An analysis of visualization processes used by selected Grade 11 and 12 learners when solving algebraic problems: A Namibian case study.

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ABSTRACT

Visualisation is gaining visibility in mathematics education research. It is a powerful tool for solving different types of problems in many areas of mathematics, including Algebra – the mathematical domain of this study. The aim of this case study was to analyse the visualisation processes that selected senior secondary school learners used to solve a set of ten algebraic problems.

The research was conducted at a secondary school in the Erongo region of Namibia. This is the school where the researcher is teaching. The sample consisted of six selected learners, three from Grade 11 and three from Grade 12. The learners were purposefully selected to participate in this study based on their mathematics performance in class and their willingness to participate. The participants were video recorded as they worked through the ten items of an Algebraic Visualisation Tasks (AVT) worksheet. They were also interviewed about the visualisation processes they employed when solving each of the tasks. The AVT, the videos and the interviews were analysed with the aid of an adapted visualisation template.

The findings of the study show that learners used visualisations in all their problem-solving processes. These visualisations were used for different purposes such as starting points, for illustrative purposes, as organizational tools and as simplification tools. Visualisations as starting points were used when the problem was wordy and had lengthy descriptions and explanations. Illustrative visualisations enabled learners to articulate in their own way mathematical notations, mathematical equations and expressions that they then used to solve the problem. Organizational pictures provided a useful structural framework for solving the problems.

This research suggests that the selected participants indeed used visualisation processes to solve algebraic problems. It is thus important for teachers to harness this aid and make the most use of these visualisation processes when teaching Algebra.

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DECLARATION OF ORIGINALITY

I **Joseane Josef (Student Number: 13j6825)** declare that this thesis is my own work written in my own words. It has not been submitted in any form for another qualification or any assessment to another University or institution. Where I have drawn on the words or ideas of others, these have been acknowledged using the reference practices according to the Rhodes University Education Department Guide to Referencing.

Joseane Josef 25 November 2016

(Signature) (Date)

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CHAPTER 1

INTRODUCTION OF THE STUDY

"People remember visual aspects of a concept better than its analytical aspects"
Vinner (1992, p. 212)

1.1 INTRODUCTION

This chapter introduces my study, which is an analysis of visualisation processes used by selected Grade 11 and 12 learners when solving algebraic problems. The chapter begins by providing the background of the study and its context. It then explains the rationale for the study. It also presents the research question and the significance of the study as well as highlighting some limitations to the study. The chapter then concludes with a brief overview of the study.

1.2 BACKGROUND TO THE STUDY

1.2.1 Context

In my experience, many students often struggle to use visualisation as a strategy especially when solving mathematical problems. This, I argue, could negatively affect their performance. Healy and Hoyles (1996) articulate, "Unlike mathematicians, students of mathematics rarely exploit the considerable potential of visual approaches to support meaningful learning. …they are reluctant to engage with visual modes of reasoning" (p. 67). Healy and Hoyles continue by stressing the advantages of being able to use particular images or diagrams in the service of mathematical generalization, and of making connections between modes of thinking. Van Garderen, Potch & Scheuermann, (2012) noted the importance of employing visualisation strategies. They argue that the use of visual aids such as sketches and diagrams are very powerful strategies in solving different types of problems for many topic areas.

The literature suggests that mathematical knowledge or computation skills are not the only key to mathematical understanding, but that there are other contributing factors, including the ability to visualise the problems that may affect achievement (Cummins et al., 1988; Hegarty et al., 1995; Kyttala & Bjorn, 2014).

My research project aimed to not only analyse the visualisation strategies employed by learners in solving algebraic problems, but also how the learners used these strategies and identified the visualisation processes that they employed. My study was inspired by Thornton (2000) who proposes, "Though much has been said about visualisation in general, there are still many issues concerning visualisation in mathematics education, which require careful attention." (p. 251). There is a need for teachers to understand how learners think visually during problem solving to be able to teach mathematical concepts such as algebraic concepts and problem solving (Makina & Wassels, 2009). It is important to listen to learners since there is a difference between what we (as teachers) want children to learn and what they actually take from our lessons (Makina & Wassels, 2009). Makina and Wassels further state that understanding the student's mind during problem solving improves the teaching of mathematics. Steenpaß, et al., (2014) resonate with Makina & Wassels that "understanding how learners come to understand and conceptualize the world around them can help teachers facilitate learning" (p. 2).

Furthermore, visualisation processes stimulate learners to ''discover the unexpected, and describe and explain the expected'' (Thomas & Cook, as cited in Rivera et al., 2014, p. 1). A visualisation process is one that involves visual imagery with or without a diagram, as an essential part of the method of solution (Presmeg, 1985, p. 298). The interaction of context, visual representational forms and using technological tools, is seen as a key strategy that supports functional understanding (Confrey & Smith, 1994). Using visualisation processes can thus assist learners to make new discoveries and solve mathematical problems. It is therefore important for teachers and learners to be aware of the type of visualisation processes and strategies used by learners that help them in solving algebraic problems. Ho (2010) articulates that "if visualisation is at the heart of mathematical problem solving, then it is vital that both teachers and students see the role of visualisation clearly and use it to help them in their problem-solving process" (p. 3).

Threlfall (2009) defines problem solving strategies in a mathematical context, as the "different ways" that mathematical problems are solved (p. 154).When learners are given a mathematical problem the overall approach to the problem is what Threlfall (2009) and Ashcraft (1990, as cited in Threlfall, 2009) refer to as strategies. For example, we can examine the possible responses of learners to a problem such as "*a number is trebled and then 7 is added to it. If the total is 28, find the number*". When learners are asked how they solved the problem, their responses will be manifold. They may say, "*I used the method that I was taught by my teacher*" or "*I came up with an equation that I solved to get the answer*" or "I *did the calculation mentally*" or "*I used the try and error method*" or "I *used a sketch to visualise the problem*". All these are legitimate strategies that learners can use to solve problems; the conclusion is that there are many different approaches that lead to the same solution.

Good and appropriate strategies and representations enable learners to solve problems efficiently and accurately (Heinze, Star & Verschaffel, 2009). Rivera (2003, p.59) postulates, "visual strategies play a mediating role in the emergence of children's sophisticated, structured and necessary understandings of mathematical objects". Children use visual strategies to help them conduct explorations, organize relevant data and anticipate an intended analysis (*ibid.*).

Draper (2010) argues that "visualisation strategies help learners recall facts, get the main idea, make an inference, draw a conclusion, predict/extend and evaluate" (p.11). Leaners use visualisation strategies to "analyse and make conjectures about information, to analyse situations to make connections and plan solutions" (Draper, 2012, p. 2). However, individual learners should acquire the ability to solve mathematical tasks flexibly by a diversity of meaningful acquired strategies and representation (Heinze et al., 2009). Learners with strategic competence can not only come up with several approaches to a non-routine problem but can also choose flexibly the method to suit the demands presented by the problem and the situation in which it was posed (Kilpatrick.et al., 2001, p. 129).

Learners can only use visualisation processes flexibly and adaptively when they know when to use them, why to use them and where to use them. In short, the choice of strategy is an important consideration. My research therefore aimed to explore the visual processes that selected learners use to solve algebraic problems. I believe that this study will also benefit other teachers and learners in recognising the importance of using visualisation processes in their practices.

1.2.2 Rationale

Watson (2007) asserts that the use of diagrams, visual representations and mental images are important elements in solving mathematical problems. There are many ways to solve algebraic problems and learners need the skill to choose an appropriate visualisation strategy for a specific problem (Polya 1957; Ho, 2010). Despite the abundant evidence provided by literature about the importance of visualisations, the use of visualisations in the teaching and learning in Namibia is, in my view, not sufficiently recognized.

The purpose of my study is therefore to inform the curriculum developers about the importance of visualisations and the impact that it has on the teaching and learning of Mathematics. This could contribute to the revision processes of the current Namibian curriculum. The research also aims to encourage teachers to make use of visual representations in their lessons, and to motivate learners to make use of visual representations when solving mathematical problems.

1.3 RESEARCH GOALS AND QUESTIONS

The aim of the study is to analyse selected learners' visualisation processes in solving algebraic problems and find out how they are used. The two research questions that frame the study are:

- What problem solving strategies using visualisation processes are employed by selected Grade 11 and 12 learners when solving algebraic problems?

- How do these Grade 11 and 12 learners use the identified problem solving processes to solve algebraic problems?

1.4 RESEARCH METHODOLOGY

1.4.1 Orientation

My research project was oriented within an interpretive paradigm as I wished to gain a deeper understanding of how learners solve algebraic problems with respect to the visualisation processes that they use. According to Bertram & Christiansen (2014), interpretivist researchers "aim to understand how people make sense of their worlds" (p. 26). My approach was qualitative.

1.4.2 Research design

The study was conducted in one school only and a small sample of three Grade 11 and three Grade 12 learners was used. These learners were purposely selected based on their participation in their class, their performance in their tests and in examinations. Thus, this study was framed as a case study. According to Rule & John, as cited in Bertram, & Christiansen (2014), "a case study is a systematic and in-depth study of one particular case in its context" (p. 42).

To collect data for this research I used different research tools such as interviews (that were conducted when learners were working on the algebraic tasks), learners' work (responses to the Algebraic Visualisation Tasks-AVT) and observation. The interviews were conducted in order to find out and interpret the visualisation processes that the learners employed. Observations were collected to further identify the visualisation processes used and the manner in which they were used. A template was designed to assist in identifying and categorising the visualisation processes that were observed.

1.4.3 Research processes

The data generation took place in three phases. In the first phase, I selected the site and the participants. I also refined the algebraic visualisation tasks (AVT) and piloted them. In the second phase, I implemented the AVT. The interviews and the observations took place in this phase. The interactions were video recoded to capture what participants said and drew. In the third phase, video recordings were analysed.

1.5 SIGNIFICANCE OF THE STUDY

Students use visualisation processes as tools to support their mathematical understandings. For teachers, the strategic use of visualisation processes in their lessons could influence students' acquisition of knowledge and have a bearing on their mathematical achievement.

The research could provide insightful information on algebraic problem solving strategies that could be useful to teachers, learners and curriculum developers. It is hoped that as I disseminate the findings of this research, teachers will appreciate the significance of recognizing the visualisation processes that learners use when they engage in algebraic problems. For curriculum developers this study could be informative in foregrounding the need to use appropriate visualisations in class in general and for algebraic problem solving in particular. This understanding can lead to the enhancement of the curriculum.

1.6 LIMITATIONS

Only six learners participated in this research case study. This is a very small sample and consequently the findings of this study cannot be generalized. The analysis of a higher number of learners' work would have possibly revealed more visualisation processes and provided a more comprehensive picture.

1.7 STRUCTURE OF THESIS

1.7.1 Chapter two

This chapter provides an in-depth contextual background to the study. It starts with the definition of visualisation in a detailed way where visualisations in mathematics are discussed. It also deals with the role and processes of visualisation in mathematics. The chapter discusses the terms *problem solving* and *algebra*. Thereafter a brief discussion of visualisations and Algebra in the Namibian context is provided. The chapter is concluded by providing a theoretical backdrop to the study.

1.7.2 Chapter three

This chapter presents and discusses the research methodology used in this study. The chapter discusses the research goals, research orientation, sampling, data generation and data analysis, as well as issues relating to ethics and validity. The final drafts of the algebraic visualisation tasks and the analytical tool are also presented in this chapter.

1.7.3 Chapter four

This chapter consists of the data analysis and findings of this study. The chapter provides an analysis of the participants' AVT work and the interviews. It starts with a description of the video clips and the analysis of the AVT. In conclusion, it gives details of the visualisations used by the learners and the manner in which they were used.

1.7.4 Chapter five

Chapter five concludes the study by providing a summary of the findings of the study, making certain recommendations arising from it, describing the limitations of the study and making suggestions for future research. Moreover, the chapter also includes some personal reflections on my journey as a novice researcher.

CHAPTER 2

LITERATURE REVIEW

"Mathematics requires representations. In fact, because of the abstract nature of mathematics, people have access to mathematical ideas only through the representation of those ideas"
(Kilpatrick, Swafford, & Findell, 2001, p. 94).

2.1 INTRODUCTION

This chapter reviews the literature about visualisations. It defines visualisation and discusses how visualisation influences the teaching and learning of mathematics. Problem solving is also described and the difference between problem solving tasks and exercise tasks is presented. Additionally reasons as to why problem solving is part of the curriculum are discussed. Thereafter, I elaborate on the role that visualisation plays in algebra, and particularly in the Namibian context. The chapter will conclude by reviewing my theoretical framework, which is constructivism and visualisation.

2.2 VISUALISATION

2.2.1 Visual representations in Mathematics

Within the domain of mathematics, the terms *visualisation(s)* and *representation(s)* are used interchangeably. However, for the purpose of this thesis I will make no distinction between the two. When quoting or referencing other authors' works, I will use their terminology.

Previous research defined and explained visualisation(s)/representation(s) as follows: Duval (2014, pp.159-160) define visual representations as "all kinds of representations that are used in mathematics and in the teaching of mathematics to fulfil quite different functions such as mathematical treatment, heuristic exploration in problem solving, and as educational tool for helping the acquisition of mathematical concepts". Palmer (1977 as cited in Goldin & Kaput, 1996) defines a visual representation as a thing which is produced that symbolises, stands for, is associated with, or otherwise represents something else. Furthermore, Stylianou (2013) echoes, "*representation* includes the choices we make for expressing and depicting mathematical ideas and the ways in which we put them to use. Representations can be drawings, diagrams, physical models, and also mathematical symbols – in short, the range of symbolic tools that can be used for representing aspects of the world". (p. 23). Visual representations are types of external representations that are used extensively in mathematics textbooks and are considered to enhance problem solving in all the phases of a particular process (Larkin & Simon, 1987).

Moreover, Shulman (1986) acknowledged visual representations as being part of teachers' pedagogical knowledge. He defined these representations as "including analogies, illustrations, examples, explanations, and demonstrations in a word, the ways of representing and formulating the subject that make it comprehensible to others" (p.9). However, representing is not a matter of simply copying what we see. Instead, it involves discovering or adapting conventions of a representational system for the purpose at hand. Thus, Stylianou (2013) argues, "the development of students' ability to represent ideas in different ways is fundamental to mathematical work" (p.23).

2.2.2 Types of mathematical visualisations

Zimmermann and Cunningham (2010) see mathematical visualisation as a broad field that comprises non-computer based visualisation as well as visualisation based on computers or other technologies. They further state that mathematical visualisation includes "the ability to draw a simple figure to represent a mathematical problem, to interpret such figures with understanding, and to use such figures as an aid in problem solving" (p.5). Zimmermann and Cunningham specify that the ability to draw mathematical representations and interpret them with understanding is a fundamental visualisation skill. "Without such fundamental skills, it is unlikely that computer-based visualisation can be used efficiently, or even meaningfully. Vision is not visualisation; to see is not necessarily to understand" (p.5).

When solving mathematical problems, learners go through different processes. They generate diagrams and other representations that aid them in finding the solution. For example, to solve a mathematics problem, the problem solver (learner) must first construct an internal representation of the problem and build a mental model of the problem situation (Casey, 1978, cited in Clements, 1980; Kintsch & Greeno, 1985; Mayer, 1992). These mental and internal constructions is what Goldin and Steingold (2001) refer to as internal and external visualisations or 'mental structures' and 'notation systems' respectively as referred to by Kaput (1991).

In an attempt to distinguish between the two visualisations, Goldin and Steingold (2001) suggest that **external** systems of representation include conventional representations that are usually symbolic in nature, such as our numeration system, mathematical equations, algebraic expressions, graphs, geometric figures, and number lines. Goldin and Kaput (1996, p. 400) define external representations as “physically embodied, observable configurations such as words, graphs, pictures, equations, or computer micro worlds”. Janvier, Girardon, & Morand (1993) echo that “representations such as numerals, algebraic equations, a graph, tables, diagrams, and charts are external manifestations of mathematical concepts.” (p. 81). According to Chiappini and Bottino (2010) external representations “are two- or three-dimensional representations of some aspects of a mathematical structure” (p.1).

In addition, Goldin & Shteingold (2001) provide other examples of external representation. They state that some external systems of representation are mainly notational and formal and these include our system of numeration; our ways of writing and manipulating algebraic expressions and equations; our conventions for denoting functions, derivatives, and integrals in calculus; and computer languages such as Logo. They further state that other external systems are designed to exhibit relationships visually or spatially, such as number lines, graphs based on Cartesian, polar, or other coordinate systems, box plots of data, geometric diagrams, and computer-generated images of fractals. Finally, they indicate that words and sentences, written or spoken, are also external representations. They can denote and describe material objects, physical properties, actions and relations, or things that are far more abstract.

On the other hand, **internal** systems of representation are created within a person’s mind and used to assign mathematical meaning (Goldin and Steingold, 2001). Kosslyn (1995) suggests that internal representations are visual imagery, which he defines as mental representations of the appearance of objects and manipulation of these representations in the mind. Similarly, internal visualisation is defined by Goldin (2002) as “the internal systems of representation that are created within a person’s mind and used to assign mathematical meaning”. (p. 178). Goldin and Kaput (1996) agree that “internal configurations are those characteristics of the reasoning individual that are encoded in the human brain and nervous system and are to be inferred from observation”. (p.402).

In the past, the area of mental representations was widely ignored during the period where the dominant school of thought was linked to behaviourist theories (Gardner, 1985), but with the increase of cognitive science, mental representations are increasingly being recognised and seen as an important part of learning (Gardner, 1985). Goldin (1998, p. 194) identifies the following internal representations:

- Verbal or syntactic: capacities related to the use of natural language by individuals, mathematical and non-mathematical vocabulary, including the use of grammar and syntax.
- Figural (imagistic) and gestural systems, including spatial and visual cognitive configurations, or mental images, gestural and body schema.
- Mental manipulation of formal notations (numerals, arithmetic operations, visualisation of symbolic steps to solve an equation)
- Strategic and heuristic processes: trial and error, breakdown into stages, etc.
- Affective systems of representation: emotions, attitudes, beliefs and values with respect to mathematics, or about themselves in relation to mathematics.

According to Schnotz & Bannert (2003), text and visual displays belong to classes of representations called descriptive and depictive representations respectively. Descriptions are regarded as more powerful in representing different kinds of subject matter while depictions are better suited to draw inferences (Johnson-Laird, 1983; Johnson-Laird & Byrne, 1991).

Schnotz & Bannert (2003) define descriptive representations as representations that are spoken words or written texts such as mathematical equations and logical expressions. According to Elia, Gagatsis, Monoyiou & Spagnolo (2014), descriptive representations consist of symbols that have a subjective structure and are associated with the content they represent simply by means of a convention. A depictive representation consists of iconic signs such as pictures, sketches, or drawn models (p. 143). Depictive representations include iconic signs that are associated with the content they represent through common structural features on either a concrete or more abstract level (Elia et al., 2014). Schnotz et al. (2003) provide a simple example, "if the descriptive representation of a function is made by the term $2x + y = 0$ then a corresponding depictive can be a straight line graph in a Cartesian plane that passes through the origin" (p. 32). In other words $2x + y = 0$ is a descriptive representation and the straight line diagram is the depictive representation.

However, this study will not go into details of what internal and external visualisations are, but rather advocate the significance of both internal and external visualisations in understanding mathematical concepts. English and Halford (1995) suggest, "The essence of understanding a concept is to have an internal representation or internal model that faithfully reflects the structure of that concept". (p. 57). However, learners have to use external representation to explain and show their understanding. The New Zealand Ministry of Education (2008) states, "a student's ability to illustrate their mental strategy with materials is evidence of strong understanding of the number properties involved". This implies that none of these types of visualisation (external and internal) can be used in isolation.

2.2.3 Interaction between external and internal representations

Understanding of a mathematical concept is based on the internal representations of a concept, which are influenced by the external representations of the concept that are presented to learners (Hiebert & Wearne, 1992). This implies that there is a relationship between internal and external representations in developing students' understanding of mathematical concepts. One example illustrates the interplay between internal and external visualisations as that of a six years old child playing with several toys (Pape & Tchoshanov, 2001). As the child plays, she names the first toy as *one*, the second as *two,* and so on. Pape & Tchoshanov (2001) say that "these number words may simply be words the child has learned to utter as she touches each object in a series of objects" and this child has no idea that "these words are symbols for the position of the toys in the series the child is enumerating" (p.119). As the child gets older she begins to understand that the last number named in this game is the number of toys that are in the set and finally that there is a numeral that represents the number of elements in the set e.g. *three* and *3.*

The interplay between internal and external representations is important as "it supports the development of an effective model" (Piggott & Woodham, 2010. p, 2). It is considered to be fundamental for teaching and learning (Godino & Font, 2010). If teachers want learners to utilise and improve their capacity to visualise, they need to understand why visualisation is important (the purposes of visualising) (Piggott & Woodham, 2010).

2.2.4 Importance of visualisation

Visualisation has long been thought to play an important role in mathematics problem solving (e.g., Hadamard, 1945). Some literature indicates that, when mathematical problems are particularly difficult, or when solutions must be shared with others, problem solvers may externalize these visualisations by making inscriptions on paper or other media (e.g., Clement, Lochhead, & Monk, 1981; Corter & Zahner,2007; Latour & Woolgar, 1986; Roth & McGinn, 1998; Russell, 2000; Schreiber, 2004). Thus, "visualisation processes help the individual to solve a problem or provide an explanation, prediction, or justification" (Perkins & Unger, 1994, pp. 6-7). In the context of word problems, Harvey and Goudvis (2000, p. 7) describe the benefits of visualising as follow:

- allows the reader to create mental images from the words in the problem
- enhances meaning with mental imagery
- links past experience to the words and ideas in the text
- enables readers to place themselves in the story
- strengthens a reader's relationship to the text
- stimulates imaginative thinking
- heightens engagement with text
- brings joy to reading

It is now well accepted that the use of particular modes of representations (e.g. visual or concrete) leads to improvement of students' mathematical abilities and development of their advanced problem solving and reasoning skills (Krutetskii, 1976; Yakimanskaya, 1991; Presmeg, 1999). That is, the use of multiple representations facilitates students' development of mathematical concepts (e.g. Brenner et al., 1997) and their efforts to carry out tasks such as problem solving (Greeno & Hall, 1997).

Visual representations are important in mathematics education because they enhance an intuitive view and an understanding in many areas of mathematics (Krutetskii, 1976; Usiskin, 1987). Parnafes & Disessa (2004) expresses, "when the students use several representations, they develop a more flexible understanding of the concept" (p. 251). Corter & Zahner (2007) echo that external sketches or diagrams may sometimes be created by a problem solver to aid in understanding the problem text.

Understanding of a mathematical concept is often based on the internal representations of a concept, which are influenced by the external representations of the concept that are presented to learners (Hiebert & Wearne, 1992). Wood (1999) also stated that conceptual understanding rests on a multiple system of 'signs' or representations. The argument has been further justified by Lesh, Landau, & Hamilton (1983) that a student understands a mathematical concept if he or she could 'translate' or move between multiple representations.

Mathematics is composed of a large set of highly related abstractions (Fennema & Franke, 1992, p.153). Visualisation is an important way to concretize mathematical concepts, and may be used to represent mathematical objects that do not have a real existence. Zimmerman and Cunningham (1991) insist that visualisation supplies depth and meaning to understanding, serving as a reliable guide to problem solving, and inspiring creative discoveries. However, in order to achieve this understanding, they propose that visualisation cannot be isolated from the rest of mathematics, implying that symbolical, numerical and visual representations of ideas must be formulated and connected.

Rösken & Rolka (2006) also acknowledge the role that visualisation plays in mathematics learning. They state that visualisation can be a powerful tool to explore mathematical problems and to give meaning to mathematical concepts and the relationship between them. They further articulate, "visualisation allows for reducing complexity when dealing with a multitude of information" (Rösken & Rolka, 2006, p. 458).

Some researchers have outlined the role that visualisations play in linking abstract mathematics to the concrete experiences of learners (Bruner & Kenney, 1965; Post & Cramer, 1989; Fennema & Franke, 1992; Duval, 1999). Fennema & Franke (1992) are of the view that:

> Mathematics is composed of a large set of highly related abstractions, and if teachers do not know how to translate those abstractions into a form that enables learners to relate the mathematics to what they already know, they will not learn with understanding. (Fennema & Franke, 1992, p.153)

Furthermore, Van Garderen, Scheuermann, & Jackson (2012) observe that when a visual diagram is used in strategic ways (e.g. not only to understand and solve the problem but also to monitor problem solving), it has been positively correlated to higher performance in solving word problems. More broadly, “Visualisations can be considered as useful tools for constructing understanding and for communicating information and understanding” (Greeno & Hall, 1997, p.362). Diagrams are powerful ways to facilitate communication about critical ideas in mathematics as well as provide a platform for sharing problem-solving strategies with others (Stylianou, 2010). Duval (1999) summarises by saying that representations play an epistemological and pedagogical role in teaching and learning.

Regarding specifically the domain of algebra, Rivera (2011) asserts that visually drawn constructions of some mathematical objects, concepts or processes can effectively assist in developing what Mason et al. (2009) name as a structural awareness of the corresponding abstract knowledge, despite being in an incomplete form. More researchers have outlined the role that representations play in linking abstract mathematics to the concrete experiences of learners (Bruner & Kenney, 1965; Post & Cramer, 1989; Fennema & Franke, 1992; Duval, 1999).

In addition, the National Council of Teachers of Mathematics (2003) articulate that in some areas of mathematics, such as geometry, understanding and using pictures and diagrams is considered to be an integral part of the domain knowledge. In other areas of mathematics, such as algebra, external visualisations may not be an inherent part of the domain knowledge, but may still be frequently used as a means of solving problems or pursuing mathematical discovery (e.g., English, 1997; Presmeg, 2006). Polya (1957) echoes that the use of visual representations is not always considered as an inherent part of the target domain knowledge; rather it may be considered more as a general technique in the mathematician’s toolbox. This is relevant to this study, which focuses on visualisations in algebra. Even though algebra is often only perceived as a set of numbers and letters (D’Emiljo, 2006), there is a possibility that learners use visualisation as a strategy to solve algebraic problems. Thus this study seeks to identify the visual processes that learners use when solving algebra and explain how they are used.

A representation is a tool that can be used in multiple ways during the problem solving process. Stylianou (2011, p. 329) suggests some of the uses and functions of visualisations in students' problem solving work. He proposes that a visual representation can be used:

- *As a tool to process information* – one might use a representation as a means of putting together the various aspects of the problem and of examining how they contribute to the problem solving process.
- *As a tool for recording information* – one might use a representation as a tool that combines all the information instead of keeping it "all in the mind".
- *As tools that allow exploration* of tasks or concepts – one might use the representation as an adaptable tool device that allows for experimentation with concepts and provides more information.
- *As monitoring and assessing tools* that evaluate progress in problem solving – representations may be used to monitor problem-solving progress and to make informed decisions when selecting subsequent goals and maintaining or revising current plans.
- *As conscription tools* – devices to negotiate and co-create meaning and strategy with fellow problem solvers. The representations form a shared interactive space that facilitates communication, as they may be used as a common language tool.
- *As presentation tools* – students use these representations to share information both formally and informally, regarding both the process and the end result.

Piggott and Woodham (2010) identify three purposes for visualising where the first one is to **step into the problem**, the second is to **model** and the third is to **plan ahead**. **Visualising to step into the problem** is when learners use visualisations to help with understanding what the problem is about. The visualisation used by learners gives them the space to go deep into the situation. It helps them to clarify and support their understanding before any generalisation can happen.

Visualising to model a situation, Piggott and Woodham (2010) indicate that this purpose of visualisation is particularly useful when the situation is physically unattainable, in other words to try to see the 'unseen-able', for example the inside of a 3D object.

Visualising to plan ahead involves “using visualising during the problem-solving process to anticipate. In other words asking yourself: 'what will be the consequence if I do this?’ This is related to problem posing ' what would happen if ...?’ It is not possible to ask the question 'What if?', if you have not thought ahead and any thinking ahead necessarily includes visualisation” (Piggott and Woodham, 2010, p.5).

Recent research by Kashefi, Alias, Kahar, Buhari & Zakaria (2015) identified the multi- functions of pictures (external visualisation) in problem solving as decorational, representational, organizational, and informational.

According to Carney & Levin (2002), **decorative pictures** simply decorate the page, bearing little or no relationship to the text content. Decorative pictures do not give any actual information concerning the solution of the problem (Elia & Philippou, 2004). **Representational pictures** represent the whole or part of the content of the problem, they illustrate what is described in the problem and so far are the most commonly used type of illustration. **Organizational pictures** provide a useful structural framework for the text content. Elia & Philippou (2004) echo that organizational pictures provide directions for drawing or written work that support the solution procedure.

Zahner and Corter (2007) articulate that reorganization of the given information is used across all problem topics because it is a very general strategy that helps problem solvers extract the necessary mathematical information from the given word problem. Florida Department of Education (2010) indicates that graphic organizers and tables are some of the strategies that are commonly used in organising ideas. Graphic organizers are diagrammatic illustrations designed to assist students in representing patterns, interpreting data, and analysing information relevant to problem-solving (Lovitt, 1994, Ellis, & Sabornie, 1990). Moreover, interpretational pictures help to clarify difficult text (e.g. representing blood pressure in terms of a pump system). Finally, **informational pictures** provide information that is essential for the solution of the problem; in other words, the problem is based on the picture.

Similarly, Spiliotopoulou & Triantafillou (2014) conducted a study to explore the co-deployment of visual representations (VRs) and reasoning in Mathematics and Physics texts in specific topics related to the notion of periodicity. Seven categories were identified concerning the role of visual representations: Illustrative, Exemplifying, Starting point, Fundamental, Product of reasoning, Organizing tool and Complementary - See table 2.1 below.

Table 2.1: Role of visual representations

Categories	Mathematics %	Physical Science %
Illustrative	9.01	26.73
Exemplifying	6.08	24.75
Starting point of reasoning	43.20	15.84
Fundamental tool in reasoning	43.20	20.79
Product of reasoning	22.73	9.90
Organizing tool	9.09	0.99
Complementary	0.00	7.92

Table 2 above presents the frequencies of the categories of the function of visual representations in reasoning as they are encountered in Mathematics and Physics texts in secondary Education. One notes the more frequent occurrence of the illustrative, exemplifying and reasoning functions of visual representations in the Physics texts. On the other hand, visual representations seem to play a more significant role only in aspects of reasoning in the Mathematics texts. For my study, this table was adapted and used as a data analytical tool (in chapter 4) which was used to analyse the data collected.

Ho (2010) conducted research aimed to find out the importance of visualisation in mathematics, the factors that influence students' choices of problem solving methods and how visualisation helps students in mathematical problem solving. She then identified different roles or functions that visualisation play as students use it to solve mathematical problems:

- to **understand** the problem – Here visualisation helps learners to understand the problem by representing the problem visually, by so doing, learners can understand how the elements in the problem relate to each other.

- to **simplify** the problem – Visualisation allows students to identify a simpler version of the problem, solving the problem and then formalizing the understanding of the given problem and identifying a method that works for all such problems.

- to see **connections** to a related problem – This involves relating the given problem to previous problem-solving experiences.

- to cater to individual **learning styles** – Each student has his or her own preference when it comes to the use of visual representations when solving problems.

- as a **substitute** for computation – The answer to the problem can be obtained directly from the visual representation itself, without the need for computation.

- as a tool to **check** the solution – The visual representation may be used to check for the reasonableness of the answer obtained.

- to **transform** the problem into a mathematical form – Mathematical forms may be obtained from the visual representation to solve the problem.

(Ho, 2010, p.4)

On the teachers' side, researchers have highlighted the role that representations play in the explanations of mathematical concepts by teachers (Leinhardt, Putnam, Stein, & Baxter, 1991; Brophy, 1991; Fennema & Franke, 1992). Brophy (1991) articulates:

> Skilled teachers have a repertoire of such representations available for use when needed to elaborate on their instruction in response to student comments or questions or to provide alternative explanations for students who were unable to follow the initial instruction. (p. 352)

In the same line, Shulman (1986, p. 9) relates, “Teachers need to be able to draw on a variety of representations as there is “no single most powerful form of representation”.

Unfortunately, dispute the current views of researchers about the importance of visualisation, there is still a tendency for visualisation to be unrecognised in mathematics classrooms. Because of that some students, though able to visualise mathematically, often opt for non-visual processes (Presmeg, 1995). Moreover, Presmeg's findings (1986) indicate that an ability to apply and interchange both visual and non-visual methods in problem solving is particularly advantageous for students. However, the teaching of school mathematics is predominately non-visual and 'visualisers are seriously under-represented amongst high mathematical achievers' (ibid.). The undervaluing of Mathematics visualisations in Namibia is one of the reasons this study was conducted. The findings of my study will hopefully contribute to teachers’, policy makers’ and curriculum designers’ understanding of how to harness visualisation processes in the effective teaching of algebra.

2.3 PROBLEM SOLVING

> Solving a problem means finding a way out of a difficulty, a way around an obstacle, attaining an aim that was not immediately attainable. Solving problems is the specific achievement of intelligence, and intelligence is the specific gift of mankind; solving problems can be regarded as the most characteristically human activity. (Doorman et al. 2007, p. 406)

Carpenter, et al. (1989) reported that when emphasis is placed on problem solving then (1) students’ attention on ideas and sense-making increases (2) there is space to construct own strategies, (3) other skills are enhanced, and (4) there is a difference in how the teacher knew their students’ thinking and the students growth in problem solving (p. 213). Van de Walle (2000) argues that problem solving:

- places the focus of the students’ attention on ideas and sense-making
- develops ‘mathematical power’
- develops belief that they are capable of doing mathematics and mathematics makes sense, and

- provides on-going assessment data that can be used to make and change instructional decisions, help students succeed, and inform parents (p. 41).

The National Council of Teachers of Mathematics (1980) have called for the adoption of a problem solving approach to the teaching of mathematics. Halmos (1980) views that "the mathematician's main reason for existence is to solve problems, therefore, what mathematics really consists of, is problems and solutions" (p. 519). Yeap, Ferrucci & Carter (2006) confirm the centrality of problem solving in Mathematics - that mathematical problem solving should be a focus of school mathematics internationally.

2.3.1 What is problem solving? / What are problems?

Even though problem solving has been prominent in mathematics education for several decades, it seems that its definition and classroom implementation are far from being consensual (Arcavi & Friedlander, 2007). Arcavi & Friedlander articulate that even people within the same culture or within the same educational system (e.g. curriculum developers, teachers), researchers in mathematics learning/teaching, and mathematicians do not necessarily share the same views on what constitutes a problem and what to teach under problem solving.

As stated above, there are different definitions of 'problems' used in literature. A problem is a situation in which a goal is to be attained, but there is no readily accessible solution for problem solvers to obtain the answer to the problem (Charles & Lester, 1984; Lester, 1980; Po´lya, 1980). Cash (1979, p. 1434) gives two definitions of a problem; he states, "in mathematics a problem is anything required to be done, or requiring the doing of something." He further gives a second definition that a problem is "a question... that is perplexing or difficult." In Stanic & Kilpatrick's (1989) historical review of problem solving, they identified three main themes regarding problem-solving usage. They state that problem solving is a **context**, it is a **skill** and thirdly, it is **"art".** In the same line, Doorman et al (2007) consider problem solving as the 'art' of dealing with non-trivial problems which do not yet have a known, routine solution strategy to the student, but which provide opportunities for the student to develop new solution strategies.

Brownell (1942) says:

> ... problem solving refers (a) only to perceptual and conceptual tasks, (b) the nature of which the subject by reason of original nature, of previous learning, or of organization of the task, is able to understand, but (c) for which at the time he knows no direct means of satisfaction. (d) The subject experiences perplexity in the problem situation, but he does not experience utter confusion. … problem solving becomes the process by which the subject extricates himself from his problem. (p. 416)

Moreover, according to Pólya (1957), problem solving is learning to deal with new and unfamiliar tasks, when the relevant solution methods are not known, or only partly mastered. Heller and Hungate (1985) take their definition of "problem solving" to mean, "...being able to solve the exercises at the end of a standard textbook chapter". Schoenfeld (1992, p. 334) reviews in detail how the meanings of problem solving range from ‘‘working rote exercises’’ to ‘‘doing mathematics as a professional’’, including goals for problem solving as diverse as ‘‘to train students to think creatively … to prepare students for problem competitions… to learn standard techniques in particular domains, most frequently in mathematical modelling … to provide a new approach to remedial mathematics (basic skills).’’

Wheatley and Cobb state:

> Mathematical problem solving is often a matter of reasoning analytically, constructing an image, using the image to support additional conceptual reasoning… a process of building from images to analysis and analysis to images [that] may continue through many cycles. (1990, p. 161)

Ponte (2007, pp. 419-420), presented the notion of a problem as a ‘‘question in which the students do not have a ready-made routine process to solve it, but that stimulates their curiosity and their will to work on it’’. At the same time, this author suggested that, in solving a problem, the student ‘‘is called to have an active participation. He/she must be a mathematician. He/she must face each new situation, think for him/herself, take his/her decisions and evaluate the work done’’ (p. 420).It is for these and other reasons that in my study learners (participants) will be actively involved; they are going to think for themselves as to what visual problem solving process they will use to solve specific tasks. Learners will also be given tasks that are not familiar to them, which I believe will rouse their curiosity.

Arcavi & Friedlander (2007) recently conducted a study in Israel, the main aim of which was to find out the explicit and implicit views on problems and problem solving and their study focused on the team leaders of elementary mathematics curriculum developers. They compiled the different responses and below are some of the quotes:

- ''A problem consists of data, a goal and a way to achieve this goal. While you solve a problem, you do not know in advance the way to achieve the goal.''

- ''A problem is a situation which the student did not encounter previously. It is possible to approach it in several ways and to 'wet your hands' while coping with it. At this stage, children don't have an algorithmic way to deal with it.''

- ''The characteristics of a problem are: there is no 'paved way' to a solution, it makes sense to the solver who has at least partial tools to cope with it, the solver can understand the constraints, the solver is aware that she has some tools to solve it, and she can know whether she solved it correctly or not.'' (p.358)

Erickson & Flowers (1999) agree that problem-solving tasks often present situations where no readily known or accessible procedures or algorithm determines the method of solution. They argue further that students make sense of mathematical situations where no well-defined routines or procedures exist. Thus, to help children reflect on their actions and create relationships, a problem-solving environment should be used and children be encouraged to use self-validation of ideas (Kamii, 1990; Yackel, Cobb, Wood, Wheatly & Merkel, 1990).

2.3.2 What kind of tasks are not problem solving?

Schoenfeld (1992) articulates that when students are tested on solving quadratic equations, for example, students know that they will be using the quadratic formula. However, when students are doing real problem solving, working on unfamiliar problems out of context, such behaviour is not always observed. Trying to differentiate a problem from an exercise, Schoenfeld (2013) proposes that complexity or difficulty alone does not make a task a problem; solving a system of 100 linear equations in 100 unknowns without the use of technology might be a real challenge, but it is not a problem in the sense that one knows how to go about getting an answer, even if it might take a very long time and one struggles over the calculations.

The research conducted by Arcavi & Friedlander (2007) indicates that an exercise where rules have to be applied is not a problem. The participants in their research were asked to explain the kind of tasks that are not problems. Their responses were recorded and some of their replies are presented below:

- ‘‘What is not a problem is an exercise, and an exercise is the application of a rule.’’
- ‘‘Exercises and straightforward word problems.’’
- ‘‘It all depends on who is the solver.’’
- ‘‘There are plenty of activities that are not problem solving for example, defining and enhancing the meaning of concepts (for example, identifying and classifying different meanings of subtraction), formulating and explaining ideas…”

(Arcavi & Friedlander, 2007, p.358)

From the definitions and views of previous researchers above, there seems to be a general agreement that exercises are not problems. It is also clear that the phrase *problem solving* is ambiguous. Hence, for the purpose of this study I was guided by the definitions above in the setting of the algebraic tasks that will be solved by the participants. I considered the “wide-scope tasks” as referred to by Arcavi & Friedlander (2007, p. 361) instead of the “small-sized riddles”. Stanic and Kilpatrick (1989, p. 20) view that ‘‘problem solving is for everyone’’ but since the participants in this study are the learners that usually do well in Mathematics, “problems for the advanced students” (as referred to by Stanic and Kilpatrick, 1989, p. 20) were considered. An example of such problems is shown below:

> Out of 721 airport maintenance workers, 257 go to work by car, 210 by bike, and 176 by either car or bike. Thirty senior workers should have reserved parking places. The parking

lot has 360 parking places. One parking place can accommodate a car or 8 bikes. Plan the workers' parking lot: Decide how many places should be allotted to cars, and how many to bikes. Describe your thinking.

(Arcavi & Friedlander, 2007, p. 361)

2.3.3 Why a problem-solving approach is used in School Mathematics

It is argued that inquiry-oriented mathematics instruction, which includes problem-solving and problem-posing tasks and activities, can assist students to develop approaches that are more creative.

Namibia is a knowledge-based society. According to the Namibian Curriculum for Basic Education, "a knowledge-based society is one where knowledge is created, transformed, and used for innovation to improve the quality of life" (p.2). Thus, through The Namibian Basic Education curriculum, learners should develop the competencies, attitudes and values needed for full participation in society by learning to use, acquire, construct, evaluate and transform knowledge. Teahen (2015) echoes that competence in mathematics is important for students to enable them to be successful participants in society. It is important for students to relate what they are learning in class to what they may do in their lives in the real world so that the learning is real and has meaning (Teahen, 2015). This is an important factor for learning to be transferred to long-term memory, a crucial step if knowledge and concepts are to be retained (Sousa, 2008).

As Cobb et al. (1991) suggested, the purpose of engaging in problem solving is not just to solve specific problems, but to "encourage the internalization and reorganization of the involved schemes as a result of the activity" (p.187). Moreover, not only does the problem solving approach develop students' confidence in their own ability to think mathematically (Schifter & Fosnot, 1993), it is also a vehicle for students to construct, evaluate and refine their own theories about mathematics and the theories of others (NCTM, 1989).

Resnick (1989) expresses the belief that "school should focus its efforts on preparing people to be good adaptive learners, so that they can perform effectively when situations are unpredictable and tasks demand change" (p.18). Cockcroft (1982) also advocated problem solving as a means of developing mathematical thinking as a tool for daily living. He states that problem solving ability lies "at the heart of mathematics" (p.73) because it is the means by which mathematics can be applied to a variety of unfamiliar situations. The National Council of Teachers of Mathematics (NCTM, 1980) recommended also that problem solving be the focus of mathematics teaching because it involves skills and functions that are an important part of everyday life. They furthermore stated that it could help people to adapt to changes and unexpected problems in their careers and other aspects of their lives. These recommendations were endorsed by the Council (NCTM, 1989) with the statement that problem solving should underlie all aspects of mathematics teaching in order to give students experience of the power of mathematics in the world around them.

In addition, one of the aims of teaching through problem solving is to encourage students to refine and build on their own processes over a period of time as their experiences allow them to remove some ideas and become aware of further possibilities (Carpenter, 1989). Carpenter (1989) further indicates that through problem solving, students are developing knowledge and an understanding of when it is appropriate to use particular strategies. Thus, through using a problem solving approach in schools, the emphasis is on making the students more responsible for their own learning rather than letting them feel that the algorithms they use are the inventions of some external and unknown 'expert'. There is considerable importance placed on exploratory activities, observation and discovery, and trial and error. Students need to develop their own theories, test them, test the theories of others, discard them if they are not consistent, and try something else (NCTM, 1989).

Furthermore, students can become even more involved in problem solving by formulating and solving their own problems, or by rewriting problems in their own words in order to facilitate understanding (Thompson, 1985). It is of particular importance to note that they are encouraged to discuss the processes that they are undertaking, in order to improve understanding, gain new insights into the problem and communicate their ideas (Thompson, 1985, Stacey & Groves, 1985). However, although the involvement of students in problem solving is important, it is still necessary for certain techniques to be available for the involvement to continue successfully. Hence, more needs to be understood about what these techniques are and how they can best be made available (Olkin & Schoenfeld, 1994).

2.3.4 Process Models of Mathematics Problem Solving

According to Adams & Wu (2006), many classification schemes for problem solving processes are derived from Polya's conception of mathematics problem solving as a four-phase [1]heuristic process: understand the problem, devise a plan, carry out the plan, look back and check (Polya, 1973). Briefly, understanding the problem includes reading and clarifying a problem to identify the known, the unknown and the goal. Devising a plan is the stage of choosing a strategy for a solution to the problem. Carrying out means the problem-solver will execute this plan and write out a solution. Looking back is when the problem-solver needs to check the legitimacy of this solution for the problem (Polya, 1973).

However, every problem-solver will notice that when tackling a problem, it is not just a simple top-down process of the above four stages. In practice, the phases are muddled up and are carried out in parallel, each new discovery tending to modify the overall plan. (Pólya, 1973, p. *xix)*. Figure 2.1 below illustrates the four steps and all the processes involved in each step (Florida Department of Education, 2010, p. 4).

Figure 2.1 Four basic principles of problem solving (Florida Department of Education, 2010, p. 4).

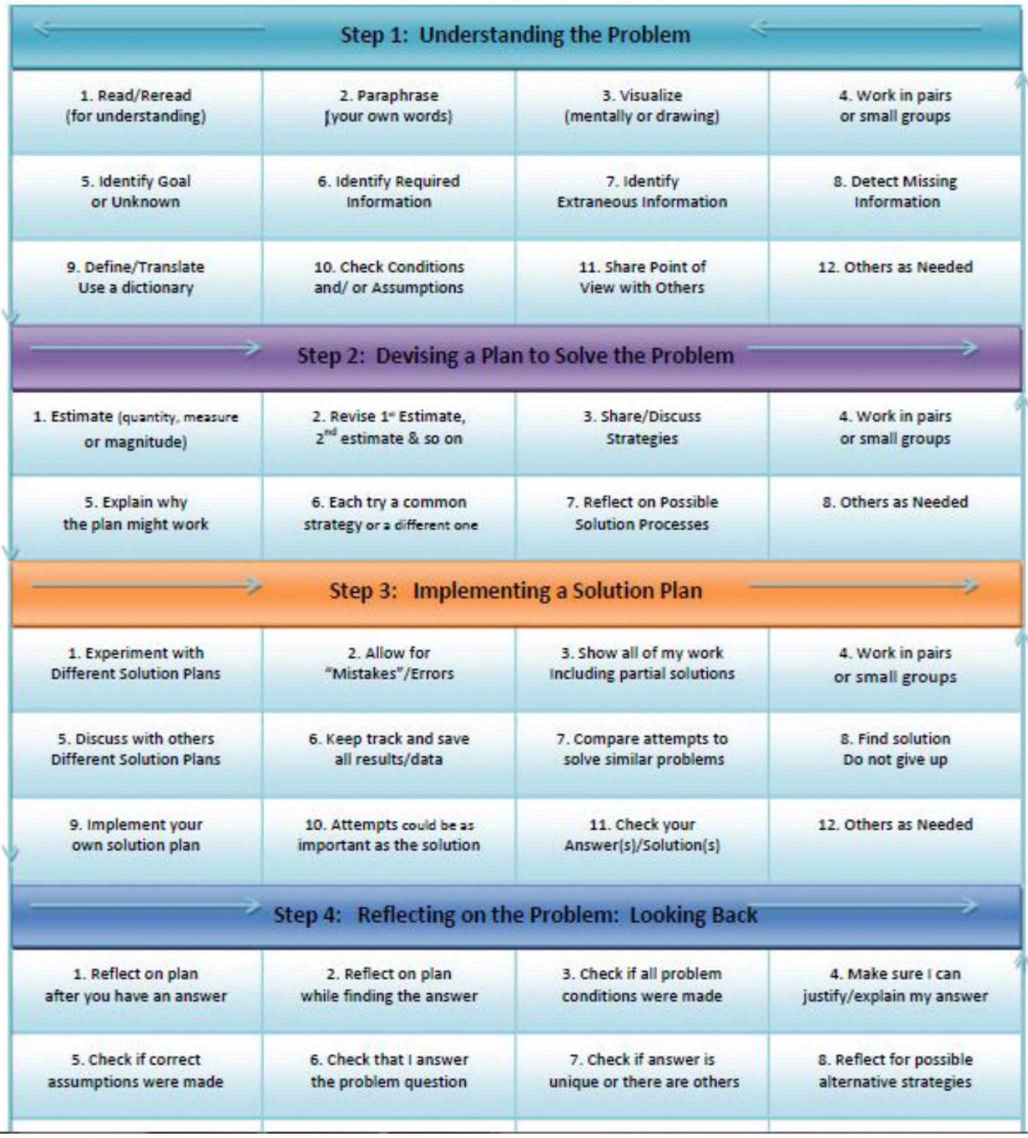


Figure 2.1 illustrates several problem-solving processes, as suggested by Poly (1957) that can be used when solving problems. The four steps mentioned above have become the framework often recommended for teaching and assessing problem-solving skills. Looking at the processes in Figure 1 above, Step 1 reveals that there are many processes that can be used to solve problems, for example **visualisation** is a process that is used by learners/problem solvers, and is an important strategy for solving problems, as it allows learners to obtain a clearer understanding of what the problem is asking. The process of visualisation is considered to be important in mathematics learning and more specifically in mathematics problem solving (Elia & Philippou, 2004).

In addition, Polya (1958) gave more examples of visual strategies that can be used in the first step (understanding the problem) of problem solving, such as graphic organisers. Graphic organizers are diagrammatic illustrations designed to assist students in representing patterns, interpreting data, and analyzing information relevant to problem-solving (Lovitt, 1994, Ellis, & Sabornie, 1990). Examples of graphic organizers are **hierarchical diagramming, sequence charts, and compare and contrast charts.** Hierarchical diagramming is a graphic organizer that begins with a main topic or idea. All information related to the main idea is connected by branches, much like those found in a tree. Sequence charts are charts designed to symbolize a sequence of procedures or events in a content area. Compare and contrast charts are charts designed to compare information across two or three groups or ideas - see Figure 2.2 below.

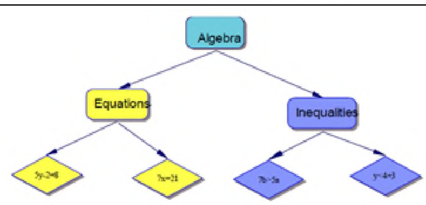
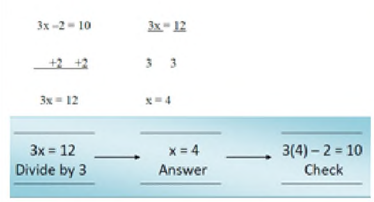
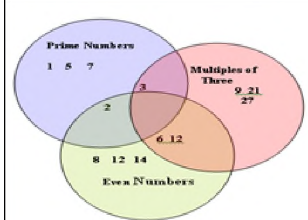
1. Hierarchical diagramming	2. Sequence charts	3. compare and contrast
		

Figure 2.2 Three types of graphic organisers (Florida Department of Education (2010, pp. 12-14)).

Furthermore, Schoenfeld (1983) developed a model based on research findings from by information-processing theorists. His model incorporated Polya’s structure and described mathematical problem solving in five stages: reading, analysis, exploration, planning/implementation and verification. The point according to Schoenfeld (1987) is that students should wisely divide their time among (a) understanding the problem, (b) planning, (c) making decisions on what to do, and (d) executing the decisions for a solution within the time-frame. In the process of solving a problem, they should be monitoring and keeping track of the progress to a solution. When the decisions seem not to work, they should try alternatives or make some adjustment. Once new alternatives are chosen, the work done should not be thrown away. There is always a chance that the curtailed efforts might have led to success.

Mayer and Hegarty (1996) examined mathematics problem solving in terms of four components: translating, integrating, planning and executing. They postulated how expert problem solvers differ from novice problem solvers in their use of strategies in these four components.

Reusser (1996) proposed a stage-wise processing model of mathematics problem solving that includes five consecutive stages: (1) constructing a propositional representation of the problem, (2) creating a situational model, (3) transforming the situation model into a formal mathematical representation, (4) applying the operations to calculate the solution, and (5) interpreting the solution in a meaningful way.

Furthermore, Casey (1978, as cited in Clements, 1980) proposed a step-wise model for the solution of mathematics word problems. Even though his model is for mathematics word problems, I believe it can be used in any mathematical problem solving. His model consists of the following steps or stages: (1) question reading, (2) question comprehension, (3) strategy selection, (4) skills selection, and (5) skills manipulation. In this model, the problem solver can “cycle back” to a previous stage to correct errors or try another solution path.

Moreover, Lithner (2008) proposed four steps that can be carried out when solving a task: (1) Problematic situation (this is when a task/ problem is encountered), (2) strategy choice (this is when a problem solver chooses a strategy from a range of strategies), (3) strategy implemented and (4) conclusion.

Ho (2010) notes that students go through the following processes when solving maths problems:

- **Understanding** – Learners try to understand the spatial relations of the elements in the problem.
- **Connecting** – Learners connecting the current problem to a previously solved problem.
- **Constructing** – Learners constructing a visual representation (in the mind, on paper, or through the use of technological tools).
- **Using** – Learners using the visual representation to solve the problem.

- **Encoding** – Learners give the answer to the problem.

These processes identified by Ho are comparable to the four basic principles of problem solving identified by Polya (1957).

In general, information-processing theorists are less concerned about the existence of separate ability factors corresponding to the stages of problem-solving processes; rather, they identify the stages of problem solving so that these can be used to teach students how to approach a problem and what to do when they encounter difficulties (Adams & Wu, 2006). In particular, these problem-solving stages can serve as useful prompts for students to monitor and evaluate their own thought processes (Silver, 1982). Without clearly identified problem-solving stages, the problem-solving activities carried out in the classroom can be somewhat *ad hoc* and disorganised. Approaching problem solving in a systematic way using steps defined through the information processing approach can help students acquire skills that are transferable to a wider range of problems.

Hitt (1998) claims, "a central goal of mathematics teaching is taken to be that the students be able to pass from one representation type to another without falling into contradictions" (p. 134). However, the choice of representation, in addition to understanding, is also influential to success. That means there are so many problem-solving models as indicated above and the ability to select, use, move between and compare representations is a crucial mathematical skill (Even, 1998).

The process of visualisation is considered crucial in mathematics learning and more specifically in mathematics problem solving. Brown and Wheatly (1997) stress the important role of visualisation in Mathematics problem solving. They indicate that to visualise during problem solving helps learners to form images of mathematical relationships, which is a necessary presupposition for effective Mathematics problem solving.

2.4 ALGEBRA

"Algebra encompasses the relationships among quantities, the use of symbols, the modeling of phenomena, and the mathematical study of change" (National Council of Teachers of Mathematics, 2000, p. 37). It is the way we express generalisations about numbers, quantities, relations and functions

(Kilpatrick, Swafford & Findell, (2001). For this reason, a good understanding of connections between numbers, quantities and relations is related to success in using algebra.

A number of different characterizations of school algebra can be found in the mathematics education literature. Usiskin (1988) described four conceptions of algebra: generalized arithmetic; the set of procedures used for solving certain problems; the study of relationships among quantities; and the study of structures. Kaput (1995) identified five aspects of algebra: generalization and formalization; syntactically guided manipulations; the study of structure; the study of functions, relations, and joint variation; and a modelling language. A discussion document published by the National Council of Teachers of Mathematics (1998) describes four organizing themes for school algebra: functions and relations, modelling, structure, and language and representation.

Kieran (2004) suggests that algebra is a focus on, (1) relations and not merely on the calculation of a numerical answer; (2) operations as well as their inverses, and on the related idea of doing/undoing; (3) both representing and solving a problem rather than merely solving it; (4) both numbers and letters, rather than numbers alone; and (5) a refocusing on the meaning of the equal sign.

Carraher, Martinez & Schliemann (2007) define algebra as a subject dealing with expressions, symbols and the extended numbers beyond the whole numbers in order to solve equations, to analyse functional relations, and to determine the structure of the representational system, which consists of expressions and relations. However, they further state that activities such as solving equations, analysing functional relations and determining structure are not the purpose of algebra, but tools for modelling of real world phenomena and problem solving related to the various situations. Furthermore, algebra is much more than a set of knowledge and techniques. It is a way of thinking. Success in algebra depends on at least six kinds of mathematical thinking abilities as follows: generalization, abstraction, analytic thinking, dynamic thinking, modelling, and organization (Carraher et al., p. 146, 2007).

2.4.1 Visual representations in algebra

Algebra is abstract. Its elements such as "unknowns, variables and other algebraic objects can only be represented *indirectly*, through means of constructions based on signs" (Kant, 1929, p. 579). These visually drawn constructions of mathematical objects, concepts or processes can effectively assist in developing a structural awareness of the corresponding abstract knowledge, despite being in an incomplete form (Mason et al. 2009; Rivera, 2011). Furthermore, visual representations are a powerful way for students to access abstract maths ideas (David, Tomaz & Ferreira, 2013).They further state that drawing a situation, graphing lists of data, or placing numbers on a number line help to make algebra abstract concepts more concrete (David et al., 2013). However, a central issue is that in most contexts for any representations, the learner must understand what is being represented.

Kieran (1996) classifies structure in algebra as surface structure of expression, arrangement of symbols and signs, systemic, operations within an expression and their actions, order, use of brackets, structure of an equation: equality of expressions and equivalence. As stated earlier, because of the abstract nature of algebra, a variety of visual representations are used to represent algebraic objects.These visual representations include graphs and diagrams, tables and grids, formulas, symbols,words, gestures, software code, videos, concrete models, physical and virtual manipulations pictures, and sounds (David et al., 2013).

There are many symbols and signs in algebra that learners need to understand, to avoid relating arithmetical meanings to algebraic expressions inappropriately (Kilpatrick et al., 2001). Cooper & Warren (2007) articulate that it is important to see algebra in the way it represents the principles (such as commutative principle and balance principle) and structures of mathematics (such as field, group and equivalence class) and not in terms of the ‘‘behaviors’’ of algebra (such as simplification and factorization), which being algorithmic, are capable of having their solution process programmed into calculators or computers. According to Radford (2008), algebra is not the manipulation of letters but rather a system characterized by indeterminacy of objects, an analytic nature of thinking and symbolic ways of designating objects.

In algebra, a number must also be seen as a symbol. Kilpatrick et al. (2001) stipulate that a critical shift is from seeing a letter as representing an unknown, or ‘hidden’, number defined within a number sentence such as: $3 + x = 8$, to seeing it as a variable. Moreover, an expression such as 47-16+20 must be seen as a structure of related relationships between numbers, so calculation must be avoided, and 4b + 6 must be seen as both the answer to a question, an object in itself, and also an algorithm or process for calculating a particular value.

An ‘equal’ sign is another visual representation that is commonly used in algebra. Expressions linked by the ‘equal’ sign might be not only numerically equal, but also equivalent. An example given by Kilpatrick et al. (2010) is that $10x - 5 = 5(2x - 1)$ is a statement about equivalence, and x is a variable, but $10x - 5 = 2x + 1$ defines a value of the variable for which this equality is true. Thus, x in the second case can be seen as an unknown to be found, whereas in the first case it is a variable. They further state that the use of graphs (external visualisation) can show the difference between the two expressions visually and powerfully because the first situation is represented by one line, and the second by two intersecting lines, i.e. one point.

2.5 THE NAMIBIAN CONTEXT

2.5.1 Visualisation in the Namibian curriculum

> Mathematics will be most relevant and meaningful for the learners if it is related to their lives. For example, two- and three-dimensional shapes/figures can be found in the immediate environment. Although mathematics is a universal language, it is only by local contextualization and application that younger learners will understand and appreciate the uses of mathematics. Where textbooks can only generalize, it is the teacher's responsibility to use local examples such as concrete materials found in the environment, e.g. stones, sticks, bottle tops, etc.
>
> (Mathematics Syllabus, NIED 2015, p .37)

The Namibian curriculum for basic education does not really say much about visualisation but it emphasises the use of teaching aids, wall display and concrete material, and indicates that textbooks are the main source of information; therefore every schoolchild is expected to have his/her own textbook. These textbooks have visual presentations such as diagrams and different kinds of pictures, which help learners to understand mathematics concepts (Namibia. MBEC, 2010). It further states that those visual presentations help learners to make the appropriate connections so as to gain a broad view of common notions presented in different textbooks.

The Namibian curriculum recommends teachers having wall displays in their classrooms. "Wall displays are pictures, wall charts and/or artefacts displayed on the walls of the classroom that make learning interesting" (Namibia. MBEC, 2010, p.7). It further indicates that learners' learning is improved when charts and pictures are displayed because they can see the same thing over a period of time, which makes it easier to remember and understand. The Namibia Learner-Centred policy echoes that knowledge and knowledge production are shared through displays of learners' work, charts, posters, and easily accessible information sources. "When learners spend time reading and discussing ideas in an interesting display, learning is better and more fun" (Namibia. MBEC, 2010, p.7). Nonetheless, these displays should be changed regularly and it is the teacher's responsibility to change the displays regularly.

2.5.2 Algebra in the Namibian curriculum

Mathematics is compulsory in the Namibian curriculum for basic education, meaning every learner in Namibia (from pre-primary to secondary) must study Mathematics. The curriculum caters for a wide range of learner abilities, including those who are going to continue their studies in Mathematics and other disciplines for which Mathematics is a prerequisite. All Namibian school graduates are therefore expected to be numerate and the study of mathematics at pre-primary, junior and secondary levels contributes to the learner's ability to think logically, work systematically and accurately and solve real-world problems.

"Algebra encompasses the relationships among quantities, the use of symbols, the modelling of phenomena and the Mathematical study of change" (National Council of Teachers of Mathematics, 2000, p. 37). In the Namibian basic education curriculum, Algebra is one of the main domains that is covered throughout the curriculum from Grade 1 to Grade 12. Even though the word *algebra* is not commonly heard in pre-primary and junior school classrooms, the mathematical investigations and conversations of learners in these grades frequently include elements of algebraic reasoning. The formal introduction of algebra in the lower grades is important because the experiences that the learners get from doing algebra in the early grades "presents rich contexts for advancing mathematical understanding and are an important pioneer to the more formalized study of algebra in the middle and secondary grades" (National Council of Teachers of Mathematics, 2000, p. 37).

The Namibian curriculum for basic education indicates clearly the aims of the curriculum and the basic competencies to be acquired. It details what learners are expected to know in algebra on the completion of each phase. To start with, at the end of the pre-primary phase every learner is expected to be able to express orally their understanding of number concepts and mathematical symbols. For the junior primary phase (Grades 1-3), learners are expected to recognise and describe patterns, relationships and shapes, and solve simple problems in everyday contexts by adding, subtracting, multiplying and dividing, and estimating and measuring, within the required range.

In the Senior Primary level, "learners are expected to generalise number relationships and number patterns, generate and solve simple equations and understand that in algebra letters are used as placeholders for numbers" (*Mathematics Syllabus Grade 4 – 7, NIED 2015).* Additionally, learners should have an understanding of the concept of rational numbers and carry out the basic operations, and they should be able to solve everyday problems involving number, measurement, and spatial relationships.

Moreover, on completion of the junior secondary phase learners are expected to use algebraic notation; simplify expressions applying the four basic operations, and; construct and solve simple equations (JSC Mathematics and Additional Mathematics NIED 2010*).* In addition, they should be able to collect, interpret and present simple data. They should also be able to use real numbers to estimate, approximate, and calculate to relevant degrees of accuracy. Learners should be able to solve problems using a range of methods, including algebra, ratio, rate and proportion, and graphic representations. It is also expected that by the end of the secondary phase (Grades 11-12), learners know how to express basic arithmetic processes algebraically, how to substitute letters in formulae with numbers, and how to transform simple formulae. They should also be able to manipulate algebraic expressions and polynomials (NIED, 2010).

2.6 THEORETICAL FRAMEWORK

2.6.1 Constructivism

Theorists like Simon (1986) describe constructivism as the process that engages the learner in the construction of new knowledge from perceptions and experiences. Bruning, Schraw, and Ronning (1999) in particular describe constructivism as, "structuring, construction, self-awareness, and selfregulation of knowledge where social interaction and the contextual nature of knowledge play an important role”.

According to the constructivist view of learning mathematics, students construct their own mathematical knowledge rather than receiving it in complete form from the teacher or a textbook (Perry, Goeghegan, Howe, & Owens, 1995).That learners actively construct their own meaning and knowledge by interacting with their environment (Confrey, 1990; Kamii, 1990; Orton, 1987; Silver, 1985; Simon & Schifter, 1991). Individuals construct their own knowledge from experiences for example in the school setting; those experiences often result from activities arranged by the teacher. Whether the individual modifies her or his ideas or not is dependent on the challenge to the viability of her or his existing conceptual structures when faced with the new experiences (Bruning et al, 1999).

This knowledge construction or reconstruction is largely independent of the way students are taught. Students construct their own knowledge for themselves resulting in their own understanding (Ndlovu, 2013). Von Glasersfeld (2001) argues that from a constructivist point of view “human beings can only know what they themselves have made” (p. 4). (Kant (1989) as quoted by Von Glasersfeld, 2001) echoes, “human reason can grasp only what she herself has produced according to her own design (p. 4)”. Duit (1996) asserts that:

> The common constructivist core is a view of human knowledge as a process of personal cognitive construction, or invention, undertaken by the individual who is trying for whatever purpose to make sense of her social or nature environment. (p. 41).

Constructivists do not see learners as passive receiver but as active constructors of knowledge. “Knowledge is not passively received either through the senses or by way of communication. Knowledge is actively built up by the cognizing subject” (Von Glasersfeld, 1990, pp.22-23). Furthermore, constructivism enables learners to participate in the production of knowledge in the classroom (Flynn, Vermette, Mesibov, & Smith, 2004). Learners learn through interaction with problems and develop what Vygotsk call the tools of intellectual interpretations, which consequently become an essential part of the learning process (Vygotsky, 1978).

Additionally Simon and Schifter (1991), echo that the theory of constructivism is drawn from the empirical and theoretical work of Piaget. They maintain further that this theory was extended into mathematics education by Cobb, Confrey, Steffe, von Glaserfeld and others. Piaget's theory on the development of thinking is grounded on the notion that people construct their own knowledge. The construction of knowledge comprises two processes namely assimilation and accommodation. To understand our environment and the world, people organize new experiences and adapt new ideas into schemas (cognitive structures). Assimilation occurs when these new experiences and ideas are incorporated into our existing knowledge (Simon and Schifter, 1991).

On the other hand, accommodation occurs when a person adjusts her/his knowledge to a new idea. That is, when we encounter an unfamiliar experience or idea, the schema tries to fit it into on the existing knowledge. If no similar structure exists, a new schema is created to accommodate the unfamiliar experiences (Piaget, 1983; Wadsworth, 1996). Thus according to Orton, (1987); Simon and Schifter, (1991), and Wadsworth, (1996), the construction of new knowledge occurs when new ideas disturb the current organization of knowledge. This disturbance is known as disequilibrium, which leads to mental activity and adaptation of new ideas.

Learners construct their meanings either mentally or symbolically. They interact with others to clarify their understanding. For example, in a mathematics classroom individual learners might us visualisation processes to construct their knowledge and their understanding of Mathematics. This knowledge has to be expressed and articulated for teachers to have access to the learners' understanding.

2.6.2 Constructivism and visualisation

The fundamental principle of constructivism is that learning is very much a constructive activity that the students themselves have to carry out. From this point of view, the task of the teacher is not to dispense knowledge but to provide students with opportunities and incentives to build it up (von Glasersfeld, 1996).

In a constructivist classroom, a learner is independent and constantly interacts with manipulative and physical materials (external visualisations). It is the learner's thinking during problem solving that drives the lesson, modifies content, and shifts instructional strategies (Brooks & Brooks, 1993).

Therefore, learners are encouraged to explore and discover mathematical concepts instead of passively learning mathematics. The core of the constructivist view is construction of knowledge using old knowledge and materials at hand. Learners create visual representations and explanations of new information that will meaningfully connect with prior knowledge.

Proponents of this perspective argue that understanding is enhanced when construction of knowledge is encouraged within the classroom discussion (Ball, 1991; Erickson, 1999). That is, construction does not occur in vacuum. The learner interacts with the environment, objects or persons (Wadsworth, 1996). The nature of the object (visual representation) causes a reaction in the schema, which according to Piaget's theory, causes either an equilibrium or disequilibrium state. If it matches, then there is assimilation or if not, the schema is rearranged for new information to be accommodated (Brooks & Brooks, 1993).

Furthermore, the constructivist advocates discussion during interactions. The teacher encourages the learners to engage in a dialogue, both with the teacher and with one another. Learners communicate about their interpretation of the visual representation. They elaborate and justify their interpretations. This enables learners to reorganize their existing knowledge and accommodate new constructed information (Ball, 1991; Brooks & Brooks, 1993; Mikala & Lewellen, 1999).

As stated earlier on, constructivism originates from a theoretical position that human beings construct their own knowledge from their personal perceptions and experiences, which are mediated through their previous knowledge. According to Goldin & Kaput (1994), external visualisations permit us to talk about mathematical relations and meaning apart from inferences concerning the individual learner or problem solver. Internal visualisations give us the framework for describing individual knowledge structures and problem-solving processes. Interactions between external and internal visualisation systems provide the means for making inferences about individuals, and for describing learning and development as a consequence of the learning environment and contingencies in that environment (Goldin & Kaput,1994).

Furthermore, constructivism enables learners to participate in the production of knowledge in the classroom (Flynn, Vermette, Mesibov, & Smith, 2004). A student learns through interaction with problems and develops what Vygotsk call the tools of intellectual interpretations, which consequently become an essential part of the learning process (Vygotsky, 1978). Constructivists view learning as constructive and friendly. For learning to take place there should be an interaction among what the learners know already, the new information and the learners' action as they learn. Learning is seen as an improvement in the learners' world, a discovery whose raw materials are learners' ideas, strategies as well as knowledge they encounter along the way (Brooks & Brooks, 1993).

In addition to that, Brooks & Brooks (1993) gave an example of learners dealing with a problem like: *I have eleven pencils, how many more do I need to have twenty-three?* They state that learners need to engage in and guide themselves in cognitive operations. They will either picture the problem mentally (internal visualisation) or use physical objects (external visualisation) to construct meaning and organize these thoughts into actions. The problem requires them to draw from knowledge in memory. It must be noted that it is the mental process of the learner that reorganizes and structures information when solving a problem (Bruning et al., 1999). Bruning et al. maintain further, "once the problem has been represented, its solution requires knowledge of action schemata" (p. 331). Subsequently the schemata contain much different information; the learners just need to select the appropriate strategy that matches the problem that requires "regulation of cognitive knowledge, a form of metacognitive knowledge" (p. 332).

Additionally, "Mathematics requires representations. In fact, because of the abstract nature of mathematics, people have access to mathematical ideas only through the representation of those ideas." (Kilpatrick, Swafford, & Findell, 2001, p. 94). Bruner (1966) believes that young children learn through manipulation and action (**enactive representation**), older children learn through perceptual organization and imagery (**iconic representation**), and adolescents learn through the use of language and symbolic thought (**symbolic representation**). Clements (1999) suggests that all three types of representation (enactive, iconic and symbolic representations) should be used in parallel to facilitate student learning. He further states that when students make connections between concrete, pictorial, and symbolic representations, their learning is enhanced and increased.

Moreover, in mathematics classes learners can use different visualisation processes such as objects, actions, pictures, symbols, and words to represent mathematical ideas. These could be likened to Bruner three types of representations, with objects and actions being enactive, pictures being **iconic**, and symbols and words being **symbolic**.

2.7 CONCLUSION

In this chapter, I looked at the literature that relates to my study. Visualisation and the important that it plays in problem solving was explored. It has been argued that visualisation plays an important role in mathematical problem solving as it is processes enables the individual to solve an algebraic problem. Algebra and visualisation in algebra was also touched since algebra is also the main aspect of my study. In addition, the constructivism theoretical framework that inform this study was also briefly discussed. Particularly the role that constructivism plays in problem solving. In the next chapter, I discuss the research methodology I employed in my research study.

CHAPTER 3

RESEARCH METHODOLOGY

3.1 INTRODUCTION

The purpose of this study was to analyze the visualisation processes used by selected learners in solving algebraic tasks. This chapter describes the research methodology of this study, explains the sample selection, describes the procedure used in designing the instrument (AVT) and collecting the data, and provides an explanation of and justification for the procedures used to analyze the data.

3.2. ORIENTATION

According to Bertram & Christiansen (2014), interpretivist researchers "aim to understand how people make sense of their worlds" (p. 26). Cohen, Manion and Morrison (2011) explain that:

> …the central endeavour in the context of the interpretive paradigm is to understand the subjective world of human experience. To retain the integrity of the phenomena being investigated, efforts are made to get inside the person and to understand from within. The imposition of external form and structure is resisted, since this reflects the viewpoint of the observer as opposed to that of the actor directly involved (p. 17).

When solving algebraic problems, students may use multiple visualisation processes that may lead to possible solutions of the problem. Some might visualise the problem externally for example by drawing pictures, graphs and diagrams on paper and some might visualise the problem internally by making visual representations in the mind. The students' use of visualisations can only be understood when interpreting them.

My research project was therefore oriented within the interpretive paradigm as I wished to gain a deeper understanding of how learners solve algebraic problems with respect to the visualisation processes that they use. My approach was qualitative. According to De Vos et al. (2002, p. 79), the qualitative research paradigm refers to "research that elicits the participant's accounts of meaning, experience or perceptions. It produces descriptive data in the participant's own written or spoken words and is concerned with non-statistical methods and small purposively selected samples."

3.3 RESEARCH METHOD

The site for this research consisted of one school only. A small group of 3 Grade 11 and 3 Grade 12 learners was selected to participate. Thus, a qualitative case study was deemed to be the most suitable method for my research. According to Rule & John, as cited in Bertram, & Christiansen, (2014), "a case study is a systematic and in-depth study of one particular case in its context" (p. 42). The case for my particular study was a group of high ability Grades 11 and 12 learners. The unity of analysis was the elicited responses of the learners to selected algebraic tasks. In particular, I sought evidence of the visualisation processes that they used and how they used them when solving the given algebraic tasks.

To collect data for this study, I used different research tools such as interviews, learners' work (responses to the Visualisation Algebraic Tasks - AVT) and observations. The interviews were conducted to find out and interpret the visualisation processes that the learners employed. Learners' work was used to analyse the visualisation processes used to solve the AVT tasks. Observations were used to further identify the visualisation processes used and how they were used. A template was designed to assist in identifying and categorising the visualisation processes.

3.4 RESEARCH DESIGN

The data generation took place in three different phases. During the **first phase**, I identified the site and selected the participants. I also refined the Algebraic Visualisation Tasks (AVT) activity sheet after testing the validity of this instrument by doing a pilot with 4 Grades 11 and 12 learners. The AVT comprised of ten algebraic problems.

The **second phase** was the implementation of the AVT with the selected Grades 11 and 12 learners. In this phase, learners worked on the AVT and were observed and interviewed while tackling the individual AVT items. I interacted with each participant as he/she "talked through" each of the items whilst solving them. I video recorded these interactions to capture what the participants were saying and drawing. Pictures of the learners' work were also collected.

In **phase three,** I transcribed all the videos before I started analysing them. I analysed the video recordings and identified the visualisation strategies that the participants used. To address research question two, a detailed analyses of the videotapes was done, guided by the literature review. This helped me to note what learners experienced when solving algebraic problems and how they used the visualisation processes to solve the problems.

3.5 PARTICIPANTS

My research participants were six high ability Grades 11 & 12 learners (3 learners from Grade 11 and 3 from Grade 12). Initially I wished to have a balanced gender mix; however, the boys in these classes were not willing to take part. I specifically selected the participation of high achievers because I wanted learners who were proficient algebraic problem solvers and who could articulate their use of different visualisation processes. It was important for my data set that my participants were good communicators and were able to articulate their thoughts and their learning processes precisely. Therefore, the learners were selected based on their capability, their performance in the class, the test and in the exams, and also willingness to participate in the study. My selection strategy was thus purposeful. According to Bertram & Christiansen (2014), purposeful sampling denotes when the researcher makes specific choices about which people, groups and objects to include in the sample. Furthermore, my aim was to collect rich in-depth data. Hence, my focus was only on one school. I used the school that I teach at because I am familiar with the learners and the ethos of the school. I have a good relationship with my colleagues and other stakeholders at the school. We have a good understanding of mutual trust and respect.

3.6 DATA COLLECTION TOOLS

3.6.1 Algebraic visualisation tasks (AVT)

The AVT consisted of 10 different algebraic problems that were set and given to the participants. They were encouraged to solve the problems using visualisation processes such as graphs, drawings and sketches. I sat with the learners whilst they engaged with the AVT items and they were required to give reasons of their choice of drawings/diagrams used, as well as to explain how such diagrams assisted them in solving the given algebraic problems. They were also asked to explain why they opted to use these diagrams/drawings over any others. Learners were encouraged to talk and explain their problem-solving processes while tackling the algebraic problems. This gave me (as a researcher) a deep insight of their thoughts and the visualisation processes they underwent as they attempted to solve the AVT items. Learners were videotaped while working on the problems and engaging with me.

Below are the 10 algebraic tasks that formed the AVT. They were adopted from the Namibia senior secondary certificate mathematics higher-level examination paper 1 of 2012.

Task 1

A hitchhiker set out on a journey of 60 km. He walked the first 5 km and then got a lift from a lorry [truck] driver. When the driver dropped him off, he still had half of his journey to travel. How far had he travelled in the lorry?

Task 2

Penouwa bought a pizza and cut it into three pieces. When she weighed the pieces, she found that one piece was 7g lighter than the largest piece and 4g heavier than the smallest piece. The mass of the whole pizza was 300g. What was the mass of each piece?

Task 3

A man was very overweight and his doctor told him to lose 36 kg. If he loses 11 kg the first week, 9 kg the second week, and 7 kg the third week, and he continues losing at this rate, how long will it take him to lose 36 kg?

Task 4

One side of a rectangle is 3 cm shorter than the other side. If we increase the length of each side by 1 cm, then the area of the rectangle will increase by 18 cm^2. Find the lengths of all sides.

Task 5

Hafoletu is putting up a tent for a family reunion. The tent measures 16 m by 5 m. Each 4-m section of tent needs a post except the sides that are 5 m long. How many posts will he need?

Task 6

A moving company is hired to take 578 clay pots to a florist shop. The florist will pay the moving company a N$200 fee, plus N$1 for every pot that is delivered safely. The moving company must pay the florist N$4 each for any pots that are lost or broken. If two pots are lost, four pots are broken, and the rest are delivered safely, how much should the moving company be paid?

Task 7

A ladder of length 8 m rests against a wall so that the angle between the ladder and the wall is 31°. How far is the base of the ladder from the wall?

Task 8

f is a quadratic function whose graph has a vertex at the point (-3 , 2) and a y-intercept at the point (0 , -16). Find the x-intercepts of the graph of f.

Task 9

My brother is two years older than me, my sister is five years younger than me; she is 12, how old will my brother be in three years' time?

Task 10

A shop owner has room in her shop for up to 20 television sets. She can buy either type **A** for **N$ 150** each or type **B** for **N$ 300** each. She has a total of **N$ 4500** she can spend and she must have at least 6 of each type in the stock. She makes a profit of **N$80** on each television of type **A** and a profit of **N$ 100** on each of type **B.** How many of each type should she buy so that she makes a maximum profit?

According to our regional scheme of work, algebra is supposed to be covered in the first term of Grade 11. At the time this research was conducted, algebra had already been taught. Therefore, all algebra concepts in the Grades 11 and 12 curriculum were included in the AVT. The learners were provided with ample blank paper to scribble on whilst solving the 10 tasks.

3.6.2 Interviews

The main source of data for this research was the transcripts of the solution strategies, and the interviews and conversations I had with the participants. According to Bertram and Christiansen (2014), interviews are used in working towards the aim of "exploring and describing people's perceptions and understanding that might be unique to them" (p. 82). They further state that interpretivist research uses the interview method extensively, since it allows the researcher to ask probing and clarifying questions and to discuss research participants' understanding with them. For this study, the interviews with the individual learners took the form of a conversation whereby the learners were asked to explain how they used their visualisation processes to solve each of the AVT tasks. According to Cohen and Manion (1994), the unstructured interview is an open-ended approach to interviewing, in which questions flow from the immediate context.

During these conversations, learners were asked to explain their drawings or any visualisation processes they used. Their visual representations may not have all be self-evident (Steinbring, 2005).

I had to ask for clarifications and elaborations from the learners as they engaged with these representations.

3.6.3 Observations

In conjunction with conversing with the participants as articulated above, I also video recorded these interactions and used the video recordings as supplementary data. I used the video data to corroborate the conversation data and subjected it to a similar analysis process – see data analysis below. Simpson & Tuson (2003, p. 48) encourage that "if we are dealing with people, video recording can be a great help as it allows the same observation to be reviewed many times, with each viewing having the potential to elicit additional information".

3.7 DATA ANALYSIS

3.7.1 Algebraic visualisation tasks (responses)

In order to identify, classify and code the visualisation processes that the participants used, I analysed the interview/conversation and video data concurrently with the aid of the analytic templates which I had constructed from selected key resources in the literature, see Table 3.1 and Table 3.2. In the analysis of the data, I used Table 3.2 to categorise the visual processes used for each task of the AVT that each participant solved.

Table 3.1 is my visualisation analysis template that I used to analyse the visualisation processes of each participant for each AVT task.

Table 3.1 Analytic Template A - Categories of visualisation processes

Observable indicator	**coding**		
	1	**2**	**3**
External visualisation *These are observable shapes such as diagrams, graphs and pictures. They also include representations such as equations and words.*			
Internal visualisation *These are representations that are drawn in the mind and are imagined. They are only made explicit when verbalised and manifest themselves in conversation.*			
Descriptive representation *This is similar to the internal visualisation indictor above and refers to those representations that are spoken words or written texts such as mathematical equations and other expressions.*			
Depictive representation *These are iconic signs such as pictures, sketches, or drawn models and they are similar to the external indicators above.*			
Fundamental tool *This is when an overall visualisation representation is used throughout the problem.*			
Illustrative tool *This is when a visual representation is used to illustrate the problem in a familiar way.*			
Simplification tool *This is when a visual representation is used to make the problem easier.*			
Organizational tool *This is when a visual representation is used to deconstruct the problem into smaller components.*			
As a starting point *This is when a visual representation is used as an initial starting point that is developed further.*			

Table 3.2 Analytic Template B - Coding visualisation processes used in each AVT task

Coding template
1- very strong evidence When the visualisation process used is indicated clearly, **it is visible** and **observable**. There is strong evidence that the visualisation process was used. **2- mediocre evidence** This is when there is **mediocre evidence** that the visualisation process was used. The representations are **tentative**. The representations are **not clear** – they are mere scribbles and very rough sketches. **3-weak evidence** This is when the student claims to have used a certain visualisation process but there is no evidence that it was really used. No picture, drawing, graph or table is drawn, or the student could not explain the picture that had been drawn in mind.

Table 3.2 above illustrates the coding categories that applied to the observable indicators in Table 3.1.

3.7.2 Interviews and video recordings

As previously indicated, I used the above templates as I engaged with learners while they were solving each task of the AVT, and used the same templates to analyse the video recordings. The analysis of the video recording ensured that I captured all the visualisation processes evident for each learner for each task as comprehensively as possible.

In terms of the second research question, how the learners use the visualisation strategies, the verbal responses to each word problem were transcribed and read several times to establish a good sense of the data. The same template was then used to identify how the visualisations were used, for example if they were used to simplify, to organise, as a starting point or to illustrate.

Table 3.3 Summary of data generation and analysis phases

Phase	**Instrument**	**Purpose**	**Data**	**Analyses**
Phase 1				
Selection of the site and the participants	N/A	To gain permission to collect data and get potential participants for the research.	N/A	N/A
Seeking consent	N/A	To gain acceptance and consent from intended participants.	N/A	N/A
Development and piloting of AVT	AVT	To make any improvements and adjustments to the AVT	N/A	N/A
Phase 2	observations Interviews video recordings	To generate data	The verbal, written and drawn responses to the questions in the AVT.	N/A
Phase 3 Data analysis	Own designed data analysis tool	To answer the research questions. To analyse the findings. To suggest conclusion.	The verbal, written and drawn responses to the questions in the AVT.	Qualitative analyses

3.8 VALIDITY

Researchers need to ensure that their data collection methods and data are as valid or trustworthy as possible (Bertram & Christiansen, 2014, p.176). Gregory (1992, p.117) defines validity as "the extent to which a test (in this case the AVT) measures what it claims to measure". I had a sized sample of six learners (three in Grade 11 and three in Grade 12). These learners were selected according to their capability in mathematics and their communication skills to ensure they would be able to solve the AVT using visualisation strategies and explain their solution process while doing so. Before administering my AVT to the research participants, I piloted the AVT to enhance the validity of my instrument and made amendments. The participants were my learners, so we were familiar with each other. This helped them to be free to express their ideas and explain their processes. Furthermore, all the interviews and observation were video recorded.

3.9 ETHICS

3.9.1 Respect and dignity

The study was done at the school that I teach at because I am familiar with the learners and the ethos of the school. I have a good relationship with my colleagues and other stakeholders at the school. We have a good understanding of mutual trust and respect. Moreover, participants were ensured of their anonymity and were informed that this would be respected and retained throughout the study. It was also made clear to them that they had the right to withdraw from the study at any time. Participants were also informed that all sessions would be conducted after school to ensure they would not miss their normal classes.

3.9.2 Transparency and honesty

Permission in the form of written consent was obtained from the Department of Education and the school principal of the school where the research was conducted - see Appendix A and B. The participants as well as parents/guardians of the learners who agreed to participate in the research were also issued with a consent form that they were required to sign and return agreeing to be part of the study - see Appendix C and D. Moreover, when interviews were transcribed, the interview transcripts were shared with participants to ensure that data was appropriately and ethically collected and reported.

3.9.3 Accountability and responsibility

My participants were interviewed and the fact that they were my learners could have affected their way of answering the questions. Before I started with interviews, I made it clear to the participants that despite me being their teacher, they must take me just as an ordinary researcher and answer my questions honestly and as truthfully as possible.

3.9.4 Integrity, academic professionalism and researcher positionality

The study followed a qualitative case study due to the fact that the site for the intervention consisted of only one school and a small sample of 6 Grades 11 and 12 learners. To ensure validity, numerous sources of data were used such as interviews, observations and learners' work. All interactions with the learners were video recorded and pictures of their work were also collected. I also declared that the study is my original work and I adhered to the Rhodes Education Department Referencing Guide in acknowledging the work of others.

3.10 CONCLUSION

My study was oriented in an interpretive paradigm and I used a case study method as a research approach. Interviews were the main instrument used in this study. This chapter highlighted the interview process, how interviewees were contacted, where the interviews took place and who the interviewees were. Furthermore, all interviews were video recorded and the video recordings were used as supplementary data. The chapter was then concluded with a discussion of the data analysis procedure and the issues relating to ethics and validity. The findings of the collected data are analysed and described in the next chapter.

CHAPTER 4

DATA PRESENTATION AND RESULT ANALYSIS

4.1 INTRODUCTION

This chapter focuses on the presentation and analysis of data obtained from the AVT and the research interviews. A description of the respondents who took part in the research is also provided.

4.2 THE ALGERAIC VISUALISATION TASKS

The AVT is a set of 10 algebraic tasks that were set by the researcher and answered by the learners. In this section, I analyse the participants' responses to the AVT using the analytical tool that I developed, which consists of observable indicators and a framework for coding:

Observable indicator	coding		
	1	2	3
External visualisation *These are observable shapes such as diagrams, graphs and pictures. It also includes representations such as equations and words.*			
Internal visualisation *These representations are drawn in the mind and are imagined. These are only made explicit when verbalized..*			
Descriptive representation *This is similar to the internal visualisation indictor above and refers to those representations that are spoken words or written texts such as mathematical equations and other expressions.*			
Depictive representation *These are iconic signs such as pictures, sketches, or drawn models and they are similar to the external indicators above.*			
Fundamental tool *This is when an overall visualisation representation is used throughout the problem.*			
Illustrative tool *This is when a visual representation is used to illustrate the problem in a familiar way.*			
Simplification tool *This is when a visual representation is used to make the problem easier.*			
Organizational tool *This is when the visual representation is used to deconstruct the problem into smaller components.*			
As a starting point *This is when the visual representation is used as an initial starting point and then developed further.*			

1 is awarded when the visualisation process used is clear, **visible** and **observable** i.e. there is evidence of strong visualisation processes that were used. **2** is indicative of **mediocre evidence that** a visualisation process was used. The representations are thus **tentative** or **not clear** – they are mere scribbles and very rough sketches. Lastly, a 3 is awarded when there is **weak** evidence of visualisation processes. This is when the learner claims to have used a certain visualisation process but there is no evidence that it was really used, i.e. there are no pictures, drawings, graphs or tables. Further, the learner cannot explain the picture that he or she has drawn. For the analysis, learners will be coded as L1, L2 etc. for Learner 1, Learner 2 etc. respectively.

At the end of each task, there is a bar chart indicating the number of learners who used representations to solve that specific task. At the end of the chapter, there is also a chart of combined results for all 60 tasks, which is also briefly interpreted.

4.2.1 Task 1

Task 1
A hitchhiker set out on a journey of 60 km. He walked the first 5 km and then got a lift from a lorry [truck] driver. When the driver dropped him off, he still had half of his journey to travel. How far had he travelled in the lorry?

Learner 1

Observable indicator	coding		
	1	2	3
External visualisation *These are observable shapes such as diagrams, graphs and pictures. It also includes representations such as equations and words.*			√
Internal visualisation *These are representations that are drawn in the mind and are imagined. These are only made explicit verbally and manifest themselves in conversation.*	√		
Descriptive representation *This is similar to the internal visualisation indictor above and refers to those representations that are spoken words or written texts such as mathematical equations and other expressions.*	√		
Depictive representation *These are iconic signs such as pictures, sketches, or drawn models and they are similar to the external indicators above.*			
Fundamental tool *This is when an overall visualisation representation is used throughout the problem.*			
Illustrative tool *This is when a visual representation is used to illustrate the problem in a familiar way.*			
Simplification tool *This is when a visual representation is used to make the problem easier.*			
Organizational tool *This is when the visual representation is used to deconstruct the problem into smaller components.*			
As a starting point *This is when the visual representation is used as an initial starting point that is developed further.*			

L1 did not use any external visualisations. After reading the question several times, L1 closed her eyes and used her fingers to help with subtraction and addition. Then the learner opened her eyes and wrote down the final answer. Only the final answer was written on the paper - no other calculations were seen. When asked how L1 got the answer the explanation was that the question is clear and all calculations were done mentally. L1 said, "*I have done all the calculations in mind Mrs., the final answer is 25, no drawing is needed. Everything is clear Mrs. The man walked the first 5 km but when the driver dropped him off, he still had half of his journey to travel. That means the answer is 25*" (line 10).

Learner 2

Observable indicator	**coding**		
	1	**2**	**3**
External visualisation *These are observable shapes such as diagrams, graphs and pictures. It also includes representations such as equations and words.*	√		
Internal visualisation *These are representations that are drawn in the mind and are imagined. These are only made explicit verbally and manifest themselves in conversation.*	√		
Descriptive representation *This is similar to the internal visualisation indictor above and refers to those representations that are spoken words or written texts such as mathematical equations and other expressions.*	√		
Depictive representation *These are iconic signs such as pictures, sketches, or drawn models and they are similar to the external indicators above.*		√	
Fundamental tool *This is when an overall visualisation representation is used throughout the problem.*			
Illustrative tool *This is when a visual representation is used to illustrate the problem in a familiar way.*	√		
Simplification tool *This is when a visual representation is used to make the problem easier.*			
Organizational tool *This is when the visual representation is used to deconstruct the problem into smaller components.*			
As a starting point *This is when the visual representation is used as an initial starting point that is developed further.*			

L2 started by drawing a line (i.e. an external representation), which was labelled line AB (see Figure 4.1). The learner clearly indicated that line AB represented the distance which is 60 km. L2 then divided 60 by 2 to get half of the distance. She then subtracted 5 from 30 to get 25 which according to L2 is the distance travelled by the lorry. When asked how the diagram helped to solve the task, L2 explained, "*the line helped me to understand the question*" (Line)

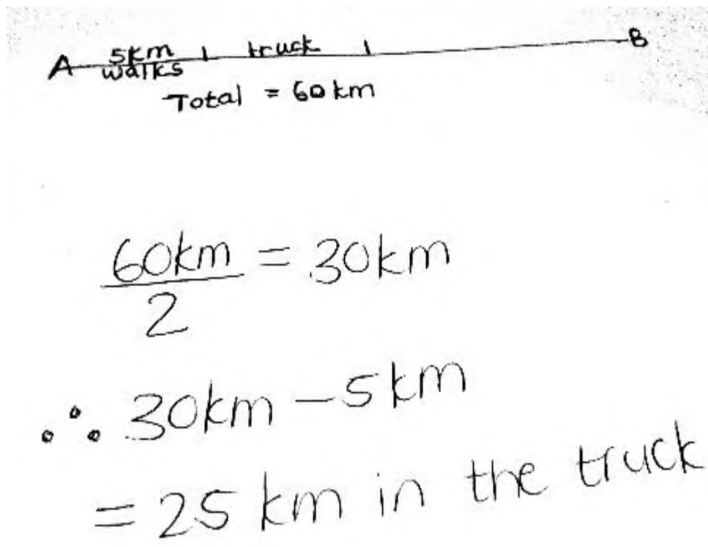


Figure 4.1 Picture of L2's response to Task 1

Learner 3

Observable indicator	coding		
	1	2	3
External visualisation *These are observable shapes such as diagrams, graphs and pictures. It also includes representations such as equations and words.*			√
Internal visualisation *These are representations that are drawn in the mind and are imagined. These are only made explicit verbally and manifest themselves in conversation.*			
Descriptive representation *This is similar to the internal visualisation indictor above and refers to those representations that are spoken words or written texts such as mathematical equations and other expressions.*	√		
Depictive representation *These are iconic signs such as pictures, sketches, or drawn models and they are similar to the external indicators above.*			
Fundamental tool *This is when an overall visualisation representation is used throughout the problem.*			
Illustrative tool *This is when a visual representation is used to illustrate the problem in a familiar way.*			
Simplification tool *This is when a visual representation is used to make the problem easier.*			
Organizational tool *This is when the visual representation is used to deconstruct the problem into smaller components.*			
As a starting point *This is when the visual representation is used as an initial starting point that is developed further.*			

L3 did not draw any diagram or picture but just started by writing the descriptive visualisation (an algebraic equation) $5 + x = 30$. L3 clarified that x represents the distance the hitchhiker travelled in the lorry plus five, which is the distance he walked. This equation was then solved to get the final answer. "*A drawing is not needed here, this is too simple and I can solve it without any sketch*" said L3 (line 394).

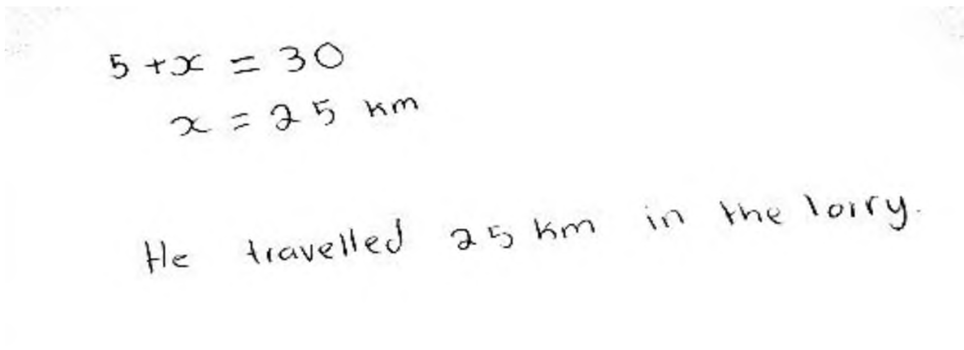


Figure 4.2 Picture of L3's response to task 1

Learner 4

Observable indicator	coding		
	1	2	3
External visualisation *These are observable shapes such as diagrams, graphs and pictures. It also includes representations such as equations and words.*			
Internal visualisation *These are representations that are drawn in the mind and are imagined. These are only made explicit verbally and manifest themselves in conversation.*			
Descriptive representation *This is similar to the internal visualisation indictor above and refers to those representations that are spoken words or written texts such as mathematical equations and other expressions.*			
Depictive representation *These are iconic signs such as pictures, sketches, or drawn models and they are similar to the external indicators above.*			
Fundamental tool *This is when an overall visualisation representation is used throughout the problem.*			
Illustrative tool *This is when a visual representation is used to illustrate the problem in a familiar way.*			
Simplification tool *This is when a visual representation is used to make the problem easier.*			
Organizational tool *This is when the visual representation is used to deconstruct the problem into smaller components.*			
As a starting point *This is when the visual representation is used as an initial starting point that is developed further.*			

1After reading the question once and without drawing any pictures, L4 wrote down the equation to represent the information given algebraically (see Figure 4.3). The learner subtracted 5 km from the total kilometers and divided the answer by 2. L4 said, "*if the total journey is 60 and a hitchhiker walked 5 km and got a lorry which dropped him off when he still had half of his journey to travel, you simply divided 55 by 2 to get the answer. No diagram is needed*" (Line 680).

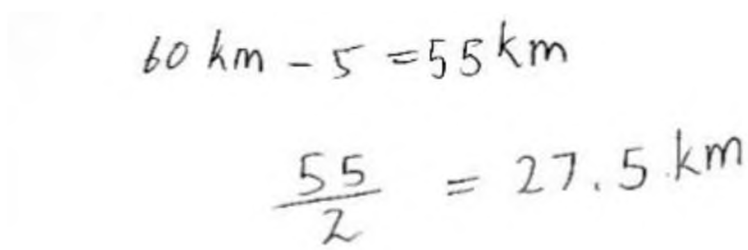


Figure 4.3 Picture of L4's response Task 1

Learner 5

Observable indicator	**coding**		
	1	**2**	**3**
External visualisation *These are observable shapes such as diagrams, graphs and pictures. It also includes representations such as equations and words.*	√		
Internal visualisation *These are representations that are drawn in the mind and are imagined. These are only made explicit verbally and manifest themselves in conversation.*	√		
Descriptive representation *This is similar to the internal visualisation indictor above and refers to those representations that are spoken words or written texts such as mathematical equations and other expressions.*	√		
Depictive representation *These are iconic signs such as pictures, sketches, or drawn models and they are similar to the external indicators above.*	√		
Fundamental tool *This is when an overall visualisation representation is used throughout the problem.*			
Illustrative tool *This is when a visual representation is used to illustrate the problem in a familiar way.*			
Simplification tool *This is when a visual representation is used to make the problem easier.*			
Organizational tool *This is when the visual representation is used to deconstruct the problem into smaller components.*			
As a starting point *This is when the visual representation is used as an initial starting point that is developed further.*	√		

A line which represented the whole journey (60 km) was drawn. It was then divided into the first 5 kilometers that the hitchhiker walked, and then 30 kilometers was indicated, which is half of his journey. (See Figure 4.4). "*Since he walked 5 kilometers and when he was dropped off by the lorry he still had half of his journey to travel, I will simply subtract 5 from 30. That means he still had 25 kilometers to travel*" said L5 (line 883). Even though L5 started with the drawing, she also used the algebraic equation to obtain the final answer.

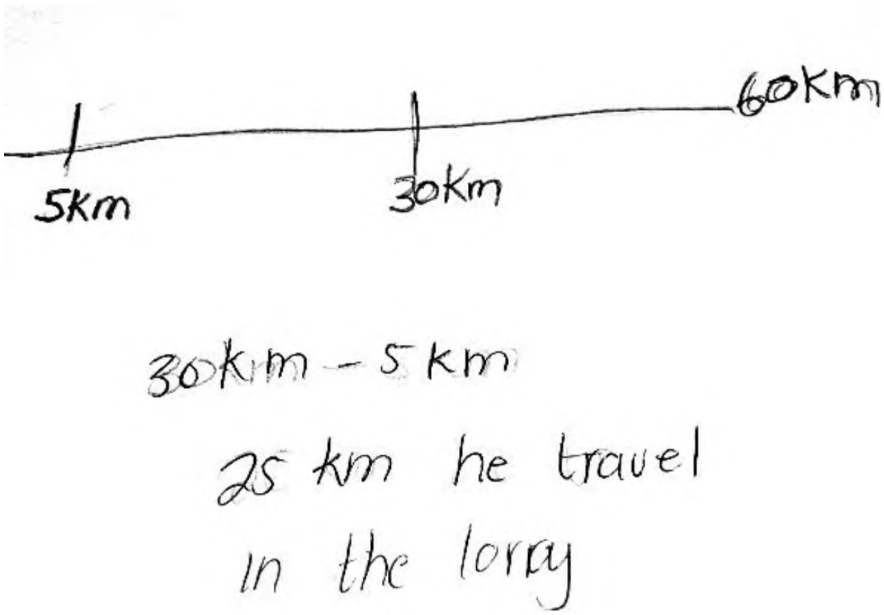


Figure 4.4 Picture of L5's response to Task 1

Learner 6

Observable indicator	coding		
	1	2	3
External visualisation *These are observable shapes such as diagrams, graphs and pictures. It also includes representations such as equations and words.*			
Internal visualisation *These are representations that are drawn in the mind and are imagined. These are only made explicit verbally and manifest themselves in conversation.*	√		
Descriptive representation *This is similar to the internal visualisation indictor above and refers to those representations that are spoken words or written texts such as mathematical equations and other expressions.*	√		
Depictive representation *These are iconic signs such as pictures, sketches, or drawn models and they are similar to the external indicators above.*			
Fundamental tool *This is when an overall visualisation representation is used throughout the problem.*			
Illustrative tool *This is when a visual representation is used to illustrate the problem in a familiar way.*			
Simplification tool *This is when a visual representation is used to make the problem easier.*			
Organizational tool *This is when the visual representation is used to deconstruct the problem into smaller components.*			
As a starting point *This is when the visual representation is used as an initial starting point that is developed further.*			

L6 stated that when he read this question the first thing he thought of was ‘distance, time and speed’. L6 then wrote Distance = Speed x Time. Just when the learner was busy substituting the given information in the equation he had written, L6 realized that none of time or speed is given so he decided to read the question again. “*I think I am taking the wrong direction. We don’t have speed and we don’t* *have time*", said L6 (Line 1085). After reading the question again, L6 generated an equation as follows: 60-5=55, then 55-30=25. L6 stated that the 5 km is the distance that the man walked subtracted from the total, which is 60. 55 is the distance that he still had to walk when he was picked up by the lorry, which dropped him off when he still had half of the total distance to reach his destination. So here L6 used descriptive representations only and indicated, "*I don't need a diagram for this problem, what's needed here is just basics mathematics*" (Line 1090).

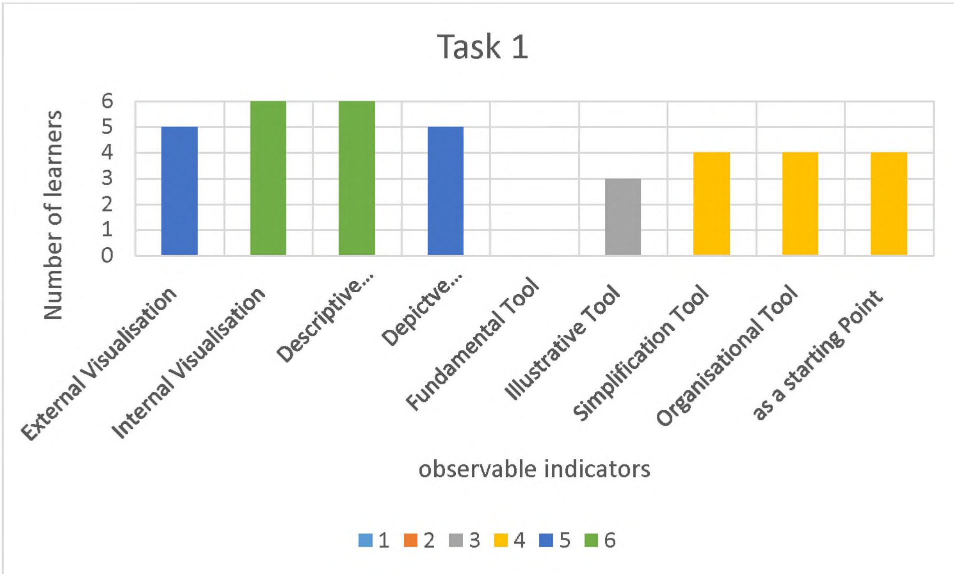


Figure 4.5 The types of visualisations used in task 1.

As indicated in Fig 4.5 above, out of the six learners who answered this question, 5 used external visualisations to solve the question. Only one learner who did not use external visualisation claimed to have done the calculations in her mind. Furthermore, four learners used visualisations as starting points. The table indicates also that four learners used visualisation to organize the given data and to simplify the task. However, no learner used visualisation as a fundamental tool.

4.2.2 Task 2

Task 2
Penouwa bought a pizza and cut it into three pieces. When she weighed the pieces, she found that one piece was 7g lighter than the larger piece and 4g heavier than the smallest piece. The mass of the whole pizza was 300g. What was the mass of each piece?

Learner 1

Observable indicator	**coding**		
	1	**2**	**3**
External visualisation *These are observable shapes such as diagrams, graphs and pictures. It also includes representations such as equations and words.*			√
Internal visualisation *These are representations that are drawn in the mind and are imagined. These are only made explicit verbally and manifest themselves in conversation.*			√
Descriptive representation *This is similar to the internal visualisation indictor above and refers to those representations that are spoken words or written texts such as mathematical equations and other expressions.*			
Depictive representation *These are iconic signs such as pictures, sketches, or drawn models and they are similar to the external indicators above.*			√
Fundamental tool *This is when an overall visualisation representation is used throughout the problem.*			
Illustrative tool *This is when a visual representation is used to illustrate the problem in a familiar way.*			
Simplification tool *This is when a visual representation is used to make the problem easier.*			
Organizational tool *This is when the visual representation is used to deconstruct the problem into smaller components.*			
As a starting point *This is when the visual representation is used as an initial starting point that is developed further.*			

For this question, L1 drew a circle, which according to her represented a pizza that was cut by Penouwa. The circle was divided into 3 parts, which represent the larger piece, the medium and the smallest piece. After that, L1 paused for some minutes and when she was asked what she was thinking. L1 replied, *"I am trying to figure out what each piece is weighing"*. (Line 15). However, this was all L1 could draw and did not proceed further. "*I am stuck Miss, I don't know if I should let the smaller or the bigger piece be x"* (line 17).

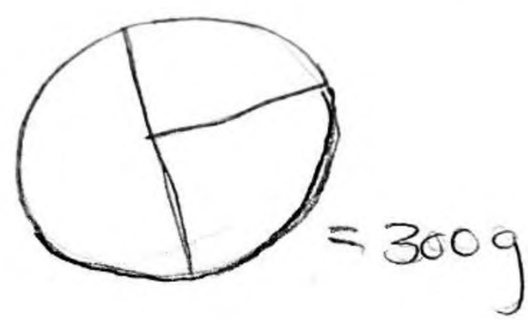


Figure 4.6 Picture of L1's response to Task 2

Learner 2

Observable indicator	coding		
	1	2	3
External visualisation *These are observable shapes such as diagrams, graphs and pictures. It also includes representations such as equations and words.*	√		
Internal visualisation *These are representations that are drawn in the mind and are imagined. These are only made explicit verbally and manifest themselves in conversation.*	√		
Descriptive representation *This is similar to the internal visualisation indictor above and refers to those representations that are spoken words or written texts such as mathematical equations and other expressions.*	√		
Depictive representation *These are iconic signs such as pictures, sketches, or drawn models and they are similar to the external indicators above.*	√		
Fundamental tool *This is when an overall visualisation representation is used throughout the problem.*			
Illustrative tool *This is when a visual representation is used to illustrate the problem in a familiar way.*			
Simplification tool *This is when a visual representation is used to make the problem easier.*	√		
Organizational tool *This is when the visual representation is used to deconstruct the problem into smaller components.*			
As a starting point *This is when the visual representation is used as an initial starting point that is developed further.*	√		

L2 also started by drawing a circle. L2 explained, "*here I am thinking of drawing a diagram Mrs. to represent the pizza that Penouwa bought*" (Line 263). This circle was then divided into three parts and labeled as $x, x-7 \text{ and } x-7+4$. According to L2 x represents the bigger piece, $x-7 \text{ the smaller piece and } x-7+4$ the medium piece. An equation was then formed and used throughout the processes. Since the total weight for the pizza is 300g then the sum of the three pieces must be 300. So $x + x - 7 + x - 7 + 4$ =300. L2 solved this equation to get the value of x, which was substituted in the expressions $x, x - 7$ and $x - 7 + 4$ to get the size of each piece. (See Figure 4.7)

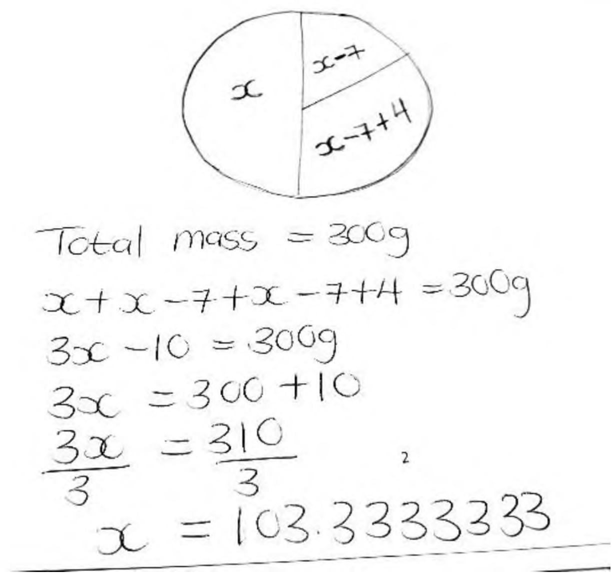


Figure 4.7 Picture of L2's response to Task 2

Learner 3

Observable indicator	**coding**		
	1	**2**	**3**
External visualisation *These are observable shapes such as diagrams, graphs and pictures. It also includes representations such as equations and words.*			
Internal visualisation *These are representations that are drawn in the mind and are imagined. These are only made explicit verbally and manifest themselves in conversation.*			
Descriptive representation *This is similar to the internal visualisation indictor above and refers to those representations that are spoken words or written texts such as mathematical equations and other expressions.*	√		
Depictive representation *These are iconic signs such as pictures, sketches, or drawn models and they are similar to the external indicators above.*			
Fundamental tool *This is when an overall visualisation representation is used throughout the problem.*			
Illustrative tool *This is when a visual representation is used to illustrate the problem in a familiar way.*			
Simplification tool *This is when a visual representation is used to make the problem easier.*			
Organizational tool *This is when the visual representation is used to deconstruct the problem into smaller components.*			
As a starting point *This is when the visual representation is used as an initial starting point that is developed further.*			

From the given information, L3 constructed an algebraic equation $x + x - 7 + x + 4 = 300g$ (see Figure 4.8). This descriptive representation was used throughout the process. For this question, L3 did not use any other visual representations. L3 indicated that no sketch, diagram or table was needed for her to solve this question. "*All I need is just an equation and solve it*" said L3 (Line 491).

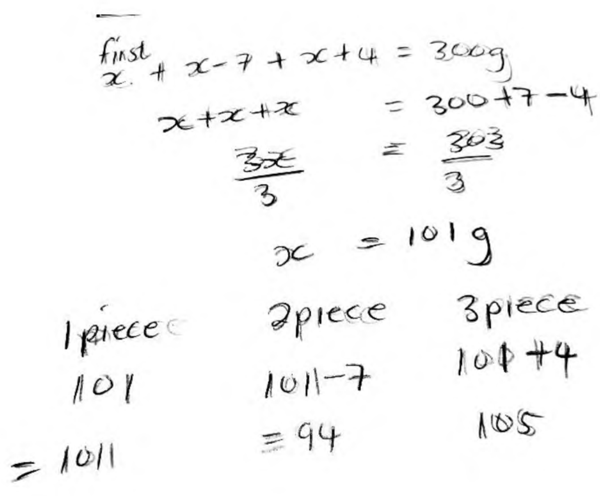


Figure 4.8 Picture of L3's response to Task 2

Learner 4

Observable indicator	coding		
	1	2	3
External visualisation *These are observable shapes such as diagrams, graphs and pictures. It also includes representations such as equations and words.*	√		
Internal visualisation *These are representations that are drawn in the mind and are imagined. These are only made explicit verbally and manifest themselves in conversation.*	√		
Descriptive representation *This is similar to the internal visualisation indictor above and refers to those representations that are spoken words or written texts such as mathematical equations and other expressions.*	√		
Depictive representation *These are iconic signs such as pictures, sketches, or drawn models and they are similar to the external indicators above.*			
Fundamental tool *This is when an overall visualisation representation is used throughout the problem.*			
Illustrative tool *This is when a visual representation is used to illustrate the problem in a familiar way.*	√		
Simplification tool *This is when a visual representation is used to make the problem easier.*	√		
Organizational tool *This is when the visual representation is used to deconstruct the problem into smaller components.*			
As a starting point *This is when the visual representation is used as an initial starting point that is developed further.*	√		

A circle as a visual representation was used by L4. This was divided into three parts, which were labelled as $x, x+4$ *and* $x+11$. L4 used both descriptive and depictive representations. These were seen clearly on the learner's answer script (see Figure 4.9). After reading the question L4 decided to let the smallest piece be x, middle sized piece= $x+4$ and largest piece =$x+11$. After that the equation was then formed ($x + 4 + x + x + 11 = 300$) which was used to get the size of the smallest piece which is x. The value of x was then used to calculate the values of the other two pieces.

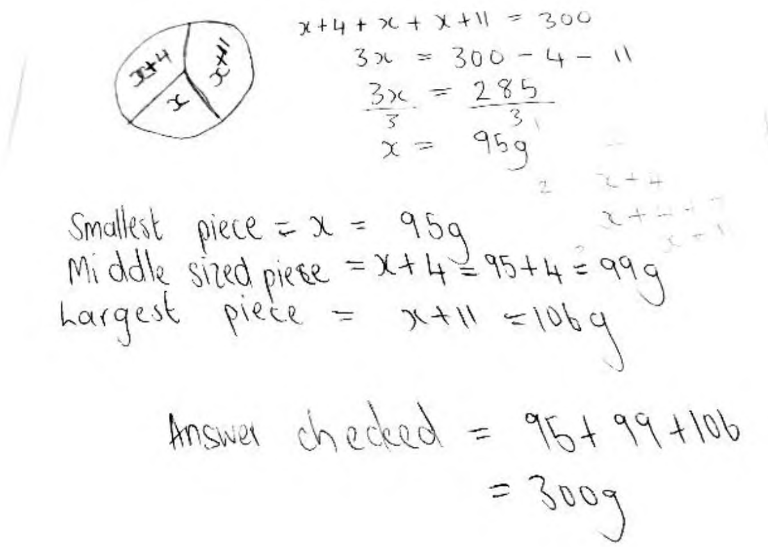


Figure 4.9 Picture of L4's response to Task 2

Learner 5

Observable indicator	**coding**		
	1	**2**	**3**
External visualisation *These are observable shapes such as diagrams, graphs and pictures. It also includes representations such as equations and words.*	√		
Internal visualisation *These are representations that are drawn in the mind and are imagined. These are only made explicit verbally and manifest themselves in conversation.*	√		
Descriptive representation *This is similar to the internal visualisation indictor above and refers to those representations that are spoken words or written texts such as mathematical equations and other expressions.*	√		
Depictive representation *These are iconic signs such as pictures, sketches, or drawn models and they are similar to the external indicators above.*			
Fundamental tool *This is when an overall visualisation representation is used throughout the problem.*			
Illustrative tool *This is when a visual representation is used to illustrate the problem in a familiar way.*			
Simplification tool *This is when a visual representation is used to make the problem easier.*			
Organizational tool *This is when the visual representation is used to deconstruct the problem into smaller components.*			
As a starting point *This is when the visual representation is used as an initial starting point that is developed further.*			

L5 started by writing the given information in algebraic form such as $x, x+7$ and $x-4$. with this information an equation was formed and solved for $x+(x+7)+(x-4)=300$. L5 claimed, "*I think I don't need any sketches for this question. All I need is just an equation that I will solve to get the answer*" (line 905). For this question, unlike other learners, L5 only used descriptive representations, which were equations and expressions (see Figure 4.10).

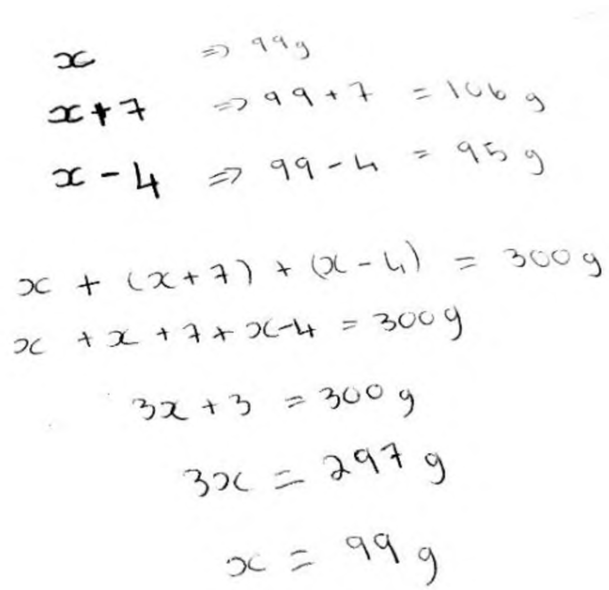


Figure 4.10 Picture of L5’s response to Task 2

Learner 6

Observable indicator	coding		
	1	2	3
External visualisation *These are observable shapes such as diagrams, graphs and pictures. It also includes representations such as equations and words.*		√	
Internal visualisation *These are representations that are drawn in the mind and are imagined. These are only made explicit verbally and manifest themselves in conversation.*			
Descriptive representation *This is similar to the internal visualisation indictor above and refers to those representations that are spoken words or written texts such as mathematical equations and other expressions.*	√		
Depictive representation *These are iconic signs such as pictures, sketches, or drawn models and they are similar to the external indicators above.*			
Fundamental tool *This is when an overall visualisation representation is used throughout the problem.*			
Illustrative tool *This is when a visual representation is used to illustrate the problem in a familiar way.*	√		
Simplification tool *This is when a visual representation is used to make the problem easier.*			
Organizational tool *This is when the visual representation is used to deconstruct the problem into smaller components.*			
As a starting point *This is when the visual representation is used as an initial starting point that is developed further.*			

"*Let me first draw a circle to represent the pizza and cut it into three pieces and see what's going to happen*", suggested L6 (Lines 1021-1022). When the circle was drawn and divided into three pieces, L6 constructed an equation from the given information, which he wrote algebraically: $x - 7 + x + 4 = 300$. The equation was then solved to get the value of x. When L6 was asked if he had used the circle that he had drawn, he replied, "*a "circle was just something that I did to get into my head to see* *if I could map it out like this will be my larger piece and this will be my smaller piece. Just to see the sizes of the pieces*" (lines 1102-1103).

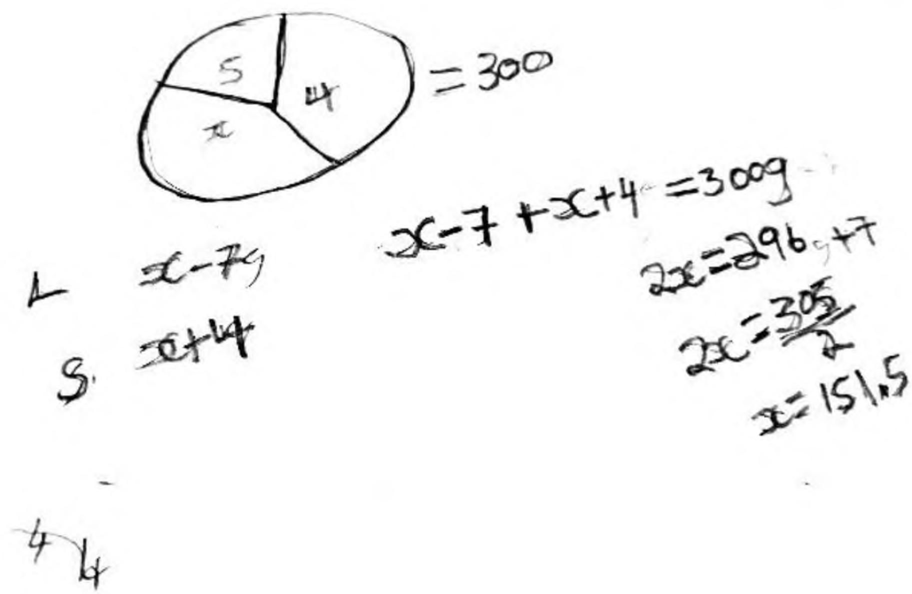


Figure 4.11 Picture of L6's response to Task 2

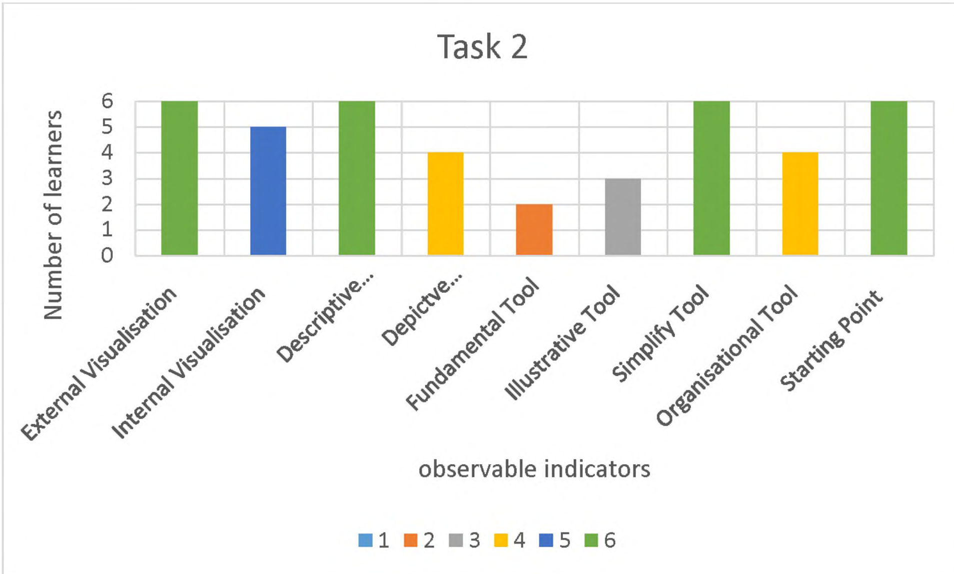


Figure 4.12 The types of visualisations used in Task 2

Figure 4.12 shows that all six participants used external and descriptive visualisation. Further, all participants used visualisations to simplify the task and used these visualisations as starting points. Only two participants used visualisations as a fundamental tool. Additionally, it shows that four participants used visualisations to organize the information and three used them as illustrative tools.

4.2.3 Task 3

Task 3
A man was very overweight and his doctor told him to lose 36 kg. If he loses 11 kg the first week, 9 kg the second week, and 7 kg the third week, and he continues losing at this rate, how long will it take him to lose 36 kg?

Learner 1

Observable indicator	coding		
	1	2	3
External visualisation *These are observable shapes such as diagrams, graphs and pictures. It also includes representations such as equations and words.*			
Internal visualisation *These are representations that are drawn in the mind and are imagined. These are only made explicit verbally and manifest themselves in conversation.*	√		
Descriptive representation *This is similar to the internal visualisation indictor above and refers to those representations that are spoken words or written texts such as mathematical equations and other expressions.*		√	
Depictive representation *These are iconic signs such as pictures, sketches, or drawn models and they are similar to the external indicators above.*			
Fundamental tool *This is when an overall visualisation representation is used throughout the problem.*			
Illustrative tool *This is when a visual representation is used to illustrate the problem in a familiar way.*			
Simplification tool *This is when a visual representation is used to make the problem easier.*			
Organizational tool *This is when the visual representation is used to deconstruct the problem into smaller components.*			
As a starting point *This is when the visual representation is used as an initial starting point that is developed further.*			

L1 started writing down the weeks and the number of kg lost in each week (see Figure 4.13) e.g. 1st week =11kg, 2nd week =9 , 3rd week =7 , 4th week = 5, 5th week=3 and 6th =1. According to L1, the numbers 11, 9, 7, 5, 3 and 1 form a sequence. Since the rate is the same, learner 1 subtracted 9 from 11 and kept subtracting two from each term until she reached one. L1 stated that "*If I add 11 plus 9 plus 7 plus 5 plus 3 plus 1, I will get 36, and the last kilogram will be lost in the 6th week , that means it will take him six weeks*". Therefore, for L1 it was easier to solve this problem without any diagram, sketch or drawing. L1 indicated, "*I prefer solving sequences that way than drawing a table*" (Line 88)

Figure 4.13 Picture of L1's response to Task 3

Learner 2

Observable indicator	coding		
	1	2	3
External visualisation *These are observable shapes such as diagrams, graphs and pictures. It also includes representations such as equations and words.*	√		
Internal visualisation *These are representations that are drawn in the mind and are imagined. These are only made explicit verbally and manifest themselves in conversation.*	√		
Descriptive representation *This is similar to the internal visualisation indictor above and refers to those representations that are spoken words or written texts such as mathematical equations and other expressions.*			
Depictive representation *These are iconic signs such as pictures, sketches, or drawn models and they are similar to the external indicators above.*			
Fundamental tool *This is when an overall visualisation representation is used throughout the problem.*	√		
Illustrative tool *This is when a visual representation is used to illustrate the problem in a familiar way.*			
Simplification tool *This is when a visual representation is used to make the problem easier.*			
Organizational tool *This is when the visual representation is used to deconstruct the problem into smaller components.*			
As a starting point *This is when the visual representation is used as an initial starting point that is developed further.*			

L2 started by drawing a bar graph and labelled the horizontal axis as number of days and the vertical axis as mass (kg). When L2 was done with the first three bars, she read the question again and decided to use a table instead of a bar chart. According to L2 when she read the question, she noticed that all the given numbers are odd numbers "*so there should be a sequence*" (line 387). L2 then drew a table consisting of two rows; one for the number of weeks and the other one for the kilograms (see Figure 4.14). This learner started filling in the information given from the first week and the kilograms lost in that week up to the six weeks where the learner stopped. L2 only used the table to get the final answer.

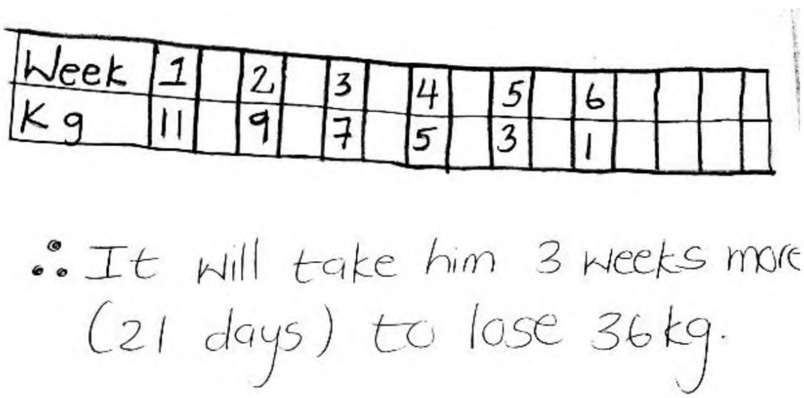


Figure 4.14 Picture of L2’s response to Task 3

Learner 3

Observable indicator	**coding**		
	1	**2**	**3**
External visualisation *These are observable shapes such as diagrams, graphs and pictures. It also includes representations such as equations and words.*			
Internal visualisation *These are representations that are drawn in the mind and are imagined. These are only made explicit verbally and manifest themselves in conversation.*	√		
Descriptive representation *This is similar to the internal visualisation indictor above and refers to those representations that are spoken words or written texts such as mathematical equations and other expressions.*	√		
Depictive representation *These are iconic signs such as pictures, sketches, or drawn models and they are similar to the external indicators above.*			
Fundamental tool *This is when an overall visualisation representation is used throughout the problem.*			
Illustrative tool *This is when a visual representation is used to illustrate the problem in a familiar way.*			
Simplification tool *This is when a visual representation is used to make the problem easier.*			
Organizational tool *This is when the visual representation is used to deconstruct the problem into smaller components.*			
As a starting point *This is when the visual representation is used as an initial starting point that is developed further.*			

After reading the question several times, L3 came up with a number sequence, 11, 9, 7… And then she calculated the difference between the terms such as 9-11=-2 and 7-9= -2. Common difference =-2 and first term =11.With that information the learner wrote down an equation $S_n = \frac{n}{2}[2a + (n-1)d]$.When asked if L 3 can't use a picture, a diagram or a table, the learners indicated that she prefers using what she was taught in the class and it is much easier and shorter than using a diagram or a sketch. "*I prefer using a formula. It is much shorter*", said L3 (line 444) .When L3 wrote down the formula, she started substituting the given information in the formula $36 = \frac{n}{2}[2(11) + (n-1) - 2$ and solved it to get the final solution.

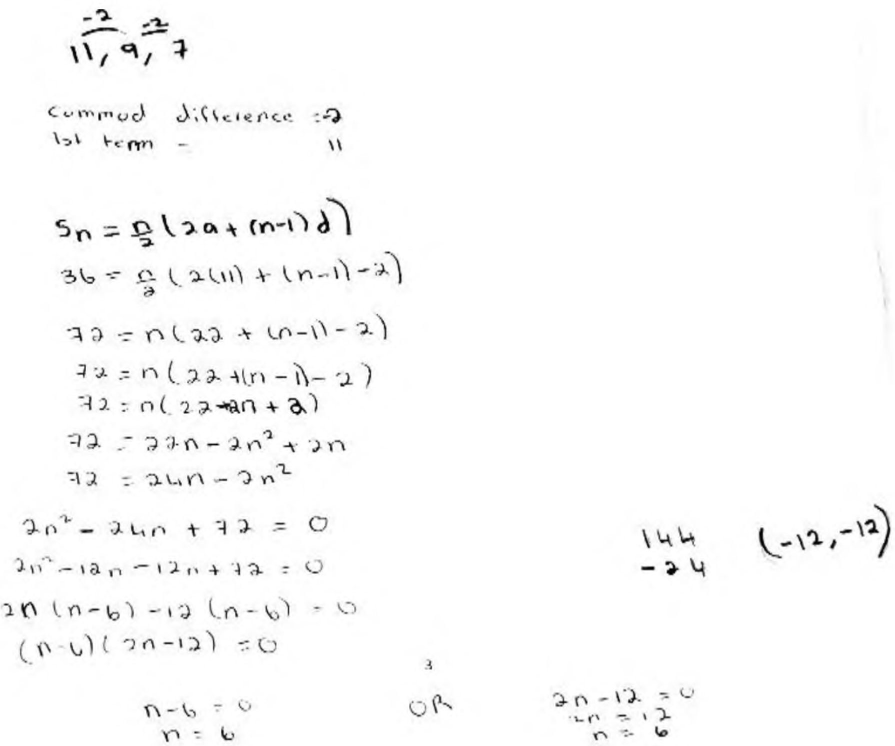


Figure 4.15 Picture of L2's response to Task 3

Learner 4

Observable indicator	coding		
	1	2	3
External visualisation *These are observable shapes such as diagrams, graphs and pictures. It also includes representations such as equations and words.*			
Internal visualisation *These are representations that are drawn in the mind and are imagined. These are only made explicit verbally and manifest themselves in conversation.*	√		
Descriptive representation *This is similar to the internal visualisation indictor above and refers to those representations that are spoken words or written texts such as mathematical equations and other expressions.*	√		
Depictive representation *These are iconic signs such as pictures, sketches, or drawn models and they are similar to the external indicators above.*			
Fundamental tool *This is when an overall visualisation representation is used throughout the problem.*			
Illustrative tool *This is when a visual representation is used to illustrate the problem in a familiar way.*			
Simplification tool *This is when a visual representation is used to make the problem easier.*			
Organizational tool *This is when the visual representation is used to deconstruct the problem into smaller components.*			
As a starting point *This is when the visual representation is used as an initial starting point that is developed further.*			

L4 approached the question a bit differently than the others by first calculating what he referred to as the rate, rate = $x - 2$. This means subtract 2 from the previous week's kilograms to get the kg for the next week? Since the man had to lose 36 kilograms, learner 4 let that be the total kilograms so that 36-11-9-7=9 kg. L4 indicated that in week three the man still had 9 kilograms to lose, 9-5-3-1=0. L4 kept subtracting 2 from the previous kilograms until he got 0. This learner did not use any diagrams but he used equations which are descriptive representations (see figure 4.16).

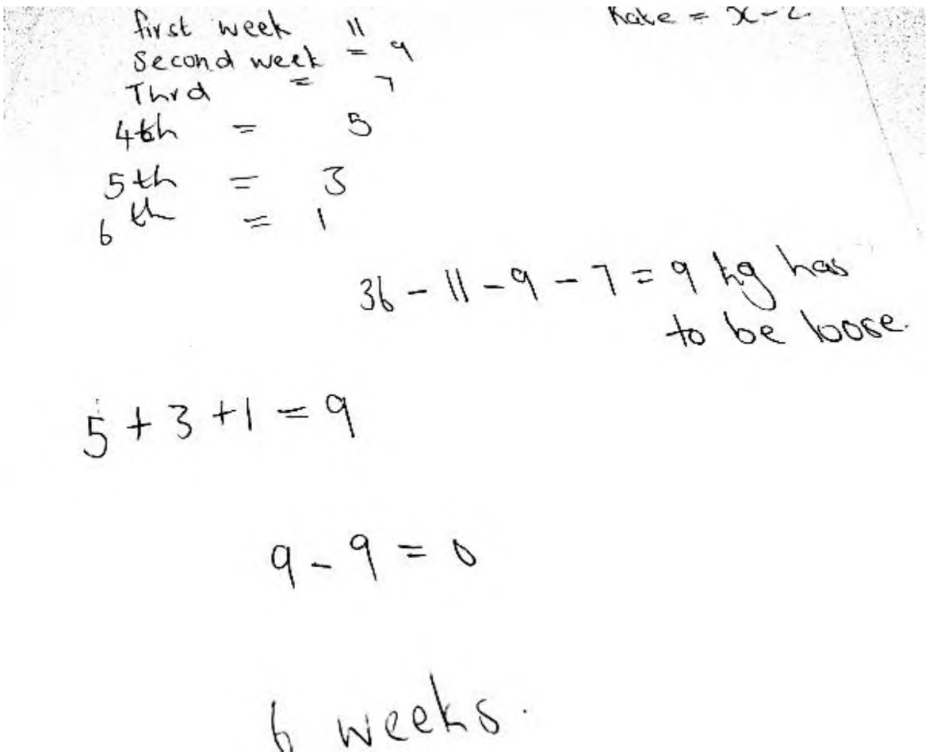


Figure 4.16 Picture of L4's response to Task 3

Learner 5

Observable indicator	**coding**		
	1	**2**	**3**
External visualisation *These are observable shapes such as diagrams, graphs and pictures. It also includes representations such as equations and words.*	√		
Internal visualisation *These are representations that are drawn in the mind and are imagined. These are only made explicit verbally and manifest themselves in conversation.*	√		
Descriptive representation *This is similar to the internal visualisation indictor above and refers to those representations that are spoken words or written texts such as mathematical equations and other expressions.*			
Depictive representation *These are iconic signs such as pictures, sketches, or drawn models and they are similar to the external indicators above.*			
Fundamental tool *This is when an overall visualisation representation is used throughout the problem.*			
Illustrative tool *This is when a visual representation is used to illustrate the problem in a familiar way.*			
Simplification tool *This is when a visual representation is used to make the problem easier.*	√		
Organizational tool *This is when the visual representation is used to deconstruct the problem into smaller components.*			
As a starting point *This is when the visual representation is used as an initial starting point that is developed further.*			

L5, just like some of the previous learners started by writing down the kilograms the man lost in the first three weeks, 11+9+7 and subtracted the total kilograms lost in three weeks from 36. According to L5, first she had to solve for x. Thus, an equation was formed and solved $x + 11 + 9 + 7 = 36$, $x = 36 - 11 - 9 - 7, x = 9$. L5 found that in the third week the man still had to lose 9 kilograms and using the same method was not going to help her get to the final answer. L5 decided to draw a table. The table illustrates the weeks and the number of kilograms lost in each week. This table was then used to obtain the final answer.

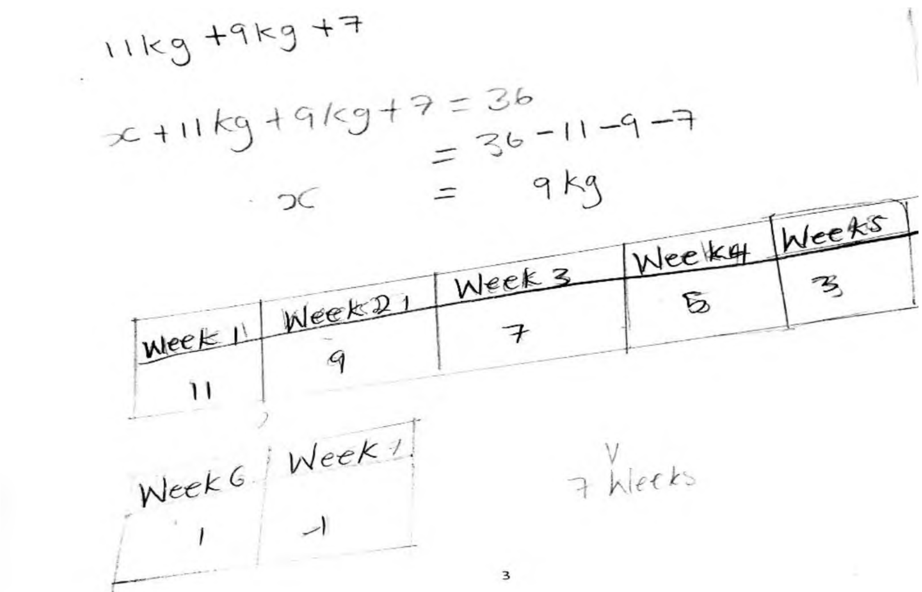


Figure 4.17 Picture of L5's response to Task 3

Learner 6

Observable indicator	**coding**		
	1	**2**	**3**
External visualisation *These are observable shapes such as diagrams, graphs and pictures. It also includes representations such as equations and words.*	√		
Internal visualisation *These are representations that are drawn in the mind and are imagined. These are only made explicit verbally and manifest themselves in conversation.*			
Descriptive representation *This is similar to the internal visualisation indictor above and refers to those representations that are spoken words or written texts such as mathematical equations and other expressions.*			
Depictive representation *These are iconic signs such as pictures, sketches, or drawn models and they are similar to the external indicators above.*			
Fundamental tool *This is when an overall visualisation representation is used throughout the problem.*	√		
Illustrative tool *This is when a visual representation is used to illustrate the problem in a familiar way.*			
Simplification tool *This is when a visual representation is used to make the problem easier.*	√		
Organizational tool *This is when the visual representation is used to deconstruct the problem into smaller components.*	√		
As a starting point *This is when the visual representation is used as an initial starting point that is developed further.*			

L6 also started writing down the given information and then calculated the amount of kilograms that the man loses every week which according to L6 is the "*common difference or rate*"L6 (line 1111). After that, L6 got stuck and was asked to try a table or a sketch or a diagram. He then opted to draw a table with the number of weeks and the kilograms (see figure 4-15). The information was then entered starting from week one until he reached the week that the man lost one kg. According to L6, the table really helped him because it made things simpler than what he wanted to do. "*Like here (*pointing at his first calculations*) I was a bit confused, I confused myself but when the table came in, it organized everything*", explained L6 (lines 1130-1131).

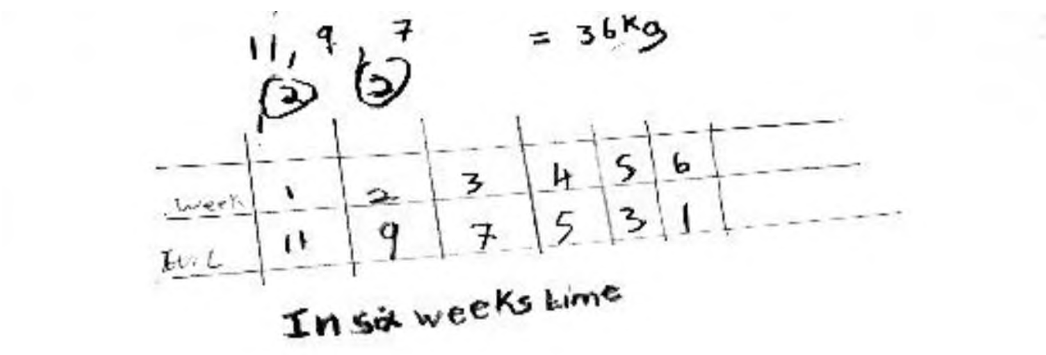


Figure 4.18 Picture of L6's response to Task 3

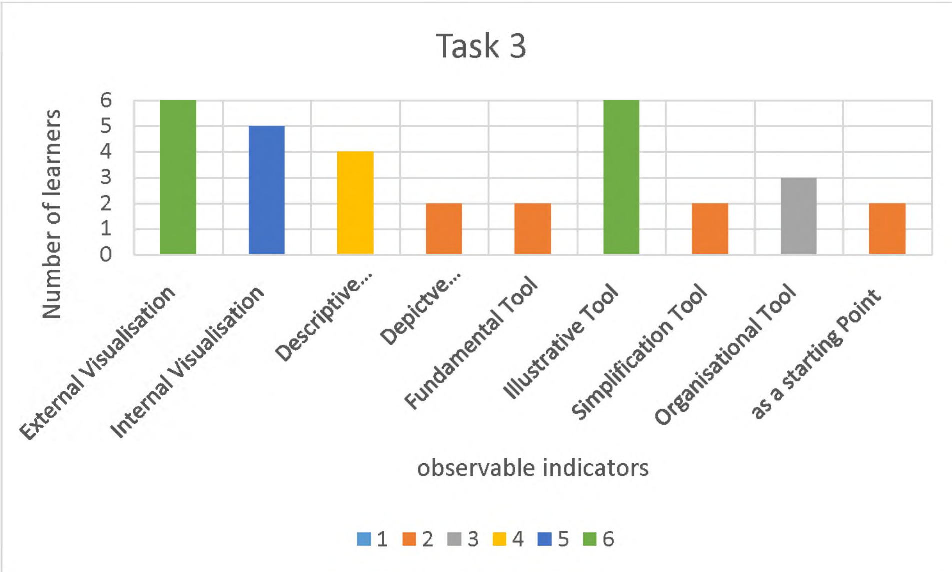


Figure 4.19 The types of visualisations used in Task 3

Figure 4.19 indicates that all six learners used external visualisations and 5 used internal visualisations. It also indicates that 4 participants used descriptive visualisations whereas 2 used depictive visualisations. In addition, two learners used visualisations as simplification tools and as organizational tools and the other three used them as starting points.

4.2.4 Task 4

Task 4
One side of a rectangle is 3 cm shorter than the other side. If we increase the length of each side by 1 cm, then the area of the rectangle will increase by 18 cm^2. Find the lengths of all sides.

Learner 1

Observable indicator	coding		
	1	2	3
External visualisation *These are observable shapes such as diagrams, graphs and pictures. It also includes representations such as equations and words.*	√		
Internal visualisation *These are representations that are drawn in the mind and are imagined. These are only made explicit verbally and manifest themselves in conversation.*	√		
Descriptive representation *This is similar to the internal visualisation indictor above and refers to those representations that are spoken words or written texts such as mathematical equations and other expressions.*	√		
Depictive representation *These are iconic signs such as pictures, sketches, or drawn models and they are similar to the external indicators above.*			
Fundamental tool *This is when an overall visualisation representation is used throughout the problem.*			
Illustrative tool *This is when a visual representation is used to illustrate the problem in a familiar way.*			
Simplification tool *This is when a visual representation is used to make the problem easier.*	√		
Organizational tool *This is when the visual representation is used to deconstruct the problem into smaller components.*			
As a starting point *This is when the visual representation is used as an initial starting point that is developed further.*	√		

L1 first drew a rectangle and label the length and breadth as $x\ and\ x-3$ respectively. L1 then drew another triangle but with different dimensions, $x+1\ and\ x-3+1$. L1 indicated, "*I have to draw another triangle with different dimensions because the instruction says the lengths were increased by 1*" (Line 88). From the two triangles the learner generated the equation $(x-3+1)(x+1)=x+18$, which was used to calculate the value of x. When x was obtained, L1 substituted the value in the expression $x-3$ to get the dimension of the other side.

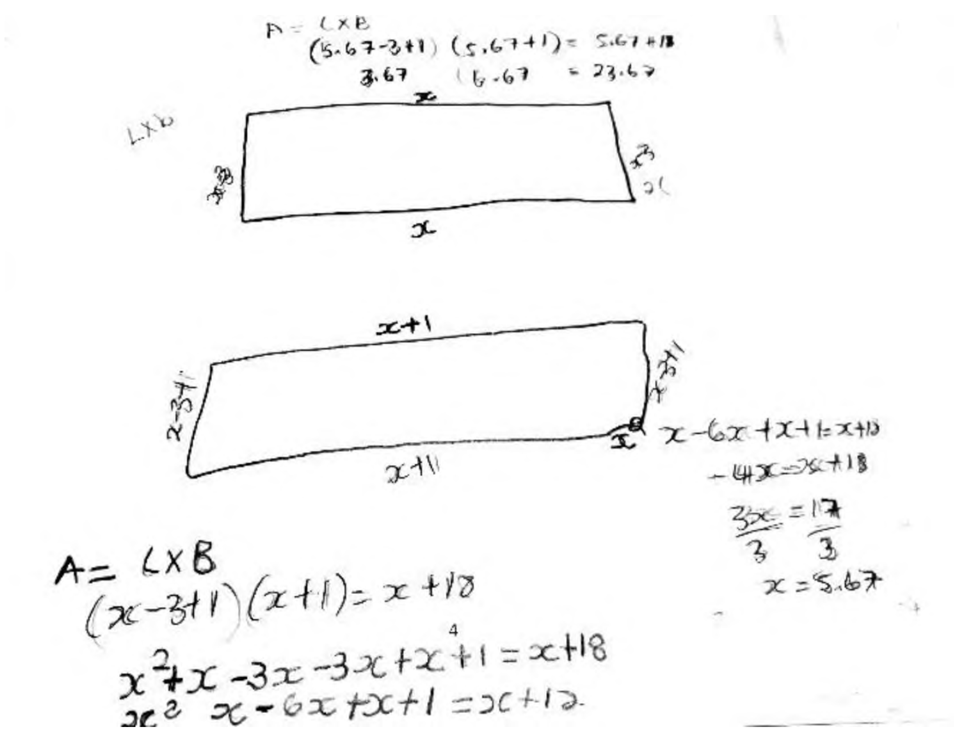
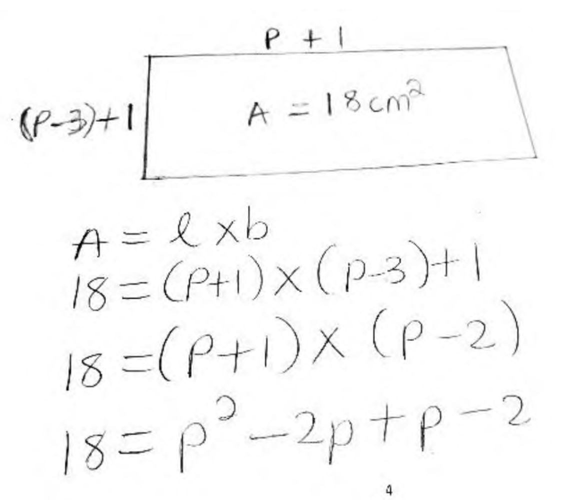


Figure 4.20 Picture of L1's response to Task 4

Learner 2

Observable indicator	coding		
	1	2	3
External visualisation *These are observable shapes such as diagrams, graphs and pictures. It also includes representations such as equations and words.*	√		
Internal visualisation *These are representations that are drawn in the mind and are imagined. These are only made explicit verbally and manifest themselves in conversation.*	√		
Descriptive representation *This is similar to the internal visualisation indictor above and refers to those representations that are spoken words or written texts such as mathematical equations and other expressions.*	√		
Depictive representation *These are iconic signs such as pictures, sketches, or drawn models and they are similar to the external indicators above.*			
Fundamental tool *This is when an overall visualisation representation is used throughout the problem.*			
Illustrative tool *This is when a visual representation is used to illustrate the problem in a familiar way.*	√		
Simplification tool *This is when a visual representation is used to make the problem easier.*			
Organizational tool *This is when the visual representation is used to deconstruct the problem into smaller components.*			
As a starting point *This is when the visual representation is used as an initial starting point that is developed further.*	√		

After reading the question several times, L2 drew a rectangle and label the length $p + 1\ and\ the\ width\ (p - 3) + 1$ and write inside the triangle A=18 cm^2. According to L2 since the area of a rectangle is $A = l \times b, then\ 18 = (p + 1) \times (p - 3) + 1$. The equation was simplified and solved to get the final solution. So L2 used both external representations and the descriptive representations.



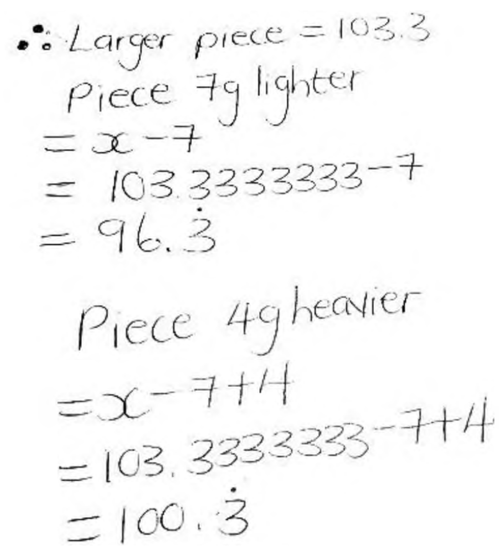


Figure 4.21 and Figure 4.22 Pictures of L2's response to Task 4

Learner 3

Observable indicator	coding		
	1	2	3
External visualisation *These are observable shapes such as diagrams, graphs and pictures. It also includes representations such as equations and words.*	√		
Internal visualisation *These are representations that are drawn in the mind and are imagined. These are only made explicit verbally and manifest themselves in conversation.*	√		
Descriptive representation *This is similar to the internal visualisation indictor above and refers to those representations that are spoken words or written texts such as mathematical equations and other expressions.*	√		
Depictive representation *These are iconic signs such as pictures, sketches, or drawn models and they are similar to the external indicators above.*			
Fundamental tool *This is when an overall visualisation representation is used throughout the problem.*			
Illustrative tool *This is when a visual representation is used to illustrate the problem in a familiar way.*		√	
Simplification tool *This is when a visual representation is used to make the problem easier.*	√		
Organizational tool *This is when the visual representation is used to deconstruct the problem into smaller components.*			
As a starting point *This is when the visual representation is used as an initial starting point that is developed further.*	√		

L3 drew a triangle and labelled the two sides $x \; and \; x - 3$, and the Area =y. L3 then wrote an equation taking information from the rectangle $y = x \times (x - 3)$. The learner read the second sentence of the question (If we increase the length of each side by 1 cm, then the area of the rectangle will increase by 18 cm^2 and constructed another equation from there, which is $(x + 1) \times [(x - 3) + 1] = y + 18$.

"*Since I have two unknowns in my equations, I will try to solve them simultaneously*", said L3 (Line 397). The substitution method was then used to solve the two equations.

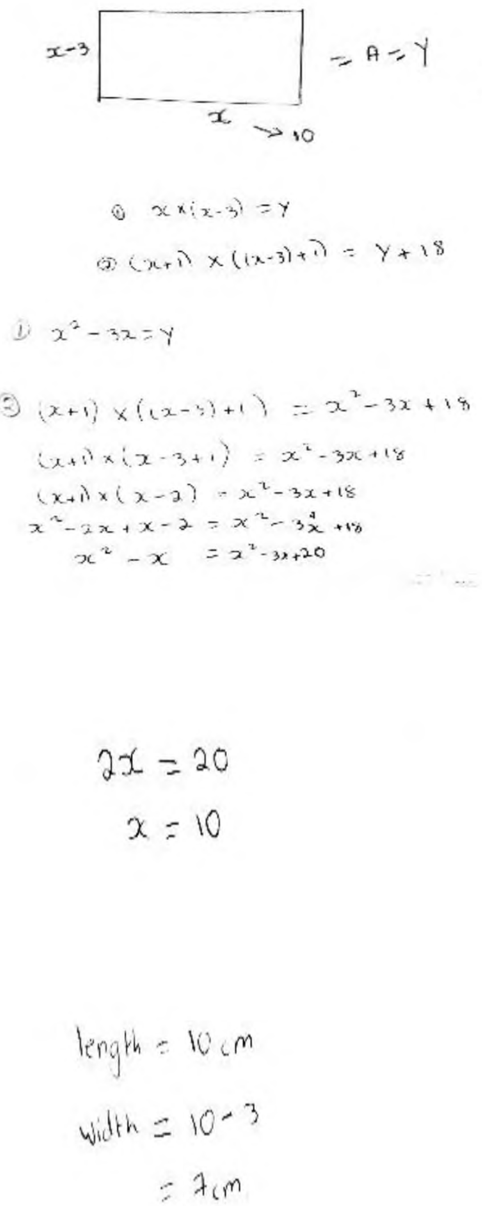


Figure 4.23 and Figure 4.24. Pictures of L3's response to Task 4

Learner 4

Observable indicator	coding		
	1	2	3
External visualisation *These are observable shapes such as diagrams, graphs and pictures. It also includes representations such as equations and words.*	√		
Internal visualisation *These are representations that are drawn in the mind and are imagined. These are only made explicit verbally and manifest themselves in conversation.*	√		
Descriptive representation *This is similar to the internal visualisation indictor above and refers to those representations that are spoken words or written texts such as mathematical equations and other expressions.*	√		
Depictive representation *These are iconic signs such as pictures, sketches, or drawn models and they are similar to the external indicators above.*			
Fundamental tool *This is when an overall visualisation representation is used throughout the problem.*			
Illustrative tool *This is when a visual representation is used to illustrate the problem in a familiar way.*		√	
Simplification tool *This is when a visual representation is used to make the problem easier.*	√		
Organizational tool *This is when the visual representation is used to deconstruct the problem into smaller components.*			
As a starting point *This is when the visual representation is used as an initial starting point that is developed further.*	√		

L4 sketched a rectangle and labelled the length as x and the breadth as $x-3$. L4 then extended the sides of the length and breadth and name them $x+1$ and $(x-3)+1$ respectively. From that information L4 formed an equation using the formula for the area of a rectangle: $A=lb$. $18cm^2 = (x+1)(x-2)$. This equation was solved. The learner then read through the question and looked at the first drawing and realized that it was wrong and decided to draw a new sketch. This time L4 decided to compare the areas of the two shapes and wrote a descriptive representation that stated, area of new sides = area of previous sides +18. That is $(x+1)(x-2) = x(x-3)+18.$ This was solved to get the value of x which was then used to get all the dimensions.

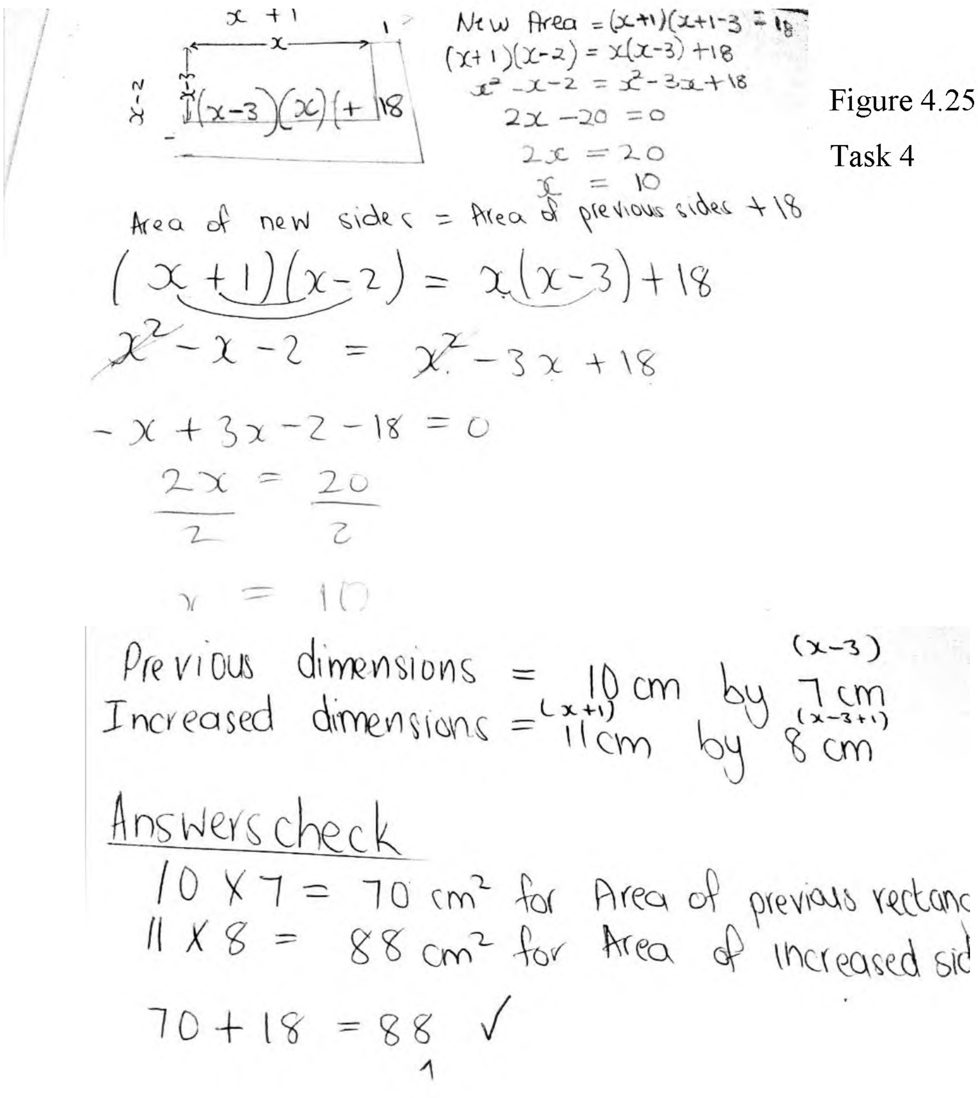


Figure 4.25 Picture of L4’s response to Task 4

Learner 5

Observable indicator	coding		
	1	2	3
External visualisation *These are observable shapes such as diagrams, graphs and pictures. It also includes representations such as equations and words.*			√
Internal visualisation *These are representations that are drawn in the mind and are imagined. These are only made explicit verbally and manifest themselves in conversation.*		√	
Descriptive representation *This is similar to the internal visualisation indictor above and refers to those representations that are spoken words or written texts such as mathematical equations and other expressions.*			
Depictive representation *These are iconic signs such as pictures, sketches, or drawn models and they are similar to the external indicators above.*			
Fundamental tool *This is when an overall visualisation representation is used throughout the problem.*			
Illustrative tool *This is when a visual representation is used to illustrate the problem in a familiar way.*			
Simplification tool *This is when a visual representation is used to make the problem easier.*			
Organizational tool *This is when the visual representation is used to deconstruct the problem into smaller components.*			
As a starting point *This is when the visual representation is used as an initial starting point that is developed further.*			

L5 read the question several times and drew a rectangle. L5 spent some more minutes reading the question and trying to figure out where to put the given information. However, the learner could not proceed and decided to move to the next question. "*The question is a bit tricky, I don't know what to do next*", testified L5 (Line 999).

Learner 6

Observable indicator	coding		
	1	2	3
External visualisation *These are observable shapes such as diagrams, graphs and pictures. It also includes representations such as equations and words.*	√		
Internal visualisation *These are representations that are drawn in the mind and are imagined. These are only made explicit verbally and manifest themselves in conversation.*	√		
Descriptive representation *This is similar to the internal visualisation indictor above and refers to those representations that are spoken words or written texts such as mathematical equations and other expressions.*	√		
Depictive representation *These are iconic signs such as pictures, sketches, or drawn models and they are similar to the external indicators above.*			
Fundamental tool *This is when an overall visualisation representation is used throughout the problem.*			
Illustrative tool *This is when a visual representation is used to illustrate the problem in a familiar way.*	√		
Simplification tool *This is when a visual representation is used to make the problem easier.*			
Organizational tool *This is when the visual representation is used to deconstruct the problem into smaller components.*	√		
As a starting point *This is when the visual representation is used as an initial starting point that is developed further.*	√		

L6 explained that he first had to draw a rectangle, label the sides, and see if it would work out. L6 explained, "*using a triangle will help me organize the question and clarify the question more than just reading. It will actually help me to see where I am going wrong*" (Lines 1124-1125). He then drew a rectangle, labelled the length and width as $x\ and\ x - 3$ respectively. "*x represents the length of the rectangle that we do not know*" explained L6 (line 1133). After that, he increased the sides by 1 as per instruction. L6 then came up with an equation: $l \times b = area, (x + 1)(x - 3 + 1) = 18cm^2$, which was simplified to $x^2 - x - 19 = 0$. L6 further stated, *"from here I will then use either the quadratic formula or solve by completing the squares. But completing the square is much faster than using the formula."* (Lines 1135-1136). This equation was solved by completing the square.

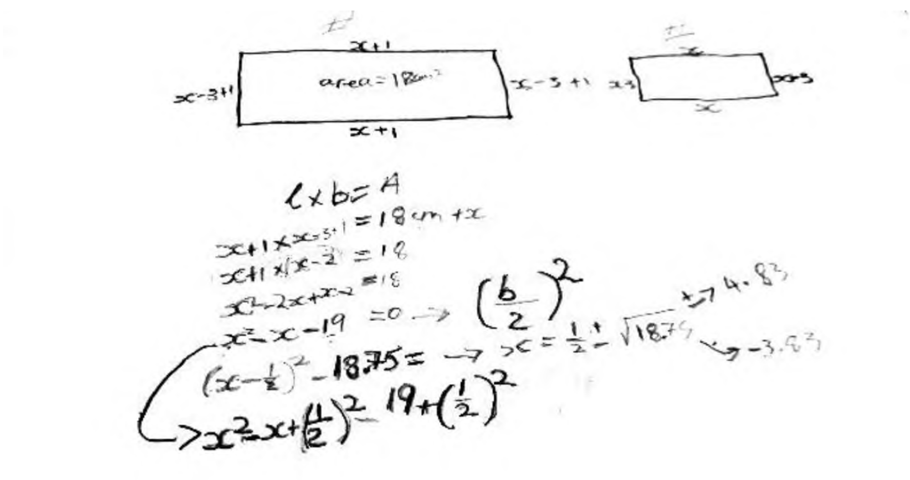


Figure 4.26 Picture of L6's response to Task 4

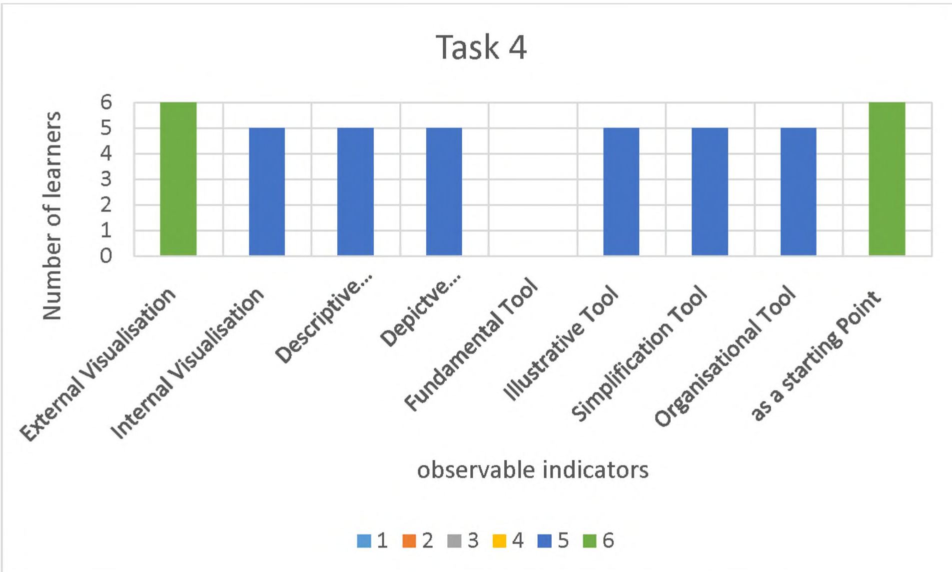


Figure 4.27 The types of visualisations used in Task 4

Figure 4.27 illustrates that six learners used external visualisations as starting points. It also indicates that five participants used internal visualisations. Moreover, five learners used descriptive and depictive visualisations to illustrate, to organize and simplify the task. However, no participant used visualisations as a fundamental tool.

4.2.5 Task 5

Task 5 Hafoletu is putting up a rectangular tent for a family reunion. The tent measures 16 m by 5 m. Each 4 m length of tent needs a post except on the sides that are 5 m long. How many posts will he need?

Learner 1

Observable indicator	**coding**		
	1	**2**	**3**
External visualisation *These are observable shapes such as diagrams, graphs and pictures. It also includes representations such as equations and words.*		√	
Internal visualisation *These are representations that are drawn in the mind and are imagined. These are only made explicit verbally and manifest themselves in conversation.*			
Descriptive representation *This is similar to the internal visualisation indictor above and refers to those representations that are spoken words or written texts such as mathematical equations and other expressions.*			
Depictive representation *These are iconic signs such as pictures, sketches, or drawn models and they are similar to the external indicators above.*		√	
Fundamental tool *This is when an overall visualisation representation is used throughout the problem.*			
Illustrative tool *This is when a visual representation is used to illustrate the problem in a familiar way.*	√		
Simplification tool *This is when a visual representation is used to make the problem easier.*			
Organizational tool *This is when the visual representation is used to deconstruct the problem into smaller components.*			
As a starting point *This is when the visual representation is used as an initial starting point that is developed further.*			

Since the question is about a rectangular tent, L1 started by drawing a rectangle and then divided the lengths of the rectangle into 4 meters. According to the question, each length is 16 meters, so L1 wrote an equation 16+16=32 and divided 32 by 4 to get 8. So the learner used a rectangle which is an external visualisation, however to get the solution an equation was constructed and solved (see Figure 4.27).

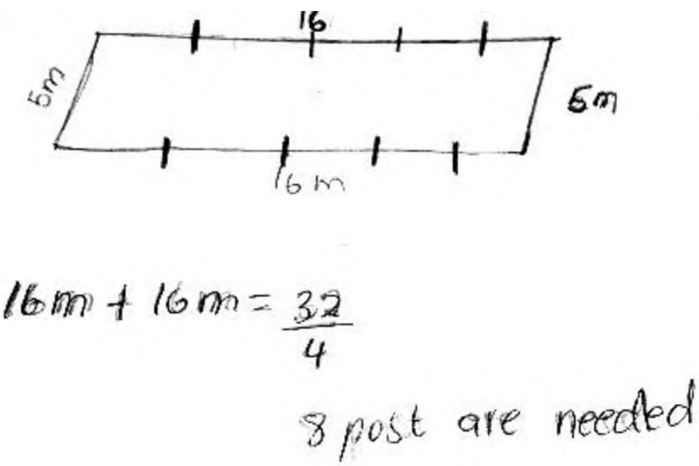


Figure 4.28 Picture of L1's response to Task 5

Learner 2

Observable indicator	coding		
	1	2	3
External visualisation *These are observable shapes such as diagrams, graphs and pictures. It also includes representations such as equations and words.*			
Internal visualisation *These are representations that are drawn in the mind and are imagined. These are only made explicit verbally and manifest themselves in conversation.*			
Descriptive representation *This is similar to the internal visualisation indictor above and refers to those representations that are spoken words or written texts such as mathematical equations and other expressions.*			
Depictive representation *These are iconic signs such as pictures, sketches, or drawn models and they are similar to the external indicators above.*			
Fundamental tool *This is when an overall visualisation representation is used throughout the problem.*			
Illustrative tool *This is when a visual representation is used to illustrate the problem in a familiar way.*			
Simplification tool *This is when a visual representation is used to make the problem easier.*			
Organizational tool *This is when the visual representation is used to deconstruct the problem into smaller components.*			
As a starting point *This is when the visual representation is used as an initial starting point that is developed further.*			

During this question L2 did not write or draw anything. When asked if she could not think of a diagram or a sketch L2 replied that "*I cannot think of anything and I think is because I don't understand the question Mrs.*" (Line 1009). This question was left blank by L2.

Learner 3

Observable indicator	**coding**		
	1	**2**	**3**
External visualisation *These are observable shapes such as diagrams, graphs and pictures. It also includes representations such as equations and words.*	√		
Internal visualisation *These are representations that are drawn in the mind and are imagined. These are only made explicit verbally and manifest themselves in conversation.*	√		
Descriptive representation *This is similar to the internal visualisation indictor above and refers to those representations that are spoken words or written texts such as mathematical equations and other expressions.*	√		
Depictive representation *These are iconic signs such as pictures, sketches, or drawn models and they are similar to the external indicators above.*			
Fundamental tool *This is when an overall visualisation representation is used throughout the problem.*			
Illustrative tool *This is when a visual representation is used to illustrate the problem in a familiar way.*	√		
Simplification tool *This is when a visual representation is used to make the problem easier.*			
Organizational tool *This is when the visual representation is used to deconstruct the problem into smaller components.*			
As a starting point *This is when the visual representation is used as an initial starting point that is developed further.*			

L 3 drew a rectangle and labelled the length 16 and the breadth 5. The learner then added 16 to 16 to get 32. Then 32 was divided by 4 to get the 8. L3 indicated, "*I am drawing a rectangle just to get a clear picture but to get the final answer I have to add the two sides and divide the answer by 4 to get the number of poles*". (Line 515)

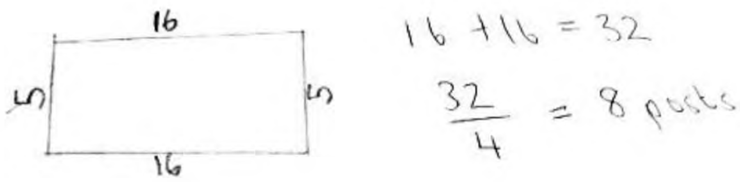


Figure 4.29 Picture of L3's response to Task 5

Learner 4

Observable indicator	**coding**		
	1	**2**	**3**
External visualisation *These are observable shapes such as diagrams, graphs and pictures. It also includes representations such as equations and words.*	√		
Internal visualisation *These are representations that are drawn in the mind and are imagined. These are only made explicit verbally and manifest themselves in conversation.*			
Descriptive representation *This is similar to the internal visualisation indictor above and refers to those representations that are spoken words or written texts such as mathematical equations and other expressions.*			
Depictive representation *These are iconic signs such as pictures, sketches, or drawn models and they are similar to the external indicators above.*			
Fundamental tool *This is when an overall visualisation representation is used throughout the problem.*	√		
Illustrative tool *This is when a visual representation is used to illustrate the problem in a familiar way.*	√		
Simplification tool *This is when a visual representation is used to make the problem easier.*			
Organizational tool *This is when the visual representation is used to deconstruct the problem into smaller components.*			
As a starting point *This is when the visual representation is used as an initial starting point that is developed further.*			

A rectangle was drawn. The instruction indicates that each 4 m length of a tent needs a post except on the sides that are 5 m long. L4 divided the sides that are 16 meters into four equal parts. The learner used dots to indicate the position of the posts. She started in the corner of the rectangle so she got five dots on each side, which gave a total of 10. L4 indicated that the number of dots represented the number of the poles needed. For this equation, L4 used only the diagram to solve the equation. No equation or expression was formed.

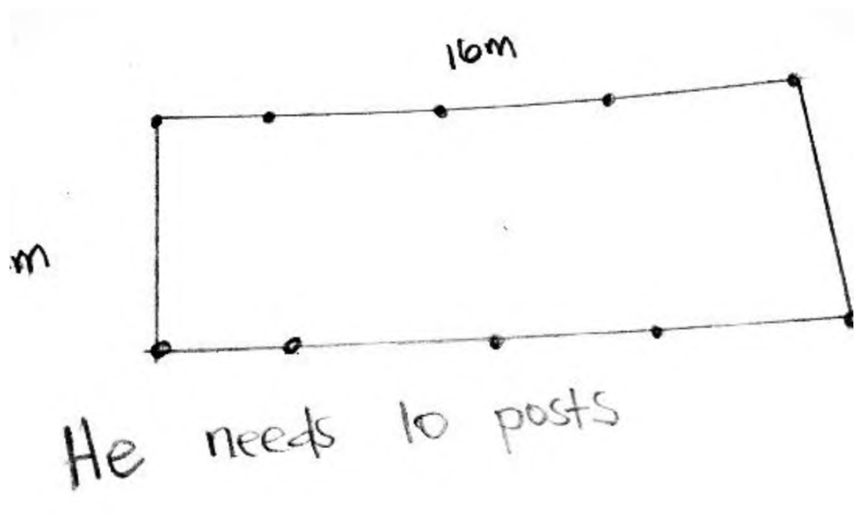


Figure 4.30. Picture of L4's response to Task 5

Learner 5

Observable indicator	coding		
	1	2	3
External visualisation *These are observable shapes such as diagrams, graphs and pictures. It also includes representations such as equations and words.*		√	
Internal visualisation *These are representations that are drawn in the mind and are imagined. These are only made explicit verbally and manifest themselves in conversation.*	√		
Descriptive representation *This is similar to the internal visualisation indictor above and refers to those representations that are spoken words or written texts such as mathematical equations and other expressions.*	√		
Depictive representation *These are iconic signs such as pictures, sketches, or drawn models and they are similar to the external indicators above.*			
Fundamental tool *This is when an overall visualisation representation is used throughout the problem.*			
Illustrative tool *This is when a visual representation is used to illustrate the problem in a familiar way.*			
Simplification tool *This is when a visual representation is used to make the problem easier.*			
Organizational tool *This is when the visual representation is used to deconstruct the problem into smaller components.*			
As a starting point *This is when the visual representation is used as an initial starting point that is developed further.*			

After reading the question L5 took a ruler, drew a rectangle, and labelled the length 16 and breadth 5. The learner then started dividing the length of the rectangle into four equal parts, which the learner then erased and instead wrote an equation $\frac{16}{4} = 4\,poles\ \times 2\,sides$ .When asked why L5 erased the lines L5 responded, "*it is easier to get the answer by just dividing than going a long way of drawing*". (Line 773)

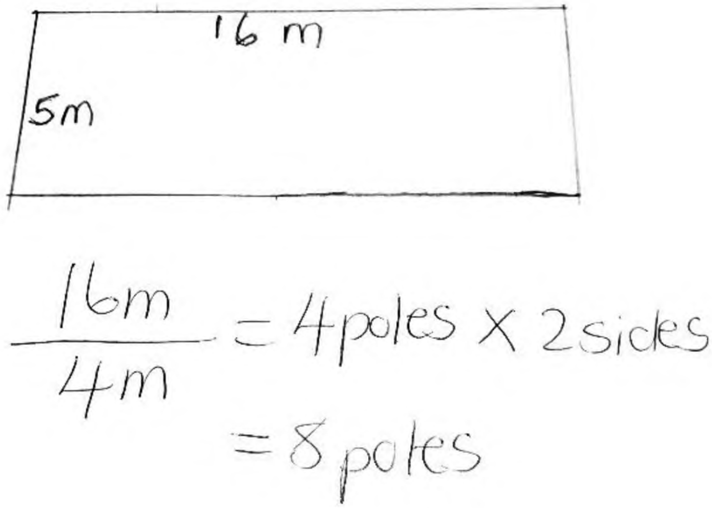


Figure 4.31. Picture of L5's response to Task 5

Learner 6

Observable indicator	**coding**		
	1	**2**	**3**
External visualisation *These are observable shapes such as diagrams, graphs and pictures. It also includes representations such as equations and words.*			√
Internal visualisation *These are representations that are drawn in the mind and are imagined. These are only made explicit verbally and manifest themselves in conversation.*	√		
Descriptive representation *This is similar to the internal visualisation indictor above and refers to those representations that are spoken words or written texts such as mathematical equations and other expressions.*		√	
Depictive representation *These are iconic signs such as pictures, sketches, or drawn models and they are similar to the external indicators above.*			
Fundamental tool *This is when an overall visualisation representation is used throughout the problem.*			
Illustrative tool *This is when a visual representation is used to illustrate the problem in a familiar way.*			
Simplification tool *This is when a visual representation is used to make the problem easier.*			
Organizational tool *This is when the visual representation is used to deconstruct the problem into smaller components.*			
As a starting point *This is when the visual representation is used as an initial starting point that is developed further.*			

For this question, L6 drew a rough sketch and stopped for some minutes. He then divided L6 by 4 to get 4, which he multiplied by two ($4 \times 2 = 8$). L6 indicated that he multiplied 4×2 because he has two lengths that are 16 (line 1140). When asked why he did not use the diagram that he had drawn, L6 explained, "*it is because this question is pretty clear*" (line 1145).

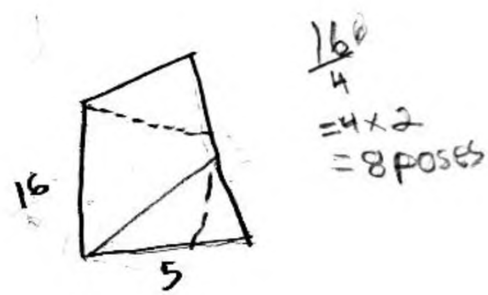


Figure 4.32 Picture of L6's response to Task 5

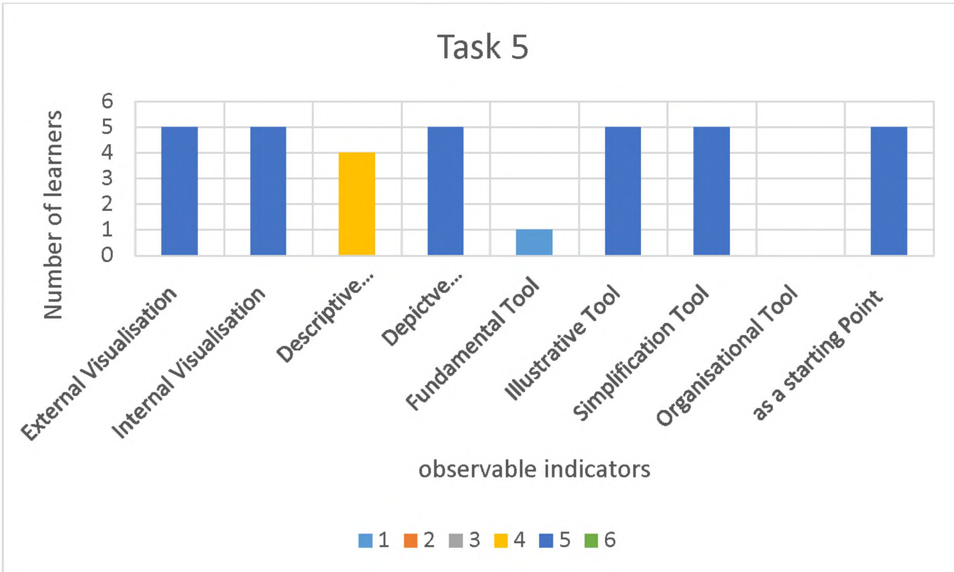


Figure 4.33 The type of visualisations used in Task 5

In Figure 4.33, five participants used external and internal visualisations. Descriptive visualisations were used by four participants while depictive visualisations were used by five. Further, five learners used visualisations to illustrate, to simplify and to organize given information. Only one participant used visualisation as a fundamental tool.

4.2.6 Task 6

Task 6
A moving company is hired to take 578 clay pots to a florist shop. The florist will pay the moving company a N$200 fee, plus N$1 for every pot that is delivered safely. The moving company must pay the florist N$4 each for any pots that are lost or broken. If two pots are lost, four pots are broken, and the rest are delivered safely, how much should the moving company be paid?

Learner 1

Observable indicator	coding		
	1	2	3
External visualisation *These are observable shapes such as diagrams, graphs and pictures. It also includes representations such as equations and words.*			
Internal visualisation *These are representations that are drawn in the mind and are imagined. These are only made explicit verbally and manifest themselves in conversation.*	√		
Descriptive representation *This is similar to the internal visualisation indictor above and refers to those representations that are spoken words or written texts such as mathematical equations and other expressions.*	√		
Depictive representation *These are iconic signs such as pictures, sketches, or drawn models and they are similar to the external indicators above.*			
Fundamental tool *This is when an overall visualisation representation is used throughout the problem.*			
Illustrative tool *This is when a visual representation is used to illustrate the problem in a familiar way.*			
Simplification tool *This is when a visual representation is used to make the problem easier.*			
Organizational tool *This is when the visual representation is used to deconstruct the problem into smaller components.*			
As a starting point *This is when the visual representation is used as an initial starting point that is developed further.*			

First L1 read the question several times and started noting down some information such as 2 (lost pots) x N$ 4 =N$8, 4 (broken pots) x N$4=16, 578-6=572 pots, 572+200=N$ 772. From the given information, 2 pots are lost and 4 pots are broken, and a broken or lost pot each costs N$ 4.Thus L1 multiplied 2 by 4 and 4 by 4 to get the amount that the company had to pay the florist. L1 then subtracted the lost and broken pots (6 pots in total) from the total number of pots. Since 572 pots delivered safely, L1 multiplied that by 1 and added the guaranteed 200 to give the total amount of N$772 (see Figure 4.34). During this process, L1 did not use any pictures, table or diagram. She only used a depictive representation to solve the problem.

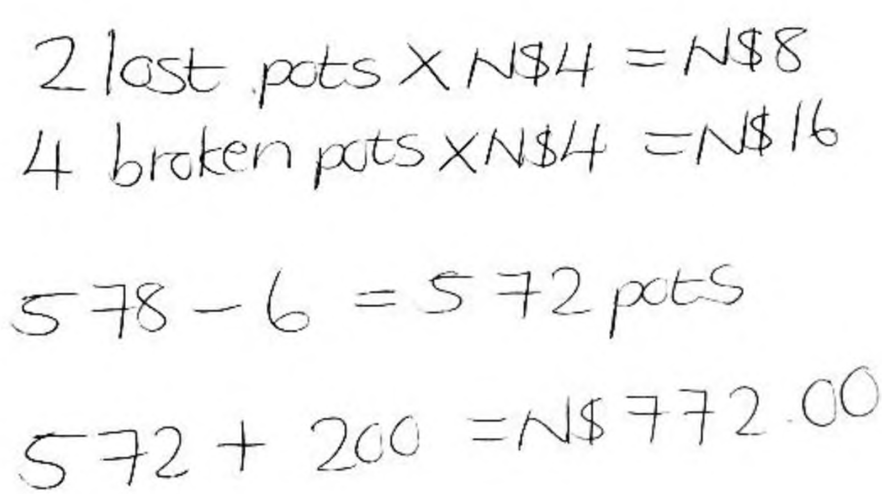


Figure 4.34. Picture of L1's response to Task 6

Learner 2

Observable indicator	coding		
	1	2	3
External visualisation *These are observable shapes such as diagrams, graphs and pictures. It also includes representations such as equations and words.*	√		
Internal visualisation *These are representations that are drawn in the mind and are imagined. These are only made explicit verbally and manifest themselves in conversation.*			
Descriptive representation *This is similar to the internal visualisation indictor above and refers to those representations that are spoken words or written texts such as mathematical equations and other expressions.*	√		
Depictive representation *These are iconic signs such as pictures, sketches, or drawn models and they are similar to the external indicators above.*			
Fundamental tool *This is when an overall visualisation representation is used throughout the problem.*	√		
Illustrative tool *This is when a visual representation is used to illustrate the problem in a familiar way.*			
Simplification tool *This is when a visual representation is used to make the problem easier.*			
Organizational tool *This is when the visual representation is used to deconstruct the problem into smaller components.*	√		
As a starting point *This is when the visual representation is used as an initial starting point that is developed further.*			

L2 wrote down information as she was reading the question such as 578 pots, N$ 200 was guaranteed. After reading the question, L2 decided to draw a table. According to L2, the table would help to organize the information. In the table the amount was grouped into different categories such as broken 4x4, lost 2x4, safe=572x1, guaranteed amount=200 and their income (the moving company) =748. Since information was clearly and nicely arranged, L2 simply added the safe to guaranteed amount and subtracted the amount for the broken and lost pots to get the amount paid to the moving company. So for L2 the table was used as a fundamental tool.

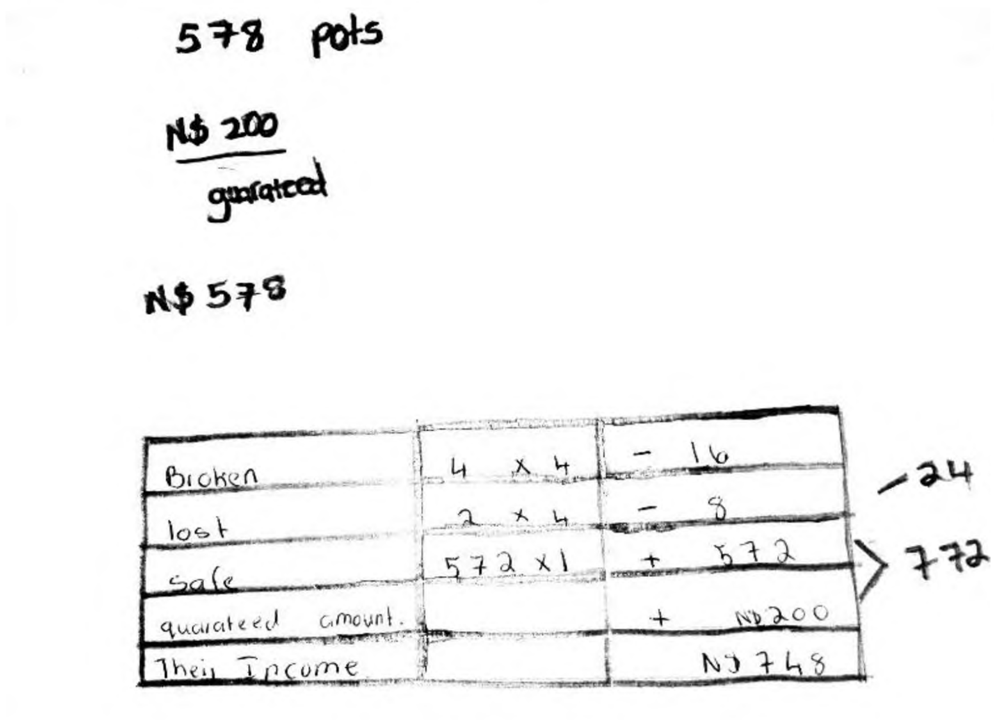


Figure 4.35. Picture of L2's response to Task 6

Learner 3

Observable indicator	coding		
	1	2	3
External visualisation *These are observable shapes such as diagrams, graphs and pictures. It also includes representations such as equations and words.*			
Internal visualisation *These are representations that are drawn in the mind and are imagined. These are only made explicit verbally and manifest themselves in conversation.*	√		
Descriptive representation *This is similar to the internal visualisation indictor above and refers to those representations that are spoken words or written texts such as mathematical equations and other expressions.*	√		
Depictive representation *These are iconic signs such as pictures, sketches, or drawn models and they are similar to the external indicators above.*			
Fundamental tool *This is when an overall visualisation representation is used throughout the problem.*			
Illustrative tool *This is when a visual representation is used to illustrate the problem in a familiar way.*			
Simplification tool *This is when a visual representation is used to make the problem easier.*			
Organizational tool *This is when the visual representation is used to deconstruct the problem into smaller components.*			
As a starting point *This is when the visual representation is used as an initial starting point that is developed further.*			

L3 decided not to use any depictive visualisation for this question. However, descriptive representations were used throughout. First L3 calculated the pots that were delivered safely 578 (total pots) -6 (lost and broken pots) = 572. Since each broken or lost pot cost N$4, L3 multiplied 6 by 4 to get 24 and indicated that 24 has to be subtracted from the amount the moving company had to be paid. Therefore the moving company had to be paid 572-24= N$548. L3 did not use any diagrams, sketches or tables to solve the answer.

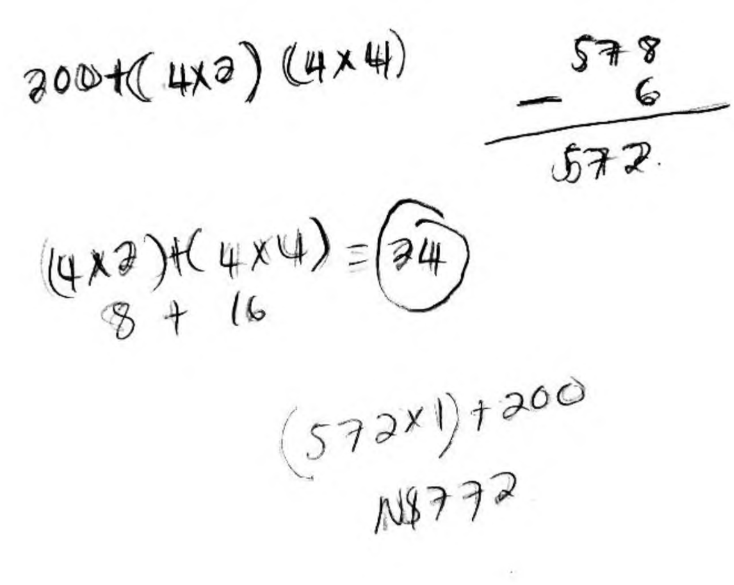


Figure 4.36. Picture of L3's response to Task 6

Learner 4

Observable indicator	coding		
	1	2	3
External visualisation *These are observable shapes such as diagrams, graphs and pictures. It also includes representations such as equations and words.*			
Internal visualisation *These are representations that are drawn in the mind and are imagined. These are only made explicit verbally and manifest themselves in conversation.*		√	
Descriptive representation *This is similar to the internal visualisation indictor above and refers to those representations that are spoken words or written texts such as mathematical equations and other expressions.*			√
Depictive representation *These are iconic signs such as pictures, sketches, or drawn models and they are similar to the external indicators above.*			
Fundamental tool *This is when an overall visualisation representation is used throughout the problem.*			
Illustrative tool *This is when a visual representation is used to illustrate the problem in a familiar way.*			
Simplification tool *This is when a visual representation is used to make the problem easier.*			
Organizational tool *This is when the visual representation is used to deconstruct the problem into smaller components.*			
As a starting point *This is when the visual representation is used as an initial starting point that is developed further.*			

L4 wrote down some information, such as the total price for the broken and lost pots, which is$(4 \times 4) + (2 \times 4) = 22$. "*From here I will just subtract the lost and broken pots from the total pots to get the pots that were delivered safely*", aid L4 (line 757). For the whole process, L4 used descriptive representations. "*That's the only way I can think of Mrs. Drawing a table or diagram? I don't know how I will draw it Mrs.*", confessed L4 (line 760).

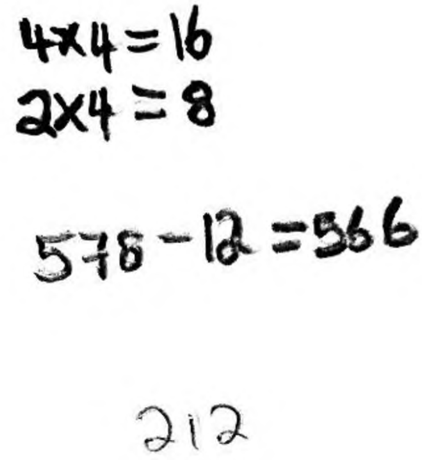


Figure 4.37 Picture of L4's response to Task 6

Learner 5

Observable indicator	coding		
	1	2	3
External visualisation *These are observable shapes such as diagrams, graphs and pictures. It also includes representations such as equations and words.*			
Internal visualisation *These are representations that are drawn in the mind and are imagined. These are only made explicit verbally and manifest themselves in conversation.*	√		
Descriptive representation *This is similar to the internal visualisation indictor above and refers to those representations that are spoken words or written texts such as mathematical equations and other expressions.*			
Depictive representation *These are iconic signs such as pictures, sketches, or drawn models and they are similar to the external indicators above.*	√		
Fundamental tool *This is when an overall visualisation representation is used throughout the problem.*			
Illustrative tool *This is when a visual representation is used to illustrate the problem in a familiar way.*			
Simplification tool *This is when a visual representation is used to make the problem easier.*			
Organizational tool *This is when the visual representation is used to deconstruct the problem into smaller components.*			
As a starting point *This is when the visual representation is used as an initial starting point that is developed further.*			

L5 used equations and expressions (descriptive representations) to solve this problem, first the total number of broken and lost pots were calculated and subtracted from the total number of pots 578-6= 572, since each pot that was delivered safely costs N$ 1 that means the cost is 572 dollars. Then L5 subtracted the amount for the lost and broken pots from 572 to get the solution. According to L5, "*Mrs.*

for this one I just figured it out and I don't need any diagrams since the information is clear" (line 1019)

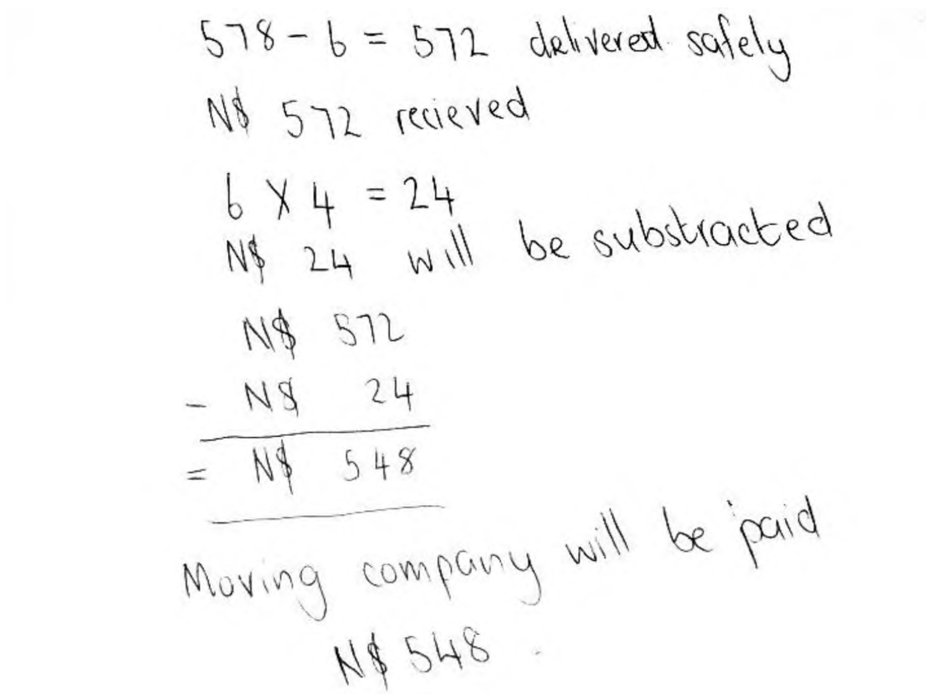


Figure 4.38 Picture of L5's response to Task 6

Learner 6

Observable indicator	**coding**		
	1	**2**	**3**
External visualisation *These are observable shapes such as diagrams, graphs and pictures. It also includes representations such as equations and words.*			
Internal visualisation *These are representations that are drawn in the mind and are imagined. These are only made explicit verbally and manifest themselves in conversation.*	√		
Descriptive representation *This is similar to the internal visualisation indictor above and refers to those representations that are spoken words or written texts such as mathematical equations and other expressions.*	√		
Depictive representation *These are iconic signs such as pictures, sketches, or drawn models and they are similar to the external indicators above.*			
Fundamental tool *This is when an overall visualisation representation is used throughout the problem.*			
Illustrative tool *This is when a visual representation is used to illustrate the problem in a familiar way.*			
Simplification tool *This is when a visual representation is used to make the problem easier.*			
Organizational tool *This is when the visual representation is used to deconstruct the problem into smaller components.*			
As a starting point *This is when the visual representation is used as an initial starting point that is developed further.*			

L6 read the question and he underlined what he considered were important points/information. The learner first subtracted the lost and broken pots from the total ($578 - 6 = 572.$ 572) is the number of pots that were delivered safely and the company was paid N$1 per pot. L6 then added the fixed amount

N$200 to 572 to get the final answer. According to L6, this question was also very clear and he did not need a sketch or diagram.

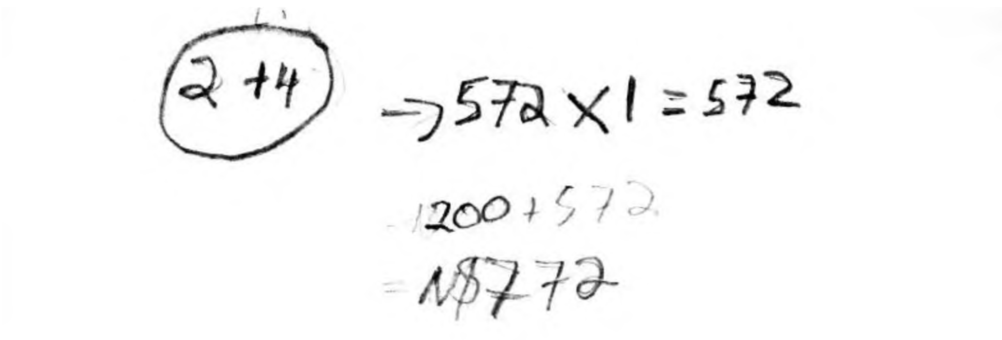


Figure 4.39. Picture of L6's response to Task 6

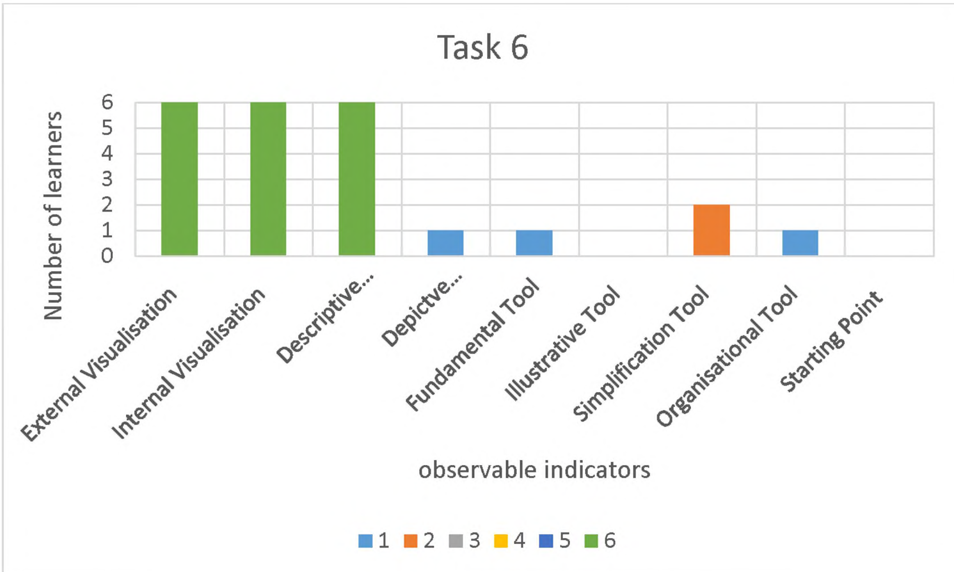


Figure 4.40 The types of visualisations used in Task 6

As illustrated in Figure 4.40, all six participants used external and internal visualisations. Six learners used descriptive visualisations and only two learners used depictive visualisations. Two participants used visualisations as fundamental tools and as organizational tools. Moreover, three participants used visualisations to simplify the task and none of the participants used visualisations as illustrative tools or as starting points.

4.2.7 Task 7

Task 7 A ladder of length 8m rests against a wall so that the angle between the ladder and the wall is 31°. How far is the base of the ladder from the wall?

Learner 1

Observable indicator	**coding**		
	1	**2**	**3**
External visualisation *These are observable shapes such as diagrams, graphs and pictures. It also includes representations such as equations and words.*	√		
Internal visualisation *These are representations that are drawn in the mind and are imagined. These are only made explicit verbally and manifest themselves in conversation.*	√		
Descriptive representation *This is similar to the internal visualisation indictor above and refers to those representations that are spoken words or written texts such as mathematical equations and other expressions.*	√		
Depictive representation *These are iconic signs such as pictures, sketches, or drawn models and they are similar to the external indicators above.*			
Fundamental tool *This is when an overall visualisation representation is used throughout the problem.*			
Illustrative tool *This is when a visual representation is used to illustrate the problem in a familiar way.*	√		
Simplification tool *This is when a visual representation is used to make the problem easier.*			
Organizational tool *This is when the visual representation is used to deconstruct the problem into smaller components.*			
As a starting point *This is when the visual representation is used as an initial starting point that is developed further.*	√		

L1 drew a right-angled triangle, and labelled the longest side 8m and the base of the triangle as x. Inside the triangle the learner named one of the angles 31, "*this is the angle between the ladder and the wall*", explained L1 (line 145). The side that was 8 m was named the Hypotenuse and the base that was x was named the Opposite. Then the learner came up with an equation $\sin\theta = \frac{opp}{hyp}$ and substituted the given information into that equation. $\text{Sin } 31 = \frac{8}{x}$, which was solved to get the final answer.

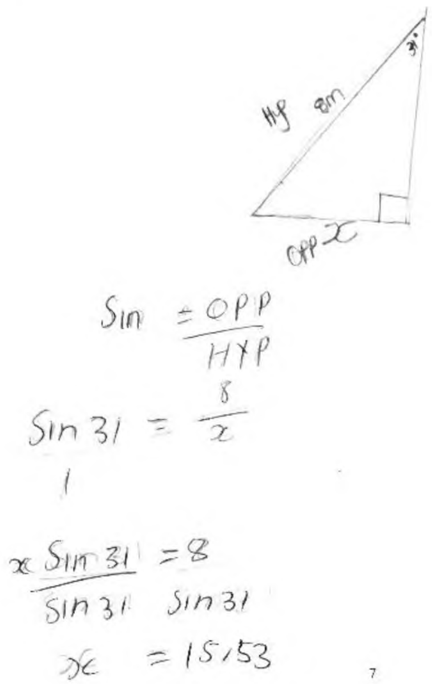


Figure 4.41. Picture of L1's response to Task 7

Learner 2

Observable indicator	coding		
	1	2	3
External visualisation *These are observable shapes such as diagrams, graphs and pictures. It also includes representations such as equations and words.*			√
Internal visualisation *These are representations that are drawn in the mind and are imagined. These are only made explicit verbally and manifest themselves in conversation.*			√
Descriptive representation *This is similar to the internal visualisation indictor above and refers to those representations that are spoken words or written texts such as mathematical equations and other expressions.*			
Depictive representation *These are iconic signs such as pictures, sketches, or drawn models and they are similar to the external indicators above.*			
Fundamental tool *This is when an overall visualisation representation is used throughout the problem.*			
Illustrative tool *This is when a visual representation is used to illustrate the problem in a familiar way.*			
Simplification tool *This is when a visual representation is used to make the problem easier.*			
Organizational tool *This is when the visual representation is used to deconstruct the problem into smaller components.*			
As a starting point *This is when the visual representation is used as an initial starting point that is developed further.*			

A ladder leaning on a wall was drawn. However, L2 stopped and claimed not to have any idea of what to do next. When asked to read the question again and try to figure it out L2 stated, "*this is a bit tricky Mrs. I don't know what to do next. Where to put an angle, even the distance that I should calculate*". (line 328)

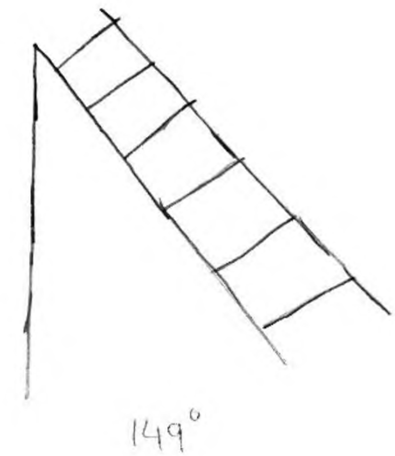


Figure 4.42 Picture of L2's response to Task 7

Learner 3

Observable indicator	coding		
	1	2	3
External visualisation *These are observable shapes such as diagrams, graphs and pictures. It also includes representations such as equations and words.*	√		
Internal visualisation *These are representations that are drawn in the mind and are imagined. These are only made explicit verbally and manifest themselves in conversation.*	√		
Descriptive representation *This is similar to the internal visualisation indictor above and refers to those representations that are spoken words or written texts such as mathematical equations and other expressions.*	√		
Depictive representation *These are iconic signs such as pictures, sketches, or drawn models and they are similar to the external indicators above.*			
Fundamental tool *This is when an overall visualisation representation is used throughout the problem.*			
Illustrative tool *This is when a visual representation is used to illustrate the problem in a familiar way.*	√		
Simplification tool *This is when a visual representation is used to make the problem easier.*			
Organizational tool *This is when the visual representation is used to deconstruct the problem into smaller components.*			
As a starting point *This is when the visual representation is used as an initial starting point that is developed further.*	√		

After reading the question, L3 drew a triangle and labelled the sides as the hypotenuse (8m), the adjacent and the opposite (x). According to L3, the triangle represents the wall and the ladder whereby the hypotenuse is the ladder, the adjacent is the wall and the opposite is the base of the ladder from the wall. From that information, L3 formed an equation $\sin 31 = \frac{x}{8}$ and played around with the question until the final answer was obtained. For this question, L3 used external visualisations (both depictive and descriptive).

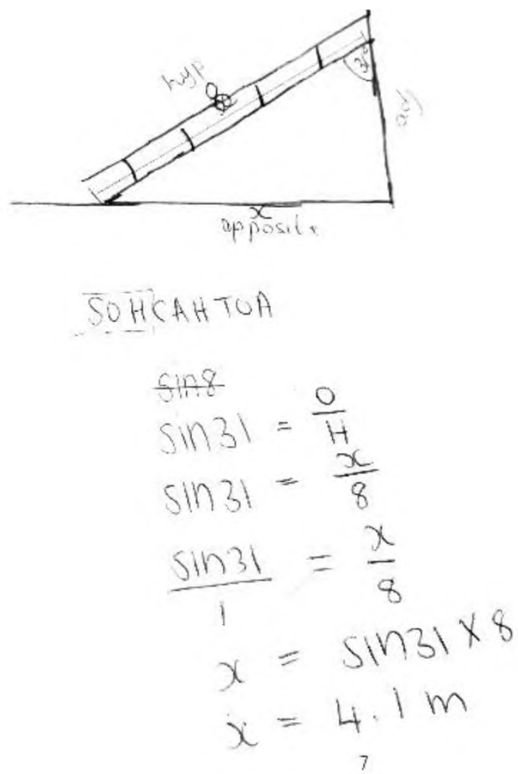


Figure 4.43 Picture of L3's response to Task 7

Learner 4

Observable indicator	coding		
	1	2	3
External visualisation *These are observable shapes such as diagrams, graphs and pictures. It also includes representations such as equations and words.*	√		
Internal visualisation *These are representations that are drawn in the mind and are imagined. These are only made explicit verbally and manifest themselves in conversation.*	√		
Descriptive representation *This is similar to the internal visualisation indictor above and refers to those representations that are spoken words or written texts such as mathematical equations and other expressions.*	√		
Depictive representation *These are iconic signs such as pictures, sketches, or drawn models and they are similar to the external indicators above.*			
Fundamental tool *This is when an overall visualisation representation is used throughout the problem.*			
Illustrative tool *This is when a visual representation is used to illustrate the problem in a familiar way.*	√		
Simplification tool *This is when a visual representation is used to make the problem easier.*	√		
Organizational tool *This is when the visual representation is used to deconstruct the problem into smaller components.*			
As a starting point *This is when the visual representation is used as an initial starting point that is developed further.*	√		

After reading the question, L4 indicated that a sketch was needed. "*As for this question, I think I need a sketch just to have a clear picture of what they are asking for and also to be able to identify the trigonometry ratio that I am going to use*", stated L4 (line 500). L4 first drew a straight vertical line, which according to L4 represented the wall. L4 then drew a slanting line (the ladder) and named it 8m. After that, the wall and the ladder were joined by a straight line, which was named x. The right-angled triangle was formed and L4 substituted the given information in the equation $\frac{sin31}{x} = \frac{\sin 90}{8}$. , which was then solved to get the final answer.

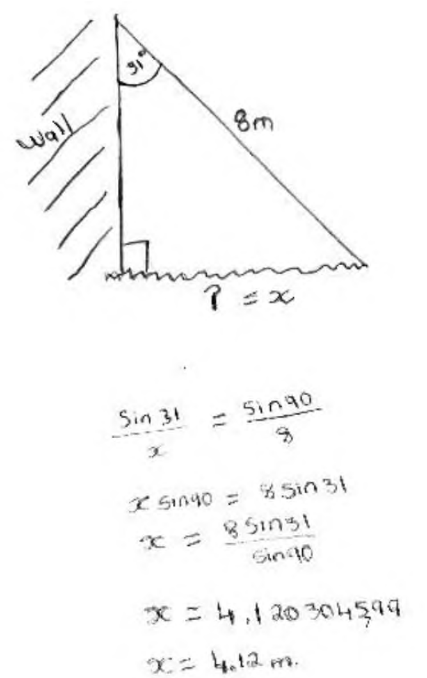


Figure 4.44 Picture of L4's response to Task 7

Learner 5

Observable indicator	coding		
	1	2	3
External visualisation *These are observable shapes such as diagrams, graphs and pictures. It also includes representations such as equations and words.*	√		
Internal visualisation *These are representations that are drawn in the mind and are imagined. These are only made explicit verbally and manifest themselves in conversation.*		√	
Descriptive representation *This is similar to the internal visualisation indictor above and refers to those representations that are spoken words or written texts such as mathematical equations and other expressions.*			
Depictive representation *These are iconic signs such as pictures, sketches, or drawn models and they are similar to the external indicators above.*			
Fundamental tool *This is when an overall visualisation representation is used throughout the problem.*			
Illustrative tool *This is when a visual representation is used to illustrate the problem in a familiar way.*	√		
Simplification tool *This is when a visual representation is used to make the problem easier.*			
Organizational tool *This is when the visual representation is used to deconstruct the problem into smaller components.*			
As a starting point *This is when the visual representation is used as an initial starting point that is developed further.*			

L5 read the question several times and drew a rectangle, which was clearly labelled as one side of 8 m that was the ladder and the other side was the wall. The angle was also indicated in the triangle. L5 then indicated that trigonometry was involved and she could not remember the formulae they learned in the class. “*Here it involves trigonometry but I cannot remember what we learned Mrs.*” confessed L5 (line 1030)

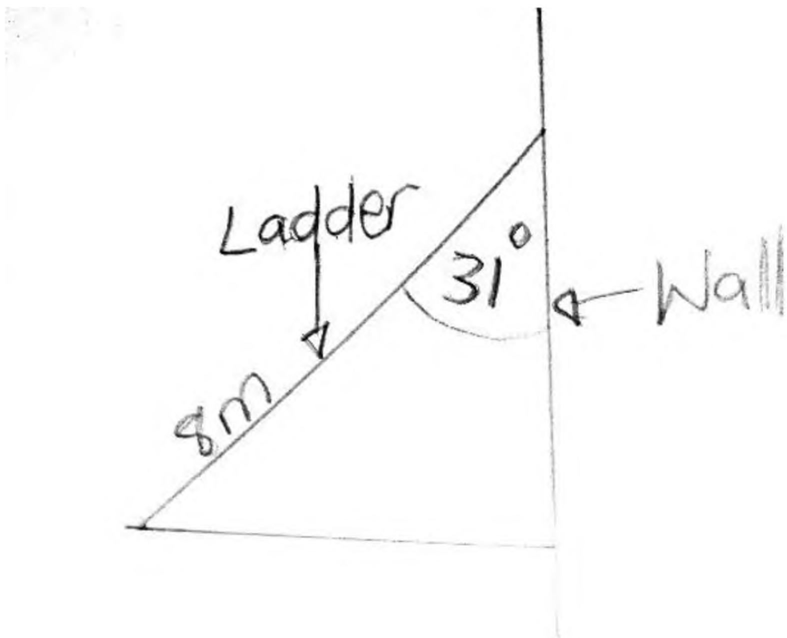


Figure 4.45. Picture of L5’s response to Task 7

Learner 6

Observable indicator	coding		
	1	2	3
External visualisation *These are observable shapes such as diagrams, graphs and pictures. It also includes representations such as equations and words.*	√		
Internal visualisation *These are representations that are drawn in the mind and are imagined. These are only made explicit verbally and manifest themselves in conversation.*			
Descriptive representation *This is similar to the internal visualisation indictor above and refers to those representations that are spoken words or written texts such as mathematical equations and other expressions.*	√		
Depictive representation *These are iconic signs such as pictures, sketches, or drawn models and they are similar to the external indicators above.*			
Fundamental tool *This is when an overall visualisation representation is used throughout the problem.*			
Illustrative tool *This is when a visual representation is used to illustrate the problem in a familiar way.*	√		
Simplification tool *This is when a visual representation is used to make the problem easier.*	√		
Organizational tool *This is when the visual representation is used to deconstruct the problem into smaller components.*			
As a starting point *This is when the visual representation is used as an initial starting point that is developed further.*	√		

Just by reading the first part of the question, L6 indicated that for this question a sketch would be needed. According to L6, "*a sketch will be needed because a ladder leaning on a wall that's already a triangle to be formed and there is an angle involved that means, I have to identify the given sides and the trig ratio that I am going to use such as SOH CAH TOA)*" *(*line 1154-1156). After drawing the triangle and nicely labelling it, L6 indicated that since the Hypotenuse is given and he was looking for the opposite then he was going to use the sin function. An equation was then formed: $\sin 31 = \frac{opposite}{8}$ and solved to get the answer.

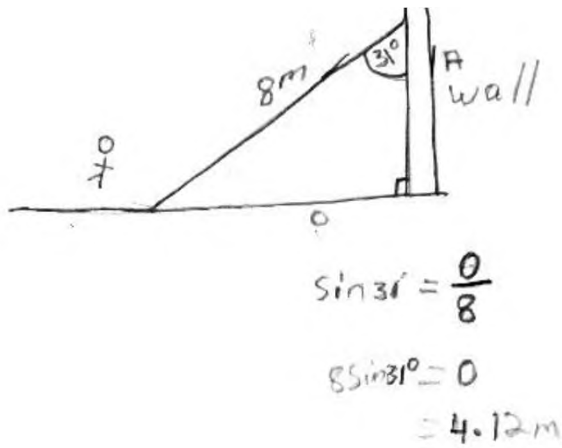


Figure 4.46. Picture of L6’s response to Task 7

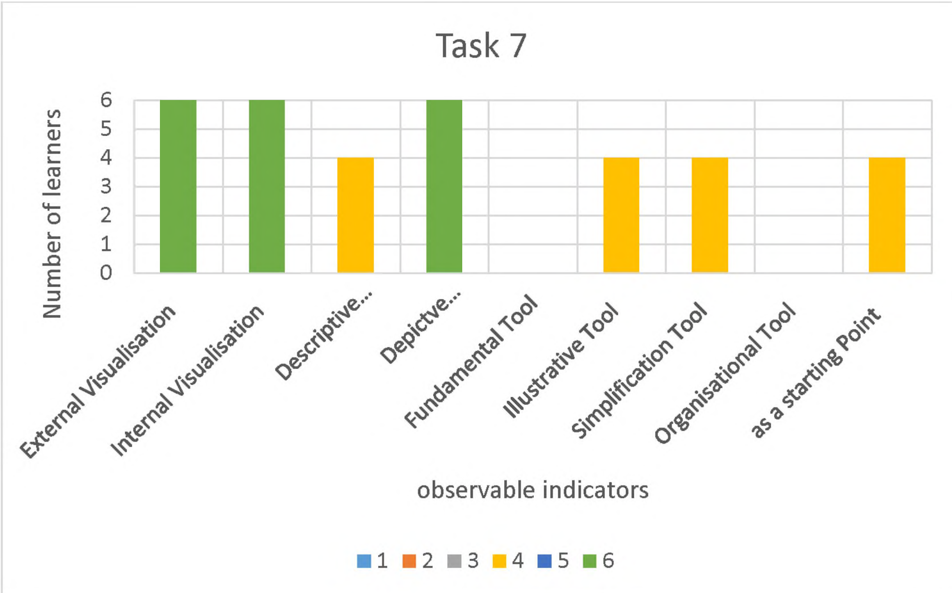


Figure 4.47 The types of visualisations used in Task 7

Figure 4.47 indicates that all six participants used external and internal visualisations. It further shows that four participants used descriptive visualisations while six used depictive visualisations. None of the participants used visualisations as a fundamental tool nor as an organizational tool. However, five participants used visualisations as illustrative tools, as simplification tools and as starting points.

4.2.8 Task 8

Task 8
f is a quadratic function whose graph has a vertex at the point (-3 , 2) and a y-intercept at the point (0 , -16). Find the x-intercepts of the graph of f.

Learner 1

Observable indicator	coding		
	1	2	3
External visualisation *These are observable shapes such as diagrams, graphs and pictures. It also includes representations such as equations and words.*			
Internal visualisation *These are representations that are drawn in the mind and are imagined. These are only made explicit verbally and manifest themselves in conversation.*			
Descriptive representation *This is similar to the internal visualisation indictor above and refers to those representations that are spoken words or written texts such as mathematical equations and other expressions.*			
Depictive representation *These are iconic signs such as pictures, sketches, or drawn models and they are similar to the external indicators above.*			
Fundamental tool *This is when an overall visualisation representation is used throughout the problem.*			
Illustrative tool *This is when a visual representation is used to illustrate the problem in a familiar way.*			
Simplification tool *This is when a visual representation is used to make the problem easier.*			
Organizational tool *This is when the visual representation is used to deconstruct the problem into smaller components.*			
As a starting point *This is when the visual representation is used as an initial starting point that is developed further.*			

For this task, L1 read the question several times and claimed not to have any idea of what to do. "*This is a graph Mrs. I know I have to use a graph to get the answer but how do I use it? I have no idea*", said L1 (line 201). When asked to draw just a sketch and see if she could come up with something, L1 responded, "*even if I draw a Cartesian plane Mrs. I don't think that will help because after drawing then what Mrs.*?" (line 203).

Learner 2

Observable indicator	**coding**		
	1	**2**	**3**
External visualisation *These are observable shapes such as diagrams, graphs and pictures. It also includes representations such as equations and words.*			√
Internal visualisation *These are representations that are drawn in the mind and are imagined. These are only made explicit verbally and manifest themselves in conversation.*		√	
Descriptive representation *This is similar to the internal visualisation indictor above and refers to those representations that are spoken words or written texts such as mathematical equations and other expressions.*			
Depictive representation *These are iconic signs such as pictures, sketches, or drawn models and they are similar to the external indicators above.*			
Fundamental tool *This is when an overall visualisation representation is used throughout the problem.*			
Illustrative tool *This is when a visual representation is used to illustrate the problem in a familiar way.*			
Simplification tool *This is when a visual representation is used to make the problem easier.*			
Organizational tool *This is when the visual representation is used to deconstruct the problem into smaller components.*			
As a starting point *This is when the visual representation is used as an initial starting point that is developed further.*			

L2 drew a sketch of a Cartesian plane and plotted the two points given. L2 stopped for some minutes and when asked what she was thinking, L2 said, "*I am trying to figure out what to do next. Should I connect the two points? If I connect the two points, it will just be a straight-line graph. Oh, there is a* *formula Mrs. but the formula doesn't wanna come Mrs. I am lost*", claimed L2 (lines 337-339). This task was therefore not completed.

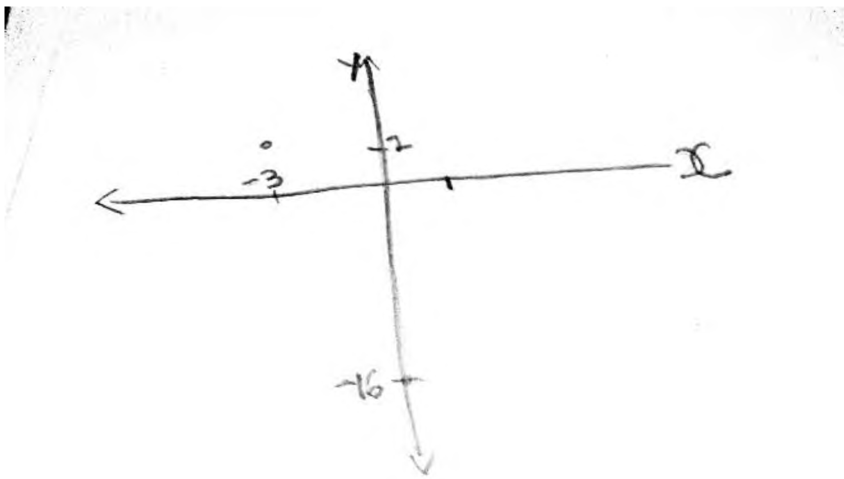


Figure 4.48. Picture of L2's response to Task 8

Learner 3

Observable indicator	coding		
	1	2	3
External visualisation *These are observable shapes such as diagrams, graphs and pictures. It also includes representations such as equations and words.*	√		
Internal visualisation *These are representations that are drawn in the mind and are imagined. These are only made explicit verbally and manifest themselves in conversation.*	√		
Descriptive representation *This is similar to the internal visualisation indictor above and refers to those representations that are spoken words or written texts such as mathematical equations and other expressions.*			
Depictive representation *These are iconic signs such as pictures, sketches, or drawn models and they are similar to the external indicators above.*			
Fundamental tool *This is when an overall visualisation representation is used throughout the problem.*	√		
Illustrative tool *This is when a visual representation is used to illustrate the problem in a familiar way.*	√		
Simplification tool *This is when a visual representation is used to make the problem easier.*	√		
Organizational tool *This is when the visual representation is used to deconstruct the problem into smaller components.*			
As a starting point *This is when the visual representation is used as an initial starting point that is developed further.*	√		

"*I think for this question a simple graph will be needed*", said L3 (line 560). L3 roughly sketched the quadratic function using the two given points. After studying the graph, L3 then wrote down the final answer without any calculations. When asked how L3 got the solution, L3 responded, "*I think the y-axis is the line of symmetry of the graph. Meaning if this point* (pointing at the point on the graph) *is -3; 2 then the other one is 3; 2. So my points of interception are -3 and 3 on the x-axis* ", explained L3 (line 571).

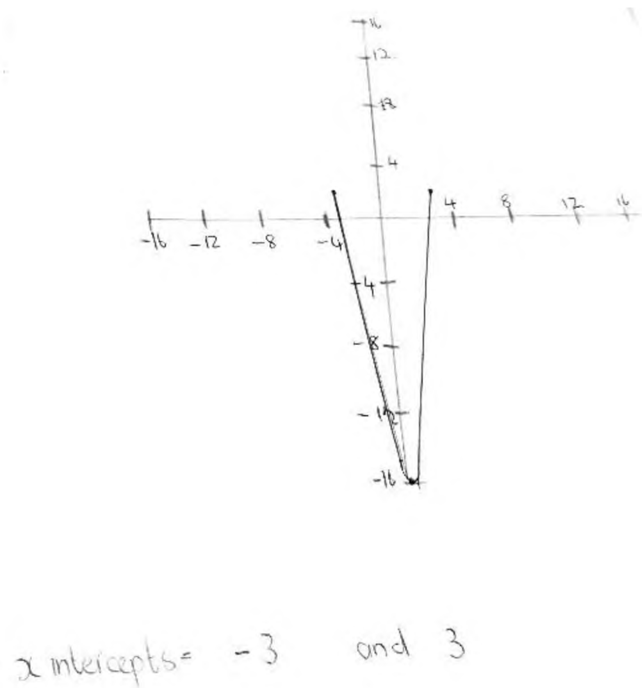


Figure 4.49. Picture of L3's response to Task 8

Learner 4

Observable indicator	coding		
	1	2	3
External visualisation *These are observable shapes such as diagrams, graphs and pictures. It also includes representations such as equations and words.*			√
Internal visualisation *These are representations that are drawn in the mind and are imagined. These are only made explicit verbally and manifest themselves in conversation.*			√
Descriptive representation *This is similar to the internal visualisation indictor above and refers to those representations that are spoken words or written texts such as mathematical equations and other expressions.*			
Depictive representation *These are iconic signs such as pictures, sketches, or drawn models and they are similar to the external indicators above.*			
Fundamental tool *This is when an overall visualisation representation is used throughout the problem.*			
Illustrative tool *This is when a visual representation is used to illustrate the problem in a familiar way.*			
Simplification tool *This is when a visual representation is used to make the problem easier.*			
Organizational tool *This is when the visual representation is used to deconstruct the problem into smaller components.*			
As a starting point *This is when the visual representation is used as an initial starting point that is developed further.*			

For this task, L4 drew a Cartesian plane. "*I think if I draw a Cartesian plane I will somehow get a picture*", claimed L4 (line 811). After the graph was drawn and labelled the learner claimed to have forgotten a formula that should be used to find the x intersection. "*I know some calculations have to be done but I forgot Mrs. I just don't know how*", *stated* L4 (line 814).

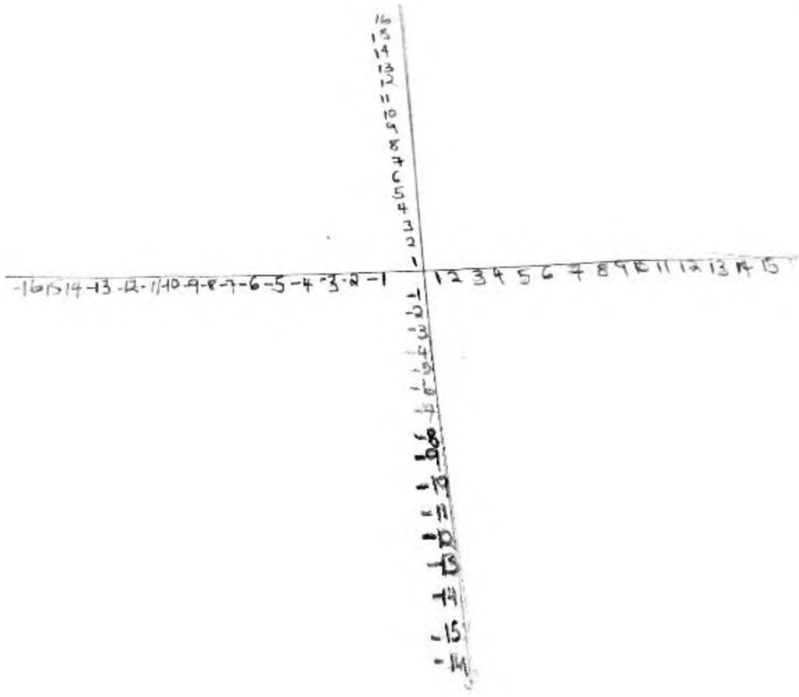


Figure 4.50. Picture of L4's response to Task 8

Learner 5

Observable indicator	coding		
	1	2	3
External visualisation *These are observable shapes such as diagrams, graphs and pictures. It also includes representations such as equations and words.*			
Internal visualisation *These are representations that are drawn in the mind and are imagined. These are only made explicit verbally and manifest themselves in conversation.*			
Descriptive representation *This is similar to the internal visualisation indictor above and refers to those representations that are spoken words or written texts such as mathematical equations and other expressions.*			
Depictive representation *These are iconic signs such as pictures, sketches, or drawn models and they are similar to the external indicators above.*			
Fundamental tool *This is when an overall visualisation representation is used throughout the problem.*			
Illustrative tool *This is when a visual representation is used to illustrate the problem in a familiar way.*			
Simplification tool *This is when a visual representation is used to make the problem easier.*			
Organizational tool *This is when the visual representation is used to deconstruct the problem into smaller components.*			
As a starting point *This is when the visual representation is used as an initial starting point that is developed further.*			

L5 read the question so many times and then indicated that if it is a quadratic function she had no idea. L5 was asked to just draw a rough sketch but L5 stated, "*I don't know where to start, a straight line graph Mrs. is better and easier than a quadratic. I just have no idea*". (line 2020).

Learner 6

Observable indicator	**coding**		
	1	**2**	**3**
External visualisation *These are observable shapes such as diagrams, graphs and pictures. It also includes representations such as equations and words.*	√		
Internal visualisation *These are representations that are drawn in the mind and are imagined. These are only made explicit verbally and manifest themselves in conversation.*			
Descriptive representation *This is similar to the internal visualisation indictor above and refers to those representations that are spoken words or written texts such as mathematical equations and other expressions.*			
Depictive representation *These are iconic signs such as pictures, sketches, or drawn models and they are similar to the external indicators above.*	√		
Fundamental tool *This is when an overall visualisation representation is used throughout the problem.*	√		
Illustrative tool *This is when a visual representation is used to illustrate the problem in a familiar way.*	√		
Simplification tool *This is when a visual representation is used to make the problem easier.*			
Organizational tool *This is when the visual representation is used to deconstruct the problem into smaller components.*			
As a starting point *This is when the visual representation is used as an initial starting point that is developed further.*	√		

Reading the question, L6 noticed that he needed to sketch the graph just to see what the graph looked like. L6 indicated also that since the graph has a vertex at the point (-3, 2) then "*that vertex can be a minimum or maximum turning point of the graph, that means I need to draw a sketch of the graph to see how it looks*" (lines 1170-1171). A Cartesian plane was drawn and the two given points were plotted. After plotting the points, L6 pointed to the graph and said, "*you can see this is a negative quadratic equation since it is going down. This part is half of the graph. Meaning it's not complete. Let me complete it and find the points of intersection*" (lines 1191-1193). After he had completed the graph, L6 got the x-intercepts of the graph of **f** by taking the readings where the graph and the x-axis intersect. L6 did not use any equations or expressions to solve this problem; he only used the graph.

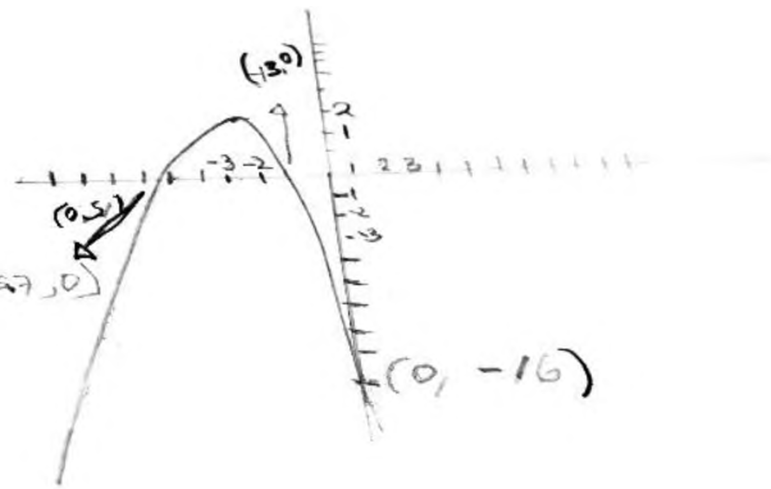


Figure 4.51. Picture of L6's response to Task 8

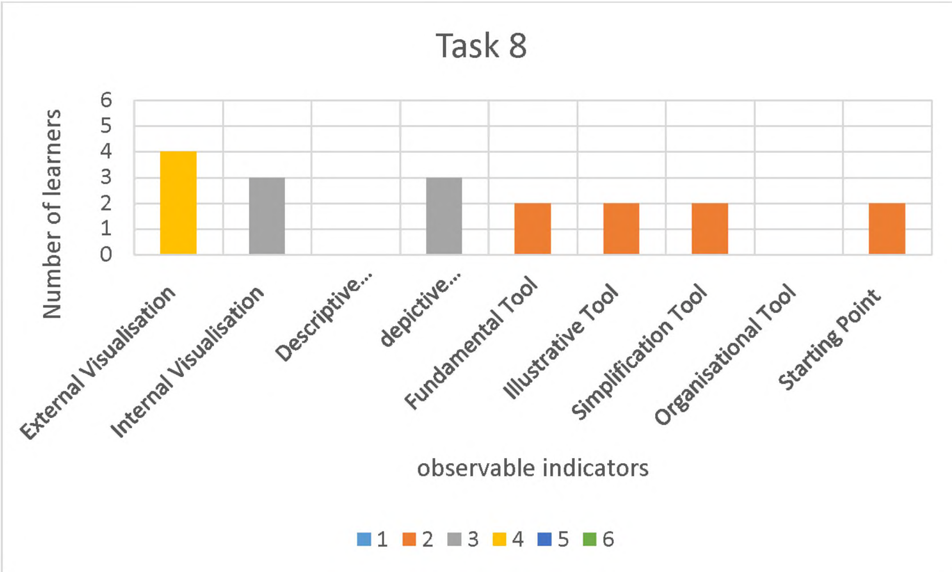


Figure 4.52 The types of visualisations used in Task 8

Fig 4.52 shows that four participants used external visualisations and three used internal visualisations. No descriptive visualisations were used and depictive visualisations were used by three learners. Four learners used visualisations as fundamental tools, illustrative tools, and simplification tools, as well as starting points. However, no visualisation was used as an organizational tool.

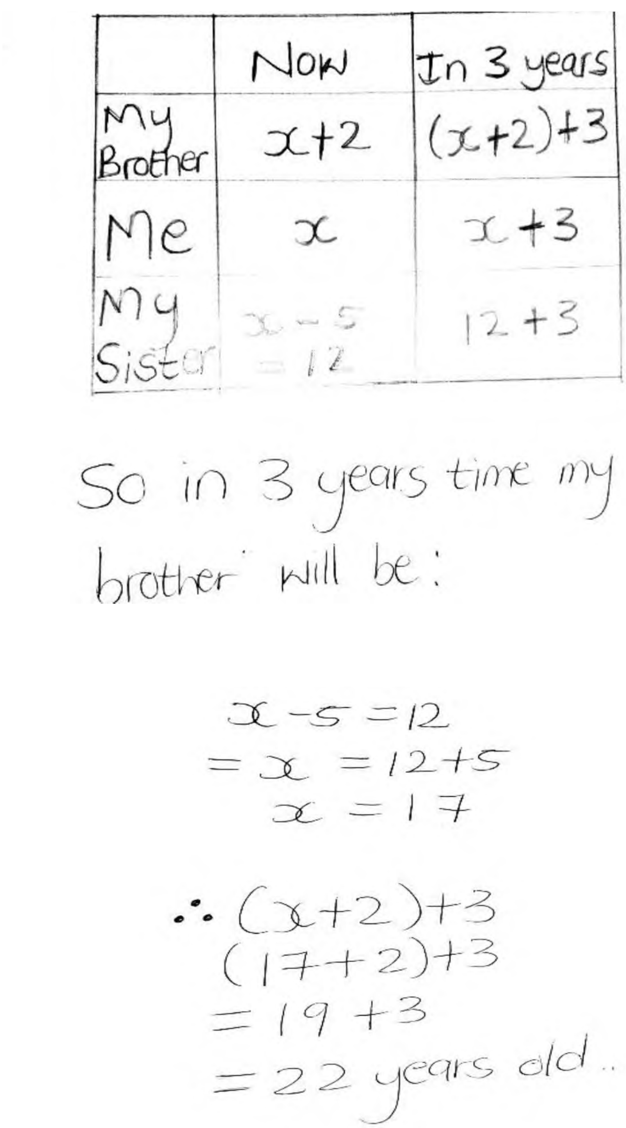
4.2.9 Task 9

Task 9 My brother is two years older than me, my sister is five years younger than me; she is 12, how old will my brother be in three years' time?

Learner 1

Observable indicator	coding		
	1	2	3
External visualisation *These are observable shapes such as diagrams, graphs and pictures. It also includes representations such as equations and words.*	√		
Internal visualisation *These are representations that are drawn in the mind and are imagined. These are only made explicit verbally and manifest themselves in conversation.*			
Descriptive representation *This is similar to the internal visualisation indictor above and refers to those representations that are spoken words or written texts such as mathematical equations and other expressions.*	√		
Depictive representation *These are iconic signs such as pictures, sketches, or drawn models and they are similar to the external indicators above.*			
Fundamental tool *This is when an overall visualisation representation is used throughout the problem.*			
Illustrative tool *This is when a visual representation is used to illustrate the problem in a familiar way.*			
Simplification tool *This is when a visual representation is used to make the problem easier.*	√		
Organizational tool *This is when the visual representation is used to deconstruct the problem into smaller components.*	√		
As a starting point *This is when the visual representation is used as an initial starting point that is developed further.*	√		

L1 started by drawing a table with 3 columns and 4 rows. The first row was named my brother, second was named me, and the last one my sister. For the columns, one was for their current ages and the other for their ages in three years' time. All information given was then filled in the table in algebraic form. L1 then constructed an equation using the sister's age, which is known $x - 5 = 12$. According to L1, x represents my current age. This equation was then solved to find x. When L1 got the value of x, an expression was formed$(x + 2) + 3$; the value of x was substituted in the expression to get the ages of the brother after three years.



Figures 4.53 and 4.54 Pictures of L1's response to Task 9

Learner 2

Observable indicator	coding		
	1	2	3
External visualisation *These are observable shapes such as diagrams, graphs and pictures. It also includes representations such as equations and words.*		√	
Internal visualisation *These are representations that are drawn in the mind and are imagined. These are only made explicit verbally and manifest themselves in conversation.*			
Descriptive representation *This is similar to the internal visualisation indictor above and refers to those representations that are spoken words or written texts such as mathematical equations and other expressions.*	√		
Depictive representation *These are iconic signs such as pictures, sketches, or drawn models and they are similar to the external indicators above.*			
Fundamental tool *This is when an overall visualisation representation is used throughout the problem.*			
Illustrative tool *This is when a visual representation is used to illustrate the problem in a familiar way.*			
Simplification tool *This is when a visual representation is used to make the problem easier.*			
Organizational tool *This is when the visual representation is used to deconstruct the problem into smaller components.*	√		
As a starting point *This is when the visual representation is used as an initial starting point that is developed further.*			

L2 drew a table but it was a bit different from L1's table. L2's table consisted of two columns and three rows. L2 used only the current ages for the brother, the sister and me. After the information was filled in the table, L2 calculated the current age for me, which is 12 +5=17. Since the brother is 2 years older than me, L2 formed an equation $x + 2 = 17$ and solved for x. When L2 obtained the value of x, 3 was then added to get the current age of the brother. Therefore, in the whole process L2 used visual representations such as equations, expressions (descriptive) and the table (which is depictive).

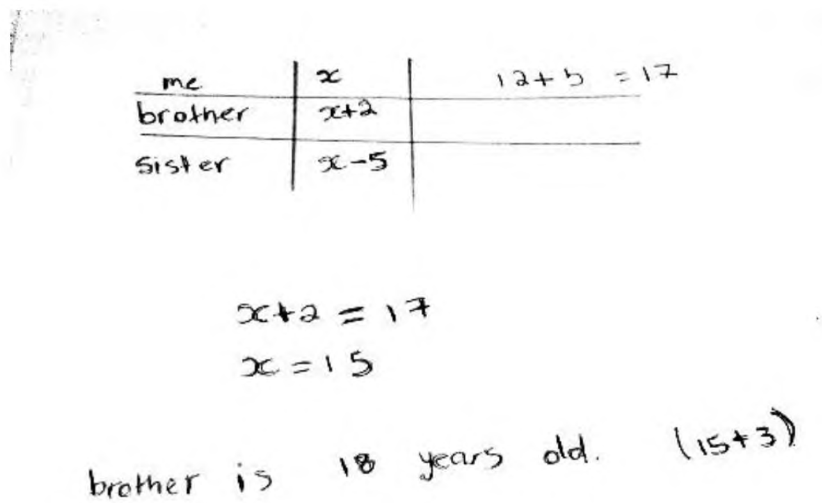


Figure 4.55 Picture of L2's response to Task 9

Learner 3

Observable indicator	**coding**		
	1	**2**	**3**
External visualisation *These are observable shapes such as diagrams, graphs and pictures. It also includes representations such as equations and words.*	√		
Internal visualisation *These are representations that are drawn in the mind and are imagined. These are only made explicit verbally and manifest themselves in conversation.*	√		
Descriptive representation *This is similar to the internal visualisation indictor above and refers to those representations that are spoken words or written texts such as mathematical equations and other expressions.*			
Depictive representation *These are iconic signs such as pictures, sketches, or drawn models and they are similar to the external indicators above.*	√		
Fundamental tool *This is when an overall visualisation representation is used throughout the problem.*			
Illustrative tool *This is when a visual representation is used to illustrate the problem in a familiar way.*			
Simplification tool *This is when a visual representation is used to make the problem easier.*	√		
Organizational tool *This is when the visual representation is used to deconstruct the problem into smaller components.*			
As a starting point *This is when the visual representation is used as an initial starting point that is developed further.*			

L3 started writing down information in a table such as me=x, brother = $x + 2$ and sister = $x - 5$ (but my sister is currently 12 years). L3 then formed an equation $x - 5 = 12$, and solved it to get the current age of (me). The table was drawn when all those calculations were done. L3's table is also different from the tables of the first two learners; the content and the way their ages were written in algebraic form were also different. The brother's age after 3 years was written as $x + 2 + 3$, where by x=17. L3 substituted the value of x and simplified the expression to get the answer.

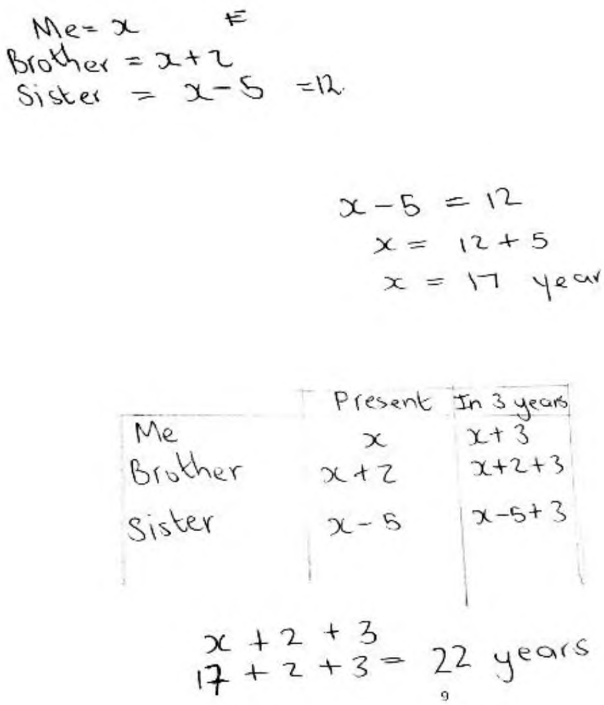


Figure 4.56 Picture of L3's response to Task 9

Learner 4

Observable indicator	coding		
	1	2	3
External visualisation *These are observable shapes such as diagrams, graphs and pictures. It also includes representations such as equations and words.*			√
Internal visualisation *These are representations that are drawn in the mind and are imagined. These are only made explicit verbally and manifest themselves in conversation.*			√
Descriptive representation *This is similar to the internal visualisation indictor above and refers to those representations that are spoken words or written texts such as mathematical equations and other expressions.*			
Depictive representation *These are iconic signs such as pictures, sketches, or drawn models and they are similar to the external indicators above.*			
Fundamental tool *This is when an overall visualisation representation is used throughout the problem.*			
Illustrative tool *This is when a visual representation is used to illustrate the problem in a familiar way.*			
Simplification tool *This is when a visual representation is used to make the problem easier.*			
Organizational tool *This is when the visual representation is used to deconstruct the problem into smaller components.*			
As a starting point *This is when the visual representation is used as an initial starting point that is developed further.*			

Like the previous learners, L4 did not draw a table. She read the question and tried to come up with algebraic expressions such as $2 + x$ and $x - 5$. Later on L4 confessed that she was not familiar with such questions and she could not think of a picture or diagram to assist in getting the answer (line 827).

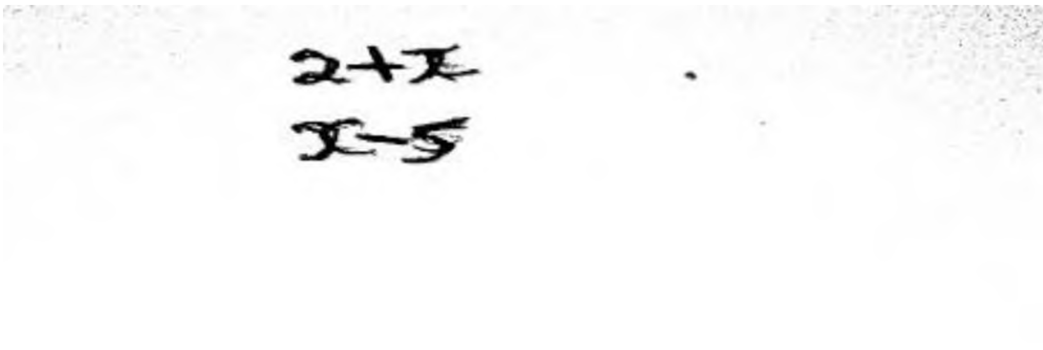


Figure 4.57 Picture of L4's response to Task 9

Learner 5

Observable indicator	**coding**		
	1	**2**	**3**
External visualisation *These are observable shapes such as diagrams, graphs and pictures. It also includes representations such as equations and words.*			
Internal visualisation *These are representations that are drawn in the mind and are imagined. These are only made explicit verbally and manifest themselves in conversation.*			
Descriptive representation *This is similar to the internal visualisation indictor above and refers to those representations that are spoken words or written texts such as mathematical equations and other expressions.*	√		
Depictive representation *These are iconic signs such as pictures, sketches, or drawn models and they are similar to the external indicators above.*			
Fundamental tool *This is when an overall visualisation representation is used throughout the problem.*			
Illustrative tool *This is when a visual representation is used to illustrate the problem in a familiar way.*			
Simplification tool *This is when a visual representation is used to make the problem easier.*			
Organizational tool *This is when the visual representation is used to deconstruct the problem into smaller components.*			
As a starting point *This is when the visual representation is used as an initial starting point that is developed further.*			

No table or diagram was drawn as L5 preferred not to use one. L5 stated, "*here I really don't need a diagram. I can get the answer by generating equations from the given information.*" (line 1023). .Then she started writing down the given information. .L5 used x for my age, and then $x + 2$ for the brother's age, since the sister is five years younger an equation was constructed $x - 5 = 12$. "*To get my* *brother's age I have to find my age first using the sister's current age*", said L5 (line 1030). The equation $x - 5 = 12$ was solved to find the value of x. The value of x was then used to get the age of the brother.

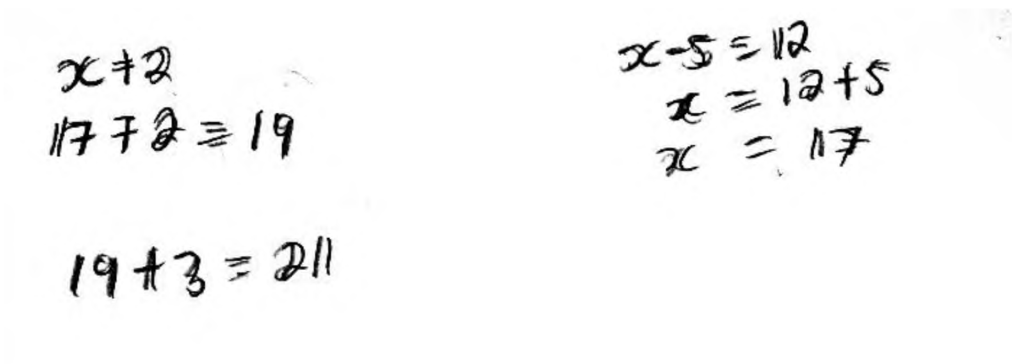


Figure 4.58 Picture of L5's response to Task 9

Learner 6

Observable indicator	**coding**		
	1	**2**	**3**
External visualisation *These are observable shapes such as diagrams, graphs and pictures. It also includes representations such as equations and words.*	√		
Internal visualisation *These are representations that are drawn in the mind and are imagined. These are only made explicit verbally and manifest themselves in conversation.*	√		
Descriptive representation *This is similar to the internal visualisation indictor above and refers to those representations that are spoken words or written texts such as mathematical equations and other expressions.*	√		
Depictive representation *These are iconic signs such as pictures, sketches, or drawn models and they are similar to the external indicators above.*			
Fundamental tool *This is when an overall visualisation representation is used throughout the problem.*			
Illustrative tool *This is when a visual representation is used to illustrate the problem in a familiar way.*			
Simplification tool *This is when a visual representation is used to make the problem easier.*			
Organizational tool *This is when the visual representation is used to deconstruct the problem into smaller components.*	√		
As a starting point *This is when the visual representation is used as an initial starting point that is developed further.*	√		

"*Here I am thinking of a diagram to show their age now and their age in 3 years' time*" explained L6 (line 1198). According to L6 the table would help him organize the information "otherwise I will be writing xs and other things all over the paper which is a waste of time" (line 1200). L6 drew the table and present the ages of the brother, and sister in algebraic form. L3 indicated that x represents the age of the person who's speaking (me), brother's age is $x + 2$ and the sister is $x - 5$. However the sister is also 12, that means $x - 5 = 12$. The ages of "me" was then calculated using $x - 5 = 12$. When he got the value of x it was substituted into the expression for my brother's age in three years to get his age in three years time.

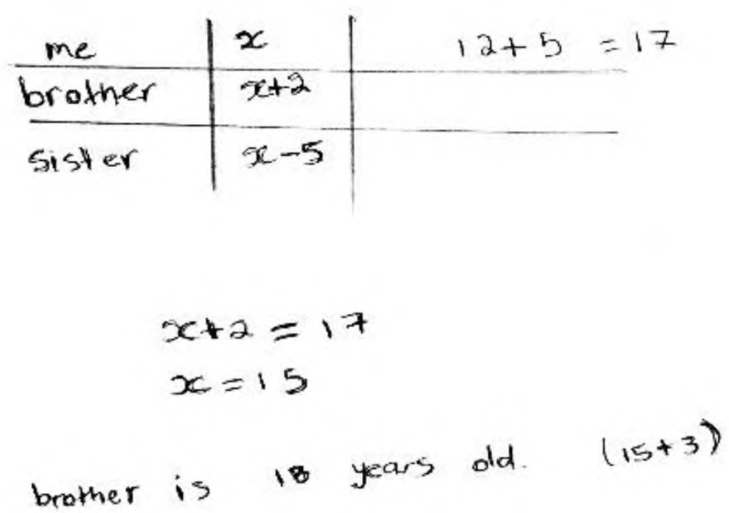


Figure 4.59 Picture of L6's response to Task 9

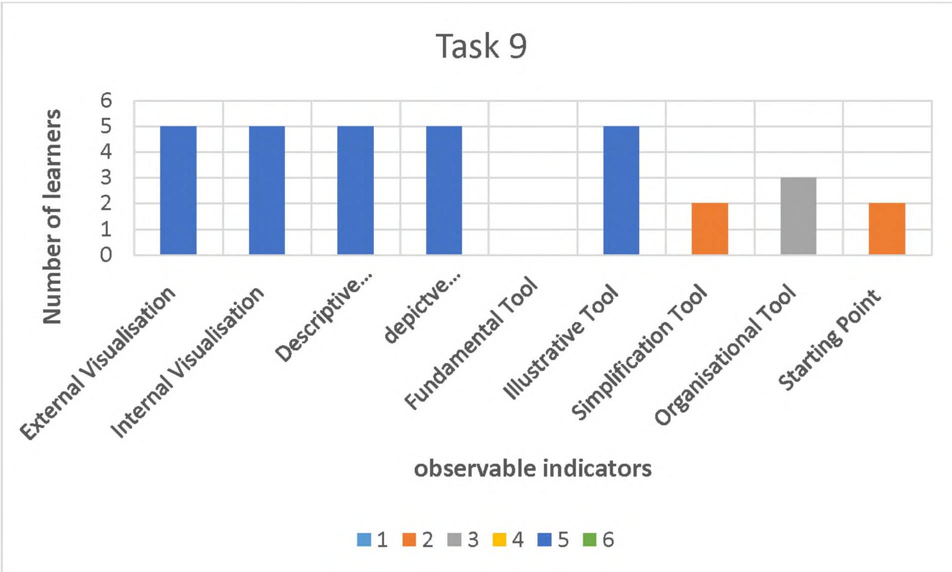


Figure 4.60 The types of visualisations used in task 9

Figure 4.60 indicates that external and internal visualisations were used by five learners. It also illustrates that five learners used descriptive and depictive visualisations. Additionally, five visualisations were used to illustrate, two to simplify a task and as a starting point and three used visualisations to organize data.

4.2.10 Task 10

Task 10 A shop owner has room in her shop for up to 20 television sets. She can buy either type **A** for **N$ 150** each or type **B** for **N$ 300** each. She has a total of **N$ 4500** she can spend and she must have at least 6 of each type in stock. She makes a profit of **N$80** on each television of type **A** and a profit of **N$ 100** on each of type **B.** How many of each type should she buy so that she makes a maximum profit?

Learner 1

Observable indicator	**coding**		
	1	**2**	**3**
External visualisation *These are observable shapes such as diagrams, graphs and pictures. It also includes representations such as equations and words.*			√
Internal visualisation *These are representations that are drawn in the mind and are imagined. These are only made explicit verbally and manifest themselves in conversation.*			√
Descriptive representation *This is similar to the internal visualisation indictor above and refers to those representations that are spoken words or written texts such as mathematical equations and other expressions.*			
Depictive representation *These are iconic signs such as pictures, sketches, or drawn models and they are similar to the external indicators above.*			
Fundamental tool *This is when an overall visualisation representation is used throughout the problem.*			
Illustrative tool *This is when a visual representation is used to illustrate the problem in a familiar way.*			
Simplification tool *This is when a visual representation is used to make the problem easier.*			
Organizational tool *This is when the visual representation is used to deconstruct the problem into smaller components.*			
As a starting point *This is when the visual representation is used as an initial starting point that is developed further.*			

L1 read the question several times. L1 indicated, "*I am trying to understand the question and figure out how I am going to tackle this question*" (Line 211). However, L1 could not construct any equations nor draw a table or a diagram. "*I am stuck Mrs. and there is just too much information I cannot figure out any best method to solve this*". (line 217). When asked to draw a table L1 had no idea how nor what information to write in the table.

Learner 2

Observable indicator	coding		
	1	2	3
External visualisation *These are observable shapes such as diagrams, graphs and pictures. It also includes representations such as equations and words.*			
Internal visualisation *These are representations that are drawn in the mind and are imagined. These are only made explicit verbally and manifest themselves in conversation.*			
Descriptive representation *This is similar to the internal visualisation indictor above and refers to those representations that are spoken words or written texts such as mathematical equations and other expressions.*			
Depictive representation *These are iconic signs such as pictures, sketches, or drawn models and they are similar to the external indicators above.*			
Fundamental tool *This is when an overall visualisation representation is used throughout the problem.*			
Illustrative tool *This is when a visual representation is used to illustrate the problem in a familiar way.*			
Simplification tool *This is when a visual representation is used to make the problem easier.*			
Organizational tool *This is when the visual representation is used to deconstruct the problem into smaller components.*			
As a starting point *This is when the visual representation is used as an initial starting point that is developed further.*			

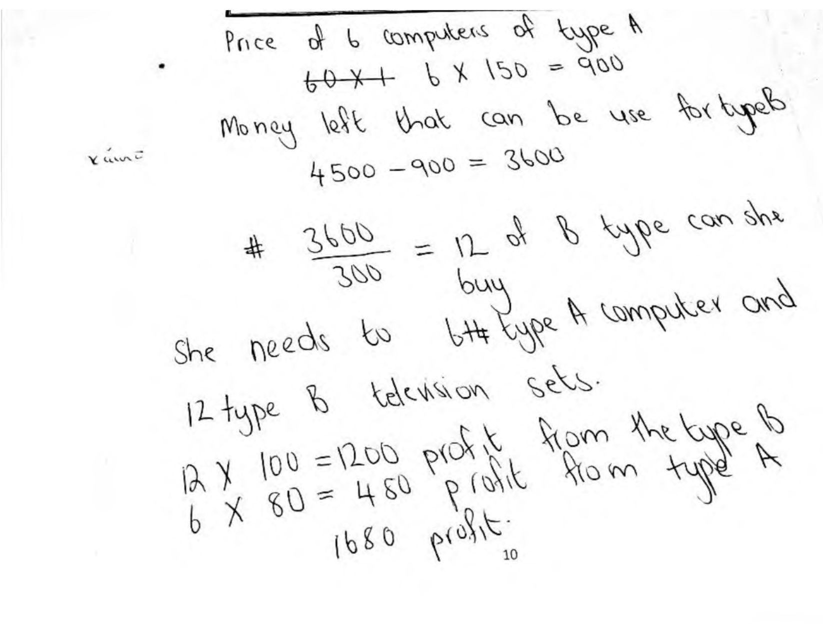
L2 was the same as L1. The question was read several times but she had no idea of what to do. According to L2, the question was not for her level. "*I am lost Mrs. totally lost*", stated L2 (line 373).

Learner 3

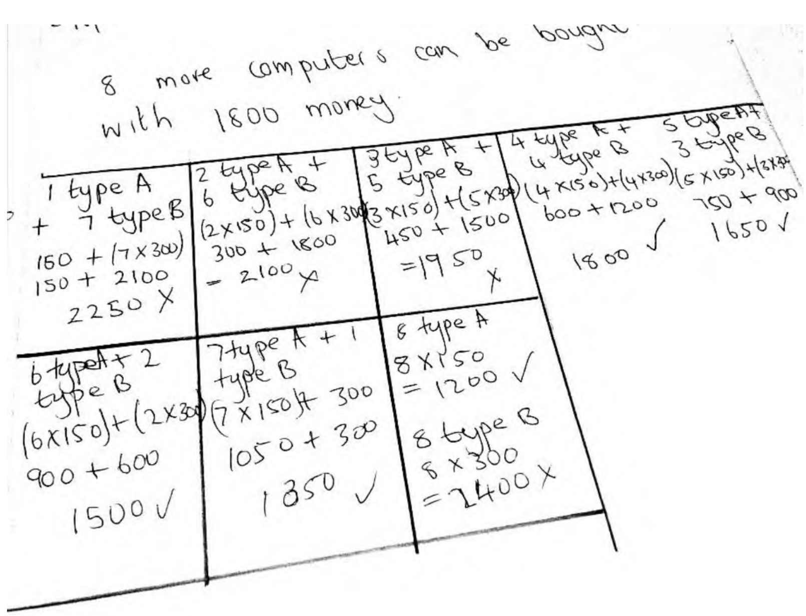
Observable indicator	coding		
	1	2	3
External visualisation *These are observable shapes such as diagrams, graphs and pictures. It also includes representations such as equations and words.*	√		
Internal visualisation *These are representations that are drawn in the mind and are imagined. These are only made explicit verbally and manifest themselves in conversation.*	√		
Descriptive representation *This is similar to the internal visualisation indictor above and refers to those representations that are spoken words or written texts such as mathematical equations and other expressions.*	√		
Depictive representation *These are iconic signs such as pictures, sketches, or drawn models and they are similar to the external indicators above.*	√		
Fundamental tool *This is when an overall visualisation representation is used throughout the problem.*			
Illustrative tool *This is when a visual representation is used to illustrate the problem in a familiar way.*			
Simplification tool *This is when a visual representation is used to make the problem easier.*	√		
Organizational tool *This is when the visual representation is used to deconstruct the problem into smaller components.*	√		
As a starting point *This is when the visual representation is used as an initial starting point that is developed further.*			

L3 used both descriptive and depictive representations to answer this question (external representations). L3 started by writing descriptive representations such as equations and expressions. Since the instruction says the shop owner must have at least 6 of each type in the stock, L3 subtracted the amount for 6 of both types from the total amount that the shop owner had [4500-(6x150)-(6x 300).

Then he found the remaining amount that the shop owner still had to use in the rest of the TVs, which is 1800. The learner also noted that the shop owner only had space for 8 TVs. However, L3 decided to use a table since it was a long process. L3 then organized the information in a table where the information was taken from the table to form equations. The answers from different equations were then compared to get the final answer.



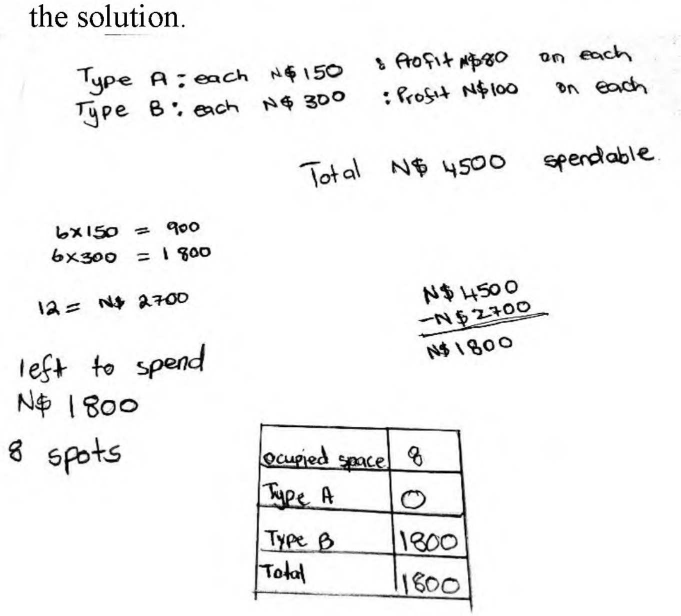
Figures 4.61 and 4.62 Pictures of L3’s response to Task 10

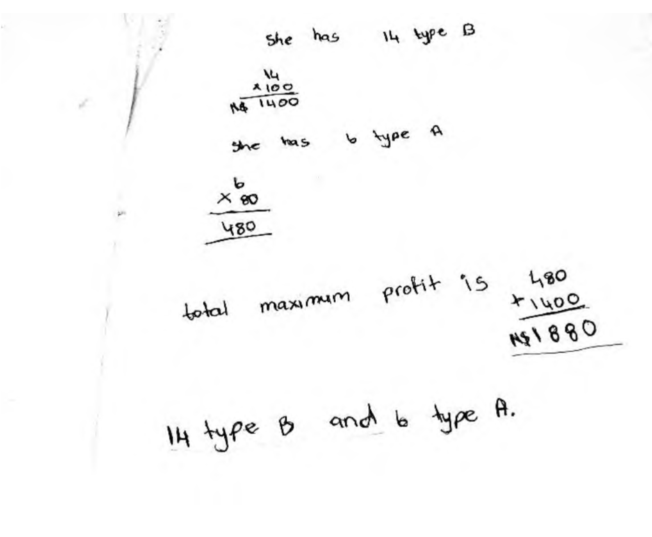


Learner 4

Observable indicator	coding		
	1	2	3
External visualisation *These are observable shapes such as diagrams, graphs and pictures. It also includes representations such as equations and words.*			
Internal visualisation *These are representations that are drawn in the mind and are imagined. These are only made explicit verbally and manifest themselves in conversation.*	√		
Descriptive representation *This is similar to the internal visualisation indictor above and refers to those representations that are spoken words or written texts such as mathematical equations and other expressions.*	√		
Depictive representation *These are iconic signs such as pictures, sketches, or drawn models and they are similar to the external indicators above.*			
Fundamental tool *This is when an overall visualisation representation is used throughout the problem.*			
Illustrative tool *This is when a visual representation is used to illustrate the problem in a familiar way.*			
Simplification tool *This is when a visual representation is used to make the problem easier.*	√		
Organizational tool *This is when the visual representation is used to deconstruct the problem into smaller components.*			
As a starting point *This is when the visual representation is used as an initial starting point that is developed further.*			

L4 started taking notes of the given information e.g. each type and its cost and the profit on each Type A: each N$150: profit N$80 on each and Type B: each N$300: profit N$100 on each. Then L4 constructed equations and expressions to find the amount of money that the shop owner still had to spend if she needs to have at least six of each type. L3 indicated that, the shop owner still had to spent N$1800, and had space for eight TVs only. According to L3, "*since the shop owner makes more profit from type B, I think it will be advisable to buy more type B TVs than type A. But before I decide on that let me calculate the amount she gets when she has at least 6 TVs of each type*", said L4 (lines 853-856). L4 then calculated the space occupied which is 12 spaces and the amount she spent on 12 TVs which is N$2700. L4 spent some minutes thinking of the equation that she could use and later on L4 decided to draw a table. L4 stated, "*I think a table will help me reorganize the information and get a clear picture of what to do next*" (line 874). Therefore, a table was then drawn and L4 managed to get the solution.





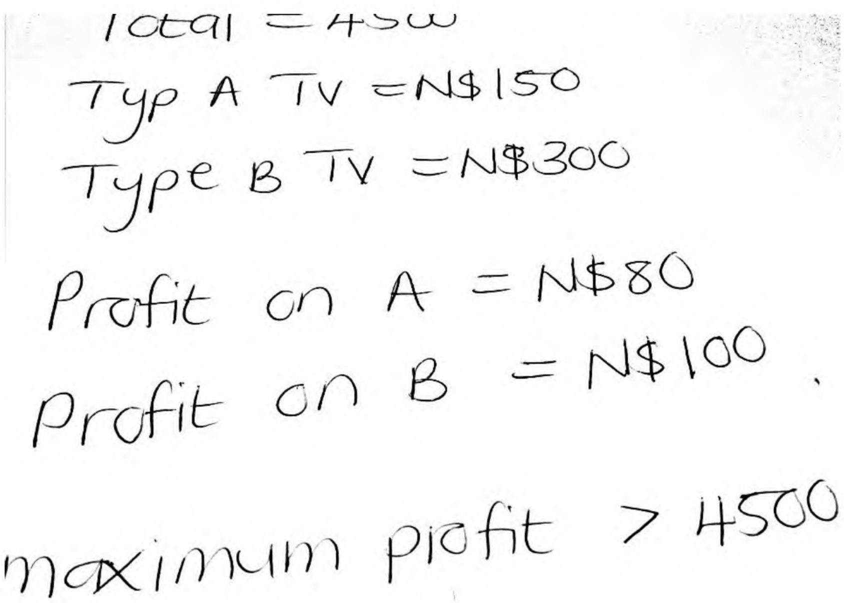
Figures 4.63 and 4.64 Pictures of L4's response to Task 10

Learner 5

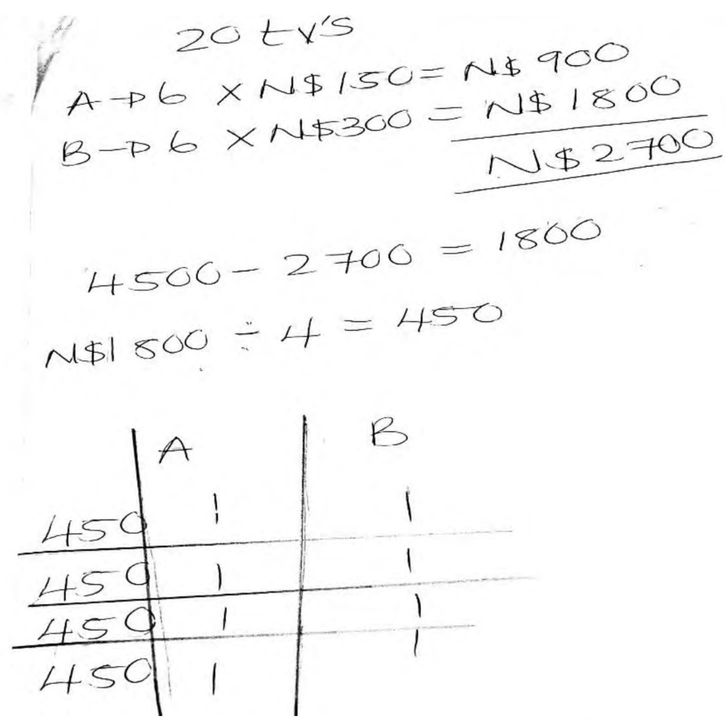
Observable indicator	coding		
	1	2	3
External visualisation *These are observable shapes such as diagrams, graphs and pictures. It also includes representations such as equations and words.*	√		
Internal visualisation *These are representations that are drawn in the mind and are imagined. These are only made explicit verbally and manifest themselves in conversation.*	√		
Descriptive representation *This is similar to the internal visualisation indictor above and refers to those representations that are spoken words or written texts such as mathematical equations and other expressions.*			
Depictive representation *These are iconic signs such as pictures, sketches, or drawn models and they are similar to the external indicators above.*			
Fundamental tool *This is when an overall visualisation representation is used throughout the problem.*			
Illustrative tool *This is when a visual representation is used to illustrate the problem in a familiar way.*			
Simplification tool *This is when a visual representation is used to make the problem easier.*	√		
Organizational tool *This is when the visual representation is used to deconstruct the problem into smaller components.*	√		
As a starting point *This is when the visual representation is used as an initial starting point that is developed further.*			

L5 started by writing down the total amount that the shop owner could spend which is N$4500. After that, L5 decided to write down the rest of the given information; "this question is tough for me I really have to think. So I must first write down all the information given" (line 1034). L5 then wrote down the information such as: type A TV=N$ 150, type B TV=N$300, profit on A=N$80, profit on

B=N$100. L5 indicated also that the maximum profit should be more than N$4500 (Maximum profit >N$4500). She then calculated the amount of having at least 6 TVs of each type= 2700. The remaining amount N$1800, added the cost of type A TV +type B TV=450 and divided 1800 by 450 to get four. This information was then indicated or displayed in a table at the end of the task. L5 clearly showed how she got 4 TVs of each type.



Figures 4.65 and 4.66 Pictures of L5’s response to Task 10

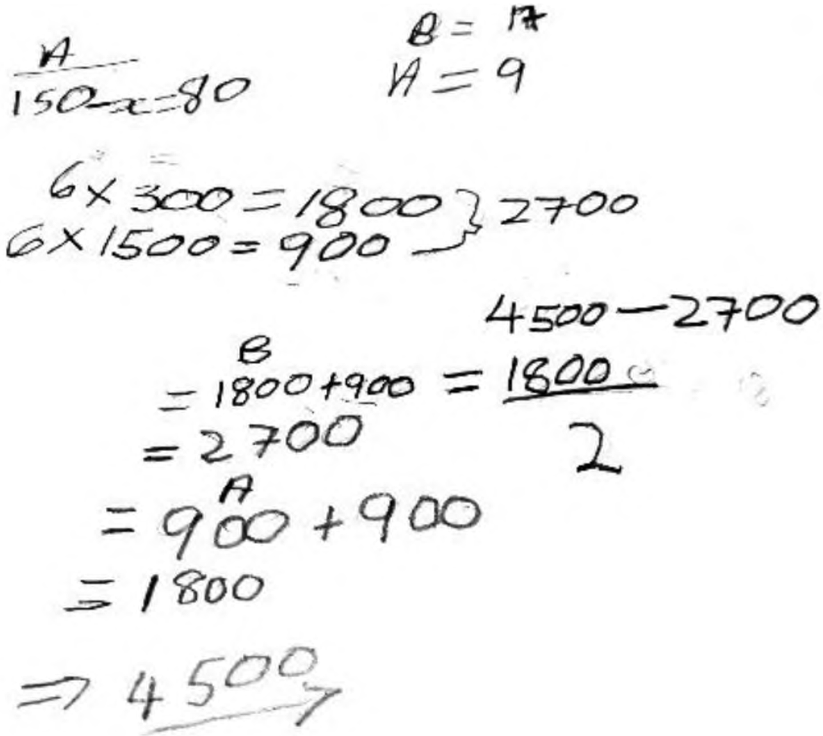


Learner 6

Observable indicator	coding		
	1	2	3
External visualisation *These are observable shapes such as diagrams, graphs and pictures. It also includes representations such as equations and words.*			
Internal visualisation *These are representations that are drawn in the mind and are imagined. These are only made explicit verbally and manifest themselves in conversation.*			
Descriptive representation *This is similar to the internal visualisation indictor above and refers to those representations that are spoken words or written texts such as mathematical equations and other expressions.*	√		
Depictive representation *These are iconic signs such as pictures, sketches, or drawn models and they are similar to the external indicators above.*			
Fundamental tool *This is when an overall visualisation representation is used throughout the problem.*			
Illustrative tool *This is when a visual representation is used to illustrate the problem in a familiar way.*			
Simplification tool *This is when a visual representation is used to make the problem easier.*			
Organizational tool *This is when the visual representation is used to deconstruct the problem into smaller components.*			
As a starting point *This is when the visual representation is used as an initial starting point that is developed further.*			

L6 took some minutes, underlining and re-reading the question, before he started with the calculations. He grouped the given information into two groups, the type A and type B groups. Just like others, L6 calculated the amount for the 12 TVs (at least 6 TVs from each type that the shop owner must have), which is 4500 -$[(6 \times 300) + (6 \times 150)] = 1800$. L3 divided 1800 by 2 to allocate equal amounts to both types of TV. According to him, it means that the shop owner had 2700 to spend on type B and 1800 to spend on type A. He then divided the amount of each type by the cost of one TV.

For this question, L6 did not use any diagram, sketch or table. L6 clearly stated, "*this type of question just needs understanding. I don't really need a sketch for this one*" (line 1281.)



Figures 4.67 Picture of L6's response to Task 10

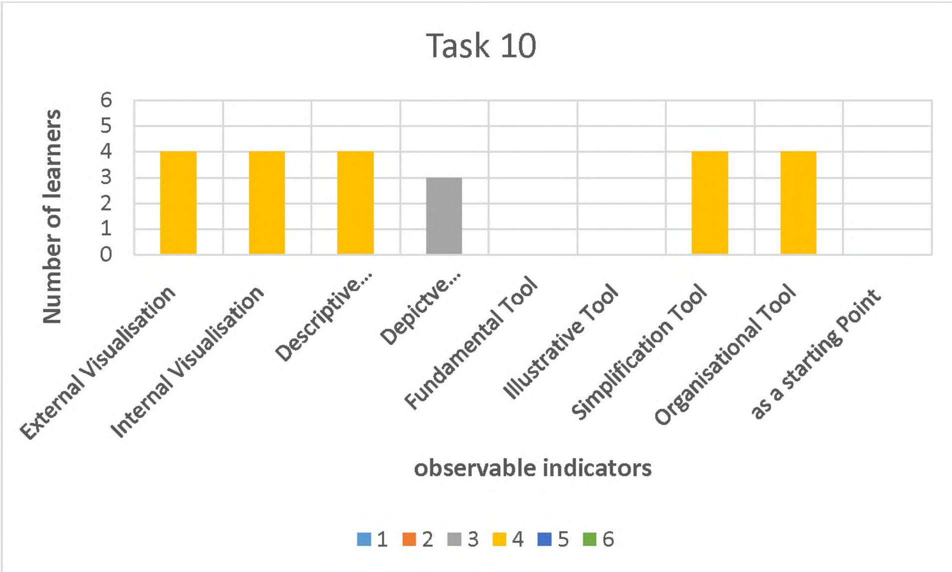


Figure 4.68 The types of visualisations used in task 10

Figure 4.68 shows that four participants used external and internal visualisations to simplify the task and organize given information. It further indicates that 4 participants used descriptive visualisations and three used depictive visualisations. However, none of the participants used visualisations as fundamental tools, as illustrative tools, or as starting points.

4.3 THE COMBINED RESULTS FOR ALL TASKS

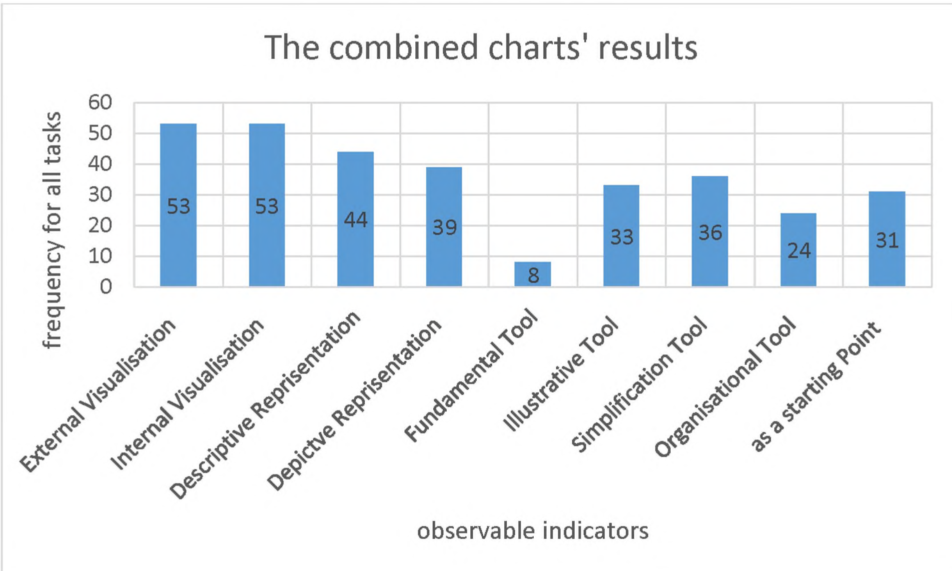


Figure 4.69 the use of all visualisations for all tasks

The aim of the study was to identify the nature of the visualisations that selected learners used to solve the 10 algebraic tasks of the AVT. Each learner answered 10 algebraic tasks. The types of visualisations that were used are displayed in the chart above – see Figure 4.69. As indicated above, the visualisations with the highest frequencies are external and internal visualisations. This implies that for almost every task that was solved learners used both external and internal visualisations. When interviewed learners indicated that, they first had to figure out the question. They tried to understand the question and think of the best method to tackle the question before they put anything on paper. As L6 commented, "*I always figure out the method or methods that I am going to use before I tackle the question"* (line 1195). L3 agreed with L6 (line 1195) that "*Whatever I am writing I first figure it out, be it a picture or a diagram*". Thus on the graph above, external visualisation was used 53 times, as were internal visualisations.

As shown on the graph above the second most used visualisation was the descriptive visualisation; it was used 44 times. The third most was depictive, at 39 times. Most learners used algebraic representations in almost every task they tackled. Some used equations and expressions only throughout problem solving while some opted to use both algebraic equations and a picture or a diagram or a table. Some learners indicated that there are some questions that one had no other option but to use both depictive and descriptive visualisations. This was particularly true for task 7. L4 said, "*this question Mrs is obvious I have to draw a rectangle to be able to identify the trigonometry ratio that I am going to use*" (line 761). This remark shows that learners felt that there are questions where they had no other option but to use both types of visualisations to obtain the solution. L6 comfirmed this by remarking, "*a sketch will be needed here because a ladder leaning on a wall that's already a triangle to be formed and there is an angle involved that means, I have to identify the given sides and the trig ratio. That I can only get by drawing a rough sketch*" (lines 1168-1171).

Moreover, out of the 60 visualisations that were used, only 8 of them were used as fundamental tools. In most cases, learners opted to use both descriptive and depictive visualisations. Mostly a picture or diagram was used only at the beginning or at the end of the problem solving process. Looking at the example I gave earlier of Task 7, all six learners started by drawing a triangle but ended up using algebraic equations to obtain the final solution.

Additionally, Figure 4.69 shows that visual representations were used as illustrative tools 33 times. This was mostly done at the beginning of the task when learners were trying to understand the task. They used different representations such as diagrams, tables, expressions and equations to rewrite the questions. Furthermore, visual representations were used 22 times as simplification tools. L5 (line 1078) stated, "*this question looks difficult and challenging but I think drawing a graph will make it simple and clear*". After drawing a graph, this learner managed to complete this task.

Figure 4.69 shows that 22 visualisations were used as organizational tools. This tool was either used at the beginning, in the middle or at the end of the problem solving process. These organizational tools assisted learners to choose strategies that helped them to solve the problems. L3 supported this by saying, "*I was stuck and did not know what to do. But using a table organized the information in a simpler way that gave me a clear picture of what to do next*" (line 674). L5 (line 880) stated, "*first I have to organize the given information to be able to get the approach or the method, just to have a plan*".

The last bar in Figure 4.69 shows the visualisation tools that were used as starting points occurred 33 times. The visualisations that were used by the learners ranged from drawing graphs, pictures, tables etc. These visual representations as starting points helped the learners to understand the problems. Learners commented that knowing where to start encouraged them to go ahead and solve the problems but "*not knowing where to start discourages us learners*", said L2 (line 242).

For all the tasks, learners were expected to use any visualisations of their choice to solve the problems. They were also asked to explain how they used these visualisations. As indicated above different visualisations were used for different purposes and these were categorized using the themes that emerged from the literature review.

4.4 CONCLUSION

This chapter presented the data collected from the interviews and the solving of the algebraic tasks. Each task with its interview was individually interpreted and analysed. At the end of each task, the results for that particular task were synthesized and shown on one chart. At the end of the chapter, I provided a bar chart that summarized all the types of visualisations that were used.

In the next chapter, the findings are discussed. I also articulate conclusions, make recommendations, and discuss the limitations of my research study.

CHAPTER 5

FINDINGS AND CONCLUSION

5.1 INTRODUCTION

The objectives of this study were to identify the nature of the visualisation processes that selected Grades 11 and 12 learners used to solve the algebraic tasks of the AVT. Specifically the study sought to answer the following questions:

- What problem solving strategies using visualisation processes are employed by selected Grades 11 and 12 learners when solving algebraic problems?
- How do these Grades 11 and 12 learners use the identified problem solving processes to solve algebraic problems?

This chapter presents a summary of the findings, discusses the significance of the study and presents some recommendations as a result of this study. The chapter also includes some limitations, suggestions for further research, personal reflections and a final conclusion.

5.2 SUMMARY OF FINDINGS

5.2.1 External visualisation

The evidence that emerged when the data of the AVT tasks were analysed was that learners used external visualisations in all their problem-solving processes. The study revealed that learners drew external visualisations such as geometrical shapes, graphs, diagrams, sketches or a set of symbols to illustrate some mathematical procedure. Learners indicated that the use of external visualisations helped them understand not only the questions that they were familiar with, but even with the unfamiliar questions. Learners pointed out that when using visualisations one is not restricted to one type of visualisation, one can switch from one external visualisation to another. Some learners emphasised that using visualisations in the problem solving processes lessened the time spent on solving a particular problem. They supported this assertion by suggesting that visualisations assisted in organising information and simplified the problem.

5.2.2 Internal visualisations

Learners emphasised the importance of internal visualisations in solving problems. Learners indicated that they could not write or draw a diagram, table or picture without visualising it in their minds first. They had to think and visualise internally about the problem and its diagram and draw it mentally before drawing it on paper. One interesting finding was that learners' said that their internal visualisations allowed them to mentally manipulate, rotate and twist any objects. The study also revealed that learners who failed to visualise mentally struggled to understand the tasks and would often not attempt the tasks as a consequence. Learners strongly asserted that to solve any problem fully, they needed to have an image or at least a partial image in their mind. This image, they said was the starting point to solving a problem. Failing to do this hindered the successful completion of the problem.

5.2.3 Use of illustrative visualisations

The study found that illustrative visualisations were important as they guided the learners in their problem solving process. Learners stated that using illustrative diagrams, graphs or sketches helped them to understand the requirements of the task more clearly. They specified that these visualisations assisted them to identify vital information and features of the problem. Moreover, it was also discovered that using illustrative visualisations enabled learners to articulate in their own way mathematical notation and mathematical equations and expressions that they then used to solve the problem. Learners also indicated that illustrational visualisations assisted them in forming a conceptual picture of the task. This applied particularly to the trigonometry tasks.

5.2.4 The use of organizational visualisations

The study revealed that organizational visualisations also played an important role in the problem solving process. Organizational pictures provided a useful structural framework for solving the problems. Another finding was that organizing critical information of the problem helped them to identify where they had applied inappropriate methods or patterns in their solution processes. Learners indicated that they reorganized the information either mentally or practically, with a sketch or diagram. They also used tables and some organized the information in sequences of patterns that then led to the solutions. There was strong evidence that most learners' strategies involved the use of either visual or analytic methods that matched their abilities.

5.2.5 The use of visualisations as starting points

Many learners said that they particularly used visualisations as starting points, when the problem was wordy and had lengthy descriptions and explanations. They also indicated that they found it difficult to come up with a starting point visualisation for those problem statements that were too difficult to understand. The study also revealed that the learners used different approaches to create the representations they needed to solve the problem.

5.2.6 The use of visualisations as simplification tools

Another interesting finding that was revealed by this study was that visual representations could be used for different purposes. Learners indicated that when the question was difficult or tricky, they used visual representations to simplify them. In the middle of the problem solving processes when learners became stuck or confused, they switched to another visualisation that was simpler to work from. For example, when the diagram was confusing or complicated they switched to equations and used those to get the solution.

5.3 SIGNIFICANCE OF THE STUDY

This study showed that students use visualisation processes as tools to support their mathematical understanding and problem solving strategies. It is thus important for teachers to harness this and make the most use of these visualisation processes in assisting learners to solve algebraic problems. This could potentially have a positive impact on learners' mathematical achievement. It is also important that curriculum developers are aware of how teachers can harness learners' visualisation processes to enrich and enhance their teaching. This study thus argues that it is important for teachers, learners, textbook authors and curriculum developers to understand and recognise the significance of visualisations as an effective tool to solve mathematical problems.

5.4 RECOMMENDATIONS

The following recommendations are based on the findings and conclusion of the study:

- Teachers should be encouraged to make use of visualisations to strengthen the conceptual understanding of mathematical concepts and improve the learning process in the classroom. As Whiteley (2004, p. 3) suggests, " we learn to see; we create what we see; visual reasoning or 'seeing to think' is learned, it can also be taught and it is important to teach it". I therefore recommend that teachers teach their learners how to use visualisations in solving algebraic problems.

- Teachers, curriculum developers and learners should be made aware of the potential of using visualisations, not only in algebra but also in other mathematical domains such as geometry. Most fields of mathematics can be visually reinforced and interpreted.
- Teachers should recognize that algebra and its symbolic logic is a form of visualisation on its own.
- Teachers should also emphasise that imagery is a powerful tool for perception and understanding. It is therefore the teachers' responsibility to continually encourage learners try to visualise a problem before they tackle it. This is supported by the Open University (1988) that "being able to "see" something mentally is a common metaphor for understanding it." (p. 10).
- Learners must also be encouraged to make use of visualisation as it can have a positive impact on their problem solving strategies.

Despite the notable evidence in research and texts on the benefits of using visualisations, the potential pitfalls and difficulties involved in using these in the classroom must also be acknowledged. For example teachers should not assume that students recognize representations in the manner they expected (Hall, 1998). Teachers must also take into consideration that the meaning that particular representations have for the teacher may be quite different to the meaning they have for the student (Cobb et al., 1992). Therefore, if certain representations are to be used in the classroom, teachers need to support learners in learning how to interpret these (Flevares & Perry, 2001). They can do this by providing "effective transitional experiences" (Boulton-Lewis, 1998, p. 222) to support learners.

5.5 LIMITATIONS

Only six learners participated in this research case study. This is a very small sample and consequently the findings of this study cannot be generalized. The analysis of a higher number of learners' work would have possibly revealed more visualisation processes and provided a more comprehensive picture.

5.6 SUGGESTIONS FOR FURTHER RESEARCH

The participants in this research were above average learners who were selected based on their performances in their tests, exams and participation in the class. I therefore suggest the same research be conducted with below average and average learners.

Visualisation plays an important role in problem solving. This research focused on algebra only. It would be interesting to broaden the scope of this research to include geometry and other domains of school mathematics.

One can learn how to visualise, but it can also be taught. It would be interesting to research how teachers use visualisations to teach algebraic problem solving or the visualisation processes that teachers use in their own teaching practice.

5.7 PERSONAL REFLECTION

5.7.1 My own experience of visualisations in algebra at school

In the years of my schooling, mathematics teachers were generally underqualified and as a result, they opted to teach only those mathematical domains that they were comfortable with. I attended one of the previously disadvantaged schools in the north of Namibia and I do not recall anything I learnt in algebra. It was not until I got a chance to improve my mathematics at UNAM that I appreciated working with algebra. Even though our tutor was good, he never really had time to emphasise the importance of visualisations in problem solving processes. The emphasis was more on memorizing the facts and methods if you wanted to become proficient. However, I came to realise the importance of visualisation when I went to Zimbabwe. Their curriculum emphasises the use of visualisations by teachers in delivering their lessons.

Algebra is one of the most challenging domains in the Namibian curriculum. Namibian learners have difficulty solving algebraic problems and I believe that as teachers we really need to change our way of teaching. We need to incorporate visual representations in the teaching and learning of algebra. I like quoting Whiteley (2004) that, "visualising can be learned, it can also be taught and it is important to teach it" (p.3). Thus, teachers need to understand the importance of visualisations and their positive impact in the teaching and learning of algebra.

5.7.2 My research experience

When I received an email that I had been accepted for a Masters degree in education, I was so delighted because this is what I had always wanted to do. However, when the classes started I soon realized how hectic and stressful doing a Masters could be. The pressure of reading articles, meeting the due dates and still maintaining family and sustaining my work commitments, was difficult to handle. It was not easy and there was a point at which I was totally discouraged and thought of dropping out. Later I realized that nothing comes on a silver platter and became more motivated. I realized that the more you read and write the better teacher/person you become. I have also learned that one has to be dedicated, hardworking, and motivated, and must enjoy what you are doing in order to succeed. It is all about the will to succeed.

Writing was never easy but as my writing skills improved and grew with time, I managed. I enjoyed collecting data and interacting with the learners. I found it fascinating to see how they were solving the algebraic tasks. It made me realize how important it is to emphasize the significance of visualisations to my own learners. The data analysis phase was challenging and I advise my fellow researchers to start analyzing data as soon as it comes in.

Overall, conducting this research taught me a lot about being a researcher, how to come up with a research problem and research questions and the difference between these two. I also developed a love for reading and it made me a critical thinker and analyzer. Most importantly, it improved my own styles of teaching. Now I try to use visual representation as much as possible not only in the teaching of algebra but in all mathematical domains where they can be used.

5.8 CONCLUSION

In this chapter, I presented a summary of the findings, discussed the research limitations, identified areas for future research, and provided recommendations and some personal reflections.

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APPENDICES

APPENDIX A – LETTER TO THE PRINCIPAL

P O BOX 5158
Oshikuku
5 January 2016

The Principal
Kolin Foundation SS
P O BOX 97
Arandis

Dear Sir

RE: Request for permission to carry out my research with selected grade 11 and 12 learners at Kolin Foundation secondary school.

I am Joseane Josef, a teacher at Kolin Foundation secondary School. I am currently pursuing a course of study leading to a master degree in Mathematics education from Rhodes University in South Africa. This is a two-year course and I am in my second year, the research year.

I hereby seek your consent to conduct my research project at Kolin Foundation SS. I wish to carry out a research study with six grade 11 and 12 learners. The research is seeking to find out the visualisation processes used by the grade 11 and 12 learners when solving algebraic problems.

The use of visualisation is gaining visibility in mathematics education research. It is a powerful tool for solving different types of problems in mathematical domains such as Algebra, the main focus of this study. I wish to select 6 Grade 11 and Grade 12 learners and engage with them for a period of 2 weeks. I wish to observe and analyse how these 6 learners use visualisation processes to solve 10 tasks that I will present to them. These sessions will occur after school hours during mutually agreed times. The findings of the study will hopefully contribute to teachers', policy makers' and curriculum designers' understanding of how to harness visualisation processes in the effective teaching of Algebra.

My proposal for this study was approved by the Rhodes University Higher Degrees Committee at the beginning of December 2015.
In anticipation, I thank you for your cooperation and look forward to hearing from you.

Yours Sincerely

………………………. ……………………………………

Joseane Josef Prof Marc Schafer (Supervisor)
Mathematics teacher
Student Number: 13j6825
Cell number : 0812089595

Kolin Foundation Secondary School
P.O. Box 97 Arandis
Tel No: +264 64 510166/Fax No: +264 64 510417
Email: kolinfschool@gmail.com

ERONGO REGIONAL COUNCIL
DIRECTORATE OF EDUCATION, ARTS AND CULTURE – SWAKOPMUND CIRCUIT

01 February 2016

Dear Sir/Madam

RE: Approved

Your request for permission to carry out research has been approved.

The research seeking to find out the visualization process used by the grade 11 and 12 learners when solving algebra problems.

Yours Sincerely

....
Mr. R. Geiseb
Principal

MINISTRY OF EDUCATION
KOLIN FOUNDATION SEC SHCOOL
2016 -02- 01
PO BOX 97, ARANDIS
TEL: 064 - 510166
FAX: 064 - 510417

APPENDIX B – LETTER TO INSPECTOR

P O BOX 97
Arandis
5 January 2016

The Inspector of Education
Swakopmund Circuit
Erongo Region

Dear Sir

RE: Request for permission to carry out my research with selected grade 11 and 12 learners at Kolin Foundation secondary school.

I am Joseane Josef, a teacher at Kolin Foundation secondary School. I am currently pursuing a course of study leading to a master degree in Mathematics education from Rhodes University in South Africa. This is a two-year course and I am in my second year, the research year.

I hereby seek your consent to conduct my research project at Kolin Foundation SS. I wish to carry out a research study with six grade 11 and 12 learners. The research is seeking to find out the visualisation processes used by the grade 11 and 12 learners when solving algebraic problems.

The use of visualisation is gaining visibility in mathematics education research. It is a powerful tool for solving different types of problems in mathematical domains such as Algebra, the main focus of this study. I wish to select 6 Grade 11 and Grade 12 learners and engage with them for a period of 2 weeks. I wish to observe and analyse how these 6 learners use visualisation processes to solve 10 tasks that I will present to them. These sessions will occur after school hours during mutually agreed times. The findings of the study will hopefully contribute to teachers', policy makers' and curriculum designers' understanding of how to harness visualisation processes in the effective teaching of Algebra.

My proposal for this study was approved by the Rhodes University Higher Degrees Committee at the beginning of December 2015.

In anticipation, I thank you for your cooperation and look forward to hearing from you.

Yours Sincerely

………………………. ………………………………….

Joseane Josef | Prof Marc Schafer (Supervisor)

Mathematics teacher Kolin Foundation SS
Student Number: 13j6825
Cell number : 0812089595

ERONGO REGIONAL COUNCIL

DIRECTORATE OF EDUCATION, ARTS AND CULTURE
SWAKOPMUND CIRCUIT OFFICE

PRIVATE BAG 5024
SWAKOPMUND

TELEPHONE: 064-415 461
FAX: 064-415 459

Enquiries: Mr. E.J.J Olivier
Inspswk@moe.org.na

Date: 25 January 2016

The Principal

Permission is herewith granted to Mrs. J. Josef, a Mathematics teacher at Kolin Foundation SS, to carry out a research with selected Grade 11 and 12 leaners at the school.

The research is seeking to find out the visualization process used by the grade 11 and 12 learners when solving algebraic problems.

Yours sincerely

....

EJJ OLIVIER
INSPECTOR OF EDUCATION, ARTS AND CULTURE
SWAKOPMUND CIRCUIT

APPENDIX C: LETTER TO THE PARENTS

Parents/Guardians Letter of Introduction and Informed Consent Form

I, Mrs Joseane Josef am conducting educational research as part of my Master's degree. I humbly invite your child to be part of my research project. As this research project ultimately contributes to the improvement of mathematics teaching, his/her participation is very important and will be greatly valued. My study is entitled: *An analysis of visualization processes used by selected Grade 11 and 12 learners when solving algebraic problems.*

In a nutshell I wish to research how young Namibian learners use images, diagrams and other visual images when solving algebraic problems. I wish to engage with the learners as they solve 10 selected algebraic problems. They will be expected to explain and describe to me how they solve each of the 10 tasks. This is not to evaluate their mathematical proficiency, but to gain insight into how they approach an algebra problem. As we engage with each other our conversation will be video-taped so that I can analyse this conversation in detail at a later stage. Our engagement will happen after school hours at mutually convenient times. It is anticipated that our engagement sessions will last one afternoon.

Your child's participation in this project will be entirely voluntary and he/she may withdraw from the project at any time she/he wish without any penalty and prejudice. His/her name will not be reveal in the final write-up of this project. You are welcome to ask me (the researcher) any questions before the interview is conducted or once the interview is completed.

I ______________________________, in my capacity as Parent/guardian, hereby give consent for ______________________________ to take part in the research study to be conducted by Mrs. Joseane Josef, an Med research student in the Education Department of Rhodes University, Grahamstown. I understand what will be required of my child in his/her role as research participant. Furthermore, I am aware that my child retains the right to withdraw from the research project at any point without explanation.

APPENDIX D: CONSENT FORM

Learners Letter of Introduction and Informed Consent Form

I, Mrs Joseane Josef am conducting educational research as part of my Master's degree. I humbly invite you to be part of my research project. As this research project ultimately contributes to the improvement of mathematics teaching, your participation is very important and will be greatly valued. My study is entitled: *An analysis of visualization processes used by selected Grade 11 and 12 learners when solving algebraic problems.*

In a nutshell I wish to research how young Namibian learners like yourself are using images, diagrams and other visual images when solving algebraic problems. I wish to engage with you as you solve 10 selected algebraic problems. You will be expected to explain and describe to me how you solve each of the 10 tasks. This is not to evaluate your mathematical proficiency, but to gain insight into how you approach an algebra problem. As we engage with each other our conversation will be video-taped so that I can analyse this conversation in detail at a later stage. Our engagement will happen after school hours at mutually convenient times. It is anticipated that our engagement sessions will last one afternoon.

Your participation will be entirely voluntary and you may withdraw from the project at any time you wish without any penalty and prejudice. Your name will not be reveal in the final write-up of this project. You are welcome to ask the researcher any questions that occur to you during the interview. If you have further questions once the interview is completed, you are encouraged to contact me (the researcher).

I, [redacted] (name; please print clearly), have read the above information. I freely agree to participate in this study. I understand that I am free to refuse to answer any question and to withdraw from the study at any time. I understand that my responses will be kept anonymous.

____________________ 19 January 2016

Participant Signature Date

APPENDEX E: THE ALGEBRAIC VISUALISATION TASKS

Task 1

A hitchhiker set out on a journey of 60 km. He walked the first 5 km and then got a lift from a lorry [truck] driver. When the driver dropped him off, he still had half of his journey to travel. How far had he travelled in the lorry?

Task 2

Penouwa bought a pizza and cut it into three pieces. When she weighed the pieces, she found that one piece was 7g lighter than the largest piece and 4g heavier than the smallest piece. The mass of the whole pizza was 300g. What was the mass of each piece?

Task 3

A man was very overweight and his doctor told him to lose 36 kg. If he loses 11 kg the first week, 9 kg the second week, and 7 kg the third week, and he continues losing at this rate, how long will it take him to lose 36 kg?

Task 4

One side of a rectangle is 3 cm shorter than the other side. If we increase the length of each side by 1 cm, then the area of the rectangle will increase by 18 cm^2. Find the lengths of all sides.

Task 5 Hafoletu is putting up a tent for a family reunion. The tent measures 16 m by 5 m. Each 4-m section of tent needs a post except the sides that are 5 m long. How many posts will he need?

Task 6

A moving company is hired to take 578 clay pots to a florist shop. The florist will pay the moving company a N$200 fee, plus N$1 for every pot that is delivered safely. The moving company must pay the florist N$4 each for any pots that are lost or broken. If two pots are lost, four pots are broken, and the rest are delivered safely, how much should the moving company be paid?

Task 7

A ladder of length 8 m rests against a wall so that the angle between the ladder and the wall is 31°. How far is the base of the ladder from the wall?

Task 8

f is a quadratic function whose graph has a vertex at the point (-3 , 2) and a y-intercept at the point (0 , -16). Find the x-intercepts of the graph of f.

Task 9

My brother is two years older than me, my sister is five years younger than me; she is 12, how old will my brother be in three years' time?

Task 10

A shop owner has room in her shop for up to 20 television sets. She can buy either type **A** for **N$ 150** each or type **B** for **N$ 300** each. She has a total of **N$ 4500** she can spend and she must have at least 6 of each type in the stock. She makes a profit of **N$80** on each television of type **A** and a profit of **N$ 100** on each of type **B.** How many of each type should she buy so that she makes a maximum profit?.

APPENDEX F: ANALYTIC TEMPLATE A - CATEGORIES OF VISUALISATION PROCESSES.

Observable indicator	**coding**		
	1	**2**	**3**
External visualisation *These are observable shapes such as diagrams, graphs and pictures. They also include representations such as equations and words.*			
Internal visualisation *These are representations that are drawn in the mind and are imagined. They are only made explicit when verbalised and manifest themselves in conversation.*			
Descriptive representation *This is similar to the internal visualisation indictor above and refers to those representations that are spoken words or written texts such as mathematical equations and other expressions.*			
Depictive representation *These are iconic signs such as pictures, sketches, or drawn models and they are similar to the external indicators above.*			
Fundamental tool *This is when an overall visualisation representation is used throughout the problem.*			
Illustrative tool *This is when a visual representation is used to illustrate the problem in a familiar way.*			
Simplification tool *This is when a visual representation is used to make the problem easier.*			
Organizational tool *This is when a visual representation is used to deconstruct the problem into smaller components.*			
As a starting point *This is when a visual representation is used as an initial starting point that is developed further.*			

APPENDIX G: ANALYTIC TEMPLATE B - CODING VISUALISATION PROCESSES USED IN EACH AVT TASK

Coding template
1- very strong evidence When the visualisation process used is indicated clearly, **it is visible** and **observable**. There is strong evidence that the visualisation process was used. **2- mediocre evidence** This is when there is **mediocre evidence** that the visualisation process was used. The representations are **tentative**. The representations are **not clear** – they are mere scribbles and very rough sketches. **3-weak evidence** This is when the student claims to have used a certain visualisation process but there is no evidence that it was really used. No picture, drawing, graph or table is drawn, or the student could not explain the picture that had been drawn in mind.