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ON β-NORMAL SPACES

by

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Abstract

This thesis revisits some types of normal spaces, namely, β-normal spaces and α-normal spaces, which were introduced by Arkhangel'skii and Ludwig in 2001. We study some properties of these spaces with a focus on improving some of the already existing properties and exploring new properties that are not available in the literature. Under β-normal spaces, among other things, we characterize these spaces using some types of open sets. We use the ultrafilter space to answer Murtinova's question about the existence of a β-normal and regular space which is not Tychonoff. α-normal spaces are described in terms of countable open sets, a result imitating that of normality. It turns out that continuous functions which are onto, open and closed preserve β-normality, while those which are injective, open and closed reflect α-normality. The notion of β-normal spaces is extended to the category of bitopological spaces where we characterize these bitopological spaces simultaneously in terms of i-open sets, (i, j)-preopen and (i, j)-α-open sets. We study the interrelations of these spaces with other bitopological spaces.

Contents

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Psalm 73:26 My flesh and my heart may fail, but God is the strength of my heart and my portion forever.

Declaration

I declare that the work in this thesis entitled *On β-normal spaces* which I hereby submit for the degree of Master of Science in Mathematics at Rhodes University is my own work. I also declare that this research has not been submitted anywhere before and that all the sources that I have used or quoted have been indicated and acknowledged by means of complete references.

	DECEMBER 2024
Thobile Ngcamphalala	Date signed

Basic notations

(X, τ)	Topological space
(X, τ_1, τ_2)	Bitopological space
$\subset$	Subset
$\cap$	Intersection
$\cup$	Union
$-$	Set difference
$\mathrm{int}(A)$	Interior of A in (X, τ)
$\overline{A}$	Closure of A in (X, τ)
$\mathrm{int}_i(A)$	Interior of A in (X, τ_i, τ_j) with respect to τ_i where $i, j \in \{1, 2\}, i \neq j$
$cl_i(A)$	Closure of A in (X, τ_i, τ_j) with respect to τ_i where $i, j \in \{1, 2\}, i \neq j$
τ_d	Discrete topology
τ_s	Semi-regularization

Chapter 1

Introduction

1.1 A brief history of β-normal and α-normal spaces

In the category of topological spaces, separation axioms are known for distinguishing points and subsets in a topological space. The notion of separation properties plays a central role in understanding the structure of topological spaces. These separation properties, encapsulated in axioms such as normality and regularity provide insights into how points and closed sets within a space can be "separated" by open sets. The concept of normality in topological spaces, first formalized in the early 20th century, describes spaces in which disjoint closed sets can be separated by disjoint open sets. This fundamental property has led to the development of various extensions and generalizations, such as almost normal spaces [39], mildly normal [41] (or κ-normal [38]) spaces and p-normal spaces [28]. In 2001, Arkhangel'skii and Ludwig [7] further advanced the theory by introducing the notions of β-normality and α-normality. They called a topological space (X, τ) *β-normal* (resp. *α-normal*) if for any two disjoint closed subsets F and K of X, there are open subsets U and V of X such that $F \cap U$ is dense in F, $K \cap V$ is dense in K and $\overline{U} \cap \overline{V} = \emptyset$ (resp. $U \cap V = \emptyset$). Since their introduction, these notions have been extensively studied by several authors. Relative versions of β-normality and α-normality were studied in [10] and one of the most recent works on β-normal spaces was published in 2023 by Singh and Rana [40] where the authors investigate a weaker variant of normality called $\beta\kappa$-normality which is a generalization of normality, κ-normality and almost β-normality.

Another important area of research in modern topology involves the study of bitopological spaces. While Kelly [18] was researching distance functions that do not have to be symmetric, he discovered that a set X can be endowed with two topologies τ_1 and τ_2. He called the triple (X, τ_1, τ_2) a *bitopological space*. Additionally, he introduced the idea of separation axioms in bitopological spaces where he defined, among others, pairwise regularity and pairwise normality. Reilly [35] discussed separation axioms of bitopological spaces in detail in his doctoral dissertation.

Arkhangel'skii and Ludwig [7] characterized β-normal spaces as follows: A topological space (X, τ) is β-normal if and only if for any disjoint closed sets F, K of X, there exist open sets U, V of X such that $\overline{F \cap U} = F$, $\overline{K \cap V} = K$ and $\overline{U} \cap \overline{V} = \emptyset$ if and only if for each closed set $A \subset X$ and each open set $U \subset X$ such that $A \subset U$, there exists an open set $V \subset X$ such that

$$\overline{A \cap V} = A \subset \overline{V} \subset U.$$

This thesis aims to extend the initiated characterizations by Arkhangel'skii and Ludwig [7] in 2001. We characterize β-Normal spaces using some types of open sets such as α-open sets and preopen sets. We use the ultrafilter space to answer Murtinova's question in [27] about the existence of a β-normal and regular space which is not Tychonoff. α-Normal spaces are described in terms of countable open sets, a result imitating that of normality. It turns out that continuous functions which are onto, open and closed preserve β-normality, while those which are injective, open and closed reflect α-normality. The notion of β-normal spaces is extended to the category of bitopological spaces where we characterize these bitopological spaces simultaneously in terms of i-open sets, (i, j)-preopen and (i, j)-α-open sets. We study the interrelations of these spaces with other bitopological spaces.

This work contributes to the ongoing development of separation theory in topology, shedding light on new connections and generalizations in the field.

1.2 Synopsis of thesis and some contributions

This thesis is organized as follows. The second chapter provides foundational information which is needed throughout the thesis. It begins with a review of set concepts, followed by a section defining topological spaces and discussing relevant results. Here, we also discuss normal spaces and provide some examples. It turns out that every almost discrete space is normal − a result not previously documented in the literature. The chapter explores different types of open sets, starting with α-open sets, followed by preopen sets, and concluding with regular open sets. This chapter ends with an introduction of bitopological spaces.

The third chapter consists of three sections. We begin the first section with characterizing β-normal spaces in terms of α-open sets as well as preopen sets. The second section discusses results related to other separation axioms where we show, among other results, that a topological space is normal if and only if is β-normal and almost normal. We use the ultrafilter space to answer Murtinova's [27] question about the existence of a β-normal and regular space which is not Tychonoff. We conclude the chapter with a section exploring how β-normal spaces are preserved and reflected by continuous mappings.

The fourth chapter discusses properties of α-normal spaces. A characterization of these spaces is presented in terms of countable open subsets. We establish several equivalent conditions for a space to be hereditary α-normal spaces. Additionally, we discuss the sending forward and backward of α-normality by continuous functions.

In chapter five we transfer the concept of a β-normal space to bitopological spaces and explore its properties. We study the interrelations of this space with ordinary separation axioms in the category of bitopological spaces.

The last chapter contains a summary of what we have achieved in the thesis.

Some of the contributions in this thesis include, but not limited to, (i) a characterization β-normal spaces in terms of α-open sets and preopen sets, (ii) an answer to Murtinova's question about about the existence of a β-normal and regular space which is not Tychonoff, (iii) a result which relates β-normality of a space with its T_0-reflection: We show that a space is β-normal if and only if its T_0-reflection is β-normal, (iv) a characterization of α-normal spaces in terms of countable open subsets, and (v) the introduction of β-normality in bitopological spaces. These contributions formed integral part in the development of the manuscript: "T. Ngcamphalala and M. Nxumalo, *β-normality in locales*, Applied General Topology, In Press." The above manuscript only covers results on β-normal spaces and α-normal spaces, excluding β-normality in bitopological spaces.

Throughout the text, we have included citations on every result (i,e., definition, corollary, proposition, theorem and observation) that is not original.

Chapter 2

Preliminaries

In this chapter, we introduce some notions that will be used throughout the thesis. The book [45] is our main reference for terminologies and notations used in this chapter. The reader can also consult [26].

2.1 Review on sets and functions

We write $x \in A$ to say that x is *an element of* A and $x \notin A$ to indicate that x is *not an element* of a set A. For any sets A and B, we write $A \subset B$ to mean that A is a *subset* of B, $A \cap B$ for the *intersection* of A and B, and $A \cup B$ for the *union* of A and B. We will denote the *set difference* of the set B with respect to A by $A - B$.

For any set X and any subsets A and B of X, the following statements are equivalent:

1. $A \cap B = \emptyset$
2. $A \subset X - B$
3. $B \subset X - A$.

Definition 2.1.1. [45] Let X and Y be sets, $A \subset X$, $B \subset Y$ and $f : X \to Y$ be a function. Then:

1. f is *injective* if for all $x, y \in X$, $f(x) = f(y)$ implies $x = y$. Hence, f is injective if and only if $f^{-1}(f(A)) = A$.
2. f is *onto* if for each $y \in Y$, there is $x \in X$ such that $f(x) = y$. Hence, f is onto if and only if $f(f^{-1}(B)) = B$.

2.2 Topological spaces

We now discuss topological spaces. We begin by defining a topology on a set.

Definition 2.2.1. [45] Let X be a set. A *topology* τ on X is a collection of subsets of X satisfying the following axioms:

1. $X \in \tau$
2. $\emptyset \in \tau$
3. Any union of elements of τ belong to τ
4. Every finite intersection of members of τ belong to τ.

For a set X and a topology τ on X, the pair (X, τ) is called a *topological space*. We will sometimes use *space* instead of topological space. We may drop τ if the topology is clear from context. Any $U \subset X$ is said to be *open* if $U \in \tau$. A subset A of X is *closed* if $X - A$ is open.

We provide the following definitions of the closure and interior of a set.

Definition 2.2.2. [45] Let (X, τ) be a topological space. The *closure* of a set A of X, denoted by $\overline{A}$, and the *interior* of A, denoted by $\text{int}(A)$ are the sets

$$\overline{A} = \bigcap \{F \subset X \mid F \text{ is closed and } A \subset F\}$$

and

$$\text{int}(A) = \bigcup \{U \subset X \mid U \text{ is open and } U \subset A\},$$

respectively.

It is clear that every set is contained in its closure and that every set contains its interior.

In the following proposition, we give some results on closure and interior. We only prove 6.

Proposition 2.2.3. *[45] Let (X, τ) be a topological space and A, B be subsets of X. The following statements hold:*

1. *If $A \subset B$, then $\overline{A} \subset \overline{B}$.*

2. If $A \subset B$ and B is closed, then $\overline{A} \subset B$.

3. $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$.

4. A is closed if and only if $A = \overline{A}$.

5. If $A \subset B$, then $\operatorname{int}(A) \subset \operatorname{int}(B)$.

6. If $A \subset B$ with A open, then $A \subset \operatorname{int}(B)$.

7. A is open if and only if $A = \operatorname{int}(A)$.

Proof. 6. Suppose $A \subset B$ with A an open set and let $x \in A$. Then $x \in A \subset B$. By Definition 2.2.2, we have $x \in \operatorname{int}(B)$. □

Definition 2.2.4. [26] Let (X, τ) be a topological space and $A, B \subset X$. Then A and B are called *separated* if

$$A \cap \overline{B} = B \cap \overline{A} = \emptyset.$$

It is clear that if $A, B \subset X$ are open (resp. closed) sets with $A \cap B = \emptyset$, then A and B are separated.

Proposition 2.2.5. *[45] Let (X, τ) be a topological space and $A \subset X$. Then*

$$X - \overline{A} = \operatorname{int}(X - A)$$

and

$$X - \operatorname{int}(A) = \overline{X - A}.$$

Proof. Let $x \in X$. Then

$$\begin{aligned}
x \in X - \operatorname{int}(A) &\Leftrightarrow x \notin \operatorname{int}(A) \\
&\Leftrightarrow x \notin \bigcup \{U \subset X | U \text{ is open, } U \subset A\} \\
&\Leftrightarrow x \notin U \text{ for each open } U \subset A \\
&\Leftrightarrow x \in X - U \text{ for each open } U \subset A \\
&\Leftrightarrow x \in X - U \text{ for every closed } X - U \text{ such that } X - A \subset X - U \\
&\Leftrightarrow x \in \overline{X - A}
\end{aligned}$$

and

$$
\begin{aligned}
x \in X - \overline{A} &\Leftrightarrow x \notin \overline{A} \\
&\Leftrightarrow x \notin \bigcap\{F \subset X | F \text{ is closed, } A \subset F\} \\
&\Leftrightarrow x \notin F \text{ for some closed } F, A \subset F \\
&\Leftrightarrow x \in X - F \text{ for some closed } F, A \subset F \\
&\Leftrightarrow x \in X - F \text{ for some open } X - F \subset X - A \\
&\Leftrightarrow x \in \text{int}(X - A)
\end{aligned}
$$

which prove the result. □

Definition 2.2.6. [26] A subset A of a topological space (X, τ) is *dense* if $\overline{A} = X$.

Definition 2.2.7. [45] Let (X, τ) be a topological space. A collection $\mathcal{B} \subset \tau$ is called a *base* for τ if each non-empty member of τ can be presented as a union of members of $\mathcal{B}$.

Consequently, $\mathcal{F}$ is base for the closed sets in (X, τ) if every closed set is an intersection of some subfamily of $\mathcal{F}$.

Proposition 2.2.8. *[45] Let (X, τ) be a topological space and $\mathcal{B} \subset \tau$. Then $\mathcal{B}$ is a base for τ if and only if for each open U of a point $x \in X$, there is a $B \in \mathcal{B}$ such that $x \in B \subset U$.*

Proposition 2.2.9. *[26] $\mathcal{B}$ is a base for some topology on X if and only if:*

1. *For all $x \in X$, there exists $B \in \mathcal{B}$ such that $x \in B$.*

2. *If $B_1, B_2 \in \mathcal{B}$ with $p \in B_1 \cap B_2$, then there is $B_3 \in \mathcal{B}$ such that*

$$p \in B_3 \subset B_1 \cap B_2.$$

Definition 2.2.10. [45] If (X, τ) is a topological space and Y is a subset of X, then the collection

$$\tau_Y = \{Y \cap U | U \in \tau\}$$

is topology on Y, called the *subspace topology* for Y.

A pair (Y, τ_Y) is called a *subspace* of X. For $A \subset Y$, we write $\overline{A}^Y$ for the closure of A taken in Y. We shall write Y-open and Y-closed to represent open and closed subsets of (Y, τ_Y), respectively. We also note that subspace topologies are topologies in their own rights.

We give the following result with a not hard proof. We only prove statements 3. and 4.

Proposition 2.2.11. *[45] Let (X, τ) be a topological space and $A \subset X$. Then:*

1. *$H \subset A$ is open in A iff $H = U \cap A$ for some $U \in \tau$.*
2. *$F \subset A$ is closed in A iff $F = K \cap A$ for some K closed in X.*
3. *If $A \subset B$, then $\overline{A}^B = \overline{A} \cap B$.*
4. *$A \subset B$ implies that $\operatorname{int}(A) \cap B \subset \operatorname{int}_B(A)$*

Proof. 3. Suppose that $A \subset B$. Since $A \subset \overline{A}$, we have that $A \subset B \cap \overline{A}$. Therefore

$$\overline{A}^B \subset \overline{B \cap \overline{A}}^B = B \cap \overline{A}$$

where the latter equality follows since $B \cap \overline{A}$ is closed in B.

On the other hand, let $K \subset B$ be closed in B such that $A \subset K$. It follows from 2. that $K = B \cap F$ for some F closed in X. Therefore

$$A \subset B \cap F \subset F.$$

This implies that $\overline{A} \subset \overline{F} = F$. Therefore

$$B \cap \overline{A} \subset B \cap F = K.$$

Thus $B \cap \overline{A} \subset \overline{A}^B$ and hence $\overline{A}^B = \overline{A} \cap B$.

4. Assume that $A \subset B$. We know that $\operatorname{int}(A) \subset A$. Therefore

$$\operatorname{int}(A) \cap B \subset A \cap B = A.$$

Because $\operatorname{int}(A) \cap B$ is open in B we get that $\operatorname{int}(A) \cap B \subset \operatorname{int}_B(A)$. □

Definition 2.2.12. [45] A property of topological spaces is called *hereditary* if, whenever a space possesses the property, then so must all of its subspaces.

Definition 2.2.13. [45] Let (X, τ) be a topological space and $x \in X$. A set $N \subset X$ is called a *neighborhood* (or simply *nhood*) of x if there is an open set G such that $x \in G \subset N$.

Definition 2.2.14. [45] Let (f, τ) and (Y, ρ) be topological spaces, and

$$f : (X, \tau) \to (Y, \rho)$$

be a function. Then f is continuous if A open open in Y implies $f^{-1}(A)$ is open in X.

Proposition 2.2.15. *[45] Let $f : (X, \tau) \to (Y, \rho)$ be a function between topological spaces (X, τ) and (Y, ρ). The following statements are equivalent:*

1. *f is continuous.*
2. *$f^{-1}(F)$ is closed in X for each closed F in Y.*
3. *$f\left(\overline{A}\right) \subset \overline{f(A)}$ for all $A \subset X$.*
4. *$\overline{f^{-1}(B)} \subset f^{-1}\left(\overline{B}\right)$ for every $B \subset Y$.*

Definition 2.2.16. [45] A function $f : (X, \tau) \to (Y, \rho)$ between two topological spaces is *open* (resp. *closed*) if for any A open (resp. closed) in X, $f(A)$ is open (resp. closed) in Y.

We provide the following definitions of some separation axioms.

Definition 2.2.17. [45] A topological space (X, τ) is said to be T_0 if, for every distinct elements x and y of X (i.e., $x \neq y$), there is a nhood of one element not containing the other.

Definition 2.2.18. [45] A topological space (X, τ) is said to be T_1 if, for every distinct elements x and y of X, there is a nhood of each element not containing the other.

Definition 2.2.19. [45] A topological space (X, τ) is said to be T_2 or *Hausdorff* if, for every distinct elements $x, y \in X$, there are disjoint open sets U and V of X such that $x \in U$ and $y \in V$.

Proposition 2.2.20. *[45] Let (X, τ) be a topological space. The following statements are equivalent:*

1. *(X, τ) is T_1.*

2. *Each singleton subset of X is closed.*

Definition 2.2.21. [45] A topological space (X, τ) is *regular* if, for all $x \in X$ and for all closed $F \subset X$ with $x \notin F$, there are disjoint open sets U and V with $x \in U$ and $A \subset V$.

Definition 2.2.22. [45] A topological space (X, τ) is said to be *normal* if, for any two disjoint closed sets F, K of X, there exist disjoint open subsets U and V of X such that $F \subset U$ and $K \subset V$.

Theorem 2.2.23. *[45] Let (X, τ) be a topological space. Then the following are equivalent:*

1. *(X, τ) is normal.*

2. *For each closed set $A \subset X$ and open set $U \subset X$ such that $A \subset U$, there exists an open set $V \subset X$ such that*

$$A \subset V \subset \overline{V} \subset U.$$

Proof. Suppose (X, τ) is normal. Let A be a closed set and U an open set such that $A \subset U$. Since (X, τ) is normal, for the closed set A and the closed set $X - U$ disjoint from A, there exist disjoint open sets V and W such that $A \subset V$ and $X - U \subset W$. Now, since $V \cap W = \emptyset$, we have $\overline{V} \cap W = \emptyset$. Therefore $\overline{V} \subset X - W \subset U$. Thus, we have found an open set V such that

$$A \subset V \subset \overline{V} \subset U.$$

Conversely, suppose 2. holds. Let F and K be two disjoint closed sets. Then $F \subset X - K$. By assumption, there exists an open set V such that

$$F \subset V \subset \overline{V} \subset X - K.$$

Now, let $W = X - \overline{V}$. Then W is open with $F \subset V$ and $K \subset W$. Moreover, $V \cap W = \emptyset$. Thus (X, τ) is normal. □

We give some examples of normal spaces.

Example 2.2.24. (1) Consider the topology

$$\tau = \{\emptyset, X, \{a\}, \{d\}, \{b,c\}, \{a,d\}, \{a,b,c\}, \{b,c,d\}\}$$

on the set $X = \{a,b,c,d\}$. Observe that the closed sets are exactly the open ones. This means that any pair of disjoint closed sets is the required pair of disjoint open sets to make (X, τ) a normal space.

(2) Consider the topology

$$\tau = \{\emptyset, X, \{b\}, \{d\}, \{b,d\}, \{a,b\}, \{c,d\}, \{a,b,d\}, \{b,c,d\}\}$$

on the set $X = \{a,b,c,d\}$. The closed sets are:

$$\emptyset, \{X\}, \{a,c,d\}, \{a,b,c\}, \{a,c\}, \{c,d\}, \{c\}, \{a\} \text{ and } \{a,b\}$$

with only $\emptyset, \{a\}, \{a,b\}, \{c,d\}$ and $\{c\}$ as disjoint closed sets. Thus (X, τ) is a normal space.

Observation 2.2.1. The author in [19] defines an *almost discrete* space as one in which every open set is closed, (equivalently, every closed set is open). The argument we used in Example 2.2.24(1) raises a question as to whether every almost discrete space is normal. It turns out that this is always true and since we have not seen this result in the literature, we verify it here: Let (X, τ) be an almost discrete space and choose disjoint closed subsets F and K of X. Since X is almost discrete, F and K are open. Thus, (X, τ) is normal. The converse is not always true. For instance, the space in Example 2.2.24(2) is normal but not almost discrete since $\{b\}$ is an open set which is not closed.

2.3 Types of open sets

In this section we aim to discuss some types of open sets called α-open sets, preopen sets and regular open sets.

2.3.1 α-Open sets

We begin with the following definition.

Definition 2.3.1. [29] A subset A of a topological space (X, τ) is *α-open* if

$$A \subset \text{int}(\overline{\text{int}\,(A)}).$$

Following is an example showing α-open sets.

Example 2.3.2. Let $X = \{a, b, c\}$ be a set endowed with the topology

$$\tau = \{\emptyset, X, \{a\}\}.$$

The α-open sets of (X, τ) are:

$$\emptyset, X, \{a\}, \{a, b\} \text{ and } \{a, c\}.$$

We give the following result about α-open sets which we shall use below.

Proposition 2.3.3. *[29] Every open set is α-open.*

Proof. Consider a topological space (X, τ) and A an open subset of X. Then $A = \text{int}(A) \subset \text{int}(\overline{A})$. Therefore

$$A \subset \text{int}(A) \subset \text{int}(\overline{\text{int}(A)}),$$

which gives $A \subset \text{int}(\overline{\text{int}(A)})$. □

Observation 2.3.1. The converse of Proposition 2.3.3 does not always hold. For instance, the sets

$$\{a, c\} \text{ and } \{a, b\}$$

in Example 2.3.2 are α-open but not open.

2.3.2 Preopen sets

This subsection introduces the notion of preopen sets.

Definition 2.3.4. [22] A subset A of a topological space (X, τ) is *preopen* if

$$A \subset \text{int}(\overline{A}).$$

We now give the following example.

Example 2.3.5. Let $X = \{a, b, c, d\}$ be a set and

$$\tau = \{\emptyset, X, \{b, d\}, \{a, b, d\}, \{b, c, d\}\}$$

be a topology on X. The preopen sets of (X, τ) are:

$$\emptyset, X, \{b\}, \{d\}, \{a, b\}, \{a, d\}, \{b, d\}, \{b, c\},$$
$$\{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\} \text{ and } \{b, c, d\}.$$

The following result is an application of the definition of preopen sets.

Proposition 2.3.6. *Let (X, τ) be a topological space and $A, U \subset X$. If $A \subset \overline{U}$ and U is preopen, then $A \subset \overline{\text{int}(\overline{U})}$.*

Proof. Assume $A \subset \overline{U}$. Since $U \subset \text{int}(\overline{U})$, it follows that $A \subset \overline{\text{int}(\overline{U})}$. □

2.3.3 Regular open sets

In this subsection we introduce the concept of regular open sets. We begin with the following definition.

Definition 2.3.7. [43] A subset A of a topological space (X, τ) is *regular open* if

$$A = \mathrm{int}(\overline{A}).$$

The complement of a regular open set is called a *regular closed* set.

Example 2.3.8. Let

$$\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$$

be a topology on the set $X = \{a, b, c\}$. Then the regular open sets are $\{a\}$ and $\{b\}$.

Proposition 2.3.9. *The closure of a preopen set is regular closed.*

Proof. Let U be preopen set. Then $U \subset \mathrm{int}(\overline{U})$. Therefore

$$\overline{U} \subset \overline{\mathrm{int}(\overline{U})} \subset \overline{\overline{U}} = \overline{U}.$$

Thus $\overline{U} = \overline{\mathrm{int}(\overline{U})}$. □

We find a relationship between regular open sets, preopen sets and α-open sets.

Proposition 2.3.10. *[2] We have the following implications:*

$$\textit{Regular open} \quad \Longrightarrow \quad \textit{open} \quad \Longrightarrow \quad \alpha\textit{-open} \quad \Longrightarrow \quad \textit{preopen}.$$

Proof. To prove the first implication, consider a regular open set A satisfying that $A = \mathrm{int}(\overline{A})$. Then

$$\mathrm{int}(A) = \mathrm{int}(\mathrm{int}(\overline{A})) = \mathrm{int}(\overline{A}).$$

But A is regular open, so $\mathrm{int}(A) = A$. Therefore A is open.

The second implication follows from Proposition 2.3.3.

For the third implication, let A be an α-open subset of a topological space (X, τ) such that

$$A \subset \mathrm{int}(\overline{\mathrm{int}\, A}).$$

Since $A = \mathrm{int}(A)$, we get that $A \subset \mathrm{int}(\overline{A})$. Hence A is preopen. □

None of the above implications is reversible as shown in the following example.

Example 2.3.11. (1) Not every open set is regular open: The set $\{a, b\}$ in Example 2.3.8 is open but not regular open.

(2) See Observation 2.3.1 for an example of an α-open set which is not open.

(3) Not every preopen set is α-open: In Example 2.3.5 the set $\{b, c\}$ is preopen but not α-open.

(4) Not every preopen set is regular open: The sets verified in Example 2.3.5 are preopen but not regular open.

2.4 Bitopological spaces

A *bitopological space* (X, τ_1, τ_2) is a set X endowed with two topologies, τ_1 and τ_2. Given a bitopological space (X, τ_1, τ_2) and a topological property $\mathcal{P}$, just like in [12], we use the notation (i, j)-$\mathcal{P}$ to represent properties related to topologies τ_i and τ_j. We will denote, for a subset A of (X, τ_1, τ_2), the *interior* and *closure* of A with respect to τ_i, where $i \in \{1, 2\}$, as $\mathrm{int}_i(A)$ and $cl_i(A)$, respectively. For a τ_i-open set and a τ_i-closed set, we use i-open and i-closed, respectively.

A generalization of regular open sets, preopen sets and α-open sets to bitopological spaces occurred in the late 80's and early 90's. In 1987, Banerjee [8] introduced an (i, j)-*regular open* set as a subset A of a bitopological space (X, τ_i, τ_j) such that $A = \mathrm{int}_i(cl_j(A))$. In 1990, Jelic [17] introduced the concepts of (i, j)-*preopen* sets and (i, j)-α-*open* sets such that $A \subset \mathrm{int}_i(cl_j(A))$ and $A \subset \mathrm{int}_i(cl_j(\mathrm{int}_i(A)))$ for any subset A of (X, τ_i, τ_j). These sets have also been referenced in other articles (see [3]).

The following results are from Kelly [18] and Rielly [35].

Definition 2.4.1. Let (X, τ_1, τ_2) be a bitopological space. We say that τ_1 is regular with respect to τ_2 if for each point $x \in X$ and each 1-closed set A such that $x \notin A$, there is a 1-open set U and a 2-open set V disjoint from U such that $x \in U$ and $A \subset V$.

A bitopological space (X, τ_1, τ_2) is *pairwise regular* if τ_1 is regular with respect to τ_2 and τ_2 is regular with respect to τ_1.

Proposition 2.4.2. *For a bitopological space* (X, τ_1, τ_2)*, the following are equivalent:*

1. τ_1 *is regular with respect to* τ_2*.*

2. *For each $x \in X$ and 1-open set U containing x, there is a 1-open set V such that*
$$x \in V \subset cl_2(V) \subset U.$$

Proof. (1. $\Rightarrow$ 2.): Let $x \in X$ and U be an 1-open set such that $x \in U$. Then $X - U$ is a 1-closed sets such that $x \notin X - U$. By 1., there is a 1-open set V and a 2-open set V such that $x \in V$ and $X - U \subset W$. Because $V \subset X - W$, $cl_2(V) \subset X - W$. Thus

$$x \in V \subset cl_2(V) \subset X - U.$$

(2. $\Leftarrow$ 1.): Let A be a 1-closed set and $x \in X$ with $x \notin A$. So $x \in X - A$ where $X - A$ is a 1-open set. By hypothesis, there exists a 1-open sets U such that
$$x \in U \subset cl_2(U) \subset X - A.$$
Therefore, $A \subset X - cl_2(U)$ with $X - cl_2(U)$ disjoint from U. □

Definition 2.4.3. A bitopological space (X, τ_i, τ_j) is *pairwise normal* if for any i-closed set F and any j-closed set K such that $F \cap K = \emptyset$, there is a j-open set U containing F, a i-open set V containing K such that $U \cap V = \emptyset$ for $i, j = 1, 2; i \neq j$.

Proposition 2.4.4. *Let (X, τ_1, τ_2) be a bitopological space. Then the following statements are equivalent:*

1. *(X, τ_i, τ_j) is pairwise normal.*

2. *For each i-closed set F and each j-closed set K disjoint from F, there is a j-open set U containing F such that $cl_i(U) \cap K = \emptyset$.*

Proof. (1. $\Rightarrow$ 2.): Suppose (X, τ_i, τ_j) is pairwise normal. Let F and K be disjoint closed sets such that F is i-closed and K is j-closed. By hypothesis, there is a j-open set U and an i-open set V such that $F \subset U$, $K \subset V$ and $U \cap V = \emptyset$. Therefore, $cl_i(U) \cap V = \emptyset$ so that

$$K \subset V \subset X - cl_i(U),$$

making $K \cap cl_i(U) = \emptyset$.

(2. $\Leftarrow$ 1.): Assume 2. holds. Pick an i-closed set F and a j-closed K such that F and K are disjoint. Then by assumption, there is a j-open set U such that $F \subset U$ and $cl_i(U) \cap K = \emptyset$. So, it follows that $X - cl_i(U)$ is an i-open set containing K and disjoint from U. Therefore (X, τ_i, τ_j) is pairwise normal. □

Chapter 3

β-Normal spaces

This chapter aims to introduce β-normal spaces and examine some of their properties. We shall give characterizations of these spaces some of which will be in terms of the types of open sets discussed in section 2.3, explore some relationships between β-normal spaces with other separation axioms and examine functions sending β-normality back and forth.

3.1 Characterizing β-normal spaces

We begin this section by recalling the following definition from [7].

Definition 3.1.1. A topological space (X, τ) is called *β-normal* if, for any two disjoint closed sets F, K of X, there exist open sets U, V of X such that $F \cap U$ is dense in F, $K \cap V$ is dense in K and $\overline{U} \cap \overline{V} = \emptyset$.

We give the following example.

Example 3.1.2. Let $X = \{a, b, c\}$ be a set endowed with the topology

$$\tau = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}\}.$$

Then (X, τ) is β-normal. To see this, we have that the closed sets of X are

$$\emptyset, X, \{b, c\}, \{a, b\} \text{ and } \{b\}.$$

It is clear that the conditions of β-normality are satisfied when two disjoint closed sets, with one of them being the empty set, are considered. Since, here, in each pair of disjoint closed subsets of X one of them is the empty set, it follows that (X, τ) is β-normal.

Observation 3.1.1. The argument used in the above example also shows that (X, τ) is normal. However, the space (X, τ) above being both normal and β-normal does not mean that these two concepts are always equivalent. For instance, Murtinová in [27, Example 1] gives an example of a β-normal space which is not normal. In Proposition 3.2.1, we will show that every normal space is β-normal.

We shall need the following lemma to prove the next theorem.

Lemma 3.1.3. *Let (X, τ) be a topological space, U a preopen subset of X and $F \subset X$ a closed set. If $\overline{U \cap F} = F$, then $\overline{\mathrm{int}(\overline{U}) \cap F} = F$.*

Proof. $\overline{\mathrm{int}(\overline{U}) \cap F} \subset F$: It is clear that $\mathrm{int}(\overline{U}) \cap F \subset F$. Since F is closed, we get that $\overline{\mathrm{int}(\overline{U}) \cap F} \subset F$ by Proposition 2.2.3(2).

$F \subset \overline{\mathrm{int}(\overline{U}) \cap F}$: Let $x \in F$ and choose a neighbourhood N of x. Then there is an open $G \subset X$ such that $x \in G \subset N$. Since $F = \overline{U \cap F}$, $x \in \overline{U \cap F}$. This implies that $N \cap (U \cap F) \neq \emptyset$ because N is a neighbourhood of x. Because $U \subset \mathrm{int}(\overline{U})$, we have that

$$N \cap (\mathrm{int}(\overline{U}) \cap F) \neq \emptyset.$$

It follows that $x \in \overline{\mathrm{int}(\overline{U}) \cap F}$. Thus $F \subset \overline{\mathrm{int}(\overline{U}) \cap F}$.

Hence $\overline{\mathrm{int}(\overline{U}) \cap F} = F$. □

In what follows we characterize β-normal spaces. Some of the characterizations include some types of open sets.

Theorem 3.1.4. *Let (X, τ) be a topological space. Then the following are equivalent:*

1. *(X, τ) is β-normal.*

2. *For any disjoint closed sets F, K of X, there exist open sets U, V of X such that $\overline{F \cap U} = F$, $\overline{K \cap V} = K$ and $\overline{U} \cap \overline{V} = \emptyset$.*

3. *For each closed set $A \subset X$ and each open set $U \subset X$ such that $A \subset U$, there exists an open set $V \subset X$ such that*

$$\overline{A \cap V} = A \subset \overline{V} \subset U.$$

4. *For any disjoint closed sets F, K of X, there are α-open sets U, V of X such that $\overline{F \cap U} = F$, $\overline{K \cap V} = K$ and $\overline{U} \cap \overline{V} = \emptyset$.*

5. *For every closed set $A \subset X$ and every open set $U \subset X$ such that $A \subset U$, there exists an α-open set $V \subset X$ such that*

$$\overline{A \cap V} = A \subset \overline{V} \subset U.$$

6. *If F and K are disjoint closed subsets of X, then there are preopen sets U, V of X such that $\overline{F \cap U} = F$, $\overline{K \cap V} = K$ and $\overline{U} \cap \overline{V} = \emptyset$.*

7. *For each closed set $A \subset X$ and each open set $U \subset X$ such that $A \subset U$, there exists a preopen set $V \subset X$ such that*

$$\overline{A \cap V} = A \subset \overline{V} \subset U.$$

Proof. (1. $\Rightarrow$ 2.): Let $F, K \subset X$ be disjoint closed sets. Since X is β-normal, there exist open sets U, V of X such that $F \cap U$ is dense in F, $K \cap V$ is dense in K and $\overline{U} \cap \overline{V} = \emptyset$.

Therefore $\overline{F \cap U}^F = F$ and $\overline{K \cap V}^K = K$. By Proposition 2.2.11,

$$\overline{F \cap U} \cap F = F$$

and

$$\overline{K \cap V} \cap K = K.$$

Therefore $F \subset \overline{F \cap U} \cap F$, making $F \subset \overline{F \cap U}$. Similarly, $K \subset \overline{K \cap V} \cap K$, making $K \subset \overline{K \cap V}$. Since $F \cap U \subset F$ and $K \cap V \subset K$ and both F and K are closed, it follows from Proposition 2.2.3 that

$$\overline{F \cap U} \subset \overline{F} = F$$

and

$$\overline{K \cap V} \subset \overline{K} = K.$$

Thus $\overline{F \cap U} = F$ and $\overline{K \cap V} = K$.

(2. $\Rightarrow$ 3.): Let $A \subset X$ be a closed set and $U \subset X$ be an open set such that $A \subset U$. We have that A and $X - U$ are disjoint closed sets. By 2., there exist open sets V, W such that $\overline{A \cap V} = A$,

$$\overline{(X - U) \cap W} = X - U,$$

and $\overline{V} \cap \overline{W} = \emptyset$. Therefore

$$A = \overline{A \cap V} \subset \overline{A} \cap \overline{V} \subset \overline{V},$$

$$X - U = \overline{(X - U) \cap W} \subset \overline{X - U} \cap \overline{W} \subset \overline{W}$$

and $\overline{V} \subset X - \overline{W}$.

We get that

$$X - \overline{W} \subset X - (X - U) = U,$$

making $\overline{V} \subset U$. Thus

$$A = \overline{A \cap V} \subset \overline{V} \subset U.$$

(3. $\Rightarrow$ 4.): Let F and K be disjoint closed subsets of X. Then $X - K$ is open in X and $F \subset X - K$. By 3., there exists an open subset U of X such that

$$\overline{U \cap F} = F \subset \overline{U} \subset X - K.$$

Therefore

$$K = X - (X - K) \subset X - \overline{U}$$

and $X - \overline{U}$ is open in X. So by 3. again, there exists an open subset V of X such that

$$\overline{V \cap K} = K \subset \overline{V} \subset X - \overline{U}.$$

Therefore $\overline{V} \cap \overline{U} = \emptyset$. Since every open set is α-open by Proposition 2.3.3, we have that U and V are the required α-open subsets of X.

(4. $\Rightarrow$ 5.): Let A and U be subsets of X such that A is closed, U is open and $A \subset U$.

Knowing that $A \cap (X - U) = \emptyset$, we have, by 4., α-open sets V and W satisfying that $\overline{A \cap V} = A$,

$$\overline{X - U \cap W} = X - U,$$

and $\overline{V} \cap \overline{W} = \emptyset$. Therefore

$$A = \overline{A \cap V} \subset \overline{V},$$

$$X - U = \overline{(X - U) \cap W} \subset \overline{W}$$

and $\overline{V} \subset X - \overline{W}$.

This implies that $X - \overline{W} \subset U$ so that $\overline{V} \subset U$. Therefore,

$$A = \overline{A \cap V} \subset \overline{V} \subset U.$$

(5. $\Rightarrow$ 6.): Let F and K be disjoint closed subsets of X such that $X - K$ is open in X and $F \subset X - K$. Then by 5., there exists an α-open set U of X such that

$$\overline{U \cap F} = F \subset \overline{U} \subset X - K.$$

Therefore $K \subset X - \overline{U}$ and $X - \overline{U}$ is open in X. Applying 5. again gives that there exists an α-open subset V of X such that

$$\overline{V \cap K} = K \subset \overline{V} \subset X - \overline{U},$$

where the last containment gives $\overline{V} \cap \overline{U} = \emptyset$. Because every α-open set is preopen by Proposition 2.3.10, it follows that U and V are preopen subsets of X.

(6. $\Rightarrow$ 7.): Let $A \subset X$ be a closed set and $U \subset X$ an open set such that $A \subset U$. We have that A and $X - U$ are disjoint closed sets. By 6., there exists preopen sets V, W such that

$$\overline{A \cap V} = A, \quad \overline{(X - U) \cap W} = X - U,$$

and $\overline{V} \cap \overline{W} = \emptyset$. Therefore $A \subset \overline{V}$, $X - U \subset \overline{W}$ and $\overline{V} \subset X - \overline{W}$. We get that $X - \overline{W} \subset U$, making $\overline{V} \subset U$. Thus

$$A = \overline{A \cap V} \subset \overline{V} \subset U.$$

(7. $\Rightarrow$ 1.): Let F and K be disjoint closed subsets of X. Then $X - K$ is open in X and $F \subset X - K$. By 7., there exists a preopen set U of X such that

$$\overline{U \cap F} = F \subset \overline{U} \subset X - K.$$

A combination of Lemma 3.1.3 and Proposition 2.3.6 gives

$$\overline{\operatorname{int}(\overline{U}) \cap F} = F \subset \overline{\operatorname{int}(\overline{U})}.$$

We also have that $K \subset X - \overline{U}$ where $X - \overline{U}$ is open in X, so by 7., there exists a preopen set V of X such that

$$\overline{V \cap K} = K \subset \overline{V} \subset X - \overline{U}. \tag{3.1.1}$$

Using Lemma 3.1.3 and Proposition 2.3.6 again yields

$$\overline{\operatorname{int}(\overline{V}) \cap K} = K \subset \overline{\operatorname{int}(\overline{V})}.$$

From Equation 3.1.1, we get that $\overline{V} \cap \overline{U} = \emptyset$. Since $\text{int}(\overline{V}) \subset \overline{V}$ and $\text{int}(\overline{U}) \subset \overline{U}$, we have that $\overline{\text{int}(\overline{V})} \subset \overline{V}$ and $\overline{\text{int}(\overline{U})} \subset \overline{U}$, making

$$\overline{\text{int}(\overline{V})} \cap \overline{\text{int}(\overline{U})} = \emptyset.$$

We are left to show that

$$\overline{\text{int}(\overline{U}) \cap F}^{F} = F \text{ and } \overline{\text{int}(\overline{V}) \cap K}^{K} = K:$$

Since $\overline{\text{int}(\overline{U}) \cap F} = F$, we have that

$$\overline{\text{int}(\overline{U}) \cap F} \cap F = F \cap F = F,$$

making $\overline{\text{int}(\overline{U}) \cap F}^{F} = F$ by Proposition 2.2.11. The case of

$$\overline{\text{int}(\overline{V}) \cap K}^{K} = K$$

follows a similar argument.

Thus (X, τ) is β-normal. $\square$

Recall from [28] that a topological space is *p-normal* if, for any pair of disjoint closed sets F and K of X, there are disjoint preopen sets $U, V \subset X$ such that $F \subset U$ and $K \subset V$.

Observation 3.1.2. These characterizations in Theorem 3.1.4 depict a behavior of β-normality which is different from that of normality in the sense that, for example, the property of separating closed subsets with preopen subsets is not equivalent to normality, in fact, it yields a weaker variant of normality called p-normality. For instance, the space in Example 2.3.5 is p-normal but not normal.

It is mentioned in [7] that β-normality is not generally a hereditary property. The following proposition shows that β-normality is a hereditary property with respect to closed subspaces.

Proposition 3.1.5. *[7] A closed subspace of a β-normal space is β-normal.*

Proof. Let (X, τ) be a β-normal space, A a subspace of X and choose disjoint closed sets $F, K \subset A$. By Proposition 2.2.11 it follows that F and K are closed subsets of X. Since (X, τ) is β-normal, there exist open sets U, V of X such that $F \cap U$ is dense in F, $K \cap V$ is dense in K and $\overline{U} \cap \overline{V} = \emptyset$. Now $U \cap A$ and $V \cap A$ are open in A such that

$$\overline{F \cap (U \cap A)} = F, \quad \overline{K \cap (V \cap A)} = K$$

and

$$\overline{V \cap A} \cap \overline{U \cap A} = \emptyset.$$

Therefore A is β-normal. □

Following our characterization of β-normal spaces in terms of certain types of open sets, we also characterize hereditarity of β-normal spaces, where some of the statements are in terms of certain types of open sets.

We start by giving the following lemma.

Lemma 3.1.6. *Let (X, τ) be a topological space and $A, U, F \subset X$ be sets with U open in X. If $A \subset \overline{U \cap \overline{F}}$, then $A \subset \overline{U \cap F}$.*

Proof. Let $x \in A$ and choose N an open nhood of x. Then

$$\emptyset \neq (U \cap \overline{F}) \cap N = \overline{F} \cap (U \cap N).$$

It follows that

$$\emptyset \neq F \cap (U \cap N) = (U \cap F) \cap N$$

implying that $x \in \overline{U \cap F}$. □

Proposition 3.1.7. *The following are equivalent for any topological space (X, τ).*

1. *(X, τ) is hereditarily β-normal.*
2. *Every preopen subspace of (X, τ) is β-normal.*
3. *Every α-open subspace of (X, τ) is β-normal.*
4. *Every open subspace of (X, τ) is β-normal.*

Proof. (1. $\Rightarrow$ 2.): Trivial.

(2. $\Rightarrow$ 3.): This result follows since every α-open set is preopen, by Proposition 2.3.10.

(3. $\Rightarrow$ 4.): Follows since every open sets is α-open by Proposition 2.3.3.

(4. $\Rightarrow$ 1.): Let A be a subspace of X and choose F and K disjoint A-closed sets. Define

$$Y = X - (\overline{F} \cap \overline{K})$$

an open subspace of X. We have $\overline{F}^Y$ and $\overline{K}^Y$ are disjoint closed subsets of Y. To see this, since

$$(\overline{F} \cap A) \cap (\overline{K} \cap A) = F \cap K = \emptyset,$$

we have that

$$(X - \overline{F}) \cup (X - A) \cup (X - \overline{K}) = X$$

which implies that $Y \cup (X - A) = \emptyset$.

Therefore $A \subset Y$. So, $F \subset A \subset Y$ and $K \subset A \subset Y$. Furthermore, we have that

$$\begin{aligned}(X - \overline{F}^Y) \cup (X - \overline{K}^Y) &= \big(X - (\overline{F} \cap Y)\big) \cup \big(X - (\overline{K} \cap Y)\big) \\ &= (X - \overline{F}) \cup (X - Y) \cup (X - \overline{K}) \cup (X - Y) \\ &= (X - \overline{F}) \cup (X - \overline{K}) \cup (X - Y) \\ &= Y \cup (X - Y) \\ &= X.\end{aligned}$$

So, $\overline{F}^Y \cap \overline{K}^Y = \emptyset$. Since Y is β-normal, there exist open sets U and V of Y such that

$$\overline{\overline{F}^Y \cap U} \cap Y = \overline{F}^Y, \quad \overline{\overline{K}^Y \cap V} \cap Y = \overline{K}^Y$$

and $\overline{U}^Y \cap \overline{V}^Y = \emptyset$. Both U and V are open subsets of X since Y is also open in X. We have that

$$\overline{\overline{F}^Y \cap U} \cap Y \cap A = \overline{F}^Y \cap A,$$

$$\overline{\overline{K}^Y \cap V} \cap Y \cap A = \overline{K}^Y \cap A$$

and

$$(\overline{U}^Y \cap A) \cap (\overline{V}^Y \cap A) = \emptyset.$$

Since $A \subset Y$,

$$A \cap \overline{\overline{F}^Y \cap U} = \overline{F} \cap A = F,$$

$$A \cap \overline{\overline{K}^Y \cap V} = \overline{K} \cap A = K$$

and

$$\overline{U} \cap A \cap \overline{V} = \emptyset.$$

So,

$$A \cap \overline{U \cap \overline{F} \cap Y} = A \cap \overline{U \cap \overline{F}} = F,$$

$$A \cap \overline{V \cap \overline{K} \cap Y} = A \cap \overline{V \cap \overline{K}} = K$$

and

$$\overline{A \cap U} \cap A \cap \overline{A \cap V} = \emptyset.$$

Therefore,

$$F \subset \overline{F \cap U} \cap A = \overline{F \cap U}^A,$$

$$F \subset \overline{K \cap V} \cap A = \overline{K \cap V}^A$$

and

$$\overline{A \cap U}^A \cap \overline{A \cap V}^A = \emptyset,$$

where the first two statements follow from Lemma 3.1.6.

Thus (X, τ) is hereditary β-normal.

□

3.2 β-Normality and other topological spaces

In this section, we will consider how β-normality relates with other topological spaces.

We start by showing that every normal space is β-normal, as mentioned in Observation 3.1.1. The following result was mentioned by the authors of [7] with no proof. We give a proof here.

Proposition 3.2.1. *[7] Every normal space is β-normal.*

Proof. Assume that (X, τ) is a normal space and choose a closed subset A of X and an open $U \subset X$ such that $A \subset U$. Then $A \cap (X - U) = \emptyset$ where both A and $X - U$ are closed subsets of X.

Since X is normal, there are disjoint open subsets V and W of X such that $A \subset V$ and $X - U \subset W$.

1. $\overline{A \cap V} = A$: Because $A \cap V \subset A$, we have that

   $$\overline{A \cap V} \subset \overline{A} = A$$

   where the latter equality follows since A is closed.

   On the other hand, since $A \subset V$ and $A \subset A$, $A \subset V \cap A$. Therefore $A \subset \overline{A \cap V}$.

   Thus $\overline{A \cap V} = A$.

2. $A \subset \overline{V}$: Follows since $A \subset V$.

3. $\overline{V} \subset U$: Since $X - U \subset W$, $X - W \subset U$. But W is open so $X - W$ is closed, making $\overline{X - W} = X - W$. Therefore $\overline{X - W} \subset U$.

   Because $W \cap V = \emptyset$, we have that $V \subset X - W$ so that $\overline{V} \subset \overline{X - W}$.

   Therefore $\overline{V} \subset U$.

The combination of 1.,2., and 3. above shows that V is an open subset of X such that

$$\overline{V \cap A} = A \subset \overline{V} \subset U.$$

Hence (X, τ) is β-normal. □

Proposition 3.2.2. *[7] If a β-normal space (X, τ) satisfies the T_1 seperation axiom, then the space (X, τ) is regular.*

Proof. Let $x \in X$, F be a closed subset of X and $x \notin F$. Since (X, τ) is T_1, $\{x\}$ is closed and disjoint from F. By β-normality of (X, τ), there exist open subsets U, V of X such that

$$\overline{F \cap U} = F, \quad \overline{\{x\} \cap V} = \{x\} \text{ and } \overline{U} \cap \overline{V} = \emptyset.$$

It follows that $F \subset \overline{U} \subset X - \overline{V}$ and $\{x\} \subset \overline{V}$. Because $V \subset \overline{V}$, we have that $V \cap (X - \overline{V}) = \emptyset$ so that $X - \overline{V}$ and V are disjoint open sets such that $F \subset X - \overline{V}$ and $x \in V$.

Thus, (X, τ) is regular. □

Recall from [38] that a topological space is *κ-normal* if for every pair of disjoint regular closed sets $F, K \subset X$, there exist disjoint open sets U and V such that $F \subset U$ and $K \subset V$.

In [7], they investigate conditions that can be applied to β-normality to ensure that normality is achieved. The authors proved that a topological space (X, τ) is normal if and only if (X, τ) is β-normal and κ-normal. In the following theorem we improve this result. Let us recall that a topological space (X, τ) is *almost normal* if any pair of disjoint closed sets F and K of X, one of which is regular closed, there are disjoint open sets $U, V \subset X$ such that $F \subset U$ and $K \subset V$. Equivalently, a space (X, τ) is almost normal if and only if for every closed set F and every regular open set U containing F, there exists an open set V such that

$$F \subset V \subset \overline{V} \subset U.$$

Theorem 3.2.3. *A topological space (X, τ) is normal if and only if it is β-normal and almost normal.*

Proof. (⇒): <u>Normal implies β-normal</u>: This implication is shown in Proposition 3.2.1.

<u>Normal implies almost normal</u>: Follows since every regular closed set is closed.

(⇐): Let $A \subset X$ be closed and $U \subset X$ be an open set such that $A \subset U$. By β-normality of (X, τ), there exists an open set V such that

$$\overline{A \cap V} = A \subset \overline{V} \subset U.$$

Now $\overline{V}$ is regular closed and hence $X - \overline{V}$ is a regular open set containing $X - U$. By almost normality of (X, τ), there is an open set W satisfying

$$X - U \subset W \subset \overline{W} \subset X - \overline{V}.$$

We get that $X - \overline{W}$ is a regular open set containing $\overline{V}$. By almost normality again, there exists an open set H such that

$$\overline{V} \subset H \subset \overline{H} \subset X - \overline{W}.$$

But $A \subset \overline{V}$ and $X - \overline{W} \subset U$, so

$$A \subset H \subset \overline{H} \subset U.$$

Therefore (X, τ) is normal. □

We recall that a topological space (X, τ) is *almost p-normal* [28] if for each closed set F and each regular closed set K disjoint from F, there exist disjoint preopen sets U and V such that $F \subset U$ and $K \subset V$, and is *mildly p-normal* [28] if, for every pair of disjoint regular closed sets $F, K \subset X$, there exist disjoint preopen sets U and V such that $F \subset U$ and $K \subset V$.

We have the following implication.

Proposition 3.2.4. *A topological space (X, τ) is p-normal only if (X, τ) is β-normal and mildly p-normal.*

Proof. Let (X, τ) be β-normal and mildly p-normal. Pick disjoint closed subsets F, K of X. By Theorem 3.1.4, there exist preopen subsets U, V of X such that $\overline{F \cap U} = F$, $\overline{K \cap V} = K$ and $\overline{U} \cap \overline{V} = \emptyset$. It follows that $F \subset \overline{U}$ and $K \subset \overline{V}$ where $\overline{U}$ and $\overline{V}$ are regular closed sets by Proposition 2.3.9. By mildly p-normality, there are disjoint preopen subsets H, G of X such that $F \subset \overline{U} \subset H$ and $K \subset \overline{V} \subset G$.

Thus (X, τ) p-normal. □

Observation 3.2.1. The equivalence of the above proposition is not always true because p-normality sometimes fails to imply β-normality: To see this, consider the topological space in Example 2.3.5. The space is p-normal but not β-normal since we cannot find preopen sets U and V satisfying that $\overline{U} \cap \overline{V} = \emptyset$.

We recall that a space (X, τ) *submaximal* if every preopen set is open [13].

Theorem 3.2.5. *In a submaximal and β-normal space (X, τ) the following statements are equivalent:*

1. *(X, τ) is normal*
2. *(X, τ) almost normal*
3. *(X, τ) is κ-normal*
4. *(X, τ) is p-normal*
5. *(X, τ) is almost p-normal*
6. *(X, τ) is mildly p-normal.*

Proof. 1. $\Rightarrow$ 2. $\Rightarrow$ 3. and 4. $\Rightarrow$ 5. $\Rightarrow$ 6. follow from definitions.

(3. $\Rightarrow$ 4.): Similar to the proof of Proposition 3.2.4.

(6. $\Rightarrow$ 1.): This proof is similar to that of Proposition 3.2.4 including the fact that in a submaximal space every preopen set is open. □

The *semi-regularization* space (X, τ_s) is a topology on X whose basis is the family of regular open sets in (X, τ) [24]. In the following result, we provide a relationship between a topological space (X, τ) and β-normality of its semi-regularization space (X, τ_s). For any $U \in \tau_s$, $U = \bigcup_{i \in I} V_i$ with each V_i a regular open set in (X, τ). Recall that a topological space is *seminormal* [44] if, for every closed set A and each open set U containing A, there exists a regular open set V such that

$$A \subset V \subset U.$$

A subset A of a space (X, τ) is said be to π-*closed* [46] if it is a finite intersection of regular closed sets. A topological space is *partial normal* [5] if, for every pair of disjoint closed subsets F and K of X, one of which is π-closed and the other is regular closed, there exist disjoint preopen subsets U and V of X such that $F \subset U$ and $K \subset V$. [4, Theorem 2.11] states that (X, τ) is partial normal if and only if (X, τ_s) is almost normal.

Lemma 3.2.6. *[9] Let (X,τ) be a topological space and (X,τ_s) be the semi-regularization space. Then $\overline{U}^{\tau} = \overline{U}^{\tau_s}$ for any $U \subset X$.*

Proof. $\overline{U}^{\tau} \subset \overline{U}^{\tau_s}$: Let $x \in \overline{U}^{\tau}$ and assume that $x \in X - \overline{U}^{\tau_s}$. Then

$$x \in \text{int}_{\tau_s}(X - U).$$

For any τ_s-open set W, $x \in W \subset X - U$. Since

$$W = \bigcup \{V | V \text{ regular open in } X\}$$

is an open set in (X,τ), we have that $x \in \text{int}(X - U)$. Therefore, $x \in X - \overline{U}$, which is a contradiction.

$\overline{U}^{\tau_s} \subset \overline{U}$: Let $x \in \overline{U}^{\tau_s}$. Then $x \in U \subset K$ for all τ_s-closed sets. Because

$$K = \bigcap \{F | F \text{ regular closed in } X\}$$

is a closed subset of (X,τ), we obtain that $x \in \overline{U}$. □

Corollary 3.2.7. *Let (X,τ) be a topological space and (X,τ_s) be the semi-regularization space. Then $\text{int}_{\tau_s}(U) = \text{int}(U)$ for any $U \subset X$.*

Proof. Let $U \subset X$. Then

$$\begin{aligned}\text{int}(U) &= \text{int}(X - X - U)\\ &= X - \overline{X - U}\\ &= X - \overline{X - U}^{\tau_s} \text{ by Lemma 3.2.6}\\ &= \text{int}_{\tau_s}(X - X - U)\\ &= \text{int}_{\tau_s}(U)\end{aligned}$$

which proves the result. □

Part of the proof for the following proposition is from [16].

Proposition 3.2.8. *[16] If (X,τ) is a seminormal and partial normal space, then (X,τ_s) is β-normal.*

Proof. Let A and B be disjoint closed subsets of (X,τ_s). Then

$$A = \bigcap \{F | F \text{ regular closed in X}\}$$

and
$$B = \bigcap \{K | K \text{ regular closed in } X\}$$
are closed subsets of (X, τ). So, $A \subset X - B$, and by seminormality, there is a regular open set $V \subset X$ such that
$$A \subset V \subset X - B.$$
We have that A is closed in (X, τ_s) and V is open in (X, τ_s). Since (X, τ) is partial normal, using [4, Theorem 2.11], there exists a τ_s-open set U such that
$$A \subset U \subset \overline{U} \subset \text{int}(\overline{V}) \subset X - B.$$
Therefore,
$$\overline{A \cap U} = A \subset \overline{U} \subset X - B$$
since A is closed. But $B \subset X - \overline{U}$ so, again by seminormality, there is a regular open set W of X such that
$$B \subset W \subset X - \overline{U}.$$
Because $\text{int}(\overline{W})$ is regular open and using [4, Theorem 2.11], there exists a τ_s-open set G such that
$$B \subset G \subset \overline{G} \subset \text{int}(\overline{W}) \subset X - \overline{U}.$$
It follows that
$$\overline{B \cap G} = B \subset \overline{G} \subset X - \overline{U}.$$
Therefore $\overline{U} \cap \overline{G} = \emptyset$. Thus (X, τ_s) is β-normal. □

We move our focus to discussing β-normality with respect to the ultrafilter space given in [36, 33].

Definition 3.2.9. [23] A non-empty family $\mathcal{F}$ of subsets of a non-empty set X is called a *filter* on X if it satisfies the following conditions:

1. $\emptyset \notin \mathcal{F}, X \in \mathcal{F}$;
2. $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$ and
3. $A \subset B \subset X$ and $A \in \mathcal{F}$ implies $B \in \mathcal{F}$.

Definition 3.2.10. [23] If $\mathcal{F}_1$ and $\mathcal{F}_2$ are filters on a set X such that $\mathcal{F}_1 \subset \mathcal{F}_2$, then we call $\mathcal{F}_2$ a *refinement* of $\mathcal{F}_1$.

Definition 3.2.11. [45] Let X be a set. A filter $\mathcal{G}$ on X is called an *ultrafilter* on X if, whenever $\mathcal{H}$ is a filter and $\mathcal{H}$ is a refinement of $\mathcal{G}$, then $\mathcal{G} = \mathcal{H}$.

The set $\mathcal{U}X$ denotes the set of all ultrafilters on X for a topological space (X, τ) and each ultrafilter on X is considered as a point in $\mathcal{U}X$, denoted by lowercase letters (e.g., p, q etc) [33].

Definition 3.2.12. [36] Let (X, τ) be a topological space and suppose $A \subset X$. We define

$$A^* = \{p \in \mathcal{U}X | A \in p\}.$$

In [33], it is shown that the collection

$$\mathcal{B} = \{G^* | G \in \tau\}$$

is a base for some topology on a space $\mathcal{U}X$. The topology generated by this collection is denoted by $\mathcal{U}\tau$, and the topological space $(\mathcal{U}X, \mathcal{U}\tau)$ is called the *ultrafilter space* of the topological space (X, τ) [36].

We shall use the following properties of A^* which are proved in [33]:

1. $\emptyset^* = \emptyset$
2. $X^* = \mathcal{U}X$
3. $A \subset B$ if and only if $A^* \subset B^*$
4. $A \cap B = \emptyset$ if and only if $A^* \cap B^* = \emptyset$
5. $(A \cap B)^* = A^* \cap B^*$

We also recall from [31, Lemma 2.3] that

$$\left(\overline{\bigcup_{V \in \mathcal{B}} V}\right)^* = \overline{\bigcup_{V \in \mathcal{B}} (V^*)}$$

for a collection $\mathcal{B}$ of subsets of X.

For the coming proposition, we shall need the following two results.

Corollary 3.2.13. *[31] Let (X, τ) be a topological space. Then for all $A \subset X$, $\left(\overline{A}\right)^* = \overline{A^*}$.*

Proof. (⇒): Let $p \in \left(\overline{A}\right)^*$ and $p \notin \overline{A^*}$. Then $p \in (X^* - \overline{A^*})$. Therefore there is $G \in \tau$ such that $p \in G^* \subset X^* - \overline{A^*}$ which implies that $G^* \cap A^* = \emptyset$. Thus, $G \cap A = \emptyset$, making $G \cap \overline{A} = \emptyset$, which is a contradiction.

(⇐): Let $p \in \overline{A^*}$ and assume that $p \notin \left(\overline{A}\right)^*$. Then $\overline{A} \notin p$ implying that $X - \overline{A} \in p$. Therefore, $p \in X^* - \left(\overline{A}\right)^*$ is an open nhood of p and thus

$$A^* \cap \left(X^* - \left(\overline{A}\right)^*\right) \neq \emptyset.$$

Since $A^* \subset \left(\overline{A}\right)^*$ we get that

$$\left(\overline{A}\right)^* \cap \left(X^* - \left(\overline{A}\right)^*\right) \neq \emptyset,$$

a contradiction. □

Lemma 3.2.14. *Let (X, τ) be a topological space. Then $\overline{U} \cap \overline{V} = \emptyset$ if and only if $\overline{U^*} \cap \overline{V^*} = \emptyset$ for all $U, V \subset X$.*

Proof. We have that

$$\begin{aligned} \overline{U} \cap \overline{V} = \emptyset &\Leftrightarrow (\overline{U} \cap \overline{V})^* = \emptyset^* \\ &\Leftrightarrow (\overline{U})^* \cap (\overline{V})^* = \emptyset \\ &\Leftrightarrow \overline{U^*} \cap \overline{V^*} = \emptyset \end{aligned}$$

which ends the proof. □

The following result tells us that β-normality of the ultrafilter space $(\mathcal{U}X, \mathcal{U}\tau)$ implies β-normality of (X, τ).

Proposition 3.2.15. *(X, τ) is β-normal only if $(\mathcal{U}X, \mathcal{U}\tau)$ is β-normal.*

Proof. Choose F and K disjoint closed sets of X. Then F^* and K^* are disjoint and closed. By assumption, there exist $\mathcal{U}\tau$-open sets U and V, such that $\overline{F^* \cap U} = F^*$, $\overline{K^* \cap V} = K^*$ and $\overline{U} \cap \overline{V} = \emptyset$. We have that $U = \bigcup_{i \in I} U_i^*$ where each $U_i^* \in \tau$ and $V = \bigcup_{j \in I} V_j^*$ where each $V_j^* \in \tau$. Therefore

$$\overline{F^* \cap \bigcup_{i \in I} U_i^*} = \overline{\left(F \cap \bigcup_{i \in I} U_i\right)^*} = F^*,$$

$$\overline{K^* \cap \bigcup_{j \in J} V_j^*} = \overline{\left(K \cap \bigcup_{j \in J} V_j\right)^*} = K^*$$

and

$$\left(\overline{\bigcup_{i\in I} U_i^*}\right) \cap \left(\overline{\bigcup_{j\in J} V_j^*}\right) = \emptyset.$$

We get that

$$\overline{F \cap \bigcup_{i\in I} U_i} = F, \quad \overline{K \cap \bigcup_{i\in J} V_j} = K$$

and

$$\overline{\bigcup_{i\in I} U_i} \cap \overline{\bigcup_{j\in J} V_j} = \emptyset.$$

Thus (X, τ) is β-normal. □

We remind a reader that a topological space (X, τ) is *discrete* if every subset of X is open in X. A space (X, τ) *compact* [45] if every open cover has a finite subcover.

Recall from [36] that for a topological space (X, τ);

1. $(\mathcal{U}X, \mathcal{U}\tau)$ is compact.

2. If (X, τ) is discrete, then $(\mathcal{U}X, \mathcal{U}\tau)$ is Hausdorff.

It is well known that every compact Hausdorff space is normal, hence the following observation.

Observation 3.2.2. A case where β-normality of (X, τ) implies β-normality of $(\mathcal{U}X, \mathcal{U}\tau)$ is when (X, τ) is discrete: Indeed, if (X, τ) is discrete, then $(\mathcal{U}X, \mathcal{U}\tau)$ is Hausdorff. But $(\mathcal{U}X, \mathcal{U}\tau)$ is always compact and every compact Hausdorff space is normal, so $(\mathcal{U}X, \mathcal{U}\tau)$ is normal. Since from Proposition 3.2.1 we have that normality implies β-normality, we get that $(\mathcal{U}X, \mathcal{U}\tau)$ is β-normal.

Recall the following definitions from [45]. Let (X, τ) be a topological space.

1. (X, τ) is said to be *completely regular* if, whenever A is a closed set in X and $x \notin A$, there is a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(A) = \{1\}$.

2. (X, τ) is said to be *Tychonoff* if it is T_1 and completely regular.

In [27, Example 1], the author asked whether there does exist a regular β-normal non-Tychonoff space. This was answered in the same paper. We also give an answer here. We shall need the following two results.

Lemma 3.2.16. *Almost discrete spaces are regular and β-normal.*

Proof. Let (X, τ) be an almost discrete space, $x \in X$ and choose a closed set F of X such that $x \notin F$. Then $X - F$ is an open set containing x. Since (X, τ) is almost discrete, F is an open set containing itself. Therefore F and $X - F$ are disjoint open sets and (X, τ) is regular.

β-normality follows, since, by Observation 2.2.1., every almost discrete space is normal, and also knowing that normality implies β-normality. □

Let us recall from [37] that:

1. $(\mathcal{U}X, \mathcal{U}\tau)$ is T_1 if and only if (X, τ) is discrete and finite.
2. $(\mathcal{U}X, \mathcal{U}\tau)$ is (completely) regular if and only if every open set in (X, τ) is closed (i.e. (X, τ) is almost discrete).
3. $(\mathcal{U}X, \mathcal{U}\tau)$ is normal if and only if (X, τ) is normal.

Lemma 3.2.17. *If (X, τ) is almost discrete, then $(\mathcal{U}X, \mathcal{U}\tau)$ is β-normal*

Proof. Since (X, τ) is almost discrete we have that (X, τ) is normal from Observation 2.2.1. This implies that $(\mathcal{U}X, \mathcal{U}\tau)$ is normal. Thus $(\mathcal{U}X, \mathcal{U}\tau)$ is β-normal. □

We now answer the question in [27, Example 1].

Example 3.2.18. Let (X, τ) be a non-discrete and almost discrete space (See Example 2.2.24(1)). It follows from Lemma 3.2.17 that $(\mathcal{U}X, \mathcal{U}\tau)$ is (completely) regular and β-normal. But $(\mathcal{U}X, \mathcal{U}\tau)$ is not T_1 because (X, τ) is not discrete and therefore $(\mathcal{U}X, \mathcal{U}\tau)$ is not Tychonoff.

3.3 β-Normality and continuous functions

Our next goal is to study preservation and reflection of β-normal spaces by continuous mappings.

Lemma 3.3.1. *Let $f : X \to Y$ be an onto continuous function. If*

$$\overline{f^{-1}(A) \cap U} = f^{-1}(A),$$

then $\overline{A \cap f(U)} = A$ for every $U \subset X$ and all closed $A \subset Y$.

Proof. Let $A \subset Y$ be closed and $U \subset X$. Observe that $A \cap f(U) \subset A$, therefore $\overline{A \cap f(U)} \subset A$ since A is closed.

Since f is continuous and A is closed, $f^{-1}(A)$ is closed. But

$$f^{-1}(A) \cap U \subset f^{-1}(A),$$

it follows that $\overline{f^{-1}(A) \cap U} \subset f^{-1}(A)$.

From hypothesis, we have that

$$f^{-1}(A) \subset \overline{f^{-1}(A) \cap U}$$

which implies that

$$\begin{aligned} f(f^{-1}(A)) \subset f(\overline{f^{-1}(A) \cap U}) &\Rightarrow A \subset \overline{f(f^{-1}(A) \cap U)} \text{ since f is onto} \\ &\Rightarrow A \subset \overline{f(f^{-1}(A)) \cap f(U)} \\ &\Rightarrow A \subset \overline{A \cap f(U)}. \end{aligned}$$

Thus $\overline{A \cap f(U)} = A$. □

Proposition 3.3.2. *[7] Let $f : X \to Y$ be a continuous function between two topological spaces such that f is onto, open and closed. If X is β-normal, then so is Y.*

Proof. Let A be closed in Y and V open in Y with $A \subset V$. Since f is continuous, $f^{-1}(A)$ is closed in X and $f^{-1}(V)$ open in X such that $f^{-1}(A) \subset f^{-1}(V)$.

As (X, τ) is β-normal, there exists an open set U of X such that

$$\overline{f^{-1}(A) \cap U} = f^{-1}(A) \subset \overline{U} \subset f^{-1}(V).$$

By Lemma 3.3.1, $\overline{A \cap f(U)} = A$. Therefore

$$f(f^{-1}(A)) \subset f(\overline{U}) \subset f(f^{-1}(V))$$

and $\overline{f(U)} = f(\overline{U})$ implying that $A \subset \overline{f(U)} \subset V$ since f is onto.

Therefore Y is β-normal. □

We digress to discuss preservation of β-normality by the T_0-reflection map. Recall the following three definitions from [1].

Definition 3.3.3. A *category* $\mathcal{A}$ is a quadruple $= (\mathcal{O}, hom, id, \circ)$ consisting of

1. a class $\mathcal{O}$, whose members are called $\mathcal{A}$-objects,

2. for each pair (A, B) of $\mathcal{A}$-objects, a set $hom_{\mathcal{A}}(A, B)$, whose members are called $\mathcal{A}$-morphisms from A to B,

3. for each $\mathcal{A}$-object A, a morphism $id_{\mathcal{A}} : A \to A$, called the $\mathcal{A}$-identity on $\mathcal{A}$, and

4. a composition law associating with each $\mathcal{A}$-morphism $f : A \to B$ and each $\mathcal{A}$-morphism $g : B \to C$ an $\mathcal{A}$-morphism $g \circ f : A \to C$, called the composite of f and g, subject to the following conditions:

   (a) composition is associative; i.e., for $\mathcal{A}$-morphisms $f : A \to B$, $g : B \to C$ and $h : C \to D$. the equation $h \circ (g \circ f) = (h \circ g) \circ f$ holds,

   (b) $\mathcal{A}$-identities act as identities to composition: i.e., for $\mathcal{A}$-morphisms $f : A \to B$, we have $id_B \circ f = f$ and $f \circ id_A = f$,

   (c) the sets $hom_{\mathcal{A}}(A, B)$ are pairwise disjoint.

Definition 3.3.4. A category $\mathcal{B}$ is said to be a *subcategory* of a category $\mathcal{A}$ provided that the following conditions are satisfied:

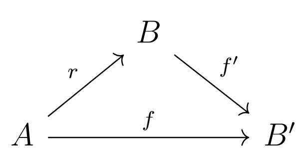
1. for each $B \in \mathcal{B}, B \in \mathcal{A}$,

2. for each $B, B' \in \mathcal{B}, hom_{\mathcal{B}}(B, B') \subset hom_{\mathcal{A}}(B, B')$,

3. for each $\mathcal{B}$-object B, the $\mathcal{A}$-identity on B is the $\mathcal{B}$-identity on B,

4. the composition law in $\mathcal{B}$ is the restriction of the composition law in $\mathcal{A}$ to the morphisms of $\mathcal{B}$.

Definition 3.3.5. Let $\mathcal{B}$ be a subcategory of a category $\mathcal{A}$. We say that $\mathcal{B}$ is a *reflective subcategory* of $\mathcal{A}$ if for each $A \in \mathcal{A}$, there is a $\mathcal{B}$-object B and a $\mathcal{B}$-morphism $r : A \to B$ with the following universal property: for any $\mathcal{A}$-morphism $f : A \to B'$ from A into some $\mathcal{B}$-object B', there exists a unique $\mathcal{B}$-morphism $f' : B \to B'$ such that the triangle



commutes.

The pair $(\mathcal{B}, r)$ is called a $\mathcal{B}$-reflection for $\mathcal{A}$. Furthermore, $\mathcal{B}$ is called a reflective subcategory of $\mathcal{A}$ provided that each $\mathcal{A}$-object has a $\mathcal{B}$-reflection.

Proposition 3.3.6. *[45] Let $\sim$ be an equivalence relation on X given by $x \sim y$ if and only if $\overline{\{x\}} = \overline{\{y\}}$, for each $x, y \in X$. The quotient space $X - (\sim)$ is the T_0-reflection of X and $r : X \to X - (\sim)$, given by $x \mapsto [x]$, is the T_0-reflection map.*

We denote the set X_0 as the T_0-reflection of X. It is worth noting that the T_0-reflection mapping is continuous, open, closed and onto [36]. We also have that $r(r^{-1}(A)) = A$ for each open (resp. closed) A in X_0.

Lemma 3.3.7. *Let $f : (X, \tau) \to (Y, \rho)$ be a continuous function. If*

$$\overline{U \cap f(A)} = f(A),$$

for every $U \subset Y$ and all closed $A \subset X$, then

$$\overline{A \cap f^{-1}(U)} = A.$$

Proof. Let $A \subset X$ be closed and $U \subset Y$. Since $A \cap f^{-1}(U) \subset A$ and A is closed, $\overline{A \cap f^{-1}(U)} \subset A$.

Let $x \in A$ and choose N an open nhood of x. Then $f(x) \in f(A) = \overline{U \cap f(A)}$. This implies that

$$\begin{aligned} f(N) \cap f(A) \cap U \neq \emptyset &\Rightarrow f^{-1}(f(N)) \cap f^{-1}(f(A)) \cap f^{-1}(U) \neq \emptyset \\ &\Rightarrow N \cap A \cap f^{-1}(U) \neq \emptyset, \end{aligned}$$

meaning that $x \in \overline{A \cap f^{-1}(U)}$. Therefore $\overline{A \cap f^{-1}(U)} = A$. □

Proposition 3.3.8. *Let $r : X \to X_0$ be the T_0-reflection map for a space X. Then X is β-normal if and only if X_0 is β-normal.*

Proof. ($\Rightarrow$): Since r is continuous, onto, open and closed, it follows from Proposition 3.3.2 that X_0 is β-normal.

($\Leftarrow$): Let A be a closed subset of X and U be an open subset of X with $A \subset U$. From hypothesis, we have that $r(A)$ is closed in X_0 and $r(U)$ in X_0 is open such that $r(A) \subset r(U)$. Since X_0 is β-normal, there exists an open set $V \subset X_0$ such that

$$\overline{r(A) \cap V} = r(A) \subset \overline{V} \subset r(U).$$

By Lemma 3.3.7, $\overline{A \cap r^{-1}(V)} = A$.

$\overline{r^{-1}(V)} \subset U$: Let $x \in \overline{r^{-1}(V)}$. Then

$$[x] \in r(\overline{r^{-1}(V)}) \subset \overline{r(r^{-1}(V))} = \overline{V}.$$

Therefore $[x] \in \overline{V} \subset r(U)$ implying that $x \in r^{-1}(r(U)) = U$.

Thus X is β-normal. □

We now consider functions which pull β-normality backwards.

Proposition 3.3.9. *Let $f : X \to Y$ be a continuous function between two topological spaces such that such that f is injective, open and closed. If Y is β-normal, then so is X.*

Proof. Let $A \subset X$ be closed and $U \subset X$ be open such that $A \subset U$. Then $f(A)$ is closed in Y and $f(U)$ is open in Y with $f(A) \subset f(U)$. By hypothesis there is a Y-open set V such that

$$\overline{V \cap f(A)} = f(A) \subset \overline{V} \subset f(U).$$

From Lemma 3.3.7 we get that $\overline{A \cap f^{-1}(V)} = A$.

We show that $\overline{f^{-1}(V)} \subset U$: Since $\overline{V} \subset f(U)$,

$$f^{-1}(\overline{V}) \subset f^{-1}(f(U)) = U$$

by injectivity of f. But $\overline{f^{-1}(V)} \subset f^{-1}(\overline{V})$ since f is continuous, so $\overline{f^{-1}(V)} \subset U$.

Therefore X is β-normal. □

We show that the conditions hypothesized in Proposition 3.3.9 do not necessarily make f a homeomorphism.

Example 3.3.10. Let $X = \{a, b\}$ and $Y = \{x, y, z\}$ be endowed with discrete topologies τ_d and ρ_d, respectively. Define $f : (X, \tau_d) \to (Y, \rho_d)$ as $f(a) = x$ and $f(b) = y$. Then f is continuous, open, closed and injective. But f is not homeomorphic.

Chapter 4

α-Normal spaces

This chapter aims to introduce and investigate properties of α-normal spaces.

4.1 Characterizing α-normal spaces

In this section will give characterizations of α-normal spaces. We begin with the following definition.

Definition 4.1.1. [7] A topological space (X, τ) is *α-normal* if for any pair of disjoint closed sets $F, K \subset X$, there are disjoint open sets U, V of X such that $F \cap U$ is dense in F and $K \cap V$ is dense in K.

We note the following example.

Example 4.1.2. Let $X = \{a, b, c, d\}$ be a set endowed with the topology

$$\tau = \{\emptyset, X, \{a, c, d\}, \{a, b, c\}, \{a, c\}, \{c, d\}, \{c\}, \{a\}, \{a, b\}\}.$$

Then (X, τ) is α-normal. We have that the closed sets of X are

$$\emptyset, X, \{b\}, \{d\}, \{b, d\}, \{a, b\}, \{c, d\}, \{a, b, d\} \text{ and } \{b, c, d\}.$$

We recall that a subset Y of a topological space (X, τ) is *nowhere dense* if $\mathrm{int}(\overline{Y}) = \emptyset$ [45]. Equivalently, Y is nowhere dense if and only if the complement of its closure is dense. Recall from [14] that $\mathrm{int}_A(Y) = A - \overline{A - Y}^A$. For $Y \subset A \subset X$, the subset Y of A is called *A-nowhere dense* if Y is nowhere dense in A.

The following lemma is an adaptation to spaces of a result given in [32, 30] in terms of locales.

Lemma 4.1.3. *Let (X, τ) be a topological space. Then a subspace Y of a subspace A of X is A-nowhere dense if and only if $A \subset \overline{A \cap (X - \overline{Y})}$.*

Proof. We have that

$$
\begin{aligned}
Y \text{ is nowhere dense in } A &\Leftrightarrow \operatorname{int}_A(\overline{Y}^A) = \emptyset \\
&\Leftrightarrow \operatorname{int}_A(\overline{Y} \cap A) = \emptyset \\
&\Leftrightarrow A - \overline{(A - (\overline{Y} \cap A))}^A = \emptyset \\
&\Leftrightarrow A - \left(\overline{A - (\overline{Y} \cap A)} \cap A\right) = \emptyset \\
&\Leftrightarrow A \cap \left(X - \left(\overline{A - (\overline{Y} \cap A)} \cap A\right)\right) = \emptyset \\
&\Leftrightarrow A \cap \left(\left(X - \overline{(A - (\overline{Y} \cap A))}\right) \cup (X - A)\right) = \emptyset \\
&\Leftrightarrow \left(A \cap \left(X - \overline{(A - (\overline{Y} \cap A))}\right)\right) \cup (A \cap (X - A)) = \emptyset \\
&\Leftrightarrow A \cap \left(X - \overline{(A \cap (X - (\overline{Y} \cap A)))}\right) = \emptyset \\
&\Leftrightarrow A \cap \left(X - \overline{(A \cap ((X - \overline{Y}) \cup (X - A)))}\right) = \emptyset \\
&\Leftrightarrow A \cap \left(X - \overline{(A \cap (X - \overline{Y})) \cup (A \cap (X - A))}\right) = \emptyset \\
&\Leftrightarrow A \cap \left(X - \overline{(A \cap (X - \overline{Y}))}\right) = \emptyset \\
&\Leftrightarrow A \subset \overline{A \cap (X - \overline{Y})}
\end{aligned}
$$

which concludes the proof. □

In the following proposition, we characterize α-normal spaces. The first two statements are from [21, Lemma 2.2].

Proposition 4.1.4. *The following statements are equivalent for a topological space (X, τ).*

1. *(X, τ) is α-normal.*
2. *For any pair of disjoint closed subsets F and K of X there exists an open subset U of X such that $\overline{F \cap U} = F$ and $K \cap \overline{U}$ is nowhere dense in K.*
3. *For every open subsets U and V of X satisfying $U \cup V = X$, there is an open set $G \subset X$ such that*

$$U = \operatorname{int}(U \cup (X - G))$$

and

$$V = \operatorname{int}(V \cup \overline{G}).$$

4. For any collection $\{U_i\}_{i\in\mathbb{N}}$ *and* $\{V_i\}_{i\in\mathbb{N}}$ *of open subsets of* X *such that*

$$U_i \cup \operatorname{int}\left(\bigcap V_i\right) = X$$

and

$$V_i \cup \operatorname{int}\left(\bigcap U_i\right) = X,$$

$i \in \mathbb{N}$, *there exists* $G \in \tau$ *such that*

$$U_i = \operatorname{int}(U_i \cup (X - G))$$

and

$$V_i = \operatorname{int}(V_i \cup \overline{G})$$

for each $i \in \mathbb{N}$.

Proof. (1. $\Rightarrow$ 2.): Let F and K be disjoint closed subsets of X. Since (X, τ) is α-normal there exist disjoint open sets $U, V \subset X$ such that $F \cap U$ is dense in F and $K \cap V$ is dense in K. Since $U \cap V = \emptyset$, $\overline{U} \cap V = \emptyset$ which implies that $K \cap \overline{U} \cap V = \emptyset$. Therefore, $V \subset X - (K \cap \overline{U})$ so that

$$K \cap V \subset K \cap [X - (\overline{U} \cap K)] = K - (\overline{U} \cap K).$$

But $K \cap V$ is dense in K meaning that $K - (\overline{U} \cap K)$ is dense in K and $\overline{U} \cap K$ is nowhere dense in K.

(2. $\Rightarrow$ 3.): Let U and V be open subsets of X whose union is X. Then $X - U$ and $X - V$ are disjoint closed subsets of X. By 2., there is an open set $G \subset X$ such that $\overline{(X - U) \cap G} = X - U$ and $\overline{G} \cap (X - V)$ is nowhere dense in $X - V$. Then

$$U = X - \left(\overline{(X - U) \cap G}\right) = \operatorname{int}(U \cup (X - G)).$$

By Lemma 4.1.3 we get that

$$X - V \subset \overline{(X - V) \cap \left(X - \left(\overline{\overline{G} \cap (X - V)}\right)\right)} \subset \overline{X - V} = X - V$$

since $X - V$ is closed. Therefore,

$$\begin{aligned}
X - V &= \overline{(X - V) \cap \left(X - \overline{\overline{G} \cap (X - V)}\right)} \\
&= \overline{(X - V) \cap \left(X - (\overline{G} \cap (X - V))\right)} \text{ since } \overline{G} \cap (X - V) = \overline{\overline{G} \cap (X - V)} \\
&= \overline{(X - V) \cap \left((X - \overline{G}) \cup V)\right)} \\
&= \overline{\left((X - V) \cap (X - \overline{G})\right) \cup ((X - V) \cap V)} \\
&= \overline{(X - V) \cap (X - \overline{G})} \text{ since } (X - V) \cap V = \emptyset \\
&= \overline{X - (V \cup \overline{G})}
\end{aligned}$$

so that

$$V = X - (\overline{X - (V \cup \overline{G})}) = \text{int}(V \cup \overline{G}).$$

(3. $\Rightarrow$ 4.): Let $\{U_i\}_{i \in \mathbb{N}}$ and $\{V_i\}_{i \in \mathbb{N}}$ be a collection of open subsets of X satisfying

$$U_i \cup \text{int}\left(\bigcap V_i\right) = X$$

and

$$V_i \cup \text{int}\left(\bigcap U_i\right) = X$$

for each $i \in \mathbb{N}$. Then, for every $i \in \mathbb{N}$,

$$U_i \cup \text{int}\left(\bigcap V_i\right) = X$$

which implies by 3. that there is an open set $G_i \in \tau$ satisfying

$$U_i = \text{int}(U_i \cup (X - G_i))$$

and

$$\text{int}\left(\bigcap V_i\right) = \text{int}\left(\text{int}\left(\bigcap V_i\right) \cup \overline{G_i}\right).$$

Similarly, from

$$V_i \cup \text{int}\left(\bigcap U_i\right) = X,$$

it follows that there is $H_i \in \tau$ such that

$$V_i = \text{int}(V_i \cup (X - H_i))$$

and

$$\text{int}\left(\bigcap U_i\right) = \text{int}\left(\text{int}\left(\bigcap U_i\right) \cup \overline{H_i}\right).$$

For each $i \in \mathbb{N}$, set

$$G = \bigcup_{i=1}^{\infty} \left(G_i \cap \operatorname{int} \left(\bigcap_{k=1}^{i} X - \overline{H_k} \right) \right)$$

and

$$H = \bigcup_{i=1}^{\infty} \left(H_i \cap \operatorname{int} \left(\bigcap_{k=1}^{i} X - \overline{G_k} \right) \right).$$

We show that $\operatorname{int}(U_i \cup (X - G)) \subset U_i$: Indeed,

$$\begin{aligned}
\operatorname{int}(U_i \cup (X - G)) &= \operatorname{int} \left(U_i \cup \left(X - \bigcup_{i=1}^{\infty} \left(G_i \cap \operatorname{int} \left(\bigcap_{k=1}^{i} X - \overline{H_k} \right) \right) \right) \right) \\
&= \operatorname{int} \left(U_i \cup \bigcap_{i=1}^{\infty} \left((X - G_i) \cup \left(X - \operatorname{int} \left(\bigcap_{k=1}^{i} (X - \overline{H_k}) \right) \right) \right) \right) \\
&\subset \operatorname{int} \left(U_i \cup (X - G_i) \cup \overline{\bigcup_{k=1}^{i} \overline{H_k}} \right) \\
&= \operatorname{int} \left(U_i \cup (X - G_i) \cup \bigcup_{k=1}^{i} \overline{H_k} \right) \\
&= \bigcup_{k=1}^{i} \left(\operatorname{int}(U_i \cup (X - G_i) \cup \overline{H_k}) \right) \\
&\subset \bigcup_{i=1}^{\infty} \left(\operatorname{int} \left(U_i \cup (X - G_i) \cup \overline{H_i} \right) \right)
\end{aligned}$$

We claim that

$$\operatorname{int}(U_i \cup (X - G_i) \cup \overline{H_i}) \subset U_i.$$

To see this, choose

$$x \in \operatorname{int}(U_i \cup (X - G_i) \cup \overline{H_i}).$$

Then

$$x \in Q \subset U_i \cup (X - G_i) \cup \overline{H_i}$$

for some open $Q \subset X$. Therefore,

$$Q \cap (X - \overline{H_i}) \subset U_i \cup (X - G_i)$$

so that

$$Q \cap (X - \overline{H_i}) \subset \operatorname{int}(U_i \cup (X - G_i))$$

since $Q \cap (X - \overline{H_i})$ is open. Because

$$U_i = \operatorname{int}(U_i \cup (X - G_i)),$$

we have that $Q \cap (X - \overline{H_i}) \subset U_i$. Therefore $Q \subset U_i \cup \overline{H_i}$, which implies that

$$\begin{aligned} Q &\subset \bigcap_{i \in \mathbb{N}} (U_i \cup \overline{H_i}) \\ &= \bigcap_{i \in \mathbb{N}} U_i \cup \bigcap_{i \in \mathbb{N}} \overline{H_i} \\ &\subset \left(\bigcap_{i \in \mathbb{N}} U_i \right) \cup \overline{H_i}. \end{aligned}$$

This makes

$$\begin{aligned} Q &\subset \operatorname{int}\left(\left(\bigcap_{i \in \mathbb{N}} U_i \right) \cup \overline{H_i} \right) \\ &\subset \operatorname{int}\left(\operatorname{int}\left(\bigcap_{i \in \mathbb{N}} U_i \right) \cup \overline{H_i} \right) \\ &= \operatorname{int}\left(\bigcap_{i \in \mathbb{N}} U_i \right) \\ &\subset \operatorname{int} U_i \\ &= U_i. \end{aligned}$$

Thus $x \in U_i$ and hence

$$\operatorname{int}(U_i \cup (X - G)) \subset U_i.$$

Similarly,

$$\operatorname{int}(V_i \cup (X - H)) = V_i.$$

Observe that $G \cap H = \emptyset$:

$$\begin{aligned} &\bigcup_{i=1}^{\infty} \left(G_i \cap \operatorname{int}\left(\bigcap_{k=1}^{i} X - \overline{H_k} \right) \right) \cap \bigcup_{i=1}^{\infty} \left(H_j \cap \operatorname{int}\left(\bigcap_{l=1}^{i} X - \overline{G_l} \right) \right) \\ = &\bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} \left[G_i \cap \operatorname{int}\left(\bigcap_{k=1}^{i} X - \overline{H_k} \right) \cap H_j \cap \operatorname{int}\left(\bigcap_{l=1}^{i} X - \overline{G_l} \right) \right] = \emptyset. \end{aligned}$$

Therefore $\overline{G} \cap H = \emptyset$ so that $\overline{G} \subset (X - H)$. This implies that

$$\text{int}(V_i \cup \overline{G}) \subset \text{int}(V_i \cup (X - H)) \subset V_i.$$

Thus $\text{int}(V_i \cup \overline{G}) \subset V_i$ as required.

(4. $\Rightarrow$ 1.): Let F and K be closed subsets of X such that $F \cap K = \emptyset$. Then $X - F$ and $X - K$ are open subsets of X such that $(X - F) \cup (X - K) = X$. Set $X - F = U_i$ and $X - K = V_i$ for each $i \in \mathbb{N}$. Then $\{U_i\}_{i\in\mathbb{N}}$ and $\{V_i\}_{i\in\mathbb{N}}$ are a collection of open subsets of X satisfying $U_i \cup \text{int}\left(\bigcap_{i\in\mathbb{N}} V_i\right) = X$ and $V_i \cup \text{int}\left(\bigcap_{i\in\mathbb{N}} U_i\right) = X$ for all $i \in \mathbb{N}$. It follows from 4. that there exists an open set $G \subset X$ satisfying $U_i = \text{int}(U_i \cup (X - G))$ and $V_i = \text{int}(V_i \cup \overline{G})$ for each $i \in \mathbb{N}$. Therefore

$$X - F = \text{int}((X - F) \cup (X - G))$$

and

$$X - K = \text{int}((X - K) \cup \overline{G}),$$

which imply that $\overline{F \cap G} = F$ and

$$K = X - \text{int}\left((X - K) \cup \overline{G}\right) = \overline{K \cap (X - \overline{G})}.$$

The required disjoint open sets are G and $X - \overline{G}$.

Thus (X, τ) is an α-normal space. □

Similar to β-normal spaces, α-normal spaces are closed hereditary.

Proposition 4.1.5. *[7] α-Normality is closed hereditary.*

Proof. This proof is similar to that of Proposition 3.1.5. □

We need the following lemma to state the next theorem.

Lemma 4.1.6. *Let (X, τ) be topological space. If A and B are separated subsets of X and*

$$Y = \text{int}\left(\overline{(X - \overline{A}) \cup (X - \overline{B})}\right),$$

then $A, B \subset Y$.

Proof. Let $A, B \subset X$ be separated sets. Then $A \subset X - \overline{B}$ and $B \subset X - \overline{A}$ so that

$$\begin{aligned} A \cup B &\subset (X - \overline{A}) \cup (X - \overline{B}) \\ &= \text{int}((X - \overline{A}) \cup (X - \overline{B})) \\ &\subset \text{int}\left(\overline{(X - \overline{A}) \cup (X - \overline{B})}\right) \end{aligned}$$

which concludes the proof. □

Building on the previous lemma, we can now characterize hereditary α-normal spaces.

Theorem 4.1.7. *The following are equivalent for any topological space (X, τ).*

1. *(X, τ) is hereditary α-normal.*
2. *Every preopen subspace of (X, τ) is α-normal.*
3. *Every α-open subspace of (X, τ) is α-normal.*
4. *Every open subspace of (X, τ) is α-normal.*
5. *Every regular open subspace of (X, τ) is α-normal.*
6. *For any pair of separated sets $A, B \subset X$, there are disjoint open sets U and V of X such that $\overline{A \cap U} = A$ and $\overline{B \cap V} = B$.*

Proof. (1. $\Rightarrow$ 2.): Trivial.

(2. $\Rightarrow$ 3.): Let Y be an α-open subset of X. Since every α-open set is preopen, we have that Y is a preopen subset of X.

(3. $\Rightarrow$ 4.): Consider the open set $Y \subset X$. It follows that Y is a preopen subset of X since every open set is preopen.

(4. $\Rightarrow$ 5.): Let Y be a regular open subset of X. We have that Y is open since every regular open set is open. Therefore Y is the required open set.

(5. $\Rightarrow$ 6.): Let A and B be separated sets of X. Consider

$$Y = \text{int}(\overline{(X - \overline{A}) \cup (X - \overline{B})})$$

a regular open subspace containing A and B. Since A and B are disjoint and Y-closed and Y is α-normal, there exist disjoint open sets $U, V \subset Y$ such that $\overline{A \cap U}^Y = A$ and $\overline{B \cap U}^Y = B$. It is clear that U and V are open subsets of X.

(6. $\Rightarrow$ 1.): [7] Consider Y a subset of X and choose a pair of disjoint Y-closed sets A and B. Then A and B are separated in X. By hypothesis, there are disjoint open subsets of X such that $\overline{A \cap U} = A$ and $\overline{B \cap V} = B$. Now $U \cap Y$ and $V \cap Y$ are open in Y such that

$$\overline{A \cap (U \cap Y)} = A, \quad \overline{B \cap (V \cap Y)} = B$$

and

$$(U \cap Y) \cap (V \cap Y) = \emptyset.$$

Thus, Y is α-normal. $\square$

4.2 More on α-normality

In the following result, we demonstrate that every normal space is α -normal by employing statement 3. of Proposition 4.1.4.

Proposition 4.2.1. *[7] Every normal space is α-normal.*

Proof. Suppose that (X, τ) is a normal space and choose U, V open subsets of X such that $U \cup V = X$. Then

$$(X - U) \cap (X - V) = \emptyset$$

where $X - U$ and $X - V$ are closed sets. By normality of (X, τ), there are disjoint open sets H and G of X such that $X - U \subset H$ and $X - V \subset G$. It follows that $X - H \subset U$ and therefore $(X - H) \cup U = U$. Since U is open, we have that $U = \text{int}((X - H) \cup U)$. Because $H \cap G = \emptyset$, we get that $\overline{H} \cap G = \emptyset$ which implies that $\overline{H} \subset V$ since $X - G \subset V$. Hence, $\overline{H} \cup V = V$ and it follows that $V = \text{int}(V \cup \overline{H})$ by openness of V.

Thus (X, τ) is α-normal. □

In what follows, we give relations of α-normal spaces with other axioms of separation.

Proposition 4.2.2. *[7] If an α-normal space (X, τ) satisfies the T_1 separation axiom, then the space X is Hausdorff.*

Proof. This is proved in [6]. □

Proposition 4.2.3. *If (X, τ) is a seminormal and partial normal space, then (X, τ_s) is α-normal.*

Proof. Follows from Proposition 3.2.8. □

In Chapter 3 we looked at how the ultrafilter space $(\mathcal{U}X, \mathcal{U}\tau)$ implies β-normality of (X, τ). We now do the same for α-normal spaces.

Proposition 4.2.4. *(X, τ) is α-normal only if $(\mathcal{U}X, \mathcal{U}\tau)$ is α-normal.*

Proof. This proof is similar to Proposition 3.2.15. □

Remark 4.2.5. Note that Observation 3.2.2 holds for α-normal spaces.

This section ends with some results demonstrating how α-normality is sent back and forth by continuous functions.

In the next result, we show that α-normality can be preserved by conditions weaker than homeomorphisms.

Proposition 4.2.6. *Let $f : (X, \tau) \to (Y, \rho)$ be a continuous function between two topological spaces such that f is onto, open and closed satisfying that $f^{-1}(f(A)) = A$ for all open $A \subset Y$. If X is α-normal, then so is Y.*

Proof. Let $F, K \subset Y$ be closed and disjoint. Then $f^{-1}(F)$ and $f^{-1}(K)$ are closed and disjoint in X. Since (X, τ) is α-normal, there exist disjoint open sets U and V of X such that

$$\overline{f^{-1}(F) \cap U} = f^{-1}(F) \text{ and } \overline{f^{-1}(K) \cap V} = f^{-1}(K).$$

It follows from Lemma 3.3.1 that $\overline{F \cap f(U)} = F$ and $\overline{K \cap f(V)} = K$.

We now show that $f(U) \cap f(V) = \emptyset$:

$$\begin{aligned} U \cap V = \emptyset &\Rightarrow f^{-1}(f(U)) \cap f^{-1}(f(V)) = \emptyset \\ &\Rightarrow f^{-1}(f(U) \cap f(V)) = \emptyset \\ &\Rightarrow f(f^{-1}(f(U) \cap f(V))) = \emptyset \\ &\Rightarrow f(U) \cap f(V) = \emptyset \text{ since } f \text{ is onto} \end{aligned}$$

which concludes the proof. □

Observation 4.2.1. Not every continuous, onto, open, and closed function $f : X \to Y$ that satisfies $f^{-1}(f(A)) = A$ for all open $A \subset Y$, is a homeomorphism. For instance, the T_0-reflection map is a counterexample.

We give the following well-known result. We have not seen its proof in literature, so we give it here.

Lemma 4.2.7. *Let $f : X \to Y$ be an injective function. If $A \cap B = \emptyset$, then $f(A) \cap f(B) = \emptyset$ for all $A, B \subset X$.*

Proof. Suppose $f(A) \cap f(B) \neq \emptyset$. This means that there exists some

$$y \in f(A) \cap f(B).$$

Then $y \in f(A)$ and $y \in f(B)$. Therefore $y = f(a) = f(b)$ for some $a \in A$ and for some $b \in B$. By the injectivity of f, it follows that $a = b$. Therefore $a \in A \cap B$. Hence a contradiction. □

Proposition 4.2.8. *Let $f : (X, \tau) \to (Y, \rho)$ be a continuous function between two topological spaces such that f is injective, open and closed. If Y is α-normal, then so is X.*

Proof. Let F and K be disjoint closed subsets of X. Since f is closed and injective, $f(F)$ and $f(K)$ are disjoint Y-closed sets. By α-normality of Y, there are disjoint open sets $U, V \subset Y$ such that

$$\overline{f(F) \cap U} = f(F) \text{ and } \overline{f(K) \cap V} = f(K).$$

Since f is continuous, it follows that $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint open subsets of X. Hence, by Lemma 3.3.7, we get that

$$\overline{F \cap f^{-1}(U)} = F \text{ and } \overline{K \cap f^{-1}(V)} = K.$$

Therefore (X, τ) is α-normal □

Chapter 5

β-Normality of bitopological spaces

To make the reading of this chapter easy for a reader, we start by recalling some of the bitopological notions in Chapter 2.

5.1 Introduction

Kelly [18] defined a *bitopological space* (X, τ_1, τ_2) as a set X endowed with two topologies, τ_1 and τ_2. Throughout this chapter we will denote, for a subset A of (X, τ_1, τ_2), the *interior* and *closure* of A with respect to τ_i, where $i \in \{1, 2\}$, as $\text{int}_i(A)$ and $cl_i(A)$, respectively. Let $cl_{iB}(A)$ represent the *closure of subset A in B relative to τ_i* where $i \in \{1, 2\}$. By i-open and i-closed, we shall mean open and closed with respect to the topology τ_i where $i \in \{1, 2\}$.

We refer to [18] and [35] for the following terminologies.

Definition 5.1.1. For a bitopological space (X, τ_1, τ_2), τ_1 *is regular with respect to τ_2* if for each point $x \in X$ and each 1-closed set A such that $x \notin A$, there is a 1-open set U and a 2-open set V disjoint from U such that $x \in U$ and $A \subset V$.

(X, τ_1, τ_2) is *pairwise regular* if τ_1 is regular with respect to τ_2 and τ_2 is regular with respect to τ_1.

Definition 5.1.2. A bitopological space (X, τ_i, τ_j) is *pairwise normal* if for any given i-closed set F and j-closed set K such that $F \cap K = \emptyset$ there is a j-open set U containing F and an i-open set V containing K where $i \neq j, i, j \in \{1, 2\}$.

5.2 $(1, 2)$-β-normal bitopological spaces

This section introduces the notion of a β-normal bitopological space and looks at the interrelations with other separation axioms in bitopological spaces.

We introduce the following definition.

Definition 5.2.1. A bitopological space (X, τ_1, τ_2) is $(1, 2)$-β-normal if for any 1-closed set F and any 2-closed set K such that $F \cap K = \emptyset$, there exist a 2-open set U and a 1-open set V such that $F \cap U$ is 1-dense in F, $K \cap V$ is 2-dense in K and $cl_1(U) \cap cl_2(V) = \emptyset$. A $(2, 1)$-β-normal space is defined in a similar way.

Example 5.2.2. Consider $X = \{a, b, c\}$ with topologies

$$\tau_1 = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}\}$$

and

$$\tau_2 = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$$

defined on X. The 1-closed sets are $\{b, c\}, \{a, b\}, \{b\}, \emptyset$ and X, and the 2-closed sets are $\{b, c\}, \{a, c\}, \{c\}, \{a\}, \emptyset$ and X. (X, τ_1, τ_2) is $(1, 2)$-β-normal: Observe that for the 1-closed set $\{b\}$ disjoint from the 2-closed sets $\{c\}, \{a\}$ and $\{a, c\}$ we can always find a 2-open set $\{b\}$ and a 1-open set $\{a, c\}$ such that the conditions are met. This holds for all 1-closed sets disjoint from the 2-closed sets. This is also an example of a $(2, 1)$-β-normal.

The following three lemmas shall be used in the proof of the subsequent theorem.

Lemma 5.2.3. *We have the following implications:*

$$i\text{-open} \implies (i, j)\text{-}\alpha\text{-open} \implies (i, j)\text{-preopen}.$$

Proof. Let (X, τ_i, τ_j) be a bitopological space.

First implication: Choose A an i-open subset of X. Then $A = \text{int}_i(A)$. Since $A \subset cl_j(A)$, it follows that

$$\text{int}_i(A) \subset \text{int}_i(cl_j(A)).$$

Therefore

$$A \subset \text{int}_i(A) \subset \text{int}_i(cl_j(\text{int}_i(A))),$$

which gives $A \subset \text{int}_i(cl_j(\text{int}_i(A)))$.

Second implication: Let A be an (i, j)-α-open set. Since

$$A \subset \text{int}_i(cl_j(\text{int}_i(A))),$$

we get that $A \subset \text{int}_i(cl_j(A))$, making A an (i, j)-preopen set. □

In the following example, we show that that the converse of the above lemma need not be true.

Example 5.2.4. Let $X = \{a, b, c, d\}$ be a set with the topologies

$$\tau_1 = \{\emptyset, X, \{d\}, \{a, d\}, \{b, c\}, \{b, c, d\}\}$$

and $\tau_2 = \{\emptyset, X, \{d\}, \{a, b, c\}\}$. Then the set $\{a, c, d\}$ is $(1, 2)$-α-open but it is not 1-open. The set $\{b\}$ is $(1, 2)$-preopen but it is neither 1-open nor $(1, 2)$-α-open.

Lemma 5.2.5. *For any bitopological space (X, τ_i, τ_j), where U is a (j, i)-preopen subset and F is an i-closed subset of X, if $cl_i(F \cap U) = F$, then*

$$cl_i(F \cap \text{int}_j(cl_i(U)) = F.$$

Proof. $\underline{cl_i(F \cap \text{int}_j(cl_i(U)) \subset F}$: Since $F \cap \text{int}_j(cl_i(U)) \subset F$ and F is i-closed, we have that

$$cl_i(F \cap \text{int}_j(cl_i(U)) \subset F.$$

$\underline{F \subset cl_i(F \cap U)\,\text{int}_j(cl_i(U))}$: Since

$$F = cl_i(F \cap U) \subset cl_i(F \cap \text{int}_j(cl_i(U))).$$

Therefore, $cl_i(F \cap \text{int}_j(cl_i(U)) = F$. □

Lemma 5.2.6. *Let (X, τ_1, τ_2) be a bitopological space, U an (j, i)-α-open subset of X and $F \subset X$ an i-closed set. If $cl_i(F \cap U) = F$, then*

$$cl_i(F \cap \text{int}_j(cl_i(\text{int}_j(U)))) = F.$$

Proof. Similar to the proof of Lemma 5.2.5. □

Recall from [3] that a subset A of a bitopological space (X, τ_i, τ_j) is called *i-dense* if $cl_i(A) = X$ for $i \in \{1, 2\}$.

We characterize $(1, 2)$-β-normal bitopological spaces. One will notice that, unlike in the case of β-normal spaces, here some equivalences do not necessarily hold. We shall give a condition where all the statements are equivalent.

Theorem 5.2.7. *Let (X, τ_1, τ_2) be a bitopological space. Then consider the following statements:*

1. *(X, τ_1, τ_2) is $(1, 2)$-β-normal.*

2. *For any* 1*-closed set* F *and a* 2*-closed set* K *disjoint from* F*, there exist a* 2*-open set* U *and a* 1*-open set* V *such that* $cl_1(F \cap U) = F$*,* $cl_2(K \cap V) = K$ *and* $cl_1(U) \cap cl_2(V) = \emptyset$*.*

3. *If* F *and* K *are disjoint sets of* X *such that* F *is* 1*-closed and* K *is* 2*-closed, there exist a* $(2,1)$*-*α*-open set* U *and a* $(1,2)$*-*α*-open set* V *such that* $cl_1(F \cap U) = F$*,* $cl_2(K \cap V) = K$ *and* $cl_1(U) \cap cl_2(V) = \emptyset$*.*

4. *For every* 1*-closed set* F *and every* 2*-closed set* K *such that* $F \cap K = \emptyset$*, there exist a* $(2,1)$*-preopen set* U *and a* $(1,2)$*-preopen set* V *such that* $cl_1(F \cap U) = F$*,* $cl_2(K \cap V) = K$ *and* $cl_1(U) \cap cl_2(V) = \emptyset$*.*

5. *For each* 1*-closed set* $A \subset X$ *and each* 2*-open set* $U \subset X$ *such that* $A \subset U$*, there exists a* $(2,1)$*-preopen set* $V \subset X$ *such that*
$$cl_1(A \cap V) = A \subset cl_1(V) \subset U.$$

6. *For every* 1*-closed set* $A \subset X$ *contained in a* 2*-open set* $U \subset X$*, there is a* 2*-open set* $V \subset X$ *satisfying that*
$$cl_1(A \cap V) = A \subset cl_1(V) \subset U.$$

7. *For any* 1*-closed set* $A \subset X$ *and every* 2*-open set* $U \subset X$ *such that* $A \subset U$*, there exist a* $(2,1)$*-*α*-open set* $V \subset X$ *such that*
$$cl_1(A \cap V) = A \subset cl_1(V) \subset U.$$

Then 1. $\Rightarrow$ 2. $\Rightarrow$ 3. $\Rightarrow$ 4. $\Rightarrow$ 5. $\Rightarrow$ 6. $\Leftrightarrow$ 7. *Moreover, all these statements are equivalent whenever* (X, τ_1, τ_2) *is* $(2,1)$*-*β*-normal.*

Proof. (1. $\Rightarrow$ 2.): Let F and K be disjoint subsets of X such that F is 1-closed and K is 2-closed. By 1. there is a 2-open set U and a 1-open set V so that $F \cap U$ is 1-dense in F, $K \cap V$ is 2-dense in K and $cl_1(U) \cap cl_2(V) = \emptyset$. This means that $cl_{1F}(F \cap U) = F$ and $cl_{2K}(K \cap V) = K$. Then by Proposition 2.2.11 we obtain
$$cl_1(F \cap U) \cap F = F$$
and
$$cl_2(K \cap V) \cap K = K.$$
It follows that
$$F \subset cl_1(F \cap U) \cap F$$

so that $F \subset cl_1(F \cap U)$. Since $F \cap U \subset F$ we get

$$cl_1(F \cap U) \subset cl_1(F) = F.$$

Thus $cl_1(F \cap U) = F$. Similarly, it can be shown that $cl_2(K \cap V) = K$.

(2. $\Rightarrow$ 3.): Follows from Lemma 5.2.3.

(3. $\Rightarrow$ 4.): Follows since every (i,j)-α-open set is (i,j)-preopen from Lemma 5.2.3.

(4. $\Rightarrow$ 5.): Let U be a 2-open set containing a 1-closed set A. Then

$$A \cap (X - U) = \emptyset.$$

By 4., there is a $(1,2)$-preopen set W and a $(2,1)$-preopen V so that

$$cl_1(A \cap V) = A, cl_2((X - U) \cap W) = X - U$$

and $cl_2(W) \cap cl_1(V) = \emptyset$. Having that $X - cl_2(W) \subset U$ and $cl_1(V) \subset X - cl_1(W)$, we obtain $cl_1(V) \subset U$. So

$$cl_1(A \cap V) = A \subset cl_1(V) \subset U.$$

(5. $\Rightarrow$ 6.): Let A be 1-closed and U be 2-open such that $A \subset U$. By 5. there exist a $(2,1)$-preopen set V such that

$$cl_1(A \cap V) = A \subset cl_1(V) \subset U.$$

It follows that $V \subset \text{int}_2(cl_1(V))$ (since V is $(2,1)$-preopen), making

$$cl_1(V) \subset cl_1(\text{int}_2(cl_1(V))).$$

Therefore, from Lemma 5.2.5, we get that

$$cl_1(A \cap \text{int}_2(cl_1(V)) = A \subset cl_1(\text{int}_2(cl_1(V)).$$

But

$$cl_1(\text{int}_2(cl_1(V)) \subset cl_1(cl_1(V)) = cl_1(V) \subset U$$

which proves the result.

(6. $\Rightarrow$ 7.): Follows since every i-open set is (i,j)-α-open from Lemma 5.2.3.

(7. $\Rightarrow$ 6.): Let A be 1-closed and U be 2-open such that $A \subset U$. From 7. we have a $(2,1)$-α-open set V such that

$$cl_1(A \cap V) = A \subset cl_1(V) \subset U.$$

Since V is $(2,1)$-α-open we get that $V \subset \text{int}_2(cl_1(\text{int}_2(V)))$. From Lemma 5.2.6 we have that

$$cl_1(A \cap \text{int}_2(cl_1(\text{int}_2(V)))) = A \subset cl_1(\text{int}_2(cl_1(\text{int}_2(V)))).$$

It follows that

$$cl_1(\text{int}_2(cl_1(\text{int}_2(V)))) \subset cl_1(\text{int}_2(cl_1(V))) \subset cl_1(cl_1(V)) = cl_1(V) \subset U$$

which concludes the proof.

For the special case, assume, that (X, τ_1, τ_2) is $(2,1)$-β-normal.

(6. $\Rightarrow$ 1.): Let F and K be disjoint subsets of X such that F is 1-closed and K is 2-closed. Then $F \subset X - K$. By 6. there is a 2-open set U such that

$$cl_1(F \cap U) = F \subset cl_1(U) \subset X - K.$$

We have $K \subset X - cl_1(U)$ so that $cl_1(U) \cap K = \emptyset$. Since (X, τ_1, τ_2) is $(2,1)$-β-normal, there is a 2-open set V and a 1-open set W such that

$$cl_1(cl_1(U) \cap V) = cl_1(U), cl_2(K \cap W) = K \text{ and } cl_1(V) \cap cl_2(W) = \emptyset.$$

Therefore $cl_1(U) \cap cl_2(W) = \emptyset$ since

$$cl_1(U) \subset cl_1(V) \subset X - cl_2(W)$$

and (X, τ_1, τ_2) is $(1,2)$-β-normal as desired. □

In the following example we show that without $(2,1)$-β-normality, statement 6. Theorem 5.2.7 does not always imply statement 1. This demonstrates that $(1,2)$-β-normality behaves differently than β-normality of topological spaces.

Example 5.2.8. Consider $X = \{a, b, c, d\}$ with topologies

$$\tau_1 = \{\emptyset, X, \{a, b\}, \{c, d\}\}$$

and

$$\tau_2 = \{\emptyset, X, \{b\}, \{c\}, \{b, c\}, \{b, c, d\}\}$$

defined on X. The 1-closed subsets of X are $\emptyset, X, \{a, b\}$ and $\{c, d\}$, and the 2-closed subsets of X are $\emptyset, X, \{a, c, d\}, \{a, b, d\}, \{a, d\}$ and $\{a\}$. Hence (X, τ_1, τ_2) satisfies statement 6.. However (X, τ_1, τ_2) does not satisfy statement 1. because for the 1-closed set $\{c, d\}$ disjoint from the 2-closed set $\{a\}$, we cannot find a 2-open set U and a 1-open set V such that $cl_1(U) \cap cl_2(V) = \emptyset$. Also note that the bitopological space (X, τ_1, τ_2) is not pairwise normal for the same reason.

5.3 More on $(1,2)$-β-normality

In this section we transfer some of the properties in Chapter 3 to bitopological spaces. We begin by demonstrating that pairwise normality implies $(1,2)$-β-normality.

Proposition 5.3.1. *Every pairwise normal space is* $(1,2)$-β-*normal.*

Proof. Let (X,τ_1,τ_2) be a pairwise normal bitopological space. Choose disjoint subsets of F and K of X such that F is 1-closed and K is 2-closed. Since (X,τ_1,τ_2) is pairwise normal, there is a 2-open set U such that $F \subset U$ and $cl_1(U) \cap K = \emptyset$. Again by hypothesis, there is a 2-open set W such that $cl_1(U) \subset W$ and $cl_1(W) \cap K = \emptyset$. Let $V = X - cl_1(W)$ so that $K \subset V$. Then

$$V \subset X - W \subset X - cl_1(U)$$

implying that

$$cl_2(V) \subset X - W \subset X - cl_1(U)$$

since $X - W$ is 2-closed. Therefore, $cl_2(V) \cap cl_1(U) = \emptyset$. Because $F \subset U$ we get that $F \cap U = F$ and F being 1-closed we obtain that $cl_1(F \cap U) = F$. Similarly for $cl_2(K \cap V) = K$.

Thus (X,τ_1,τ_2) is $(1,2)$-β-normal. □

Recall from [35] that a bitopological space (X,τ_1,τ_2) is *pairwise* T_1 if for each pair of distinct points x, y, there exists a 1-open set U and a 2-open set V such that $x \in U, y \notin U$ and $y \in V, x \notin V$.

(X,τ_1,τ_2) is pairwise T_1 if and only if each singleton set is both 1-closed and 2-closed [35].

Proposition 5.3.2. *Every* $(1,2)$-β-*normal and pairwise* T_1 *bitopological space is pairwise regular.*

Proof. We prove that τ_1 is regular with respect to τ_2:

Let $x \in X$ and pick a 1-open set U such that $x \in U$. Now $\{x\}$ is a 2-closed set such that $\{x\} \cap (X - U) = \emptyset$. By hypothesis, there is a 2-open set W and a 1-open set V such that $cl_2(\{x\} \cap V) = \{x\}$,

$$cl_1((X - U) \cap W) = X - U$$

and $cl_2(V) \cap cl_1(W) = \emptyset$. Since

$$\{x\} \subset cl_2(V) \subset X - cl_1(W)$$

and $X - U \subset cl_1(W)$, we get that

$$\{x\} \subset cl_2(V) \subset U.$$

Having that $\{x\} \cap V$ is 2−dense in $\{x\}$ implies that $\{x\} \cap V \neq \emptyset$. Therefore $x \in V$ and

$$x \in V \subset cl_2(V) \subset U.$$

The proof is similar for τ_2 is regular with respect to τ_1. Therefore (X, τ_1, τ_2) pairwise regular. □

Next, we give conditions under which $(1,2)$-β-normality implies pairwise normality. We first recall from [11] the following definition: A bitopological space (X, τ_1, τ_2) is called (i,j)-*extremally disconnected* if the i-closure of every j-open set is j-open. (X, τ_1, τ_2) is called *pairwise extremally disconnected* if it is both $(1,2)$-extremally disconnected and $(2,1)$-extremally disconnected.

Proposition 5.3.3. *Every pairwise extremally disconnected* $(1,2)$-β-*normal bitopological space is pairwise normal.*

Proof. Let F and K be disjoint subsets of X such that F is 1-closed and K is 2-closed. By $(1,2)$-β-normality, there exist a 1-open V and a 2-open set U such that $cl_1(F \cap U) = F$, $cl_2(K \cap V) = K$ and $cl_1(U) \cap cl_2(V) = \emptyset$. Since (X, τ_1, τ_2) is pairwise extremally disconnected, $cl_1(U)$ is a 2-open set containing F and $cl_2(V)$ is a 1-open containing K.

Therefore (X, τ_1, τ_2) is pairwise normal. □

The following results give a decomposition of pairwise normality.

We recall the following definitions from [42]: A bitopological space (X, τ_1, τ_2) is *almost pairwise normal* if for each i-closed set F and for each (j,i)-regular closed set K disjoint from F, there exist a j-open set U and an i-open set V disjoint from U such that $F \subset U$ and $K \subset V$. A bitopological space (X, τ_1, τ_2) is said to be *mildly pairwise normal* if for each (i,j)-regular closed set F and for each (j,i)-regular closed set K disjoint from F, there exist a j-open set U and an i-open set V disjoint from U such that $F \subset U$ and $K \subset V$.

Lemma 5.3.4. *Every* (i,j)-*regular closed set is* i-*closed.*

Proof. Let A be an (i,j)-regular closed set. Then $A = cl_i(\text{int}_j(A))$. So

$$cl_i(A) = cl_i(cl_i(\text{int}_j(A))) = cl_i(\text{int}_j(A)) = A.$$

Therefore $A = cl_i(A)$. □

Proposition 5.3.5. *A bitopological space* (X, τ_1, τ_2) *is pairwise normal if and only if it is* $(1,2)$*-*β*-normal and mildly pairwise normal.*

Proof. ($\Rightarrow$): Pairwise normal $\Rightarrow$ $(1,2)$-β-normal: Follows from Proposition 5.3.1.

Pairwise normal $\Rightarrow$ mildly pairwise normal: This follows from Lemma 5.3.4.

($\Leftarrow$): Let (X, τ_1, τ_2) be a $(1,2)$-β-normal and mildly pairwise normal space. Pick F a 1-closed set and K a 2-closed set such that $F \cap K = \emptyset$. Since (X, τ_1, τ_2) is $(1,2)$-β-normal, there exist a 2-open set U and a 1-open set V such that $cl_1(F \cap U) = F$, $cl_2(K \cap V) = K$ and $cl_2(V) \cap cl_1(U) = \emptyset$. So $cl_1(U)$ is a $(1,2)$-regular closed set and $cl_2(V)$ is a $(2,1)$-regular closed set. Therefore, by pairwise mildly normality of (X, τ_1, τ_2), there exist a 2-open set H and a 1-open set G such that $F \subset cl_1(U) \subset H$ and $K \subset cl_2(V) \subset G$.

Therefore (X, τ_1, τ_2) is pairwise normal. □

Proposition 5.3.6. *A bitopological space* (X, τ_1, τ_2) *is pairwise normal if and only if it is* $(1,2)$*-*β*-normal and almost pairwise normal.*

Proof. ($\Rightarrow$): Pairwise normal $\Rightarrow$ $(1,2)$-β-normal: Follows from Proposition 5.3.1.

Pairwise normal $\Rightarrow$ almost pairwise normal: Since every (i,j)-regular closed set is i-closed from Lemma 5.3.4.

($\Leftarrow$): Let (X, τ_1, τ_2) be a $(1,2)$-β-normal and almost pairwise normal space, A be 1-closed and U be 2-open with $A \subset U$. Then $A \cap X - U = \emptyset$. As (X, τ_1, τ_2) is $(1,2)$-β-normal, there is a 2-open set V and a 1-open set W such that

$$cl_1(A \cap V) = A, cl_2((X - U) \cap W) = X - U$$

and $cl_1(V) \cap cl_2(W) = \emptyset$. We get that $X - cl_2(W)$ is a $(2,1)$-regular open set since $cl_2(W)$ is $(2,1)$-regular closed and $cl_1(V) \subset X - cl_2(W)$. So by almost pairwise normality, there is a 2-open set H such that

$$cl_1(V) \subset H \subset cl_1(H) \subset X - cl_2(W).$$

Moreover,

$$A \subset H \subset cl_1(H) \subset U$$

since $A \subset cl_1(V)$ and $X - cl_2(W) \subset U$.

Therefore (X, τ_1, τ_2) is pairwise normal. □

In the following arrows diagram, the implications hold but they are not reversible.

$$\begin{array}{ccc} \text{Pairwise normal} & \longrightarrow & (1,2)\text{-}\beta\text{-Normal} \\ \downarrow & & \\ \text{Almost pairwise normal} & \longrightarrow & \text{Mildly pairwise normal} \end{array}$$

Example 5.3.7. (1) In [27, Example 1] they have a non-normal, β-normal space. If we endow the set in [27, Example 1] with the same topology, (i.e., $\tau_1 = \tau_2$), then we get a $(1, 2)$-β-normal space which is not pairwise normal (resp. almost pairwise normal and mildly pairwise normal).

(2) [25] Let $\mathbb{R}$ be the set of real numbers and $a, b \in \mathbb{R}$ with $a < b$. Define

$$\tau_1 = \{\emptyset, \mathbb{R}, (-\infty, a], (-\infty, b)\}$$

and

$$\tau_2 = \{\emptyset, \mathbb{R}, [a, \infty), [b, \infty)\}.$$

The bitopological space is $(\mathbb{R}, \tau_1, \tau_2)$ is mildly pairwise normal but the space is not $(1, 2)$-β-normal.

Theorem 5.3.8. *Let (X, τ_1, τ_2) be a $(1, 2)$-β-normal bitopological space. The following statements are equivalent:*

1. *(X, τ_1, τ_2) is pairwise normal.*
2. *(X, τ_1, τ_2) is almost pairwise normal.*
3. *(X, τ_1, τ_2) is mildly pairwise normal.*

Proof. (1. $\Rightarrow$ 2. $\Rightarrow$ 3.): Follow from definitions.

(3. $\Rightarrow$ 1.): Since (X, τ_1, τ_2) is $(1, 2)$-β-normal and mildly pairwise normal, it follows from Proposition 5.3.5 that (X, τ_1, τ_2) is pairwise normal. $\square$

To close this section, we present a preservation result of $(1, 2)$-β-normality and $(2, 1)$-β-normality. We remind a reader that a function

$$f : (X, \tau_1, \tau_2) \to (Y, \rho_1, \rho_2)$$

is said to be *pairwise continuous* (resp. *pairwise open* and resp. *pairwise closed*) [34, 15] if the induced functions

$$f : (X, \tau_1) \to (Y, \rho_1)$$

and

$$f : (X, \tau_2) \to (Y, \rho_2)$$

are continuous (resp. open and resp. closed).

Similar to Proposition 3.3.9, we have the following result.

Proposition 5.3.9. *Let $f : (X, \tau_1, \tau_2) \to (Y, \rho_1, \rho_2)$ be a pairwise continuous function such that f is onto, pairwise open and pairwise closed. If (X, τ_1, τ_2) is $(1, 2)$-β-normal and $(2, 1)$-β-normal, then so is Y.*

Proof. Let A be 1-closed in Y and U be 2-open in Y. Then $f^{-1}(A)$ is a 1-closed subset of X contained in a 2-open set $f^{-1}(U)$ of X. By hypothesis, there is a 2-open set V such that

$$cl_1(f^{-1}(A) \cap V) = f^{-1}(A) \subset cl_1(V) \subset f^{-1}(U).$$

From Lemma 3.3.1 we obtain $cl_1(A \cap f(V)) = A$. Now,

$$f(f^{-1}(A)) \subset f(cl_1(V)) \subset f(f^{-1}(U))$$

which implies that $A \subset cl_1(f(V)) \subset U$ since f is onto and closed.

Therefore (X, τ_1, τ_2) is $(1, 2)$-β-normal and $(2, 1)$-β-normal. □

From the above theorem we obtain the following corollary.

Corollary 5.3.10. *Let $f : (X, \tau_1, \tau_2) \to (Y, \rho_1, \rho_2)$ be a pairwise continuous function such that f is onto, pairwise open and pairwise closed. If X is $(1, 2)$-β-normal and $(2, 1)$-β-normal, then Y is $(1, 2)$-β-normal.*

Remark 5.3.11. Most of the results presented in this chapter can be reproduced when $(2, 1)$-β-normality is replaced with $(1, 2)$-β-normality.

Conclusion

In this thesis, we were able to characterize β-normal spaces using different types of open sets and compare them with other spaces. With the use of the ultrafilter space, we were able to provide a counterexample to a question posed by Murtinova about the existence of a β-normal and regular space which is not Tychonoff. We showed that β-normal spaces are preserved by continuous, onto, open and closed maps and are continuous, injective, open and closed maps.

α-Normal spaces were characterized using countable open sets. Relation to other topological spaces were also investigated. We proved that α-normal spaces are preserved by continuous, onto, open, and closed functions, $f : X \to Y$, that satisfy $f^{-1}(f(A)) = A$ for all open $A \subset Y$. These spaces are reflected by continuous, injective, open and closed maps.

We also introduced the notion of β-normality to bitopological spaces and gave characterizations of these bitopological spaces in terms of (i, j)-α-open and (i, j)-preopen sets.

As part of future work, we will further explore β-normality in bitopological spaces and extend these to bilocales.

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