_1(\cdot) = 0$, $v_2(\cdot) = 0$, $v_3(\cdot) = 0$ and $v_4(\cdot) = u_2(\cdot)$. Hence $\dot{x}(\cdot) = \dot{y}(\cdot) = \dot{z}(\cdot) = 0$ and so $x(\cdot) = x$, $y(\cdot) = y$ and $z(\cdot) = z$ for some constants $x, y, z \in \mathbb{R}$. Furthermore we get that $\dot{\theta}(t) = u_2(t)$, yielding $\theta(t) = \int_0^t u_2(s)ds + \theta$ for some constant $\theta \in \mathbb{R}$.

We now show that the described triplet is indeed an extremal triplet. As $p_3 \neq 0$ the condition $(\nu, p(\cdot)) \neq 0$ is satisfied. Also, as $H_u^0\big(g(t), p(t)\big) = 0$ the condition $H_{u(t)}^0\big(g(t), p(t)\big) = \max_{u\in\mathbb{R}^\ell} H_u^0\big(g(t), p(t)\big)$ is satisfied. Now, by proposition A.3.9, we have that $\vec{H}_{u(t)}^0 = (X, Y^*)$, $X(g(t), p(t)) = \Xi_{u(t)}(\mathbf{1})$, $Y^*(g(t), p(t)) = -\mathrm{ad}^*\Xi_{u(t)}(\mathbf{1}) \cdot p(t)$. Thus we get that

$$\begin{aligned}
\dot{g}(t) &= \tfrac{d}{dt}\widetilde{m}\Big(x, y, z, \textstyle\int_0^t u_2(s)ds + \theta\Big) = g(t)\,\Xi_{u(t)}(\mathbf{1}) = g(t)\,X(g(t), p(t)) \\
\dot{p}(t) &= 0 = -\mathrm{ad}^* u_2(t)E_4 \cdot p_3(t)E_3^* = -\mathrm{ad}^*\,\Xi_{u(t)}(\mathbf{1}) \cdot p(t) = Y^*(g(t), p(t)).
\end{aligned}$$

That is, the condition $\frac{d}{dt}\big(g(t), p(t)\big) = \vec{H}_{u(t)}^\nu\big(g(t), p(t)\big)$ is satisfied. □

4.1.2 Normal extremals

We consider extremal triplets for which $\nu \neq 0$. We will conveniently choose $\nu = -\frac{1}{2}$. We start by considering the expression $\max_{u\in\mathbb{R}^\ell} H_u^\nu\big(g, p\big)$. In this case we have that

$$H_u^\nu\big(g, p\big) = p_1 u_1 + p_4 u_4 - \tfrac{1}{2}\left(\beta_1 u_1^2 + \beta_2 u_2^2\right).$$

Note that $H_u^\nu\big(g, p\big)$ is smooth, defines a paraboloid and has a global maximum (w.r.t. $u \in \mathbb{R}^\ell$, and for a fixed $p \in (\widetilde{\mathfrak{h}}_3^\diamond)^*$) at $\frac{\partial H_u^\nu}{\partial u} = 0$. But $\frac{\partial H_u^\nu}{\partial u} = 0$ implies that

$$\begin{bmatrix} p_1 - \beta_1 u_1 & p_4 - \beta_2 u_2 \end{bmatrix} = 0.$$

That is, $u_1 = \frac{1}{\beta_1}p_1$ and $u_2 = \frac{1}{\beta_2}p_4$. (So in particular, the condition $H_{u(t)}^\nu\big(g(t), p(t)\big) = \max_{u\in\mathbb{R}^\ell} H_u^\nu(g(t), p(t))$ implies that $u(t) = \big(\frac{1}{\beta_1}p_1(t), \frac{1}{\beta_2}p_4(t)\big)$.) Substituting these values back into $H_u^\nu\big(g, p\big)$ we get a maximised Hamiltonian

$$H_{\max}(g, p) = \tfrac{1}{2}\left(\tfrac{1}{\beta_1}p_1^2 + \tfrac{1}{\beta_2}p_4^2\right).$$

Thus, by proposition A.3.21, an extremal triplet $\big(g(\cdot), p(\cdot), u(\cdot)\big)$ satisfies (for $t \in [0, T]$)

$$\tfrac{d}{dt}\big(g(t), p(t)\big) = \vec{H}_{\max}\big(g(t), p(t)\big), \quad t \in [0, T].$$

But as $H_{\max}$ is $\widetilde{\mathsf{H}}_3^\diamond$-invariant, we can consider it as a Hamiltonian on $(\widetilde{\mathfrak{h}}_3^\diamond)^*$ (with Lie-Poisson structure) and consequently get that $\dot{p}(t) = \vec{H}_{\max}(p(t))$. Now notice that $\frac{\partial H_{\max}}{\partial p} = \frac{1}{\beta_1}p_1E_1 + \frac{1}{\beta_2}p_4E_4$ (where we identify $(\widetilde{\mathfrak{h}}_3^\diamond)^{**}$ with $\widetilde{\mathfrak{h}}_3^\diamond$ canonically). Then, applying proposition A.3.15, we get (in coordinates, suppressing evaluation at t) that

$$\begin{aligned}
\dot{p}_1 &= -p \cdot [E_1, \tfrac{1}{\beta_1}p_1E_1 + \tfrac{1}{\beta_2}p_4E_4] = -p \cdot (\tfrac{1}{\beta_2}p_4E_2) = -\tfrac{1}{\beta_2}\,p_2\,p_4 \\
\dot{p}_2 &= -p \cdot [E_2, \tfrac{1}{\beta_1}p_1E_1 + \tfrac{1}{\beta_2}p_4E_4] = -p \cdot (-\tfrac{1}{\beta_1}p_1E_3 - \tfrac{1}{\beta_2}p_4E_1) = p_1\left(\tfrac{1}{\beta_1}\,p_3 + \tfrac{1}{\beta_2}\,p_4\right) \\
\dot{p}_3 &= \{P_3, H_{\max}\} = 0 \\
\dot{p}_4 &= -p \cdot [E_4, \tfrac{1}{\beta_1}p_1E_1 + \tfrac{1}{\beta_2}p_4E_4] = -p \cdot (-\tfrac{1}{\beta_1}p_1E_2) = \tfrac{1}{\beta_1}\,p_1\,p_2.
\end{aligned}$$

So, in summary, we get the following result.

4.1.3 Proposition. *The normal extremal triplets $(g(\cdot), p(\cdot), u(\cdot))$ are of the form*

$$\dot{g}(t) = \widetilde{\Xi}_1^{(2,0)}\big(g(t), u(t)\big), \qquad u(t) = \big(\tfrac{1}{\beta_1} p_1(t), \tfrac{1}{\beta_2} p_4(t)\big), \qquad \dot{p}(t) = \vec{H}_{\max}(p(t)),$$

where $H_{\max} \in C^\infty\big((\widetilde{\mathfrak{h}}_3^\diamond)^*\big)$, $H_{\max}(p) = \frac{1}{2}\left(\frac{1}{\beta_1} p_1^2 + \frac{1}{\beta_2} p_4^2\right)$ and $\vec{H}_{\max}(p) = (\dot{p}_1, \dot{p}_2, \dot{p}_3, \dot{p}_4)$ is given (in coordinates) by

$$\dot{p}_1 = -\tfrac{1}{\beta_2}\, p_2\, p_4, \qquad \dot{p}_2 = p_1\left(\tfrac{1}{\beta_1}\, p_3 + \tfrac{1}{\beta_2}\, p_4\right), \qquad \dot{p}_3 = 0, \qquad \dot{p}_4 = \tfrac{1}{\beta_1}\, p_1\, p_2.$$

Proof. All that is left to show is that $\dot{g}(t)$ takes the specified form. But this follows from the condition $\frac{d}{dt}\big(g(t), p(t)\big) = \vec{H}^{\nu}_{u(t)}\big(g(t), p(t)\big)$ and propositions A.3.9 and A.3.12. □

Thus the problem (of studying normal extremal triplets) is effectively reduced to studying the properties of the Hamiltonian $H_{\max} \in C^\infty\big((\widetilde{\mathfrak{h}}_3^\diamond)^*\big)$ and its associated vector field $\vec{H}_{\max}$. Specifically, we are interested in the reduced normal extremals $p(\cdot)$, i.e., integral curves of $\vec{H}_{\max}$. *The rest of this chapter will be dedicated to the investigation of this topic.* However, before we continue on to do so, we briefly investigate how the problem of finding the "extremal" integral curves $g(\cdot)$ of $\widetilde{\Xi}_1^{(2,0)}(\cdot, u(t))$ may be approached (given an integral curve $p(\cdot)$ of $\vec{H}_{\max}$, defining an "extremal control" $u(\cdot)$). Specifically we show that one may reduce this problem to solving a system of four "parametric" differential equations. Furthermore, we show that under the assumption $p_3(0) \neq 0$, we may further reduce the problem to solving a system of two "parametric" differential equations. (Section C.8 contains the supporting Mathematica code.)

We recall (again) that in chapter 3 (equation (3.2.1)) we showed that we may parametrise an integral curve $g(\cdot)$ of a (left-invariant) vector field $\Xi : g \mapsto g\sum_{i=1}^4 v_i(t) E_i$ as

$$g(t) = \widetilde{m}(x(t), y(t), z(t), \theta(t)).$$

(For the definition of $\widetilde{m} : \mathbb{R}^4 \to \widetilde{\mathsf{H}}_3^\diamond$, see section 1.3.1.) Accordingly we got a system of "parametric" differential equations

$$\begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix} = \begin{bmatrix} \cos\theta(t) & \sin\theta(t) \\ -\sin\theta(t) & \cos\theta(t) \end{bmatrix} \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} \quad \dot{z}(t) = \tfrac{1}{2}\big(x(t)\dot{y}(t) - y(t)\dot{x}(t)\big) + v_3(t) \quad \dot{\theta}(t) = v_4(t).$$

So in this case we have that $v_1(\cdot) = \frac{1}{\beta_1} p_1(\cdot)$, $v_2(\cdot) = 0$, $v_3(\cdot) = 0$ and $v_4(\cdot) = \frac{1}{\beta_2} p_4(\cdot)$. Hence, suppressing the evaluation at t, we get a system of four differential equations

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \tfrac{1}{\beta_1} p_1 \begin{bmatrix} \cos\theta \\ -\sin\theta \end{bmatrix} \qquad \dot{z} = \tfrac{1}{2}\big(x\dot{y} - y\dot{x}\big) \qquad \dot{\theta} = \tfrac{1}{\beta_2} p_4.$$

4.1.4 Remark. Suppose we choose to rather parametrise $g(t)$ by the inverse of our usual element, i.e., $g(t) = \widetilde{m}(\bar{x}(t), \bar{y}(t), \bar{z}(t), \bar{\theta}(t))^{-1}$. (This may be safely done as the inverse operation is a diffeomorphism.) Then we get, by the same process followed in chapter 3, that

$$\dot{\bar{x}}(t) = -v_1(t) - y(t)v_4(t), \qquad \dot{\bar{y}}(t) = -v_2(t) + x(t)v_4(t),$$
$$\dot{\bar{z}}(t) = \tfrac{1}{2}\left(x(t)v_2(t) - y(t)v_1(t)\right) - v_3(t), \qquad \dot{\bar{\theta}}(t) = -v_4(t).$$

Therefore, in this case, we would get that

$$\dot{x}(t) = -\tfrac{1}{\beta_1} p_1(t) - \tfrac{1}{\beta_2} y(t) p_4(t), \quad \dot{y}(t) = \tfrac{1}{\beta_2} x(t) p_4(t), \quad \dot{z}(t) = -\tfrac{1}{2\beta_1} y(t) p_1(t), \quad \dot{\theta}(t) = -\tfrac{1}{\beta_2} p_4(t).$$

Next we show, that under the assumption that the constant component $p_3(\cdot) = p_3(0) \neq 0$, we can reduce the above four differential equations to two. We base this reduction on the result $p(t) = \mathrm{Ad}^* g(t)^{-1} \cdot \mu$, (for some $\mu \in (\widetilde{\mathfrak{h}}_3^\diamond)^*$) as presented in proposition A.3.11 (or using propositions A.3.9 and A.3.12). At this stage we will conveniently assume that $g(0) = \mathbf{1}$ (this can always be arranged by using a suitable left translation). Then we have that $\mu = p(0)$. Now recall that in section 1.4.3 we calculated $\mathrm{Ad}^*\, m(x, y, z, \theta)$ w.r.t. the dual basis of $\mathfrak{h}_3^\diamond$ for an element $m(x, y, z, \theta) \in \mathsf{H}_3^\diamond$. In the same way we can calculate $\mathrm{Ad}^*\, g(t)$ (or alternatively use the Lie group covering homomorphism $q : \widetilde{\mathsf{H}}_3^\diamond \to \mathsf{H}_3^\diamond$, $\widetilde{m}(x, y, z, \theta) \mapsto m(x, y, z, \theta)$). We get that

$$\mathrm{Ad}^* \left(\widetilde{m}(x, y, z, \theta)\right)^{-1} = \begin{bmatrix} \cos\theta & -\sin\theta & -x\sin\theta - y\cos\theta & 0 \\ \sin\theta & \cos\theta & x\cos\theta - y\sin\theta & 0 \\ 0 & 0 & 1 & 0 \\ -y & x & \frac{1}{2}\left(x^2 + y^2\right) & 1 \end{bmatrix}.$$

Consequently (from the condition $p(t) = \mathrm{Ad}^* g(t)^{-1} \cdot p(0)$) we find that

$$\begin{bmatrix} -p_3(0)\big(x(t)\sin\theta(t) + y(t)\cos\theta(t)\big) + p_1(0)\cos\theta(t) - p_2(0)\sin\theta(t) \\ p_3(0)\big(x(t)\cos\theta(t) - y(t)\sin\theta(t)\big) + p_1(0)\sin\theta(t) + p_2(0)\cos\theta(t) \\ p_3(0) \\ \frac{1}{2} p_3(0)\left(x(t)^2 + y(t)^2\right) - y(t)p_1(0) + x(t)p_2(0) + p_4(0) \end{bmatrix} = \begin{bmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \\ p_4(t) \end{bmatrix}$$

Equating the first two components we find that

$$\begin{bmatrix} \cos\theta(t) & -\sin\theta(t) \\ \sin\theta(t) & \cos\theta(t) \end{bmatrix} \begin{bmatrix} p_1(0) - p_3(0)y(t) \\ p_2(0) + p_3(0)x(t) \end{bmatrix} = \begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix}.$$

Therefore we get that

$$\begin{bmatrix} -y(t) \\ x(t) \end{bmatrix} = \frac{1}{p_3(0)} \left(-\begin{bmatrix} p_1(0) \\ p_2(0) \end{bmatrix} + \begin{bmatrix} \cos\theta(t) & \sin\theta(t) \\ -\sin\theta(t) & \cos\theta(t) \end{bmatrix} \begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix} \right).$$

In other words we get that

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -y(t) \\ x(t) \end{bmatrix} = \frac{1}{p_3(0)} \left(\begin{bmatrix} -p_2(0) \\ p_1(0) \end{bmatrix} - \begin{bmatrix} \sin\theta(t) & -\cos\theta(t) \\ \cos\theta(t) & \sin\theta(t) \end{bmatrix} \begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix} \right).$$

That is to say, having solved $\dot{\theta}(t) = \frac{1}{\beta_2} p_4(t)$, we immediately have solutions for the parameters $x(t)$ and $y(t)$ (under the assumption we have solved for $p(\cdot)$ and the constant coordinate $p_3(\cdot) = p_3(0) \neq 0$). Making use of this expression (and the expressions for $\dot{x}(t), \dot{y}(t), \dot{z}(t)$), we can write $\dot{z}(t)$ solely in terms of $p(t)$ and $\theta(t)$. This calculation was performed in Mathematica; see section C.8. We get that

$$\dot{z}(t) = \frac{p_1(t)\left(p_1(t) - p_1(0)\cos\theta(t) + p_2(0)\sin\theta(t)\right)}{2p_3(0)\beta_1}.$$

Thus we have effectively reduced the system of four parametric differential equations to two. We summarise these results in a proposition.

4.1.5 Proposition. *If $(g(\cdot), p(\cdot), u(\cdot))$ is a normal extremal triplet (as described in proposition 4.1.3) then $g(t) = \widetilde{m}(x(t), y(t), z(t), \theta(t))$ for some parametric functions $x(\cdot), y(\cdot), z(\cdot), \theta(\cdot) : \mathbb{R} \to \mathbb{R}$, which satisfy*

$$\begin{bmatrix}\dot{x}(t)\\ \dot{y}(t)\end{bmatrix} = \tfrac{1}{\beta_1} p_1(t) \begin{bmatrix}\cos\theta(t)\\ -\sin\theta(t)\end{bmatrix} \qquad \dot{z}(t) = \tfrac{1}{2}\big(x(t)\dot{y}(t) - y(t)\dot{x}(t)\big) \qquad \dot{\theta}(t) = \tfrac{1}{\beta_2} p_4(t).$$

If in addition $g(0) = \mathbf{1}$ and $p_3(\cdot) = p_3(0) \neq 0$, then we have that

$$\begin{bmatrix}x(t)\\ y(t)\end{bmatrix} = \frac{1}{p_3(0)} \left(\begin{bmatrix}-p_2(0)\\ p_1(0)\end{bmatrix} - \begin{bmatrix}\sin\theta(t) & -\cos\theta(t)\\ \cos\theta(t) & \sin\theta(t)\end{bmatrix} \begin{bmatrix}p_1(t)\\ p_2(t)\end{bmatrix} \right)$$

$$\dot{z}(t) = \frac{p_1(t)\,(p_1(t) - p_1(0)\cos\theta(t) + p_2(0)\sin\theta(t))}{2p_3(0)\beta_1}.$$

4.2 Qualitative Analysis of (Reduced) Normal Extremals

We now study the qualitative properties of $\vec{H}_{\max}$. We start with an analysis of the equilibrium points (with regard to stability). We then continue on to describe and classify the integral curves of $\vec{H}_{\max}$. Finally we suggest a subdivision of typical cases for further analysis, specifically with regard to integration. We recall, for use in this section, that $H_{\max}$ and $\mathcal{C}$ are given (in coordinates) by

$$H_{\max}(p) = \tfrac{1}{2}\left(\tfrac{1}{\beta_1} p_1^2 + \tfrac{1}{\beta_2} p_4^2\right), \qquad \mathcal{C}(p) = p_1^2 + p_2^2 - 2p_3p_4$$

and (from proposition 4.1.3), that $\vec{H}_{\max}(p) = (\dot{p}_1, \dot{p}_2, \dot{p}_3, \dot{p}_4)$ is given (in coordinates) by

$$\dot{p}_1 = -\tfrac{1}{\beta_2}\, p_2\, p_4, \qquad \dot{p}_2 = p_1 \left(\tfrac{1}{\beta_1}\, p_3 + \tfrac{1}{\beta_2}\, p_4\right), \qquad \dot{p}_3 = 0, \qquad \dot{p}_4 = \tfrac{1}{\beta_1}\, p_1\, p_2.$$

4.2.1 Stability

4.2.1 Proposition. *The set of equilibrium points E (of $\vec{H}_{\max}$) is given, in coordinates, by*

$$E = \{(a, 0, b, -\tfrac{\beta_2}{\beta_1} b), (0, a, b, 0), (0, 0, a, b) \in (\widetilde{\mathfrak{h}}_3^\diamond)^* \mid a, b \in \mathbb{R}\}.$$

Proof. By definition, $p \in (\widetilde{\mathfrak{h}}_3^\diamond)^*$ is an equilibrium point if $\vec{H}_{\max}(p) = 0$, i.e.,

$$-\tfrac{1}{\beta_2}\, p_2\, p_4 = 0 \qquad p_1\left(\tfrac{1}{\beta_1}\, p_3 + \tfrac{1}{\beta_2}\, p_4\right) = 0 \qquad \tfrac{1}{\beta_1}\, p_1\, p_2 = 0.$$

We break this down into subcases. Suppose $p \in E$ and $p_1 \neq 0$. Then we have $p_2 = 0$ and $\frac{p_3}{\beta_1} + \frac{p_4}{\beta_2} = 0$, implying $p_4 = -\frac{\beta_2}{\beta_1} p_3$. Thus $p = (a, 0, b, -\frac{\beta_2}{\beta_1} p_3)$ for some $a, b \in \mathbb{R}$. Moreover, for any $a, b \in \mathbb{R}$ we have that $(a, 0, b, -\frac{\beta_2}{\beta_1} p_3)$ is an equilibrium point. On the other hand assume $p \in E$ and $p_1 = 0$. Then we require that $p_2 p_4 = 0$. If $p_2 = 0$, then $p = (0, 0, a, b)$ for some $a, b \in \mathbb{R}$ and for any $a, b \in \mathbb{R}$ we have that $(0, 0, a, b)$ is an equilibrium point. If $p_2 \neq 0$, then $p_4 = 0$ and $p = (0, a, b, 0)$ for some $a, b \in \mathbb{R}$ and for any $a, b \in \mathbb{R}$ we have that $(0, a, b, 0)$ is an equilibrium point. □

With the equilibrium points at hand, we now turn to an investigation of their stability. Specifically, we partition the set of equilibrium points into the (Lyapunov) stable and unstable equilibrium points. (For more details see section A.4.2.)

4.2.2 Proposition. *The sets of (Lyapunov) stable equilibrium points E_s and unstable equilibrium points E_u are given by*

$$E_s = \{(0,0,0,a),(0,b,d,0),(0,0,b,c) \mid a,b,c,d \in \mathbb{R}, b \neq 0, c\,(c\beta_1 + b\beta_2) > 0\}$$
$$E_u = \{(0,0,b,c),(b,0,a,-\tfrac{\beta_2}{\beta_1}a) \mid a,b,c \in \mathbb{R}, b \neq 0, c\,(c\beta_1 + b\beta_2) \leq 0\}.$$

Proof. First notice that $E_s \cup E_u = E$ and $E_s \cap E_u = \emptyset$. Thus E_s and E_u partition E. For negative results (i.e., E_u) we use theorem A.4.4, i.e., we prove linear instability; for positive results (i.e., E_s) we use theorem A.4.5 (an extended Casimir Energy method). We use Mathematica (see section C.9) to assist our calculations. Before starting to investigate individual cases we briefly make some definitions and general calculations. The linearisation of $\vec{H}_{\max}$ is given (at some point p, in coordinates) by

$$D\vec{H}_{\max}(p) = \begin{bmatrix} 0 & -\frac{p_4}{\beta_2} & 0 & -\frac{p_2}{\beta_2} \\ \frac{p_3}{\beta_1} + \frac{p_4}{\beta_2} & 0 & \frac{p_1}{\beta_1} & \frac{p_1}{\beta_2} \\ 0 & 0 & 0 & 0 \\ \frac{p_2}{\beta_1} & \frac{p_1}{\beta_1} & 0 & 0 \end{bmatrix}.$$

We define a family of (Casimir energy) functions $F_\lambda : (\widetilde{\mathfrak{h}}_3^\diamond)^* \to \mathbb{R}$ by $F_\lambda(p) = \lambda_0 H_{\max}(p) + \lambda_1 P_3(p) + \lambda_2 \mathcal{C}(p)$, where P_3 and $\mathcal{C}$ are the Casimir functions defined in proposition 4.1.1. Then we have that

$$dF_\lambda(p) = \begin{bmatrix} \frac{p_1\lambda_0}{\beta_1} + 2p_1\lambda_2 \\ 2p_2\lambda_2 \\ \lambda_1 - 2p_4\lambda_2 \\ \frac{p_4\lambda_0}{\beta_2} - 2p_3\lambda_2 \end{bmatrix} \qquad d^2F_\lambda(p) = \begin{bmatrix} \frac{\lambda_0}{\beta_1} + 2\lambda_2 & 0 & 0 & 0 \\ 0 & 2\lambda_2 & 0 & 0 \\ 0 & 0 & 0 & -2\lambda_2 \\ 0 & 0 & -2\lambda_2 & \frac{\lambda_0}{\beta_2} \end{bmatrix}$$

and

$$dH_{\max}(p) = \begin{bmatrix} \frac{p_1}{\beta_1} \\ 0 \\ 0 \\ \frac{p_4}{\beta_2} \end{bmatrix} \qquad dP_3(p) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad d\mathcal{C}(p) = \begin{bmatrix} 2p_1 \\ 2p_2 \\ -2p_4 \\ -2p_3 \end{bmatrix}.$$

Define $W(p) = \ker dH_{\max}(p) \cap \ker dP_3(p) \cap \ker d\mathcal{C}(p)$.

We will start with the stable equilibrium points. Suppose $p = 0$. Set $\lambda_1 = 0$ to get $dF_\lambda(0) = 0$. Choosing $\lambda_0 = \lambda_2 = 1$ we then get (w.r.t. the ordered basis $\{E_1^*, E_2^*, E_4^*\}$) that

$$d^2F_\lambda\big|_{\ker dP_3(p)\times\ker dP_3(p)}(p) = \operatorname{diag}\left(\tfrac{1}{\beta_1} + 2, 2, \tfrac{1}{\beta_2}\right)$$

is positive definite, and thus so is $d^2F_\lambda\big|_{W(p)\times W(p)}(p)$ (as $W(p)$ is a subspace of $\ker dP_3(p)$). Thus we conclude that $p = 0$ is a stable equilibrium point.

Next we consider a point p of the form $p = (0, 0, 0, a)$ for $a \neq 0$ (we have already dealt with $a = 0$). In this case we have that

$$dF_\lambda(p) = \begin{bmatrix} 0 \\ 0 \\ \lambda_1 - 2a\lambda_2 \\ \frac{a\lambda_0}{\beta_2} \end{bmatrix}.$$

Setting $\lambda_0 = 0$ and $\lambda_1 = 2a\lambda_2$ we have that $dF_\lambda(p) = 0$. Then

$$d^2F_\lambda(p) = \begin{bmatrix} 2\lambda_2 & 0 & 0 & 0 \\ 0 & 2\lambda_2 & 0 & 0 \\ 0 & 0 & 0 & -2\lambda_2 \\ 0 & 0 & -2\lambda_2 & 0 \end{bmatrix}.$$

Now we have that

$$dH_{\max}(p) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{a}{\beta_2} \end{bmatrix} \qquad dP_3(p) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad d\mathcal{C}(p) = \begin{bmatrix} 0 \\ 0 \\ -2a \\ 0 \end{bmatrix}.$$

Hence $W(p)$ has (ordered) basis $\{E_1^*, E_2^*\}$, and so $d^2F_\lambda\big|_{W(p)\times W(p)}(p) = \operatorname{diag}(2\lambda_2, 2\lambda_2)$ with respect to this basis. Choosing $\lambda_2 = 1$ then yields $d^2F_\lambda\big|_{W(p)\times W(p)}(p)$ positive define. Thus we conclude that $(0, 0, 0, a)$ is a stable equilibrium point for $a \neq 0$.

Now consider an equilibrium point of the form $p = (0, b, d, 0)$ for $b \neq 0$ and $d \in \mathbb{R}$. In this case we have that

$$dF_\lambda(p) = \begin{bmatrix} 0 \\ 2b\lambda_2 \\ \lambda_1 \\ -2d\lambda_2 \end{bmatrix}.$$

Setting $\lambda_1 = \lambda_2 = 0$ we have that $dF_\lambda(p) = 0$. Then $d^2F_\lambda(p) = \operatorname{diag}\left(\frac{\lambda_0}{\beta_1}, 0, 0, \frac{\lambda_0}{\beta_2}\right)$ and

$$dH_{\max}(p) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad dP_3(p) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad d\mathcal{C}(p) = \begin{bmatrix} 0 \\ 2b \\ 0 \\ -2d \end{bmatrix}.$$

Hence $W(p) = \operatorname{span}\{E_1^*, E_2^*, E_4^*\} \cap \operatorname{span}\{E_1^*, E_3^*, dE_2^* + bE_4^*\}$ and as such has ordered basis $\{E_1^*, dE_2^* + bE_4^*\}$. Hence (w.r.t. this basis) we get that $d^2F_\lambda\big|_{W(p)\times W(p)}(p) = \operatorname{diag}\left(\frac{\lambda_0}{\beta_1}, \frac{b^2\lambda_0}{\beta_2}\right)$. Choosing $\lambda_0 = 1$ then yields $d^2F_\lambda\big|_{W(p)\times W(p)}(p)$ positive define. Therefore we conclude that $(0, b, d, 0)$ is a stable equilibrium point for $b \neq 0$ and $d \in \mathbb{R}$.

Next we consider equilibrium points of the form $p = (0, 0, b, c)$ satisfying $b \neq 0$ and $c(c\beta_1 + b\beta_2) > 0$ (note that we have already dealt with the case $b = 0$ and $c \in \mathbb{R}$). In particular note that $c \neq 0$. In this case we have that

$$dF_\lambda(p) = \begin{bmatrix} 0 \\ 0 \\ \lambda_1 - 2c\lambda_2 \\ \frac{c\lambda_0}{\beta_2} - 2b\lambda_2 \end{bmatrix}.$$

Let $\lambda_2 = \frac{c\lambda_0}{2b\beta_2}$ and $\lambda_1 = 2c\lambda_2$. Then we get that $dF_\lambda(p) = 0$. Furthermore, we get that

$$d^2F_\lambda(p) = \begin{bmatrix} \frac{\lambda_0}{\beta_1} + \frac{c\lambda_0}{b\beta_2} & 0 & 0 & 0 \\ 0 & \frac{c\lambda_0}{b\beta_2} & 0 & 0 \\ 0 & 0 & 0 & -\frac{c\lambda_0}{b\beta_2} \\ 0 & 0 & -\frac{c\lambda_0}{b\beta_2} & \frac{\lambda_0}{\beta_2} \end{bmatrix}$$

and

$$dH_{\max}(p) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{c}{\beta_2} \end{bmatrix} \qquad dP_3(p) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad d\mathcal{C}(p) = \begin{bmatrix} 0 \\ 0 \\ -2c \\ -2b \end{bmatrix}.$$

Hence $W(p)$ has ordered basis $\{E_1^*, E_2^*\}$ and so (w.r.t. this basis) we get that

$$d^2F_\lambda\big|_{W(p)\times W(p)}(p) = \operatorname{diag}\left(\frac{\lambda_0}{\beta_1} + \frac{c\lambda_0}{b\beta_2}, \frac{c\lambda_0}{b\beta_2}\right).$$

Taking $\lambda_0 = \frac{b}{c}\beta_2$ we get that $d^2F_\lambda\big|_{W(p)\times W(p)}(p) = \operatorname{diag}\left(1 + \frac{b\beta_2}{c\beta_1}, 1\right)$. Form our assumption $c(c\beta_1 + b\beta_2) > 0$, we have that $c^2\beta_1 + cb\beta_2 > 0$, and so (dividing through by $c^2\beta_1$) we get that $1 + \frac{b\beta_2}{c\beta_1} > 0$. Hence we conclude that $d^2F_\lambda\big|_{W(p)\times W(p)}(p)$ is positive definite and therefore get that $p = (0, 0, b, c)$ is a stable equilibrium point for $b \neq 0$ and $c(c\beta_1 + b\beta_2) > 0$.

That completes the proof of the stable equilibrium points. We now continue on to the unstable equilibrium points. First consider an equilibrium point of the form $p = (0, 0, b, c)$ for $b \neq 0$ and $c\,(c\beta_1 + b\beta_2) \leq 0$. In this case, we get that

$$D\vec{H}_{\max}(p) = \begin{bmatrix} 0 & -\frac{c}{\beta_2} & 0 & 0 \\ \frac{b}{\beta_1} + \frac{c}{\beta_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and so the eigenvalues of $D\vec{H}_{\max}(p)$ are given by

$$\left\{0, 0, -\frac{\sqrt{-c^2\beta_1 - bc\beta_2}}{\sqrt{\beta_1}\beta_2}, \frac{\sqrt{-c^2\beta_1 - bc\beta_2}}{\sqrt{\beta_1}\beta_2}\right\}.$$

Now as $c\,(c\beta_1 + b\beta_2) \leq 0$ we have that $-c^2\beta_1 - bc\beta_2 > 0$ or $-c^2\beta_1 - bc\beta_2 = 0$. If $-c^2\beta_1 - bc\beta_2 > 0$, then one of the eigenvalues is (real and) positive and so p is an unstable equilibrium point. Now suppose that $-c^2\beta_1 - bc\beta_2 = -c(c\beta_1 + b\beta_2) = 0$. Thus either $c = 0$ or $c = -\frac{\beta_2}{\beta_1}$. If $c = 0$, we get that 0 is an eigenvalue (of algebraic multiplicity four), with eigenvectors $\{E_2, E_3, E_4\}$. Hence, the algebraic multiplicity is greater than the geometric multiplicity, and so p is unstable. On the other hand, if $c = -\frac{\beta_2}{\beta_1}$, we get that 0 is an eigenvalue (of algebraic multiplicity four), with eigenvectors $\{E_1, E_3, E_4\}$. Therefore (again as the algebraic multiplicity is greater than the geometric multiplicity) p is an unstable equilibrium point. Thus we have shown that any equilibrium point $p = (0, 0, b, c)$ for $b \neq 0$ and $c\,(c\beta_1 + b\beta_2) \leq 0$ is unstable.

It remains to be shown that a point of the form $p = (b, 0, a, -\frac{\beta_2}{\beta_1}a)$, $b \neq 0$, $a \in \mathbb{R}$ is unstable. For such a point, we get that

$$D\vec{H}_{\max}(p) = \begin{bmatrix} 0 & \frac{a}{\beta_1} & 0 & 0 \\ 0 & 0 & \frac{b}{\beta_1} & \frac{b}{\beta_2} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{b}{\beta_1} & 0 & 0 \end{bmatrix}.$$

Consequently the eigenvalues of $D\vec{H}_{\max}(p)$ are $\left\{0, 0, -\frac{b}{\sqrt{\beta_1\beta_2}}, \frac{b}{\sqrt{\beta_1\beta_2}}\right\}$. As $b \neq 0$, we have that $-\frac{b}{\sqrt{\beta_1\beta_2}} > 0$ or $\frac{b}{\sqrt{\beta_1\beta_2}} > 0$. Hence we conclude that p is unstable. □

To aid in the plotting these equilibrium points, we give an alternative description of the condition $c\,(c\beta_1 + b\beta_2) \leq 0$.

4.2.3 PROPOSITION. *Assuming $p_3 \neq 0$, the inequality $p_4\,(p_4\beta_1 + p_3\beta_2) \leq 0$ is equivalent to*

$$(p_3 > 0 \text{ and } -\tfrac{\beta_2}{\beta_1}p_3 \leq p_4 \leq 0) \text{ or } (p_3 < 0 \text{ and } 0 \leq p_4 \leq -\tfrac{\beta_2}{\beta_1}p_3).$$

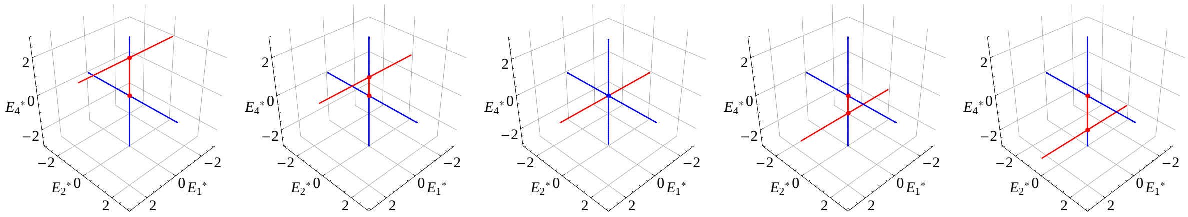
PROOF. If $p_4 = 0$ the equivalence is true. Next suppose $p_3 > 0$. If $p_4 > 0$, then the equivalence is true (as $p_4\beta_1 + p_3\beta_2 > 0$ an so $p_4\,(p_4\beta_1 + p_3\beta_2) > 0$, yielding both statements necessarily false). Then again, for the case $p_4 < 0$, we get that

$$(p_4\,(p_4\beta_1 + p_3\beta_2) \leq 0 \,\wedge\, p_4 < 0) \quad\Leftrightarrow\quad (p_4 \geq -\tfrac{\beta_2}{\beta_1}p_3 \,\wedge\, p_4 < 0) \quad\Leftrightarrow\quad -\tfrac{\beta_2}{\beta_1}p_3 \leq p_4 < 0.$$

On the other hand suppose $p_3 < 0$. If $p_4 < 0$ the equivalence is true (as $p_4\beta_1 + p_3\beta_2 < 0$ and so $p_4\,(p_4\beta_1 + p_3\beta_2) > 0$, yielding both statements necessarily false). Then again, for the case $p_4 > 0$, we get that

$$(p_4\,(p_4\beta_1 + p_3\beta_2) \leq 0 \,\wedge\, p_4 > 0) \quad\Leftrightarrow\quad (p_4 \leq -\frac{\beta_2}{\beta_1}p_3 \,\wedge\, p_4 > 0) \quad\Leftrightarrow\quad 0 < p_4 \leq -\frac{\beta_2}{\beta_1}p_3. \;\square$$

Making use of the "sliding window" $P_3^{-1}(p_3)$, $p_3 \in \mathbb{R}$ we graph the equilibrium sets E_s and E_u in figure 4.1. E_s is plotted in blue and E_u is plotted in red. We used $\beta_1 = \beta_2 = 1$ for generating these figures. (We will use this conversion of plotting stable equilibrium points in blue and unstable equilibrium points in red throughout this section.)



(a) $p_3 = -2$ (b) $p_3 = -1$ (c) $p_3 = 0$ (d) $p_3 = 1$ (e) $p_3 = 2$

Figure 4.1: Equilibrium points of reduced system (blue=stable, red=unstable)

4.2.2 Classification of integral curves

Our goal in this subsection is to show that any integral curve of $\vec{H}_{\max}$ is either constant, periodic or bounded (with singletons as α- and ω-limit sets). This is done in theorem 4.2.16. As $H_{\max}$, P_3 and $\mathcal{C}$ are constants of motion for $\vec{H}_{\max}$ (see corollary A.3.8) we have that the integral curves of $\vec{H}_{\max}$ are contained in the (intersection of) the level sets defined by $H_{\max}$, P_3 and $\mathcal{C}$. As such, it is of use to study (the intersection of) these level sets. We start by showing that, within the sliding window $P_3^{-1}(\rho)$, $\rho \in \mathbb{R}$, the level sets $\mathcal{C}$ and $H_{\max}$ are closed embedded submanifolds.

4.2.4 Remark. We note that we use the term *closed embedded submanifold* (as used by J. Lee [22] and Boothby [8]) to denote an embedded submanifold that is in addition topologically closed (as a subset of the ambient manifold). It may be shown that any closed embedded submanifold is exactly a *proper submanifold* (as defined by Sharpe [35]), i.e., the inclusion map is in addition proper.

4.2.5 Proposition. *The (level) sets*

$$P_3^{-1}(\rho),\ \rho \in \mathbb{R} \qquad\qquad P_3^{-1}(\rho) \cap H_{\max}^{-1}(h),\ \rho \in \mathbb{R},\ h > 0$$
$$P_3^{-1}(\rho) \cap \mathcal{C}^{-1}(c),\ \rho \neq 0,\ c \in \mathbb{R} \qquad\qquad P_3^{-1}(0) \cap \mathcal{C}^{-1}(c),\ c > 0$$

are closed embedded submanifolds of $(\widetilde{\mathfrak{h}}_3^\diamond)^*$. Furthermore, all but $P_3^{-1}(\rho)$ are two-dimensional.

Proof. We make use of constant-rank level set theorem (as for instance presented in [22]). Note that (as a linear map) P_3 is smooth. Next (w.r.t. the ordered basis of $(\widetilde{\mathfrak{h}}_3^\diamond)^*$) we have that

$$DP_3(p) = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}$$

is of full rank for all $p \in (\widetilde{\mathfrak{h}}_3^\diamond)^*$. Thus $P_3^{-1}(\rho)$ is a closed embedded 3-submanifold of $(\widetilde{\mathfrak{h}}_3^\diamond)^*$ for all $\rho \in \mathbb{R}$. Henceforth, for a fixed $\rho \in \mathbb{R}$, a point $(p_1, p_2, \rho, p_4) \in P_3^{-1}(\rho)$ will simply be written as (p_1, p_2, p_4). (That is $P_1 \times P_2 \times P_4 : P_3^{-1}(\rho) \subset (\widetilde{\mathfrak{h}}_3^\diamond)^* \to \mathbb{R}^3$ is used as a global coordinate chart.) Then, for some fixed $\rho \in \mathbb{R}$, we have that $\mathcal{C}|_{P_3^{-1}(\rho)}, H_{\max}|_{P_3^{-1}(\rho)} : P_3^{-1}(\rho) \to \mathbb{R}$ are smooth functions and we get (in coordinates) that

$$D\,\mathcal{C}|_{P_3^{-1}(\rho)}(p_1, p_2, p_4) = \begin{bmatrix} 2p_1 & 2p_2 & -2\rho \end{bmatrix}$$
$$D\,H_{\max}|_{P_3^{-1}(\rho)}(p_1, p_2, p_4) = \begin{bmatrix} \frac{1}{\beta_1} p_1 & 0 & \frac{1}{\beta_2} p_4 \end{bmatrix}.$$

Suppose $\rho \in \mathbb{R}$ and $h > 0$. Observe that $V = P_3^{-1}(\rho)\backslash\{(0,r,0) \mid r \in \mathbb{R}\}$ is an open subset of $P_3^{-1}(\rho)$ and hence an embedded submanifold of $P_3^{-1}(\rho)$. Then note that for $p \in P_3^{-1}(\rho)$, we have that $p \in V$ if and only if $H_{\max}|_{P_3^{-1}(\rho)}(p) > 0$. That is to say $H_{\max}|_{P_3^{-1}(\rho)}^{-1}(h) \subset V$. Furthermore, for $p \in V$, we have that $p_1 \neq 0$ or $p_4 \neq 0$ and consequently that $D\, H_{\max}|_{P_3^{-1}(\rho)}(p)$ is of full rank. Thus we get that $H_{\max}^{-1}(h) \cap P_3^{-1}(\rho) = H_{\max}|_{P_3^{-1}(\rho)}^{-1}(h)$ is an embedded 2-submanifold of V. Consequently $H_{\max}^{-1}(h) \cap P_3^{-1}(\rho)$ is an embedded 2-submanifold of $P_3^{-1}(\rho)$ and hence of $(\widetilde{\mathfrak{h}}_3^\diamond)^*$. As $\{h\}$ is closed in $\mathbb{R}$, and $H_{\max}$ is continuous, we get that $H_{\max}^{-1}(h)$ is closed. Similarly, $P_3^{-1}(\rho)$ is closed as P_3 is continuous and $\{\rho\}$ is closed in $\mathbb{R}$. Thus $P_3^{-1}(\rho) \cap H_{\max}^{-1}(h)$ is closed.

If $\rho \neq 0$ then $D\,\mathcal{C}|_{P_3^{-1}(\rho)}(p)$ is of full rank for $p \in P_3^{-1}(\rho)$ and hence $\mathcal{C}|_{P_3^{-1}(\rho)}^{-1}(c)$ is a closed embedded 2-submanifold of $P_3^{-1}(\rho)$ for $c \in \mathbb{R}$. Accordingly $P_3^{-1}(\rho) \cap \mathcal{C}^{-1}(c)$ is a closed embedded 2-submanifold of $(\widetilde{\mathfrak{h}}_3^\diamond)^*$ for $\rho \neq 0$ and $c \in \mathbb{R}$.

For the last case, suppose $c > 0$. We proceed in a similar fashion as to the case of $P_3^{-1}(\rho) \cap H_{\max}^{-1}(h)$. Observe that $V = P_3^{-1}(0)\backslash\{(0,0,r) \mid r \in \mathbb{R}\}$ is an open subset of $P_3^{-1}(0)$ and hence an embedded submanifold of $P_3^{-1}(0)$. Then notice that, for $p \in P_3^{-1}(0)$, we have that $p \in V$ if and only if $\mathcal{C}|_{P_3^{-1}(0)}(p) > 0$. That is to say $\mathcal{C}|_{P_3^{-1}(0)}^{-1}(c) \subset V$. Furthermore for $p \in V$ we have that $p_1 \neq 0$ or $p_2 \neq 0$. Consequently, for $p \in V$, we get that $D\,\mathcal{C}|_{P_3^{-1}(0)}(p)$ is of full rank. Thus, for $c > 0$, we have that $P_3^{-1}(0) \cap \mathcal{C}^{-1}(c) = \mathcal{C}|_{P_3^{-1}(0)}^{-1}(c)$ is a embedded 2-submanifold of V. It therefore follows that $P_3^{-1}(0) \cap \mathcal{C}^{-1}(c)$ is an embedded 2-submanifold of $(\widetilde{\mathfrak{h}}_3^\diamond)^*$. Finally, as $\{0\}$ and $\{c\}$ are closed in $\mathbb{R}$, we get that $P_3^{-1}(0) \cap \mathcal{C}^{-1}(c)$ is closed. □

Next we define the intersection of the levels sets of P_3, $\mathcal{C}$ and $H_{\max}$ as

$$\mathcal{K}_{(\rho,c,h)} = P_3^{-1}(\rho) \cap \mathcal{C}^{-1}(c) \cap H_{\max}^{-1}(h),$$

where $\rho = p_3$, $c = \mathcal{C}(p)$ and $h = H_{\max}(p)$ for some $p \in (\widetilde{\mathfrak{h}}_3^\diamond)^*$. (Notice that, for any $\bar{p} \in \mathcal{K}_{(\rho,c,h)}$, we get that $P_3(\bar{p}) = \bar{p}_3 = \rho$, $\mathcal{C}(\bar{p}) = \bar{p}_1^2 + \bar{p}_2^2 - 2\bar{p}_3\bar{p}_4 = c$ and $H_{\max}(\bar{p}) = \frac{1}{2}\big(\frac{1}{\beta_1}\bar{p}_1^2 + \frac{1}{\beta_2}\bar{p}_4^2\big) = h$.)

4.2.6 REMARK. At this time we note that, as a finite-dimensional vector space, the topology on $(\widetilde{\mathfrak{h}}_3^\diamond)^* \cong \mathbb{R}^4$ is generated by any norm on $(\widetilde{\mathfrak{h}}_3^\diamond)^*$. We will conveniently use the following norm:

$$||p|| = \sqrt{p_1^2 + p_2^2 + p_3^2 + p_4^2}.$$

4.2.7 PROPOSITION. *The subset $\mathcal{K}_{(\rho,c,h)}$ of $(\widetilde{\mathfrak{h}}_3^\diamond)^*$ is compact.*

PROOF. By continuity of P_3, $\mathcal{C}$ and $H_{\max}$, we get that $\mathcal{K}_{(\rho,c,h)}$ is closed. Next for $p \in \mathcal{K}_{(\rho,c,h)}$ we have that $p_3^2 = \rho^2$, $p_1^2 + p_2^2 - 2p_3p_4 = c$ and $\frac{p_1^2}{\beta_1} + \frac{p_4^2}{\beta_2} = 2h$. From the condition $\frac{p_1^2}{\beta_1} + \frac{p_4^2}{\beta_2} = 2h$ we get that $p_1^2 \leq 2h\beta_1$ and $p_4^2 \leq 2h\beta_2$. Hence, from $p_2^2 = c + 2p_3p_4 - p_1^2$, we get that $p_2^2 \leq c + 2\sqrt{2\rho^2 h\beta_2}$. Hence, $||p||^2 \leq \rho^2 + 2h(\beta_1 + \beta_2) + c + 2\sqrt{2\rho^2 h\beta_2}$. We therefore have that $\mathcal{K}_{(\rho,c,h)}$ is a closed and bounded subset of $(\widetilde{\mathfrak{h}}_3^\diamond)^*$, and hence compact. □

4.2.8 COROLLARY. *The vector field $\vec{H}_{\max}$ is complete.*

Proof. Let $p(\cdot)$ be any integral curve of $\vec{H}_{\max}$, such that $p \in \operatorname{im} p(\cdot)$. Let $\rho = p_3$, $c = p_1^2 + p_2^2 - 2p_3p_4$ and $h = \frac{1}{2}(\frac{1}{\beta_1}p_1^2 + \frac{1}{\beta_2}p_4^2)$. Then (as P_3, $\mathcal{C}$ and $H_{\max}$ are constants of motion for $\vec{H}_{\max}$) we get that $\operatorname{im} p(\cdot) \subseteq \mathcal{K}_{(\rho,c,h)}$. The result then follows from corollary A.4.2. □

Note that, as $\vec{H}_{\max}$ is complete, any integral curve of $\vec{H}_{\max}$ has maximal domain given by $\mathbb{R}$. We will restrict ourselves to considering such integral curves for the remainder of this section. We now proceed on to showing that $\mathcal{K}_{(\rho,c,h)}$ is a embedded 1-submanifold (under some conditions). However, before we do so we have the following (supporting) lemma.

4.2.9 Lemma. *We have the following equivalence*

$$p \in E \quad \Leftrightarrow \quad \begin{bmatrix} 2p_1 & 2p_2 & -2p_3 \\ \frac{1}{\beta_1}p_1 & 0 & \frac{1}{\beta_2}p_4 \end{bmatrix} \text{ is not of full rank.}$$

Proof. We have that $p \in E$ if and only if

$$-\tfrac{1}{\beta_2}p_2p_4 = 0 \qquad p_1\left(\tfrac{1}{\beta_1}p_3 + \tfrac{1}{\beta_2}p_4\right) = 0 \qquad \tfrac{1}{\beta_1}p_1p_2 = 0.$$

This in turn is equivalent to

$$\left(\tfrac{1}{\beta_2}p_2p_4\right)^2 + \left(p_1\left(\tfrac{1}{\beta_1}p_3 + \tfrac{1}{\beta_2}p_4\right)\right)^2 + \left(\tfrac{1}{\beta_1}p_1p_2\right)^2 = 0$$

But, using Mathematica, we get that

$$\left(\tfrac{1}{\beta_2}p_2p_4\right)^2 + \left(p_1\left(\tfrac{1}{\beta_1}p_3 + \tfrac{1}{\beta_2}p_4\right)\right)^2 + \left(\tfrac{1}{\beta_1}p_1p_2\right)^2 = \frac{p_2^2p_4^2\beta_1^2 + p_1^2p_2^2\beta_2^2 + {p_1}^2\left(p_4\beta_1 + p_3\beta_2\right)^2}{\beta_1^2\beta_2^2}.$$

On the other hand let

$$M = \begin{bmatrix} 2p_1 & 2p_2 & -2p_3 \\ \frac{1}{\beta_1}p_1 & 0 & \frac{1}{\beta_2}p_4 \end{bmatrix}.$$

Then M is not of full rank if and only if $\det(MM^\top) = 0$. But (again using Mathematica for calculations) we have that

$$\det(MM^\top) = \frac{4\left(p_2^2p_4^2\beta_1^2 + p_1^2p_2^2\beta_2^2 + p_1^2\left(p_4\beta_1 + p_3\beta_2\right)^2\right)}{\beta_1^2\beta_2^2}$$

Notice that the two conditions are exactly the same to get the result. □

4.2.10 Proposition. *Let $\rho = p_3$, $c = \mathcal{C}(p)$ and $h = H_{\max}(p)$ for some $p \in (\widetilde{\mathfrak{h}}_3^\diamond)^*$. Further suppose that $h > 0$ and that $c > 0$ whenever $\rho = 0$. If $E \cap \mathcal{K}_{(\rho,c,h)} = \emptyset$, then $\mathcal{K}_{(\rho,c,h)}$ is a compact embedded 1-submanifold of $(\widetilde{\mathfrak{h}}_3^\diamond)^*$. If $E \cap \mathcal{K}_{(\rho,c,h)} \neq \emptyset$, then $\mathcal{K}_{(\rho,c,h)} \backslash E$ is a bounded embedded 1-submanifold of $(\widetilde{\mathfrak{h}}_3^\diamond)^*$.*

PROOF. By proposition 4.2.5 we have that $P_3^{-1}(\rho)$ is a closed embedded 3-submanifold of $(\widetilde{\mathfrak{h}}_3^\diamond)^*$. (Again we use $P_1 \times P_2 \times P_4 : P_3^{-1}(\rho) \to \mathbb{R}^3$ as a global chart.) Moreover, as E is a closed subset of $(\widetilde{\mathfrak{h}}_3^\diamond)^*$ (as a union of a finite number of 2D linear subspaces), we have that $P_3^{-1}(\rho)\backslash E$ is an open subset of $P_3^{-1}(\rho)$, and hence an embedded 3-submanifold of $P_3^{-1}(\rho)$. Consider the smooth map $F : P_3^{-1}(\rho)\backslash E \to \mathbb{R}^2$, given by

$$F(p) = \begin{bmatrix} \mathcal{C}(p) \\ H_{\max}(p) \end{bmatrix} \qquad\qquad DF(p) = \begin{bmatrix} 2p_1 & 2p_2 & -2\rho \\ \frac{1}{\beta_1}p_1 & 0 & \frac{1}{\beta_2}p_4 \end{bmatrix}.$$

Note that $F^{-1}(c, h) = P_3^{-1}(\rho)\backslash E \cap \mathcal{C}^{-1}(c) \cap H_{\max}^{-1}(h) = \mathcal{K}_{(\rho,c,h)}\backslash E$. Now by the preceding lemma we have that $DF(p)$ is of full rank for $p \in P_3^{-1}(\rho)\backslash E$. Hence we have that $F^{-1}(c, h)$ is an embedded 1-submanifold of $P_3^{-1}(\rho)\backslash E$ and hence of $(\widetilde{\mathfrak{h}}_3^\diamond)^*$. That is to say $\mathcal{K}_{(\rho,c,h)}\backslash E$ is an embedded 1-submanifold of $(\widetilde{\mathfrak{h}}_3^\diamond)^*$.

If $E \cap \mathcal{K}_{(\rho,c,h)} = \emptyset$, then $\mathcal{K}_{(\rho,c,h)}\backslash E = \mathcal{K}_{(\rho,c,h)}$ and as $\mathcal{K}_{(\rho,c,h)}$ is compact (proposition 4.2.7) we then have that $\mathcal{K}_{(\rho,c,h)}$ is a compact embedded 1-submanifold of $(\widetilde{\mathfrak{h}}_3^\diamond)^*$. On the other hand (if $E \cap \mathcal{K}_{(\rho,c,h)} \neq \emptyset$), we no longer have that $\mathcal{K}_{(\rho,c,h)}\backslash E$ is compact. However, we do have that $\mathcal{K}_{(\rho,c,h)}\backslash E$ is bounded as it is contained in $\mathcal{K}_{(\rho,c,h)}$, which is compact (proposition 4.2.7). □

4.2.11 PROPOSITION. *Let $p(\cdot) : \mathbb{R} \to (\widetilde{\mathfrak{h}}_3^\diamond)^*$ be an integral curve of $\vec{H}_{\max}$, $\rho = p_3(0)$, $c = \mathcal{C}(p(0))$ and $h = H_{\max}(p(0))$. If $p(0) \notin E$ (i.e., $p(\cdot)$ is not constant) then*

$$\operatorname{im} p(\cdot) \subseteq \mathcal{K}^0 \subseteq \mathcal{K}_{(\rho,c,h)}\backslash E \subseteq \mathcal{K}_{(\rho,c,h)},$$

where $\mathcal{K}^0$ is the connected component of $\mathcal{K}_{(\rho,c,h)}\backslash E$ containing $p(0)$. Moreover $\mathcal{K}^0$ is a bounded connected embedded 1-submanifold of $(\widetilde{\mathfrak{h}}_3^\diamond)^*$.

PROOF. $\mathcal{K}_{(\rho,c,h)}\backslash E \subseteq \mathcal{K}_{(\rho,c,h)}$ trivially. By proposition 4.2.10 we have that $\mathcal{K}_{(\rho,c,h)}\backslash E$ is a bounded embedded 1-submanifold of $(\widetilde{\mathfrak{h}}_3^\diamond)^*$. Hence $\mathcal{K}^0$ is a bounded connected embedded 1-submanifold of $(\widetilde{\mathfrak{h}}_3^\diamond)^*$. We are left to show that $\operatorname{im} p(\cdot) \subseteq \mathcal{K}^0$. Now as

$$\mathcal{K}_{(\rho,c,h)} = P_3^{-1}(\rho) \cap \mathcal{C}^{-1}(c) \cap H_{\max}^{-1}(h)$$

and P_3, $\mathcal{C}$ and $H_{\max}$ are constants of motion for $\vec{H}_{\max}$, it follows that $\operatorname{im} p(\cdot) \subseteq \mathcal{K}_{(\rho,c,h)}$. Next as $p(0) \notin E$ it follows that $\operatorname{im} p(\cdot) \subseteq \mathcal{K}_{(\rho,c,h)}\backslash E$ (if $p(t_1) \in E$ then $\operatorname{im} p(\cdot) \subseteq \{p(t_1)\} \subseteq E$). Then, as $p(\cdot)$ is an integral curve, it follows that $\operatorname{im} p(\cdot)$ is connected and contains $p(0)$. Therefore we get that $\operatorname{im} p(\cdot)$ is contained in the connected component of $\mathcal{K}_{(\rho,c,h)}\backslash E$ containing $p(0)$, namely $\mathcal{K}^0$. □

Having described the submanifolds on which the integral curves of $H_{\max}$ develop, we are almost ready to make our classification. However, before we do so we develop a (supporting) result concerning integrals curves on compact connected 1-manifolds; present two (technical supporting) lemmas; and recall a topological result regarding connectedness (as a lemma).

4.2.12 PROPOSITION. *Let $\mathcal{M}$ be a connected compact smooth 1-manifold and $X : \mathcal{M} \to T\mathcal{M}$ be a complete smooth vector field which is never zero (i.e., $\forall m \in \mathcal{M}, X(m) \neq 0$). Then any (maximal) integral curve $c(\cdot) : \mathbb{R} \to \mathcal{M}$ of X is periodic and surjective.*

Proof. Suppose $c(\cdot) : \mathbb{R} \to \mathcal{M}$ is an integral curve of X. As $\mathcal{M}$ is a connected compact smooth 1-manifold, we have that $\mathcal{M}$ is diffeomorphic to $\mathsf{SO}(2)$ (or equivalently to $\mathbb{S}^1$). That is, we have a diffeomorphism $\phi : \mathcal{M} \to \mathsf{SO}(2)$. Then the integral curve $\phi(c(\cdot)) : \mathbb{R} \to \mathsf{SO}(2)$ of (the push forward) $\phi_* X$ can be written as

$$\phi(c(t)) = \begin{bmatrix} \cos\theta(t) & -\sin\theta(t) \\ \sin\theta(t) & \cos\theta(t) \end{bmatrix},$$

for some smooth function $\theta : \mathbb{R} \to \mathbb{R}$. (This follows as $\phi(c(\cdot))$ is required to be a smooth curve on $\mathsf{SO}(2)$.) Now as $\frac{d}{dt}\phi(c(t)) = \phi_* X(\phi(c(t)))$ we have that

$$\begin{bmatrix} \dot\theta(t) & 0 \\ 0 & \dot\theta(t) \end{bmatrix} = \begin{bmatrix} -\sin\theta(t) & -\cos\theta(t) \\ \cos\theta(t) & -\sin\theta(t) \end{bmatrix}^{-1} \phi_* X(\phi(c(t)))$$

This is equivalent to $\dot\theta(t) = Y(\theta(t))$ for some smooth vector field $Y : \mathbb{R} \to \mathbb{R}$. (Effectively we lift the dynamics to the universal covering $\widetilde{\mathsf{SO}}(2) \cong \mathbb{R}$.) Note that Y is periodic, specifically $Y(\theta + 2\pi) = Y(\theta)$ for $\theta \in \mathbb{R}$. Next note that as X is never zero we have that Y is never zero (if Y were zero at $\theta(t_1)$, then $\phi_* X(\phi(c(t_1))) = 0$ and so $X(c(t_1)) = 0$). Thus, as Y is smooth, we have that either $\forall t \in \mathbb{R}$, $Y(\theta(t)) > 0$ or $\forall t \in \mathbb{R}$, $Y(\theta(t)) < 0$. W.l.o.g. we assume $Y(\theta(t)) > 0$ for $t \in \mathbb{R}$. Now, as $[0, 2\pi]$ is compact, $|Y(\cdot)| = Y(\cdot)$ attains a minimum $y > 0$ on this interval. Moreover, as Y is periodic, this is a global minimum (for $\theta \notin [0, 2\pi]$ there exists $n \in \mathbb{Z}$ such that $\theta + 2n\pi \in [0, 2\pi]$, then $Y(\theta) = Y(\theta + 2n\pi) \geq y$). Thus $\theta : \mathbb{R} \to \mathbb{R}$ is strictly increasing. Moreover, as $\dot\theta(t) \geq y$, we get that $\theta(t) - \theta(0) \geq yt$ for $t \in [0, \infty)$. Hence we have that $\theta(\frac{2\pi}{y}) - \theta(0) \geq 2\pi$, and as $\theta(\cdot)$ is continuous we have that $\forall r \in [\theta(0), \theta(0) + 2\pi]$, $\exists t \in [0, \frac{2\pi}{y}]$ such that $\theta(t) = r$. Thus $\phi(c([0, \frac{2\pi}{y}])) = \mathsf{SO}(2)$ proving surjective property. Also there exists $t \in [0, \frac{2\pi}{y}]$, $t \neq 0$ such that $\theta(t) = \theta(0) + 2\pi$. So $\phi(c(0)) = \phi(c(t))$ and hence $c(0) = c(t)$, proving $c(\cdot)$ is periodic. □

4.2.13 Lemma. *Let $X : \mathbb{R} \to \mathbb{R}$ be a complete smooth vector field which is strictly positive (i.e., $\forall r \in \mathbb{R}$, $X(r) > 0$); $c(\cdot) : \mathbb{R} \to \mathbb{R}$ be a (maximal) integral curve of X; and $\{t_n\}_{n \in \mathbb{N}}$ be an unbounded sequence (i.e., $t_n \to \pm\infty$ as $n \to \infty$). Then $\{c(t_n)\}_{n \in \mathbb{N}}$ does not converge in $\mathbb{R}$.*

Proof. Suppose $c(t_n) \to s \in \mathbb{R}$ as $n \to \infty$. Then there exists $N \in \mathbb{N}$ such that $\forall n \geq N$, $c(t_n) \in [s-1, s+1]$. Then, as $[s-1, s+1]$ is compact, X attains a minimum $x_{\min}$ and a maximum $x_{\max}$ on $[s-1, s+1]$. Moreover, as X is strictly positive, $0 < x_{\min} \leq x_{\max}$. Hence we get that $x_{\min} \leq \dot c(t) \leq x_{\max}$ and so (for $t \in \mathbb{R}$) we have that

$$(t - t_N)x_{\min} \leq c(t) - c(t_N) \leq (t - t_N)x_{\max}.$$

Suppose $t_n \to \infty$ as $n \to \infty$. Then there exists $n_1 \in \mathbb{N}$ such that $n_1 > N$ and $t_{n_1} - t_N > \frac{2}{x_{\min}}$. Hence we get that $2 < c(t_1) - c(t_N)$ implying that $c(t_1) > 2 + c(t_N)$. But as $c(t_N) \in [s-1, s+1]$, we get that $c(t_1) > s+1$ and so $c(t_1) \notin [s-1, s+1]$, a contradiction. On the other hand, suppose $t_n \to -\infty$ as $n \to \infty$. Then there exists $n_2 \in \mathbb{N}$ such that $n_2 > N$ and $t_{n_2} - t_N < -\frac{2}{x_{\max}}$. Therefore we get that $c(t_2) - c(t_N) < -2$ implying $c(t_1) < c(t_N) - 2$. But as $c(t_N) \in [s-1, s+1]$, we get that $c(t_2) < s-1$ and so $c(t_2) \notin [s-1, s+1]$, a contradiction. Thus we get that $\{c(t_n)\}_{n \in \mathbb{N}}$ does not converge in $\mathbb{R}$. □

4.2.14 Lemma. *The set $P_3^{-1}(\rho) \cap H_{\max}^{-1}(h) \cap E$ contains between two and four points (thus it is finite and hence totally disconnected) for $\rho \in \mathbb{R}, h > 0$. Specifically*

$$|P_3^{-1}(\rho) \cap H_{\max}^{-1}(h) \cap E| = 3 + \operatorname{sgn}\left(2h\beta_1 - \tfrac{\beta_2}{\beta_1}\rho^2\right).$$

Proof. Recall that $H_{\max}(p) = \frac{1}{2}\left(\frac{p_1^2}{\beta_1} + \frac{p_4^2}{\beta_2}\right)$ and

$$E = \{(a, 0, b, -\tfrac{\beta_2}{\beta_1}b), (0, a, b, 0), (0, 0, a, b) \mid a, b \in \mathbb{R}\}.$$

As $h > 0$, for $p \in H_{\max}^{-1}(h)$, we have that $p_1 \neq 0$ or $p_4 \neq 0$. Thus there are no equilibrium points of the form $(0, a, b, 0)$ in $P_3^{-1}(\rho) \cap H_{\max}^{-1}(h) \cap E$.

Next suppose $p = (0, 0, a, b)$ for some $a, b \in \mathbb{R}$ and $p \in P_3^{-1}(\rho) \cap H_{\max}^{-1}(h)$. Then $a = \rho$ and $H_{\max}(p) = \frac{b^2}{2\beta_2} = h > 0$. Thus $b = \pm\sqrt{2h\beta_2}$. That is to say there are exactly two equilibrium points of this form, namely $(0, 0, \rho, \sqrt{2h\beta_2})$ and $(0, 0, \rho, -\sqrt{2h\beta_2})$.

For the last case suppose $p = (a, 0, b, -\frac{\beta_2}{\beta_1}b)$ for some $a, b \in \mathbb{R}$ and $p \in P_3^{-1}(\rho) \cap H_{\max}^{-1}(h)$. Then $b = \rho$, and as $H_{\max}(p) = h$, we get that $a^2 = 2h\beta_1 - \frac{\beta_2}{\beta_1}\rho^2$. Now if $2h\beta_1 - \frac{\beta_2}{\beta_1}\rho^2 < 0$ then there is no real solution for a, that is to say there are no equilibrium points of this form. If $2h\beta_1 - \frac{\beta_2}{\beta_1}\rho^2 = 0$ then there is exactly one equilibrium point of this form, namely $(0, 0, \rho, -\frac{\beta_2}{\beta_1}\rho)$. If $2h\beta_1 - \frac{\beta_2}{\beta_1}\rho^2 > 0$ then there are exactly two equilibrium points of this form, namely $(\sqrt{2h\beta_1 - \frac{\beta_2}{\beta_1}\rho^2}, 0, \rho, -\frac{\beta_2}{\beta_1}\rho)$ and $(-\sqrt{2h\beta_1 - \frac{\beta_2}{\beta_1}\rho^2}, 0, \rho, -\frac{\beta_2}{\beta_1}\rho)$. □

4.2.15 Lemma. ([38]) *If V is a connected subset of a topological space $\mathcal{M}$ and $V \subseteq W \subseteq \operatorname{cl} V$, then W is connected.*

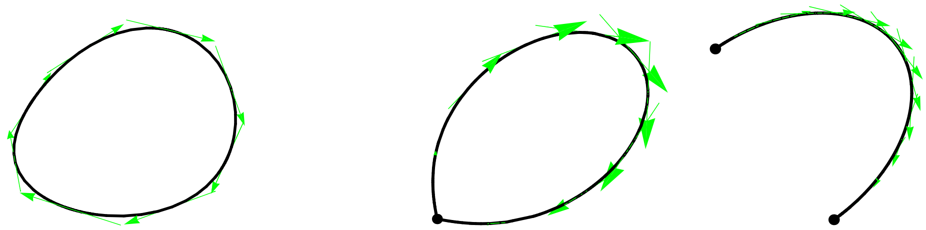
We now produce the main result of this subsection. (For the definition of α- and ω-limit sets see section A.4.1.)

4.2.16 Theorem. *Let $p(\cdot) : \mathbb{R} \to (\widetilde{\mathfrak{h}}_3^\diamond)^*$ be an integral curve of $\vec{H}_{\max}$. Let $c = \mathcal{C}(p(0))$, $h = H_{\max}(p(0))$ and $\mathcal{K}^0$ be the connected component of $\mathcal{K}_{(p_3(0),c,h)} \backslash E$ containing $p(0)$. We classify the behaviour of $p(\cdot)$ as one of three types:*

1. *if $p(0) \in E$ then $p(\cdot)$ is constant;*
2. *if $p(0) \notin E$ and $\mathcal{K}^0$ is closed in $(\widetilde{\mathfrak{h}}_3^\diamond)^*$ (or equivalently $\operatorname{cl}\mathcal{K}^0 \cap E = \emptyset$), then $p(\cdot)$ is periodic (and has image $\mathcal{K}^0$); figure 4.2a;*
3. *if $p(0) \notin E$ and $\mathcal{K}^0$ is not closed in $(\widetilde{\mathfrak{h}}_3^\diamond)^*$ (or equivalently $\operatorname{cl}\mathcal{K}^0 \cap E \neq \emptyset$), then $p(\cdot)$ is bounded and the limit sets $\lim_\omega p(\cdot)$ and $\lim_\alpha p(\cdot)$ are singletons (possibly the same) in E; figure 4.2b.*

We illustrate the above typical cases (excluding that of the constant curve) in figure 4.2.

Proof. We start by proving the equivalence: $\mathcal{K}^0$ is closed if and only if $\operatorname{cl}\mathcal{K}^0 \cap E = \emptyset$. If $\mathcal{K}^0$ is closed, then $\operatorname{cl}\mathcal{K}^0 = \mathcal{K}^0 \subseteq \mathcal{K}_{(p_3(0),c,h)} \backslash E$ and hence $\operatorname{cl}\mathcal{K}^0 \cap E = \emptyset$. Conversely if $\operatorname{cl}\mathcal{K}^0 \cap E = \emptyset$,



(a) $\mathcal{K}^0$ is closed (b) $\mathcal{K}^0$ is not closed

Figure 4.2: Typical integral curves of $\vec{H}_{\max}$

then $\operatorname{cl}\mathcal{K}^0 \subset (\widetilde{\mathfrak{h}}_3^\diamond)^* \backslash E$ and $\operatorname{cl}\mathcal{K}^0 \subseteq \mathcal{K}_{(p_3(0),c,h)}$ (as $\mathcal{K}_{(p_3(0),c,h)}$ is closed by proposition 4.2.7). Hence, $\mathcal{K}^0 \subseteq \operatorname{cl}\mathcal{K}^0 \subseteq \mathcal{K}_{(p_3(0),c,h)} \backslash E$. But as $\mathcal{K}^0$ is connected so is $\operatorname{cl}\mathcal{K}^0$ (lemma 4.2.15), implying that $\operatorname{cl}\mathcal{K}^0$ is the connected component of $\mathcal{K}_{(p_3(0),c,h)} \backslash E$ containing $p(0)$ and hence $\operatorname{cl}\mathcal{K}^0 = \mathcal{K}^0$.

Item 1 is trivial. For item 2 we have that $\mathcal{K}^0$ is a bounded connected 1-submanifold of $(\widetilde{\mathfrak{h}}_3^\diamond)^*$ by proposition 4.2.11. Hence, as $\mathcal{K}^0$ is closed, we have that $\mathcal{K}^0$ is a connected compact embedded 1-submanifold. Let $X : \mathcal{K}^0 \to T\mathcal{K}^0$ be the restriction of $\vec{H}_{\max}$ to $\mathcal{K}^0$. Then (as $\mathcal{K}^0 \cap E = \emptyset$) X is non-zero everywhere. As $\operatorname{im} p(\cdot) \subseteq \mathcal{K}^0$ (proposition 4.2.11), we have that $p(\cdot)$ is an integral curve X on $\mathcal{K}^0$. The result then follows from proposition 4.2.12.

We now proceed to item 3. We have that $\operatorname{im} p(\cdot) \subseteq \mathcal{K}^0 \subseteq \mathcal{K}_{(p_3(0),c,h)}$ (proposition 4.2.11) and that $\mathcal{K}_{(p_3(0),c,h)}$ is compact (proposition 4.2.7). So, by theorem A.4.3, we have that $\lim_\omega p(\cdot)$ and $\lim_\alpha p(\cdot)$ are non-empty and connected subsets of $\widetilde{\mathfrak{h}}_3^\diamond$. We claim that $\lim_\omega p(\cdot) \cap \mathcal{K}^0 = \emptyset$ and $\lim_\alpha p(\cdot) \cap \mathcal{K}^0 = \emptyset$. Let $X : \mathcal{K}^0 \to T\mathcal{K}^0$ be the restriction of $\vec{H}_{\max}$ to $\mathcal{K}^0$. As $\mathcal{K}^0 \subseteq \mathcal{K}_{(p_3(0),c,h)} \backslash E$ we have that $\mathcal{K}^0 \cap E = \emptyset$ and hence that X is non-zero everywhere. Now suppose $\lim_\omega p(\cdot) \cap \mathcal{K}^0 \neq \emptyset$ or $\lim_\alpha p(\cdot) \cap \mathcal{K}^0 \neq \emptyset$. That is, we have an unbounded sequence $\{t_n\}_{n\in\mathbb{N}}$ (i.e., $t_n \to \pm\infty$ as $n \to \infty$) such that $p(t_n) \to \bar{p} \in \mathcal{K}^0$. Now as $\mathcal{K}^0$ is a connected non-compact (as not closed) embedded 1-submanifold it is diffeomorphic to $\mathbb{R}$. That is we have a diffeomorphism $\phi : \mathcal{K}^0 \to \mathbb{R}$. Let $Y : \mathbb{R} \to \mathbb{R}$ be the push forward of X by ϕ, that is $Y = \phi_* X$. Then

$$Y(\phi(p(t))) = \tfrac{d}{dt}\phi(p(t)) = X(p(t)).$$

Notice that $\phi(p(\cdot))$ is an integral curve of Y. As X is non-zero everywhere, so is Y. Thus either $\forall r \in \mathbb{R}$, $Y(r) > 0$ or $\forall r \in \mathbb{R}$, $Y(r) < 0$. W.l.o.g. we assume $Y(r) > 0$ for $r \in \mathbb{R}$ (i.e., $\phi(p(\cdot))$ is strictly increasing). So, by lemma 4.2.13, we have that $\{\phi(p(t_n))\}_{n\in\mathbb{N}}$ does not converge in $\mathbb{R}$. But, as ϕ is continuous and $p(t_n) \to \bar{p}$ (by assumption), we have that $\phi(p(t_n)) \to \phi(\bar{p})$ as $n \to \infty$, a contradiction. Thus we have that $\lim_\omega p(\cdot) \cap \mathcal{K}^0 = \emptyset$ and $\lim_\alpha p(\cdot) \cap \mathcal{K}^0 = \emptyset$.

Next we show that $\lim_\omega p(\cdot) \subseteq \mathcal{K}_{(p_3(0),c,h)} \cap E$ and $\lim_\alpha p(\cdot) \subseteq \mathcal{K}_{(p_3(0),c,h)} \cap E$. Suppose we have an unbounded sequence $\{t_n\}_{n\in\mathbb{N}}$ (i.e., $t_n \to \pm\infty$ as $n \to \infty$) such that $p(t_n) \to \bar{p}$. We have shown that $\bar{p} \notin \mathcal{K}^0$. But as $\{p(t_n)\}_{n\in\mathbb{N}}$ is a sequence in $\mathcal{K}^0$ we have that $\bar{p} \in \operatorname{cl}\mathcal{K}^0$. Now we claim that $\bar{p} \notin \mathcal{K}_{(p_3(0),c,h)} \backslash E$. Suppose that $\bar{p} \in \mathcal{K}_{(p_3(0),c,h)} \backslash E$. Now as $\mathcal{K}^0 \subseteq \mathcal{K}^0 \cup \{\bar{p}\} \subseteq \operatorname{cl}\mathcal{K}^0$ we have that $\mathcal{K}^0 \cup \{\bar{p}\}$ is connected (lemma 4.2.15). Then $\mathcal{K}^0 \cup \{\bar{p}\} \subseteq \mathcal{K}_{(p_3(0),c,h)} \backslash E$ and hence $\mathcal{K}^0 \cup \{\bar{p}\}$ is a connected set containing $p(0)$. Thus $\mathcal{K}^0 \cup \{\bar{p}\} \subseteq \mathcal{K}^0$ (as $\mathcal{K}^0$ is the connected component of $\mathcal{K}_{(p_3(0),c,h)} \backslash E$ containing $p(0)$). That is to say $\bar{p} \in \mathcal{K}^0$, a contradiction. Hence, $\bar{p} \notin \mathcal{K}_{(p_3(0),c,h)} \backslash E$. However, as $\mathcal{K}_{(p_3(0),c,h)}$ is compact (proposition 4.2.7) and $\mathcal{K}^0 \subseteq \mathcal{K}_{(p_3(0),c,h)}$, we get that $\operatorname{cl} \mathcal{K}^0 \subseteq \mathcal{K}_{(p_3(0),c,h)}$ and so $\bar{p} \in \mathcal{K}_{(p_3(0),c,h)}$. That is to say

$$\bar{p} \in \mathcal{K}_{(p_3(0),c,h)} \cap (\widetilde{\mathfrak{h}}_3^\diamond)^* \backslash \Big(\mathcal{K}_{(p_3(0),c,h)} \backslash E \Big) = \mathcal{K}_{(p_3(0),c,h)} \cap E.$$

Thus it follows that $\lim_\omega p(\cdot) \subseteq \mathcal{K}_{(p_3(0),c,h)} \cap E$ and $\lim_\alpha p(\cdot) \subseteq \mathcal{K}_{(p_3(0),c,h)} \cap E$.

Thus $\lim_\omega p(\cdot)$ and $\lim_\alpha p(\cdot)$ are both contained in E. We are left to show that both limit sets are singletons. Now we have that

$$\begin{aligned} \mathcal{K}_{(p_3(0),c,h)} \cap E &= P_3^{-1}(p_3(0)) \cap \mathcal{C}^{-1}(c) \cap H_{\max}^{-1}(h) \cap E \\ &\subseteq P_3^{-1}(p_3(0)) \cap H_{\max}^{-1}(h) \cap E. \end{aligned}$$

But, by lemma 4.2.14, we have that $P_3^{-1}(p_3(0)) \cap H_{\max}^{-1}(h) \cap E$ is completely disconnected. Thus $\lim_\omega p(\cdot)$ and $\lim_\alpha p(\cdot)$ are subsets of a completely disconnected set, but as both limit sets are connected and non-empty, this means that both sets are singletons. □

4.2.3 Subdivision of typical cases

In this subsection we suggest a subdivision of typical cases for further analysis, especially pertaining to integrating $\vec{H}_{\max}$ (i.e., finding explicit expressions for the integral curves of $\vec{H}_{\max}$). Specifically, we subdivide values of (ρ, c, h) which yield qualitatively different structure to $\mathcal{K}_{(\rho,c,h)}$ (as defined in the preceding subsection). Integral curves developing on sets $\mathcal{K}_{(\rho,c,h)}$ of the same "type", are expected to have similar properties.

The first separation we make is between $\rho = 0$ and $\rho \neq 0$. This is motivated by the fact that the sets $\mathcal{C}\big|_{P_3^{-1}(\rho)}^{-1}(c)$ are paraboloids when $\rho \neq 0$ and are cylinders when $\rho = 0$. Indeed (in the global chart $P_1 \times P_2 \times P_4$ for $P_3^{-1}(\rho)$) we have (assuming $\rho \neq 0$) that

$$\begin{aligned} \mathcal{C}_{P_3^{-1}(0)}^{-1}(c) &= \{(x, y, z) \in P_3^{-1}(0) \mid x, y, z \in \mathbb{R},\, x^2 + y^2 = c\} \\ \mathcal{C}_{P_3^{-1}(\rho)}^{-1}(c) &= \{(x, y, z) \in P_3^{-1}(\rho) \mid x, y, z \in \mathbb{R},\, x^2 + y^2 - 2\rho z = c\} \\ &= \{(r\cos\theta, r\sin\theta, \tfrac{r^2 - c}{2\rho}) \in P_3^{-1}(\rho) \mid r > 0,\, \theta \in \mathbb{R}\}. \end{aligned}$$

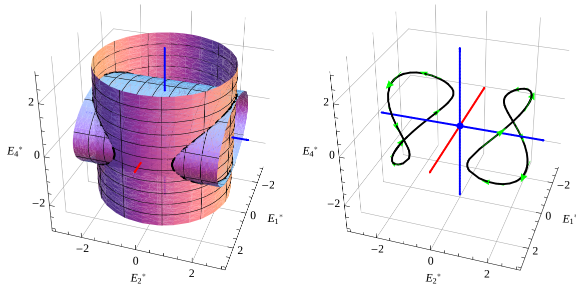
Note that these level sets correspond to (some of) the coadjoint orbits, which are in a linear bijection with the adjoint orbits (which were shown to be points, paraboloids and cylinders), as presented in section 1.4.

Further separations are then made by considering at which "point" the closed embedded submanifolds $P_3^{-1}(\rho) \cap H_{\max}^{-1}(h)$ and $P_3^{-1}(\rho) \cap \mathcal{C}^{-1}(c)$ (see proposition 4.2.5) are tangent to one another. This appears to be exactly the "point" at which bifurcations occur and at which integral curves of the periodic type, degenerate into the other two types (see theorem 4.2.16).

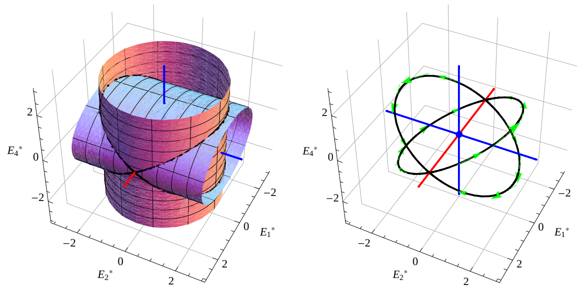
We will start by considering the cases for which $\rho = 0$, i.e., intersections of the form $\mathcal{K}_{(0,c,h)}$. Note that $P_3^{-1}(0) \cap \mathcal{C}^{-1}(c)$ and $P_3^{-1}(0) \cap H_{\max}^{-1}(h)$ are cylinders in the hyperplane $P_3^{-1}(0)$ defined by $p_1^2 + p_2^2 = c$ and $\frac{p_1^2}{\beta_1} + \frac{p_4^2}{\beta_2} = 2h$. Let $p \in \mathcal{K}_{(0,c,h)}$. If $h = 0$, then $p_1 = p_4 = 0$ and thus $p \in E$. Similarly if $c = 0$, then $p_1 = p_2 = 0$ and we again we have $p \in E$. We rule these cases out and restrict ourselves to cases for which $h > 0$ and $c > 0$. We now consider at which point (and for what values of c and h) the cylinders $P_3^{-1}(0) \cap H_{\max}^{-1}(h)$ and $P_3^{-1}(0) \cap \mathcal{C}^{-1}(c)$ are tangent to one another. If they are tangent at a point $(\bar{p}_1, \bar{p}_2, \bar{p}_4)$ in $P_3^{-1}(0)$, then there gradients at $\bar{p}$ must be parallel, i.e.,

$$\begin{bmatrix} 2\bar{p}_1 & 2\bar{p}_2 & 0 \end{bmatrix} = r \begin{bmatrix} \frac{1}{\beta_1}\bar{p}_1 & 0 & \frac{1}{\beta_4}\bar{p}_4 \end{bmatrix}$$

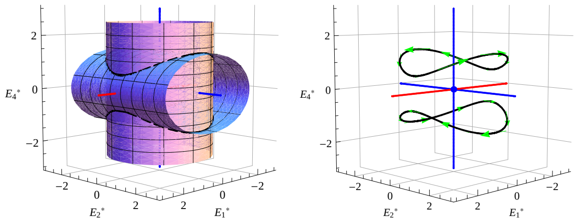
for some $r \in \mathbb{R}$. If $r = 0$, then $\bar{p}_1 = \bar{p}_2 = 0$ and so $c = 0$, but we have restricted ourselves to $c > 0$. If $r \neq 0$ we have $\bar{p}_2 = \bar{p}_4 = 0$ and as $\bar{p} \in H_{\max}^{-1}(h) \cap \mathcal{C}^{-1}(c)$ we get $c = \bar{p}_1^2 = 2h\beta_1$. This motivates us to distinguish between the three cases: $\boldsymbol{c > 2h\beta_1}$, $\boldsymbol{c = 2h\beta_1}$ and $\boldsymbol{c < 2h\beta_1}$. By choosing suitable values of c, h, β_1 and β_2 we illustrate the situation of each case in figure 4.3.



(a) Case: $c > 2h\beta_1$



(b) Case: $c = 2h\beta_1$



(c) Case: $c < 2h\beta_1$

Figure 4.3: Typical cases of $\mathcal{K}_{(0,c,h)}$

Having dealt with the case where $\rho = 0$, i.e., with intersections of the form $\mathcal{K}_{(0,c,h)}$, we now turn our investigation to the case where $\rho \neq 0$. First we show than we can restrict ourselves to $\rho > 0$.

4.2.17 PROPOSITION. *The mapping* $\zeta : (\widetilde{\mathfrak{h}}_3^\diamond)^* \to (\widetilde{\mathfrak{h}}_3^\diamond)^*$, $p \mapsto (-p_1, p_2, -p_3, -p_4)$ *is a diffeomorphism taking integral curves of* $\vec{H}_{\max}$ *to integral curves of* $\vec{H}_{\max}$. *Furthermore,* ζ *is an involution (i.e.,* $\zeta \circ \zeta = id$*),* $H_{\max} \circ \zeta = H_{\max}$, $\mathcal{C} \circ \zeta = \mathcal{C}$, *but* $P_3 \circ \zeta = -P_3$.

PROOF. From the expression of ζ in coordinates we easily get that ζ is a diffeomorphism. Also, as $\zeta(\zeta(p)) = (-(-p_1), p_2, -(-p_3), -(-p_4)) = p$, we have that ζ is an involution. Further observe that $(H_{\max} \circ \zeta)(p) = \frac{1}{\beta_1}(-p_1)^2 + \frac{1}{\beta_2}(-p_4)^2 = H_{\max}(p)$, $(\mathcal{C} \circ \zeta)(p) = (-p_1)^2 + p_2^2 - 2(-p_3)(-p_4) = \mathcal{C}(p)$ and $(P_3 \circ \zeta)(p) = -p_3 = -P_3(p)$ for $p \in (\widetilde{\mathfrak{h}}_3^\diamond)^*$. Now let $p(\cdot)$ be an integral curve of $\vec{H}_{\max}$ and $\bar{p}(\cdot) = \zeta(p(\cdot))$. Then (suppressing evaluation at t) we have that

$$\tfrac{d}{dt}p_1 = -\tfrac{1}{\beta_2}\, p_2\, p_4, \qquad \tfrac{d}{dt}p_2 = p_1 \left(\tfrac{1}{\beta_1}\, p_3 + \tfrac{1}{\beta_2}\, p_4 \right), \qquad \tfrac{d}{dt}p_3 = 0, \qquad \tfrac{d}{dt}p_4 = \tfrac{1}{\beta_1}\, p_1\, p_2.$$

and consequently get that

$$\begin{aligned}
&\tfrac{d}{dt}\bar{p}_1 = -\tfrac{d}{dt}p_1 = \tfrac{1}{\beta_2}\,p_2p_4 = -\tfrac{1}{\beta_2}p_2(-p_4) = -\tfrac{1}{\beta_2}\bar{p}_2\bar{p}_4 \\
&\tfrac{d}{dt}\bar{p}_2 = \tfrac{d}{dt}p_2 = p_1\left(\tfrac{1}{\beta_1}\,p_3 + \tfrac{1}{\beta_2}\,p_4\right) = (-p_1)\left(\tfrac{1}{\beta_1}\,(-p_3) + \tfrac{1}{\beta_2}\,(-p_4)\right) = \bar{p}_1\left(\tfrac{1}{\beta_1}\,\bar{p}_3 + \tfrac{1}{\beta_2}\,\bar{p}_4\right) \\
&\tfrac{d}{dt}\bar{p}_3 = -\tfrac{d}{dt}p_3 = 0 \\
&\tfrac{d}{dt}\bar{p}_4 = -\tfrac{d}{dt}p_4 = -\tfrac{1}{\beta_1}\,p_1\,p_2 = \tfrac{1}{\beta_1}\,p_1\,(-p_2) = \tfrac{1}{\beta_1}\,\bar{p}_1\,\bar{p}_2.
\end{aligned}$$

That is to say $\frac{d}{dt}\bar{p}(t) = \vec{H}_{\max}(\bar{p}(t))$. Thus $\zeta(p(\cdot)) = \bar{p}(\cdot)$ is an integral curve of $\vec{H}_{\max}$. □

Accordingly, to find explicit expressions for integral curves of $H_{\max}$ developing on $P_3^{-1}(\rho)$ for $\rho < 0$, we need only find the integral curves developing on $P_3^{-1}(-\rho)$ and apply ζ. In particular, to solve the Cauchy problem $p(0) = p$, $\dot{p}(t) = \vec{H}(p(t))$, with $p_3 < 0$, we may equivalently solve the problem $p(0) = \zeta(p)$, $\dot{p}(t) = \vec{H}(p(t))$. We would then only need to apply ζ to this solution to find the solution of the original problem. Thus (without loss of generality) we restrict ourselves to the cases for which $\rho > 0$.

Suppose $p \in \mathcal{K}_{(\rho,c,h)}$ and $\rho > 0$ (so in particular $p_3 > 0$). Now if $h = 0$ then $p_1 = p_4 = 0$ and hence $p \in E$. Thus we restrict ourselves to $h > 0$. Next observe that we have the following (two sequences of) implications

$$\begin{aligned}
&\tfrac{1}{\beta_1}p_1^2 + \tfrac{1}{\beta_2}p_4^2 = 2h \quad \Rightarrow \quad p_4^2 \leq 2h\beta_2 \quad \Rightarrow \quad -\sqrt{2h\beta_2} \leq p_4 \leq \sqrt{2h\beta_2} \\
&p_1^2 + p_2^2 - 2p_3p_4 = c \quad \Rightarrow \quad -2p_3p_4 \leq c \quad \Rightarrow \quad p_4 \geq \tfrac{c}{-2p_3}.
\end{aligned}$$

Hence, we have that $\frac{c}{-2p_3} \leq \sqrt{2h\beta_2}$, implying that $c \geq -2p_3\sqrt{2h\beta_2}$. (This condition may be shown to be equivalent to $\mathcal{K}_{(\rho,c,h)} \neq \emptyset$.) Note that if $c = -2p_3\sqrt{2h\beta_2}$, then

$$\tfrac{-2p_3\sqrt{2h\beta_2}}{-2p_3} = \sqrt{2h\beta_2} \leq p_4 \leq \sqrt{2h\beta_2}$$

thus $p_4 = \sqrt{2h\beta_2}$ and

$$\begin{aligned}
&\tfrac{1}{\beta_1}p_1^2 + \tfrac{1}{\beta_2}p_4^2 = 2h \quad \Rightarrow \quad \tfrac{1}{\beta_1}p_1^2 + 2h = 2h \quad \Rightarrow \quad p_1^2 = 0 \\
&p_1^2 + p_2^2 - 2p_3p_4 = c \quad \Rightarrow \quad p_2^2 - 2p_3\sqrt{2h\beta_2} = -2p_3\sqrt{2h\beta_2} \quad \Rightarrow \quad p_2^2 = 0.
\end{aligned}$$

So again we get that $p \in E$. In summary we have the following.

4.2.18 **Lemma.** *Suppose $p \in (\widetilde{\mathfrak{h}}_3^\diamond)^*$, $p_3 = \rho > 0$, $c = \mathcal{C}(p)$ and $h = H_{\max}(p)$ (and so $p \in \mathcal{K}_{(\rho,c,h)}$). Then*

$$c \geq -2\rho\sqrt{2h\beta_2} \qquad\qquad -\sqrt{2h\beta_2} \leq p_4 \leq \sqrt{2h\beta_2}.$$

Additionally: if $h = 0$, then $p \in E$; if $c = -2\rho\sqrt{2h\beta_2}$, then $\mathcal{K}_{(\rho,c,h)} = \{(0,0,\rho,0)\} \subset E$.

Next we consider at which points (and for what values of ρ, c and h) the cylinder $P_3^{-1}(\rho) \cap H_{\max}^{-1}(h)$ and the paraboloid $P_3^{-1}(\rho) \cap \mathcal{C}^{-1}(c)$ are tangent to one another. If they are tangent at a point $p = (p_1, p_2, p_4) \in P_3^{-1}(\rho)$, then the gradients of the functions defining these level surfaces at p must be parallel, i.e.,

$$\begin{bmatrix} \frac{1}{\beta_1} p_1 & 0 & \frac{1}{\beta_2} p_4 \end{bmatrix} = r \begin{bmatrix} 2p_1 & 2p_2 & -2\rho \end{bmatrix}$$

for some $r \in \mathbb{R}$. There are two distinct cases

$$\begin{cases} p_1 = 0 \\ p_2 = 0 \\ p_4 = -2r\beta_2\rho \end{cases} \qquad \begin{cases} p_1 \neq 0 \\ p_2 = 0 \\ p_4 = -\frac{\beta_2}{\beta_1}\rho\,. \end{cases}$$

For the first case (as $H_{\max}(p) = h$ and $\mathcal{C}(p) = c$) we have that $p_4^2 = 2h\beta_2$ and $p_4 = \frac{c}{-2\rho}$, i.e., $c^2 = (2\rho)^2(2h\beta_2)$. We dismiss the case $c = -2\rho\sqrt{2h\beta_2}$, as $\mathcal{K}_{(\rho,-2\rho\sqrt{2h\beta_2},h)}$ is a singleton in E (lemma 4.2.18). Thus if our two level surfaces are (non-trivially) tangent at a point p such that $p_1 = 0$, then $c = 2\rho\sqrt{2h\beta_2}$. This motivates us to distinguish between the cases $\boldsymbol{c < 2\rho\sqrt{2h\beta_2}}$, $\boldsymbol{c = 2\rho\sqrt{2h\beta_2}}$ and $\boldsymbol{c > 2\rho\sqrt{2h\beta_2}}$. We now proceed to the second type of tangent point. Supposing such a tangent point exists we have that (at that point)

$$2h = \tfrac{1}{\beta_1}p_1^2 + \tfrac{1}{\beta_2}\left(-\tfrac{\beta_2}{\beta_1}\rho\right)^2 \quad \Rightarrow \quad 2h\beta_1 = p_1^2 + \tfrac{\beta_2}{\beta_1}\rho^2 \quad \Rightarrow \quad p_1^2 = 2h\beta_1 - \tfrac{\beta_2}{\beta_1}\rho^2.$$

So as $p_1^2 > 0$ this means that $2h\beta_1 > \frac{\beta_2}{\beta_1}\rho^2$ is a necessary (and may be shown to be sufficient) condition for such a tangent point to exists. Furthermore, at that point we have that

$$c = p_1^2 - 2\rho\left(-\tfrac{\beta_2}{\beta_1}\rho\right) = \left(2h\beta_1 - \tfrac{\beta_2}{\beta_1}\rho^2\right) + 2\tfrac{\beta_2}{\beta_1}\rho^2 = 2h\beta_1 + \tfrac{\beta_2}{\beta_1}\rho^2$$

That is, if our two level surfaces are tangent at a point with $p_1 \neq 0$ then $2h\beta_1 > \frac{\beta_2}{\beta_1}\rho^2$ and $c = 2h\beta_1 + \frac{\beta_2}{\beta_1}\rho^2$. This motivates us to distinguish between the cases $\boldsymbol{2h\beta_1 \leq \frac{\beta_2}{\beta_1}\rho^2}$ and $\boldsymbol{2h\beta_1 > \frac{\beta_2}{\beta_1}\rho^2}$, and in the second case between $\boldsymbol{c < 2h\beta_1 + \frac{\beta_2}{\beta_1}\rho^2}$, $\boldsymbol{c = 2h\beta_1 + \frac{\beta_2}{\beta_1}\rho^2}$ and $\boldsymbol{c > 2h\beta_1 + \frac{\beta_2}{\beta_1}\rho^2}$. We use the above subdivisions to subdivide the cases for which $\rho > 0$, however before we do so, we prove an inequality that will allow us to eliminate some cases.

4.2.19 Lemma. *If $2h\beta_1 > \frac{\beta_2}{\beta_1}\rho^2$ and $c \geq 2h\beta_1 + \frac{\beta_2}{\beta_1}\rho^2$, then $c > 2\rho\sqrt{2h\beta_2}$.*

Proof. We will prove that $2h\beta_1 + \frac{\beta_2}{\beta_1}\rho^2 > 2\rho\sqrt{2h\beta_2}$. The result will then follow from the assumption that $c \geq 2h\beta_1 + \frac{\beta_2}{\beta_1}\rho^2$. We have the following sequence of equivalences

$$\begin{aligned} & 2h\beta_1 + \tfrac{\beta_2}{\beta_1}\rho^2 > 2\rho\sqrt{2h\beta_2} \\ \Leftrightarrow\quad & \left(2h\beta_1 + \tfrac{\beta_2}{\beta_1}\rho^2\right)^2 > \left(2\rho\sqrt{2h\beta_2}\right)^2 \\ \Leftrightarrow\quad & 4h^2\beta_1^2 + 4h\beta_2\rho^2 + \left(\tfrac{\beta_2}{\beta_1}\rho^2\right)^2 > 8\rho^2 h\beta_2 \\ \Leftrightarrow\quad & 4h^2\beta_1^2 - 4h\beta_2\rho^2 + \left(\tfrac{\beta_2}{\beta_1}\rho^2\right)^2 > 0 \\ \Leftrightarrow\quad & \left(2h\beta_1 - \tfrac{\beta_2}{\beta_1}\rho^2\right)^2 > 0. \end{aligned}$$

But the last inequality is true by the assumption that $2h\beta_1 > \frac{\beta_2}{\beta_1}\rho^2$. □

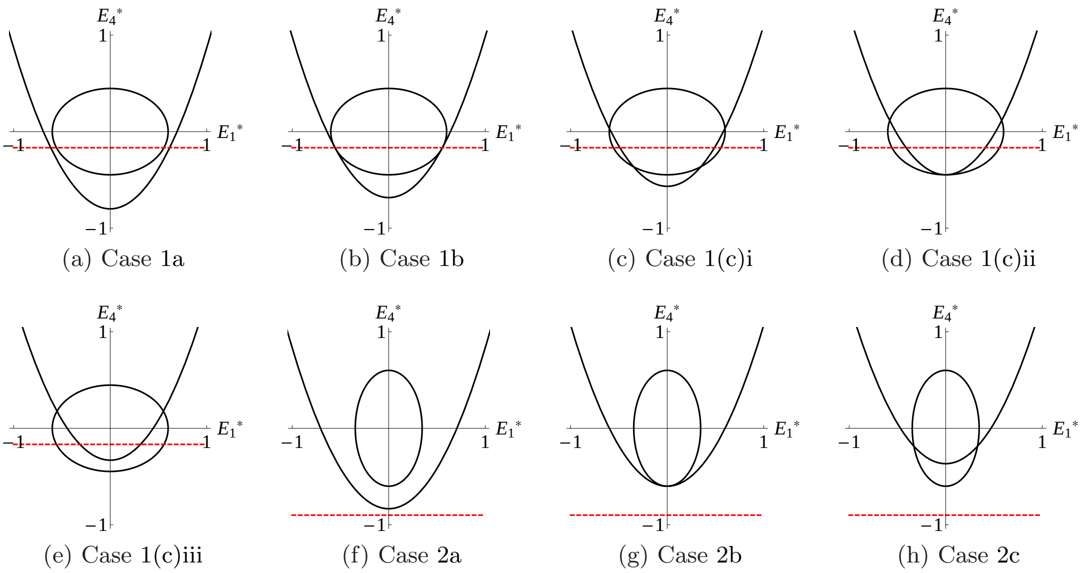
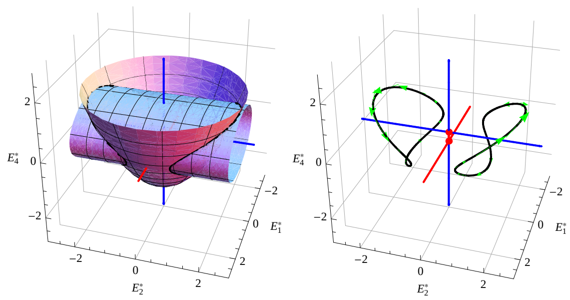


Figure 4.4: Qualitative separation of cases for $\rho > 0$

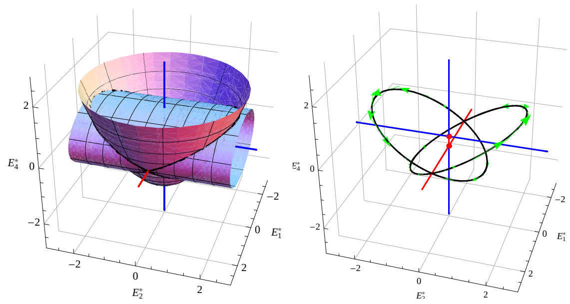
We now suggest the following subdivision of cases:

1. $2h\beta_1 > \frac{\beta_2}{\beta_1}\rho^2$,
   (a) $c > 2h\beta_1 + \frac{\beta_2}{\beta_1}\rho^2$; figures 4.4a and 4.5a;
   (b) $c = 2h\beta_1 + \frac{\beta_2}{\beta_1}\rho^2$; figures 4.4b and 4.5b;
   (c) $c < 2h\beta_1 + \frac{\beta_2}{\beta_1}\rho^2$,
      i. $c > 2\rho\sqrt{2h\beta_2}$; figures 4.4c and 4.5c;
      ii. $c = 2\rho\sqrt{2h\beta_2}$; figures 4.4d and 4.5d;
      iii. $c < 2\rho\sqrt{2h\beta_2}$; figures 4.4e and 4.5e;
2. $2h\beta_1 \leq \frac{\beta_2}{\beta_1}\rho^2$,
   (a) $c > 2\rho\sqrt{2h\beta_2}$; figures 4.4f and 4.5f;
   (b) $c = 2\rho\sqrt{2h\beta_2}$; figures 4.4g and 4.5g;
   (c) $c < 2\rho\sqrt{2h\beta_2}$; figures 4.4h and 4.5h.

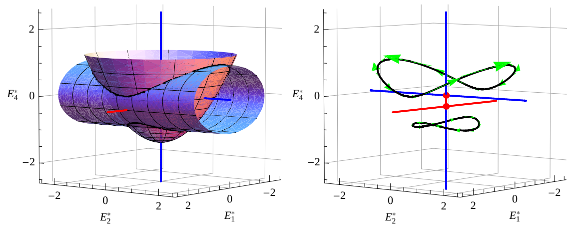
We will refer to the above cases as case 1a, 1b, 1(c)i, 1(c)ii, 1(c)iii, 2a, 2b and 2c respectively. Figure 4.4 shows the projection of the paraboloid $P_3^{-1}(p_3) \cap \mathcal{C}^{-1}(c)$ and the cylinder $P_3^{-1}(\rho) \cap H_{\max}^{-1}(h)$ onto $P_2^{-1}(0) \cap P_3^{-1}(\rho)$ (i.e., the E_1^*, E_4^* plane in $P_3^{-1}(\rho)$) for each case (i.e., for some suitable values of ρ, c, h, β_1 and β_2). Note that by lemma 4.2.19 we need not subdivide cases 1a and 1b, as the only subcase possible is $c > 2\rho\sqrt{2h\beta_2}$. By choosing suitable values of ρ, c, h, β_1 and β_2 we graph (within the 3D hyperplane $P_3^{-1}(\rho)$) the level surfaces $P_3^{-1}(\rho) \cap H_{\max}^{-1}(h)$, $P_3^{-1}(\rho) \cap \mathcal{C}^{-1}(c)$, and their intersection $\mathcal{K}_{(\rho,c,h)}$ (along with the vector field $\vec{H}_{\max}$ restricted to



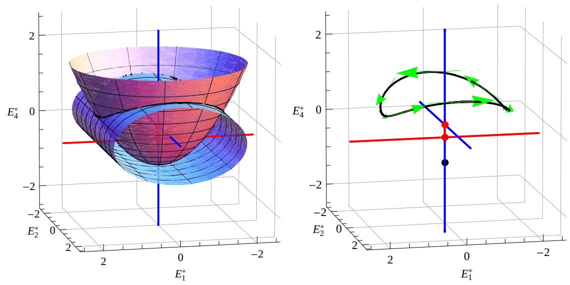
(a) Case 1a



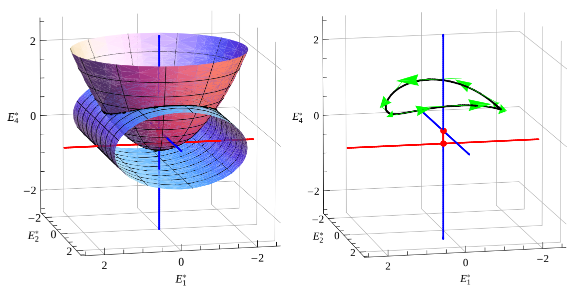
(b) Case 1b



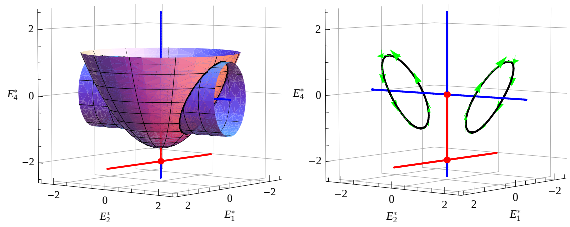
(c) Case 1(c)i



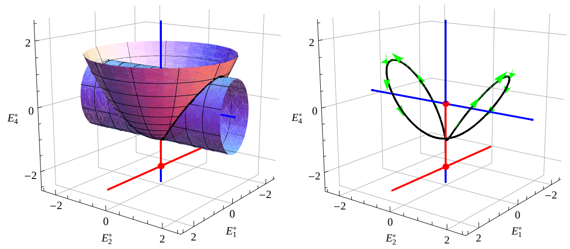
(d) Case 1(c)ii



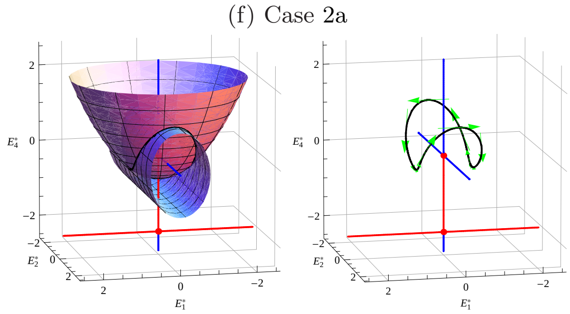
(e) Case 1(c)iii



(f) Case 2a



(g) Case 2b



(h) Case 2c

Figure 4.5: Typical cases of $\mathcal{K}_{(\rho,c,h)}$ with $\rho > 0$

$\mathcal{K}_{(\rho,c,h)}$) in figure 4.5. As an example, section C.10 contains the code used to generate figure 4.5a.

4.3 Integration of Some (Reduced) Normal Extremals

We now proceed onto the integration of $\vec{H}_{\max}$. That is, finding explicit expressions for the integral curves of $\vec{H}_{\max}$. We recall (again) that $H_{\max}$ and $\mathcal{C}$ are given (in coordinates) by

$$H_{\max}(p) = \tfrac{1}{2}\left(\tfrac{1}{\beta_1}p_1^2 + \tfrac{1}{\beta_2}p_4^2\right), \qquad\qquad \mathcal{C}(p) = p_1^2 + p_2^2 - 2p_3p_4,$$

and (from proposition 4.1.3), that $\vec{H}_{\max}(p) = (\dot{p}_1, \dot{p}_2, \dot{p}_3, \dot{p}_4)$ is given (in coordinates) by

$$\dot{p}_1 = -\tfrac{1}{\beta_2}\,p_2\,p_4, \qquad \dot{p}_2 = p_1\left(\tfrac{1}{\beta_1}\,p_3 + \tfrac{1}{\beta_2}\,p_4\right), \qquad \dot{p}_3 = 0, \qquad \dot{p}_4 = \tfrac{1}{\beta_1}\,p_1\,p_2. \tag{4.3.1}$$

We will find explicit expressions for the reduced extremal curves for a subset of initial conditions. Specifically we will cover all cases for which $p_3 = \rho = 0$, but only cover case 1a (as described in section 4.2.3) when $p_3 = \rho \neq 0$. A summary of our results appear in appendix B.

We start by transforming the system of differential equations (for an integral curve $p(\cdot)$ as described in equation (4.3.1)) to a single separable differential equation, which we will attempt to solve by use of elliptic integrals. Suppose $p(\cdot) : \mathbb{R} \to (\widetilde{\mathfrak{h}}_3^\diamond)^*$ is an integral curve of $\vec{H}_{\max}$. Let $\rho = P_3(p(0))$, $c = \mathcal{C}(p(0))$ and $h = \vec{H}_{\max}(p(0))$ (i.e., $p(\cdot)$ develops on $\mathcal{K}_{(\rho,c,h)}$). Note that as P_3 is a constant of the motion, we trivially have that $p_3(\cdot) = \rho$ and $\frac{d}{dt}p_3(t) = 0$ (in all cases). Using the constants of motion for $\vec{H}_{\max}$, we get (in coordinates, suppressing evaluation at t) that

$$\begin{aligned}\dot{p}_4^2 = \tfrac{p_1^2p_2^2}{\beta_1^2} &= \tfrac{1}{\beta_1^2}\left(2h\beta_1 - \tfrac{\beta_1}{\beta_2}p_4^2\right)\left(c + 2\rho p_4 - \left(2h\beta_1 - \tfrac{\beta_1}{\beta_2}p_4^2\right)\right)\\ &= \tfrac{1}{\beta_2^2}\left(2h\beta_2 - p_4^2\right)\left(\tfrac{c\beta_2}{\beta_1} - 2h\beta_2 + 2\tfrac{\beta_2}{\beta_1}\rho p_4 + p_4^2\right).\end{aligned} \tag{4.3.2}$$

In each of the cases to be investigated, we transform this equation into a standard form, as described in section A.5.2. We then use the integral formulae (A.5.1), (A.5.2) and (A.5.3) in section A.5.1 to solve the (transformed) separable differential equation. Finally, we transform this solution into a solution for $p_4(\cdot)$, and use the constants of motion for $\vec{H}_{\max}$ to solve for the other coordinates.

4.3.1 Type $\rho = 0$

In this case, equation (4.3.2) becomes

$$\dot{p}_4^2 = \tfrac{1}{\beta_2^2}\left(2h\beta_2 - p_4^2\right)\left(\tfrac{c\beta_2}{\beta_1} - 2h\beta_2 + p_4^2\right)$$

which is already in standard form. Hence we have that

$$\tfrac{\sigma_1}{\beta_2}dt = \frac{dp_4}{\sqrt{\left(2h\beta_2 - p_4^2\right)\left(\left(\frac{c}{\beta_1} - 2h\right)\beta_2 + p_4^2\right)}} \tag{4.3.3}$$

for some $\sigma_1 \in \{-1, 1\}$. If $h = 0$, then $p_1(\cdot) = p_4(\cdot) = 0$ and hence $p(\cdot) = p(0) \in E$. Similarly, if $c = 0$, then $p_1(\cdot) = p_2(\cdot) = 0$ and so $p(\cdot) = p(0) \in E$. Accordingly, we can restrict ourselves to $h > 0$ and $c > 0$. As discussed in the previous subsection we distinguish between the cases $c > 2h\beta_1$, $c = 2h\beta_1$ and $c < 2h\beta_1$. (Note that this separation is supported by the expression we wish to integrate, as we have that $2h\beta_2 > 0$, but the sign of $(\frac{c}{\beta_1} - 2h)\beta_2 = \frac{\beta_2}{\beta_1}(c - 2h\beta_1)$ is undetermined.)

We start with the case $c > 2h\beta_1$. Thus we have that $(\frac{c}{\beta_1} - 2h)\beta_2 > 0$. So, for $0 \leq p_4(t) \leq \sqrt{2h\beta_2}$ and assuming $p_4(0) = 0$, we get, by use of formula (A.5.1), that

$$\begin{aligned}
\frac{\sigma_1}{\beta_2}t &= \int_0^{p_4(t)} \frac{dp_4}{\sqrt{\left(\left(\frac{c}{\beta_1}-2h\right)\beta_2+p_4^2\right)\left(2h\beta_2-p_4^2\right)}} \\
&= \frac{1}{\sqrt{\left(\frac{c}{\beta_1}-2h\right)\beta_2+2h\beta_2}}\mathrm{sd}^{-1}\left(\frac{\sqrt{\left(\frac{c}{\beta_1}-2h\right)\beta_2+2h\beta_2}\; p_4(t)}{\sqrt{\left(\frac{c}{\beta_1}-2h\right)\beta_2 2h\beta_2}}, \frac{\sqrt{2h\beta_2}}{\sqrt{\left(\frac{c}{\beta_1}-2h\right)\beta_2+2h\beta_2}}\right) \\
&= \frac{1}{\sqrt{\frac{c\beta_2}{\beta_1}}}\mathrm{sd}^{-1}\left(\frac{\sqrt{\frac{c\beta_2}{\beta_1}}p_4(t)}{\sqrt{2h\beta_2^2\left(\frac{c}{\beta_1}-2h\right)}}, \frac{\sqrt{2h\beta_2}}{\sqrt{\frac{c\beta_2}{\beta_1}}}\right) \\
&= \frac{1}{\sqrt{\frac{c\beta_2}{\beta_1}}}\mathrm{sd}^{-1}\left(\frac{p_4(t)}{\sqrt{\frac{2h\beta_1\beta_2}{c}\left(\frac{c}{\beta_1}-2h\right)}}, \sqrt{\frac{2h\beta_1}{c}}\right).
\end{aligned}$$

Hence, we have that

$$\begin{aligned}
p_4(t) &= \sqrt{\frac{2h\beta_1\beta_2}{c}\left(\frac{c}{\beta_1} - 2h\right)}\;\mathrm{sd}\left(\sqrt{\frac{c\beta_2}{\beta_1}}\frac{\sigma_1}{\beta_2}t, \sqrt{\frac{2h\beta_1}{c}}\right) \\
&= \sqrt{2h\beta_2\left(1 - \frac{2h\beta_1}{c}\right)}\;\mathrm{sd}\left(\sqrt{\frac{c}{\beta_1\beta_2}}\sigma_1 t, \sqrt{\frac{2h\beta_1}{c}}\right).
\end{aligned}$$

Fix the modulus $k = \sqrt{\frac{2h\beta_1}{c}}$ (notice that by the assumption $c > 2h\beta_1 > 0$ we get that $0 < k < 1$) and complimentary modulus $k' = \sqrt{1 - k^2}$, and write $\mathrm{sd}\,(x)$ for $\mathrm{sd}\,(x, k)$. Also let $\Omega = \sqrt{\frac{c}{\beta_1\beta_2}}$. Then (noting that $\mathrm{sd}\,(x)$ is an odd function for a fixed modulus k) we have that

$$p_4(t) = \sqrt{\frac{2h\beta_1}{c}}\sqrt{\frac{c\beta_2}{\beta_1}\left(1 - \frac{2h\beta_1}{c}\right)}\;\mathrm{sd}\,(\Omega\sigma_1 t) = \sigma_1 k k' \sqrt{\frac{c\beta_2}{\beta_1}}\;\mathrm{sd}\,(\Omega t)$$

Now as $H_{\max}(p(t)) = \frac{1}{2}\left(\frac{1}{\beta_1}p_1(t)^2 + \frac{1}{\beta_2}p_4(t)^2\right) = h$ we have that

$$\begin{aligned}
p_1(t)^2 &= 2h\beta_1 - \frac{\beta_1}{\beta_2}p_4(t)^2 = 2h\beta_1 - ck^2(k')^2\,\mathrm{sd}^2\,(\Omega t) = 2h\beta_1\left(1 - (k')^2\,\mathrm{sd}^2\,(\Omega t)\right) \\
&= 2h\beta_1\,\mathrm{cd}^2\,(\Omega t).
\end{aligned}$$

Hence we get that $p_1(t) = \sigma_2\sqrt{c}k\,\mathrm{cd}\,(\Omega t)$ for some $\sigma_2 \in \{-1, 1\}$. Next as $\mathcal{C}(p(t)) = p_1(t)^2 + p_2(t)^2 = c$ we have that

$$p_2(t)^2 = c - p_1(t)^2 = c - ck^2\mathrm{cd}^2\,(\Omega t) = c\left(1 - k^2\mathrm{cd}^2\,(\Omega t)\right) = c(k')^2\mathrm{nd}^2\,(\Omega t).$$

Thus we get that $p_2(t) = \sigma_3\sqrt{c}k' \operatorname{nd}(\Omega t)$ for some $\sigma_3 \in \{-1, 1\}$.

In summary we have that

$$\begin{cases} p_1(t) = \sigma_2\sqrt{c}k \operatorname{cd}(\Omega t) \\ p_2(t) = \sigma_3\sqrt{c}k' \operatorname{nd}(\Omega t) \\ p_3(t) = 0 \\ p_4(t) = \sigma_1 kk'\sqrt{\frac{c\beta_2}{\beta_1}} \operatorname{sd}(\Omega t) \end{cases} \qquad \begin{cases} k = \sqrt{\frac{2h\beta_1}{c}} \\ \Omega = \sqrt{\frac{c}{\beta_1\beta_2}} \\ \sigma_i \in \{-1, 1\},\ i = \overline{1, 3}. \end{cases}$$

Notice that $p(t)$ is defined for all $t \in \mathbb{R}$, i.e., we have a smooth curve $p(\cdot) : \mathbb{R} \to (\widetilde{\mathfrak{h}}_3^\diamond)^*$ as a prospective candidate for a general integral curve of $\vec{H}_{\max}$. We now verify exactly when (specifically for which σ) $p(\cdot)$ is an integral curve of $\vec{H}_{\max}$. Using equation (4.3.1), we investigate when $\frac{d}{dt}(p(t)) - \vec{H}_{\max}(p(t)) = 0$. Now

$$\begin{aligned} \dot{p}_1(t) + \tfrac{p_2(t)p_4(t)}{\beta_2} &= -\sigma_2\sqrt{c}k(k')^2\Omega \operatorname{nd}(\Omega t) \operatorname{sd}(\Omega t) + \tfrac{1}{\beta_2}\sigma_3\sqrt{c}k' \operatorname{nd}(\Omega t)\, \sigma_1 kk'\sqrt{\tfrac{c\beta_2}{\beta_1}} \operatorname{sd}(\Omega t) \\ &= \left(-\sigma_2\sqrt{\tfrac{c}{\beta_1\beta_2}} + \tfrac{1}{\beta_2}\sigma_3\sigma_1\sqrt{\tfrac{c\beta_2}{\beta_1}}\right)\sqrt{c}k(k')^2 \operatorname{nd}(\Omega t) \operatorname{sd}(\Omega t) \\ &= -\left(\sigma_2 - \sigma_3\sigma_1\right)\Omega\sqrt{c}k(k')^2 \operatorname{nd}(\Omega t) \operatorname{sd}(\Omega t) \\ \dot{p}_2(t) - \tfrac{p_1(t)p_4(t)}{\beta_2} &= \sigma_3\sqrt{c}k'k^2\Omega \operatorname{cd}(\Omega t) \operatorname{nd}(\Omega t) - \tfrac{1}{\beta_2}\sigma_2\sqrt{c}k \operatorname{cd}(\Omega t)\, \sigma_1 kk'\sqrt{\tfrac{c\beta_2}{\beta_1}} \operatorname{sd}(\Omega t) \\ &= \left(\sigma_3 - \sigma_1\sigma_2\right)\Omega\sqrt{c}k'k^2 \operatorname{cd}(\Omega t) \operatorname{nd}(\Omega t) \\ \dot{p}_4(t) - \tfrac{p_1(t)p_2(t)}{\beta_1} &= \sigma_1 kk'\sqrt{\tfrac{c\beta_2}{\beta_1}}\sqrt{\tfrac{c}{\beta_1\beta_2}} \operatorname{cd}(\Omega t) \operatorname{nd}(\Omega t) - \tfrac{1}{\beta_1}\sigma_2\sqrt{c}k \operatorname{cd}(\Omega t)\, \sigma_3\sqrt{c}k' \operatorname{nd}(\Omega t) \\ &= \left(\sigma_1 - \sigma_2\sigma_3\right)\tfrac{c}{\beta_1}kk' \operatorname{cd}(\Omega t) \operatorname{nd}(\Omega t). \end{aligned}$$

That is to say, provided that $\sigma_1 = \sigma_2\sigma_3$, $\sigma_2 = \sigma_3\sigma_1$ and $\sigma_3 = \sigma_1\sigma_2$, or equivalently

$$\sigma \in \{(1, 1, 1), (1, -1, -1), (-1, -1, 1), (-1, 1, -1)\},$$

we have that $p(\cdot)$ is an integral curve of $\vec{H}_{\max}$ (for any $c > 0$ and $h > 0$ such that $c > 2h\beta_1$). We now make an explicit claim regarding the integral curves of $\vec{H}_{\max}$ in this case.

4.3.1 PROPOSITION. *Suppose $p(\cdot) : \mathbb{R} \to (\widetilde{\mathfrak{h}}_3^\diamond)^*$ is an integral curve of $\vec{H}_{\max}$ such that $P_3(p(0)) = 0$, $H_{\max}(p(0)) = h$, $\mathcal{C}(p(0)) = c$ and $c > 2h\beta_1 > 0$. Then there exists $t_0 \in [-\frac{K}{\Omega}, \frac{3K}{\Omega}]$ and $\sigma \in \{-1, 1\}$ such that $p(t) = \bar{p}(t + t_0)$ for $t \in \mathbb{R}$, where ($4K$ is the period of $\operatorname{cn}(\cdot, k)$ and)*

$$\begin{cases} \bar{p}_1(t) = \sigma\sqrt{c}k \operatorname{cd}(\Omega t) \\ \bar{p}_2(t) = \sigma\sqrt{c}k' \operatorname{nd}(\Omega t) \\ \bar{p}_3(t) = 0 \\ \bar{p}_4(t) = kk'\sqrt{\frac{c\beta_2}{\beta_1}} \operatorname{sd}(\Omega t) \end{cases} \qquad \begin{cases} \Omega = \sqrt{\frac{c}{\beta_1\beta_2}} \\ k = \sqrt{\frac{2h\beta_1}{c}}. \end{cases}$$

PROOF. Let $\varsigma = \operatorname{sgn}(p_1(0))$ and $\sigma = \operatorname{sgn}(p_2(0))$. Notice that if $p_2(0) = 0$, then the condition $c > 2h\beta_1$ is equivalent to $p_1(0)^2 > p_1(0)^2 + \frac{\beta_1}{\beta_2}p_4(0)^2$, a contradiction. Thus $\sigma \neq 0$. Next observe that

$$\bar{p}_4(\tfrac{K}{\Omega}) = kk'\sqrt{\tfrac{c\beta_2}{\beta_1}} \operatorname{sd}(K) = \sqrt{\tfrac{2h\beta_1}{c}}k'\sqrt{\tfrac{c\beta_2}{\beta_1}}\tfrac{1}{k'} = \sqrt{2h\beta_2}.$$

Similarly we get (along with the above result) that

$$\bar{p}_4(-\tfrac{K}{\Omega}) = -\sqrt{2h\beta_2} \qquad \bar{p}_4(\tfrac{K}{\Omega}) = \sqrt{2h\beta_2} \qquad \bar{p}_4(\tfrac{3K}{\Omega}) = -\sqrt{2h\beta_2}.$$

Thus, as $\bar{p}_4(\cdot)$ is continuous, for all $r \in \mathbb{R}$ such that $-\sqrt{2h\beta_2} \leq r \leq \sqrt{2h\beta_2}$, there exists a $t_0 \in [-\frac{K}{\Omega}, \frac{K}{\Omega}]$ and a $t_0 \in [\frac{K}{\Omega}, \frac{3K}{\Omega}]$, such that $\bar{p}_4(t_0) = r$. However, observe that $\operatorname{cn}(x,k) \geq 0$ for $x \in [-K, K]$ and $\operatorname{cn}(x,k) \leq 0$ for $x \in [K, 3K]$. Thus, for all $r \in \mathbb{R}$ such that $-\sqrt{2h\beta_2} \leq r \leq \sqrt{2h\beta_2}$, there exists a $t_0 \in [-\frac{K}{\Omega}, \frac{3K}{\Omega}]$, such that $\bar{p}_4(t_0) = r$ and $\varsigma\sigma \operatorname{cd}(t_0) \geq 0$. Now as $\frac{1}{\beta_1}p_1(0)^2 + \frac{1}{\beta_2}p_4(0)^2 = 2h$, we have that $-\sqrt{2h\beta_2} \leq p_4(0) \leq \sqrt{2h\beta_2}$. Therefore there then exists a $t_0 \in [-\frac{K}{\Omega}, \frac{3K}{\Omega}]$ such that $\bar{p}_4(t_0) = p_4(0)$ and $\varsigma\, \bar{p}_1(t_0) \geq 0$. As $\frac{1}{\beta_1}p_1(0)^2 + \frac{1}{\beta_2}p_4(0)^2 = 2h$, we have that

$$p_1(0)^2 = \beta_1\big(2h - \tfrac{1}{\beta_2}p_4(0)^2\big) = \beta_1\big(2h - \tfrac{1}{\beta_2}\bar{p}_4(t_0)^2\big) = \bar{p}_1(t_0)^2.$$

But we have that $\operatorname{sgn}(p_1(0))\bar{p}_1(t_0) \geq 0$, thus $p_1(0) = \bar{p}_1(t_0)$. Next, as $p_1(0)^2 + p_2(0)^2 = c$, we have that

$$p_2(0)^2 = c - \bar{p}_1(t_0)^2 = \bar{p}_2(t_0)^2.$$

But as $\operatorname{sgn}(\bar{p}_2(t_0)) = \sigma = \operatorname{sgn}(p_2(0))$, we get that $p_2(0) = \bar{p}_2(0)$. Thus $p(0) = \bar{p}(t_0)$. By the preceding discussion we have that $\bar{p}(\cdot)$ is also an integral curve of $\vec{H}_{\max}$. Thus as $t \mapsto p(t)$ and $t \mapsto \bar{p}(t + t_0)$ are both integral curves of $\vec{H}_{\max}$ and go through the same point at $t = 0$ (i.e., they solve the same Cauchy problem), they must be the same. □

4.3.2 Remark. Note that the two values for σ in the above proposition correspond to the two connected components of $\mathcal{K}_{(0,c,h)}$, as illustrated in figure 4.3a. We verified the above claim by solving the Cauchy problem $p(0) = \bar{p}(0)$, $\dot{p}(t) = \vec{H}_{\max}$ numerically (using the Mathematica function `NDSolve`) over a suitable interval and for some suitable c, h, β_1 and β_2. (Specifically, for the choice of $h = 1; \beta_1 = 2.5; \beta_2 = 1; p_3 = 0; c = 2h\beta_1 + 5$, we found that $\max_{t\in[0,10]} \sum_{i=1}^{4}(p_i(t) - \bar{p}_i(t))^2$ was of the order of 10^{-13}.)

We now continue on to the case $c = 2h\beta_1$. Here we have that $(\frac{c}{\beta_1} - 2h)\beta_2 = 0$. Thus equation (4.3.3) becomes

$$\tfrac{\sigma_1}{\beta_2}dt = \frac{dp_4}{\sqrt{(2h\beta_2 - p_4^2)p_4^2}}.$$

So, for $0 \leq p_4(t) \leq \sqrt{2h\beta_2}$ and assuming $p_4(0) = 0$, we get, by use of formula (A.5.2), that

$$\tfrac{\sigma_1}{\beta_2}t = \int_{p_4(t)}^{\sqrt{2h\beta_2}} \frac{dp_4}{\sqrt{(2h\beta_2 - p_4^2)p_4^2}} = \tfrac{1}{\sqrt{2h\beta_2}} \operatorname{dn}^{-1}\left(\tfrac{p_4(t)}{\sqrt{2h\beta_2}}, 1\right)$$

Hence we get that (noting that $\operatorname{sech}(\cdot)$ is even)

$$p_4(t) = \sqrt{2h\beta_2}\operatorname{sech}(\Omega\sigma_1 t) = \sqrt{2h\beta_2}\operatorname{sech}(\Omega t),$$

where $\Omega = \sqrt{\frac{2h}{\beta_2}}$. Now as $\frac{1}{2}\left(\frac{p_1^2}{\beta_1} + \frac{p_4^2}{\beta_2}\right) = h$ we have that

$$p_1(t)^2 = 2h\beta_1 - \tfrac{p_4(t)^2\beta_1}{\beta_2} = 2h\beta_1\left(1 - \operatorname{sech}^2(\Omega t)\right) = 2h\beta_1 \tanh^2(\Omega t).$$

Thus we get that $p_1(t) = \sigma_2\sqrt{2h\beta_1}\tanh(\Omega t)$ for some $\sigma_2 \in \{-1, 1\}$.

Next, as $c = 2h\beta_1$, we have that $p_2(t)^2 = \frac{\beta_1}{\beta_2}p_4(t)^2$ and so get that $p_2(t)^2 = 2h\beta_1\operatorname{sech}^2(\Omega t)$. Thus $p_2(t) = \sigma_3\sqrt{2h\beta_1}\operatorname{sech}(\Omega\sigma_1 t)$ for some $\sigma_3 \in \{-1, 1\}$

In summary (generalising the sign of $p_4(t)$ with $\sigma_4 \in \{-1, 1\}$) we have that

$$\begin{cases} p_1(t) = \sigma_2\sqrt{2h\beta_1}\tanh(\Omega t) \\ p_2(t) = \sigma_3\sqrt{2h\beta_1}\operatorname{sech}(\Omega t) \\ p_3(t) = 0 \\ p_4(t) = \sigma_4\sqrt{2h\beta_2}\operatorname{sech}(\Omega t) \end{cases} \qquad \Omega = \sqrt{\tfrac{2h}{\beta_2}}$$

Notice that $p(t)$ is defined for all $t \in \mathbb{R}$, i.e., we have a smooth curve $p(\cdot) : \mathbb{R} \to (\widetilde{\mathfrak{h}}_3^\diamond)^*$ as a prospective candidate for a general integral curve of $\vec{H}_{\max}$. We again verify exactly when (specifically for which σ) $p(\cdot)$ is an integral curve of $\vec{H}_{\max}$. Using equation (4.3.1), we investigate when $\frac{d}{dt}(p(t)) - \vec{H}_{\max}(p(t)) = 0$. We get that

$$\begin{aligned}
\dot{p}_1(t) + \tfrac{p_2(t)p_4(t)}{\beta_2} &= \sigma_2\sqrt{2h\beta_1}\Omega\operatorname{sech}^2(\Omega t) + \tfrac{1}{\beta_2}\sigma_3\sqrt{2h\beta_1}\operatorname{sech}(\Omega t)\,\sigma_4\sqrt{2h\beta_2}\operatorname{sech}(\Omega t) \\
&= (\sigma_2 + \sigma_3\sigma_4)2h\sqrt{\tfrac{\beta_1}{\beta_2}}\operatorname{sech}^2(\Omega t) \\
\dot{p}_2(t) - \tfrac{p_1(t)p_4(t)}{\beta_2} &= -\sigma_3\sqrt{2h\beta_1}\Omega\operatorname{sech}(\Omega t)\,\tanh(\Omega t) - \tfrac{1}{\beta_2}\sigma_2\sqrt{2h\beta_1}\tanh(\Omega t)\,\sigma_4\sqrt{2h\beta_2}\operatorname{sech}(\Omega t) \\
&= -(\sigma_3 + \sigma_4\sigma_2)2h\sqrt{\tfrac{\beta_1}{\beta_2}}\operatorname{sech}(\Omega t)\,\tanh(\Omega t) \\
\dot{p}_4(t) - \tfrac{p_1(t)p_2(t)}{\beta_1} &= -\sigma_4\sqrt{2h\beta_2}\Omega\operatorname{sech}(\Omega t)\,\tanh(\Omega t) - \tfrac{1}{\beta_1}\sigma_2\sqrt{2h\beta_1}\tanh(\Omega t)\,\sigma_3\sqrt{2h\beta_1}\operatorname{sech}(\Omega t) \\
&= -(\sigma_4 + \sigma_2\sigma_3)2h\operatorname{sech}(\Omega t)\,\tanh(\Omega t)
\end{aligned}$$

That is to say, provided that $\sigma_2 = -\sigma_3\sigma_4$, $\sigma_3 = -\sigma_4\sigma_2$ and $\sigma_4 = -\sigma_2\sigma_3$, or equivalently

$$(\sigma_2, \sigma_3, \sigma_4) \in \{(1, -1, 1), (1, 1, -1), (-1, 1, 1), (-1, -1, -1)\},$$

we have that $p(\cdot)$ is an integral curve of $\vec{H}_{\max}$ (for any $h > 0$). We now make an explicit claim regarding the integral curves of $\vec{H}_{\max}$ in this case.

4.3.3 PROPOSITION. *Suppose $p(\cdot) : \mathbb{R} \to (\widetilde{\mathfrak{h}}_3^\diamond)^*$ is an integral curve of $\vec{H}_{\max}$ such that $p(0) \notin E$, $P_3(p(0)) = 0$, $H_{\max}(p(0)) = h$, $\mathcal{C}(p(0)) = c$ and $c = 2h\beta_1$. Then there exists $t_0 \in \mathbb{R}$ and $\sigma_1, \sigma_2 \in \{-1, 1\}$ such that $p(t) = \bar{p}(t + t_0)$ for $t \in \mathbb{R}$, where*

$$\begin{cases} \bar{p}_1(t) = \sigma_1\sqrt{2h\beta_1}\tanh(\Omega t) \\ \bar{p}_2(t) = \sigma_2\sqrt{2h\beta_1}\operatorname{sech}(\Omega t) \\ \bar{p}_3(t) = 0 \\ \bar{p}_4(t) = -\sigma_1\sigma_2\sqrt{2h\beta_2}\operatorname{sech}(\Omega t) \end{cases} \qquad \Omega = \sqrt{\frac{2h}{\beta_2}}.$$

PROOF. We start by claiming that $p_4(0) \neq 0$, $p_2(0) \neq 0$ and $p_1(0) \neq \pm\sqrt{2h\beta_1}$. If $p_4(0) = 0$, then $p(0) = (p_1(0), p_2(0), 0, 0) \in E$ (see proposition 4.2.1). If $p_1(0) = \pm\sqrt{2h\beta_1}$, then (as $\frac{1}{\beta_1}p_1(0)^2 + \frac{1}{\beta_2}p_4(0)^2 = 2h$) we get that $p_4(0) = 0$ and hence that $p(0) \in E$. If $p_2(0) = 0$, then (as $p_1(0)^2 + p_2(0)^2 = c = 2h\beta_1$) we get that $p_1(0) = \pm\sqrt{2h\beta_1}$ and hence that $p(0) \in E$.

Let $\sigma_2 = \operatorname{sgn}(p_2(0))$ and $-\sigma_1\sigma_2 = \operatorname{sgn}(p_4(0))$. As $\frac{1}{\beta_1}p_1(0)^2 + \frac{1}{\beta_2}p_4(0)^2 = 2h$ and $p_1 \neq \pm\sqrt{2h\beta_1}$, we have that $-\sqrt{2h\beta_1} < p_1(0) < \sqrt{2h\beta_1}$. Now, as $\operatorname{im}\tanh(\cdot) = (-1,1)$, there exists $t_0 \in \mathbb{R}$ such that $\sigma_1\sqrt{2h\beta_1}\tanh(\Omega t_0) = \bar{p}_1(t_0) = p_1(0)$. Then, as $p_2(0)^2 = c - p_1(0)^2 = c - \bar{p}_1(t_0)^2 = \bar{p}_2(t_0)^2$ and $\operatorname{sgn}(\bar{p}_2(t_0)) = \sigma_2 = \operatorname{sgn}(p_2(0))$, we get that $p_2(0) = \bar{p}_2(t_0)$. Similarly, as $p_4(0) = 2h\beta_2 - \frac{\beta_2}{\beta_1}p_1(0)^2 = \bar{p}_4(0)^2$ and $\operatorname{sgn}(\bar{p}_4(t_0)) = -\sigma_1\sigma_2 = \operatorname{sgn}(p_4(0))$, we get that $p_4(0) = \bar{p}_4(t_0)$. That is to say, there exists values for t_0 and σ (as described above) such that $p(0) = \bar{p}(t_0)$. Thus. as $t \mapsto p(t)$ and $t \mapsto \bar{p}(t + t_0)$ are both integral curves of $\vec{H}_{\max}$ and go through the same point at $t = 0$ (i.e., they solve the same Cauchy problem), they are identical.□

4.3.4 Remark. Note that the four possibilities for σ in the above proposition correspond to the four components of $\mathcal{K}_{(0,c,h)}\backslash E$, as illustrated in figure 4.3b. We verified the above claim by solving the Cauchy problem $p(0) = \bar{p}(0)$, $\dot{p}(t) = \vec{H}_{\max}$ numerically (using the Mathematica function `NDSolve`) over a suitable interval and for some suitable c, h, β_1 and β_2. We also note that the above proposition may be arrived at by limiting the cases of $c < 2h\beta_1$ and $c > 2h\beta_1$ to $c = 2h\beta_1$ (i.e., limiting $k \to 1$), each providing (two) connected components (of the four connected components) of $\mathcal{K}_{(0,c,h)}\backslash E$.

We are left to investigate the case $c < 2h\beta_1$. Note that in this case $(\frac{c}{\beta_1} - 2h)\beta_2 < 0$. So, for $\sqrt{(2h - \frac{c}{\beta_1})\beta_2} \leq p_4(t) \leq \sqrt{2h\beta_2}$ and assuming $p_4(0) = 0$, we get, by use of formula (A.5.2), that

$$\begin{aligned}
\tfrac{\sigma_1}{\beta_2}t &= \int_{p_4(t)}^{\sqrt{2h\beta_2}} \frac{dp_4}{\sqrt{\left(2h\beta_2 - p_4^2\right)\left(p_4^2 - \left(2h - \frac{c}{\beta_1}\right)\beta_2\right)}} \\
&= \tfrac{1}{\sqrt{2h\beta_2}}\operatorname{dn}^{-1}\left(\frac{p_4}{\sqrt{2h\beta_2}}, \frac{\sqrt{2h\beta_2 - \left(2h - \frac{c}{\beta_1}\right)\beta_2}}{\sqrt{2h\beta_2}}\right) \\
&= \tfrac{1}{\sqrt{2h\beta_2}}\operatorname{dn}^{-1}\left(\frac{p_4}{\sqrt{2h\beta_2}}, \sqrt{\frac{c}{2h\beta_1}}\right)
\end{aligned}$$

Hence we get (noting that $\operatorname{dn}(x,k)$ is even for a fixed modulus k) that

$$p_4(t) = \sqrt{2h\beta_2}\operatorname{dn}\left(\sqrt{\tfrac{2h}{\beta_2}}t, \sqrt{\tfrac{c}{2h\beta_1}}\right) = \sqrt{2h\beta_2}\operatorname{dn}(\Omega t),$$

where $\Omega = \sqrt{\frac{2h}{\beta_2}}$ and the modulus $k = \sqrt{\frac{c}{2h\beta_1}}$ has been fixed. Now as $\frac{1}{\beta_1}p_1(t)^2 + \frac{1}{\beta_2}p_4(t)^2 = 2h$ we get that

$$p_1(t)^2 = 2h\beta_1 - \tfrac{p_4(t)^2\beta_1}{\beta_2} = 2h\beta_1\left(1 - \operatorname{dn}^2(\Omega t)\right) = 2h\beta_1 k^2\operatorname{sn}^2(\Omega t) = c\operatorname{sn}^2(\Omega t).$$

Therefore we have that $p_1(t) = \sigma_2\sqrt{c}\operatorname{sn}\left(\sqrt{\frac{2h}{\beta_2}}\sigma_1 t\right)$ for some $\sigma_2 \in \{-1,1\}$. Next, as $p_1^2 + p_2^2 = c$, we get that

$$p_2(t)^2 = c - p_1(t)^2 = c(1 - \operatorname{sn}^2(\Omega t)) = c\operatorname{cn}^2(\Omega t)$$

and so $p_2(t) = \sigma_3\sqrt{c}\,\mathrm{cn}\left(\sqrt{\frac{2h}{\beta_2}}\sigma_1 t\right)$ for some $\sigma_3 \in \{-1, 1\}$.

In summary (generalising the sign of $p_4(t)$ with $\sigma_4 \in \{-1, 1\}$) we have that

$$\begin{cases} p_1(t) = \sigma_2\sqrt{c}\,\mathrm{sn}\,(\Omega t) \\ p_2(t) = \sigma_3\sqrt{c}\,\mathrm{cn}\,(\Omega t) \\ p_3(t) = 0 \\ p_4(t) = \sigma_4\sqrt{2h\beta_2}\,\mathrm{dn}\,(\Omega t) \end{cases} \qquad \begin{cases} \Omega = \sqrt{\frac{2h}{\beta_2}} \\ k = \sqrt{\frac{c}{2h\beta_1}} \\ \sigma_i \in \{-1, 1\},\ i = \overline{2, 4}. \end{cases}$$

Notice that $p(t)$ is defined for all $t \in \mathbb{R}$, i.e., we have a smooth curve $p(\cdot) : \mathbb{R} \to (\widetilde{\mathfrak{h}}_3^\diamond)^*$ as a prospective candidate for a general integral curve of $\vec{H}_{\max}$ of this type. We again verify exactly when (specifically for which σ) $p(\cdot)$ is an integral curve of $\vec{H}_{\max}$. Using equation (4.3.1), we investigate when $\frac{d}{dt}(p(t)) - \vec{H}_{\max}(p(t)) = 0$. We get that

$$\begin{aligned} \dot{p}_1(t) + \tfrac{p_2(t)p_4(t)}{\beta_2} &= \sigma_2\sqrt{c}\Omega\,\mathrm{cn}\,(\Omega t)\ \mathrm{dn}\,(\Omega t) + \tfrac{1}{\beta_2}\sigma_3\sqrt{c}\,\mathrm{cn}\,(\Omega t)\,\sigma_4\sqrt{2h\beta_2}\,\mathrm{dn}\,(\Omega t) \\ &= (\sigma_2 + \sigma_3\sigma_4)\sqrt{\tfrac{2hc}{\beta_2}}\,\mathrm{cn}\,(\Omega t)\ \mathrm{dn}\,(\Omega t) \\ \dot{p}_2(t) - \tfrac{p_1(t)p_4(t)}{\beta_2} &= -\sigma_3\sqrt{c}\Omega\,\mathrm{sn}\,(\Omega t)\ \mathrm{dn}\,(\Omega t) - \tfrac{1}{\beta_2}\sigma_2\sqrt{c}\,\mathrm{sn}\,(\Omega t)\,\sigma_4\sqrt{2h\beta_2}\,\mathrm{dn}\,(\Omega t) \\ &= -(\sigma_3 + \sigma_4\sigma_2)\sqrt{\tfrac{2hc}{\beta_2}}\,\mathrm{sn}\,(\Omega t)\ \mathrm{dn}\,(\Omega t) \\ \dot{p}_4(t) - \tfrac{p_1(t)p_2(t)}{\beta_1} &= -\sigma_4\sqrt{2h\beta_2}\Omega k^2\,\mathrm{sn}\,(\Omega t)\ \mathrm{cn}\,(\Omega t) - \tfrac{1}{\beta_1}\sigma_2\sqrt{c}\,\mathrm{sn}\,(\Omega t)\,\sigma_3\sqrt{c}\,\mathrm{cn}\,(\Omega t) \\ &= -(\sigma_4 + \sigma_2\sigma_3)\tfrac{c}{\beta_1}\,\mathrm{sn}\,(\Omega t)\ \mathrm{cn}\,(\Omega t) \end{aligned}$$

That is to say, provided that $\sigma_2 = -\sigma_3\sigma_4$, $\sigma_3 = -\sigma_4\sigma_2$ and $\sigma_4 = -\sigma_2\sigma_3$, or equivalently

$$(\sigma_2, \sigma_3, \sigma_4) \in \{(1, -1, 1), (1, 1, -1), (-1, 1, 1), (-1, -1, -1)\},$$

we have that $p(\cdot)$ is an integral curve of $\vec{H}_{\max}$ (for any $h > 0$ and $c > 0$ such that $c < 2h\beta_1$). We now make an explicit claim regarding the integral curves of $\vec{H}_{\max}$ in this case.

4.3.5 Proposition. *Suppose $p(\cdot) : \mathbb{R} \to (\widetilde{\mathfrak{h}}_3^\diamond)^*$ is an integral curve of $\vec{H}_{\max}$ such that $P_3(p(0)) = 0$, $H_{\max}(p(0)) = h$, $\mathcal{C}(p(0)) = c$ and $0 < c < 2h\beta_1$. Then there exists $t_0 \in [-\frac{K}{\Omega}, \frac{3K}{\Omega}]$ and $\sigma \in \{-1, 1\}$ such that $p(t) = \bar{p}(t + t_0)$ for $t \in \mathbb{R}$, where ($4K$ is the period of $\mathrm{cn}(\cdot, k)$ and)*

$$\begin{cases} p_1(t) = \sqrt{c}\,\mathrm{sn}\,(\Omega t) \\ p_2(t) = -\sigma\sqrt{c}\,\mathrm{cn}\,(\Omega t) \\ p_3(t) = 0 \\ p_4(t) = \sigma\sqrt{2h\beta_2}\,\mathrm{dn}\,(\Omega t) \end{cases} \qquad \begin{cases} \Omega = \sqrt{\frac{2h}{\beta_2}} \\ k = \sqrt{\frac{c}{2h\beta_1}}. \end{cases}$$

Proof. We follow an approach very similar to that of the proof of proposition 4.3.1. Let $\varsigma = \mathrm{sgn}(p_2(0))$ and $\sigma = \mathrm{sgn}(p_4(0))$. Notice that if $p_4(0) = 0$, then the condition $c < 2h\beta_1$ is equivalent to $p_1(0)^2 + p_2(0)^2 < p_1(0)^2$, a contradiction. Thus $\sigma \neq 0$.

Next observe that

$$\bar{p}_1(-\tfrac{K}{\Omega}) = -\sqrt{c} \qquad \bar{p}_1(\tfrac{K}{\Omega}) = \sqrt{c} \qquad \bar{p}_1(\tfrac{3K}{\Omega}) = -\sqrt{c}.$$

Thus, as $\bar{p}_1(\cdot)$ is continuous, for all $r \in \mathbb{R}$ such that $-\sqrt{c} \leq r \leq \sqrt{c}$, there exists a $t_0 \in [-\frac{K}{\Omega}, \frac{K}{\Omega}]$ and a $t_0 \in [\frac{K}{\Omega}, \frac{3K}{\Omega}]$, such that $\bar{p}_1(t_0) = r$. However, observe that $\mathrm{cn}(x,k) \geq 0$ for $x \in [-K, K]$ and $\mathrm{cn}(x,k) \leq 0$ for $x \in [K, 3K]$. Thus, for all $r \in \mathbb{R}$ such that $-\sqrt{c} \leq r \leq \sqrt{c}$, there exists a $t_0 \in [-\frac{K}{\Omega}, \frac{3K}{\Omega}]$, such that $\bar{p}_1(t_0) = r$ and $-\varsigma\sigma\,\mathrm{cn}\,(t_0) \geq 0$. Now as $p_1(0)^2 + p_2(0)^2 = c$, we have that $-\sqrt{c} \leq p_1(0) \leq \sqrt{c}$. Therefore there then exists a $t_0 \in [-\frac{K}{\Omega}, \frac{3K}{\Omega}]$ such that $\bar{p}_1(0) = p_1(0)$ and $\varsigma\,\bar{p}_2(t_0) \geq 0$. Next, as $p_1(0)^2 + p_2(0)^2 = c$, we have that

$$p_2(0)^2 = c - \bar{p}_1(t_0)^2 = \bar{p}_2(t_0)^2.$$

But we have that $\mathrm{sgn}(p_2(0))\bar{p}_2(t_0) \geq 0$, thus $p_2(0) = \bar{p}_2(t_0)$. Next, as $\frac{1}{\beta_1}p_1(0)^2 + \frac{1}{\beta_2}p_4(0)^2 = 2h$, we have that

$$p_4(0)^2 = \beta_2(2h - \tfrac{1}{\beta_1}\bar{p}_1(t_0)^2) = \bar{p}_4(t_0)^2.$$

But as $\mathrm{sgn}(\bar{p}_4(t_0)) = \sigma = \mathrm{sgn}(p_4(0))$, we get that $p_4(0) = \bar{p}_4(0)$. Thus we have that $p(0) = \bar{p}(t_0)$. By the preceding discussion we have that $\bar{p}(\cdot)$ is also an integral curve of $\vec{H}_{\max}$. Thus as $t \mapsto p(t)$ and $t \mapsto \bar{p}(t + t_0)$ are both integral curves of $\vec{H}_{\max}$ and go through the same point at $t = 0$ (i.e., they solve the same Cauchy problem), they are identical. □

4.3.6 Remark. Note that the two values for σ in the above proposition correspond to the two connected components of $\mathcal{K}_{(0,c,h)}$, as illustrated in figure 4.3c. We verified the above claim by solving the Cauchy problem $p(0) = \bar{p}(0)$, $\dot{p}(t) = \vec{H}_{\max}$ numerically (using the Mathematica function `NDSolve`) over a suitable interval and for some suitable c, h, β_1 and β_2.

4.3.2 Type $\rho > 0$, case 1a

We now move on to considering integral curves developing on $\mathcal{K}_{(\rho,c,h)}$ where $\rho > 0$ and the conditions of case 1a are satisfied, as described in section 4.2.3. (That is $2h\beta_1 > \frac{\beta_2}{\beta_1}\rho^2$ and $c > 2h\beta_1 + \frac{\beta_2}{\beta_1}\rho^2$. In particular, notice that these assumptions imply that $h > 0$ and $c > 0$.) Calculations in this section become quite cumbersome. As such we will make extensive use of Mathematica; the code (and output) for supporting calculations may be found in section C.11.

Under the above assumptions (specifically $\rho > 0$), we have that

$$\dot{p}_4^2 = \tfrac{1}{\beta_2^2}\left(2h\beta_2 - p_4^2\right)\left(\tfrac{c\beta_2}{\beta_1} - 2h\beta_2 + 2\tfrac{\beta_2}{\beta_1}\rho p_4 + p_4^2\right),$$

i.e., equation (4.3.2), is not in standard form. We use the method described in subsection A.5.2 to transform this equation into standard form. However, before we do this, we prove some (useful) inequalities for this case. We claim that:

$$c^2 - 8h\rho^2\beta_2 > 0; \quad \sqrt{c^2 - 8h\rho^2\beta_2} - c < 0; \quad c - 2\tfrac{\beta_2}{\beta_1}\rho^2 > 0; \quad \sqrt{c^2 - 8h\rho^2\beta_2} - c + 2\tfrac{\beta_2}{\beta_1}\rho^2 > 0.$$

By lemma 4.2.19, we get that $c > 2\rho\sqrt{2h\beta_2}$, proving the first inequality. Next, as $8h\rho^2\beta_2 > 0$, we get that $c^2 - 8h\rho^2\beta_2 < c^2$ and hence (as both sides are positive) that $\sqrt{c^2 - 8h\rho^2\beta_2} - c < 0$. Now observe that (by using the conditions $2h\beta_1 > \frac{\beta_2}{\beta_1}\rho^2$ and $c > 2h\beta_1 + \frac{\beta_2}{\beta_1}\rho^2$) we get that

$$c - 2\tfrac{\beta_2}{\beta_1}\rho^2 > 2h\beta_1 + \tfrac{\beta_2}{\beta_1}\rho^2 - 2\tfrac{\beta_2}{\beta_1}\rho^2 > \tfrac{\beta_2}{\beta_1}\rho^2 - \tfrac{\beta_2}{\beta_1}\rho^2 = 0,$$

yielding the third inequality. Hence, for the last case we have that

$$\begin{aligned} &\sqrt{c^2 - 8h\rho^2\beta_2} - c + 2\tfrac{\beta_2}{\beta_1}\rho^2 > 0 \\ \Leftrightarrow\quad & c^2 - 8h\rho^2\beta_2 > \left(c - 2\tfrac{\beta_2}{\beta_1}\rho^2\right)^2 \\ \Leftrightarrow\quad & -8h\rho^2\beta_2 + 4c\tfrac{\beta_2}{\beta_1}\rho^2 - 4\tfrac{\beta_2^2}{\beta_1^2}\rho^4 > 0, \end{aligned}$$

but we have (using the inequality $c > 2h\beta_1 + \frac{\beta_2}{\beta_1}\rho^2$) that

$$-8h\rho^2\beta_2 + 4c\tfrac{\beta_2}{\beta_1}\rho^2 - 4\tfrac{\beta_2^2}{\beta_1^2}\rho^4 > -8h\rho^2\beta_2 + 4\left(2h\beta_1 + \tfrac{\beta_2}{\beta_1}\rho^2\right)\tfrac{\beta_2}{\beta_1}\rho^2 - 4\tfrac{\beta_2^2}{\beta_1^2}\rho^4 = 0$$

thus yielding the fourth and last inequality. To simplify future expressions define, we define a constant δ as

$$\delta = \sqrt{c^2 - 8h\rho^2\beta_2}.$$

In summary we then have the following.

4.3.7 Lemma. *If $\rho > 0$, $2h\beta_1 > \frac{\beta_2}{\beta_1}\rho^2$ and $c > 2h\beta_1 + \frac{\beta_2}{\beta_1}\rho^2$, then*

$$\delta^2 > 0 \qquad \delta - c < 0 \qquad c - 2\rho^2\tfrac{\beta_2}{\beta_1} > 0 \qquad \delta - c + 2\tfrac{\beta_2}{\beta_1}\rho^2 > 0.$$

We now proceed to transform equation (4.3.2) into standard form. We give the results for the main steps; full details may be found in section C.11. Let

$$X_1 = \tfrac{c\beta_2}{\beta_1} - 2h\beta_2 + 2\tfrac{\beta_2}{\beta_1}p_3p_4 + p_4^2 \qquad\qquad X_2 = 2h\beta_2 - p_4^2.$$

Then we have that $\dot{p}_4^2 = \frac{1}{\beta_2^2}X_1X_2$. The values for which $X_1 - \lambda X_2$ is a perfect square are $\lambda_1 = \frac{c-4h\beta_1-\delta}{4h\beta_1}$ and $\lambda_2 = \frac{c-4h\beta_1+\delta}{4h\beta_1}$. Then we have that

$$X_1 - \lambda_1X_2 = (1+\lambda_1)(x-r_1)^2 \qquad\qquad X_1 - \lambda_2X_2 = (1+\lambda_2)(x-r_2)^2$$

where $r_1 = -\frac{c+\delta}{2\rho}$ and $r_2 = -\frac{4h\rho\beta_2}{c+\delta}$. Consequently we have that

$$X_1X_2 = \left(A_1(x-r_1)^2 + B_1(x-r_2)^2\right)\left(A_2(x-r_1)^2 + B_2(x-r_2)^2\right),$$

where

$$A_1 = \tfrac{\delta - c + 2\rho^2\frac{\beta_2}{\beta_1}}{2\delta} > 0 \qquad A_2 = \tfrac{c-\delta}{2\delta} > 0 \qquad B_1 = \tfrac{\delta + c - 2\rho^2\frac{\beta_2}{\beta_1}}{2\delta} > 0 \qquad B_2 = \tfrac{-c-\delta}{2\delta} < 0.$$

Thus we have that (for some $\sigma_1 \in \{-1, 1\}$)

$$\tfrac{1}{\beta_2}\sigma_1 dt = \frac{dp_4}{\sqrt{(A_1(x-r_1)^2+B_1(x-r_2)^2)(A_2(x-r_1)^2+B_2(x-r_2)^2)}}.$$

Then using the change of variables $s = \frac{p_4-r_1}{p_4-r_2}$ we get that

$$\sigma_1 dt = \frac{1}{\frac{1}{\beta_2}(r_1-r_2)\sqrt{A_1A_2}}\frac{ds}{\sqrt{\left(s^2+\frac{B_1}{A_1}\right)\left(s^2+\frac{B_2}{A_2}\right)}},$$

where $(r_1 - r_2)\sqrt{A_1A_2} = -\frac{\delta}{\rho}\sqrt{A_1A_2} < 0$, $\frac{B_1}{A_1} > 0$ and $\frac{B_2}{A_2} < 0$.

Having reduces our equation into standard form we now proceed to integration. Using formula (A.5.3), we get (under some constraints for $p_4(t)$) that

$$\begin{aligned}\tfrac{1}{\beta_2}(r_1 - r_2)\sqrt{A_1A_2}\sigma_1 t &= \int_{\frac{p_4(t)-r_1}{p_4(t)-r_2}}^{-\frac{B_2}{B_1}} \frac{ds}{\sqrt{\left(s^2-\left(-\frac{B_2}{A_2}\right)\right)\left(s^2+\frac{B_1}{A_1}\right)}}\\ &= \frac{1}{\sqrt{\frac{B_1}{A_1}-\frac{B_2}{A_2}}}\,\mathrm{nc}^{-1}\left(\frac{1}{\sqrt{-\frac{B_2}{A_2}}}\frac{p_4(t)-r_1}{p_4(t)-r_2}, \frac{\sqrt{\frac{B_1}{A_1}}}{\sqrt{\frac{B_1}{A_1}-\frac{B_2}{A_2}}}\right).\end{aligned}$$

Define Ω, and fix the modulus k as

$$\begin{aligned}\Omega &= (r_2 - r_1)\tfrac{1}{\beta_2}\sqrt{A_1A_2}\sqrt{\tfrac{B_1}{A_1} - \tfrac{B_2}{A_2}} = (r_2 - r_1)\tfrac{1}{\beta_2}\sqrt{A_2B_1 - A_1B_2}\\ k &= \frac{\sqrt{\frac{B_1}{A_1}}}{\sqrt{\frac{B_1}{A_1}-\frac{B_2}{A_2}}} = \sqrt{\tfrac{A_2B_1}{A_2B_1-A_1B_2}}.\end{aligned}$$

Then (noting that $\mathrm{cn}(\cdot, k)$ is even for a fixed modulus k) we get that

$$\begin{aligned} & \mathrm{nc}(-\Omega\sigma_1 t) = \frac{1}{\sqrt{-\frac{B_2}{A_2}}}\frac{p_4(t)-r_1}{p_4(t)-r_2}\\ \Rightarrow\quad & \sqrt{-\tfrac{B_2}{A_2}}\,\mathrm{nc}(\Omega t)p_4(t) - r_2\sqrt{-\tfrac{B_2}{A_2}}\,\mathrm{nc}(\Omega t) = p_4(t) - r_1\\ \Rightarrow\quad & r_1 - r_2\sqrt{-\tfrac{B_2}{A_2}}\,\mathrm{nc}(\Omega t) = p_4(t)\left(1 - \sqrt{-\tfrac{B_2}{A_2}}\,\mathrm{nc}(\Omega t)\right)\\ \Rightarrow\quad & p_4(t) = \frac{r_1 - r_2\sqrt{-\frac{B_2}{A_2}}\,\mathrm{nc}(\Omega t)}{1-\sqrt{-\frac{B_2}{A_2}}\,\mathrm{nc}(\Omega t)}\\ \Rightarrow\quad & p_4(t) = \frac{r_2\sqrt{-\frac{B_2}{A_2}} - r_1\,\mathrm{cn}(\Omega t)}{\sqrt{-\frac{B_2}{A_2}} - \mathrm{cn}(\Omega t)}\end{aligned}$$

Using Mathematica, we get the following simplifications

$$\begin{aligned}\Omega &= \sqrt{\tfrac{\delta}{\beta_1\beta_2}} & k &= \sqrt{\tfrac{\delta-c+4h\beta_1}{2\delta}} & k' &= \sqrt{\tfrac{c+\delta-4h\beta_1}{2\delta}}\\ r_1 &= -\tfrac{c+\delta}{2\rho} & \sqrt{-\tfrac{B_2}{A_2}} &= \sqrt{\tfrac{c+\delta}{c-\delta}} & r_2\sqrt{-\tfrac{B_2}{A_2}} &= -\tfrac{\sqrt{c^2-\delta^2}}{2\rho}.\end{aligned}$$

Using the above simplifications we get that

$$p_4(t) = \frac{-\frac{\sqrt{c^2-\delta^2}}{2\rho} + \frac{c+\delta}{2\rho}\,\mathrm{cn}(\Omega t)}{\sqrt{\frac{c+\delta}{c-\delta}} - \mathrm{cn}(\Omega t)} = \frac{\sqrt{c^2-\delta^2}}{2\rho}\,\frac{-\sqrt{c-\delta}+\sqrt{c+\delta}\,\mathrm{cn}(\Omega t)}{\sqrt{c+\delta}-\sqrt{c-\delta}\,\mathrm{cn}(\Omega t)}.$$

We claim that $p_4(t)$ (as defined above) is defined for all $t \in \mathbb{R}$ and any ρ, c and h such that $\rho > 0$, $2h\beta_1 > \frac{\beta_2}{\beta_1}\rho^2$ and $c > 2h\beta_1 + \frac{\beta_2}{\beta_1}\rho^2$. Assume $\rho > 0$, $2h\beta_1 > \frac{\beta_2}{\beta_1}\rho^2$ and $c > 2h\beta_1 + \frac{\beta_2}{\beta_1}\rho^2$. First notice (by in part using lemma 4.3.7) that: $c > 0$; $\delta^2 = c^2 - 8h\rho^2\beta_2 > 0$ and so δ is defined; $c^2 - \delta^2 = 8h\rho^2\beta_2 > 0$; $\delta - c > 0$; $\delta + c > 0$; and $\Omega = \sqrt{\frac{\delta}{\beta_1\beta_2}}$ is defined. Next consider $k = \sqrt{\frac{\delta - c + 4h\beta_1}{2\delta}}$. We have (applying assumption $2h\beta_1 > \frac{\beta_2}{\beta_1}\rho^2$ and using lemma 4.3.7) that

$$\delta - c + 2(2h\beta_1) > \delta - c + 2\tfrac{\beta_2}{\beta_1}\rho^2 > 0.$$

Furthermore we have that $\frac{\delta - c + 4h\beta_1}{2\delta} < 1$ if and only if $c + \delta - 4h\beta_1 > 0$. But (by applying the assumptions $2h\beta_1 > \frac{\beta_2}{\beta_1}\rho^2$ and $c > 2h\beta_1 + \frac{\beta_2}{\beta_1}\rho^2$) we get that

$$\delta + c - 4h\beta_1 > \delta + (2h\beta_1 + \tfrac{\beta_2}{\beta_1}\rho^2) - 4h\beta_1 = \delta + (\tfrac{\beta_2}{\beta_1}\rho^2 - 2h\beta_1) > \delta > 0.$$

That is to say, our modulus k is defined (i.e., $0 < k < 1$). Finally we argue that the denominator of $p_4(t)$ is non-zero for all $t \in \mathbb{R}$. We observe that $c + \delta > c > \sqrt{8h\rho^2\beta_2} = \sqrt{c^2 - \delta^2}$, implying that $\frac{c+\delta}{\sqrt{c^2-\delta^2}} = \frac{\sqrt{c+\delta}}{\sqrt{c-\delta}} > 1 \geq \operatorname{cn}(\Omega t)$ and so yielding $\operatorname{cn}(\Omega t) - \frac{\sqrt{c+\delta}}{\sqrt{c-\delta}} < 0$. Hence, as $\sqrt{c+\delta} > 0$, we have that $\sqrt{c+\delta} - \sqrt{c-\delta}\operatorname{cn}(\Omega t) > 0$.

Having established that $p_4(t)$ is defined for all $t \in \mathbb{R}$, we now proceed to calculate the other coordinates by use of the constants of motion $\mathcal{C}$ and $H_{\max}$ for $\vec{H}_{\max}$. As $\frac{1}{2}\left(\frac{p_1^2}{\beta_1} + \frac{p_4^2}{\beta_2}\right) = h$ we get (again using Mathematica for simplifications, and noting $c^2 - \delta^2 = 8h\rho^2\beta_2$) that

$$\begin{aligned} p_1(t)^2 &= 2h\beta_1 - \tfrac{p_4(t)^2\beta_1}{\beta_2} = \tfrac{(c^2-\delta^2)}{4\rho^2\beta_2}\beta_1 - \tfrac{p_4(t)^2\beta_1}{\beta_2} \\ &= \tfrac{\delta(c^2-\delta^2)\beta_1}{2\rho^2\beta_2}\tfrac{1-\operatorname{cn}^2(\Omega t)}{\left(\sqrt{c+\delta}-\sqrt{c-\delta}\operatorname{cn}(\Omega t)\right)^2} \\ &= \tfrac{\delta(c^2-\delta^2)\beta_1}{2\rho^2\beta_2}\tfrac{\operatorname{sn}^2(\Omega t)}{\left(\sqrt{c+\delta}-\sqrt{c-\delta}\operatorname{cn}(\Omega t)\right)^2}. \end{aligned}$$

Thus $p_1(t) = \sigma_2\frac{\sqrt{\delta(c^2-\delta^2)\beta_1}}{\rho\sqrt{2\beta_2}}\frac{\operatorname{sn}(\Omega t)}{\sqrt{c+\delta}-\sqrt{c-\delta}\operatorname{cn}(\Omega t)}$ for some $\sigma_2 \in \{-1, 1\}$. Next as $p_1(t)^2 + p_2(t)^2 - 2\rho p_4(t) = c$ we get (making use of Mathematica) that

$$p_2^2 = c - p_1^2 + 2\rho p_4 = \tfrac{2\delta^2\operatorname{dn}^2(\Omega t)}{\left(\sqrt{c+\delta}-\sqrt{c-\delta}\operatorname{cn}(\Omega t)\right)^2}.$$

Thus $p_2(t) = \sigma_3\frac{\sqrt{2}\delta\operatorname{dn}(\Omega t)}{\sqrt{c+\delta}-\sqrt{c-\delta}\operatorname{cn}(\Omega t)}$ for some $\sigma_3 \in \{-1, 1\}$.

In summary we have that

$$\begin{cases} p_1(t) = \sigma_2\frac{\sqrt{\delta(c^2-\delta^2)\beta_1}}{\rho\sqrt{2\beta_2}}\frac{\operatorname{sn}(\Omega t)}{\sqrt{c+\delta}-\sqrt{c-\delta}\operatorname{cn}(\Omega t)} \\ p_2(t) = \sigma_3\frac{\sqrt{2}\delta\operatorname{dn}(\Omega t)}{\sqrt{c+\delta}-\sqrt{c-\delta}\operatorname{cn}(\Omega t)} \\ p_3(t) = \rho > 0 \\ p_4(t) = \frac{\sqrt{c^2-\delta^2}}{2\rho}\frac{-\sqrt{c-\delta}+\sqrt{c+\delta}\operatorname{cn}(\Omega t)}{\sqrt{c+\delta}-\sqrt{c-\delta}\operatorname{cn}(\Omega t)} \end{cases} \qquad \begin{cases} \delta = \sqrt{c^2 - 8h\rho^2\beta_2} \\ \Omega = \sqrt{\frac{\delta}{\beta_1\beta_2}} \\ k = \sqrt{\frac{\delta-c+4h\beta_1}{2\delta}} \\ \sigma_i \in \{-1, 1\},\ i = \overline{2,3}. \end{cases}$$

Notice that $p(t)$ is defined for all $t \in \mathbb{R}$, i.e., we have a smooth curve $p(\cdot) : \mathbb{R} \to (\widetilde{\mathfrak{h}}_3^\diamond)^*$ as a prospective candidate for a general integral curve of $\vec{H}_{\max}$ of this type. We again verify exactly when (specifically for which σ) $p(\cdot)$ is an integral curve of $\vec{H}_{\max}$. That is, we investigate when $\frac{d}{dt}(p(t)) - \vec{H}_{\max}(p(t)) = 0$. This calculation gets quite involved and was therefore done with Mathematica. We found, for $(\sigma_2, \sigma_3) \in \{(1, -1), (-1, 1)\}$, that $\frac{d}{dt}(p(t)) - \vec{H}_{\max}(p(t)) = 0$. For details see section C.11. We now make an explicit claim regarding the integral curves of $\vec{H}_{\max}$ in this case.

4.3.8 PROPOSITION. *Suppose $p(\cdot) : \mathbb{R} \to (\widetilde{\mathfrak{h}}_3^\diamond)^*$ is an integral curve of $\vec{H}_{\max}$ such that $P_3(p(0)) = \rho > 0$, $H_{\max}(p(0)) = h$, $\mathcal{C}(p(0)) = c$, $2h\beta_1 > \frac{\beta_2}{\beta_1}\rho^2$ and $c > 2h\beta_1 + \frac{\beta_2}{\beta_1}\rho^2$. Then there exists $t_0 \in [0, \frac{4K}{\Omega}]$ and $\sigma \in \{-1, 1\}$ such that $p(t) = \bar{p}(t + t_0)$ for $t \in \mathbb{R}$, where ($4K$ is the period of $\mathrm{cn}(\cdot, k)$ and)*

$$\begin{cases} \bar{p}_1(t) = -\sigma \frac{\sqrt{\delta(c^2-\delta^2)\beta_1}}{\rho\sqrt{2\beta_2}} \frac{\mathrm{sn}(\Omega t)}{\sqrt{c+\delta} - \sqrt{c-\delta}\,\mathrm{cn}(\Omega t)} \\ \bar{p}_2(t) = \sigma \frac{\sqrt{2}\delta\,\mathrm{dn}(\Omega t)}{\sqrt{c+\delta} - \sqrt{c-\delta}\,\mathrm{cn}(\Omega t)} \\ \bar{p}_3(t) = \rho > 0 \\ \bar{p}_4(t) = \frac{\sqrt{c^2-\delta^2}}{2\rho} \frac{-\sqrt{c-\delta} + \sqrt{c+\delta}\,\mathrm{cn}(\Omega t)}{\sqrt{c+\delta} - \sqrt{c-\delta}\,\mathrm{cn}(\Omega t)} \end{cases} \qquad \begin{cases} \delta = \sqrt{c^2 - 8h\rho^2\beta_2} \\ \Omega = \sqrt{\frac{\delta}{\beta_1\beta_2}} \\ k = \sqrt{\frac{\delta - c + 4h\beta_1}{2\delta}}. \end{cases}$$

PROOF. We follow an approach very similar to that of the proof of proposition 4.3.1. Let $\varsigma = \mathrm{sgn}(p_1(0))$ and $\sigma = \mathrm{sgn}(p_2(0))$. We claim that $\sigma \neq 0$, i.e., $\sigma \in \{-1, 1\}$. Suppose $p_2(0) = 0 = \sigma$. Then (as $p_1(0)^2 + p_2(0)^2 - 2\rho p_4(0) = c$ and $\frac{1}{\beta_1}p_1(0)^2 + \frac{1}{\beta_2}p_4(0)^2 = 2h$) we get that $p_1(0)^2 = c + 2\rho p_4(0)$ and $p_1(0)^2 = 2h\beta_1 - \frac{\beta_1}{\beta_2}p_4(0)^2$. Thus we have that $2h\beta_1 - \frac{\beta_1}{\beta_2}p_4(0)^2 = c + 2\rho p_4(0)$ and so $\frac{\beta_1}{\beta_2}p_4(0)^2 + 2\rho p_4(0) + (c - 2h\beta_1) = 0$. But as this is a quadratic equation in $p_4(0)$ we have that the discriminant is non-negative. That is, we have that $4(\rho^2 - \frac{\beta_1}{\beta_2}(c - 2h\beta_1)) \geq 0$. Thus we get that $\frac{\beta_2}{\beta_1}\rho^2 + 2h\beta_1 \geq c$. But this contradicts the assumption $c > 2h\beta_1 + \frac{\beta_2}{\beta_1}\rho^2$. Therefore $\sigma \neq 0$.

Next observe that

$$\bar{p}_4(0) = \frac{\sqrt{c^2-\delta^2}}{2\rho} = \sqrt{2h\beta_2} \qquad \bar{p}_4(\tfrac{2K}{\Omega}) = -\sqrt{2h\beta_2} \qquad \bar{p}_4(\tfrac{4K}{\Omega}) = \sqrt{2h\beta_2}.$$

Thus, as $\bar{p}_4(\cdot)$ is continuous, for all $r \in \mathbb{R}$ such that $-\sqrt{2h\beta_2} \leq r \leq \sqrt{2h\beta_2}$, there exists a $t_0 \in [0, \frac{2K}{\Omega}]$ and a $t_0 \in [\frac{2K}{\Omega}, \frac{4K}{\Omega}]$, such that $\bar{p}_4(t_0) = r$. However, observe that $\mathrm{sn}(x, k) \geq 0$ for $x \in [0, 2K]$ and $\mathrm{sn}(x, k) \leq 0$ for $x \in [2K, 4K]$. Thus, for all $r \in \mathbb{R}$ such that $-\sqrt{2h\beta_2} \leq r \leq \sqrt{2h\beta_2}$, there exists a $t_0 \in [0, \frac{4K}{\Omega}]$, such that $\bar{p}_4(t_0) = r$ and $-\varsigma\sigma\,\mathrm{sn}\,(t_0) \geq 0$. Now as $\frac{1}{\beta_1}p_1(0)^2 + \frac{1}{\beta_2}p_4(0)^2 = 2h$, we have that $-\sqrt{2h\beta_2} \leq p_4(0) \leq \sqrt{2h\beta_2}$. Therefore there then exists a $t_0 \in [0, \frac{4K}{\Omega}]$ such that $\bar{p}_4(0) = p_4(0)$ and $\varsigma\,\bar{p}_1(t_0) \geq 0$. Thus we get that

$$p_1(0)^2 = 2h\beta_2 - \frac{\beta_1}{\beta_2}p_4(0)^2 = 2h\beta_2 - \frac{\beta_1}{\beta_2}\bar{p}_4(0)^2 = \bar{p}_1(0)^2$$

and so (as $\mathrm{sgn}(p_1(0))\bar{p}_1(t_0) \geq 0$) we have that $p_1(0) = \bar{p}_1(t_0)$. Next, as $p_1(0)^2 + p_2(0)^2 - 2\rho p_4(0) = c$, we have that

$$p_2(0)^2 = c - p_1(0)^2 + 2\rho p_4(0) = c - \bar{p}_1(t_0)^2 + 2\rho\bar{p}_4(t_0) = \bar{p}_2(t_0)^2.$$

But we have that $\operatorname{sgn}(p_2(0)) = \sigma = \operatorname{sgn}\bar{p}_2(t_0)$, thus $p_2(0) = \bar{p}_2(t_0)$. Hence we have that $p(0) = \bar{p}(t_0)$. Furthermore, by the preceding discussion we have that $\bar{p}(\cdot)$ is an integral curve of $\vec{H}_{\max}$. Thus as $t \mapsto p(t)$ and $t \mapsto \bar{p}(t + t_0)$ are both integral curves of $\vec{H}_{\max}$ and go through the same point at $t = 0$ (i.e., they solve the same Cauchy problem), they are identical. □

4.3.9 Remark. Note that the two values for σ in the above proposition correspond to the two connected components of $\mathcal{K}_{(\rho,c,h)}$, as illustrated in figure 4.5a. We verified the above claim by solving the Cauchy problem $p(0) = \bar{p}(0)$, $\dot{p}(t) = \vec{H}_{\max}$ numerically (using the Mathematica function `NDSolve`) over a suitable interval and for some suitable c, h, β_1 and β_2. Details may be found in section C.11.

By applying proposition 4.2.17 (and noting ρ is replaced by $-\rho$), we get the following corollary.

4.3.10 Corollary. *Suppose $p(\cdot) : \mathbb{R} \to (\widetilde{\mathfrak{h}}_3^\diamond)^*$ is an integral curve of $\vec{H}_{\max}$ such that $P_3(p(0)) = \rho < 0$, $H_{\max}(p(0)) = h$, $\mathcal{C}(p(0)) = c$, $2h\beta_1 > \frac{\beta_2}{\beta_1}\rho^2$ and $c > 2h\beta_1 + \frac{\beta_2}{\beta_1}\rho^2$. Then there exists $t_0 \in [0, \frac{4K}{\Omega}]$ and $\sigma \in \{-1, 1\}$ such that $p(t) = \bar{p}(t + t_0)$ for $t \in \mathbb{R}$, where ($4K$ is the period of $\operatorname{cn}(\cdot, k)$ and)*

$$\begin{cases} \bar{p}_1(t) = -\sigma \frac{\sqrt{\delta(c^2-\delta^2)\beta_1}}{\rho\sqrt{2\beta_2}} \frac{\operatorname{sn}(\Omega t)}{\sqrt{c+\delta} - \sqrt{c-\delta}\operatorname{cn}(\Omega t)} \\ \bar{p}_2(t) = \sigma \frac{\sqrt{2}\delta \operatorname{dn}(\Omega t)}{\sqrt{c+\delta} - \sqrt{c-\delta}\operatorname{cn}(\Omega t)} \\ \bar{p}_3(t) = \rho < 0 \\ \bar{p}_4(t) = \frac{\sqrt{c^2-\delta^2}}{2\rho} \frac{-\sqrt{c-\delta} + \sqrt{c+\delta}\operatorname{cn}(\Omega t)}{\sqrt{c+\delta} - \sqrt{c-\delta}\operatorname{cn}(\Omega t)} \end{cases} \qquad \begin{cases} \delta = \sqrt{c^2 - 8h\rho^2\beta_2} \\ \Omega = \sqrt{\frac{\delta}{\beta_1\beta_2}} \\ k = \sqrt{\frac{\delta - c + 4h\beta_1}{2\delta}}. \end{cases}$$

Conclusion

This thesis investigated four-dimensional oscillator Lie groups; classified (locally and globally) left-invariant control affine systems evolving on these groups (under detached feedback equivalence); and then finally considered a general optimal control problem (with quadratic cost) on one of the equivalence classes. A connected four-dimensional linear oscillator Lie group $\mathsf{H}_3^\diamond$ was realised as a semi-direct product of H_3 and $\mathsf{SO}(2)$. Then the universal covering Lie group $\widetilde{\mathsf{H}}_3^\diamond$ of $\mathsf{H}_3^\diamond$ was found and thereafter all connected Lie groups with isomorphic Lie algebras were classified and briefly investigated. Additionally the adjoint and coadjoint orbits of the oscillator Lie algebra $\mathfrak{h}_3^\diamond$ were explicitly calculated. This then lead to finding an invariant scalar product on $\mathfrak{h}_3^\diamond$ which in turn lead us to show that $\mathfrak{h}_3^\diamond$ is a double extension of the abelian Lie algebra $\mathbb{R}^2$ by $\mathbb{R}$. In doing so we also calculated all the ideals of $\mathfrak{h}_3^\diamond$, showing that there are exactly two, namely the centre and the Heisenberg subalgebra.

We then proceeded to consider left-invariant control affine systems evolving on these groups. We started by classifying, locally, all such detached feedback equivalent systems. This was accomplished by reducing the problem to classifying $\mathfrak{L}$-related affine subspaces of the oscillator Lie algebra $\mathfrak{h}_3^\diamond$. From this local classification we then easily obtained a global classification (under detached feedback equivalence) of systems evolving on the universal covering Lie group $\widetilde{\mathsf{H}}_3^\diamond$. We then determined which of the these equivalence classes consists of controllable systems, which in turn allowed us to find a controllability criteria for systems evolving on $\widetilde{\mathsf{H}}_3^\diamond$. Now note that any system evolving on a connected four-dimensional oscillator Lie group is covered (in the sense of a **LiCAS** covering morphism, see [6]) by an equivalence representative of one of the classes of systems evolving on $\widetilde{\mathsf{H}}_3^\diamond$. This then allows for future study of such systems (especially w.r.t. local properties) to be reduced to the study of a finite list of types of equivalence representatives. Also, as we have already seen in the study of systems on the n-fold coverings $\mathsf{H}_3^\diamond(n)$ of $\mathsf{H}_3^\diamond$, some global properties (such as controllability) are propagated by such covering morphisms.

Next we considered (left-invariant control affine) systems evolving on the n-fold coverings $\mathsf{H}_3^\diamond(n)$ of $\mathsf{H}_3^\diamond$. In order to classify such systems (globally) under detached feedback equivalence, we needed to calculate the subgroup $d\mathrm{Aut}\,\mathsf{H}_3^\diamond(n)$ of $\mathrm{Aut}\,\mathfrak{h}_3^\diamond(n) \cong \mathrm{Aut}\,\mathfrak{h}_3^\diamond$. After having completed this classification, we again determined which equivalence representatives were controllable (making use of some of the results already obtained for systems evolving on $\widetilde{\mathsf{H}}_3^\diamond$). It then turned out that all full rank systems evolving on $\mathsf{H}_3^\diamond(n)$ are controllable. During the course of the research done for this thesis, we also considered systems evolving on the other types of four-dimensional connected oscillator Lie groups, namely $\widetilde{\mathsf{H}}_3^\diamond/\mathsf{N}_2$ and $\widetilde{\mathsf{H}}_3^\diamond/(\mathsf{N}_{1,n} \oplus \mathsf{N}_2)$ (corollary 1.3.8). The same approach as used for systems evolving on $\mathsf{H}_3^\diamond(n)$ proved to be successful in classifying (globally) systems evolving on these groups. However, as these groups have no faithful linear representation, we resorted to making calculations for these groups on $\widetilde{\mathsf{H}}_3^\diamond$ by means of an appropriate Lie group covering homomorphism. For the sake of brevity, these results were omitted in the presentation of this thesis.

In the last chapter of this thesis we then considered a general optimal control problem (with fixed terminal time and quadratic cost) on the two-input homogeneous system evolving on $\widetilde{\mathsf{H}}_3^\diamond$ (classification showed there is only one such system, up to equivalence). We briefly investigated the abnormal extremals and showed that the case of the normal extremals reduces to the study of single Hamiltonian on the dual space $(\widetilde{\mathfrak{h}}_3^\diamond)^*$ equipped with the Lie-Poisson bracket. The study of the integral curves of this Hamiltonian was then the main focus of the last chapter. First, we established the stability nature of all equilibrium states. Then, we investigated these integral curves qualitatively and showed that they are either constant, periodic or bounded curves; in the last case, limiting to and from some possibly distinct equilibrium points. The approach developed in making this qualitative study may very well prove successful in the study of similar problems (already in [2] and [3] the same qualitative behaviour was observed). Next, we proceeded to find explicit expressions for the integral curves of this Hamiltonian. Although we only did so for a subclass of these integral curves, the approach developed in doing so promises to be quite effective. Already in [2] and [3] this approach has proved to be successful in finding explicit expressions for all reduced normal extremals for some classes of optimal control problems associated to single- and two-input systems on $\mathsf{SE}(2)$. This indicates that similar results may be possible in the case of the four-dimensional oscillator Lie groups, although one expects them to be more complicated.

In accordance with the methods employed in this thesis to classify detached feedback equivalent left-invariant control affine systems evolving on four-dimensional oscillator Lie groups, we have already classified (almost all) such systems evolving on three-dimensional Lie groups. All these results suggest that it would be feasible to consider such a classification on four-dimensional Lie groups. Also, by adapting orbital feedback equivalence (see, e.g., [18]) to invariant control systems on Lie groups, we may be able to enlarge the class of equivalence relations to be considered. We also note that the approach developed in this thesis for finding explicit expressions of reduced extremals may very well be applied in the cases of other Lie groups (most notably, for three-dimensional Lie groups).

Appendix A

Review of Prerequisites

We briefly review the essential prerequisites for the development of this thesis. We provide proofs for statements, either when proof is very short (but still illustrative), or when a suitable reference (for the exact statement) could not be found. A basic familiarity with smooth manifolds and topology is assumed.

A.1 Lie Groups and Lie Algebras

We review some basic (real) Lie theory as needed for the presentation of this thesis. This text is largely based on [14]. We also make use of [21] and [30].

A.1.1 Basic notions

A **Lie Group** G is a group equipped with the structure of a (finite-dimensional) real smooth manifold such that the product map $\mu : \mathsf{G} \times \mathsf{G} \to \mathsf{G}$, $(g, h) \mapsto gh$ is smooth. (By the implicit function theorem, it then follows that the inverse map $g \mapsto g^{-1}$ is also smooth.) Note that our definition of a Lie group is usually termed a real Lie group. A subgroup H of G is a **Lie subgroup** if it is an immersed submanifold of G (i.e., H is the image of an injective immersion). If H is additionally an embedded submanifold we will say that it is an embedded Lie subgroup. It may be shown (Cartan's theorem, [14]) that embedded Lie subgroups are exactly topologically closed (abstract) subgroups. As such, embedded Lie subgroups will be referred to as **closed Lie subgroups**. A Lie group G is then said to be a **(closed) linear Lie group** if it is a (closed) Lie subgroup of the Lie group $\mathsf{GL}(n, \mathbb{R})$ (of linear transformations of $\mathbb{R}^n$) for some $n \in \mathbb{N}$.

A map $\phi : \mathsf{G} \to \mathsf{H}$ between Lie groups G and H is a **Lie group homomorphism** if it is simultaneously a homomorphism of abstract groups and a smooth map. A Lie group homomorphism $\phi : \mathsf{G} \to \mathsf{H}$ is called a **Lie group isomorphism** if it is simultaneously an isomorphism of abstract groups and a diffeomorphism of manifolds. (However note that a bijective Lie group homomorphism is a Lie group isomorphism. Also note that the image of a Lie group under a Lie group homomorphism is always a Lie subgroup, but not necessarily a closed Lie subgroup.) We have the following result showing that Lie group homomorphisms are just continuous group homomorphisms.

A.1.1 Theorem. ([17]) *Let $\phi : \mathsf{G} \to \mathsf{H}$ be an abstract group homomorphism of the Lie groups G and H. If ϕ is continuous at $\mathbf{1}$, then ϕ is smooth and thus a group homomorphism.*

A **Lie algebra** $\mathfrak{g}$ is a finite-dimensional vector space over $\mathbb{R}$ equipped with a bilinear skew-symmetric binary operation $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called the **Lie bracket**, satisfying the Jacobi identity $[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0$ for all $A, B, C \in \mathfrak{g}$. To every Lie group G we can naturally associate a Lie algebra $\mathfrak{g}$ (sometimes called the tangent algebra) as the vector space $T_{\mathbf{1}}\mathsf{G}$ (tangent space at identity) with a Lie bracket defined by $[A,B] = \frac{\partial^2}{\partial t \partial s}(g(t), h(t))\big|_{t=s=0}$, where $g(t)$ and $h(t)$ are smooth paths on G such that $g(0) = h(0) = \mathbf{1}$, $\dot{g}(0) = A$, and $\dot{h}(0) = B$. (Note that $(\cdot,\cdot) : \mathsf{G} \times \mathsf{G} \to \mathsf{G}$, $(g,h) \mapsto ghg^{-1}h^{-1}$ denotes the group commutator here.)

A map $\psi : \mathfrak{g} \to \mathfrak{h}$ between Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ is a **Lie algebra homomorphism** if it is a linear map preserving the Lie bracket (i.e., $\psi \cdot [A,B]_{\mathfrak{g}} = [\psi \cdot A, \psi \cdot B]_{\mathfrak{h}}$). A Lie algebra homomorphism $\phi : \mathsf{G} \to \mathsf{H}$ is called a **Lie algebra isomorphism** if it is a linear isomorphism.

In the language of categories we have a functor (called the Lie functor) between the category of Lie groups and the category of Lie algebras. The Lie functor assigns to each Lie group homomorphism $\phi : \mathsf{G} \to \mathsf{H}$ its tangent map at identity $T_{\mathbf{1}}\phi : \mathfrak{g} \to \mathfrak{h}$, which is a Lie algebra homomorphism.

With the help of left and right translations one can construct natural isomorphisms between the tangent spaces of a Lie group G at different points. Let $L_g : \mathsf{G} \to \mathsf{G}$, $h \mapsto gh$ and $R_g : \mathsf{G} \to \mathsf{G}$, $h \mapsto hg$ denote the **left** and **right translations** respectively. Then for $\xi \in T_h\mathsf{G}$ let

$$g\xi = T_h L_g \cdot \xi \in T_{gh}\mathsf{G} \qquad \xi g = T_h R_g \cdot \xi \in T_{hg}\mathsf{G}.$$

From the associativity of group multiplication we may derive the following identities:

$$(gh)\xi = g(h\xi) \qquad (g\xi)h = g(\xi h) \qquad (\xi g)h = \xi(gh),$$

for any $g, h \in \mathsf{G}$ and $\xi \in T\mathsf{G}$. In particular for a linear Lie group the "products" $g\xi$ and ξg coincide with the products of these matrices.

A subspace $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is called an **ideal** of $\mathfrak{g}$ if $[\mathfrak{h}, \mathfrak{g}] \leq \mathfrak{h}$. We have the following elementary result.

A.1.2 Proposition. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of a Lie algebra $\mathfrak{g}$. Then $\mathfrak{a} \cap \mathfrak{b}$ is an ideal of $\mathfrak{g}$.*

Proof. We already have that $\mathfrak{a} \cap \mathfrak{b}$ is a vector subspace (and has dimension given by $\dim(\mathfrak{a} \cap \mathfrak{b}) = \dim \mathfrak{a} + \dim \mathfrak{b} - \dim(\mathfrak{a} + \mathfrak{b})$). Now for $A \in \mathfrak{a} \cap \mathfrak{b}$ and $C \in \mathfrak{g}$ we have $[A, C] \in \mathfrak{a}$ ($\mathfrak{a}$ is an ideal) and $[A, C] \in \mathfrak{b}$ ($\mathfrak{b}$ is an ideal). Hence $[A, C] \in \mathfrak{a} \cap \mathfrak{b}$. Thus $[\mathfrak{a} \cap \mathfrak{b}, \mathfrak{g}] \leq \mathfrak{a} \cap \mathfrak{b}$, that is to say $\mathfrak{a} \cap \mathfrak{b}$ is an ideal. □

We will denote the **centre of a Lie group** G as $\mathrm{Z}(\mathsf{G})$ and the **centre of a Lie algebra** as $\mathrm{Z}(\mathfrak{g})$. We note that these centres are given by

$$\mathrm{Z}(\mathsf{G}) = \{g \in \mathsf{G} \mid \forall h \in \mathsf{G},\, g\,h = h\,g\} \qquad \mathrm{Z}(\mathfrak{g}) = \{A \in \mathfrak{g} \mid \forall B \in \mathfrak{g},\, [A,B] = 0\}.$$

Further note that $\mathrm{Z}(\mathsf{G})$ is a closed normal Lie subgroup with Lie algebra $\mathrm{Z}(\mathfrak{g})$, an ideal of $\mathfrak{g}$.

We will customarily refer to bijective, surjective and injective homomorphisms as bimorphisms, epimorphisms and monomorphisms, respectively. We will also call isomorphisms, with matching domain and codomain, automorphisms.

A.1.2 Quotients of Lie groups

The set of cosets of a closed Lie subgroup of a Lie group can naturally be organised as a smooth manifold. Specifically, we have the following theorem.

A.1.3 Theorem. ([14]) *Let* H *be a closed Lie subgroup of a Lie group* G*. The set* G/H *of left cosets of* H *in* G *possesses a unique differentiable structure for which the canonical map* $p : \mathsf{G} \to \mathsf{G}/\mathsf{H}$*,* $g \mapsto g\mathsf{H}$ *is a quotient map. In addition, the map* p *is a locally trivial fibre bundle and the canonical action of the group* G *on* G/H *(by left translations) is differentiable.*

Consequently one can sensibly speak of Lie quotient groups.

A.1.4 Theorem. ([14]) *Let* N *be a closed normal Lie subgroup of a Lie group* G*. Then the quotient group* G/N *with the differentiable structure defined in theorem A.1.3 is a Lie group.*

Corresponding to the first isomorphism theorem of abstract groups, we have the following result, referred to as the epimorphism theorem.

A.1.5 Theorem. ([14]) *Let* $\phi : \mathsf{G} \to \mathsf{H}$ *be a Lie group epimorphism and let* $\mathsf{N} = \ker \phi$*. Then the map*

$$\phi' : \mathsf{G}/\mathsf{N} \to \mathsf{H}, \quad g\mathsf{N} \to \phi(g)$$

is a isomorphism of Lie groups.

(For a Lie group homomorphism $\phi : \mathsf{G} \to \mathsf{H}$ it may be shown that $\ker \phi$ is a closed normal Lie subgroup of G.)

A.1.3 Connectedness and simply-connectedness of Lie groups

A topological space X is **disconnected** if there are disjoint non-empty open sets H and K in X such that $X = H \cup K$. When no such disconnection exists, X is **connected**. A space X is **pathwise connected** if for any two points x and y in X, there is a continuous function $f : [0, 1] \to X$ such that $f(0) = x$ and $f(1) = y$ (called a path from x to y). A space X is **locally path connected** if X has a basis of path connected open sets.

A.1.6 Proposition. ([22]) *Let* X *be a smooth manifold, then* X *is locally path connected. Additionally* X *is connected if and only if it is path connected.*

A Hausdorff space X is called **simply connected** if X is pathwise connected and the **fundamental group** (see, e.g., [15]) of X, denoted $\pi_1(X)$, is the trivial group (i.e., $\pi_1(X) = \{\mathbf{1}\}$).

A.1.7 Proposition. ([15]) *Let* X *and* Y *be pathwise connected Hausdorff spaces and* $\mathbb{S}^1$ *denote the circle (as a topological group). Then*

1. *if* X *and* Y *are homeomorphic then* $\pi_1(X) \cong \pi_1(Y)$ *as groups;*
2. $\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$ *as groups;*
3. $\pi_1(\mathbb{S}^1) \cong \mathbb{Z}$.

A Lie group epimorphism $q : \mathsf{G} \to \mathsf{H}$ is called a **covering homomorphism** if it satisfies any of the following equivalent conditions:

1. q maps diffeomorphically some neighbourhood of the identity of the group G onto a neighbourhood of the identity of H;
2. the kernel of q is discrete;
3. q is a covering map in the topological sense (i.e., it is a locally trivial fibre bundle with a discrete fibre);
4. $T_{\mathbf{1}}q$ is an isomorphism of the tangent spaces.

We note that for every Lie group G (or equivalently any Lie algebra $\mathfrak{g}$) there exists a simply connected Lie group $\widetilde{\mathsf{G}}$ with isomorphic Lie algebra. Moreover we have the following.

A.1.8 Theorem. ([14]) *Every connected Lie group* G *is isomorphic to a quotient* $\widetilde{\mathsf{G}}/\mathsf{N}$ *where* $\widetilde{\mathsf{G}}$ *is a simply connected Lie group and* N *is a discrete normal subgroup. The pair* $(\widetilde{\mathsf{G}}, \mathsf{N})$ *is determined by these conditions up to an isomorphism, i.e., if* $(\widetilde{\mathsf{G}}, \mathsf{N})$ *and* $(\widetilde{\mathsf{G}}', \mathsf{N}')$ *are two such pairs, then there exists an isomorphism* $\widetilde{\mathsf{G}} \to \widetilde{\mathsf{G}}'$*, taking* N *to* N'*.*

A Lie group $\widetilde{\mathsf{G}}$ satisfying the conditions of theorem A.1.8 is called the **universal covering Lie group** of the Lie group G. A covering homomorphisms $q : \widetilde{\mathsf{G}} \to \mathsf{G}$ is then called a **universal covering** (homomorphism) of G. Note that a simply connected Lie group is determined up to isomorphism by its Lie algebra. Thus we have that the universal covering Lie group $\widetilde{\mathsf{G}}$ of some Lie group G (or Lie algebra $\mathfrak{g}$) is uniquely determined, up to Lie group isomorphism. We have the following related results.

A.1.9 Proposition. ([14]) *Every discrete normal subgroup* N *of a connected Lie group* G *is contained in its centre.*

A.1.10 Proposition. *Suppose* $\widetilde{\mathsf{G}}$ *is a simply connected Lie group with Lie algebra* $\widetilde{\mathfrak{g}}$*. Further suppose* N *and* N' *are discrete central subgroups of* $\widetilde{\mathsf{G}}$*. Then* $\widetilde{\mathsf{G}}/\mathsf{N}$ *is Lie group isomorphic to* $\widetilde{\mathsf{G}}/\mathsf{N}'$ *if and only if there exists an automorphism* $\widetilde{\phi} : \widetilde{\mathsf{G}} \to \widetilde{\mathsf{G}}$ *such that* $\widetilde{\phi}(\mathsf{N}) = \mathsf{N}'$*.*

Proof. If $\widetilde{\mathsf{G}}/\mathsf{N}$ is isomorphic to $\widetilde{\mathsf{G}}/\mathsf{N}'$, then the result follows from theorem A.1.8. For the converse assume that $\widetilde{\phi} : \widetilde{\mathsf{G}} \to \widetilde{\mathsf{G}}$ is an automorphism such that $\widetilde{\phi}(\mathsf{N}) = \mathsf{N}'$. Let $q : \widetilde{\mathsf{G}} \to \widetilde{\mathsf{G}}/\mathsf{N}$ and $q' : \widetilde{\mathsf{G}} \to \widetilde{\mathsf{G}}/\mathsf{N}'$ be the canonical covering homomorphisms. Then $q' \circ \widetilde{\phi} : \widetilde{\mathsf{G}} \to \widetilde{\mathsf{G}}/\mathsf{N}'$ is a covering homomorphism. Now observe that $\ker(q' \circ \widetilde{\phi}) = \widetilde{\phi}^{-1}(\ker q') = \widetilde{\phi}^{-1}(\mathsf{N}') = \mathsf{N}$. Hence by the epimorphism theorem (theorem A.1.5) we have that $\widetilde{\mathsf{G}}/\mathsf{N}$ is Lie group isomorphic to $\widetilde{\mathsf{G}}/\mathsf{N}'$.□

Consequently, in order to classify all connected Lie groups with Lie algebra (isomorphic to) $\mathfrak{g}$, one need only enumerate all discrete central subgroups of $\widetilde{\mathsf{G}}$. The above proposition shows that the discrete subgroups need to be classified up to being related by a Lie group automorphisms (i.e., $\mathsf{N} \sim \mathsf{N}' \Leftrightarrow \exists \phi \in \operatorname{Aut}\widetilde{\mathsf{G}},\ \phi(\mathsf{N}) = \mathsf{N}'$).

Note that the centre $Z(\mathsf{G})$ of a Lie group G is an abelian Lie group. It is therefore of use to describe the discrete subgroups of $\mathbb{R}^n$. We have the following result.

A.1.11 Theorem. (Bourbaki 1975, Cassels 1959, see [31]) *Every discrete subgroup* N *of* $\mathbb{R}^n$ *is of the form* $\mathbb{Z}e_1 + \cdots + \mathbb{Z}e_\ell$*, where* $\{e_i\}$ *is some linearly independent system of vectors in* $\mathbb{R}^n$*.*

We conclude this section with two useful results.

A.1.12 Theorem. ([21]) *Let $\widetilde{\mathsf{G}}$ be a simply connected Lie group and let N be a discrete normal subgroup of $\widetilde{\mathsf{G}}$. Then the fundamental group of the quotient group $\widetilde{\mathsf{G}}/\mathsf{N}$ is isomorphic to the group N as abstract groups.*

A.1.13 Proposition. *Let $q : \mathsf{G} \to \mathsf{H}$ be a covering homomorphism with G a connected Lie group. Then $\mathrm{Z}(\mathsf{H}) = q(\mathrm{Z}(\mathsf{G}))$.*

Proof. Let $g' \in \mathrm{Z}(\mathsf{G})$, then $\forall g \in \mathsf{G}, g'g = gg'$. Thus $\forall g \in \mathsf{G}, q(g')q(g) = q(g)q(g')$ and as q is epimorphic we thus have that $\forall h \in \mathsf{H}, q(g')h = hq(g')$, which is to say $q(g') \in \mathrm{Z}(\mathsf{H})$. Thus we have that $q(\mathrm{Z}(\mathsf{G})) \subseteq \mathrm{Z}(\mathsf{H})$. Now suppose $h \in \mathrm{Z}(\mathsf{H})$. Then we have that $\forall g \in \mathsf{G}, hq(g)h^{-1}q(g^{-1}) = \mathbf{1}$. Now we have that there $\exists g' \in \mathsf{G}$ such that $q(g') = h$. Then we have that $\forall g \in \mathsf{G}, q(g'g(g')^{-1}g^{-1}) = \mathbf{1}$, that is to say, $\forall g \in \mathsf{G}, g'g(g')^{-1}g^{-1} \in \ker q$. Now consider the map $\mathsf{G} \to \ker q$, $g \mapsto g'g(g')^{-1}g^{-1}$. It has a connected image (containing $\mathbf{1} = g'\mathbf{1}(g')^{-1}\mathbf{1}$), but as $\ker q$ is discrete this means that the image is a single point, namely $\mathbf{1}$. Thus we have that $\forall g \in \mathsf{G}, g'g(g')^{-1}g^{-1} = \mathbf{1}$ and hence that $g' \in \mathrm{Z}(\mathsf{G})$. That is to say we have that $\mathrm{Z}(\mathsf{H}) \subseteq q(\mathrm{Z}(\mathsf{G}))$. □

A.1.4 Group actions and linear representations

Lie group actions

A homomorphism ϕ from a Lie group G to the group Diff X of diffeomorphisms of a smooth manifold X is called its **action** on X if the map $\mathsf{G} \times X \to X$, $(g, x) \mapsto \phi(g)x$ is smooth. (Recall that an abstract group action of a group G on a set X is a mapping $\mathsf{G} \times X \to \mathsf{G}$, $(g, x) \mapsto \phi(g)x$ such that $\forall x \in X,\ \phi(\mathbf{1})x = x$ and $\forall g, g' \in \mathsf{G}, x \in X,\ \phi(g)\phi(g')x = \phi(gg')x$. We note that a Lie group action satisfies the conditions of a abstract group action.) Let ϕ be an action of a group G on a smooth manifold X. Then the **orbit** of a point $x \in X$ is the set $\phi(\mathsf{G})x = \{\phi(g)x \mid g \in \mathsf{G}\}$; the **stabilizer** of the point x is the set $\mathsf{G}_x = \{g \in \mathsf{G} \mid \phi(g)x = x\}$.

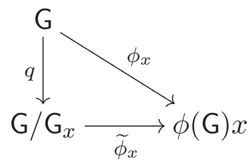
Consider the map $\phi_x : \mathsf{G} \to X, \quad g \mapsto \phi(g)x$. Its image is precisely the orbit $\phi(\mathsf{G})x$ of the point x and the preimage of the point x is its stabilizer G_x. From the definition of a Lie group action it follows that the map ϕ_x is smooth. Then, as $(\phi_x \circ L_g)(\cdot) = \phi(g)\phi_x(\cdot)$ for any $g \in \mathsf{G}$, we get that ϕ_x has constant rank. We then have the following theorem.

A.1.14 Theorem. ([14]) *Let ϕ be an action of a Lie group G on a smooth manifold X. For any point $x \in X$ the map ϕ_x has a constant rank and if this constant rank is k, then:*

1. *the stabilizer G_x is a closed Lie subgroup of codimension k in G and $T_e(\mathsf{G}_x) = \ker T_{\mathbf{1}}\phi_x$;*
2. *for some neighbourhood U of the identity in the group G the orbit $\phi(U)x$ is a submanifold of dimension k in X, and $T_x(\phi(U)x) = T_{\mathbf{1}}\phi_x(T_{\mathbf{1}}(\mathsf{G}))$;*
3. *if the orbit $\phi(\mathsf{G})x$ is a submanifold in X (this is not always the case), then $\dim \phi(\mathsf{G})x = k$.*

As G_x is a closed Lie subgroup it follows that G/G_x possesses a unique differentiable structure for which the canonical map $q : \mathsf{G} \to \mathsf{G}/\mathsf{G}_x$, $g \mapsto g\mathsf{G}_x$ is a quotient map. That is to say, we have a "natural" differential structure on G/G_x. With this in mind we have the following result.

A.1.15 Proposition. ([1]) *Let* $\phi : \mathsf{G} \times X \to X$ *be an action. Define, for* $x \in X$, $\widetilde{\phi}_x : \mathsf{G}/\mathsf{G}_x \to \phi(\mathsf{G})x \subseteq X$, $g\mathsf{G}_x \mapsto \phi_x(g)$. *That is,* $\widetilde{\phi}_x$ *is the unique injective (smooth) map for which the triangle*



is commutative. Then $\widetilde{\phi}_x$ *is an injective immersion.*

A.1.16 Remark. As $\widetilde{\phi}_x$ is an injective immersion we immediately have that the any orbit $\phi(\mathsf{G})x$ is an immersed submanifold of the manifold X (although not necessarily an embedded submanifold).

Linear representations

Let G be a Lie group, V be a vector space over $\mathbb{R}$ and $\mathsf{GL}(V)$ be the Lie group of invertible linear transformations of V. Then a Lie group homomorphism $\rho : \mathsf{G} \to \mathsf{GL}(V)$ is called a **linear representation** of G in the space V. We say that a linear representation ρ is **faithful** if it is injective. By the **dual representation** of a linear representation ρ, we mean the linear representation ρ^* of the group G in the space V^* (the dual space of V) defined by $(\rho^*(g) \cdot \mu)(v) = \mu(\rho(g)^{-1}v)$ for $\mu \in V^*$ and $v \in V$.

Any linear representation of a Lie group defines a natural action over its vector space. That is, $\mathsf{G} \times V \to V$, $(g, v) \mapsto \rho(g) \cdot v$, where $\rho(g) \in \mathsf{GL}(V)$. The condition that ρ is a homomorphism from G to $\mathsf{GL}(V)$ ensures that this is indeed a Lie group action.

The adjoint (and coadjoint) linear representation and action

For a Lie group G we have a natural linear representation (and associated dual representation) of the group in its Lie algebra $\mathfrak{g}$. The adjoint representation (and associated dual representation, called the coadjoint representation) of a Lie group G in its Lie algebra $\mathfrak{g}$, denoted by Ad (and Ad* respectively) are linear representations given by

$$\begin{array}{rrcl} \mathrm{Ad}: & \mathsf{G} & \to & \mathsf{GL}(\mathfrak{g}) \\ & g & \mapsto & \mathrm{Ad}\, g \\ \mathrm{Ad}\, g: & \mathfrak{g} & \to & \mathfrak{g} \\ & A & \mapsto & g\, A\, g^{-1} \end{array} \qquad \begin{array}{rrcl} \mathrm{Ad}^*: & \mathsf{G} & \to & \mathsf{GL}(\mathfrak{g}^*) \\ & g & \mapsto & \mathrm{Ad}^*\, g \\ \mathrm{Ad}^*\, g: & \mathfrak{g}^* & \to & \mathfrak{g}^* \\ & \mu(\cdot) & \mapsto & \mu(\mathrm{Ad}\,(g^{-1})(\cdot)). \end{array}$$

We denote their tangent maps at identity by ad and ad* and note that they are given by

$$\begin{array}{rrcl} \mathrm{ad}: & \mathfrak{g} & \to & \mathrm{Der}(\mathfrak{g}) \\ & A & \mapsto & \mathrm{ad}A \\ \mathrm{ad}A: & \mathfrak{g} & \to & \mathfrak{g} \\ & B & \mapsto & [A, B] \end{array} \qquad \begin{array}{rrcl} \mathrm{ad}^*: & \mathfrak{g} & \to & \mathrm{Der}(\mathfrak{g}^*) \\ & A & \mapsto & \mathrm{ad}^* A \\ \mathrm{ad}^* A: & \mathfrak{g}^* & \to & \mathfrak{g}^* \\ & \mu(\cdot) & \mapsto & -\mu(\mathrm{ad}(A)(\cdot)). \end{array}$$

The **adjoint orbit** through a point $A \in \mathfrak{g}$ is simply the orbit of the adjoint action through that point, i.e., $\mathrm{Ad}\,\mathsf{G} \cdot A$. The **coadjoint orbit** through a point $A^* \in \mathfrak{g}^*$ is similarly given by $\mathrm{Ad}^* \mathsf{G} \cdot A^*$. Note that these orbits are independent of the connected Lie group chosen (i.e., they are solely properties of the Lie algebra). We note that the adjoint and coadjoint orbits are immersed submanifolds of $\mathfrak{g}$ and $\mathfrak{g}^*$, respectively (see remark A.1.16). Moreover the coadjoint orbits are symplectic manifolds ([24]) and hence even-dimensional.

A.1.5 Classes of Lie algebras and Lie groups

The **lower central series** for a Lie algebra, $\mathfrak{g}$, is the descending series of ideals

$$\mathfrak{g} = \mathfrak{g}_0 \trianglerighteq \mathfrak{g}_1 \trianglerighteq \cdots \trianglerighteq \mathfrak{g}_n \trianglerighteq \cdots$$

where $\mathfrak{g}_{n+1} = [\mathfrak{g}, \mathfrak{g}_n]$, the ideal of $\mathfrak{g}$ generated by all Lie brackets $[A, B]$ with $A \in \mathfrak{g}$ and $B \in \mathfrak{g}_n$. (Alternatively one may define the concepts of a central series and a upper or ascending central series, in terms of which some of the subsequent definitions may be restated.) The **derived** series of a Lie algebra $\mathfrak{g}$, is the descending series of ideals

$$\mathfrak{g} \trianglerighteq \mathfrak{g}^{(1)} \trianglerighteq \mathfrak{g}^{(2)} \trianglerighteq \cdots \trianglerighteq \mathfrak{g}^{(n)} \trianglerighteq \cdots$$

where $\mathfrak{g}^{(n+1)} = [\mathfrak{g}^{(n)}, \mathfrak{g}^{(n)}]$, the ideal of $\mathfrak{g}^{(n)}$ generated by all Lie brackets $[A, B]$ with $A, B \in \mathfrak{g}^{(n)}$.

A Lie algebra $\mathfrak{g}$ is **solvable** if its derived series stabilizes at zero. A **nilpotent** Lie algebra, $\mathfrak{g}$, is one whose lower central series terminates in the zero ideal after finitely many steps. A Lie algebra is called **triangular** (or supersolvable or completely solvable or fully-solvable) if all adjoint operators $\mathrm{ad}A$, $A \in \mathfrak{g}$, have only real eigenvalues. (A triangular Lie algebra may alternatively be characterised as one that has a faithful linear representation being triangular w.r.t. some basis, or one such that there exists a complete or full flag in $\mathfrak{g}$ preserved by $\mathrm{ad}\,\mathfrak{g}$; [30].) A connected Lie group G is said to be nilpotent, solvable or triangular if its Lie algebra is nilpotent, solvable or triangular, respectively. (Alternatively these concepts may be defined by by means of the lower central series and the derived series for a Lie group, and characterised as above.) A Lie group G is said to be **exponential** if the exponential mapping $\exp : \mathfrak{g} \to \mathsf{G}$ is a diffeomorphism. A Lie algebra is said to be exponential if the simply connected Lie group corresponding to it is exponential. We note that exponential Lie groups (algebras) are sometimes called Lie groups (algebras) of type (E). We have the following characterisations of exponential Lie groups and algebras.

A.1.17 Theorem. (Dixmier, Nono, Gorbatsevich, [30]) *Let G be a simply connected Lie group with Lie algebra $\mathfrak{g}$. Then the following conditions are equivalent:*

1. *the Lie group is exponential;*
2. *the mapping $\exp : \mathfrak{g} \to \mathsf{G}$ is injective;*
3. *for any element $A \in \mathfrak{g}$, the operator $\mathrm{ad}\,A$ does not have any purely imaginary eigenvalues;*
4. *for any $g \in \mathsf{G}$ the operator $\mathrm{Ad}\,g$ has no eigenvalues equal to 1 in absolute value but different from 1;*
5. *there is no ideal $\mathfrak{h}$ in the algebra $\mathfrak{g}$ such that the quotient algebra $\mathfrak{g}/\mathfrak{h}$ has a subalgebra isomorphic to the 3D Euclidean Lie algebra $\mathfrak{e}(2)$.*

We have the following result, adapted from results in [14], [21] and [30], regarding the relative containment (i.e., partial ordering w.r.t. $\subseteq$) of the above classes of Lie algebras.

A.1.18 Theorem. *For a Lie algebra $\mathfrak{g}$, the following implications hold*

$$\mathfrak{g} \text{ is nilpotent} \quad \Rightarrow \quad \mathfrak{g} \text{ is triangular} \quad \Rightarrow \quad \mathfrak{g} \text{ is exponential} \quad \Rightarrow \quad \mathfrak{g} \text{ is solvable.}$$

A non-abelian Lie algebra $\mathfrak{g}$ is called **simple** if it contains no proper ideals (i.e., distinct from 0 and $\mathfrak{g}$). A Lie Algebra $\mathfrak{g}$ is called **semisimple** if it contains no nonzero abelian ideals. A Lie group G is called **simple** (resp., **semisimple**) if its Lie algebra $\mathfrak{g}$ is simple (resp., semisimple).

A Lie group is said to be **unimodular** if its left-invariant Haar measure is also right-invariant. We have the following characterisation of unimodular Lie groups.

A.1.19 Proposition. ([29]) *A Lie group* G *is unimodular if and only if the linear transformation* $\operatorname{Ad} g$ *has determinant* ± 1 *for every* $g \in \mathsf{G}$.

A.1.20 Proposition. ([29]) *A connected Lie group* G *is unimodular if and only if the linear transformation* $\operatorname{ad} A$ *has trace zero for every* $A \in \mathfrak{g}$.

A.1.6 Semidirect products

In this section we discuss the construction of a semi-direct product of two Lie Groups. We start by examining the concept for abstract groups.

Let H, K be two abstract groups. Let $\mu : \mathsf{K} \to \operatorname{Aut} \mathsf{H}$ be a homomorphism. Then by the **semi-direct product** of H and K with respect to μ, written $\mathsf{H} \rtimes_\mu \mathsf{K}$, we mean the set $\mathsf{H} \times \mathsf{K}$ with the binary operation given by

$$(h_1, k_1)\,(h_2, k_2) = (h_1\, \mu(k_1) \cdot h_2, k_1\, k_2).$$

Note that $\mathsf{H} \rtimes_\mu \mathsf{K}$ with above product is indeed a group. Elements of the form $(h, \mathbf{1})$ (resp. $(\mathbf{1}, k)$) form a subgroup of $\mathsf{H} \rtimes_\mu \mathsf{K}$ isomorphic to H (resp. K). The subgroup H is normal and

$$(\mathbf{1}, k) \cdot (h, \mathbf{1}) \cdot (\mathbf{1}, k)^{-1} = \mu(k) \cdot h. \tag{A.1.1}$$

The subgroup K is normal if and only if the homomorphism μ is trivial, in which case the semi-direct product is simply a direct product. A group G **decomposes** as a semi-direct product of subgroups H and K (written $\mathsf{G} = \mathsf{H} \rtimes \mathsf{K}$), if

1. $\mathsf{H} \trianglelefteq \mathsf{G}$;   2. $\mathsf{H} \cap \mathsf{K} = \{\mathbf{1}\}$;   3. $\mathsf{HK} = \mathsf{G}$.

A.1.21 Proposition. ([14]) *If* G *decomposes as a semi-direct product of subgroups* H *and* K *there is indeed an isomorphism*

$$\mathsf{H} \rtimes_\mu \mathsf{K} \to \mathsf{G}, \quad (h, k) \mapsto h\, k,$$

where $\mu : \mathsf{K} \to \operatorname{Aut}(\mathsf{H})$ *is the homomorphism defined above by (A.1.1).*

We now specialise the above two definitions and proposition for Lie groups (cf. [14]). The **semi-direct product** of Lie groups is defined as the semi-direct product of the underlying abstract groups, with the differentiable structure of the direct product of smooth manifolds (in particular it carries the product topology). Moreover, the homomorphism μ is required to define a Lie group action of the group H on G. In particular the automorphism $\mu(k)$ of the group H should be differentiable for any $k \in \mathsf{K}$. This ensures differentiability of the product in $\mathsf{H} \rtimes_\mu \mathsf{K}$. We say that a Lie Group G **decomposes** as a semi-direct product of closed Lie subgroups H and K if it decomposes as their semi-direct product as an abstract group.

A.1.22 PROPOSITION. ([14]) *If a Lie Group G decomposes as a semi-direct product of closed Lie subgroups H and K, then the action μ of the group K on H, defined above by (A.1.1), is differentiable and the abstract isomorphism defined in proposition A.1.21 is a Lie group isomorphism.*

A.1.7 Some results on the relation between Lie groups and Lie algebras

We review some definitions and results regarding the relation between Lie groups (and Lie group homomorphisms) and Lie algebras (and Lie algebra homomorphisms). Again we mainly depend on [14].

Integration of Lie algebras homomorphisms

A.1.23 THEOREM. (UNIQUENESS, [14]) *A Lie group homomorphism ϕ from a connected Lie group G to a Lie group H, is uniquely determined by its tangent map at identity $T_{\mathbf{1}}\phi$.*

A.1.24 THEOREM. (EXISTENCE, [14]) *Let G and H be Lie groups with G simply connected. Then for every Lie algebra homomorphism $\psi : \mathfrak{g} \to \mathfrak{h}$ there exists a Lie group homomorphism $\phi : \mathsf{G} \to \mathsf{H}$ such that $T_{\mathbf{1}}\phi = \psi$.*

A.1.25 THEOREM. ([14]) *Let G_1 and G_2 be Lie subgroups of a Lie group G. If $\mathsf{G}_1 \subseteq \mathsf{G}_2$, then G_1 is a Lie subgroup of the Lie group G_2 and $\mathfrak{g}_1 \subseteq \mathfrak{g}_2$. Conversely, if $\mathfrak{g}_1 \subseteq \mathfrak{g}_2$ and the group G_1 is connected, then $\mathsf{G}_1 \subseteq \mathsf{G}_2$.*

A.1.26 THEOREM. ([14]) *A connected Lie subgroup is uniquely determined by its Lie algebra (as a subalgebra of the Lie algebra of the ambient Lie group). Moreover every subalgebra of the Lie algebra of a Lie group is the Lie algebra of some (uniquely defined) connected Lie subgroup.*

A.1.27 THEOREM. (YAMABE 1950, [14]) *Every path connected subgroup of a real Lie group is a Lie subgroup.*

A.1.28 PROPOSITION. ([14]) *A connected Lie subgroup H of a connected Lie group G is normal if and only if the subalgebra $\mathfrak{h}$ is an ideal of the algebra $\mathfrak{g}$.*

A.1.29 THEOREM. ([14]) *The commutator subgroup (G,G) of a connected Lie group G is a connected Lie subgroup with tangent algebra $[\mathfrak{g},\mathfrak{g}]$. If the group G is simply connected, then (G,G) is a closed Lie subgroup.*

The exponential map

For any smooth path $g(\cdot)$ in a Lie group G we have that $\dot{g}(t) = g(t)A(t)$, where $A(t) \in \mathfrak{g}$. Moreover given a smooth map $t \mapsto A(t)$ from a connected subset of $\mathbb{R}$ to the algebra $\mathfrak{g}$, there exists a solution of the above equation on the said interval. We refer to $A(\cdot)$ as the **velocity** of the path. A smooth path $t \mapsto g(t)$ in a Lie group G, defined for all $t \in \mathbb{R}$, is called a **one parameter subgroup** if $g(t+s) = g(t)\,g(s)$. It turns out that a smooth path $g(\cdot)$ is a one-parameter subgroup if and only if its velocity $A(\cdot)$ is constant and $g(0) = \mathbf{1}$.

For any $A \in \mathfrak{g}$ we shall denote by $g_A(\cdot)$ the one parameter subgroup with velocity $A(\cdot) = A$. If G is a linear Lie group then $g_A(t) = \exp tA$, where the exponential is understood as $\exp A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n$. For any Lie group G we define the **exponential map** $\exp : \mathfrak{g} \to \mathsf{G}$ as $A \mapsto g_A(1)$.

Note that for a Lie group homomorphism $\phi : \mathsf{G} \to \mathsf{H}$ we have that $\phi(\exp(A)) = \exp(T_{\mathbf{1}}\phi \cdot A)$.

The group of automorphisms

Let G be a Lie group, $\operatorname{Aut}\mathsf{G}$ be the group of its automorphisms (as a Lie group) and $\operatorname{Aut}\mathfrak{g}$ be the group of automorphisms of its Lie algebra. If G is connected, then the map $d : \operatorname{Aut}\mathsf{G} \to \operatorname{Aut}\mathfrak{g}$, $\phi \mapsto T_{\mathbf{1}}\phi$ is an injection. Moreover, if G is simply connected, then it is a Lie group isomorphism. The group $\operatorname{Aut}\mathfrak{g}$ is a linear Lie group. Hence, in the case of a simply connected Lie group G, one can transfer the Lie group structure of $\operatorname{Aut}\mathfrak{g}$ to $\operatorname{Aut}\mathsf{G}$. In general we have the following.

A.1.30 Proposition. ([14]) *For any connected Lie group G, the group $d\operatorname{Aut}\mathsf{G}$ is a closed Lie subgroup of the Lie group $\operatorname{Aut}\mathfrak{g}$.*

If G is connected, then $\operatorname{Ad}\mathsf{G}$ depends only on the algebra $\mathfrak{g}$ and is a normal subgroup of $\operatorname{Aut}\mathfrak{g}$. It is called the **group of inner automorphisms of the algebra $\mathfrak{g}$**. Being the image of the group G under the adjoint representation, the group $\operatorname{Ad}\mathsf{G}$ is a Lie subgroup (but not necessarily a closed Lie subgroup) of $\operatorname{Aut}\mathfrak{g}$.

A.1.8 Linear Lie groups

A Lie group G is said to be a **(closed) linear Lie group** if it is a (closed) Lie subgroup of $\mathsf{GL}(n,\mathbb{R})$ for some $n \in \mathbb{N}$. Note that if G admits faithful finite-dimensional linear representation, then it is isomorphic to a linear Lie group. We have the following results.

A.1.31 Theorem. ([30]) *Let G be a connected solvable Lie group. Then G admits a faithful finite-dimensional linear representation if and only if the commutator subgroup (G,G) of the Lie group G is simply connected.*

A.1.32 Theorem. (Djokovic 1976, [30]) *Let G be a connected Lie group admitting a faithful finite-dimensional linear representation. Then there exists a faithful finite-dimensional linear representation $\rho : \mathsf{G} \to \mathsf{GL}(V)$ such that the subgroup $\rho(\mathsf{G})$ is a closed Lie subgroup of $\mathsf{GL}(V)$.*

A.1.33 Theorem. ([30]) *Any Lie group is locally isomorphic to a linear Lie group. Specifically, for any simply connected Lie group G, there exists a discrete central subgroup N such that the group G/N admits a faithful finite-dimensional linear representation.*

A.2 Left-Invariant Control Affine Systems

We briefly define left-invariant control affine systems and recall how they may be organised into a category, denoted **LiCAS** (see [6]). We then make an investigation of controllability of **LiCAS**-objects, based on the work done in [20]. Finally, we briefly review equivalence in **LiCAS**, specialising the results of [7].

A.2.1 Definitions

A **left-invariant control affine system** is a pair $\Sigma = (\mathsf{G}, \Xi)$. The **state space** G is a (real, finite-dimensional) Lie group. The dynamics $\Xi : \mathsf{G} \times \mathbb{R}^\ell \to T\mathsf{G}$ is left-invariant; i.e., for any $g \in \mathsf{G}$ and any $u \in \mathbb{R}^\ell$, $\Xi(g, u) = g\,\Xi(\mathbf{1}, u)$. Furthermore the **parametrisation map** $\Xi(\mathbf{1}, \cdot) : \mathbb{R}^\ell \to \mathfrak{g}$ is an affine embedding, i.e.,

$$\Xi(\mathbf{1}, \cdot) : \mathbb{R}^\ell \to \mathfrak{g}, \quad u \mapsto A + u_1 B_1 + \cdots + u_\ell B_\ell$$

where the set $\{B_i\}_{i=\overline{1,\ell}}$ is linearly independent. Hence the **trace** $\Gamma = \operatorname{im}\Xi(\mathbf{1}, \cdot) = \operatorname{span}\{B_i\}_{i=\overline{1,\ell}}$ is an affine subspace of the Lie algebra $\mathfrak{g}$. By identifying (the left-invariant vector field) $\Xi(\cdot, u) \in \mathfrak{X}^L(\mathsf{G})$ with $\Xi(\mathbf{1}, u) \in \mathfrak{g}$, we have that $\Gamma = \left\{\Xi_u \,|\, u \in \mathbb{R}^\ell\right\}$.

Admissible controls will be piecewise continuous maps $u(\cdot) : [0, T] \to \mathbb{R}^\ell$, $0 < T < \infty$. A **trajectory** for an admissible control $u(\cdot)$ is an absolutely continuous curve $g(\cdot) : [0, T] \to \mathsf{G}$ such that $\dot{g}(t) = g(t)\,\Xi(\mathbf{1}, u(t))$ for almost every $t \in [0, T]$. The Carathéodory existence and uniqueness theorem of ordinary differential equations implies the local existence and global uniqueness of trajectories. A remarkable property of left-invariant systems is that a left translation of a trajectory is a trajectory.

The **attainable set** (from the identity $\mathbf{1} \in \mathsf{G}$) is the set $\mathcal{A}$ of all terminal points $g(T)$ of all trajectories $g(\cdot) : [0, T] \to \mathsf{G}$ starting at $\mathbf{1}$. We say that a system Σ is **controllable** if for any $g_0, g_1 \in \mathsf{G}$, there exists a $T \geq 0$ and a trajectory $g(\cdot) : [0, T] \to \mathsf{G}$ such that $g(0) = g_0$ and $g(T) = g_1$. A system is controllable if and only if $\mathcal{A} = \mathsf{G}$. Necessary conditions for controllability are that the group G be connected (we will say Σ is connected) and that the Lie algebra $\operatorname{Lie}\Gamma$ generated by the trace $\Gamma \subseteq \mathfrak{g}$ be the whole space $\mathfrak{g}$ (we will say Σ or Γ is of full rank). Systems satisfying these necessary conditions are called **proper systems**.

The **orbit** (through the identity $\mathbf{1}$) is the unique connected Lie subgroup $\mathcal{O}$ of the state space G with Lie algebra $\operatorname{Lie}\Gamma$. We then have that $\mathcal{A} \subseteq \mathcal{O}$. (The orbit may be described as the smallest connected Lie subgroup containing the attainable set.)

An affine subspace Γ will be termed **homogeneous** if $0 \in \Gamma$ and **inhomogeneous** if not. We denote the linear subspace associated to Γ by Γ^0, i.e., $\Gamma^0 = \{A - B \mid A, B \in \Gamma\}$. Thus if $A \in \Gamma$ we have that $\Gamma = A + \Gamma^0$.

A.2.2 The category LiCAS

The left-invariant control affine systems (evolving on real, finite-dimensional Lie groups) can be organised into a category, denoted by **LiCAS**. (Note that **LiCAS** may be realised as a full subcategory of **LiCS**, the category of left-invariant control systems.) We briefly review (and specialise) the definitions and results presented in [6], as needed for this thesis. An **object** in

LiCAS is a left-invariant control affine system $\Sigma = (\mathsf{G}, \Xi)$, as described above. A **morphism** $\Phi = (\phi, \varphi) : \Sigma \to \Sigma'$ in **LiCAS** is a mapping $\Phi : \mathsf{G} \times \mathbb{R}^\ell \to \mathsf{G}' \times \mathbb{R}^{\ell'}, \quad (g, u) \mapsto (\phi(g), \varphi(g, u))$, where the **state component** $\phi : \mathsf{G} \to \mathsf{G}'$ and the **feedback component** $\varphi : \mathsf{G} \times \mathbb{R}^\ell \to \mathbb{R}^{\ell'}$ are smooth maps such that the diagram

$$\begin{array}{ccc} \mathsf{G} \times \mathbb{R}^\ell & \xrightarrow{\quad\Phi\quad} & \mathsf{G}' \times \mathbb{R}^{\ell'} \\ {\scriptstyle \Xi}\big\downarrow & & \big\downarrow{\scriptstyle \Xi'} \\ T\mathsf{G} & \xrightarrow{\quad T\phi\quad} & T\mathsf{G}' \end{array}$$

commutes. The feedback component φ is said to be G**-invariant** if it does not depend on G explicitly.

We say that a system Σ' **covers** a system Σ if there exists an **LiCAS**-epimorphism $\Theta = (\theta, \vartheta) : \Sigma' \to \Sigma$ such that (the state component) $\theta : \mathsf{G}' \to \mathsf{G}$ is a Lie group covering homomorphism (i.e., a Lie group epimorphism with discrete kernel). Such a **LiCAS**-morphism is called a **covering morphism** and the pair (Σ', Θ) will be referred to as a **covering** of Σ. Let us note that for a covering morphism $\Theta = (\theta, \vartheta)$ the feedback component ϑ is G'-invariant and $T_{\mathbf{1}}\theta \cdot \Gamma' = \Gamma$

Given a connected system $\Sigma = (\mathsf{G}, \Xi)$ and a Lie group covering homomorphism $\theta : \mathsf{G}' \to \mathsf{G}$ (with G' connected), a covering (Σ', Θ) of Σ can be constructed, such that θ is the state component of Θ, as follows:

$$\Xi' : \mathsf{G}' \times \mathbb{R}^\ell \to T\mathsf{G}', \quad (g', u) \mapsto (T_{g'}\theta)^{-1} \cdot \Xi\,(\theta(g'), u)$$
$$\Theta : \mathsf{G}' \times \mathbb{R}^{\ell'} \to \mathsf{G} \times \mathbb{R}^\ell, \quad (g', u) \mapsto (\theta(g'), u).$$

We have the following pertinent results.

A.2.1 Proposition. ([6]) *Let Σ and Σ' be proper systems and $\phi : \mathsf{G} \to \mathsf{G}'$ be a smooth map such that $\phi(\mathbf{1}) = \mathbf{1}$. Then there exists a unique morphism $\Phi = (\phi, \varphi) : \Sigma \to \Sigma'$ such that (the feedback component) φ is G-invariant if and only if ϕ is a Lie group homomorphism such that $T_{\mathbf{1}}\phi \cdot \Gamma \subseteq \Gamma'$.*

A.2.2 Proposition. ([6]) *Let $\Phi = (\phi, \varphi) : \Sigma \to \Sigma'$ be a morphism with (the state space component) ϕ surjective. If Σ is controllable, then so is Σ'.*

A.2.3 Attainability and controllability

We investigate some properties of the attainable set (of a **LiCAS**-object Σ) in order to provide sufficient conditions for controllability. We mainly restate and adapt some of the work done in [20] (for right-invariant control affine systems). We denote the interior of $\mathcal{A}$ relative to $\mathcal{O}$ as $\operatorname{int}_{\mathcal{O}} \mathcal{A}$. Similarly, we denote the closure of $\mathcal{A}$ relative to $\mathcal{O}$ as $\operatorname{cl}_{\mathcal{O}} \mathcal{A}$.

A.2.3 Proposition. ([4]) *The interior of $\mathcal{A}$ relative to $\mathcal{O}$ is non-empty, i.e.,* $\operatorname{int}_{\mathcal{O}}\mathcal{A} \neq \emptyset$.

A.2.4 Proposition. ([20]) *The sets $\mathcal{A}$, $\operatorname{int}_{\mathcal{O}} \mathcal{A}$ and $\operatorname{cl}_{\mathcal{O}} \mathcal{A}$ are all semi-groups.*

Proof. Let $g_1, g_2 \in \mathcal{A}$. Then there exists trajectories $g_1(\cdot) : [0, T_1] \to \mathsf{G}$ and $g_2(\cdot) : [0, T_2] \to \mathsf{G}$ (with controls $u_1(\cdot)$ and $u_2(\cdot)$) such that $g_1(0) = \mathbf{1} = g_2(0)$, $g_1(T_1) = g_1$ and $g_2(T_2) = g_2$. We define a curve $g_3(\cdot) : [0, T_1 + T_2] \to \mathsf{G}$ as

$$g_3(t) = \begin{cases} g_1(t) & 0 \leq t \leq T_1 \\ g_1\, g_2(t - T_1) & T_1 < t \leq T_1 + T_2 \end{cases}$$

Then $g_3(\cdot)$ is a trajectory of Σ such that $g_3(0) = \mathbf{1}$ and $g_3(T_1 + T_2) = g_1\, g_2$. Hence $g_1\, g_2 \in \mathcal{A}$ and so $\mathcal{A}$ is a semi-group.

Now suppose $g_1, g_2 \in \operatorname{int}_{\mathcal{O}}\mathcal{A}$, then we have open (relative to $\mathcal{O}$) subsets $V, W \subseteq \mathcal{A}$ such that $g_1 \in V$ and $g_2 \in W$. Hence $V\,W$ is open (relative to $\mathcal{O}$), $V\,W \subseteq \mathcal{A}$, as $\mathcal{A}$ is a semi-group, and $g_1\, g_2 \in V\,W$. Thus we have that $g_1 g_2 \in \operatorname{int}_{\mathcal{O}} \mathcal{A}$ and so $\operatorname{int}_{\mathcal{O}} \mathcal{A}$ is a semi-group.

Recall that for a Lie group G, the group product $\mu : \mathsf{G} \times \mathsf{G} \to \mathsf{G}$ is in particular continuous. Also recall that for a continuous function $f : X \to Y$ between two topological spaces we have that $f(\operatorname{cl} W) \subseteq \operatorname{cl} f(W)$ for any subset W of X ([38]). Now, as $\mathcal{A}$ is semi-group, we have that $\mu(\mathcal{A}, \mathcal{A}) \subseteq \mathcal{A}$. Thus we have that $\mu(\operatorname{cl}_{\mathcal{O}} \mathcal{A} \times \operatorname{cl}_{\mathcal{O}} \mathcal{A}) \subseteq \operatorname{cl}_{\mathcal{O}} \mu(\mathcal{A}, \mathcal{A}) \subseteq \operatorname{cl}_{\mathcal{O}} \mathcal{A}$. Hence we have that $\operatorname{cl}_{\mathcal{O}} \mathcal{A}$ is a semi-group. □

A.2.5 Lemma. *Suppose Σ has a connected state space and the trace Γ is homogeneous. Then $\mathcal{A} = \mathcal{O}$.*

Proof. As $\mathcal{A}$ is a collection of trajectory endpoints, it is path connected. By lemma A.2.4 we have that $\mathcal{A}$ is a semi-group. We show it is a group. Let $g \in \mathcal{A}$, then there exists a trajectory $g(\cdot) : [0, T] \to \mathsf{G}$ such that $g(0) = \mathbf{1}$ and $g(T) = g$. We define a curve $g'(\cdot) : [0, T] \to \mathsf{G}$ as $g'(t) = g^{-1}\, g(T - t)$ for $t \in [0, T]$. Then $g'(\cdot)$ is a trajectory (as Γ is homogeneous) such that $g'(0) = \mathbf{1}$ and $g'(T) = g^{-1}$. Thus we get that $g^{-1} \in \mathcal{A}$ and so $\mathcal{A}$ is a group. Hence, as $\mathcal{A}$ is a path connected subgroup of G, we get that $\mathcal{A}$ is a connected Lie subgroup (by theorem A.1.27). Then observe that Lie algebra of $\mathcal{A}$ is $\operatorname{Lie}\Gamma$ (as $\exp(tA) \in \mathcal{A}$ for $A \in \Gamma$). The result then follows from theorem A.1.26. □

To every system Σ we associate the following two Lie subalgebras: the subalgebra $\operatorname{Lie}\Gamma$ generated by Γ; the subalgebra $\operatorname{Lie}\Gamma^0$ generated by Γ^0. The corresponding connected Lie subgroups (see theorem A.1.26) are denoted by $\mathcal{O}$ (the orbit) and $\mathcal{O}^0$. It then follows that $\operatorname{Lie}\Gamma^0 \subseteq \operatorname{Lie}\Gamma$ and consequently (by theorem A.1.25) that $\mathcal{O}^0 \subseteq \mathcal{O}$.

A.2.6 Lemma. *The set $\operatorname{cl}_{\mathcal{O}}\mathcal{O}^0$ is a group.*

Proof. Recall that the group product $\mu : \mathsf{G} \times \mathsf{G} \to \mathsf{G}$ and the group inverse $\iota : \mathsf{G} \to \mathsf{G}$ are smooth. So as $\mathcal{O}^0$ is a group we have that $\mu(\mathcal{O}^0 \times \mathcal{O}^0) \subseteq \mathcal{O}^0$ and $\iota(\mathcal{O}^0) \subseteq \mathcal{O}^0$. Also recall that for a continuous function $f : X \to Y$ between two topological spaces we have that $f(\operatorname{cl} W) \subseteq \operatorname{cl} f(W)$ for any subset W of X ([38]). Hence we get that $\mu(\operatorname{cl}_{\mathcal{O}} \mathcal{O}^0 \times \operatorname{cl}_{\mathcal{O}} \mathcal{O}^0) \subseteq \operatorname{cl}_{\mathcal{O}} \mu(\mathcal{O}^0 \times \mathcal{O}^0) \subseteq \operatorname{cl}_{\mathcal{O}} \mathcal{O}^0$ and $\iota(\operatorname{cl}_{\mathcal{O}} \mathcal{O}^0) \subseteq \operatorname{cl}_{\mathcal{O}} \iota(\mathcal{O}^0) \subseteq \operatorname{cl}_{\mathcal{O}} \mathcal{O}^0$. Thus the result follows. □

A.2.7 Lemma. ([20]) *Let G be a connected Lie group, and let $B_1, B_2, \ldots, B_\ell$ be elements of $\mathfrak{g}$ that generate $\mathfrak{g}$ (i.e. $\operatorname{Lie}\{B_i\}_{i=\overline{1,\ell}} = \mathfrak{g}$). Then every $g \in \mathsf{G}$ is a finite product of elements of the form $\exp(tB_i)$, where t is real and $i = \overline{1, \ell}$.*

Proof. The set H of all finite products of elements of the form $\exp(tB_i)$ is a path-connected (abstract) subgroup of G. Therefore, H is a connected Lie subgroup of G (see theorem A.1.27). Obviously, H contains $\exp(\mathbb{R}B_i)$, $i = \overline{1,n}$ and therefore $B_1, \dots, B_n$ belong to $\mathfrak{h}$. Thus $\mathfrak{g} = \mathfrak{h}$ and hence we get that $\mathsf{G} = \mathsf{H}$ (by theorem A.1.26), completing our proof. □

A.2.8 Lemma. ([20]) *If $\mathcal{A}$ is dense in $\mathcal{O}$, then $\mathcal{A} = \mathcal{O}$.*

Proof. Let $g \in \operatorname{int}\mathcal{A} \neq \emptyset$ and assume $\mathcal{A}$ is dense in $\mathcal{O}$. The we have an open (relative to $\mathcal{O}$) subset $V \subseteq \mathcal{A}$ such that $g \in V$. Let $W = \{h^{-1} : h \in V\}$. Then W is a non-empty relatively open subset of $\mathcal{O}$. Our assumption implies that W contains an element $h \in \mathcal{A}$. So the set $h\,V$ is relatively open in $\mathcal{O}$ and is contained in $\mathcal{A}$ (as $\mathcal{A}$ is semi-group, see proposition A.2.4). Moreover $h\,V$ contains the identity. Therefore, the semi-group $\mathcal{A}$ contains a neighbourhood of the identity in $\mathcal{O}$. Since $\mathcal{A}$ and $\mathcal{O}$ are connected, we have that $\mathcal{A} = \mathcal{O}$. □

A.2.9 Lemma. ([20]) *We have that* $\mathrm{cl}_{\mathcal{O}}\mathcal{O}^0 \subseteq \mathrm{cl}_{\mathcal{O}}\mathcal{A}$.

Proof. Let $\Gamma = A + \Gamma_0 = A + \operatorname{span}\{B_1, \dots, B_\ell\}$. By lemma A.2.7, every element of $\mathcal{O}^0$ is a product of elements of the form $\exp(tB_i)$, $t \in \mathbb{R}, i = \overline{1,\ell}$. We show that $\exp(tB_i)$ belongs to $\mathrm{cl}_{\mathcal{O}}\mathcal{A}$ for every real t and for every $i = \overline{1,\ell}$. Since $\mathrm{cl}_{\mathcal{O}}\mathcal{A}$ is a semi-group (see proposition A.2.4), this will imply that $\mathcal{O}^0 \subseteq \mathrm{cl}_{\mathcal{O}}\mathcal{A}$, and the desired result will follow immediately.

Let $t \in \mathbb{R}$ and fix $i \in \{1, \dots, \ell\}$. Consider $g_n(t) = \exp(t(A + nB_i))$. We have that $g_n(0) = \mathbf{1}$ and $\dot{g}_n(t) = g_n(t)\,(A + nB_i)$. That is $g_n(\cdot)$ is a trajectory and $g_n(t) \in \mathcal{A}$ for $n \in \mathbb{N}$ and $t \in [0, \infty)$. Now we have that $g_n(\frac{t}{n}) = \exp(\frac{t}{n}A + tB_i)$. Letting $n \to \infty$, we get that $\frac{t}{n}A + tB_i \to tB_i$ and so, as the exponential map is continuous, $\exp(\frac{t}{n}A + tB_i) \to \exp(tB_i)$. We conclude that $\exp(tB_i) \in \mathrm{cl}_{\mathcal{O}}\mathcal{A}$. □

A.2.10 Theorem. ([20]) *If $\mathcal{O}$ is compact, then $\mathcal{A} = \mathcal{O}$.*

Proof. Let $\mathsf{H} = \mathrm{cl}_{\mathcal{O}}\mathcal{A}$. Then H is a semi-group (see proposition A.2.4). We show that H is a group. Let $h \in \mathsf{H}$. Then for every $n \in \mathbb{N}$, $h^n \in \mathsf{H}$. The sequence $\{h^n\}_{n\in\mathbb{N}}$ has a convergent subsequence $\{h^{n(k)}\}_{k\in\mathbb{N}}$ (where $n(k) < n(k+1)$ for all k). Now as $k \to \infty$, $h^{-1} = \lim h^{n(k+1)-n(k)-1} = \lim h_k$. Since $n(k+1) - n(k) - 1$ is non-negative, it follows that h_k belongs to H for each k. Since H is closed, $h^{-1} \in \mathsf{H}$. Therefore H is a group. Since $\mathcal{A} \subseteq \mathsf{H}$ and $\mathcal{A}$ has a non-empty interior relative to $\mathcal{O}$, the same is true for H. Since H is a group, $\mathsf{H} \subseteq \mathcal{O}$ and $\mathcal{O}$ is connected, we conclude that $\mathsf{H} = \mathcal{O}$. Therefore $\mathcal{A}$ is dense in $\mathcal{O}$ and our result follows from lemma A.2.8. □

A.2.11 Theorem. ([20]) *Suppose $u \in \mathbb{R}^\ell$ and $\{t_n\}_{n\in\mathbb{N}}$ is sequence satisfying $t_n \geq \epsilon > 0$ for some $\epsilon \in \mathbb{R}$. If $\lim \exp(t_n\,\Xi(\mathbf{1}, u))$ exists and belongs to $\mathrm{cl}_{\mathcal{O}}\mathcal{O}^0$, then $\mathcal{A} = \mathcal{O}$.*

Proof. Let $A = \Xi(\mathbf{1}, u)$. We first show that $\exp(tA) \in \mathrm{cl}_{\mathcal{O}}\mathcal{A}$ for every $t \in \mathbb{R}$. If $\{t_n\}_{n\in\mathbb{N}}$ is bounded there exists a positive number T such that $\exp(TA) \in \mathrm{cl}_{\mathcal{O}}\mathcal{O}^0$. Let t be any real number and n be a natural number such that $nT + t > 0$. Since $\mathrm{cl}_{\mathcal{O}}\mathcal{O}^0$ is a group (lemma A.2.6) we have that $\exp(-TA) \in \mathrm{cl}_{\mathcal{O}}\mathcal{O}^0$, and hence that $\exp(-nTA) \in \mathrm{cl}_{\mathcal{O}}\mathcal{O}^0$. Consequently by lemma A.2.9 it follows that $\exp(-nTA) \in \mathrm{cl}_{\mathcal{O}}\mathcal{A}$. We also have that $\exp((nT + t)A) \in \mathcal{A} \subseteq \mathrm{cl}_{\mathcal{O}}\mathcal{A}$. Then, as

$\exp(tA) = \exp(-nTA)\exp((nT+t)A)$ and $\mathrm{cl}_{\mathcal{O}}\mathcal{A}$ is semi-group (proposition A.2.4), we get that $\exp(tA) \in \mathrm{cl}_{\mathcal{O}}\mathcal{A}$. If on the other hand $\{t_n\}_{n\in\mathbb{N}}$ is unbounded, let $\{t_{n_k}\}_{k\in\mathbb{N}}$ be a subsequence of $\{t_n\}_{n\in\mathbb{N}}$ with $t_{n_{k+1}} - t_{n_k} > k$, and let $\tau_k = t_{n_{k+1}} - t_{n_k}$. Then we have that $\tau_k \to \infty$ and $\exp(\tau_k A) \to \mathbf{1}$ as $k \to \infty$. Thus for any real number t, $\exp(tA) = \lim_{k\to\infty} \exp((t+\tau_k)A)$. If k is sufficiently large, then $t+\tau_k$ is positive and hence $\exp((t+\tau_k)A)$ belongs to $\mathcal{A}$. Consequently it follows that $\exp(tA) \in \mathrm{cl}_{\mathcal{O}}\mathcal{A}$.

Let $\Gamma^0 = \mathrm{span}\,\{B_i\}_{i=\overline{1,\ell}}$. Then $\exp(tB_i)$ belongs to $\mathrm{cl}_{\mathcal{O}}\mathcal{A}$ for $t \in \mathbb{R}$ and $i = \overline{1,\ell}$ (by lemma A.2.9). Since $\mathrm{cl}_{\mathcal{O}}\mathcal{A}$ is a semi-group (proposition A.2.4), it follows that every product of the elements of the form $\exp(tC)$, $t \in \mathbb{R}$, $C \in \{A, B_1, \ldots, B_\ell\}$ belongs to $\mathrm{cl}_{\mathcal{O}}\mathcal{A}$. Clearly, the elements $\{A, B_1, \ldots, B_\ell\}$ generate $\mathrm{Lie}\,\Gamma$. Then, by lemma A.2.7 (and as $\mathrm{cl}_{\mathcal{O}}\mathcal{A} \subseteq \mathcal{O}$), we get that $\mathrm{cl}_{\mathcal{O}}\mathcal{A} = \mathcal{O}$. Then by lemma A.2.8 we have that $\mathcal{A} = \mathcal{O}$. □

Controllability

We collect our results for the attainable set and apply them to find sufficient conditions for controllability of a left-invariant control affine system.

A.2.12 Theorem. *If a system Σ has a connected state space, is of full rank and any of the following statements are true, then Σ is controllable.*

1. G *is compact.*
2. Γ *is homogeneous.*
3. *There exists an element $u \in \mathbb{R}^\ell$ and a sequence $\{t_n\}_{n\in\mathbb{N}}$, $t_n \geq \epsilon > 0$ for some ϵ, such that $\lim \exp(t_n \Xi(\mathbf{1}, u))$ exists and belongs to $\mathrm{cl}\mathcal{O}^0$. In particular this statement is true if any of the following statements is true.*
   - *(a)* $\mathrm{Lie}\,\Gamma^0 = \mathfrak{g}$.
   - *(b)* $\exists u \in U$ *such that* $\exp(t\Xi(\mathbf{1}, u))$ *is periodic.*
   - *(c)* $\exists u \in U,\ \exists T \in \mathbb{R},\ T > 0$ *such that* $\exp(T\Xi(\mathbf{1}, u)) \in \mathrm{cl}\,\mathcal{O}^0$.

Proof. Item 1 follows from theorem A.2.10, item 2 follows from lemma A.2.5 and item 3 from lemma A.2.11. Next we show that the statements 3a, 3b and 3c all imply 3. Starting with 3c, if $\exists u \in \mathbb{R}^\ell$, $\exists T \in \mathbb{R}$, $T > 0$ such that $\exp(T\Xi(\mathbf{1}, u)) \in \mathcal{O}^0$ then simply take the sequence $\{t_n\}$ as constant, $t_n = T$ to obtain result. For item 3b we have that if $\exp(t\Xi(\mathbf{1}, u))$ is periodic (for some $u \in \mathbb{R}^\ell$) then there exists a $T > 0$ such that $\exp(T\Xi(\mathbf{1}, u)) = \exp(0) = \mathbf{1} \in \mathrm{cl}\,\mathcal{O}^0$, thus satisfying 3c. For item 3a we have that $\mathcal{O}^0 = \mathsf{G}$ and hence that for any $T > 0$ that $\exp(T\Xi(\mathbf{1}, u)) \in \mathcal{O}^0$, thus satisfying 3c. □

A.2.4 Detached feedback equivalence

We briefly recall the definition of detached feedback equivalence, its characterisations and some properties. For more details see [7].

Let $\Sigma = (\mathsf{G}, \Xi)$ and $\Sigma' = (\mathsf{G}', \Xi')$ be **LiCAS**-objects. Then Σ and Σ' are called **locally detached feedback equivalent** (shortly DF_{loc}-equivalent) at points $a \in \mathsf{G}$ and $a' \in \mathsf{G}'$ if there exist open neighbourhoods N and N' of a and a', respectively, and a (local) diffeomorphism $\Phi : N \times \mathbb{R}^\ell \to N' \times \mathbb{R}^{\ell'}$, $(g, u) \mapsto (\phi(g), \bar{\varphi}(u))$ such that $\phi(a) = a'$ and

$T_g\phi \cdot \Xi\,(g,u) = \Xi'\,(\phi(g), \bar{\varphi}(u))$ for $g \in N$ and $u \in \mathbb{R}^\ell$ (i.e., the diagram

$$\begin{array}{ccc} N \times \mathbb{R}^\ell & \xrightarrow{\phi\times\bar{\varphi}} & N' \times \mathbb{R}^{\ell'} \\ \Xi \downarrow & & \downarrow \Xi' \\ TN & \xrightarrow[T\phi]{} & TN' \end{array}$$

commutes). Σ and Σ' are called **detached feedback equivalent** (shortly DF-equivalent) if this happens globally (i.e., $N = \mathsf{G}$ and $N' = \mathsf{G}'$).

It may be shown that if Φ defines a detached feedback equivalence, then the state component $\phi : \mathsf{G} \to \mathsf{G}'$ of Φ maps trajectories to trajectories and preserves left-invariant vector fields (see [6]). (By preserving left-invariant vector fields we mean that the push forward of a left-invariant vector field, by ϕ, is left-invariant). Moreover detached feedback equivalence is the most general feedback equivalence (i.e., largest subgroup of feedback equivalences) that does so (see [6] and [7]). This motivates detached feedback equivalence as a natural equivalence for left-invariant control systems.

Any DF_{loc}-equivalence between two system can be reduced to a local equivalence between neighbourhoods of identity.

A.2.13 Proposition. ([7]) *Σ and Σ' are DF_{loc}-equivalent at $a \in \mathsf{G}$ and $a' \in \mathsf{G}'$ if and only if they are DF_{loc}-equivalent at $\mathbf{1} \in \mathsf{G}$ and $\mathbf{1} \in \mathsf{G}'$.*

We then have the following characterisations of local and global detached feedback equivalence.

A.2.14 Theorem. ([7]) *Two systems Σ and Σ' of full rank are DF_{loc}-equivalent if and only if there exists a Lie algebra isomorphism $\psi : \mathfrak{g} \to \mathfrak{g}'$ such that $\psi \cdot \Gamma = \Gamma'$.*

A.2.15 Theorem. ([7]) *Two proper systems Σ and Σ' are DF-equivalent if and only if there exists a Lie group isomorphism $\phi : \mathsf{G} \to \mathsf{G}'$ such that $T_{\mathbf{1}}\phi \cdot \Gamma = \Gamma'$.*

Then as the map $d : \operatorname{Aut} \mathsf{G} \to \operatorname{Aut} \mathfrak{g}$, $\phi \mapsto T_{\mathbf{1}}\phi$ is a bijection for simply connected Lie groups (section A.1.7) we get the following.

A.2.16 Corollary. ([7]) *If two proper systems Σ and Σ' are DF_{loc}-equivalent and both have simply connected state spaces, then they are DF-equivalent.*

A.2.17 Remark. Notice that controllability is invariant under DF-equivalence. (This may be justified by noting that a DF-equivalence is in particular a **LiCAS**-isomorphism, and applying proposition A.2.2.)

A.3 Optimal Control on Lie Groups

We briefly review the basic tools for dealing with a class of optimal control problems on Lie Groups. This section is based on results presented in [4], [19] and [24].

A.3.1 Remark. Throughout this section ad (and ad*) are interpreted as the tangent maps (at identity) of the adjoint linear representation Ad (and its dual representation Ad*) of a Lie group G in its Lie algebra $\mathfrak{g}$ (and its dual $\mathfrak{g}^*$). See section A.1.4 for details. In particular note that $\mathrm{ad}^* A : \mathfrak{g}^* \to \mathfrak{g}^*$, $p(\cdot) \mapsto -p(\mathrm{ad}(A)(\cdot))$ is defined with a negative sign.

A.3.1 Trivialisation of the cotangent bundle of a Lie group

Recall that left translations on a Lie group G form a family of diffeomorphisms $L_g : \mathsf{G} \to \mathsf{G}$, $h \mapsto g\,h$. The tangent map of a left translation L_g at $h \in \mathsf{G}$ is then of the form $T_h L_g : T_h\mathsf{G} \to T_{gh}\mathsf{G}$. The dual of such a linear map is then given by $(T_h L_{g^{-1}})^* : T_h^*\mathsf{G} \to T_{gh}^*\mathsf{G}$, $\xi(\cdot) \mapsto (\xi \circ T_h L_{g^{-1}})(\cdot)$. In particular we have that $(T_{\mathbf{1}} L_{g^{-1}})^* : \mathfrak{g}^* \to (T_g^*\mathsf{G})$.

We define a diffeomorphism $\Psi : \mathsf{G} \times \mathfrak{g}^* \to T^*\mathsf{G}$, $(g,p) \mapsto (T_{\mathbf{1}} L_{g^{-1}})^* \cdot p$. This provides a trivialisation of $T^*\mathsf{G}$ as: $\pi \circ \Phi(g,p) = g$, where $\pi : T^*\mathsf{G} \to \mathsf{G}$ is the natural projection $\pi : T_g^*\mathsf{G} \to \{g\}$; for any $g \in \mathsf{G}$ the mapping $\Phi(g,\cdot) : \mathfrak{g}^* \to T_g^*\mathsf{G}$, $p \mapsto \Phi(g,p)$ is a linear isomorphism. Thus $\mathfrak{g}^*$ is identified with any vertical fiber $T_g^*\mathsf{G}$ and as such can be viewed as a typical fiber of the cotangent bundle $T^*\mathsf{G}$. Note that the diffeomorphism $\Phi : \mathsf{G} \times \mathfrak{g} \to T\mathsf{G}$, $(g, A) \mapsto T_{\mathbf{1}} L_g \cdot A$ similarly trivialises $T\mathsf{G}$.

Next we consider the tangent bundle $T(T^*\mathsf{G})$ of $T^*\mathsf{G}$. Having identified $T^*\mathsf{G}$ with $\mathsf{G} \times \mathfrak{g}^*$ we can identify $T(T^*\mathsf{G})$ with $T(\mathsf{G} \times \mathfrak{g}^*)$. Then we can identify $T(\mathsf{G} \times \mathfrak{g}^*)$ with $T\mathsf{G} \times T\mathfrak{g}^*$ which in turn can be identified with $(\mathsf{G} \times \mathfrak{g}) \times (\mathfrak{g}^* \times \mathfrak{g}^*)$. In summary, we get an identification

$$
\begin{aligned}
((\mathsf{G} \times \mathfrak{g}) \times (\mathfrak{g}^* \times \mathfrak{g}^*)) &\longleftrightarrow T(T^*\mathsf{G}) \\
((g, A), (p, B^*)) &\longleftrightarrow T_{(g,p)}\Psi \cdot \left(TL_g \cdot A,\ \frac{d}{dt}(p + tB^*)\bigg|_{t=0} \right).
\end{aligned}
$$

Using these conventions, vector fields on $T^*\mathsf{G}$ will be represented as

$$
(X, Y^*) : (g,p) \mapsto \Big(\big(g, X(g,p)\big), \big(p, Y^*(g,p)\big) \Big),
$$

where $X : \mathsf{G} \times \mathfrak{g}^* \to \mathfrak{g}$, $Y^* : \mathsf{G} \times \mathfrak{g}^* \to \mathfrak{g}^*$.

Having identified $T^*\mathsf{G}$ with $\mathsf{G} \times \mathfrak{g}^*$, functions on $T^*\mathsf{G}$ become functions on $\mathsf{G} \times \mathfrak{g}^*$. For each such function F, $\frac{\partial F}{\partial g}$ and $\frac{\partial F}{\partial p}$ will denote the partial differentials of F; $\frac{\partial f}{\partial g}$ is the differential of the restriction of f to p =constant, and $\frac{\partial F}{\partial p}$ is the differential of the restriction of F to g =constant. Then in our identification we have that $(\frac{\partial F}{\partial g}, \frac{\partial F}{\partial p}) \in T_g^*\mathsf{G} \times T_p^*\mathfrak{g}^*$. Furthermore we can canonically identify $\frac{\partial F}{\partial p}$ with an element of $\mathfrak{g}$, as $T^*\mathfrak{g}^* \cong (\mathfrak{g}^*)^* \cong \mathfrak{g}$.

A.3.2 Lemma. ([19]) *In terms of the above notations, for any vector field $V = (X, Y^*)$ on $T^*\mathsf{G}$ and any function F on $T^*\mathsf{G}$ we have that*

$$
V(F)(g,p) = T_{(g,p)}F \cdot V(g,p) = \left((T_{\mathbf{1}} L_g)^* \cdot \frac{\partial F}{\partial g}(g,p) \right) \cdot X(g,p) + Y^*(g,p) \cdot \left(\frac{\partial F}{\partial p}(g,p) \right).
$$

Proof. We have that

$$\begin{aligned}
T_{(g,p)}F \cdot V(g,p) &= \left(\frac{\partial F}{\partial g}(g,p), \frac{\partial F}{\partial p}(g,p)\right) \cdot ((g, X(g,p)), (p, Y^*(g,p))) \\
&= \Big(\underbrace{\frac{\partial F}{\partial g}(g,p)}_{\in T_g^*\mathsf{G}}, \underbrace{\frac{\partial F}{\partial p}(g,p)}_{\in T_p^*\mathfrak{g}^* \cong \mathfrak{g}}\Big) \cdot \Big(\underbrace{T_{\mathbf{1}}L_g \cdot X(g,p)}_{\in T_g\mathsf{G}}, \underbrace{Y^*(g,p)}_{\in \mathfrak{g}^*}\Big) \\
&= \frac{\partial F}{\partial g}(g,p) \cdot T_{\mathbf{1}}L_g \cdot X(g,p) + \frac{\partial F}{\partial p}(g,p) \cdot Y^*(g,p) \\
&= \left((T_{\mathbf{1}}L_g)^* \cdot \frac{\partial F}{\partial g}(g,p)\right) \cdot X(g,p) + Y^*(g,p) \cdot \frac{\partial F}{\partial p}(g,p).
\end{aligned}$$

□

A.3.2 Symplectic structure on $T^*\mathsf{G}$

A **symplectic manifold** is a pair $(\mathcal{M}, \omega)$ where $\mathcal{M}$ is a manifold and ω is a closed non-degenerate two form on $\mathcal{M}$.

We briefly recall the definitions of a (differentiable) 2-form and the exterior derivative of a 1-form. A (differentiable) 2-form ω on a manifold $\mathcal{M}$ is any mapping $x \mapsto \omega_x$, $x \in \mathcal{M}$, such that $\omega_x : T_x\mathcal{M} \times T_x\mathcal{M} \to \mathbb{R}$ is a bilinear, antisymmetric mapping, and the dependence $x \mapsto \omega_x$ is smooth. A 2-form ω is called non-degenerate if ω_x is non-degenerate for each x, i.e., $\forall w \in T_x\mathcal{M},\ \omega_x(v,w) = 0 \Rightarrow v = 0$.

The exterior derivative $d\theta$ of a 1-form θ on a manifold $\mathcal{M}$ is a 2-form given by the following expression:

$$d\theta_x(X(x), Y(x)) = X(\theta(Y)) - Y(\theta(X)) - \theta([X,Y]),$$

for any vector fields X and Y on $\mathcal{M}$. In this notation, $\theta(Y)$ is regarded as a function on $\mathcal{M}$, and $X(\theta(Y))$ is the (Lie) derivative of $\theta(Y)$ in the direction of X. (Note that we use the conventions of [24], specifically $[X,Y] = \mathbf{D}Y \cdot X - \mathbf{D}X \cdot Y$ locally, for vector fields X and Y.) Recall that ω is closed if its exterior derivative $d\omega$ is equal to zero. Any differential form that is an exterior derivative (of some differential form) is closed.

Let $\pi : T^*\mathsf{G} \to \mathsf{G}, (g,p) \mapsto g$ be the projection carrying each covector in $T^*\mathsf{G}$ to its base point. The tangent map of π is then of the form $T\pi : T(T^*\mathsf{G}) \to T\mathsf{G}$. We define a differential one form $\theta : T^*\mathsf{G} \to T^*(T^*\mathsf{G})$ as the dual map of $T\pi$. That is, using above identifications (specifically $(g,p) \leftrightarrow (T_{\mathbf{1}}L_{g^{-1}})^*(p) \in T^*\mathsf{G}$), we have that

$$\begin{aligned}
\theta\big((g,p)\big) \cdot \big((g,A),(p,B^*)\big) &= (g,p) \cdot (T\pi \cdot ((g,A),(p,B^*))) = (g,p) \cdot (g,A) \\
&= (T_{\mathbf{1}}^* L_{g^{-1}} \cdot p) \cdot (T_{\mathbf{1}}L_g \cdot A) = p \cdot (T_{\mathbf{1}}L_{g^{-1}} \cdot T_{\mathbf{1}}L_g \cdot A) \\
&= p \cdot A.
\end{aligned}$$

A.3.3 Lemma. *The exterior derivative $d\theta$ is given by*

$$d\theta_{(g,p)}(X(g,p), Y(g,p)) = B_1^* \cdot A_2 - B_2^* \cdot A_1 - p \cdot [A_1, A_2].$$

where $X : (g,p) \mapsto ((g,A_1),(p,B_1^*))$, $Y : (g,p) \mapsto ((g,A_2),(p,B_2^*))$ are vector fields on $T^*\mathsf{G}$.

Proof. Recall that $d\theta_x(X(x), Y(x)) = X(\theta(Y)) - Y(\theta(X)) - \theta([X,Y])$. Now we have that $\theta(Y) = \theta(g,p)\cdot((g,A_2),(p,B_2^*)) = p\cdot A_2$ and similarly $\theta(X) = p\cdot A_1$. So then, using lemma A.3.2, we get that $X(\theta(Y)) = B_1^*\cdot A_2$ and $Y(\theta(X)) = B_2^*\cdot A_1$ Next (by for instance considering expressions in local coordinates) we have that $[X,Y](g,p) = \big((g,[A_1,A_2]),(p,0)\big)$. Therefore $\theta_{(g,p)}([X,Y](g,p)) = p\cdot[A_1,A_2]$. Thus the result follows. □

A.3.4 Proposition. *The two form $\omega = -d\theta$, given by*

$$\begin{aligned}\omega_{(g,p)}\Big(\big((g,A_1),(p,B_1^*)\big),\big((g,A_2),(p,B_2^*)\big)\Big) &= B_2^*\cdot A_1 - B_1^*\cdot A_2 + p\cdot[A_1,A_2]\\ &= B_2^*\cdot A_1 - B_1^*\cdot A_2 - (\mathrm{ad}^* A_1\cdot p)\cdot A_2\end{aligned}$$

defines a (strong) symplectic form on $T^*\mathsf{G}$. In other words, $\big(T^*\mathsf{G},\omega\big)$ is a symplectic manifold.

A vector field V on $(T^*\mathsf{G},\omega)$ is called **Hamiltonian** if there exists a smooth function $H\in C^\infty(T^*\mathsf{G})$ such that $\omega(V,\cdot) = TH$. That is, for all $\xi\in T_{(g,p)}(T^*\mathsf{G})$, we have the identity $\omega_{(g,p)}(V(g,p),\xi) = T_{(g,p)}H\cdot\xi$. Thus a vector field (X,Y^*) on $T^*\mathsf{G}$ is Hamiltonian if there exists a smooth function $H\in C^\infty(T^*\mathsf{G})$ such that

$$\omega_{(g,p)}\left(\Big(\big(g,X(g,p)\big),\big(p,Y^*(g,p)\big)\Big),\Big(\big(g,A\big),\big(p,B^*\big)\Big)\right) = T_{(g,p)}H\cdot((g,A),(p,B^*))$$

for all $g\in G$, $A\in\mathfrak{g}$ and $p,B^*\in\mathfrak{g}^*$. We note that (by non-degeneracy and finite dimension) for any given function $H\in C^\infty(T^*G)$ there exists a Hamiltonian vector field (X,Y^*) for which H is a suitable function.

A.3.5 Proposition. ([19]) *Let $\vec{H} = (X,Y^*)$, i.e., $\vec{H}(g,p) = \Big(\big(g,X(g,p)\big),\big(p,Y^*(g,p)\big)\Big)$, denote the Hamiltonian vector field corresponding to a smooth function H on $\mathsf{G}\times\mathfrak{g}^*\cong T^*\mathsf{G}$. Then*

$$X(g,p) = \frac{\partial H}{\partial p}(g,p), \qquad Y^*(g,p) = -(T_{\mathbf{1}}L_g)^*\cdot\left(\frac{\partial H}{\partial g}(g,p)\right) - \mathrm{ad}^* X(g,p)\cdot p.$$

A.3.3 Poisson structure on $T^*\mathsf{G}$

A **Poisson bracket** (or a Poisson structure) on a manifold $\mathcal{M}$ is a bilinear operation $\{,\}: C^\infty(\mathcal{M})\times C^\infty(\mathcal{M})\to C^\infty(\mathcal{M})$ such that $(C^\infty(\mathcal{M}),\{,\})$ is a Lie algebra and $\{,\}$ is a derivation in each factor, i.e., $\{FG,H\} = \{F,H\}G + F\{G,H\}$ for all $F,G,H\in C^\infty(\mathcal{M})$.

A.3.6 Proposition. (see, e.g., [24]) *As $(T^*\mathsf{G},\omega)$ is a symplectic manifold it is a Poisson manifold with Poisson bracket given by*

$$\{H_1,H_2\}(g,p) = \omega_{(g,p)}(\vec{H}_1(g,p),\vec{H}_2(g,p)) = TH_1\cdot\vec{H}_2,$$

where $\vec{H}_1$ and $\vec{H}_2$ are the Hamiltonian vector fields on $T^*\mathsf{G}$ corresponding to the smooth functions H_1 and H_2 on $T^*\mathsf{G}$ respectively.

A vector field $\vec{H}$ on $\left(T^*\mathsf{G}, \omega\right)$ is called **Hamiltonian** if there exists a smooth function $H : T^*\mathsf{G} \to \mathbb{R}$ such that $TF \cdot \vec{H} = \{F, H\}$ for all $F \in C^\infty(T^*\mathsf{G})$. Note that for any $H \in C^\infty(T^*\mathsf{G})$ the exists a unique corresponding Hamiltonian vector field $\vec{H}$. Also note that this definition of a Hamiltonian vector field is equivalent to our usual one on a symplectic manifold. The **flow** of a vector field V is the collection of maps $\mathscr{F}_t : T^*\mathsf{G} \to T^*\mathsf{G}$ such that $t \mapsto \mathscr{F}_t(g, p)$ is the integral curve of V with initial condition (g, p), i.e., $\frac{d}{dt}\mathscr{F}_t(g, p) = V(\mathscr{F}_t(g, p))$ and $\mathscr{F}_0(g, p) = (g, p)$.

A.3.7 Proposition. ([24]) *Let $H, F \in C^\infty(\mathcal{M})$ and $\mathscr{F}_t$ be the flow of $\vec{H}$, then*

$$\frac{d}{dt}(F \circ \mathscr{F}_t) = \{F \circ \mathscr{F}_t, H\} = \{F, H\} \circ \mathscr{F}_t.$$

A.3.8 Corollary. *$F \in C^\infty(\mathcal{M})$ is a constant of motion for $\vec{H}$ (i.e., $F \circ \mathscr{F}_t$ =constant) if and only if $\{F, H\} = 0$.*

A.3.4 The Hamiltonian lift

The Hamiltonian H_Ξ associated to a vector field Ξ on G is a function on $T^*\mathsf{G}$ defined by

$$H_\Xi(g, p) = (g, p) \cdot \Xi(g) = p \cdot (T_g L_{g^{-1}} \cdot \Xi(g))$$

The vector field $\vec{H}_\Xi$ is called the **Hamiltonian lift** of Ξ.

Our interest lies in left-invariant control affine systems (see section A.2). That is, we wish to consider families of left-invariant vector fields (parametrised by $u \in \mathbb{R}^\ell$) of the form $\Xi : \mathsf{G} \times \mathbb{R}^\ell \to T\mathsf{G}$, $(g, u) \mapsto g\left(A + \sum_{i=1}^{\ell} u_i B_i\right)$. So for a fixed $u \in \mathbb{R}^\ell$ we have a left-invariant vector filed $\Xi_u : \mathsf{G} \to T\mathsf{G}, g \mapsto \Xi(\mathbf{1}, u)$. The Hamiltonian lift H_{Ξ_u} of such a vector field is then given by $H_{\Xi_u}(g, p) = p \cdot \Xi_u(\mathbf{1})$. Notice that H_{Ξ_u} does not depend on G explicitly. We will refer to such Hamiltonians as **G-invariant**.

In anticipation of dealing with the optimal control problem we generalise slightly to a family of cost extended Hamiltonians. We consider (G-invariant positive definite diagonal) costs of the form $\chi : \mathbb{R}^\ell \mapsto \sum_{i=1}^{\ell} \beta_i u_i^2$ for some $\beta_i > 0$, $i = \overline{1, \ell}$. Let Σ be a left-invariant control affine system. Then we define a family of **cost extended Hamiltonians** $\{H_u^\nu \mid \nu \in \mathbb{R}, u \in \mathbb{R}^\ell\}$ associated to a pair (Σ, χ) as smooth functions on $T^*\mathsf{G}$ given by

$$H_u^\nu(g, p) = (g, p) \cdot \Xi_u(g) + \nu\chi(u) = p \cdot \Xi_u(\mathbf{1}) + \nu\chi(u).$$

Note that the Hamiltonian functions $H_u^\nu(g, p) = H_u^\nu(p)$ for a pair (Σ, χ) are G-invariant. Moreover they are affine functions on $\mathfrak{g}^*$. (Also note that $H_u^\nu(g, p) = H_u^0(p) + \nu\chi(u)$.)

We now present some results, adapted from [19], concerning the Hamiltonian vector fields of cost extended Hamiltonian functions associated to a pair (Σ, χ).

A.3.9 Proposition. *The Hamiltonian vector field $\vec{H}_u = (X, Y^*)$ corresponding to a Hamiltonian function $H_u^\nu(g, p)$ for some fixed $\nu \in \mathbb{R}$, $u \in \mathbb{R}^\ell$ is independent of the choice of ν and is given by $X(g, p) = \Xi_u(\mathbf{1})$, $Y^*(g, p) = -\mathrm{ad}^*\Xi_u(\mathbf{1}) \cdot p$.*

PROOF. By proposition A.3.5 we have that $\vec{H}_u(g,p) = (X(g,p), Y^*(g,p))$ is given by $X(g,p) = \frac{\partial H}{\partial p}(g,p)$, $Y^*(g,p) = -(T_{\mathbf{1}}L_g)^*\left(\frac{\partial H}{\partial g}(g,p)\right) - (\mathrm{ad}^* X(g,p))(p)$. Now $\frac{\partial H}{\partial p}(g,p) = \Xi_u(\mathbf{1})$ and $\frac{\partial H}{\partial g}(g,p) = 0$ (as H_u^ν is G-invariant), thus yielding result. □

A.3.10 REMARK. Let $\pi : T^*\mathsf{G} \to \mathsf{G}, (g,p) \to g$. Then $T\pi \cdot \vec{H}_u(g,p) = \Xi_u(g)$, as expected. This follows as $T\pi \cdot \vec{H}_u(g,p) = T\pi \cdot \Big(\big(g, \Xi_u(\mathbf{1})\big), \big(p, -\mathrm{ad}^*\Xi_u(\mathbf{1}) \cdot p\big)\Big) = \big(g, \Xi_u(\mathbf{1})\big) = T_{\mathbf{1}}L_g \cdot \Xi_u(\mathbf{1}) = \Xi_u(g)$.

A.3.11 PROPOSITION. *Suppose that $t \mapsto (g(t), p(t))$ is an integral curve of a G-invariant vector field $\vec{H} = (X, Y^*), (g,p) \mapsto ((g, X(p)), (p, Y^*(p)))$. Then (for some $\mu \in \mathfrak{g}^*$)*

$$\dot{g}(t) = T_{\mathbf{1}}L_g \cdot \frac{\partial H}{\partial p}(g,p) \qquad\qquad p(t) = \mathrm{Ad}^*\left(g(t)\right)^{-1} \cdot \mu.$$

Thus $p(t)$ is contained in the coadjoint orbit through $p(0) = \mathrm{Ad}^*\left(g(0)\right)^{-1} \cdot \mu$. That is to say, $p(t) \in \mathrm{Ad}^*\mathsf{G} \cdot p(0)$.

PROOF. From proposition A.3.5 we have that $\vec{H} = (X, Y^*)$ is given by $X(g,p) = \frac{\partial H}{\partial p}(g,p)$, $Y^*(g,p) = -(T_{\mathbf{1}}L_g)^*\big(\frac{\partial H}{\partial g}(g,p)\big) - (\mathrm{ad}^* X(g,p))(p)$. Now as H is G-invariant we have that $\frac{\partial H}{\partial g}(g,p) = 0$ and hence that $Y^*(g,p) = -(\mathrm{ad}^* X(p))(p)$. Then, as $t \mapsto (g(t), p(t))$ is an integral curve of $\vec{H}$, we have that $\dot{g}(t) = T_{\mathbf{1}}L_{g(t)} \cdot \frac{\partial H}{\partial p}(p)$ and $\dot{p}(t) = -\mathrm{ad}^*\frac{\partial H}{\partial p}(p) \cdot p(t)$. Let $\mu = \mathrm{Ad}^*\left(g(0)\right) \cdot p(0)$. Then $\mathrm{Ad}^*\left(g(0)\right)^{-1} \cdot \mu = p(0)$ as required. Next we show that $\mathrm{Ad}^*\left(g(0)\right)^{-1} \cdot \mu$ satisfies the required differential equation. Now first notice that, as $\mathrm{Ad} : \mathsf{G} \to \mathrm{Aut}(\mathfrak{g})$ is a Lie group homomorphism, we have that $\mathrm{Ad}g \circ \mathrm{Ad} \circ L_{g^{-1}} = \mathrm{Ad}$. Thus we have that $T_g\mathrm{Ad} = \mathrm{Ad}g \cdot T_{\mathbf{1}}\mathrm{Ad} \cdot T_gL_{g^{-1}} = \mathrm{Ad}g \cdot \mathrm{ad} \cdot T_gL_{g^{-1}}$. So then for $A \in \mathfrak{g}$ we get that

$$\begin{aligned}
\tfrac{d}{dt}\left((\mathrm{Ad}^*\left(g(t)\right)^{-1} \cdot \mu) \cdot A\right) &= \mu \cdot \left(\tfrac{d}{dt}\mathrm{Ad}\, g(t) \cdot A\right) \\
&= \mu \cdot \left(\left(T_{g(t)}\mathrm{Ad} \cdot \dot{g}(t)\right) \cdot A\right) \\
&= \mu \cdot \Big(\Big(\mathrm{Ad}\, g(t) \cdot \mathrm{ad} \cdot T_{g(t)}L_{(g(t))^{-1}} \cdot T_{\mathbf{1}}L_{g(t)} \cdot \tfrac{\partial H}{\partial p}(p)\Big) \cdot A\Big) \\
&= \mu \cdot \Big(\mathrm{Ad}\, g(t) \cdot \mathrm{ad}\ \tfrac{\partial H}{\partial p}(p) \cdot A\Big) \\
&= \Big(-\mathrm{ad}^*\ \tfrac{\partial H}{\partial p}(p) \cdot \left(\mathrm{Ad}^*\left(g(t)\right)^{-1} \cdot \mu\right)\Big) \cdot A.
\end{aligned}$$

That is to say that $\frac{d}{dt}\left(\mathrm{Ad}^*(g(t))^{-1} \cdot \mu\right) = -\mathrm{ad}^*\frac{\partial H}{\partial p}(p) \cdot \left(\mathrm{Ad}^*(g(t))^{-1} \cdot \mu\right)$. By uniqueness of the solution to the Cauchy problem, we then get the result. □

We can adapt the above result for the case of a time varying Hamiltonian vector field.

A.3.12 PROPOSITION. *Suppose that $t \mapsto (g(t), p(t))$ is an integral curve of the time varying vector field $\vec{H}_{u(t)} = (X, Y^*)$, $X(g,p) = \Xi_{u(t)}(\mathbf{1})$, $Y^*(g,p) = -\mathrm{ad}^*\Xi_{u(t)}(\mathbf{1}) \cdot p$ for some admissible control $u(\cdot)$. Then (for some $\mu \in \mathfrak{g}^*$)*

$$\dot{g}(t) = \Xi(g(t), u(t)), \qquad\qquad p(t) = \mathrm{Ad}^*(g(t))^{-1} \cdot \mu$$

for some $p(0) \in \mathfrak{g}^*$. Thus $p(t)$ is contained in the coadjoint orbit through $p(0) = \mathrm{Ad}^*\left(g(0)\right)^{-1} \cdot \mu$, i.e., $p(t) \in \mathrm{Ad}^*\,\mathsf{G} \cdot p(0)$.

Proof. As $t \mapsto (g(t), p(t))$ is an integral curve of $\vec{H}_{u(t)}$ we have (almost everywhere) that $\dot{g}(t) = T_{\mathbf{1}}L_{g(t)} \cdot \Xi_{u(t)}(\mathbf{1}) = \Xi(g(t), u(t))$ and $\dot{p}(t) = -\mathrm{ad}^*\Xi_{u(t)}(\mathbf{1}) \cdot p(t)$. Let $\mu = \mathrm{Ad}^*(g(0)) \cdot p(0)$. Then $\mathrm{Ad}^*(g(0))^{-1} \cdot \mu = p(0)$ as required. Next we show that $\mathrm{Ad}^*(g(0))^{-1} \cdot \mu$ satisfies the required differential equation. As shown in the preceding proposition's proof we have that $T_g\mathrm{Ad} = \mathrm{Ad}\, g \cdot \mathrm{ad} \cdot T_g L_{g^{-1}}$. So then, for $A \in \mathfrak{g}$, we get that

$$\begin{aligned}
\tfrac{d}{dt}\big((\mathrm{Ad}^*(g(t))^{-1} \cdot \mu) \cdot A\big) &= \mu\left(\tfrac{d}{dt}\mathrm{Ad}\, g(t) \cdot A\right) \\
&= \mu\left(\left(T_{g(t)}\mathrm{Ad} \cdot \dot{g}(t)\right) \cdot A\right) \\
&= \mu\left(\left(\mathrm{Ad}\, g(t) \cdot \mathrm{ad} \cdot T_{g(t)}L_{(g(t))^{-1}} \cdot T_{\mathbf{1}}L_{g(t)} \cdot \Xi_{u(t)}(\mathbf{1})\right) \cdot A\right) \\
&= \mu\left(\mathrm{Ad}\, g(t) \cdot \mathrm{ad}\ \Xi_{u(t)}(\mathbf{1}) \cdot A\right) \\
&= \left(-\mathrm{ad}^*\, \Xi_{u(t)}(\mathbf{1}) \cdot \left(\mathrm{Ad}^*(g(t))^{-1} \cdot \mu\right)\right) \cdot A.
\end{aligned}$$

That is to say that $\frac{d}{dt}\left(\mathrm{Ad}^*(g(t))^{-1} \cdot \mu\right) = -\mathrm{ad}^*\Xi_{u(t)}(\mathbf{1}) \cdot \left(\mathrm{Ad}^*(g(t))^{-1} \cdot \mu\right)$. By uniqueness of the solution to the Cauchy problem, we then get the result. □

A.3.5 The Lie-Poisson structure on $\mathfrak{g}^*$

From the symplectic structure on $T^*\mathsf{G}$ we have a Poisson structure on $T^*\mathsf{G}$ and, through our identification, thus on $\mathsf{G} \times \mathfrak{g}^*$. The Hamiltonian functions $(g, p) \mapsto H(g, p)$ of interest for us were G-invariant, i.e., of the form $(g, p) \mapsto H(p)$. It is thus natural to consider our Poisson bracket $\{,\}$ on such functions.

A.3.13 Proposition. *For two G invariant Hamiltonian functions H_1 and H_2 on $\mathsf{G} \times \mathfrak{g}^*$ we have that*

$$\{H_1, H_2\}(g, p) = -p \cdot \left[\frac{\partial H_1}{\partial p}(g, p), \frac{\partial H_2}{\partial p}(g, p)\right].$$

Proof. By G-invariance, we have that $\frac{\partial H_i}{\partial g}(g, p) = 0$, $i = \overline{1, 2}$. By applying the definition of $\{,\}$ in terms of the symplectic form ω, and applying proposition A.3.5, we get that

$$\begin{aligned}
&\{H_1, H_2\}(g, p) \\
&= \omega_{(g,p)}\big(\vec{H}_1(g, p), \vec{H}_2(g, p)\big) \\
&= \omega_{(g,p)}\Big(\big(\big(g, \tfrac{\partial H_1}{\partial p}(g, p)\big), \big(p, -\big(\mathrm{ad}^* \tfrac{\partial H_1}{\partial p}(g, p)\big)(p)\big)\big), \\
&\qquad\qquad \big(\big(g, \tfrac{\partial H_2}{\partial p}(g, p)\big), \big(p, -\big(\mathrm{ad}^* \tfrac{\partial H_2}{\partial p}(g, p)\big)(p)\big)\big)\Big) \\
&= -\big(\mathrm{ad}^* \tfrac{\partial H_2}{\partial p}(g, p)\big)(p) \cdot \tfrac{\partial H_1}{\partial p}(g, p) + \big(\mathrm{ad}^* \tfrac{\partial H_1}{\partial p}(g, p)\big)(p) \cdot \tfrac{\partial H_2}{\partial p}(g, p) + p \cdot \big[\tfrac{\partial H_1}{\partial p}(g, p), \tfrac{\partial H_2}{\partial p}(g, p)\big] \\
&= p \cdot \big[\tfrac{\partial H_2}{\partial p}(g, p), \tfrac{\partial H_1}{\partial p}(g, p)\big] - p \cdot \big[\tfrac{\partial H_1}{\partial p}(g, p), \tfrac{\partial H_2}{\partial p}(g, p)\big] + p \cdot \big[\tfrac{\partial H_1}{\partial p}(g, p), \tfrac{\partial H_2}{\partial p}(g, p)\big] \\
&= -p \cdot \big[\tfrac{\partial H_1}{\partial p}(g, p), \tfrac{\partial H_2}{\partial p}(g, p)\big].
\end{aligned}$$

□

A.3.14 Proposition. *The mapping $\{,\}_{\mathfrak{g}^*} : C^\infty(\mathfrak{g}^*) \times C^\infty(\mathfrak{g}^*) \to C^\infty(\mathfrak{g}^*)$ defined by*

$$\{H_1, H_2\}_{\mathfrak{g}^*}(p) = -p \cdot \left[\frac{\partial H_1}{\partial p}(p), \frac{\partial H_2}{\partial p}(p)\right]$$

*defines a Poisson structure on $\mathfrak{g}^*$, called the **Lie-Poisson bracket**.*

PROOF. By the preceding proposition we have closure, i.e., $\{H_1, H_2\}_{\mathfrak{g}^*} \in C^\infty(\mathfrak{g}^*)$. The required properties of $\{,\}_{\mathfrak{g}^*}$ to be a Poisson structure then follow from the properties for $\{,\}$ on $T^*\mathsf{G}$.□

When the context is clear, we will simply write $\{,\}$ for $\{,\}_{\mathfrak{g}^*}$. We identify G-invariant Hamiltonians on $T^*\mathsf{G}$ with Hamiltonians on $\mathfrak{g}^*$. Specifically, we make the identification

$$(g, p) \mapsto H(\mathbf{1}, p) \quad \longleftrightarrow \quad p \mapsto H(\mathbf{1}, p).$$

Then, for G-invariant Hamiltonians $H_1, H_2 \in C^\infty(T^*\mathsf{G})$, we have that $\{H_1, H_2\} = \{H_1, H_2\}_{\mathfrak{g}^*}$.

Suppose we have a fixed Lie algebra $\mathfrak{g}$. Let $\{E_i\}_{i=\overline{1,n}}$ be a basis for $\mathfrak{g}$, $\{E_i^*\}_{i=\overline{1,n}}$ be the dual basis (for $\mathfrak{g}^*$) and $\{P_i\}_{i=\overline{1,n}}$ be the double dual basis (for $\mathfrak{g}^{**} \cong \mathfrak{g}$). (Specifically, P_i is the linear function on $\mathfrak{g}^*$ such that $P_i(E_j^*) = \delta_{ij}$ for $i, j = \overline{1, n}$.) Then any $p \in \mathfrak{g}^*$ may be written as $p = \sum_{i=1}^n p_i E_i^*$. In other words, we define the i'th component p_i of an element $p \in \mathfrak{g}^*$ as $p_i = P_i(p)$.

A.3.15 PROPOSITION. *Suppose $H \in C^\infty(T^*\mathsf{G})$ is a G-invariant Hamiltonian and that $t \mapsto (g(t), p(t))$ is an integral curve of $\vec{H}$. Then, in component form, we have that $p(t) = \sum_{i=1}^n p_i(t)E_i^*$ and*

$$\dot{p}_i(t) = \{P_i, H\}(p(t)) = -p(t) \cdot \left[E_i, \frac{\partial H}{\partial p}(g(t), p(t))\right], \quad i = \overline{1, n}.$$

PROOF. By proposition A.3.7, we get that $\frac{d}{dt}P_i(\mathscr{F}_t(p(0))) = \{P_i, H\}(\mathscr{F}_t(p(0)))$, where $\mathscr{F}_t$ is the flow of $\vec{H}$. But we have that $p(t) = \mathscr{F}_t(p(0))$. Thus we get that $\frac{d}{dt}p_i(t) = \{P_i, H\}(p(t))$ as required. Alternatively, by proposition A.3.5 (noting $\frac{\partial H}{\partial g} = 0$), we have that $\dot{p}(t) = -\mathrm{ad}^* \frac{\partial H}{\partial p}(g, p) \cdot p(t)$. Thus $\dot{p}_i(t) = P_i \cdot \dot{p}(t) = \left(-\mathrm{ad}^* \frac{\partial H}{\partial p}(g, p) \cdot p(t)\right) \cdot E_i = p(t) \cdot \left(\mathrm{ad} \frac{\partial H}{\partial p}(g, p) \cdot E_i\right) = -p(t) \cdot [E_i, \frac{\partial H}{\partial p}(g, p)]$. □

A.3.16 COROLLARY. *Suppose $H \in C^\infty(\mathfrak{g}^*)$ is a Hamiltonian and that $t \mapsto p(t)$ is an integral curve of $\vec{H}$. Then, in component form, we have that $\dot{p}_i(t) = \{P_i, H\}(p(t)) = -p\left(\left[E_i, \frac{\partial H}{\partial p}(p(t))\right]\right)$, $i = \overline{1, n}$.*

Again we have a corresponding result for the time varying case.

A.3.17 PROPOSITION. *Suppose that $t \mapsto (g(t), p(t))$ is an integral curve of the time varying vector field $\vec{H}_{u(t)} = (X, Y^*)$, $X(g, p) = \Xi_{u(t)}(\mathbf{1})$, $Y^*(g, p) = -\mathrm{ad}^* \Xi_{u(t)}(\mathbf{1}) \cdot p$ for some admissible control $u(\cdot)$. Then, in component form, we have that $p(t) = \sum_{i=1}^n p_i(t)E_i^*$ and*

$$\dot{p}_i(t) = -p(t) \cdot \left[E_i, \Xi_{u(t)}(\mathbf{1})\right], \quad i = \overline{1, n}.$$

PROOF. By definition of $\vec{H}_{u(t)}$ we have that $\dot{p} = -\mathrm{ad}^* \Xi_{u(t)}(\mathbf{1}) \cdot p(t)$. Thus, as before, we get that $\dot{p}_i(t) = P_i \cdot \dot{p}(t) = (-\mathrm{ad}^* \Xi_{u(t)}(\mathbf{1}) \cdot p(t)) \cdot E_i = p(t) \cdot (\mathrm{ad}\, \Xi_{u(t)}(\mathbf{1}) \cdot E_i) = -p(t) \cdot [E_i, \Xi_{u(t)}(\mathbf{1})]$.□

Casimir Functions

A function $F \in C^\infty(\mathcal{M})$ on the Poisson manifold $(\mathcal{M}, \{,\})$ is a **Casimir function** if one of the following equivalent conditions hold

1. for every $H \in C^\infty(\mathcal{M})$ we have that $\{F, H\} = 0$;
2. C generates trivial dynamics, i.e., $\vec{C} = 0$;
3. C is constant along the flow of all Hamiltonian vector fields.

A.3.18 Proposition. *Any function $C \in C^\infty(\mathfrak{g}^*)$ that is invariant under the coadjoint action, i.e., $\forall g \in \mathsf{G}, p \in \mathfrak{g}^*,\ C(\mathrm{Ad}^* g \cdot p) = C(p)$, is a Casimir function on $(\mathfrak{g}^*, \{,\})$.*

Proof. Suppose $\forall g \in \mathsf{G}, p \in \mathfrak{g}^*,\ C(\mathrm{Ad}^* g \cdot p) = C(p)$. Let $H \in C^\infty(\mathfrak{g}^*)$. Then $(g, p) \mapsto H(p)$ is a Hamiltonian function on $T^*\mathsf{G}$ and from proposition A.3.11 we have that any integral curve $p(\cdot)$ of $\vec{H}$ on $\mathfrak{g}^*$ is contained in the coadjoint orbit through $p(0)$, i.e., $p(t) \in \mathrm{Ad}^*\,\mathsf{G} \cdot p(0)$. Thus, if $\mathscr{F}_t : \mathfrak{g}^* \to \mathfrak{g}^*$ is the flow of $\vec{H}$ we have that $\mathscr{F}_t(p) \in \mathrm{Ad}^*\,\mathsf{G} \cdot p$. So then $C(\mathscr{F}_t(p)) = C(p)$, that is to say, C is constant along the flow of $\vec{H}$. □

A.3.19 Proposition. *For any $A \in Z(\mathfrak{g})$, $A^{**} : \mathfrak{g}^* \to \mathbb{R}$, $B^* \mapsto B^*(A)$ is a Casimir function.*

Proof. For any $H \in C^\infty(\mathcal{M})$ we have that $\{A^{**}, H\}(p) = -p \cdot \left[A, \frac{\partial H}{\partial p}\right] = -p \cdot 0 = 0.$ □

A.3.6 The optimal control problem

Let $\Sigma = (\mathsf{G}, \Xi)$ be a left-invariant control system, as described in section A.2.2. Given an admissible control $u(\cdot)$, we associate a non-autonomous Cauchy problem $\dot{g}(t) = \Xi(g(t), u(t))$, $g(0) = g_0$ to our system. (By Carathéodory's theorem this problem has a unique solution.) We refer to a solution $g(\cdot)$, for some admissible control $u(\cdot)$, as a trajectory of the system Σ, and to the pair $(g(\cdot), u(\cdot))$ as a control trajectory pair. In order to compare admissible controls on a segment $[0, T]$, we introduce a cost functional: $J(u(\cdot)) = \int_0^T \chi(g(t), u(t))dt$. We assume that the cost (or Lagrangian) χ takes the form $\chi : \mathbb{R}^\ell \to \mathbb{R}$, $u \mapsto \sum_{i=1}^{\ell} \beta_i u_i^2$, for some $\beta_i > 0$, $i = \overline{1, \ell}$. The problem is then to minimise the functional J among all possible admissible controls $u(\cdot) : [0, T] \to \mathbb{R}^\ell$, for which the corresponding solution $g(\cdot)$ of the Cauchy problem associated to Σ satisfies the boundary condition $g(T) = g_1$. In other words, we seek to find a control trajectory pair of minimal cost taking g_0 to g_1. We will restrict ourselves to only considering the case where the terminal time T is fixed. The problem may be succinctly written as

$$\begin{aligned}
&\dot{g}(t) = \Xi(g(t), u(t)), \quad g(\cdot) : [0, T] \to \mathsf{G}, \quad u(\cdot) : [0, T] \to \mathbb{R}^\ell, \\
&g(0) = g_0, \quad g(T) = g_1, \quad g_0, g_1 \in \mathsf{G} \text{ fixed}, \\
&\quad T > 0 \text{ fixed}, \\
J(u(\cdot)) = &\int_0^T \chi(u(t))dt \to \min. \qquad \text{(A.3.1)}
\end{aligned}$$

(Note that given such a problem, existence of a solution is not assured.)

Pontryagin Maximum Principle

Pontryagin Maximum Principle (shortly PMP) is the fundamental necessary condition of optimality for optimal control problems. The following result is written in our notation and specialised for left-invariant control systems.

A.3.20 Theorem. ([4]) *Let* $t \mapsto \widetilde{u}(t)$, $t \in [0, T]$ *be an optimal control for problem* (A.3.1) *and* $t \mapsto \widetilde{g}(t)$ *be the solution to the Cauchy problem* $\frac{d}{dt}\widetilde{g}(t) = \Xi(\widetilde{g}(t), \widetilde{u}(t))$, $\widetilde{g}(0) = g_0$ *for* $t \in [0, T]$. *Define a family* $\{H_u^\nu \mid u \in \mathbb{R}^\ell, \nu \in \mathbb{R}\}$ *of (cost extended* G*-invariant) Hamiltonian functions by*

$$H_u^\nu : T^*\mathsf{G} \to \mathbb{R}, \quad (g, p) \mapsto p \cdot \Xi(\mathbf{1}, u) + \nu\chi(u).$$

Then there exists a non-trivial pair: $(\nu, p(\cdot)) \neq 0$, $\nu \in \mathbb{R}$, $p(\cdot) : [0, T] \to \mathfrak{g}^*$ *such that the following conditions hold:*

$$\begin{aligned} &\tfrac{d}{dt}\big(\widetilde{g}(t), p(t)\big) = \vec{H}^\nu_{\widetilde{u}(t)}\big(\widetilde{g}(t), p(t)\big), \\ &H^\nu_{\widetilde{u}(t)}\big(\widetilde{g}(t), p(t)\big) = \max_{u \in \mathbb{R}^\ell} H^\nu_u\big(\widetilde{g}(t), p(t)\big) \quad \forall \text{ a.e. } t \in [0, T], \\ &\nu \leq 0. \end{aligned}$$

The next result shows that, when the maximised Hamiltonian is smooth, PMP reduces the problem to the study of solutions of just one Hamiltonian system.

A.3.21 Proposition. ([4]) *Assume that the maximized Hamiltonian of PMP*

$$H_{max}(g, p) = \max_{u \in \mathbb{R}^\ell} H^\nu_u(g, p)$$

is defined and smooth on $T^*\mathsf{G}\backslash\{p = 0\}$. *If a pair* $\big(\widetilde{u}(t), p(t)\big)$, $t \in [0, T]$, *satisfies PMP, then*

$$\tfrac{d}{dt}\big(\widetilde{g}(t), p(t)\big) = \vec{H}_{max}\big(\widetilde{g}(t), p(t)\big), \quad t \in [0, T].$$

In particular note that, if we consider the G-invariant Hamiltonian H_{max} in the above proposition as a smooth function on $\mathfrak{g}^*$, then we get that $\frac{d}{dt}p(t) = \vec{H}_{max}(p(t))$.

Extremal curves

There are two distinct possibilities for the constant parameter ν in theorem A.3.20:

1. if $\nu \neq 0$, then the curve $(\widetilde{g}(\cdot), p(\cdot))$ is called a **normal extremal**. Since the pair $(\nu, p(\cdot))$ can be multiplied by any positive number (see [4]), we can normalise $\nu < 0$ and assume that $\nu = -1$ in the normal case;
2. if $\nu = 0$, then $(\widetilde{g}(\cdot), p(\cdot))$ is called a **abnormal extremal**

We will customarily call the component $p(\cdot)$ of an extremal curve a **reduced extremal**. A triplet $(g(\cdot), p(\cdot), u(\cdot))$ will then be referred to as an **extremal triplet** for the family H^ν_u if it satisfies the necessary conditions of theorem A.3.20.

A.4 Qualitative Analysis of Dynamical Systems

In this section we briefly review some theory regarding limit sets and the stability of equilibrium points (of a dynamical system). However, before doing so, we briefly recall the definition of a complete vector field (and its flow) and give a sufficient condition for a smooth vector field to be complete.

Let $\mathcal{M}$ be a manifold and $V : \mathcal{M} \to T\mathcal{M}$ a smooth vector field on $\mathcal{M}$. The vector field V is said to be **complete** if the maximum interval of existence for any integral curve of V, is $\mathbb{R}$. An integral curve $c(\cdot)$ with domain being the maximum interval of existence, is called a **maximal integral curve**. The **flow** of a vector field V is the collection of maps $\mathscr{F}_t : \mathcal{M} \to \mathcal{M}$ such that $t \mapsto \mathscr{F}_t(x)$ is an integral curve of V with initial condition x, i.e., $\frac{d}{dt}\mathscr{F}_t(x) = V(\mathscr{F}_t(x))$ and $\mathscr{F}_0(x) = (x)$. So, for a complete vector field V, the flow of V is of the form

$$\mathscr{F} : \mathbb{R} \times \mathcal{M} \to \mathcal{M}, \quad (t, x) \mapsto \mathscr{F}_t(x) = \mathscr{F}(t, x).$$

Note that $\mathscr{F}_0(x) = x$ for all $x \in \mathcal{M}$ and $\mathscr{F}_t \circ \mathscr{F}_s(x) = \mathscr{F}_{t+s}(x)$ for all $s, t \in \mathbb{R}$ and $x \in \mathcal{M}$. We have the following sufficient condition for a vector field to be complete.

A.4.1 Theorem. ([1]) *Let V be a smooth vector field (on a manifold $\mathcal{M}$). Let $c(\cdot)$ be a maximal integral curve of V such that for every open finite interval (a, b) in the domain of $c(\cdot)$, $c[(a, b)]$ lies in a compact subset of $\mathcal{M}$. Then $c(\cdot)$ is defined for all $t \in \mathbb{R}$.*

A.4.2 Corollary. *Suppose that, for every integral curve $c(\cdot)$ of a smooth vector field V (on a manifold $\mathcal{M}$), there exists compact subset $K \subseteq \mathcal{M}$ containing the image of $c(\cdot)$. Then V is complete.*

A.4.1 Limit points

We review some basic facts regarding limit points as presented in [16] and [33], specifically for vector fields on $\mathbb{R}^n$. (As we wish to apply this theory to trajectories on the dual of a Lie algebra, which is isomorphic to $\mathbb{R}^n$ as a vector space, there is no need to specialise this theory to manifolds in this thesis.) Let V be a complete smooth vector field (on $\mathbb{R}^n$) and let $\mathscr{F}$ be the flow of V. Then a point $p \in \mathbb{R}^n$ is a **ω-limit point** of the integral curve $\mathscr{F}(\cdot, x)$ if there exists a sequence $t_n \to \infty$ such that $\lim_{n\to\infty} \mathscr{F}(t_n, x) = p$. Similarly, if there is a sequence $t_n \to -\infty$ such that $\lim_{n\to\infty} \mathscr{F}(t_n, x) = q$, then the point q is called an **α-limit point** of the integral curve $\mathscr{F}(\cdot, x)$. The set of all ω-limit points of a integral curve $\mathscr{F}(\cdot, x)$ is called the **ω-limit set** of $\mathscr{F}(\cdot, x)$ and denoted $\lim_\omega \mathscr{F}(\cdot, x)$. Similarly the set of all α-limit points of a integral curve $\mathscr{F}(\cdot, x)$ is called the **α-limit set** of $\mathscr{F}(\cdot, x)$ and denoted $\lim_\alpha \mathscr{F}(\cdot, x)$.

A.4.3 Theorem. ([16],[33]) *The α- and ω-limit sets of an integral curve $\mathscr{F}(\cdot, x)$ are closed subsets of $\mathbb{R}^n$. Moreover, if (the image of) the integral curve $\mathscr{F}(\cdot, x)$ is contained in a compact subset of $\mathbb{R}^n$, then both limits sets are non-empty, connected, compact subsets of $\mathbb{R}^n$.*

A.4.2 Stability

We review the theory of stability of equilibrium points as presented in [34] and [32]. We will restrict ourselves to considering Hamiltonian vector fields on Poisson manifolds. For a

Hamiltonian vector field $\vec{H}$ on a Poisson manifold $(\mathcal{M}, \{,\})$ we define the set E of **equilibrium points** as

$$E = \{x_e \in \mathcal{M} \mid \vec{H}(x_e) = 0\}.$$

Let $\mathscr{F}$ be the flow of $\vec{H}$. An equilibrium point $x_e \in E$ is **stable** (or Lyapunov stable) if for any neighbourhood N of x_e in $\mathcal{M}$ there exists an open neighbourhood $N' \subseteq N$ of x_e such that $\mathcal{F}_t(x) \in N$ for any $x \in N'$ and for any $t > 0$. (An equilibrium point $x_e \in E$ is **unstable** if it is not stable.)

The **linearisation** of $\vec{H}$ at a equilibrium point x_e is a linear map $\psi : T_{x_e}\mathcal{M} \to T_{x_e}\mathcal{M}$ defined by $\psi(v) = \frac{d}{dt}\big(T_{x_e}\mathscr{F}_t \cdot v\big)\big|_{t=0} = D\vec{H}(x_e) \cdot v$. An equilibrium point x_e is called **linearly stable** (respectively **unstable**) if the origin is a stable (respectively unstable) equilibrium point for the linear dynamical system on $T_{x_e}\mathcal{M}$ defined by ψ. We note that (Lyapunov) stability (at a point x_e) implies linear stability (at x_e). Hence if x_e is linearly unstable then it is unstable. We have the following characterisation of linear instability.

A.4.4 Theorem. *For each eigenvalue λ of $D\vec{H}(x_e)$ let m_λ denote the algebraic multiplicity of λ and d_λ denote the geometric multiplicity of λ. Then x_e is linearly unstable if and only if some eigenvalue of $D\vec{H}(x_e)$ has positive real part, or there is an eigenvalue λ of $D\vec{H}(x_e)$ with zero real part and $d_\lambda < m_\lambda$.*

This provides enough tools for our purposes to prove instability. To prove stability we have the following result.

A.4.5 Theorem. ([32]) *Let $(\mathcal{M}, \{,\})$ be a Poisson manifold and $\vec{H}$ be a Hamiltonian vector field corresponding to Hamiltonian function H. Further let x_e be a equilibrium point of $\vec{H}$ and $C_1, \dots C_k : \mathcal{M} \to \mathbb{R}$ be conserved quantities of $\vec{H}$, i.e., $\{C_i, H\} = 0$, $i = \overline{1,k}$. Assume that there exist constants $\{\lambda_0, \lambda_1, \dots, \lambda_k\}$ such that*

$$d(\lambda_0 H + \lambda_1 C_1 + \cdots + \lambda_k C_k)(x_e) = 0$$

and the quadratic form

$$d^2\left(\lambda_0 H + \lambda_1 C_1 + \cdots + \lambda_k C_k\right)\big|_{W\times W}(x_e)$$

is positive definite with

$$W = \ker dH(x_e) \cap \ker dC_1(x_e) \cap \cdots \cap \ker dC_k(x_e).$$

Then x_e is a (Lyapunov) stable equilibrium point.

A.5 Elliptic Functions and Integrals

A.5.1 Jacobi elliptic functions

We briefly recall the (system) definition of the Jacobi elliptic functions, as presented in [28], and discuss some basic properties of these functions.

Let $k \in (0,1)$, and let t denote a real variable that we interpret as time. The Jacobi elliptic functions $\mathrm{sn}(t,k)$, $\mathrm{cn}(t,k)$ and $\mathrm{dn}(t,k)$ are defined as the solutions of the system of differential equations

$$\dot{x} = yz \qquad \dot{y} = -zx \qquad \dot{z} = -k^2 xy$$

that satisfy the initial conditions

$$\mathrm{sn}(0,k) = x(0) = 0, \qquad \mathrm{cn}(0,k) = y(0) = 1, \qquad \mathrm{dn}(0,k) = z(0) = 1.$$

The parameter k is known as the **modulus** and satisfies $0 < k < 1$; the **complementary modulus** is $k' = \sqrt{1-k^2}$. The definition immediately gives the derivatives for the functions, namely

$$\tfrac{d}{dt}\mathrm{sn}(t,k) = \mathrm{cn}(t,k)\,\mathrm{dn}(t,k), \quad \tfrac{d}{dt}\mathrm{cn}(t,k) = -\mathrm{dn}(t,k)\,\mathrm{sn}(t,k), \quad \tfrac{d}{dt}\mathrm{dn}(t,k) = -k^2\mathrm{sn}(t,k)\,\mathrm{cn}(t,k).$$

It may be shown that, for a fixed k, $\mathrm{sn}(t,k)$ is an odd function of t; $\mathrm{cn}(t,k)$ and $\mathrm{dn}(t,k)$ are even functions of t.

The Jacobi elliptic function degenerate into the usual trigonometric functions and the hyperbolic functions as k approaches 0 and 1 respectively. Specifically we have that as $k \to 0$ from the right

$$\mathrm{sn}(t,k) \to \sin t, \qquad \mathrm{cn}(t,k) \to \cos t, \qquad \mathrm{dn}(t,k) \to 1,$$

and as $k \to 1$ from the left

$$\mathrm{sn}(t,k) \to \tanh t, \qquad \mathrm{cn}(t,k) \to \operatorname{sech} t, \qquad \mathrm{dn}(t,k) \to \operatorname{sech} t.$$

Next we note that $\mathrm{sn}(t,k)$, $\mathrm{cn}(t,k)$ and $\mathrm{dn}(t,k)$ satisfy the following identities

$$\mathrm{sn}^2(t,k) + \mathrm{cn}^2(t,k) \equiv 1, \qquad k^2\mathrm{sn}^2(t,k) + \mathrm{dn}^2(t,k) \equiv 1.$$

Consequently (as $\mathrm{dn}(t,k)$ may be shown to be positive) we have the following inequalities

$$-1 \le \mathrm{sn}(t,k) \le 1, \qquad -1 \le \mathrm{cn}(t,k) \le 1, \qquad k' \le \mathrm{dn}(t,k) \le 1.$$

We now discuss the periodic properties of the Jacobi elliptic functions (for a fixed k). Let $K > 0$ be the time that $\mathrm{cn}(t,k)$ takes to decrease to zero, i.e., $\mathrm{cn}(K,k) = 0$ and $\mathrm{cn}(t,k) > 0$ for $0 < t < K$. Then

$$\begin{array}{lll} \mathrm{sn}(0,k) = 0, & \mathrm{sn}(K,k) = 1, & 0 < \mathrm{sn}(t,k) < 1 \text{ for } 0 < t < K, \\ \mathrm{cn}(0,k) = 1, & \mathrm{cn}(K,k) = 0, & 0 < \mathrm{cn}(t,k) < 1 \text{ for } 0 < t < K, \\ \mathrm{dn}(0,k) = 1, & \mathrm{dn}(K,k) = k', & k' < \mathrm{dn}(t,k) < 1 \text{ for } 0 < t < K. \end{array}$$

As functions of t, $\mathrm{sn}(t,k)$ and $\mathrm{dn}(t,k)$ are even about K and $\mathrm{cn}(t,k)$ is odd about K, i.e., for a fixed k, $0 < k < 1$, and all $t \in \mathbb{R}$ we have that

$$\mathrm{sn}(K+t,k) = \mathrm{sn}(K-t,k), \quad \mathrm{cn}(K+t,k) = -\mathrm{cn}(K-t,k), \quad \mathrm{dn}(K+t,k) = \mathrm{dn}(K-t,k).$$

Hence, $\mathrm{sn}(t,k)$ and $\mathrm{cn}(t,k)$ are $4K$ periodic in t and $\mathrm{dn}(t,k)$ is $2K$ periodic in t.

We use the following convenient notation (invented by Glaisher) to express reciprocals and quotients of the Jacobi elliptic functions. The reciprocals are denoted by reversing the order of the letters which express the function, thus

$$\mathrm{ns}(t,k) = \tfrac{1}{\mathrm{sn}(t,k)}, \qquad \mathrm{nc}(t,k) = \tfrac{1}{\mathrm{cn}(t,k)}, \qquad \mathrm{nd}(t,k) = \tfrac{1}{\mathrm{dn}(t,k)}.$$

Quotients are denoted by writing (in order) the first letters of the numerator and denominator, thus

$$\begin{aligned} \mathrm{sc}(t,k) &= \tfrac{\mathrm{sn}(t,k)}{\mathrm{cn}(t,k)}, & \mathrm{sd}(t,k) &= \tfrac{\mathrm{sn}(t,k)}{\mathrm{dn}(t,k)}, & \mathrm{cd}(t,k) &= \tfrac{\mathrm{cn}(t,k)}{\mathrm{dn}(t,k)}, \\ \mathrm{cs}(t,k) &= \tfrac{\mathrm{cn}(t,k)}{\mathrm{sn}(t,k)}, & \mathrm{ds}(t,k) &= \tfrac{\mathrm{dn}(t,k)}{\mathrm{sn}(t,k)}, & \mathrm{dc}(t,k) &= \tfrac{\mathrm{dn}(t,k)}{\mathrm{cn}(t,k)}. \end{aligned}$$

In terms of this notation we then have the following two identities (which may be derived from the identities for $\mathrm{sc}(t,k)$, $\mathrm{cn}(t,k)$ and $\mathrm{dn}(t,k)$)

$$1 - (k')^2 \mathrm{sd}^2\,(t,k) = \mathrm{cd}^2\,(t,k) \qquad\qquad 1 - k^2\,\mathrm{cd}^2\,(t,k) = (k')^2\,\mathrm{nd}^2\,(t,k)$$

(We only list the identities needed for this thesis, in the form that they are needed.)

The Jacobi elliptic functions are of particular interest to us in their application to (elliptic) integrals. Without delving into the theory of elliptic integrals, we simply quote the results needed for this thesis, as may be found in [5]:

$$\int_0^x \frac{dt}{\sqrt{(a^2+t^2)(b^2-t^2)}} = \frac{1}{\sqrt{a^2+b^2}}\,\mathrm{sd}^{-1}\left(\frac{\sqrt{a^2+b^2}}{ab}\,x,\ \frac{b}{\sqrt{a^2+b^2}}\right), \qquad 0 \le x \le b \tag{A.5.1}$$

$$\int_x^a \frac{dt}{\sqrt{(a^2-t^2)(t^2-b^2)}} = \frac{1}{a}\,\mathrm{dn}^{-1}\left(\frac{1}{a}\,x,\ \frac{\sqrt{a^2-b^2}}{a}\right), \qquad b \le x \le a \tag{A.5.2}$$

$$\int_a^x \frac{dt}{\sqrt{(t^2-a^2)(t^2+b^2)}} = \frac{1}{\sqrt{a^2+b^2}}\,\mathrm{nc}^{-1}\left(\frac{1}{a}\,x,\ \frac{b}{\sqrt{a^2+b^2}}\right), \qquad a \le x. \tag{A.5.3}$$

A.5.2 Reduction to the standard form

For the purposes of this thesis, we are interested in solving an integral of the form $\int \frac{dx}{\omega}$, where ω^2 is a quartic (or cubic) in x. We briefly outline how an integral of this form may be reduced to a standard form, as discussed in [5] and [37]. (Having reduced to such a standard form, we can apply formulae (A.5.1), (A.5.2) and (A.5.3) in order to solve our problem.)

First we discuss how any quartic (or cubic) ω^2 in x (with no repeated factors) can be expressed in the form

$$\left(A_1(x-r_1)^2 + B_1(x-r_2)^2\right)\left(A_2(x-r_1)^2 + B_2(x-r_2)^2\right)$$

where $A_1, B_1, A_2, B_2, r_1, r_2 \in \mathbb{R}$. We can express ω^2 as the product of two quadratics (which, if the roots are real, are arranged to have non-interlacing roots)

$$\omega^2 = X_1 X_2 = (a_1x^2 + 2b_1x + c_1)(a_2x^2 + 2b_2x + c_2)$$

with $a_i, b_i, c_i \in \mathbb{R}$, $i = \overline{1,2}$. Consider the polynomial $X_1 - \lambda X_2$. We seek to find values of λ that make a perfect square in x, which is equivalent to the condition

$$(a_1 - \lambda a_2)(c_1 - \lambda c_2) - (b_1 - \lambda b_2)^2 = 0.$$

Let the roots of this quadratic (in λ) be λ_1 and λ_2. Then there exists r_1 and r_2 such that

$$X_1 - \lambda_1 X_2 = (a_1 - \lambda_1 a_2)(x - r_1)^2, \qquad X_1 - \lambda_2 X_2 = (a_1 - \lambda_2 a_2)(x - r_2)^2.$$

(We note that λ_1 and λ_2 can be shown to be real and distinct under our assumptions, unless $a_1 b_2 = a_2 b_1$, in which case $X_1 = a_1(x - r_1)^2 + B_1, X_2 = a_2(x - r_1)^2 + B_2$.) That is to say that

$$\begin{bmatrix} 1 & -\lambda_1 \\ 1 & -\lambda_2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} (a_1 - \lambda_1 a_2)(x - r_1)^2 \\ (a_1 - \lambda_2 a_2)(x - r_2)^2 \end{bmatrix}$$

and thus we have that

$$\begin{aligned} X_1 &= \tfrac{-\lambda_2(a_1 - a_2\lambda_1)(x-r_1)^2 + \lambda_1(a_1 - a_2\lambda_2)(x-r_2)^2}{\lambda_1 - \lambda_2} \\ X_2 &= \tfrac{-(a_1 - a_2\lambda_1)(x-r_1)^2 + (a_1 - a_2\lambda_2)(x-r_2)^2}{\lambda_1 - \lambda_2}. \end{aligned}$$

In other words our quartic (or cubic) takes the given form

$$X_1 X_2 = \left(A_1(x - r_1)^2 + B_1(x - r_2)^2\right)\left(A_2(x - r_1)^2 + B_2(x - r_2)^2\right),$$

where

$$A_1 = \tfrac{\lambda_2(a_1 - a_2\lambda_1)}{\lambda_2 - \lambda_1} \qquad A_2 = \tfrac{a_1 - a_2\lambda_1}{\lambda_2 - \lambda_1} \qquad B_1 = \tfrac{\lambda_1(a_1 - a_2\lambda_2)}{\lambda_1 - \lambda_2} \qquad B_2 = \tfrac{a_1 - a_2\lambda_2}{\lambda_1 - \lambda_2}.$$

So at this stage we have that

$$\int \tfrac{dx}{\omega} = \int \tfrac{dx}{\sqrt{(A_1(x-r_1)^2 + B_1(x-r_2)^2)(A_2(x-r_1)^2 + B_2(x-r_2)^2)}}.$$

Consider the change in variable $s = \frac{x-r_1}{x-r_2}$, i.e., $x = \frac{r_2 s - r_1}{s-1}$. Then (provided $A_1 \neq 0$ and $A_2 \neq 0$) we have that

$$\begin{aligned} dx &= \tfrac{r_1 - r_2}{(s-1)^2} ds \\ \tfrac{dx}{\omega} &= \tfrac{(r_1 - r_2)ds}{(s-1)^2 \sqrt{\left(A_1(\frac{r_2 s - r_1}{s-1} - r_1)^2 + B_1(\frac{r_2 s - r_1}{s-1} - r_2)^2\right)\left(A_2(\frac{r_2 s - r_1}{s-1} - r_1)^2 + B_2(\frac{r_2 s - r_1}{s-1} - r_2)^2\right)}} \\ &= \tfrac{(r_1 - r_2)ds}{(s-1)^2 \sqrt{\frac{1}{(s-1)^4}} \sqrt{(A_1(r_2 - r_1)^2 s^2 + B_1(r_2 - r_1)^2)(A_2(r_2 - r_1)^2 s^2 + B_2(r_2 - r_1)^2)}} \\ &= \tfrac{(r_1 - r_2)ds}{(r_1 - r_2)^2 \sqrt{(A_1 s^2 + B_1)(A_2 s^2 + B_2)}} \\ &= \tfrac{ds}{(r_1 - r_2)\sqrt{\sigma A_1 A_2} \sqrt{\sigma\left(s^2 + \frac{B_1}{A_1}\right)\left(s^2 + \frac{B_2}{A_2}\right)}} \end{aligned}$$

for some $\sigma \in \{-1, 1\}$. Thus we can (under some assumptions) express the integral $\int \frac{dx}{\omega}$ as

$$\int \tfrac{dx}{\omega} = \tfrac{1}{(r_1 - r_2)\sqrt{\sigma A_1 A_2}} \int \tfrac{ds}{\sqrt{\sigma\left(s^2 + \frac{B_1}{A_1}\right)\left(s^2 + \frac{B_2}{A_2}\right)}}.$$

Appendix B

Tabulation of Results

We collect some of the main results of this thesis (especially the classifications) in table form.

Table B.1 lists the (tangent maps at identity of the) automorphisms of $\widetilde{\mathsf{H}}_3^\diamond$ and $\mathsf{H}_3^\diamond(n)$, as produced in proposition 1.2.27 and theorem 3.3.2, respectively. Table B.2 is a full classifying table of $\mathfrak{L}$-related (by $\psi \in \operatorname{Aut}\mathfrak{h}_3^\diamond$) affine subspaces of $\mathfrak{h}_3^\diamond$, as developed in propositions 2.2.4, 2.3.1, 2.4.5, 2.5.3 and 2.6.1 (and summarised in theorem 2.7.1). Table B.3 then lists the invariants that were involved in making the above classification, as developed in chapter 2.

Table B.4 lists a classification of the (traces of the) controllable systems on $\widetilde{\mathsf{H}}_3^\diamond$ and $\mathsf{H}_3^\diamond(n)$ respectively, as developed in chapter 3. This table is organised so that each system (with trace) in the right-hand column, is covered (in the sense of a **LiCAS**-covering, see section A.2.2) by a system (with trace) in the left-hand column and the same row. (This is the reason for the dashed horizontal lines in the case of $(2,1)$; each system on the right is covered by a system in the same row, but necessarily in the same "dashed row.")

Finally, table B.5 summarises the integral curves of $\vec{H}_{\max}$ we were able to find explicit expressions for, as presented in propositions 4.3.1, 4.3.3, 4.3.5, 4.3.8 and corollary 4.3.10.

Group G	$d\operatorname{Aut}\mathsf{G}$, $k \in \{-1,1\}$, $x^2+y^2 \neq 0$
$\widetilde{\mathsf{H}}_3^\diamond$	$\begin{bmatrix} x & y & 0 & u \\ -ky & kx & 0 & v \\ kux - vy & kuy + xv & k(x^2+y^2) & w \\ 0 & 0 & 0 & k \end{bmatrix}$
$\mathsf{H}_3^\diamond(n)$	$\begin{bmatrix} x & y & 0 & u \\ -ky & kx & 0 & v \\ kux - vy & kuy + xv & k(x^2+y^2) & \frac{1}{2}k(u^2+v^2) \\ 0 & 0 & 0 & k \end{bmatrix}$

Table B.1: Automorphisms of $\widetilde{\mathsf{H}}_3^\diamond$ and $\mathsf{H}_3^\diamond(n)$

<table>
<tr><th>(ℓ,ε)</th><th colspan="3">Conditions</th><th>Equivalence representative</th></tr>
<tr><td rowspan="2">$(1,1)$</td><td colspan="3">$\pi_4(\Gamma^0)\neq\{0\}$, $\mathfrak{P}(\Gamma)=\alpha$, $\alpha\geq 0$</td><td>$E_1+\alpha E_3+\langle E_4\rangle$</td></tr>
<tr><td colspan="3">$\pi_4(\Gamma^0)=\{0\}$, $\pi_4(A)=\pm\alpha$, $\alpha>0$</td><td>$\alpha E_4+\langle E_1\rangle$</td></tr>
<tr><td>$(2,0)$</td><td colspan="3"></td><td>$\langle E_1,E_4\rangle$</td></tr>
<tr><td rowspan="5">$(2,1)$</td><td rowspan="3">$\pi_4(\Gamma)\neq\{0\}$</td><td rowspan="2">$\dim\mathrm{Lie}(\Gamma^0)=4$</td><td>$\mathfrak{T}(\Gamma)=1$
$\mathfrak{S}(\Gamma)=\alpha$
$\alpha\geq 0$</td><td>$E_2+\alpha E_3$
$+\langle E_1,E_4\rangle$</td></tr>
<tr><td>$\mathfrak{T}(\Gamma)=0$</td><td>$E_3+\langle E_1,E_4\rangle$</td></tr>
<tr><td colspan="2">$\dim\mathrm{Lie}(\Gamma^0)\neq 4$</td><td>$E_1+\langle E_3,E_4\rangle$</td></tr>
<tr><td colspan="2" rowspan="2">$\pi_4(\Gamma^0)=\{0\}$
$\pi_4(A)=\pm\alpha,\ \alpha>0$</td><td>$\mathfrak{T}(\Gamma)=1$</td><td>$\alpha E_4+\langle E_1,E_2\rangle$</td></tr>
<tr><td>$\mathfrak{T}(\Gamma)=0$</td><td>$\alpha E_4+\langle E_1,E_3\rangle$</td></tr>
<tr><td rowspan="2">$(3,0)$</td><td colspan="3">$\mathfrak{R}(\Gamma)=1$</td><td>$\langle E_1,E_2,E_4\rangle$</td></tr>
<tr><td colspan="3">$\mathfrak{R}(\Gamma)=0$</td><td>$\langle E_1,E_3,E_4\rangle$</td></tr>
<tr><td rowspan="3">$(3,1)$</td><td rowspan="2">$\pi_4(\Gamma^0)\neq\{0\}$</td><td colspan="2">$\mathfrak{R}(\Gamma^0)=0$</td><td>$E_2+\langle E_1,E_3,E_4\rangle$</td></tr>
<tr><td colspan="2">$\mathfrak{R}(\Gamma^0)=1$</td><td>$E_3+\langle E_1,E_2,E_4\rangle$</td></tr>
<tr><td colspan="3">$\pi_4(\Gamma^0)=\{0\}$, $\pi_4(A)=\pm\alpha$, $\alpha>0$</td><td>$\alpha E_4+\langle E_1,E_2,E_3\rangle$</td></tr>
<tr><td>$(4,0)$</td><td colspan="3"></td><td>$\langle E_1,E_2,E_3,E_4\rangle$</td></tr>
<tr><td colspan="5">$[E_1,E_2]=E_3$ $[E_1,E_4]=E_2$ $[E_2,E_4]=-E_1$ Others zero</td></tr>
</table>

Table B.2: Classifying table for ($\mathfrak{L}$-related) affine subspaces of $\mathfrak{h}_3^\diamond$

<table>
<tr><th>Invariant</th><th>Γ</th><th>Value</th></tr>
<tr><td>$\mathfrak{P}$</td><td>$A+\langle B\rangle$</td><td>$\left|\frac{b_4(-a_1b_1-a_2b_2+a_3b_4)+a_4\left(b_1^2+b_2^2-b_3b_4\right)}{a_4^2\left(b_1^2+b_2^2\right)-2a_4(a_1b_1+a_2b_2)b_4+\left(a_1^2+a_2^2\right)b_4^2}\right|$</td></tr>
<tr><td>$\mathfrak{T}$</td><td>$A+\langle B,C\rangle$</td><td rowspan="2">$\left|\operatorname{sgn}\left(\begin{vmatrix} a_1 & a_2 & a_4 \\ b_1 & b_2 & b_4 \\ c_1 & c_2 & c_4 \end{vmatrix}\right)\right|$</td></tr>
<tr><td>$\mathfrak{R}$</td><td>$\langle A,B,C\rangle$</td></tr>
<tr><td>$\mathfrak{S}$</td><td>$A+\langle B,C\rangle$
$\pi_4(B)=0$
$\pi_4(C)\neq 0$</td><td>$\left|\frac{c_4\left(-(a_2b_1-a_1b_2)(-b_2c_1+b_1c_2)+\left(a_3\left(b_1^2+b_2^2\right)-(a_1b_1+a_2b_2)b_3\right)c_4\right)+a_4\left((b_2c_1-b_1c_2)^2+\left(b_3(b_1c_1+b_2c_2)-\left(b_1^2+b_2^2\right)c_3\right)c_4\right)}{\begin{vmatrix} a_1 & a_2 & a_4 \\ b_1 & b_2 & 0 \\ c_1 & c_2 & c_4 \end{vmatrix}^2}\right|$</td></tr>
</table>

Table B.3: Invariants of affine subspaces of $\mathfrak{h}_3^\diamond$

(ℓ, ε)	$\widetilde{\mathsf{H}}_3^\diamond$	$\mathsf{H}_3^\diamond(n)$
$(1,1)$	$E_1 + \alpha E_3 + \langle E_4 \rangle$ $\alpha > 0$	$E_1 + \alpha_1 E_3 + \langle \alpha_2 E_3 + E_4 \rangle$ $\alpha_1 \geq 0, \alpha_2 \in \mathbb{R}$
		$\sigma E_3 + \alpha E_4 + \langle E_1 \rangle$ $\alpha > 0,\ \sigma \in \{-1, 0, 1\}$
$(2,0)$	$\langle E_1, E_4 \rangle$	$\langle E_1, \sigma E_3 + E_4 \rangle$ $\sigma \in \{-1, 0, 1\}$
$(2,1)$	$E_2 + \alpha E_3 + \langle E_1, E_4 \rangle$ $\alpha \geq 0$	$\alpha_1 E_2 + \alpha_2 E_3 + \langle E_1, \varsigma E_3 + E_4 \rangle$ $\alpha_1, \alpha_2 \geq 0,\ \alpha \neq 0,\ \varsigma \in \{-1, 1\}$
	$E_3 + \langle E_1, E_4 \rangle$	$E_2 + \alpha E_3 + \langle E_1, E_4 \rangle,\ \alpha \geq 0$
		$E_3 + \langle E_1, E_4 \rangle$
	$E_1 + \langle E_3, E_4 \rangle$	$E_1 + \langle E_3, E_4 \rangle$
		$\sigma E_3 + \alpha E_4 + \langle E_1, E_2 \rangle$ $\alpha > 0,\ \sigma \in \{-1, 0, 1\}$
		$\alpha E_3 + \langle E_1, E_3 \rangle,\ \alpha > 0$
$(3,0)$	$\langle E_1, E_2, E_4 \rangle$	$\langle E_1, E_2, \sigma E_3 + E_4 \rangle$ $\sigma \in \{-1, 0, 1\}$
	$\langle E_1, E_3, E_4 \rangle$	$\langle E_1, E_3, E_4 \rangle$
$(3,1)$	$E_2 + \langle E_1, E_3, E_4 \rangle$	$E_2 + \langle E_1, E_3, E_4 \rangle$
	$E_3 + \langle E_1, E_2, E_4 \rangle$	$\alpha E_3 + \langle E_1, E_2, \varsigma E_3 + E_4 \rangle$ $\alpha > 0, \varsigma \in \{-1, 1\}$
		$E_3 + \langle E_1, E_2, E_4 \rangle$
		$\alpha E_4 + \langle E_1, E_2, E_3 \rangle,\ \alpha > 0$
$(4,0)$	$\langle E_1, E_2, E_3, E_4 \rangle$	$\langle E_1, E_2, E_3, E_4 \rangle$

Table B.4: Classification of DF-equivalent controllable systems on $\widetilde{\mathsf{H}}_3^\diamond$ and $\mathsf{H}_3^\diamond(n)$

Conditions	Integral Curve	Parameters
$\rho = 0$ $c > 2h\beta_1 > 0$	$\bar{p}_1(t) = \sigma\sqrt{c}k\,\mathrm{cd}\,(\Omega t)$ $\bar{p}_2(t) = \sigma\sqrt{c}k'\,\mathrm{nd}\,(\Omega t)$ $\bar{p}_3(t) = 0$ $\bar{p}_4(t) = kk'\sqrt{\frac{c\beta_2}{\beta_1}}\ \mathrm{sd}\,(\Omega t)$	$\Omega = \sqrt{\frac{c}{\beta_1\beta_2}}$ $k = \sqrt{\frac{2h\beta_1}{c}}$ $\sigma \in \{-1, 1\}$
$\rho = 0$ $c = 2h\beta_1 > 0$	$\bar{p}_1(t) = \sigma_1\sqrt{2h\beta_1}\tanh\,(\Omega t)$ $\bar{p}_2(t) = \sigma_2\sqrt{2h\beta_1}\,\mathrm{sech}\,(\Omega t)$ $\bar{p}_3(t) = 0$ $\bar{p}_4(t) = -\sigma_1\sigma_2\sqrt{2h\beta_2}\,\mathrm{sech}\,(\Omega t)$	$\Omega = \sqrt{\frac{2h}{\beta_2}}$ $\sigma_1 \in \{-1, 1\}$ $\sigma_2 \in \{-1, 1\}$
$\rho = 0$ $0 < c < 2h\beta_1$	$p_1(t) = \sqrt{c}\,\mathrm{sn}\,(\Omega t)$ $p_2(t) = -\sigma\sqrt{c}\,\mathrm{cn}\,(\Omega t)$ $p_3(t) = 0$ $p_4(t) = \sigma\sqrt{2h\beta_2}\,\mathrm{dn}\,(\Omega t)$	$\Omega = \sqrt{\frac{2h}{\beta_2}}$ $k = \sqrt{\frac{c}{2h\beta_1}}$ $\sigma \in \{-1, 1\}$
$\rho \neq 0$ $2h\beta_1 > \frac{\beta_2}{\beta_1}\rho^2$ $c > 2h\beta_1 + \frac{\beta_2}{\beta_1}\rho^2$	$\bar{p}_1(t) = -\sigma\frac{\sqrt{\delta(c^2-\delta^2)\beta_1}}{\rho\sqrt{2\beta_2}}\frac{\mathrm{sn}(\Omega t)}{\sqrt{c+\delta}-\sqrt{c-\delta}\,\mathrm{cn}(\Omega t)}$ $\bar{p}_2(t) = \sigma\frac{\sqrt{2\delta}\,\mathrm{dn}(\Omega t)}{\sqrt{c+\delta}-\sqrt{c-\delta}\,\mathrm{cn}(\Omega t)}$ $\bar{p}_3(t) = \rho$ $\bar{p}_4(t) = \frac{\sqrt{c^2-\delta^2}}{2\rho}\ \frac{-\sqrt{c-\delta}+\sqrt{c+\delta}\,\mathrm{cn}(\Omega t)}{\sqrt{c+\delta}-\sqrt{c-\delta}\,\mathrm{cn}(\Omega t)}$	$\delta = \sqrt{c^2 - 8h\rho^2\beta_2}$ $\Omega = \sqrt{\frac{\delta}{\beta_1\beta_2}}$ $k = \sqrt{\frac{\delta-c+4h\beta_1}{2\delta}}$ $\sigma \in \{-1, 1\}$

Table B.5: Some integral curves of $\vec{H}_{\max}$ for reduced extremals of $\left(\widetilde{\Sigma}_1^{(2,0)}, \chi_1^{(2,0)}\right)$

Appendix C

Mathematica Notebooks

Wolfram Mathematica 7 was used to perform numerous calculations and to plot some graphs (as stereographic projections for three-dimensional graphs). We present a selection of notebooks (i.e., code). Input is presented in **bold**, and output not bold. (Each section represents a single notebook.)

C.1 Basic Calculations for $\mathsf{H}_3^\diamond$

Note that these calculation have been made on the universal covering Lie group $\widetilde{\mathsf{H}}_3^\diamond$ rather than on $\mathsf{H}_3^\diamond$. This is preferable as $\widetilde{\mathsf{H}}_3^\diamond$ permits a faithful parametrisation in the form of a diffeomorphism $\widetilde{m} : \mathbb{R}^4 \to \widetilde{\mathsf{H}}_3^\diamond$.

```
ClearAll["Global'*"]
```

Basic setup

```
cc[A_, B_]:=A.B − B.A;
m[x_, y_, z_, θ_]:=
{{1, −xCos[θ] + ySin[θ], xSin[θ] + yCos[θ], −2z, 0},
{0, Cos[θ], −Sin[θ], y, 0}, {0, Sin[θ], Cos[θ], x, 0},
{0, 0, 0, 1, 0}, {0, 0, 0, 0, Exp[θ]}};
M[x_, y_, z_, θ_]:=
{{0, −x, y, −2z, 0}, {0, 0, −θ, y, 0}, {0, θ, 0, x, 0},
{0, 0, 0, 0, 0}, {0, 0, 0, 0, θ}};
E1 = M[1, 0, 0, 0]; E2 = M[0, 1, 0, 0]; E3 = M[0, 0, 1, 0];
E4 = M[0, 0, 0, 1]; Base = {E1, E2, E3, E4};
Minv[MM_]:= {MM[[3, 4]], MM[[2, 4]], −1/2 MM[[1, 4]],
MM[[3, 2]]};
minv[MM_]:= {MM[[3, 4]], MM[[2, 4]], −1/2 MM[[1, 4]],
Log[MM[[5, 5]]]};
mt = m[x, y, z, θ];
Mt = M[x, y, z, θ];
MatrixForm[mt]
Simplify[minv[mt], Element[θ, Reals]]
```

$$\begin{pmatrix} 1 & -x\mathrm{Cos}[\theta] + y\mathrm{Sin}[\theta] & y\mathrm{Cos}[\theta] + x\mathrm{Sin}[\theta] & -2z & 0 \\ 0 & \mathrm{Cos}[\theta] & -\mathrm{Sin}[\theta] & y & 0 \\ 0 & \mathrm{Sin}[\theta] & \mathrm{Cos}[\theta] & x & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & e^\theta \end{pmatrix}$$

$\{x, y, z, \theta\}$

```
MatrixForm[Mt]
Minv[Mt]
```

$$\begin{pmatrix} 0 & -x & y & -2z & 0 \\ 0 & 0 & -\theta & y & 0 \\ 0 & \theta & 0 & x & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \theta \end{pmatrix}$$

$\{x, y, z, \theta\}$

Commutator table

```
SBase = Table [E_i, {i, 4}] ;
sct = Table[0, {i, 4}, {i, 4}];
For[i = 1, i ≤ 4, i++,
For[j = 1, j ≤ 4, j++,
sct[[i, j]] = SBase.Minv[cc[Base[[i]], Base[[j]]]];
]
];
TableForm[sct, TableHeadings → {SBase, SBase}]
```

	E_1	E_2	E_3	E_4
E_1	0	E_3	0	E_2
E_2	$-E_3$	0	0	$-E_1$
E_3	0	0	0	0
E_4	$-E_2$	E_1	0	0

Simple calculations

```
MatrixForm[m[0, 0, 0, 0]]
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Temp = FullSimplify[m[x,y,z,θ].m[x',y',z',θ']];
vt = FullSimplify[minv[Temp], Element[{θ,θ'}, Reals]]

$\{x+\mathrm{Cos}[\theta]x'+\mathrm{Sin}[\theta]y', y-\mathrm{Sin}[\theta]x'+\mathrm{Cos}[\theta]y', \frac{1}{2}\big(-(y\mathrm{Cos}[\theta]+x\mathrm{Sin}[\theta])x' +(x\mathrm{Cos}[\theta]-y\mathrm{Sin}[\theta])y'+2(z+z')\big), \theta+\theta'\}$

MatrixForm[
FullSimplify[m[vt[[1]], vt[[2]], vt[[3]], vt[[4]]]−
Temp]]

$$\begin{pmatrix} 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix}$$

Simplify[minv[m[x,y,z,0].m[0,0,0,θ]],
Element[θ, Reals]]

$\{x,y,z,\theta\}$

FullSimplify[vt[[1]] + $i$vt[[2]]]

$x+iy+e^{-i\theta}(x'+iy')$

Collect[Expand[vt[[3]]], {Cos[θ], Sin[θ]}]

$z+\mathrm{Cos}[\theta]\left(-\frac{yx'}{2}+\frac{xy'}{2}\right)+\mathrm{Sin}[\theta]\left(-\frac{xx'}{2}-\frac{yy'}{2}\right)+z'$

FullSimplify[
ComplexExpand[
$z+z'+\frac{1}{2}$Im[Exp[$-i\theta$]($x-iy$)($x'+iy'$)] − vt[[3]]] ,
Element[{$x,y,z,\theta,x',y',z',\theta'$}, Reals]]

0

FullSimplify[minv[Inverse[m[x,y,z,θ]]],
Element[θ, Reals]]

$\{-x\mathrm{Cos}[\theta]+y\mathrm{Sin}[\theta], -y\mathrm{Cos}[\theta]-x\mathrm{Sin}[\theta], -z, -\theta\}$

MatrixForm[
FullSimplify[m[vt[[1]], vt[[2]], vt[[3]], vt[[4]]]−
m[x,y,z,θ].m[x',y',z',θ']]]

$$\begin{pmatrix} 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix}$$

(rhr = m[0,0,0,θ].m[x,y,z,0].m[0,0,0,$-\theta$]//
FullSimplify)//MatrixForm
minv[rhr]//FullSimplify

$$\begin{pmatrix} 1 & -x\mathrm{Cos}[\theta]-y\mathrm{Sin}[\theta] & y\mathrm{Cos}[\theta]-x\mathrm{Sin}[\theta] & -2z & 0\\ 0 & 1 & 0 & y\mathrm{Cos}[\theta]-x\mathrm{Sin}[\theta] & 0\\ 0&0&1& x\mathrm{Cos}[\theta]+y\mathrm{Sin}[\theta] & 0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix}$$

$\{x\mathrm{Cos}[\theta]+y\mathrm{Sin}[\theta], y\mathrm{Cos}[\theta]-x\mathrm{Sin}[\theta], z, 0\}$

Diffeomorphism

D[mt, x]//MatrixForm
D[mt, y]//MatrixForm
D[mt, z]//MatrixForm
D[mt, θ]//MatrixForm

$$\begin{pmatrix} 0&-\mathrm{Cos}[\theta]&\mathrm{Sin}[\theta]&0&0\\ 0&0&0&0&0\\ 0&0&0&1&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix}$$

$$\begin{pmatrix} 0&\mathrm{Sin}[\theta]&\mathrm{Cos}[\theta]&0&0\\ 0&0&0&1&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix}$$

$$\begin{pmatrix} 0&0&0&-2&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix}$$

$$\begin{pmatrix} 0&y\mathrm{Cos}[\theta]+x\mathrm{Sin}[\theta]&x\mathrm{Cos}[\theta]-y\mathrm{Sin}[\theta]&0&0\\ 0&-\mathrm{Sin}[\theta]&-\mathrm{Cos}[\theta]&0&0\\ 0&\mathrm{Cos}[\theta]&-\mathrm{Sin}[\theta]&0&0\\ 0&0&0&0&0\\ 0&0&0&0&e^{\theta} \end{pmatrix}$$

Centre

CD = m[x,y,z,θ].m[x',y',z',θ']−
m[x',y',z',θ'].m[x,y,z,θ];
CD[[3, 4]]/.{$x'\to 0, y'\to 0, \theta'\to \pi/2$}

$x-y$

CD[[3, 4]]/.{$x'\to 0, y'\to 0, \theta'\to 3\pi/2$}

$x+y$

CD[[3, 4]]/.{$x'\to 1, y'\to 0, x\to 0, y\to 0$}

$-1+\mathrm{Cos}[\theta]$

CD[[3, 4]]/.{$x'\to 0, y'\to 1, x\to 0, y\to 0$}

$\mathrm{Sin}[\theta]$

Norm[
Simplify[
(m[0, 0, z, $2n\pi$].m[x',y',z',θ']−
m[x',y',z',θ'].m[0, 0, z, $2n\pi$]), $n\in$ Integers]]

0

(Alternative: used for universal covering group)

CD[[3, 4]]/.{$\theta'\to 0$}

$-x'+\mathrm{Cos}[\theta]x'+\mathrm{Sin}[\theta]y'$

Simplify[CD[[1, 4]]/.{$\theta'\to 0$, Cos[θ] → 1, Sin[θ] → 0},
{Element[n, Integers]}]

$2yx'-2xy'$

Quotient group SE(2)

ϕ[o_]:={{1, 0, 0}, {o[[3, 4]], o[[2, 2]], o[[3, 2]]},
{o[[2, 4]], o[[2, 3]], o[[3, 3]]}};

mt//MatrixForm

$$\begin{pmatrix} 1 & -x\text{Cos}[\theta]+y\text{Sin}[\theta] & y\text{Cos}[\theta]+x\text{Sin}[\theta] & -2z & 0 \\ 0 & \text{Cos}[\theta] & -\text{Sin}[\theta] & y & 0 \\ 0 & \text{Sin}[\theta] & \text{Cos}[\theta] & x & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & e^{\theta} \end{pmatrix}$$

ϕ[mt]//MatrixForm

$$\begin{pmatrix} 1 & 0 & 0 \\ x & \text{Cos}[\theta] & \text{Sin}[\theta] \\ y & -\text{Sin}[\theta] & \text{Cos}[\theta] \end{pmatrix}$$

$\phi[m[x,y,z,\theta].m[x',y',z',\theta']]-$
$\phi[m[x,y,z,\theta]].\phi[m[x',y',z',\theta']]$//
FullSimplify//MatrixForm

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Exponential

MatrixForm[
FullSimplify[MatrixExp[xE1 + yE2 + zE3 + θE4]]]
MatrixForm[FullSimplify[MatrixExp[xE1 + yE2 + zE3]]]

$$\begin{pmatrix} 1 & \frac{y-y\text{Cos}[\theta]-x\text{Sin}[\theta]}{\theta} & \frac{x-x\text{Cos}[\theta]+y\text{Sin}[\theta]}{\theta} & \frac{\theta\left(x^2+y^2-2z\theta\right)-\left(x^2+y^2\right)\text{Sin}[\theta]}{\theta^2} & 0 \\ 0 & \text{Cos}[\theta] & -\text{Sin}[\theta] & \frac{x(-1+\text{Cos}[\theta])+y\text{Sin}[\theta]}{\theta} & 0 \\ 0 & \text{Sin}[\theta] & \text{Cos}[\theta] & \frac{y-y\text{Cos}[\theta]+x\text{Sin}[\theta]}{\theta} & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & e^{\theta} \end{pmatrix}$$

$$\begin{pmatrix} 1 & -x & y & -2z & 0 \\ 0 & 1 & 0 & y & 0 \\ 0 & 0 & 1 & x & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Mexp[v_]:=
FullSimplify[MatrixExp[aE1 + bE2 + cE3 + dE4]]/.
$\{a \to v[[1]], b \to v[[2]], c \to v[[3]], d \to v[[4]]\}$;

$$A = \left\{ \frac{\frac{1}{2}\theta x(\text{Cos}[\theta]+1)}{\text{Sin}[\theta]} - \frac{1}{2}y\theta, \frac{\frac{1}{2}\theta y(\text{Cos}[\theta]+1)}{\text{Sin}[\theta]} + \frac{1}{2}x\theta, \frac{4z(\text{Cos}[\theta]-1)+\left(x^2+y^2\right)(\text{Sin}[\theta]-\theta)}{4(\text{Cos}[\theta]-1)}, \theta \right\};$$

MatrixForm[FullSimplify[Mexp[A]]]

$$\begin{pmatrix} 1 & -x\text{Cos}[\theta]+y\text{Sin}[\theta] & y\text{Cos}[\theta]+x\text{Sin}[\theta] & -2z & 0 \\ 0 & \text{Cos}[\theta] & -\text{Sin}[\theta] & y & 0 \\ 0 & \text{Sin}[\theta] & \text{Cos}[\theta] & x & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & e^{\theta} \end{pmatrix}$$

Limit[$A, \theta \to 0$]
MatrixForm[MatrixExp[%.Base]]
MatrixForm[$m[x,y,z,0]$]

$\{x, y, z, 0\}$

$$\begin{pmatrix} 1 & -x & y & -2z & 0 \\ 0 & 1 & 0 & y & 0 \\ 0 & 0 & 1 & x & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -x & y & -2z & 0 \\ 0 & 1 & 0 & y & 0 \\ 0 & 0 & 1 & x & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Limit[$A, \theta \to \pi$]
MatrixForm[MatrixExp[%.Base]]
MatrixForm[$m[x,y,z,\pi]$]

$\left\{-\frac{\pi y}{2}, \frac{\pi x}{2}, \frac{1}{8}\left(\pi\left(x^2+y^2\right)+8z\right), \pi\right\}$

$$\begin{pmatrix} 1 & x & -y & -2z & 0 \\ 0 & -1 & 0 & y & 0 \\ 0 & 0 & -1 & x & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & e^{\pi} \end{pmatrix}$$

$$\begin{pmatrix} 1 & x & -y & -2z & 0 \\ 0 & -1 & 0 & y & 0 \\ 0 & 0 & -1 & x & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & e^{\pi} \end{pmatrix}$$

Limit[$A, \theta \to -\pi$]
MatrixForm[MatrixExp[%.Base]]
MatrixForm[$m[x,y,z,-\pi]$]

$\left\{\frac{\pi y}{2}, -\frac{\pi x}{2}, \frac{1}{8}\left(-\pi\left(x^2+y^2\right)+8z\right), -\pi\right\}$

$$\begin{pmatrix} 1 & x & -y & -2z & 0 \\ 0 & -1 & 0 & y & 0 \\ 0 & 0 & -1 & x & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & e^{-\pi} \end{pmatrix}$$

$$\begin{pmatrix} 1 & x & -y & -2z & 0 \\ 0 & -1 & 0 & y & 0 \\ 0 & 0 & -1 & x & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & e^{-\pi} \end{pmatrix}$$

Note that beyond [-π,π] things can go wrong:

Limit[$A, \theta \to 2\pi$]

$\{\text{DirectedInfinity}[x], \text{DirectedInfinity}[y], \text{DirectedInfinity}\left[x^2+y^2\right], 2\pi\}$

For Universal Cover:

minv[Simplify[MatrixExp[$M[x,y,z,2n\pi]$]], $n \in$ Integers]]

$\left\{0, 0, -\frac{x^2+y^2-4n\pi z}{4n\pi}, \text{Log}\left[e^{2n\pi}\right]\right\}$

minv $\left[\text{Simplify}\left[\text{MatrixExp}\left[M\left[x, y, \frac{x^2+y^2}{4n\pi}, 2n\pi\right]\right]\right.\right.$,
$n \in$ Integers]]

$\left\{0, 0, 0, \text{Log}\left[e^{2n\pi}\right]\right\}$

Automorphisms

Necessary Conditions

```
e1 = {1, 0, 0, 0}; e2 = {0, 1, 0, 0}; e3 = {0, 0, 1, 0};
e4 = {0, 0, 0, 1};
ψ = {{x, y, a3, u}, {b1, b2, b3, v}, {c1, c2, c3, w},
{d1, d2, d3, d4}};
cd1L = ψ.Minv[cc[e1.Base, e2.Base]];
cd1R = Minv[cc[ψ.e1.Base, ψ.e2.Base]];
cd2L = ψ.Minv[cc[e1.Base, e3.Base]];
cd2R = Minv[cc[ψ.e1.Base, ψ.e3.Base]];
cd3L = ψ.Minv[cc[e1.Base, e4.Base]];
cd3R = Minv[cc[ψ.e1.Base, ψ.e4.Base]];
cd4L = ψ.Minv[cc[e2.Base, e3.Base]];
cd4R = Minv[cc[ψ.e2.Base, ψ.e3.Base]];
cd5L = −ψ.Minv[cc[e2.Base, e4.Base]];
cd5R = −Minv[cc[ψ.e2.Base, ψ.e4.Base]];
cd6L = ψ.Minv[cc[e3.Base, e4.Base]];
cd6R = Minv[cc[ψ.e3.Base, ψ.e4.Base]];
cdsL = {cd1L, cd2L, cd3L, cd4L, cd5L, cd6L};
cdsR = {cd1R, cd2R, cd3R, cd4R, cd5R, cd6R};
MatrixForm[cdsL[[{1, 2, 3, 4, 5, 6}, 4]]]==
MatrixForm[cdsR[[{1, 2, 3, 4, 5, 6}, 4]]]
```

$$\begin{pmatrix} d_3 \\ 0 \\ d_2 \\ 0 \\ d_1 \\ 0 \end{pmatrix} == \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

```
sub = {d1 → 0, d2 → 0, d3 → 0};
MatrixForm[cdsL/.sub]
MatrixForm[cdsR/.sub]//Simplify
```

$$\begin{pmatrix} a_3 & b_3 & c_3 & 0 \\ 0 & 0 & 0 & 0 \\ y & b_2 & c_2 & 0 \\ 0 & 0 & 0 & 0 \\ x & b_1 & c_1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & -yb_1 + xb_2 & 0 \\ 0 & 0 & -a_3b_1 + xb_3 & 0 \\ -b_1d_4 & xd_4 & vx - ub_1 & 0 \\ 0 & 0 & -a_3b_2 + yb_3 & 0 \\ b_2d_4 & -yd_4 & -vy + ub_2 & 0 \\ -b_3d_4 & a_3d_4 & va_3 - ub_3 & 0 \end{pmatrix}$$

```
sub2 =
Thread[Flatten[cdsL/.sub] → Flatten[cdsR/.sub]];
ψ/.sub/.sub2//Simplify//MatrixForm
```

$$\begin{pmatrix} b_2d_4 & -b_1d_4 & 0 & u \\ -yd_4 & xd_4 & 0 & v \\ -vy + ub_2 & vx - ub_1 & -yb_1 + xb_2 & w \\ 0 & 0 & 0 & d_4 \end{pmatrix}$$

Verification

```
ψa = {{x, y, 0, u}, {−ky, kx, 0, v},
{kux − vy, kuy + xv, k (x^2 + y^2), w}, {0, 0, 0, k}};
ψa.Minv[cc[M[x, y, z, θ], M[x′, y′, z′, θ′]]]−
Minv[cc[(ψa.{x, y, z, θ}).Base,
(ψa.{x′, y′, z′, θ′}).Base]];
Simplify[%, k^2 == 1]
```

$\{0, 0, 0, 0\}$

C.2 Adjoint Representation

Basic setup

```
ClearAll["Global′*"]
M[x_, y_, z_, θ_]:=
{{0, −x, y, −2z}, {0, 0, −θ, y}, {0, θ, 0, x},
{0, 0, 0, 0}};
MatrixForm[M[x, y, z, θ]]
E1 = M[1, 0, 0, 0]; E2 = M[0, 1, 0, 0]; E3 = M[0, 0, 1, 0];
E4 = M[0, 0, 0, 1]; Base = {E1, E2, E3, E4};
CC[M_, N_]:=M.N − N.M;
Minv[MM_]:={MM[[3]][[4]], MM[[2]][[4]],
1/−2 MM[[1]][[4]], MM[[3]][[2]]};
Minv[M[x, y, z, θ]]
```

$$\begin{pmatrix} 0 & -x & y & -2z \\ 0 & 0 & -\theta & y \\ 0 & \theta & 0 & x \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\{x, y, z, \theta\}$

```
m[x_, y_, z_, θ_]:=
{{1, −xCos[θ] + ySin[θ], xSin[θ] + yCos[θ], −2z},
{0, Cos[θ], −Sin[θ], y}, {0, Sin[θ], Cos[θ], x},
{0, 0, 0, 1}};
MatrixForm[m[x, y, z, θ]]
```

$$\begin{pmatrix} 1 & -x\mathrm{Cos}[\theta] + y\mathrm{Sin}[\theta] & y\mathrm{Cos}[\theta] + x\mathrm{Sin}[\theta] & -2z \\ 0 & \mathrm{Cos}[\theta] & -\mathrm{Sin}[\theta] & y \\ 0 & \mathrm{Sin}[\theta] & \mathrm{Cos}[\theta] & x \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Calculation of $\mathrm{Ad}\, g$

```
a1 = FullSimplify[
Minv[m[x, y, z, θ].E1.Inverse[m[x, y, z, θ]]]];
a2 = FullSimplify[
Minv[m[x, y, z, θ].E2.Inverse[m[x, y, z, θ]]]];
a3 = FullSimplify[
Minv[m[x, y, z, θ].E3.Inverse[m[x, y, z, θ]]]];
a4 = FullSimplify[
Minv[m[x, y, z, θ].E4.Inverse[m[x, y, z, θ]]]];
Ad = Transpose[{a1, a2, a3, a4}];
MatrixForm[Ad]
```

$$\begin{pmatrix} \mathrm{Cos}[\theta] & \mathrm{Sin}[\theta] & 0 & -y \\ -\mathrm{Sin}[\theta] & \mathrm{Cos}[\theta] & 0 & x \\ -y\mathrm{Cos}[\theta] - x\mathrm{Sin}[\theta] & x\mathrm{Cos}[\theta] - y\mathrm{Sin}[\theta] & 1 & \frac{1}{2}\left(x^2 + y^2\right) \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

```
MatrixForm[
FullSimplify[FullSimplify[(Ad.{xa, ya, za, θa}).Base]−
FullSimplify[m[x, y, z, θ].M[xa, ya, za, θa].
Inverse[m[x, y, z, θ]]]]]
```

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Det[Ad]

$\text{Cos}[\theta]^2 + \text{Sin}[\theta]^2$

Calculation of ad A

```
b1 = FullSimplify[Minv[CC[M[x, y, z, θ], E1]]];
b2 = FullSimplify[Minv[CC[M[x, y, z, θ], E2]]];
b3 = FullSimplify[Minv[CC[M[x, y, z, θ], E3]]];
b4 = FullSimplify[Minv[CC[M[x, y, z, θ], E4]]];
ad = Transpose[{b1, b2, b3, b4}];
MatrixForm[ad]
```

$$\begin{pmatrix} 0 & \theta & 0 & -y \\ -\theta & 0 & 0 & x \\ -y & x & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

```
MatrixForm[
FullSimplify[FullSimplify[(ad.{xa, ya, za, θa}).Base]−
FullSimplify[CC[M[x, y, z, θ], M[xa, ya, za, θa]]]]]
```

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

```
Simplify[Det[ad − λIdentityMatrix[4]]]
```

$\lambda^2 (\theta^2 + \lambda^2)$

C.3 Graphs of Adjoint Orbits

```
ClearAll["Global'*"]
```

Paraboloids

```
za = Table[i, {i, −1, 0.5, 1.5/4}];
θa = 1.3;
Orb1 = {};
For[j = 1, j < Length[za] + 1, j++,
Orb1 = Append[Orb1,
ParametricPlot3D[
{rCos[θ], rSin[θ], r^2/(2θa) + za[[j]]},
{θ, −π, π}, {r, 0, 1.5}, Mesh → 10]]]
```

Cylinders

```
ra = Table[i, {i, 0.4, 1.5, (1.5 − 0.4)/2}];
Orb2 = {};
For[j = 1, j < Length[ra] + 1, j++,
Orb2 = Append[Orb2,
ParametricPlot3D[
{ra[[j]]Cos[θ], ra[[j]]Sin[θ], z},
{θ, −π, π}, {z, −1, 1}, Mesh → 10]]];
Orb2 = Append[Orb2,
ListPointPlot3D[
Table[{0, 0, i}, {i, −1, 1, 2/6}],
PlotStyle → Directive[Black,
PointSize[Large]]]];
```

Sliding window

```
θa = {−10, −1, 1, 10};
Orbs = {};
For[j = 1, j < Length[θa] + 1, j++,
Orbs = Append[Orbs,
ParametricPlot3D[
{rCos[θ], rSin[θ], r^2/(2θa[[j]])}, {θ, −π, π},
{r, 0, 1.5}, Mesh → 10]]];
Orbs = Append[Orbs,
ParametricPlot3D[{Cos[θ], Sin[θ], z},
{θ, −π, π}, {z, −1, 1}, Mesh → 10]];
```

Stereographic plots

```
pr = 1.5;
Viewv = {0, 0, 1};
View1 = {3Pi/2.8, Pi/3, 2};
View2 = {.3Pi.8, Pi/3, 2};
Views = {6Pi/2.8, 0.3Pi, 2};
Opts = {ViewVertical → Viewv, Axes → True,
BoxRatios → {1, 1, 1},
PlotRange → {{−pr, pr}, {−pr, pr}, {−pr, pr}},
Boxed → False, ImageSize → Medium,
AxesLabel → {"E1", "E2", "E3"},
LabelStyle → Directive[Medium],
AxesEdge → {{1, −1}, {1, −1}, {1, −1}},
FaceGrids → {{−1, 0, 0}, {0, −1, 0}, {0, 0, −1}},
TicksStyle → Directive[Medium]};
Optss = {ViewVertical → Viewv, Axes → False,
BoxRatios → {1, 1, 1}, Boxed → False,
PlotRange → {{−pr, pr}, {−pr, pr}, {−pr, pr}},
ImageSize → Small, ViewPoint → Views};
Opts1 = Append[Opts, ViewPoint → View1];
Opts2 = Append[Opts, ViewPoint → View2];
Show[Orb1, Opts1]
Show[Orb2, Opts2]
{Show[Orbs[[1]], Optss],
Show[Orbs[[2]], Optss],
Show[Orbs[[5]], Optss],
Show[Orbs[[3]], Optss],
Show[Orbs[[4]], Optss]}
```

Graphics shown in figures 1.1 and 1.2.

C.4 Coadjoint Calculations

Basic setup

ClearAll["Global'*"]
M[x_, y_, z_, θ_]:=
$\{\{0,-x,y,-2z\},\{0,0,-\theta,y\},\{0,\theta,0,x\},$
$\{0,0,0,0\}\}$;
E1 = M[1, 0, 0, 0]; E2 = M[0, 1, 0, 0];
E3 = M[0, 0, 1, 0]; E4 = M[0, 0, 0, 1];
Base = {E1, E2, E3, E4};
iBase = {{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0},
{0, 0, 0, 1}};
CC[M_, N_]:=$M.N - N.M$;
Minv[MM_]:=
$\left\{\text{MM[[3,4]]}, \text{MM[[2,4]]}, \frac{1}{-2}\text{MM[[1,4]]},\right.$
MM[[3, 2]]};
Minv[M[x, y, z, θ]]
$\{x, y, z, \theta\}$
m[x_, y_, z_, θ_]:=
$\{\{1, -x\text{Cos}[\theta] + y\text{Sin}[\theta], x\text{Sin}[\theta] + y\text{Cos}[\theta],$
$-2z\}, \{0, \text{Cos}[\theta], -\text{Sin}[\theta], y\},$
$\{0, \text{Sin}[\theta], \text{Cos}[\theta], x\}, \{0, 0, 0, 1\}\}$;

Calculation of $\mathrm{Ad}^* g$

Recall:
$\mathrm{Ad}^*g.\mu(A) = \mu\left(\mathrm{Ad}g^{-1}.A\right)$
$\mathrm{Ad}^*g = \left(E_j{}^*\left(\mathrm{Ad}g^{-1}.E_i\right)\right)_{ij}$

gg = m [x_g, y_g, z_g, θ_g] ;
Ads = IdentityMatrix[4];
For[i = 1, i < 5, i++,
For[j = 1, j < 5, j++,
Ads[[i, j]] =
FullSimplify[
iBase[[j]].
(Minv[Inverse[gg].Base[[i]].gg])]
]
];
MatrixForm[Ads]

$$\begin{pmatrix} \text{Cos}[\theta_g] & \text{Sin}[\theta_g] & y_g & 0 \\ -\text{Sin}[\theta_g] & \text{Cos}[\theta_g] & -x_g & 0 \\ 0 & 0 & 1 & 0 \\ \text{Sin}[\theta_g]\,x_g + \text{Cos}[\theta_g]\,y_g & -\text{Cos}[\theta_g]\,x_g + \text{Sin}[\theta_g]\,y_g & \frac{1}{2}\left(x_g^2 + y_g^2\right) & 1 \end{pmatrix}$$

Collect [Expand [Ads. {x_c, y_c, z_c, θ_c}] ,
{Cos [θ_g] , Sin [θ_g] , z_c}]
$\{\text{Cos}[\theta_g]\,x_c + \text{Sin}[\theta_g]\,y_c + y_g z_c,$
$-\text{Sin}[\theta_g]\,x_c + \text{Cos}[\theta_g]\,y_c - x_g z_c, z_c,$
$\text{Cos}[\theta_g]\,(-x_g y_c + x_c y_g)$
$+\text{Sin}[\theta_g]\,(x_c x_g + y_c y_g) + \left(\frac{x_g^2}{2} + \frac{y_g^2}{2}\right) z_c + \theta_c\}$

Dual equivalence

ψ = {{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 0, −1},
{0, 0, −1, 0}};
Adg = {{Cos [θ_g] , Sin [θ_g] , 0, $-y_g$} ,
{−Sin [θ_g] , Cos [θ_g] , 0, x_g} ,
{$-y_g$Cos [θ_g] − x_gSin [θ_g] , x_gCos [θ_g] − y_gSin [θ_g] ,
1, $\frac{1}{2}\left(x_g{}^2 + y_g{}^2\right)$} , {0, 0, 0, 1}} ;
Ads.ψ − ψ.Adg//Simplify//MatrixForm

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

C.5 Supporting Calculations for Local Classification

ClearAll["Global'*"]
$A = \{a_1, a_2, a_3, a_4\}$; $B = \{b_1, b_2, b_3, b_4\}$;
CC = $\{c_1, c_2, c_3, c_4\}$; $\tilde{A} = \{\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, \tilde{a}_4\}$;
$\tilde{B} = \left\{\tilde{b}_1, \tilde{b}_2, \tilde{b}_3, \tilde{b}_4\right\}$;
$\tilde{\text{CC}} = \{\tilde{c}_1, \tilde{c}_2, \tilde{c}_3, \tilde{c}_4\}$;

Type (1,1) - the invariant $\mathfrak{P}$

Motivation

x2y2 = FullSimplify[
{1, 0}.MatrixPower[
Inverse $\left[\left\{\left\{a_1 - \frac{a_4 b_1}{b_4}, a_2 - \frac{a_4 b_2}{b_4}\right\},\right.\right.$
$\left.\left.\left\{a_2 - \frac{a_4 b_2}{b_4}, -a_1 + \frac{a_4 b_1}{b_4}\right\}\right\}\right]$, 2] .{1, 0}] ;
cc = $\frac{-b_1}{b_4}\left(a_1 - \frac{a_4 b_1}{b_4}\right) + \frac{-b_2}{b_4}\left(a_2 - \frac{a_4 b_2}{b_4}\right) +$
$\left(a_3 - \frac{a_4 b_3}{b_4}\right)$;
Simplify [cc x2y2, {$b_4 \neq 0$}]
$\frac{b_4(-a_1 b_1 - a_2 b_2 + a_3 b_4) + a_4\left(b_1^2 + b_2^2 - b_3 b_4\right)}{a_4^2\left(b_1^2 + b_2^2\right) - 2a_4(a_1 b_1 + a_2 b_2) b_4 + \left(a_1^2 + a_2^2\right) b_4^2}$

Parametrisation invariant

$\mathfrak{P}$ =
$(b_4(-a_1 b_1 - a_2 b_2 + a_3 b_4) +$
$a_4\left(b_1^2 + b_2^2 - b_3 b_4\right))/$
$\left(a_4^2\left(b_1^2 + b_2^2\right) - 2a_4(a_1 b_1 + a_2 b_2) b_4 +\right.$
$\left.\left(a_1^2 + a_2^2\right) b_4^2\right)$;
Subs = {};
Subs =

Flatten $\left[\text{Append}\left[\text{Subs, Thread}\left[A \to \tilde{A} + \nu\tilde{B}\right]\right]\right]$;
Subs = Flatten $\left[\text{Append}\left[\text{Subs, Thread}\left[B \to \mu\tilde{B}\right]\right]\right]$;
Simplify $\left[\mathfrak{P}/.\text{Subs}, \left\{\tilde{b}_4 \neq 0, \mu \neq 0\right\}\right]$

$$\frac{\tilde{b}_4\left(-\tilde{a}_1\tilde{b}_1-\tilde{a}_2\tilde{b}_2+\tilde{a}_3\tilde{b}_4\right)+\tilde{a}_4\left(\tilde{b}_1^2+\tilde{b}_2^2-\tilde{b}_3\tilde{b}_4\right)}{\tilde{a}_4^2\left(\tilde{b}_1^2+\tilde{b}_2^2\right)-2\tilde{a}_4\left(\tilde{a}_1\tilde{b}_1+\tilde{a}_2\tilde{b}_2\right)\tilde{b}_4+\left(\tilde{a}_1^2+\tilde{a}_2^2\right)\tilde{b}_4^2}$$

Automorphism invariant

$\psi = \{\{x, y, 0, u\}, \{-ky, kx, 0, v\},$
$\{kux - vy, kuy + xv, k\left(x^2 + y^2\right), w\},$
$\{0, 0, 0, k\}\}$;
Subs = {};
Subs =
Flatten[Append[Subs, Thread[$A \to \psi.A$]]];
Subs =
Flatten[Append[Subs, Thread[$B \to \psi.B$]]];
$\psi\mathfrak{P}$ **= Simplify** $[\mathfrak{P}/.\text{Subs}, \{b_4 \neq 0, k^2 == 1\}]$

$$\frac{b_4(-a_1b_1-a_2b_2+a_3b_4)+a_4\left(b_1^2+b_2^2-b_3b_4\right)}{k\left(a_4^2\left(b_1^2+b_2^2\right)-2a_4(a_1b_1+a_2b_2)b_4+\left(a_1^2+a_2^2\right)b_4^2\right)}$$

Simplify $[\psi\mathfrak{P}^2 - \mathfrak{P}^2, k^2 == 1]$
0

Type (2,1) - invariants $\mathfrak{T}$ and $\mathfrak{S}$

Definitions

$\mathfrak{T} =$
AS [Det [{{a_1, a_2, a_4}, {b_1, b_2, b_4},
{c_1, c_2, c_4}}]];
$\mathfrak{S} =$
$(c_4\,(-\,(a_2b_1 - a_1b_2)\,(-b_2c_1 + b_1c_2) +$
$\left(a_3\left(b_1^2 + b_2^2\right) - (a_1b_1 + a_2b_2)\,b_3\right)c_4) +$
$a_4\,((b_2c_1 - b_1c_2)^2 +$
$\left(b_3\,(b_1c_1 + b_2c_2) - \left(b_1^2 + b_2^2\right)c_3\right)c_4))/$
(Det [{{a_1, a_2, a_4}, {$b_1, b_2, 0$},
{c_1, c_2, c_4}}])^2;
Note that $\mathfrak{S}$ appears here without the absolute value.
$\mathfrak{S}$/.
({$A \to \{0, a_2, a_3, 0\}$ //Thread,
$B \to \{1, 0, 0, 0\}$//Thread,
CC $\to \{0, 0, 0, 1\}$//Thread}//Flatten)
$\frac{a_3}{a_2^2}$

Parametrisation invariant

Subs = {};
Subs =
Flatten[Append[Subs,
Thread $\left[A \to \tilde{A} + \eta_1\tilde{B} + \eta_2\tilde{\text{CC}}\right]\big]\big]$;
Subs =
Flatten[Append[Subs,
Thread $\left[B \to \nu_1\tilde{B} + \nu_2\tilde{\text{CC}}\right]\big]\big]$;
Subs =
Flatten[Append[Subs,
Thread $\left[\text{CC} \to \mu_1\tilde{B} + \mu_2\tilde{\text{CC}}\right]\big]\big]$;
FullSimplify[$\mathfrak{T}$/.Subs]
$\text{AS}\Big[-(\mu_2\nu_1 - \mu_1\nu_2)\left(\tilde{a}_4\left(\tilde{b}_2\tilde{c}_1 - \tilde{b}_1\tilde{c}_2\right)\right.$
$\left.+\,\tilde{a}_2\left(-\tilde{b}_4\tilde{c}_1 + \tilde{b}_1\tilde{c}_4\right) + \tilde{a}_1\left(\tilde{b}_4\tilde{c}_2 - \tilde{b}_2\tilde{c}_4\right)\right)\Big]$
BB = {$b_1, b_2, b_3, 0$}; $\tilde{\text{BB}}$ = $\left\{\tilde{b}_1, \tilde{b}_2, \tilde{b}_3, 0\right\}$;
Subs1 = {};
Subs1 =
Flatten $\left[\text{Append}\left[\text{Subs1, Thread}\left[A \to \tilde{A}\right]\right]\right]$;
Subs1 =
Flatten $\left[\text{Append}\left[\text{Subs1, Thread}\left[\text{BB} \to \tilde{\text{BB}}\right]\right]\right]$;
Subs1 =
Flatten $\left[\text{Append}\left[\text{Subs1, Thread}\left[\text{CC} \to \tilde{\text{CC}}\right]\right]\right]$;
Subs2 = {};
Subs2 =
Flatten[Append[Subs2,
Thread $\left[A \to \tilde{A} + \eta_1\tilde{\text{BB}} + \eta_2\tilde{\text{CC}}\right]\big]\big]$;
Subs2 =
Flatten $\left[\text{Append}\left[\text{Subs2, Thread}\left[\text{BB} \to \nu_1\tilde{\text{BB}}\right]\right]\right]$;
Subs2 =
Flatten[Append[Subs2,
Thread $\left[\text{CC} \to \mu_1\tilde{\text{BB}} + \mu_2\tilde{\text{CC}}\right]\big]\big]$;
Simplify[($\mathfrak{S}$/.Subs1) − ($\mathfrak{S}$/.Subs2),
{$\nu_1 \neq 0, \mu_2 \neq 0, \tilde{c}_4 \neq 0$}]
0

Automorphism invariant

$\psi = \{\{x, y, 0, u\}, \{-ky, kx, 0, v\},$
$\{kux - vy, kuy + xv, k\left(x^2 + y^2\right), w\},$
$\{0, 0, 0, k\}\}$;
Subs = {};
Subs =
Flatten[Append[Subs, Thread[$A \to \psi.A$]]];
Subs =
Flatten[Append[Subs, Thread[$B \to \psi.B$]]];
Subs =
Flatten[Append[Subs, Thread[CC $\to \psi$.CC]]];
$\psi\mathfrak{T}$ **= FullSimplify[$\mathfrak{T}$/.Subs]**
$\text{AS}\Big[-k^2\left(x^2 + y^2\right)\left(a_4\,(b_2c_1 - b_1c_2)\right.$
$\left.+\,a_2\,(-b_4c_1 + b_1c_4) + a_1\,(b_4c_2 - b_2c_4)\right)\Big]$
FullSimplify [($\mathfrak{S}$/.Subs/.$b_4 \to 0$)/$\mathfrak{S}$,
{$k^2 == 1$}]
$\frac{1}{k}$
Collect[Reduce [{($\mathfrak{S}$/.Subs/.$b_4 \to 0$)^2 − $\mathfrak{S}^2$ == Tv,
$k^2 == 1, a_4b_2c_1 - a_4b_1c_2 + a_2b_1c_4 - a_1b_2c_4 \neq 0$,

$x^2+y^2\neq 0\}]\,,x^2,y^2]$

$\left(\text{Tv} == 0\&\&k == -1\&\&x^2\left(a_4b_2c_1 - a_4b_1c_2 + a_2b_1c_4 - a_1b_2c_4\right) + y^2\left(a_4b_2c_1 - a_4b_1c_2 + a_2b_1c_4 - a_1b_2c_4\right) \neq 0\right)\|$
$\left(\text{Tv} == 0\&\&k == 1\&\&x^2\left(a_4b_2c_1 - a_4b_1c_2 + a_2b_1c_4 - a_1b_2c_4\right) + y^2\left(a_4b_2c_1 - a_4b_1c_2 + a_2b_1c_4 - a_1b_2c_4\right) \neq 0\right)$

Type (3,0) - the invariant $\mathfrak{R}$

Definition

$\mathfrak{R} =$
$\text{AS}\left[\text{Det}\left[\{\{a_1,a_2,a_4\},\{b_1,b_2,b_4\},\{c_1,c_2,c_4\}\}\right]\right];$

Parametrisation invariant

Subs = {};
Subs = Flatten[Append[Subs,
 Thread $\left[A \to \eta_1\tilde{A} + \eta_2\tilde{B} + \eta_3\tilde{\text{CC}}\right]$]];
Subs = Flatten[Append[Subs,
 Thread $\left[B \to \nu_1\tilde{A} + \nu_2\tilde{B} + \nu_3\tilde{\text{CC}}\right]$]];
Subs = Flatten[Append[Subs,
 Thread $\left[\text{CC} \to \mu_1\tilde{A} + \mu_2\tilde{B} + \mu_3\tilde{\text{CC}}\right]$]];
FullSimplify[$\mathfrak{R}$/.Subs]

$\text{AS}\left[(\eta_3(\mu_2\nu_1 - \mu_1\nu_2) + \eta_2(-\mu_3\nu_1 + \mu_1\nu_3) + \eta_1(\mu_3\nu_2 - \mu_2\nu_3))\left(\tilde{a}_4\left(-\tilde{b}_2\tilde{c}_1 + \tilde{b}_1\tilde{c}_2\right) + \tilde{a}_2\left(\tilde{b}_4\tilde{c}_1 - \tilde{b}_1\tilde{c}_4\right) + \tilde{a}_1\left(-\tilde{b}_4\tilde{c}_2 + \tilde{b}_2\tilde{c}_4\right)\right)\right]$

Automorphism invariant

$\psi = \{\{x,y,0,u\},\{-ky,kx,0,v\},\left\{kux - vy, kuy + xv, k\left(x^2+y^2\right), w\right\},\{0,0,0,k\}\};$
Subs = {};
Subs = Flatten[Append[Subs, Thread[$A \to \psi.A$]]];
Subs = Flatten[Append[Subs, Thread[$B \to \psi.B$]]];
Subs = Flatten[Append[Subs, Thread[CC $\to \psi$.CC]]];
$\psi\mathfrak{R}$ = FullSimplify[$\mathfrak{R}$/.Subs]

$\text{AS}\left[-k^2\left(x^2+y^2\right)\left(a_4(b_2c_1 - b_1c_2) + a_2(-b_4c_1 + b_1c_4) + a_1(b_4c_2 - b_2c_4)\right)\right]$

Type (3,1) - some calculations

ClearAll["Global`*"]
cc[A_, B_]:=$A.B - B.A$;
M[x_, y_, z_, θ_]:=
$\{\{0,-x,y,-2z,0\},\{0,0,-\theta,y,0\},\{0,\theta,0,x,0\},\{0,0,0,0,0\},\{0,0,0,0,\theta\}\};$
E1 = M[1,0,0,0];
E2 = M[0,1,0,0];
E3 = M[0,0,1,0];
E4 = M[0,0,0,1];
Base = {E1, E2, E3, E4};
Minv[MM_]:={MM[[3]][[4]], MM[[2]][[4]], $\frac{-1}{2}$MM[[1]][[4]], MM[[3]][[2]]} ;
$A = \{a_1,a_2,a_3,0\}$;
$B = \{b_1,b_2,b_3,0\}$;
CC = $\{c_1,c_2,c_3,1\}$;
AC = Minv[cc[A.Base, CC.Base]];
AC.$\{E_1,E_2,E_3,E_4\}$

$-a_2E_1 + a_1E_2 + \frac{1}{2}(-2a_2c_1 + 2a_1c_2)E_3$

AAC = Minv[cc[A.Base, AC.Base]];
AAC.$\{E_1,E_2,E_3,E_4\}$ //Simplify

$\left(a_1^2+a_2^2\right)E_3$

R1 = {A, AC, AAC, CC};
R1//MatrixForm

$$\begin{pmatrix} a_1 & a_2 & a_3 & 0\\ -a_2 & a_1 & \frac{1}{2}(-2a_2c_1+2a_1c_2) & 0\\ 0 & 0 & \frac{1}{2}\left(2a_1^2+2a_2^2\right) & 0\\ c_1 & c_2 & c_3 & 1\end{pmatrix}$$

Det[R1]//FullSimplify

$\left(a_1^2+a_2^2\right)^2$

C.6 Controllability on $\widetilde{\mathsf{H}}_3^\diamond$

Basic setup

ClearAll["Global`*"]
cc[A_, B_]:=$A.B - B.A$;
m[x_, y_, z_, θ_]:=
$\{\{1,-x\text{Cos}[\theta]+y\text{Sin}[\theta], x\text{Sin}[\theta]+y\text{Cos}[\theta], -2z,0\},\{0,\text{Cos}[\theta],-\text{Sin}[\theta],y,0\},\{0,\text{Sin}[\theta],\text{Cos}[\theta],x,0\},\{0,0,0,1,0\},\{0,0,0,0,\text{Exp}[\theta]\}\};$
M[x_, y_, z_, θ_]:=
$\{\{0,-x,y,-2z,0\},\{0,0,-\theta,y,0\},\{0,\theta,0,x,0\},\{0,0,0,0,0\},\{0,0,0,0,\theta\}\};$
E1 = M[1,0,0,0]; E2 = M[0,1,0,0];
E3 = M[0,0,1,0]; E4 = M[0,0,0,1];
Minv[A_]:= {A[[3,4]], A[[1,3]], $\frac{-1}{2}A$[[1,4]], A[[5,5]]}
Mv[vv_]:=M[vv[[1]], vv[[2]], vv[[3]], vv[[4]]];
Minv[$M[x,y,z,\theta]$]
Norm[Mv[Minv[$M[x,y,z,\theta]$]] − $M[x,y,z,\theta]$]

$\{x,y,z,\theta\}$
0

Ξ11[u_]:=E1 + αE3 + uE4;
Ξ11s[u_]:=E1 + uE4;
Ξ12[u_]:=αE4 + uE1;
Ξ23[u_, v_]:=E1 + uE3 + vE4;
Ξ24[u_, v_]:=αE4 + uE1 + vE2;
Ξ25[u_, v_]:=αE4 + uE1 + vE3;
Ξ33[u_, v_, w_]:=αE4 + uE1 + vE2 + wE3;

Investigation of controllability

For $\Sigma_{1,\alpha}^{(1,1)} : E_1 + \alpha E_3 + uE_4$

Assume $\alpha \neq 0$.

```
MatrixForm[
FullSimplify [MatrixExp [α4πΞ11 [1/(2α)]]]]
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & e^{2\pi} \end{pmatrix}$$

```
MatrixForm[
FullSimplify[MatrixExp[Ξ11[2π] − Ξ11[0]]]]
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & e^{2\pi} \end{pmatrix}$$

Hence controllable. Now case α=0.

```
F = MatrixExp[πΞ11s[s1]].MatrixExp[ πΞ11s[s2]].
MatrixExp[3/2 πΞ11s[−s3]].
MatrixExp[1/2πΞ11s[−s4]];
Pnt = {s1 → 1, s2 → 1, s3 → 1, s4 → 1};
MatrixForm[F/.Pnt//Simplify]
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

```
D1 = FullSimplify[D[F, s1]/.Pnt];
D2 = FullSimplify[D[F, s2]/.Pnt];
D3 = FullSimplify[D[F, s3]/.Pnt];
D4 = FullSimplify[D[F, s4]/.Pnt];
D1A = Minv[D1];
D2A = Minv[D2];
D3A = Minv[D3];
D4A = Minv[D4];
{Norm[Mv[D1A] − D1//FullSimplify],
Norm[Mv[D2A] − D2//FullSimplify],
Norm[Mv[D3A] − D3//FullSimplify],
Norm[Mv[D4A] − D4//FullSimplify]}
```

$\{0, 0, 0, 0\}$

```
(R = {D1A, D2A, D3A, D4A})//MatrixForm
```

$$\begin{pmatrix} \pi & 2 & \pi & \pi \\ \pi & -2 & \pi & \pi \\ 1+\frac{3\pi}{2} & -1 & \frac{1}{2}(-2-3\pi) & -\frac{3\pi}{2} \\ \frac{1}{2}(-2+\pi) & 1 & \frac{2-\pi}{2} & -\frac{\pi}{2} \end{pmatrix}$$

```
Det[R]
```

$-16\pi^2$

Hence locally controllable at identity and so controllable.

For $\Sigma_3^{(2,1)} : E_1 + u_1E_3 + u_2E_4$

```
MatrixForm [FullSimplify [MatrixExp [Ξ23 [1/(4π), 2π]]]]
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & e^{2\pi} \end{pmatrix}$$

```
MatrixForm[MatrixExp[Ξ23[0, 2π] − Ξ23[0, 0]]]
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & e^{2\pi} \end{pmatrix}$$

Hence controllable.

Parametric equations

```
LL = FullSimplify[D[m[x[t], y[t], z[t], θ[t]], t]];
RR = FullSimplify[m[x[t], y[t], z[t], θ[t]].
 M [v1[t], v2[t], v3[t], v4[t]]] ;
FullSimplify[
Solve[LL == RR, {x'[t], y'[t], z'[t], θ'[t]}]]
```

$$\Big\{\Big\{z'[t] \to \tfrac{1}{2}\Big(-(\mathrm{Sin}[\theta[t]]x[t] + \mathrm{Cos}[\theta[t]]y[t])v_1[t] + (\mathrm{Cos}[\theta[t]]x[t] - \mathrm{Sin}[\theta[t]]y[t])v_2[t] + 2v_3[t]\Big),$$
$$x'[t] \to \mathrm{Cos}[\theta[t]]v_1[t] + \mathrm{Sin}[\theta[t]]v_2[t],$$
$$y'[t] \to -\mathrm{Sin}[\theta[t]]v_1[t] + \mathrm{Cos}[\theta[t]]v_2[t],$$
$$\theta'[t] \to v_4[t]\Big\}\Big\}$$

C.7 Control Systems on $\mathsf{H}_3^{\diamond}(n)$

Basic setup

```
ClearAll["Global'*"]
cc[A_, B_]:=A.B − B.A;
m[x_, y_, z_, θ_]:=
{{1, −xCos[θ] + ySin[θ], xSin[θ] + yCos[θ],
−2z, 0}, {0, Cos[θ], −Sin[θ], y, 0},
{0, Sin[θ], Cos[θ], x, 0}, {0, 0, 0, 1, 0},
{0, 0, 0, 0, Exp [iθ/n]}} ;
M[x_, y_, z_, θ_]:=
{{0, −x, y, −2z, 0}, {0, 0, −θ, y, 0},
{0, θ, 0, x, 0}, {0, 0, 0, 0, 0},
{0, 0, 0, 0, iθ/n}} ;
E1 = M[1, 0, 0, 0]; E2 = M[0, 1, 0, 0];
E3 = M[0, 0, 1, 0]; E4 = M[0, 0, 0, 1];
Base = {E1, E2, E3, E4};
Minv[A_]:= {A[[3, 4]], A[[1, 3]], −1/2 A[[1, 4]],
n/i A[[5, 5]]}
Mv[vv_]:=M[vv[[1]], vv[[2]], vv[[3]], vv[[4]]]
```

Minv[$M[x, y, z, \theta]$]
Norm[Mv[Minv[$M[x, y, z, \theta]$]] − $M[x, y, z, \theta]$]

$\{x, y, z, \theta\}$

0

The group $d\mathrm{Aut}\mathsf{H}_3^\diamond(n)$

Necessary conditions

$\psi = \{\{a, b, 0, u\}, \{-kb, ka, 0, v\}, \{kua - vb, kub + av, k(a\hat{}2 + b\hat{}2), w\}, \{0, 0, 0, k\}\};$
$\omega = \{\{1, 0, 0, 0\}, \{0, 1, 0, 0\}, \{0, 0, 0, -1\}, \{0, 0, -1, 0\}\};$
$(\psi\omega =$ Simplify $[$Transpose$[\psi].\omega.\psi, k^2 == 1])$ // MatrixForm

$$\begin{pmatrix} a^2+b^2 & 0 & 0 & 0 \\ 0 & a^2+b^2 & 0 & 0 \\ 0 & 0 & 0 & -a^2-b^2 \\ 0 & 0 & -a^2-b^2 & u^2+v^2-2kw \end{pmatrix}$$

Simplify[$\{x, y, z, \theta\}.\psi\omega.\{x, y, z, \theta\}$]

$$\left(u^2+v^2-2kw\right)\theta^2 + a^2\left(x^2+y^2-2z\theta\right) + b^2\left(x^2+y^2-2z\theta\right)$$

Sufficient conditions

FullSimplify[MatrixExp[$(\psi.\{x, y, z, 0\})$.Base], $k^2 == 1$] //MatrixForm
FullSimplify[MatrixExp[$(\psi.\{0, 0, 0, \theta\})$.Base], $k^2 == 1$] //MatrixForm

$$\begin{pmatrix} 1 & -ax-by & k(-bx+ay) & -2\left(akux-bvx+bkuy+avy+\left(a^2+b^2\right)kz\right) & 0 \\ 0 & 1 & 0 & k(-bx+ay) & 0 \\ 0 & 0 & 1 & ax+by & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & \frac{v-v\mathrm{Cos}[k\theta]-u\mathrm{Sin}[k\theta]}{k} & \frac{u-u\mathrm{Cos}[k\theta]+v\mathrm{Sin}[k\theta]}{k} & \left(k\left(u^2+v^2\right)-2w\right)\theta-\left(u^2+v^2\right)\mathrm{Sin}[k\theta] & 0 \\ 0 & \mathrm{Cos}[k\theta] & -\mathrm{Sin}[k\theta] & \frac{u(-1+\mathrm{Cos}[k\theta])+v\mathrm{Sin}[k\theta]}{k} & 0 \\ 0 & \mathrm{Sin}[k\theta] & \mathrm{Cos}[k\theta] & \frac{v-v\mathrm{Cos}[k\theta]+u\mathrm{Sin}[k\theta]}{k} & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & e^{\frac{ik\theta}{n}} \end{pmatrix}$$

Classification calculations

$(\psi = \{\{x, y, 0, u\}, \{-ky, kx, 0, v\}, \{kux - vy, kuy + xv, k\left(x^2+y^2\right), \frac{1}{2}k\left(u^2+v^2\right)\}, \{0, 0, 0, k\}\})$ //MatrixForm

$$\begin{pmatrix} x & y & 0 & u \\ -ky & kx & 0 & v \\ kux-vy & vx+kuy & k\left(x^2+y^2\right) & \frac{1}{2}k\left(u^2+v^2\right) \\ 0 & 0 & 0 & k \end{pmatrix}$$

Type (1,1)

$\Gamma_{1,\alpha} \not\sim \Gamma_{1,\beta}$
MatrixForm [$\psi.\ \{1, 0, \alpha_1, 0\}$] ==
MatrixForm [$\{1, 0, \beta_1, 0\} + r_1\ \{0, 0, \beta_2, 1\}$]
MatrixForm [$\psi.\ \{0, 0, \alpha_2, 1\}$] ==
MatrixForm [$r_2\ \{0, 0, \beta_2, 1\}$]

$$\begin{pmatrix} x \\ -ky \\ kux-vy+k\left(x^2+y^2\right)\alpha_1 \\ 0 \end{pmatrix} == \begin{pmatrix} 1 \\ 0 \\ \beta_1+r_1\beta_2 \\ r_1 \end{pmatrix}$$

$$\begin{pmatrix} u \\ v \\ \frac{1}{2}k\left(u^2+v^2\right)+k\left(x^2+y^2\right)\alpha_2 \\ k \end{pmatrix} == \begin{pmatrix} 0 \\ 0 \\ r_2\beta_2 \\ r_2 \end{pmatrix}$$

$\Gamma_{2,\alpha,\sigma} \not\sim \Gamma_{2,\beta,\varsigma}$
MatrixForm[$\psi.\{0, 0, \sigma, \alpha\}$] ==
MatrixForm [$\{0, 0, \varsigma, \beta\} + r_1\{1, 0, 0, 0\}$]
MatrixForm[$\psi.\{1, 0, 0, 0\}$]==
MatrixForm [$r_2\{1, 0, 0, 0\}$]

$$\begin{pmatrix} u\alpha \\ v\alpha \\ \frac{1}{2}k\left(u^2+v^2\right)\alpha+k\left(x^2+y^2\right)\sigma \\ k\alpha \end{pmatrix} == \begin{pmatrix} r_1 \\ 0 \\ \varsigma \\ \beta \end{pmatrix}$$

$$\begin{pmatrix} x \\ -ky \\ kux-vy \\ 0 \end{pmatrix} == \begin{pmatrix} r_2 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Type (2,0)

$\Gamma_{1,\sigma} \not\sim \Gamma_{1,\varsigma}$
MatrixForm[$\psi.\{1, 0, 0, 0\}$] ==
MatrixForm [$r_1\{1, 0, 0, 0\} + r_2\{0, 0, \varsigma, 1\}$]
MatrixForm[$\psi.\{0, 0, \sigma, 1\}$]==
MatrixForm [$r_3\{1, 0, 0, 0\} + r_4\{0, 0, \varsigma, 1\}$]

$$\begin{pmatrix} x \\ -ky \\ kux-vy \\ 0 \end{pmatrix} == \begin{pmatrix} r_1 \\ 0 \\ \varsigma r_2 \\ r_2 \end{pmatrix}$$

$$\begin{pmatrix} u \\ v \\ \frac{1}{2}k\left(u^2+v^2\right)+k\left(x^2+y^2\right)\sigma \\ k \end{pmatrix} == \begin{pmatrix} r_3 \\ 0 \\ \varsigma r_4 \\ r_4 \end{pmatrix}$$

Type (2,1)

$\Gamma_{1,\alpha,\sigma} \not\sim \Gamma_{1,\beta,\varsigma}$
MatrixForm [$\psi.\ \{0, \alpha_1, \alpha_2, 0\}$] ==
MatrixForm [$\{0, \beta_1, \beta_2, 0\} + r_1\{1, 0, 0, 0\} + r_2\{0, 0, \varsigma, 1\}$]
MatrixForm[$\psi.\{1, 0, 0, 0\}$] ==
MatrixForm [$r_3\{1, 0, 0, 0\} + r_4\{0, 0, \varsigma, 1\}$]

MatrixForm[ψ.{0, 0, σ, 1}] ==
MatrixForm [r_5{1, 0, 0, 0} + r_6{0, 0, ς, 1}]

$$\begin{pmatrix} y\alpha_1 \\ kx\alpha_1 \\ (vx+kuy)\alpha_1 + k\left(x^2+y^2\right)\alpha_2 \\ 0 \end{pmatrix} == \begin{pmatrix} r_1 \\ \beta_1 \\ \varsigma r_2 + \beta_2 \\ r_2 \end{pmatrix}$$

$$\begin{pmatrix} x \\ -ky \\ kux - vy \\ 0 \end{pmatrix} == \begin{pmatrix} r_3 \\ 0 \\ \varsigma r_4 \\ r_4 \end{pmatrix}$$

$$\begin{pmatrix} u \\ v \\ \frac{1}{2}k\left(u^2+v^2\right) + k\left(x^2+y^2\right)\sigma \\ k \end{pmatrix} == \begin{pmatrix} r_5 \\ 0 \\ \varsigma r_6 \\ r_6 \end{pmatrix}$$

$\Gamma_{1,\alpha,\sigma} \nsim \Gamma_{2,\beta}, \Gamma_3$
MatrixForm[ψ.{1, 0, 0, 0}] ==
MatrixForm [r_1{1, 0, 0, 0} + r_2{0, 0, 0, 1}]
MatrixForm[ψ.{0, 0, σ, 1}] ==
MatrixForm [r_3{1, 0, 0, 0} + r_4{0, 0, 0, 1}]

$$\begin{pmatrix} x \\ -ky \\ kux - vy \\ 0 \end{pmatrix} == \begin{pmatrix} r_1 \\ 0 \\ 0 \\ r_2 \end{pmatrix}$$

$$\begin{pmatrix} u \\ v \\ \frac{1}{2}k\left(u^2+v^2\right) + k\left(x^2+y^2\right)\sigma \\ k \end{pmatrix} == \begin{pmatrix} r_3 \\ 0 \\ 0 \\ r_4 \end{pmatrix}$$

$\Gamma_{2,\alpha} \nsim \Gamma_{2,\beta}$
MatrixForm[ψ.{0, 1, α, 0}] ==
MatrixForm [{0, 1, β, 0} + r_1{1, 0, 0, 0}+
r_2{0, 0, 0, 1}]
MatrixForm[ψ.{1, 0, 0, 0}] ==
MatrixForm [r_3{1, 0, 0, 0} + r_4{0, 0, 0, 1}]
MatrixForm[ψ.{0, 0, 0, 1}] ==
MatrixForm [r_5{1, 0, 0, 0} + r_6{0, 0, 0, 1}]

$$\begin{pmatrix} y \\ kx \\ vx + kuy + k\left(x^2+y^2\right)\alpha \\ 0 \end{pmatrix} == \begin{pmatrix} r_1 \\ 1 \\ \beta \\ r_2 \end{pmatrix}$$

$$\begin{pmatrix} x \\ -ky \\ kux - vy \\ 0 \end{pmatrix} == \begin{pmatrix} r_3 \\ 0 \\ 0 \\ r_4 \end{pmatrix}$$

$$\begin{pmatrix} u \\ v \\ \frac{1}{2}k\left(u^2+v^2\right) \\ k \end{pmatrix} == \begin{pmatrix} r_5 \\ 0 \\ 0 \\ r_6 \end{pmatrix}$$

$\Gamma_{5,\alpha,\sigma} \nsim \Gamma_{5,\beta,\varsigma}$
MatrixForm[ψ.{0, 0, σ, α}] ==
MatrixForm [{0, 0, ς, β} + r_1{1, 0, 0, 0}+
r_2{0, 1, 0, 0}]
MatrixForm[ψ.{1, 0, 0, 0}] ==
MatrixForm [r_3{1, 0, 0, 0} + r_4{0, 1, 0, 0}]
MatrixForm[ψ.{0, 1, 0, 0}] ==
MatrixForm [r_5{1, 0, 0, 0} + r_6{0, 1, 0, 0}]

$$\begin{pmatrix} u\alpha \\ v\alpha \\ \frac{1}{2}k\left(u^2+v^2\right)\alpha + k\left(x^2+y^2\right)\sigma \\ k\alpha \end{pmatrix} == \begin{pmatrix} r_1 \\ r_2 \\ \varsigma \\ \beta \end{pmatrix}$$

$$\begin{pmatrix} x \\ -ky \\ kux - vy \\ 0 \end{pmatrix} == \begin{pmatrix} r_3 \\ r_4 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} y \\ kx \\ vx + kuy \\ 0 \end{pmatrix} == \begin{pmatrix} r_5 \\ r_6 \\ 0 \\ 0 \end{pmatrix}$$

$\Gamma_{6,\alpha} \nsim \Gamma_{6,\beta}$
MatrixForm[ψ.{0, 0, 0, α}] ==
MatrixForm [{0, 0, 0, β} + r_1{1, 0, 0, 0}+
r_2{0, 0, 1, 0}]

$$\begin{pmatrix} u\alpha \\ v\alpha \\ \frac{1}{2}k\left(u^2+v^2\right)\alpha \\ k\alpha \end{pmatrix} == \begin{pmatrix} r_1 \\ 0 \\ r_2 \\ \beta \end{pmatrix}$$

Type (3,0)

$\Gamma_{1,\sigma} \nsim \Gamma_{1,\varsigma}$
MatrixForm[ψ.{1, 0, 0, 0}] ==
MatrixForm [r_1{1, 0, 0, 0} + r_2{0, 1, 0, 0}+
r_3{0, 0, ς, 1}]
MatrixForm[ψ.{0, 1, 0, 0}] ==
MatrixForm [r_4{1, 0, 0, 0} + r_5{0, 1, 0, 0}+
r_6{0, 0, ς, 1}]
MatrixForm[ψ.{0, 0, σ, 1}] ==
MatrixForm [r_7{1, 0, 0, 0} + r_8{0, 1, 0, 0}+
r_9{0, 0, ς, 1}]

$$\begin{pmatrix} x \\ -ky \\ kux - vy \\ 0 \end{pmatrix} == \begin{pmatrix} r_1 \\ r_2 \\ \varsigma r_3 \\ r_3 \end{pmatrix}$$

$$\begin{pmatrix} y \\ kx \\ vx + kuy \\ 0 \end{pmatrix} == \begin{pmatrix} r_4 \\ r_5 \\ \varsigma r_6 \\ r_6 \end{pmatrix}$$

$$\begin{pmatrix} u \\ v \\ \frac{1}{2}k\left(u^2+v^2\right) + k\left(x^2+y^2\right)\sigma \\ k \end{pmatrix} == \begin{pmatrix} r_7 \\ r_8 \\ \varsigma r_9 \\ r_9 \end{pmatrix}$$

Type (3,1)

$\Gamma_{2,\alpha,\sigma} \nsim \Gamma_{2,\beta,\varsigma}$
MatrixForm[(ψ.{0, 0, α, 0})[[{3, 4}]]] ==
MatrixForm[
({0, 0, β, 0} + r_5{1, 0, 0, 0} + r_6{0, 1, 0, 0}+

$r_1\{0,0,\varsigma,1\})$ [[{3,4}]]]
MatrixForm[(ψ.{1,0,0,0})[[{3,4}]]] ==
MatrixForm[
$(r_7\{1,0,0,0\}+r_8\{0,1,0,0\}+r_2\{0,0,\varsigma,1\})$ [[
{3,4}]]]
MatrixForm[(ψ.{0,1,0,0})[[{3,4}]]] ==
MatrixForm[
$(r_9\{1,0,0,0\}+r_{10}\{0,1,0,0\}+r_3\{0,0,\varsigma,1\})$ [[
{3,4}]]]
MatrixForm[(ψ.{0,0,σ,1})[[{3,4}]]] ==
MatrixForm[
$(r_{11}\{1,0,0,0\}+r_{12}\{0,1,0,0\}+r_4\{0,0,\varsigma,1\})$ [[
{3,4}]]]

$$\begin{pmatrix} k\left(x^2+y^2\right)\alpha \\ 0 \end{pmatrix} == \begin{pmatrix} \beta+\varsigma r_1 \\ r_1 \end{pmatrix}$$
$$\begin{pmatrix} kux-vy \\ 0 \end{pmatrix} == \begin{pmatrix} \varsigma r_2 \\ r_2 \end{pmatrix}$$
$$\begin{pmatrix} vx+kuy \\ 0 \end{pmatrix} == \begin{pmatrix} \varsigma r_3 \\ r_3 \end{pmatrix}$$
$$\begin{pmatrix} \frac{1}{2}k\left(u^2+v^2\right)+k\left(x^2+y^2\right)\sigma \\ k \end{pmatrix} == \begin{pmatrix} \varsigma r_4 \\ r_4 \end{pmatrix}$$

Controllability

Systems

Ξ11c2[u_, σ_]:=σE3 + αE4 + u(E1);
Ξ21c4[u_, v_]:=E1 + uE3 + vE4;

Case $\alpha E_4 \in \Gamma$

Simplify [MatrixExp [$\frac{2n\pi}{\alpha}$(αE4)] , n ∈ Integers] //
MatrixForm

$$\begin{pmatrix} 1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1 \end{pmatrix}$$

For $\Sigma^{(1,1)}_{2,\alpha,\sigma} : \sigma E_3 + \alpha E_4 + uE_1$

FullSimplify [MatrixExp [$\frac{2n\pi}{\alpha}$Ξ11c2[u,σ]] ,
n ∈ Integers]//MatrixForm

$$\begin{pmatrix} 1&0&0&\frac{2n\pi\left(u^2-2\alpha\sigma\right)}{\alpha^2}&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1 \end{pmatrix}$$

Case: $\sigma = 1$
FullSimplify [MatrixExp [$\frac{2n\pi}{\alpha}$Ξ11c2 [$\sqrt{2\alpha}$, 1]] ,
n ∈ Integers]//MatrixForm

$$\begin{pmatrix} 1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1 \end{pmatrix}$$

Case: $\sigma = 0$
FullSimplify [MatrixExp [$\frac{2n\pi}{\alpha}$Ξ11c2[0, 0]] ,
n ∈ Integers]//MatrixForm

$$\begin{pmatrix} 1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1 \end{pmatrix}$$

For $\Sigma^{(2,1)}_4 : E_1 + u_1E_3 + u_2E_4$

FullSimplify [MatrixExp [2πnΞ21c4 [$\frac{1}{2}$, 1]] ,
n ∈ Integers]//MatrixForm

$$\begin{pmatrix} 1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1 \end{pmatrix}$$

For $\Sigma^{(1,1)}_{2,\alpha,-1} : -E_3 + \alpha E_4 + uE_1$

Ξ[u_]:= − E3 + αE4 + u(E1);
F1 = FullSimplify [MatrixExp [$\frac{t_1\pi}{6\alpha}$Ξ[u]]] ;
F2 = FullSimplify [MatrixExp [$\frac{t_2\pi}{3\alpha}$Ξ[−u]]] ;
F3 = FullSimplify [MatrixExp [$\frac{t_3\pi}{3\alpha}$Ξ[u]]] ;
F4 = FullSimplify [MatrixExp [$\frac{\pi}{6\alpha}$Ξ[−u]]] ;
F = Simplify[F1.F2.F3.F4];
Pnt = $\left\{t_1\to 1, t_2\to 1, t_3\to 1, u\to\sqrt{\frac{2\pi}{2\sqrt{3}-\pi}}\sqrt{\alpha}\right\}$;
(FatP = F/.Pnt//Simplify)//MatrixForm

$$\begin{pmatrix} 1&0&0&0&0\\0&-1&0&0&0\\0&0&-1&0&0\\0&0&0&1&0\\0&0&0&0&e^{\frac{i\pi}{n}} \end{pmatrix}$$

FullSimplify[MatrixPower[FatP, 2n],
{n ∈ Integers, n > 0}]//MatrixForm

$$\begin{pmatrix} 1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1 \end{pmatrix}$$

D1 = Simplify [D [F, t_1] /.Pnt] ;
D2 = Simplify [D [F, t_2] /.Pnt] ;
D3 = Simplify [D [F, t_3] /.Pnt] ;
D4 = Simplify[D[F, u]/.Pnt];
D1A = Minv[Inverse[FatP].D1//FullSimplify];
D2A = Minv[Inverse[FatP].D2//FullSimplify];
D3A = Minv[Inverse[FatP].D3//FullSimplify];

D4A = Minv[Inverse[FatP].D4//FullSimplify];
{Norm[Mv[D1A] − Inverse[FatP].D1//FullSimplify],
Norm[Mv[D2A] − Inverse[FatP].D2//FullSimplify],
Norm[Mv[D3A] − Inverse[FatP].D3//FullSimplify],
Norm[Mv[D4A] − Inverse[FatP].D4//FullSimplify]}

$\{0,0,0,0\}$

R = {D1A, D2A, D3A, D4A};
R//MatrixForm

$$\begin{pmatrix} -\frac{\pi^{3/2}}{3\sqrt{4\sqrt{3}\alpha-2\pi\alpha}} & 0 & -\frac{\pi}{6\alpha} & \frac{\pi}{6} \\ \frac{\left(-1+\sqrt{3}\right)\pi^{3/2}}{3\sqrt{\sqrt{3}\alpha-\frac{\pi\alpha}{2}}} & -\frac{\pi^{3/2}}{3\sqrt{\sqrt{3}\alpha-\frac{\pi\alpha}{2}}} & -\frac{\pi\left(2\sqrt{3}-5\pi+2\sqrt{3}\pi\right)}{6\sqrt{3}\alpha-3\pi\alpha} & \frac{\pi}{3} \\ \frac{\left(-1+\sqrt{3}\right)\pi^{3/2}}{3\sqrt{\sqrt{3}\alpha-\frac{\pi\alpha}{2}}} & \frac{\pi^{3/2}}{3\sqrt{\sqrt{3}\alpha-\frac{\pi\alpha}{2}}} & -\frac{\pi\left(2\sqrt{3}-5\pi+2\sqrt{3}\pi\right)}{6\sqrt{3}\alpha-3\pi\alpha} & \frac{\pi}{3} \\ 0 & 0 & \frac{\sqrt{2\left(2\sqrt{3}-\pi\right)\pi}}{\alpha^{3/2}} & 0 \end{pmatrix}$$

Det[R]//FullSimplify

$$-\frac{2\sqrt{\frac{2}{6\sqrt{3}-3\pi}}\pi^{9/2}}{9\alpha^{5/2}}$$

N[Det[R]]

$$-\frac{55.1627}{\alpha^{5/2}}$$

C.8 Parametric Reduction of Normal Extremals

Basic setup

ClearAll["Global\`*"]
M[x_, y_, z_, θ_]:=
{{0, −x, y, −2z, 0}, {0, 0, −θ, y, 0},
{0, θ, 0, x, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, θ}};
E1 = M[1, 0, 0, 0]; E2 = M[0, 1, 0, 0];
E3 = M[0, 0, 1, 0]; E4 = M[0, 0, 0, 1];
Base = {E1, E2, E3, E4};
iBase = {{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0},
{0, 0, 0, 1}};
CC[M_, N_]:=M.N − N.M;
Minv[MM_]:={MM[[3, 4]], MM[[2, 4]],
$\frac{1}{-2}$MM[[1, 4]], MM[[3, 2]]};
Minv[M[x, y, z, θ]]

$\{x, y, z, \theta\}$

m[x_, y_, z_, θ_]:=
{{1, −xCos[θ] + ySin[θ], xSin[θ] + yCos[θ],
−2z, 0}, {0, Cos[θ], −Sin[θ], y, 0},
{0, Sin[θ], Cos[θ], x, 0}, {0, 0, 0, 1, 0},
{0, 0, 0, 0, Exp[θ]}};

Calculations for reduction

In matrix form: $\mathrm{Ad}^* g = \left(E_j{}^* \left(\mathrm{Ad}g^{-1}.E_i\right)\right)_{ij}$

gg = m[x[t], y[t], z[t], θ[t]];
Ads = IdentityMatrix[4];
For[i = 1, i < 5, i++,
For[j = 1, j < 5, j++,
Ads[[i, j]] =
FullSimplify[
iBase[[j]].(Minv[Inverse[gg].Base[[i]].gg])]
]
];
MatrixForm[iAds = Simplify[Inverse[Ads]]]

$$\begin{pmatrix} \mathrm{Cos}[\theta[t]] & -\mathrm{Sin}[\theta[t]] & -\mathrm{Sin}[\theta[t]]x[t]-\mathrm{Cos}[\theta[t]]y[t] & 0 \\ \mathrm{Sin}[\theta[t]] & \mathrm{Cos}[\theta[t]] & \mathrm{Cos}[\theta[t]]x[t]-\mathrm{Sin}[\theta[t]]y[t] & 0 \\ 0 & 0 & 1 & 0 \\ -y[t] & x[t] & \frac{1}{2}\left(x[t]^2+y[t]^2\right) & 1 \end{pmatrix}$$

(iAdsp0 = Simplify [iAds. {p₁[0], p₂[0], p₃, p₄[0]}]) //
MatrixForm

$$\begin{pmatrix} -p_3(\mathrm{Sin}[\theta[t]]x[t]+\mathrm{Cos}[\theta[t]]y[t])+\mathrm{Cos}[\theta[t]]p_1[0]-\mathrm{Sin}[\theta[t]]p_2[0] \\ p_3(\mathrm{Cos}[\theta[t]]x[t]-\mathrm{Sin}[\theta[t]]y[t])+\mathrm{Sin}[\theta[t]]p_1[0]+\mathrm{Cos}[\theta[t]]p_2[0] \\ p_3 \\ \frac{1}{2}p_3\left(x[t]^2+y[t]^2\right)-y[t]p_1[0]+x[t]p_2[0]+p_4[0] \end{pmatrix}$$

R = {{Cos[θ[t]], −Sin[θ[t]]},
{Sin[θ[t]], Cos[θ[t]]}};
$v = \{p_1[0] - p_3 y[t], p_2[0] + p_3 x[t]\}$;
R.v − iAdsp0[[{1, 2}]]//Simplify

$\{0, 0\}$

LHS = {−y[t], x[t]};
RHS =
$\frac{1}{p_3}$ (− {p₁[0], p₂[0]} + Inverse[R]. {p₁[t], p₂[t]}) //
Simplify

$$\left\{\frac{-p_1[0]+\mathrm{Cos}[\theta[t]]p_1[t]+\mathrm{Sin}[\theta[t]]p_2[t]}{p_3}, -\frac{\mathrm{Sin}[\theta[t]]p_1[t]+p_2[0]-\mathrm{Cos}[\theta[t]]p_2[t]}{p_3}\right\}$$

Subs = {−LHS[[1]] → −RHS[[1]],
LHS[[2]] → RHS[[2]]}

$$\left\{y[t]\to -\frac{-p_1[0]+\mathrm{Cos}[\theta[t]]p_1[t]+\mathrm{Sin}[\theta[t]]p_2[t]}{p_3}, x[t]\to -\frac{\mathrm{Sin}[\theta[t]]p_1[t]+p_2[0]-\mathrm{Cos}[\theta[t]]p_2[t]}{p_3}\right\}$$

iAdsp0/.Subs//FullSimplify//MatrixForm

$$\begin{pmatrix} p_1[t] \\ p_2[t] \\ p_3 \\ \frac{-p_1[0]^2+p_1[t]^2-p_2[0]^2+p_2[t]^2}{2p_3}+p_4[0] \end{pmatrix}$$

Sw = {{0, 1}, {−1, 0}};
Sw.LHS//MatrixForm
(xy = Sw.RHS)//MatrixForm

$$\begin{pmatrix} x[t] \\ y[t] \end{pmatrix}$$

$$\begin{pmatrix} -\frac{\mathrm{Sin}[\theta[t]]p_1[t]+p_2[0]-\mathrm{Cos}[\theta[t]]p_2[t]}{p_3} \\ -\frac{-p_1[0]+\mathrm{Cos}[\theta[t]]p_1[t]+\mathrm{Sin}[\theta[t]]p_2[t]}{p_3} \end{pmatrix}$$

$\mathrm{Subs2} = \left\{\theta'[t]\to \frac{1}{\beta_2}p_4[t], (p_1)'[t]\to \frac{-1}{\beta_2}p_2[t]p_4[t], (p_2)'[t]\to p_1[t]\left(\frac{1}{\beta_1}p_3+\frac{1}{\beta_2}p_4[t]\right)\right\}$;

dxy = D[Sw.RHS, t]/.Subs2//Simplify

$\left\{\frac{\mathrm{Cos}[\theta[t]]p_1[t]}{\beta_1}, -\frac{\mathrm{Sin}[\theta[t]]p_1[t]}{\beta_1}\right\}$

dz = $\frac{1}{2}$(xy[[1]]dxy[[2]] − xy[[2]]dxy[[1]])// Simplify

$\frac{p_1[t](-\mathrm{Cos}[\theta[t]]p_1[0]+p_1[t]+\mathrm{Sin}[\theta[t]]p_2[0])}{2p_3\beta_1}$

Alternative parametrisation

(gin = Inverse[m[x[t], y[t], z[t], θ[t]]]// Simplify)//MatrixForm

$$\begin{pmatrix} 1 & x[t] & -y[t] & 2z[t] & 0 \\ 0 & \mathrm{Cos}[\theta[t]] & \mathrm{Sin}[\theta[t]] & -\mathrm{Sin}[\theta[t]]x[t]-\mathrm{Cos}[\theta[t]]y[t] & 0 \\ 0 & -\mathrm{Sin}[\theta[t]] & \mathrm{Cos}[\theta[t]] & -\mathrm{Cos}[\theta[t]]x[t]+\mathrm{Sin}[\theta[t]]y[t] & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & e^{-\theta[t]} \end{pmatrix}$$

LL = FullSimplify[D[gin, t]];
RR = FullSimplify[
gin.M [w_1[t], w_2[t], w_3[t], w_4[t]]] ;
LL[[1, {2, 3, 4}]]

$\{x'[t], -y'[t], 2z'[t]\}$

RR[[1, {2, 3, 4}]]

$\{-w_1[t]-y[t]w_4[t], w_2[t]-x[t]w_4[t], -y[t]w_1[t]+x[t]w_2[t]-2w_3[t]\}$

{LL[[5, 5]], RR[[5, 5]]}

$\left\{-e^{-\theta[t]}\theta'[t], e^{-\theta[t]}w_4[t]\right\}$

Solve[LL == RR, {x'[t], y'[t], z'[t], θ'[t]}]

$$\left\{\left\{z'[t]\to\frac{1}{2}\left(-y[t]w_1[t]+x[t]w_2[t]-2w_3[t]\right), x'[t]\to -w_1[t]-y[t]w_4[t], y'[t]\to -w_2[t]+x[t]w_4[t], \theta'[t]\to -w_4[t]\right\}\right\}$$

C.9 Stability Analysis

Basic setup

Ca = $p_1{}^2+p_2{}^2-2p_3p_4$;

Hm = $\frac{1}{2}\left(\frac{p_1^2}{\beta_1}+\frac{p_4^2}{\beta_2}\right)$;

Hv = $\left\{-\frac{1}{\beta_2}p_2p_4, p_1\left(\frac{1}{\beta_1}p_3+\frac{1}{\beta_2}p_4\right), 0, \frac{1}{\beta_1}p_1p_2\right\}$;

HvL = D [Hv, {{p_1, p_2, p_3, p_4}}] ;
MatrixForm[HvL]

$$\begin{pmatrix} 0 & -\frac{p_4}{\beta_2} & 0 & -\frac{p_2}{\beta_2} \\ \frac{p_3}{\beta_1}+\frac{p_4}{\beta_2} & 0 & \frac{p_1}{\beta_1} & \frac{p_1}{\beta_2} \\ 0 & 0 & 0 & 0 \\ \frac{p_2}{\beta_1} & \frac{p_1}{\beta_1} & 0 & 0 \end{pmatrix}$$

CEf = λ_0Hm + $\lambda_1 p_3$ + λ_2Ca
dCEf = D [CEf, {{p_1, p_2, p_3, p_4}}] ;
dCEf//MatrixForm

$\frac{1}{2}\left(\frac{p_1^2}{\beta_1}+\frac{p_4^2}{\beta_2}\right)\lambda_0+p_3\lambda_1+\left(p_1^2+p_2^2-2p_3p_4\right)\lambda_2$

$$\begin{pmatrix} \frac{p_1\lambda_0}{\beta_1}+2p_1\lambda_2 \\ 2p_2\lambda_2 \\ \lambda_1-2p_4\lambda_2 \\ \frac{p_4\lambda_0}{\beta_2}-2p_3\lambda_2 \end{pmatrix}$$

ddCEf = D [dCEf, {{p_1, p_2, p_3, p_4}}] ;
ddCEf//MatrixForm

$$\begin{pmatrix} \frac{\lambda_0}{\beta_1}+2\lambda_2 & 0 & 0 & 0 \\ 0 & 2\lambda_2 & 0 & 0 \\ 0 & 0 & 0 & -2\lambda_2 \\ 0 & 0 & -2\lambda_2 & \frac{\lambda_0}{\beta_2} \end{pmatrix}$$

Wk0 = {0, 0, 0, 0}; Wk1 = Wk0; Wk2 = Wk1;
Wk0 = D [Hm, {{p_1, p_2, p_3, p_4}}] ;
Wk1 = D [p_3, {{p_1, p_2, p_3, p_4}}] ;
Wk2 = D [Ca, {{p_1, p_2, p_3, p_4}}] ;
{MatrixForm[Wk0], MatrixForm[Wk1], MatrixForm[Wk2]}

$$\left\{\begin{pmatrix} \frac{p_1}{\beta_1} \\ 0 \\ 0 \\ \frac{p_4}{\beta_2} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2p_1 \\ 2p_2 \\ -2p_4 \\ -2p_3 \end{pmatrix}\right\}$$

Case $p = 0$

Cs = Thread [{p_1, p_2, p_3, p_4} → {0, 0, 0, 0}] ;
dCEf/.Cs//MatrixForm

$$\begin{pmatrix} 0 \\ 0 \\ \lambda_1 \\ 0 \end{pmatrix}$$

ddCEfr3 = ddCEf[[{1, 2, 4}, {1, 2, 4}]];
MatrixForm[ddCEfr3]

$$\begin{pmatrix} \frac{\lambda_0}{\beta_1}+2\lambda_2 & 0 & 0 \\ 0 & 2\lambda_2 & 0 \\ 0 & 0 & \frac{\lambda_0}{\beta_2} \end{pmatrix}$$

MatrixForm [ddCEfr3/. {λ_0 → 1, λ_1 → 1, λ_2 → 1}]

$$\begin{pmatrix} 2+\frac{1}{\beta_1} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \frac{1}{\beta_2} \end{pmatrix}$$

Stable.

Case $p = (0, 0, 0, a)$, $a \neq 0$

Cs = Thread [{p_1, p_2, p_3, p_4} → {0, 0, 0, a}] ;
dCEf/.Cs//MatrixForm

$$\begin{pmatrix} 0 \\ 0 \\ \lambda_1-2a\lambda_2 \\ \frac{a\lambda_0}{\beta_2} \end{pmatrix}$$

dCEf/.Cs/. $\{\lambda_0 \to 0, \lambda_1 \to 2a\lambda_2\}$ **//MatrixForm**

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

ddCEf/.Cs/. $\{\lambda_0 \to 0, \lambda_1 \to 2a\lambda_2\}$ **//MatrixForm**

$$\begin{pmatrix} 2\lambda_2 & 0 & 0 & 0 \\ 0 & 2\lambda_2 & 0 & 0 \\ 0 & 0 & 0 & -2\lambda_2 \\ 0 & 0 & -2\lambda_2 & 0 \end{pmatrix}$$

{Wk0//MatrixForm, Wk1//MatrixForm, Wk2//MatrixForm}/.Cs/. $\{\lambda_0 \to 0, \lambda_1 \to 2a\lambda_2\}$

$$\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{a}{\beta_2} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -2a \\ 0 \end{pmatrix} \right\}$$

ddCEfW = ddCEf[[{1,2},{1,2}]]/.Cs/. $\{\lambda_0 \to 0, \lambda_1 \to 2a\lambda_2\}$**;**
MatrixForm[ddCEfW]

$$\begin{pmatrix} 2\lambda_2 & 0 \\ 0 & 2\lambda_2 \end{pmatrix}$$

Stable (take $\lambda_2 = 1$).

Case $p = (0, b, d, 0)$, $b \neq 0$

Cs = Thread $[\{p_1, p_2, p_3, p_4\} \to \{0, b, d, 0\}]$**;**
dCEf/.Cs//MatrixForm

$$\begin{pmatrix} 0 \\ 2b\lambda_2 \\ \lambda_1 \\ -2d\lambda_2 \end{pmatrix}$$

dCEf/.Cs/. $\{\lambda_1 \to 0, \lambda_2 \to 0\}$ **//MatrixForm**

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

ddCEf/.Cs/. $\{\lambda_1 \to 0, \lambda_2 \to 0\}$ **//MatrixForm**

$$\begin{pmatrix} \frac{\lambda_0}{\beta_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\lambda_0}{\beta_2} \end{pmatrix}$$

{Wk0//MatrixForm, Wk1//MatrixForm, Wk2//MatrixForm}/.Cs/. $\{\lambda_0 \to 0, \lambda_1 \to 0\}$

$$\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2b \\ 0 \\ -2d \end{pmatrix} \right\}$$

B **= {{q1, q3}, {q3, q2}};**
$v = \{x, dy, 0, by\}$**;**
$v.$**(ddCEf/.Cs/.** $\{\lambda_1 \to 0, \lambda_2 \to 0\}$**)** $.v$
Resolve[
ForAll[$\{x, y\}$**,**
$v.$**(ddCEf/.Cs/.** $\{\lambda_1 \to 0, \lambda_2 \to 0\}$**)** $.v ==$
$\{x, y\}.B.\{x, y\}$**]]**

$$\frac{x^2\lambda_0}{\beta_1} + \frac{b^2y^2\lambda_0}{\beta_2}$$

$\mathrm{q3} == 0 \&\& \beta_2 \neq 0 \&\& -\mathrm{q2}\beta_2 + b^2\lambda_0 == 0$
$\&\& \beta_1 \neq 0 \&\& -\mathrm{q1}\beta_1 + \lambda_0 == 0$

Solve[%, {q1, q2, q3}]
ddCEfW = B**/.%[[1]];**
MatrixForm[ddCEfW]

$$\left\{\left\{\mathrm{q3} \to 0, \mathrm{q2} \to \frac{b^2\lambda_0}{\beta_2}, \mathrm{q1} \to \frac{\lambda_0}{\beta_1}\right\}\right\}$$

$$\begin{pmatrix} \frac{\lambda_0}{\beta_1} & 0 \\ 0 & \frac{b^2\lambda_0}{\beta_2} \end{pmatrix}$$

Stable (take $\lambda_0 = 1$)

Case $p = (0, 0, b, c)$, $b \neq 0$, $c\,(c\beta_1 + b\beta_2) > 0$

Cs = Thread $[\{p_1, p_2, p_3, p_4\} \to \{0, 0, b, c\}]$**;**
dCEf/.Cs//MatrixForm

$$\begin{pmatrix} 0 \\ 0 \\ \lambda_1 - 2c\lambda_2 \\ \frac{c\lambda_0}{\beta_2} - 2b\lambda_2 \end{pmatrix}$$

dCEf/.Cs/. $\lambda_1 \to 2c\lambda_2$**/.** $\lambda_2 \to \frac{c\lambda_0}{2\beta_2 b}$**//MatrixForm**

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

ddCEf/.Cs/. $\lambda_1 \to 2c\lambda_2$**/.** $\lambda_2 \to \frac{c\lambda_0}{2\beta_2 b}$**//MatrixForm**

$$\begin{pmatrix} \frac{\lambda_0}{\beta_1} + \frac{c\lambda_0}{b\beta_2} & 0 & 0 & 0 \\ 0 & \frac{c\lambda_0}{b\beta_2} & 0 & 0 \\ 0 & 0 & 0 & -\frac{c\lambda_0}{b\beta_2} \\ 0 & 0 & -\frac{c\lambda_0}{b\beta_2} & \frac{\lambda_0}{\beta_2} \end{pmatrix}$$

{Wk0//MatrixForm, Wk1//MatrixForm, Wk2//MatrixForm}/.Cs/. $\lambda_1 \to 2c\lambda_2$**/.** $\lambda_2 \to \frac{c\lambda_0}{2\beta_2 b}$

$$\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{c}{\beta_2} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -2c \\ -2b \end{pmatrix} \right\}$$

ddCEfW =
ddCEf[[{1,2},{1,2}]]/.Cs/. $\lambda_1 \to 2c\lambda_2$**/.** $\lambda_2 \to \frac{c\lambda_0}{2\beta_2 b}$**;**
MatrixForm[ddCEfW]

$$\begin{pmatrix} \frac{\lambda_0}{\beta_1} + \frac{c\lambda_0}{b\beta_2} & 0 \\ 0 & \frac{c\lambda_0}{b\beta_2} \end{pmatrix}$$

MatrixForm $\left[\mathbf{ddCEfW}/.\lambda_0 \to \frac{b}{c}\beta_2\right]$

$$\begin{pmatrix} 1 + \frac{b\beta_2}{c\beta_1} & 0 \\ 0 & 1 \end{pmatrix}$$

$c^2\beta_1 + cb\beta_2 > 0$ implies that $1+\frac{b\beta_2}{c\beta_1}>0$ (divide through by $c^2\beta_1$). Stable.

Case $p = (0, 0, b, c)$, $b \neq 0$, $c\,(c\beta_1 + b\beta_2) \leq 0$

Cs = Thread$[\{p_1, p_2, p_3, p_4\} \to \{0, 0, b, c\}]$;
MatrixForm[HvL/.Cs]
CharacteristicPolynomial[HvL/.Cs, x**]**
Eigenvalues[HvL/.Cs]

$$\begin{pmatrix} 0 & -\frac{c}{\beta_2} & 0 & 0 \\ \frac{b}{\beta_1} + \frac{c}{\beta_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$x^4 + \frac{c^2x^2}{\beta_2^2} + \frac{bcx^2}{\beta_1\beta_2}$

$\left\{0, 0, -\frac{\sqrt{-c^2\beta_1 - bc\beta_2}}{\sqrt{\beta_1}\beta_2}, \frac{\sqrt{-c^2\beta_1 - bc\beta_2}}{\sqrt{\beta_1}\beta_2}\right\}$

If $-c\,(c\beta_1 + b\beta_2)>0$, then we have positive eigenvalue. Hence unstable.
Assume $-c\,(c\beta_1 + b\beta_2)=0$

Subcase $c = 0$

Cs = Thread$[\{p_1, p_2, p_3, p_4\} \to \{0, 0, b, 0\}]$;
MatrixForm[HvL/.Cs]
ES = Eigensystem[HvL/.Cs];
Evalues = ES[[1]]
Evectors = {MatrixForm[ES[[2, 1]]],
MatrixForm[ES[[2, 2]]], MatrixForm[ES[[2, 3]]],
MatrixForm[ES[[2, 4]]]}

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{b}{\beta_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\{0, 0, 0, 0\}$

$$\left\{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right\}$$

Alg. Mult. greater than Geom. Mult. Unstable.

Subcase $c = -\frac{\beta_2}{\beta_1}b$

Cs = Thread$\left[\{p_1, p_2, p_3, p_4\} \to \left\{0, 0, b, -\frac{\beta_2}{\beta_1}b\right\}\right]$;
MatrixForm[HvL/.Cs]
ES = Eigensystem[HvL/.Cs];
Evalues = ES[[1]]
Evectors = {MatrixForm[ES[[2, 1]]],
MatrixForm[ES[[2, 2]]], MatrixForm[ES[[2, 3]]],
MatrixForm[ES[[2, 4]]]}

$$\begin{pmatrix} 0 & \frac{b}{\beta_1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\{0, 0, 0, 0\}$

$$\left\{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right\}$$

Alg. Mult. greater than Geom. Mult. Unstable.

Case $p = \left(b, 0, a, -\frac{\beta_2}{\beta_1}a\right)$, $b \neq 0$

Cs = Thread$\left[\{p_1, p_2, p_3, p_4\} \to \left\{b, 0, a, \frac{-\beta_2}{\beta_1}a\right\}\right]$;
MatrixForm[HvL/.Cs]
CharacteristicPolynomial[HvL/.Cs, x**]**
Eigenvalues[HvL/.Cs]

$$\begin{pmatrix} 0 & \frac{a}{\beta_1} & 0 & 0 \\ 0 & 0 & \frac{b}{\beta_1} & \frac{b}{\beta_2} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{b}{\beta_1} & 0 & 0 \end{pmatrix}$$

$x^4 - \frac{b^2x^2}{\beta_1\beta_2}$

$\left\{0, 0, -\frac{b}{\sqrt{\beta_1}\sqrt{\beta_2}}, \frac{b}{\sqrt{\beta_1}\sqrt{\beta_2}}\right\}$

Positive eigenvalue. Hence unstable.

C.10 Graph for Case 1a

ClearAll["Global'*"]

$h = 0.5$;
$\beta_1 = 3$;
$\beta_2 = 1$;
$p_3 = 1$;
$c = 2h\beta_1 + \frac{\beta_2}{\beta_1}p_3{}^2 + 0.5$;
Os = 0.08;

Type test

$\left\{2h\beta_1, \frac{\beta_2}{\beta_1}p_3{}^2\right\}$
$2h\beta_1 > \frac{\beta_2}{\beta_1}p_3{}^2$
$\{3., \frac{1}{3}\}$
True

Equilibrium points

el = 2.5;
Es1 = ParametricPlot3D[
{{0, z, 0}, {0, $-z$, 0}}, {z, Os, 3},

```
PlotStyle → Directive[Blue, Thick],
Boxed → False];
Esn0 = ParametricPlot3D[
{{0, 0, z}, {0, 0, −z − (β2/β1) p3}}, {z, Os, el},
PlotStyle → Directive[Blue, Thick],
Boxed → False];
Eu1 = ParametricPlot3D[
{{z, 0, (−β2/β1) p3}, {−z, 0, (−β2/β1) p3}},
{z, Os, el},
PlotStyle → Directive[Thick, Red],
Boxed → False];
Eun0 = ParametricPlot3D[{0, 0, z},
{z, −(β2/β1) p3, 0},
PlotStyle → Directive[Thick, Red],
Boxed → False];
Eun0p = ListPointPlot3D[
{{0, 0, 0}, {0, 0, −(β2/β1) p3}},
PlotStyle → Directive[Red,
PointSize[Large]]];
```

Hamiltonian

```
hl = 2.5;
Ha = ParametricPlot3D[
{√(2hβ1) Cos[θ], z, √(2hβ2) Sin[θ]},
{θ, −π, π}, {z, −hl, hl}, Mesh → 8];
```

Casimir

```
cl = 2.6;
Ca = ParametricPlot3D[
{rCos[θ], rSin[θ], (r^2−c)/(2p3)}, {θ, −π, π},
{r, 0, cl}, Mesh → 10];
```

Intersection

```
IntR = {√(2hβ1) Cos[θ], y, √(2hβ2) Sin[θ]} /.
Solve[
2hβ1(Cos[θ])^2 + y^2−
2p3 √(2hβ2) Sin[θ] == c, {y}]
```

$\{\{1.73205\mathrm{Cos}[\theta], -1.\sqrt{3.83333 - 3.\mathrm{Cos}[\theta]^2 + 2.\mathrm{Sin}[\theta]}, 1.\mathrm{Sin}[\theta]\}, \{1.73205\mathrm{Cos}[\theta], \sqrt{3.83333 - 3.\mathrm{Cos}[\theta]^2 + 2.\mathrm{Sin}[\theta]}, 1.\mathrm{Sin}[\theta]\}\}$

```
Int1 = ParametricPlot3D[IntR, {θ, −π, π},
Axes → False,
PlotStyle → Directive[Thick, Black],
Axes → False, Boxed → False];
```

Vector field on intersection

```
Vpa = Flatten[Table[IntR, {θ, −π, π, π/5}], 1];
Vp = {};
For[i = 1, i ≤ Length[Vpa], i++,
If[Im[Vpa[[i]]] == {0, 0, 0},
Vp = Append[Vp, Vpa[[i]]]]
];
Field = VectorPlot3D[
{−yz/β2, x (p3/β1 + z/β2),
xy/β1}, {x, −5, 5}, {y, −5, 5}, {z, −5, 5},
VectorPoints → Vp, VectorScale → Medium,
VectorStyle → Green];
```

Stereographic plots

```
pr = 2.8;
View = {3Pi/2.8, Pi/3, 2};
Viewv = {0, 0, 1};
Opts = {ViewVertical → Viewv,
ViewPoint → View, Axes → True,
BoxRatios → {1, 1, 1},
PlotRange → {{−pr, pr}, {−pr, pr},
{−pr, pr}}, Boxed → False,
ImageSize → Medium,
AxesLabel → {"E1*", "E2*", "E4*"},
LabelStyle → Directive[Medium],
AxesEdge → {{1, −1}, {1, −1}, {1, −1}},
FaceGrids → {{−1, 0, 0}, {0, −1, 0},
{0, 0, −1}},
TicksStyle → Directive[Medium]};
Show[Eq, Ca, Ha, Int1, Opts]
Show[Eq, Int1, Field, Opts]
```

Graphics shown in figure 4.5a.

C.11 Integration of Case 1a

Note that the Jacobi elliptic functions in Mathematica use the modulus k squared as their second parameter. For example $\mathrm{sn}(x, k)$ is represented as **JacobiSN[x, k^2]** in Mathematica.

```
ClearAll["Global'*"]
```

Basic setup

```
cc[A_, B_]:=A.B − B.A;
m[x_, y_, z_, θ_]:=
{{1, −xCos[θ] + ySin[θ],
xSin[θ] + yCos[θ], −2z, 0},
{0, Cos[θ], −Sin[θ], y, 0},
```

$\{0, \mathrm{Sin}[\theta], \mathrm{Cos}[\theta], x, 0\},$
$\{0, 0, 0, 1, 0\}, \{0, 0, 0, 0, \mathrm{Exp}[\theta]\}\};$
$M[\mathrm{x\_}, \mathrm{y\_}, \mathrm{z\_}, \theta\_]:=$
$\{\{0, -x, y, -2z, 0\}, \{0, 0, -\theta, y, 0\},$
$\{0, \theta, 0, x, 0\}, \{0, 0, 0, 0, 0\},$
$\{0, 0, 0, 0, \theta\}\};$
$\mathrm{E1} = M[1, 0, 0, 0]; \mathrm{E2} = M[0, 1, 0, 0];$
$\mathrm{E3} = M[0, 0, 1, 0]; \mathrm{E4} = M[0, 0, 0, 1];$
$\mathrm{Base} = \{\mathrm{E1}, \mathrm{E2}, \mathrm{E3}, \mathrm{E4}\};$
$\mathrm{Minv}[\mathrm{MM\_}]:=$
$\{\mathrm{MM}[[3]][[4]], \mathrm{MM}[[2]][[4]],$
$\frac{-1}{2}\mathrm{MM}[[1]][[4]], \mathrm{MM}[[3]][[2]]\};$
$\mathrm{Minv}[M[x, y, z, \theta]]$
$\{x, y, z, \theta\}$
$\Xi = \{u_1, u_2\};$
$\chi = \{u_1, u_2\}.\mathrm{DiagonalMatrix}[\{\beta_1, \beta_2\}].$
$\{u_1, u_2\}$
$H = \{p_1, p_4\}.\Xi + \nu\chi /. \nu \to \frac{-1}{2}$
$\mathrm{Ca} = p_1{}^2 + p_2{}^2 - 2p_3p_4$
$u_1^2\beta_1 + u_2^2\beta_2$
$p_1u_1 + p_4u_2 + \frac{1}{2}\left(-u_1^2\beta_1 - u_2^2\beta_2\right)$
$p_1^2 + p_2^2 - 2p_3p_4$

Extremal equations

$\{D[H, u_1], D[H, u_2]\}$
$\{p_1 - u_1\beta_1, p_4 - u_2\beta_2\}$
$\mathrm{Us} = \mathrm{Solve}[\{D[H, u_1] == 0, D[H, u_2] == 0\},$
$\{u_1, u_2\}]$
$\left\{\left\{u_1 \to \frac{p_1}{\beta_1}, u_2 \to \frac{p_4}{\beta_2}\right\}\right\}$
$\mathrm{Hmax} = (\mathrm{FullSimplify}[H/.\mathrm{Us}[[1]]])$
$\frac{1}{2}\left(\frac{p_1^2}{\beta_1} + \frac{p_4^2}{\beta_2}\right)$
$\mathrm{DHmax} = \mathrm{FullSimplify}[$
$\{D[\mathrm{Hmax}, p_1], D[\mathrm{Hmax}, p_2], D[\mathrm{Hmax}, p_3],$
$D[\mathrm{Hmax}, p_4]\ \}];$
$\mathrm{De1} = \mathrm{FullSimplify}[$
$\mathrm{Minv}[-\mathrm{cc}[\mathrm{E1}, \mathrm{DHmax.Base}]].$
$\{p_1, p_2, p_3, p_4\}];$
$\mathrm{De2} = \mathrm{FullSimplify}[$
$\mathrm{Minv}[-\mathrm{cc}[\mathrm{E2}, \mathrm{DHmax.Base}]].$
$\{p_1, p_2, p_3, p_4\}];$
$\mathrm{De3} = \mathrm{FullSimplify}[$
$\mathrm{Minv}[-\mathrm{cc}[\mathrm{E3}, \mathrm{DHmax.Base}]].$
$\{p_1, p_2, p_3, p_4\}];$
$\mathrm{De4} = \mathrm{FullSimplify}[$
$\mathrm{Minv}[-\mathrm{cc}[\mathrm{E4}, \mathrm{DHmax.Base}]].$
$\{p_1, p_2, p_3, p_4\}];$
$\mathrm{MatrixForm}[\{\dot{p}_1 == \mathrm{De1}, \dot{p}_2 == \mathrm{De2},$
$\dot{p}_3 == \mathrm{De3}, \dot{p}_4 == \mathrm{De4}\}]$
$\mathrm{UE} == \mathrm{Us}[[1]]$
$\mathrm{GD} == \Xi/.\mathrm{Us}[[1]]$

$$\begin{pmatrix} \dot{p}_1 == -\frac{p_2p_4}{\beta_2} \\ \dot{p}_2 == p_1\left(\frac{p_3}{\beta_1} + \frac{p_4}{\beta_2}\right) \\ \dot{p}_3 == 0 \\ \dot{p}_4 == \frac{p_1p_2}{\beta_1} \end{pmatrix}$$

$\mathrm{UE} == \left\{u_1 \to \frac{p_1}{\beta_1}, u_2 \to \frac{p_4}{\beta_2}\right\}$
$\mathrm{GD} == \left\{\frac{p_1}{\beta_1}, \frac{p_4}{\beta_2}\right\}$

Assumptions

$\mathrm{Assump} = \{c > 0, h > 0, \rho > 0, \beta_1 > 0,$
$\beta_2 > 0, 2h\beta_1 > \frac{\beta_2}{\beta_1}\rho^2, c > 2h\beta_1 + \frac{\beta_2}{\beta_1}\rho^2,$
$c^2 - 8h\rho^2\beta_2 > 0, \sqrt{c^2 - 8h\rho^2\beta_2} - c < 0,$
$c - 2\rho^2\frac{\beta_2}{\beta_1} > 0,$
$\sqrt{c^2 - 8h\rho^2\beta_2} - c + 2\frac{\beta_2}{\beta_1}\rho^2 > 0, \delta > 0,$
$\delta - c < 0, \delta - c + 2\frac{\beta_2}{\beta_1}\rho^2 > 0\Big\};$
$\delta\mathrm{v} = \sqrt{c^2 - 8h\rho^2\beta_2};$

Reduction into standard form

$\mathrm{coeff} = \frac{1}{\beta_2{}^2};$
$\mathrm{X1} = p_4^2 + (c + 2\rho p_4 - 2h\beta_1)\frac{\beta_2}{\beta_1};$
$\mathrm{X2} = -p_4^2 + 2h\beta_2;$

$\omega 2 = \mathrm{coeffX1X2}$
$\frac{\left(-p_4^2+2h\beta_2\right)\left(p_4^2+\frac{(c+2\rho p_4-2h\beta_1)\beta_2}{\beta_1}\right)}{\beta_2^2}$

$\mathrm{a1} = \mathrm{Coefficient}[\mathrm{X1}, p_4, 2];$
$\mathrm{b1} = \frac{1}{2}\mathrm{Coefficient}[\mathrm{X1}, p_4, 1];$
$\mathrm{c1} = \mathrm{Coefficient}[\mathrm{X1}, p_4, 0]//\mathrm{Simplify};$
$\mathrm{a2} = \mathrm{Coefficient}[\mathrm{X2}, p_4, 2];$
$\mathrm{b2} = \frac{1}{2}\mathrm{Coefficient}[\mathrm{X2}, p_4, 1];$
$\mathrm{c2} = \mathrm{Coefficient}[\mathrm{X2}, p_4, 0];$
$\mathrm{Simplify}\left[\left(\mathrm{a1} - \frac{\mathrm{a1}}{\mathrm{a2}}\mathrm{a2}\right)\left(\mathrm{c1} - \frac{\mathrm{a1}}{\mathrm{a2}}\mathrm{c2}\right) -\right.$
$\left.\left(\mathrm{b1} - \frac{\mathrm{a1}}{\mathrm{a2}}\mathrm{b2}\right)\hat{}2\right]$
$-\frac{\rho^2\beta_2^2}{\beta_1^2}$
$\mathrm{Rts} = \mathrm{FullSimplify}[$
$\mathrm{Solve}[$
$(\mathrm{a1} - \lambda\mathrm{a2})(\mathrm{c1} - \lambda\mathrm{c2}) - (\mathrm{b1} - \lambda\mathrm{b2})\hat{}2 ==$
$0, \lambda], \mathrm{Assump}];$
$\lambda_1 = \lambda/.\mathrm{Rts}[[1]][[1]];$
$\lambda_2 = \lambda/.\mathrm{Rts}[[2]][[1]];$
$\{\lambda_1/.\delta\mathrm{v} \to \delta, \lambda_2/.\delta\mathrm{v} \to \delta\}$
$\left\{\frac{c-\delta-4h\beta_1}{4h\beta_1}, \frac{c+\delta-4h\beta_1}{4h\beta_1}\right\}$
$(\mathrm{rts1} = \mathrm{FullSimplify}[$
$\mathrm{Solve}\left[\left(\mathrm{FullSimplify}\left[\frac{(\mathrm{X1}-\lambda_1\mathrm{X2})}{\mathrm{a1}-\lambda_1\mathrm{a2}}\right]\right) ==\right.$
$0, p_4], \mathrm{Assump}])/.\delta\mathrm{v} \to \delta$
$(\mathrm{rts2} = \mathrm{FullSimplify}[$
$\mathrm{Solve}\left[\left(\mathrm{FullSimplify}\left[\frac{(\mathrm{X1}-\lambda_2\mathrm{X2})}{\mathrm{a1}-\lambda_2\mathrm{a2}}\right]\right) ==\right.$

$0, p_4]$, Assump]) /.δv → δ
($r_1 = p_4$/.rts1[[1]][[1]]) /.δv → δ
($r_2 = p_4$/.rts2[[1]][[1]]) /.δv → δ
$\left\{\left\{p_4 \to -\frac{c+\delta}{2\rho}\right\}, \left\{p_4 \to -\frac{c+\delta}{2\rho}\right\}\right\}$
$\left\{\left\{p_4 \to \frac{-c+\delta}{2\rho}\right\}, \left\{p_4 \to \frac{-c+\delta}{2\rho}\right\}\right\}$
$-\frac{c+\delta}{2\rho}$
$\frac{-c+\delta}{2\rho}$
(X1 − λ_1X2) − (a1 − λ_1a2) ($p_4 - r_1$) ^2// Simplify
0
X1 − λ_2X2 − (a1 − λ_2a2) ($p_4 - r_2$) ^2// Simplify
0
A_1 = Together [FullSimplify [$\frac{\lambda_2(\text{a1}-\text{a2}\lambda_1)}{\lambda_2-\lambda_1}$, Assump]];
Together [Simplify [%/.δv^2 → δ^2, Assump]]
$\frac{-c\beta_1+\delta\beta_1+2\rho^2\beta_2}{2\delta\beta_1}$
A_2 = Together [FullSimplify [$\frac{(\text{a1}-\text{a2}\lambda_1)}{\lambda_2-\lambda_1}$, Assump]];
Together [Simplify [%/.δv^2 → δ^2, Assump]]
$\frac{c-\delta}{2\delta}$
B_1 = Together [FullSimplify [$\frac{\lambda_1(\text{a1}-\text{a2}\lambda_2)}{\lambda_1-\lambda_2}$, Assump]];
Together [Simplify [%/.δv^2 → δ^2, Assump]]
$\frac{c\beta_1+\delta\beta_1-2\rho^2\beta_2}{2\delta\beta_1}$
B_2 = Together [FullSimplify [$\frac{(\text{a1}-\text{a2}\lambda_2)}{\lambda_1-\lambda_2}$, Assump]];
Together [Simplify [%/.δv^2 → δ^2, Assump]]
$\frac{-c-\delta}{2\delta}$
(A_1 ($p_4 - r_1$) ^2 + B_1 ($p_4 - r_2$) ^2)
(A_2 ($p_4 - r_1$) ^2 + B_2 ($p_4 - r_2$) ^2) − X1X2// Simplify
0

Simplifications for final form

FullSimplify [($r_1 - r_2$) $\sqrt{\tilde{A}_1\tilde{A}_2}$, Assump] ;
Simplify [%/.δv^2 → δ^2, Assump]
$-\frac{\delta\sqrt{\tilde{A}_1\tilde{A}_2}}{\rho}$
Ωv = Simplify [($r_2 - r_1$) $\frac{1}{\beta_2}\sqrt{A_2B_1 - A_1B_2}$, Assump];
Simplify [%/.δv^2 → δ^2, Assump]
$\sqrt{\frac{\delta}{\beta_1\beta_2}}$
kv = Simplify [$\sqrt{\frac{A_2B_1}{A_2B_1-A_1B_2}}$, Assump] ;
Simplify [%/.δv^2 → δ^2, Assump]
$\frac{\sqrt{\frac{-c+\delta+4h\beta_1}{\delta}}}{\sqrt{2}}$
kdv = Simplify [$\sqrt{1-\text{kv}^2}$, Assump] ;
Simplify [%/.δv^2 → δ^2, Assump]
$\frac{\sqrt{\frac{c+\delta-4h\beta_1}{\delta}}}{\sqrt{2}}$
Simplify [r_1/.δv^2 → δ^2, Assump]
$-\frac{c+\delta}{2\rho}$
Simplify [Simplify [$\sqrt{\frac{-B_2}{A_2}}$, Assump] /. δv^2 → δ^2, Assump]
$\sqrt{\frac{c+\delta}{c-\delta}}$
Simplify [Simplify [$r_2\sqrt{\frac{-B_2}{A_2}}$, Assump] /. δv^2 → δ^2, Assump]
$-\frac{\sqrt{c^2-\delta^2}}{2\rho}$

Prospective integral curves

p4 = $\frac{\sqrt{c^2-\delta^2}}{2\rho}$ $\frac{-\sqrt{c-\delta}+\sqrt{c+\delta}\text{JacobiCN}[\Omega t,k^2]}{\sqrt{c+\delta}-\sqrt{c-\delta}\text{JacobiCN}[\Omega t,k^2]}$;

Substitution rules

JsubsDN =
{JacobiSN [$\Omega t, k^2$]^2 →
$\frac{1}{k^2}$ (1 − JacobiDN [$\Omega t, k^2$]^2) ,
JacobiCN [$t\Omega, k^2$]^2 →
$\frac{1}{k^2}$ (JacobiDN [$\Omega t, k^2$]^2 + k^2 − 1)} ;
JsubsCN =
{JacobiSN [$\Omega t, k^2$]^2 →
1 − JacobiCN [$\Omega t, k^2$]^2 ,
JacobiDN [$t\Omega, k^2$]^2 →
k^2 JacobiCN [$\Omega t, k^2$]^2 + 1 − k^2} ;
Example:
JacobiDN [$\Omega t, k^2$]^2 /.JsubsCN
$1 - k^2 + k^2$JacobiCN $[t\Omega, k^2]^2$
%/.JsubsDN
JacobiDN $[t\Omega, k^2]^2$

Solving for p_1

$\frac{c^2-\delta^2}{4\rho^2\beta_2}$/.$\delta$ → δv//Simplify
$2h$
Simplify [$\frac{c^2-\delta^2}{4\rho^2\beta_2}\beta_1 - \frac{\beta_1}{\beta_2}$p4^2, Assump]
$\frac{\delta(-c^2+\delta^2)(-1+\text{JacobiCN}[t\Omega,k^2]^2)\beta_1}{2\rho^2(\sqrt{c+\delta}-\sqrt{c-\delta}\text{JacobiCN}[t\Omega,k^2])^2\beta_2}$

$$\mathbf{p1} = \sigma_2 \frac{\sqrt{\delta(c^2-\delta^2)\beta_1}}{\rho\sqrt{2\beta_2}} \frac{\text{JacobiSN}[\Omega t, k^2]}{\sqrt{c+\delta}-\sqrt{c-\delta}\text{JacobiCN}[\Omega t,k^2]};$$

$$\text{Simplify}\left[\frac{\mathbf{p1}^2}{\beta_1}+\frac{\mathbf{p4}^2}{\beta_2}-\frac{c^2-\delta^2}{4\rho^2\beta_2}/.\sigma_2\to 1,\ \text{Assump}\right]$$

$$\frac{\delta(c^2-\delta^2)\left(-1+\text{JacobiCN}[t\Omega,k^2]^2+\text{JacobiSN}[t\Omega,k^2]^2\right)}{2\rho^2\left(\sqrt{c+\delta}-\sqrt{c-\delta}\text{JacobiCN}[t\Omega,k^2]\right)^2\beta_2}$$

Solving for p_2

$$\text{Simplify}\left[\text{Together}\left[c+2\rho\mathbf{p4}-\mathbf{p1}^2\right],\ \text{Assump}\right]/.\text{JsubsDN}/.\sigma_2^2\to 1/.$$
$$k\to \text{kv}/.\delta\to\delta\text{v};$$
$$\text{Simplify}[\%,\text{Assump}];$$
$$\text{Simplify}\left[\%/.\text{kv}^2\to k^2/.\delta\text{v}^2\to\delta^2,\ \text{Assump}\right]$$

$$\frac{2\delta^2\text{JacobiDN}[t\Omega,k^2]^2}{\left(\sqrt{c+\delta}-\sqrt{c-\delta}\text{JacobiCN}[t\Omega,k^2]\right)^2}$$

$$\mathbf{p2}=\sigma_3\frac{\sqrt{2}\delta\text{JacobiDN}[\Omega t,k^2]}{\sqrt{c+\delta}-\sqrt{c-\delta}\text{JacobiCN}[\Omega t,k^2]};$$

$$\text{Simplify}\left[\mathbf{p1}^2+\mathbf{p2}^2-2\rho\mathbf{p4}-c/.\ \{\sigma_2\to 1,\sigma_3\to 1\},\ \text{Assump}\right]/.$$
$$\text{JsubsCN}/.k\to\text{kv}/.\delta\to\delta\text{v}//$$
$$\text{Simplify}$$

0

Integral curve test

$$\text{Simplify}\left[\left(D[\mathbf{p1},t]+\left(\frac{1}{\beta_2}\mathbf{p2p4}\right)\right),\ \text{Assump}\right]/.$$
$$\text{JsubsCN};$$
$$\text{Simplify}\left[\%/.k^2\to\text{kv}^2/.\delta\to\delta\text{v}/.\Omega\to\Omega\text{v},\ \text{Assump}\right];$$
$$\text{Simplify}\left[\%/.h\to\frac{c^2-\delta^2}{8\rho^2\beta_2},\ \text{Assump}\right];$$
$$\text{DCond1}=\%;$$
$$\text{Simplify}\left[\left(D[\mathbf{p4},t]-\left(\frac{1}{\beta_1}\mathbf{p1p2}\right)\right),\ \text{Assump}\right]/.$$
$$\text{JsubsCN};$$
$$\text{Simplify}\left[\%/.k^2\to\text{kv}^2/.\delta\to\delta\text{v}/.\Omega\to\Omega\text{v},\ \text{Assump}\right];$$
$$\text{Simplify}\left[\%/.h\to\frac{c^2-\delta^2}{8\rho^2\beta_2},\ \text{Assump}\right];$$
$$\text{DCond2}=\%;$$
$$\text{Simplify}\left[\left(D[\mathbf{p2},t]-\left(\mathbf{p1}\left(\frac{1}{\beta_1}\rho+\frac{1}{\beta_2}\mathbf{p4}\right)\right)\right),\ \text{Assump}\right]/.\text{JsubsCN};$$
$$\text{Simplify}[\%/.k\to\text{kv}/.\delta\to\delta\text{v}/.\Omega\to\Omega\text{v},\ \text{Assump}];$$
$$\text{Simplify}\left[\%/.h\to\frac{c^2-\delta^2}{8\rho^2\beta_2},\ \text{Assump}\right];$$
$$\text{DCond3}=\text{Simplify}[\%,\text{Assump}];$$
$$\text{DConds}=\{\text{DCond1},\text{DCond2},\text{DCond3}\};$$
$$\text{Simplify}[$$
$$\text{DConds}/.\ \{\{\sigma_2\to 1,\sigma_3\to -1\},$$
$$\{\sigma_2\to -1,\sigma_3\to 1\}\},\ \delta>0]$$

$$\{\{0,0,0\},\{0,0,0\}\}$$

Specifically:

$$\text{SimpSub}=\left\{\frac{t}{\sqrt{\frac{\beta_1\beta_2}{\delta}}}\to\Omega t,\ \frac{(c-\delta)\left((c+\delta)\beta_1-2\rho^2\beta_2\right)}{4\delta\rho^2\beta_2}\to k^2\right\};$$

$$\text{DCond1}/.\text{SimpSub}$$

$$\frac{\delta\left((-c+\delta)\sqrt{c+\delta}+\sqrt{c-\delta}(c+\delta)\text{JacobiCN}[t\Omega,k^2]\right)}{\sqrt{2}\rho\left(\sqrt{c+\delta}-\sqrt{c-\delta}\text{JacobiCN}[t\Omega,k^2]\right)^2\beta_2}\times\text{JacobiDN}[t\Omega,k^2](\sigma_2+\sigma_3)$$

$$\text{DCond2}/.\text{SimpSub}$$

$$-\frac{\delta^{3/2}\text{JacobiDN}[t\Omega,k^2]\text{JacobiSN}[t\Omega,k^2]\sqrt{\frac{c^2-\delta^2}{\rho^2\beta_1\beta_2}}(1+\sigma_2\sigma_3)}{\left(\sqrt{c+\delta}-\sqrt{c-\delta}\text{JacobiCN}[t\Omega,k^2]\right)^2}$$

$$\text{DCond3}/.\text{SimpSub}$$

$$-\Bigg(\text{JacobiSN}[t\Omega,k^2]\left(\beta_2\left(c\sqrt{(c-\delta)\delta\beta_2}+\sqrt{(c-\delta)\delta^3\beta_2}\right)\sigma_2\right.$$
$$-\frac{1}{2\rho^2\beta_2}\text{JacobiCN}[t\Omega,k^2]\Big(\beta_1\Big(-c^2\sqrt{\delta(c+\delta)\beta_2^3}$$
$$+\sqrt{\delta^5(c+\delta)\beta_2^3}\Big)-2\rho^2\left(-c\sqrt{\delta(c+\delta)\beta_2^5}+\sqrt{\delta^3(c+\delta)\beta_2^5}\right)\Big)$$
$$\times(\sigma_2+\sigma_3)-\frac{1}{2\rho^2}(c+\delta)\sqrt{(c-\delta)\delta\beta_2}\Big(-2\rho^2\beta_2\sigma_3$$
$$\left.+(c-\delta)\beta_1(\sigma_2+\sigma_3)\Big)\right)\Bigg)$$
$$\Big/\left(\sqrt{2}\left(\sqrt{c+\delta}-\sqrt{c-\delta}\text{JacobiCN}[t\Omega,k^2]\right)^2\sqrt{\beta_1}\beta_2^2\right)$$

Numerical verification

```
ClearAll["Global`*"]
```

$$h=0.5;$$
$$\beta_1=3;$$
$$\beta_2=1;$$
$$\rho=1;$$
$$c=2h\beta_1+\frac{\beta_2}{\beta_1}\rho^2+0.5;$$
$$\delta=\sqrt{c^2-8h\rho^2\beta_2};$$
$$\Omega=\sqrt{\frac{\delta}{\beta_1\beta_2}};$$
$$\sigma=1;$$
$$k=\sqrt{\frac{-c+\delta+4h\beta_1}{2\delta}};$$
$$\text{denom}=\sqrt{c+\delta}-\sqrt{c-\delta}\text{JacobiCN}[\Omega t,k^2];$$
$$\mathbf{p1}=-\sigma\frac{\sqrt{\delta(c^2-\delta^2)\beta_1}}{\rho\sqrt{2\beta_2}}\frac{\text{JacobiSN}[\Omega t,k^2]}{\text{denom}};$$
$$\mathbf{p2}=\sigma\frac{\sqrt{2}\delta\text{JacobiDN}[\Omega t,k^2]}{\text{denom}};$$
$$\mathbf{p4}=\frac{\sqrt{c^2-\delta^2}}{2\rho}\frac{-\sqrt{c-\delta}+\sqrt{c+\delta}\text{JacobiCN}[\Omega t,k^2]}{\text{denom}};$$

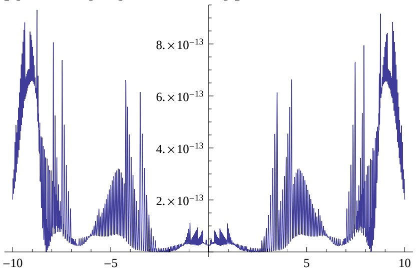
Numerical solution

$$\mathbf{p0}=\{\mathbf{p1},\mathbf{p2},\mathbf{p4}\}/.\{t\to 0\};$$

ns = NDSolve[
$\{D[\text{p1n}[t], t] == \frac{-1}{\beta_2}\text{p2n}[t]\text{p4n}[t]$,
$D[\text{p2n}[t], t] == \text{p1n}[t]\left(\frac{\rho}{\beta_1} + \frac{\text{p4n}[t]}{\beta_2}\right)$,
$D[\text{p4n}[t], t] == \frac{\text{p1n}[t]\text{p2n}[t]}{\beta_1}$,
p1n[0] == p0[[1]], p2n[0] == p0[[2]],
p4n[0] == p0[[3]]},
{p1n[*t*], p2n[*t*], p4n[*t*]},
{*t*, −10, 10}];

Comparison

ERRn =
$(\text{p1n}[t] - \text{p1})^2 + (\text{p2n}[t] - \text{p2})^2 +$
$(\text{p4n}[t] - \text{p4})^2$/.ns;
Plot[{ERRn}, {*t*, −10, 10}]



Max[Table[ERRn, {*t*, −10, 10, 0.01}]]
9.3022×10^{-13}
$MachineEpsilon
2.22045×10^{-16}
($MachineEpsilon gives the difference between 1.0 and the next-nearest number representable as a machine-precision number.)

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