CONTRIBUTIONS TO THE THEORY OF GROUP RINGS

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To Lily

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CHAPTER 1

INTRODUCTION (A BRIEF SURVEY)

1.1 INTRODUCTION

Let G be a multiplicative group and R a ring. The elements of the group ring RG consist of all formal sums $r = \sum_{\alpha} r_{g_\alpha} g_\alpha$, $r_{g_\alpha} \in R$, $g_\alpha \in G$ and $S_r = \{g_\alpha \in G | r_{g_\alpha} \neq 0\}$ is finite. S_r is called the support of r. The operations of addition, scalar multiplication and multiplication are defined as follows:

(i) $\sum_{\alpha} r_{g_\alpha} g_\alpha + \sum_{\alpha} s_{g_\alpha} g_\alpha = \sum (r_{g_\alpha} + s_{g_\alpha}) g_\alpha$

(ii) $a \sum r_{g_\alpha} g_\alpha = \sum [a r_{g_\alpha}] g_\alpha$, $a \in R$

(iii) $([\sum r_{g_\alpha} g_\alpha] \cdot [\sum s_{g_\beta} g_\beta]) = \sum [r_{g_\alpha} \cdot s_{g_\beta}] g_\alpha g_\beta = \sum t_{g_\gamma} g_\gamma$

where $t_{g_\gamma} = \sum_{g_\alpha g_\beta = g_\gamma} r_{g_\alpha} \cdot s_{g_\beta}$

If R is a ring with identity, then the identification of $g \in G$ with $1 \cdot g$ yields a natural embedding of G into this ring.

As the name group ring implies, this subject is a meeting place for two essentially different disciplines and indeed the results are frequently a rather nice blending of group theory and ring theory. In this chapter a short review is offered of the main results in some areas of the theory of group rings. Indications will be given where we succeeded in making some contribution to the problems mentioned.

If G is finite then the group ring can be studied using trace functions and the fairly strong structure theorems for finite dimensional algebras. On the other hand, if G is infinite then these methods are no longer available and the problems are therefore correspondingly more difficult. Except for a few exceptions, the first papers on infinite group rings appeared in the early 1950's. Important impetus was given to this subject by the inclusion of group ring problems in Kaplansky's Ram's Head Inn problems (1957 [24], 1970 [25]) and by the inclusion of group ring material in the books of Lambek (1966 [27]), Ribenboim (1969 [39]) and the monograph of Herstein (1971 [21]). The only books devoted totally to this subject are those of Passman (1972 [35], 1977 [38]), Michalev and Zalesskii (1973 [31]), and Bovdi (1974 [7]). Since group rings are rings after all, the questions we ask about them must necessarily be ring theoretic in nature. On the other hand, the answers and techniques involved usually exhibit a strong group theoretic flavour.

1.2 ZERO DIVISORS AND UNITS

The question of the existence of proper zero divisors in group rings is probably the oldest and least understood problem in the field. There has been some exciting progress recently on this problem ([8] and [13]). We define a *torsion free group* as a group in which every element, except the identity, has infinite order.

If R is a field, then all the elements of the form $\alpha = kx$ with $k \in R$, $k \neq 0$, $x \in G$ are units and we consider these to be trivial. In this section we also discuss the existence of nontrivial units in group rings. If G is a finite cyclic group of order n, generated by an element a, then the zero divisors of ZG, Z denoting the ring of rational integers, are multiples of elements of the form f(a), where f(x) is a non-trivial polynomial, with integral coefficients, which divides the polynomial $x^n - 1$. (See [4], Theorem 2.) In [4] the following result is proved to give a characterization of the zero divisors of ZG in terms of the coefficients, G finite cyclic of order n: If $b = \sum r_g g$ is a right zero divisor of ZG then the determinant

$$\begin{vmatrix} r_{g_1^{-1}g_1} & r_{g_2^{-1}g_1} & \cdots & r_{g_n^{-1}g_1} \\ r_{g_1^{-1}g_2} & r_{g_2^{-1}g_2} & \cdots & r_{g_n^{-1}g_2} \\ \vdots & & & \\ r_{g_1^{-1}g_n} & r_{g_1^{-1}g_n} & \cdots & r_{g_n^{-1}g_n} \end{vmatrix}$$

is zero and conversely, where $g_1, g_2, \ldots, g_n$ are the elements of G in some order.

Let K denote a field.

Lemma 1.2.1 (Passman [35 , p.110])

Let G be a group which is not torsion free. Then KG, K a field, has proper divisors of zero. Moreover, if $|K| > 3$, then KG has nontrivial units.

The question of interest here is whether the converse of the above lemma is true. Namely if G is torsionfree, does it follow that KG has no nonzero zero divisors and only trivial units? The answer is not known and we shall investigate some special cases here.

A group G is said to be a *u.p.-group* (unique product group) if given any two nonempty finite subsets A and B of G, then there exists at least one element $x \in G$ which has a unique representation in the form $x = ab$ with $a \in A$ and $b \in B$.

A group G is said to be a *t.u.p.-group* (two unique products group) if given any two nonempty finite subsets A and B of G with $|A| + |B| > 2$, then there exists at least two distinct elements x and y of G with unique representations in the form $x = ab$, $y = cd$ with $a, c \in A$ and $b, d \in B$. Clearly every t.u.p.-group is a u.p.-group.

Theorem 1.2.2 (Passman [35], p.111)
Let K be an arbitrary field. If G is a u.p.-group, then KG has no proper divisors of zero. If G is a t.u.p.-group, then KG has only trivial units.

In view of the previous lemma, this implies that every u.p.-group is torsion free. The following result shows that such groups form a fairly large class.

Theorem 1.2.3 (G. Higman [22], Rudin-Schneider [43])
Let G be a group. Then any of the following implies that G is a u.p.-group (respectively, a t.u.p.-group):

(i) G has a normal subgroup H such that both H and G/H are u.p.-groups (respectively, t.u.p.-groups).

(ii) G has a family of normal subgroups H_β such that $\cap_\beta H_\beta = \langle 1 \rangle$ and such that G/H_β is a u.p.-group (respectively, a t.u.p.-group).

(iii) Every finitely generated nonidentity subgroup of G can be mapped homomorphically onto a nonidentity u.p.-group (respectively, a nonidentity t.u.p.-group).

The above theorem shows how to construct new t.u.p-groups from known ones but it does not show that such groups exist. We do this below.

A group G is an *ordered group* if it admits a strict linear ordering < such that $x < y$ implies $xz < yz$ and $zx < zy$ for all $z \in G$.

Lemma 1.2.4 (Passman [35, p.113])
Any ordered group is a t.u.p.-group.

Proof: Suppose G is an ordered group and A and B are finite non-empty subsets of G with $|A| + |B| > 2$. Let a^+ and b^+ be the maximal elements of A and B, respectively, and let a^- and b^- be their minimal elements. Then clearly $x = a^+b^+$ and $y = a^-b^-$ are distinct uniquely represented elements of AB. □

There are many group theoretic theorems which guarantee that certain large classes of groups can be ordered. We shall content ourselves here with just a few examples.

Example 1. Any torsion free Abelian group can be ordered (Newmann [32]).

A decending chain of subgroups $G = C_0 \supset C_1 \supset C_2 \supset \ldots C_\alpha \supset C_{\alpha+1} \supset \ldots$ with α varying over ordinals less than a fixed τ is called a *transfinite central series* of G if $C_{\alpha+1}$ is a (normal) subgroup of C_α such that the commutator $[G,C_\alpha]$ is contained in $C_{\alpha+1}$ and, for a limit ordinal α, C_α is the intersection of all C_β with $\beta < \alpha$. Clearly, the C_α are normal in G.

Example 2 (i) If a group G has a transfinite central series ending with $C_\tau = \{e\}$ such that all factor groups are torsion-free then G is an ordered group (Newmann [33]).

(ii) If the lower central series (upper central series) of a group G is of length ω, ω denoting the first infinite ordinal, and the factor groups are torsion free, then G is an ordered group (Newmann [33]).

(iii) Any finitely generated torsion free nilpotent group is an ordered group (Jennings [23]).

Example 3. All free groups (of finite or infinite rank) are ordered groups (Newmann [33]).

Theorem 1.2.5 (Bovdi [6])

Let G be a group and suppose that $G = G_o \supseteq G_1 \supseteq G_2 \supseteq \ldots \supseteq G_n = \langle 1 \rangle$ where G_{i+1} is normal in G_i and G_i/G_{i+1} is torsion free Abelian. If K is any field, then KG has no proper divisors of zero and no nontrivial units.

Proof: By Lemma 1.2.4 and Example 1, G_i/G_{i+1} is a t.u.p.-group. Thus by Theorem 1.2.3(i) and induction, G is a t.u.p.-group. The result follows from Theorem 1.2.2. □

The following example shows that not every t.u.p.-group is an ordered group.

Example 4 (Passman [35, p.114]) Let $G = \langle x \rangle \times_\sigma \langle y \rangle$ be the semi-direct product of two infinite cyclic groups with $y^{-1}xy = x^{-1}$. Since both $\langle x \rangle$ and $G/\langle x \rangle \simeq \langle y \rangle$ are ordered groups by Example 1, it follows from Theorem 1.2.3(i) and Lemma 1.2.4 that G is a t.u.p.-group. On the other hand G is not an ordered group since $x > x^{-1}$ implies

$$x^{-1} = y^{-1}xy > y^{-1}x^{-1}y = x$$

and this in turn implies $x < x^{-1}$, a contradiction.

The following result is a generalization of Theorem 1.2.2.

Theorem 1.2.6 (Passman [36, p.95])

Let G be a u.p.-group and let characteristic (char)K = 0. Then KG has only trivial units.

Now recent work in this field has centered upon groups which are not orderable.

Theorem 1.2.7 (Formanek 1973 [14])

Let K be a field and G a torsion free supersolvable group. Then the group ring KG has no zero divisors.

There has been some encouraging progress recently on the zero divisor problem, in particular in the case of polycyclic-by-finite groups.

If $\mathcal{A}$ and $\mathcal{B}$ are two classes of groups, then G is called a *$\mathcal{A}$-by-$\mathcal{B}$ group* if there is a normal subgroup N of G such that $N \in \mathcal{A}$ and $G/N \in \mathcal{B}$ (Robinson [40, p.2]). A group G is called a *poly-$\mathcal{A}$-group* if G has a series of subgroups $G = G_1 \supseteq G_2 \supseteq \dots \supseteq G_{n+1} = \langle 1 \rangle$, such that for each i, G_{i+1} is a normal subgroup of G_i and the factor group $G_i/G_{i+1} \in \mathcal{A}$.

In a recent remarkable paper, [8], K. Brown shows that KG has no zero divisors, when char K = 0 and G is torsion free Abelian-by-finite. Inspired by the techniques of Brown and by using K-theoretic methods, D.R. Farkas and R.L. Snider [13], proved that if K is a field of char 0 and G is a torsion free polycyclic-by-finite group, then KG has no zero divisors.

If R is commutative then the prime radical is denoted by N(R) or, if there is no ambiguity, by N. A commutative ring R is called *completely primary* if $\overline{R} = R/N$ is a field. The aim of Chapter 2 of this thesis is to determine the ideal theoretic structure of the group ring RG where G is the direct product of a finite Abelian group and an ordered group with R a completely primary ring. Our choice of rings and groups entails that the study centres mainly on zero divisor ideals of group rings and hence it contributes in a small way to the zero divisor problem. We show that if R is a completely primary ring, then there exists a one-one correspondence of the prime zero divisor ideals in RG and $\overline{R}(G)$, G finite cyclic of order n. In [28] the following theorem is proved: $f(x) \in R[x]$, R a commutative ring, is a divisor of zero in R[x] if and only if there exists a nonzero element $c \in R$ such that $cf(x) = 0$. If R is a ring with the property, if $\alpha\beta = 0$ then $\beta\alpha = 0$ for every $\alpha, \beta \in R$, and S is an ordered semigroup, we show that if $\sum \alpha_i s_i \in RS$, $\alpha_i \in R$, is a divisor of zero, then all the coefficients α_i belong to a zero divisor ideal in R. The converse is proved in the case where R is a commutative Noetherian ring. These results are applied to give an account of the divisors of zero in the group ring over the direct product of finite Abelian groups and an ordered group with coefficients in a completely primary ring. Finally we determine the units of the group ring RG where R is not necessarily commutative and G is an ordered group.

If R is a ring such that x, y $\in$ R and xy = 0 then yx = 0 and G an ordered group, then we show that $\sum \alpha_g g$ is a unit in RG if and only if there exists $\sum \beta_h h$ in RG such that $\sum \alpha_g \beta_{g^{-1}} = 1$ and $\alpha_g \beta_h$ is nilpotent whenever gh $\neq$ 1. We also show that if R is a ring with no nilpotent elements $\neq 0$ and no idempotents $\neq 0,1$ then RG has only trivial units. Corresponding results for the group ring, with R commutative, have been obtained by Parmenter [34] and for polynomial rings by Coleman and Enochs [10].

1.3 CHAIN CONDITIONS

A ring is *right Artinian* (*right Noetherian*) if it has the minimum (maximum) condition on right ideals. Similarly, left Artinian and left Noetherian rings are defined. A right (left) Artinian ring is right (left) Noetherian. Either property is inherited by factors.

Proposition 1.3.1 (Connell [11])

The group ring RG is right Artinian if and only if R is right Artinian and G is finite.

Proposition 1.3.2 (Connell [11])

(i) RG is right Noetherian if R is right Noetherian and G is finite.

(ii) If RG is right Noetherian, then R is right Noetherian and every ascending sequence of subgroups of G becomes ultimately stationary.

(iii) If G is Abelian, then RG is right Noetherian if and only if R is right Noetherian and G is finitely generated.

In [35] the following research problem is posed:
Let K be a field. Is it true that KG is Noetherian if and only if G has a series of subgroups $G = G_o \supseteq G_1 \supseteq \ldots \supseteq G_n = \langle 1 \rangle$ such that G_{i+1} is normal in G_i and G_i/G_{i+1} is either a finite group or infinite cyclic?

Lemma 1.3.2 (Passman [35, p.136])
Let H be a normal subgroup of G and suppose that KH is right Noetherian. If G/H is either a finite group or infinite cyclic, then KG is right Noetherian.

The fact that KG is Noetherian if G has the above series is now a consequence of induction and Lemma 1.3.2.

In any ring R, for a nonempty subset T of R, let $\ell(T) = \{x \in R : xt = 0 \text{ for all } t \in T\}$. We call $\ell(T)$ the *left annihilator* of T and term the left ideal L of R a left annihilator if $L = \ell(T)$ for some appropriate T in R. We similarly define the *right annihilator* $r(T)$ of T and speak of a right ideal as a right annihilator.

Definition ([21, p.171])
A ring R is said to be a (left) Goldie ring if:

(i) R satisfies the ascending chain condition on left annihilators.

(ii) R contains no infinite direct sums of left ideals.

A ring R is *right finite dimensional* in the sense of Goldie if it contains no infinite direct sum of right ideals. For a right finite dimensional ring R, there exists a unique integer n such that R contains a direct sum of n-summunds, and the number of summunds in any direct sum is at most n. In this case we write *dim R = n.*

Let A_o denote the class of torsion free Abelian groups.

Proposition 1.3.3 (Brown [9])

Let R be a ring, H a subgroup of G. Then

(i) dim RH $\leq$ dim R

(ii) dim RG = $\sup_T$\{dim RT\}, as T ranges over all subgroups of G such that $T = \langle H, x_1, \ldots, x_m \rangle$, where $x_i \in G$, $1 \leq i \leq m$.

(iii) if there exists a series

$$H = H_o \subset H_1 \subset \ldots \subset H_\alpha \subset H_{\alpha+1} \subset \ldots \subset H_\rho = G$$

where ρ is an ordinal, and for all ordinals α, $0 \leq \alpha < \rho$, dim RH_α = dim $RH_{\alpha+1}$ it follows that dim RG = dim RH.

Proposition 1.3.4 (Brown [9])

Let R be a ring, and let G be a group with a normal subgroup H such that $G/H \in A_o$. Then dim RH = dim RG.

1.4 ON THE RADICALS

Let R be any ring. An ideal P of R is a *prime ideal* if for any two ideals A and B in R, AB $\subseteq$ P implies A $\subseteq$ P or B $\subseteq$ P. A *ring is prime* if 0 is a prime ideal. The *prime radical* P(R) is the intersection of all the prime ideals in R. The *Jacobson radical*

$J(R)$ is the intersection of the maximal right ideals; equivalently $J(R)$ consists of all $x \in R$ such that for all $y \in R$, $1 - xy$ has a right inverse. It turns out that $J(R)$ is also the intersection of the maximal left ideals, and if $x \in J(R)$, then $1 - xy$ is actually a unit for all $y \in R$. An ideal J of R is *nilpotent* if for some integer n, $J^n = 0$. $P(R)$ contains all nilpotent ideals; in fact R is semiprime, $P(R) = 0$, if and only if it contains no nilpotent $J \neq 0$. J is *nil* if every $x \in J$ is nilpotent, i.e., for every $x \in J$ there exists an $n = n(x)$ such that $x^n = 0$. By $\mathcal{U}(R)$ we denote the *upper nil radical*, namely the union of all nil ideals in R. The following relationships hold:

$$P(R) \subseteq \mathcal{U}(R) \subseteq J(R) .$$

Proposition 1.4.1 (Connell [11])

Let R_1 be a subring of R and let H be a subgroup of G, so that R_1G and RH are subrings of RG. Then

(i) $P(R_1G) \supseteq R_1G \cap P(RG)$, with equality if R_1 is contained in the centre of R. Also

(ii) $P(RH) \supseteq RH \cap P(RG)$, with equality if $H \subseteq C$, C the centre of G. For example, putting $H = \{1\}$,

(iii) $P(R) = R \cap P(RG)$,

thus if RG is semiprime, so is R. Next

(iv) $J(RH) \supseteq RH \cap J(RG)$

Putting $H = \{1\}$, we have

(v) $J(R) \supseteq R \cap J(RG)$,

there being equality in (v) if either

(a) R is Artinian, or

(b) G is locally finite (every finitely generated subgroup is finite).

We now consider some relations between certain radicals of the ring R and the group ring RG. In [29] McCoy proved that if R is an arbitrary ring, and R[x] is the ring of polynomials in one indeterminate, then P(R)[x] = P(R[x]). Tan [49] extended this to group rings. He proved the following:

Theorem 1.4.2 (Tan [49])

If R is left Goldie and G is torsion free, then P(RG) = P(R)G.

Theorem 1.4.3 (Tan [48])

If R is a left Goldie ring (with identity) and G the infinite cyclic group, then P(RG) is nilpotent and

$$P(RG) = J(RG) = P(R)G = NG$$

where $N = J(R[x]) \cap R$.

We show that if R is any ring (with identity), then P(R)G = P(RG) if and only if the order of no finite normal subgroup of G is a zero divisor in $\overline{R} = R/P(R)$. Tan's result follows as a corollary from this. In [41] De La Rosa defined quasi-semiprime ideals. We prove that if Q is the quasi-radical of the ring R, i.e., the intersection of all quasi-semiprime ideals in R, then QS is the quasi-radical of RS, R any ring and S a semigroup with unity. We also show that if R is a commutative ring (with identity), then $N(RG) = N(R)G = P(R)G = P(RG)$ if and only if the order of no finite normal subgroup of G is a divisor of zero in R. Here $N(R)$ denotes the nilpotent radical of R, i.e., the union of all two-sided nilpotent ideals of the ring R.

If G is a finite group of order n and R a commutative ring, then we prove that $J(RG) = J(R)G$ if and only if n is not a zero divisor in $\overline{R} = R/J(R)$. If G is locally finite and R commutative then we show that $J(RG) = J(R)G$ if and only if $J(\overline{R}G) = (0)$.

In [44] the class of $2\Omega\Gamma$-semigroups is defined. This class contains the class of ordered groups. We use the results of [44] to get among other results also an extension of Theorem 1.4.3.

1.5 PRIME AND SEMIPRIME RINGS

Theorem 1.5.1 (Connell [11])

RG is semiprime if and only if R is semiprime and the order of no finite normal subgroup is a zero divisor in R.

With any element g of the group G one associates its centralizer $C(g) = \{h \in G | hg = gh\}$. Clearly, C(g) is a subgroup of G. It is not difficult to show that C(g) has finite index if and only if the total number of distinct conjugates of g in G is finite. We shall put

$$G^* = \{g \in G | C(g) \text{ has finite index}\}$$

G^* is a normal subgroup of G. Let G^+ denote the set of elements in G^* of finite order.

Theorem 1.5.2 (Connell [11])

RG is prime if and only if R is prime and $G^+ = 1$.

In [18] Handelman and Lawrence introduced strongly prime rings. A ring R is *right strongly prime* if, given $a \in R$, there exists a finite set $\{x_1, x_2, \ldots, x_n\}$, called a *right insulator* of a, in R such that the set $\{ax_1, \ldots, ax_n\}$ has zero annihilator. Left strongly prime rings are defined analogously, and a ring is said to be strongly prime if it is both right and left strongly prime. Although prime is a symmetric notion, strongly prime is not (for an example see [18]). However, we will generally work on the right, and 'strongly prime' and 'insulator' will denote right strongly prime and right insulator, respectively. We have the following result:

Proposition 1.5.3 (Handelman and Lawrence [18])

(a) If RG is strongly prime, R is strongly prime and G contains no locally finite normal subgroups other than $\{1\}$;

(b) If G is torsion free Abelian and R is strongly prime then RG is strongly prime.

The authors of [18] tried the following generalization of Connell's Theorem 1.5.2: The group ring RG is strongly prime if and only if R is strongly prime and G contains no locally finite normal subgroups (other than 1). Unfortunately, efforts to prove the converse of (a) seem to run into difficulties, paralleling those of the zero divisor problem.

Proposition 1.5.4 [18]

If R is strongly prime and $G = A*B$ is a free product of nontrivial groups A and B (not both of order 2), and $|G| \geq |R|$, then the group ring RG is strongly prime.

In [18] the following questions are posed:

(i) Is there an elementwise characterization of the strongly prime radical?

(ii) Characterize strongly prime group rings.

In connection with question (ii) we proved that, if R is a strongly prime ring and G is an u.p.-group, then RG is a strongly prime ring. If H is a normal subgroup of the group G such that G/H is right ordered, then we show that RG is strongly prime if RH is strongly prime.

We could not characterize the strongly prime radical elementwise, but we succeeded in determining some relations between the strongly prime radical of R and RG.

We introduced the concept of a strongly prime ideal, and defined the strongly prime radical s(R) as the intersection of all the strongly prime ideals in the ring R. If s(R) is the strongly prime radical of the ring R, R a ring with identity, then s(RG) = s(R)G if G is an u.p.-group. If G is a group as defined in Proposition 1.5.4 we also have s(RG) = s(R)G. For the above groups we have proved that RG is semi-strongly prime if R is semi-strongly prime. If R is a commutative semisimple ring and G is a solvable group with no non-trivial locally finite normal subgroups, then RG is a semi-strongly prime ring.

1.6 SEMISIMPLICITY

As is to be expected, the problem of simplicity and semisimplicity of group rings was one of the first to be investigated and a considerable amount of work has been done in this field. Although these problems are only obliquely related to the work done in this thesis, we include a short review of the most important results in this field, mainly since we occassionly refer to results mentioned in this paragraph and also to give a more comprehensive review of the whole field.

Theorem 1.6.1 (Maschke, see [21, p.26])

If K is a field, then KG is semisimple if and only if G is finite and the characteristic of K does not divide the order of G.

In 1958 Villamayor [54] established sufficient conditions for a group algebra over a commutative ring to be semisimple. The main results are: *Theorem 1.6.2:*

(i) If G is free Abelian and R a commutative ring without non-zero nilpotent elements, then RG is semisimple,

(ii) If G is an Abelian group and K is a semisimple commutative algebra over the rationals, then KG is semisimple, and

(iii) If G/Z is locally finite, where Z is the centre of G, and if K is a semisimple commutative algebra over the rationals, then KG is semisimple.

This was followed by Amitsur [3] who established the following results:

Let F denote a field of characteristic 0.

Proposition 1.6.3

If G is any Abelian group and if FH is semisimple for every finitely generated subgroup H of G, then FG is semisimple. Hence, if G is any Abelian group then FG is semisimple for F any field of characteristic 0.

Let G be a group and let Ω be a totally ordered set. A set $(\wedge_\sigma, \vee_\sigma; \sigma \in \Omega)$ of pairs of subgroups $\wedge_\sigma$, $\vee_\sigma$ of G is called a series of G if:

(i) $V_\sigma \lhd \Lambda_\sigma$, all $\sigma \in \Omega$,

(ii) $\Lambda_\sigma \leq V_\tau$, whenever $\sigma < \tau$,

(iii) $G - \{1\} = \bigcup_{\sigma\in\Omega} (\Lambda_\sigma - V_\sigma)$

If all the factors $\Lambda_\sigma / V_\sigma$ are Abelian, then the series is called an Abelian series of G. Kurosh [26] calls a group with an Abelian series an SN-group. If all the factors $\Lambda_\sigma / V_\sigma$ are locally finite over their centre, then the group is called a GSN-group. The class of GSN-groups includes the class of SN-groups.

In 1959 Villamayor [55] extended his result, Theorem 1.6.2(iii), to the class of GSN-groups. In 1962 an article of Connell [11] was published. The author makes a detailed study of the connection between properties of G and R and of RG. A group G is *not torsion* if it has at least one element of infinite order. In connection with the semisimplicity problem he proved the following.

Theorem 1.6.4 (Connell [11, p.671])
Let R be an arbitrary commutative ring and G an Abelian group. If G is a torsion group [not a torsion group], then RG is semisimple if and only if R is semisimple [semiprime] and the order of each finite subgroup of G is regular in R.

As indicated by Passman [36], the high water mark of our knowledge on the semisimplicity problem is the solution of this problem for group rings of solvable groups. In [36] we get the following notation: Let H be a subgroup of G. We say H has locally finite index in G and write $[G:H] = lf$ if $[L : L \cap H] < \infty$

for all finitely generated subgroups $L \subseteq G$. We define

$$\Delta = \Delta(G) = \{x \in G \,|\, [G : C_G(x)] < \infty\}$$

where Δ is a characteristic subgroup of G, the so-called f.c.-subgroup of G. (See def. of G^* p.14).

For solvable groups G, Zalesskii (1973 [59]) proved an intersection theorem of tremendous strength.

Theorem 1.6.5 (Zalesskii [59])

If F is a field of characteristic $p > 0$ and G is a solvable group, then G has a characteristic subgroup $Z(G)$, with $\Delta(Z(G)) = Z(G)$, such that for any nonzero ideal I of FG we have $I \cap FZ(G) \neq 0$.

We call G a Δ-group if $G = \Delta(G)$. In [36] the following result is proved: If F is a field of characteristic $p > 0$ and if, furthermore G has a normal Δ-subgroup H and an element $h \in H$ of order p with $[G : C_G(h)] =$ 1f. Then $J(FG) \neq 0$. Furthermore, at least in the case of solvable groups, the converse also holds. This was proved in a series of three major steps by Hampton-Passman and Zalesskii to give the following:

Theorem 1.6.5 (Hampton-Passman-Zalesskii [17] and [59])

Let F be a field of characteristic $p > 0$ and let G be a solvable group. Then $J(FG) \neq 0$ if and only if $Z(G)$ has an element h of order p with $[G : C_G(h)] =$ 1f.

Finally we remark that Formanek (1972 [15]) has proved the semisimplicity of the group ring of the infinite symmetric group and (Passman [37] and Zalesskii [58] and [60]) studied the semisimplicity problem for locally solvable and linear groups.

CHAPTER 2

THE IDEAL THEORETIC STRUCTURE AND UNITS OF A CLASS OF GROUP RINGS

2.1 INTRODUCTION

The object of the first part of this chapter is to consider the ideal theoretic structure of particular group rings. Our choice of rings and groups entails that the study centres mainly on zero divisor ideals of the group rings. We show that if R is a completely primary ring, then there exists a one-one correspondence of the prime zero divisor ideals in RG and $\overline{R}G$, G finite cyclic of order n.

It is also shown that if G is a finite Abelian group and R a completely primary ring, then RG is the direct sum of completely primary rings.

If S is an ordered semigroup, R a ring such that α, $\beta \in R$ and $\alpha\beta = 0$ implies $\beta\alpha = 0$, we show that if $\sum \alpha_i s_i \in RS$, $\alpha_i \in R$, is a divisor of zero, then all the coefficients α_i belong to a zero divisor ideal in R. The converse is proved in the case where R is a commutative Noetherian ring.

These results are applied to give an account of the divisors of zero in group rings over a group which are the direct product of a finite number of finite cyclic groups and an ordered group, which includes a finitely generated Abelian group, with coefficients in a completely primary ring. This provides information about the ideal theoretic structure of these rings.

In [18] Handelman and Lawrence introduced strongly prime rings. Strongly prime rings are prime and if R is a commutative strongly prime ring, then R is an integral domain. In the first half of this chapter we prove that if R is an integral domain and G an ordered group, then RG is an integral domain. Handelman and Lawrence proved that if R is a strongly prime ring and G is torsion free Abelian, then RG is a strongly prime ring. The aim is now to extend this to a class of groups containing the torsion free Abelian groups and also to determine some more group rings which are strongly prime.

We show that if $H \lhd G$ such that G/H is a right ordered group, then RG is strongly prime if RH is strongly prime. We also prove that if R is a strongly prime ring and G is an unique product group, then RG is strongly prime.

Finally, we consider the units of the group ring RG. If R is a ring, such that x, y $\in$ R and xy = 0 implies yx = 0, and G is an ordered group, we prove that $\sum\alpha_g g$ is a unit in RG if and only if there exists $\sum\beta_h h$ in RG such that $\sum\alpha_g\beta_{g^{-1}} = 1$ and $\alpha_g\beta_h$ is nilpotent whenever $gh \neq 1$. Furthermore, we show that RG has only trivial units if R has no nonzero nilpotent elements and no idempotents $\neq 0,1$. This extends a result of Segal [45], who proved a similar result for R a commutative ring. We apply the above results on units to show that if R and S are local rings (see p.75 of [27]) with no nonzero nilpotent elements and $\sigma : RG \to SG$ is an isomorphism, then $\sigma(R) = S$. Corresponding results for the group ring, with R commutative, have been obtained by Parmenter [34] and for polynomial rings by Coleman and Enochs [10].

2.2 DIVISORS OF ZERO IN THE GROUP RING RG, G A FINITE ABELIAN GROUP

Unless otherwise stated, R shall always be a commutative ring with identity element. It is well known that if R is a commutative ring, then the prime radical coincides with the set of nilpotent elements of the ring. The prime radical is then denoted by N(R) or, if there is no ambiguity, by N. N(R) is then the ideal consisting of all nilpotent elements of R. If R is not commutative, then as in Chapter 1, the prime radical of R will be denoted by P(R) and again, if there is no ambiguity, by P. The word 'radical' shall always refer to the prime radical.

ψ denotes the canonical homomorphism $R \to R/N(R)$ and $\overline{R}$ denotes the ring $R/N(R)$.

ψ induces a homomorphism $\psi^*: R[x] \to \overline{R}[x]$, defined by

$$\psi^*\left(\sum_{i=o}^{n} a_i x^i\right) = \sum_{i=o}^{n} \psi(a_i)x^i, \quad a_i \in R .$$

It is well known that $N(R[x]) = N(R)[x]$. Since $N(R)[x]$ is the kernel of ψ^*, it follows that $R[x]/N(R[x]) \cong \overline{R}[x]$.

For an account of the ideal theory we use, we refer to [28] and [61].

If S is a subset of R, we shall denote the image set of S under the map ψ by $\psi(S)$ or $\overline{S}$, and we shall denote the largest set in R mapped upon $\overline{S}$ by $\psi^{-1}(\overline{S}) = (S, N(R))$.

Lemma 2.2.1. Let ψ be a homomorphism of a ring R onto a ring R' with kernel N. If U is an ideal in R containing N, then U is respectively prime or maximal if and only if $\psi(U)$ is prime or maximal. If U' is an ideal in R', then U' is respectively prime or maximal if and only if $\psi^{-1}(U')$ is prime or maximal.

Proof. See [61] Theorem 11, page 151. □

A commutative ring R is called *completely primary* if $\overline{R}$ is a field.

Let R be a completely primary ring. The prime ideals of $R[x]$ are maximal and of the form $(h(x), N[x])$ where $\overline{h}(x) \in \overline{R}[x]$ is an irreducible polynomial. For since $\overline{R}[x]$ is a P.I.D. and hence every prime ideal is maximal in $\overline{R}[x]$, the remark immediately follows from Lemma 2.2.1.

Proposition 2.2.2. Let R be a completely primary ring. The prime ideals belonging to the ideal (x^n-1) generated by $x^n - 1 \in R[x]$ are all maximal and of the form $(h_i(x), N(R[x]))$ where $\overline{h}_i(x) = \psi^*(h_i(x))$ are irreducible factors of $x^n - \overline{1}$ in $\overline{R}[x]$.

Proof. An immediate consequence of the fact that if $\overline{R}$ is of characteristic p and $n \not\equiv o(p)$ is that there exists an unique factorization $\overline{h}_1(x)\cdot\ldots\cdot\overline{h}_r(x)$ of $x^n - \overline{1}$ into different irreducible polynomials in $\overline{R}[x]$, and if $n \equiv o(p)$, the unique factorization of $x^n - \overline{1}$ is of the form $[\overline{h}_1(x)\cdot\ldots\cdot\overline{h}_r(x)]^{p^s}$ for some positive integer s, all the $\overline{h}_i(x)$ being different.

Note that in the latter case, if $n = mp^s$, $m \not\equiv o(p)$, then $x^n - \overline{1} = (x^m-\overline{1})^{p^s}$ and $\overline{h}_1(x)\cdot\ldots\cdot\overline{h}_r(x) = x^m - \overline{1}$. □

Remark. In the proof all we need of $\overline{R}[x]$ is that it should be a P.I.D. However, the assumption that $\overline{R}[x]$ is a P.I.D. implies

R to be completely primary.

Proposition 2.2.3. Let R be any ring with identity and G a finite group of order n. $P(R)G = P(RG)$ if and only if the order of no normal subgroup of G is a divisor of the characteristic of $\overline{R}$.

To prove this, we need the following:

Lemma 2.2.4. If K is an ideal of R, R not necessarily commutative, such that R/K is semiprime, then $K \supseteq P(R)$.

Proof. See Lambek [27], page 56. □

Lemma 2.2.5 (Connell-Passman)
RG is semiprime if and only if R is semiprime and the order of no finite normal subgroup of G is a zero divisor in R.

Proof. See Lambek [27] Proposition 8, page 162. □

We now prove Proposition 2.2.3.

Suppose the order of no normal subgroup of G is a zero divisor in $\overline{R}$. By Proposition 1.4.1 $P(R) = P(RG) \cap R$. Hence $P(R)G \subseteq P(RG)$, since $P(RG)$ is the intersection of all prime ideals P_i in RG, and if $P(R) \subseteq P_i$, then $P(R)G \subseteq P_i$ for all i.

On the other hand, we have $RG/P(R)G \cong \overline{R}G$. From Lemmas 2.2.4 and 2.2.5 it follows that $P(R)G \supseteq P(RG)$. (The condition on G in Lemma 2.2.5 is satisfied since $n \nmid$ characteristic of $\overline{R}$.) Hence $P(R)G = P(RG)$.

The converse follows from Lemma 2.2.5 and the fact that if $P(R)G = P(RG)$, then $\overline{R}G \cong RG/P(RG)$ and that, consequently, $\overline{R}G$ is semiprime and hence, by Lemma 2.2.5, the order of no normal subgroup of G is a zero divisor in $\overline{R}$. □

Proposition 2.2.6. Let R be a completely primary ring. If G is a finite cyclic group of order n and $\overline{R}$ is of characteristic p, p prime, and $n \equiv o(p)$, say $n = mp^s$, $m \not\equiv o(p)$, then $N(RG) = (g^m - e, N(R)G)$ where g is a generating element of G.

Proof. By Proposition 2.2.2 the intersection of the prime ideals belonging to (x^n-1) is $(x^m-1, N(R[x]))$. Considering the map $\chi : R[x] \to RG$, defined by $\chi(f(x)) = f(g)$, it follows that $N(RG) = (g^m-e, N(R)G)$. □

Theorem 2.2.7. If R is a completely primary ring, then there exists a one-one correspondence of the prime zero divisor ideals in RG and $\overline{R}G$, G finite cyclic of order n. The prime zero divisor ideals in RG are all of the form $(h(g), N(R)G)$ where $\overline{h}(x)$ is an irreducible factor of $x^n - \overline{1}$ in $\overline{R}[x]$ with $\psi^*(h(x)) = \overline{h}(x)$, and they correspond to the ideals $(\overline{h}(g))$ in $\overline{R}G$.

Proof

1) Let the characteristic of $\overline{R}$ be zero or p, p prime, and in the latter case suppose $n \not\equiv o(p)$. Then by Proposition 2.2.3

$$\frac{RG}{N(RG)} = \frac{RG}{N(R)G} \cong \overline{R}G$$

From [11, Theorem 1] $\overline{R}G$ is Artinian and hence every proper ideal in $\overline{R}G$ is a zero divisor ideal. It is now easy to show that every proper ideal of RG also consists of zero divisors. Considering the map χ, we have by Proposition 2.2.2 that the prime ideals are of the required form. Also, since clearly

$\overline{R}G \cong \overline{R}[x]/(x^n-\overline{1})$, the corresponding prime ideals in $\overline{R}G$ are of the form as stated.

2) Let the characteristic of $\overline{R}$ be p and suppose $n = mp^s$, $m \not\equiv o(p)$. Proposition 2.2.3 does not hold and hence RG/N(RG) is not isomorphic to $\overline{R}G$. The proof of the theorem in this case, therefore, needs some modification.

It is well known that if $\gamma : R \to R'$ is a homomorphism between any two rings with kernel I in R then an ideal P' in R' is prime if and only if its pre-image $\gamma^{-1}(P')$, which contains I, is prime. Now clearly $\overline{R}G \cong \overline{R}[x]/(x^n-\overline{1})$. Since $\overline{R}[x]$ is a P.I.D. and $x^n - \overline{1} = (x^m-\overline{1})^{p^s}$, an ideal in $\overline{R}[x]$ containing the ideal $(x^n-\overline{1})$ is prime if and only if it is of the form $(\overline{h}(x))$ where $\overline{h}(x)$ is an irreducible factor of $x^m - \overline{1}$ (and hence of $x^n - \overline{1}$) in $\overline{R}[x]$. Hence ideals in $\overline{R}G$ are prime if and only if they are of the form $(\overline{h}(g))$.

Consider the maps: $\alpha : R[x] \to RG; \quad f(x) \to f(g)$

$$\beta : RG \to \overline{R}G \quad ; \quad f(g) \to \overline{f}(g).$$

α and β have the ideals (x^n-1) in R[x] and N(R)G in RG as kernels respectively, while the composite map $\beta\circ\alpha : R[x] \to \overline{R}G$ has $(N(R)[x], x^n-1)$ as kernel. Let $(\overline{h}(g))$ be an ideal in $\overline{R}G$ and let $\beta(h(g)) = \overline{h}(g)$. Then $(\overline{h}(g))$ is prime if and only if (h(g), N(R)G) in RG is prime. Let $\beta\circ\alpha(h(x)) = \overline{h}(g)$. Then $(\overline{h}(g))$ is prime in $\overline{R}G$ if and only if $(h(x), N(R)[x], x^n-1)$ is prime in R[x] and clearly $\psi^*(h(x)) = \overline{h}(x)$. □

Note that since the ideals $(\overline{h}(x))$ in $\overline{R}[x]$ are maximal, $\overline{h}(x)$ an irreducible factor of $x^n - 1$, $(\overline{h}(g))$ is maximal in $\overline{R}G$ and hence

$(h(g), N(R)G)$ is maximal in RG.

Lemma 2.2.8. Let $f(x) = \sum_{i=o}^{n} a_i x^i \in R[x]$, R a commutative ring with identity. Put $f'(x) = na_n x^{n-1} + (n-1)a_{n-1}x^{n-1} + \ldots + a_1$. Denote $\psi^*(f(x))$ by $\overline{f}(x)$ and $\psi(a)$, $a \in R$, by $\overline{a}$. If $\overline{f}(x) = \overline{0}$ has a solution $\overline{\lambda} \in \overline{R}$ and if $\overline{f}'(\overline{\lambda})$ is a unit in $\overline{R}$, then there exists an element $\lambda^* \in R$ such that $f(\lambda^*) = 0$ and $\psi(\lambda^*) = \overline{\lambda}$. λ^* is uniquely determined.

Proof. Suppose $\lambda \in R$ is such that $\psi(\lambda) = \overline{\lambda}$. Then $f(\lambda) \in N(R)$, say $f(\lambda) = \delta$. Since $\overline{f}'(\overline{\lambda})$ is a unit in $\overline{R}$, $f'(\lambda)$ is a unit in R, and hence $f'(\lambda)\,[f'(\lambda)]^{-1}\delta = \delta$. Put $[f'(\lambda)]^{-1}\delta = \eta$. Then $(\eta) \subseteq (\delta)$. Let $\lambda' = \lambda - \eta$, and expand $f(\lambda-\eta)$ in powers of η:

$$f(\lambda') = f(\lambda) - f'(\lambda)\eta + \frac{f''(\lambda)}{2!}\eta^2 + \ldots + (-1)^n a_n \eta^n$$

$$= \frac{f''(\lambda)}{2!}\eta^2 + \ldots + (-1)^n a_n \eta^n$$

Hence $f(\lambda') \equiv 0 \quad (\eta^2)$ and $(\eta^2) \subseteq (\delta^2)$.

Putting $f(\lambda') = \delta'$, $\delta' \in (\eta^2)$, it follows similarly that $\lambda'' \in R$ exists such that $f(\lambda'') \equiv 0\ (\delta'^2)$ and $(\delta'^2) \subseteq (\delta^4)$. Suppose $\delta^k = 0$. Then we eventually obtain an element $\lambda^* \in R$ such that $f(\lambda^*) \equiv 0(\delta^m)$ where $m \geq k$, and hence $f(\lambda^*) = 0$.

Let λ^* and $\lambda^* + \mu$, $\mu \in N(R)$ be two different solutions of $f(x) = 0$ in R. From $f(\lambda^*+\mu) = f(\lambda^*) = 0$ follows that $f'(\lambda^*)\mu = \alpha$, $\alpha \in (\mu^2)$. Consequently $\mu \in (\mu^{2k})$ for all positive integers k. Therefore, $\mu = 0$ and the solution λ^* is unique. □

Remark. Let $f(x) = x^2 - x$ and suppose $\overline{e} \in \overline{R}$ is an idempotent, i.e. $\overline{e}$ satisfies $\overline{f}(x) = \overline{0}$ $(\overline{f}(x) = \overline{1}\cdot x^2 - \overline{1}x)$. Then $\overline{f}'(\overline{e}) = \overline{2}\,\overline{e} - \overline{1}$ and since $(\overline{2}\,\overline{e} - \overline{1})^2 = \overline{1}$ we have that $\overline{f}'(\overline{e})$ is a unit in $\overline{R}$. Consequently there exists by Lemma 2.2.8 an unique element $e^* \in R$ such that $e^{*2} = e^*$.

Corollary 2.2.9. Let $\psi^*(x^n - 1) = x^n - \overline{1} \in \overline{R}[x]$ and suppose $\overline{\alpha}$ is a root of $x^n - \overline{1} = 0$ in $\overline{R}$.

(1) If $\overline{R}$ is of characteristic q and $(n,q) = 1$, then there exists an element $\alpha \in R$, $\psi(\alpha) = \overline{\alpha}$ which is an uniquely determined root of $x^n - 1 = 0$.

(2) If $\overline{R}$ is of characteristic zero and $\overline{n\alpha^{n-1}}$ is a unit of $\overline{R}$, the same conclusion as in (1) holds for $x^n - \overline{1}$.

(3) If $\overline{R}$ is of characteristic p, p prime, and, say, $n = mp^s$, $m \not\equiv o(p)$ then there exists an element $\alpha \in R$, $\psi(\alpha) = \overline{\alpha}$, which is an uniquely determined root of $x^m - 1 = 0$, and hence is a root of $x^n - 1 = 0$.

Proof

(1) Since $(n,q) = 1$, a positive integer t exists such that $nt \equiv 1(q)$. Also $\overline{\alpha}$ is a unit in $\overline{R}$ since $\overline{\alpha}^n = \overline{1}$. Consequently $\overline{n\alpha^{n-1}}$ is a unit of $\overline{R}$ and the assertion immediately follows from Lemma 2.2.8.

(2) is a direct application of Lemma 2.2.8.

(3) Let $\overline{\alpha} \in \overline{R}$ be a root of $x^n - \overline{1} = \overline{0}$. Then

$$\overline{\alpha}^n - \overline{1} = \overline{\alpha}^{mp^s} - \overline{1} = (\overline{\alpha}^m - \overline{1})^{p^s} = \overline{0}.$$

Since $\overline{R}$ has no nonzero nilpotent elements, it follows that

$\overline{\alpha}^m - \overline{1} = 0$. From $m \not\equiv o(p)$ and (1), we now have that $x^m - 1 = 0$ has a uniquely determined root α in R such that $\psi(\alpha) = \overline{\alpha}$. But since $\alpha^m = 1$ implies $(\alpha^m)^{p^s} = 1$, α is also a root of $x^n - 1 = 0$. □

As a simple application of the previous theorem and its corollary, we prove that if $\overline{R}$ contains all the zeros $\overline{\alpha}_i$ of $x^n - \overline{1}$, there exists unique corresponding zeros α_i of $x^n - 1$ in R such that the product of all the elements $g - \alpha_i \in RG$ is either zero or nilpotent, G cyclic of order n.

Theorem 2.2.10. Let R be a completely primary ring and let G be a cyclic group of order n, generated by g. Suppose $\overline{R}$ contains all the roots of $x^n - \overline{1} = \overline{0}$. Let $\overline{\alpha}_1$ be one such root.

(1) If the characteristic of $\overline{R}$ is zero, or its characteristic is p, p prime, and $n \not\equiv o(p)$, then there exists an unique $\alpha_1^* \in R$, $\overline{\alpha}_1^* = \overline{\alpha}_1$, such that $g - \alpha_1^* e$ is a zero divisor of RG and there exist uniquely determined elements $\alpha_i \in R$, $(i = 2,\ldots,n)$ such that

$$(g-\alpha_1^*)(g-\alpha_2) \ldots (g-\alpha_n) = 0.$$

(2) If $\overline{R}$ is of characteristic p, and $n = mp^s$, $m \not\equiv o(p)$, then similarly α_1^* exists together with uniquely determined elements $\alpha_i \in R$, $(i = 2,\ldots,m)$ such that, for some k,

$$[(g-\alpha_1^*)(g-\alpha_2) \ldots (g-\alpha_m)]^{kp^s} = 0.$$

Proof

(1) By Corollary 2.2.9 we have $\alpha_1^{*n} - 1 = 0$, $\alpha_1^* \in R$. Hence $x - \alpha_1^*$ is a factor of $x^n - 1$, say $x^n - 1 = q(x)(x-\alpha_1^*)$, q(x) a monic polynomial. Again, by Corollary 2.2.9 if $\overline{\alpha}_2$ is a

root different from $\overline{\alpha}_1$ of $x^n - \overline{1} = \overline{0}$, we can find $\alpha_2 \in R$ such that α_2 is a root of $x^n - 1 = 0$. Therefore, $q(\alpha_2)(\alpha_2 - \alpha_1^*) = 0$. Since $\overline{\alpha}_2 - \overline{\alpha}_1$ is a unit in $\overline{R}$, $\alpha_2 - \alpha_1^*$ is a unit in R. Hence $q(\alpha_2) = 0$ and consequently $x - \alpha_2$ is a factor of $q(x)$ and hence of $x^n - 1$. Since $(n,p) = 1$, $nx^{n-1} \neq 0$ for all $x \neq 0$, $x \in \overline{R}$ and hence all the roots of $x^n - 1$ are simple (of multiplicity 1). Eventually $x^n - 1 = (x-\alpha_1^*)(x-\alpha_2) \dots (x-\alpha_n)$. Hence

$$(g-\alpha_1^* e)(g-\alpha_2 e) \dots (g-\alpha_n e) = 0.$$

(2) If $n = mp^s$, $m \not\equiv o(p)$ we can find, as in (1), elements $\alpha_1^*, \alpha_2, \dots, \alpha_m$ of R such that $x^m - 1 = (x-\alpha_1^*)(x-\alpha_2) \dots (x-\alpha_m)$. But $x^n - \overline{1} = (x^m - \overline{1})^{p^s} = [(x-\overline{\alpha}_1)\dots(x-\overline{\alpha}_m)]^{p^s}$, and $RG/N(RG) \cong \overline{R}[x]/(x^m-1) \cong \overline{R}G'$, and therefore,

$$[(g'-\overline{\alpha}_1 e) \dots (g'-\overline{\alpha}_m e)]^{p^s} = 0,$$

where g' is a generator of G' and the image of g under the homomorphism $RG \to \overline{R}G'$. Therefore, $[(g-\alpha_1^* e) \dots (g-\alpha_m e)]^{p^s} = r(g) \in N(RG)$. If k is a positive integer such that $[r(g)]^k = 0$, then

$$[(g-\alpha_1^* e) \dots (g-\alpha_m e)]^{kp^s} = (g^m - e)^{kp^s} = 0. \qquad \square$$

2.3 AN APPLICATION TO DETERMINANTS

Let F be any prime field and G a cyclic group of order n. Suppose $a(g) = \sum_{i=o}^{n-1} a_i g^i$ and $b(g) = \sum_{i=o}^{n-1} b_i g^i$ are divisors of zero of the group ring FG where $a_i \in F$ and $b_i \in F$ $(i = 0,1,\dots,n-1)$

It follows that:

$$\begin{bmatrix} b_o & b_{n-1} & b_{n-2} & \cdots & b_1 \\ b_1 & b_o & b_{n-1} & \cdots & b_2 \\ \vdots & \vdots & \vdots & & \vdots \\ b_{n-1} & b_{n-2} & b_{n-3} & \cdots & b_o \end{bmatrix} \begin{bmatrix} a_o \\ a_1 \\ \vdots \\ a_{n-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \ldots(1)$$

is a necessary and sufficient condition for $a(g)\cdot b(g) = 0$. Hence $b(g)$ is a divisor of zero if and only if the system of equations (1) has a nontrivial solution, i.e. if and only if

$$D = \begin{vmatrix} b_o & b_{n-1} & b_{n-2} & \cdots & b_1 \\ b_1 & b_o & b_{n-1} & & b_2 \\ \vdots & \vdots & \vdots & & \vdots \\ b_{n-1} & b_{n-2} & b_{n-3} & \cdots & b_o \end{vmatrix} = 0 \quad 1)$$

Theorem 2.3.1. $D = 0$ if and only if the nonzero elements b_i are the coefficients of the powers x^i $(i = 0,1,2,\ldots,n-1)$ of a polynomial in $F[x]$ which is a multiple of a product $\phi_{d_1}(x)\cdot\phi_{d_2}(x)\ldots\phi_{d_k}(x)$ of cyclotomic polynomials $\phi_{d_j}(x)$, which are the irreducible factors of $x^n - 1$ in $F[x]$.

Proof. The map χ of the polynomial ring $F[x]$ onto FG, defined by $\chi(f(x)) = f(g)$, $f(x) \in F[x]$, is a homomorphism with kernel the ideal generated by $x^n - 1$. Hence $FG \cong F[x]/(x^n-1)$, and consequently the zero divisor ideals of FG are exactly the images under χ of the ideals generated by the irreducible factors of $x^n - 1$. If F is of characteristic zero or of p, $n \not\equiv o(p)$, then the cyclotomic polynomials $\phi_d(x)$ are the different irreducible factors of $x^n - 1$ and $x^n - 1 = \prod_{d_j|n} \phi_{d_j}(x)$. If $n = mp^s$, $m \not\equiv o(p)$ then $x^m - 1 = \prod_{d_j|m} \phi_{d_j}(x)$

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1) cf. [4], p.46

and $x^n - 1 = (x^m-1)^{p^s}$. Consequently each $\phi_d(x)$ is an irreducible factor of $x^n - 1$. Therefore, the maximal prime ideals related to the ideal (x^n-1) in $F[x]$ are the principal ideals generated by the polynomials $\phi_d(x)$. Hence, by the homomorphism χ, $b(g) \in FG$ is a divisor of zero if and only if $b(g)$ is the image of a polynomial which is a multiple of a product $\phi_{d_1}(x) \dots \phi_{d_k}(x)$. Reducing this polynomial modulo $x^n - 1$ gives $b(x) = b_o + b_1 x + \dots + b_{n-1} x^{n-1}$ as a pre-image of $b(g)$ under the map χ, and clearly $b(x)$ is a multiple of a product of some of the polynomials $\phi_{d_j}(x)$. Since $D = 0$ is a necessary and sufficient condition for $b(g)$ to be a divisor of zero, the theorem is proved. □

Corollary 2.3.2. $b_o + b_1 g + b_2 g^2 + \dots + b_{n-1} g^{n-1}$ is a zero divisor in FG, G cyclic of order n, if and only if $b_i \in F$ $(i = 1,2,\dots,n-1)$ are the coefficients of powers x^i of the polynomial $b(x) \in F[x]$ which is a multiple of a product of some of the irreducible factors of $x^n - 1$.

Remark. The above theorem is immediately deducible from known properties of determinants for the case where the characteristic of the prime field is zero. It is well known[2)] that if F is of characteristic zero, the factors of the circulant:

$$D = \begin{vmatrix} b_o & b_{n-1} & b_{n-2} & \dots & b_1 \\ b_1 & b_o & b_{n-1} & \dots & b_2 \\ \vdots & & & & \\ b_{n-1} & b_{n-2} & b_{n-3} & \dots & b_o \end{vmatrix}$$

are of the form $b_{n-1}\omega^{n-1} + b_{n-2}\omega^{n-2} + \dots + b_o$ ...(2)

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2) [1], p.123

where ω is an n-th root of one. Hence D = 0 if and only if the numbers b_i $(i = 1,2,\ldots,n-1)$ are so chosen as to make one of these factors equal to zero. Hence D = 0 if and only if the polynomials $f(x) = b_{n-1}x^{n-1} + \ldots + b_o$ and $x^n - 1$ have at least one irreducible factor in common.

If F is a prime field, it follows that D = 0 if and only if $b_i \in F$ $(i = 0,1,2,\ldots,n-1)$ are coefficients of a polynomial f(x) which is a multiple of a product of cyclotomic polynomials $\phi_d(x)$ which are the irreducible factors of $x^n - 1$.

2.4 DIVISORS OF ZERO IN RS, S AN ORDERED SEMIGROUP

In [28] the following theorem is proved: $f(x) \in R[x]$ is a divisor of zero in R[x] if and only if there exists a nonzero element $c \in R$, R a commutative ring with identity, such that $cf(x) = 0$. We shall prove a corresponding result for the semigroup ring RS, S ordered.

Theorem 2.4.1. Let S be an ordered semigroup and let R be a ring such that if $\alpha, \beta \in R$ and $\alpha\beta = 0$ then $\beta\alpha = 0$. If $a = \sum_{i=1}^{m} \alpha_i s_i$, $\alpha_i \in R$, $s_i \in S$ and $b = \sum_{i=1}^{n} \beta_j t_j$, $\beta_j \in R$, $t_j \in S$ are two nonzero elements of RS such that ab = 0, n being chosen to be as small as possible and compatible with ab = 0, then $\alpha_i\beta_j = 0$ for all i and j.

Proof. Suppose $s_1 < s_2 < \ldots < s_m$ and $t_1 < t_2 < \ldots < t_n$. If n = 1 then $\alpha_i\beta_1 = 0$, $i = 1,2,\ldots,m$. Suppose n > 1. Then ab = 0 implies $\alpha_m\beta_n = 0$. From our assumption we have $\beta_m\alpha_n = 0$. Now

$a(b\alpha_m) = (ab)\alpha_m = 0$, where

$$b\alpha_m = \beta_1\alpha_m t_1 + \beta_2\alpha_m t_2 + \ldots + \beta_{n-1}\alpha_m t_{n-1}$$

By choice of b we must have $b\alpha_m = 0$. Thus $\beta_j\alpha_m = \alpha_m\beta_j = 0$, $j = 1,2,\ldots,n$. Suppose $\alpha_i\beta_j = 0$, $i = d+1, \ldots, m$; $j = 1,2,\ldots,n$ and $\alpha_d\beta_j \neq 0$ for some positive integer d and some j. Then

$$(\alpha_1 s_1 + \alpha_2 s_2 + \ldots + \alpha_d s_d)(\beta_1 t_1 + \beta_2 t_2 + \ldots + \beta_n t_n)$$

$$= (\alpha_1 s_1 + \alpha_2 s_2 + \ldots + \alpha_d s_d + \alpha_{d+1} s_{d+1} + \ldots + \alpha_m s_m) \times$$
$$(\beta_1 t_1 + \beta_2 t_2 + \ldots + \beta_n t_n)$$
$$= ab = 0\ .$$

Hence $\alpha_d\beta_n = 0$ and consequently $a(b\alpha_d) = (ab)\alpha_d = 0$, where $b\alpha_d = \beta_1\alpha_d t_1 + \beta_2\alpha_d t_2 + \ldots + \beta_{n-1}\alpha_d t_{n-1}$. This contradicts the choice of n. Hence, in fact, $\alpha_i\beta_j = 0$ for $i = 1,2,\ldots,m$ and $j = 1,2,\ldots,n$. □

Remark. The class of rings for which the condition $\alpha\beta = 0$ implies $\beta\alpha = 0$ holds, includes the class of all reduced rings.

An immediate corollary, for R, as in Theorem 2.4.1, is:

Corollary 2.4.2. An element a of RS is a divisor of zero if and only if there exists an element $\beta \in R$, $\beta \neq 0$ such that $\beta a = 0$.

Theorem 2.4.3. Let $a = \sum_{i=1}^{m} \alpha_i s_i$, $s_1 < s_2 < \ldots < s_m$, $\alpha_i \in R$, be a divisor of zero in RS, R a ring such that if $\alpha, \beta \in R$ and $\alpha\beta = 0$ then $\beta\alpha = 0$. Then the ideal $A = (\alpha_1,\alpha_2,\ldots,\alpha_m)$ is a zero divisor ideal of R. If R is a commutative Noetherian ring and $A = (\alpha_1,\ldots,\alpha_m)$, $\alpha_i \in R$, is a zero divisor ideal in R, then conversely $a = \sum_{i=1}^{m} \alpha_i s_i$, $s_1 < s_2 < \ldots < s_m$, is a divisor of zero in RS. In this case

$a \in RS$ is a divisor of zero if and only if $\alpha_i \in A$, $i = 1,2,\ldots,n$ where A is a maximal zero divisor ideal of R.

Proof. If a is a divisor of zero in RS, then, by Corollary 2.4.2, an element $\beta \in R$, $\beta \neq 0$, exists such that $\beta\alpha_i = 0$, $i = 1,\ldots,m$. Let $r \in (\alpha_1,\ldots,\alpha_m) = (\alpha_1) + \ldots + (\alpha_m)$. Then $r = \sum_{i=1}^{m} [(\sum_{j=1}^{n_i} t_{ij}\alpha_i r_{ij}) + t_i\alpha_i + \alpha_i r_i + c_i\alpha_i]$ with t_{ij}, r_{ij}, t_i, $r_i \in R$ and c_i an integer. Since $\beta\alpha_i = 0$ it follows, from our assumption on R, that $\alpha_i\beta = 0$. From this, we get $(t_{ij}\alpha_i)\beta = 0$ and $(t_i\alpha_i)\beta = 0$ and consequently we have $\beta t_{ij}\alpha_i = 0$ and $\beta t_i\alpha_i = 0$, $i = 1,2,\ldots,m$ and $j = 1,2,\ldots,n_i$. Hence

$$\beta[(\sum_{j=1}^{n_i} t_{ij}\alpha_i r_{ij}) + t_i\alpha_i + \alpha_i r_i + c_i\alpha_i] = \sum_{j=1}^{n_i} [(\beta t_{ij}\alpha_i)r_{ij}] + \beta t_i\alpha_i + \beta\alpha_i r_i + c_i\beta\alpha_i = 0.$$

Consequently $\beta r = 0$ and hence $(\alpha_1,\alpha_2,\ldots,\alpha_m)$ is a zero divisor ideal.

Conversely, let R be a commutative Noetherian ring and suppose $A = (\alpha_1,\ldots,\alpha_m)$ is a zero divisor ideal. Since the zero divisor ideals of R are the ideals related to the null ideal of R, A is contained in a maximal zero divisor ideal of R which is a prime ideal belonging to (0). From [61], Corollary 1, p.214, the quotient ideal $[(0):A] = \{c \in R : cA = (0)\} \neq (0)$. Consequently there exists an element $\beta \in R$, $\beta \neq 0$, such that $\beta A = (0)$, and it follows trivially that $\sum_{i=1}^{m} \alpha_i s_i$ is a divisor of zero in RS. □

__Remark__. An immediate consequence of Theorem 2.4.3 is that RS, R an integral domain, and S an ordered semigroup, is again an integral domain (see [46], page 653).

The direct product of a finite number of infinite cyclic groups is an ordered group. Therefore, a finitely generated Abelian group is the direct product of a number of cyclic groups of finite order and an ordered group.

2.5 DIVISORS OF ZERO IN RG, G A DIRECT PRODUCT OF FINITE CYCLIC GROUPS AND AN ORDERED GROUP

Remark. Finitely generated Abelian groups are of the abovementioned type.

Proposition 2.5.1. If R is a commutative ring and I an ideal contained in the prime radical, then if R/I is an Artinian ring, R is the direct sum of completely primary rings.

Proof. $R/N \cong (R/I)/(N/I)$, and since R/I is Artinian, it follows that R/N is Artinian. Hence R/N has only a finite number of prime ideals which are all maximal in R. Consequently R also has only a finite number of prime ideals which are maximal in R, say $M_1, M_2, \ldots, M_k$.

Therefore, $N = M_1 \cap M_2 \cap \ldots \cap M_k$ and M_i $(i = 1,\ldots,k)$ are the prime ideals belonging to the zero ideal (0) of R. It follows that (0) is the product of primary ideals Q_i $(i = 1,2,\ldots,k)$ such that $(Q_i,Q_j) = R$ for $i \neq j$ and M_i is the unique prime ideal belonging to Q_i. The Q_i are uniquely determined. Hence $(0) = Q_1 \cdot Q_2 \cdot Q_3 \cdot \ldots \cdot Q_k$ and R is the direct sum of completely primary rings R_i where $R_i \cong R/Q_i$ $(i = 1,2,\ldots,k)$. (See [50], §§89, 90 and [61], Chapter III, §13.) □

Theorem 2.5.2. If R is a completely primary ring and G a finite Abelian group, then RG is the direct sum of completely primary rings.

Proof. We have $RG/N(R)G \cong \overline{R}G$. Since $\overline{R}$ is a field, it follows from Proposition 1.3.1 that $\overline{R}G \cong RG/N(R)G$ is Artinian. Furthermore, RG is commutative and we know that $N(R)G \subseteq N(RG)$. The result follows by applying Proposition 2.5.1. □

Theorem 2.5.3. Let R be the direct sum of completely primary rings: $R = R_1 + \ldots + R_s$, and let G be the direct product of a finite Abelian group A and an ordered group H : $G = A \times H$. Then $R(G) = S_1(H) + \ldots + S_t(H)$ where S_i $(i = 1,2,\ldots,t)$ is completely primary, and $S = S_1 + \ldots + S_t \cong R(A)$.

If P_i is a radical of S_i, then

$$Q_i = S_1 + \ldots + S_{i-1} + P_i + S_{i+1} + \ldots + S_t$$

is a zero divisor ideal which is maximal in the ring S. If S is Artinian, then $Q_i(H)$ is a maximal zero divisor ideal of R(G).

Proof. To prove the first part of the theorem it is sufficient to refer to Theorem 2.5.2 and to point out that $R(G) = R_1(G) + \ldots + R_s(G)$ while $R_i(A \times H) \cong R_i(A)(H)$.

Clearly Q_i is a zero divisor ideal in S which is maximal in S (see Żariski and Samuel [61], page 175).

If S is Artinian, S not only has a finite number of maximal ideals, but also satisfies the a.c.c. and hence Theorem 2.4.3 implies that $\alpha \in S(H)$ is a divisor of zero if and only if $\alpha \in Q_i(H)$ for some i. Clearly $Q_i(H)$ is an ideal in S(H), and $S(H) = R(G)$.

It is now easy to see that the ideals $Q_i(H)$ are the maximal zero divisor ideals of S(H). This proves the theorem. □

2.6 STRONGLY PRIME GROUP RINGS

In this section all rings are associative, with identity, and usually denoted by R. Strongly prime (SP) rings have been studied in [18]. We say a ring R is *right strongly prime* if, given $a \in R$, $a \neq 0$, there exists a finite set $\{x_1, x_2, \ldots, x_n\}$, called a *right insulator* of a, in R such that the set $\{ax_1, ax_2, \ldots, ax_n\}$ has zero right annihilator. Left strongly prime rings are defined analogously, and a ring is said to be strongly prime if it is both right and left strongly prime.

In [42] Rubin introduced the so-called absolutely torsion free (ATF) rings and in [18] it is shown that the ATF rings are exactly the SP rings. From this remark and [42], Proposition 1.8, it follows that every strongly prime ring is a prime ring. From [42], Proposition 1.9, we also have that if R is a commutative strongly prime ring, then R is an integral domain. Other properties of SP and consequently of ATF rings may be found in [18] and [42].

In [18], Handelman and Lawrence raised the problem of characterising SP-group algebras. They show that if R is a SP ring and G is torsion free Abelian, then RG is a SP ring. The aim of this section is to extend this to a class of groups containing the torsion free Abelian groups and also to determine some more SP-group rings.

In what follows, strongly prime and insulator will denote right strongly prime and right insulator, respectively.

Let G be any group and R a ring with identity. For each $x \in G$ and $\alpha \in RG$ we define ${}^x : RG \to RG$ by $\alpha^x = x^{-1}\alpha x$. From [38] we have that conjugation is always an automorphism of RG. Now, if S is a subset of RG, then S is said to be *G-invariant* if $S^x = S$ for all $x \in G$. In particular, if $H \lhd G$, then RH is G-invariant and if I is an ideal of RG, then I is G-invariant.

A group G is said to be *right ordered* if the elements of G are linearly ordered with respect to the relation < and if, for all $x, y, z \in G$, $x < y$ implies $xz < yz$. Every ordered group is right orderd (see [38], page 586).

Remark. Let H be a normal subgroup of the group G such that G/H can be ordered. Then the elements of any transversal of H in G, i.e. a complete set of coset representatives, can also be ordered: For, if $g_iH < g_jH$ then, by definition, $g_i < g_j$. Let now $g_iH < g_jH$ (i.e., $g_i < g_j$) and $g_kH < g_\ell H$ (i.e., $g_k < g_\ell$), then $g_iHg_kH < g_jHg_\ell H$, i.e. $g_ig_kH < g_jg_\ell H$ and hence, by definition, $g_ig_k < g_jg_\ell$.

Theorem 2.6.1. Let R be any ring with identity. If H is a normal subgroup of the group G such that G/H is right ordered, then RG is a strongly prime ring if RH is strongly prime.

Proof. Let T be a transversal of H in G and suppose RH is strongly prime. We show that every element of RG has an insulator. Let $a \in RG$, then, from [38, Lemma 1.3], a can be written uniquely in the form $\alpha_1 g_1 + \alpha_2 g_2 + \dots + \alpha_m g_m$, $\alpha_i \in RH$, $i = 1,2,3,\dots,m$, and $g_i \in T$ with $g_1 < g_2 < \dots < g_m$. Let $\{\beta_1, \beta_2, \dots, \beta_n\}$ be an insulator for α_m in RH. We show that the finite set $\{(\beta_1)^{g_m}, (\beta_2)^{g_m}, \ldots, (\beta_n)^{g_m}\}$ is an insulator for a. Suppose there exists an element $b \in RG$, $b \neq 0$, such that $a(\beta_i)^{g_m}b = 0$, $i = 1,2,\ldots,n$. Without loss of generality we can assume $b = \sum_{i=1}^{t} \gamma_i v_i$ where $\gamma_i \in RH$, $\gamma_i \neq 0$, $i = 1,2,\ldots,t$; and $v_i \in T$ with $v_1 < v_2 < \ldots < v_t$. Let $g_m v_p$ be the maximal element of the set $\{g_m v_1, \ldots, g_m v_t\}$. Then $g_i v_j < g_m v_p$ for $(i,j) \neq (m,p)$, $1 \leq i \leq m$ and $1 \leq j \leq t$. Since $a(\beta_i)^{g_m}b = 0$ we have $\alpha_m g_m (\beta_i)^{g_m} \gamma_p v_p = 0$, $1 \leq i \leq n$. Hence $\alpha_m g_m (g_m^{-1} \beta_i g_m) \gamma_p v_p =$ $\alpha_m \beta_i (\gamma_p)^{g_m^{-1}} g_m v_p = 0$. Consequently $\alpha_m \beta_i (\gamma_p)^{g_m^{-1}} = 0$, $1 \leq i \leq n$. Furthermore, $(\gamma_p)^{g_m^{-1}} \in RH$ and since conjugation is an automorphism, we have $(\gamma_p)^{g_m^{-1}} \neq 0$. This is impossible since $\{\beta_1, \beta_2, \ldots, \beta_n\}$ is an insulator for α_m. Hence $b = 0$ and, therefore, $\{(\beta_1)^{g_m}, \ldots, (\beta_n)^{g_m}\}$ is an insulator for a. Since the element a was arbitrary, we conclude that RG is strongly prime. □

Theorem 2.6.3. Let R be a strongly prime ring with identity. If G is an u.p. group, then RG is a strongly prime ring.

Proof. Let R be strongly prime and suppose RG is not strongly prime. Then there exists an element $r \in RG$, $r \neq 0$ such that for every finite subset I of RG, we have $\text{Ann}(\{rs : s \in I\}) \neq (0)$. Let $r = \sum_{i=1}^{n} \alpha_i g_i$, $\alpha_i \in R$, $g_i \in G$. Since R is strongly prime, every α_i, $1 \leq i \leq n$, has an insulator S_i in R. Put $J = \bigcup_{i=1}^{n} S_i$. Since J is a finite subset of RG, it follows from our assumption that $\text{Ann}(\{r\sigma : \sigma \in J\}) \neq (0)$. Let t be a nonzero element of RG such that $r\sigma t = 0$ for every $\sigma \in J$, say $t = \sum_{j=1}^{m} \beta_j h_j$, $\beta_j \in R$ and $h_j \in G$. Furthermore, put $A = \text{supp.}r$ and $B = \text{supp.}t$. Since G is an u.p. group, there exists an $x \in G$ which can be represented uniquely in the form $x = g_p h_q$ where $g_p \in A$ and $h_q \in B$. Consequently $\alpha_p \sigma \beta_q = 0$ where σ is any element of S_p. Since $\beta_q \neq 0$ we have a contradiction and consequently RG is strongly prime. □

2.7 UNITS OF RG

In this section, we find the units of the group ring RG where R is a ring with identity and G an ordered group. Let $U(RG)$ denote the units of RG.

Proposition 2.7.1. Let R be any ring with identity and let G be an ordered group. Then the following are equivalent.

(i) $U(RG) = \{\sum \alpha_g g \mid$ there exist elements β_g in R with $\sum \alpha_g \beta_{g^{-1}} = 1$ and $\alpha_g \beta_h = 0$ whenever $gh \neq 1\}$.

(ii) R has no nonzero nilpotent elements.

Proof. Assume (i) holds and let $\gamma \in R$ be nilpotent. Say $\gamma^t = 0$. Then $(1+\gamma g)(1-\gamma g+\gamma^2 g^2-\gamma^3 g^3+\ldots\pm\gamma^{t-1} g^{t-1}) = 1$.
Hence $1 + \gamma g$ is a unit in RG. If $\gamma \neq 0$, $1 + \gamma g$ does not satisfy condition (i). Hence $\gamma = 0$ and (ii) holds.

Conversely, assume (ii) holds and let $ab = 1$ where $a = \sum_{i=1}^{n} \alpha_i g_i$ and $b = \sum_{i=1}^{m} \beta_i h_i$. We will show that $\alpha_i \beta_j = 0$ whenever $g_i h_j \neq 1$. The other statement follows immediately.

Suppose that $g_1 < g_2 < \ldots < g_n$ and $h_1 < h_2 < \ldots < h_m$. For $i \neq n$ or $j \neq m$ we have $g_i h_j < g_n h_m$ and, hence, $g_i h_j \neq g_n h_m$. We want to show $\alpha_i\beta_j = 0$ whenever $g_ih_j > 1$. If $g_nh_m \leq 1$ there is nothing to show. If $g_nh_m > 1$ we have $\alpha_n\beta_m = 0$ from the above. Assume that we know that $\alpha_r\beta_s = 0$ whenever $g_rh_s > g_{i_1}h_{k_1} = g_{i_2}h_{k_2} = \ldots = g_{i_p}h_{k_p} > 1$ (the $g_{i_\ell}h_{k_\ell}$ being a complete list of products equal to $g_{i_1}h_{k_1}$). We see that $\alpha_{i_1}\beta_{k_1} + \ldots + \alpha_{i_p}\beta_{k_p} = 0$ and we may assume that $g_{i_1} < g_{i_2} < g_{i_3} < \ldots < g_{i_p}$. Since $\alpha_r\beta_s = 0$ whenever $g_rh_s > g_{i_1}h_{k_1}$, we have $(\beta_s\alpha_r)^2 = \beta_s\alpha_r\beta_s\alpha_r = 0$. From our assumption, that R has no nonzero nilpotent elements, it follows that $\beta_s\alpha_r = 0$ whenever $g_rh_s > g_{i_1}h_{k_1}$. Now, multiplying the above equation on the right by α_{i_p} we obtain:

$$\alpha_{i_1}\beta_{k_1}\alpha_{i_p} + \alpha_{i_2}\beta_{k_2}\alpha_{i_p} + \ldots + \alpha_{i_p}\beta_{k_p}\alpha_{i_p} = 0$$

For $t < p$, $g_{i_p}h_{k_t} > g_{i_t}h_{k_t}$. Hence, by the remark above, $\alpha_{i_p}\beta_{k_t} = 0$ and hence $\beta_{k_t}\alpha_{i_p} = 0$. We conclude that $\alpha_{i_p}\beta_{k_p}\alpha_{i_p} = 0$. Hence $(\alpha_{i_p}\beta_{k_p})^2 = 0$ and $\alpha_{i_p}\beta_{k_p} = 0$, using (ii). Working back, we obtain $\alpha_{i_t}\beta_{k_t} = 0$ for $1 \leq t \leq p$. Therefore, we have shown that $\alpha_i\beta_j = 0$ whenever $g_ih_j > 1$. To complete the proof, we show that $\alpha_i\beta_j = 0$ whenever $g_ih_j < 1$. For $i, j \neq 1$ we have $g_1h_1 < g_ih_j$ and, hence $g_ih_j \neq g_1h_1$. If $g_1h_1 \geq 1$, there is nothing to show. If $g_1h_1 < 1$ we have $\alpha_1\beta_1 = 0$ from the above. Assume that we know that $\alpha_r\beta_s = 0$ whenever $g_rh_s < g_{i_1}h_{k_1} = g_{i_2}h_{k_2} = \ldots = g_{i_q}h_{k_q} < 1$. We see that $\alpha_{i_1}\beta_{k_1} + \ldots + \alpha_{i_q}\beta_{k_q} = 0$ and we may assume that $g_{i_1} < g_{i_2} < \ldots < g_{i_p}$. As above, we have $\beta_s\alpha_r = 0$ whenever $g_rh_s < g_{i_t}h_{k_t}$, $1 \leq t \leq q$. Now, multiplying the above equation on the right by α_{i_1}, we obtain:

$$\alpha_{i_1}\beta_{k_1}\alpha_{i_1} + \alpha_{i_2}\beta_{k_2}\alpha_{i_1} + \ldots + \alpha_{i_q}\beta_{k_q}\alpha_{i_1} = 0$$

For $t > 1$, $g_{i_1}h_{k_t} < g_{i_t}h_{k_t}$. Hence $\beta_{k_t}\alpha_{i_1} = 0$. As above, we get $\alpha_{i_1}\beta_{k_1} = 0$ and consequently, $\alpha_{i_2}\beta_{k_2} + \ldots + \alpha_{i_q}\beta_{k_q} = 0$. By repeating this argument we get $\alpha_{i_t}\beta_{k_t} = 0$ for $1 \leq t \leq q$. Therefore, we have shown that $\alpha_i\beta_j = 0$ whenever $g_ih_j \neq 1$. □

Lemma 2.7.2. Suppose R is a ring such that if $x, y \in R$ and $xy = 0$ then $yx = 0$. Then the set of nilpotent elements of R forms an ideal.

Proof. [10], Lemma 2. □

Theorem 2.7.3. Suppose R is a ring with identity which satisfies the hypothesis of Lemma 2.7.2. Then $\sum\alpha_g g$ is a unit in RG if and only if there exists $\sum\beta'_h h$ in RG such that $\sum\alpha_g\beta'_{g^{-1}} = 1$ and $\alpha_g\beta'_h$ is nilpotent whenever $gh \neq 1$.

Proof. First assume $\sum\alpha_g g$ is a unit in RG. Let N denote the set of nilpotent elements of R. From Lemma 2.7.2, N is an ideal. Passing from RG to (R/N)G, $\sum\bar{\alpha}_g g$ is a unit in (R/N)G. Proposition 2.7.1 then says that there exists $\sum\bar{\beta}_h h$ in (R/N)G such that $\sum\bar{\alpha}_g\bar{\beta}_{g^{-1}} = \bar{1}$ and $\bar{\alpha}_g\bar{\beta}_h = 0$ whenever $gh \neq 1$. Hence $\sum\alpha_g\beta_{g^{-1}} = 1+n$ where $n \in N$ and $\alpha_g\beta_h$ is nilpotent whenever $gh \neq 1$. If $n^s = 0$, we see that $\sum\alpha_g\beta_{g^{-1}}(1-n+n^2-\ldots\pm n^{s-1}) = (1+n)(1-n+n^2-\ldots\pm n^{s-1}) = 1$ and $\alpha_g\beta_h(1-n+n^2-\ldots\pm n^{s-1})$ is nilpotent whenever $gh \neq 1$, since N is an ideal. Putting $\beta'_h = \beta_h(1-n+n^2-\ldots\pm n^{s-1})$, then $\sum\beta'_g g$ satisfy the required conditions.

Before proving the converse, we show that NG is a nil ideal. Let $a = \sum_{i=1}^{n}\alpha_i g_i \in NG$. The elements α_i are nilpotent with exponents k_i. Let $m = \sum_{i=1}^{n} k_i$. Now, using the property, $\alpha_i\alpha_j = 0$ implies $\alpha_j\alpha_i = 0$, it is easy to show that $a^m = 0$. Hence NG is a nil ideal.

Suppose $\sum \alpha_g g$ satisfy the conditions, then from Proposition 2.7.1 $\sum \bar{\alpha}_g g$ is a unit in (R/N)G. Since NG is a nil ideal and $(R/N)G \cong RG/NG$, we conclude that $\sum \alpha_g g$ is a unit in RG. □

Corollary 2.7.4. Let R be a ring with identity satisfying hypothesis of Lemma 2.7.2 with no idempotents $\neq 0,1$. Then $\sum \alpha_g g$ is a unit in RG if and only if for some g, α_g is a unit and all the other α_g's are nilpotent.

Proof. Suppose $\sum \alpha_g g$ is a unit in RG. Then by Theorem 2.7.3 there exist elements β_g in R such that $\sum \alpha_g \beta_{g^{-1}} = 1$ and $\alpha_g \beta_h$ is nilpotent whenever $gh \neq 1$. Hence $(\sum_g \alpha_g \beta_{g^{-1}})\alpha_h = \alpha_h \beta_{h^{-1}} \alpha_h + \sum_{g \neq h} \alpha_g \beta_{g^{-1}} \alpha_h = \alpha_h$ for any α_h. For $h \neq g$ we have $\alpha_g \beta_{h^{-1}}$ nilpotent, say $(\alpha_g \beta_{h^{-1}})^p = 0$. Then, using the property that $xy = 0$ implies $yx = 0$ and the fact that $(\alpha_g \beta_{h^{-1}})^p = 0$, we get $(\beta_{h^{-1}} \alpha_g)^p = 0$. Hence $\beta_{h^{-1}} \alpha_g$ is also nilpotent and consequently, since the set of nilpotent elements is an ideal, $\alpha_h \beta_{h^{-1}} \alpha_h = \alpha_h + n$ where n is nilpotent. Furthermore, $\alpha_h \beta_{h^{-1}} \alpha_h \beta_{h^{-1}} = \alpha_h \beta_{h^{-1}} + n\beta_{h^{-1}}$ and $\beta_{h^{-1}} \alpha_h \beta_{h^{-1}} \alpha_h = \beta_{h^{-1}} \alpha_h + (\beta_{h^{-1}})n$ and consequently $(\alpha_h \beta_{h^{-1}})^2 = \alpha_h \beta_{h^{-1}} + m_1$ and $(\beta_{h^{-1}} \alpha_h)^2 = \beta_{h^{-1}} \alpha_h + m_2$ where $m_1, m_2 \in N$. Therefore, $\alpha_h \beta_{h^{-1}}$ and $\beta_{h^{-1}} \alpha_h$ are idempotent modulo N, and we conclude that $\alpha_h \beta_{h^{-1}} \in N$ or $\alpha_h \beta_{h^{-1}} - 1 \in N$ and $\beta_{h^{-1}} \alpha_h \in N$ or $\beta_{h^{-1}} \alpha_h - 1 \in N$, since idempotents can be lifted modulo N and R has no idempotents $\neq 0,1$ (see [27], Proposition 1, p.72). If $\alpha_h \beta_{h^{-1}} \in N$, then $\alpha_h \beta_{h^{-1}} \alpha_h \in N$ and by the above $\alpha_h = \alpha_h \beta_{h^{-1}} \alpha_h - n \in N$. Since $\sum \alpha_g \beta_{g^{-1}} = 1$, all the $\alpha_g \beta_{g^{-1}}$ cannot be elements of N (this would imply that $1 \in N$). Say $\alpha_h \beta_{h^{-1}} \notin N$ for some specific h. Then $\alpha_h \beta_{h^{-1}} - 1 \in N$ and hence we have $\alpha_h \beta_{h^{-1}} \alpha_k - \alpha_k \in N$ for each α_k. If $k \neq h$, then $\alpha_k \beta_{h^{-1}} \in N$ and consequently $\beta_{h^{-1}} \alpha_k \in N$ as was shown above. Hence $\alpha_h \beta_{h^{-1}} \alpha_k \in N$

and, therefore $\alpha_k \in N$. Furthermore, from the fact that $\alpha_h\beta_{h^{-1}} \notin N$ we also have $\beta_{h^{-1}}\alpha_h \notin N$, and consequently $\beta_{h^{-1}}\alpha_h - 1 \in N$. Say $\alpha_h\beta_{h^{-1}} = 1 + n_1$ and $\beta_{h^{-1}}\alpha_h = 1 + n_2$, $n_1, n_2 \in N$. From this it now follows easily that $\alpha_h\beta_{h^{-1}}$ and $\beta_{h^{-1}}\alpha_h$ are units in R, say, $\alpha_h\beta_{h^{-1}}\gamma = 1$ and $\gamma'\beta_{h^{-1}}\alpha_h = 1$. Hence α_h has a left as well as a right inverse in R. Hence in $\sum\alpha_g g$ we have α_h a unit in R for some $h \in G$ and all the other α_g's are nilpotent.

The converse follows from Theorem 2.7.3. □

Corollary 2.7.5. Let R be a ring with no nilpotent elements ≠0 and no idempotents ≠0,1. Then the only units in RG are of the form ug where u is a unit of R.

Proof. Let a, b ∈ R such that ab = 0, then baba = $(ba)^2$ = 0. Since R has no nilpotent elements ≠0 the hypothesis of Lemma 2.7.2 is satisfied. The result follows from Corollary 2.7.4. □

For R a commutative ring, a result equivalent to Corollary 2.7.5 have been obtained in [45].

We now apply the above results to study some isomorphic group rings. Recall that a ring R with 1 is called *local* if the non-units of R form an ideal.

Proposition 2.7.6. Let R and S be local rings with no non-zero nilpotent elements. Let G be ordered. If σ : RG → SG is a homomorphism, then $\sigma(R) \subseteq S$.

Proof. Since R is local, for all r in R either r or 1-r is a unit. Hence it is enough to prove that if r is a unit in R, then $\sigma(r)$

belongs to S.

Since R and S are local, they contain no idempotents $\neq 0, 1$. Since they also have no nilpotent elements, Corollary 2.7.5 says that if r is a unit of R, then $\sigma(r) = ug$ for some unit u of S and some g in G. Note that if $g \neq 1$, then $g \neq g^{-1}$ since G is ordered.

Now $\sigma(r+r^{-1}) = ug + u^{-1}g^{-1}$ which is not a unit of SG unless $g = 1$ by Corollary 2.7.5. Hence $r + r^{-1}$ is not a unit of R. Since R is local, $1 - (r+r^{-1})$ is a unit of R. But $\sigma(1-r-r^{-1}) = 1 - ug - u^{-1}g^{-1}$ is not a unit of SG unless $g = 1$. Hence $g = 1$ and $\sigma(R) \subseteq S$ as required. □

Corollary 2.7.7. Let R, S be local with 1, and no non-zero nilpotent elements. Let G be ordered. If $\sigma : RG \to SG$ is an isomorphism, then $\sigma(R) = S$.

For R and S commutative, results equivalent to Proposition 2.7.6 and Corollary 2.7.7 have been obtained in [34].

CHAPTER 3

A STUDY OF RELATIONS BETWEEN IDEALS IN A GROUP RING AND IDEALS IN ITS COEFFICIENT RING

3.1 INTRODUCTION

In this chapter we shall derive results to indicate the relations between certain classes of ideals in R and RG. We also obtain certain relations between different radicals of the ring R and the group ring RG. If S is a semigroup with unity then we get the following relation between the prime radical of R and the prime radical of the semigroup ring RS. If S is an u.p.-semigroup with unity, then $P(R)S = P(RS)$. We also prove some results about the relation between the ideals and the radicals of the group rings RH and RG where H is a central subgroup of G. Here we show that if G/H is an u.p.-group then $P(RH)\cdot RG = P(RG)$ where R is any ring with identity.

Section 3.4 deals with the upper nil radical, which we denote by $\mathcal{U}(R)$ for the ring R. We show that if H is a central subgroup of the group G such that G/H can be ordered, then

$$\mathcal{U}(RG) \subseteq \mathcal{U}(RH)\cdot RG ,$$

where R is a ring with identity. If R is any ring and S an ordered semigroup with unity then we have $\mathcal{U}(RS) \subseteq \mathcal{U}(R)S$.

If R is any ring, $\mathcal{L}(R)$ denotes the Levitzki nil radical of R, i.e. the union of all the locally nilpotent ideals of R. In Section 3.5 we show that if R is any ring and S an ordered semigroup with unity then $\mathcal{L}(RS) = \mathcal{L}(R)S$. We also prove that for any ring and any semigroup with unity $\mathcal{L}(R) = \mathcal{L}(RS) \cap R$. If R is a ring with identity and H is a central subgroup of the group G such that G/H can be ordered, then we have $L(RH)\cdot RG = L(RG)$. Furthermore, we show that $L(RH) = L(RG) \cap RH$ if H is any central subgroup of G.

In Section 3.6 the quasi-semiprime ideals, defined by De La Rosa in [41], are considered. We prove that if Q is the quasi-radical of the ring R, i.e. the intersection of all quasi-semiprime ideals in R, then QS is the quasi-radical of RS where R is any ring and S any semigroup with unity.

In the last section the concept of a strongly prime ring is used to introduce a strongly prime ideal. We define the strongly prime radical s(R) as the intersection of all the strongly prime ideals in the ring R. We show that if G is an u.p.-group or if G is a free product of nontrivial groups (not both order 2) and $|G| \geq |R|$, then s(RG) = s(R)G. If s(R) = (0) we say R is semi-strongly prime. If R is a commutative semisimple ring and G is a solvable group with no nontrivial locally finite normal subgroups, then we prove that RG is a semi-strongly prime ring.

3.2 A GENERAL CLOSURE PROPERTY

Let R be any ring and S any semigroup with unity. The following terminology of McCoy [29] will be used. Let σ be a property of ideals defined for ideals in R and the semigroup ring RS. An ideal with property σ may be called a σ-*ideal*. If A is a subset of R, let $AS = \{a \in RS : a = \sum \alpha_i s_i,\ \alpha_i \in A \text{ and } s_i \in S\}$. Clearly AS is an ideal in RS if and only if A is an ideal in R. If σ is such that A, a σ-ideal in R, implies AS a σ-ideal in RS, we say the *"going up" condition* holds for σ-ideals. If Q, a σ-ideal in RS, implies that $Q \cap R$ is a σ-ideal in R, we shall say the *"going down" condition* holds for σ-ideals. If R is itself a σ-ideal, the σ-property may be used to define a closure operation on the set of all ideals of R. If A is any ideal in R, the *closure* of A relative to σ, which we denote by $\overline{A}$, is the intersection of all σ-ideals which contain A. The *closed ideals* (relative to σ) are the ideals that are intersections of σ-ideals. Everything said of ideals in this paragraph hold mutatis mutandis for right ideals.

In what follows, we shall use the concepts of prime ideal and m-system as defined in [30].

Theorem 3.2.1. Let R be any ring (which need not have an identity) and S a semigroup with unity. Let σ be a property of ideals defined in R and in the semigroup ring RS such that the following are true:

(a) R is a σ-ideal.

(b) If A is any ideal (right ideal) in RS and B a σ-ideal in RS such that $A \cdot RS \subseteq B$, then $A \subseteq B$.

(c) The "going up" and "going down" conditions hold for σ-ideals.

Then for each ideal (right ideal) Q in R, $\overline{QS} = \overline{Q}S$.

Proof. Clearly (a) and (c) implies RS is a σ-ideal in RS. Moreover, it follows easily from (c) that if Q is a closed ideal in R, then QS is a closed ideal in RS and if A is a closed ideal in RS then $A \cap R$ is a closed ideal in R. Hence the "going up" and "going down" conditions hold also for closed ideals.

Since $Q \subseteq \overline{Q}$, we have $QS \subseteq \overline{Q}S$. But by the "going up" condition for closed ideals $\overline{Q}S$ is a closed ideal in RS and hence $\overline{QS} \subseteq \overline{Q}S$. To obtain inclusion in the other direction, suppose $QS \subseteq B$ where B is any closed ideal in RS. Then $B \cap R$ is a closed ideal in R. Since $Q \subseteq B \cap R$ we have $\overline{Q} \subseteq B \cap R$. It follows that $\overline{Q}S \cdot RS \subseteq B$ and (b) implies that $\overline{Q}S \subseteq B$ since $\overline{Q}S \cdot RS$, and hence $\overline{Q}S$ itself, is contained in each σ-ideal containing B. Applying this result to the case in which $B = \overline{QS}$, we have $\overline{Q}S \subseteq \overline{QS}$ and the proof is completed. □

Theorem 3.2.1 holds for polynomial rings R[x], since R[x] is just the semigroup ring of R over a cyclic semigroup (cf. [29]).

3.3 THE PRIME RADICAL OF THE SEMIGROUP RING

In this section we give a simple application of the preceding theorem to the case in which σ-ideal means, prime ideal.

If T is a subset of the ring R, $C_R(T)$ shall denote its complement in R.

As in Section 1.2, a semigroup S is said to be an *u.p.-semigroup* if given any two nonempty finite subsets A and B of S, then there exists at least one element $x \in S$ which has a unique representation in the form $x = ab$ with $a \in A$ and $b \in B$.

Proposition 3.3.1. Let R be any ring and S an u.p.-semigroup with unity. Then the "going up" condition holds for prime ideals.

Proof. Take Q to be a prime ideal in R and suppose QS not prime in RS. Consequently we can find elements $a, b \in C_{RS}(QS)$ such that $a(RS)b \subseteq QS$. Say $a = \sum_{i=1}^{n} \alpha_i t_i$ and $b = \sum_{i=1}^{m} \beta_i s_i$ where $\alpha_i, \beta_i \in R$

and s_i, $t_j \in S$ $(i = 1,2,\ldots,m)$ and $(j=1,2,\ldots,n)$. Since $a, b \in C_{RS}(QS)$ we have that $\alpha_i, \beta_j \notin Q$ for at least one i and j.

Let now $a = a_1 + a_2$ and $b = b_1 + b_2$ where all the coefficients of a_1 and b_1 are in $C_R(Q)$ and all the coefficients of a_2 and b_2 are in Q. Say $a_1 = \alpha'_1 t'_1 + \ldots + \alpha'_\ell t'_\ell$, $b_1 = \beta'_1 s'_1 + \ldots + \beta'_k s'_k$. Let $A = \text{Supp } a_1$ and $B = \text{Supp } b_1$. Since S is an u.p.-semigroup there exists at least one element $s \in S$ which has a unique representation in the form $s = t'_j s'_i$ with $t'_j \in A$ and $s'_i \in B$. Since S has an identity, R is a subring of RS and since QS is an ideal in RS, we have $a_1 R b_1 \subseteq QS$. From the above we have $\alpha'_j R \beta'_i \subseteq Q$ with $\alpha'_j, \beta'_i \notin Q$, contradicting the fact that Q is a prime ideal in R. Therefore, QS must be prime. □

Example. The "going up" condition is not true for any group ring, for if R is a completely primary ring, then since $\overline{R} = R/N(R)$ is a field, N(R) must be a prime ideal in R. From Proposition 2.2.3 we have that if the characteristic of $\overline{R}$ is zero or p, p prime, and $n \not\equiv o(p)$, then $N(R)G = N(RG)$ where G is a finite group of order n. Hence, for the prime ideal $Q = N(R)$, QG need not be a prime ideal.

Proposition 3.3.2. Let R be any ring and S a semigroup with unity. The "going down" condition holds for prime ideals.

Proof. Let A be a prime ideal in RS. To show that $A \cap R$ is a prime ideal, we only need to show that if $\alpha, \beta \in R$ such that $\alpha R \beta \subseteq A \cap R$, then $\alpha \in A \cap R$ or $\beta \in A \cap R$. If $\alpha R \beta \subseteq A \cap R$, then each element of $\alpha RS \beta RS$ is a sum of terms of the form rs, $r \in R$ and $s \in S$, belonging to A and since A is an ideal $\alpha RS\beta RS \subseteq A$. Since

A is a prime ideal in RS, it follows that $\alpha RS \subseteq A$ or $\beta RS \subseteq A$ and consequently $\alpha RS\alpha \subseteq A$ or $\beta RS\beta \subseteq A$. Applying again the fact that A is a prime ideal in RS, it follows that $\alpha \in A$ or $\beta \in A$ and consequently $\alpha \in A \cap R$ or $\beta \in A \cap R$. □

We recall that the *prime radical* of the ring R is the intersection of all prime ideals in R and that a ring is a prime ring if the zero ideal is prime.

Corollary 3.3.3 *(i)* If S is an u.p.-semigroup with unity, then RS is a prime ring if and only if R is a prime ring.

3.3.3 (ii) If R is any ring and S a semigroup with unity then $P(R) = P(RS) \cap R$.

Proof. (i) If R is a prime ring then (0), the zero ideal of R, is a prime ideal and from Proposition 3.3.1 the zero ideal in RS, i.e. (0)S, is a prime ideal. Consequently RS is a prime ring. If the zero ideal (0) of RS is a prime ideal, then we have from Proposition 3.3.2 that $(0) \cap R$, the zero ideal of R, is a prime ideal.

(ii) It is well known that $P(R_1) \supseteq R_1 \cap P(R)$ where R_1 is a subring of some ring R. Consequently $P(R) \supseteq R \cap P(RS)$. To prove inclusion in the other direction, we have

$$\begin{aligned} P(R) &= \cap \{Q : Q \text{ prime ideal in } R\} \\ &\subseteq \cap \{R \cap J : J \text{ prime ideal in } RS\} \text{ (by Proposition 3.3.2)} \\ &= R \cap P(RS) \end{aligned}$$ □

Corollary 3.3.3(ii) is an extension of a result of Connell [11], Proposition 9.

Theorem 3.3.4. If P(R) is the prime radical of the ring R, then the prime radical of RS, S an u.p.-semigroup with unity, is P(R)S, i.e. P(R)S = P(RS).

Proof. Properties (a), (b) and (c) of Theorem 3.2.1 are satisfied with prime ideals as σ-ideals. Since $P(R) = (\overline{0})$ where (0) is the zero ideal in R and since $\overline{(0)}S = \overline{(0)S}$, by Theorem 3.2.1, and the closure of the zero ideal in RS, i.e. (0)S, is P(RS) we have P(R)S = P(RS). □

Corollary 3.3.5. If S is an u.p.-semigroup with unity, then R is a semiprime ring if and only if RS is a semiprime ring.

Let G be any group and H a subgroup of G. Clearly, the group ring RH is a subring of the group ring RG. In what follows, we shall determine the relation between the prime radical of RG and the prime radical of RH.

Lemma 3.3.6. Let R be any ring with identity and H a normal subgroup of the group G. If L is a G-invariant ideal of RH, then $L \cdot RG = \{\sum \alpha_i r_i : \alpha_i \in L \text{ and } r_i \in RG\}$ is a two-sided ideal in RG and $L \cdot RG = RG \cdot L = \{\sum r_i \alpha_i : r_i \in RG, \alpha_i \in L\}$. Furthermore, if A is a two-sided ideal in RH, then $A \cdot RG = \{c \in RG : c = \sum \alpha_i g_i$ where $\alpha_i \in A$ and $g_i \in T$, T a transversal of H in G$\}$ and the expression $\sum \alpha_i g_i$ is unique.

Proof. For the proof of the first part of the lemma, see [38], Lemma 1.5, Ch. I. To prove the second half, let $b = \sum \beta_i g_i$, $\beta_i \in A$ and $g_i \in T$. Then clearly, since R has an identity element we have that $g_i \in RG$ and consequently $b \in A \cdot RG$. By definition of

$A \cdot RG$, $b = \beta_1 r_1 + \beta_2 r_2 + \ldots + \beta_n r_n$, where $\beta_i \in A$ and $r_i \in RG$, $i = 1,2,\ldots,n$. Consider the term $\beta_i r_i$, then, since r_i can be written uniquely in the form $r_i = \sum_k r_{i_k} g_k$ where $r_{i_k} \in RH$ and $g_k \in T$, we have $\beta_i r_i = \sum_k (\beta_i r_{i_k}) g_k$, with $\beta_i r_{i_k} \in A$ since A is an ideal in RH. If now in $b = \beta_1 r_1 + \ldots + \beta_n r_n$ the terms involving the same element of T are grouped together, we have that every coefficient is a sum of terms belonging to A. Consequently every coefficient belongs to A and since $A \subseteq RH$ and b can be written uniquely in the form $b = \sum_k \gamma_k h_k$, $\gamma_k \in RH$ and $h_k \in T$, we have that the linear expression $b = \sum_i \beta_i g_i$, $\beta_i \in A$ and $g_i \in T$, is unique. □

Remark. If Q is an ideal of RH and $a \in RG$, $a = \sum_{i=1}^{n} \alpha_i g_i$, $\alpha_i \in RH$, $g_i \in T$, is the unique representation of the element a, then if $a \notin Q \cdot RG$ we have from Lemma 3.3.6 that there must exist at least one k such that $\alpha_k \in RH$ and $\alpha_k \notin Q$. This will be used frequently in what follows.

The following proprosition has been proved in [11] but our proof differs from that of Connell.

Proposition 3.3.7 ([11], Proposition 3)
Let R be a ring with identity and H a central subgroup of the group G. If A is a prime ideal in RG, then $RH \cap A$ is a prime ideal in RH.

Proof. Let $a, b \in RH$ such that $aRHb \subseteq RH \cap A$. We show that $a \in RH \cap A$ or $b \in RH \cap A$. Let c be any element of aRGbRG, i.e. $c = a(\sum_{i=1}^{m} \alpha_i g_i) b (\sum_{j=1}^{n} \beta_j k_j)$ where $\alpha_i, \beta_j \in RH$ and $g_i, k_j \in T$, T a transversal of H in G. A typical term of c is $a\alpha_i g_i b\beta_j k_j$ and since H is central in G, we have $a\alpha_i g_i b\beta_j k_j = a\alpha_i b\beta_j g_i k_j$ with

$a\alpha_i b \in aRHb \subseteq A$ and $\beta_j g_i k_j \in RG$. Since A is an ideal in RG, $c \in A$ and hence $aRGbRG \subseteq A$. But A is a prime ideal in RG, hence $aRG \subseteq A$ or $bRG \subseteq A$. Now, since $aRGa \subseteq A$ or $bRGb \subseteq A$, again from the fact that A is a prime ideal, it follows that $a \in A \cap RH$ or $b \in A \cap RH$. Hence $A \cap RH$ is a prime ideal in RH. □

Proposition 3.3.8. Let R be any ring with identity and let H be a normal subgroup of G. If Q is a prime ideal in RH which is also G-invariant, and if G/H is an u.p.-group, then Q•RG is a prime ideal in RG.

Proof. From Lemma 3.3.6 we have that Q•RG is an ideal in RG if Q is a G-invariant ideal of RH. Take Q to be a prime ideal in RH and suppose Q•RG is not a prime ideal in RG. This implies that there exist elements $a, b \in RG - Q\cdot RG$ such that $aRGb \subseteq Q\cdot RG$. Let T be a transversal of H in G. Since $a, b \notin Q\cdot RG$ we can write uniquely $a = \sum_{i=1}^{n} \alpha_i g_i + \sum_{j=1}^{m} \beta_j h_j$ and $b = \sum_{i=1}^{p} \alpha'_i g'_i + \sum_{j=1}^{q} \beta'_j h'_j$ where $g_1, g_2, \ldots, g_n, h_1, h_2, \ldots, h_m$ and $g'_1, g'_2, \ldots, g'_p, h'_1, \ldots, h'_q$ are elements of T and where $\alpha_i, \alpha'_i \in RH - Q$, $\beta_j, \beta'_j \in Q$. By assumption, n and $p \neq 0$. Since $aRHb \subseteq aRGb \subseteq Q\cdot RG$ and since Q•RG is an ideal, it follows that $(\sum_{i=1}^{n} \alpha_i g_i)RH(\sum_{i=1}^{p} \alpha'_i g'_i) \subseteq Q\cdot RG$. Let $B = \{g_1, g_2, \ldots, g_n\}$ and $B' = \{g'_1, \ldots, g'_p\}$. Since $\overline{G}$ is an u.p.-group, there exists an uniquely represented element $\overline{g} = \overline{g}_i\overline{g}'_k$ in the product $\overline{B}\,\overline{B}'$, where $\overline{B} = \{\overline{g}_i\}$, $\overline{B}' = \{\overline{g}'_i\}$ and $\overline{g}_i$ $(\overline{g}'_i)$ is the image of g_i (g'_i) under the canonical homomorphism $G \to G/H$. We also have $g_i g'_k = hg$ where $g \in T$ and $h \in H$.

Let r be an arbitrary element of RH. Since conjugation by elements of G is an automorphism of RG and since RH is G-invariant, we have $c = (r)^{g_i} = g_i^{-1} r g_i \in RH$. From the uniqueness of the product $\overline{g_i}\overline{g_k'}$ it follows that the summand $\alpha_i g_i \cdot c \cdot \alpha_k' g_k'$, which reduces to $\alpha_i r (\alpha_k')^{g_i^{-1}} g_i g_k'$ with $(\alpha_k')^{g_i^{-1}} \in RH$ (since RH is G-invariant), cannot be cancelled by any other terms in the product $(\sum_{i=1}^{n} \alpha_i g_i) \cdot c \cdot (\sum_{k=1}^{p} \alpha_k' g_k')$. Furthermore, $\alpha_i r (\alpha_k')^{g_i^{-1}} g_i g_k' = \alpha_i r (\alpha_k')^{g_i^{-1}} hg$ and since $(\sum_{i=1}^{n} \alpha_i g_i) \cdot c \cdot (\sum_{k=1}^{p} \alpha_k' g_k') \in Q \cdot RG$, we have from the uniqueness of the representation of elements of $Q \cdot RG$, as a linear expression $\sum \alpha_i g_i$, $\alpha_i \in Q$ and $g_i \in T$, that $\alpha_i r (\alpha_k')^{g_i^{-1}} h \in Q$. Since $\alpha_k' \notin Q$ and Q is G-invariant we have $(\alpha_k')^{g_i^{-1}} \notin Q$. It is now easy to see that $(\alpha_k')^{g_i^{-1}} h \notin Q$. Since r was arbitrary, $\alpha_i RH (\alpha_k')^{g_i^{-1}} h \subseteq Q$ with α_i, $(\alpha_k')^{g_i^{-1}} h \notin Q$. This contradicts the fact that Q is a prime ideal in RH and consequently $Q \cdot RG$ is a prime ideal in RG. □

The following result was proved in [11]. Since the result follows easily from our previous results, we give the proof.

Proposition 3.3.9. If R is a ring with identity and G a group with central subgroup H, then $P(RH) = P(RG) \cap RH$.

Proof. $P(RH) = \cap \{Q : Q \text{ prime ideal in } RH\}$

$\subseteq \cap \{RH \cap A : A \text{ prime ideal in } RG\}$ (Proposition 3.3.7)

$= RH \cap (\cap \{A : A \text{ prime ideal in } RG\})$

$= RH \cap P(RG)$

Since RH is a subring of RG we have $P(RH) \supseteq RH \cap P(RG)$. Hence $P(RH) = P(RG) \cap RH$. □

Theorem 3.3.10. Let R be a ring with identity and H a normal subgroup of the group G. Let σ be a property of ideals defined in

RH and in RG such that the following are true:

(a) RH is a σ-ideal (in RH).

(b) If A is a σ-ideal in RH which is G-invariant then $A \cdot RG$ is a σ-ideal in RG ("going up" condition).

(c) If Q is a σ-ideal in RG, then $Q \cap RH$ is a G-invariant σ-ideal in RH ("going down" condition).

Then for each ideal D in RH, $\overline{D \cdot RG} = \overline{D} \cdot RG$.

Proof. Remark: A closed ideal in RH is defined as the intersection of G-invariant σ-ideals in RH. Note that consequently a closed ideal in RH is also G-invariant.

From (a) we have that RH is a G-invariant σ-ideal of RH and from (b) it follows that $RH \cdot RG = RG$ is a σ-ideal in RG. Firstly, we show that the "going up" and "going down" conditions also hold for closed ideals. Let B be a closed ideal in RH, say $B = \cap B_j$ where B_j are G-invariant σ-ideals in RH. Furthermore, let T be a transversal of H in G and suppose a is an arbitrary element of $B \cdot RG$, say $a = \sum_{i=1}^{n} \alpha_i g_i$ with $\alpha_i \in B$ and $g_i \in T$. Now,

$$\begin{aligned} a \in B \cdot RG &\Leftrightarrow \alpha_i \in B_j \text{ for all } j \text{ and } i = 1,2,\dots,n \\ &\Leftrightarrow a \in B_j \cdot RG \text{ for all } j \\ &\Leftrightarrow a \in \bigcap_j (B_j \cdot RG) \end{aligned}$$

Hence $B \cdot RG = \cap(B_j \cdot RG)$ and from (b) we have that $B \cdot RG$ is a closed ideal in RG.

For the "going down" condition, let P be a closed ideal in RG where $P = \cap P_j$, P_j a σ-ideal in RG. From (c) we have that $P_j \cap RH$

is a σ-ideal for each j and clearly $P \cap RH = \bigcap(P_j \cap RH)$. Since P_j and RH are G-invariant, $P_j \cap RH$ is G-invariant. Hence $P \cap RH$ is a closed ideal. Therefore, the "going up" and "going down" conditions hold also for closed ideals.

Let A be an arbitrary ideal in RH. Since $A \subseteq \overline{A}$, we have $A \cdot RG \subseteq \overline{A} \cdot RG$. But, by the "going up" condition for closed ideals, $\overline{A} \cdot RG$ is a closed ideal in RG and hence $\overline{A \cdot RG} \subseteq \overline{A} \cdot RG$. To obtain inclusion in the other direction, suppose $A \cdot RG \subseteq F$ where F is any closed ideal in RG. Then $F \cap RH$ is a closed ideal in RG and since R has an identity element it follows that $A \subseteq F$ and consequently $A \subseteq F \cap RH$. Hence $\overline{A} \subseteq F \cap RH$ and, therefore, $\overline{A} \subseteq F$. Since F is an ideal in RG we have $\overline{A} \cdot RG \subseteq F$. Applying this result to the case in which $F = \overline{A \cdot RG}$ we have $\overline{A} \cdot RG = \overline{A \cdot RG}$. □

Remark. If R is any ring and H a central subgroup of the group G, then clearly each ideal in RH is G-invariant.

Theorem 3.3.11. Let R be any ring with identity and H a central subgroup of G such that G/H is an u.p.-group. Then $P(RG) = P(RH) \cdot RG$.

Proof. Properties (a), (b) and (c) of Theorem 3.3.10 are satisfied for prime ideals as σ-ideals (Propositions 3.3.7 and 3.3.8). Since $P(RH) = \overline{(0)}$ where (0) is the zero ideal in RH and since $\overline{(0)} \cdot RG = \overline{(0) \cdot RG}$, by Theorem 3.3.10, and the closure of the zero ideal in RG, i.e. $\overline{(0) \cdot RG}$, is P(RG) we have $P(RH) \cdot RG = P(RG)$. □

3.4 THE UPPER NIL RADICAL OF THE GROUP RING

In [52] Van der Walt introduced a stronger class of prime ideals, called s-prime ideals. Murata et al. [32] extended this further to f-prime ideals.

Van der Walt defines s-prime ideals as follows:

Definition. A set V of elements of the ring R is called an *s-system* if and only if V contains a multiplicatively closed system S, called a kernel of V, such that for every ideal Q of R with $Q \cap V \neq \phi$ we also have $Q \cap S \neq \phi$. The void set ϕ is defined to be an s-system.

Definition. An ideal Q in R is an *s-prime ideal* if and only if its complement C(Q) in R is an s-system.

From [52] we have that an s-prime ideal is also a prime ideal. The s-radical T of the ring R is defined as the intersection of all the s-prime ideals in R. Clearly $P(R) \subseteq T$.

Definition. An ideal Q in the ring R is called a nil radical if and only if (i) Q is nil (ii) R/Q contains no nonzero nilpotent ideals.

Baer [5] showed that the sum U of all the nil radicals of R is a nil radical. U is called the *upper nil radical.* The lower nil radical, intersection of all the nil radicals of R, is known to be the intersection of all the prime ideals in R.

Remark. Van der Walt [52] proved that U is equal to the intersection of all the s-prime ideals in the ring R.

Theorem 3.4.1. Let R be a ring with identity and let H be a normal subgroup of G. If G/H is ordered and Q is a G-invariant s-prime ideal in RH, then Q·RG is an s-prime ideal in RG.

Proof. Suppose Q is an s-prime ideal in RH and let S be the kernel of the s-system $C_{RH}(Q)$. Let E be a transversal of H in G and put $K = \{a : a \in RG : \alpha_1 \in S$ where $a = \sum \alpha_i h_i$, with $\alpha_i \in RH$, $h_i \in E$ with $e = h_1 < h_2 < \ldots < h_n$, e identity element of $G\}$. K is a multiplicative system, for if a, b $\in$ K, say $a = \sum_{i=1}^{n} \alpha_i g_i$, $b = \sum_{i=1}^{m} \beta_j h_j$ with $\alpha_i, \beta_j \in RH$, $g_i, h_j \in E$, $e = g_1 < g_2 < \ldots < g_n$ and $e = h_1 < h_2 < \ldots < h_m$, then $\alpha_1 g_1 \beta_1 h_1 = \alpha_1 \beta_1$ is the "smallest" summand in the product $a \cdot b = \alpha_1 \beta_1 g_1 h_1 + \ldots + \ldots$ Since $\alpha_1 \beta_1 \in S$, we have ab $\in$ K. Clearly K is contained in $C(Q \cdot RG)$, the complement of $Q \cdot RG$ in RG, since the expression $\sum \alpha_i h_i$ for each $a \in K$ is unique and $\alpha_1 \notin Q$. If A is an ideal in RG such that $A \cap C_{RG}(Q\ RG) \neq \phi$, then there exists an element $b \in A$, $b = \sum_{i=1}^{n} \beta_i g_i$ with $\beta_i \in RH$, $g_i \in E$ and $g_1 < g_2 < \ldots < g_n$, such that $\beta_1 \notin Q$. To prove this we only need to prove that $A + Q \cdot RG$ contains such an element. For if $d = \sum_{i=1}^{n} \delta_i g_i$, $\delta_i \in RH$, $g_i \in E$, is an element of $A + Q \cdot RG$ such that $\delta_1 \notin Q$, then $\delta_1 g_1 \notin Q \cdot RG$ and we can write $d = d_1 + d_2$ where $d_1 \in A$ and $d_2 \in Q \cdot RG$ with, say $d_1 = \delta_1' g_1' + \delta_2' g_2' + \ldots + \delta_s' g_s'$, $1 \leq s \leq n$, $g_1 = g_1' < g_2' < \ldots < g_s'$ and $\delta_1 = \delta_1'$.

Let $a = \sum_{i=1}^{n} \alpha_i g_i \in A$, $\alpha_i \in RH$ and $g_i \in E$ with $g_1 < g_2 < \ldots < g_n$, and $a \notin Q \cdot RG$. This is possible since $A \nsubseteq Q \cdot RG$. Thus a contains at least one coefficient $\alpha_k \notin Q$. If k = 1 then we are done. If $\alpha_1 \in Q$, then $a - \alpha_1 g_1 = \alpha_2 g_2 + \ldots + \alpha_n g_n \in A + Q \cdot RG$ with $\alpha_k \notin Q$ for at least one k, $2 \leq k \leq n$. So, again if k = 2 we are done. If, however, $\alpha_2 \in Q$, then $\alpha_3 g_3 + \ldots + \alpha_n g_n \in A + Q \cdot RG$. After a finite number of steps we must have $\alpha_k g_k + \ldots + \alpha_n g_n \in A + Q \cdot RG$ with $\alpha_k \notin Q$, since otherwise $a \in Q \cdot RG$. Hence from the remark above, A contains an element with first coefficient not in Q. Let this element be $b = \sum_{i=1}^{m} \beta_i g_i$, $\beta_i \in RH$ and $\beta_1 \notin Q$, $g_i \in E$ and $g_1 < g_2 < \ldots < g_m$. Furthermore, since A is an ideal in RG, $b' = bg_1^{-1} = \beta_1 + \beta'_2 g'_2 + \ldots + \beta'_m g'_m \in A$ with $\beta'_i \in RH$, $g'_i \in E$, $e = g'_1 < g'_2 < \ldots < g'_m$.

Now, since $(\beta_1) \cap C_{RH}(Q) \neq \phi$, we have $(\beta_1) \cap S \neq \phi$. Let $\lambda \in (\beta_1) \cap S$, then $\lambda = \sum \delta_i \beta_1 \delta'_i$ where $\delta_i, \delta'_i \in RH$. Since A is an ideal in RG, it contains the element $b'' = \sum \delta_i b' \delta'_i$ with λ the coefficient of e. Thus $A \cap K \neq \phi$, and this implies that $C_{RG}(Q \cdot RG)$ is an s-system with kernel K. Hence if Q is an s-prime ideal in RH, then $Q \cdot RG$ is an s-prime ideal in RG. □

If R is any ring and S any semigroup then the elements of RS can be written uniquely as $\sum \alpha_i s_i$, $\alpha_i \in R$, $s_i \in S$. Hence we have the following proposition:

Proposition 3.4.2. If R is any ring and S an ordered semigroup with identity then if Q is an s-prime ideal in R, QS is an s-prime ideal in RS.

Proof. The proof uses the same technique as the proof of Theorem 3.4.1. □

Theorem 3.4.3. Let R be any ring and S an ordered semigroup with unity. Then $\mathcal{U}(RS) \subseteq \mathcal{U}(R)S$.

Proof. Let S' be the set of s-prime ideals in RS and $\mathcal{S}$ the set of s-prime ideals in R.

$$\begin{aligned} \mathcal{U}(RS) &= \cap \{Q : Q \in S'\} && \text{(Remark)} \\ &\subseteq \cap \{PS : P \in S\} && \text{(Proposition 3.4.2)} \\ &= (\cap \{P : P \in S\})S \\ &= \mathcal{U}(R)S \quad \square \end{aligned}$$

Theorem 3.4.4. Let R be any ring with identity and H a central subgroup of G such that G/H is an ordered group. Then $\mathcal{U}(RG) \subseteq \mathcal{U}(RH)\cdot RG$.

Proof.

$$\begin{aligned} \mathcal{U}(RG) &= \cap \{P : P \text{ s-prime ideal in } RG\} && \text{(Remark)} \\ &\subseteq \cap \{Q\cdot RG : Q \text{ s-prime ideal in } RH\} && \text{(Theorem 3.4.1)} \\ &= (\cap \{Q : Q \text{ s-prime ideal in } RH\})\cdot RG \\ &= \mathcal{U}(RH)\cdot RG \quad \square \end{aligned}$$

3.5 THE LEVITZKI RADICAL OF THE GROUP RING

A ring is *locally nilpotent* if any finite set of elements generates a subring which is nilpotent. We say that an ideal I of a ring R is locally nilpotent if, thought of as a ring, I is locally nilpotent. We see that every nilpotent ring is locally nilpotent and every locally nilpotent ring is nil.

The *Levitzki radical* of the ring R, $L(R)$, is defined as the union of all the locally nilpotent ideals of the ring R. $L(R)$ is a locally nilpotent ideal which contains all the locally nilpotent ideals of the ring R (cf. [12], Chapter 6).

In [53] Van der Walt introduced ℓ-prime ideals to give another characterization of the Levitzki radical. Van der Walt defines *ℓ-prime ideals* as follows:

Definition (i) A set L of elements of the ring R is called an *ℓ-system* if for every ideal Q of R such that $Q \cap L \neq \phi$ the ideal Q contains a finite number of elements $b_1, b_2, \ldots, b_{m(Q)}$ such that the following condition is satisfied: If Q and B are ideals such that $Q \cap L \neq \phi$ and $B \cap L \neq \phi$, then for every positive integer $n > 1$, there exists a product of n factors, formed out of $b_1, b_2, \ldots, b_{m(Q)} \in Q$ and $c_1, c_2, \ldots, c_{m(B)} \in B$, which lies in L.

(ii) An ideal Q in R is an *ℓ-prime* ideal if and only if its complement $C_R(Q)$ is an ℓ-system.

From the definitions of s-prime ideal, ℓ-prime ideal and prime ideal we have that every s-prime ideal is an ℓ-prime ideal and every ℓ-prime ideal is a prime ideal.

Proposition 3.5.1. If R is any ring then the Levitzki radical of the ring, $L(R)$, is equal to the intersection of all the ℓ-prime ideals.

Proof. See [53], Theorems 1 and 2. □

Proposition 3.5.2. Let R be a ring with identity and let H be a normal subgroup of G. If G/H is ordered and Q is an ℓ-prime ideal in RH, then Q·RG is an ℓ-prime ideal in RG ("going up" condition).

Proof. Let Q be an ℓ-prime ideal in RH, i.e. $C_{RH}(Q)$ is an ℓ-system. Assume further that $A \cap C_{RG}(Q \cdot RG) \neq \phi$ and $B \cap C_{RG}(Q \cdot RG) \neq \phi$, where A and B are ideals in RG. If T is a transversal of H in G, then there exist, as in Theorem 3.4.1, $a \in A \cap C_{RG}(Q \cdot RG)$ and $b \in B \cap C_{RG}(Q \cdot RG)$ with $a = \sum_{i=1}^{m} \alpha_i g_i$, $b = \sum_{j=1}^{n} \beta_j g_j'$; $\alpha_i, \beta_j \in RH$, $g_i, g_j' \in T$ with $e = g_1 < g_2 < \ldots < g_m$ and $e = g_1' < g_2' < \ldots < g_n'$, such that

α_1, $\beta_1 \notin Q$. Now, since $C_{RH}(Q)$ is an ℓ-system, the principal ideals (α_1) and (β_1), in RH, contains a finite number of elements $\delta_1, \delta_2, \ldots, \delta_r$ and $\gamma_1, \gamma_2, \ldots, \gamma_s$ respectively, such that for every positive integer $p > 1$ there exists a product of p factors, formed out of the δ's and γ's, which lies in $C_{RH}(Q)$.

We may write:

$$\delta_i = \sum_j \pi_{ij}\alpha_1\pi'_{ij}, \quad \pi_{ij}, \pi'_{ij} \in RH, \quad 1 \leq i \leq r,$$

and $$\gamma_j = \sum_i \tau_{ij}\beta_1\tau'_{ij}, \quad \tau_{ij}, \tau'_{ij} \in RH, \quad 1 \leq j \leq s .$$

Let $$a_i = \sum_j \pi_{ij}a\pi'_{ij} \in (a) \subseteq A, \qquad 1 \leq i \leq r,$$

and $$b_j = \sum_i \tau_{ij}b\tau'_{ij} \in (b) \subseteq B, \qquad 1 \leq j \leq s .$$

Clearly, the coefficient of $g_1 = e$ in a_i is δ_i and the coefficient of $g'_1 = e$ in b_j is γ_j.

To complete the proof, we show that for every positive integer $p > 1$, there exists a product of p factors, formed out of the a_i's and b_j's, which lies in $C_{RG}(Q\cdot RG)$. To show this, consider for example, the case where $p = 4$. We consider the product $a_ib_ja_kb_h$ where $i, k \in \{1,2,\ldots,r\}$ and $j, h \in \{1,2,\ldots,s\}$. Since G/H is an ordered group, and consequently the elements of T are ordered, the "smallest" summand in the product $a_ib_ja_kb_h$ is $\delta_ie\gamma_je\delta_ke\cdot\gamma_he = \delta_i\gamma_j\delta_k\gamma_he$. Furthermore, since $\delta_i\gamma_j\delta_k\gamma_h \in C_{RH}(Q)$, we have $a_ib_ja_kb_h \in C(Q\cdot RG)$. From this it is clear that we have the same for any $p > 1$. Hence $C_{RG}(Q\cdot RG)$ is an ℓ-system and consequently $Q\cdot RG$ is an ℓ-prime ideal. □

Proposition 3.5.3. Let R be any ring and S an ordered semigroup with unity. If Q is an ℓ-prime ideal in R, then QS is an ℓ-prime ideal in RS.

Proof. The proof uses the same technique as that of Proposition 3.5.2. □

Proposition 3.5.4. Let R be a ring with identity and H a central subgroup of the group G. If A is an ℓ-prime ideal in RG, then $Q = A \cap RH$ is an ℓ-prime ideal in RH ("going down" condition).

Proof. Let A be an ℓ-prime ideal in RG, i.e. $C_{RG}(A)$ is an ℓ-system. Assume further $D \cap C_{RH}(Q) \neq \phi$ and $B \cap C_{RH}(Q) \neq \phi$, where D and B are ideals in RH and $Q = A \cap RH$. Clearly, $D \cap C_{RH}(Q) \subseteq D\cdot RG \cap C_{RG}(A) \neq \phi$ and $B \cap C_{RH}(Q) \subseteq B\cdot RG \cap C_{RG}(A) \neq \phi$. Since $C_{RG}(A)$ is an ℓ-system, $D\cdot RG$ and $B\cdot RG$ contain a finite number of elements $a_1, a_2, \ldots, a_p$ and $b_1, b_2, \ldots, b_q$ respectively, such that for every positive integer $n > 1$ there exists a product of n factors, formed out of $a_1, a_2, \ldots, a_p$ and $b_1, b_2, \ldots, b_q$, which lies in $C_{RG}(A)$.

Let T be a transversal of H in G. Furthermore, let $a_j = \sum_{i=1}^{r_j} \alpha_{ij} g_{ij}$, $1 \leq j \leq p$, and $b_j = \sum_{i=1}^{s_j} \beta_{ij} g'_{ij}$, $1 \leq j \leq q$, with $\alpha_{ij} \in D$, $g_{ij} \in T$, $1 \leq i \leq r_j$ and also $\beta_{ij} \in B$, $g'_{ij} \in T$, $1 \leq i \leq s_j$. For every $n > 1$ there exists a product of n factors, formed out of the α_{ij}'s and β_{ij}'s that lies in $C_{RH}(Q)$. To show this, consider for example the case where $n = 4$. Since, for example, $a_r b_s a_\nu b_\omega \in C_{RG}(A)$, $r, \nu \in \{1,2,\ldots,p\}$ and $s, \omega \in \{1,2,\ldots,q\}$, and $Q\cdot RG \subseteq A$, we have $a_r b_s a_\nu b_\omega \in C_{RG}(Q\cdot RG)$. Furthermore, since the elements of $Q\cdot RG$ can be written uniquely in the form $\sum \gamma_i g_i$ where $\gamma_i \in Q$ and $g_i \in T$, it follows from $a_r b_s a_\nu b_\omega \notin Q\cdot RG$ that at least one of the coefficients of the product $a_r b_s a_\nu b_\omega$ lies in $C_{RH}(Q)$. Furthermore, since H is a central subgroup of G, such a coefficient is a sum of terms of the form δh where δ is a product of four factors, two out of the α_{ij}'s and two out of the β_{ij}'s, and h is an element of H. Clearly, at least one of the terms in this sum lies in $C_{RH}(Q)$. If this term is $\delta'h'$, then, since Q is an ideal in RH and $h' \in RH$, we have $\delta' \in C_{RH}(Q)$. From this it is clear that we have the same for any $n > 1$. Hence $C_{RH}(Q)$ is an ℓ-system and consequently $Q = A \cap RH$ is an ℓ-prime ideal in RH. □

Proposition 3.5.5. If R is a ring with identity and H is a central subgroup of G, then $L(RH) = L(RG) \cap RH$.

Proof. Since $L(RG)$ is locally nilpotent, $L(RG) \cap RH$ is a locally nilpotent ideal in RH and consequently $L(RG) \cap RH \subseteq L(RH)$.

To prove inclusion in the other direction, we have

$$\begin{aligned} L(RH) &= \cap \{P : P\ \ell\text{-prime ideal in } RH\} \quad \text{(Proposition 3.5.1)} \\ &\subseteq \cap \{Q \cap RH : Q\ \ell\text{-prime ideal in } RG\} \quad \text{(Proposition 3.5.4)} \\ &= L(RG) \cap RH. \quad \square \end{aligned}$$

Proposition 3.5.6. Let R be any ring and S a semigroup with unity. If Q is an ℓ-prime ideal in RS, then $Q \cap R$ is an ℓ-prime ideal in R.

Proof. The proof uses the same technique as that of Proposition 3.5.4. □

Proposition 3.5.7. Let R be any ring and S a semigroup with unity. Then $L(R) = L(RS) \cap R$.

Proof. As in Proposition 3.5.5, $L(RS) \cap R \subseteq L(R)$. For the inclusion in the other direction, we have

$$\begin{aligned} L(R) &= \cap \{P : P\ \ell\text{-prime in } R\} \quad \text{(Proposition 3.5.1)} \\ &\subseteq \cap \{Q \cap R : Q\ \ell\text{-prime in } RS\} \quad \text{(Proposition 3.5.6)} \\ &= L(RS) \cap R. \quad \square \end{aligned}$$

Definition. The ring R is called an *ℓ-prime ring* if and only if the zero ideal of R is an ℓ-prime ideal.

Corollary 3.5.8. Let R be any ring and S an ordered semigroup with unity. Then RS is an ℓ-prime ring if and only if R is an ℓ-prime ring.

Proof. This follows from Propositions 3.5.3 and 3.5.6. □

Theorem 3.5.9. Let R be a ring with identity and H a central subgroup of the group G such that G/H can be ordered. Then $L(RH)\cdot RG = L(RG)$.

Proof. Properties (a), (b) and (c) of Theorem 3.3.10 are satisfied for ℓ-prime ideals as σ-ideals (Propositions 3.5.2 and 3.5.4). Since $L(RH) = \overline{(0)}$ where (0) is the zero ideal in RH and since $\overline{(0)}\cdot RG = \overline{(0)\cdot RG}$, by Theorem 3.3.10, and the closure of the zero ideal in RG, i.e. $(0)\cdot RG$, is $L(RG)$ we have $L(RH)\cdot RG = L(RG)$. □

Theorem 3.5.10. Let R be any ring and S an ordered semigroup with unity. Then $L(R)S = L(RS)$.

Proof. By applying Theorem 3.2.1, the proof follows by a similar argument as in the proof of Theorem 3.5.9. □

Corollary 3.5.11 (i) Let R be a ring with identity and H a central subgroup such that G/H can be ordered. Then $L(RH) = (0)$ if and only if $L(RG) = (0)$.

(ii) Let R be any ring and S an ordered semigroup with unity. Then $L(R) = (0)$ if and only if $L(RS) = (0)$.

Proof. This is clear from Theorems 3.5.9 and 3.5.10. □

3.6 THE QUASI-RADICAL OF THE SEMIGROUP RING RS

In [41] a quasi-semiprime ideal is defined which is a generalization of the concept of a semiprime ideal.

Definition. An ideal Q in a ring R may be called a *quasi-semiprime* (q-semiprime) ideal if from $RAR \subseteq Q$ where A is an ideal in R, it follows that $A \subseteq Q$.

The following theorem, as an easy consequence of this definition, was given in [41].

Theorem 3.6.1. For an ideal Q of the ring R the following statements are equivalent:

(i) Q is a q-semiprime ideal in R.

(ii) If α is an element of R such that $R\alpha R \subseteq Q$, then $\alpha \in Q$.

Proposition 3.6.2. Let R be any ring which need not have an identity and S a semigroup with identity. If Q is a proper ideal in R, then QS is a q-semiprime ideal in RS if and only if Q is a q-semiprime ideal in R ("going up condition).

Proof. Take Q to be a q-semiprime ideal in R. Let $a = \sum_{i=1}^{n} \alpha_i s_i$, $\alpha_i \in R$, $s_i \in S$, be an element of RS such that $RSaRS \subseteq QS$. Suppose $a \notin QS$. Because $a \notin QS$ there exists a k, $1 \leq k \leq n$, such that $\alpha_k \notin Q$. From the fact that Q is a q-semiprime ideal in R we have $R\alpha_k R \not\subseteq Q$, which implies that $\beta\alpha_k\delta \notin Q$ for some $\beta, \delta \in R$. Consider the elements βe and δe in RS, where e is the identity element of S. Now $(\beta e)a(\delta e) = \sum_{i=1}^{n} \beta\alpha_i\delta s_i \in QS$ and we have $\beta\alpha_k\delta \in Q$. This gives a contradiction, and hence $a \in QS$. Consequently if Q is q-semiprime, then QS is q-semiprime.

Conversely, let QS be a q-semiprime ideal in RS and suppose Q not q-semiprime in R. Thus there exists an element β in R, $\beta \notin Q$ such that $R\beta R \subseteq Q$. Hence $RS(\beta e)RS \subseteq QS$ with $\beta e \notin QS$. This contradicts the fact that QS is a q-semiprime ideal, and hence Q is q-semiprime. □

Proposition 3.6.3. Let A be a q-semiprime ideal in RS, R any ring and S a semigroup with identity. Then $A \cap R$ is a q-semiprime ideal in R ("going down" condition).

Proof. Put $Q = A \cap R$ and let α be an element of R such that $R\alpha R \subseteq Q$. Consider an arbitrary element d of RS(QS)RS. Then $d = \sum_{j=1}^{m} a_j q_j b_j$ where $a_j = \sum_{i=1}^{n} \alpha_{ij} t_{ij}$, $b_j = \sum_{i=1}^{k} \beta_{ij} s_{ij} \in RS$ and $q_j = \sum_{i=1}^{\ell} \gamma_{ij} r_{ij} \in QS$. A typical term of $a_j q_j b_j$ is $\alpha_{ij}\gamma_{uj}\beta_{pj}t_{ij}r_{uj}s_{pj}$ with $\alpha_{ij}\gamma_{uj} \in A$, since $\gamma_{uj} \in Q \subseteq A$, and $\beta_{pj}t_{ij}r_{uj}s_{pj} \in RS$. Since A is an ideal in RS, $a_j q_j b_j$ and consenquently d is an element of A. Since d was an arbitrary element of RS(QS)RS we have $RS(QS)RS \subseteq A$ and the fact that A is a q-semiprime ideal implies that $QS \subseteq A$. Furthermore, since each element of $RS(RS\alpha RS)RS$ is a sum of terms belonging to QS, $RS(RS\alpha RS)RS \subseteq QS \subseteq A$. But A is a q-semiprime ideal, hence $RS\alpha RS \subseteq A$ and consequently $\alpha \in A$. We have proved that if $\alpha \in R$ such that $R\alpha R \subseteq Q$, then $\alpha \in A \cap R = Q$, hence Q is a q-semiprime ideal in R. □

Definition. The quasi-radical of the ring R is defined to be the intersection of all q-semiprime ideals in R. The quasi-radical of a ring R is contained in the prime radical of R (see [41]).

Theorem 3.6.4. If q(R) is the quasi-radical of the ring R and q(RS) the quasi-radical of RS, R any ring and S a semigroup with unity, then q(R)S = q(RS).

Proof. If σ-ideal means q-semiprime ideal, condition (c) of Theorem 3.2.1 is satisfied by Propositions 3.6.2 and 3.6.3. Clearly R is a q-semiprime ideal and condition (a) of Theorem 3.2.1 is also satisfied. To show that condition (b) is satisfied, let ARS $\subseteq$ B, A any ideal and B a q-semiprime ideal in RS. Then RSARS $\subseteq$ B $\Rightarrow$ A $\subseteq$ B. □

3.7 THE STRONGLY PRIME RADICAL

In Section 2.6 we considered strongly prime rings. In [18] Handelman and Lawrence considered strongly prime rings in relation of characterizing the strongly prime radical elementwise. We succeeded in determining some relation between the strongly prime radical of a ring R and the strongly prime radical of the group ring RG.

We define a *strongly prime ideal* as follows:

Definition. The ideal Q of the ring R is strongly prime if and only if for every x $\in$ R − Q there exists a finite subset I of R such that if r $\in$ R and xIr $\subseteq$ Q then r $\in$ Q.

It is clear from the definition of a strongly prime ring, that R/Q is a strongly prime ring if and only if Q is a strongly prime ideal.

We now define the *strongly prime radical* of the ring R, s(R), as the intersection of all the strongly prime ideals in the ring R.

From Section 2.6 it follows that every strongly prime ideal is a prime ideal and hence $P(R) \subseteq s(R)$.

Proposition 3.7.1. Let R be a ring with identity and S a semigroup with identity element e. If A is a strongly prime ideal in RS, then $A \cap R$ is a strongly prime ideal in R ("going down" condition).

Proof. Let A be a strongly prime ideal in RS and suppose $A \cap R$ is not strongly prime in R. Then there exists an element x in $C_R(A \cap R) = C_{RS}(A) \cap R$ such that for every finite subset I of R there exists $r \in C_{RS}(A) \cap R$ with $xIr \subseteq A \cap R$. Let J be any finite subset of RS. Say $J = \{z_1, \ldots, z_n\}$, with $z_i = \sum_{j=1}^{n_i} \alpha_{ij} s_{ij}$, $\alpha_{ij} \in R$ and $s_{ij} \in S$, $i = 1, \ldots, n$. Furthermore, let $I_i = \{\alpha_{ij} : j = 1, 2, \ldots, n_i\}$ and $I = \bigcup_{i=1}^{n} I_i$. Then I is a finite subset of R and from our assumption there exists $r \in C_{RS}(A) \cap R$ such that

$$xIr \subseteq A \cap R \subseteq A \qquad \ldots(1)$$

Let $z_i \in J$ be arbitrary. Then

$$xz_i r = x(\alpha_{i1} s_{i1} + \ldots + \alpha_{in_i} s_{in_i})r$$

$$= x\alpha_{i1} r s_{i1} + \ldots + x\alpha_{in_i} r s_{in_i}$$

and from (1) we have $x\alpha_{ij} r \in A$, $j = 1, 2, \ldots, n_i$. Since A is an ideal in RS and $s_i \in RS$ we have $xz_i r \in A$. But z_i was arbitrary, hence $xJr \subseteq A$. This contradicts the fact that A is a strongly prime ideal since $r \notin A$ and consequently $A \cap R$ is a strongly prime ideal in R. □

Definition (Free Product). Let G_1 and G_2 be two multiplicative groups. The group A is said to be the free product of G_1 and G_2 when there exist monomorphisms $f_i : G_i \to A$ $(i = 1, 2)$ so that one has: if an arbitrary group H and homomorphisms $\rho_i : G_i \to H$ are given, then there exists exactly one homomorphism $\psi : A \to H$ such that $\rho_i = \psi f_i$ holds $(i = 1,2)$ (see Kurosh [26]).

Example: A non-cyclic free group is the free product of infinite cyclic groups (Kurosh [26]).

Proposition 3.7.2. Let R be any ring with identity.

(i) If G is an u.p.-group and Q is a strongly prime ideal in R then QG is a strongly prime ideal in RG.

(ii) If $G = A * B$ is a free product of nontrivial groups A and B (not both order 2), and $|G| \geq |R|$ then QG is a strongly prime ideal in RG if Q is a strongly prime ideal in R.

Proof (i) This follows from Theorem 2.6.3, the fact that R/Q is a strongly prime ring if and only if Q is a strongly prime ideal, and the isomorphism $RG/QG \cong (R/Q)G$.

(ii) This is clear from the above isomorphism and Proposition 1.5.4. □

Remark. Let R be any ring with identity and G any group. Then property (b) of Theorem 3.2.1 is satisfied for strongly prime ideals in RG. For let A be any ideal and B a strongly prime ideal in RG such that $ARG \subseteq B$. Suppose $A \not\subseteq B$, then there exists $a \in A$ with $a \notin B$. Furthermore, since B is a strongly prime ideal there exists a finite subset I_a of RG such that $aI_a b \subseteq B \Rightarrow b \in B$. Hence for every $r \notin B$ there exists $s \in I_a$ such that $asr \notin B$. But $a \in A$ and $sr \in RG$, hence $asr \in A \cdot RG \subseteq B$. This contradiction implies that $A \subseteq B$.

Theorem 3.7.3 (i) If s(R) is the strongly prime radical of the ring R, R a ring with identity, then s(RG) = s(R)G if G is an u.p.-group.

(ii) If $G = A*B$ is a free product of nontrivial groups A and B (not both order 2), and $|G| \geq |R|$ then we also have s(RG) = s(R)G.

Proof. From Propositions 3.7.1 and 3.7.2, and the above remark, properties (b) and (c) of Theorem 3.2.1 are satisfied for strongly prime ideals in both cases. Clearly property (a) is also satisfied and consequently the result follows from a direct application of Theorem 3.2.1. □

If the strongly prime radical of a ring R is zero then we call R a *semi-strongly prime ring* (semi-SP ring).

Theorem 3.7.4. A ring R is isomorphic to a subdirect sum of strongly prime rings if and only if R is a semi-SP ring.

Proof. (cf. [30], Theorem 4.27) □

Corollary 3.7.5. If G is a group as in Theorem 3.7.3 then R is a semi-SP ring if and only if RG is a semi-SP ring.

Proof. This is a direct consequence of Theorem 3.7.3. □

Proposition 3.7.6 ([19]). If F is a field and G is a solvable group with no nontrivial locally finite normal subgroups, then FG is strongly prime.

Lemma 3.7.7 ([21]). A commutative semisimple ring is a subdirect sum of fields.

Proof. See [21], page 54 □

Proposition 3.7.8. If R is a commutative semisimple ring and G is a solvable group with no nontrivial locally finite normal subgroups, then RG is a semi-SP ring.

Proof. From Lemma 3.7.7, $R \cong \bigoplus_{\alpha\in\Delta}{}^{s} F_\alpha$, a subdirect sum of some collection of fields F_α. Thus

$$RG \cong (\bigoplus_{\alpha\in\Delta}{}^{s} F_\alpha)G \cong \bigoplus_{\alpha\in\Delta}{}^{s} F_\alpha G$$

However, $F_\alpha G$ is strongly prime for each α, by Proposition 3.7.6, and consequently from Theorem 3.7.4 we have $s(RG) = (0)$. □

CHAPTER 4

RELATIONS BETWEEN VARIOUS RADICALS IN CERTAIN CLASSES OF GROUP RINGS

4.1 INTRODUCTION

In Section 4.2 we define the concept of a radical property. We show that if R is any associative ring and σ is a radical property then the σ-radical of R is the smallest ideal K such that the σ-radical of the ring R/K is zero. This result, applied to specific radical properties, is used in the proofs of many of the theorems that follow.

The aim of Section 4.3 is to extend a result of Tan.. In [49] Tan proved that if R is a left Goldie ring with identity and G a torsion free group, then P(R)G = P(RG), where P(R) denotes the prime radical of the ring R. We show that if R is any ring with identity, then P(R)G = P(RG) if and only if the order of no finite normal subgroup of G is a zero divisor in $\overline{R}$ = R/P(R). Tan's result follows as a corollary from this.

If R is any ring, $\mathcal{N}$(R) denotes the nilpotent radical of R, i.e. the union of all two-sided nilpotent ideals of R. In Section 4.4 we show that if R is commutative or left Goldie, then $\mathcal{N}$(RG) = $\mathcal{N}$(R)G = P(RG) if and only if the order of no finite normal subgroup of G is a divisor of zero in $\overline{R}$ = R/P(R). If R is any ring with identity and H a normal subgroup of G such that G/H is an ordered group then we show that $\mathcal{N}$(RH)·RG = $\mathcal{U}$(RG) = $\mathcal{N}$(RG) if $\mathcal{U}$(RH) is nilpotent. If R is any ring and S an ordered semigroup then $\mathcal{N}$(R)S = $\mathcal{N}$(RS) = $\mathcal{U}$(RS) = P(RS) if $\mathcal{U}$(R) is nilpotent. Let N(R) denote the set of nilpotent elements of the ring R. If S is an ordered semigroup, R a ring such that $\alpha, \beta \in R$ and $\alpha\beta = 0$ implies $\beta\alpha = 0$, then we show that $N(R)S = N(RS) = \mathcal{U}(R)S = \mathcal{U}(RS)$.

In Section 4.5 we consider the Jacobson radical which we denote by J(R) for the ring R. We prove that if G is a finite group of order n and R a commutative ring, then $J(RG) = J(R)G$ if and only if n is not a zero divisor in $\overline{R} = R/J(R)$. We also show that if G is locally finite and R commutative, then $J(RG) = J(R)G$ if and only if $J(\overline{R}G) = 0$.

Section 4.6 deals with the Brown McCoy radical, denoted by $\mathcal{B}(R)$ for the ring R. We show that if R is a simple ring with identity and G a finitely generated torsion free Abelian group, then $\mathcal{B}(RG) = (0)$. We also prove that RG is Brown McCoy semisimple if R is Brown McCoy semisimple, G a finitely generated torsion free Abelian group.

In the last section we determine further relations between some of the previously defined radicals, in particular between P(R), $\mathcal{U}(R)$ and J(R). In [48] Tan proved that if R is a left Goldie ring with identity and G an infinite cyclic group, then $P(RG) = J(RG) = P(R)G$. Connell proved in [11] (Proposition 11) that if G is an ordered group and R a commutative ring, then $J(RG) = (0)$ if and only if $P(R) = (0)$. We introduce the class of $2\Omega\Gamma$ semigroups (defined in [44]), which contains the class of ordered groups, to extend the results mentioned above. We show that if R is a left Goldie ring with identity and S is an ordered semigroup with unity, then $\mathcal{U}(RS) = \mathcal{U}(R)S = P(RS) = J(RS)$. We also show that if R is a commutative ring and S is a 2Ω semigroup then RS is semisimple if and only if R is semiprime.

4.2 A RESULT ON GENERAL RADICAL THEORY

Let σ be a certain property that a ring may possess. A ring R is called a *σ-ring* if it has the property σ. An ideal of a given ring is called a *σ-ideal* if A, viewed as a ring, is a σ-ring. A ring which does not contain any nonzero σ-ideals is said to be *σ-semisimple*.

Definition ([12], page 3)

A property σ is called a *radical property* if the following three conditions are satisfied:

(i) A homomorphic image of a σ-ring is a σ-ring.

(ii) Every ring R contains a σ-ideal S which contains every other σ-ideal in R.

(iii) The factor ring R/S is σ-semisimple.

The unique maximal σ-ideal S of R is called the *σ-radical* of R and is denoted by $\sigma(R)$. A σ-ring is its own σ-radical. Such a ring shall be termed a σ-radical ring. The requirement (ii) ensures that (0) is a σ-radical ring with respect to any given radical property σ. Obviously a ring is σ-semisimple if and only if $\sigma(R) = (0)$.

Lemma 4.2.1. Let R and S be two rings and let ψ be a homomorphism of R onto S. If σ is a radical property then:

(a) $\psi(\sigma(R)) \subseteq \sigma(S)$.

(b) The σ-radical of the ring R is the smallest ideal K such that $\sigma(R/K) = (0)$.

Proof. (a) Since $\sigma(R)$ is a σ-ideal in R it follows from property (i) of the definition that $\psi(\sigma(R))$ is a σ-ideal in S. From property (ii) $\sigma(S)$ is the unique maximal σ-ideal in S. Hence $\psi(\sigma(R)) \subseteq \sigma(S)$.

(b) Let K be an ideal in R such that $\sigma(R/K) = (0)$. Furthermore, let ψ be the canonical homomorphism of R onto R/K. Take r to be an arbitrary element of $\sigma(R)$. From the first part of the lemma it follows that $\psi(r) \in \sigma(R/K) = (0)$. Hence $r \in K$, and consequently $\sigma(R) \subseteq K$. This completes the proof. □

We shall apply this result to some of the radical properties which will be introduced in the rest of this chapter.

4.3 THE PRIME RADICAL OF THE GROUP RING

In [49] Tan proved that if R is a left Goldie ring with identity and G a torsion free group then $P(R)G = P(RG)$. We shall prove a more general result which gives the result of Tan as a corollary.

Proposition 4.3.1. Let R be any ring. The prime radical of the ring R is the smallest ideal K such that $P(R/K) = (0)$.

Proof. Let σ be the lower radical property determined by all the nilpotent rings. Then, if R is any ring $\sigma(R) = P(R)$ (cf. [12], Theorems 17 and 18). Hence if K is any ideal in R such that $P(R/K) = (0)$ it follows from Lemma 4.2.1 that $P(R) \subseteq K$. □

Theorem 4.3.2. Let R be any ring with identity. Then $P(R)G = P(RG)$ if and only if the order of no finite normal subgroup of G is a zero divisor in $\overline{R} = R/P(R)$.

Proof. Suppose the order of no finite normal subgroup of G is a zero divisor in $\overline{R}$. From Corollary 3.3.3 (ii) we have $P(R) = P(RG) \cap R$ and consequently $P(R)G \subseteq P(RG)$ since $P(RG)$ is an ideal.

On the other hand, we have $RG/P(R)G \cong [R/P(R)]G = \overline{R}G$. However, $\overline{R}G$, and consequently $RG/P(R)G$, is semiprime by Lemma 2.2.5. It now follows from Proposition 4.3.1 that $P(RG) \subseteq P(R)G$. Hence $P(R)G = P(RG)$.

For the converse, let $P(R)G = P(RG)$. Then $RG/P(RG) = RG/P(R)G \cong \overline{R}G$ and consequently $\overline{R}G$ is semiprime. The result follows from Lemma 2.2.5. □

Proposition 2.2.3 and the following corollaries are easy consequences of this result.

Proposition 2.2.3. Let R be any ring with identity and G a finite group of order n. $P(R)G = P(RG)$ if and only if the order of no normal subgroup is a zero divisor in $\overline{R}$.

Corollary 4.3.3. If G is any group and R any ring with identity then $P(RG) = P(R)G$ if and only if $P(\overline{R}G) = (0)$.

Proof. $P(\overline{R}G) = (0)$ implies $P(RG/P(R)G) = (0)$ and consequently $P(RG) \subseteq P(R)G$. Furthermore $P(R)G \subseteq P(RG)$ for any group G. Hence $P(R)G = P(RG)$. □

Corollary 4.3.4. If G is torsion free and R any ring with identity, then $P(RG) = P(R)G$.

Proof. G has no non-trivial finite normal subgroups. □

4.4 THE NILPOTENT AND UPPER NIL RADICAL

Definition. The union of all two sided nilpotent ideals of a ring R is a two sided nil ideal $N(R)$ and is called the *nilpotent radical* of R.

For the following properties of the nilpotent radical we refer the reader to [11], page 676 and [39], §6.

(i) $N(R) \subseteq P(R)$ for any ring R.

(ii) In general $N(R)$ is not a nil radical since it may happen that $N(R/N(R)) \neq (0)$ and therefore, in general $P(R) \neq N(R)$. Recall that if R is a Goldie ring, then every nil ideal is nilpotent (cf. [49]). Consequently we have $N(R) = P(R) = U(R)$ if R is commutative or Goldie.

(iii) $N(R) = (0)$ if and only if $P(R) = (0)$ for any ring R.

Definition. An element x of the ring R is called *strongly nilpotent* if every sequence $\{x_n\}$, where $x_o = x$, $x_{n+1} = x_n y_n x_n$, $y_n \in R$ arbitrary, is ultimately 0.

Remark (cf. [11]). Let R be any ring. Then $x \in N(R)$ if and only if the principal ideal J generated by x is nilpotent; equivalently, there exists an n such that $xa_1 \cdot xa_2 \cdot \ldots \cdot xa_n = 0$ for all choices of the $a_i \in R$. Thus if R_1 is a subring of the ring R, $N(R_1) \supseteq R_1 \cap N(R)$.

In [11] Connell proved that $N(R) = N(RG) \cap R$ for R any ring with identity and G any group. By a similar argument we get the following result for semigroup rings:

Proposition 4.4.1. Let R be any ring and S a semigroup with unity. Then $N(R) = N(RS) \cap R$.

Proof. Since S has an identity element, R is a subring of RS. Hence, from the remark above, we have

$$N(R) \supseteq N(RS) \cap R .$$

For inclusion in the other direction, let α be an arbitrary element of $N(R)$. From the remark there exists an n such that $\alpha\beta_1 \cdot \ldots \cdot \alpha\beta_n = 0$ for all $\beta_i \in R$, now $\alpha r_1 \cdot \ldots \cdot \alpha r_n$, where $r \in RS$, is a sum of terms of the form $\alpha\beta_1 \cdot \ldots \cdot \alpha\beta_n s$, $s \in S$, and therefore vanishes. Hence $\alpha = \alpha e \in N(RS)$ and consequently $N(R) = N(RS) \cap R$. □

Proposition 4.4.2. Let G be a group such that the order of no finite normal subgroup is a divisor of zero in $\overline{R} = R/P(R)$, where R is commutative or left Goldie. Then $N(RG) = N(R)G = P(R)G = P(RG)$.

Proof. From Proposition 4.4.1 we have $N(R)G \subseteq N(RG)$ for any ring R and any group G. Since R is commutative or left Goldie we have $N(R) = P(R)$ and consequently $[R/P(R)]G = [R/N(R)]G \cong RG/N(R)G$. Since $P(R/P(R)) = (0)$ it follows from $[R/P(R)]G \cong RG/N(R)G$ and Lemma 2.2.5 that $P(RG/N(R)G) = (0)$. Applying Proposition 4.3.1 we get $N(RG) \subseteq P(RG) \subseteq N(R)G$. From Theorem 4.3.2 $P(RG) = P(R)G$. But from the above $N(R)G \subseteq N(RG)$ and $N(RG) \subseteq N(R)G$. Hence we have $N(RG) = N(R)G = P(R)G = P(RG)$. □

Proposition 4.4.3. Let R be a ring with identity and H a normal subgroup of the group G such that $N(RH)$ is nilpotent and G/H is an ordered group. Then $N(RH) \cdot RG = N(RG)$.

Proof. Since $N(RH)$ is nilpotent, $N(RH) \cdot RG$ is also nilpotent (cf. [9]) and consequently $N(RH) \cdot RG \subseteq N(RG)$. Suppose $N(RG) \not\subseteq N(RH) \cdot RG$ and choose $\beta \in N(RG) - N(RH) \cdot RG$ with $|\text{supp } \beta|$ minimal. Let T be a transversal of H in G. Without loss of generality, we may assume $\beta = \sum_{i=1}^{n} \beta_i g_i$ with $\beta_i \in RH$, $\beta_i \neq 0$, $i = 1,2,\ldots,n$; $g_i \in T$ and $g_1 = e < g_2 < \ldots < g_n$ where e is the identity element of G. Since $\beta \in N(RG)$, $RG\beta RG$ is a nilpotent ideal, say $(RG\beta RG)^m = (0)$. Hence we have for all α_i, $\gamma_i \in RH$, $i = 1,2,\ldots,m$,

$$0 = \prod_{i=1}^{m} (\alpha_i \beta \gamma_i) = \prod_{i=1}^{m} (\alpha_i \beta_1 \gamma_i) + \eta_2 g_2' + \ldots + \eta_t g_t'$$

with $\eta_i \in RH$, $i = 2,3,\ldots,t$; $g_i' \in T$ and $e < g_2' < \ldots < g_t'$, since R/H is ordered, and consequently $\prod_{i=1}^{m} \alpha_i \beta_1 \gamma_i = 0$. Hence $RH\beta_1 RH$ is nilpotent and therefore $\beta_1 \in N(RH)$. Since $\beta_1 \in N(RH)\cdot RG \subseteq N(RG)$ and $\beta \in N(RG)$, we have $\beta - \beta_1 e \in N(RG) - N(RH)\cdot RG$ and since $|\mathrm{supp}(\beta-\beta_1)| < |\mathrm{supp}\ \beta|$, by choice of β we have a contradiction. □

Corollary 4.4.4. If R is a ring with identity and G an ordered group, then $N(R)G = N(RG)$.

Proof. From Proposition 4.4.1 we have $N(R)G \subseteq N(RG)$ for any group and if we put $H = \{e\}$ in the proof of Proposition 4.4.3, we have by a similar argument that $N(RG) \subseteq N(R)G$. Hence $N(R)G = N(RG)$. □

Proposition 4.4.5. Let R be a ring and H a subgroup of G such that $N(RH)$ is nilpotent. If there exists an ascending series $H = H_o \subset H_1 \subset \ldots \subset H_\alpha \subset H_{\alpha+1} \subset \ldots \subset H_\rho = G$, where ρ is an ordinal, $H_\alpha \triangleleft H_{\alpha+1}$, and $H_{\alpha+1}/H_\alpha$ an ordered group, $0 \leq \alpha < \rho$, then $N(RH)\cdot RG = N(RG)$.

Proof. This is a direct consequence of [9] Lemma 2.2 and Proposition 4.4.3. □

Lemma 4.4.6. If R is any ring and H a normal subgroup of the group G, Then $[\sigma(RH)\cdot RG]^n \subseteq [\sigma(RH)]^n\cdot RG$ for every positive integer and any radical property σ.

Proof. Since conjugation is an automorphism of RH, we have from Lemma 4.2.1(a) that $[\sigma(RH)]^x \subseteq \sigma(RH)$ for every $x \in G$. Let $n = 2$ and suppose $\gamma = \sum_i r_i s_i \in [\sigma(RH)\cdot RG]^2$. Then we can write $r_i s_i = (\alpha_1 g_1 + \ldots + \alpha_m g_m)(\beta_1 g_1' + \ldots + \beta_p g_p')$ with α_j, $\beta_k \in \sigma(RH)$ and g_j, $g_k' \in T$, T a transversal of H in G; $j = 1,2,\ldots,m$ and $k = 1,2,\ldots,p$. Hence $r_i s_i = \alpha_i(\beta_1)^{g_1^{-1}} g_1 g_1' + \ldots + \alpha_m(\beta_p)^{g_m^{-1}} g_m g_p'$ and since $(\beta_k)^{g_j^{-1}} \in \sigma(RH)$, $k = 1,2,\ldots,p$ and $j = 1,2,\ldots m$, we have $r_i s_i \in [\sigma(RH)]^2\cdot RG$ and consequently $\gamma \in \sigma(RH)^2\cdot RG$. The proof now follows by complete induction, using the fact that $([\sigma(RH)]^n)^x \subseteq [\sigma(RH)]^n$ for every $x \in G$. □

Corollary 4.4.7. Let R be any ring with identity and H a normal subgroup of the group G. If σ is any radical property, then $\sigma(RH)\cdot RG$ is nilpotent if $\sigma(RH)$ is nilpotent.

Proposition 4.4.8. Let R be a ring with identity and H a normal subgroup of the group G such that G/H is an ordered group. If $\mathcal{U}(RH)$ is nilpotent, then $\mathcal{U}(RG) = \mathcal{U}(RH)\cdot RG$.

Proof. From Corollary 4.4.7 we have that $\mathcal{U}(RH)\cdot RG$ is nilpotent if $\mathcal{U}(RH)$ is nilpotent and consequently $\mathcal{U}(RH)\cdot RG \subseteq \mathcal{U}(RG)$. To prove inclusion in the other direction, suppose $\mathcal{U}(RG) \not\subseteq \mathcal{U}(RH)\cdot RG$ and choose $\beta \in \mathcal{U}(RG) - \mathcal{U}(RH)\cdot RG$ with $|\text{supp}\beta|$ minimal. Let T be a transversal of H in G. Without loss of generality, we can assume $\beta = \sum_{i=1}^{n} \beta_i g_i$ with $\beta_i \in RH$, $\beta_i \neq 0$, $i = 1,2,\ldots,n$; $g_i \in T$ and $e = g_1 < g_2 < \ldots < g_n$ where e is the identity element of G. Put $K = \{\alpha_1 : a = \alpha_1 e + \alpha_2 g_2' + \alpha_3 g_3' + \ldots + \alpha_m g_m' \in \mathcal{U}(RG);\ g_i' \in T$ with $e < g_1' < \ldots < g_m'$ and $\alpha_i \in RH$, $i = 1,2,\ldots,m\}$. It is easy to check that K is a nil ideal in RH. Since $\beta_1 \in K$, we have

$\beta_1 \in \mathcal{U}(RH) \subseteq \mathcal{U}(RH)\cdot RG \subseteq \mathcal{U}(RG)$ and consequently $\beta - \beta_1 \in \mathcal{U}(RG) - \mathcal{U}(RH)\cdot RG$. Since $|\text{supp}(\beta-\beta_1)| < |\text{supp}\ \beta|$, by choice of β we have a contradiction. This completes the proof. □

As in [9], let $\mathcal{Y}$ be a class of groups. If G is a group, $G \in L\mathcal{Y}$ if and only if every finitely generated subgroup of G is contained in a subgroup of G which is a member of $\mathcal{Y}$.

Proposition 4.4.9. Let R be a ring. Let $\mathcal{Y}$ be any class of groups satisfying the hypothesis that if L is a normal subgroup of a group M such that $M/L \in \mathcal{Y}$, and $\mathcal{U}(RL)$ is nilpotent, then $\mathcal{U}(RM) = \mathcal{U}(RL)\cdot RM$.

Then (i) if H is a subgroup of a group G such that

(a) there exists an ascending series

$$H = H_0 \subset H_1 \subset \dots \subset H_\alpha \subset H_{\alpha+1} \subset \dots \subset H_\rho = G,$$

where ρ is an ordinal, $H_\alpha \triangleleft H_{\alpha+1}$ and $H_{\alpha+1}/H_\alpha \in \mathcal{Y}$, $0 \leq \alpha < \rho$, and

(b) $[\mathcal{U}(RH)]^n = 0$, for some integer $n \geq 1$,

$$\mathcal{U}(RG) = \mathcal{U}(RH)\cdot RG \quad \text{and} \quad [\mathcal{U}(RG)]^n = 0.$$

(ii) if H is a normal subgroup of a group G, such that $G/H \in L\mathcal{Y}$, and $\mathcal{U}(RH)$ is nilpotent,

$$\mathcal{U}(RG) = \mathcal{U}(RH)\cdot RG.$$

Proof. The proof uses the same technique as that of [9] Lemma 2.2.

Theorem 4.4.10. Let R be a ring, and H a subgroup of a group G, such that $\mathcal{U}(RH)$ is nilpotent. If there exists an ascending series

$$H = H_0 \subset H_1 \subset \dots \subset H_\alpha \subset H_{\alpha+1} \subset \dots \subset H_\rho = G,$$

where ρ is an ordinal, $H_\alpha \triangleleft H_{\alpha+1}$, and $H_{\alpha+1}/H_\alpha$ is an ordered group, then $\mathcal{U}(RH)\cdot RG = \mathcal{U}(RG) = \mathcal{N}(RG)$.

Proof. From Propositions 4.4.8 and 4.4.9 it follows that $\mathcal{U}(RH)\cdot RG = \mathcal{U}(RG)$. Furthermore, since $\mathcal{U}(RH)$ is nilpotent, we have $\mathcal{U}(RH) = \mathcal{N}(RH)$ and consequently $\mathcal{U}(RH)\cdot RG = \mathcal{N}(RH)\cdot RG$. From Proposition 4.4.5 we have $\mathcal{N}(RG) = \mathcal{N}(RH)\cdot RG$ and from the above we have $\mathcal{N}(RH)\cdot RG = \mathcal{U}(RH)\cdot RG = \mathcal{U}(RG)$. This completes the proof. □

Remark. The class of groups in Theorem 4.4.10 contains the class of SN-groups (Kurosh [26], pp.171-) with torison free Abelian factors.

Proposition 4.4.11. If R is any ring and S an ordered semigroup with unity, then

$$\mathcal{N}(R)S = \mathcal{N}(RS) = \mathcal{U}(RS) = P(RS)$$

if $\mathcal{U}(R)$ is nilpotent.

Proof. It is easy to show that $[\mathcal{U}(R)S]^m \subseteq [\mathcal{U}(R)]^m S$ for any positive integer m. Consequently if $\mathcal{U}(R)$ is nilpotent then $\mathcal{U}(R)S \subseteq \mathcal{U}(RS)$. From Theorem 3.4.3 we have $\mathcal{U}(RS) \subseteq \mathcal{U}(R)S$ and consequently $\mathcal{U}(RS) = \mathcal{U}(R)S$. Furthermore, since $\mathcal{U}(R)$ is nilpotent we have $\mathcal{U}(R) = \mathcal{N}(R) = P(R)$ and consequently from Theorem 3.3.4 and the above, it follows that $\mathcal{U}(RS) = \mathcal{U}(R)S = \mathcal{N}(R)S = P(R)S = P(RS)$. To complete the proof, we only have to show that $\mathcal{N}(R)S = \mathcal{N}(RS)$. From Proposition 4.4.1 we have $\mathcal{N}(R)S \subseteq \mathcal{N}(RS)$ and since $\mathcal{U}(RS) \subseteq \mathcal{U}(RS) = \mathcal{N}(R)S$ we have $\mathcal{N}(R)S = \mathcal{N}(RS)$. □

Corollary 4.4.12. If R is a left Goldie ring and S an ordered semigroup then $\mathcal{N}(R)S = \mathcal{N}(RS) = \mathcal{U}(RS) = P(RS)$.

Proof. $\mathcal{U}(R)$ is a nil ideal in R and since R is left Goldie it is also nilpotent. □

As in Chapter 2, let N(R) denote the set of nilpotent elements of the ring R. From [39], Theorem 12 we have $\mathcal{N}(R) \subseteq \mathcal{U}(R) \subseteq N(R)$.

Proposition 4.4.13. Let S be an ordered semigroup and let R be a ring with identity such that if $\alpha, \beta \in R$ and $\alpha\beta = 0$ then $\beta\alpha = 0$. Then $N(R)S = N(RS) = \mathcal{U}(R)S = \mathcal{U}(RS)$.

Proof. From our assumption that $\alpha\beta = 0$ implies $\beta\alpha = 0$ and Lemma 2.7.2, it follows that N(R) is an ideal in R and consequently $N(R) = \mathcal{U}(R)$. Hence $N(R)S = \mathcal{U}(R)S$. Let $a, b \in RS$ such that $ab = 0$. If $a = \sum_{i=1}^{n} \alpha_i s_i$ and $b = \sum_{i=1}^{n} \beta_i t_i$ then we have from Theorem 2.4.1 that $\alpha_i \beta_j = 0$ for all i and j. From our assumption we also have that $\beta_j \alpha_i = 0$ for all i and j and consequently $ba = 0$. As above, we have from Lemma 2.7.2 that N(RS) is an ideal and hence $N(RS) = \mathcal{U}(RS)$. Furthermore, from the proof of Theorem 2.7.3 we have that N(R)S is a nil ideal and consequently $N(R)S \subseteq \mathcal{U}(RS)$. Inclusion in the other direction follows from Theorem 3.4.3. Hence we have $\mathcal{U}(R)S = N(R)S = \mathcal{U}(RS) = N(RS)$. □

4.5 THE JACOBSON RADICAL OF THE GROUP RING

Let J(R) denote the Jacobson radical of the ring R.

Proposition 4.5.1. If R is any ring then J(R) is the smallest two sided ideal K of R such that R/K has null Jacobson radical.

Proof. Let σ be the upper radical property determined by all the primitive rings. Then if R is any ring $\sigma(R) = J(R)$ (cf. [12]). Hence if K is any ideal in R such that $J(R/K) = (0)$ it follows from Lemma 4.2.1 that $J(R) \subseteq K$. □

Lemma 4.5.2. Let R be a commutative ring with identity and let G be a finite group of order n. Then J(RG) = (0) if and only if J(R) = (0) and n is not a zero divisor in R.

Proof. See [11], Theorem 7. □

Proposition 4.5.3. Let R be a commutative ring with identity. If G is a finite group of order n, then J(R)G = J(RG) if and only if n is not a zero divisor in $\overline{R}$.

Proof. From Proposition 1.4.1 J(R) = J(RG) ∩ R if G is locally finite. Hence J(R)G ⊆ J(RG). Futhermore, we have [R/J(R)]G ≅ RG/J(R)G and from Lemma 4.5.2 it follows that J(RG/J(R)G) = (0). By applying Proposition 4.5.1 we have J(RG) ⊆ J(R)G and consequently J(RG) = J(RG). Conversely, suppose J(R)G = J(RG). Then [R/J(R)]G ≅ RG/J(R)G = RG/J(RG) and from Lemma 4.5.2 n is not a divisor of zero in $\overline{R}$. □

Proposition 4.5.4. If G is locally finite and R commutative then J(RG) = J(R)G if and only if J($\overline{R}$G) = (0).

Proof. As in the previous proposition J(R)G ⊆ J(RG). Since $\overline{R}$G ≅ RG/J(R)G, it follows from our assumption that J(RG/J(R)G) = (0) and Proposition 4.5.1 implies J(RG) ⊆ J(R)G. Consequently J(RG) = J(R)G. The converse is clear. □

4.6 THE BROWN McCOY RADICAL

If R is a ring with identity, we denote the Brown McCoy radical of R by $\mathcal{B}$(R). In this section we show that if G is a finitely generated torsion free Abelian group, then $\mathcal{B}$(RG) ⊆ $\mathcal{B}$(R)G. For the following definitions and results, see McCoy [30].

With each element a of R let us associate the right ideal F(a), defined as follows:

$$F(a) = \{ar - r : r \in R\} .$$

Then a is *right quasi-regular* if and only if $a \in F(a)$. A right ideal or left ideal in R is said to be right quasi-regular (left quasi-regular) if each of its elements is right quasi-regular (left quasi-regular). Recall, that the Jacobson radical J(R) of a ring R is defined as follows: $J(R) = \{a : a \in R, aR \text{ is right quasi-regular}\}$.

If $a \in R$, we define G(a) to be the two sided ideal in R generated by the elements of the right ideal F(a). Then

$$G(a) = \{ar - r + \sum(x_i a y_i - x_i y_i) : r, x_i, y_i \in R\} .$$

An *element* a of the ring R is said to be *G-regular* if $a \in G(a)$. An *ideal* is said to be *G-regular* if each of its elements is G-regular.

Definition. The Brown McCoy radical $\mathcal{B}(R)$ of the ring R is defined as follows:

$$\mathcal{B}(R) = \{b : b \in R, (b) \text{ is G-regular}\} .$$

A ring with more than one element is said to be a *simple ring* if and only if its only ideals are the two trivial ideals, namely (0) and R. An ideal M in R is a *modular maximal* ideal if and only if R/M is a simple ring with identity.

Theorem 4.6.1. Let R be a ring such that $\mathcal{B}(R) \neq R$, and let M_i, $i \in \mathcal{U}$, be all the modular maximal ideals in R. Then $\mathcal{B}(R) = \bigcap_{i \in \mathcal{U}} M_i$.

Proof. See [30], Theorem 7.26. □

Theorem 4.6.2. If R is a ring with more than one element, then $\mathcal{B}(R) = (0)$ if and only if R is isomorphic to a subdirect sum of simple rings with identity.

Proof. See [30], Theorem 7.27. □

Proposition 4.6.3. If R is any ring, then $\mathcal{B}(R)$ is the smallest two sided ideal K of R such that $\mathcal{B}(R/K) = (0)$.

Proof. Let σ be the radical property determined by the class of all simple rings with identity. Then $\sigma(R) = \mathcal{B}(R)$ for any ring R (cf. [12]). Hence if K is any ideal in R such that $\mathcal{B}(R/K) = (0)$, it follows from Lemma 4.2.1 that $\mathcal{B}(R) \subseteq K$. □

Proposition 4.6.4. If R is a simple ring with identity and $G = \langle x \rangle$, the infinite cyclic group generated by x, then RG is a principal ideal ring.

Proof. Let A be any ideal in RG. Since R[x] is a subring of RG, $A \cap R[x]$ is an ideal in R[x]. We can pick a nonzero element $a(x) \in A \cap R[x]$ with minimal degree. Since the leading coefficients of all the elements of $A \cap R[x]$ with minimal degree, (say n), together with (0), forms an ideal in R, and R is a simple ring with identity, we can without loss of generality, assume that a(x) is monic. Firstly, we show that $A \cap R[x] = \langle a(x) \rangle$. To this purpose we prove that $a(x)R[x] = R[x]a(x) = \langle a(x) \rangle$. Let $r \in R$, then $(a(x)r - ra(x)) \in A \cap R[x]$ and degree $(a(x)r - ra(x)) < n$. Hence $a(x)r = ra(x)$ for every $r \in R$ and consequently $a(x)R[x] = R[x]a(x)$. From this and the definition of $\langle a(x) \rangle$ it follows that

$\langle a(x)\rangle = a(x)R[x] = R[x]a(x)$. Let $f(x)$ be an arbitrary element of $A \cap R[x]$ of degree k with leading coefficient β. If $n = k$, then $[a(x)\beta - f(x)] \in A \cap R[x]$ and degree $(a(x)\beta - f(x)) < n$. Consequently $f(x) = a(x)\beta \in a(x)R[x] = \langle a(x)\rangle$. Suppose that every element of $A \cap R[x]$ of degree k, $n \leq k \leq m$, is an element $\langle a(x)\rangle$, then if $f(x)$ is of degree m+1 we have $g(x) = f(x) - a(x)\beta x^{m+1-n} \in A \cap R[x]$ with degree $g(x) \leq m$. From our assumption there exists $h(x) \in R[x]$ such that $g(x) = a(x)h(x)$ and consequently $f(x) = a(x)p(x)$, where $p(x) = h(x) - \beta x^{m+1-n} \in R[x]$. Hence $A \cap R[x] \subseteq \langle a(x)\rangle$. However, since $A \cap R[x]$ is an ideal in $R[x]$ we have $\langle a(x)\rangle \subseteq A \cap R[x]$. Consequently $A \cap R[x] = \langle a(x)\rangle$. We claim that $A = a(x)RG = RGa(x)$. Clearly $a(x)RG \subseteq A$. Next, let $y \in A$, $y \neq 0$. We can write $y = x^j f(x)$ for some integer j and $f(x) \in R[x]$. Then $yx^{-j} = f(x) \in A$. Hence $f(x) \in \langle a(x)\rangle$ and we can write $f(x) = a(x)k(x)$ where $k(x) \in R[x]$. Hence $y = f(x)x^j = a(x)k(x)x^j \in a(x)RG$. Therefore, $a(x)RG = RGa(x) = A$. Thus we have proved that A is a principal ideal in RG, generated by $a(x)$. □

Lemma 4.6.5. Let R be a simple ring with identity and G an infinite cyclic group. Then $\mathcal{B}(RG) = (0)$.

Proof. Let $G = \langle x\rangle$ be the infinite cyclic group generated by x. Suppose now I is the Brown McCoy radical of RG. From Proposition 4.6.4 there exists a monic polynomial $a(x)$ of degree n, say, in $I \cap R[x]$ such that $I = a(x)RG = RGa(x)$. Then $I = \langle a(x)\rangle$ and $a(x)$ is G-regular in RG, that is

$$a(x) \in G(a(x)) = \{a(x)y - y + \sum(g_i a(x)h_i - g_i h_i)\}$$

where the summation is over a finite range and $y, g_i, h_i \in RG$.

Since $a(x)RG = RGa(x)$ we have $G(a(x)) = F(a(x)) = \{a(x) - y\}$. Hence there is $s \in RG$, $s \neq 0$, such that $a(x)s - a(x) - s = 0$. By comparing degrees we see that either degree $s = 0$ or degree $a(x) = 0$. If degree $s = 0$ but degree $a(x) \neq 0$ then, for the coefficient of x^n in $a(x)s$ and $a(x)$ to cancel, we must have $s = 1$. This is impossible for it will imply $1 = 0$. Similarly we can prove that neither degree $s = 0$ and degree $a(x) = 0$ nor degree $s \neq 0$ and degree $a(x) = 0$. Hence $s = 0$ and hence $a(x) = 0$. Consequently $I = (0)$, i.e. $\mathcal{B}(RG) = (0)$. □

Lemma 4.6.6. Let R be a ring with identity and G an infinite cyclic group. If $\mathcal{B}(R) = (0)$ then $\mathcal{B}(RG) = (0)$.

Proof. Since $\mathcal{B}(R) = (0)$ it follows from Theorem 4.6.1 that $\bigcap_{i\in\mathcal{U}} M_i = (0)$ where $\{M_i : i \in \mathcal{U}\}$ is the family of all the modular maximal ideals in R. Hence for each $i \in \mathcal{U}$, R/M_i is a simple ring with identity. Now $RG/M_i(G) \cong R/M_iG$ and from Lemma 4.6.5 R/M_iG has zero Brown McCoy radical. Put $R/M_i = \overline{R}_i$. From Theorem 4.6.2 it now follows that for each $i \in \mathcal{U}$, $\overline{R}_iG$ is a subdirect sum of simple rings with identity. Say $\overline{R}_iG = \bigoplus_{j}{}_s T_{ij}$. Furthermore, $\bigcap_{i\in\mathcal{U}} (M_iG) = (\bigcap_{i\in\mathcal{U}} M_i)G = (0)$ and consequently it follows from [30], Theorem 3.9 that RG is isomorphic to a subdirect sum of the rings $\overline{R}_iG$. Hence $RG \cong \bigoplus_{i}{}_s \overline{R}_iG \cong \bigoplus_{i,j}{}_s T_{ij}$. Since for each i and j, T_{ij} is a simple ring with identity, it follows from Theorem 4.6.2 that $\mathcal{B}(RG) = (0)$. This completes the proof. □

Theorem 4.6.7. If R is a simple ring with identity and G a finitely generated torsion free Abelian group, then $\mathcal{B}(RG) = (0)$.

Proof. If $G = C_1 \times C_2 \times \ldots \times C_n$, C_i infinite cyclic, then $RG \cong (RC_1)(C_2 \times C_3 \times \ldots \times C_n)$. By Lemma 4.6.5 $B(RC_1) = (0)$ and by Lemma 4.6.6 and complete induction the result follows. □

Corollary 4.6.8. Let R be a ring with identity and G a finitely generated torsion free Abelian group. If $B(R) = (0)$ then $B(RG) = 0$.

Proof. Put $G = C_1 \times C_2 \times \ldots \times C_n$, C_i infinite cyclic. Then $RG \cong (RC_1)(C_2 \times \ldots \times C_n)$. By Lemma 4.6.6 the result follows by complete induction. □

Proposition 4.6.9. Let R be any ring with identity and G a finitely generated torsion free Abelian group. Then $B(RG) \subseteq B(R)G$.

Proof. Consider the isomorphism $[R/B(R)]G \cong RG/B(R)G$. From Corollary 4.6.8 we have $B(RG/B(R)G) = (0)$ and hence from Proposition 4.6.3 it follows that $B(RG) \subseteq B(R)G$. □

4.7 RELATIONS BETWEEN SOME OF THE PREVIOUSLY DEFINED RADICALS

The aim of this section is to determine relations between P(RG), U(RG) and J(RG) for certain R and G. We show that under certain conditions these three radicals coincide.

We first state some definitions and theorems we need.

Proposition 4.7.1. Let R be any ring. The upper nil radical $U(R)$ of the ring R is the smallest ideal K such that $U(R/K) = (0)$.

Proof. Let σ be the lower radical property determined by the class of all nil rings. Then $\sigma(R) = U(R)$ (cf. [12]). The proof now follows from Lemma 4.2.1. □

Definition ([44], Definition 2.1)

Let D be a semigroup. A nonempty subset G of D is grouplike (in D) if

(i) $a, b \in G$ imply $ab \in G$

(ii) $a, ab \in G$ imply $b \in G$.

Definition ([44], Definition 2.2)

Let D be a semigroup. If $A \subseteq D$, $A \neq \phi$, then $<<A>>$ is defined as $\cap \{G : A \subseteq G \subseteq D$ and G is grouplike$\}$.

Definition ([44], Definition 2.4)

A semigroup D is said to be a Γ-semigroup if and only if it satisfies the following condition: For all nonempty finite sets A contained in D, there exists g in D, such that for every finite set B contained in $<<Ag>>$ there exists an integer $n = n(B)$, for which $(Ag)^n B \cap B = \phi$.

Definition ([44], Definition 2.17)

A semigroup is said to be a 2Ω semigroup if and only if for all pairs of finite nonempty subsets A, B of D with $|A| + |B| \geq 3$, there exists at least two elements c in AB which admit exactly one representation $c = ab$, with $a \in A$, $b \in B$. We say that such an element $c \in AB$ is uniquely expressible with respect to A and B. Every 2Ω-semigroup is an u.p.-semigroup ([38], page 588).

Definition ([44], Definition 2.19)

The semigroup D is a $2\Omega\Gamma$-semigroup if it is both 2Ω and Γ.

Example 1. Every ordered semigroup D with more than one element is a $2\Omega\Gamma$-semigroup. Hence every ordered group is a $2\Omega\Gamma$-semigroup ([44], Theorem 2.22).

Example 2. If G is a group with a normal series with ordered factor groups, then G is a $2\Omega\Gamma$-semigroup. This example includes the case of SN-groups (Kurosh [26], pp.171-) with torsion free Abelian factors ([44], Theorem 2.23).

Definition (i) An integer n is called cancelable in the ring R if, for any $\alpha \in R$, $n\alpha = 0$ implies $\alpha = 0$.

(ii) A group G is cancelable with respect to R if and only if every integer n such that G has an element of order n is cancelable in R.

Proposition 4.7.2. Let D be a $2\Omega\Gamma$-semigroup with 1, and let R be a ring. Then $\mathcal{U}(R) = (0)$ implies $J(RD) = (0)$. If R is commutative $\mathcal{U}(R) = (0)$ implies $J(RD) = (0)$ holds for 2Ω-semigroups with 1.

Proof. (See [44], Theorem 3.6). □

Proposition 4.7.3. Let R be a ring and G a group. Suppose that G is cancelable with respect to R. If $\mathcal{U}(R) = (0)$ then $\mathcal{U}(RG) = (0)$.

Proof. (See [44], Theorem 4.4). □

Proposition 4.7.4. Let R be a ring and let G be an Abelian group with at least one element of infinite order. If $\mathcal{U}(R) = (0)$ and G is cancelable with respect to R, then $J(RG) = (0)$.

Proof. (See [44], Theorem 4.7). □

Proposition 4.7.5. Let R be a commutative ring without nonzero nilpotent elements. Suppose the additive group of R is torsion free. Then for any group G, $\mathcal{U}(RG) = (0)$.

Proof. (See [44], Corollary 4.6). □

The following result is an extension of a result proved by Tan ([48], Theorem 1).

Theorem 4.7.6. If R is a left Goldie ring with identity and S is an ordered semigroup with unity, then

$$P(R)S = P(RS) = \mathcal{U}(RS) = J(RS) .$$

Proof. Since $\mathcal{U}(R)$ is a nil ideal and R is left Goldie, $\mathcal{U}(R)$ is also nilpotent and we have $\mathcal{U}(R) = P(R)$. As in the proof of Proposition 4.4.11 we get $\mathcal{U}(R)S = \mathcal{U}(RS) = P(RS)$. Furthermore, $RS/\mathcal{U}(R)S \cong (R/\mathcal{U}(R))S$ and from Proposition 4.7.2 and Example 2, we have $J(RS/\mathcal{U}(R)S) = (0)$ and consequently $J(RS) \subseteq \mathcal{U}(R)S$. To complete the proof, it suffices to notice that $\mathcal{U}(R)S = \mathcal{U}(RS) \subseteq J(RS)$. □

Proposition 4.7.7. If R is a commutative ring (with identity) and S a 2Ω-semigroup with 1, then $P(R)S = P(RS) = \mathcal{U}(RS) = J(RS)$.

Proof. Since R is commutative and considering that $RS/\mathcal{U}(R)S \cong (R/\mathcal{U}(R))S$ while $(R/\mathcal{U}(R)) = (0)$, it follows from Proposition 4.7.2 that

$J(RS/\mathcal{U}(R)S) = (0)$. Consequently $J(RS) \subseteq \mathcal{U}(R)S = P(R)S = P(RS)$. But $P(RS) \subseteq \mathcal{U}(RS) \subseteq J(RS) \subseteq \mathcal{U}(R)S = P(RS) \subseteq \mathcal{U}(RS) \subseteq J(RS)$ and the result follows. □

Corollary 4.7.8. If R is a left Goldie ring with identity and S is an ordered semigroup with unity, then RS is semisimple if and only if R is semiprime.

Proof. First J(RS) = (0) implies P(RS) = (0) and since P(R) = R ∩ P(RS), we have P(R) = (0). The converse is clear from Theorem 4.7.6. □

Corollary 4.7.9. If R is a commutative ring (with identity) and S is a 2Ω-semigroup with 1, then RS is semisimple if and only if R is semiprime.

Proof. This is clear from Proposition 4.7.7 and the proof of the previous corollary. □

Corollary 4.7.10 ([11]). If R is a commutative ring with identity and G is an ordered group, then RG is semisimple if and only if R is semiprime.

Proof. This is clear. □

Corollary 4.7.10 coincides with Proposition 11 of Connell [11], and hence Corollaries 4.7.8 and 4.7.9 are extensions of this Proposition.

Proposition 4.7.11. If R is a left Goldie ring (with identity) and G an SN group with a normal series whose factors are Abelian torsion free, then

$$\mathcal{U}(R)G = \mathcal{U}(RG) = P(R)G = P(RG) = J(RG) .$$

Proof. Since G has no finite subgroups, we have from Theorem 4.3.2 that P(R)G = P(RG). Furthermore, Proposition 4.7.2 and Example 2 implies that $J(RG/\mathcal{U}(R)G) = (0)$ and consequently $J(RG) \subseteq \mathcal{U}(R)G$. The rest of the proof follows from Theorem 4.4.10. □

Corollary 4.7.12. If R is left Goldie and G an SN group with torsion free Abelian factors, then RG is semisimple if and only if R is semiprime.

Proof. This is clear. □

Amitsur [2] has proved that $J(R[x]) = N[x]$, where $N = J(R[x] \cap R)$ and N is a nil ideal in R. Thus the following proposition is of some interest.

Proposition 4.7.13. Suppose R is a ring with no idempotents $\neq 0,1$, and whose nilpotent elements form an ideal N. Then $J(RG) = NG$ where G is an ordered group.

Proof. Let $x = \sum_{i=1}^{n} \alpha_i g_i \in J(RG)$. Then $1-x$ is a unit. From Corollary 2.7.4 we have α_i, $i = 1,2,\ldots,n$, are nilpotent. Hence $x \in NG$. If $y = \sum_{i=1}^{m} \beta_i g_i \in NG$ then $\beta_i \in N$ and from Corollary 2.7.4, $1-y$ is a unit. Consequently $y \in J(RG)$. □

Proposition 4.7.14. Let R be a left Goldie ring and G a group that is cancelable with respect to $\overline{R} = R/P$, then $P(R)G = P(RG) = \mathcal{U}(RG)$.

Proof. From Propositions 4.7.1 and 4.7.3 and the isomorphism $RG/\mathcal{U}(R)G \cong (R/\mathcal{U}(R))G$, we have $\mathcal{U}(RG) \subseteq \mathcal{U}(R)G = P(R)G$. The rest then follows from the fact that $P(R)G \subseteq P(RG) \subseteq \mathcal{U}(RG)$ for any R and G. □

Proposition 4.7.15. Let G be an Abelian group with at least one element of infinite order and R a left Goldie ring such that G is cancelable with respect to $\overline{R} = R/\mathcal{U}(R)$, then $\mathcal{U}(R)G = J(RG) = \mathcal{U}(RG)$.

Proof. $\mathcal{U}(R)G \subseteq \mathcal{U}(RG) \subseteq J(RG)$ since R is left Goldie. The rest follows from Propositions 4.7.4 and 4.5.1 together with the isomorphism $(R/\mathcal{U})G \cong RG/\mathcal{U}(R)G$. □

Proposition 4.7.16. Let R and G be commutative and suppose G has at least one element of infinite order. If P(R) = 0 and G is cancelable with respect to R then J(RG) = (0).

Proof. See [14], Corollary 4.8. □

Proposition 4.7.17. Let R and G be commutative and suppose G has at least one element of infinite order. If G is cancelable with respect to $\overline{R}$ = R/P, then P(R)G = P(RG) = $\mathcal{U}$(RG) = J(RG).

Proof. Immediate by Proposition 4.7.16 and the isomorphism (R/P)G ≅ RG/P(R)G. □

Corollary 4.7.18. Let R and G be commutative and suppose G has at least one element of infinite order. Then, if G is cancelable with respect to $\overline{R}$ = R/P, J(RG) = 0 if and only if P(R) = 0.

Proof. This is clear.

Proposition 4.7.19. Let R be a commutative ring such that the additive group of R/P is torsion free. Then, for all G, P(R)G = P(RG) = $\mathcal{U}$(RG).

Proof. From Propositions 4.7.1 and 4.7.5 together with the isomorphism (R/P)G ≅ RG/P(R)G we have $\mathcal{U}$(RG) ⊆ P(R)G. The rest is clear. □

SUMMARY

Chapter 1 is a short review of the main results in some areas of the theory of group rings.

In the first half of Chapter 2 we determine the ideal theoretic structure of the group ring RG where G is the direct product of a finite Abelian group and an ordered group with R a completely primary ring. Our choice of rings and groups entails that the study centres mainly on zero divisor ideals of group rings and hence it contributes in a small way to the zero divisor problem. We show that if R is a completely primary ring, then there exists a one-one correspondence of the prime zero divisor ideals in RG and $\overline{R}G$, G finite cyclic of order n. If R is a ring with the property $\alpha, \beta \in R$, then $\alpha\beta = 0$ implies $\beta\alpha = 0$, and S is an ordered semigroup, we show that if $\sum \alpha_i s_i \in RS$ is a divisor of zero, then the coefficients α_i belong to a zero divisor ideal in R. The converse is proved in the case where R is a commutative Noetherian ring. These results are applied to give an account of the zero divisors in the group ring over the direct product of a finite Abelian group and an ordered group with coefficients in a completely primary ring.

In the second half of Chapter 2 we determine the units of the group ring RG where R is not necessarily commutative and G is an ordered group. If R is a ring such that if $\alpha, \beta \in R$ and $\alpha\beta = 0$, then $\beta\alpha = 0$, and if G is an ordered group, then we show that $\sum \alpha_g g$ is a unit in RG if and only if there exists $\sum \beta_h h$ in RG such that $\sum \alpha_g \beta_{g^{-1}} = 1$ and $\alpha_g \beta_h$ is nilpotent whenever $GH \neq 1$. We also show that if R is a ring with no nilpotent elements $\neq 0$ and no idempotents $\neq 0,1$, then RG has only trivial units.

In this chapter we also consider strongly prime rings. We prove that RG is strongly prime if R is strongly prime and G is an unique product (u.p.) group. If $H \triangleleft G$ such that G/H is right ordered, then it is shown that RG is strongly prime if RH is strongly prime.

In Chapter 3 results are derived to indicate the relations between certain classes of ideals in R and RG. If σ is a property of ideals defined for ideals in R and RG, then the "going up" condition holds for σ-ideals if Q being a σ-ideal in R implies that QG is a σ-ideal in RG. The "going down" condition is satisfied if P being a σ-ideal in RG implies that $P \cap R$ is a σ-ideal in R. We proved that the "going up" and "going down" conditions are satisfied for prime ideals, ℓ-prime ideals, q-semiprime ideals and strongly prime ideals. These results are then applied to obtain certain relations between different radicals of the ring R and the group ring (semigroup ring) RG (RS). Similarly, results about the relation between the ideals and the radicals of the group rings RH and RG, where H is a central subgroup of G, are obtained. For the upper nil radical we prove that $\mathcal{U}(RG) \subseteq \mathcal{U}(RH) \cdot RG$, H a central subgroup of G, if G/H is an ordered group. If S is an ordered semigroup, however, then $\mathcal{U}(RS) \subseteq \mathcal{U}(R)S$ for any ring R.

In Chapter 4 we determine relations between various radicals in certain classes of group rings. In Section 4.3, as an extension of a result of Tan, we prove that $P(R)G = P(RG)$, R a ring with identity, if and only if the order of no finite normal subgroup of G is a zero divisor in $R/P(R)$.

If R is any ring with identity and H a normal subgroup of G such that G/H is an ordered group, we show that $\mathcal{N}(RH) \cdot RG = \mathcal{U}(RG) = \mathcal{N}(RG)$, if

$\mathcal{U}$(RH) is nilpotent. Similar results are obtained for the semigroup ring RS, S ordered.

It is also shown if R is commutative and G finite of order n, then J(R)G = J(RG) if and only if n is not a zero divisor in R/J(R), J(R) being the Jacobson radical of R.

For the Brown McCoy radical we determine the following: If R is Brown McCoy semisimple or if R is a simple ring with identity, then $\mathcal{B}$(RG) = (0), where G is a finitely generated torsion free Abelian group.

In the last section we determine further relations between some of the previously defined radicals, in particular between P(R), $\mathcal{U}$(R) and J(R). Among other results, the following relations between the abovementioned radicals are obtained: $\mathcal{U}$(RS) = $\mathcal{U}$(R)S = P(RS) = J(RS) where R is a left Goldie ring and S an ordered semigroup with unity.

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