

CONTINUITY AND GENERALIZED
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AND OTHER APPLICATIONS

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Abstract

The topological dynamics of continuous and noncontinuous dynamical systems are investigated. Various definitions of chaos are studied, as well as notions of stability. Results are obtained on asymptotically stable sets and the perturbation stability of such sets. The primary focus is on the traditional point sets of topological dynamics, including the chain recurrent set, omega-limit sets and attractors. The basic setting is that of a continuous function on a compact metric space, sometimes with additional properties on the space. The investigation includes results on the dynamical properties of typical continuous functions in the sense of Baire category. Results are also developed concerning dynamical systems involving quasi-continuous functions. An invariance property for the omega-limit sets of such functions is given. Omega-limit sets are characterized for Riemann integrable derivatives and derivatives which are continuous almost everywhere. Techniques used in the investigation and formulation of results include finding theorems which relate the rather disparate notions of dynamical properties and generalized continuity. In addition to dynamical systems, numerous other applications of generalized continuity are investigated. Techniques used include application of the Baire Category Theorem and the notion of semi-closure. For example, results are formulated concerning functions determined by dense sets, including separately continuous functions, thus generalizing the classical result for continuous functions on dense subsets of the domain. The uniform

boundedness theorem is extended to functions which are not necessarily continuous, including various derivatives. The closed graph theorem is strictly generalized in two separate ways, and applications are presented using these generalizations. An invariance property of separately continuous functions is given. Cluster sets are studied in connection with separate continuity, and various results are presented concerning locally bounded functions.

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CHAPTER 1

INTRODUCTION

All results labeled as theorems, propositions, corollaries and lemmas are the author's results, except for Theorems 44, 49, 53 and 55, which are the work of D.A. Rose and are included for the full development of the subject. All other material is set forth in narrative form with references to the bibliography. That is, known results of other authors are cited throughout the thesis, but they are not set out in the text as numbered theorems, propositions, corollaries, and lemmas. Definitions and examples are also presented in narrative form, and are not numbered. The aim of this thesis is to investigate the role of continuity and generalized continuity in dynamical systems and in other applications in mathematics.

The substantive material is divided into two parts. Part I, comprised of Chapters 2 and 3, deals with dynamical systems, both continuous and non-continuous. Chapter 2 of Part I presents results on the dynamics of continuous functions on compact domain spaces. In particular, the author investigates various definitions of chaos, as well as asymptotic attractors and other point sets in topological dynamics. Results are developed concerning the dynamical properties of residual subsets of the set of all continuous functions on a compact n -cube in Euclidean n -space. For example, Theorem 2 of Chapter 2 shows that typi-

cal continuous functions (in the sense of Baire category) are *not* chaotic in the sense of Devaney. In fact, it is shown that the set of functions which are not chaotic in the sense of Devaney comprises a dense open subset of the set of all continuous functions. It then follows, by Corollary 4, that there is a dense open subset of the set of all continuous functions on a compact interval in the real line, such that every function in the dense open set is Block-Coppell chaotic but not chaotic in the sense of Devaney. This same result is then proved for the set of all polynomials on a compact interval of the real line. A similar result is given in Corollary 3 for functions which are chaotic in the sense of Li and Yorke. In Section 2 of Chapter 2 results are obtained concerning asymptotically stable sets, again with emphasis on the properties of residual subsets of function spaces of continuous functions. Proposition 6 gives the result that all functions in a dense open subset of the set of all continuous functions have an asymptotically stable set. The chain recurrent set is also investigated, as well as Milnor attractors. Theorem 12 deals with the relationship between an asymptotically stable set and its basin of attraction.

Chapter 3 of Part I investigates the dynamics of noncontinuous functions. In particular, the author investigates the dynamics of quasi-continuous functions and related families of noncontinuous functions. There are not very many papers on the subject of dynamical systems for noncontinuous functions. However, there are a surprising number of areas of applied mathematics in which noncontinuity appears in dynamics. For example, noncontinuity can appear in dynamical

systems with impulse effect. In such systems the iterated point can “jump” at certain points of the state space. Noncontinuous functions can also appear in dynamics in the iteration of derivatives. The first result in Chapter 3 is Theorem 13 which gives an equivalence between a generalized-continuity condition known as “bilateral quasi-continuity” and a notion from dynamical systems known as the “dense mapping property.” In essence, then, Theorem 13 marries two rather disparate fields and permits the investigation in Chapter 3 to approach the problem of noncontinuous dynamics using the structure drawn from the field of generalized continuity. In Theorem 18 and Theorem 19 the application of Theorem 13 enables us to characterize the omega-limit sets of Riemann-integrable derivatives (not necessarily continuous) and the omega-limit sets of derivatives which are continuous almost everywhere. Theorem 23 shows that omega-limit sets for quasi-continuous functions possess an invariance property.

Topological dynamics can trace its lineage to the late nineteenth century with the work of Henri Poincaré and the famous n -body problem, and somewhat later to the work of G.D. Birkhoff. However, a much more modern theory has emerged in the latter part of the twentieth century with the work of A.N. Šarkovskii, S. Smale and many others. Here we study discrete dynamics which came of age with the advent of digital computing. This is due to the simple fact that discrete dynamics is essentially the iteration of functions and of course digital computers perform such iterations very efficiently.

It should also be said that the results presented here deal with nonsmooth

dynamics. Nonsmooth dynamics has a definite place in the literature and an important role to play. The early investigations by Smale into hyperbolicity developed powerful methods, but there are many dynamical systems in which hyperbolicity can play no part. In fact, as is well known, the typical continuous function is far from being smooth in any sense. On the other hand, although most of Part I deals with nonsmooth dynamics, some results which are presented in Chapter 2 below deal with the dynamics of polynomials.

Part II of the thesis, which is comprised of Chapters 4 and 5, takes up generalized continuity as a subject in its own right and also deals with other applications of these concepts. In particular, Chapter 4 contains applications to approximately differentiable functions, functions determined by dense sets, and the uniform boundedness theorem. For example, Lemma 25 and Proposition 26 allow us to generalize the classical result in topology that continuous functions which agree on a dense subset of their domain, are identical. Subsequent results show that certain specific families of noncontinuous functions are determined by their values on dense subsets of the domain. These generalizations use the notion of semi-closure and enable us, for example, to extend the classical result to show that real-valued separately continuous functions on the product of two Hausdorff spaces, which agree on a dense subset of their domain, are identical everywhere. Similar reasoning in Sections 3 and 4 of Chapter 4 produces the result in Theorem 33 showing that the uniform boundedness theorem can be extended to Riemann integrable derivatives on a closed interval of the real numbers.

Chapter 5 of Part II takes up the investigation of separately continuous functions, the closed-graph property, and related notions. Theorem 37 sets forth an invariance property for separately continuous functions. Results are also presented on cluster sets. A significant part of Chapter 5 deals with the notion of local w^* -continuity, which was introduced by David Rose in 1978. The investigation into this notion of generalized continuity began with the discovery, by the author, that w^* -continuity and local w^* -continuity are strict generalizations of the topological closed-graph property. See Theorem 43. It is then shown that this more general property can be substituted for the closed-graph property in a number of applications. For example, it is known that separately continuous functions with the closed-graph property are continuous; and in Proposition 42 it is proved that functions which are separately continuous and w^* -continuous, are continuous. Another strict generalization of the closed-graph property, called local relative continuity, is also investigated. The last section of Chapter 5 concerns the investigation of functions which are locally bounded but not necessarily continuous. That is, local boundedness is investigated as a strict generalization of continuity. Theorem 60 presents a generalization of the well known theorem that a continuous real-valued function on a compact space is bounded. That is, Theorem 60 shows that if $f : X \rightarrow \mathbf{R}$ is locally bounded (not necessarily continuous), where X is a compact space, then f is bounded on X by a constant M . Once again, in order to find the result in Theorem 60, the investigation began by finding an equivalence between a kind of generalized continuity (subcontinuity)

and the seemingly unrelated notion of local boundedness. See Theorem 58.

Continuity is one of the most fundamental notions in mathematics. One of the first detailed studies of generalized continuity is to be found in the thesis of R. Baire. The notion of separately continuous functions was investigated by Baire in [B]. This investigation and his well known “classes of Baire” comprise one of the first detailed investigations of generalized continuity. As is well known, separately continuous functions are not necessarily continuous but are of the first class of Baire. The notion of quasi-continuity, which is the subject of much of Chapter 4 and part of Chapter 3, is related to the idea of separate continuity, since all separately continuous functions are quasi-continuous. The property of quasi-continuity was evidently first observed by Volterra [P2], but it was Kempisty [Kem] who extended results of Hahn and Baire and conducted the first detailed investigation of quasi-continuity.

CHAPTER 2

DEVANEY'S CHAOS, ATTRACTORS AND THE CHAIN RECURRENT SET

For general background and notation, we refer the reader to [BC]. In this chapter we shall refer to certain known results in the monograph of Block and Coppel [BC] by chapter and number and ask the reader to consult the bibliography of the monograph for information on the original papers. Euclidean n -space with the usual metric will be denoted by E^n . We let $f : K \rightarrow K$ represent a continuous function on a compact n -cube K in E^n . That is, K is the Cartesian product of n nondegenerate compact intervals in the real line. Let $C(K, K)$ denote the set of all continuous functions of the form $f : K \rightarrow K$. By "typical continuous functions," we mean all functions in some residual subset S of $C(K, K)$, where the complement of the residual set S is the countable union of nowhere dense subsets of $C(K, K)$. We put the uniform topology on the function space $C(K, K)$. Then since the range space K is metrizable, we have the supremum metric; so that for elements f and h in $C(K, K)$, $D(f, h) = \sup\{d((f(x), h(x)) : x \in K)\}$. Then $C(K, K)$ is a complete metric space and the residual subset S is dense in $C(K, K)$.

We now set forth some fundamental notions for continuous functions of the form $f : X \rightarrow X$, where X is a compact metric space, unless otherwise indicated. The iterates of a function $f : X \rightarrow X$ are defined inductively such that $f(f^n(x)) =$

$f^{n+1}(x)$, where f^n is the n -fold composition of f . The *trajectory* of a point x in X is the sequence $\{f^n(x)\}_{n \geq 0}$ where we define $f^0(x) = x$. The *orbit* of f at x , denoted by $\gamma(x, f)$, is the point set $\{f^n(x) : n \geq 0\}$. The *omega-limit set* of f at x , denoted by $\omega(x, f)$, is the set of all subsequential limit points of the sequence $\{f^n(x)\}_{n \geq 0}$. We denote the *closure* of a set A by $Cl(A)$. A continuous function on a compact interval will be denoted by $f : I \rightarrow I$ where f has no additional properties unless otherwise indicated. We shall denote the set of all continuous functions on a compact interval by $C(I, I)$. Omega-limit sets for continuous functions of the form $f : I \rightarrow I$ were characterized in [Šar] and [ABCP]. In particular, for any $x \in I$, $\omega(x, f)$ is either a closed nowhere dense set or is the union of finitely many disjoint closed subintervals of I . This result is extended in Chapter 3 below to a broader class of functions.

A function $f : X \rightarrow X$ is (*topologically*) *transitive* if, for every pair of non-empty open sets U and V in X , there is a positive integer k such that $f^k(U) \cap V \neq \emptyset$. If X is a separable and second category metric space with no isolated points, then an equivalent condition is the following: $f : X \rightarrow X$ is transitive iff there is a point x in X such that $\omega(x, f) = X$. That is, $\gamma(x, f)$ is dense in X . [Silv] There are many examples in the literature of transitive functions, including the tent map on the unit interval and the logistic function (sometimes called the population equation) $h(x) = rx(1 - x)$ for $r = 4$. A function $f : X \rightarrow X$ has *sensitive dependence on initial conditions* if there exists $\delta > 0$ such that for any x in X , and for any open set U containing x , there exists a point y in U and

a positive integer n such that $d(f^n(x), f^n(y)) > \delta$. In Devaney's definition of chaos [De] a function $f : X \rightarrow X$ is *chaotic* if f has sensitive dependence on initial conditions, has a dense set of periodic points and is transitive. Today Devaney's definition of chaos is known to contain a redundant condition. That is, if X is infinite, then transitivity and dense periodic points, taken together, imply sensitive dependence on initial conditions. See [BBCDS].

2.1 Typical Continuous Functions are not Chaotic in the Sense of Devaney

Interest in the phenomenon known as "chaos" goes back at least to the paper of Li and Yorke in 1975.[LY] However, there is no single universally accepted definition of chaos. Instead, over a period of time several notions of chaos have appeared in the literature and some of them are quite different from one another. Topological transitivity implies that the dynamical system passes as close as we like to every state of the system. Sensitive dependence on initial conditions goes to the issue of nonpredictability. This is perhaps the most easily recognized aspect of chaos appearing in experimental work. Other characteristics which have been associated with the idea of chaos are random-like behavior and having large sets of periodic points such that the length of the periods are of every magnitude. Chaos has also been associated with the movement of large sets of points in the domain or state space into close proximity to one another, followed by movement away from one another. These phenomena have been observed in applications in many fields, including economics, neural networks, physics and

biology.

In this chapter we are interested in finding which forms of chaos are residual in the set of all continuous functions on various compact domain spaces. Any property of dynamical systems which appears only in the complement of a residual set will presumably have only a fleeting appearance in nature. This is indeed the case for Devaney's chaos. The first result, Lemma 1, will enable us to show that the set of all transitive functions in $C(K, K)$ is a nowhere dense subset of $C(K, K)$.

Lemma 1 *Let K be a compact n -cube in E^n where E^n has the usual metric d . Let $C(K, K)$ denote the set of all continuous functions of the form $f : K \rightarrow K$ with the supremum metric. Then the set W of all functions f such that f has the property that for some nonempty open subset $U \subset K$, where $U \neq K$, $f(Cl(U)) \subset U$, is a dense open subset of $C(K, K)$.*

Proof. To show that W is open in $C(K, K)$, let h be any function in W . Then for some open set $U \subset K$, $h(Cl(U)) \subset U$, where $U \neq K$ and U is not dense in K . Since $h(Cl(U))$ and ∂U are both compact subsets of K , there is a positive distance $\mu > 0$ between them. Let $N_\mu(h)$ be an μ -neighborhood of h in $C(K, K)$. Then for any f in $N_\mu(h)$, $f(Cl(U)) \subset U$. Hence, W is open in $C(K, K)$.

To show that W is dense in $C(K, K)$, fix $\epsilon > 0$, and let h be any element in $C(K, K)$ where h is not necessarily an element of W . Since K has the fixed-point

property, h has a fixed point, say x_0 , in K . Since h is uniformly continuous on K , then there exists $\delta^* > 0$ such that for all x_1, x_2 in K , $d(x_1, x_2) < \delta^*$ implies that $d(h(x_1), h(x_2)) < \epsilon$.

Choose $\delta > 0$ such that $\delta \leq \delta^*$ and $\delta < \text{diam}(K)$. Let $B(x_0, \delta/2) = B_0$ be the open ball of radius $\delta/2$ about x_0 . Let $B(x_0, \delta) = B_1$ be the open ball of radius δ about x_0 . We now define a continuous function $f : K \rightarrow K$ as follows:

$$f(x) = \begin{cases} x_0 & \text{if } x \in Cl(B_0) \\ (2 - 2t)x_0 + (2t - 1)h(x) & \text{if } x \in Cl(B_1) \setminus B_0 \\ h(x) & \text{if } x \in K \setminus B_1 \end{cases},$$

where t is defined so that for any $x \in Cl(B_1) \setminus B_0$, $t = \frac{d(x, x_0)}{\delta}$

Observe that for all x in ∂B_1 , $f(x) = h(x)$, and for all x in ∂B_0 , $f(x) = x_0$, so that the “pasting lemma” gives us the continuity of f on K . Also, observe that $D(h, f) < \epsilon$. Finally, by letting $B_0 = U$, we have $f(Cl(U)) \subset U$, $U \neq K$. Hence, W is a dense open subset of $C(K, K)$. ■

Hereafter in Chapter 2, we will continue to let $C(K, K)$ denote the set of all continuous functions $f : K \rightarrow K$, where K is a compact n -cube in E^n , and where $C(K, K)$ has the supremum metric. We are now able to present a result on Devaney’s definition of chaos.

Theorem 2 *Let K be an n -cube in E^n where E^n has the usual metric. Then there exists a dense open subset W in $C(K, K)$ such that every function in W is not topologically transitive and, hence, not chaotic in the sense of Devaney*

Proof. Let f be an element of the set $W \subset C(K, K)$. Then there exists an

open non-empty set $U \subset K$ such that $f(Cl(U)) \subset U$ where $U \neq K$. It follows that $U \cap K$ is a proper subset of K . Then no iterate $f^k(U)$ of U meets any other nonempty open set in K which is disjoint from U . Hence, f is not transitive on K ; and f is not chaotic in the sense of Devaney. ■

The result for Li-Yorke chaos is quite different from that given in Theorem 2 above for Devaney's chaos. Following [LY], we say that a continuous function $f : I \rightarrow I$ has a *scrambled set*, call it S , if, for every pair of points x and y in S , $\limsup_{n \rightarrow \infty} |f^n(x) - f^n(y)| > 0$, and $\liminf_{n \rightarrow \infty} |f^n(x) - f^n(y)| = 0$. Again following [LY], we say that the function $f : I \rightarrow I$ is *Li-Yorke chaotic* if it possesses an uncountable scrambled set. It is shown in [LY] that if the function $f : I \rightarrow I$ has a point which is periodic of period three, then f is Li-Yorke chaotic. In an intuitive sense, then, a function is Li-Yorke chaotic if there is a large set of points whose orbits regularly get close to one other and then become far apart. Subsequently, in [KS], it was shown that the existence of a two-point scrambled set implies the existence of an uncountable scrambled set. In contrast to the result in Theorem 2 above, in [BP] and [K] it is shown that functions which are chaotic in the sense of Li and Yorke are residual in $C(I, I)$. Hence, we have:

Corollary 3 *There is a residual subset E of $C(I, I)$ such that every function in E is Li-Yorke chaotic but is not chaotic in the sense of Devaney.*

Proof. The subset W of $C(I, I)$ of all functions not chaotic in the sense of Devaney is open and dense in $C(I, I)$ and, therefore, residual in $C(I, I)$. Let

A be the residual subset of all Li-Yorke chaotic functions in $C(I, I)$. Since the intersection of two residual subsets is again residual, $E = W \cap A$ is residual in $C(I, I)$. ■

In the case of chaos in the sense of Block and Coppel, the result is also quite different from the result in Theorem 2 above. Following [BC], we say that a continuous function $f : I \rightarrow I$ is *turbulent* if there exist compact subintervals J and L in I , with at most one point in common, such that $J \cup L \subset f(J) \cap f(L)$. The function f is said to be *chaotic in the sense of Block and Coppel* if f^n is turbulent for some positive integer n . Block-Coppel chaos is sometimes called “strong chaos.” In Corollary II.20 of [BC], it is shown that functions which are chaotic in the sense of Block and Coppel comprise an open subset of $C(I, I)$. In [K], although different terminology is used, Kloeden essentially shows that the set of functions of the form $f : I \rightarrow I$ which are Block-Coppel chaotic comprise a dense subset of $C(I, I)$. In summary, the set of functions in $C(I, I)$ which are Block-Coppel chaotic are a dense open subset of $C(I, I)$. Since the intersection of a finite number of dense open sets is again dense and open, we now have:

Corollary 4 *There is a dense open subset D in $C(I, I)$ such that every function in D is Block-Coppel chaotic but not chaotic in the sense of Devaney.*

For the sake of completeness, we give an example of a function which is Block-Coppel chaotic but not chaotic in the sense of Devaney. Let $f : I \rightarrow I$ be a continuous function defined as follows:

$$f(x) = \begin{cases} 3x & \text{if } 0 \leq x \leq 1/3 \\ 1 & \text{if } 1/3 < x < 2/3 \\ 3 - 3x & \text{if } 2/3 \leq x \leq 1 \end{cases}$$

Then f is Block-Coppel chaotic; for if we let $J = [0, \frac{1}{3}]$ and let $L = [\frac{2}{3}, 1]$, then $J \cup L \subset f(J) \cap f(L)$. On the other hand, f is not transitive because $f^2(Cl(\frac{1}{3}, \frac{2}{3})) = 0$, and all iterates $f^i, i > 2$, remain at zero.

In [BrC] a notion of chaos lying strictly between Li-Yorke and Block-Coppel is set forth. Bruckner and Ceder studied the map $\omega_f : I \rightarrow C$ where C is the space of all compact subsets of the closed unit interval I with the Hausdorff metric, and where ω_f maps each point x to its omega-limit set. A continuous function $f : I \rightarrow I$ is *chaotic in the sense of Bruckner and Ceder* if the function $x \mapsto \omega(x, f)$ is no better than second class of Baire.

We now turn our attention to the set of all polynomials on a compact interval I of the real line. We shall denote the set of all such polynomials by $P(I, I)$. Surprisingly, although polynomials, being analytic functions, are among the “nicest” functions which we encounter in analysis, the typical polynomial is Block-Coppel chaotic as shown by the next result.

Theorem 5 *Let $P(I, I)$ denote the set of all polynomials on a compact interval I . There exists a dense open subset $T \subset P(I, I)$ such that all polynomials in T are chaotic in the sense of Block and Coppel but not chaotic in the sense of Devaney.*

Proof. The set of polynomials $P(I, I)$ is dense in $C(I, I)$. By Corollary 4 above, the set D of all Block-Coppel chaotic functions which are not chaotic in the sense of Devaney is a dense open subset of $C(I, I)$. It follows that, in the relative topology on $P(I, I)$, the set T is open and dense in $P(I, I)$. ■

2.2 Asymptotic Attractors and The Chain Recurrent Set

Let $f : X \rightarrow X$ be a continuous function, where X is a compact metric space. Following [BC], we say that a nonempty closed set $A \subset X$ is *Lyapunov-stable* if, for each open set U containing A , there exists an open set V containing A such that $\gamma(x, f) \subset U$ for every x in V . Clearly, $V \subset U$. The set A is an *attractor* if there exists an open set U_0 containing A such that $\omega(x, f) \subset A$ for every $x \in U_0$. If a nonempty closed set A is both Lyapunov-stable and an attractor, we say that A is an *asymptotically stable set*. By Corollary V.14 of [BC], if $U \subset X$ is a nonempty open set such that $f(Cl(U)) \subset U$, then the set $A = \bigcap_{n \geq 0} f^n(Cl(U))$ is asymptotically stable.

The *basin* $B(A)$ for an asymptotically stable set A is the point set $B(A) = \{y \in X : \omega(y, f) \subset A\}$. It is easy to show that $B(A)$ is open in X for A asymptotically stable. Let $x_0 \in B(A)$. By asymptotic stability, there exists an open set U_0 , $A \subset U_0$, such that for every $y \in U_0$, $\omega(y, f) \subset A$. Then there exists a non-negative integer n_0 such that $f^{n_0}(x_0) \in U_0$. Since f is continuous, there is a neighborhood G of x_0 such that $f^{n_0}(G) \subset U_0$. Hence, $G \subset B(A)$, and $B(A)$ is open in X .

The word “basin” is used in two different contexts in this chapter. The basin of an asymptotically stable set should be distinguished from the basin of a Milnor attractor as described in the discussion preceding Proposition 8 below.

Proposition 6 *Let $C(K, K)$ denote the set of all continuous functions of the form $g : K \rightarrow K$, and let W denote the set of all functions in $C(K, K)$ which have an asymptotically stable set. Then W is a dense open subset of $C(K, K)$.*

Proof. The proof of Lemma 1 above shows that all functions in $C(K, K)$ which have an inward set make up a dense open subset W of $C(K, K)$. Then for any $g \in W$, $g(Cl(U)) \subset U$ for some nonempty open set U . It follows by Corollary V.14 of [BC] that every element of W has an asymptotically stable set.

■

Corollary 7 *Let D denote the set of all functions in $C(I, I)$ such that every function $f \in D$ is Block-Coppel chaotic and possesses an asymptotically stable set. Then D is a dense open subset of $C(I, I)$.*

Observe that for any $f \in C(K, K)$, and for any $x_0 \in K$, the compactness of K guarantees that there exists an omega-limit set $\omega(x_0, f)$. However, this is no guarantee that f has an asymptotically stable set. An example using the Feigenbaum limit is given below; but first we briefly discuss a different kind of attractor known as a “Milnor attractor.”

Milnor attractors (See [Mil1] and [Mil2]) for a function $f : X \rightarrow X$, where X is a compact metric space, can be defined as follows: A compact invariant set

A is said to be a *Milnor attractor* if (1) its basin of attraction $B(A)$ has positive Lebesgue measure and (2) there is no proper closed subset $A' \subset A$ such that $B(A)$ and $B(A')$ differ by a set of zero measure. Although we use the term “basin of attraction” for both asymptotically stable sets and for Milnor attractors, the reader will observe that these notions are different, and we shall always specify which meaning is intended. As shown above, basins for asymptotically stable sets are always open. However, basins for Milnor attractors are not necessarily open. For example, consider the logistic equation $f(x) = \lambda x(1 - x)$ on the unit interval for $\lambda = 3.57\dots$ This is the Feigenbaum limit (See [Bue] at page 23.) At the Feigenbaum limit the function has an invariant attractor which is a transitive Cantor set whose basin of attraction is the entire unit interval, except for a countable set of points. For this map the Milnor attractor is the only attractor; and it is not an asymptotically stable attractor. On the other hand, all asymptotically stable attractors are Milnor attractors because the open basin of an asymptotically stable attractor has positive measure.

Proposition 8 *The set of functions in $C(K, K)$ which have only Milnor attractors, none of which are asymptotically stable attractors, is a nowhere dense subset of $C(K, K)$.*

Proof. By Proposition 6 above, the set of functions in $C(K, K)$ which have an asymptotically stable attractor is a dense open subset of $C(K, K)$. ■

Proposition 9 below deals with the concept of chain recurrence. Let $f : K \rightarrow$

K be continuous, and let x and y be points in K such that $x \neq y$. An ϵ -chain from x to y is a finite sequence of points $\{x_0, x_1, \dots, x_n\}$ where $x = x_0$ and $y = x_n$ and $d(f(x_{i-1}), x_i) < \epsilon$ for $i = 1, \dots, n$. If for any $\epsilon > 0$ there exists an ϵ -chain from x to y , then we say that x can be chained to y . If x can be chained to itself, then x is a *chain recurrent point*. The set of all chain recurrent points is called the *chain recurrent set*. We denote the chain recurrent set of a function f by $CR(f)$.

Before presenting the next result, we recall Proposition V.15 of [BC], which says that if A is an asymptotically stable set for a function of the form $f : X \rightarrow X$, X a compact metric space, then there exists an open set U containing A such that $f[Cl(U)] \subset U$. Proposition V.15 also provides that for any open set G containing A , U can be chosen so that $Cl(U) \subset G$.

Proposition 9 *Let S be the set of functions in $C(K, K)$ such that for every function g in S , $CR(g) \neq K$. Then S contains an open dense subset of $C(K, K)$.*

Proof. By Lemma 1 of [BCo], since K is connected, if $CR(g) = K$ for some continuous function g , then there is no nonempty open set $U \neq K$ such that $g(Cl(U)) \subset U$. Then by the contrapositive of V.15 of [BC], g does not have an asymptotically stable set. Now, by application of Proposition 6, we conclude that S contains an open and dense subset of $C(K, K)$. ■

We remark that a much stronger result than Proposition 9 above is known for functions of the form $f : I \rightarrow I$, where I is a compact interval in the real line.

In particular, see [Fr]. Also, see theorem 0(i) of [Mi] which states that there is a residual subset D of $C(I, I)$ such that for every function f in D , $CR(f)$ is of zero Lebesgue measure and is nowhere dense.

For the next result, we let $C(X, X)$ denote the set of continuous functions of the form $f : X \rightarrow X$ with the sup metric where X is a compact metric space. Lemma 10 shows that if any two functions in $C(X, X)$ are sufficiently close, then their n th iterates remain close. We shall apply Lemma 10 and Proposition 11 in the proof of Theorem 12 below.

Lemma 10 *Let f and g be elements of $C(X, X)$. Then for any positive integer n , and for any $\epsilon > 0$, there exists $\delta > 0$ such that if $D(f, g) < \delta$, then $D(f^n, g^n) < \epsilon$.*

Proof. The case for $n = 1$ is trivial. We shall prove the lemma for $n = 2$; then, assuming it is true for $n = k$, we show it is true for $n = k + 1$. Let $\epsilon > 0$ be given. Since $f(X)$ and $g(X)$ are compact, by the uniform continuity of f , we can choose $\delta > 0$ such that if $d(f(x), g(x)) < \delta$, then $d(f(f(x)), f(g(x))) < \epsilon/2$ for all $x \in X$. Let $D(f, g) < \min\{\delta, \epsilon/2\}$. Then for all $x \in X$,

$$\begin{aligned} d(f^2(x), g^2(x)) &= d(f(f(x)), g(g(x))) \leq d(f(f(x)), f(g(x))) + d(f(g(x)), g(g(x))) \\ &< \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

It follows that $D(f^2, g^2) < \epsilon$. Now suppose the lemma is true for $n = k$. Since f^k and g^k are continuous functions for any non-negative integer k , the argument

is the same for any positive integer $k > 1$. This proves the second part of the induction argument. ■

Proposition 11 *Let f and g be elements of $C(X, X)$. Then for any positive integer n , and for any $\epsilon > 0$, there exists $\delta > 0$ such that if $D(f, g) < \delta$, then $D(f^k, g^k) < \epsilon$ for all $k, k = 1, 2, \dots, n$.*

Proof. Since Lemma 10 provides a $\delta_k > 0$ such that for any particular k , $D(f, g) < \delta_k$ implies that $D(f^k, g^k) < \epsilon$, choose $\delta = \min\{\delta_k : k = 1, 2, \dots, n\}$. ■

We now turn our attention to the question of perturbation stability of asymptotically stable sets for functions in $C(K, K)$. In the proof of Theorem 12 below we shall apply the following result set forth in [Pil]. Suppose that $f : X \rightarrow X$ is a continuous function on the compact metric space X with asymptotically stable set $A \subset X$. Let $B(A)$ denote the basin of A and let F be a compact subset of X contained in $B(A)$. Then Lemma 3.1.2 of [Pil] says that, for any open set V containing A , there exists a natural number n_0 such that $f^n(F) \subset V$, for all $n \geq n_0$.

In the proof of Theorem 12 below, we shall also make use of the following notation: For any function $f \in C(K, K)$, where f has at least one asymptotically stable set A_f , we let $\tilde{A}_f = \{A_f : A_f \text{ is an asymptotically stable set for } f\}$.

Further, in the proof of Theorem 12, we shall make use of the following metric space. For the metric space (F, H) , we let F denote the family of all nonempty closed subsets in K and let H denote the Hausdorff metric, so that for elements

A and C in F , the distance between A and C in the Hausdorff metric is

$$H(A, C) = \inf\{\delta > 0 : A \subset B_\delta(C) \text{ and } C \subset B_\delta(A)\}.$$

, where for any set E , $B_\delta(E) = \{x \in K : d(x, y) < \delta, \text{ for some } y \in E\}$.

Essentially, the next result shows that small perturbations of a function cause its attractor to shrink and its basin of attraction to grow.

Theorem 12 *Let $f \in C(K, K)$. Let A_f be an asymptotically stable set for f with basin $B(A_f)$. Then for any $\epsilon > 0$, there exists $\delta > 0$ such that for any $g \in C(K, K)$, $D(f, g) < \delta$, there exists A_g in \tilde{A}_g such that $A_g \subset S(A_f, \epsilon)$ and $B(A_f) \subset S(B(A_g), \epsilon)$ where $S(A_f, \epsilon)$ denotes the open ball in (F, H) centered at A_f , with radius ϵ , and where $S(B(A_g), \epsilon)$ denotes the open ball centered at $B(A_g)$ of radius ϵ .*

Proof. Since A_f is an asymptotically stable set for f , there exists an open set G containing A_f such that $f(Cl(G)) \subset G$, and $Cl(G) \subset S(A_f, \epsilon)$. [BC, V.15] Let V be an open set contained in $B(A_f)$ such that $H(V, B(A_f)) < \epsilon$ and $Cl(V) \subset B(A_f)$. Then by Theorem 3.12 of [Pil], there exists a natural number n_0 such that $f^{n_0}(Cl(V)) \subset G$, and such that $f^{n_0+1}(Cl(V)) \subset f(G) \subset f(Cl(G)) \subset G$. Let $\gamma = \min\{d(f(x), \partial G) : x \in Cl(G)\}$ where ∂G denotes the boundary of G . By Proposition 11, we can choose $\delta > 0$ so that $D(f, g) < \delta \implies D(f^k, g^k) < \gamma$ for $1 \leq k \leq N + 1$. Then $g^{n_0+1}(Cl(V)) \subset G$ and $g(Cl(G)) \subset G$. It follows that $A_g = \bigcap_{n \geq 0} g^n(Cl(G))$ is an asymptotically stable set for g ; and by choosing G so

that $Cl(G) \subset S(A_f, \epsilon)$, we see that $A_g \subset Cl(G) \subset S(A_f, \epsilon)$. Also, if $x \in V$, then $\omega(x, g) \subset A_g$ so that $V \subset B(A_g)$. Hence, $B(A_f) \subset S(B(A_g), \epsilon)$. ■

The reader will note that since typical continuous functions are Block-Coppel chaotic, even the most minute perturbation of a nonchaotic function can result in a sudden shift to strong chaos. On the other hand, as this chapter shows, the typical continuous function has at least one asymptotically stable set, and each such asymptotically stable set enjoys the kind of perturbation stability described in Theorem 12.

CHAPTER 3

DYNAMICAL SYSTEMS AND QUASI-CONTINUITY

3.1 Introduction and Preliminaries

For general background and notation, we refer the reader to [BC], [BCP], and [Kel]. We will let $f : \mathbf{R} \rightarrow \mathbf{R}$ denote a real-valued function on the real line. We investigate omega-limit sets of Darboux-Baire-1 functions, quasi-continuous functions and other noncontinuous functions. A different approach is taken to quasi-continuous dynamics in [CK]. See also [CM]. Noncontinuous functions arise in such familiar applications as Newton's method of finding the zeros of a function which is differentiable, but not C^1 . See [Kel]. Following the notation in Chapter 2 above, we let $\gamma(x, f)$ denote the orbit of a point x and we let $\omega(x, f)$ denote the omega-limit set of f at x . For any omega-limit set $\omega(x_0, f)$, we observe that $\omega(x, f) = \bigcap_{m \geq 0} Cl(\bigcup_{n \geq m} f^n(x))$. A function $f : X \rightarrow Y$, where X and Y are topological spaces, is *quasi-continuous* at a point x in X if for any open set V containing $f(x)$, and for any open set U containing x , there exists an open nonempty set $G \subset U$ such that $f(G) \subset V$. [Mar]. For more information on the nature of quasi-continuity, see Chapter 4 below and [Neu]. Darboux-Baire-1 functions may not be quasi-continuous. [Mar]. However, Darboux-Baire-1 functions which have Banach's condition T_2 are quasi-continuous. See page 123 of [Bru] and page 277 and chapters VII and IX of [S]. A function $f : \mathbf{R} \rightarrow \mathbf{R}$ is said to satisfy *Banach's*

condition T_2 if almost every value taken by f is taken at most a denumerable number of times. Functions which are both Darboux and quasi-continuous are bilaterally quasi-continuous; but bilaterally quasi-continuous functions are not necessarily Darboux. [Bor]. A function $f : \mathbf{R} \rightarrow \mathbf{R}$ is *bilaterally quasi-continuous* if f is both right-hand-sided quasi-continuous and left-hand-sided quasi-continuous; and f is *left-hand-sided (right-hand-sided) quasi-continuous* at x if for every $\delta > 0$ and for every open neighborhood V of $f(x)$ there exists an open nonempty set

$$W \subset (x - \delta, x) \cap f^{-1}(V) \quad (W \subset (x, x + \delta) \cap f^{-1}(V)).$$

An example of a function which is bilaterally quasi-continuous but not continuous is the following function of the form $f : [-1, 1] \rightarrow [-1, 1]$, given by:

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ \sin \frac{1}{x} & \text{if } x \neq 0 \end{cases}$$

A subset S of a topological space X is called *semi-open* if there is an open set G in X such that $G \subset S \subset Cl(G)$. See [CH] and [CH2], as well as Chapter 4 below. A subset C of X is called *semi-closed* if its complement is semi-open. For any set $E \subset X$ the *semi-closure* of E , denoted by $SCl(E)$, is defined to be the intersection of all semi-closed sets containing E . Then for any set E the set $SCl(E)$ is semi-closed. As an example, consider the open unit interval $(0, 1)$. The sets $[0, 1)$, $(0, 1]$, $(0, 1)$ and $[0, 1]$ are all semi-closed sets. However, the semi-closure of $(0, 1)$, being the intersection of all four of the preceding sets, is exactly the set $(0, 1)$.

In his seminal paper, Kempisty defined quasi-continuity at a point for real

valued functions of several real variables. Basically, this is the definition given in the introduction to this chapter. The notion of semi-open sets was first introduced in [L] in connection with “semi-continuity” (not to be confused with the usual definition of upper semi-continuity or lower semi-continuity). Although semi-continuity was introduced independently of Kempisty’s earlier paper on quasi-continuity, the two notions were subsequently shown to be equivalent for single-valued functions of the form $f : X \rightarrow Y$, where X and Y are topological spaces. See [Neub]. We shall use the notion of quasi-continuity throughout this thesis. Arbitrary intersections of semi-closed sets are semi-closed, and a set C is semi-closed iff $\text{Int}(\text{Cl}(C)) \subset C$. A function $f : X \rightarrow Y$ is quasi-continuous iff, for any subset E of X , $f(\text{Scl}(E)) \subset \text{Cl}(f(E))$. See [CH2], Theorem 1.16. Semi-open (semi-closed) sets are also related to quasi-continuity in the following way: Let $f : X \rightarrow Y$ be a function, where X and Y are topological spaces. Then f is quasi-continuous iff inverse images of open (closed) sets are semi-open (semi-closed).

We recall that the *boundary* of a set S in a topological space X , denoted by ∂S , is defined as follows: $\partial S = \text{Cl}(S) \cap \text{Cl}(X \setminus S)$. A function $f : \mathbf{R} \rightarrow \mathbf{R}$ has a *perfect road* if for each $x \in \mathbf{R}$ there exists a perfect set P having x as a bilateral limit point such that $f|_P$ is continuous at x .

Finally, a function $f : \mathbf{R} \rightarrow \mathbf{R}$ has the *Young property* if for each $x \in \mathbf{R}$, there exist sequences $x_n \uparrow x$ and $y_n \downarrow x$ such that $f(x) = \lim_{n \rightarrow \infty} f(x_n) = \lim_{n \rightarrow \infty} f(y_n)$.

3.2 The Dense Mapping Property in Dynamics and Applications of
Quasi-continuity

In [Kel] Keller introduced the concept of “dense mapping property” (DMP). A function $f : X \rightarrow Y$, where X and Y are topological spaces, has *the dense mapping property* if for every subset D of X with $Cl(D)$ connected, $f(Cl(D)) \subset Cl(f(D))$. Keller shows that Darboux-Baire-1 (D-B1) functions may not have the DMP. The DMP is more general than continuity, but any Darboux function with the DMP can have only those omega-limit sets which are possible for continuous functions. In particular, Keller proves (Theorem 2.5 of [Kel]) that if $f : \mathbf{R} \rightarrow \mathbf{R}$ is Darboux and has the DMP, then for any x in \mathbf{R} the set $\omega(x, f)$ is either a nowhere dense set or is the union of finitely many closed intervals. That is, the characterization of the omega-limit sets for such functions is the same as for continuous functions of the form $f : I \rightarrow I$. See [Šar] and [ABCP]. Keller also shows that functions with the DMP are closed under composition of functions.

Theorem 13 *Let $f : \mathbf{R} \rightarrow \mathbf{R}$, where \mathbf{R} has the usual topology. Then f is bilaterally quasi-continuous iff f has the DMP.*

Proof. For the sufficiency, assume that f is not left-hand-sided quasi-continuous at some point x_0 in X . Then there is an open set V containing $f(x_0)$ and there exists $\delta > 0$ such that there is no open nonempty set $G \subset (x_0 - \delta, x_0)$ such that $f(G) \subset V$. Evidently, the set $D = \{x \in (x_0 - \delta, x_0) : f(x) \notin V\}$ is dense in $(x_0 - \delta, x_0)$. By the DMP, $f(Cl(D)) \subset Cl(f(D))$. Since $x_0 \in Cl(D)$,

$V \cap f(Cl(D)) \neq \emptyset$, and there is a contradiction. It follows that f is left-hand-sided quasi-continuous. Similarly, we can show that f is right-hand-sided quasi-continuous. Hence, f is bilaterally quasi-continuous.

For the necessity, let D be a subset of X such that $Cl(D)$ is connected. We may suppose that $Cl(D)$ is an interval. Therefore, $SCL(D)$ is also an interval, since $Int(Cl(D)) \subset SCL(D) \subset Cl(D)$. Since f is quasi-continuous, $f(SCL(D)) \subset Cl(f(D))$. The claim now is that $f(Cl(D)) \subset Cl(f(D))$. Assume, to the contrary, that there exists x in $Cl(D)$ such that $f(x) \notin Cl(f(D))$. Therefore, there exists an open set H containing $f(x)$ such that $H \cap f(D) = \emptyset$. Now by bilateral quasi-continuity, for any open set U containing x , U contains an open nonempty set G such that $f(G) \subset H$. If x is an endpoint, the set G can be chosen on the left side or right side of x , so that G meets the set D . Then $H \cap f(D)$ is not empty, and we have a contradiction. Hence, f has the DMP. ■

Corollary 14 *Suppose that $f : \mathbf{R} \rightarrow \mathbf{R}$ is a $D-B_1$ function. Then if f has Banach's T_2 condition, f has the DMP.*

Proof. Since f is $D-B_1$ and satisfies Banach's condition T_2 , then f is quasi-continuous. Since f is Darboux and quasi-continuous, then f is bilaterally quasi-continuous, and hence has the DMP. ■

Proposition 15 *Suppose that $f : \mathbf{R} \rightarrow \mathbf{R}$ is Darboux and has the DMP. Then f has a perfect road. Moreover, f has the Young property.*

Proof. Since f is Darboux and quasi-continuous, f has a perfect road [GR]. Also, f has the Young property [Bor]. ■

We observe that functions of the form $f : \mathbf{R} \rightarrow \mathbf{R}$ which are lower or upper semi-continuous do not have the DMP. For example, let $f : \mathbf{R} \rightarrow \mathbf{R}$ be defined as follows:

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } 0 < x \end{cases}$$

Let $D = (0, 1)$. Then

$$f(Cl(D)) = \{0, 1\} \not\subseteq Cl(f(0, 1)) = Cl(\{1\}) = \{1\} = Cl(f(D)).$$

Hence, f is lower semi-continuous but does not have the DMP. Similarly, if we define f as follows, then f is upper semi-continuous but does not have the DMP:

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } 0 \leq x \end{cases}$$

Theorem 13 above also gives us a new result concerning bilaterally quasi-continuous functions. The following result might be more appropriately included in Chapter 4 below but we include it here because it follows from Theorem 13.

Theorem 16 *Bilaterally quasi-continuous functions of the form $f : \mathbf{R} \rightarrow \mathbf{R}$ are closed under composition of functions.*

Proof. Bilaterally quasi-continuous functions have the DMP. In [Kel] it is shown that functions with the DMP are closed under composition of functions.

■

Lemma 17 *Let $f : \mathbf{R} \rightarrow \mathbf{R}$ be a $D - B_1$ function satisfying Banach's T_2 condition. Then for any x in \mathbf{R} , $\omega(x, f)$ is either a nowhere dense set, or is the union of finitely many disjoint closed intervals.*

Proof. Since f has the DMP, Keller's Theorem 2.5 completes the proof. ■

In [Mar] Marcus proves the result that a derivative which is continuous almost everywhere, is quasi-continuous. Hence, we have:

Theorem 18 *Let $f : [a, b] \rightarrow [a, b]$ be a derivative which is continuous almost everywhere. Then for any x in $[a, b]$, $\omega(x, f)$ is either a nowhere dense set or is the union of finitely many disjoint closed intervals.*

Proof. The function f is quasi-continuous, and since we also have the Darboux property, f is bilaterally quasi-continuous. The result follows from Theorem 13 and Keller's Theorem. ■

Theorem 19 *Let $f : [a, b] \rightarrow [a, b]$ be a Riemann integrable derivative. Then for any x in $[a, b]$, $\omega(x, f)$ is either a nowhere dense set, or is the union of finitely many disjoint closed intervals.*

3.3 Connectedness Properties and a Fixed-point Property

Let $f : X \rightarrow Y$ be a function, where X and Y are topological spaces. We say that f is a *connectivity function* if for any connected subset $C \subset X$, the graph of f restricted to C is a connected subset of $X \times Y$. That is, a connectivity function

is a function from X to Y such that the induced map g from X into $X \times Y$, defined by $g(x) = (x, g(x))$ transforms connected subsets of X onto connected subsets of $X \times Y$. See [Ha].

Of course, fixed points are one kind of omega-limit set. Since Darboux quasi-continuous functions of the form $f : \mathbf{R} \rightarrow \mathbf{R}$ are not necessarily connectivity functions [GR], we cannot necessarily conclude that DMP functions with reasonable properties are connectivity or have a fixed point, although either is a possibility. On the other hand, it is known that D-B₁ functions of the form $f : I \rightarrow I$ have a fixed point.[BCP] We can prove a slight variation of this result. Further results on connectedness properties are presented in Chapters 4 and 5 below.

Before presenting the next result, we recall the notion of “maximum additive family,” which deals with the way D-B₁ functions behave with respect to algebraic operations with continuous functions. Let Γ be a family of functions of the form $f : \mathbf{R} \rightarrow \mathbf{R}$. A subfamily $F \subset \Gamma$ is the *maximal additive family* for Γ if F is the set of all functions in Γ such that $f + g \in \Gamma$ whenever $f \in F$ and $g \in \Gamma$. See Chapter 2, Section 3 of [Bru].

Proposition 20 *Let $f : I \rightarrow I$ be a D-B₁ function where I is a closed bounded interval. If $J = [a, b]$ is a compact subinterval of I such that $f(J) \supset J$, then f has a fixed point in J .*

Proof. Define $g(x) = f(x) - x$. Then g is Darboux because the continuous

functions form the maximum additive family for the set of D-B₁ functions. Since $f(J) \supset J$, there are points x_1 and x_2 in J such that $f(x_1) = a$ and $f(x_2) = b$. It is easy to show that $g(x_1) \leq 0$ and $g(x_2) \geq 0$. Hence, by the intermediate value property, $g(y) = 0$ for some y in J , which implies that $f(y) = y$. ■

3.4 An Invariance Property of Omega-limit Sets of Riemann Integrable

Derivatives and Derivatives which are Continuous Almost Everywhere.

It is well known that omega-limit sets for continuous functions are invariant. That is, for a continuous function $f : \mathbf{R} \rightarrow \mathbf{R}$, $f(\omega(x, f)) \subset \omega(x, f)$. (Actually, for continuous functions omega-limit sets are *strongly invariant*; that is, for $f : X \rightarrow X$ continuous on a compact metric space X , for any $x \in X$, $f(\omega(x, f)) = \omega(x, f)$.)

The invariance of omega-limit sets for continuous functions can be shown as follows:

$$f(\omega(x, f)) = f(\bigcap_{m \geq 0} Cl(\bigcup_{n \geq m} f^n(x))) \subset \bigcap_{m \geq 0} f(Cl(\bigcup_{n \geq m} f^n(x))).$$

Then

$$\begin{aligned} & \bigcap_{m \geq 0} f(Cl(\bigcup_{n \geq m} f^n(x))) \subset \\ & \subset \bigcap_{m \geq 0} Cl(f(\bigcup_{n \geq m} f^n(x))) \subset \bigcap_{m \geq 0} Cl(\bigcup_{n \geq m} f^n(x)) = \omega(x, f). \end{aligned}$$

The reader will note that the above proof depends upon the following property of continuous functions: For any continuous function $f : X \rightarrow Y$, where X, Y

are topological spaces, if $E \subset X$ is any subset of X , then $f(Cl(E)) \subset Cl(f(E))$. In the case of quasi-continuous functions, we do not have this nice property, but we do have the property that for $f : X \rightarrow Y$ quasi-continuous, $f(SCl(E)) \subset Cl(f(E))$. This property will be applied in Section 3.5 below.

Corollary 2.6 of [Kel], applied to a real-valued function $f : \mathbf{R} \rightarrow \mathbf{R}$, states that if f is Darboux and has the DMP, then for any component K of $\omega(x, f)$, with nonempty interior, $Cl(f^n(K))$ is also a component of $\omega(x, f)$ for any nonnegative integer n . We now slightly extend Keller's result by applying Theorems 18 and 19 to self-maps of a compact interval.

Theorem 21 *Let $f : I \rightarrow I$ be a Riemann integrable derivative or a derivative continuous almost everywhere, where I is a compact interval in the real line. Then for any omega-limit set $\omega(x, f)$ for f , where $\omega(x, f)$ is the union of a finite number of nondegenerate closed intervals, $\omega(x, f)$ is invariant.*

Proof. By the theorems of [Mar], f is quasi-continuous. Since f is Darboux and quasi-continuous, f is bilaterally quasi-continuous. Then by Theorem 13 above, f has the DMP. Now since f is Darboux and has the DMP, by Theorem 2.5 of [Kel], $\omega(x, f)$ can be the union of a finite number of nondegenerate closed intervals as supposed in the statement of the theorem. Let $\epsilon > 0$ be given. Let M_ϵ be the closed ϵ -neighborhood of $\omega(x, f)$. Let K_i be a component of $\omega(x, f)$. Observe that $\gamma(x, f) \cap K_i$ is dense in K_i . The trajectory $\{f^n(x)\}_{n \geq 0}$ of x can have only finitely many points which are not in M_ϵ , for otherwise by the compactness

of I , the set $Cl(I \setminus M_\epsilon)$ would contain a limit point of the trajectory and hence a point in $\omega(x, f)$, a contradiction. Therefore, we can remove a finite number of points from $\gamma(x, f) \cap K_i$ and obtain a dense subset $D_i^\epsilon \subset K_i$ such that $f(D_i^\epsilon) \subset M_\epsilon$. To do this, since $\gamma(x, f)$ is infinite, then for every point $f^r(x), r = 1, 2, \dots$, there is only one k such that $f(f^k(x)) = f^r(x)$, namely $k = r - 1$. Observe that the set D_i^ϵ obtained in this way is still dense in K_i . Therefore, we now have that $f(D_i^\epsilon) \subset M_\epsilon$. Then by the DMP, $Cl(f(D_i^\epsilon)) \subset M_\epsilon$, because M_ϵ is closed. To summarize, $f(K_i) \subset M_\epsilon$, for any ϵ . Hence, $f(K_i) \subset \omega(x, f) \implies f(\omega(x, f)) \subset \omega(x, f)$ ■

So far as the author knows, the question of whether Theorem 21 is true for omega-limit sets which are nowhere dense sets, is an open question.

3.5 An Invariance Property of Omega-limit Sets of Quasi-continuous Functions

In this section we investigate the question of invariance of the omega-limit sets of quasi-continuous functions. We observe that, although bilaterally quasi-continuous functions of the form $f : \mathbf{R} \rightarrow \mathbf{R}$ have the DMP, quasi-continuous functions in general do not, as shown by the following elementary example: Define the function $f : \mathbf{R} \rightarrow \mathbf{R}$ as follows:

$$f(x) = \begin{cases} x & \text{if } x < 1 \\ x + 1 & \text{if } 1 \leq x \end{cases},$$

Then let $D = (\frac{1}{2}, 1)$. Then $f(Cl(D))$ contains the point $f(x) = f(1) = 2$, but $Cl(f(D))$ does not. On the other hand, f is clearly quasi-continuous.

In Theorem 23 below we present an invariance theorem for omega-limit sets of quasi-continuous functions. We prove Theorem 23 for functions of the form $f : \mathbf{R} \rightarrow \mathbf{R}$ and then show how the result can be generalized to more general spaces. For information on the kinds of sets which can be omega-limit sets for various families of noncontinuous functions, see [BCP]. Before proceeding with the proof of the theorem, we prove a lemma concerning “regular open” sets. A subset G of a topological space X is *regular open* if G is open in X and $G = \text{Int}(\text{Cl}(G))$. In \mathbf{R} with the usual topology, it is trivial that connected open sets (intervals) are regular open. Also, since \mathbf{R} with the usual topology is locally connected, each component of each open set in \mathbf{R} is open.

Lemma 22 *Let X be a topological space. If E is regular open in X , then $S\text{Cl}(E) = E$.*

Proof. Let E be regular open. By Theorem 1.2 of [CH], a set A is semi-closed iff $\text{Int}(\text{Cl}(A)) \subset A$. Since E is regular open, then $\text{Int}(\text{Cl}(E)) = E \implies \text{Int}(\text{Cl}(E)) \subset E$. Then E is semi-closed and hence $S\text{Cl}(E) = E$. ■

In the next proof we shall apply the fact that for $f : X \rightarrow Y$ quasi-continuous, $f(S\text{Cl}(E)) \subset \text{Cl}(f(E))$ for any $E \subset X$. We shall also make use of Lemma 25 in Chapter 4 below which states that for any space X , if $D \subset X$ is dense in X and D is semi-closed, then $D = X$. Observe that the proof of Theorem 23 relies entirely on properties of the quasi-continuous function f and does not require that any iterate f^n be quasi-continuous for $n > 1$, but only that f be quasi-

continuous. As is well known, the composition of two quasi-continuous functions is not necessarily quasi-continuous.

Theorem 23 *Let $f : \mathbf{R} \rightarrow \mathbf{R}$ be quasi-continuous on \mathbf{R} . Let $x \in \mathbf{R}$. Then $f(\text{Int}(\omega(x, f))) \subset \omega(x, f)$.*

Proof. Let $x \in \mathbf{R}$ and let $G = \text{Int}(\omega(x, f))$. If $G = \emptyset$, the result is trivially true. Suppose that $G \neq \emptyset$. Let J be any component of G . Then $f(J) = \bigcap_{m \geq 0} f(J) = \bigcap_{m \geq 0} f(\text{SCl}(J) \cap \bigcup_{n \geq m} f^n(x))$. Since $f(\text{SCl}(E) \cap \bigcup_{n \geq m} f^n(x)) \subset \text{Cl}(f(E))$ for any set E , it follows that

$$f(J) \subset \bigcap_{m \geq 0} \text{Cl}(f(J) \cap \bigcup_{n \geq m} f^n(x)) \subset \bigcap_{m \geq 0} \text{Cl}(\bigcup_{n \geq m} f^n(x)) = \omega(x, f).$$

Hence, since the component J is arbitrary, $f(\text{Int}(\omega(x, f))) \subset \omega(x, f)$. ■

We remark that Theorem 23 can be generalized to quasi-continuous functions of the form $f : X \rightarrow X$, where X is a compact locally connected metric space with no isolated points, such that any open connected set in X is the interior of its closure. For let $x \in X$ and let $G = \text{Int}(\omega(x, f))$ and let U be any component of G . Since U is open and connected, then $\text{SCl}(U) = U$. Since $\gamma(x, f)$, restricted to U , is dense in U , and since $U = \text{SCl}(\text{Int}(U))$, then

$$f(U) = f[\bigcap_{m \geq 0} (\text{SCl}(U \cap \bigcup_{n \geq m} f^n(x)))] \subset \bigcap_{m \geq 0} f(\text{SCl}(U \cap (\bigcup_{n \geq m} f^n(x)))).$$

Since $f(\text{SCl}(U \cap \bigcup_{n \geq m} f^n(x)) \subset \text{Cl}(f(U \cap \bigcup_{n \geq m} f^n(x)))$, it follows that

$$\begin{aligned} f(U) &\subset \bigcap_{m \geq 0} \text{Cl}(f(U \cap \bigcup_{n \geq m} f^n(x))) \subset \bigcap_{m \geq 0} \text{Cl}(\bigcup_{n \geq m} f^n(x)) = \omega(x, f) \implies \\ &\implies f(\text{Int}(\omega(x, f))) \subset \omega(x, f). \end{aligned}$$

So far as this author knows, it is an open question whether or not $f(\omega(x, f)) \subset \omega(x, f)$ for f a quasi-continuous function. The question of strong invariance, that is, $f(\omega(x, f)) = \omega(x, f)$ is also open so far as this author knows.

As indicated above, for continuous real-valued functions invariance can be proved using the property that $f(Cl(E)) \subset Cl(f(E))$ for any subset E . Of course, for noncontinuous functions, such is not possible. In fact, as the next result shows, for a nowhere dense set N , if $f(Cl(N)) \subset Cl(f(N))$, then f is continuous.

Theorem 24 *Let $f : \mathbf{R} \rightarrow \mathbf{R}$ be a real-valued function. Suppose that for any nowhere dense subset N of X , $f(Cl(N)) \subset Cl(f(N))$. Then f is continuous.*

Proof. Assume that f is not continuous at some point x in X . Then there exists an open set V containing $f(x)$ such that for every open set U containing x , there exists a point z in U such that $f(z) \notin V$. Choose a positive integer n such that $\frac{1}{n} < |x - z|$. It is easy to see that we can construct a sequence $\{x_n\}_{n=1}^{\infty}$ converging to x , such that the set $N = \{x_n : n \geq 1\}$ is nowhere dense in X . Since there is no subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ such that $\{f(x_{n_k})\}_{k=1}^{\infty}$ converges to $f(x)$, then $f(Cl(N)) \not\subset Cl(f(N))$, a contradiction. Hence, f is a continuous function. ■

CHAPTER 4
QUASICONTINUOUS FUNCTIONS, SEMI-CLOSED SETS, AND
APPLICATIONS

In Chapter 3 above we investigated the dynamics of quasi-continuous functions and bilaterally quasi-continuous functions. In this chapter and the next we now investigate these and related notions in a more general context. The field of continuity and generalized continuity has attracted a great deal of interest in the latter half of the twentieth century, in particular in the fields of analysis and general topology. Our results emphasize applications of generalized continuity to various areas of mathematics.

The concept of quasi-continuity was introduced by Kempisty in [Kem] in 1932. There is now a large mathematical literature on the subject. For example, see the survey paper by Neubrunn at [Neu]. For additional detail we refer the reader to the beginning of Chapter 5 below on separate continuity. In 1961 the notions of semi-open sets in topological spaces was introduced by Levine. [L] As is well known, quasi-continuity can be defined very naturally in terms of semi-open sets. Semi-closed sets were also introduced by Levine, but the thorough development of the concept of semi-closure is given in [CH2].

4.1 Application of Quasicontinuity to Functions Determined by Dense Sets

As is well known, if $f : X \rightarrow Y$ and $g : X \rightarrow Y$ are continuous functions on a space X into a Hausdorff space Y , then the set $D = \{x \in X : f(x) = g(x)\}$ is closed in X . Then if D is dense in X , f and g must agree on all of X . In this section we generalize this notion in various ways. In this section we are concerned with collections or families of functions which are uniquely determined by their values on dense subsets of their domain.

In addition to the family of continuous functions, several authors have shown that real-valued separately continuous functions are also determined by dense sets. Separately continuous functions are functions which are continuous in each variable separately, but not necessarily continuous as a function of two or more variables. For more information on separate continuity we refer the reader to Chapter 5 below. Sierpinski [Sie] and Tolstoff [Tol2] proved that separately continuous functions of the form $f : \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R}$ are determined by their values on dense sets. Goffman and Neugebauer [GN2] proved the case for separately continuous real-valued functions on \mathbf{R}^n . In proposition 27 we prove a more general theorem on separately continuous functions and determination on dense sets. In fact, the primary result of this section presents a larger class of functions which are determined by dense sets, a class which includes separately continuous functions. In a preliminary result we show that, in general, if X is a topological space, and D is a dense semi-closed subset of X , then D is the whole space X .

Applying this result, we also obtain a more general test to find new families of functions which are determined by dense sets.

We recall that a set A in a topological space X is *semi-open* if there exists an open set G such that $G \subset A \subset Cl(G)$, and a set B in X is *semi-closed* if its complement is semi-open. A well known property of quasi-continuous functions is that inverse images of open sets are semi-open and inverse images of closed sets are semi-closed. [P]. Following [CH2], we say that the *semi-closure* of a set S in a topological space X , denoted by $SCl(S)$, is the intersection of all semi-closed sets containing S . Then the set S is semi-closed iff $S = SCl(S)$.

4.2 Families of Functions Determined by Dense Sets

Lemma 25 *Let X be a topological space. Let D be a dense subset of X . If D is semi-closed in X , then $D = X$.*

Proof. Since D is semi-closed, the complement $(X \setminus D)$ is semi-open. Then there exists an open set G in X such that $G \subset (X \setminus D) \subset Cl(G)$. Since G is open and D is dense in X , then if G is not empty, $G \cap D$ is not empty. But this is impossible. So, G must be the empty set. Hence, $\emptyset \subset (X \setminus D) \subset Cl(\emptyset) = \emptyset$ and $D = X$. ■

We provide an additional proof of Lemma 25 in order to illustrate the power of the results set forth in [CH2]. Theorem 0.5 of [CH2] states that a set $D \subset X$, where X is a topological space, is semi-closed iff $Int[Cl(D)] \subset D$. Let $D \subset X$

be dense and semi-closed in X . Then

$$\text{Int}[Cl(D)] \subset D \implies \text{Int}(X) \subset D \implies X \subset D \implies X = D.$$

The result in Lemma 25 now follows.

From Lemma 25 we immediately have the following:

Proposition 26 *Let S be a set of real-valued functions of the form $f : X \rightarrow \mathbf{R}$, where X is a topological space. Suppose that for any functions f and g in S , the set $A = \{x \in X : f(x) = g(x)\}$ is semi-closed in X . Then the functions in S are determined by their values on dense sets.*

Proposition 27 *Let $f : X \times Y \rightarrow \mathbf{R}$ and $g : X \times Y \rightarrow \mathbf{R}$ be separately continuous real-valued functions, where X and Y are Hausdorff topological spaces. If $f((x, y)) = g((x, y))$ for all points (x, y) in some dense subset $D \subset X \times Y$, then $f = g$ everywhere on $X \times Y$.*

Proof. Let $D \subset X \times Y$ be a dense subset of $X \times Y$, and suppose that $f((x, y)) = g((x, y))$ for all $(x, y) \in D$. Since the set of real-valued separately continuous functions is closed under subtraction, we may define the function $h : X \times Y \rightarrow \mathbf{R}$ so that $h((x, y)) = f((x, y)) - g((x, y))$ for all (x, y) in $X \times Y$. It follows that $h((x, y)) = 0$ for all $(x, y) \in D$. Since h is separately continuous, and hence quasi-continuous [P], then $h^{-1}(\{0\})$ is semi-closed in $X \times Y$ because the set $\{0\}$ is closed in \mathbf{R} . But $D = h^{-1}(\{0\})$. ■

Quasi-continuous functions are not determined by dense sets. [D2] Also, bilaterally quasi-continuous functions are not determined by dense sets. [D]. For example, consider the function $f : [-1, 1] \rightarrow \mathbf{R}$, defined as follows:

$$f(x) = \begin{cases} \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases},$$

If we now define a function $g : [-1, 1] \rightarrow \mathbf{R}$ which differs from the function f above only at the point zero, say $g(0) = 1$, then f and g are bilaterally quasi-continuous and are identical on the dense set $[-1, 1] \setminus \{0\}$, but not identical everywhere.

It is known [N] that approximately continuous functions are not determined by dense sets. See [GW] for a discussion of approximate continuity. We note that separately continuous functions are not approximately continuous. For example, consider the separately continuous function $f : \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R}$, defined by:

$$f((x, y)) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

To see this, let V be the open interval $(-\frac{1}{4}, \frac{1}{4})$. Then the point $(0, 0)$ is not a point of density one with respect to the set $f^{-1}(V)$.

In addition to separately continuous functions, the class of functions described in Proposition 26 above contains other sets of functions. For example, let S be the set of functions of the form $f : [0, 1] \rightarrow \mathbf{R}$, where for each f in S , there is a finite subdivision, $0 = a_1 < a_2 < \dots < a_n = 1$, of the interval $[0, 1]$, and there are real numbers $k_i, 1 \leq i \leq n - 1$, such that $f(x) = k_i$ for $a_i \leq x < a_{i+1}$,

and $f(x) = k_{n-1}$, for $x = 1$. Then since the functions in S are quasi-continuous and are closed under subtraction, by Proposition 26 they are determined by their values on dense sets. The reader will readily find similar examples of sets of functions of the form $f : \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R}$, where the functions are determined by dense sets but are not separately continuous. For the sake of completeness, we remark that C. Neugebauer [N] has shown that approximately differentiable real-valued functions on the unit interval are determined by their values on dense sets. We refer the reader to [N] for the necessary definitions.

4.3 Application of Semi-closed Sets to the Uniform Boundedness Theorem

The main result in this section deals with the uniform boundedness theorem for real-valued functions on a Baire space. We recall that a space X is said to be a *Baire space* if the complement of any countable family of closed nowhere dense subsets of X , is dense in X . We recall that a function $f : X \rightarrow Y$, where X and Y are topological spaces, is called *quasi-continuous* if the inverse image of every open (closed) set in Y is semi-open (semi-closed) in X . Clearly, continuous functions are quasi-continuous but not conversely. Arbitrary intersections of semi-closed sets are semi-closed, and a set A is semi-closed iff $\text{Int}[Cl(A)] \subset A$. [CH] The empty set is both semi-open and semi-closed. Throughout this section, X and Y will denote arbitrary topological spaces with no particular properties unless otherwise indicated. The complement of a set S will be denoted by S^C , and the closed interval $[0, 1]$ of the real line will be represented by the symbol I .

It is well known, and indeed trivial, that nowhere dense sets are not necessarily closed. However, nowhere dense sets are always semi-closed. Theorem 1.3 of [CH] states that if $E \subset X$ is a nowhere dense subset of a space X , then E is semi-closed in X . To see this, observe that if E is nowhere dense, then $\text{Int}(Cl(E)) = \emptyset$, which implies that $\text{Int}(Cl(E)) \subset E$, which implies that E is semi-closed.

Theorem 1.3 of [CH] provides a link between the notions of Baire category and measure. Theorem 1.6 of [O] says that the real line can be decomposed into two complementary sets A and B such that A is of first category and B is of measure zero. Consider the closed interval I . As is well known, we can construct a compact nowhere dense set of positive measure in I analogous to the Cantor ternary set. In fact, by choosing the length of the complementary intervals as small as we like, we can make the measure of this nowhere dense set as close to one as we like. However, it is easy to see that the measure of this nowhere dense set cannot equal one. Hence, Theorem 1.6 of [O] does not apply to nowhere dense sets.

Proposition 28 *Let $E \subset I$ be nowhere dense in I . Then E cannot have measure one.*

Proof. E^C is nonempty and semi-open. Hence, E^C contains an interval of positive measure. ■

The following result also follows from Theorem 1.3 of [CH]:

Proposition 29 *Let S be a subset of a topological space X , where S is the complement of the countable union of nowhere dense subsets of X . Then S is the countable intersection of dense semi-open subsets of X .*

In Proposition 30 we are now able to give a new proof of Theorem 9.2 of [O]. In the proof we apply Theorem 7 of [L] which provides that every semi-open subset of a space X is the union of two sets G and B , where G is open, B is nowhere dense, and $G \cap B = \emptyset$.

Proposition 30 *In a Baire space X , a set E is residual iff E contains a dense G_δ subset of X .*

Proof. For the necessity, let E be residual in X . Then E contains the complement of the countable union $\cup_{i=1}^{\infty} N_i$ of nowhere dense subsets of X . It follows, by Theorem 1.3 of [CH], that each N_i is semi-closed, and thus $E = \cap_{i=1}^{\infty} (X \setminus N_i)$, where each set $(X \setminus N_i)$ is semi-open. By Theorem 7 of [L], for each i , $X \setminus N_i = G_i \cup B_i$, where G_i is open, $G_i \cap B_i = \emptyset$, and B_i is nowhere dense. Then each G_i is open and dense in X . Hence, $E = \cap_{i=1}^{\infty} (X \setminus N_i) \supset \cap_{i=1}^{\infty} G_i$, and E contains a dense G_δ set. The sufficiency is immediate. ■

The following lemma is motivated by the often-used fact that a closed subset of a topological space which is not nowhere dense, has a nonempty interior.

Proposition 31 *Let E be a semi-closed subset of a topological space X . If E is not nowhere dense in X , then $\text{Int}(E)$ is not empty.*

Proof. Since E is semi-closed, $\text{Int}[Cl(E)] \subset E$. Since E is not nowhere dense, $\text{Int}[Cl(E)]$ is not empty. Hence, $\text{Int}(E)$ is not empty. ■

In applications of the Baire category theorem, the fact that a set is closed and not nowhere dense implies that the set has a nonempty interior. Proposition 31 above permits the same conclusion when the set is semi-closed. We illustrate this fact in the proof of the next result, which follows the classic proof of the uniform boundedness theorem.

Theorem 32 *Let F be a family of real-valued quasi-continuous functions on a Baire space X . Suppose that for each x in X , there is a real number $M_x \geq 0$ such that $|f(x)| \leq M_x$ for all f in F . Then there is a nonempty open set G in X and a constant $M \geq 0$ such that $|f(x)| \leq M$ for all x in G and for all f in F .*

Proof. For each positive integer m and each $f \in F$, let $E_{m,f} = \{x \in X : |f(x)| \leq m\}$ and let $E_m = \bigcap_{f \in F} E_{m,f}$. Then $E_{m,f}$ is semi-closed for all m and for all f in F . Also, E_m is semi-closed for every m . Now $X = \bigcup_{m=1}^{\infty} E_m$, and since X is a Baire space, there is a set $E_{m'}$ which is not nowhere dense. By Proposition 31, since $E_{m'}$ is semi-closed and not nowhere dense, it contains a nonempty open set G . ■

There are conditions on a function which are strictly stronger than quasi-continuity but strictly weaker than continuity for which the conclusion of Theorem 32 above would also hold. An example is α -continuity. See [RV].

4.4 Applications of the Generalized Uniform Boundedness Theorem to
Riemann Integrable Derivatives and Derivatives which are Differentiable
Almost Everywhere

In [Mar] S. Marcus showed that Riemann integrable derivatives of the form $f : [a, b] \rightarrow \mathbf{R}$ are quasi-continuous. Using Marcus's result and applying Theorem 32 above, we have:

Theorem 33 *Let F be a family of Riemann integrable derivatives of the form $f : [a, b] \rightarrow \mathbf{R}$. Suppose that for each x in $[a, b]$, there is a real number $M_x \geq 0$ such that $|f(x)| \leq M_x$ for all f in F . Then there is a nonempty open interval $I \subset [a, b]$ on which F is uniformly bounded.*

We can now present an elementary application of Theorem 33. As is well known, in the context of the convergence theorem for Riemann integrals, uniform convergence can be replaced by pointwise convergence if it is assumed that the limit function f is Riemann integrable. Hence, by application of Theorem 33, we have:

Corollary 34 *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of Riemann integrable derivatives of the form $f_n : [a, b] \rightarrow \mathbf{R}$. Suppose that $\{f_n\}_{n=1}^{\infty}$ converges pointwise to a Riemann integrable derivative $f : [a, b] \rightarrow \mathbf{R}$. Then there exists a nonempty closed interval $I \subset [a, b]$ such that $\int_I f_n \rightarrow \int_I f$.*

Proof. By Theorem 33 there is an open interval $J \subset [a, b]$ on which the sequence $\{f_n\}_{n=1}^{\infty}$ is bounded. Then there is a closed interval $I \subset J \subset [a, b]$ on which $\{f_n\}_{n=1}^{\infty}$ is bounded. Since each f_n is Riemann integrable and thus Lebesgue integrable, by the Bounded Convergence Theorem, the result follows.

■

We note that although Riemann integrable derivatives are bounded, there is no uniform bound. Therefore, the result in Theorem 33 is needed in the proof above.

Theorem 32 may be thought of as a nonlinear version of the uniform boundedness theorem. For a linear functional on a topological vector space there is no essential difference between continuity and quasi-continuity. We direct the interested reader to [Ham], where the following result is presented: Let h be a linear functional on a topological vector space X . Assume that $h(x) \neq 0$ for some x in X . Then the following are equivalent:

- h is continuous
- h is quasi-continuous
- the null space $N(h)$ is semi-closed
- $N(h)$ is not dense in X ; and
- h is bounded on some semi-neighborhood of 0, where a *semi-neighborhood* of a point x is simply a semi-open set containing x .

Corollary 35 *Let F be a family of real-valued functions which are differentiable*

almost everywhere on the closed interval $[a, b]$. Suppose that for each x in $[a, b]$, there is a real number $M_x \geq 0$ such that $|f(x)| \leq M_x$ for all f in F . Then there is a nonempty open interval $I \subset [a, b]$ on which F is uniformly bounded.

CHAPTER 5

SEPARATE CONTINUITY, RELATED NOTIONS AND APPLICATIONS

It is well known that separately continuous functions are not necessarily continuous at all points of the domain. That is, separately continuous functions are continuous in each variable separately but are not necessarily continuous as a function of all variables. Hence, separate continuity is a generalization of continuity. The detailed study of separately continuous functions was introduced by R. Baire in [B]. One of the properties of separately continuous functions is that they are quasi-continuous. This property was first observed by Volterra. See [B] at page 95. Another important property of separately continuous functions, which is shown in [B], is that all such functions of the form $f : \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R}$ are of the first class of Baire, and hence are continuous except on a set of first category.

Throughout this chapter a function f from a space X into a space Y will be denoted by $f : X \rightarrow Y$ where X and Y have no additional properties unless otherwise indicated, and where the function is not necessarily continuous unless so indicated. We now need a notation for separately continuous functions. Let X, Y , and Z denote arbitrary topological spaces. Consider a function $f : X \times Y \rightarrow Z$, where $X \times Y$ is the usual product of the spaces X and Y . For a given point (x_0, y_0) in the domain of f , the function $f_{x_0}(y)$, given by $f_{x_0}(y) = f(x_0, y)$, is called an x -section of f ; and the function $f^{y_0}(x)$, given by $f^{y_0}(x) = f(x, y_0)$, is

called a y -section of f . If all x -sections and all y -sections of f are continuous, we say that f is *separately continuous*. In the context of separate continuity, ordinary continuity is sometimes called “joint continuity” to emphasize that the function is continuous as a function of all of its variables.

5.1 An Invariance Property of Separately Continuous Functions

The function $f : X \rightarrow Y$ is *connected* if the image of every connected subset of X is a connected subset of Y . In this chapter we present an invariance theorem for separately continuous functions. In order to do so, we now formulate a notion which we call “O-connectedness.” We say that a function $f : X \rightarrow Y$ is *O-connected* if the image of every connected open subset of X is a connected subset of Y . O-connectedness is strictly weaker than connectedness (that is, the Darboux property) of a function, as shown by the following example: Let $f : [-1, 1] \rightarrow \mathbf{R}$ be the real-valued function defined by:

$$f(x) = \begin{cases} \sin \frac{1}{x} & \text{if } 0 < x \\ 1 & \text{if } x = 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Then the image $f([-1, 0])$ is not connected, but the image under f of every open interval in $[-1, 1]$ is connected. In short, the condition that a real-valued function f takes open intervals into connected sets does not imply that f has the Darboux property.

The following elementary example shows that separately continuous functions are not connected functions: Define $f : \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R}$, as follows:

$$f((x, y)) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

To see this, let $E = \{(x, y) : x \geq 0, y \geq 0 \text{ and } \frac{1}{3}x \leq y \leq 3x\}$. Then the image of E is not a connected set.

We now show that real-valued separately continuous functions map the product of two connected sets onto connected sets. First, we present a preliminary lemma which is quite similar to Theorem 3.5 of [PL]. Since the proof of Theorem 3.5 is omitted in [PL], we provide a proof of Lemma 36 for the convenience of the reader.

Lemma 36 *Let $f : X \times Y \rightarrow Z$ be a separately continuous function, where X , Y , and Z are topological spaces. Let $A \subset X$ and $B \subset Y$ be connected sets in the topologies on X and Y respectively. Then $f(A \times B)$ is a connected subset of Z .*

Proof. Assume, to the contrary, that $f(A \times B) = Z_1 \cup Z_2$ is a separation. (Hence, if Z^* is a connected set in Z such that $Z^* \subset f(A \times B)$, then either $Z^* \subset Z_1$ or $Z^* \subset Z_2$.) Take points $(a_1, b_1), (a_2, b_2) \in A \times B$ such that $f(a_1, b_1) \in Z_1$ and $f(a_2, b_2) \in Z_2$. The sets $H = A \times \{b_1\}$ and $V = \{a_2\} \times B$ are connected in $X \times Y$ and the restriction of f to each of these two sets is continuous. Hence, $f(H), f(V)$ are connected in Z . Since $f(H) \ni f((a_1, b_1))$, we have that $f(H) \subset Z_1$. Similarly, $f(V) \subset Z_2$. But the point (a_2, b_1) belongs to both H and V . Therefore, $f(a_2, b_1) \in Z_1 \cap Z_2 = \emptyset$, a contradiction. ■



Before presenting the next result, we recall that if O is an open cover of a connected set S in a space X , then any two points a and b of S can be connected by a simple chain consisting of elements of O . (See, for example, Theorem 26.15 of [W] the proof of which is readily adapted to the subspace topology.) The next theorem presents an invariance property for separately continuous functions.

Theorem 37 *Let $f : X \times Y \rightarrow Z$ be a separately continuous function, where X and Y are locally connected spaces and Z is a space. Then f is O -connected.*

Proof. Assume, to the contrary, that f is not O -connected. Then there is a connected open subset G of $X \times Y$ such that $f(G)$ is not connected in Z . That is, there is a separation $Z_1|Z_2$ of Z such that $f(G) = Z_1|Z_2$. Let (x_1, y_1) and (x_*, y_*) be points in G such that $f(x_1, y_1) \in Z_1$ and $f(x_*, y_*) \in Z_2$. Since G is connected, $G = \cup_{\lambda \in \Lambda} (U_\lambda \times V_\lambda)$, where each $U_\lambda \subset X$ is a connected basis element and each $V_\lambda \subset Y$ is a connected basis element. Since $\{U_\lambda \times V_\lambda\}_{\lambda \in \Lambda}$ is an open cover of G , then the points (x_1, y_1) and (x_*, y_*) can be joined by a simple chain of basis elements $[U_{\lambda_1} \times V_{\lambda_1}, U_{\lambda_2} \times V_{\lambda_2}, \dots, U_{\lambda_n} \times V_{\lambda_n}]$ such that $(x_1, y_1) \in (U_{\lambda_1} \times V_{\lambda_1})$ only and $(x_*, y_*) \in (U_{\lambda_n} \times V_{\lambda_n})$ only and any two successive sets $(U_{\lambda_i} \times V_{\lambda_i})$ and $(U_{\lambda_{i+1}} \times V_{\lambda_{i+1}})$ have at least one common point. Let $U = \cup_{i=1}^n U_{\lambda_i}$ and let $V = \cup_{i=1}^n V_{\lambda_i}$. Then U and V are connected, and hence $U \times V$ is connected. Observe that $Z_1|Z_2$ must be a separation for $f(U \times V)$ because $f(x_1, y_1) \in Z_1$ and $f(x_*, y_*) \in Z_2$. The sets $A = U \times \{y_1\}$ and $B = \{x_*\} \times V$ are connected in $X \times Y$ and the restriction of f to each of these two sets is continuous. Hence,

$f(A)$ and $f(B)$ are connected subsets of Z . Since $f(A) \ni f((x_1, y_1))$, it follows that $f(A) \subset Z_1$. Similarly, $f(B) \subset Z_2$. But the point (x_*, y_1) belongs to both A and B . Therefore, $f((x_*, y_1)) \in Z_1 \cap Z_2 = \emptyset$, which is a contradiction. ■

5.2 Cluster Sets and Connectedness Properties

For a function $f : X \rightarrow Y$ where X and Y are first-countable spaces, we say that the *cluster set of f at $x \in X$* , denoted by $C(f; x)$, is the set of all y in Y such that there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in X converging to x and $\{f(x_n)\}_{n=1}^{\infty}$ converges to y . It is easy to show that the set $C(f; x)$ is always closed. Also, $C(f; x)$ is never empty, since $f(x)$ is always an element of $C(f; x)$. In [PL], W. Pervin and N. Levine showed that for a connected function $f : X \rightarrow Y$, where X is first countable and locally connected, and Y is first countable and compact Hausdorff, the cluster set $C(f; x)$ is connected for every $x \in X$. Only slight modifications of the proof of Pervin and Levine are needed to prove the next result.

Theorem 38 *Let X be a locally connected and first countable space, and let Y be a compact Hausdorff and first countable space. Suppose that $f : X \rightarrow Y$ is O -connected. Then for any x in X , $C(f; x)$ is a connected subset of Y .*

Corollary 39 *Let $f : \mathbf{R} \times \mathbf{R} \rightarrow I$ be a separately continuous function from the real plane into a closed interval I . Then for any point (x, y) in the domain of f , the cluster set of f at (x, y) is connected.*

Proof. Apply Theorem 37 and Theorem 38. ■

In Corollary 39 the cluster set is degenerate at points of joint continuity. We also remark that the converse of Corollary 39 is not true, as illustrated by the following function of the form $f : \mathbf{R} \times \mathbf{R} \rightarrow [-1, 1]$:

$$f((x, y)) = \begin{cases} \sin((x^2 + y^2))^{-1} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

Now, by application of Theorem 37 and Corollary 39 above, we obtain the following:

Theorem 40 *Let $f : \mathbf{R} \times \mathbf{R} \rightarrow I$ be a separately continuous function from the real plane into a closed interval I . Let (x_0, y_0) be any point in $\mathbf{R} \times \mathbf{R}$. Then in any open set containing (x_0, y_0) , f takes on every value in $C(f; (x_0, y_0))$, except possibly the end points if $C(f; (x_0, y_0))$ is an interval.*

Proof. If $C(f; (x_0, y_0)) = \{f((x_0, y_0))\}$, we are done. If $C(f; (x_0, y_0))$ is a closed interval $[a, b]$, then any open set containing (x_0, y_0) contains the tail of a sequence $\{(x_n, y_n)\}_{n=1}^{\infty}$ such that the sequence $\{f((x_n, y_n))\}_{n=1}^{\infty}$ converges to a . A similar sequence converges to b . Now apply Theorem 37. ■

5.3 Separate Continuity, The Closed Graph Property, and Related Notions

A function $f : X \rightarrow Y$ has a *closed graph* if the graph of f , denoted by $G(f) = \{(x, f(x)) : x \in X\}$, is a closed subset of the product space $X \times Y$. Certain families of functions which have a closed graph, are said to have the

closed graph property. As is well known, if f is a continuous function, then f has the closed graph property if Y is Hausdorff. That is, if X and Y are topological spaces, Y is Hausdorff, and $f : X \rightarrow Y$ is continuous, then $G(f)$ is closed in $X \times Y$. Of course, a function can have the closed graph property and not be continuous. For example, consider the following function of the form $f : \mathbf{R} \rightarrow \mathbf{R}$:

$$f(x) = \begin{cases} \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

In this section we turn our attention to functions which have the closed graph property but are not necessarily continuous. If a function has the closed graph property and is not continuous, an important problem is to determine what additional conditions on the function are necessary to make the function continuous. The monograph by Hamlett and Herrington [HH] is devoted to this subject, as well as numerous papers published during the past two decades.

We are particularly interested in the role played by the closed graph property with respect to separately continuous functions. In this section we show that with suitable restrictions on the range space of a function, the property of “local w^* -continuity”, which was first introduced in [R] by David Rose, is a generalization of the closed graph property. A strictly weaker condition is also presented which is the notion of “local relative continuity”. That is, local w^* -continuity is strictly weaker than the closed graph property, and local relative continuity is strictly weaker than local w^* -continuity. As an example of our application of

these results, Theorem 44 below gives the result that a function which is both separately continuous and locally w^* -continuous, is continuous. Then Corollary 46 immediately follows, giving the previously known result that separately continuous functions with the closed graph property are continuous. In Theorem 55 and Corollary 56 below we present similar results for local relative continuity.

Propositions 41 and 42 are the first results obtained by the author in the investigation of separate continuity and the closed graph property. They are included here because the proofs contain the seeds of a number of the results in this section and illustrate some of the techniques applied in this chapter.

A well-known condition equivalent to the closed graph property for a function $f : X \rightarrow Y$, is that whenever the net $\{x_\lambda\}_{\lambda \in \Lambda}$ converges to x , then $\{f(x_\lambda)\}_{\lambda \in \Lambda}$ converges to y , where $y = f(x)$. [HH] Obviously, if Y is Hausdorff, then this condition is equivalent to continuity.

We define a space X to be *Bolzano-Weierstrass* if every infinite subset of X has at least one limit point. Clearly, every compact space is a Bolzano-Weierstrass space, but the converse is not true.

Proposition 41 *Let $f : X \times Y \rightarrow Z$ be a function, where X and Y are first-countable Hausdorff spaces and Z is a Bolzano-Weierstrass space. If $G(f)$ is closed in $X \times Y \times Z$, then f is continuous.*

Proof. Assume, to the contrary, that f is not continuous. Then there exists a point $(x, y) \in X \times Y$ such that $\{(x_n, y_n)\}_{n=1}^\infty$ converges to (x, y) , but

$\{f((x_n, y_n))\}_{n=1}^{\infty}$ does not converge to $f((x, y))$. Let V be any open subset of Z such that V contains $f((x, y))$. Since $\{f((x_n, y_n))\}_{n=1}^{\infty}$ does not converge to $f((x, y))$, there is an infinite set A consisting entirely of points of $\{f((x_n, y_n))\}_{n=1}^{\infty}$ such that $V \cap A = \emptyset$. Since Z is Bolzano-Weierstrass, the set A has at least one limit point; call it z . It follows that there is a sequence of elements of A converging to z , and that sequence can be represented as the subsequence $\{f((x_{n_i}, y_{n_i}))\}_{i=1}^{\infty}$. Since all subsequences of a convergent sequence of real numbers converge to the same limit as the main sequence, then $\{(x_{n_i}, y_{n_i})\}_{i=1}^{\infty}$ converges to (x, y) . But clearly, $z \neq f((x, y))$, and this is a contradiction of the closed graph property. Hence, f is continuous. ■

Proposition 41 is a variation on Theorem XI.2.7 of [Dug] which states that $f : X \rightarrow Y$, where X is Hausdorff and Y is compact, is continuous iff f has the closed graph property. In Theorem 61 and Corollary 62 below, we present results similar to Proposition 41, but for locally bounded functions.

We are now interested in finding conditions on a function which constitute generalizations of the closed graph property.

We say that a function $f : X \rightarrow Y$ is *w*-continuous* if for any open set V in Y , $f^{-1}[Fr(V)]$ is closed in X , where Fr denotes the frontier operator. [L2] A function $f : X \rightarrow Y$ is *locally w*-continuous* if there is an open basis \mathbf{B} for the topology on Y such that $f^{-1}[Fr(V)]$ is closed in X for each V in \mathbf{B} .

Proposition 42 *Let X and Y be locally connected spaces and let \mathbf{R} denote the*

real numbers. Then the function $f : X \times Y \rightarrow \mathbf{R}$ is continuous iff f is separately continuous and w^* -continuous.

Proof. Necessity is clear. Sufficiency: Let (x_0, y_0) be a point in the domain of f . Let W be an open set containing the point $f(x_0, y_0)$. Then there is an open interval $(a, b) \subset W$ such that $f(x_0, y_0) \in (a, b)$. The frontier of the interval (a, b) is the set $\{a, b\}$, which does not contain the point $f(x_0, y_0)$. Since f is w^* -continuous, the inverse image of the frontier of the interval (a, b) is closed, and thus its complement, which we denote by G , is open. Since G is open and contains (x_0, y_0) , and since X and Y are locally connected, we can find open connected sets $U \subset X$ and $V \subset Y$ such that $(x_0, y_0) \in U \times V \subset G$. Also, since all of the x -sections and y -sections of f are continuous, we can find open connected sets U_0 and V_0 such that $x_0 \in U_0 \subset U$ and $y_0 \in V_0 \subset V$ such that for all x in U_0 , $f^{y_0}(x) = f(x, y_0)$ is contained in (a, b) ; and such that for all y in V_0 , $f_{x_0}(y) = f(x_0, y)$ is contained in (a, b) . The claim is that for all (x, y) in the open set $U_0 \times V_0$, $f(x, y) \in (a, b) \subset W$. Assume, to the contrary, that there exists a point $(x_1, y_1) \in U_0 \times V_0$ such that $f(x_1, y_1) \notin (a, b)$. We may assume, without loss of generality, that $f(x_1, y_1) > b$. Now since U_0 is connected and $f^{y_1}(x)$ is a continuous real-valued function, and since $f(x_1, y_1) > b$, and $f(x_0, y_1) < b$, then there is a point $x_2 \in U_0$ such that $f(x_2, y_1) = b$. But this is impossible, because the set G contains no points of the set $f^{-1}(\{a, b\})$. Hence, f is continuous. ■

An example taken from [PW] shows that the condition of local connectedness

is essential in Theorem 42 above: Let $X = Y = [0, 1] \setminus \{\frac{1}{n} : n \text{ is a natural number}\}$ with the usual topology. Observe that X is not locally connected since 0 does not have a connected neighborhood. Define the function $f : X \times Y \rightarrow R$ so that $f(x, y) = n$, if $x, y \in (\frac{1}{n+1}, \frac{1}{n})$ for some n ; and $f(x, y) = 0$, otherwise. Then f is separately continuous and has a closed graph, but is discontinuous at the point $(0, 0)$.

The next result shows that local w^* -continuity is a strict generalization of the closed graph property. For the proof, we shall need the result in Theorem I.4.3 of [HH], which says that for any function $f : X \rightarrow Y$, if f has a closed graph, then for any open set $V \subset Y$ such that $(Y \setminus V)$ is compact, $f^{-1}(V)$ is open in X (a condition known as *c-continuity*.) Of course, Theorem 43 below could be proved for Y locally compact. However, we prove it for the more general condition of “rim-compactness.” A topological space Y is *rim-compact* if there is an open basis \mathbf{B} for the topology on Y such that $Fr(V)$ is compact for each V in \mathbf{B} . A locally compact Hausdorff space is rim-compact. See [R].

Theorem 43 *Let $f : X \rightarrow Y$ be a function, where Y is a rim-compact space. If f has the closed graph property, then f is locally w^* -continuous.*

Proof. Since Y is rim-compact, there is an open basis \mathbf{B} for the topology on Y such that $Fr(V)$ is compact for each V in \mathbf{B} . Now let V be any element in \mathbf{B} . Since $[Y \setminus Fr(V)]$ is open in Y , and its complement is compact, then by the *c-continuity* of f , $f^{-1}[Y \setminus Fr(V)]$ is open in X . Hence, $f^{-1}[Fr(V)]$ is closed in

X and f is locally w^* -continuous. ■

The converse of Theorem 43 is not true, as shown by the function $f : \mathbf{R} \rightarrow \mathbf{R}$, defined by:

$$f(x) = \begin{cases} x & \text{if } x < 1 \\ x + 1 & \text{if } 1 \leq x \end{cases}$$

Observe that for any open interval J , $f^{-1}(Fr(J))$ is closed. However, f clearly does not have a closed graph.

Theorem 44 (D.A. Rose) *Let X be a locally connected space and let Y and Z be topological spaces. Then the function $f : X \times Y \rightarrow Z$ is continuous iff f is both separately continuous and locally w^* -continuous.*

Proof. The necessity is clear. For the sufficiency, let (x_0, y_0) be any point in the domain of f , and let $W \subset Z$ be an open set containing $f((x_0, y_0))$. Then there exists a basic open set W_0 with $f((x_0, y_0)) \in W_0 \subset W$. Since f is locally w^* -continuous, $f^{-1}[Fr(W_0)]$ is closed. Therefore, the complement of $f^{-1}[Fr(W_0)]$, which we shall call G , is open and contains (x_0, y_0) . Let $Z_1|Z_2$ be a separation for $Z \setminus Fr(W_0)$, where Z_1 contains W_0 and Z_2 contains $Z \setminus Cl(W_0)$. Then all points in G are mapped into either Z_1 or Z_2 . Let $U \subset X$ be a basic open connected set containing x_0 such that $f(U \times \{y_0\}) \subset W_0$. Let $V \subset Y$ be a basic open set containing y_0 such that $f(\{x_0\} \times V) \subset W_0$. The claim now is that for all (x, y) in $(U \times V)$, $f((x, y)) \in W_0 \subset W$. Assume that there exists a point (x_1, y_1) in $U \times V$ such that $f((x_1, y_1)) \notin W_0$. Then $f(x_1, y_1) \in Z_2$. Since $f(\{x_0\} \times V) \subset W_0$, then $f((x_0, y_1)) \in W_0 \subset Z_1$. Since U is connected, $(U \times \{y_1\})$

is also connected, and since $f((x_1, y_1)) \in Z_2$, it follows that $f(x_0, y_1) \in Z_2$. Hence, $f((x_0, y_1)) \in Z_1 \cap Z_2 = \emptyset$, a contradiction. ■

Theorem 45 *Let X be a locally connected space and let Y and Z be topological spaces. Then the function $f : X \times Y \rightarrow Z$ is continuous iff f is both separately continuous and w^* -continuous.*

Proof. Every w^* -continuous function is locally w^* -continuous. ■

Corollary 46 *Let X and Y be topological spaces with X locally connected and let Z be a rim-compact space. Suppose that $f : X \times Y \rightarrow Z$ is separately continuous and has the closed graph property. Then f is continuous.*

We now return to the notion of O -connectedness. The next theorem and its corollary generalize the well known result that a connected function with a closed graph, is continuous.

Theorem 47 *Let X be a locally connected space and let Y be a space. Let $f : X \rightarrow Y$ be locally w^* -continuous. If f is O -connected, then f is continuous.*

Proof. Let $x \in X$ and let $W \subset Y$ be an open set containing $f(x)$. There exists a basic open set $V \subset W$ such that $f(x) \in V \subset W$. By local w^* -continuity, $f^{-1}[Fr(V)]$ is closed in X . Then the complement of $f^{-1}[Fr(V)]$ is open and contains x . The set $Y \setminus Fr(V)$ is a disconnected subset of Y . Let $Y_1|Y_2$ be a separation for $Y \setminus Fr(V)$, where Y_1 contains V and Y_2 contains $Y \setminus Cl(V)$. Since

X is locally connected, there exists a connected open set $U \subset X$ containing x . The claim is that $f(U) \subset V \subset W$. Assume, to the contrary, that there exists $x_0 \in U$ such that $f(x_0) \notin V$. Since f is O -connected, $f(U)$ is connected $\implies f(U)$ is not contained in $V \implies f(x) \notin V \implies f(x) \in Y_2$. But $V \subset Y_1$. Hence, $f(x) \in Y_1 \cap Y_2 = \emptyset$, a contradiction. ■

Theorem 48 *Let $f : X \rightarrow Y$ be a function where X is locally connected and Y is rim-compact. Suppose that f has the closed graph property. Then if f is O -connected, f is continuous.*

Proof. By Theorem 43, the function f is locally w^* -continuous. Now by application of Theorem 47 above, the result follows. ■

In Remark 1 of [PW], the following result is presented: If X is locally connected, Y is locally compact, and $f : X \rightarrow Y$ is a connected mapping with a closed graph, then f is continuous. This result can now be generalized as follows. The proof is an immediate consequence of Theorem 47 and is omitted.

Corollary 49 (D.A. Rose) *Let X and Y be topological spaces, with X locally connected. If a function $f : X \rightarrow Y$ is connected and locally w^* -continuous, then f is continuous.*

Theorem 47 above also enables us to generalize the result which states that a derivative with a closed graph is continuous. See [HH,I.4.12,13]

Corollary 50 *Let $f : \mathbf{R} \rightarrow \mathbf{R}$ be a real-valued function with derivative f' . Then f is continuous iff f' is locally w^* -continuous.*

5.4 Further Generalizations of the Closed Graph Property

“Local relative continuity” is a strict generalization of local w^* -continuity. This means that local relative continuity is also a generalization of the closed graph property with applications to connected functions and separate and joint continuity. We say that a function $f : X \rightarrow Y$ is *locally relatively continuous* if there exists an open basis \mathbf{B} for the topology on Y such that $f^{-1}(V)$ is open in the subspace $f^{-1}[Cl(V)]$ for any $V \in \mathbf{B}$.

Theorem 51 *If $f : X \rightarrow Y$ is locally w^* -continuous, then f is locally relatively continuous.*

Proof. By local w^* -continuity, there is an open basis \mathbf{B} for the topology on Y such that for any $V \in \mathbf{B}$, $f^{-1}[Fr(V)]$ is closed in X . Let V be any set in \mathbf{B} . Then

$$\begin{aligned} V &= Cl(V) \setminus Fr(V), \text{ and } f^{-1}(V) = f^{-1}[Cl(V)] \setminus f^{-1}[Fr(V)] \\ &= f^{-1}[Cl(V)] \cap (X \setminus f^{-1}[Fr(V)]). \end{aligned}$$

Since f is locally w^* -continuous, the set $\{X \setminus f^{-1}[Fr(V)]\}$ is open in X . Hence, $f^{-1}(V)$ is open in the subspace $f^{-1}[Cl(V)]$ ■

An example by Rose in [MR2], shows that local relative continuity is strictly weaker than local w^* -continuity: Consider the function $f : [0, 1] \rightarrow [-2, 2]$ given by:

$$f(x) = \begin{cases} \frac{3}{2} & \text{if } x = 0 \\ \sin \frac{1}{x} & \text{if } x \neq 0 \end{cases}$$

If we choose a basis consisting of open intervals of length at most $\frac{1}{4}$ relativized to $[-2, 2]$, we see that f is locally relatively continuous. However, f is not locally w^* -continuous for any open basis. To see this, consider the open interval $(-\frac{1}{2}, \frac{1}{2})$ in the range of f , and let y be a point in $(-\frac{1}{2}, \frac{1}{2})$. Then there exists an open basis element V , contained in $(-\frac{1}{2}, \frac{1}{2})$, such that $y \in V$. Since V is open, it is a union of open intervals. Let α be the infimum of the left endpoints of these intervals and let β be the supremum of the right endpoints of these intervals. Then α and β belong to the set $Fr(V)$, and we see that $f^{-1}[Fr(V)]$ has zero as a limit point, but zero is not an element of $f^{-1}[Fr(V)]$. It follows that local w^* -continuity implies local relative continuity, but not conversely.

Theorem 52 *Let $f : X \rightarrow Y$ be a function, where Y is a rim-compact space. If f has the closed graph property, then f is locally relatively continuous.*

Proof. Theorem 43 and Theorem 51. ■

Theorem 53 (D.A. Rose) *Let X be a locally connected space and let $f : X \rightarrow Y$ be a connected function. If f is locally relatively continuous, f is continuous.*

Proof. Let \mathbf{B} be an open basis for the topology on Y such that for any $V \in \mathbf{B}$, $f^{-1}(V)$ is open in the subspace $f^{-1}[Cl(V)]$. It is sufficient to show that for any $V \in \mathbf{B}$, $f^{-1}(V)$ is open in X . Assume that there exists $V \in \mathbf{B}$ such that $f^{-1}(V)$

is not open in X . The set $(Y \setminus Fr(V))$ is a disconnected subset of Y . Let $Y_1|Y_2$ be a separation for $(Y \setminus Fr(V))$, so that $V \subset Y_1$ and $(Y \setminus Cl(V)) \subset Y_2$. There is a point $x_0 \in f^{-1}(V)$ such that any open set in X containing x_0 meets $X \setminus f^{-1}(V)$. In the subspace topology for the space $f^{-1}[Cl(V)]$, there is an open basis element G containing x_0 such that $G \subset f^{-1}(V)$. Now $G = H \cap f^{-1}[Cl(V)]$, where H is open in X . Since X is locally connected, there is a connected open subset U of H containing x_0 . Since U is open in X and $x_0 \in U$, $U \cap [X \setminus f^{-1}(V)] \neq \emptyset$. Therefore, there is a point $x_1 \in U$ such that $f(x_1) \notin V \implies f(x_1) \in Y_2$. But $f(x_0) \in V \implies f(x_0) \in Y_1$. Since U is connected and f is a connected function, we see that $f(U) \subset Y_1$ and $f(U) \subset Y_2 \implies f(U) \subset Y_1 \cap Y_2 = \emptyset$, a contradiction.

■

Corollary 54 *Let $f : \mathbf{R} \rightarrow \mathbf{R}$ be a real-valued function with derivative f' . Then f' is continuous iff f' is locally relatively continuous.*

Proof. Since the derivative f' is a connected function, apply Theorem 53 above. ■

We now return to the central theme of this chapter, viz. separate continuity.

Theorem 55 (D.A. Rose) *Let X be a locally connected space, and let Y and Z be topological spaces. Suppose that a function $f : X \times Y \rightarrow Z$ has continuous x -sections and connected y -sections. Then if f is locally relatively continuous, f is continuous.*

Proof. There is an open basis \mathbf{B} for the topology on Z such that for any $B \in \mathbf{B}$, $f^{-1}(B)$ is open in the subspace $f^{-1}[Cl(B)]$. It is sufficient to show that for any $B \in \mathbf{B}$, $f^{-1}(B)$ is open in $X \times Y$. Assume that there exists a B in \mathbf{B} such that $f^{-1}(B)$ is not open in $X \times Y$. Then there is a point $(x_0, y_0) \in f^{-1}(B)$ such that any open set in $X \times Y$ containing (x_0, y_0) meets $(X \times Y) \setminus f^{-1}(B)$. In the subspace topology for $f^{-1}[Cl(B)]$ there exists an open basis element G containing (x_0, y_0) such that $G \subset f^{-1}(B)$. Now $G = G_0 \cap f^{-1}[Cl(B)]$, where G_0 is open in $X \times Y$. It follows that there is an open basis element U for the topology on X , and there is an open basis element V for the topology on Y such that $(x_0, y_0) \in (U \times V) \subset G_0$. Since f is continuous in y for every fixed x , there is an open set V_0 in Y such that $y_0 \in V_0 \subset V$ and for all $y \in V_0$, $f(x_0, y) \in B$. Since X is locally connected, there is a connected open set U_0 in X such that $x_0 \in U_0 \subset U$, and for every x in U_0 , $f(x, y_0) \in B$. Since $U_0 \times V_0$ is open in $X \times Y$ and contains (x_0, y_0) , then $\{(U_0 \times V_0) \setminus f^{-1}(B)\} \neq \emptyset$. That is, we can find a point $(x_1, y_1) \in U_0 \times V_0$ such that $f((x_1, y_1)) \notin Cl(B)$. Now the space $Z \setminus Fr(B)$ is disconnected. Let $Z_1|Z_2$ be a separation for $Z \setminus Fr(B)$, where $B \subset Z_1$ and $Z \setminus Cl(B) \subset Z_2$. Since U_0 is connected, the set $(U_0 \times \{y_1\})$ is also connected. Hence, since $f((x_1, y_1)) \in (Z \setminus Cl(B)) \subset Z_2$, then $f((x_0, y_1)) \in (Z_1 \setminus Cl(B)) \subset Z_2$. But since $f((x_0, y_1))$ is also contained in $B \subset Z_1$, we have $f((x_0, y_1)) \in Z_1 \cap Z_2 = \emptyset$, a contradiction. ■

Corollary 56 *Let X be a locally connected space. Then the function $f : X \times$*

$Y \rightarrow Z$ is continuous iff f is both separately continuous and locally relatively continuous.

5.5 Locally Bounded Functions

In this chapter we investigate a well known property of continuous real-valued functions of the form $f : X \rightarrow \mathbf{R}$, where X is a topological space. We call the property “local boundedness.” In particular, we are interested in the fact that for any $x \in X$, f is bounded on some neighborhood containing x . This property is given as an exercise in [Bo] at page 84. Here we investigate this property in a more general context. We show that certain kinds of noncontinuous functions have this property. In fact, we show that Fuller’s notion of “subcontinuity” [F] is equivalent to local boundedness. We also investigate the properties of locally bounded functions. The classical theorem that a continuous real-valued function on a compact space is bounded, is generalized and shown to be true for the larger class of locally bounded functions.

A function $f : X \rightarrow M$, where X is a space, and M is a metric space, is *locally bounded* if for any $x \in X$, there exists an open set U containing x such that f is bounded on U .

Theorem 57 *If M is a metric space and $f : X \rightarrow M$ is a continuous function, then f is locally bounded.*

Proof. The function f is bounded on the open set $f^{-1}[B(f(x), 1)]$, where $B(f(x), 1)$ denotes an open ball in M of radius 1. ■

The notion of subcontinuity was introduced in [F] and has been further studied in [No2]. A function $f : X \rightarrow Y$, where X and Y are topological spaces, is *subcontinuous* if for any net (x_λ) in X such that (x_λ) converges to p in X , there is a subnet, $(f(x_{\lambda_\mu}))$, of $(f(x_\lambda))$ which converges to some point q in Y .

Theorem 58 *Let $f : X \rightarrow \mathbf{R}$ be a real-valued function on a first countable Hausdorff space X . Then f is locally bounded iff f is subcontinuous.*

Proof. For the necessity, let $\{x_n\}_{n=1}^\infty$ be a sequence converging to a point x_0 in X . Let U be any open set in X containing the point x_0 , and such that f is bounded on U . Then U contains the tail of the sequence $\{x_n\}_{n=1}^\infty$, and since all but finitely many of the x_n are contained in U , the sequence $\{f(x_n)\}_{n=1}^\infty$ is bounded in \mathbf{R} . Hence, $\{f(x_n)\}_{n=1}^\infty$ contains a convergent subsequence, and f is subcontinuous. For the sufficiency, assume that f is not locally bounded. Then there exists a point x_0 in X such that f is unbounded on every open set containing x_0 . Let $\{U_n\}_{n=1}^\infty$ be a neighborhood system for x_0 . Order $\{U_n\}_{n=1}^\infty$ by reverse inclusion so that for natural numbers j and k , $j < k \implies U_k \subset U_j$. Since f is unbounded on every U_n , for each natural number n choose x_n so that $x_n \in U_n$ and $|f(x_n)| > n$. Then no subnet of $\{f(x_n)\}_{n=1}^\infty$ can converge, and f is not subcontinuous. ■

In [MW] Wingler generalized the sufficiency part of the above result to functions of the form $f : X \rightarrow M$, where X is a first countable Hausdorff space and M is a metric space.

We say that a function $f : X \rightarrow Y$ has at worst a removable discontinuity at a point $x \in X$ if there is a point $y \in Y$ such that for each open set V containing y , there is an open set U containing x such that $f(U \setminus \{x\}) \subset V$.

Theorem 59 *Let $f : X \rightarrow M$ be a function, where M is a metric space and X is a topological space. If f has at worst a removable discontinuity at a point x in X , then f is locally bounded at x .*

Proof. If f has at worst a removable discontinuity at the point x , then there exists an open ball $B(y, 1)$ in M and an open set U containing x such that $f(U \setminus \{x\}) \subset B(y, 1)$. Since $B(y, 1)$ is a bounded subset of M , the set $B(y, 1) \cup \{f(x)\}$ is also a bounded subset of M . Hence, f is locally bounded at x . ■

There exist real-valued functions which are not locally bounded at any point of their domain. The following example is given by Wingler in [MW]: Let Q denote the set of rational numbers $\frac{p}{q}$, where p and q are relatively prime natural numbers, and let $f : (0, 1) \rightarrow \mathbf{R}$ be defined as follows:

$$f(x) = \begin{cases} 0 & \text{if } x \notin Q \\ q & \text{if } x = \frac{p}{q} \end{cases}$$

Then f is not locally bounded at any point in $(0, 1)$.

It is evident that since a continuous function is not necessarily bounded on its domain, a locally bounded function need not be bounded on its domain. However, if the domain of a continuous function is compact, then the function is

bounded. As the next result shows, locally bounded functions have this same property.

Theorem 60 *Let X be a compact space, and let $f : X \rightarrow \mathbf{R}$ be a locally bounded function. Then f is bounded on X by a constant M .*

Proof. For any $x \in X$, there exists an open set U_x containing x such that f is bounded on U_x . The collection $\{U_x\}_{x \in X}$ of all such open sets is an open cover for X , and since X is compact, there is a finite subcover $\{U_i\}_{i=1}^n$ covering X . For each i , there exists a real number $M_i \geq 0$ such that $|f(x)| \leq M_i$ for all $x \in U_i$. Let $M = \max_{1 \leq i \leq n} \{M_i\}$. Hence, f is bounded on X by a constant M . ■

Since continuous functions of the form $f : X \rightarrow \mathbf{R}$ are not bounded for X locally compact, it is clear that Theorem 60 does not hold for X locally compact.

Theorem 61 *If $f : X \rightarrow \mathbf{R}$ is locally bounded and has a closed graph, where X is a first countable topological space, then f is continuous.*

Proof. Assume, to the contrary, that f is not continuous. Then there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in X such that $x_n \rightarrow x$ but $f(x_n)$ does not converge to $f(x)$. By local boundedness, there exists an open set U containing x such that f is bounded on U . Then U contains all but finitely many of the points in $\{x_n\}_{n=1}^{\infty}$. We may assume [Case 1] that $\{f(x_n)\}_{n=1}^{\infty}$ contains infinitely many distinct elements. Since $\{f(x_n)\}_{n=1}^{\infty}$ does not converge to $f(x)$, there is an open set V containing $f(x)$

and there is an infinite set A of points of $\{f(x_n)\}_{n=1}^{\infty}$ such that $A \cap V = \emptyset$. Since U contains all but finitely many of the preimages of the points in A , A contains an infinite subset B which is bounded. Then B has at least one limit point; call it y . It follows that B contains a subsequence $\{f(x_{n_i})\}_{i=1}^{\infty}$ which converges to y . Clearly, $y \neq f(x)$. Since $\{x_{n_i}\}_{i=1}^{\infty}$ is a subsequence of $\{x_n\}_{n=1}^{\infty}$, and since $\{x_n\}_{n=1}^{\infty}$ converges to x , then $\{x_{n_i}\}_{i=1}^{\infty}$ also converges to x . But this contradicts the closed graph property, because $\{x_{n_i}\} \rightarrow x$ and $\{f(x_{n_i})\} \rightarrow y$, but $y \neq f(x)$. For [Case 2], if the sequence $\{f(x_n)\}_{n=1}^{\infty}$ becomes constant after some natural number n_0 , or if it oscillates, then the contradiction of the closed graph property is immediate. ■

Corollary 62 *Suppose that the function $f : \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R}$ has a closed graph and that each x -section and each y -section of f is locally bounded. Then f is continuous.*

Proof. By Theorem 61 above, f is separately continuous. Hence, f is continuous by Corollary 46 above. ■

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