On the Gravitational Dual to Strongly Coupled Fluids

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Abstract

This thesis discusses the prospect of finding the gravitational dual to the strongly coupled conformal fluids, with a special interest in the quark-gluon plasma. Such a task can be achieved by matching certain physical observables of two apparently different theories that are dually related owing to the fact that the same string theory can be viewed in two different ways. This is particularly useful when one of the theories is intractable while its dual is manageable. We begin by postulating a particular type of gravitational theory from which we determine graviton scattering amplitudes in a special regime of high momentum. Using the gauge–gravity duality dictionary, the graviton scattering amplitudes can be mapped to stress-tensor correlation functions in the gauge theory. One of the outcomes of high-energy scattering experiments involving the quark-gluon plasma is stress-tensor correlator data. This thesis provides an algorithm for matching graviton scattering amplitudes with stress-tensor correlator data which, in principle, can be used to identify the gravitational dual to the quark-gluon plasma.

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To my mother and my siblings “natotela saana mukwai”.

Statement of declaration

I, Mark Musonda Webster Shawa, declare that the thesis entitled *On the gravitational dual to strongly coupled fluids* and the work presented in the thesis is my own and has been generated by me as a result of my own original research. To the best of my knowledge, this thesis contains no material previously published or written by another person, except where due reference is made in the text.

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Contents

Chapter 1

Introduction

The gauge–gravity duality is one of the hallmark features of string theory [1]. It embodies the very essence of the holographic principle which states that under special circumstances, the dynamics of a $(d+1)$-dimensional theory of quantum gravity can be described by the dynamics of a d-dimensional quantum field theory [2]. In other words, given some spacetime dimension, there is a correspondence between certain physical quantities of a quantum theory of gravity defined in some bulk spacetime and the physical quantities of a gauge theory defined at what can be surmised as the boundary of that spacetime.

The ideas that inspired the holographic principle emerged from the study of black hole entropy by Bekenstein [3] who proposed that the entropy of a black hole is proportional to its surface area. Such an achievement was fuelled by the need to understand the mechanical nature of black holes [4]. However, it is worth mentioning that while thermodynamic entropy is an expression of our ignorance of a system's microscopic configuration when macroscopic data is available, black hole entropy expresses our ignorance of the interior of the black hole.

Following Bekenstein's discovery, ‘t Hooft took a step further by suggesting an alternative interpretation of the black hole entropy law [5]. Using thermodynamics and quantum mechanics, he determined that at the Planck scale the observable degrees of freedom of a $(3+1)$-dimensional gravitationally collapsed system are encoded on a closed 2-dimensional surface evolving in time. This meant that information contained within a black hole is projected onto its surface.

String theory was initially developed to study strong nuclear interactions but was replaced by Quantum Chromodynamics (QCD). What followed instead was that it became viewed as a potential quantum theory of gravity due to the frequent occurrence of spin-2 excitations in all its formulations. By 1973, ‘t Hooft [6] had already captured the connection between the string and gauge theories through his double-line formulation of perturbation theory. In addition, the work by Susskind and Klebanov [7] showed that string behaviour was present in field theoretic systems, particularly (lattice) gauge theory.

The idea that a quantum theory of gravity must be holographic was only later refined in [2], but the full realization of this principle came even later on through the work by Maldacena [1]. Using string theory, he came to the conclusion that a gravitational theory defined in a certain black hole background has a dual relationship with a conformally-invariant gauge theory. The duality implies a one-to-one correspondence of some physical quantities between the two theories via a special dictionary.

However, Maldacenas original formulation is an ideal and highly symmetric case of the duality. The real world is far less symmetric and so one makes modifications by considering extensions that include broken symmetries (particularly, supersymmetry and conformal symmetry) as shown in [8, 9]. These extensions have come to be known as *gauge–gravity* dualities and have, previously, been constructed in one of two ways; *top-down* and *bottom-up*.

The top-down construction starts with some low-energy, weakly coupled string theory in a black hole background dual to a strongly coupled field theory. An explicit description of a quantum field theory may not exist in this construction as the gravitational sector of the string theory dictates what is allowed and observable in the dual theory.

The bottom-up construction requires one to postulate a weakly coupled theory of gravity dual to a strongly coupled gauge theory. This is because the weakly coupled gravity is much easier to understand than strongly coupled gauge theories. However, the constituents of gravitational theory depend on the observables from the gauge theory [10]. Regardless of whether one uses a bottom-up or top-down approach, the gauge–gravity dictionary takes centre stage when relating the dynamics, symmetries, fields, operators and states of one theory to its dual.

In terms of real world applications, the duality has so far made some noteworthy predictions. The most cited of these is based on hydrodynamic studies of strongly coupled perfect fluids that also conform to the rules of QCD. Particularly, the universal lower-bound for the ratio of shear viscosity to entropy density as $\eta/s \geq 1/4\pi$ [11]. String corrections to classical gravity also confirm this bound [12] as do hydrodynamic calculations from high-energy scattering experiments [13].

From an experimental point of view, string theory is very difficult to verify. As of the writing of this thesis, there has been no observed phenomenon that could be viewed as uniquely stringy. This is because string phenomenon operates at an extremely high energy (where at strong coupling it is estimated to be close to the Planckian scale $\sim 10^{19}$ GeV) [14] and no man-made machine is even close to that scale.

In principle, the gauge–gravity duality could provide the means to the gravitational sector of the string scale since all we would need to really understand is the gauge theory. The result of this approach might not correspond to the string theory that governs our physical reality but, at the very least, it could provide useful insights into physics close to the Planck scale.

Studying string theory using the gauge–gravity duality would require a gauge theory that has a consistent description at energy scales close to string phenomenon, and preferably one that is accessible by experiment. High-energy scattering experiments such as those performed at the Large hadron Collider (LHC) located at the European Organization for Nuclear Research (CERN) and the Relativistic Heavy Ion Collider (RHIC) at the Brookhaven National Laboratory (BNL) are a good place to start.

High-energy ion collisions at RHIC and CERN produce the quark-gluon plasma (QGP) [15]. Though we should point out that the QGP is directly not observed. What is detected in the apparatus is hadrons which only emerge after quarks and gluons have cooled down; a process known as hadronization. Until the next great discovery in Physics, the QGP is as close as we get to strongly coupled matter with energies close to the Planck scale [16]. From a theoretical point of view, the QGP is presumed to obey the rules of QCD. However, the gravitational dual to QCD is yet to be discovered. There have been attempts such as the Sakai-Sugimoto model [9] that works well in the low energy sector of QCD but not sufficient in the high energy sector, where the QGP is concerned.

The production of QGP captures all sorts of data from which theoretically relevant information, such as stress-energy correlation functions, can be extracted. In principle, this information can be mapped through the gauge–gravity dictionary to an appropriate gravitational theory [17].

The goal of this thesis is to find the gravitational dual to a strongly coupled fluid such as the quark-gluon plasma. This will be achieved by first postulating some generalized theory of gravity constructed out of Riemann tensors only and a set of unknown numerical constants. Then, inspired by a special regime of graviton momentum, first established in [18], we will find the graviton scattering amplitudes for the generalized theory of gravity. Using the gauge–gravity dictionary, our scattering amplitudes are mapped to stress-tensor correlation functions that can be matched with those obtained in high-energy scattering experiments. The experimental data sets the values of the unknown numerical constants in the gravitational theory. This algorithm gives us a way to study the gravitational dual to the QGP.

The thesis is structured as follows: Chapter 2 provides a quick summary of open and closed strings, their dualities and D-branes. We follow this up with a description of Maldacena's correspondence and its dictionary. Chapter 3 briefly discusses some known applications of the gauge–gravity duality such as the viscosity/entropy bound. This is followed by a discussion on higher-point functions in relation to QCD. This is followed by a discussion on a special regime of high-momentum gravitons we have termed the Brustein-Medved (BM) regime. From there on, the regime is activated for the rest of the thesis. Chapter 4 gives a detailed analysis of our work based on [19]. In the BM regime, we calculate tree-level 1-particle irreducible (1PI) $2n$-point graviton scattering amplitudes corresponding to a certain class of generalized theory of gravity. The calculation is performed with the aide of what we call basis amplitudes and a touch of combinatorics. The goal of Chapter 5 is to calculate the 1-particle reducible correlation functions (1PR) which, together with the 1PIs, can be used to make a connection with experimental data. This starts with taking the 1PI amplitudes from the bulk to the boundary, followed by the holographic renormalization of the bulk-to-boundary amplitudes. Based on [20], the 1PR functions and their corresponding Witten diagrams are then determined here through convolution of the renormalized 1PIs. Chapter 6 puts everything together by providing an explicit interpretation of the graviton scattering amplitudes as stress-tensor correlation functions through the gauge–gravity dictionary. It is also here that we show the algorithm for finding the gravitational dual to the quark-gluon plasma. This is then followed by a discussion on the implication of some of our results and some predictions on what we would expect to see from experiments. Appendix A is meant to help clarify matters concerning the origins of certain scattering amplitudes.

Chapter 2

Strings and the gauge–gravity duality

It was mentioned in Chapter 1 that the gauge–gravity duality maps certain physical quantities between two unlikely theories in different dimensions. This chapter aims to explain how the mapping is achieved. It may be prudent to give some technical description of open and closed string dynamics.

2.1 Open and closed strings

Starting from first principles [21, 22, 23], the arena is a D-dimensional 'target' spacetime with a flat metric given by $\eta_{\mu\nu}$, where the indices run as $\mu, \nu = 0, 1, 2, ..., D-1$. As a string propagates, it sweeps out a two-dimensional area (open strings) or volume (closed strings) called the 'world-sheet' with intrinsic coordinates $(\sigma^1, \sigma^2) \equiv (\tau, \sigma)$, where $0 \leq \tau < \infty$ and $0 \leq \sigma \leq \pi$. The path of the string in spacetime is parameterized by $X^\mu(\tau, \sigma)$.

The induced metric on the world-sheet is given by

$$h_{ab} = \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}, \tag{2.1}$$

where $\partial_a = \frac{\partial}{\partial \sigma^a}$ such that $a = 1, 2$. The area (or volume) of the world-sheet is given by

$$S = -T \int d\sigma^1 d\sigma^2 \sqrt{-h}, \tag{2.2}$$

where $h = \det(h_{ab})$, $T = \frac{1}{2\pi \ell_s^2}$ is the string tension and ℓ_s is its length. For historical reasons [1], one usually sets $\alpha' = \ell_s^2$. The world-sheet volume given by Eq. (2.2) is known as the Nambu-Goto action and, on-shell, is equivalent to the Polyakov action [24, 25] given by,

$$S = -\frac{T}{2} \int d\sigma^1 d\sigma^2 \sqrt{-\gamma} \gamma^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}, \tag{2.3}$$

where γ_{ab} is a dynamical world-sheet metric and γ is its determinant. The Polyakov action admits both diffeomorphism and Weyl invariance. These symmetries ensure that the two actions stay equivalent and that they describe the correct string dynamics [23]. The equations of motion are determined by varying with respect to X^μ Eq. (2.3):

$$\frac{\delta S}{\delta X^\mu} = \eta_{\mu\nu} \partial_a (\sqrt{-\gamma} \gamma^{ab} \partial_b X^\nu) = 0. \tag{2.4}$$

Boundary conditions play a significant role in string dynamics. If periodic boundary conditions [2] are imposed on $X^\mu, \partial_\sigma X^\mu$ and γ_{ab} along the σ

[1] α' was known as the Regge slope at a time when string theory was thought to be the theory of strongly interacting sub-atomic particles.

[2] Periodic boundary conditions are defined such that for a function $A(x)$, there is an interval $a \leq x \leq b$ where $A(a) = A(b)$.

direction, then the string is closed. If we choose to impose Neumann boundary conditions, for which $\partial_\sigma X^\mu(\tau, 0) = \partial_\sigma X^\mu(\tau, \pi) = 0$, and/or Dirichlet boundary conditions, $X^\mu = constant$ at the endpoints, then the string is open.

If the the endpoints of an open string satisfy Dirichlet boundary conditions along certain target space directions, then their movement is restricted to a hyperplane called a Dp-brane, where p is the spatial dimension of the hyperplane and the "D" implies Dirichlet boundary conditions. Historically, D-branes were studied in geometry, particularly through the generalization of Maxwell's theory of electromagnetism; p-form electrodynamics. It can be shown that D-branes have a property similar to charge [26]. Thus, each Dp-brane can also carry a $(p+1)$-form charge, in much the same way a D0-brane (or particle) can be associated to a 1-form (A_μ, the electromagnetic potential), as is the case in Maxwell's electrodynamics.

2.2 Low-energy effective actions

The Polyakov action may be considered a two-dimensional conformal field theory of D-dimensional bosonic fields, X^μ. It is a conformal field theory precisely because the Polyakov action is invariant under Weyl (or conformal) transformations. Such a symmetry is particularly useful if we wish to have a quantum field theory with couplings that do not dependent on a scale. In other words, such theories that appear the same at any momentum or energy scale are fixed points of a renormalization group (RG) flow in a space of couplings [23, 27]. We can further clarify this statement by realizing that a theory may look different depending on the behaviour of its couplings at different energy scales. Thus, if at different scales the couplings do not change, then theory is a fixed point.

The spectrum of quantized bosonic strings includes a tachyonic state with a negative mass squared, $M^2 = -\frac{a}{\alpha'}$ $(a > 0)$. This unrealistic state can be dealt with through the implementation of the Becchi, Rouet, Stora and Tyutin (BRST) formalism [27]. However, this formalism requires a 26-dimensional target space. An alternative approach is to consider super-symmetry, in which case, the fermionic super-partners of the X^μs (commonly referred to as spacetime fermions) are included in the theory. This approach ensures the removal of the tachyonic state and has the added effect of changing the conformal symmetry of world-sheet to a super-conformal symmetry [28].

Thus far, there are five ways to implement super-symmetry in string theory. All five implementations are packaged with a critical dimension, $D = 10$, in which non-physical states such as tachyons completely vanish. In addition, these theories are invariant under both global Poincare and local Weyl transformations [29]. The five formulations are known as Type I, Type IIA, Type IIB, Heterotic $SO(32)$ and Heterotic $E_8 \times E_8$ superstring theories. These theories differ through their symmetry properties, brane configurations and string orientations.

Coming back to the bosonic string theory, there are three types of massless background fields that couple to the closed string. They are the dilaton $\Phi(X)$, the metric tensor $G_{\mu\nu}(X)$ and the 2-form $B_{\mu\nu}(X)$, which goes by the name Kalb-Ramond field. The 2-form is a gauge potential, analogous to the electromagnetic potential vector A_μ (a 1-form itself) in electromagnetism. The dilaton couples to the Euler density of the string world-sheet, which in this case is the two-dimensional Ricci scalar. The asymptotic value of the dilaton, Φ_0, sets the string coupling via $g_s = e^{\Phi_0}$.

Quantum states corresponding to these background fields emerge naturally during the quantization of the string [23, 29]. The fields can be viewed as coherent states of quantum excitations. For instance, a non-flat metric tensor can be viewed as a coherent state of excited gravitons.

In a generalized Polyakov action, one can always set $\eta_{\mu\nu} \to G_{\mu\nu}(X)$. The generalized closed string action can then be written, with a Euclidean dynamical world-sheet metric g_{ab}, as

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{g} \Big[\alpha' \Phi(X) R^{(2)} + \big\{ g^{ab} G_{\mu\nu}(X) + i\epsilon^{ab} B_{\mu\nu}(X) \big\} \partial_a X^\mu \partial_b X^\nu \Big], \tag{2.5}$$

where ϵ^{ab} is an anti-symmetric tensor and $R^{(2)}$ is the two-dimensional Ricci scalar.

In order to preserve the conformal symmetry of the quantum theory, the stress tensor from the action given by Eq. (2.5) must be traceless. Without giving an explicit derivation [3], the trace of the stress tensor can be written as

$$T = -\frac{1}{2\alpha'} \beta^G_{\mu\nu} g^{ab} \partial_a X^\mu \partial_b X^\nu - \frac{i}{2\alpha'} \beta^B_{\mu\nu} \epsilon^{ab} \partial_a X^\mu \partial_b X^\nu - \frac{1}{2} \beta^\Phi R^{(2)}. \tag{2.6}$$

[3] See [28] for a detailed derivation.

The beta functions, for small α', can be written as

$$\begin{aligned}
\beta^G_{\mu\nu} &= \alpha'\bigg(R_{\mu\nu} + 2\nabla_\mu\nabla_\nu\Phi - \frac{1}{4}H_{\mu\nu\rho}H^{\mu\nu\rho}\bigg) + \mathcal{O}(\alpha'^2),\\
\beta^B_{\mu\nu} &= \alpha'\bigg(-\frac{1}{2}\nabla^\rho H_{\rho\mu\nu} + \nabla^\rho\Phi H_{\rho\mu\nu}\bigg) + \mathcal{O}(\alpha'^2),\\
\beta^\Phi &= \alpha'\bigg(\frac{D-26}{6\alpha'} - \frac{1}{2}\nabla^2\Phi + \nabla_\rho\Phi\nabla^\rho\Phi - \frac{1}{24}H_{\mu\nu\rho}H^{\mu\nu\rho}\bigg) + \mathcal{O}(\alpha'^2), \quad (2.7)
\end{aligned}$$

where the 3-form $H = dB \equiv \partial_\rho B_{\mu\nu} + \partial_\nu B_{\rho\mu} + \partial_\mu B_{\nu\rho}$ is the field strength.

Conformal invariance[4] is preserved when the beta functions vanish. But as it turns out, the equations $\beta^G_{\mu\nu} = 0$, $\beta^B_{\mu\nu} = 0$ and $\beta^\Phi = 0$ in fact resemble the spacetime field equations for the background fields with an action given by,

$$S = \frac{1}{2\kappa_0^2}\int d^DX\sqrt{-G}e^{-2\Phi}\bigg[R + 4\nabla_\mu\Phi\nabla^\mu\Phi - \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} - \frac{2(D-26)}{3\alpha'} + \mathcal{O}(\alpha')\bigg], \quad (2.8)$$

where $\kappa_0 \sim \ell_s^{\frac{D-2}{2}}$.

Eq. (2.8) is known as the string frame action. It is also known as the *low-energy effective action* of the bosonic string since it governs the dynamics of the background spacetime fields (or massless sector) of the string theory in the limit $\alpha' \to 0$ [23, 29]. When $D = 10$, it is sometimes referred to as *supergravity*. We should note that on we will, on occasion, use “supergravity” to imply the gravitational sector of Type IIB string theory with maximal supersymmetry.

[4] A QFT is conformally invariant when the trace of the stress-energy tensor vanishes.

Suppose we make the following redefinitions

$$\bar{\Phi} = \Phi - \Phi_0,$$
$$\bar{G}_{\mu\nu} = e^{\frac{-4\bar{\Phi}}{D-2}} G_{\mu\nu}, \tag{2.9}$$

where Φ_0 is the asymptotic value of the dilaton that sets the value of the string coupling, then Eq. (2.8) becomes

$$S = \frac{1}{2\kappa^2} \int d^D X \sqrt{-\bar{G}} \Big[\bar{R} - \frac{4}{D-2} \nabla_\mu \bar{\Phi} \nabla^\mu \bar{\Phi} - \frac{1}{12} e^{-\frac{8\bar{\Phi}}{D-2}} H_{\mu\nu\rho} H^{\mu\nu\rho} - \frac{2(D-26)}{3\alpha'} e^{\frac{4\bar{\Phi}}{D-2}} + \mathcal{O}(\alpha') \Big], \tag{2.10}$$

where now $\kappa = \kappa_0 g_s = (8\pi G_D)^{\frac{1}{2}}$ and G_D is Newtons constant in D-dimensions. This known as the Einstein frame action due to its similarity with the Einstein-Hilbert action. These two frames produce equivalent physics [30].

In Type IIA and Type IIB superstring theories, the target space is 10-dimensional and the gauge potential, $B_{\mu\nu}$, is replaced by a charged $(p+1)$-form C_{p+1} (with p-even for IIA theories and p-odd for IIB theories) [31]. The field strength is then defined by a $(p+2)$-form $F_{p+2} = dC_{p+1}$.

There is a special class of solutions for the Type II string theories called black p-branes that play a significant role in realizing the duality [32, 33]. Black p-branes are extended black holes that have all the typical characteristics of classical black holes such as mass, charge, angular momentum and a curvature singularity hiding behind a horizon. Like classical black holes, an extremal black p-brane is defined as having the minimum possible mass while having an angular momentum and charge.

A similar construction occurs in the open string sector with the addition of a one-form background field from an action given by

$$S = -\frac{k}{4}\int d^D X e^{-\Phi}\mathbf{Tr}(F_{\mu\nu}F^{\mu\nu}) + \mathcal{O}(\alpha'), \tag{2.11}$$

where k is a constant and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ [23]. Open strings excitations are usually associated to gauge fields the same way closed string excitations are usually associated to excitations of gravitational theories.

2.3 Open/Closed string duality

Dai, Leigh and Polchinski [34] realised that Dp-branes are dynamical components of Type II string theories. Essentially, in the low-energy limit, the dynamics of a D-brane are described by excitations of the open string end-points. There is a world-sheet duality in which the Dp-branes, the end-points of open strings, act as a source of closed strings. For instance, one can imagine the end-points of an open string moving in a loop and make the observation that such an operation is similar to the propagation of a closed string between Dp-branes.

Another important quality of D-branes is that a stack of them can behave like a black hole. The reason is that any D-brane has tension and therefore energy density. The "mass" of a stack of D-branes thus depends on the number of D-branes in the stack. A large enough mass in a small volume inevitably leads to a black hole. This is one way to construct black p-branes from Dp-branes. And as Polchinski also pointed out, a stack of N Dp-branes results in a $(p+1)$-dimensional hyperplane carrying N units of

$(p+1)$-form charge [35]. Hence, one can construct charged black p-branes from a stack of Dp-branes.

If N Dp-branes are stacked, then the effective coupling parameter of the closed string sector becomes $g_s N$. Polchinski came to the realization that in the limit $g_s N \gg 1$, Dp-branes gave a complete description of extremal black p-branes. In the opposite limit, $g_s N \ll 1$, Dp-branes are hyperplanes in a flat closed string background upon which open strings end. This showed that, under certain limitations, Dp-branes can admit more than one kind of description.

For the purpose of illustrating the gauge–gravity duality, we consider brane configurations for which $p = 3$. In the low-energy limit of the Type II theory, the supergravity solution for a stack of N coincident D3-branes can be written as,

$$\begin{aligned} & ds^2 \equiv g_{\mu\nu} dx^\mu dx^\nu = H_3^{-\frac{1}{2}} \eta_{\mu\nu} dx^\mu dx^\nu + H_3^{\frac{1}{2}} dx^i dx^i, \\ & C_4 = (H_3^{-1} - 1) g_s^{-1} dx^0 \wedge \cdots \wedge dx^3, \\ & e^\Phi = g_s, \\ & \text{where } H_3 = 1 + \frac{4\pi g_s N (\alpha')^2}{r^4}, \end{aligned} \tag{2.12}$$

r is the radial coordinate, $\mu, \nu = 0, 1, 2, 3$ and $i = 4, \ldots, 9$. The radial coordinate can be rescaled as $r \to u = \frac{r}{\alpha'}$. Under this rescaling, the region close to the brane also known as the *near-horizon limit* described by $r \to 0$ has a metric that can be written as

$$\frac{ds^2}{\alpha'} = \frac{u^2}{L^2}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + L^2 \frac{du^2}{u^2} + L^2 d\Omega_5^2, \tag{2.13}$$

where $L^2 = \sqrt{4\pi g_s N}$ and $d\Omega_5^2$ is a metric component from a five-sphere, S^5, with radius set by $r_S^2 = \alpha' L^2$. It so happens that Eq. (2.13) is precisely the $AdS_5 \times S^5$ metric.

In the near-horizon region of the D3-brane, excitations of the closed string sector decouple from those in the asymptotically flat region near space-like infinity [31]. Which is why the $r \to 0$ limit is sometimes referred to as the *decoupling limit* [36]. Simply put, excitations in the asymptotically flat region are governed by a free theory of gravity and do not interact with those found near the brane which has a $AdS_5 \times S^5$ geometry.

From a different point of view, the low-energy dynamics of the D3-brane is described by excitations of open string endpoints, particularly by a $SU(N)$ supersymmetric gauge field theory, more commonly referred to as super-Yang-Mills theory (SYM) with $\mathcal{N} = 4$ supersymmetry in four dimensions [31]. Here N describes the rank of the gauge group and the number of colours of the gauge theory. This gauge theory is known to be conformal. In the low-energy limit ($\alpha \to 0$), the open string excitations are decoupled from the ambient flat spacetime which is described by supergravity.

The Yang-Mills theory has a coupling parameter that is related to the string coupling parameter through $g_{\mathrm{YM}}^2 = 2\pi g_s$. However, the effective coupling for this theory is given by ‘t Hoofts parameter $\lambda = g_{\mathrm{YM}}^2 N$. The $\lambda \ll 1$ limit, which corresponds to $g_{\mathrm{YM}}^2 N \sim g_s N \ll 1$, describes the regime in which a perturbative analysis of the gauge theory is known to be valid. On the other hand, $\lambda \gg 1$ describes the strong coupling regime of the gauge theory. It is also the regime of validity for the supergravity theory since closed string loop corrections have an effective coupling that scales as $g \sim 1/N$ [23]. Thus in the strong coupling regime, the super-Yang-Mills gauge theory has a description in terms of a supergravity solution given by $\mathrm{AdS}_5 \times \mathrm{S}^5$. This is a realization of the famed *AdS/CFT correspondence*.

In strict terms, in order to completely suppress string interactions so as to arrive at a theory of supergravity, the string coupling must be such that $g_s \to 0$. This means that one has to set $N \to \infty$ in order to arrive at $g_s N \gg 1$, the regime in which supergravity is valid. On the gauge theory side, this would amount to an infinite number of colours and according to 't Hooft [6], the limit in which only planar Feynman diagrams survive. The strong coupling limit of the Yang-Mills theory is notoriously difficult to study but thanks to the duality it can be studied to some degree using its gravitational dual. However, one has to map the quantities of the two theories using an appropriate *dictionary*.

2.4 Gauge–gravity duality dictionary

The AdS/CFT correspondence, as presented, shows a duality between a weakly coupled theory of gravity and a strongly coupled gauge theory. Certain physical quantities in one theory can be mapped to specific physical quantities in the dual theory. This section shows a part of the dictionary on how the mapping is done for some quantities.

Probably the simpler, yet key, entries in the dictionary are relationships between parameters of the two dual theories. For instance, the number of stacked branes plays the role of flux of the field strength at the boundary of S^5 [35],

$$N \sim \frac{1}{\ell_s} \int_{S^5} \star F_5 \frac{1}{g_s} \sim \left(\frac{r_S}{\ell_s} \right)^4 \frac{1}{g_s}, \tag{2.14}$$

From the gauge theory perspective, N is the rank of the gauge group and, as shown already, the ‘t Hooft coupling parameter is given by $\lambda = g_{YM} N \equiv 2\pi g_s N$. Thus, Eq. (2.14) shows a relationship between the ‘t Hooft coupling and the string tension ℓ_s^{-2} as

$$\lambda \sim \left(\frac{r_S}{\ell_s} \right)^4. \tag{2.15}$$

Similarly, the relationship between the five dimensional Newtons constant (G_5) for the gravitational theory and the rank of the gauge group in the conformal field theory is given by

$$G_5 = \frac{\pi}{2} \frac{r_S^3}{N^2}. \tag{2.16}$$

2.4.1 *Radius/Scale correspondence*

As it turns out, apart from describing the decoupling limit, the radial coordinate plays another key role in the correspondence. In particular, actions that require movement along the radial coordinate of AdS_5 correspond to changes in the energy scale of the gauge theory [37, 67]. So given Eq. (2.13), it can viewed as if each slice of r is conformally related to four-dimensional Minkowski space on which a gauge theory is defined. This is the reason that the radial coordinate is sometimes referred to as the holographic direction.

From a gauge theory perspective, the limits $r \to 0$ and $r \to \infty$ can be considered the infra-red (IR) and ultra-violet (UV), respectively. These limits are so defined following the behaviour of open and closed string one-loop vacuum amplitudes [39, 40].

On the other hand, while $r \to \infty$ is the natural AdS_5 boundary for the metric in Eq. (2.13), it is also the IR limit of the gravitational theory. This type of correspondence between the two theories is more commonly referred to as the *UV/IR correspondence*. One frequently encounters the phrase "the dual gauge theory lives on the boundary of Anti-de Sitter space" in the literature to imply the UV/IR correspondence. When in fact, the dual theory "lives" everywhere along the radial coordinate, one simply chooses a slice so as to focus on some effective field theory. The process of choosing to work on some slice of r is in fact equivalent to choosing some cut-off scale in the field theory [41].

One can summarize this correspondence as high energies (or short distances) in the field theory translate to moving closer to the boundary in the bulk theory. Similarly, low energies (or large distances) in the field theory translate to moving away from the boundary of the bulk theory. In a later chapter, we will discuss the handling of the infinities that arise at high-energy scales using holographic renormalization techniques.

2.4.2 *Matching observables*

The correspondence between the large-N limit gauge theory and the gravitational theory implies that the partition functions are equivalent [42, 43].

That is to say,

$$Z_{\text{gauge theory}} \equiv Z_{\text{gravity}}. \tag{2.17}$$

This expression is known as the GKPW relation after it's authors [42, 43]. However, the relation is realized only when one takes the boundary values of certain fields in the bulk theory and these are shown to have a correspondence with specific operators coupled to sources in the boundary theory. Schematically, for bulk fields denoted by $\phi(x^\mu, r)$ and boundary theory operators denoted by $\mathcal{O}(x^\mu)$, the GKPW relation is presented in Euclidean form as,

$$Z_{\text{string theory}}[\phi_b] = \int_{\phi(x^\mu,r)\to\phi(x^\mu,\infty)} \mathcal{D}\phi \; e^{-S[\phi(x^\mu,r)]} = \left\langle e^{-\int d^4x \phi_b(x^\mu)\mathcal{O}(x^\mu)} \right\rangle_{conn} = Z_{\text{CFT}}[\phi_b], \tag{2.18}$$

where the subscript b denotes that we are taking the boundary value of the bulk fields. The GKPW relation is a general statement about all gauge–gravity correspondences. Admittedly, when presented as Eq. (2.18), we have over simplified a host of inter connected ideas. To summarize the reality of things here, the source field, $\phi_b(x^\mu)$, on the CFT side is related to some field, $\Phi(x^\mu, r)$ on the gravity side. One then performs a near-boundary expansion of $\Phi(x^\mu, r)$ so that ultimately, $\lim_{r\to\infty} \Phi(x^\mu, r) = \phi_b(x^\mu)$ [31].

Furthermore, the *conn* on the CFT side implies that we are taking the connected correlation functions. So for example, one computes the two-point function (an observable of the theory) as,

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle_{conn} = -\frac{1}{Z_{\text{CFT}}} \frac{\delta^2 Z_{\text{CFT}}}{\delta\phi_b(x_1)\delta\phi_b(x_2)}\bigg|_{\phi_b=0}. \tag{2.19}$$

Higher-point functions are obtained by repeated functional differential of the partition function with respect to the source. This way the GKPW relation provides the means of gaining knowledge about observables of one side of the duality from observables on the other side.

Although the metric (gravity) is always present, scalar fields and other gauge fields may be present depending on the compactification scheme employed on the flavour of string theory under consideration. The duality implies that bulk gauge fields such as A_μ are dual to currents J_μ in the boundary theory and that metric perturbations, $h_{\mu\nu}$, are dual to the stress-energy tensor, $T_{\mu\nu}$, of the gauge theory. More importantly, the observables of the two theories share a correspondence. That is to say, the bulk amplitudes correspond to the boundary connected correlation functions,

$$\begin{aligned}\lim_{r\to\infty} \langle A_\mu(x_1, r)\cdots A_\mu(x_n, r)\rangle_{connected} &\leftrightarrow \langle J_\mu(x_1)\cdots J_\mu(x_n)\rangle_{connected}\,,\\ \lim_{r\to\infty} \langle h_{\mu\nu}(x_1, r)\cdots h_{\mu\nu}(x_n, r)\rangle_{connected} &\leftrightarrow \langle T_{\mu\nu}(x_1)\cdots T_{\mu\nu}(x_n)\rangle_{connected}\,.\end{aligned} \tag{2.20}$$

There are other entries in the AdS/CFT dictionary [1, 31, 44, 45]. However, they will not be addressed here since they do not bear immediate importance to the rest of the thesis as our goal is to relate observables between two theories.

Chapter 3

Applicability of the gauge–gravity duality to strongly coupled theories

Now that we have some idea of what the gauge–gravity duality is, we can discuss ways in which it is and could be applied.

As previously mentioned, so far there are two approaches towards the application of the gauge–gravity duality; the top-down and the bottom-up approaches. The top-down method begins with a well-defined weakly coupled theory of gravity that is dual to some unknown strongly coupled field theory. The field theory parameters, which determine the physical observables, are set by the gravitational theory. In the bottom-up approach, one postulates a gravitational theory in the bulk that is asymptotically AdS. The parameters of the bulk theory are set by the field theory.

Since our goal is to find a gravitational theory whose parameters are set by a dual theory with physical observables extracted from scattering experiments on the quark-gluon plasma (QGP), the bottom-up approach may be of particular interest. Though, we should state explicitly that our work is independent of either approach.

Lets start with some of the things that are already known about the QGP in relation to the gauge–gravity duality. From an experimental point-of-view, the plasma is produced in ultra-relativistic heavy-ion collisions [46, 47, 48, 49]. The QGP shows collective behavior, which is another way of saying the plasma behaves like a strongly coupled fluid. As a fluid, the QGP can be studied using specific hydrodynamic models from which physical parameters such as the ratio of viscosity to entropy density can be determined. Experiment shows that this ratio, η/s, lies between $0.08 - 0.24$ [13].

From a theoretical point of view, η/s can also be calculated using the AdS/CFT duality since the strong coupling regime of the gauge theory is not well understood. Using this string theory approach, Kovtun–Son–Starinets (KSS) [11] determined a universal lower bound for strongly coupled fluids as

$$\frac{\eta}{s} \geq \frac{1}{4\pi} \sim 0.07957, \tag{3.1}$$

in natural units. This bound is in agreement with experiment and represents a significant achievement for the AdS/CFT duality and string theory in general.

Originally, Kovtun–Son–Starinets adopted the Kubo formula [50] to find the shear viscosity. Kubo's formula makes use of correlators to measure the response of an observable quantity. In this case, they calculated the shear viscosity from the low-momentum limit graviton 2-point amplitudes (the dual to stress-tensor correlation functions) as,

$$\eta = \lim_{\omega \to 0} \frac{1}{2\omega} \int dt\ d\mathbf{x}\ e^{i\omega t} \left\langle [h_{xy}(t,\mathbf{x}), h_{xy}(0,\mathbf{0})] \right\rangle \propto \frac{A}{16\pi G}, \tag{3.2}$$

where $x \neq y$ are mutually perpendicular to the direction of propagation, A is the area of the black hole horizon, G is Newtons constant and ω is the momentum of the graviton. The entropy of an asymptotically AdS black hole was then used to complete the calculation of the bound. This highlights the role of correlation functions in aiding the determination of the bound.

However, the original KSS bound is the result of an application of the prototypical AdS/CFT duality, which means we still have $N \to \infty$. Hadrons, on the other hand, obey the rules of a far less symmetrical theory; (lattice) quantum chromodynamics (QCD) with $N = 3$. This does not necessarily mean that predictions by the duality do not apply in this case. In the past, there have been both top-down and bottom-up attempts at an extension of the duality known as holographic QCD or AdS/QCD [10, 51].

There is a specific construction of AdS/QCD in [10] that begins with a well known model of QCD with a fixed number of colours. Using the field/operator correspondence from the AdS/CFT dictionary (see Section 2.4.2), the bulk theory is constructed in such a manner as to reproduce measurable physical observables of the QCD, particularly, the pion mass and the pion decay constant. Yet, given that QCD is not conformal, such models deviate from the prototype construction of the duality even though the majority of them adopt many of its ideas. For the most part, they still require that N be large yet finite. However, only the 2-point functions have a description in these models since higher n-point correlators vanish [52]. Scattering experiments generate vast amounts of information, which includes a large array of higher-point functions, that cannot be utilized in holographic QCD models.

Another crucial point about holographic QCD models is that they are constructed in the low-energy domain in which fields are strongly coupled [53]. Under normal circumstances, this would be a good thing in terms of the duality since strongly coupled gauge theories are not well understood, and it also guarantees that the gravitational dual is weakly coupled and can thus be approximated to be classical or semi-classical. However, since only 2-point functions have a description, this makes holographic QCD models very limited in phenomenological applications. For instance, an important property of QCD known as spontaneous Chiral Symmetry Breaking is described by higher n-point functions. In addition, the characterization of hadron form factors also depends on n-point functions [54, 55]. At the very least, this shows that a complete picture of higher-point correlators meaningfully furthers our understanding of QCD and, by extension, our knowledge of the QGP and its dual.

In general, AdS/QCD models cannot be utilized adequately to study the QGP. Regardless, the duality still remains the most promising tool for computing real-time n-point correlators of strongly interacting gauge theories [56, 57]. This is because graviton amplitudes have proven to be much easier to compute than their dual counterparts, the stress-tensor correlators.

At the very least, any string theory that is dual to the QGP is bound to meet a few key criteria. The most obvious one being that we would require that the stack of branes be much less than infinity, preferably $N = 3$. This, in effect, would mean a deviation from supergravity since the ‘t Hooft coupling would be too small to suppress the string interaction terms. In the mean time, as suggested in [58], we can work with a large-yet-finite N as most features of the duality do not directly depend on the size of N. That being said, any deviation from $N \to \infty$ would require a departure from classical gravity in the form of a perturbative expansion.

Scattering amplitudes (or correlation functions) are essential in the application of duality, though finding the full set of correlators on the gauge theory side borders between notoriously difficult and impossible. The same task on the gravity side, while difficult, is achievable. Alternatively, we could choose to work in a particular regime of energy/momentum which would also help reduce the complexity of the calculation as long as useful information can be extracted.

3.1 String theory in a special regime

As it turns out, there is a special regime of high-momentum that significantly reduces the difficulty of calculating scattering amplitudes while still being relevant for the application of the duality. This kinematic regime was heavily utilized by Brustein and Medved [18] as part of their quest to find meaningful string theory predictions that are experimentally accessible in a dual gauge theory. To avoid any confusion here, we shall call this kinematic region the Brustein-Medved (BM) regime, the details of which we shall now discuss.

Originally, the BM regime was described using an asymptotically AdS_5 black brane metric and a graviton. We should mention that the black brane metric was used for the express purposes of making the connection with hydrodynamics in the dual theory. The metric is given by,

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -f(r)dt^2 + \frac{1}{g(r)}dr^2 + \frac{r^2}{L^2}(dx^2 + dy^2 + dz^2), \tag{3.3}$$

where L is the AdS curvature length, the radial coordinate ranges from the black brane horizon r_h to the AdS boundary at $r \to \infty$ and $f(r), g(r) = (\frac{r}{L})^2(1 - (\frac{r_h}{r})^4)$ exhibit the behaviour,

$$\begin{aligned} &\lim_{r\to r_h} f(r) = 0, \quad \lim_{r\to r_h} g(r) = 0, \\ &\lim_{r\to\infty} \frac{f(r)}{r^2} = \frac{1}{L^2}, \quad \lim_{r\to\infty} \frac{g(r)}{r^2} = \frac{1}{L^2}. \end{aligned} \tag{3.4}$$

On the other hand, gravitons are set to propagate along the z-direction taking the plane-wave form,

$$\delta g_{\mu\nu} \equiv h_{\mu\nu} = \phi_{\mu\nu}(r)e^{i(\omega t - kz)}, \tag{3.5}$$

where $\phi_{\mu\nu}(r)$ is a well-defined function with quasi-normal behaviour at the horizon [59].

In the BM regime, we require that t- and z-derivatives of the graviton must dominate over background derivatives. Mathematically, this means expressions like $\{\partial_t\partial_t h_{\mu\nu},\ \partial_t h_{\mu\nu}\partial_t h_{\mu\nu}\} \propto \omega^2$ or $\{\partial_z\partial_z h_{\mu\nu},\ \partial_z h_{\mu\nu}\partial_z h_{\mu\nu}\} \propto k^2$ or $\{\partial_t\partial_z h_{\mu\nu},\ \partial_t h_{\mu\nu}\partial_z h_{\mu\nu}\} \propto \omega k$ need to be much greater than the background derivatives $\partial_r\partial_r h_{\mu\nu} \sim \frac{1}{L^2}$. In other words, the BM regime requires that $\{\omega^2, \omega k, k^2\} \gg \frac{1}{L^2}$.

Applying the correspondence with strongly coupled fluids in play means invoking some of the constraints from fluid dynamics. With that in mind, hydrodynamic computations with a black brane background [60] require that the magnitude of graviton momenta be much less than the Hawking temperature of the black brane. In other words, the hydrodynamic regime demands that $\{\omega, k\} \ll T$. Thus, the BM regime is given by,

$$1 \ll L\omega \ll TL. \tag{3.6}$$

Clearly, the BM regime is consistent with $TL \gg 1$, the hydrodynamic regime for which a black brane can be utilized as background [42]. We should also note that the Hawking temperature of the brane is equivalent to the temperature of the strongly coupled fluid in the dual theory.

We will show in the next chapter that this regime is especially well suited for distinguishing between contributions of terms in a perturbative expansion of the low-energy gravitational Lagrangian and is particularly useful for applications of the duality in the context of this thesis.

Chapter 4

Graviton scattering amplitudes in the AdS bulk

First, let's recall that the goal of this thesis is to find the gravitational dual to the quark-gluon-plasma (QGP). To that end, we will use the relationship between graviton scattering amplitudes of a weakly coupled string theory and stress-tensor correlators of a strongly coupled gauge theory, as formally dictated by the AdS/CFT dictionary. High-energy experimental data on stress-tensor correlation functions could in the near-future be made readily available. Thus, our plan is to find a way to match our theoretical calculations with the experimental data.

In general, graviton amplitudes are difficult to compute in a full string theory. However, this task could be greatly simplified if we restricted ourselves to a particular regime of momentum. In the previous chapter, we briefly discussed the Brustein-Medved (BM) region, which was described as the regime in which graviton momenta must be significantly greater than background contributions. Here we show how to calculate graviton amplitudes for a general higher-derivative theory of gravity in the BM regime.

4.1 Generalized higher-derivative gravity

Most theories of gravity are made up of invariant quantities constructed out of the Riemann curvature tensor and its contractions—the Ricci tensor and Ricci scalar. Generally, the Lagrangian can be expressed as a function of the curvature terms in the form of

$$I = \int d^D x \sqrt{-g}\; f(R_{abcd}), \tag{4.1}$$

where D is the dimension of the spacetime and contractions of the Riemann tensor are implied in the Lagrangian.

The most notable of the $f(R_{abcd})$ theories is Einstein's theory which contains a single Ricci scalar term and thus two-derivatives in the Lagrangian and field equations. An extension to Einstein gravity known as Gauss-Bonnet gravity contains a total of four-derivatives in the Lagrangian, yet only two-derivatives appear in the field equations. These theories are, in fact, unique among the $f(R_{abcd})$ type theories since they take on the role of topological invariants (Euler densities of the spacetime) in certain spacetime dimensions – $D = 2$ for Einstein and $D = 4$ for Gauss-Bonnet. Which means that in those particular dimensions, there are no field equations and no meaningful analysis can be drawn.

A natural generalization to Einstein and Gauss-Bonnet gravity is the higher-derivative Lovelock theory of gravity [61]. Constructed from Euler densities, Lovelock theory yields two-derivative field equations. This feature becomes handy when simplifying amplitudes, as single gravitons "carrying" two-derivatives ($\partial_\alpha \partial_\beta h_{\mu\nu}$) can be substituted for gravitons each "carrying" a single derivative ($\partial_\alpha h_{\mu\nu} \partial_\beta h_{\mu\nu}$). This particular procedure was performed in [18] to calculate the on-shell graviton amplitudes for Einstein and Gauss-Bonnet gravity in the BM regime.

Our work follows from recent findings in [62] where it was found that if one is only interested in gauge-invariant quantities such as scattering amplitudes, then all terms that contain the Ricci scalar or tensor can be gauged away with an appropriate metric redefinition of the form $\delta g_{\mu\nu} = a(g, R) g_{\mu\nu} + b(g, R) R_{\mu\nu}$, where a and b are functions of the metric and Ricci scalar. After the gauge transformation, the Lagrangian would only feature Riemann tensors. As a consequence, our scattering amplitudes will also include expressions of the form $\partial_\alpha \partial_\beta h_{\mu\nu}$ since the Lovelock field equations no longer act as constraints.

We will refer to a theory containing k Riemann tensors as Riem^k gravity or, in some cases, $2k$-derivative gravity. For example, a Riem^2 theory refers to the term $\gamma R_{abcd} R^{abcd}$ and a Riem^3 theory refers to the expression $\gamma_1 R_{abcd} {R_{mn}}^{ab} R^{mncd} + \gamma_2 R_{abcd} {R_{mn}}^{ad} R^{mncb}$, where the γ's are model-dependent constants. The Riem^3 theory, as shown here, implies all possible self-contractions of the Riemann tensors have to be considered.

We are working with a weakly coupled string based theory of gravitation, with $g_s \ll 1$. Up-to gauge transformations, we can define a five-dimensional generalized classical theory of higher-derivative gravity as,

$$S = \int d^5x \, \mathcal{L} = \int d^5x \, \frac{\sqrt{-g}}{16\pi G_5} \big(\Lambda + \gamma_0 R + \epsilon \gamma_1 R^2 + \epsilon^2 \gamma_2 R^3 + \cdots + \epsilon^n \gamma_n R^{n+1}\big), \tag{4.2}$$

where G_5 is the five-dimensional Newtons constant, Λ is the cosmological constant, R^n represents a Riem^n term, ϵ is a dimensionless perturbation parameter related to α' through $\epsilon = \frac{\ell_s^2}{L^2} \ll 1$, L is the AdS curvature radius and each γ_i is a model-dependent numerical factor of order unity and of dimension $[\gamma_i] = \text{length}^{2i}$. The γ_i's are the unknown numbers we are trying to be determine.

The field equations, determined by varying the action with respect to $\delta g^{\mu\nu}$, are given by,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \epsilon (\text{Riem}^2)_{\mu\nu} + \epsilon^2 (\text{Riem}^3)_{\mu\nu} + \cdots \epsilon^n (\text{Riem}^{q+1})_{\mu\nu} = 0, \tag{4.3}$$

where

$$(\text{Riem}^Q)_{\mu\nu} = \frac{\delta(\sqrt{-g} R^Q)}{\delta g^{\mu\nu}}, \text{ for } 2 \leq Q \leq (q+1). \tag{4.4}$$

The physical observables of Eq. (4.2) are given by scattering amplitudes (or, correlation functions, correlators in gauge theory parlance). In particular, one considers connected scattering amplitudes as opposed to their disconnected counterparts since experimental data consists of connected amplitudes/functions. We should note that connected functions are those for which the Feynman or Witten diagram consists of one component, whereas disconnected functions consist of more than one component (see, Appendix A for more details).

Connected scattering amplitudes are generated by a functional of the form $iW[J] = \ln(Z[J])$, where

$$Z[J] = \frac{\int [\mathcal{D}g]\ e^{iS+i\int J^{\mu\nu}h_{\mu\nu}}}{\int [\mathcal{D}g]\ e^{iS}}, \tag{4.5}$$

$J^{\mu\nu}$ is a source term. We can then obtain scattering amplitudes through functional differentiation of $W[J]$ with respect to the source,

$$\langle h_{\mu\nu}(x_1)h_{\mu\nu}(x_2)\cdots h_{\mu\nu}(x_n)\rangle_{\text{connected}} = i^{n-1}\frac{\delta^n W}{\delta J^{\mu\nu}(x_1)\delta J^{\mu\nu}(x_2)\cdots\delta J^{\mu\nu}(x_n)}\bigg|_{J^{\mu\nu}=0}. \tag{4.6}$$

However, in the large-N limit of the string partition function we can take the *saddle point approximation* [31, 63], from which we find that the classical action dominates the string partition function. Thus, the on-shell classical action generates the connected scattering amplitudes, that is to say $W \simeq S_{\text{on-shell}}$.

Thus, an on-shell expansion of the gravitational Lagrangian using a perturbed metric gives the on-shell 1-particle irreducible (1PI) graviton scattering amplitudes (see Appendix A for further details). This chapter focuses on finding the 1PI scattering amplitudes at each order in ϵ of Eq. (4.2) for a specific regime of graviton momentum in a classical AdS black hole background. Though we should also mention that if the length scale associated with the higher order γ_i's is the string length, then quantum corrections are to be considered in the partition function. For now, we will maintain an arbitrary length scale for the γ_i's as they are the unknowns. The action will be treated as classical since the lower order ϵ terms are dominant following the size of $\epsilon \ll 1$. And, as we shall see soon, we will be working in regime of high graviton momentum in which $s\epsilon \lesssim 1$, where s is the square of the graviton momentum, hence higher order terms will not be ignored. In the next chapter, we will determine the 1-particle-reducible (1PR) amplitudes via convolution of 1PIs.

4.1.1 Brustein-Medved regime considerations

Finding a complete account of scattering amplitudes for any given theory is a notoriously difficult problem because of the many possible variations one may encounter. Fortunately, the BM regime can be used to reduce the complexity of this task. Let's recall from Section 3.1 that in the interest of studying a 4-dimensional strongly coupled fluid, an AdS_5 black brane can act as background on the gravitational side of the duality. For our purposes, we will adopt the same AdS_5 metric given by Eq. (3.3) and gravitons propagating along the z-direction, that is $h_{\mu\nu} \sim e^{i(\omega t - kz)}$.

The BM regime requires that only amplitudes with the highest powers of momentum are of relevance so as to dominate any background contributions. This makes this regime uniquely suited at discriminating scattering amplitudes from different ϵ-orders of the Lagrangian since at every order the only relevant amplitudes are the ones with the highest powers of momentum. This is because each Riemann tensor supplies one of either ω^2 or ωk or k^2. Thus, a Riem^q term contributes momentum of the order of ω^{2q}. This simplifies the calculation in the sense that we can deal with each Riem^q

term as a separate theory. Thus, we shall proceed by finding scattering amplitudes at every order in ϵ and then add everything at the very end.

In addition, we can choose the radial gauge, for which $h_{r\nu} = 0$ for any ν. As a consequence, the graviton divides into scalar $h = \{h_{xx}, h_{yy}, h_{zz}, h_{tt}, h_{tz}\}$, vector $h_A = \{h_{tx}, h_{zx}, h_{ty}, h_{zy}\}$ and tensor $\{h_{xy}\}$ sectors [65].

The field equations Eq. (4.3) can be used to study the behaviour of these graviton sectors. Recall that the background metric (AdS metric) obeys the Einstein field equations, that is,

$$R_{\mu\nu}(g) - \frac{1}{2}g_{\mu\nu}R(g) + g_{\mu\nu}\Lambda = 0. \tag{4.7}$$

The perturbed metric conforms to the field equation,

$$\frac{1}{2}(\partial_\alpha\partial_\mu h^\alpha_\nu + \partial_\alpha\partial_\nu h^\alpha_\mu - \partial_\mu\partial_\nu h - \Box h_{\mu\nu} - g_{\mu\nu}\partial_\alpha\partial_\beta h^{\alpha\beta} + g_{\mu\nu}\Box h) + \mathcal{O}(\epsilon) = 0, \tag{4.8}$$

where $\Box\psi \equiv \partial_\alpha\partial^\alpha\psi$.

Lets recall that the BM regime requires the maximal number of derivatives acting on gravitons and, subject to the previous constraint, the minimal number of epsilons. Each term in the expanded Lagrangian must have the maximum number of ω's and/or k's and minimum number of ϵ's associated with it. Thus, on-shell one can see, from Eq. (4.8), that the scalar modes can be discarded since they would either vanish or have to be sourced by higher $\mathcal{O}(\epsilon)$ terms. Vector modes on the other hand, can always be gauged away as they are analogous to potentials in gauge theories. Even then, they would have to be sourced by higher order ϵ terms when appearing in gauge-invariant combinations. Thus, vector modes do not contribute to graviton amplitudes here. That leaves the tensor modes, h_{xy}, as the only relevant participants in the BM regime. However, any appearance of $\Box h_{xy}$ has to be discarded as it would violate conditions of the BM regime by including higher order ϵ terms through $\Box h_{\mu\nu} = 0 + \mathcal{O}(\epsilon)$ as dictated by Eq. (4.8). Therefore, tensor modes would necessarily have to come in pairs so as to satisfy general covariance of the amplitudes. Thus, the BM regime constrains all scattering amplitudes to be even.

Adopting conventions from [64] for perturbations to the determinant and contravariant metrics, respectively,

$$\sqrt{-g} \;\rightarrow\; \sqrt{-g}\left[1 + \frac{1}{2}h^{\mu}_{\mu} - \frac{1}{2^2}h^{\nu}_{\mu}h^{\mu}_{\nu} + \mathcal{O}(h^3)\right] , \tag{4.9}$$

$$g^{\mu\nu} \;\rightarrow\; g^{\mu\nu} - h^{\mu\nu} + h^{\mu}_{\rho}h^{\nu\rho} + \mathcal{O}(h^3) \; . \tag{4.10}$$

In our case, the perturbed contravariant metric and the determinant become,

$$g^{xx} \;\rightarrow\; \bar{g}^{xx} = g^{xx} + h^{x}_{y}h^{xy} + (h^{x}_{y}h^{xy})^2 + (h^{x}_{y}h^{xy})^3 + \cdots + (h^{x}_{y}h^{xy})^p, \tag{4.11}$$

$$\sqrt{-g} \;\rightarrow\; \sqrt{-\bar{g}} = \sqrt{-g}\left[1 - \frac{1}{2}h^{xy}h_{xy} - \frac{1}{2^2 2!}(h^{xy}h_{xy})^2 - \cdots + \Theta(p)(h^{xy}h_{xy})^p\right], \tag{4.12}$$

respectively, where the expression for $\bar{g}^{yy}$ is similar to Eq. (4.11) but with an interchange between x and y and

$$\Theta(p) = -\frac{\Gamma[p - \frac{1}{2}]}{2\sqrt{\pi}p!}, \text{ for } p \in \mathbb{Z}. \tag{4.13}$$

The expressions given by Eqs. (4.11) and (4.12) play an important role in the calculation of $2n$-point amplitudes as they supply undifferentiated gravitons. Differentiated gravitons are, at each order in ϵ, supplied by what we have termed *basis* amplitudes.

4.2 Bulk graviton scattering amplitudes

4.2.1 Basis multi-point amplitudes

As mentioned in the previous section, differentiated gravitons in the BM regime are supplied by what we call basis amplitude. We define the basis amplitude as a multi-point amplitude for which all included gravitons are differentiated. The first step is to expand the Riemann tensor in terms of the perturbed metric. Using $\bar{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}$, the expanded Riemann tensor takes the form,

$$\bar{R}_{abcd}(\bar{g}) = R_{abcd}(g) + \delta^{(1)} R_{abcd}(h) + \delta^{(2)} R_{abcd}(h) + f_{abcd}(g, h), \tag{4.14}$$

where

$$\delta^{(1)} R_{abcd}(h) = \partial_b \Gamma_{dac}(h) - \partial_c \Gamma_{dab}(h), \tag{4.15}$$

$$\delta^{(2)} R_{abcd}(h) = g^{ef} \Bigg(\Gamma_{eac}(h) \Gamma_{fbd}(h) - \Gamma_{ead}(h) \Gamma_{fbc}(h) \Bigg). \tag{4.16}$$

and $\Gamma_{abc}(h) = \frac{1}{2}(\partial_b h_{ac} + \partial_c h_{ab} - \partial_a h_{bc})$. The first term on the right-hand side of Eq. (4.14) is a background term and is discarded in the BM regime. Similarly, all terms containing a mixture of the background metric and gravitons are packed in the last term, $f_{abcd}(g, h)$, and are discarded in the BM regime. It should be clear that $\delta^{(1)} R_{abcd}$ represents terms with 2 derivatives acting on a single graviton and $\delta^{(2)} R_{abcd}$ represents 2 gravitons each carrying a single derivative. It might be worth mentioning that while Eq. (4.14) is an exact expression, mixed or contravariant perturbative expansions of the Riemann tensor would require careful application of terms like Eq. (4.11).

Since only t- and z-derivatives apply to gravitons in the BM regime, only a few arrangements in the indices are allowed in $\delta^{(1)} R_{abcd}$ and $\delta^{(2)} R_{abcd}$.

While being mindful of the (anti-)symmetries of the Riemann tensor, we can only have $\delta^{(1)}R_{axby}$ such that $a, b = \{t, z\}$ and $\delta^{(2)}R_{cxdx}$ such that $c, d = \{t, z, y\}$, where x and y are interchangeable.

Hence, in every Riem^k theory, a basis amplitude can be constructed from k combinations of $\delta^{(1)}R$'s and $\delta^{(2)}R$'s, which essentially extract the highest powers of momentum from gravitons. A $2n$-point amplitude constructed from a basis $2p$-point amplitude in a Riem^k theory will be labelled as $\langle h^{2n}\rangle_{(2p,\ \epsilon^{k-1})}$, for $n \geq p$. By definition, basis amplitudes themselves will be labelled as $\langle h^{2p}\rangle_{(2p,\ \epsilon^{k-1})}$.

4.2.2 Riem^3 gravity multi-point scattering amplitudes.

Starting with Riem^3 gravity, we show how one can obtain multi-point amplitudes from the basis amplitudes. Up to gauge transformations, Riem^3 is given by

$$\text{Riem}^3 = \sqrt{-g}\big(\alpha_1 R_{abcd}R^{ab}{}_{mn}R^{mncd} + \alpha_2 R_{abcd}R^{ad}{}_{mn}R^{mncb}\big), \tag{4.17}$$

here the factor $\frac{\epsilon^2 L^4}{16\pi G_5}$ has been absorbed into the model-dependent constants α_1 and α_2.

From this six-derivative theory, we can construct two types of basis amplitudes: 4- and 6-point basis amplitudes, $\langle h^4\rangle_{(4,\epsilon^2)}$ and $\langle h^6\rangle_{(6,\epsilon^2)}$ respectively. Schematically, with indices suppressed, $\langle h^4\rangle_{(4,\epsilon^2)}$ can be written as

$$\langle h^4\rangle_{(4,\epsilon^2)} \sim 3(\alpha_1 + \alpha_2)\delta^{(1)}R\ \delta^{(1)}R\ \delta^{(2)}R, \tag{4.18}$$

where the " 3 " represents the number of ways in which we can select which Riemann tensor carries 2 gravitons. In a slightly more explicit way, this basis function is given by,

$$\left\langle h^4\right\rangle_{(4,\epsilon^2)} = 3\times 2^3 \times (\alpha_1+\alpha_2)\times\Bigg(\delta^{(1)}R_{txty}\delta^{(1)}R^{txty}\delta^{(2)}R^{tx}{}_{tx} + \delta^{(1)}R_{txzy}\delta^{(1)}R^{txzy}\delta^{(2)}R^{tx}{}_{tx}$$
$$+ \delta^{(1)}R_{txzy}\delta^{(1)}R^{zxzy}\delta^{(2)}R^{tx}{}_{zx} + \{t\leftrightarrow z\}\Bigg) + \{x\leftrightarrow y\}, \qquad (4.19)$$

where $\{a\leftrightarrow b\}$ is a shorthand for interchanging a and b in all preceding terms of the amplitude and the " 2^3 " accounts for all the (anti-)symmetries of the Riemann tensors. This basis amplitude can be further expanded in terms of gravitons using Eqs. (4.15) and (4.16) and factorized into the following expression:

$$\left\langle h^4\right\rangle_{(4,\epsilon^2)} = -6(\alpha_1+\alpha_2)\sqrt{-g}(g^{xx}g^{yy})^2$$
$$\times\Big[(\omega_1 g^{tt}\omega_2 + k_1 g^{zz}k_2)(\omega_1 g^{tt}\omega_3 + k_1 g^{zz}k_3)(\omega_2 g^{tt}\omega_4 + k_2 g^{zz}k_4)\Big] h^{(1)}_{xy}h^{(2)}_{xy}h^{(3)}_{xy}h^{(4)}_{xy}, \qquad (4.20)$$

where each factor $(\omega_i g^{tt}\omega_j + k_i g^{zz}k_j)$ can be viewed as a product of the momenta of gravitons $h^{(i)}_{xy}$ and $h^{(j)}_{xy}$. For the sake of brevity, we let $C_{ij} = (\omega_i g^{tt}\omega_j + k_i g^{zz}k_j)$. Note that, in some instances C_{ij} will be written as $C_{(i,j)}$ to avoid confusion. Symmetrization of gravitons is implied, this means that the term in the square brackets is in fact equivalent to

$$C_{12}C_{13}C_{24} \to \quad C_{(12}C_{23}C_{34)} + C_{(12}C_{24}C_{34)} + C_{(12}C_{12}C_{34)}$$
$$+ C_{(12}C_{34}C_{34)} + C_{(12}C_{13}C_{34)} + C_{(12}C_{34}C_{14)}, \qquad (4.21)$$

where each term on the right-hand side is such that

$$C_{(12}C_{23}C_{34)} = C_{12}C_{23}C_{34} + C_{32}C_{23}C_{14} + C_{24}C_{23}C_{13}, \qquad (4.22)$$

and we have used the fact that terms like C_{ii} vanish since $C_{ii}\propto \Box h_{xy} = 0 + \mathcal{O}(\epsilon)$. Thus, Eq. (4.20) can be written in a much more condensed form as

$$\langle h^4 \rangle_{(4,\epsilon^2)} = -6(\alpha_1+\alpha_2)\sqrt{-g}(g^{xx}g^{yy})^2 C_{12}C_{13}C_{24}\prod_{i=1}^{4} h_{xy}^{(i)}. \tag{4.23}$$

Similarly, the 6-point basis amplitude has a schematic form which can be written as

$$\langle h^6 \rangle_{(6,\epsilon^2)} \sim (\alpha_1+\alpha_2)\ \delta^{(2)}R\ \delta^{(2)}R\ \delta^{(2)}R. \tag{4.24}$$

In terms of gravitons, this basis amplitude can be written as

$$\langle h^6 \rangle_{(6,\epsilon^2)} = -\frac{3}{2}(\alpha_1+\alpha_2)\sqrt{-g}(g^{xx}g^{yy})^3 C_{12}C_{34}C_{56}\prod_{i=1}^{6} h_{xy}^{(i)}. \tag{4.25}$$

From these basis amplitudes, all other $2n$-point amplitudes can constructed by adding pairs of gravitons from the perturbed metric determinant and contravariant metric tensors, $\bar{g}^{xx}$ and $\bar{g}^{yy}$. Starting with the 4-point basis amplitude, as an example, lets assume that p pairs are extracted from the determinant and $n-2-p$ pairs from the four contravariant metrics. Using the fact that the number of ways of drawing q identical gravitons from m distinct contravariant metrics is given by $\binom{q+m-1}{m-1}$, the $2n$-point amplitude from this basis amplitude is given by

$$\langle h^{2n} \rangle_{(4,\epsilon^2)} = -6(\alpha_1+\alpha_2)\binom{2n}{4}\sum_{p=0}^{n-2}\binom{n+1-p}{3}\Theta(p)\sqrt{-g}(g^{xx}g^{yy})^n C_{12}C_{13}C_{24}\prod_{j=1}^{n} h_{xy}^{(2j-1)}h_{xy}^{(2j)}, \tag{4.26}$$

where $n \geq 2$, $\binom{2n}{4}$ represents the number of ways of choosing 4 differentiated gravitons from $2n$ and the summation accounts for all possible ways of drawing gravitons from the determinant and the contravariant metrics. In addition, we recall that any invariant pair of undifferentiated gravitons can always be written as $h^x_y h^y_x = g^{xx} g^{yy} h_{xy} h_{xy}$.

Similarly, a $2n$-point amplitude (for $n \geq 3$) constructed from a 6-point basis is given by

$$\left\langle h^{2n} \right\rangle_{(6,\epsilon^2)} = -\frac{3}{2}(\alpha_1 + \alpha_2)\binom{2n}{6}\sum_{p=0}^{n-3}\binom{n-p+2}{5}\Theta(p)\sqrt{-g}(g^{xx}g^{yy})^n C_{12}C_{34}C_{56}\prod_{j=1}^{n} h^{(2j-1)}_{xy} h^{(2j)}_{xy}. \tag{4.27}$$

The complete $2n$-point amplitude for Riem^3 gravity is the linear combination of Eqs. (4.26) and (4.27), given by

$$\left\langle h^{2n} \right\rangle_{\text{Riem}^3} = \left\langle h^{2n} \right\rangle_{(4,\epsilon^2)} + \left\langle h^{2n} \right\rangle_{(6,\epsilon^2)}. \tag{4.28}$$

4.2.3 Riem^4 gravity multi-point scattering amplitudes

Before diving into a treatment of multi-point amplitudes of Riem^k gravity for *all* k, it maybe more beneficial for the reader to see one more type of multi-point amplitude but with an even k.

The Riem^4 theory can be written as

$$\begin{aligned}\text{Riem}^4 = {} & \beta_1 R_{abcd} R^{abmn} R_{mn}{}^{pq} R_{pq}{}^{cd} + \beta_2 R_{abcd} R^{ab}{}_{qp} R^{mndq} R_{mn}{}^{cp} + \beta_3 R_{abcd} R^{pd}{}_{mn} R^{abcq} R_{pq}{}^{mn} \\ & + \beta_4 R_{abcd} R_{mnpq} R^{ancq} R^{mbpd} + \beta_5 R_{abcd} R^{na}{}_{pq} R^{mbpd} R_{mn}{}^{qc} + \beta_6 R^{abcd} R_{abcd} R^{mnpq} R_{mnpq},\end{aligned} \tag{4.29}$$

where $\frac{\epsilon^3 L^6}{16\pi G_5}$ has been absorbed into the model-dependent β's. There are 3 types of basis amplitudes for theory; the 4-, 6- and 8-point basis amplitude. Schematically, these basis amplitudes can be written in terms of Riemann tensors as

$$\langle h^4 \rangle_{(4,\epsilon^3)} \sim (\beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5 + \beta_6)\delta^{(1)}R\ \delta^{(1)}R\ \delta^{(1)}R\ \delta^{(1)}R, \quad (4.30)$$

$$\langle h^6 \rangle_{(6,\epsilon^3)} \sim 6(\beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5 + \frac{1}{3}\beta_6)\delta^{(1)}R\ \delta^{(1)}R\ \delta^{(2)}R\ \delta^{(2)}R \quad (4.31)$$

$$\langle h^8 \rangle_{(8,\epsilon^3)} \sim (\beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5 + \beta_6)\delta^{(2)}R\ \delta^{(2)}R\ \delta^{(2)}R\ \delta^{(2)}R, \quad (4.32)$$

where the numerical factors account for all possible arrangements of Riemann tensors with respect to the number of gravitons that each of them supplies. Eq. (4.30) shows that each graviton in all Riemann tensors will carry two derivatives, while Eq. (4.32) indicates that each graviton will carry a single derivative. Following the same procedure in Section 4.2.2, the explicit forms of these basis amplitudes, in terms of gravitons, is given by

$$\langle h^4 \rangle_{(4,\epsilon^3)} = 4(\beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5 + \beta_6)\sqrt{-g}(g^{xx}g^{yy})^2 C_{12}C_{23}C_{34}C_{14} \prod_{i=1}^{4} h_{xy}^{(i)}, \quad (4.33)$$

$$\langle h^6 \rangle_{(6,\epsilon^3)} = 6(\beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5 + \frac{2}{3}\beta_6)\sqrt{-g}(g^{xx}g^{yy})^3 C_{12}C_{23}C_{34}C_{56} \prod_{i=1}^{6} h_{xy}^{(i)}, \quad (4.34)$$

$$\langle h^8 \rangle_{(8,\epsilon^3)} = \frac{1}{2}(\beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5 + 2\beta_6)\sqrt{-g}(g^{xx}g^{yy})^4 C_{12}C_{34}C_{56}C_{78} \prod_{i=1}^{8} h_{xy}^{(i)}, \quad (4.35)$$

where symmetrization of gravitons is implied in all these expressions. The $2n$-point amplitudes are constructed from these basis forms similar to Eqs. (4.26) and (4.27) in the previous section. Here they are given by,

$$\left\langle h^{2n}\right\rangle_{(4,\epsilon^3)} = 4(\beta_1+\beta_2+\beta_3+\beta_4+\beta_5+\beta_6)\binom{2n}{4}\sum_{p=0}^{n-2}\binom{n+1-p}{3}\Theta(p)\sqrt{-g}(g^{xx}g^{yy})^n$$
$$\times C_{12}C_{23}C_{34}C_{14}\prod_{j=1}^{n}h_{xy}^{(2j-1)}h_{xy}^{(2j)} \qquad (n\geq 2), \tag{4.36}$$

$$\left\langle h^{2n}\right\rangle_{(6,\epsilon^3)} = 6(\beta_1+\beta_2+\beta_3+\beta_4+\beta_5+\frac{2}{3}\beta_6)\binom{2n}{6}\sum_{p=0}^{n-3}\binom{n+2-p}{5}\Theta(p)\sqrt{-g}(g^{xx}g^{yy})^n$$
$$\times C_{12}C_{23}C_{34}C_{56}\prod_{j=1}^{n}h_{xy}^{(2j-1)}h_{xy}^{(2j)} \qquad (n\geq 3), \tag{4.37}$$

$$\left\langle h^{2n}\right\rangle_{(8,\epsilon^3)} = \frac{1}{2}(\beta_1+\beta_2+\beta_3+\beta_4+\beta_5+2\beta_6)\binom{2n}{8}\sum_{p=0}^{n-4}\binom{n+3-p}{7}\Theta(p)\sqrt{-g}(g^{xx}g^{yy})^4$$
$$\times C_{12}C_{34}C_{56}C_{78}\prod_{j=1}^{n}h_{xy}^{(2j-1)}h_{xy}^{(2j)} \qquad (n\geq 4).$$

The complete BM $2n$-point amplitude for the Riem^4 theory is the linear combination of Eqs. (4.36) to (4.38), given by

$$\left\langle h^{2n}\right\rangle_{\mathrm{Riem}^4} = \left\langle h^{2n}\right\rangle_{(4,\epsilon^3)} + \left\langle h^{2n}\right\rangle_{(6,\epsilon^3)} + \left\langle h^{2n}\right\rangle_{(8,\epsilon^3)}. \tag{4.38}$$

The explicit forms of the $2n$-point amplitudes for both Riem^3 and Riem^4 gravity shows that the basis amplitudes exhibit a pattern, in much the same way basis vectors are used to construct vectors, that could be exploited to find the amplitudes of any Riem^k theory, with arbitrary k. This is mainly due to the construction of basis amplitudes from combinations of $\delta^{(1)}R$ and $\delta^{(2)}R$ terms. The next step is to construct multi-point amplitudes for a general higher-derivative theory of gravity in the BM regime.

4.2.4 Riem^q gravity multi-point scattering amplitudes

Naturally, one expects that the number of gauge-invariant terms in a Riem^q theory grows exponentially as e^q based on the number of tensor components, $4q$. Nonetheless, up-to numerical factors and symmetrized gravitons, each of those gauge-invariant terms will roughly produce a similar *frame* for the multi-point amplitudes. This assertion gains some credence from the structural similarities between Riem^3 and Riem^4 amplitudes in the previous sections.

To construct generalized basis amplitudes, we first note that the size and parity of q gives an indication of the number and form of basis amplitudes we expect to see in a Riem^q theory. For instance, in a q-odd theory, the set of basis amplitudes is given by

$$B_{q\text{-odd}} = \left\{ \left\langle h^{q+1} \right\rangle_{(q+1,\epsilon^{q-1})}, \left\langle h^{q+3} \right\rangle_{(q+3,\epsilon^{q-1})}, \ldots, \left\langle h^{2q-2} \right\rangle_{(2q-2,\epsilon^{q-1})}, \left\langle h^{2q} \right\rangle_{(2q,\epsilon^{q-1})} \right\}. \tag{4.39}$$

$B_{q\text{-odd}}$ has cardinality $\frac{q+1}{2}$ and is such that $\left\langle h^{q+1} \right\rangle_{(q+1,\epsilon^{q-1})}$ has the most number of individual gravitons carrying two-derivatives whereas each graviton in $\left\langle h^{2q} \right\rangle_{(2q,\epsilon^{q-1})}$ carries a single derivative. In terms of Riemann tensors, these basis amplitudes can be written schematically as

$$\begin{aligned}
\left\langle h^{q+1}\right\rangle_{(q+1,\epsilon^{q-1})} &\sim \left(\prod_{i=1}^{q-1}\delta_i^{(1)}R\right)\delta^{(2)}\ R,\\
\left\langle h^{q+3}\right\rangle_{(q+3,\epsilon^{q-1})} &\sim \left(\prod_{i=1}^{q-3}\delta_i^{(1)}R\right)\left(\prod_{j=1}^{3}\delta_j^{(2)}R\right),\\
\vdots \qquad &\vdots \quad \vdots \quad \vdots \quad \vdots \qquad \vdots\\
\left\langle h^{2q-2}\right\rangle_{(2q-2,\epsilon^{q-1})} &\sim \delta^{(1)}R\ \delta^{(1)}R\ \left(\prod_{i=1}^{q-2}\delta_i^{(2)}R\right),\\
\left\langle h^{2q}\right\rangle_{(2q,\epsilon^{q-1})} &\sim \prod_{i=1}^{q}\delta_i^{(2)}R,
\end{aligned} \tag{4.40}$$

where i and j are simply labels used to count Riemann tensors.

A similar line of reasoning applies for q-even Riem^q theories. The set of basis amplitudes is

$$E_{q\text{-even}} = \left\{ \left\langle h^{q}\right\rangle_{(q,\epsilon^{q-1})}, \left\langle h^{q+2}\right\rangle_{(q+2,\epsilon^{q-1})}, \ldots, \left\langle h^{2q-2}\right\rangle_{(2q-2,\epsilon^{q-1})}, \left\langle h^{2q}\right\rangle_{(2q,\epsilon^{q-1})} \right\}, \tag{4.41}$$

with cardinality $\frac{q+2}{2}$. The difference being that the "smallest" basis amplitude will have each of its gravitons carrying two derivatives: $\langle h^q\rangle_{(q,\epsilon^{q-1})} \sim \prod_{i=1}^{q}\delta_i^{(1)}R$.

Now we can express the basis amplitudes in terms of gravitons by carefully expanding the schematic forms using Eqs. (4.15) and (4.16). The result is a polynomial of degree $2q$ composed of ω's and k's . This polynomial can then be factorized appropriately into products of pairs of gravitons. This process yields q factors of graviton momentum products which are conveniently written as C_{ij} (or $C_{(i,j)}$) $= (\omega_i g^{tt}\omega_j + k_i g^{zz} k_j)$. Then for any Riem^q

theory, the generalized basis amplitude is given by

$$\left\langle h^{2p}\right\rangle_{(2p,\epsilon^{q-1})} = \mathcal{A}_q 2^{q-2p+2}\sqrt{-g}(g^{xx}g^{yy})^p \underbrace{C_{(1,2)}C_{(3,4)}\cdots C_{(2p-1,2p)}C_{(1,2p)}}_{q \text{ momentum products}} \times \prod_{i=1}^{2p} h_{xy}^{(i)}, \tag{4.42}$$

$$(\text{ for any } p\in\mathbb{N}^+ \quad \text{such that} \quad q \leq 2p < 2q)$$

where $\mathcal{A}_q \sim \mathcal{O}(e^q)$ is a linear combination of all the model-dependent factors of the theory and graviton symmetrization is implied. For $2p = 2q$, the basis amplitude is given by

$$\left\langle h^{2q}\right\rangle_{(2q,\epsilon^{q-1})} = \mathcal{A}_q 2^{q-2p+2}\sqrt{-g}(g^{xx}g^{yy})^q \underbrace{C_{(1,2)}C_{(3,4)}\cdots C_{(2p-1,2p)}}_{q \text{ momentum products}} \times \prod_{i=1}^{2q} h_{xy}^{(i)}. \tag{4.43}$$

The difference between Eqs. (4.42) and (4.43) is that we expect more repetition in the momentum indices in the former than in the latter.

The penultimate step is finding the $2n$-point functions from each of the generalized basis amplitudes by adding undifferentiated gravitons taken from the determinant and the contravariant metrics. Using the same reasoning from Section 4.2.2, we can appropriately sort out the relevant combinatorics involving the undifferentiated gravitons so that the $2n$-point amplitude is given by

$$\begin{aligned}\left\langle h^{2n}\right\rangle_{(2p,\epsilon^{q-1})} = \mathcal{A}_q 2^{q-2p+2}\binom{2n}{2p}\sum_{r=0}^{n-p}\binom{n+p-r+1}{2p-1}\Theta(r)\sqrt{-g}(g^{xx}g^{yy})^n \\ \underbrace{C_{(1,2)}C_{(3,4)}\cdots C_{(2p-1,2p)}\cdots}_{q \text{ momentum products}} \times \prod_{i=1}^{2n} h_{xy}^{(i)}, \qquad \text{for any } n\geq p.\end{aligned} \tag{4.44}$$

Finally, we have to add all $2n$-point amplitudes from each basis to find the complete BM regime $2n$-point amplitude for the Riem^q theory. This scattering amplitude is given by

$$\left\langle h^{2n}\right\rangle_{\text{Riem}^q} = \begin{cases} \sum_{m=0}^{\frac{q}{2}} \left\langle h^{2n}\right\rangle_{(2m+q,\epsilon^{q-1})} & \forall \text{ even } q>1, \text{ such that } 2n \geq 2m+q \quad \forall m, \\ \sum_{m=0}^{\frac{q-1}{2}} \left\langle h^{2n}\right\rangle_{(2m+q+1,\epsilon^{q-1})} & \forall \text{ odd } q>1, \text{ such that } 2n \geq 2m+q+1 \quad \forall m. \end{cases} \tag{4.45}$$

Thus far, we have calculated tree-level 1PI scattering amplitudes. However, in order to make contact with the dual gauge theory we will need the full set of tree-level connected functions, that is both 1PI and 1PR amplitudes, which when evaluated at the boundary and holographically renormalized, are equivalent to stress-tensor correlation functions. In the next chapter, we connect to the dual gauge theory.

Chapter 5

Stress-tensor correlation functions at the AdS boundary

In the previous chapter, we calculated the 1-particle irreducible (1PI) graviton multi-point scattering amplitudes in a special regime of high-momentum. In this chapter, we shall continue where we left off and ultimately make the connection with the dual gauge theory by providing a method for determining all the connected stress-tensor correlators in the Brustein-Medved (BM) regime.

1-particle reducible (1PR) multi-point correlation functions can be constructed through convolution (or joining of Witten diagrams via an internal propagator, see appendix A for details) of lower ϵ-order 1PI correlation functions. In this case, it is the boundary values of holographically renormalized scattering amplitudes that will be required in the convolution process. This procedure, in terms of the BM regime, was first showcased in [66].

5.1 Holographic renormalization

First, lets recall from the the gauge–gravity duality dictionary that for every bulk field, $\phi(x^\mu, r)$, there is a dual gauge invariant operator, $\mathcal{O}_\phi$, in the boundary theory. In particular, one considers the equivalence of two generating functionals according to the GPKW relation (recalling Eq. (2.18)) given by

$$\begin{aligned} Z_{\text{string theory}}[\phi_b] &= \int_{\phi(x^\mu,r)\to\phi_b} \mathcal{D}\phi \; e^{-S[\phi(x^\mu,r)]} \\ &= \left\langle \exp\left(- \int \; \phi_b(x^\mu)\mathcal{O}(x^\mu) \right) \right\rangle_{conn} = Z_{\text{CFT}}[\phi_b], \end{aligned} \tag{5.1}$$

where ϕ_b are boundary values of $\phi(x^\mu, r)$ that act as sources for the operators in the boundary theory. In the *saddle point approximation* [63]–which is basically the classical approximation of the path integral–Eq. (5.1) becomes

$$S_{os}[\phi_b] = -W[\phi_b], \tag{5.2}$$

where S_{os} is the on-shell supergravity action and W is the generating functional of connected diagrams in the gauge theory. The correlation functions of the operators can be computed through functional differentiation of the sources ϕ_b as

$$\langle \mathcal{O}(x) \rangle = \left. \frac{\delta S_{os}}{\delta \phi_b(x)} \right|_{\phi_b=0}, \tag{5.3}$$

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2) \rangle = - \left. \frac{\delta^2 S_{os}}{\delta \phi_b(x_1)\delta \phi_b(x_2)} \right|_{\phi_b=0}, \tag{5.4}$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$\langle \mathcal{O}(x_1)\cdots\mathcal{O}(x_n) \rangle = (-1)^{n+1} \left. \frac{\delta^n S_{os}}{\delta \phi_b(x_1)\cdots\delta \phi_b(x_n)} \right|_{\phi_b=0}. \tag{5.5}$$

Unfortunately, Eqs. (5.3) to (5.5) tend to be UV divergent. However, through renormalization we can make sense of these correlation functions. In addition, we are reminded of the UV/IR correspondence from Section 2.4: UV divergences in the gauge theory correspond to IR divergences in the gravitational theory, which arise from the fact that the volume of space is infinite. In the AdS/CFT context, the IR limit of gravitational theory is found at the AdS boundary. And so the near-boundary analysis describing the handling of IR divergences is what is referred to as holographic renormalization.

With our primary goal in mind, holographic renormalization as described in [67, 68] is a three-step process.

(i) Bringing bulk quantities (in our case graviton amplitudes) to the boundary.

(ii) The second step entails multiplying the bulk-to-boundary graviton amplitudes by appropriate powers of a conformal factor. The purpose of this operation is to trace the evolution of the metric (and in turn its perturbations) along the holographic coordinate. This is analogous to regularizing a quantum field theory prior to adding counter-terms that subsequently remove any divergent terms, resulting in a renormalized action. In terms of the AdS/CFT correspondence, the regularization procedure requires one to consider the induced metric for a family of foliations as one approaches the boundary [69]. However, it should be noted that this process is largely coordinate dependent, and so is the form of the conformal factor.

(iii) Finally, one adds counter-terms to any divergent quantities from the previous step.

Here we will use the Riem^3 basis amplitudes (1PIs) to illustrate the holographic renormalization procedure. One can confirm by power counting that the same basic outcome persists for higher-order theories. We will not delve into Einstein and Gauss-Bonnet (Riem^2) amplitudes since their 1PR connected functions have been discussed at great lengths in [66]. However, our work differs from theirs in sense that we include gravitons with two derivatives acting on them.

Bulk-to-boundary amplitudes

First and foremost, the boundary geometry is described by the $r \to \infty$ limit of the metric Eq. (3.3), which is the asymptotic limit of AdS space and is given by

$$\lim_{r\to\infty} ds^2 = \frac{r^2}{L^2}(-dt^2 + dx^2 + dy^2 + dz^2) + \frac{L^2}{r^2}dr^2. \tag{5.6}$$

In this limit, the contravariant metrics are $g^{xx} = g^{yy} = g^{tt} = -g^{tt} = \frac{L^2}{r^2}$ and the metric determinant is given by $\lim_{r\to\infty} \sqrt{-g} = \left(\frac{L}{r}\right)^{-3}$. Hence, using this metric we can re-write the bulk amplitudes at the boundary.

Lets recall, from Section 4.2.2, that the 4- and 6-point basis amplitudes, from which we generate all other 1PIs, are given by

$$\langle h^4 \rangle_{(4,\epsilon^2)} = -6(\alpha_1 + \alpha_2)\sqrt{-g}(g^{xx}g^{yy})^2 C_{12}C_{23}C_{34} \;\; h^{(1)}_{xy}h^{(2)}_{xy}h^{(3)}_{xy}h^{(4)}_{xy}, \tag{5.7}$$

$$\langle h^6 \rangle_{(6,\epsilon^2)} = -\frac{3}{2}(\alpha_1 + \alpha_2)\sqrt{-g}(g^{xx}g^{yy})^3 C_{12}C_{34}C_{56} \;\; h^{(1)}_{xy}h^{(2)}_{xy}h^{(3)}_{xy}h^{(4)}_{xy}h^{(5)}_{xy}h^{(6)}_{xy}, \tag{5.8}$$

where $C_{ij} = (\omega_i g^{tt}\omega_j + k_i g^{zz}k_j)$ and, as previously stated, symmetrization of gravitons implies that

$$C_{12}C_{23}C_{34} = C_{(12}C_{23}C_{34)} + C_{(12}C_{24}C_{34)} + C_{(12}C_{12}C_{34)} \\ + C_{(12}C_{34}C_{34)} + C_{(12}C_{13}C_{34)} + C_{(12}C_{34}C_{14)}. \qquad (5.9)$$

In the boundary limit, Eqs. (5.7) and (5.8) are given by

$$\lim_{r\to\infty} \langle h^4 \rangle_{(4,\epsilon^2)} = -6(\alpha_1 + \alpha_2)\left(\frac{L}{r}\right)^{11} c_{12}c_{23}c_{34} \;\; h_{xy}^{(1)}h_{xy}^{(2)}h_{xy}^{(3)}h_{xy}^{(4)}, \qquad (5.10)$$

$$\lim_{r\to\infty} \langle h^6 \rangle_{(6,\epsilon^2)} = -\frac{3}{2}(\alpha_1 + \alpha_2)\left(\frac{L}{r}\right)^{15} c_{12}c_{34}c_{56} \;\; h_{xy}^{(1)}h_{xy}^{(2)}h_{xy}^{(3)}h_{xy}^{(4)}h_{xy}^{(5)}h_{xy}^{(6)}, \qquad (5.11)$$

where $c_{ij} = (k_ik_j - \omega_i\omega_j)$ is such that $\lim\limits_{r\to\infty} C_{ij} = \frac{L^2}{r^2}c_{ij}$. The c_{ij}'s are subject to the same symmetrization condition given by Eq. (5.9). At ϵ^2-order, the 4-point amplitude is given by the basis amplitude in Eq. (5.7), while the 6-point amplitude is the sum of the basis 6-point amplitude and a 6-point amplitude constructed from the basis 4-point amplitude. Thus, using the complete Riem^3 multi-point amplitude, Eq. (4.28), the complete boundary 6-point amplitude is the sum

$$\lim_{r\to\infty} \langle h^6 \rangle_{\mathrm{Riem}^3} = \lim_{r\to\infty} \langle h^6 \rangle_{(4,\epsilon^2)} + \lim_{r\to\infty} \langle h^6 \rangle_{(6,\epsilon^2)}\,, \\ = -(\alpha_1 + \alpha_2)\left(\frac{L}{r}\right)^{15}\left[315\; c_{12}c_{34}c_{14} + \frac{3}{2}\; c_{12}c_{34}c_{56}\right]\prod_{i=1}^{6} h_{xy}^{(i)}. \qquad (5.12)$$

In the general Riem^q theory, the complete boundary $2n$-point amplitude takes the form

$$\lim_{r\to\infty} \langle h^{2n} \rangle_{\mathrm{Riem}^q} = \left(\frac{L}{r}\right)^{4n+2q-3} \sum_p \langle h^{2n} \rangle_{(2p,\epsilon^{q-1})}\,, \qquad (5.13)$$

where terms under the summation are defined according to Eq. (4.45) and are independent of the holographic coordinate.

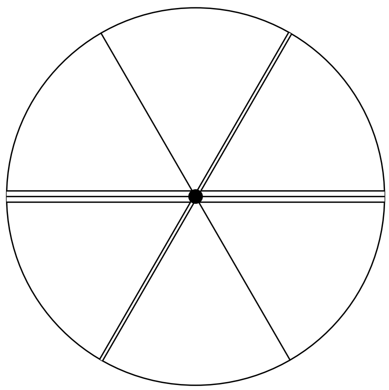
Multiplication by conformal factor & addition of counter-terms

The conformal factor is determined from the form of the asymptotic form of the AdS metric, which can be identified as $\Omega = \frac{r}{L}$. However, we have to track the evolution of every field and operator in the amplitudes. Now the holographic coordinate rescales $r \to \gamma r$ and since the line element ds^2 must be conformally invariant, both the metric and the graviton rescale as $h_{\mu\nu} \to \gamma^{-2} h_{\mu\nu}$. Therefore, the metric and the graviton have conformal dimension of 2. Similarly, the derivatives have conformal dimension of 1 while the the metric determinant has a conformal dimension of -3 [70]. Thus, we multiply the bulk-to-boundary amplitudes by $\Omega^{\Delta-3}$, where Δ is the sum of the conformal dimensions of gravitons and derivatives.

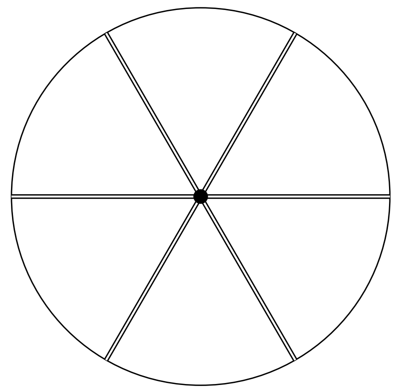
Thus, at order-ϵ^2 a "regularized" boundary 6-point amplitude takes the form

$$
\begin{aligned}
\left\langle h^6 \right\rangle_{\text{Riem}^3}^{\text{regularized, 1PI}} &= \Omega^{2\times 6+1\times 6-3} \lim_{r\to\infty} \left\langle h^6 \right\rangle_{\text{Riem}^3} , \\
&= \left(\frac{r}{L}\right)^{15} \times \left[-(\alpha_1+\alpha_2)\left(\frac{L}{r}\right)^{15} \left[315\ c_{12}c_{34}c_{14} + \frac{3}{2}\ c_{12}c_{34}c_{56} \right] \prod_{i=1}^{6} h_{xy}^{(i)} \right], \\
&= -(\alpha_1+\alpha_2)\left[315\ c_{12}c_{34}c_{14} + \frac{3}{2}\ c_{12}c_{34}c_{56} \right] \prod_{i=1}^{6} h_{xy}^{(i)} .
\end{aligned}
\tag{5.14}
$$

The final expression of Eq. (5.14) has no radial dependence and no apparent divergences as $r \to \infty$. The lack of diverging quantities only highlights the fact that the BM regime is in effect and as such the final step bears no consequence. Otherwise, under normal circumstances one has to consider subtracting divergent background bulk quantities [71], but due to the BM regime, such a subtraction is irrelevant. This means Eq. (5.14), as presented, is the renormalized 6-point 1PI amplitude. The corresponding Witten diagrams for this amplitude are given in Fig. 5.1.



(a)



(b)

Figure 5.1: (a) corresponds to $\langle h^6 \rangle^{\text{renormalized 1PI}}_{(4,\epsilon^2)}$, where the single lines denote undifferentiated gravitons, the double lines denote gravitons carrying a single derivative and the triple lines denote gravitons carrying two derivatives. (b) corresponds to $\langle h^6 \rangle^{\text{renormalized 1PI}}_{(6,\epsilon^2)}$ with all gravitons each carrying a single derivative.

A similar procedure for the Riem^q $2n$-point amplitude yields,

$$
\begin{aligned}
\langle h^{2n} \rangle^{\text{renorm, 1PI}}_{\text{Riem}^q} &= \Omega^{2n\times 2+1\times 2q-3} \lim_{r\to\infty} \langle h^{2n} \rangle_{\text{Riem}^q} \,, \\
&= \left(\frac{r}{L}\right)^{4n+2q-3} \left(\frac{L}{r}\right)^{4n+2q-3} \sum_{p}^{q} \langle h^{2n} \rangle_{(2p,\epsilon^{q-1})} \,, \\
&= \sum_{p}^{q} \langle h^{2n} \rangle_{(2p,\epsilon^{q-1})} \,. \qquad (5.15)
\end{aligned}
$$

With respect to Eq. (5.15), the number of variations of Witten diagrams of any correlation function and their complexity depends on the size of n and q. Yet still, just like Fig. 5.1, we expect a combination of single, double and triple lines for all Witten diagrams associated with the BM regime. From these renormalized 1PIs, we can now construct 1PR connected stress-tensor correlation functions.

5.2 1PR connected functions and diagrams

At the beginning of the previous section we hinted, in a general way, at how correlation functions can be derived from the generating functional. With the GPKW relation in mind, the sort of generating functional under consideration takes the form

$$\int_{h_{\mu\nu}(r\to\infty)} [\mathcal{D}h]\ e^{-S[h_{\mu\nu}]} = \left\langle e^{-S_g - \int T^{\mu\nu}h_{\mu\nu}} \right\rangle_{conn}, \tag{5.16}$$

where the left-hand side is the gravitational partition function and the right-hand side is the generating functional of the stress-tensor correlators with a gauge theory action S_g. The stress-tensor correlators are obtained through recursive functional differentiation with respect to the source, which in this case is $h_{\mu\nu}$.

Firstly, we write the left-hand side of Eq. (5.16) in the context as a Riem^q theory partition function, subject to the BM regime, given by

$$e^{-W[h]} = \int [\mathcal{D}h]\ e^{\int d^5x\ \sqrt{-g}[\gamma_0 R(h) + \gamma_1 \epsilon R^2(h) + \gamma_2 \epsilon^2 R^3(h) + \cdots + \gamma_{q-1}\epsilon^{q-1} R^q(h) + \cdots]}, \tag{5.17}$$

where the $R(h)$'s on the right-hand side are gauge-invariant quantities constructed from Riemann tensors only and the γ_i's are model-dependent constants of dimension $[\gamma_i] = (\text{length})^{2i}$. The 1PR connected functions at every order in ϵ are constructed from lower order correlators. For example, a 1PR correlator $\left\langle h^{2n} \right\rangle_{\epsilon^k}$ can be constructed from two lower order correlators as follows

$$\left\langle h^{2n} \right\rangle_{\epsilon^k} = \frac{\left\langle h^{2p} \right\rangle_{\epsilon^i} \left\langle h^{2q} \right\rangle_{\epsilon^j}}{\langle hh \rangle}, \tag{5.18}$$

where $n = p + q - 1$, $k = i + j$ and $\langle hh \rangle$ in the denominator denotes a propagator that connects the two correlators. In general, any 1PR connected function constructed from ℓ lower order 1PI functions takes the form

$$\left\langle h^{2n} \right\rangle_{\epsilon^k} = \frac{\left\langle h^{2b_1} \right\rangle_{\epsilon^{m_1}} \left\langle h^{2b_2} \right\rangle_{\epsilon^{m_2}} \cdots \left\langle h^{2b_\ell} \right\rangle_{\epsilon^{m_\ell}}}{\langle hh \rangle \langle hh \rangle \cdots \langle hh \rangle}, \tag{5.19}$$

where

$$n = -(\ell - 1) + \sum_{i=1}^{\ell} b_i, \tag{5.20}$$

$$k = \sum_{i=1}^{\ell} m_i. \tag{5.21}$$

For tree-level diagrams, Eq. (5.20) is a relationship between the number of vertices ℓ, the number of external lines $2n$ and the number of internal lines $(\ell - 1)$ irrespective of how many derivatives apply to a graviton. And as long as Eq. (5.21) is satisfied, some but not necessarily all of the auxiliary 1PIs are allowed to be Einsteinian (with $m_i = 0$). This makes the task of finding a generalized treatment (or, generalized formula) for 1PR connected correlators next to impossible as k and n become large. However, in the event that $m_i = k$ and $m_j = 0$ for all $j \neq i$, the diagram reduces to a 1PI at order-ϵ^k. There are other factors, which shall be discussed, that contribute to the difficulty of finding a generalized formula for convolution of 1PI functions.

For the moment, we can show the method of determining the 1PR functions for relatively small values of k and n. A detailed analysis of Einstein and Gauss-Bonnet (gauge transformed ϵ^0- and ϵ-order theories, in this discussion) 1PR functions was carried out in [66]. Here we shall proceed with 1PR functions for higher order theories in the context of a generalized Riem^q theory of gravity. Starting with order-ϵ^2 $2n$-point function can be written as a convolution of two 1PIs from the ϵ-order theory as follows

$$\left\langle h^{2n}\right\rangle_{\epsilon^2}^{con} = \frac{\gamma_1^2}{2!}\epsilon^2 \int d^5x \int d^5y \sqrt{-g(x)}\sqrt{-g(y)}\left[R^2(x)R^2(y)\right]_{\mathcal{O}(h^{2n})}, \quad (5.22)$$

where it should be noted that the terms in the square brackets are joined through a propagator so as to contribute to a correlation function.

One can similarly generate a $2n$-point function through convolution of ϵ^0- and ϵ^2-order theories as follows,

$$\left\langle h^{2n}\right\rangle_{\epsilon^2}^{con} = \gamma_0\gamma_2\epsilon^2 \int d^5x \int d^5y \sqrt{-g(x)}\sqrt{-g(y)}\left[R(x)R^3(y)\right]_{\mathcal{O}(h^{2n})}. \quad (5.23)$$

However, Eq. (5.23) always leads to ϵ^2-order 1PI functions. Such convolutions are trivial and will be ignored henceforth.

A non-trivial order-ϵ^2 two-vertex function was constructed in [66], where it was presented as the six-derivative correction to Einstein's theory. The outcome of that calculation was

$$\left\langle h^6\right\rangle_{\epsilon^2} = -\frac{1}{3}\int d^5x\sqrt{-g}\ \nabla_\alpha h_{ab}\nabla^\alpha h^{ab}\nabla_\beta h_{mn}\nabla^\beta h^{mn}\nabla_\gamma h_{pq}\nabla^\gamma h^{pq}. \quad (5.24)$$

Its corresponding Witten diagram is given in Fig. 5.2.

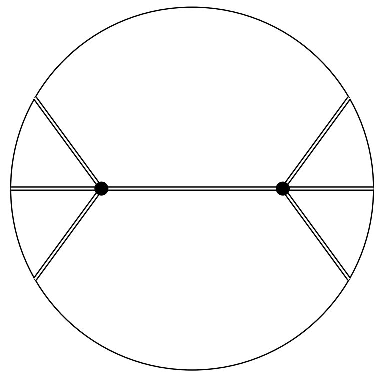


Figure 5.2: Witten diagram for a 1PR 6-point amplitude at order ϵ^2.

Here we present a detailed calculation for the 1PR 8-point function at order-ϵ^2. The calculation is three-fold: (1) determination of the Feynman weight–which compares the contribution of a 1PR connected function relative to its 1PI counterpart (see, appendix A for details), (2) the convolution of individual 1PIs and (3) determining coefficients from the convolution of the tensor structure of the gravitons. The BM regime allows us to manage this task in this fashion since derivatives do not contract with gravitons, thus reducing the complexity of the overall computation. Thus, we can deal with the derivative structure separately from the tensorial structure of the gravitons. Hence, in the second step we can temporarily ignore the tensor structure of gravitons, of which the unintended consequences are that individual gravitons temporarily become indistinguishable. The last step, which restores the uniqueness of individual gravitons, temporarily ignores the derivatives.

Just a reminder that we can schematically write a two vertex 8-point function at order-ϵ^2, constructed from a 4- and 6-point function as

$$\left\langle h^8 \right\rangle_{\epsilon^2,1PR} \sim \frac{\left[\epsilon \int_x R^2\right]_{\langle h^4 \rangle} \left[\epsilon \int_y R^2\right]_{\langle h^6 \rangle}}{\langle hh \rangle}, \tag{5.25}$$

where $\int_x = \lim_{r\to\infty} \int d^5x \sqrt{-g(r,\vec{x})}$ and $\langle hh \rangle^{-1}$ indicates the presence of an internal line propagator connecting the 4- and 6-point function. The Feynmann weight is given by

$$f_8^{\epsilon^2} = \frac{8! \cdot 4 \cdot 6}{4! \cdot 6!} = 56. \tag{5.26}$$

In the BM regime, there are are two ways to write an order-ϵ^2 8-point function. Either (a) both gravitons in the internal line are differentiated, or (b) the 6-point function has to supply an undifferentiated graviton to the internal line. We write these cases as

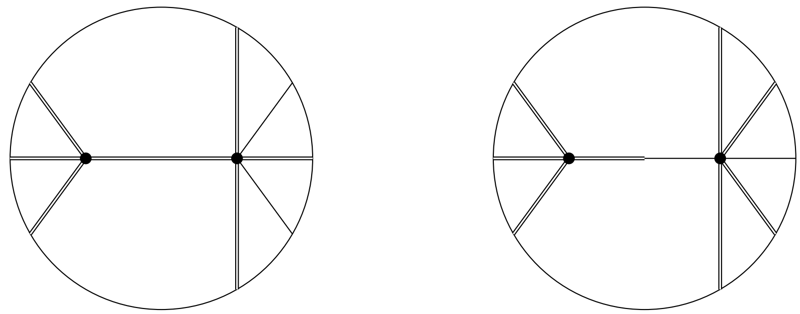
$$\left\langle h^8 \right\rangle_{\epsilon^2}^{(a)} = \int_x \int_y \; h(x)h(x)\overset{x}{\nabla}{}^{\alpha} h(x)\overset{x}{\nabla}_{\beta} h(x)\overset{x}{\nabla}{}^{\beta} h(x) \left\langle \overset{x}{\nabla}_{\alpha} h(x)\overset{y}{\nabla}_{\rho} h(y) \right\rangle \overset{y}{\nabla}{}^{\rho} h(y)\overset{y}{\nabla}_{\tau} h(y)\overset{y}{\nabla}{}^{\tau} h(y) \tag{5.27}$$

and

$$\left\langle h^8 \right\rangle_{\epsilon^2}^{(b)} = \int_x \int_y \; h(x)\overset{x}{\nabla}_{\alpha} h(x)\overset{x}{\nabla}{}^{\alpha} h(x)\overset{x}{\nabla}_{\beta} h(x)\overset{x}{\nabla}{}^{\beta} h(x) \left\langle h(x)\overset{y}{\nabla}_{\rho} h(y) \right\rangle \overset{y}{\nabla}{}^{\rho} h(y)\overset{y}{\nabla}_{\tau} h(y)\overset{y}{\nabla}{}^{\tau} h(y), \tag{5.28}$$

where $\overset{x}{\nabla}_{\alpha} = \frac{\partial}{\partial x^{\alpha}}$ denotes derivatives with respect to $x \equiv (r, \vec{x})$ coordinates. This notation also implies that $\overset{x}{\nabla}_{\beta} h(y) = 0$ and $\overset{x}{\nabla}_{\alpha}\overset{x}{\nabla}{}^{\alpha} h(x) = 0$. Without introducing additional ϵ's, the internal line is by definition bound to obey the ϵ^0-order Greens function equation given by

$$\overset{x}{\nabla}_{\alpha}\overset{x}{\nabla}{}^{\alpha} \langle h(x)h(y) \rangle \equiv \overset{y}{\nabla}_{\beta}\overset{y}{\nabla}{}^{\beta} \langle h(x)h(y) \rangle = \frac{1}{\sqrt{-g(x)}} \delta^5(x-y) = \frac{1}{\sqrt{-g(y)}} \delta^5(x-y). \tag{5.29}$$



(a) (b)

Figure 5.3: Order-ϵ^2 1PR 8-point function Witten diagrams. Figure (a) depicts $\langle h^8 \rangle_{\epsilon^2}^{(a)}$ and Fig.(b) depicts $\langle h^8 \rangle_{\epsilon^2}^{(b)}$.

While Eqs. (5.27) and (5.28) may sufficiently describe 1PR connected 8-point functions, we can evaluate these expressions further with the purpose of making numerical comparisons with the 1PI correlator of the same number of external legs. One may consider this process as an extension of idea of the Feynman weight. The difference is that only the derivatives are under consideration this time. To achieve this goal, the strategy is to drive towards a Greens function equation within Eq. (5.27) and then eliminate one integral using the delta functions.

We shall solely focus on Eq. (5.27) for the time being. First, we have to liberate one internal line graviton of its derivative by performing an integration by parts. For now, we will choose to liberate the x-coordinate graviton of its derivative. At a later point we will reverse the order of operations. Again we have to remind the reader that at this point gravitons are indistinguishable, and so effectively[1] $\nabla_\alpha h \nabla_\beta h \equiv h \nabla_\alpha \nabla_\beta h = \omega^2, \omega k, k^2$.

[1]From the perspective of the BM regime.

The derivatives are only sensitive to self-contractions of the form $\nabla_\alpha \nabla^\alpha h = \Box h = 0$, as these would violate the conditions of the BM regime by introducing additional ϵ terms. Thus, the integration by parts effectively yields

$$\left\langle h^8 \right\rangle_{\epsilon^2}^{(a)} = -4 \int_x \int_y \; h(x) \overset{x}{\nabla}_\alpha h(x) \overset{x}{\nabla}{}^\alpha h(x) \overset{x}{\nabla}_\beta h(x) \overset{x}{\nabla}{}^\beta h(x) \left\langle h(x) \overset{y}{\nabla}_\rho h(y) \right\rangle \overset{y}{\nabla}{}^\rho h(y) \overset{y}{\nabla}_\tau h(y) \overset{y}{\nabla}{}^\tau h(y), \tag{5.30}$$

where we used $\overset{x}{\nabla}_\alpha \cdot (h(x) h(x) \overset{x}{\nabla}{}^\alpha h(x) \overset{x}{\nabla}_\beta h(x) \overset{x}{\nabla}{}^\beta h(x)) \equiv 4\, h(x) \overset{x}{\nabla}_\alpha h(x) \overset{x}{\nabla}{}^\alpha h(x) \overset{x}{\nabla}_\beta h(x) \overset{x}{\nabla}{}^\beta h(x)$, in the BM regime.

Switching to the y-coordinate side, an inverse product rule $\overset{y}{\nabla}{}^\rho h \overset{y}{\nabla}_\tau h \overset{y}{\nabla}{}^\tau h \equiv \frac{1}{3} \overset{y}{\nabla}{}^\rho \cdot \left[h \overset{y}{\nabla}_\tau h \overset{y}{\nabla}{}^\tau h \right]$ followed by integration by parts leads to

$$\left\langle h^8 \right\rangle_{\epsilon^2}^{(a)} = \frac{4}{3} \int_x \int_y \; h(x) \overset{x}{\nabla}_\alpha h(x) \overset{x}{\nabla}{}^\alpha h(x) \overset{x}{\nabla}_\beta h(x) \overset{x}{\nabla}{}^\beta h(x) \overset{y}{\nabla}{}^\rho \overset{y}{\nabla}_\rho \left\langle h(x) h(y) \right\rangle h(y) \overset{y}{\nabla}_\tau h(y) \overset{y}{\nabla}{}^\tau h(y). \tag{5.31}$$

Again, we have used the fact that gravitons are momentarily indistinguishable and thus ω's and k's are indistinguishable. Using Eq. (5.29), Eq. (5.31) becomes

$$\left\langle h^8 \right\rangle_{\epsilon^2}^{(a)} = \frac{4}{3} \int_x h(x) h(x) \nabla_\alpha h(x) \nabla^\alpha h(x) \nabla_\beta h(x) \nabla^\beta h(x) \nabla_\tau h(x) \nabla^\tau h(x), \tag{5.32}$$

which is the more familiar form for a six-derivative theory of gravity.

Though we have stripped the gravitons of their individuality, their symmetrization comes into this discussion when we change the order of operations. Lets suppose between Eq. (5.27) and Eq. (5.32), we started with an integration by parts on the y side followed by an inverse product rule and then an integration by parts on the x side. The final result would have been

$$\left\langle h^8 \right\rangle_{\epsilon^2}^{(a)} = \frac{2}{5} \int_x h(x) h(x) \nabla_\alpha h(x) \nabla^\alpha h(x) \nabla_\beta h(x) \nabla^\beta h(x) \nabla_\tau h(x) \nabla^\tau h(x). \tag{5.33}$$

The difference arises due to the fact that there is a different number of gravitons to be symmetrized at each vertex in the diagram. The integration by parts over the y coordinates that led to Eq. (5.32) requires a symmetrization of 3 out of the 8 external gravitons. On the other hand, the integration over the x coordinate that leads to Eq. (5.33) requires symmetrization of 5 out of the 8 external gravitons. Thus, after symmetrization is addressed, the effective weight from this process is $\frac{3}{8} \times \frac{4}{3} + \frac{5}{8} \times \frac{2}{5} = \frac{3}{4}$.

Finally, we can now deal tensor structure of the gravitons. In the BM regime, the Riem^2 theory can represented as

$$R^{abcd} R_{abcd} = \mathcal{X}^{abmn} R_{abcd} \mathcal{X}^{cdpq} R_{mnpq}, \tag{5.34}$$

where the background term $\mathcal{X}^{abcd} = \frac{1}{2}(g^{ac} g^{bd} - g^{ad} g^{bc})$ exhibits all the (anti-)symmetrical properties of the Riemann tensor. Written this way, differentiated gravitons are contracted by the $\mathcal{X}$'s, while pairs of undifferentiated gravitons take the form $\mathcal{Y}^{abcd} h_{ac} h_{bd}$, where $\mathcal{Y}^{abcd} = \frac{1}{2}(g^{ac} g^{bd} + g^{ad} g^{bc})$.

Disregarding the derivatives, Eq. (5.27) can be written in tensor form as

$$\left\langle h^8 \right\rangle_{\epsilon^2}^{(a)} = h_{qc} h_{pq} h_{bn} \mathcal{X}^{abpq} \mathcal{X}^{mncd} \left\langle h_{an} h_{rf} \right\rangle \mathcal{X}^{efjk} \mathcal{X}^{rstu} h_{se} h_{tj} h_{ku} \mathcal{Y}^{vwkl} h_{vl} h_{wk}. \tag{5.35}$$

To evaluate this form, we need only invoke the momentum space graviton propagator [72, 73]. However, we should note that higher order ϵ terms would introduce additional ϵ's into the propagator, thus violating conditions imposed by the BM regime. Hence, only the propagator from the ϵ^0-order theory is sufficient here. It is given by,

$$\langle h_{ab}h_{cd}\rangle = \frac{1}{2}(g_{ac}g_{bd} + g_{ad}g_{bc} - g_{ab}g_{cd}). \tag{5.36}$$

The evaluation of Eq. (5.35) proceeds by using Eq. (5.36) and eliminating trace terms like $h^a{}_a$ and terms such as $h^{ab}h^c{}_a h_{cb}$ that contain hidden scalar modes. The result of this operation is

$$\left\langle h^8\right\rangle_{\epsilon^2}^{(a)} = \frac{1}{4}h_{ab}h^{ab}h_{cd}h^{cd}h_{mn}h^{mn}\mathcal{Y}^{vwkl}h_{vl}h_{wk}, \tag{5.37}$$

of which the main take away is the coefficient. In addition, we see that the properties of $\mathcal{Y}^{abcd}$ by definition vastly simplifies the last term to $\mathcal{Y}^{vwkl}h_{vl}h_{wk} \equiv h_{vl}h^{vl}$, as expected of the undifferentiated gravitons.

The evaluation of $\left\langle h^8\right\rangle_{\epsilon^2}^{(b)}$ proceeds similarly, with the second step yielding

$$\left\langle h^8\right\rangle_{\epsilon^2}^{(b)} = -\frac{3}{16}\int_x h(x)h(x)\nabla_\alpha h(x)\nabla^\alpha h(x)\nabla_\beta h(x)\nabla^\beta h(x)\nabla_\tau h(x)\nabla^\tau h(x), \tag{5.38}$$

regardless of the order in which operations take place. The evaluation of the tensor structure,

$$\left\langle h^8\right\rangle_{\epsilon^2}^{(\mathrm{b})} = h_{qc}h_{pq}h_{bn}\mathcal{X}^{abpq}\mathcal{X}^{mncd}\left\langle h_{an}h_{vl}\right\rangle\mathcal{Y}^{vwkl}\mathcal{X}^{efjk}\mathcal{X}^{rstu}h_{rf}h_{se}h_{tj}h_{ku}h_{wk}, \tag{5.39}$$

results in the following expression

$$\left\langle h^8\right\rangle_{\epsilon^2}^{(\mathrm{b})} = h_{ab}h^{ab}h_{cd}h^{cd}h_{mn}h^{mn}h^{pq}h_{pq}. \tag{5.40}$$

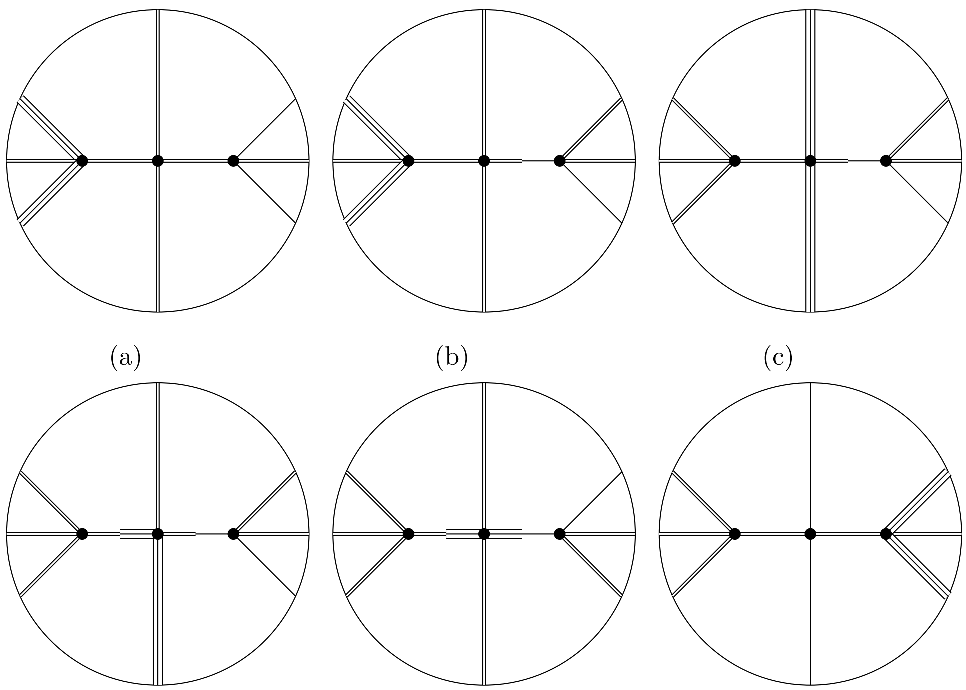
The final step is calculating the net weight of all ϵ^2-order 1PR connected 8-point correlators. Since both Figs. 5.3a and 5.3b have reflected diagrams, where the 4- and 6-point 1PI exchange positions, there are a total of four diagrams that equally contribute to the total Feynman weight. Each diagram and its reflection contribute the same extended Feynman weight and tensor structure coefficient. Hence, the contribution by Fig. 5.3a and its reflection is given by $\frac{56}{4} \times \frac{3}{4} \times \frac{1}{4} \times 2 = \frac{21}{4}$. Similarly, Fig. 5.3b and its reflection make a contribution of $\frac{56}{4} \times -\frac{3}{16} \times 1 \times 2 = -\frac{21}{4}$. This means that, at order-ϵ^2, the sum of the 1PR connected two-vertex diagrams Figs. 5.3a and 5.3b and their reflections make no net contribution to the 8-point function. Along with Eq. (5.19), this method can be used to check for (non-)vanishing multi-vertex diagrams of any multi-point function.

5.2.1 Higher-order ϵ 1PR connected multi-vertex diagrams

The diversity and complexity of diagrams only increases with higher powers of ϵ (see Fig. 5.4). Undifferentiated gravitons factor into diagrams through the metric determinant or the contravariant metrics or Einsteinian (ϵ^0) 1PI functions, which also add vertices. Though, adding an ϵ^0 diagram to a ϵ^k diagram could result in either a 1PI or a non-trivial 1PR connected diagram depending on where it is placed along the chain (see Fig. 5.4f, for example).

While each distinct diagram may not vanish, the sum of contributions from all 1PR diagrams could potentially vanish, as we have shown in the previous section. What is certain, however, is that 1PR diagrams of higher-order ϵ diagrams non-trivially contain a large number vertices. Naturally, the combinatoric calculations increase in complexity as the perturbative order increases.

Without a more general treatment of 1PR diagrams, a case-by-case approach could be sufficient in the study of multi-vertex connected diagrams at any order in the perturbation. The next chapter we discusses the implication of these connected functions with respect to the quark-gluon-plasma.



(a) (b) (c)

(d) (e) (f)

Figure 5.4: Some three-vertex diagrams at order ϵ^3 formed by convolving ϵ^0, ϵ and ϵ^2 1PIs.

In summary, we have shown how to compute multi-vertex stress-tensor correlation functions using 1-particle irreducible graviton amplitudes. This was achieved by taking the bulk 1PIs to the boundary in Section 5.1 then applying the holographic renormalization procedure from [68]. In Section 5.2, order-ϵ and ϵ^2 renormalized amplitudes were then convolved to form some ϵ^2 and ϵ^3 stress-tensor correlators. The convolution process itself, inspired by [18], required separate processes of dealing with the derivatives and tensor structure of the gravitons. Though each diagram yielded different coefficients, the net contribution of all diagrams at that order could potentially amount to zero, as was the case with the two-vertex 8-point functions associated with the Riem^3 theory.

Chapter 6

Discussion

In this chapter, we discuss our results and their implication on finding the gravitational dual to the quark-gluon-plasma (QGP). Starting with some of the scattering properties of the four-point amplitudes followed by a description of the algorithm that can be used to determine the gravitational dual to the QGP. The Brustein-Medved (BM) regime is still in effect throughout the chapter.

6.1 Scattering properties of four-point amplitudes

In quantum field theory (QFT), the scattering properties of four-point functions (usually in 2-to-2 particle events) can be expressed in terms of the three Mandelstam variables s, t and u [74]. They encode physical quantities such as energy, momentum and scattering angles and hence provide a more direct method of experimentally verifying some theoretical predictions.

In Chapter 4, we derived the BM regime four-point amplitudes for the

Riem^3 and Riem^4 theories as basis amplitudes. However, there cannot be 1PR for the four-point function as that require convolving two three-point functions, thereby violating the BM regime.

Every basis amplitude contains products of terms of the form $C_{ij} = (\omega_i g^{tt} \omega_j + k_i g^{zz} k_j)$. Incidentally, C_{ij} can be viewed as the product of the momenta of gravitons $h_{xy}^{(i)}$ and $h_{xy}^{(j)}$. At the boundary, C_{ij} is of the form $\lim_{r\to\infty} C_{ij} = \frac{L^2}{r^2} P_i^\mu P_{j\mu}$, where $P_{i\mu} = (\omega_i, \vec{k}_i)$ so that $P_i^\mu P_{j\mu} = (k_i k_j - \omega_i \omega_j)$. The constraint $C_{ii} = \Box h_{xy} = 0 + \mathcal{O}(\epsilon)$, naturally yields $P_i^\mu P_{i\mu} = 0$. It should be interesting to note that at the boundary the product $P_i^\mu P_{j\mu}$ bears a resemblance with the momentum product of two particles, labelled as i and j, propagating in Minkowski flat space $P_i^\mu P_{j\mu} = \eta^{\mu\nu} P_{i\mu} P_{j\nu}$. After holographic renormalization, we can write the boundary four-point functions from the Riem^3 theory as

$$\left\langle h_{xy}^4 \right\rangle_{\epsilon^2} = -6(\alpha_1 + \alpha_2) P_1^\mu P_{2\mu} \; P_1^\nu P_{3\nu} \; P_2^\chi P_{4\chi} \;\; h_{xy}^{(1)} h_{xy}^{(2)} h_{xy}^{(3)} h_{xy}^{(4)}. \tag{6.1}$$

But we are reminded that the gravitons have to be symmetrized, which means that the three products in Eq. (6.1) are really

$$\begin{aligned} P_1^\mu P_{2\mu} \; P_1^\nu P_{3\nu} \; P_2^\chi P_{4\chi} \to \;& P_{(1}^\mu P_{2\mu} \; P_1^\nu P_{3\nu} \; P_2^\chi P_{4\chi)} + P_{(1}^\mu P_{2\mu} \; P_3^\nu P_{4\nu} \; P_1^\chi P_{2\chi)} \\ & + P_{(1}^\mu P_{2\mu} \; P_3^\nu P_{4\nu} \; P_1^\chi P_{3\chi)} + P_{(1}^\mu P_{2\mu} \; P_3^\nu P_{4\nu} \; P_2^\chi P_{4\chi)} \\ & + P_{(1}^\mu P_{2\mu} \; P_3^\nu P_{4\nu} \; P_3^\chi P_{4\chi)} + P_{(1}^\mu P_{2\mu} \; P_3^\nu P_{4\nu} \; P_1^\chi \cdot P_{4\chi)}. \end{aligned} \tag{6.2}$$

In a typical QFT 2-to-2 scattering process, the Mandelstam variables are defined in Minkowski flat space as

$$\begin{aligned}
s &= (P_1+P_2)^\mu(P_1+P_2)_\mu = 2P_1^\mu P_{2\mu} = (P_3+P_4)^\mu(P_3+P_4)_\mu = 2P_3^\mu P_{4\mu},\\
t &= (P_1-P_3)^\mu(P_1-P_3)_\mu = -2P_1^\mu P_{3\mu} = (P_2-P_4)^\mu(P_2-P_4)_\mu = -2P_2^\mu P_{4\mu},\\
u &= (P_1-P_4)^\mu(P_1-P_4)_\mu = -2P_1^\mu P_{4\mu} = (P_2-P_3)^\mu(P_2-P_3)_\mu = -2P_2^\mu P_{3\mu},
\end{aligned} \tag{6.3}$$

where P_i denotes the momentum of the i-th particle. Massless particles obey $s+t+u=0$ and $P_i^\mu P_{i\mu}=0$. The later carries two meanings in the BM regime: (1) gravitons are massless and (2) $P_i^2 \sim C_{ii} = 0$ due to $\Box h_{xy}^{(i)} = 0 + \mathcal{O}(\epsilon)$ acts to preserve the conditions of the BM regime. Thus, one can come to the conclusion that the BM regime does not include massive string excitations.

We can use Eqs. (6.2) and (6.3) to write the boundary four-point function, up to numerical factors, in terms of Mandelstam variables as

$$\left\langle h_{xy}^4 \right\rangle_{\epsilon^2} \propto (\alpha_1+\alpha_2)(s-t-u)(s^2+t^2+u^2)\ h_{xy}^{(1)} h_{xy}^{(2)} h_{xy}^{(3)} h_{xy}^{(4)}. \tag{6.4}$$

Similarly, the four-point function from the Riem^4 theory can be written in terms of Mandelstam variables as

$$\left\langle h_{xy}^4 \right\rangle_{\epsilon^3} \propto (\sum_{i=1}^{6}\beta_i)(s^4+t^4+u^4+s^2t^2+s^2u^2+u^2t^2)\ h_{xy}^{(1)} h_{xy}^{(2)} h_{xy}^{(3)} h_{xy}^{(4)}. \tag{6.5}$$

Clearly, Eqs. (6.4) and (6.5) show that Riem^3 and Riem^4 theories, respectively, have distinct scattering profiles. Furthermore, the s-, t- and u-channels are equally distributed in each correlation function. At the very least, this suggests that no particular channel is favoured in the BM regime.

This leads to the assertion that the BM regime is the regime of large-angle scattering in gauge theory experiments since, for an angle $\theta \gg 0$, all the Mandelstam variables behave as $\{s, t, u\} \propto E^2$, where E is the centre-of-mass energy. If $s\epsilon$ is *only* slightly smaller than 1, then correlation functions from higher-curvature perturbations to Einstein and Gauss-Bonnet are expected to make a significant contribution to the dual gauge theory stress-tensor correlators.

6.2 Algorithm for the gravitational dual to a strongly coupled conformal fluid

The four-point functions give us a picture of what to expect of the scattering properties of BM regime graviton amplitudes. In particular, they hint that the BM regime could be associated to large-angle particle deflections in the dual gauge theory. However, finding the gravitational dual to the QGP, through correlation functions, requires more than just the four-point functions.

Firstly, we need a gauge theory perspective of the BM regime. In the bulk, the BM regime was defined as the region where

$$1 \ll \omega L \ll TL, \tag{6.6}$$

where again L is AdS radius, T is the Hawking temperature of the black brane and ω is a representation of the graviton momentum. Though the AdS boundary inherits a planar geometry, it can be viewed as having an S^3 geometry with radius $\mathcal{R}$ determined by the radius of curvature so that

$\mathcal{R} \sim L$ [8]. The radius $\mathcal{R}$ scales with the spatial dimensions of the conformal fluid (QGP) [75] and T describes its temperature. The same value of temperature applies to both sides of the duality. Hence, from the gauge theory perspective, the BM regime is stated as

$$1 \ll \omega\mathcal{R} \ll T\mathcal{R}. \tag{6.7}$$

So if given access to experimental data of parton scattering events consistent with the kinematic regime Eq. (6.7), we can determine the gravitational dual to the QGP through the system of equations

$$\begin{aligned}
\left\langle T_{xy}^2 \right\rangle &= \mathcal{A}_0\epsilon^0 \left\langle h_{xy}^2 \right\rangle + \mathcal{A}_1\epsilon^1 \left\langle h_{xy}^2 \right\rangle \ , \\
\left\langle T_{xy}^4 \right\rangle &= \mathcal{A}_0\epsilon^0 \left\langle h_{xy}^4 \right\rangle + \mathcal{A}_1\epsilon \left\langle h_{xy}^4 \right\rangle + \mathcal{A}_2\epsilon^2 \left\langle h_{xy}^4 \right\rangle + \mathcal{A}_3\epsilon^3 \left\langle h_{xy}^4 \right\rangle \ , \\
\left\langle T_{xy}^6 \right\rangle &= \sum_{i=0}^{5} \mathcal{A}_i\epsilon^i \left\langle h_{xy}^6 \right\rangle \ , \\
\vdots \ &\vdots \qquad \vdots\ \vdots \\
\left\langle T_{xy}^{2n} \right\rangle &= \sum_{i=0}^{2n-1} \mathcal{A}_i\epsilon^i \left\langle h_{xy}^{2n} \right\rangle \ ,
\end{aligned} \tag{6.8}$$

where the terms on the left-hand side represent experimentally determined stress-tensor correlators of the strongly coupled conformal fluid and the $\mathcal{A}_i$'s are unknown coefficients to be determined. The correlation functions on the right-hand side represent the sum of all connected diagrams that can be determined through the processes discussed in previous chapters. We can freely set $\mathcal{A}_0 = 1$ while regarding the other $\mathcal{A}$'s as weighted against $\mathcal{A}_0$.

In an ideal setting, we would like to have k equations with k unknowns. However, the reality of Eq. (6.8) shows that every $(2n+2)$-point function has 2 more unknowns than the previous $2n$-point function. This situation can be remedied if the experimental apparatus is not sensitive to all possible data types, as we now show.

In Chapter 4, we observed that in the BM regime each Riemann tensor draws out one of either ω^2, ωk, or k^2. For convenience, we can ascribe the Mandelstam variable s (where $\sqrt{s}$ is the centre-of-mass energy) to the all elements in $\{\omega^2, \omega k, k^2\}$. In the AdS_5 bulk, the gravitational Lagrangian can be written as

$$\mathcal{L} = \frac{\sqrt{-g}}{16\pi G_5}\big(\bar{\gamma}_0 R + \epsilon\bar{\gamma}_1 L^2 R^2 + \epsilon^2\bar{\gamma}_2 L^4 R^3 + \cdots + \epsilon^n \bar{\gamma}_n L^{2n} R^{n+1}\big), \tag{6.9}$$

where the $\bar{\gamma}_i$'s are to-be-determined dimensionless constants, L is the AdS radius and $\epsilon = \ell_s^2/L^2$. Note that we have redefined γ_i such that $\gamma_i = \bar{\gamma}_i L^{2i}$, so that the Lagrangian can be rephrased in terms of a boundary theory perspective, as an expansion of powers of $s\epsilon$,

$$\mathcal{L} \sim s\sum_{i=0}^{n} \bar{\gamma}_i (s\epsilon\mathcal{R}^2)^i, \tag{6.10}$$

where $\mathcal{R}$ is the radius of the boundary with an S^3 geometry and is proportional to the spatial extent of the conformal fluid. We should note that s in Eq. (6.10) serves as an average over a distribution of graviton momenta. Since $s\epsilon\mathcal{R}^2 \sim s\ell_s^2$, we expect $s\epsilon\mathcal{R}^2 < 1$. This also allows us to restate the BM regime, Eq. (6.7), as

$$\frac{1}{\mathcal{R}^2} \ll s \ll T^2. \tag{6.11}$$

Depending on the sensitivity of the experimental apparatus which we note as W, a dimensionless parameter, there could be some value s such that $(s\epsilon\mathcal{R}^2)^k < W$ but $(s\epsilon\mathcal{R}^2)^{k-1} > W$ for some, preferably small, integer k. This would allow us to eliminate terms of order $(s\epsilon)^k$ and higher, thereby reducing the number of unknown coefficients in Eq. (6.8). Finally, we can determine the model-dependent constants, $\bar{\gamma}_i$, by solving the system of equations given by $\mathcal{A}_i = \mathcal{A}_i(\bar{\gamma}_0, \bar{\gamma}_1, \bar{\gamma}_2, \cdots, \bar{\gamma}_n)$.

6.3 A discussion on the discussion

The algorithm discussed in the previous section could be used to find the gravitational dual to any strongly coupled conformal fluid not just the QGP. If the scattering properties from the four-point functions are any indication of what to expect in the real world, then we can expect to see the results presented here as large-angle jets produced in heavy-ion scattering experiments. Admittedly, our final results depend heavily on the quality of the experimental apparatus.

There is also the potential that our results correspond to stress-tensor correlation functions of parton showers prior to hadronization. Support for such a claim can be surmised in [76], where QCD computations in the finite parton energy limit corresponds to large scattering angles. In this limit, the strongly coupled QGP can be shown to emerge from an asymptotically free gauge theory.

In addition, there is a portion of jets that scatter at large angles [76]. These can be shown to be asymptotically free (with slightly weaker coupling) quarks and gluons that eventually condense into a strongly coupled liquid. Furthermore, such jets are observed in an energy regime of large-yet-finite momentum. Though rare, such kinematic events have probability amplitudes large enough to merit serious experimental consideration.

From the gauge theory perspective of probing the QGP through scattering, the only physical parameters necessary to make a connection between the theory and experiment are; the thickness of the fluid, its temperature, the energy/momentum transfer between the initial to final states and the scattering angles of the emerging partons. These parameters are predictable by string calculations, as we have shown here.

Chapter 7

Conclusion

This thesis attempts to find the gravitational dual of a strongly coupled fluid, such as the quark-gluon plasma. Starting in Chapter 2, we showed a path from the fundamentals of string theory to the celebrated AdS/CFT correspondence and its dictionary. The only aspects of the AdS/CFT dictionary that were highlighted are those that have a direct bearing on our primary objective. Chapter 3 briefly described some real world applications of the duality including a discussion on what we called the Brustein-Medved (BM) regime–a special regime of high-momentum gravitons.

Soon after that in Chapter 4, we then showed how to compute 1-particle-irreducible (1PI) graviton scattering amplitudes for a general theory of gravity in the BM regime. This calculation only became simplified once we identified the basis amplitudes. With the inclusion of gravitons carrying two derivatives, we showed how to compute all possible tree-level 1PI n-point functions using the basis amplitudes. We should note that any calculation beyond a two-point function is a victory over AdS/QCD models.

In Chapter 5, we focused on finding the renormalized stress-tensor connected Witten diagrams using the 1PI scattering amplitudes from chapter 4. We provided a method for finding connected diagrams using simpler 1PIs. However, due to the very numerous possibilities, finding a more general "formula", as we did for the 1PIs, proved a little too difficult. The fabled general formula would, in principle, help identify which diagrams make a non-zero contribution without making the tedious calculations.

In chapter 6, we discussed how our findings help find the gravitational dual to a strongly coupled fluid by solving a system of equations. This method relies heavily on data from high-energy scattering experiments, thus making whatever gravitational dual we calculate all the more realistic. Obviously, the accuracy of the gravitational dual would depend on the availability of $2n$-point functions as n increases in size.

While finding higher-point functions is always a good thing, the most remarkable thing about our work is what the four-point functions tell us. Our Riem^3 and Riem^4 four-point functions [19] add to the story that begun with [17, 18, 66] that tells us that we are working in a regime of large angle scattering. This has some significance because, according to [76], the experimental data measured in this regime suggests the energy signature of pre-hadronized matter. This means that these calculations, performed in the Brustein-Medved regime, can be experimentally linked to the quark-gluon plasma itself and not the aftermath of its existence.

The next step involves finding the general formula for connected functions and running computer simulations with real experimental data to find an actual gravitational dual.

Appendix A

Feynman diagrams & their combinatorics

This discussion is a recounting of section 4 of [66], which in itself is a reminder of Feynman diagrams in standard quantum field theory text books. The goal is two fold: to clarify why only 1-particle-irreducible amplitudes are calculated in Chapter 4 and to provide some clarity on the term *Feynman weight* in relation to the connected multi-point functions of Chapter 5.

The starting point is a D-dimensional action for a quantum field ϕ,

$$S = \int d^D x \left[\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 + \frac{c_3}{3!} \phi^3 + \frac{c_4}{4!} \phi^4 + \cdots + \frac{c_m}{m!} \phi^m \right], \quad \text{(A.1)}$$

where m is the mass of ϕ and the c_i's are model-dependent constants. For this discussion $c_i = 1$ for all values of i, however, all the c_i's will be included as they serve the useful role of keeping track of terms. The first two terms can be considered as part of a free Lagrangian $L_f = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2$ while the rest of the terms represent the interaction Lagrangian. This way the action can be written as $S = S_f + S_{interaction}$, where $S_f = \int d^D x \ L_f$ and $S_{interaction} = \int d^D x \ \frac{c_3}{3!}\phi^3 + \frac{c_4}{4!}\phi^4 + \cdots$. It should be noted that in a more general treatment, the interaction term would include higher derivative terms such as, for instance, $\mathcal{O}(\phi^3) \sim \partial_\mu\phi\partial^\mu\partial_\nu\phi\partial^\nu\phi$. However, for the purposes of illustrating the origins of certain connected functions in Chapters 4 and 5, higher derivative terms will not be included.

In the context of this thesis, we take every ϕ^k in Eq. (A.1) as a metaphorical representation of every Riem^k term in Eq. (4.2). However, the difference here is that, through $\sqrt{-g} \ g^{\mu\mu}g^{\nu\nu}$, one can generate graviton n-point function from a single Riem^k term. In this presentation, the free action S_f is the metaphorical equivalent to the Einstein-Hilbert action $\int d^5x\sqrt{-g}R$.

In the presence of an external source J, the scattering amplitudes of the field ϕ can be obtained using the generating functional given by

$$Z[J] = \frac{\int \mathcal{D}\phi \ e^{iS + i\int d^D x J\phi}}{\int \mathcal{D}\phi \ e^{iS}}, \tag{A.2}$$

where $Z[0] = 1$ is the normalization condition and $\mathcal{D}\phi = \prod_i d\phi_i$ is a measure over all field configurations. An n-point function is given by the functional differentiation of $Z[J]$ with respect to the source as follows,

$$\langle\phi_1\phi_2\ldots\phi_n\rangle = \left.\frac{\delta^n Z}{\delta J_1\delta J_2\ldots\delta J_n}\right|_{J=0}, \tag{A.3}$$

$$= \frac{\int \mathcal{D}\phi \ \phi_1\phi_2\ldots\phi_n \ e^{(iS_f + i\int d^D x \ (\frac{c_3}{3!}\phi^3 + \frac{c_4}{4!}\phi^4 + \cdots + \frac{c_m}{m!}\phi^m)}}{\int \mathcal{D}\phi \ e^{iS}}, \tag{A.4}$$

where, for accuracy purposes, we have included the subscripts in Eq. (A.3) to denote different spacetime points but subsequently dropped that notation in Eq. (A.4) since they won't be relevant for the remainder of the discussion.

The Feynman diagrams generated by Eq. (A.4) include contributions from both connected and disconnected functions. The connected diagrams/functions are those with a single component and are given by 1-particle-irreducible (1PI) and 1-particle-reducible (1PR) correlation functions. Disconnected diagrams are those that have two or more unconnected components.

The partition function that generates connected functions is defined in relation to the full partition function as $iW[J] = \ln(Z[J])$. Thus, through functional differentiation with respect to the source, we can obtain the n-point functions as

$$\begin{aligned} \langle\phi_1\phi_2\cdots\phi_n\rangle_{\text{connected}} &= i\left.\frac{\delta^n W}{\delta J_1\delta J_2\ldots\delta J_n}\right|_{J=0}, \\ &= i^{n+1}\int \mathcal{D}\phi\ \phi_1\phi_2\ldots\phi_n\ e^{(iS_f+i\int d^Dx\ (\frac{c_3}{3!}\phi^3+\frac{c_4}{4!}\phi^4+\cdots+\frac{c_m}{m!}\phi^m)}. \end{aligned} \tag{A.5}$$

From the Feynman diagram representation, the $\phi_1\phi_2\ldots\phi_n$ denote n external lines. And from the exponential term, $\int d^Dx\ c^k/k!\ \phi^k$ represents a vertex and the endpoint of k lines. A p-vertex diagram is denoted by p instances of $\int d^Dx$ in an n-point function. The exponential term can be expanded so that the n-point function can be written as,

$$\langle\phi_1\phi_2\ldots\phi_n\rangle_{\text{connected}} = \int \mathcal{D}\phi\ Z_0\ \phi_1\phi_2\ldots\phi_n\times\prod_{i=3}^{m}\left[\int d^Dx\left(1+\frac{c_i}{i!}\phi^i+\frac{1}{2!}\left(\frac{c_i}{i!}\right)^2\phi^i\phi^i+\cdots\right)\right], \tag{A.6}$$

where $Z_0 = i^{n+1}e^{iS_f}$. The idea behind these connected functions is that ϕ^i's contract either amongst themselves or with each of the $\phi_1\phi_2\ldots\phi_n$. What we mean by contractions is that we can expand the expression in terms of a series of products of Greens functions. For instance, a contraction between

$\phi_1 \equiv \phi(x_1)$ and $\phi(z)$ implies the existence of a $G(x_1 - z) \equiv \langle \phi_1 \phi(z) \rangle$. So, a single vertex 3-point function without loops can be written schematically as,

$$\langle \phi_1 \phi_2 \phi_3 \rangle_{\text{connected}} \sim \int d^D x \ \phi_1 \phi_2 \phi_3 \frac{c_3}{3!} \phi^3(x), \tag{A.7}$$

where each of the ϕ's in $c_3 \phi^3$ contract with a single ϕ in $\phi_1 \phi_2 \phi_3$. This means, to fully clarify the contractions, $\langle \phi_1 \phi_2 \phi_3 \rangle_{\text{connected}}$ can also be expanded as

$$\langle \phi_1 \phi_2 \phi_3 \rangle_{\text{connected}} \sim \int d^D x \ G(x_1 - x) G(x_2 - x) G(x_3 - x). \tag{A.8}$$

A multi-vertex diagram would include more elaborate combinations of Greens functions. For the remainder of the discussion, we will use the notation in Eq. (A.7) as it is more useful for what we are trying to accomplish, though it is to be understood that what we really mean is an expression like Eq. (A.8).

A single vertex 3-point function with loops takes the schematic form,

$$\langle \phi_1 \phi_2 \phi_3 \rangle_{\text{connected}} \sim \int d^D x \ \phi_1 \phi_2 \phi_3 \frac{1}{2!} \left(\frac{c_3}{3!} \right)^2 \phi^6(x), \tag{A.9}$$

where some ϕ's from the ϕ^6 term are allowed to contract amongst themselves (giving expressions like $G(x - x) = G(0)$) and thus forming loops. Similarly, a 2-vertex 5-point function would take the schematic form,

$$\langle \phi_1 \phi_2 \phi_3 \phi_4 \phi_5 \rangle_{\text{connected}} \sim \frac{c_3}{3!} \frac{c_4}{4!} \int d^D x \int d^D y \ \phi_1 \phi_2 \phi_3 \phi_4 \phi_5 \phi(x) \phi(x) \left\langle \phi(x) \phi(y) \right\rangle \phi(y) \phi(y) \phi(y), \tag{A.10}$$

where $\langle \phi(x) \phi(y) \rangle$ is a propagator that connects the two vertices.

In the absence of loops, there can be no contraction between two ϕ's from the same monomial. 1PI diagrams are those for which all the ϕ's in a ϕ^i term contract with all the ϕ's from $\phi_1\phi_2\ldots\phi_n$, for example Eq. (A.7). Thus, in a shorthand way, a 1PI is an expression that features exactly one c_i. Given that the Lagrangian is an expression that features exactly one c_i per term, one can thus read the 1PIs directly off the Lagrangian. Hence, tree-level 1PI graviton amplitudes can be obtained from each term of the Lagrangian Eq. (4.2) since a $\gamma_i\text{Riem}^i$ is the metaphorical equivalent to $c_i\phi^i$. In addition, these tree-level diagrams strictly require that the number of ϕ's from $c_i\phi^i$ be equal to the number of external lines, given by $\phi_1\phi_2\ldots\phi_n$. However, due to contravariant metrics and the determinant, each Riem^i in Eq. (4.2) is capable of generating a $2n$-point 1PI amplitude. Thus, the Riem^i terms, along with the contravariant metric and determinant, are capable of supplying an arbitrary number of gravitons for contraction with an arbitrary number of external lines.

Coming back to Eq. (A.5), we can always represent a 2-vertex 1PR as

$$\langle\phi_1\phi_2\ldots\phi_n\rangle_{1PR} \sim [\phi^n]\frac{c_p}{p!}\frac{c_q}{q!}\phi^p\phi^q, \tag{A.11}$$

where we have used the shorthand $[\phi^n] \equiv \phi_1\phi_2\ldots\phi_n$ and n is such that $n = p+q-2$ with a propagator implied by the -2. Eq. (A.11) also indicates that a 1PR can be obtained by joining (or convolving) 1PIs of lower order. One can think of ϕ^p and ϕ^q as two separate 1PIs joined by an internal line formed by melding two external lines supplied by each of them.

In addition, there is a relationship between the number of lines v ending on vertices, the number of internal lines i and external lines e given by the conservation of endpoints $2i + e = v$ [77]. For an arbitrary diagram from Eq. (A.6) with m-vertices generated by

$$\langle \phi_1 \phi_2 \dots \phi_n \rangle \sim [\phi^n] \frac{\phi^{j_1}}{j_1!} \frac{\phi^{j_2}}{j_2!} \cdots \frac{\phi^{j_m}}{j_m}, \tag{A.12}$$

where $e = n$ and $v = \sum_{k=1}^{m} j_k$, the number of loops is given by $\ell = i+1-m$. We can then use the conservation of endpoints so that the number of loops is given by,

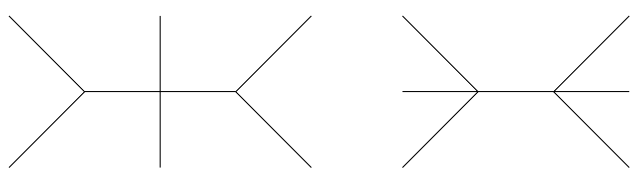
$$\ell = \frac{1}{2}\left[-2(m-1) - n + \sum_{k=1}^{m} j_k \right]. \tag{A.13}$$

From a practical perspective, we can draw as many diagrams as needed (see for example Fig. A.1), but a detailed analysis in [66] shows that,

$$-2m + \sum_{k=1}^{m} j_k = q_3 + 2q_4 + 3q_5 + \cdots , \tag{A.14}$$

where q_k counts the number of occurrences of ϕ^{j_k}. This permits us to write the loop formula Eq. (A.13) as

$$\ell = \frac{1}{2}\left[(q_3 + 2q_4 + 3q_5 + \cdots) - n + 2 \right]. \tag{A.15}$$



(a) (b)

Figure A.1: The connected six-point function in Figure (a) is the convolution of $\frac{c_3}{3!}\frac{c_4}{4!}\frac{c_3}{3!}\phi^3\phi^4\phi^3$. The variables associated with this diagram are $m = 3$, $q_3 = 2$, $q_4 = 1$ and $n = 6$. Figure (b) shows a six-point function through the convolution of $\frac{c_4}{4!}\frac{c_4}{4!}\phi^4\phi^4$ with variables $m = 2$, $q_4 = 2$ and $n = 6$. Both diagrams are tree-level as confirmed by $\ell = 0$.

Now suppose we are constrained to tree-level diagrams of $2n$-point functions (as is the case for Brustein-Medved (BM) regime graviton amplitudes and stress-tensor correlators in Chapters 4 and 5, respectively), Eq. (A.15) can give us an indication of which diagrams we would expect to see. For instance, a four-point function would require that $q_4 = 1$ and $q_i = 0$ for all $i \neq 4$, noting that $q_3 = 2$ would not be allowed now due to the $2n$-point function constraint (or the BM regime in the main text). Hence, for $n = 4$, $\langle \phi_1\phi_2\phi_3\phi_4 \rangle_{complete} = \langle \phi_1\phi_2\phi_3\phi_4 \rangle_{1PI}$ and the only diagram we expect is

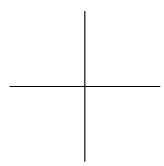


Figure A.2: A complete connected four-point function when all odd n-point functions are restricted.

For the six point function, only Fig. A.1b survives since ϕ^3's are forbidden from appearing in the 1PI correlators (a consequence of the $2n$-point function constraint, or BM regime in the main text). The complete six-point function is the sum

$$\langle \phi_1\phi_2\phi_3\phi_4\phi_5\phi_6 \rangle_{complete} = \langle \phi_1\phi_2\phi_3\phi_4\phi_5\phi_6 \rangle_{1PI} + f_6 \frac{\langle \phi_1\phi_2\phi_3\phi_4 \rangle_{1PI} \langle \phi_1\phi_2\phi_3\phi_4 \rangle_{1PI}}{\langle \phi^2 \rangle}, \tag{A.16}$$

where f_6 is the *Feynman weight* since it measures the weight of the 1PR relative to the 1PI and $\left\langle \phi^2 \right\rangle^{-1}$ is the propagator that connects the two $\langle \phi_1\phi_2\phi_3\phi_4 \rangle_{1PI}$.

The weight of the 1PI can also be calculated by realizing that it takes the form,

$$\langle \phi_1\phi_2\phi_3\phi_4\phi_5\phi_6 \rangle_{1PI} \sim [\phi^6] \; c_6 \frac{\phi^6}{6!}, \tag{A.17}$$

where, as previously stated, $c_6 = 1$ is here purely for tracking purposes. Each of the ϕ's in $[\phi^6] = \phi_1\phi_2\phi_3\phi_4\phi_5\phi_6$ are contracted with the ϕ's in ϕ^6 on the right-hand side. This can be done in 6! ways, thus the weight of the 1PI is given by $6!/6! = 1$.

Similarly, the 1PR takes the form,

$$\langle \phi_1\phi_2\phi_3\phi_4\phi_5\phi_6 \rangle_{1PR} \sim [\phi^6] \; \frac{1}{2!}\left(c_4 \frac{\phi^4}{4!} \right)\left(c_4 \frac{\phi^4}{4!} \right), \tag{A.18}$$

where the $c_4 = 1$'s are there to track the origins of the factorials in the denomenator. To determine f_6, let us recall that no two ϕ's from the same monomial can contract. We have two ϕ's from each ϕ^4 contracting to create an internal line (propagator) thus leaving a pair of ϕ^3's that must be contracted with the external lines. In the first step, there are 4 ways of drawing an ϕ from each ϕ^4. In the second step, there are 6! ways of contracting ϕ^6 with the remaining $\phi^3\phi^3$. Thus, the weight of this 1PR is given by

$$4^2 \cdot 6! \cdot \frac{1}{2!(4!)(4!)} = 10. \tag{A.19}$$

The Feynman weight f_6 of this 1PR is the ratio

$$f_6 = \frac{10}{1}, \tag{A.20}$$

the weight of the 1PR with respect to the weight of the 1PI.

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