CONDUCTIVITY PROFILES FOR A HORIZONTALLY UNIFORM EARTH

Thesis

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CONTENTS

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ABSTRACT

An investigation is made into the mathematics behind the noniterative inversion algorithm of Shimon Coen and Michael Wang-Ho Yu [1981]. The algorithm determines the conductivity profile of a horizontally uniform earth from surface measurements of apparent resistivity with a Schlumberger array. The algorithm is checked by performing the inversion on both artifical and raw field data.

SECTION 1

INTRODUCTION

The purpose of a resistivity sounding method is to determine, from surface measurements of electrical potential differences, the conductivity profile of the earth below. The earth is assumed to be a horizontally uniform infinite half-space and thus its conductivity profile is a function of depth alone. Symmetry exists about a vertical axis through a single electrode supplying direct current to the earth at its surface. In practice there must be a second current electrode, but due to the horizontal uniformity restriction and the scalar nature of potentials, the effects of the second electrode may be included after solving the single electrode problem.

In almost all problems of potential theory, the solution is a potential function which satisfies the partial differential equation involved and the prescribed boundary conditions. These problems are referred to as forward problems. The present problem is unusual in that the function to be solved for, the conductivity profile, determines the form of the partial differential equation. In this sense the present problem is referred to as an inverse problem.

Early resistivity sounding methods consisted of solving the forward problem for conductivity profiles that represent a number of discrete layers each of uniform conductivity [Stefanesco et al, 1930]. By varying the thickness and conductivities assigned to the layers it is usually possible to fit the resulting computed surface potentials to the observed surface potentials. The conductivity profile resulting in the best fit is then considered to represent reality.

A natural extension of this idea is to solve the layered forward problem for conductivity profiles that, within each layer, are simple functions of depth, e.g. polynomial and exponential functions, [Slichter, 1933]. The catalogue of forward problem solutions is thus greatly enlarged.

With the advent of computers, iterative techniques for improving on a particular forward problem solution began to appear [Oldenburg, 1978]. Noniterative inversion algorithms for the determination of conductivity profiles have only surfaced recently. Coen and Wang-Ho Yu [1981] employed Weidelt's inverse scattering techniques of theoretical physics [1972] to derive an inversion algorithm for solving the geological inverse problem for any continuous depth dependent conductivity profile. In the present work we shall make a study of their methods and test their inversion algorithm using surface data obtained from both forward problem solutions and raw field measurements.

In section 2 the partial differential equation that must be satisfied by the potential function shall be derived using elementary current theory. The form of the partial differential equation shall depend on the conductivity profile which is assumed to be a continuous function of depth. In section 3 the so-called apparent resistivity function is discussed and the boundary conditions for forward and inverse problems are defined. In section 4 the geological inverse problem is transformed into an inverse scattering problem. In sections 5,6 and 7 Weidelt's solution [1972] to the inverse scattering problem is investigated. In section 8 Coen and Wang-Ho Yu's modification [1981] of Weidelt's solution is outlined. The complete inverse algorithm is summarised in section 9. Numerical methods for implementing the inversion algorithm are discussed in section 10. Section 11 is devoted to the testing of the inversion algorithm by using it to solve the inverse problem with surface data artificially obtained from solutions to various forward problems. In section 12 the inversion algorithm is applied to field measurements of apparent resistivity, supplied by Dr. J. de Beer of the C.S.I.R.. Conclusions are drawn in section 13.

SECTION 2

ELEMENTARY CURRENT THEORY

The strength of a current, I, in a circuit is defined as the rate at which charge passes any given point in the circuit. In an extended medium, described by the Cartesian co-ordinates (x,y,z), we prefer to work with the vector quantity, current density, denoted by the symbol, $j=j(x,y,z)$. The magnitude of current density is defined to be the quantity of charge passing per second through a unit plane area normal to the direction of flow of charge. The direction of current density is defined to be the direction of flow of charge. Thus, if S is a surface within the extended medium, the current flowing through S is given by

$$(2.1) \qquad I = \int_S j.dS \ .$$

Since electric charge can neither be created or destroyed, the rate of increase of total charge inside any arbitrary volume, V, must equal the net flow of charge into this volume. Thus, if at time t, $Q=Q(x,y,z,t)$ is the charge density at the point (x,y,z), then

$$(2.2) \qquad \frac{\partial}{\partial t}\int_V Q dV = -\int_S j.dS \ .$$

Using the divergence theorem

$$(2.3) \qquad \int_V \frac{\partial Q}{\partial t}\, dV = -\int_V \mathrm{div}(j)\, dV$$

and equating integrands, since V is arbitrary,

$$(2.4) \qquad \mathrm{div}(j) = -\frac{\partial Q}{\partial t}$$

so that

$$(2.5) \qquad \mathrm{div}(j) = 0$$

if time independence is assumed.

It has been found experimentally that in conductors at constant temperature the current density is linearly proportional to the electric field at any point within the medium

$$(2.6) \qquad j = \sigma E \ .$$

Equation (2.6) is known as Ohm's law and the constant with respect to time, $\sigma(x,y,z)$, is termed the conductivity of the medium. Since the electric field $E(x,y,z)$ may be expressed as the gradient of a scalar potential field $\phi(x,y,z)$,

$$(2.7) \qquad E = -\nabla\phi \ ,$$

we find using equation (2.5) that

$$(2.8) \qquad \nabla^2\phi + \frac{1}{\sigma}(\nabla\sigma . \nabla\phi) = 0$$

where the symbol ∇ is the nabla operator defined by

$$(2.9) \qquad \nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$

so that

$$(2.10) \qquad \nabla\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z} \right)$$

and

$$(2.11) \qquad \nabla^2\phi = \nabla.\nabla\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

Equation (2.8) is the partial differential equation that must be satisfied by the potential. In the next section we shall discuss boundary conditions for the potential.

SECTION 3

THE ROLE OF APPARENT RESISTIVITY IN FORWARD AND INVERSE PROBLEMS

The forward problem

The starting point of the forward problem is the assumption that conductivity is a known function of depth,

(3.1) $\sigma = \sigma(z)$.

The assumption (3.1) motivates the consideration of the forward problem for a single electrode from which current flows with cylindrical symmetry to infinity. Later on in this section we shall discuss the effects of the second current electrode. The symmetry involved calls for the use of cylindrical co-ordinates (r,z) with the earth occupying the region z>0, while the single current electrode is situated at the surface point (0,0). The geometry of the single current electrode problem is depicted in figure 3.1 below.

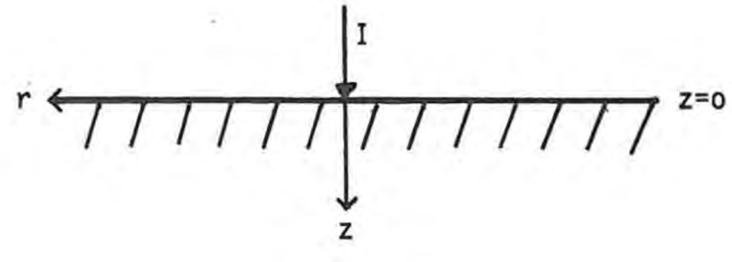

Figure 3.1 The geometry of the single electrode problem.

If the potential due to the single electrode is

$$(3.2) \qquad \phi = \phi(r,z)$$

then in the half-space, z>0, ϕ must satisfy the partial differential equation (2.8) which in cylindrical coordinates reads

$$(3.3) \qquad \{ \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} + \frac{\sigma'(z)}{\sigma(z)}\frac{\partial}{\partial z} \}\phi(r,z) = 0 .$$

The boundary conditions satisfied by the potential are that (a) the potential must be zero at infinity,

$$(3.4) \qquad \lim_{r,z\to\infty} \phi(r,z) = 0 ;$$

and (b) no current may cross the surface z=0 except at the point of entry (0,0) of the single current electrode,

$$(3.5) \qquad \int_{z=0} \sigma(0)\frac{\partial\phi}{\partial z}(r,0)\,dS = -I .$$

The integral equation (3.5) is obtained using equations (2.1), (2.6) and (2.7); integration may be avoided, by using the Dirac delta function $\delta(r)$, to rewrite the boundary condition (3.5) as

$$(3.6) \qquad \frac{\partial\phi}{\partial z}(r,0) = \frac{-I\delta(r)}{2\pi\sigma(0)r} .$$

The partial differential equation (3.3) and the boundary conditions (3.4) and (3.6) completely define the forward problem for a single current electrode. A solution to the forward problem is a potential function ϕ which satisfies the partial differential equation and the boundary conditions.

Apparent resistivity

After a solution to the forward problem has been found, potential gradients at any point on the surface may be computed by evaluating

$$(3.7) \qquad \frac{\partial\phi}{\partial r}(r,0)$$

at the surface point in question. In this manner a surface potential gradient for $0<r<\infty$ may be obtained. This surface function, however, is dependent on the strength and direction of the current, I, carried by the single electrode. At the starting point of the single electrode inverse problem we shall require a surface function that depends only on the earths electrical properties. This function is the so-called apparent resistivity which we shall define properly after a slight digression.

In section 11 we shall show that the solution to the forward problem for a single electrode carring current, I, to a completely homogeneous earth, $\sigma=\sigma(0)$, is $\phi_{hom}(r,z)$ where

$$(3.8) \qquad \phi_{hom}(r,z) = \frac{I}{2\pi\sigma(0)\sqrt{r^2+z^2}} \ .$$

Thus the surface potential gradient above a homogeneous earth is given by

$$(3.9) \qquad \frac{\partial \phi_{hom}}{\partial r}(r,0) = \frac{-I}{2\pi\sigma(0)r^2} .$$

Returning to depth dependent conductivity profiles, $\sigma=\sigma(z)$, we define apparent resistivity, $\rho_a=\rho_a(r)$, by

$$(3.10) \qquad \sigma(0)\rho_a(r) = \frac{\partial\phi/\partial r(r,0)}{\partial\phi_{hom}/\partial r(r,0)} = \frac{-2\pi\sigma(0)r^2}{I}\frac{\partial\phi}{\partial r}(r,0) .$$

$\rho_a(r)$ is then the apparent resistivity at the surface point $(r,0)$ and depends only on the earth's electrical properties. We note for future reference that since there exists some neighbourhood of the origin for which $\phi(r,z)\approx\phi_{hom}(r,z)$ we have

$$(3.11) \qquad \sigma(0)\rho_a(0) = 1 .$$

The inverse problem

We have seen that the apparent resistivity $\rho_a(r)$ of equation (3.10) may be computed from the solution to the single electrode forward problem. For the single electrode inverse problem the apparent resistivity of equation (3.10) is <u>given</u> and thus treated as a further boundary condition on the potential function ϕ. The inverse problem is solved when a conductivity profile $\sigma(z)$ for equation (3.1) is found such that there exists a potential function $\phi(r,z)$ which satisfies the partial differential equation (3.3) and the boundary conditions (3.4), (3.6) and (3.10).

In order to show that the single electrode inverse problem is of practical use it remains to demonstrate that the apparent resistivity of equation (3.10) may be derived from measurements made in the presence of two current electrodes.

Further remarks about apparent resistivity

Apparent resistivity may be calculated using various geometries for the current and potential electrodes. For the purpose of demonstration we choose the so-called Schlumberger array. Suppose we wish to calculate apparent resistivity at the surface point r=s. Using the Schlumberger array, we position the current electrodes at r=0 and r=2s. The potential electrodes are positioned at r=s-b and r=s+b where b<<s. The geometry of the Schlumberger array is depicted in figure 3.2 below.

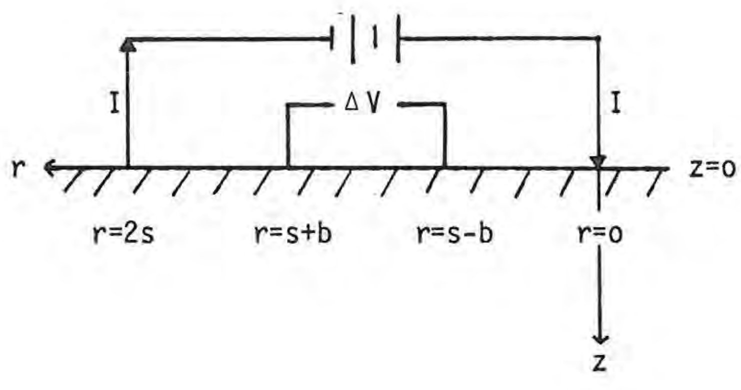

Figure 3.2 The geometry of Schlumberger's array.

Suppose that a potential difference ΔV is measured accross the potential electrodes. The horizontal uniformity assumption and the scalar nature of potentials allows the representation of the potential due to both current electrodes as the superposition of potentials due to the individual current electrodes. Thus since the current electrodes carry equal quantities of current in opposite directions

$$(3.12) \qquad \Delta V = (\ \phi(s+b,0)-\phi(s-b,0)\ ) - (\ \phi(s-b,0)-\phi(s+b,0)\ )$$

$$= 2(\ \phi(s+b,0)-\phi(s-b,0)\ ) .$$

This results in

$$(3.13) \qquad \frac{\Delta V}{2b} = 2\frac{\partial\phi}{\partial r}(s,0)$$

and

$$(3.14) \qquad \rho_a(s) = \frac{-2\pi s^2}{I}\,\frac{\Delta V}{4b} .$$

This procedure may be carried out for any surface point r=s, showing that the apparent resistivity of equation (3.10) is measurable by experiment. We note that Koefoed [1979] defines apparent resistivity for a more general geometry than the Schlumberger geometry. Our equation (3.14) is in agreement with Koefoed [1979, eq. 4.2.1] if b<<s.

SECTION 4

TRANSFORMATION TO AN INVERSE SCATTERING PROBLEM

The transformation of the geological inverse problem, defined in the previous section, to an inverse scattering problem, for which a solution exists, is accomplished by applying a Hankel transform of order zero.

If $\phi=\phi(r,z)$ is defined for $r>0$, then the function $\Phi=\Phi(s,z)$ defined by

$$(4.1) \qquad \Phi(s,z) = \int_0^\infty \phi(r,z) r J_0(rs)\,dr$$

is called the Hankel transform of order zero of the function ϕ. J_0 is the usual Bessel function of order zero and the integral on the right hand side of equation (4.1) is assumed to exist. Moreover, since we assume that ϕ is continuously differentiable with respect to r and absolutely integrable over the interval $[0,\infty)$, the Hankel transform of ϕ, Φ, is analytic with respect to the complex variable s.

Multiplying both sides of equation (3.3) by $rJ_0(rs)$ and integrating from zero to infinity with respect to r, we obtain

$$(4.2) \qquad \int_0^\infty \left(\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r}\right) rJ_0(rs)\,dr = -\int_0^\infty \left(\frac{\partial^2\phi}{\partial z^2} + \frac{\sigma'}{\sigma}\frac{\partial\phi}{\partial z}\right) rJ_0(rs)\,dr \ .$$

The left hand integral in equation (4.2) may be simplified by integrating by parts and assuming that $r\phi(r,z)$ and $r\frac{\partial\phi}{\partial r}(r,z)$ tend to zero as r tends to zero or infinity,

$$(4.3) \qquad \text{L.H.S.} = \int_0^\infty \frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right)J_0(rs)\,dr$$

$$= r\frac{\partial\phi}{\partial r}J_0(rs)\Big|_0^\infty + \int_0^\infty rs\frac{\partial\phi}{\partial r}J_1(rs)\,dr$$

$$= rs\phi J_1(rs)\Big|_0^\infty - \int_0^\infty rs^2\phi J_0(rs)\,dr$$

$$= -\int_0^\infty rs^2\phi J_0(rs)\,dr \ .$$

Using this simplification and the definition (4.1) we may now rewrite equation (4.2) as

$$(4.4) \qquad \left\{ \frac{\partial^2}{\partial z^2} + \frac{\sigma'}{\sigma}\frac{\partial}{\partial z} - s^2 \right\}\Phi(s,z) = 0 \ .$$

To simplify equation (4.2) further we define the following functions

$$(4.5) \qquad \eta(z) = \sqrt{\sigma(z)}$$

$$(4.6) \qquad q(z) = \frac{\eta''(z)}{\eta(z)}$$

$$(4.7) \qquad g(z) = \frac{\eta(z)}{\eta(0)}$$

$$(4.8) \qquad f(s,z) = \frac{\eta(z)\Phi(s,z)}{\eta(0)\Phi(s,0)} \ .$$

We then find that equation (4.4) may be replaced by the two equations (4.9) and (4.10) below:

(4.9) $$\{ \frac{\partial^2}{\partial z^2} - q(z) - s^2 \} f(s,z) = 0$$

(4.10) $$\{ \frac{d^2}{dz^2} - q(z) \} g(z) = 0 \ .$$

We note that, for any z, $f=f(s,z)$ is an analytic function of the complex variable, s, in the region, $Re(s)>0$, since in this region Φ is analytic and $\eta(0)\Phi(s,0)$ is analytic and never zero.

Defining the function $C = C(s)$ by

(4.11) $$C(s) = \frac{-\Phi(s,0)}{\partial\Phi/\partial z(s,0)} ,$$

we may write the initial conditions for the system of equations (4.9) and (4.10) as

(4.12) $$f(s,0) = 1 \qquad \frac{\partial f}{\partial z}(s,0) = \frac{\eta'(0)}{\eta(0)} - \frac{1}{C(s)}$$

(4.13) $$g(0) = 1 \qquad g'(0) = \frac{\eta'(0)}{\eta(0)} \ .$$

We now show that, for the inverse problem, C is a known function by relating $C(s)$ to the apparent resistivity $\rho_a(r)$. Firstly from equation (4.1) we have

(4.14) $$\frac{\partial\Phi}{\partial z}(s,0) = \int_0^\infty \frac{\partial\phi}{\partial z}(r,0) r J_0(rs) dr \ .$$

Using the boundary condition (3.6) we get

(4.15) $$\frac{\partial\phi}{\partial z}(s,0) = -\int_0^\infty \frac{I}{2\pi\sigma(0)} J_0(rs)\delta(r)dr = \frac{-I}{2\pi\sigma(0)}$$

and thus C may be expressed as

(4.16) $$C(s) = \frac{2\pi\sigma(0)}{I}\int_0^\infty \phi(r,0)rJ_0(rs)dr .$$

Employing Hankel's inversion theorem of order zero we find

(4.17) $$\frac{2\pi\sigma(0)}{I}\ \phi(r,0) = \int_0^\infty C(s)sJ_0(rs)ds$$

which we may write as

(4.18) $$\frac{2\pi\sigma(0)}{I}\ \phi(r,0) = \frac{1}{r} - \int_0^\infty (1-sC(s))J_0(rs)ds$$

since

(4.19) $$\int_0^\infty J_0(rs)ds = \frac{1}{r} .$$

Differentiating both sides of equation (4.18) with respect to r we obtain

(4.20) $$\frac{2\pi\sigma(0)}{I}\ \frac{\partial\phi}{\partial r}(r,0) = \frac{-1}{r^2} + \int_0^\infty (1-sC(s))sJ_1(rs)ds .$$

Multiply both sides of equation (4.20) by r^2 and use equation (3.10) to get

(4.21) $$1-\sigma(0)\rho_a(r) = r^2\int_0^\infty (1-sC(s))sJ_1(rs)ds .$$

Finally, using Hankel's inversion theorem of order one, we obtain

$$(4.22) \qquad 1-sC(s) = \int_{0}^{\infty} \frac{1-\sigma(0)\rho_a(r)}{r} J_1(rs)dr ,$$

showing that C(s) is measurable by experiment and hence a known function for the inverse problem.

To conclude this section, we can restate the inverse problem of Section 3 as follows:

We are required to find a continuous function $g=g(z)$ satisfying the initial conditions (4.13) such that if $q=q(z)$ is defined by equation (4.10) then there exists a function $f=f(s,z)$ which satisfies the partial differential equation (4.9) and the boundary conditions (4.12) where the impedance function $C=C(s)$ is known and given by equation (4.22).

At first sight it seems that the inverse problem as outlined here does not have a unique solution and that in fact any continuous function $g=g(z)$ will do. However, after observing the fact that the partial differential equation (4.9) is parabolic, we see that once the function $q=q(z)$ is specified, the boundary conditions (4.12) overdetermine the problem [Morse and Feshbach, 1953, p. 706]. Our first impressions are thus not necessarily correct. For a detailed discussion of uniqueness the reader is referred to Gel'fand and Levitan [1955].

By means of Weidelt's procedure [1972] introduced in the next section we shall construct a solution $g=g(z)$ to the inverse problem.

SECTION 5

INTRODUCTION TO WEIDELT'S PROCEDURE

Weidelt [1972] used his procedure to solve the system of equations,

$$(5.1) \qquad \left\{ \frac{\partial^2}{\partial z^2} - q(z) - s^2 \right\} f(s,z) = 0$$

$$(5.2) \qquad \left\{ \frac{d^2}{dz^2} - q(z) \right\} g(z) = 0$$

for the function g=g(z). Weidelt assumed that f=f(s,z) is an analytic function of the complex variable, s, in the half-plane, Re(s)>0. He also assumed that both f=f(s,z) and g=g(z) possess second derivatives with respect to the real variable, z, in the interval z>0. The initial conditions satisfied by f and g are:

$$(5.3) \qquad f(s,0) = 1 \qquad \frac{\partial f}{\partial z}(s,0) = k - \frac{1}{C(s)}$$

$$(5.4) \qquad g(0) = 1 \qquad \frac{dg}{dz}(0) = k$$

where k is a known constant and C=C(s), the impedance function, is known for any complex number s.

Central to Weidelt's procedure [1972], is an integral representation of two independent solutions to equation (5.1). Denoting these two solutions as $f_+(s,z)$ and $f_-(s,z)$ the envisaged representation is

$$(5.5) \qquad f_\pm(s,z) = e^{\pm sz} + \int_{-z}^{z} A(t,z)e^{\pm st}dt \ .$$

The kernel function, $A=A(t,z)$, in the above representation is assumed to be real valued and independent of s.

Weidelt obtains the function $g=g(z)$ by showing firstly that g is a limiting case of either f_+ or f_-,

$$(5.6) \qquad g(z) = \lim_{s \to o} f_\pm(s,z) = 1 + \int_{-z}^{z} A(t,z)dt \ .$$

Secondly, Weidelt relates the kernel function, $A=A(t,z)$, to the known impedance function, $C=C(s)$, by means of a number of solvable integral equations.

In the next section, section 6, we shall show that the representation (5.5) is valid. In section 7 we shall show that the representation (5.5) leads to the limiting relation (5.6) and we shall construct the integral equations that relate the kernel $A=A(t,z)$ to the impedance function $C=C(s)$.

SECTION 6

VALIDITY OF WEIDELT'S REPRESENTATIONS

If the representation suggested by equation (5.5) is to be valid then firstly, both $f_+(s,z)$ and $f_-(s,z)$ must satisfy equation (5.1). We begin by substituting f_+ into equation (5.1).

Leibnitz's rule for the differentiation of integrals yields

$$(6.1) \qquad \frac{\partial^2 f_+}{\partial z^2}(s,z) = s^2 e^{sz} + \int_{-z}^{z} \frac{\partial^2 A}{\partial z^2}(t,z) e^{st} dt$$
$$+ 2\frac{\partial A}{\partial z}(z,z)e^{sz} + 2\frac{\partial A}{\partial z}(-z,z)e^{-sz} + \frac{\partial A}{\partial t}(z,z)e^{sz}$$
$$- \frac{\partial A}{\partial t}(-z,z)e^{-sz} + sA(z,z)e^{sz} - sA(-z,z)e^{-sz} ,$$

while integration by parts gives

$$(6.2) \qquad s^2 f_+(s,z) = s^2 e^{sz} + \int_{-z}^{z} A(t,z) \frac{d^2}{dt^2}(e^{st})dt$$
$$= s^2 e^{sz} + \int_{-z}^{z} \frac{\partial^2 A}{\partial t^2}(t,z) e^{st} dt - \frac{\partial A}{\partial t}(z,z)e^{sz}$$
$$+ \frac{\partial A}{\partial t}(-z,z)e^{-sz} + sA(z,z)e^{sz} - sA(-z,z)e^{-sz} .$$

Using equations (6.1) and (6.2) the substitution of $f_+(s,z)$ into equation (5.1) results in

$$(6.3)\qquad \int_{-z}^{z} e^{st}\{ \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial t^2} - q(z) \}A(t,z)dt$$

$$+ e^{sz}(2\frac{d}{dz}A(z,z) - q(z)) + 2e^{-sz}\frac{d}{dz}A(-z,z) = 0$$

and since the kernel A=A(t,z) is independent of s, each of the three terms in equation (6.3) must vanish separately:

$$(6.4)\qquad \{ \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial t^2} - q(z) \}A(t,z) = 0$$

$$(6.5)\qquad A(z,z) = \frac{1}{2}\int_{o}^{z} q(t)dt + \text{constant}$$

$$(6.6)\qquad A(-z,z) = \text{constant} \quad .$$

The same set of three equations is obtained if $f_-(s,z)$ is substituted into equation (5.1). Thus, both f_+ and f_- satisfy equation (5.1) if the kernel function A=A(t,z) satisfies the set of equations, (6.4), (6.5) and (6.6).

To obtain the constant of integration in equations (6.5) and (6.6), we must first show that f_+ and f_- are independent functions of z.

To accomplish this task we suppose that

$$(6.7)\qquad a_1(s)f_+(s,z) + a_2(s)f_-(s,z) = 0 \quad .$$

Equation (6.7) may be written as

(6.8) $$W(s,z) = -\int_{-z}^{z} A(t,z)W(s,t)dt$$

where

(6.9) $$W(s,z) = a_1(s)e^{sz} + a_2(s)e^{-sz} .$$

Equation (6.8) gives

(6.10) $$W(s,0) = \frac{\partial W}{\partial z}(s,0) = \frac{\partial^2 W}{\partial z^2}(s,0) = \dots\dots\dots\dots = 0$$

and hence

(6.11) $$W(s,z) = 0$$

for all z. Since e^{sz} and e^{-sz} are independent functions of z, equations (6.9) and (6.11) give

(6.12) $$a_1(s) = a_2(s) = 0$$

showing that f_+ and f_- are independent functions of z.

A general solution to equation (5.1) thus takes the form

(6.13) $$f(s,z) = a_1(s)f_+(s,z) + a_2(s)f_-(s,z)$$

and using the initial conditions (5.3) we obtain

(6.14) $$a_1(s) + a_2(s) = 1$$

and

$$(6.15) \qquad k - \frac{1}{C(s)} = 2A(0,0) + s(a_1(s)-a_2(s)) .$$

Now since A(t,z) is independent of s, we can rewrite equation (6.15) as

$$(6.16) \qquad 2A(0,0) = k \qquad \text{and} \qquad sC(s)(a_2(s)-a_1(s)) = 1 .$$

The first part of equation (6.16) yields the constant of integration for equations (6.5) and (6.6) as

$$(6.17) \qquad \text{constant} = A(0,0) = \frac{k}{2} ,$$

while the second part of equation (6.16) together with equation (6.14) give

$$(6.18) \qquad a_1(s) = \frac{sC(s)-1}{2sC(s)} \qquad \text{and} \qquad a_2(s) = \frac{sC(s)+1}{2sC(s)} .$$

Equation (6.4) is a hyperbolic partial differential equation for the real function A=A(t,z) and if the function g=g(z) and hence the function q=q(z) were known, A(t,z) would be known on the pair of intersecting characteristic curves, $z \pm t = 0$. A theorem on hyperbolic equations states that under these conditions, a solution function A=A(t,z) exists in the domain z>0 and $|t| < z$, [Bernstein, 1950]. Outside this domain we choose to take A(t,z) as zero.

Thus, the function $f=f(s,z)$ given by equations (6.13) and (6.18) is a solution to equation (5.1) satisfying the initial condition (5.3), and the representation (5.5) is valid.

In the next section we shall link the conductivity profile $g=g(z)$ to the kernel $A=A(t,z)$ and the kernel $A=A(t,z)$ to the impedance $C=C(s)$.

SECTION 7

CONDUCTIVITY LINKED TO IMPEDANCE VIA WEIDELT'S KERNEL

Firstly, we note that in the limit as s tends to zero, $f_+(s,z)$, $f_-(s,z)$ and $g(z)$ satisfy the same differential equation and initial conditions. Thus we conclude that the limiting relation (5.6) is valid and provides a link between the kernel and the conductivity profile.

Secondly, we have seen in section 6 that the representation of a solution, $f(s,z)$, to equation (5.1), by a linear combination of $f_+(s,z)$ and $f_-(s,z)$, yields

$$\text{(7.1)} \qquad sC(s)f(s,z) = f_-(s,z) - b(s)(f_+(s,z)+f_-(s,z))$$

where

$$\text{(7.2)} \qquad b(s) = \frac{1}{2}(1-sC(s)) \ .$$

Substituting from equation (5.5) into equation (7.1) we obtain

$$\text{(7.3)} \qquad sC(s)f(s,z)-e^{-sz} = \int_{-z}^{z} A(t,z)e^{-st}dt$$

$$- b(s)(e^{sz}+e^{-sz}) - b(s)\int_{-z}^{z} A(t,z)(e^{st}+e^{-st})dt \ .$$

Multiply both sides of equation (7.3) by

(7.4) $$\frac{e^{sy}}{2\pi i}$$

where $|y|<z$. Integrate over the complex variable s along the path $Re(s)=\varepsilon$, with $\varepsilon>0$. The result may be abbreviated to read

(7.5) $$I_1 = I_2 + I_3 + I_4$$

where I_1 to I_4 denote the integrals resulting from the four terms of equation (7.3). We now investigate each of these four integrals separately.

Firstly,

(7.6) $$I_1 = \frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty} (sC(s)f(s,z)-e^{-sz})e^{sy}ds.$$

An asymptotic representation of $f(s,z)$, for $s\to\infty$, is given by Kamke [1959] as

(7.7) $$f(s,z) = \exp(-sz + \frac{1}{2s}\int_0^z q(t)dt + O(s^{-2})) \quad .$$

Thus, as $s\to\infty$, the integrand in equation (7.6) behaves like

(7.8) $$e^{-s(z-y)}(sC(s)-1)$$

and since $|y|<z$, the contour may be closed by a large semicircle in the half-plane, $Re(s)>0$, without affecting the value of the integral.

In the interior of this closed contour the integrand is analytic. Thus

$$(7.9) \qquad I_1 = 0 \ .$$

Secondly,

$$(7.10) \qquad I_2 = \frac{1}{2\pi i} \int_{\varepsilon-i\infty}^{\varepsilon+i\infty} \left(\int_{-z}^{z} A(t,z) e^{-st} dt \right) e^{sy} ds \ .$$

Treating I_2 as a function of y and z, the two-sided Laplace transform inversion theorem yields

$$(7.11) \qquad \int_{-z}^{z} A(t,z) e^{-st} dt = \int_{-\infty}^{\infty} I_2(y,z) e^{-sy} dy \ .$$

Since $A(t,z)=0$ for $|t|>z$, the limits of the integral on the left hand side of equation (7.11) may be extended to infinity yielding

$$(7.12) \qquad I_2 = A(y,z) \ .$$

Thirdly,

$$(7.13) \qquad I_3 = \frac{-1}{2\pi i} \int_{\varepsilon-i\infty}^{\varepsilon+i\infty} b(s) e^{s(z+y)} ds - \frac{1}{2\pi i} \int_{\varepsilon-i\infty}^{\varepsilon+i\infty} b(s) e^{-s(z-y)} ds \ .$$

Since $|y|<z$, the contour of the second integral of equation (7.13) may be closed by a large semicircle in the half-plane, $\mathrm{Re}(s)>0$, and since the integrand is analytic in its interior, the integral vanishes. Thus

(7.14) $$I_3 = -B(z+y)$$

where

(7.15) $$B(x) = \frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty} b(s)e^{sx}ds \ .$$

B(x) is a real function of the real variable, x, since

(7.16) $$\overline{b(s)} = b(\bar{s})$$

which follows from equations (7.2) and (4.22) and properties of the Hankel transform. Note that if x<0 then B(x)=0 since, in this case the contour of integration may be closed by a large semicircle in the half-plane, Re(s)>0, without altering the value of the integral, and within this closed contour the integrand is analytic.

Lastly,

(7.17) $$I_4 = \frac{-1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty} e^{sy}b(s)\left(\int_{-z}^{z}A(t,z)(e^{st}+e^{-st})dt \right)ds$$

$$= -\int_{-z}^{z}A(t,z)\left(B(y+t)+B(y-t) \right)dt \ .$$

Collecting equations (7.17), (7.14), (7.12) and (7.9) and substituting into equation (7.5), we obtain

(7.18) $$A(y,z) = B(z+y) + \int_{-z}^{z}A(t,z)\left(B(y+t)+B(y-t) \right)dt$$

which is a linear integral equation in the variables y and t for each value of the parameter z.

To conclude, equations (5.6), (7.18), (7.15) and (7.2) link the impedance function, C=C(s), to the conductivity profile g=g(z) via the kernel function, A=A(t,z).

This completes our discussion of Weidelt's procedure [1972]. In the next section we shall discuss Coen and Wang-Ho Yu's adaption [1981] of Wiedelt's procedure to suit the geological inverse problem.

SECTION 8

COEN AND WANG-HO YU'S MODIFICATION OF WEIDELT'S PROCEDURE

In section 4, we have shown that the conductivity profile problem may be transformed to an inverse scattering problem by means of a Hankel transform of order zero. The inverse scattering problem is defined by the partial differential equations (4.9) and (4.10) together with the initial conditions (4.12) and (4.13).

We have noted that the function, $f=f(s,z)$, of equation (4.9) is analytic, with respect to the complex variable s, in the region $Re(s)>0$, and twice differentiable with respect to the real variable z. We assume that the conductivity profile, $\sigma=\sigma(z)$, and hence the function, $g=g(z)$, is twice differentiable in the region $z>0$. Further, we have shown that the impedance function, $C=C(s)$, of the initial condition (4.12) is measurable by experiment and hence known for any complex number s.

We may therefore conclude that Weidelt's procedure [1972] is applicable to the inverse scattering problem and the solution is given by the limiting equation (5.6), which relates the conductivity profile to the kernel function, together with the set of integral equations (7.2), (7.15) and (7.18), which relate the kernel function to the impedance function.

Equation (4.22) provides the link between apparent resistivity and impedance. Coen and Wang-Ho Yu [1981] observed that apparent resistivity could be linked directly to the intermediate function B=B(x) of equation (7.15), as follows:

Define a real valued function of the real variable r, d=d(r), by

(8.1) $$d(r) = \frac{1}{2}\left(1-\sigma(0)\rho_a(r) \right) .$$

Equations (4.21) and (7.2) then give

(8.2) $$d(r) = r^2\int_0^\infty b(s)sJ_1(rs)ds .$$

However, using the two-sided Laplace transform inversion theorem on equation (7.15), we have

(8.3) $$b(s) = \int_0^\infty B(x)e^{-sx}dx$$

and on substituting equation (8.3) into equation (8.2) and interchanging the order of integration we obtain

(8.4) $$d(r) = r^2\int_0^\infty B(x)\left(\int_0^\infty e^{-sx}sJ_1(rs)ds \right)dx .$$

Using Gradshteyn and Ryzhik [1965, eq. 6.623.1] equation (8.4) reduces to

(8.5) $$d(r) = \int_0^\infty B(x)\frac{r^3}{(x^2+r^2)^{3/2}}dx$$

and thus the link between apparent resistivity and the kernel function is provided by equations (8.1), (8.5) and (7.18).

In the next section we shall summarise the complete inversion procedure.

SECTION 9

THE COMPLETE INVERSION ALGORITHM

To summarise, we are given $\sigma(0)\rho_a(r)$ for all $0<r<\infty$. Define a function $d=d(r)$ by

(9.1) $$d(r) = \frac{1}{2}\left(1-\sigma(0)\rho_a(r) \right) .$$

Solve the integral equation

(9.2) $$d(r) = \int_0^\infty B(x) \frac{r^3}{(x^2+r^2)^{3/2}} dx$$

for $B(x)$. Solve the integral equation

(9.3) $$A(y,z) = B(z+y) + \int_{-z}^{z} A(t,z)\left(B(y+t)+B(y-t) \right)dt$$

for the kernel $A(y,z)$. Obtain $g(z)$ from

(9.4) $$g(z) = 1 + \int_{-z}^{z} A(t,z)dt$$

and then the conductivity profile is given by

(9.5) $$\sigma(z) = \sigma(0)(g(z))^2$$

which is real and positive as expected.

The conductivity profile so obtained is unique whenever the apparent resistivity is completely and precisely specified. However, in practice, the apparent resistivity is not available for all $0<r<\infty$, but only at a finite number, M, of points on the surface of the earth. In the next section we shall discuss numerical methods for obtaining an approximation to the conductivity profile from the 2M real numbers $\{r_i, \sigma(0)\rho_a(r_i)\}_{i=1}^{M}$.

SECTION 10

NUMERICAL ANALYSIS

In this section we shall investigate how to approximate the conductivity profile $\sigma(z)$, $z>0$, using the 2M real numbers $\{r_i, \sigma(0)\rho_a(r_i)\}_{i=1}^{M}$ as input to the inversion procedure outlined in the previous section. By convention we shall assume $r_1<r_2<..<r_M$.

The first step of the inversion procedure now consists of the M equations

$$(10.1) \qquad d_i = \frac{1}{2}\left(1-\sigma(0)\rho_a(r_i) \right) .$$

The second step requires us to approximate the function $B=B(x)$ from the M equations

$$(10.2) \qquad d_i = \int_0^\infty B(x)L_i(x)dx$$

where

$$(10.3) \qquad L_i(x) = \frac{r_i^3}{(x^2+r_i^2)^{3/2}} .$$

Naturally we cannot obtain $B(x)$ uniquely since the M functions $L_i(x)$ do not form a complete set. However, if we multiply both sides of equation (10.2) by the as yet unknown MN real constants V_{ij} and sum over i from 1 to M, we obtain N equations of the form

$$(10.4) \qquad h_j = \int_0^\infty B(x)O_j(x)dx$$

where

$$(10.5) \qquad h_j = \sum_{i=1}^{M} V_{ij}d_i$$

and

$$(10.6) \qquad O_j(x) = \sum_{i=1}^{M} V_{ij}L_i(x) \ .$$

If we can choose the MN constants V_{ij} such that the N functions $O_j(x)$ form an orthonormal set on the interval $(0,\infty)$, then, by inspection, the solution to the set of equations (10.4) is given by

$$(10.7) \qquad B(x) = \sum_{k=1}^{N} h_k O_k(x) \ .$$

It therefore remains to determine V_{ij} such that

$$(10.8) \qquad \int_0^\infty O_k(x)O_j(x)dx = \delta_{kj}$$

where δ_{kj} is the N by N Kronecker delta. We thus require

$$(10.9) \qquad \sum_{i,n=1}^{M} V_{ik}\left(\int_0^\infty L_i(x)L_n(x)dx \right)V_{nj} = \delta_{kj}$$

so that the N columns of the matrix V_{ij} must be distinct eigenvectors of the matrix D_{in}, where D_{in} is the real symmetric M by M matrix

$$(10.10) \qquad D_{in} = \int_0^\infty L_i(x) L_n(x)\,dx \ .$$

We are restricted to choosing N such that $N \leq M$, but apart from this restriction we may order the eigenvectors of D_{in} in any manner we like and then take $N \leq M$ such that only the first N eigenvectors are used in computing B(x).

Letting λ_j be the eigenvalue corresponding to the j^{th} eigenvector we shall order the eigenvectors such that

$$(10.11) \qquad \lambda_1 \geq \lambda_2 \geq \cdots\cdots\cdots \lambda_N \cdots\cdots\cdots \lambda_M > 0$$

and then choose N such that $N \leq M$ and

$$(10.12) \qquad \frac{\lambda_N}{\lambda_1} > \varepsilon > 0$$

for some tuning parameter ε. We shall compute the conductivity profile for various values of the tuning parameter and we suspect that the best value for the tuning parameter may be found by reconstructing the apparent resistivity.

We conclude the numerical analysis of the second step by noting that the integral in equation (10.10) may be evaluated exactly using the table of integrals [Gradshteyn and Ryzhik, 1965, eqs. 3.153.4, 3.249.1]. The result is

$$
(10.13) \qquad D_{in} = \begin{cases} \dfrac{r_i^2\, r_n}{(r_i^2-r_n^2)^2}\left((r_i^2+r_n^2)E(q)-2r_n^2K(q) \right) , & r_i > r_n \\[2ex] \dfrac{3\pi}{16} r_i , & r_i = r_n \\[2ex] D_{ni} , & r_i < r_n \end{cases}
$$

where

$$
(10.14) \qquad q = \frac{\sqrt{r_i^2-r_n^2}}{r_i}
$$

and E(.) and K(.) are respectivly complete elliptic integrals of the first and second kind.

The numerical implementation of the third step of the inversion procedure is discussed next. In equation (9.3) we make the change of variable, $y = sz$ and $t = uz$, to obtain

$$
(10.15) \qquad A(sz,z) = B(z+sz) + z\int_{-1}^{1} A(uz,z)\left(B(sz+uz)+B(sz-uz) \right)du
$$

where $-1<s<1$ since $|y|<z$. For a fixed z, the objective is to approximate A(sz,z) for all $-1<s<1$. This is done by seeking a solution to equation (10.15) of the form

$$
(10.16) \qquad A(sz,z) = \sum_{i=o}^{L} \alpha_i(z)\psi_i(s)
$$

where $\psi_i(s)$ are the orthonormal Legendre polynomials and $\alpha_i(z)$ are coefficients to be determined for a given z. Substitution of equation (10.16) into equation (10.15) yields

$$(10.17) \qquad \sum_{i=o}^{L} \alpha_i(z)\psi_i(s) = B(z+sz)$$

$$+ z\sum_{i=o}^{L} \alpha_i(z)\int_{-1}^{1}\psi_i(u)(\ B(sz+uz)+B(sz-uz)\ )du \ .$$

Multiply both sides of equation (10.17) by $\psi_j(s)$ and integrate with respect to s over [-1,1] to obtain the L equations

$$(10.18) \qquad \sum_{i=o}^{L} (\ zW_{ij}(z)-\delta_{ij}\ )\alpha_i(z) = -U_j(z)$$

where

$$(10.19) \qquad U_j(z) = \int_{-1}^{1}\psi_j(s)B(z+sz)ds$$

and

$$(10.20) \qquad W_{ij}(z) = \int_{-1}^{1}\int_{-1}^{1}\psi_j(s)\psi_i(u)(\ B(sz+uz)+B(sz-uz)\ )duds \ .$$

Now the integrals of equations (10.19) and (10.20) can be approximated by a Gaussian quadrature of suitable order. Thus equation (10.18) becomes a set of L linear algebraic equations for the determination of the coefficients $\alpha_i(z)$ and equation (10.16) gives an approximation to $A(sz,z)$ for a given $z>0$ and any $-1<s<1$.

Using the change of variable $t = uz$ the fourth step of the inversion procedure reduces as follows

$$(10.21) \qquad g(z) = 1 + z\int_{-1}^{1} A(uz,z)\,du$$

$$= 1 + \sum_{i=o}^{L} z\alpha_i(z)\int_{-1}^{1}\psi_i(u)\,du$$

$$= 1 + \sqrt{2}\,z\alpha_o(z) \ .$$

The fifth and final step yields the conductivity profile as

$$(10.22) \qquad \sigma(z) = \sigma(0)\left(\ 1+\sqrt{2}\,z\alpha_o(z)\ \right)^2 \ .$$

This completes the numerical analysis. In the next section we shall solve the forward problem for various contrived conductivity profiles. We shall then use the inversion procedure outlined above to solve the corresponding inverse problems. By performing these calculations we hope to impart to the reader a measure of confidence in the inversion procedure.

SECTION 11

INVERSION OF FORWARD PROBLEM SOLUTIONS

Following our discussion of section 3, we may frame the forward problem as follows:

Given a particular depth dependent conductivity profile

$$(11.1) \qquad \sigma = \sigma(z) \ ,$$

we are required to find the apparent resistivity defined by

$$(11.2) \qquad \sigma(0)\rho_a(r) = \frac{-2\pi\sigma(0)r^2}{I}\frac{\partial\phi}{\partial r}(r,0)$$

where $\phi(r,z)$ is the potential induced by a single electrode carrying current I and therefore obeys equation (3.3),

$$(11.3) \qquad \{ \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} + \frac{\sigma'}{\sigma}\frac{\partial}{\partial z} \}\phi(r,z) = 0$$

where the prime denotes differentiation with respect to z. The boundary conditions satisfied by ϕ are firstly, that no current crosses the surface except at the point of entry of the electrode,

$$(11.4) \qquad \frac{\partial\phi}{\partial z}(r,0) = \frac{-I}{2\pi\sigma(0)}\frac{\delta(r)}{r}$$

and secondly, that the potential is zero at infinity,

(11.5) $$\lim_{r,z \to \infty} \phi(r,z) = 0 \ .$$

To solve this forward problem we follow Slichter [1933].

Since equation (11.3) is separable, we set

(11.6) $$\phi(r,z) = R(r)Z(z)$$

which on substitution into equation (11.3) yields a Bessel's equation

(11.7) $$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \lambda^2 R = 0$$

for R and the Sturmian equation

(11.8) $$\frac{d^2Z}{dz^2} + \frac{\sigma'}{\sigma}\frac{dZ}{dz} - \lambda^2 Z = 0$$

for Z. In equations (11.7) and (11.8), λ is an arbitary constant.

Since the potential remains finite at r=0, the appropriate solution for equation (11.7) is

(11.9) $$R(r) = J_0(\lambda r) \ .$$

The required solution of equation (11.8) depends on the form of the conductivity profile.

Test profile 1

We shall begin by investigating the simplest profile,

(11.10) $\sigma(z) = \sigma(0)$,

which represents a completely homogeneous earth. Equation (11.8) now reads

(11.11) $\frac{d^2Z}{dz^2} - \lambda^2 Z = 0$

and the appropriate solution which must vanish at infinity is given by

(11.12) $Z(z) = e^{-\lambda z}$.

Thus the required solution for the potential is of the form

(11.13) $\phi(r,z) = \int_0^\infty f(\lambda) J_0(\lambda r) e^{-\lambda z} d\lambda$

where f is an arbitary function of λ that must be fixed by the boundary condition (11.4) which reads

(11.14) $\frac{\partial\phi}{\partial z}(r,0) = -\int_0^\infty \lambda f(\lambda) J_0(\lambda r) d\lambda = \frac{-I}{2\pi\sigma(0)} \frac{\delta(r)}{r}$.

If f is taken as

$$(11.15) \qquad f(\lambda) = \frac{I}{2\pi\sigma(0)} \frac{\sin(\varepsilon\lambda)}{\varepsilon\lambda}$$

then equation (11.14) reduces to

$$(11.16) \qquad \int_0^\infty \frac{\sin(\varepsilon\lambda)}{\varepsilon\lambda} \lambda J_0(\lambda r)\, d\lambda = \frac{\delta(r)}{r}$$

which, using [Gradshteyn and Ryzhik, 1965, eq. 6.671.1], can be written as

$$(11.17) \qquad \frac{\delta(r)}{r} = \begin{cases} 0 & , \quad r > \varepsilon \\ \dfrac{1}{\varepsilon\sqrt{\varepsilon^2 - r^2}} & , \quad r < \varepsilon \end{cases}$$

showing that if ε is small our choice for the function f was correct, since on integrating both sides of equation (11.17) over the whole surface we obtain

$$\int_{z=0} \text{L.H.S. } ds = \int_0^\infty 2\pi\delta(r)\, dr = 2\pi$$

and

$$\int_{z=0} \text{R.H.S. } ds = \int_0^\varepsilon \frac{2\pi r}{\varepsilon\sqrt{\varepsilon^2 - r^2}}\, dr = 2\pi$$

as expected. We may now write equation (11.13) as

$$(11.18) \qquad \phi(r,z) = \frac{I}{2\pi\sigma(0)} \int_0^\infty \frac{\sin(\varepsilon\lambda)}{\varepsilon\lambda} J_0(\lambda r) e^{-\lambda z} d\lambda$$

and derive apparent resistivity from equation (11.2) as

$$(11.19) \qquad \sigma(0)\rho_a(r) = \frac{-2\pi\sigma(0)r^2}{I}\frac{\partial\phi}{\partial r}(r,0)$$

$$= r^2\int_0^\infty \frac{\sin(\varepsilon\lambda)}{\varepsilon\lambda}\lambda J_1(\lambda r)\,d\lambda$$

$$= \frac{r}{\sqrt{r^2-\varepsilon^2}}$$

$$= 1$$

if ε is small.

To test the numerical procedure of the last section we compute apparent resistivity from equation (11.19) at a finite number of surface points, and use this data as input to obtain a conductivity profile $\sigma=\sigma(z)$. In figure 11.1 below, we plot the reciprocal of conductivity (i.e. resistivity) versus depth.

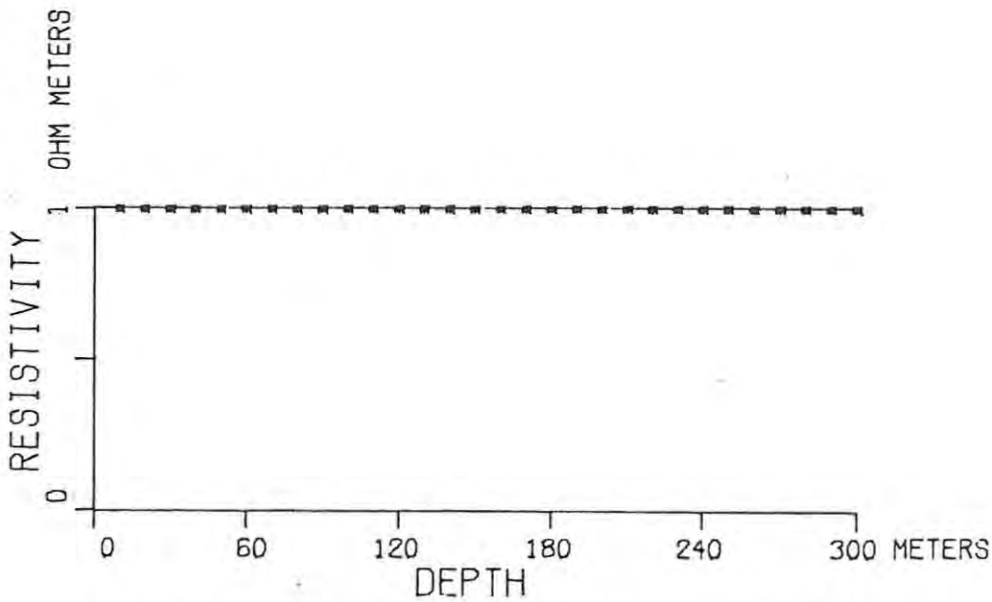

Figure 11.1 Reconstruction ** of the resistivity profile —— for a homogeneous earth.

Test profile 2

We now investigate a more complicated profile,

$$\sigma(z) = \sigma(0)(1+az)^2 \tag{11.20}$$

where a is some real constant. For this profile equation (11.8) reads

$$\frac{d^2Z}{dz^2} + \frac{2a}{1+az}\frac{dZ}{dz} - \lambda^2 Z = 0 \tag{11.21}$$

a solution of which is

$$Z(z) = \frac{e^{-\lambda z}}{1+az} \tag{11.22}$$

where λ is an arbitary constant. The required solution for the potential is thus of the form

$$\phi(r,z) = \int_0^\infty f(\lambda) J_0(\lambda r) \frac{e^{-\lambda z}}{1+az} d\lambda \ . \tag{11.23}$$

Using the same technique as used for the first test profile, the arbitary function f may be written as

$$f(\lambda) = \frac{I}{2\pi\sigma(0)} \frac{\sin(\varepsilon\lambda)}{\varepsilon\lambda} \frac{\lambda}{\lambda+a} \tag{11.24}$$

and equation (11.23) now reads

$$\phi(r,z) = \frac{I}{2\pi\sigma(0)} \int_0^\infty \frac{\sin(\varepsilon\lambda)}{\varepsilon\lambda} \frac{\lambda}{\lambda+a} J_0(\lambda r) \frac{e^{-\lambda z}}{1+az} d\lambda \tag{11.25}$$

so that the apparent resistivity, in this case, is

$$(11.26)\qquad \sigma(0)\rho_a(r) = -r^2\frac{\partial}{\partial r}\int_0^\infty \frac{\sin(\varepsilon\lambda)}{\varepsilon\lambda}\,\frac{\lambda}{\lambda+a}\,J_0(\lambda r)\,d\lambda\ .$$

Letting $\varepsilon \to 0$,

$$(11.27)\qquad \sigma(0)\rho_a(r) = -r^2\frac{\partial}{\partial r}\int_0^\infty \left(J_0(\lambda r) - \frac{aJ_0(\lambda r)}{\lambda+a} \right) d\lambda\ .$$

Using [Gradshteyn and Ryzhik, 1965, eqs. 6.511.1, 6.562.2]

$$(11.28)\qquad \sigma(0)\rho_a(r) = -r^2\frac{\partial}{\partial r}\left(\frac{1}{r} - \frac{a\pi}{2}(H_0(ra)-Y_0(ra)) \right)$$

where $H_0(.)$ and $Y_0(.)$ are respectively Sturve and Neuman functions of order zero. Using [Gradshteyn and Ryzhik, 1965, eqs. 8.553.3, 8.473.5] we may rewrite equation (11.28) as

$$(11.29)\qquad \sigma(0)\rho_a(r) = 1 - \frac{\pi a^2 r^2}{2}(H_1(ra)-Y_1(ra)-\frac{2}{\pi})$$

where $H_1(.)$ is the Sturve function of order one while $Y_1(.)$ is the Neuman function of order one.

If we take $\sigma(0)=1$ and $a=.005$ and compute apparent resistivity at a number of surface points from equation (11.29) then the inversion procedure of section 10 yields the resistivity profile of figure 11.2.

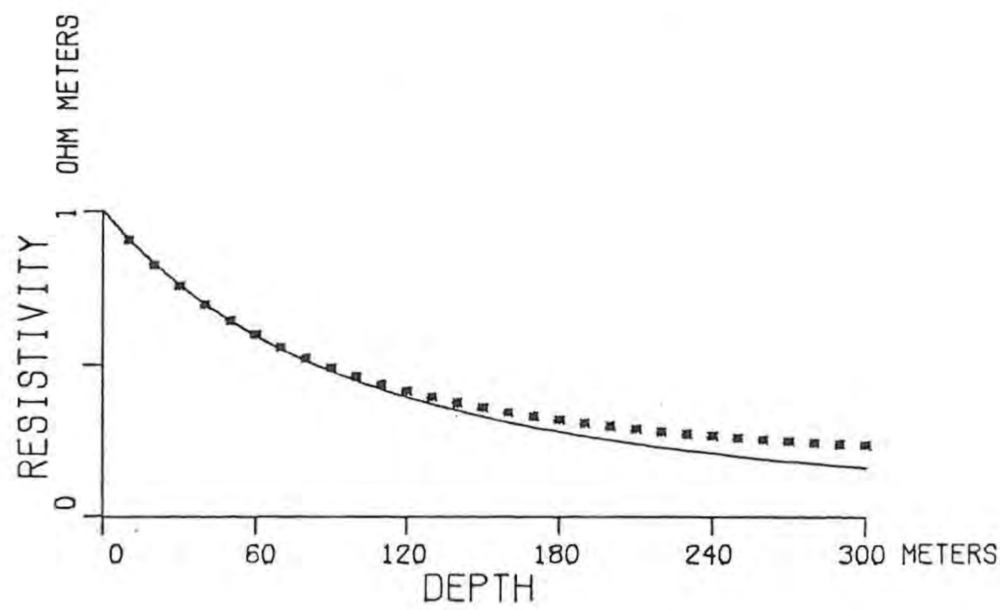

Figure 11.2 Reconstruction ** of resistivity profile —— $1/(1+.005z)^2$.

Test profile 3

The final profile investigated is that of a homogeneous layer over a homogeneous infinite half-space as depicted in figure 11.3 below.

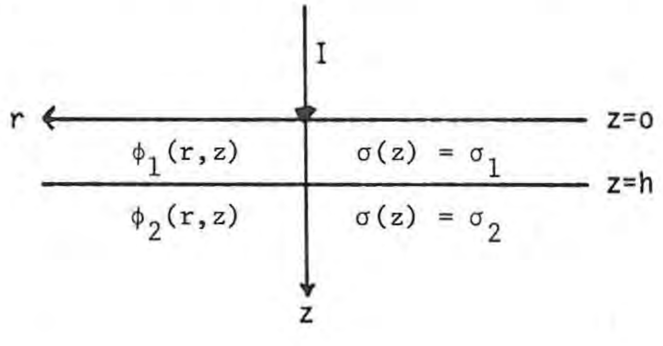

Figure 11.3 The geometry of the layer over half-space problem.

To solve this forward problem we follow the method of Stefanesco et al [1930]. Since the conductivity is constant in both the layer and the half-space, equation (11.3) reduces to

$$(11.30) \qquad \{ \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \}\phi_i(r,z) = 0 \quad , \quad i = 1,2 \ .$$

The boundary conditions satisfied by $\phi_1(r,z)$ and $\phi_2(r,z)$ are as follows:

No current crosses the surface except at the point of entry of the electrode,

$$(11.31) \qquad \frac{\partial \phi_1}{\partial z}(r,0) = \frac{-I}{2\pi\sigma_1}\frac{\delta(r)}{r} \ .$$

At the interface of the layer and the half-space the potential is continuous,

$$(11.32) \qquad \phi_1(r,h) = \phi_2(r,h) \ .$$

At the interface of the layer and the half-space the current density is continuous,

$$(11.33) \qquad \sigma_1 \frac{\partial \phi_1}{\partial z}(r,h) = \sigma_2 \frac{\partial \phi_2}{\partial z}(r,h) \ .$$

At infinite depth the potential must approach zero,

$$(11.34) \qquad \lim_{z\to\infty} \phi_2(r,z) = 0 \ .$$

For any z>0 the potential must remain finite as r approaches zero.

(11.35) $$\lim_{r\to o} \phi_i(r,z) \neq \infty \quad , \quad i = 1,2 .$$

As before, equation (11.30) is separable. We set

(11.36) $$\phi_i(r,z) = R_i(r)Z_i(z)$$

which yields the Bessel equations,

(11.37) $$\frac{d^2R_i}{dr^2} + \frac{1}{r}\frac{dR_i}{dr} + \lambda^2R_i = 0$$

for $R_i(r)$, and the equations

(11.38) $$\frac{d^2Z_i}{dz^2} - \lambda^2Z_i = 0$$

for $Z_i(z)$, where λ is an arbitary constant.

Since the potential remains finite at r=0, the appropriate solution for equation (11.37) is,

(11.39) $$R_i(r) = J_0(\lambda r) .$$

In the layer, there are no conditions on the value of the potential as depth becomes infinite, while in the half-space the potential must be zero at infinite depth. Thus the appropriate solutions for equations (11.38) are

(11.40) $$Z_1(z) = e^{\pm\lambda z} \quad , \quad Z_2(z) = e^{-\lambda z} \ .$$

General solutions for the potentials are therefore of the form

(11.41)
$$\phi_1(r,z) = \int_o^\infty (f_1^*(\lambda)e^{-\lambda z}+g_1(\lambda)e^{\lambda z})J_0(\lambda r)d\lambda$$
$$\phi_2(r,z) = \int_o^\infty f_2^*(\lambda)e^{-\lambda z}J_0(\lambda r)d\lambda$$

where f_1^* , f_2^* and g_1 are arbitary functions of λ still to be determined from the remaining boundary conditions. Since f_1^* and f_2^* are arbitary we may isolate the primary potentials, the potentials that would occur if no layer was present, by rewriting equations (11.41) as,

(11.42)
$$\phi_1(r,z) = \frac{I}{2\pi\sigma_1}\int_o^\infty (\frac{\sin(\varepsilon\lambda)}{\varepsilon\lambda}e^{-\lambda z} +f_1(\lambda)e^{-\lambda z}+g_1(\lambda)e^{\lambda z})J_0(\lambda r)d\lambda$$
$$\phi_2(r,z) = \frac{I}{2\pi\sigma_1}\int_o^\infty (\frac{\sin(\varepsilon\lambda)}{\varepsilon\lambda}e^{-\lambda z}+f_2(\lambda)e^{-\lambda z})J_0(\lambda r)d\lambda$$

where ε is an arbitarily small positive constant and f_1, f_2 and g_1 are again arbitary functions of λ. In order to apply boundary condition (11.31) we differentiate the first of equations (11.42) with respect to z, and then set z=0 and use equation (11.16) to obtain,

(11.43) $$\frac{\partial\phi_1}{\partial z}(r,0) = \frac{-I}{2\pi\sigma_1}\int_o^\infty (f_1(\lambda)-g_1(\lambda))J_0(\lambda r)d\lambda - \frac{I}{2\pi\sigma_1}\frac{\delta(r)}{r} \ .$$

Boundary condition (11.31) then yields,

(11.44) $$f_1(\lambda) = g_1(\lambda) .$$

Boundary condition (11.32) yields

(11.45) $$f_1(\lambda)e^{-\lambda h} + g_1(\lambda)e^{\lambda h} = f_2(\lambda)e^{-\lambda h} .$$

Boundary condition (11.33) yields

(11.46) $$\sigma_1(-f_1(\lambda)e^{-\lambda h} + g_1(\lambda)e^{\lambda h} - e^{-\lambda h}) = \sigma_2(-e^{-\lambda h} - f_2(\lambda)e^{-\lambda h}) .$$

Solving these three equations for $f_1=g_1$, we obtain

(11.47) $$f_1(\lambda) = g_1(\lambda) = \frac{ke^{-2\lambda h}}{1-ke^{-2\lambda h}}$$

where the constant k is called the reflection coefficient and is given by

(11.48) $$k = \frac{\sigma_1-\sigma_2}{\sigma_1+\sigma_2} .$$

The potential at the surface z=0, is now given by

(11.49) $$\phi_1(r,0) = \frac{I}{2\pi\sigma_1}\int_0^\infty\left(\frac{\sin(\varepsilon\lambda)}{\varepsilon\lambda} + \frac{2ke^{-2\lambda h}}{1-ke^{-2\lambda h}} \right)J_0(\lambda r)d\lambda .$$

Letting $\varepsilon \to 0$ and using equation (11.2), the apparent resistivity is

(11.50) $$\sigma(0)\rho_a(r) = \frac{-2\pi\sigma_1 r^2}{I}\frac{\partial\phi_1}{\partial r}(r,0)$$

$$= -r^2\frac{\partial}{\partial r}\int_0^\infty\left(1+\frac{2ke^{-2\lambda h}}{1-ke^{-2\lambda h}} \right)J_0(\lambda r)d\lambda .$$

Expanding the integrand of equation (11.52) in an infinite series we obtain

$$(11.51)\qquad \sigma(0)\rho_a(r) = -r^2 \frac{\partial}{\partial r}\left(\int_0^\infty J_0(\lambda r)\,d\lambda + 2\sum_{n=1}^{\infty} \int_0^\infty k^n e^{-2\lambda hn} J_0(\lambda r)\,d\lambda \right) .$$

Integrating term by term, with the aid of [Gradshteyn and Ryzhik, 1965, eqs. 6.511.1, 6.611.1], we arrive at

$$(11.52)\qquad \sigma(0)\rho_a(r) = 1 + 2\sum_{n=1}^{\infty} \frac{k^n}{\left(1+\frac{4h^2n^2}{r^2}\right)^{3/2}}$$

for the apparent resistivity. Setting $\sigma_1 = 1.0$, $\sigma_2=2.0$, and h=50.0 the inversion procedure of section 9 yields the reconstruction shown in figure 11.4 below as the resistivity profile of a layer over a half-space.

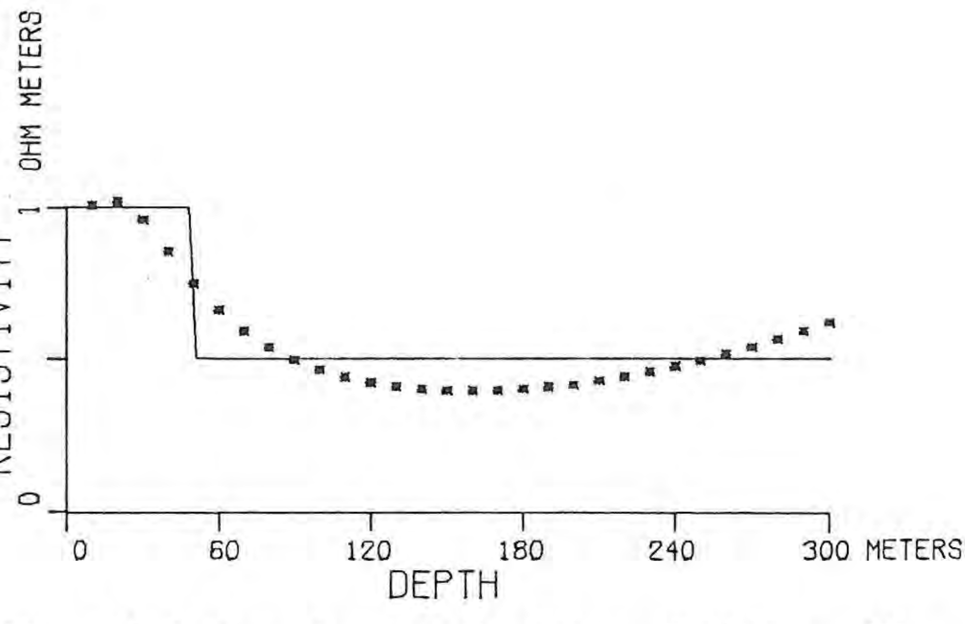

Figure 11.4 Reconstruction ** of the resistivity profile —— for a layer over an infinite half-space.

To conclude this section we note that the inversions for the test profiles 1 and 2 may be performed analytically as shown by Coen and Wang-Ho Yu [1981]. In the next section we shall apply the inversion procedure to apparent resistivity measurements supplied by Dr. J. de Beer of the C.S.I.R..

SECTION 12

INVERSION OF RAW FIELD DATA

In table 12.1 below are C.S.I.R. results of Shlumberger array field measurements of apparent resistivity, kindly made available by Dr. J. de Beer.

r	$\rho_a(r)$	r	$\rho_a(r)$	r	$\rho_a(r)$
1.000	10.001	10.000	10.389	100.000	20.464
1.259	10.001	12.589	10.699	125.893	21.548
1.585	10.002	15.849	11.201	158.489	22.696
1.995	10.004	19.953	11.946	199.526	24.067
2.512	10.008	25.119	12.947	251.189	25.843
3.162	10.015	31.623	14.158	316.228	28.172
3.981	10.029	39.811	15.488	398.107	31.090
5.012	10.057	50.119	16.838	501.187	34.484
6.310	10.110	63.096	18.135	630.957	38.135
7.943	10.209	79.433	19.343	794.328	41.809
				1000.000	45.312

Table 12.1 Raw data from Shlumberger array field measurements of apparent resistivity versus half the separation of the current electrodes.

The C.S.I.R. were able to reconstruct the apparent resistivity of table 12.1 by assuming that the earth below the electrodes consisted of two layers over an infinite half-space. The thickness of the layers and the conductivities of the layers and the half-space were varied until the reconstructed apparent resistivity fitted the measured apparent resistivity to the required degree of accuracy. The resulting model is depicted by the full line in figure 12.1 below.

Using the Coen and Wang-Ho Yu inversion algorithm we obtain the model depicted by the asterisks in figure 12.1.

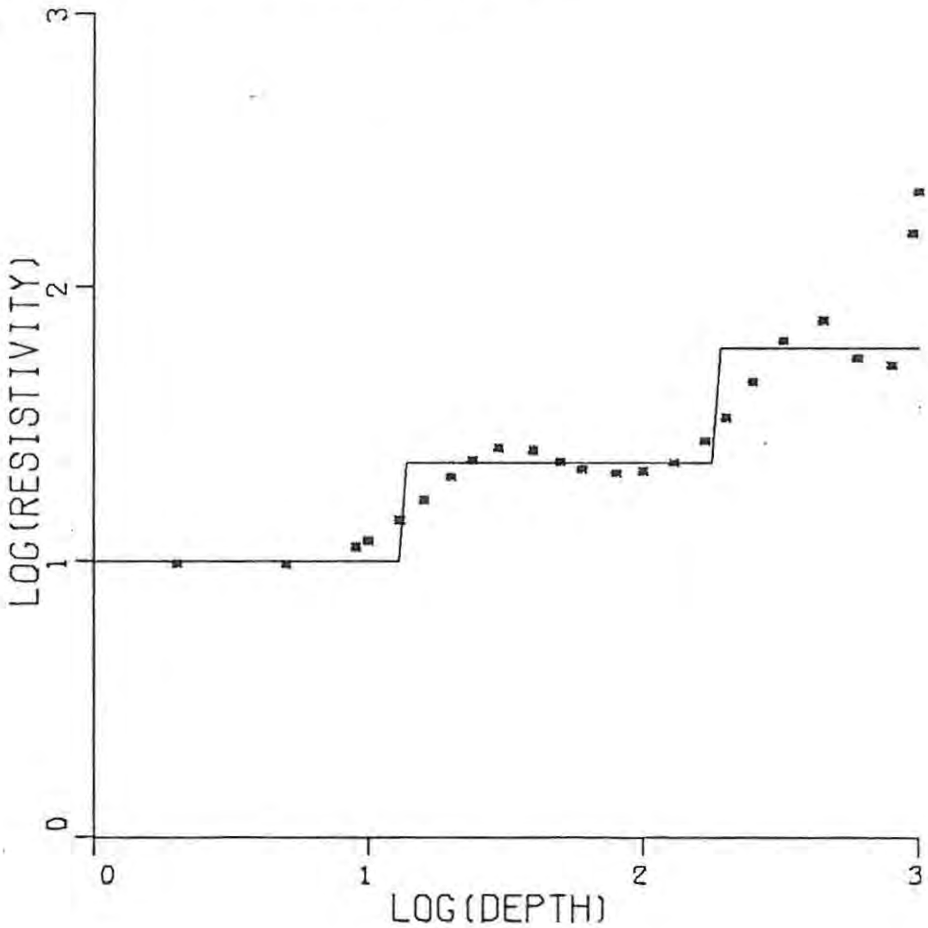

Figure 12.1 Coen and Wang-ho Yu inversion **
compared with C.S.I.R. model ——.

SECTION 13

CONCLUSION

Three features of the Coen and Wang-Ho Yu inversion algorithm [1981] have been demonstrated:

(a) If the conductivity profile is a continuous function of depth and if complete and precise surface data are given, then the inversion algorithm yields the conductivity profile as the unique solution to the inverse problem.

(b) If the conductivity profile is a continuous function of depth but only incomplete surface data are available, then the inversion algorithm yields an approximation to the conductivity profile as a particular solution of the inverse problem.

(c)If the earth consists of a number of homogeneous layers over a homogeneous infinite half-space and incomplete surface data are given, then the inversion algorithm seems to yield an approximation to the discontinuous conductivity profile.

REFERENCES

Bernstein D.L., 1950, Existance Theorems in Partial Differential Equations, Princeton University Press, Princeton.

Coen S. and Wang-Ho Yu M., 1981, The inverse problem of the direct current conductivity profile of a layered earth, Geophys., 46, 1702-1713.

Gel'fand I.M. and Levitan B.M., 1955, On the determination of a differential equation from its spectral function, Amer. Math. Soc. Transl., 1, 253-304.

Gradshteyn I.S. and Ryzhik I.M., 1965, Table of Integrals Series and Products, Translation edited by A. Jeffrey, Academic press, London.

Kamke E., 1959, Differentialgleichungen, Vol. 1, Akad. Verlagsges., Leipzig.

Koefoed O., 1979, Resistivity Sounding Measurements, Geosounding Principles 1, Elsevier Scientific Publishing Company, Oxford.

Morse P.M. and Feshbach H., 1953, Methods of Theoretical Physics, Vol. 1, McGraw-Hill, London.

Oldenburg D.W., 1978, The interpretation of direct current resistivity measurements, Geophys., 43, 610-625.

Slichter L.B., 1933, The interpretation of the resistivity prospecting method for horizontal structures, J. Appl. Phys., 4, 307-322.

Stefanesco S. and Schlumberger C. and M., 1930, Sur la distribution electrique potentielle autour d'une prise de terre ponctuelle dans un terrain a cauches horizontales homogenes et isotropes, J. d. Phys., 132, 132-140.

Weidelt P., 1972, The inverse problem of geomagnetic induction, Zeit. fur Geophys., 38, 257-289.