CONTRIBUTIONS TO THE STUDY OF NONHOLONOMIC RIEMANNIAN MANIFOLDS

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Abstract

In this thesis we consider nonholonomic Riemannian manifolds, and in particular, left-invariant nonholonomic Riemannian structures on Lie groups. These structures are closely related to mechanical systems with (positive definite) quadratic Lagrangians and nonholonomic constraints linear in velocities. In the first chapter, we review basic concepts of nonholonomic Riemannian geometry, including the left-invariant structures. We also examine the class of left-invariant structures with so-called Cartan–Schouten connections. The second chapter investigates the curvature of nonholonomic Riemannian manifolds and the Schouten and Wagner curvature tensors. The Schouten tensor is canonically associated to every nonholonomic Riemannian structure (in particular, we use it to define isometric invariants for structures on three-dimensional manifolds). By contrast, the Wagner tensor is not generally intrinsic, but can be used to characterise flat structures (i.e., those whose associated parallel transport is path-independent). The third chapter considers equivalence of nonholonomic Riemannian manifolds, particularly up to nonholonomic isometry. We also introduce the notion of a nonholonomic Riemannian submanifold, and investigate the conditions under which such a submanifold inherits its geometry from the enveloping space. The latter problem involves the concept of a geodesically invariant distribution, and we show it is also related to the curvature. In the last chapter we specialise to three-dimensional nonholonomic Riemannian manifolds. We consider the equivalence of such structures up to nonholonomic isometry and rescaling, and classify the left-invariant structures on the (three-dimensional) simply connected Lie groups. We also characterise the flat structures in three dimensions, and then classify the flat structures on the simply connected Lie groups. Lastly, we consider three typical examples of (left-invariant) nonholonomic Riemannian structures on three-dimensional Lie groups, two of which arise from problems in classical mechanics (*viz.*, the Chaplygin problem and the Suslov problem).

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Contents

Introduction

Nonholonomic Riemannian geometry has, under one guise or another, been a topic of study for over a hundred years, and has attracted the attention of numerous geometers. Among the pioneers, we mention, in particular, E. Cartan and Synge (who first introduced the nonholonomic connection), Vrănceanu (who first formalised the notion of a nonholonomic manifold) and Schouten and Wagner (who first studied the curvature of nonholonomic Riemannian structures). More recent contributions, using the language of modern differential geometry, have been made by Vershik and collaborators [70, 69, 71, 72], Lewis [46, 45] and Bloch, Crouch and collaborators [8] (and references therein), to mention but a few. (Lewis, as well as the Bloch and Crouch group, have also studied mechanical control systems on nonholonomic Riemannian manifolds.) Some standard references for nonholonomic Riemannian geometry (mainly from the viewpoint of nonholonomic mechanics) are [8, 18, 19].

Briefly, a nonholonomic Riemannian structure on a smooth manifold M consists of a pair of complementary distributions $\mathcal{D}$ and $\mathcal{D}^{\perp}$, where $\mathcal{D}$ is assumed to be nonholonomic (i.e., nonintegrable), and a positive definite metric tensor $\mathbf{g}$ (i.e., a Riemannian metric) is defined on $\mathcal{D}$. There are two (in general, inequivalent) geometries that may be defined on a nonholonomic Riemannian manifold. In the first, the dynamics are essentially specified by means of the Chetaev equations; this approach may be viewed as a special case of nonholonomic mechanics. In the second, which may be viewed as a special case of vakonomic mechanics, the dynamics follow a classical variational principle: geodesics are (local) length minimisers of the Carnot–Carathéodory distance. While the term *nonholonomic Riemannian geometry* has, in the past, been used to refer to both of these geometries (e.g., in [71, 72]), we shall use it to mean only the first. The second we shall refer to, by the usual modern terminology, as *sub-Riemannian geometry*.

From the point of view of physics, a nonholonomic Riemannian manifold is the geometric structure underlying a mechanical system that describes the motion of a free particle moving in M with kinetic energy Lagrangian, and subject to (time-independent) nonholonomic constraints linear in velocities (given by the constraint distribution $\mathcal{D}$). As such, much of the work in nonholonomic Riemannian geometry has been from the perspective of nonholonomic mechanical systems, and this remains the chief application of the field. A consequence is that much of the work done on more general (nonholonomic) mechanical systems—see, e.g., [20, 41]—also applies as a special case to nonholonomic Riemannian structures.

Left-invariant nonholonomic Riemannian structures on Lie groups (often referred to as “LL systems” in the context of nonholonomic mechanics) are the prototypes of nonholonomic Riemannian manifolds: their study is a first step toward an understanding of the general structures. In contrast to invariant sub-Riemannian structures, these structures have not been extensively studied. To the best of our knowledge, the work that has been done has been mainly devoted to reduction (see, e.g., [8, 19] and references therein), integrability (including the existence of invariant measures/volumes; see, e.g., [39, 35]) and problems from mechanics (particularly, involving symmetries and conservation laws), for instance, the Chaplygin problem [16, 52, 24, 15], the Suslov problem [63, 26] and problems with nonlinear constraints [65, 64, 60]. (Also of interest for integrability are the so-called "LR systems," where the metric is left invariant and the distribution is right invariant; see, e.g., [73, 25].) In particular, little effort has been devoted to tackling the questions of equivalence and classification of nonholonomic Riemannian structures. Cartan [14] was the first to apply his method of equivalence in nonholonomic mechanics (specifically, nonholonomic Riemannian manifolds), though only to the case of a strongly nonholonomic distribution. (The papers [38, 68] are a modern exposition of Cartan's ideas; the former paper in particular discusses some generalisations of Cartan's work to more general nonholonomic distributions.) However, regarding specific problems of equivalence and classification, we are aware only of the papers [22, 23]. In the first, the author makes use of the method of equivalence in order to determine differential invariants for three-dimensional nonholonomic Riemannian manifolds with contact distributions. (The paper does not, however, give a classification of these structures.) In the second paper, the authors again make use of the method of equivalence to derive the differential invariants, this time studying (four-dimensional) Engel manifolds.

In this thesis we consider nonholonomic Riemannian structures on smooth manifolds, and, in particular, left-invariant nonholonomic Riemannian structures on Lie groups. The thesis is organised as follows. In chapter 1 we review some preliminary concepts of nonholonomic Riemannian geometry. We explicitly define a nonholonomic Riemannian structure, prove the existence and uniqueness of its associated nonholonomic connection, and show how the connection is used to define the nonholonomic geodesics. We also discuss how nonholonomic Riemannian geometry relates to nonholonomic mechanics and sub-Riemannian geometry. In the second part of this chapter we consider left-invariant nonholonomic Riemannian structures on Lie groups, and especially a class of such structures with particularly simple dynamics (*viz.*, those with Cartan–Schouten connections).

Chapter 2 treats the curvature of nonholonomic Riemannian manifolds. As mentioned previously, Schouten and Wagner were the first to study curvature in the context of nonholonomic Riemannian geometry. In particular, the two main curvature tensors used in nonholonomic Riemannian geometry originate in the work of those two mathematicians. The chapter consists of four parts. In the first two parts, we consider the Schouten and Wagner curvature tensors. While the Schouten tensor is intrinsically associated to every nonholonomic Riemannian structure, in general the definition of the Wagner tensor relies on some additional assumptions. However, when these assumptions are met, the Wagner curvature tensor (unlike Schouten's tensor) can be used to characterise the "flat" nonholonomic Riemannian structures, i.e., those for which the associated parallel transport is path-independent. In the third part of the chapter, we revisit Schouten's and Wagner's constructions from the point of view of "restricted Ehresmann connections," showing in particular that there exists a flag of horizontal distributions associated to the Wagner tensor. In the last part of the chapter, we prove some results concerning the curvature of left-invariant nonholonomic Riemannian structures.

In chapter 3 we consider the equivalence of nonholonomic Riemannian manifolds, and introduce a suitable notion of nonholonomic Riemannian submanifolds. In the first part of the chapter, we consider three natural equivalence relations between nonholonomic Riemannian structures. In increasing strength, the equivalence relations are up to the existence of a diffeomorphism preserving: the nonholonomic geodesics; the constraint distribution and nonholonomic connection; both (complementary) distributions and the metric. Among the three equivalence relations, in this thesis we shall be chiefly concerned with the last (and strongest) one; this is also the analogue of equivalence up to isometry in Riemannian geometry. In the second part of this chapter we consider nonholonomic Riemannian submanifolds, i.e., embeddings of one nonholonomic Riemannian structure inside another. The main contribution of this section is to characterise when the nonholonomic geodesics of the embedded structure are also nonholonomic geodesics of the ambient space, or, in other words, when the embedded structure inherits its geometry from the enveloping structure. In order to characterise this occurrence, we make use of the notion of a "geodesically invariant" distribution, i.e., a distribution invariant under the nonholonomic geodesic flow. A remarkable link between geodesic invariance and curvature is also established.

Finally, in chapter 4 we specialise to the case of nonholonomic Riemannian structures on three-dimensional smooth manifolds. (The results of this chapter—apart from the characterisation and classification of flat structures and treatment of the Heisenberg problem—have been published in [4].) This is the largest chapter in the thesis, and consists of three parts. In the first part, we consider the equivalence of nonholonomic Riemannian structures in three dimensions, up to "nonholonomic isometry" and rescaling. In particular, we classify the left-invariant nonholonomic Riemannian structures on the three-dimensional simply connected Lie groups, and describe the equivalence classes in terms of isometric invariants. The second part of the chapter treats the flat nonholonomic Riemannian structures. We first characterise flatness in three dimensions, before using this characterisation to classify the flat structures. Our initial approach to the characterisation uses a direct approach; as such, after classifying the flat structures, we relate the work with the Wagner curvature tensor. In the last part of this chapter, we consider three typical problems in nonholonomic Riemannian geometry. The first—which we have called the "Heisenberg problem"—involves the study of a (left-invariant) nonholonomic Riemannian structure on the (three-dimensional) Heisenberg group H_3. (In fact, we shall see that, up to nonholonomic isometry and rescaling, there are only two equivalence classes of left-invariant nonholonomic Riemannian structures on H_3, corresponding to whether a scalar invariant $\vartheta \geq 0$ vanishes or not. The Heisenberg problem treats the equivalence class of structures with positive ϑ.) The second two problems are classical problems from nonholonomic mechanics: the Chaplygin problem and the Suslov problem.

Appendix A outlines some of our conventions, particularly regarding tensor fields and tensor derivations. It also introduces a generalisation of the Lie derivative and the exterior derivative (called the "$\mathscr{P}$-Lie derivative" and "$\mathscr{P}$-exterior derivative," respectively). Appendix B considers "restricted connections," i.e., connections whose associated parallel transport is along a restricted subset of curves in the manifold. The nonholonomic connection associated to a nonholonomic Riemannian structure is properly viewed in this context, as is a series of connections involved in the construction of the Wagner curvature tensor. We consider these connections from two different points of view, *viz.*, as a Koszul connection (i.e., a covariant derivative) and as an Ehresmann connection (essentially a horizontal distribution transversal to the vertical distribution). We also consider the parallel transport induced by such connections, as well as some special classes of these connections. In appendix C we briefly review the Bianchi–Behr classification of three-dimensional Lie algebras and their associated simply connected Lie groups, and discuss some algebraic properties distinguishing the different algebras. (This is crucial for the classification in chapter 4.) Lastly, we have also made extensive use of MATHEMATICA [81] to facilitate computations in chapter 4; the code we have written toward this end may be found in appendix D.

Original Contributions

To the best of our knowledge, the following contributions in this thesis are original.

Chapter 1: proposition 1.2.6; theorem 1.2.10; proposition 1.2.11; corollary 1.2.12; propositions 1.2.14 and 1.2.15; corollary 1.2.16; proposition 1.2.17; corollary 1.2.18.

Chapter 2: lemma 2.1.1; propositions 2.1.2, 2.1.4 and 2.1.5; proposition 2.2.2; theorem 2.2.12; proposition 2.2.13; lemma 2.3.2; proposition 2.3.3; lemmas 2.3.5, 2.3.6, 2.3.7 and 2.3.9; theorem 2.3.10; corollary 2.3.11; lemmas 2.3.12 and 2.3.13; theorems 2.3.14 and 2.3.15; corollary 2.3.16; proposition 2.4.1; corollary 2.4.2; lemmas 2.4.3 and 2.4.4; propositions 2.4.5, 2.4.6 and 2.4.7; corollaries 2.4.8 and 2.4.9.

Chapter 3: propositions 3.1.4 and 3.1.5; lemma 3.1.6; propositions 3.1.7, 3.1.8 and 3.1.9; corollary 3.1.10; lemmas 3.1.11 and 3.1.12; propositions 3.1.13, 3.1.14, 3.1.15 and 3.1.16; corollary 3.1.17; lemmas 3.2.2, 3.2.3 and 3.2.4; proposition 3.2.5; corollary 3.2.6; propositions 3.2.7, 3.2.8, 3.2.9, 3.2.10; corollary 3.2.11; proposition 3.2.12; corollary 3.2.13; propositions 3.2.14 and 3.2.15; theorem 3.2.16; proposition 3.2.17; theorem 3.2.18.

Chapter 4: propositions 3.1.2, 4.1.1 and 4.1.4; lemmas 4.1.7 and 4.1.17; proposition 4.1.18; theorem 4.1.19; corollary 4.1.20; theorem 4.1.22; corollary 4.1.23; propositions 4.1.24, 4.1.25 and 4.1.26; lemmas 4.2.1 and 4.2.2; theorem 4.2.3; corollaries 4.2.4, 4.2.5 and 4.2.6; theorems 4.2.7, 4.2.9 and 4.2.10; lemma 4.2.12; proposition 4.2.13; theorem 4.2.14; proposition 4.3.1; corollary 4.3.2; propositions 4.3.3, 4.3.5 and 4.3.6.

Appendix A and appendix B: Many of the results in the (first two) appendices are original, or at least cannot be found in the literature (though they are straightforward generalisations of known results). Specifically, in appendix A, the following results are original contributions: propositions A.3.1 and A.3.2; lemma A.3.3.

In appendix B, almost everything in section B.1 (*sans* section B.1.3.2) is original, with the exception of some results stated in [38]. (Having said that, it seems clear that all results in this section are known to the authors of [38], and most likely also to researchers in related areas.) Nevertheless, the main contribution of this appendix is to provide complete proofs for almost all results.

Conventions

We briefly mention the conventions employed in this thesis. Unless stated otherwise, we shall assume that all manifolds, tensor fields, distributions, etc. are smooth, i.e., of class $\mathcal{C}^\infty$. Appendix A lists further conventions, particularly for tensor fields and tensor derivations.

Summation convention. We follow the Einstein summation convention on repeated indices throughout this thesis. Unless stated otherwise, we shall assume the following ranges:

- i, j, k, ℓ range through $1, \ldots, n$ (or sometimes $0, 1, 2$ in chapter 4);
- a, b, c and u, w range through $1, \ldots, r$;
- λ, μ range through $r+1, \ldots, n$.

If these indices are themselves indexed (e.g., $a_1, a_2, \ldots$), then they range through the same values (e.g., $a_1, a_2, \ldots$ range through $1, \ldots, r$).

Notation. We briefly outline the notational conventions used in this thesis. Manifolds are denoted M, N, etc., in a sans serif typeface. Similarly, Lie groups are denoted G, H, etc. Their corresponding Lie algebra is denoted using a lowercase Fraktur letter (e.g., $\mathfrak{g}$, $\mathfrak{h}$, etc.). Distributions are denoted with a calligraphic typeface (e.g., $\mathcal{D}$, $\mathcal{S}$, etc.), and projections with a curly calligraphic typeface (e.g., $\mathscr{P}$, $\mathscr{Q}$, etc.). We also use the following notation:

$\mathbf{1}$	identity element of a Lie group.
$A^\flat$	induced map $A^\flat : \Gamma(\mathcal{D}) \to \Gamma(\mathcal{D}^*)$, $A^\flat(X) = A(X, \cdot)$, where A is a $(0,2)$-tensor.
$A^\sharp$	induced map $A^\sharp : \Gamma(\mathcal{D}^*) \to \Gamma(\mathcal{D})$, $A^\sharp = (A^\flat)^{-1}$, where A is a nondegenerate $(0,2)$-tensor.
$\mathcal{C}^\infty(\mathsf{M})$	the set of (real-valued) functions on a manifold M.
d	exterior derivative of a k-form.
$d_{\mathscr{P}}$	$\mathscr{P}$-exterior derivative of a k-form; see section A.3.2.
$d^{\nabla}_{\mathscr{P}}$	$\mathscr{P}$-exterior covariant derivative of a vector-valued k-form (where ∇ is a nonholonomic connection); see section 1.1.4.
$\bigwedge^k \mathcal{D}$	k^{th}-exterior power of $\mathcal{D}$: $\bigwedge^k \mathcal{D} = \bigsqcup_{q \in \mathsf{M}} \operatorname{span}\{X_1 \wedge \cdots \wedge X_k : X_1, \dots, X_k \in \mathcal{D}_q\}$.
$\bigvee^k \mathcal{D}$	k^{th}-symmetric power of $\mathcal{D}$: $\bigvee^k \mathcal{D} = \bigsqcup_{q \in \mathsf{M}} \operatorname{span}\{X_1 \vee \cdots \vee X_k : X_1, \dots, X_k \in \mathcal{D}_q\}$.
$\mathsf{Der}(\mathcal{D})$	Lie algebra of (tensor) derivations of $\mathcal{T}^k_\ell(\mathcal{D})$; see section A.2.
$\mathsf{Der}_0(\mathcal{D})$	subspace of algebraic derivations in $\mathsf{Der}(\mathcal{D})$; see section A.2.
$\mathsf{Der}_{\mathcal{S}}(\mathcal{D})$	$\mathcal{S}$-restricted derivations; see section A.2 and section 2.2.1.
$\Gamma(\mathcal{D})$	space of sections of a distribution $\mathcal{D}$. In particular, $\Gamma(T\mathsf{M})$ is the space of vector fields, and $\Gamma(T^*\mathsf{M})$ the space of 1-forms, on a manifold M.
$\Gamma^L(\mathcal{D})$	space of left-invariant sections of a left-invariant distribution $\mathcal{D}$ on a Lie group G.
$\pounds_X$	Lie derivative along a vector field $X \in \Gamma(T\mathsf{M})$.
$\pounds^{\mathscr{P}}_X$	$\mathscr{P}$-Lie derivative along a vector field $X \in \Gamma(T\mathsf{M})$; see section A.3.1.
$[\![\cdot, \cdot]\!]$	projected Lie bracket: $[\![\cdot, \cdot]\!] = \mathscr{P}([\cdot, \cdot])$.
$\langle\!\langle \cdot : \cdot \rangle\!\rangle$	symmetric bracket; see section 1.1.3.
$\Omega^k(\mathcal{D})$	space of k-forms on $\mathcal{D}$: $\Omega^k(\mathcal{D}) = \Gamma(\bigwedge^k \mathcal{D})$; by convention $\Omega^0(\mathcal{D}) = \mathcal{C}^\infty(\mathsf{M})$.
$\Omega^k(\mathcal{D}, \mathcal{S})$	space of $\mathcal{S}$-valued k-forms on $\mathcal{D}$: $\Omega^k(\mathcal{D}, \mathcal{S}) = \Omega^k(\mathcal{D}) \otimes \Gamma(\mathcal{S})$.
$T\phi$	tangent map (differential) $T\phi : T\mathsf{M} \to T\mathsf{N}$ of a map $\phi : \mathsf{M} \to \mathsf{N}$ between manifolds M and N; the tangent map at $q \in \mathsf{M}$ is denoted $T_q\phi : T_q\mathsf{M} \to T_{\phi(q)}\mathsf{N}$.
$T^k_\ell(\mathcal{D})$	bundle of (k, ℓ)-tensors on $\mathcal{D}$; see section A.1.
$\mathcal{T}^k_\ell(\mathcal{D})$	space of (k, ℓ)-tensor fields on $\mathcal{D}$: $\mathcal{T}^k_\ell(\mathcal{D}) = \Gamma(T^k_\ell(\mathcal{D}))$; see section A.1.
tr^i_j	trace (contraction) of a tensor field in the i^{th} contravariant and j^{th} covariant slot; see section A.1.1.
$X[f]$	directional derivative of $f \in \mathcal{C}^\infty(\mathsf{M})$ in the direction of a vector field X.
X^h	horizontal lift (or h-lift) of a vector field $X \in \Gamma(\mathcal{E})$ with respect to an $\mathcal{E}$-restricted Ehresmann connection h on $\mathcal{D}$; see section B.2.
X^v	vertical lift of a vector field $X \in \Gamma(\mathcal{D})$; see section B.2.
vl_{U_q}	vertical lift over (a tangent vector) U_q; see section B.2.

Chapter 1

Nonholonomic Riemannian geometry

In this chapter we lay the groundwork for our study of nonholonomic Riemannian manifolds. Most of the definitions and results in this chapter may be found in the literature, or are a straightforward generalisation of those found in Riemannian geometry. We have restricted to only the material that is required in this thesis, either directly, or for a more comprehensive understanding of the topics we present.

In section 1.1, we consider general nonholonomic Riemannian structures. In particular, we define a nonholonomic Riemannian manifold and prove the existence and uniqueness of the nonholonomic connection. Using this connection we introduce the nonholonomic geodesics, for which we prove some basic properties. Our particular approach to nonholonomic Riemannian geometry (specifically, the definition of a nonholonomic Riemannian structure as a quadruple) is somewhat new (although this approach can essentially be found in some of the early works, e.g., [75]). For this reason, in section 1.1.1 we motivate our approach by relating it to the study of nonholonomic mechanical systems with kinetic energy Lagrangians. In section 1.1.2 we continue our study of nonholonomic geodesics, introducing the nonholonomic geodesic spray, the associated (nonholonomic geodesic) flow, and the exponential map. In section 1.1.3 and section 1.1.4 we introduce two objects induced by the nonholonomic connection. The first, the so-called "symmetric bracket," is essentially the symmetric part of the connection (the skew-symmetric part of the connection is given by a projection of the Lie bracket). The second object is an exterior covariant derivative operator, induced by the connection and the projection onto the constraint distribution. Concluding the first part of the chapter, in section 1.1.5 we review some basic concepts from sub-Riemannian geometry. Every nonholonomic Riemannian structure has an associated sub-Riemannian structure. However, the geometries of the two structures are, in general, inequivalent. In this section we briefly contrast the two geometries, and mention how (in general) there is no relation between the sub-Riemannian geodesics and the nonholonomic geodesics.

In the second part of this chapter, section 1.2, we specialise to left-invariant nonholonomic Riemannian structures on Lie groups, i.e., structures invariant under left translations. As mentioned in [71], these structures model generalised problems on a rolling solid body, and are "of most interest." For these structures, we show that the associated objects (the nonholonomic connection, the nonholonomic geodesic spray, etc.) are also left invariant. As a consequence, we are able to show that every left-invariant nonholonomic Riemannian structure is geodesically complete (i.e., the domain of every nonholonomic geodesic may be extended to the entirety of $\mathbb{R}$). In section 1.2.1 we consider a special class of left-invariant nonholonomic Riemannian structures, *viz.*, those whose nonholonomic geodesics are left cosets of one-parameter subgroups (or equivalently, whose reduced dynamics is trivial). For such structures, the nonholonomic connection is called a “Cartan–Schouten connection.” The main contribution of this section is to characterise when the nonholonomic connection (of a left-invariant nonholonomic Riemannian structure) is Cartan–Schouten. Invariant nonholonomic Riemannian structures with Cartan–Schouten connections are the analogues of bi-invariant Riemannian metrics, and hence the characterisations of these structures are reminiscent of those for bi-invariant metrics in Riemannian geometry. Lastly, in section 1.2.2 we consider the question of existence of (left-invariant) nonholonomic Riemannian structures with Cartan–Schouten connections. In the Riemannian case, it is well-known (see [49]) which Lie groups admit a bi-invariant metric (and this class of Lie groups is quite small). The nonholonomic Riemannian case is not so straightforward, and there are many more examples of structures with Cartan–Schouten connections. In fact, we prove in this section that every Lie group that admits a rank two left-invariant distribution, admits a nonholonomic Riemannian structure (with the same distribution) whose nonholonomic connection is Cartan–Schouten.

1.1 Nonholonomic Riemannian structures

Let M be an n-dimensional manifold. By a *distribution* of rank r on M we shall mean a vector subbundle of $T\mathsf{M}$ with r-dimensional fibres. If $[X, Y] \in \Gamma(\mathcal{D})$ for every $X, Y \in \Gamma(\mathcal{D})$, then $\mathcal{D}$ is called *integrable* (or *holonomic*); if $\mathcal{D}$ is not integrable, then it is called *nonintegrable* (or *nonholonomic*).

Let $\mathcal{D}$ be a rank $r < n$ nonintegrable distribution on M. (The pair $(\mathsf{M}, \mathcal{D})$ is sometimes called a “nonholonomic manifold.”) The *flag of* $\mathcal{D}$ is the increasing filtration $\mathcal{D}^1 \subseteq \mathcal{D}^2 \subseteq \cdots$, where

$$\mathcal{D}^1 = \mathcal{D} \quad \text{and} \quad \mathcal{D}^{i+1} = \mathcal{D}^i + [\mathcal{D}^i, \mathcal{D}^i], \; i \geq 1. \tag{1.1.1}$$

Here $[\mathcal{E}, \mathcal{E}']_q = \{[X, Y](q) : X \in \Gamma(\mathcal{E}),\; Y \in \Gamma(\mathcal{E}')\}$, $q \in \mathsf{M}$, where $\mathcal{E}$ and $\mathcal{E}'$ are distributions on M. We will always assume that each element of the flag of a distribution is itself a distribution, i.e., the dimension of the fibres is constant. Evidently, the flag of $\mathcal{D}$ will stabilise after finitely many steps. If there exists $N \geq 2$ such that $\mathcal{D}^N = T\mathsf{M}$ and $\mathcal{D}^{N-1} \subsetneq T\mathsf{M}$, then $\mathcal{D}$ is said to be *completely nonholonomic*, and N is called the *degree of nonholonomy of* $\mathcal{D}$. The simplest case is when $N = 2$; in this case, $\mathcal{D}$ is called *strongly nonholonomic.*

Remark 1.1.1. If M is three dimensional, then a nonintegrable distribution on M must have rank two. Moreover, it is clear that every nonintegrable distribution on a three-dimensional manifold is strongly nonholonomic. □

Remark 1.1.2. Many authors define the flag of $\mathcal{D}$ as

$$\mathcal{D}^1 = \mathcal{D} \quad \text{and} \quad \mathcal{D}^{i+1} = \mathcal{D}^i + [\mathcal{D}, \mathcal{D}^i] = \mathcal{D}^i + \sum_{j+k=i} [\mathcal{D}^j, \mathcal{D}^k], \; i \geq 1. \tag{1.1.2}$$

(See, e.g., [38].) However, we prefer the definition (1.1.1), as it is more naturally used in defining the Wagner curvature tensor (see, e.g., [75, 21]), which we discuss further in chapter 2. Nevertheless, the definitions and results in this thesis that rely on the flag of $\mathcal{D}$ are readily modified to use the formulation (1.1.2). □

A curve $\gamma : I \to \mathsf{M}$ is called a *$\mathcal{D}$-curve* if it is tangent to $\mathcal{D}$, i.e., $\dot\gamma(t) \in \mathcal{D}_{\gamma(t)}$ for every $t \in I$. (Here, and henceforth, I denotes an interval of $\mathbb{R}$.) Complete nonholonomy of $\mathcal{D}$ is a sufficient condition for connectivity of M by $\mathcal{D}$-curves:

Theorem 1.1.3 (Chow–Rashevskii, [59, 17]). *Let $\mathcal{D}$ be a completely nonholonomic distribution on a connected manifold M. Then any two points in M can be joined by a $\mathcal{D}$-curve.*

A *nonholonomic Riemannian manifold* is a quadruple $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$, where M is an n-dimensional (connected) manifold, $\mathcal{D}$ is a rank $r < n$ completely nonholonomic distribution on M, $\mathcal{D}^\perp$ is a rank $n - r$ distribution complementary to $\mathcal{D}$ (so that $T\mathsf{M} = \mathcal{D} \oplus \mathcal{D}^\perp$) and $\mathbf{g}$ is a positive definite metric tensor on $\mathcal{D}$. For convenience, we shall also refer to a nonholonomic Riemannian manifold as a *nonholonomic Riemannian structure.*

Remark 1.1.4. We shall sometimes abuse the above definition slightly to include the case when $r = n$, i.e., $\mathcal{D} = T\mathsf{M}$ and $\mathcal{D}^\perp = \{0\}$. In this case, $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ is a Riemannian manifold $(\mathsf{M}, \mathbf{g})$. □

Let $\mathscr{P} : T\mathsf{M} \to \mathcal{D}$ and $\mathscr{Q} : T\mathsf{M} \to \mathcal{D}^\perp$ be the projection operators corresponding to the decomposition $T\mathsf{M} = \mathcal{D} \oplus \mathcal{D}^\perp$. For convenience, we shall denote the projected Lie bracket $\mathscr{P}([\cdot,\cdot])$ by $[\![\cdot,\cdot]\!]$. Let $\|\cdot\|$ be the norm on $\mathcal{D}$ induced by $\mathbf{g}$.

In Riemannian geometry, there exists a unique metric and torsion-free connection (the Levi-Civita connection) associated to every Riemannian structure (see, e.g., [44, 55, 53]). Remarkably, this result generalises to nonholonomic Riemannian geometry: associated to every nonholonomic Riemannian structure $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ is a unique affine connection of the form $\nabla : \Gamma(\mathcal{D}) \times \Gamma(\mathcal{D}) \to \Gamma(\mathcal{D})$ (i.e., a $\mathcal{D}$-restricted connection on $\mathcal{D}$; see appendix B) that is metric and has vanishing torsion. Here the *torsion* of ∇ (with respect to $\mathscr{P}$) is the $(1, 2)$-tensor field $T : \Gamma(\mathcal{D}) \times \Gamma(\mathcal{D}) \to \Gamma(\mathcal{D})$ given by

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [\![X, Y]\!], \qquad X, Y \in \Gamma(\mathcal{D}).$$

This connection is called the *nonholonomic connection* of $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$. We have the following result (see, e.g., [43]).

Theorem 1.1.5. *Let $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ be a nonholonomic Riemannian manifold. There exists a unique $\mathcal{D}$-connection ∇ on $\mathcal{D}$ that is both metric and torsion-free, i.e., $\nabla\mathbf{g} \equiv 0$ and $T \equiv 0$. Furthermore, ∇ is characterised by the* Koszul formula:

$$\begin{aligned}
2\,\mathbf{g}(\nabla_X Y, Z) &= (\mathscr{L}_Y^{\mathscr{P}}\mathbf{g})(X, Z) + (d_{\mathscr{P}}\mathbf{g}^\flat(Y))(X, Z) \\
&= X[\mathbf{g}(Y, Z)] + Y[\mathbf{g}(X, Z)] - Z[\mathbf{g}(X, Y)] \\
&\quad + \mathbf{g}([\![X, Y]\!], Z) - \mathbf{g}([\![X, Z]\!], Y) - \mathbf{g}([\![Y, Z]\!], X),
\end{aligned}$$

*for $X, Y, Z \in \Gamma(\mathcal{D})$. Here $\mathscr{L}_X^{\mathscr{P}}$ is the $\mathscr{P}$-*Lie derivative *(see section A.3.1) and $d_{\mathscr{P}}$ is the $\mathscr{P}$-*exterior derivative *(section A.3.2).*

Proof. Koszul's formula uniquely defines the operation $\Gamma(\mathcal{D}) \times \Gamma(\mathcal{D}) \to \Gamma(\mathcal{D})$, $(X, Y) \mapsto \nabla_X Y$. We show that this operation is in fact a $\mathcal{D}$-restricted connection on $\mathcal{D}$. Clearly, ∇ is $\mathbb{R}$-linear in both of its arguments. Let $X, Y, Z \in \Gamma(\mathcal{D})$ and $f \in \mathcal{C}^\infty(\mathsf{M})$. We have

$$\begin{aligned}
(\mathscr{L}_Y^{\mathscr{P}}\mathbf{g})(fX, Z) &= Y[\mathbf{g}(fX, Z)] - \mathbf{g}([\![Y, fX]\!], Z) - \mathbf{g}(fX, [\![Y, Z]\!]) \\
&= Y[f]\mathbf{g}(X, Z) + fY[\mathbf{g}(X, Z)] - \mathbf{g}(f[\![Y, X]\!] + Y[f]X, Z) - \mathbf{g}(fX, [\![Y, Z]\!]) \\
&= f(\mathscr{L}_Y^{\mathscr{P}}\mathbf{g})(X, Z).
\end{aligned}$$

Similarly,

$$\begin{aligned}(d_{\mathscr{P}}\mathbf{g}^\flat(Y))(fX,Z) &= fX[\mathbf{g}(Y,Z)] - Z[\mathbf{g}(Y,fX)] - \mathbf{g}(Y,\llbracket X,Z\rrbracket)\\ &= fX[\mathbf{g}(Y,Z)] - fZ[\mathbf{g}(Y,X)] - Z[f]\mathbf{g}(Y,X) - \mathbf{g}(Y,f\llbracket X,Z\rrbracket - Z[f]X)\\ &= f\,(d_{\mathscr{P}}\mathbf{g}^\flat(Y))(X,Z).\end{aligned}$$

It follows that ∇ is tensorial in its first argument. We also have

$$\begin{aligned}(\pounds^{\mathscr{P}}_{fY}\mathbf{g})(X,Z) &= fY[\mathbf{g}(X,Z)] - \mathbf{g}(\llbracket fY,X\rrbracket,Z) - \mathbf{g}(X,\llbracket fY,Z\rrbracket)\\ &= fY[\mathbf{g}(X,Z)] - \mathbf{g}(f\llbracket Y,X\rrbracket - X[f]Y,Z) - \mathbf{g}(X,f\llbracket Y,Z\rrbracket - Z[f]Y)\\ &= f(\pounds^{\mathscr{P}}_{Y}\mathbf{g})(X,Z) + X[f]\mathbf{g}(Y,Z) + Z[f]\mathbf{g}(X,Y)\end{aligned}$$

and

$$\begin{aligned}(d_{\mathscr{P}}\mathbf{g}^\flat(fY))(X,Z) &= f\,(d_{\mathscr{P}}\mathbf{g}^\flat(Y))(X,Z) + (d_{\mathscr{P}}f \wedge \mathbf{g}^\flat(Y))(X,Z)\\ &= f\,(d_{\mathscr{P}}\mathbf{g}^\flat(Y))(X,Z) + X[f]\mathbf{g}(Y,Z) - Z[f]\mathbf{g}(Y,X).\end{aligned}$$

Hence $2\,\mathbf{g}(\nabla_X fY, Z) = 2\,\mathbf{g}(X[f]Y + f\nabla_X Y, Z)$, i.e., ∇ is a derivation in its second argument. Thus ∇ is a $\mathcal{D}$-restricted connection on $\mathcal{D}$.

It remains to show that ∇ (defined via the Koszul formula) is metric and torsion-free. For metricity, we have

$$\begin{aligned}\mathbf{g}(\nabla_Z X,Y) + \mathbf{g}(X,\nabla_Z Y) &= \frac{1}{2}\big(Z[\mathbf{g}(X,Y)] + X[\mathbf{g}(Z,Y)] - Y[\mathbf{g}(Z,X)]\\ &\qquad + \mathbf{g}(\llbracket Z,X\rrbracket,Y) - \mathbf{g}(\llbracket Z,Y\rrbracket,X) - \mathbf{g}(\llbracket X,Y\rrbracket,Z)\big)\\ &\quad + \frac{1}{2}\big(Z[\mathbf{g}(Y,X)] + Y[\mathbf{g}(Z,X)] - X[\mathbf{g}(Z,Y)]\\ &\qquad + \mathbf{g}(\llbracket Z,Y\rrbracket,X) - \mathbf{g}(\llbracket Z,X\rrbracket,Y) - \mathbf{g}(\llbracket Y,X\rrbracket,Z)\big)\\ &= Z[\mathbf{g}(X,Y)].\end{aligned}$$

That is, $\nabla\mathbf{g} \equiv 0$. Similarly for the torsion, we have

$$\begin{aligned}\mathbf{g}(\nabla_X Y - \nabla_Y X, Z) &= \frac{1}{2}\big(X[\mathbf{g}(Y,Z)] + Y[\mathbf{g}(X,Z)] - Z[\mathbf{g}(X,Y)]\\ &\qquad + \mathbf{g}(\llbracket X,Y\rrbracket,Z) - \mathbf{g}(\llbracket X,Z\rrbracket,Y) - \mathbf{g}(\llbracket Y,Z\rrbracket,X)\big)\\ &\quad - \frac{1}{2}\big(Y[\mathbf{g}(X,Z)] + X[\mathbf{g}(Y,Z)] - Z[\mathbf{g}(Y,X)]\\ &\qquad + \mathbf{g}(\llbracket Y,X\rrbracket,Z) - \mathbf{g}(\llbracket Y,Z\rrbracket,X) - \mathbf{g}(\llbracket X,Z\rrbracket,Y)\big)\\ &= \mathbf{g}(\llbracket X,Y\rrbracket,Z).\end{aligned}$$

Hence $\nabla_X Y - \nabla_Y X = \llbracket X,Y\rrbracket$, i.e., ∇ is torsion free. ■

Let (X_i) be a local frame for $T\mathsf{M}$ such that (X_a) is an orthonormal frame for $\mathcal{D}$ and (X_λ) is a frame for $\mathcal{D}^\perp$. Let $c^k_{ij} \in \mathcal{C}^\infty(\mathsf{M})$ be the structure constants of this frame: $[X_i, X_j] = c^k_{ij}X_k$. Likewise, let $\Gamma^w_{ab} \in \mathcal{C}^\infty(\mathsf{M})$ be the connection coefficients of ∇: $\nabla_{X_a}X_b = \Gamma^w_{ab}X_w$.

Corollary 1.1.6. *We have* $\Gamma^w_{ab} = \frac{1}{2}(c^w_{ab} - c^b_{aw} - c^a_{bw})$. *(In particular, the connection coefficients* Γ^w_{ab} *depend only on the orthonormal frame* (X_a) *and the structure constants of* $\llbracket X_a, X_b\rrbracket$*.)*

Proof. From Koszul's formula, we have

$$\begin{aligned}
2\,\mathbf{g}(\nabla_{X_a} X_b, X_w) &= X_a[\mathbf{g}(X_b, X_w)] + X_b[\mathbf{g}(X_a, X_w)] - X_w[\mathbf{g}(X_a, X_b)] \\
&\quad + \mathbf{g}([\![X_a, X_b]\!], X_w) - \mathbf{g}([\![X_a, X_w]\!], X_b) - \mathbf{g}([\![X_b, X_w]\!], X_a) \\
&= c_{ab}^u\,\mathbf{g}(X_u, X_w) - c_{aw}^u\,\mathbf{g}(X_u, X_b) - c_{bw}^u\,\mathbf{g}(X_u, X_a) \\
&= c_{ab}^w - c_{aw}^b - c_{bw}^a.
\end{aligned}$$

That is, $\Gamma_{ab}^w = \frac{1}{2}(c_{ab}^w - c_{aw}^b - c_{bw}^a)$. ■

Using the nonholonomic connection ∇, the geodesics of a nonholonomic Riemannian structure are defined to be the self-parallel $\mathcal{D}$-curves of the connection. More precisely, a $\mathcal{D}$-curve $\gamma : I \to \mathsf{M}$ is called a *nonholonomic geodesic* of $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ (cf. [71]) if

$$\nabla_{\dot\gamma}\dot\gamma(t) = 0 \quad \text{for every } t \in I.$$

(We shall also sometimes refer to a nonholonomic geodesic as a geodesic of the nonholonomic connection.) In terms of a local frame for $\mathcal{D}$, the nonholonomic geodesics can be seen to satisfy a system of second-order ordinary differential equations. In particular, this implies the existence and uniqueness of the nonholonomic geodesic through a given initial point and in a given admissible direction.

Lemma 1.1.7. *A $\mathcal{D}$-curve $\gamma : I \to \mathsf{M}$ is a nonholonomic geodesic if and only if*

$$\ddot\gamma^w(t) + \Gamma_{ab}^w(\gamma(t))\dot\gamma^a(t)\dot\gamma^b(t) = 0$$

for every $t \in I$. (The components $\dot\gamma^a$ of $\dot\gamma$ are given by $\dot\gamma = \dot\gamma^a(X_a \circ \gamma)$.)

Proof. Let $\gamma : I \to \mathsf{M}$ be a $\mathcal{D}$-curve. With respect to the (local) frame (X_a) for $\mathcal{D}$, there exist functions $\dot\gamma^a \in \mathcal{C}^\infty(I)$ such that $\dot\gamma(t) = \dot\gamma^a(t)X_a(\gamma(t))$ for every $t \in I$. Since $\dot\gamma[\dot\gamma^a] = \ddot\gamma^a$, we have

$$\begin{aligned}
\nabla_{\dot\gamma}\dot\gamma = 0 \quad &\iff \quad \dot\gamma[\dot\gamma^a](X_a \circ \gamma) + \dot\gamma^a \nabla_{\dot\gamma}(X_a \circ \gamma) = 0 \\
&\iff \quad \ddot\gamma^a(X_a \circ \gamma) + \dot\gamma^a\dot\gamma^b \nabla_{(X_b \circ \gamma)}(X_a \circ \gamma) = 0 \\
&\iff \quad \ddot\gamma^a(X_a \circ \gamma) + \dot\gamma^a\dot\gamma^b(\Gamma_{ab}^w \circ \gamma)(X_w \circ \gamma) = 0 \\
&\iff \quad \left[\ddot\gamma^w + (\Gamma_{ab}^w \circ \gamma)\dot\gamma^a\dot\gamma^b\right](X_w \circ \gamma) = 0.
\end{aligned}$$

That is, γ is a nonholonomic geodesic if and only if $\ddot\gamma^w + \Gamma_{ab}^w(\gamma)\dot\gamma^a\dot\gamma^b = 0$. ■

Lemma 1.1.8. *Let $q \in \mathsf{M}$ and $X_q \in \mathcal{D}_q$. There exists a unique (up to domain) nonholonomic geodesic $\gamma : I \to \mathsf{M}$ (with $0 \in I$) such that $\gamma(0) = q$ and $\dot\gamma(0) = X_q$.*

Proof. If γ is a nonholonomic geodesic, then $\ddot\gamma^w = -\Gamma_{ab}^w(\gamma)\dot\gamma^a\dot\gamma^b$, where $\dot\gamma = \dot\gamma^a(X_a \circ \gamma)$. That is, the curve $t \mapsto \dot\gamma(t)$ satisfies a first-order ordinary differential equation. This ODE has a solution $\dot\gamma$, defined on some interval $I \subseteq \mathbb{R}$ containing zero; specifying the initial condition $\dot\gamma(0) = X_q$ determines a unique solution. Then $\gamma = \tau_{\mathsf{M}} \circ \dot\gamma$ is the unique nonholonomic geodesic such that $\gamma(0) = q$ and $\dot\gamma(0) = X_q$. (Here $\tau_{\mathsf{M}} : T\mathsf{M} \to \mathsf{M}$ is the canonical projection of a tangent vector onto its base point.) ■

Lemma 1.1.8 implies that, through a point $q \in \mathsf{M}$ and in a given direction $X_q \in \mathcal{D}_q$, there passes a unique nonholonomic geodesic $\gamma : I \to \mathsf{M}$ with maximal domain. That is, if $\widetilde{\gamma} : \widetilde{I} \to \mathsf{M}$ is any other nonholonomic geodesic satisfying $\widetilde{\gamma}(0) = q$ and $\dot{\widetilde{\gamma}}(0) = X_q$, then $\widetilde{I} \subseteq I$ and $\gamma|_{\widetilde{I}} = \widetilde{\gamma}$. If $I = \mathbb{R}$ for every nonholonomic geodesic with maximal domain (or, in other words, the domain of every nonholonomic geodesic may be extended to the entirety of $\mathbb{R}$), then $(\mathsf{M}, \mathcal{D}, \mathcal{D}^{\perp}, \mathbf{g})$ is said to be *geodesically complete.*

Proposition 1.1.9. *Every nonholonomic geodesic γ has constant speed, i.e., $\|\dot{\gamma}(t)\|$ is constant in t.*

Proof. Let γ be a nonholonomic geodesic. Since ∇ is metric, we have

$$0 = 2\,\mathbf{g}(\nabla_{\dot{\gamma}}\dot{\gamma}, \dot{\gamma}) = \dot{\gamma}[\mathbf{g}(\dot{\gamma}, \dot{\gamma})] = \frac{d}{dt}\,\mathbf{g}(\dot{\gamma}, \dot{\gamma}).$$

Hence $\|\dot{\gamma}(t)\| = \sqrt{\mathbf{g}_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))} = \text{constant}$. ■

Corollary 1.1.10. *Every non-constant nonholonomic geodesic $\gamma : I \to \mathsf{M}$ is regular, i.e., $\dot{\gamma}(t) \neq 0$ for every $t \in I$.*

Proposition 1.1.11. *Let $\gamma : I \to \mathsf{M}$ be a non-constant nonholonomic geodesic and $\varphi : \widetilde{I} \to I$ a diffeomorphism. The (reparametrised) $\mathcal{D}$-curve $\gamma \circ \varphi$ is a nonholonomic geodesic if and only if $\varphi(t) = at + b$ for some $a, b \in \mathbb{R}$.*

Proof. Let $\widetilde{\gamma} = \gamma \circ \varphi$. Then $\dot{\widetilde{\gamma}}(t) = \dot{\gamma}(\varphi(t))\dot{\varphi}(t) \in \mathcal{D}_{\gamma(\varphi(t))} = \mathcal{D}_{\widetilde{\gamma}(t)}$ for every $t \in I$, and so $\widetilde{\gamma}$ is a $\mathcal{D}$-curve. Furthermore, we have

$$\begin{aligned}
\nabla_{\dot{\widetilde{\gamma}}}\dot{\widetilde{\gamma}}(t) = 0 \quad &\iff \quad \nabla_{\dot{\widetilde{\gamma}}}((\dot{\gamma} \circ \varphi)\dot{\varphi})(t) = 0 \\
&\iff \quad \dot{\widetilde{\gamma}}[\dot{\varphi}](t)\dot{\widetilde{\gamma}}(t) + \dot{\varphi}(t)^2 \nabla_{(\dot{\gamma} \circ \varphi)}(\dot{\gamma} \circ \varphi)(t) = 0 \\
&\iff \quad \ddot{\varphi}(t)\dot{\widetilde{\gamma}}(t) = 0 \\
&\iff \quad \ddot{\varphi}(t) = 0.
\end{aligned}$$

That is, $\widetilde{\gamma}$ is a nonholonomic geodesic if and only if $\varphi(t) = at + b$ for some $a, b \in \mathbb{R}$. ■

Remark 1.1.12. Let $\gamma : I \to \mathsf{M}$ be a unit-speed $\mathcal{D}$-curve in M (i.e., $\|\dot{\gamma}(t)\| = 1$ for every $t \in I$). The *geodesic curvature* of γ is the function $\kappa_\gamma : I \to \mathbb{R}$ given by

$$\kappa_\gamma(t) = \|\nabla_{\dot{\gamma}}\dot{\gamma}(t)\|.$$

Evidently, γ is a nonholonomic geodesic if and only if $\kappa_\gamma \equiv 0$. That is, the nonholonomic geodesics of $(\mathsf{M}, \mathcal{D}, \mathcal{D}^{\perp}, \mathbf{g})$ are exactly those $\mathcal{D}$-curves that have vanishing geodesic curvature. In this sense, the nonholonomic geodesics can be interpreted as the "straightest" curves in M (an observation originally due to Hertz [32]). We can also view the nonholonomic geodesics as the solutions of the "instantaneous variational problem" [8]

$$\min_{\gamma} \kappa_\gamma \quad \text{subject to} \quad \dot{\gamma}(t) \in \mathcal{D}_{\gamma}(t) \text{ for every } t.$$

This approach to obtaining the nonholonomic equations of motion is referred to as *Hertz's Principle of Least Curvature* (which is a special case of *Gauss's Principle of Least Constraint*); see, e.g., [32, 46]. □

1.1.1 Motivation from nonholonomic mechanics

In this section we motivate the definition of a nonholonomic Riemannian structure by discussing how they are related to nonholonomic mechanical (specifically, Lagrangian) systems. We begin by briefly recounting the link between Riemannian manifolds and mechanical systems described by kinetic energy Lagrangians.

Let $\widetilde{\mathbf{g}}$ be a Riemannian metric on an n-dimensional (connected) manifold M. Let (q^i) be local coordinates on M, $(q^i, \dot{q}^i)$ the corresponding coordinates on $T\mathsf{M}$ and (∂_i) the coordinate frame $(\partial/\partial q^i)$. The geodesics of the Riemannian manifold $(\mathsf{M}, \widetilde{\mathbf{g}})$ are exactly the geodesics of the associated Levi-Civita connection $\widetilde{\nabla}$ (see, e.g., [44, 55, 53, 58]). These geodesics coincide with the extremal curves of the mechanical system given by the (positive definite) kinetic energy Lagrangian $L : T\mathsf{M} \to \mathbb{R}$, $(q^i, \dot{q}^i) \mapsto \frac{1}{2} g_{k\ell} \dot{q}^k \dot{q}^\ell$ (where $g_{k\ell} = \widetilde{\mathbf{g}}(\partial_k, \partial_\ell)$), i.e., the solutions of the Euler–Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = 0$$

(see, e.g., [29]). Indeed, for the kinetic energy Lagrangian, the Euler–Lagrange equations are

$$g_{ij}(\ddot{q}^j + \Gamma^j_{k\ell}\dot{q}^k\dot{q}^\ell) = 0, \tag{1.1.3}$$

where $\Gamma^j_{k\ell} \in \mathcal{C}^\infty(\mathsf{M})$ are the Christoffel symbols of $\widetilde{\nabla}$, given by $\widetilde{\nabla}_{\partial_k}\partial_\ell = \Gamma^j_{k\ell}\partial_j$. If $\gamma : t \mapsto (q^i(t))$ is a curve in M, then $\widetilde{\nabla}_{\dot{\gamma}}\dot{\gamma} = (\ddot{q}^j + \Gamma^j_{k\ell}\dot{q}^k\dot{q}^\ell)\partial_j$. Hence (1.1.3) may be written in the intrinsic form $\widetilde{\mathbf{g}}^\flat(\widetilde{\nabla}_{\dot{\gamma}}\dot{\gamma}) = 0$.

We now introduce (time-independent) linear nonholonomic constraints. Locally, these are given by $n - r$ equations of the form

$$f_j^{n-\lambda+1}(q^i)\dot{q}^j = 0, \qquad \operatorname{rank}\left[f_j^{n-\lambda+1}\right] = n - r,$$

where $f_j^{n-\lambda+1} \in \mathcal{C}^\infty(\mathsf{M})$. Geometrically, the constraints are represented by a (rank r) nonintegrable distribution $\mathcal{D}$ on M; in fact, we shall assume that $\mathcal{D}$ is completely nonholonomic.

Let $\mathcal{D}^\perp$ be the orthogonal complement of $\mathcal{D}$ with respect to the metric $\widetilde{\mathbf{g}}$. If $\widetilde{\mathbf{g}}|_{\mathcal{D}}$ denotes the restriction of $\widetilde{\mathbf{g}}$ to sections of $\mathcal{D}$, then clearly $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \widetilde{\mathbf{g}}|_{\mathcal{D}})$ is a nonholonomic Riemannian manifold. Furthermore, the projections $\mathscr{P}$ and $\mathscr{Q}$ coincide with the orthogonal projections onto $\mathcal{D}$ and $\mathcal{D}^\perp$, respectively.

It turns out that the nonholonomic connection ∇ of $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \widetilde{\mathbf{g}}|_{\mathcal{D}})$ can be written as the projection (onto $\mathcal{D}$) of the Levi-Civita connection $\widetilde{\nabla}$. (For this reason, the nonholonomic connection is sometimes also called the "truncated connection" [71].)

Proposition 1.1.13 (cf. [69, 71, 21]). *The nonholonomic connection ∇ is given by*

$$\nabla_X Y = \mathscr{P}(\widetilde{\nabla}_X Y)$$

for every $X, Y \in \Gamma(\mathcal{D})$.

Proof. Let $\overline{\nabla} : \Gamma(\mathcal{D}) \times \Gamma(\mathcal{D}) \to \Gamma(\mathcal{D})$ be defined as $\overline{\nabla}_X Y = \mathscr{P}(\widetilde{\nabla}_X Y)$ for $X, Y \in \Gamma(\mathcal{D})$. We first show that $\overline{\nabla}$ is a $\mathcal{D}$-restricted connection on $\mathcal{D}$. Clearly, $\overline{\nabla}$ is $\mathbb{R}$-linear. Let $X, Y, Z \in \Gamma(\mathcal{D})$ and $f \in \mathcal{C}^\infty(\mathsf{M})$. Then

$$\overline{\nabla}_{fX} Y = \mathscr{P}(\widetilde{\nabla}_{fX} Y) = f\mathscr{P}(\widetilde{\nabla}_X Y) = f\overline{\nabla}_X Y,$$

i.e., $\overline{\nabla}$ is tensorial in its first argument. Furthermore, we have

$$\overline{\nabla}_X(fY) = \mathscr{P}(\widetilde{\nabla}_X(fY)) = \mathscr{P}(X[f]Y + f\widetilde{\nabla}_X Y) = X[f]Y + f\overline{\nabla}_X Y,$$

and so $\overline{\nabla}$ is a derivation in its second argument. Thus $\overline{\nabla}$ is a $\mathcal{D}$-restricted connection on $\mathcal{D}$. That $\overline{\nabla}$ is metric (i.e., $\overline{\nabla}\tilde{\mathbf{g}}|_{\mathcal{D}} \equiv 0$) follows from the metricity of $\widetilde{\nabla}$ and the orthogonality of $\mathcal{D}$ and $\mathcal{D}^\perp$:

$$Z[\tilde{\mathbf{g}}(X,Y)] = \tilde{\mathbf{g}}(\widetilde{\nabla}_Z X, Y) + \tilde{\mathbf{g}}(X, \widetilde{\nabla}_Z Y) = \tilde{\mathbf{g}}(\overline{\nabla}_Z X, Y) + \tilde{\mathbf{g}}(X, \overline{\nabla}_Z Y).$$

Similarly, we have $[\![X,Y]\!] = \mathscr{P}(\widetilde{\nabla}_X Y - \widetilde{\nabla}_Y X) = \overline{\nabla}_X Y - \overline{\nabla}_Y X$, and so $\overline{\nabla}$ has vanishing torsion. By uniqueness of the nonholonomic connection (theorem 1.1.5), it follows that $\overline{\nabla} = \nabla$. ■

The following characterisation of the nonholonomic geodesics is an equivalent formulation of the Lagrange–D'Alembert Principle (cf. [19, 18, 46]).

Corollary 1.1.14. *A $\mathcal{D}$-curve $\gamma : I \to \mathsf{M}$ is a nonholonomic geodesic of* $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \tilde{\mathbf{g}}|_{\mathcal{D}})$ *if and only if*

$$\widetilde{\nabla}_{\dot\gamma}\dot\gamma(t) \in \mathcal{D}^\perp_{\gamma(t)} \quad \textit{for every } t \in I.$$

Proof. The condition $\widetilde{\nabla}_{\dot\gamma}\dot\gamma(t) \in \mathcal{D}^\perp_{\gamma(t)}$ is equivalently written as $\mathscr{P}(\widetilde{\nabla}_{\dot\gamma}\dot\gamma(t)) = 0$. Thus, for a $\mathcal{D}$-curve γ, we have $\widetilde{\nabla}_{\dot\gamma}\dot\gamma(t) \in \mathcal{D}^\perp_{\gamma(t)}$ if and only if $\nabla_{\dot\gamma}\dot\gamma(t) = \mathscr{P}(\widetilde{\nabla}_{\dot\gamma}\dot\gamma(t)) = 0$. ■

The nonholonomic geodesics of $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \tilde{\mathbf{g}}|_{\mathcal{D}})$ coincide with the nonholonomic extremals of (the kinetic energy Lagrangian) L, subject to the (linear-in-velocities) nonholonomic constraint represented by $\mathcal{D}$. Indeed, the nonholonomic extremals of L are the solutions of the Chetaev equations for L (see, e.g., [18, 8])

$$\frac{d}{dt}\frac{\partial L}{\partial \dot q^i} - \frac{\partial L}{\partial q^i} = \nu_{n-\lambda+1}\varphi^{n-\lambda+1}, \tag{1.1.4}$$

where $\varphi^{n-\lambda+1} = f_j^{n-\lambda+1} dq^j$ and $\nu_{n-\lambda+1} \in \mathcal{C}^\infty(\mathsf{M})$ are Lagrange multipliers, determined by the constraints. (In the case of linear nonholonomic constraints, the Chetaev equations are often called the Lagrange–D'Alembert equations.) Specifically for the kinetic energy Lagrangian, the equations (1.1.4) are given by

$$g_{ij}(\ddot q^j + \Gamma^j_{ab}\dot q^a \dot q^b) = \nu_{n-\lambda+1} f_i^{n-\lambda+1}. \tag{1.1.5}$$

Observing that the annihilator $\mathcal{D}^\circ = \tilde{\mathbf{g}}^\flat(\mathcal{D}^\perp)$ of $\mathcal{D}$ is spanned by $\varphi^1, \dots, \varphi^{n-r}$, we can write (1.1.5) in the invariant form $\tilde{\mathbf{g}}^\flat(\widetilde{\nabla}_{\dot\gamma}\dot\gamma) \in \mathcal{D}^\circ$. The claim now follows by corollary 1.1.14. (We should also mention that the expression of the nonholonomic equations of motion in terms of the nonholonomic connection is exactly the geometric expression of the *reduced* Chetaev equations; for more details, see, e.g., [41, 61].)

As we have seen, the nonholonomic extremals of a kinetic energy Lagrangian subject to linear nonholonomic constraints are exactly the nonholonomic geodesics of the associated nonholonomic Riemannian structure. For this reason, we view the associated nonholonomic Riemannian structure as the fundamental underlying geometric object, which motivates our study of these structures in this thesis.

1.1.2 The nonholonomic geodesic spray and exponential map

If $\gamma : I \to \mathsf{M}$ is a nonholonomic geodesic of $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$, then it satisfies the (second-order) ordinary differential equation $\nabla_{\dot\gamma}\dot\gamma(t) = 0$ (see lemma 1.1.7). On the other hand, this may be viewed as a first-order differential equation satisfied by the curve $I \ni t \mapsto \dot\gamma(t) \in \mathcal{D}_{\gamma(t)}$. Accordingly, we expect that there should exist a second-order vector field on $\mathcal{D}$ whose integral curves are exactly the tangent lifts $t \mapsto \dot\gamma(t)$ of nonholonomic geodesics γ in M.

Let $\pi : \mathcal{D} \to \mathsf{M}$ be the canonical projection of a $\mathcal{D}$-vector onto its base point and let $\iota : \mathcal{D} \to T\mathsf{M}$ be the inclusion map. A vector field $Z \in \Gamma(T\mathcal{D})$ is called a *nonholonomic semispray* (cf. [56]) if $T\pi \cdot Z = \iota$. If, in addition, Z satisfies $Z \circ \phi_t = T\phi_t \cdot e^t Z$, where $\phi_t : \mathcal{D} \to \mathcal{D}$, $U_q \mapsto e^t U_q$ is the canonical dilation on $\mathcal{D}$, then it is called a *nonholonomic spray.* (In this case, the component functions of Z with respect to a local frame can be shown to be 2-homogeneous.)

Let $h : \pi^*\mathcal{D} \to T\mathcal{D}$ be the $\mathcal{D}$-restricted Ehresmann connection on $\mathcal{D}$ associated to ∇ (see section B.2). That is,

$$h(U_q, X_q) = T_q U \cdot X_q - \mathrm{vl}_{U_q} \cdot \nabla_{X_q} U, \qquad (U_q, X_q) \in \pi^*\mathcal{D},$$

where $U \in \Gamma(\mathcal{D})$ is any vector field such that $U(q) = U_q$. Let $\Xi \in \Gamma(T\mathcal{D})$ be the vector field on $\mathcal{D}$ defined as

$$\Xi(U_q) = h(U_q, U_q), \qquad U_q \in \mathcal{D}.$$

Lemma 1.1.15. *Ξ is a nonholonomic spray.*

Proof. Let $U_q \in \mathcal{D}$. From the properties of a restricted Ehresmann connection, we have $(\tau_{\mathcal{D}} \circ \Xi)(U_q) = (\tau_{\mathcal{D}} \circ h)(U_q, U_q) = U_q$ and $T_{U_q}\pi \cdot \Xi(U_q) = T_{U_q}\pi \cdot h(U_q, U_q) = \iota(U_q)$. That is, $\tau_{\mathcal{D}} \circ \Xi = \mathrm{id}_{\mathcal{D}}$ and $T\pi \cdot \Xi = \iota$. Hence Ξ is a nonholonomic semispray. Furthermore, since h is $\mathbb{R}$-linear in its second argument and is a linear connection, we have

$$\Xi(\phi_t(U_q)) = h(\phi_t(U_q), \phi_t(U_q)) = T_{U_q}\phi_t \cdot e^t h(U_q, U_q) = T_{U_q}\phi_t \cdot e^t\, \Xi(U_q).$$

That is, Ξ is a nonholonomic spray. ■

The vector field Ξ is called the *nonholonomic geodesic spray* of $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$. The name is justified by the following result.

Proposition 1.1.16. *If γ is a nonholonomic geodesic, then $t \mapsto \dot\gamma(t)$ is an integral curve of the nonholonomic geodesic spray Ξ. Conversely, if η is an integral curve of Ξ, then $\pi \circ \eta$ is a nonholonomic geodesic.*

Proof. Let $\gamma : I \to \mathsf{M}$ be a nonholonomic geodesic and let $X \in \Gamma(\mathcal{D})$ be a local extension of $\dot\gamma$ along γ, i.e., $X(\gamma(t)) = \dot\gamma(t)$ for every $t \in I$ (possibly by shrinking I). Then

$$\Xi(\dot\gamma(t)) = h(\dot\gamma(t), \dot\gamma(t)) = T_{\gamma(t)}X \cdot \dot\gamma(t) - \mathrm{vl}_{\dot\gamma(t)} \cdot \nabla_X X(\gamma(t)) = T_{\gamma(t)}X \cdot \dot\gamma(t),$$

where in the last step we have used the fact that $\nabla_X X(\gamma(t)) = \nabla_{\dot\gamma}\dot\gamma(t) = 0$. Consequently, we get

$$\Xi(\dot\gamma(t)) = \left.\frac{d}{ds}\right|_{s=0} X(\gamma(t+s)) = \ddot\gamma(t).$$

That is, $t \mapsto \dot\gamma(t)$ is an integral curve of Ξ. Conversely, let $\eta : I \to \mathcal{D}$ be an integral curve of Ξ and let $\gamma = \pi \circ \eta$. We have

$$\dot\gamma(t) = \frac{d}{dt}\pi(\eta(t)) = T_{\eta(t)}\pi \cdot \dot\eta(t) = T_{\eta(t)}\pi \cdot \Xi(\eta(t)) = \iota(\eta(t)),$$

and so γ is a $\mathcal{D}$-curve. Furthermore, if $X \in \Gamma(\mathcal{D})$ is a local extension of $\dot\gamma$ along γ, then

$$\begin{aligned}\nabla_{\dot\gamma}\dot\gamma(t) &= \nabla_X X(\gamma(t)) \\ &= \mathrm{vl}^{-1}_{\dot\gamma(t)} \cdot \left[T_{\gamma(t)}X \cdot \dot\gamma(t) - h(\dot\gamma(t), \dot\gamma(t))\right] \\ &= \mathrm{vl}^{-1}_{\dot\gamma(t)} \cdot \left[T_{\gamma(t)}X \cdot \dot\gamma(t) - \Xi(\dot\gamma(t))\right] \\ &= \mathrm{vl}^{-1}_{\dot\gamma(t)} \cdot \left[T_{\gamma(t)}X \cdot \eta(t) - \Xi(\eta(t))\right].\end{aligned}$$

We have

$$T_{\gamma(t)}X \cdot \eta(t) = \left.\frac{d}{ds}\right|_{s=0} X(\gamma(t+s)) = \left.\frac{d}{ds}\right|_{s=0} \dot\gamma(t+s) = \left.\frac{d}{ds}\right|_{s=0} \eta(t+s) = \dot\eta(t)$$

and so $\nabla_{\dot\gamma}\dot\gamma(t) = \mathrm{vl}^{-1}_{\dot\gamma(t)} \cdot [\dot\eta(t) - \Xi(\eta(t))] = 0$. Hence γ is a nonholonomic geodesic. ■

Corollary 1.1.17. $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ *is geodesically complete if and only if* Ξ *is a complete vector field.*

The flow of Ξ, denoted $\Phi_t : \mathcal{D} \to \mathcal{D}$, is referred to as the *nonholonomic geodesic flow.* Clearly, by proposition 1.1.16 and lemma 1.1.8, the curve $t \mapsto \pi(\Phi_t(X_q))$ is the unique nonholonomic geodesic starting from $q \in \mathsf{M}$ in the direction $X_q \in \mathcal{D}_q$.

Lemma 1.1.18. *Let* $X_q \in \mathcal{D}$ *and* $k \in \mathbb{R} \setminus \{0\}$. *Then* $\Phi_t(kX_q) = \Phi_{kt}(X_q)$ *for every* t *for which both sides are defined.*

Proof. Let $\gamma : t \mapsto \pi(\Phi_t(kX_q))$ and $\widetilde\gamma : t \mapsto \pi(\Phi_{kt}(X_q))$. We have $\widetilde\gamma(0) = \gamma(0) = q$ and $\dot{\widetilde\gamma}(0) = k\, T_q\pi \cdot \Xi(\Phi_0(X_q)) = k\,\iota(X_q) = \dot\gamma(0)$. Furthermore, $\nabla_{\dot{\widetilde\gamma}}\dot{\widetilde\gamma}(t) = k^2\, \nabla_{\dot\gamma}\dot\gamma(kt) = 0$ for every t, since γ is a geodesic of ∇. Thus $\widetilde\gamma$ and γ are both nonholonomic geodesics satisfying the same initial conditions, and hence are identical. ■

The nonholonomic geodesic flow is closely related to the *exponential map* of $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$. For each $q \in \mathsf{M}$, let

$$\mathcal{O}_q = \{X_q \in \mathcal{D}_q : \text{the nonholonomic geodesic } t \mapsto \pi(\Phi_t(X_q)) \text{ is defined on } [0,1]\} \subseteq \mathcal{D}_q.$$

Clearly, if $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ is geodesically complete, then $\mathcal{O}_q = \mathcal{D}_q$ for every $q \in \mathsf{M}$.

Lemma 1.1.19. $\mathcal{O}_q$ *is star-shaped about* 0, *i.e., the line segment* $\{t X_q : t \in [0,1]\}$ *is contained in* $\mathcal{O}_q$ *for every* $X_q \in \mathcal{O}_q$.

Proof. Let $X_q \in \mathcal{O}_q$ and $s \in (0,1]$. Since $t \mapsto \pi(\Phi_t(X_q))$ is defined on $[0,1]$, we have that $t \mapsto \pi(\Phi_t(sX_q)) = \pi(\Phi_{st}(X_q))$ is defined on $[0, \frac{1}{s}] \supseteq [0,1]$. Hence $sX_q \in \mathcal{O}_q$. Since s is arbitrary, it follows that $\mathcal{O}_q$ is star-shaped. ■

The *exponential map at* q is defined as

$$\exp_q : \mathcal{O}_q \to \mathsf{M}, \quad \exp_q = \pi \circ \Phi_1|_{\mathcal{O}_q}.$$

From lemma 1.1.18 and lemma 1.1.19, we have $\Phi_t(X_q) = \Phi_1(tX_q)$, and so $\pi(\Phi_t(X_q)) = \pi(\Phi_1(tX_q)) = \exp_q(tX_q)$. Hence $t \mapsto \exp_q(tX_q)$ is the (unique) nonholonomic geodesic starting at q in the direction X_q. Combining the exponential map at each point $q \in \mathsf{M}$ into a single map, we define the *exponential map* as $\exp : \bigsqcup_{q\in\mathsf{M}} \mathcal{O}_q \to \mathsf{M}$, $\exp|_{\mathcal{O}_q} = \exp_q$.

Proposition 1.1.20 (cf. [67]). *Let* $q \in \mathsf{M}$. *The exponential map* $\exp_q : \mathcal{O}_q \to \mathsf{M}$ *is a diffeomorphism from a neighbourhood of* $0 \in \mathcal{O}_q$ *onto an* r*-dimensional submanifold of* M *containing* q.

Proof. By the inverse function theorem, it suffices to prove that $T_0 \exp_q : \mathcal{D}_q \to T_q\mathsf{M}$ has rank r at $0 \in \mathcal{D}_q$. (Here we have identified $T_0\mathcal{D}_q$ with $\mathcal{D}_q$.) If $X_q \in \mathcal{O}_q$, then

$$T_0 \exp_q(X_q) = \left.\frac{d}{dt}\right|_{t=0} \exp_q(tX_q) = \left.\frac{d}{dt}\right|_{t=0} \pi(\Phi_t(X_q)) = T_{X_q}\pi \cdot \Xi(X_q) = \iota(X_q).$$

That is, $T_0 \exp_q$ is the inclusion map $T_q\mathcal{D} \to T_q\mathsf{M}$, with rank r. ■

In particular, there is no analogue of the Hopf–Rinow theorem in nonholonomic Riemannian geometry. (On the other hand, there is such a result in sub-Riemannian geometry; see, e.g., [7, 2], and section 1.1.5.)

1.1.3 The symmetric bracket

Associated to the nonholonomic connection ∇ is a bracket (or product) defined on sections of $\Gamma(\mathcal{D})$, which in some sense is a symmetric counterpart to the projected Lie bracket $[\![\cdot,\cdot]\!]$. If $X, Y \in \Gamma(\mathcal{D})$, then the *symmetric bracket* (also called the *symmetric product*; see, e.g., [3, 46]) of X and Y, denoted $\langle\!\langle X : Y\rangle\!\rangle$, is defined as

$$\langle\!\langle X : Y\rangle\!\rangle = \nabla_X Y + \nabla_Y X.$$

Lemma 1.1.21. *Let* $X, Y \in \Gamma(\mathcal{D})$ *and* $f \in \mathcal{C}^\infty(\mathsf{M})$. *Then:*

(*i*) $\nabla_X Y = \frac{1}{2}[\![X, Y]\!] + \frac{1}{2}\langle\!\langle X : Y\rangle\!\rangle$.

(*ii*) $\langle\!\langle fX : Y\rangle\!\rangle = Y[f]X + f\langle\!\langle X : Y\rangle\!\rangle$, *i.e.,* $\langle\!\langle \cdot : \cdot\rangle\!\rangle$ *is a derivation in each argument.*

Proof. Since ∇ is torsion free, we have

$$\nabla_X Y = \frac{1}{2}(\nabla_X Y - \nabla_Y X) + \frac{1}{2}(\nabla_X Y + \nabla_Y X) = \frac{1}{2}[\![X, Y]\!] + \frac{1}{2}\langle\!\langle X : Y\rangle\!\rangle$$

for every $X, Y \in \Gamma(\mathcal{D})$, which proves item (*i*). For item (*ii*),

$$\langle\!\langle fX : Y\rangle\!\rangle = \nabla_{fX} Y + \nabla_Y(fX) = f\nabla_X Y + Y[f]X + f\nabla_Y X = Y[f]X + f\langle\!\langle X : Y\rangle\!\rangle. \quad ■$$

Notice that item (*i*) of the lemma asserts that $\frac{1}{2}\langle\!\langle X : Y\rangle\!\rangle$ is the symmetric part of $\nabla_X Y$, and $\frac{1}{2}[\![X, Y]\!]$ is the skew-symmetric part. (Consequently, a $\mathcal{D}$-curve $\gamma : I \to \mathsf{M}$ is a geodesic of ∇ if and only if $\langle\!\langle \dot\gamma, \dot\gamma\rangle\!\rangle = 0$.) We shall see in chapter 3 that the symmetric bracket plays a significant rôle in the "geodesic invariance" of vector subbundles of $\mathcal{D}$.

1.1.4 The $\mathscr{P}$-exterior covariant derivative

Every vector bundle connection induces an exterior derivative operator on vector-valued differential forms (see, e.g., [56]). A similar differential operator may be associated to the nonholonomic connection ∇. We have that $\Omega^p(\mathcal{D}, T^k_\ell(\mathcal{D})) = \Omega^p(\mathcal{D}) \otimes \mathcal{T}^k_\ell(\mathcal{D})$ denotes the space of $T^k_\ell(\mathcal{D})$-valued p-forms on $\mathcal{D}$. The *$\mathscr{P}$-exterior covariant derivative* is the differential operator

$$d^\nabla_{\mathscr{P}} : \Omega^p(\mathcal{D}, T^k_\ell(\mathcal{D})) \to \Omega^{p+1}(\mathcal{D}, T^k_\ell(\mathcal{D}))$$

defined as follows (cf. section A.3.2):

(*i*) If $T \in \Omega^0(\mathcal{D}, T^k_\ell(\mathcal{D})) = \mathcal{T}^k_\ell(\mathcal{D})$, then $d^\nabla_{\mathscr{P}} T(X) = \nabla_X T$ for every $X \in \Gamma(\mathcal{D})$.

(*ii*) If $\varphi \in \Omega^p(\mathcal{D}, T^k_\ell(\mathcal{D}))$, $p \geq 1$, then

$$\begin{aligned} d^\nabla_{\mathscr{P}} \varphi(X_0, \ldots, X_p) &= \sum_{0 \leq i \leq p} (-1)^i (\nabla_{X_i} \varphi)(X_0, \ldots, \widehat{X}_i, \ldots, X_p) \\ &= \sum_{0 \leq i \leq p} (-1)^i \nabla_{X_i} \varphi(X_0, \ldots, \widehat{X}_i, \ldots, X_p) \\ &\quad + \sum_{0 \leq i < j \leq p} (-1)^{i+j} \varphi([\![X_i, X_j]\!], X_0, \ldots, \widehat{X}_i, \ldots, \widehat{X}_j, \ldots, X_p) \end{aligned}$$

for every $X_0, \ldots, X_p \in \Gamma(\mathcal{D})$. (The hat indicates the omission of that element.)

The case of $\Omega^p(\mathcal{D}, T^1_0(\mathcal{D})) = \Omega^p(\mathcal{D}, \mathcal{D})$ is of the most interest. In particular, for a 1-form $\varphi \in \Omega^1(\mathcal{D}, \mathcal{D})$ (i.e., a $(1,1)$-tensor field), we have

$$d^\nabla_{\mathscr{P}} \varphi(X, Y) = \nabla_X \varphi(Y) - \nabla_Y \varphi(X) - \varphi([\![X, Y]\!])$$

for $X, Y \in \Gamma(\mathcal{D})$. Note that the torsion of ∇ is exactly the $\mathscr{P}$-exterior covariant derivative of the identity map $\mathrm{id}_{\mathcal{D}}$. For this reason, if $\varphi \in \Omega^1(\mathcal{D}, \mathcal{D})$, then $d^\nabla_{\mathscr{P}} \varphi$ is sometimes called the *torsion of* φ [33].

In chapter 2 we shall see how $d^\nabla_{\mathscr{P}}$ relates to the Schouten curvature tensor of ∇, and in chapter 4 we shall use it to characterise when a nonholonomic Riemannian structure on a three-dimensional manifold is flat.

1.1.5 Associated sub-Riemannian structure

To every nonholonomic Riemannian manifold $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ we can associate the *sub-Riemannian manifold* (or *sub-Riemannian structure*) $(\mathsf{M}, \mathcal{D}, \mathbf{g})$. In this section we briefly discuss some basic concepts from sub-Riemannian geometry. (For further details on sub-Riemannian geometry, refer to, e.g., [50, 2].)

Let $\gamma : [0,1] \to \mathsf{M}$ be a $\mathcal{D}$-curve. The *length* of γ is defined as $\mathrm{length}(\gamma) = \int_0^1 \|\dot\gamma(t)\|\, dt$. The *Carnot–Carathéodory distance* $d(p,q)$ between two points $p, q \in \mathsf{M}$ is defined as

$$d(p,q) = \inf_\gamma \mathrm{length}(\gamma),$$

where the infimum is taken over all $\mathcal{D}$-curves $\gamma : [0,1] \to \mathsf{M}$ such that $\gamma(0) = p$ and $\gamma(1) = q$. Since $\mathcal{D}$ is assumed to be completely nonholonomic, the Chow–Rashevskii theorem (theorem 1.1.3) ensures that there exists a $\mathcal{D}$-curve joining p to q, and hence d is well defined. The manifold M, together with the Carnot–Carathéodory distance, forms a metric space (M, d). Another consequence of the Chow–Rashevskii theorem is that d induces on M the original (manifold) topology (see, e.g., [50]).

A $\mathcal{D}$-curve $\gamma : [0,1] \to \mathsf{M}$ is called a *normal sub-Riemannian geodesic* if, for every sufficiently small interval $[t_1, t_2] \subseteq [0,1]$, the restriction $\gamma|_{[t_1,t_2]}$ is a length minimiser of the Carnot–Carathéodory distance, i.e., $d(\gamma(t_1), \gamma(t_2)) = \text{length}(\gamma|_{[t_1,t_2]})$. (In this sense sub-Riemannian geodesics are—at least, locally—the "shortest" curves in M. This is in contrast to the nonholonomic geodesics, which are the "straightest" curves; see remark 1.1.12.) There is also the concept of *abnormal* sub-Riemannian geodesics, which are not necessarily locally length-minimising; however, we shall not require this concept and shall mention it no further. (We note, however, that there are no abnormal geodesics in Riemannian geometry: every Riemannian geodesic is locally length-minimising.)

The nonholonomic geodesics are solutions of the Chetaev equations (see section 1.1.1) and do not satisfy a classical variational principle; their study may be considered to fall under the broad topic of *nonholonomic mechanics*. On the other hand, the sub-Riemannian geodesics *are* variational (in the classical sense); their study falls under the topic of *vakonomic mechanics*. (For further details on nonholonomic and vakonomic mechanics and their differences, see, e.g., [8, 13, 36, 42, 47].) As such, in general there is no relation between the nonholonomic geodesics of $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ and the sub-Riemannian geodesics of $(\mathsf{M}, \mathcal{D}, \mathbf{g})$. Having said that, there are situations in which we have the inclusion

$$\left\{ \begin{matrix} \text{nonholonomic geodesics} \\ \text{of } (\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g}) \end{matrix} \right\} \subsetneq \left\{ \begin{matrix} \text{sub-Riemannian geodesics} \\ \text{of } (\mathsf{M}, \mathcal{D}, \mathbf{g}) \end{matrix} \right\}. \tag{1.1.6}$$

(The set of sub-Riemannian geodesics is strictly richer than the set of nonholonomic geodesics, so there can be at most an inclusion; see, e.g., [36].) We mention that it is also possible for a particular nonholonomic geodesic to also be a sub-Riemannian geodesic, without having the full inclusion (1.1.6). It is of significant interest to study under what conditions the inclusion (1.1.6) exists. (We mention this, because, as a byproduct of our study of nonholonomic Riemannian embeddings, we find some sufficient conditions for (1.1.6) to occur.)

1.2 Left-invariant nonholonomic Riemannian structures on Lie groups

In this section we consider a nonholonomic Riemannian manifold $(\mathsf{G}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$, where G is a (connected) Lie group with Lie algebra $\mathfrak{g} = T_\mathbf{1}\mathsf{G}$. (We denote the identity element of G by $\mathbf{1}$.) Furthermore, we shall assume that the distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ and the metric $\mathbf{g}$ are left invariant, i.e.,

$$(L_g)_*\mathcal{D} = \mathcal{D}, \quad (L_g)_*\mathcal{D}^\perp = \mathcal{D}^\perp \quad \text{and} \quad \mathbf{g} = (L_g)^*\mathbf{g}$$

for every $g \in \mathsf{G}$, where $L_g : h \mapsto gh$ is the left translation. We call $(\mathsf{G}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ a *left-invariant nonholonomic Riemannian structure*. Evidently, such a structure is completely specified by means of a (completely nonholonomic) subspace $\mathcal{D}_\mathbf{1}$ of $\mathfrak{g}$, a complement $\mathcal{D}_\mathbf{1}^\perp$ to $\mathcal{D}_\mathbf{1}$ in $\mathfrak{g}$ and a positive-definite inner product $\mathbf{g}_\mathbf{1}$ on $\mathcal{D}_\mathbf{1}$. For convenience, we shall often identify elements of $\mathfrak{g}$ (resp. $\mathcal{D}_\mathbf{1}$, resp. $\mathcal{D}_\mathbf{1}^\perp$) with the corresponding left-invariant vector fields $\Gamma^L(T\mathsf{G})$ (resp. $\Gamma^L(\mathcal{D})$, resp. $\Gamma^L(\mathcal{D}^\perp)$).

Since left translations preserve both $\mathcal{D}$ and $\mathcal{D}^\perp$, they preserve the decomposition of $T\mathsf{G}$ as $\mathcal{D}\oplus\mathcal{D}^\perp$. In particular, the projections $\mathscr{P}$ and $\mathscr{Q}$ commute with the pushforward by a left translation:

Lemma 1.2.1. *We have* $(L_g)_*\mathscr{P}(X) = \mathscr{P}((L_g)_*X)$ *and* $(L_g)_*\mathscr{Q}(X) = \mathscr{Q}((L_g)_*X)$ *for every* $X \in \Gamma(T\mathsf{G})$ *and* $g \in \mathsf{G}$.

Proof. Let $g \in \mathsf{G}$ and $X \in \Gamma(T\mathsf{G})$. There exist vector fields $X_1 \in \Gamma(\mathcal{D})$ and $X_2 \in \Gamma(\mathcal{D}^\perp)$ such that $X = X_1 + X_2$. Furthermore, by left invariance, we have $(L_g)_*X_1 \in \Gamma(\mathcal{D})$ and $(L_g)_*X_2 \in \Gamma(\mathcal{D}^\perp)$. Consequently,

$$\mathscr{P}((L_g)_*X) = \mathscr{P}((L_g)_*X_1 + (L_g)_*X_2) = (L_g)_*X_1 = (L_g)_*\mathscr{P}(X).$$

Hence, we have $\mathscr{Q}((L_g)_*X) = (L_g)_*X - \mathscr{P}((L_g)_*X) = (L_g)_*(X - \mathscr{P}(X)) = (L_g)_*\mathscr{Q}(X)$. ■

The flag of $\mathcal{D}$ is also preserved by left translations. In particular, each element of the flag is a left-invariant distribution on G.

Lemma 1.2.2. *If* $\mathcal{D} = \mathcal{D}^1 \subsetneq \cdots \subsetneq \mathcal{D}^{N-1} \subsetneq \mathcal{D}^N = T\mathsf{G}$, $N \geq 2$ *is the flag of* $\mathcal{D}$, *then* $(L_g)_*\mathcal{D}^i = \mathcal{D}^i$ *for every* $g \in \mathsf{G}$ *and* $i = 1, \dots, N$.

Proof. We use induction on i. The result is true by assumption for $i = 1$, i.e., $(L_g)_*\mathcal{D}^1 = \mathcal{D}^1$. Suppose that $(L_g)_*\mathcal{D}^i = \mathcal{D}^i$ for some $i \geq 1$. Then

$$\begin{aligned}(L_g)_*\mathcal{D}^{i+1} &= (L_g)_*\mathcal{D}^i + (L_g)_*[\mathcal{D}^i, \mathcal{D}^i]\\ &= \mathcal{D}^i + [(L_g)_*\mathcal{D}^i, (L_g)_*\mathcal{D}^i]\\ &= \mathcal{D}^i + [\mathcal{D}^i, \mathcal{D}^i]\\ &= \mathcal{D}^{i+1}.\end{aligned}$$

Hence every element of the flag is left invariant. ■

For a left-invariant nonholonomic Riemannian structure $(\mathsf{G}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ it is always possible to find a global frame for $T\mathsf{G}$ (resp. $\mathcal{D}$) of left-invariant vector fields. Indeed, identifying $\mathfrak{g}$ (resp. $\mathcal{D}_\mathbf{1}$) with $\Gamma^L(T\mathsf{G})$ (resp. $\Gamma^L(\mathcal{D})$), any basis of $\mathfrak{g}$ (resp. $\mathcal{D}_\mathbf{1}$) is identified with a global frame for $T\mathsf{G}$ (resp. $\mathcal{D}$). Let (X_a) be a left-invariant frame for $\mathcal{D}$. By corollary 1.1.6, since the structure constants of a left-invariant frame are constant, it follows that the connection coefficients Γ^w_{ab} given by $\nabla_{X_a}X_b = \Gamma^w_{ab}X_w$ are also constant.

Proposition 1.2.3. *The nonholonomic connection of a left-invariant nonholonomic Riemannian structure is left invariant.*

Proof. Let $X, Y \in \Gamma(\mathcal{D})$ and $g \in \mathsf{G}$. We have $X = x^aX_a$ and $Y = y^aX_a$ for some functions $x^a, y^a \in \mathcal{C}^\infty(\mathsf{G})$. Accordingly,

$$\begin{aligned}\nabla_{(L_g)_*X}(L_g)_*Y &= ((L_g)_*X)[y^b \circ L_{g^{-1}}]X_b + (x^a \circ L_{g^{-1}})(y^b \circ L_{g^{-1}})\nabla_{X_a}X_b\\ &= (X[y^b] \circ L_{g^{-1}})X_b + (x^a \circ L_{g^{-1}})(y^b \circ L_{g^{-1}})\nabla_{X_a}X_b\\ &= (L_g)_*\big(X[y^b]X_b + x^ay^b\nabla_{X_a}X_b\big)\\ &= (L_g)_*\nabla_XY.\end{aligned}$$

That is, $\nabla = (L_g)^*\nabla$ for every $g \in \mathsf{G}$, and so ∇ is left invariant. ■

As a consequence of proposition 1.2.3, ∇ induces a bilinear map $\nabla : \mathcal{D}_{\mathbf{1}} \times \mathcal{D}_{\mathbf{1}} \to \mathcal{D}_{\mathbf{1}}$ given by $\nabla_U V = (\nabla_X Y)(\mathbf{1})$, where $X, Y \in \Gamma^L(\mathcal{D})$ are the left-invariant vector fields identified with $U, V \in \mathcal{D}_{\mathbf{1}}$, i.e., $X(g) = T_{\mathbf{1}}L_g \cdot U$ and $Y(g) = T_{\mathbf{1}}L_g \cdot V$. (See proposition B.1.29.)

Corollary 1.2.4. *A left translation of a nonholonomic geodesic is also a nonholonomic geodesic.*

Proof. Let $g(\cdot) : I \to \mathsf{G}$ be a nonholonomic geodesic of $(\mathsf{G}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$. We claim that the curve $\widetilde{g}_h(\cdot) = (L_h \circ g)(\cdot)$ is also a geodesic of ∇, where $h \in \mathsf{G}$. We have $\widetilde{g}_h(\cdot) = (L_h)_* g(\cdot)$, and so

$$\nabla_{\dot{\widetilde{g}}_h} \dot{\widetilde{g}}_h(t) = \nabla_{(L_h)_*\dot{g}}((L_h)_*\dot{g})(t) = (L_h)_* \nabla_{\dot{g}}\dot{g}(t) = 0$$

for every $t \in I$. Hence $\widetilde{g}_h(\cdot)$ is a nonholonomic geodesic. ■

Corollary 1.2.5. *For a left-invariant nonholonomic Riemannian structure, we have:*

(*i*) *The nonholonomic geodesic spray is left invariant, i.e., $(TL_g)_*\Xi = \Xi$ for every $g \in \mathsf{G}$.*

(*ii*) *The nonholonomic geodesic flow commutes with (the tangent map of) left translations, i.e., $TL_g \cdot \Phi_t = \Phi_t \circ TL_g|_{\mathcal{D}}$ for every $g \in \mathsf{G}$.*

Proof. We first prove item (*ii*). Let $U_h \in \mathcal{D}$, let γ be the nonholonomic geodesic $t \mapsto \pi(\Phi_t(U_h))$ and let $\widetilde{\gamma} = L_g \circ \gamma$ for $g \in \mathsf{G}$. Then $\widetilde{\gamma}$ is the unique nonholonomic geodesic starting at $L_g(h)$ in the direction $T_hL_g \cdot U_h$. That is, $\widetilde{\gamma}(t) = \pi(\Phi_t(T_hL_g \cdot U_h))$. Hence

$$T_{\gamma(t)}L_g \cdot \Phi_t(U_h) = \left.\frac{d}{ds}\right|_{s=0} L_g(\gamma(t+s)) = \left.\frac{d}{ds}\right|_{s=0} \widetilde{\gamma}(t+s) = \dot{\widetilde{\gamma}}(t) = \Phi_t(T_hL_g \cdot U_h),$$

i.e., $TL_g \cdot \Phi_t = \Phi_t \circ TL_g|_{\mathcal{D}}$ for every $g \in \mathsf{G}$. For (*i*), let $U_h \in \mathcal{D}$. Using item (*ii*), we have

$$\begin{aligned} T_{U_h}(TL_g) \cdot \Xi(U_h) &= T_{U_h}(TL_g) \cdot \Big(\left.\frac{d}{dt}\right|_{t=0} \Phi_t(U_h) \Big) \\ &= \left.\frac{d}{dt}\right|_{t=0} TL_g(\Phi_t(U_h)) \\ &= \left.\frac{d}{dt}\right|_{t=0} \Phi_t(T_hL_g \cdot U_h) = \Xi(T_hL_g \cdot U_h). \end{aligned}$$

Therefore $(TL_g)_*\Xi = \Xi$ for every $g \in \mathsf{G}$. ■

Let $g(\cdot) : I \to \mathsf{G}$ be a $\mathcal{D}$-curve in G. By left invariance, the geodesic equation $\nabla_{\dot{g}}\dot{g}(t) = 0$ may be written as

$$\begin{cases} \dot{g}(t) = T_{\mathbf{1}}L_{g(t)} \cdot U(t) \\ \dot{U}(t) = -\nabla_{U(t)}U(t), \end{cases} \tag{1.2.1}$$

where $U(\cdot)$ is a curve in $\mathcal{D}_{\mathbf{1}}$. In terms of a left-invariant orthonormal frame (X_a) for $\mathcal{D}$, the second equation in (1.2.1) takes the form $\dot{u}^w + \Gamma^w_{ab}u^a u^b = 0$, where $\Gamma^w_{ab} \in \mathbb{R}$ are the connection coefficients of ∇ with respect to the frame and $U = u^a X_a$. These equations (and their invariant form $\dot{U}(t) + \nabla_{U(t)}U(t) = 0$) are referred to as the *reduced equations of motion*, and the solutions $U(\cdot)$ are called *reduced nonholonomic geodesics*.

Proposition 1.2.6. *Every left-invariant nonholonomic Riemannian structure is geodesically complete.*

Proof. (We follow the proof for homogeneous Riemannian manifolds in [37, thm 4.5].) Let $h \in \mathsf{G}$ and let $g(\cdot)$ be a nonholonomic geodesic defined on the interval $[-a, a] \subsetneq \mathbb{R}$. We will show that $g(\cdot)$ can be extended to a nonholonomic geodesic defined on $[-a-\varepsilon, a+\varepsilon]$, where $\varepsilon > 0$. Let $\phi = L_{g(-a)h^{-1}}$, $\varphi = L_{g(a)h^{-1}}$ and $U_h = T_{g(-a)}\phi^{-1} \cdot \dot{g}(-a)$, $V_h = T_{g(a)}\varphi^{-1} \cdot \dot{g}(a)$. (Evidently, we have $U_h, V_h \in T_h\mathsf{G}$.) By left invariance, the curve $t \mapsto \phi(\exp_h(t\,U_h))$ is a nonholonomic geodesic passing through $\phi(h) = g(-a)$ at $t = -a$. Similarly, $t \mapsto \varphi(\exp_h(t\,V_h))$ is a nonholonomic geodesic passing through $g(a)$ at $t = a$. Define $\widetilde{g}(\cdot) : [-a-\varepsilon, a+\varepsilon] \to \mathsf{G}$ as

$$\widetilde{g}(t) = \begin{cases} \phi(\exp_h(t\,U_h)) & \text{for } t \in [-a-\varepsilon, -a] \\ g(t) & \text{for } t \in [-a, a] \\ \varphi(\exp_h(t\,V_h)) & \text{for } t \in [a, a+\varepsilon]. \end{cases}$$

Then $\widetilde{g}(\cdot)$ is a nonholonomic geodesic extending $g(\cdot)$. Since ε is arbitrary, it follows that we can extend $g(\cdot)$ to a nonholonomic geodesic with an arbitrarily large domain. It follows that $(\mathsf{G}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ is geodesically complete. ■

Using Koszul's formula, we can give a convenient expression for the covariant derivative of one left-invariant vector field along another. This yields an expression for the symmetric bracket, as well as another invariant form of the reduced equations of motion.

Proposition 1.2.7. *If* $X, Y \in \Gamma^L(\mathcal{D})$, *then*

$$\nabla_X Y = \frac{1}{2}(\llbracket X, Y \rrbracket - (\mathrm{ad}^{\mathscr{P}}_X)^\dagger Y - (\mathrm{ad}^{\mathscr{P}}_Y)^\dagger X).$$

Here $\mathrm{ad}^{\mathscr{P}}_X = \mathscr{P} \circ \mathrm{ad}_X \circ \iota$ (*where* $\iota : \mathcal{D} \to T\mathsf{G}$ *is the inclusion map*) *and* $(\mathrm{ad}^{\mathscr{P}}_X)^\dagger$ *is the adjoint of* $\mathrm{ad}^{\mathscr{P}}_X$, *i.e.,* $(\mathrm{ad}^{\mathscr{P}}_X)^\dagger = \mathbf{g}^\sharp \circ (\mathrm{ad}^{\mathscr{P}}_X)^* \circ \mathbf{g}^\flat$.

Proof. Let $X, Y, Z \in \Gamma^L(\mathcal{D})$. Using Koszul's formula, we have

$$\begin{aligned} 2\,\mathbf{g}(\nabla_X Y, Z) &= \mathbf{g}(\llbracket X, Y \rrbracket, Z) - \mathbf{g}(\llbracket X, Z \rrbracket, Y) - \mathbf{g}(\llbracket Y, Z \rrbracket, X) \\ &= \mathbf{g}(\llbracket X, Y \rrbracket, Z) - \mathbf{g}(\mathrm{ad}^{\mathscr{P}}_X Z, Y) - \mathbf{g}(\mathrm{ad}^{\mathscr{P}}_Y Z, X) \\ &= \mathbf{g}(\llbracket X, Y \rrbracket, Z) - \mathbf{g}(Z, (\mathrm{ad}^{\mathscr{P}}_X)^\dagger Y) - \mathbf{g}(Z, (\mathrm{ad}^{\mathscr{P}}_Y)^\dagger X) \\ &= \mathbf{g}(\llbracket X, Y \rrbracket - (\mathrm{ad}^{\mathscr{P}}_X)^\dagger Y - (\mathrm{ad}^{\mathscr{P}}_Y)^\dagger X, Z). \end{aligned}$$

Thus $\nabla_X Y = \frac{1}{2}(\llbracket X, Y \rrbracket - (\mathrm{ad}^{\mathscr{P}}_X)^\dagger Y - (\mathrm{ad}^{\mathscr{P}}_Y)^\dagger X)$. ■

Corollary 1.2.8. *If* $X, Y \in \Gamma^L(\mathcal{D})$, *then* $\langle\!\langle X : Y \rangle\!\rangle = -(\mathrm{ad}^{\mathscr{P}}_X)^\dagger Y - (\mathrm{ad}^{\mathscr{P}}_Y)^\dagger X$.

Proof. From lemma 1.1.21 and proposition 1.2.7, we have

$$\llbracket X, Y \rrbracket + \langle\!\langle X : Y \rangle\!\rangle = 2\,\nabla_X Y = \llbracket X, Y \rrbracket - (\mathrm{ad}^{\mathscr{P}}_X)^\dagger Y - (\mathrm{ad}^{\mathscr{P}}_Y)^\dagger X.$$

It follows that $\langle\!\langle X : Y \rangle\!\rangle = -(\mathrm{ad}^{\mathscr{P}}_X)^\dagger Y - (\mathrm{ad}^{\mathscr{P}}_Y)^\dagger X$. ■

Corollary 1.2.9. *If* $U(\cdot)$ *is a reduced nonholonomic geodesic, then* $\dot{U}(t) = (\mathrm{ad}^{\mathscr{P}}_{U(t)})^\dagger U(t)$.

Proof. From the second equation in (1.2.1), we have $\dot{U}(t) = -\nabla_{U(t)} U(t) = (\mathrm{ad}^{\mathscr{P}}_{U(t)})^\dagger U(t)$. ■

1.2.1 Cartan–Schouten connections

A left-invariant (tangent bundle) connection $\widetilde{\nabla}$ on a Lie group G is called a *Cartan–Schouten connection* (cf. [54, 58]) if the geodesics of $\widetilde{\nabla}$ are the left cosets of one-parameter subgroups $t \mapsto g_0 \exp(tU_0)$, $g_0 \in \mathsf{G}$, $U_0 \in \mathfrak{g}$. Following this terminology, if ∇ is the nonholonomic connection of a left-invariant nonholonomic Riemannian structure $(\mathsf{G}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$, then we shall say that ∇ is a (*nonholonomic*) *Cartan–Schouten connection* if the geodesics of ∇ are the left cosets of one-parameter subgroups $t \mapsto g_0 \exp(tU_0)$, $g_0 \in \mathsf{G}$, $U_0 \in \mathcal{D}_\mathbf{1}$. In this case, we shall also say that the structure $(\mathsf{G}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ *has a* Cartan–Schouten connection.

Theorem 1.2.10. *The following statements are equivalent:*

(*i*) ∇ *is Cartan–Schouten.*

(*ii*) $\nabla_X X = 0$ *for every* $X \in \Gamma^L(\mathcal{D})$.

(*iii*) $\langle\!\langle X : Y \rangle\!\rangle = 0$ *for every* $X, Y \in \Gamma^L(\mathcal{D})$.

(*iv*) $\nabla_X Y = \frac{1}{2}[\![X, Y]\!]$ *for every* $X, Y \in \Gamma^L(\mathcal{D})$.

(*v*) $\pounds_X^{\mathscr{P}} \mathbf{g} \equiv 0$ *for every* $X \in \Gamma^L(\mathcal{D})$.

Proof. We first show that (*i*) is equivalent to (*ii*). Let $g(\cdot)$ be the nonholonomic geodesic such that $g(0) = g_0 \in \mathsf{G}$ and $\dot{g}(0) = T_\mathbf{1} L_{g_0} \cdot U_0$, where $U_0 \in \mathcal{D}_\mathbf{1}$. Let $U(\cdot)$ be the reduced nonholonomic geodesic associated to $g(\cdot)$, i.e., $\dot{g}(t) = T_\mathbf{1} L_{g(t)} \cdot U(t)$ for every t. Then

$$\begin{aligned} g(t) = g_0 \exp(tU_0) \quad &\iff \quad U(t) = U_0 \text{ for every } t \\ &\iff \quad \nabla_{U(t)} U(t) = 0 \text{ for every } t. \end{aligned}$$

In particular, if $g(t) = g_0 \exp(tU_0)$, then $\nabla_{U_0} U_0 = 0$. Since U_0 is arbitrary, we have that (*i*) implies (*ii*). Conversely, if $\nabla_X X = 0$ for every $X \in \Gamma^L(\mathcal{D})$, then $\nabla_{U(t)} U(t) = 0$ for every t and every reduced nonholonomic geodesic $U(\cdot)$, and so every nonholonomic geodesic $g(\cdot)$ is a left coset of a one-parameter subgroup. That is, (*ii*) implies (*i*).

Suppose (*ii*) holds. From $\nabla_{X+Y}(X+Y) = 0$, we get $\langle\!\langle X : Y \rangle\!\rangle = \nabla_X Y + \nabla_Y X = 0$ for every $X, Y \in \Gamma^L(\mathcal{D})$. Consequently, $\nabla_X Y = \frac{1}{2}[\![X, Y]\!] + \frac{1}{2}\langle\!\langle X : Y \rangle\!\rangle = \frac{1}{2}[\![X, Y]\!]$. Hence we have (*ii*) $\Rightarrow$ (*iii*) $\Rightarrow$ (*iv*). Suppose (*iv*) holds. Since ∇ is metric,

$$\begin{aligned} 0 = (\nabla_X \mathbf{g})(Y, Z) &= X[\mathbf{g}(Y, Z)] - \mathbf{g}(\nabla_X Y, Z) - \mathbf{g}(Y, \nabla_X Z) \\ &= -\frac{1}{2}\mathbf{g}([\![X, Y]\!], Z) - \frac{1}{2}\mathbf{g}(Y, [\![X, Z]\!]) \\ &= -\frac{1}{2}(\pounds_X^{\mathscr{P}} \mathbf{g})(Y, Z) \end{aligned}$$

for every $X, Y, Z \in \Gamma^L(\mathcal{D})$. That is, $\pounds_X^{\mathscr{P}} \mathbf{g} \equiv 0$ for every $X \in \Gamma^L(\mathcal{D})$. Finally, if (*v*) holds, then from the Koszul formula, we get $\mathbf{g}(\nabla_X X, Y) = -\mathbf{g}([\![X, Y]\!], X) = (\pounds_Y^{\mathscr{P}} \mathbf{g})(X, X) = 0$. That is, $\nabla_X X = 0$ for every $X \in \Gamma^L(\mathcal{D})$. ■

Evidently, statement (*ii*) of theorem 1.2.10 implies that the reduced dynamics are trivial, i.e., the reduced nonholonomic geodesics are constant. Additionally, statement (*v*) implies that X is orthogonal to $[\![X, Y]\!]$ for every pair of left-invariant vector fields $X, Y \in \Gamma^L(\mathcal{D})$.

Indeed, if $X, Y \in \Gamma^L(\mathcal{D})$, then $0 = (\pounds^{\mathscr{P}}_X \mathbf{g})(Y, Y) = -2\,\mathbf{g}(\llbracket X, Y \rrbracket, Y)$. Furthermore, by polarisation this is equivalent to the condition

$$\mathbf{g}(\llbracket X, Y \rrbracket, Z) + \mathbf{g}(X, \llbracket Y, Z \rrbracket) = 0 \tag{1.2.2}$$

for every $X, Y, Z \in \Gamma^L(\mathcal{D})$. Hence, it is clear that if $\llbracket X, Y \rrbracket = 0$ for every $X, Y \in \Gamma^L(\mathcal{D})$, then ∇ is Cartan–Schouten. When $\mathcal{D}$ has rank two, the converse also holds.

Proposition 1.2.11. *If* ∇ *is Cartan–Schouten and* $\operatorname{rank}(\mathcal{D}) = 2$*, then* $\llbracket X, Y \rrbracket = 0$ *for every* $X, Y \in \Gamma^L(\mathcal{D})$*.*

Proof. Suppose that ∇ is Cartan–Schouten and $\operatorname{rank}(\mathcal{D}) = 2$. Let (X_1, X_2) be a left-invariant frame for $\mathcal{D}$. Then $\mathbf{g}(X_1, \llbracket X_1, X_2 \rrbracket) = 0$ and $\mathbf{g}(X_2, \llbracket X_2, X_1 \rrbracket) = 0$, whence $\llbracket X_1, X_2 \rrbracket = 0$. It follows that $\llbracket X, Y \rrbracket = 0$ for every $X, Y \in \Gamma^L(\mathcal{D})$. ■

Corollary 1.2.12. *Suppose* $\operatorname{rank}(\mathcal{D}) = 2$*. Then* $\nabla_X X = 0$ *for every* $X \in \Gamma^L(\mathcal{D})$ *if and only if* $\nabla_X Y = 0$ *for every* $X, Y \in \Gamma^L(\mathcal{D})$*.*

Remark 1.2.13. A distinguished class of left-invariant nonholonomic Riemannian structures are the left-invariant structures on Carnot groups. A *Carnot group* (see, e.g., [10, 50]) is a connected, simply connected, nilpotent Lie group G whose Lie algebra $\mathfrak{g}$ admits the decomposition (into vector subspaces) $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ such that

$$[\mathfrak{g}_1, \mathfrak{g}_i] = \mathfrak{g}_{i+1} \text{ for } 1 \leq i < k \quad \text{and} \quad [\mathfrak{g}_1, \mathfrak{g}_k] = \{0\}. \tag{1.2.3}$$

Define a left-invariant distribution $\mathcal{D}$ on G by specifying $\mathcal{D}_\mathbf{1} = \mathfrak{g}_1$. (From the bracket condition (1.2.3), we have that $\mathfrak{g}_1$ generates $\mathfrak{g}$ as a Lie algebra, and hence $\mathcal{D}$ is completely nonholonomic.) Likewise, let $\mathcal{D}^\perp$ be the left-invariant complement to $\mathcal{D}$ given by $\mathcal{D}^\perp_\mathbf{1} = \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_k$. Fix a left-invariant metric $\mathbf{g}$ on $\mathcal{D}$. This yields the structure $(\mathsf{G}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$, which we refer to as a *left-invariant nonholonomic Riemannian structure on a Carnot group.* It is clear that $\llbracket X, Y \rrbracket = 0$ for every $X, Y \in \Gamma^L(\mathcal{D})$, and hence ∇ is Cartan–Schouten and $\nabla_X Y = 0$ for every $X, Y \in \Gamma^L(\mathcal{D})$. As a consequence, we recover a result of [67], *viz.*, the nonholonomic geodesics of (left-invariant) nonholonomic Riemannian structures on Carnot groups are exactly the integral curves of left-invariant vector fields in $\Gamma(\mathcal{D})$. □

Proposition 1.2.14. *If* ∇ *is Cartan–Schouten, then:*

(*i*) *With respect to a left-invariant orthonormal frame for* $\mathcal{D}$*, the matrix of* $\operatorname{ad}^{\mathscr{P}}_X$ *has zero diagonal for every* $X \in \Gamma^L(\mathcal{D})$*.*

(*ii*) $\operatorname{tr}(\operatorname{ad}^{\mathscr{P}}_X) = 0$ *for every* $X \in \Gamma^L(\mathcal{D})$*.*

Proof. Let (X_i) be a left-invariant frame for $T\mathsf{G}$ such that (X_a) is an orthonormal frame for $\mathcal{D}$. Let $c^k_{ij} \in \mathbb{R}$ denote the structure constants of this frame. From item (v) of theorem 1.2.10, we have $\mathbf{g}(X, \llbracket X, Y \rrbracket) = 0$ for every $X, Y \in \Gamma^L(\mathcal{D})$. That is, $c^a_{ab} = \mathbf{g}(X_a, \llbracket X_a, X_b \rrbracket) = 0$. Consequently, with respect to the frame (X_a) we have

$$\operatorname{ad}^{\mathscr{P}}_{X_a} = \begin{bmatrix} 0 & c^1_{a2} & \cdots & c^1_{ar} \\ c^2_{a1} & 0 & \cdots & c^2_{ar} \\ \vdots & \vdots & \ddots & \vdots \\ c^r_{a1} & c^r_{a2} & \cdots & 0 \end{bmatrix}.$$

That is, the diagonal is zero. Item (i) now follows by linearity of ad. Item (ii) follows trivially from (i). ■

In the (Riemannian) case $\mathcal{D} = T\mathsf{G}$, the second item in this proposition is simply the well-known result that G must necessarily be unimodular if it is to admit a bi-invariant Riemannian metric. In the general case, it implies that the reduced nonholonomic dynamics preserves a volume form (see [39]).

Lastly, we prove a straightforward sufficient condition for a left-invariant structure to have a Cartan–Schouten connection. For the Riemannian case $\mathcal{D} = T\mathsf{G}$, this result says that, if $\mathbf{g}$ is both left- and right-invariant (i.e., bi-invariant), then ∇ is Cartan–Schouten. In fact, the bi-invariance condition is also necessary in Riemannian geometry: the Levi-Civita connection is Cartan–Schouten if and only if the metric is bi-invariant [53]. However, for nonholonomic Riemannian structures, the condition given below is only sufficient.

Proposition 1.2.15. *If* $\mathbf{g}(\mathrm{Ad}_g^{\mathscr{P}} X, \mathrm{Ad}_g^{\mathscr{P}} Y) = \mathbf{g}(X, Y)$ *for every* $X, Y \in \Gamma^L(\mathcal{D})$ *and* $g \in \mathsf{G}$, *then* ∇ *is Cartan–Schouten. (Here* $\mathrm{Ad}_g^{\mathscr{P}} = \mathscr{P} \circ \mathrm{Ad}_g \circ \iota$*.)*

Proof. Let $X, Y \in \Gamma^L(\mathcal{D})$ and suppose $\mathbf{g}(\mathrm{Ad}_g^{\mathscr{P}} X, \mathrm{Ad}_g^{\mathscr{P}} Y) = \mathbf{g}(X, Y)$ for every $g \in \mathsf{G}$. In particular, this must hold at every point along the curve $g(t) = \exp(tW)$, $W \in \Gamma^L(\mathcal{D})$. Hence, differentiating both sides at $t = 0$, we get

$$\begin{aligned}
0 &= \left.\frac{d}{dt}\right|_{t=0} \mathbf{g}(\mathrm{Ad}_{g(t)}^{\mathscr{P}} X, \mathrm{Ad}_{g(t)}^{\mathscr{P}} Y) \\
&= \mathbf{g}\Big(\left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}_{g(t)}^{\mathscr{P}} X, Y\Big) + \mathbf{g}\Big(X, \left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}_{g(t)}^{\mathscr{P}} Y\Big) \\
&= \mathbf{g}(\mathrm{ad}_W^{\mathscr{P}} X, Y) + \mathbf{g}(X, \mathrm{ad}_W^{\mathscr{P}} Y) \\
&= \mathbf{g}([\![W, X]\!], Y) + \mathbf{g}(X, [\![W, Y]\!]).
\end{aligned}$$

By theorem 1.2.10 and equation (1.2.2), it follows that ∇ is Cartan–Schouten. ■

Corollary 1.2.16. *If* $\mathbf{g}$ *is the restriction to* $\mathcal{D}$ *of a bi-invariant Riemannian metric on* G, *then* ∇ *is Cartan–Schouten.*

It is worthwhile examining why the converse to proposition 1.2.15 fails. Suppose ∇ is a Cartan–Schouten connection; in particular, equation (1.2.2) holds. Let $W, X, Y \in \Gamma^L(\mathcal{D})$ and let $g(t) = \exp(tW)$. Then

$$\begin{aligned}
&\frac{d}{dt} \mathbf{g}(\mathrm{Ad}_{g(t)}^{\mathscr{P}} X, \mathrm{Ad}_{g(t)}^{\mathscr{P}} Y) \\
&\quad= \mathbf{g}\Big(\frac{d}{dt} \mathrm{Ad}_{g(t)}^{\mathscr{P}} X, \mathrm{Ad}_{g(t)}^{\mathscr{P}} Y\Big) + \mathbf{g}\Big(\mathrm{Ad}_{g(t)}^{\mathscr{P}} X, \frac{d}{dt} \mathrm{Ad}_{g(t)}^{\mathscr{P}} Y\Big) \\
&\quad= \mathbf{g}\Big(\mathscr{P}\Big(\frac{d}{dt} \mathrm{Ad}_{g(t)} X\Big), \mathrm{Ad}_{g(t)}^{\mathscr{P}} Y\Big) + \mathbf{g}\Big(\mathrm{Ad}_{g(t)}^{\mathscr{P}} X, \mathscr{P}\Big(\frac{d}{dt} \mathrm{Ad}_{g(t)} Y\Big)\Big) \\
&\quad= \mathbf{g}\Big(\mathscr{P}(\mathrm{Ad}_{g(t)}[W, X]), \mathrm{Ad}_{g(t)}^{\mathscr{P}} Y\Big) + \mathbf{g}\Big(\mathrm{Ad}_{g(t)}^{\mathscr{P}} X, \mathscr{P}(\mathrm{Ad}_{g(t)}[W, Y])\Big) \\
&\quad= \mathbf{g}\Big([\![W, \mathrm{Ad}_{g(t)} X]\!], \mathrm{Ad}_{g(t)}^{\mathscr{P}} Y\Big) + \mathbf{g}\Big(\mathrm{Ad}_{g(t)}^{\mathscr{P}} X, [\![W, \mathrm{Ad}_{g(t)} Y]\!]\Big).
\end{aligned}$$

In general, we have $\mathrm{Ad}_{g(t)} X \notin \Gamma^L(\mathcal{D})$, and so the right-hand side is nonzero. Indeed, if $\mathrm{Ad}_{g(t)} X \in \Gamma^L(\mathcal{D})$, then in particular $[W, X] = \frac{d}{dt}\big|_{t=0} \mathrm{Ad}_{g(t)} X \in \Gamma^L(\mathcal{D})$, i.e., $\mathcal{D}_1$ is a subalgebra of $\mathfrak{g}$. This implies that $\mathcal{D}$ is integrable, a contradiction. Hence, in general we have $\mathbf{g}(\mathrm{Ad}_{g(t)}^{\mathscr{P}} X, \mathrm{Ad}_{g(t)}^{\mathscr{P}} Y) \neq \mathbf{g}(X, Y)$, and so there is no generalisation of the Riemannian "bi-invariance condition" to the case of nonholonomic Riemannian structures.

1.2.2 Existence of Cartan–Schouten connections

In this section we discuss the existence of (left-invariant) nonholonomic Riemannian structures with Cartan–Schouten connections. (We consider the question of uniqueness, up to equivalence, in chapter 3.) Not every Lie group admits a left-invariant completely nonholonomic distribution; for instance, Abelian groups, or the three-dimensional Lie group $\mathsf{G}_{3.3}$ (see appendix C). Thus, in general there does not exist a nonholonomic Riemannian structure with a Cartan–Schouten connection on a given Lie group. However, if we suppose that a Lie group G does admit at least one left-invariant distribution, we can ask the following question:

(*Q1*) Does there exist a left-invariant nonholonomic Riemannian structure on G such that the structure has a Cartan–Schouten connection?

An interesting variation on this question is:

(*Q2*) Given a left-invariant sub-Riemannian structure $(\mathsf{G}, \mathcal{D}, \mathbf{g})$, does there exist a left-invariant complement $\mathcal{D}^\perp$ to $\mathcal{D}$ such that $(\mathsf{G}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ has a Cartan–Schouten connection?

(One can of course also consider other variations of (*Q1*).) Note that, from proposition 1.2.11, when $\operatorname{rank}(\mathcal{D}) = 2$, whether ∇ is Cartan–Schouten or not does not depend on $\mathbf{g}$, but only on the decomposition $T\mathsf{G} = \mathcal{D} \oplus \mathcal{D}^\perp$. Accordingly, (*Q1*) and (*Q2*) are essentially equivalent.

We shall consider (*Q2*) further, and, in particular, show that it is always answered positively when $\operatorname{rank}(\mathcal{D}) = 2$. Let $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ be a left-invariant sub-Riemannian structure. Let (X_a) be a left-invariant orthonormal frame for $\mathcal{D}$, and extend this to a (left-invariant) frame (X_a, X_λ) for $T\mathsf{G}$. Let $c^k_{ij} \in \mathbb{R}$ be the structure constants of (X_a, X_λ).

Proposition 1.2.17. *There exists a positive answer to* (*Q2*) *if and only if there exists a solution* (*in the unknowns* f^a_λ, f^b_λ) *to the system of equations*

$$\begin{cases} c^\lambda_{ab} f^b_\lambda + c^b_{ab} = 0 & (\textit{no summation over } b) \\ c^\lambda_{bw} f^a_\lambda + c^\lambda_{aw} f^b_\lambda + c^a_{bw} + c^b_{aw} = 0 & (w \neq a,\ w \neq b). \end{cases} \tag{1.2.4}$$

Proof. Keeping in mind theorem 1.2.10, (*Q2*) can rephrased as follows: does there exist a left-invariant projection $\mathscr{P} : T\mathsf{M} \to \mathcal{D}$ (i.e., $\mathscr{P} \circ TL_g = \mathscr{P}$) such that, if $\mathcal{D}^\perp = \ker \mathscr{P}$, then for the left-invariant nonholonomic Riemannian structure $(\mathsf{G}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ we have $\langle\!\langle X : Y \rangle\!\rangle = 0$ for every $X, Y \in \Gamma^L(\mathcal{D})$?

Define a projection $\mathscr{P}$ by the following requirements: (*i*) it is tensorial; (*ii*) $\mathscr{P}(X_a) = X_a$; and (*iii*) $\mathscr{P}(X_\lambda) = f^w_\lambda X_w$, where $f^w_\lambda \in \mathbb{R}$. We have $\llbracket X_a, X_b \rrbracket = \mathscr{P}(c^w_{ab} X_w + c^\lambda_{ab} X_\lambda) = (c^\lambda_{ab} f^w_\lambda + c^w_{ab}) X_w$. Consequently,

$$\mathbf{g}((\operatorname{ad}^{\mathscr{P}}_{X_a})^\dagger X_b, X_w) = \mathbf{g}(X_b, \llbracket X_a, X_w \rrbracket) = (c^\lambda_{aw} f^u_\lambda + c^u_{aw}) \mathbf{g}(X_b, X_u) = (c^\lambda_{aw} f^b_\lambda + c^b_{aw})$$

and so $(\operatorname{ad}^{\mathscr{P}}_{X_a})^\dagger X_b = \sum_w (c^\lambda_{aw} f^b_\lambda + c^b_{aw}) X_w$. Using corollary 1.2.8, it follows that

$$\langle\!\langle X_a : X_b \rangle\!\rangle = -(\operatorname{ad}^{\mathscr{P}}_{X_a})^\dagger X_b - (\operatorname{ad}^{\mathscr{P}}_{X_b})^\dagger X_a = -\sum_w \left[c^\lambda_{bw} f^a_\lambda + c^\lambda_{aw} f^b_\lambda + c^a_{bw} + c^b_{aw} \right] X_w.$$

Since ∇ is Cartan–Schouten if and only if $\langle\!\langle X : Y \rangle\!\rangle = 0$ for every $X, Y \in \Gamma^L(\mathcal{D})$, (*Q2*) may now be rephrased as follows: does there exist a solution (in the unknowns f^a_λ, f^b_λ) to the system of equations $c^\lambda_{bw} f^a_\lambda + c^\lambda_{aw} f^b_\lambda + c^a_{bw} + c^b_{aw} = 0$? Observing that the equations with $w = a$ and $w = b$ simplify, the result is the equations (1.2.4). ■

Corollary 1.2.18. *If* $\operatorname{rank}(\mathcal{D}) = 2$*, then* $(Q2)$ *has a positive answer.*

Proof. Suppose that $\operatorname{rank}(\mathcal{D}) = 2$. Then the equations (1.2.4) simplify to

$$\begin{cases} c_{21}^{\lambda} f_{\lambda}^{1} + c_{21}^{1} = 0 \\ c_{21}^{\lambda} f_{\lambda}^{2} + c_{21}^{2} = 0. \end{cases} \tag{1.2.5}$$

At least one of the c_{21}^{λ} are nonzero, say $c_{21}^{\mu} \neq 0$. (Indeed, if $c_{21}^{\lambda} = 0$ for every λ, then $[X_2, X_1] \in \Gamma(\mathcal{D})$, which implies that $\mathcal{D}$ is integrable.) Then $f_{\mu}^{1} = -c_{21}^{1}/c_{21}^{\mu}$, $f_{\mu}^{2} = -c_{21}^{2}/c_{21}^{\mu}$ and $f_{\lambda}^{1} = f_{\lambda}^{2} = 0$ for $\lambda \neq \mu$ is a solution to (1.2.5), and hence $(Q2)$ is answered positively. ■

Chapter 2

Curvature

The study of curvature plays a central rôle in Riemannian geometry, particularly in relating local and global behaviour of the Riemannian manifold under consideration (see, e.g., [44, 53, 55]). In simple terms, the Riemannian curvature tensor measures the extent to which a Riemannian manifold fails to be Euclidean space (i.e., the prototypical flat Riemannian manifold). Curvature in Riemannian geometry has been extensively studied, and many aspects of it are well understood. In sharp contrast, the curvature of nonholonomic Riemannian manifolds has received very little attention. Although some elements of the curvature of nonholonomic Riemannian structures can be found in Synge [66], it was Schouten [62] who first explicitly considered curvature in the nonholonomic Riemannian context. In particular, Schouten introduced a curvature tensor associated to every nonholonomic Riemannian structure; this tensor is now referred to as the "Schouten curvature tensor" [21, 27]. Nevertheless, the main development in the study of curvature of nonholonomic Riemannian manifolds was due to the Russian mathematician V.V. Wagner. Wagner observed that (the vanishing of) the Schouten tensor does not characterise the *flat* nonholonomic Riemannian structures, i.e., those structures for which the parallel transport (induced by the nonholonomic connection) is path-independent. In a series of papers [74, 78, 79] (see also [75, 80, 76, 77])—which ultimately won him Kazan University's 1937 Lobachevskii prize for young Soviet mathematicians—Wagner extended Schouten's work, defining a curvature tensor (now called the "Wagner curvature tensor"), the vanishing of which *does* characterise the flat structures. Nevertheless, Wagner's construction has its limitations. In particular, in general it does not depend only on the data $(\mathsf{M}, \mathcal{D}, \mathcal{D}^{\perp}, \mathsf{g})$, but also on some additional assumptions. As a result, it is not preserved by nonholonomic isometries (which we will consider in chapter 3), and hence in this sense the Wagner tensor is not intrinsic. For modern treatments of Schouten and Wagner's work, see [21, 30]. For recent papers making use of the Schouten or Wagner curvature tensor, see, e.g., [27, 11, 28, 83].

In this chapter we consider in some detail the Schouten and Wagner tensors. In section 2.1 we introduce the Schouten curvature tensor, which is canonically associated to every nonholonomic Riemannian structure. We prove some symmetries of this tensor, and attempt to relate it (at least on an algebraic level) to a Riemannian-type curvature tensor. We do this by decomposing the Schouten tensor (in fact, the associated tensor obtained by using the metric to "lower an index") into two components: a "Riemannian" component—which satisfies all the symmetries of a Riemannian curvature tensor—and a "remainder" that can be viewed as a deviation of the Schouten tensor from a Riemannian curvature tensor. (As we shall see in section 3.2.3, this viewpoint has merit: if the distribution is strongly nonholonomic, then the "remainder" vanishes if and only if the nonholonomic Riemannian structure can be embedded inside a Riemannian manifold, in such a way that the embedded structure inherits its geometry from the enveloping space.) Using the "Riemannian" component of the Schouten tensor, we are able to introduce notions of sectional curvature, a Ricci tensor, and a scalar curvature, analogous to the corresponding concepts in Riemannian geometry.

In section 2.2 we consider the Wagner curvature tensor. The construction of this tensor is quite sophisticated, and relies on the flag of the distribution. However, as mentioned above, the construction is not intrinsic, in that it relies on some additional assumptions. (Having said that, if the distribution is strongly nonholonomic, then these assumptions are automatically satisfied.) We briefly describe Wagner's approach. Let $(\mathsf{M}, \mathcal{D}, \mathcal{D}^{\perp}, \mathbf{g})$ be a nonholonomic Riemannian structure and let $\mathcal{D} = \mathcal{D}^1 \subsetneq \mathcal{D}^2 \subsetneq \cdots \subsetneq \mathcal{D}^N = T\mathsf{M}$, $N \geq 2$ be the flag of $\mathcal{D}$. The nonholonomic connection $\nabla^1 = \nabla$ induces a parallel transport along $\mathcal{D}^1$-curves. For each component $\mathcal{D}^{i+1}$, $i = 1, \ldots, N-1$ of the flag, Wagner constructs a $\mathcal{D}^{i+1}$-restricted connection ∇^{i+1} on $\mathcal{D}$ (see appendix B). Such a connection induces a parallel transport along $\mathcal{D}^{i+1}$-curves. Furthermore, ∇^{i+1} is defined in such a way that it extends ∇^i and the set of parallel tensors of ∇^{i+1} coincides with that of ∇^i. Finally, one gets a vector bundle connection ∇^N on $\mathcal{D}$ (whose corresponding parallel transport is along any curve in M), with an associated curvature tensor K^N; this is the Wagner curvature tensor. The vanishing of K^N characterises the flatness of ∇^N, and hence (by construction of $\nabla^2, \ldots, \nabla^{N-1}$), the flatness of $(\mathsf{M}, \mathcal{D}, \mathcal{D}^{\perp}, \mathbf{g})$.

In section 2.3 we revisit both the Schouten and Wagner curvature tensors from the viewpoint of restricted Ehresmann connections (see section B.2). The main contribution of this section is to show that Wagner's construction is equivalently formulated as a flag of horizontal distributions on $\mathcal{D}$.

Lastly, in section 2.4 we consider left-invariant nonholonomic Riemannian structures on Lie groups. Specifically, we show that the Schouten and Wagner curvature tensor (and all associated tensors) are left invariant. We also characterise (in section 2.4.1) the existence of left-invariant parallel frames.

2.1 The Schouten curvature tensor

Let $(\mathsf{M}, \mathcal{D}, \mathcal{D}^{\perp}, \mathbf{g})$ be a nonholonomic Riemannian manifold, with associated nonholonomic connection ∇. As $\mathcal{D}$ is nonintegrable, the usual curvature tensor (for a vector bundle connection), *viz.*, $(X, Y, Z) \mapsto [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$, is not defined. Instead, associated to $(\mathsf{M}, \mathcal{D}, \mathcal{D}^{\perp}, \mathbf{g})$ is the *Schouten curvature tensor* [21] $K \in \mathcal{T}^1_3(\mathcal{D})$, given by

$$K(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[\![X,Y]\!]}Z - [\![\mathscr{Q}([X,Y]), Z]\!], \qquad X, Y, Z \in \Gamma(\mathcal{D}).$$

K is clearly skew-symmetric in its first two arguments, and hence we may also view it as a mapping

$$\Gamma(\textstyle\bigwedge^2 \mathcal{D}) \to \mathcal{T}^1_1(\mathcal{D}), \quad K(X \wedge Y)Z = K(X,Y)Z.$$

The associated $(0,4)$ curvature tensor (obtained by lowering an index using $\mathbf{g}$), which we denote $\widehat{K}$, is given by

$$\widehat{K}(W, X, Y, Z) = \mathbf{g}(K(W, X)Y, Z)$$

for $W, X, Y, Z \in \Gamma(\mathcal{D})$.

Lemma 2.1.1. *Let* $X, Y, Z \in \Gamma(\mathcal{D})$. *Then:*

(*i*) $(d^{\nabla}_{\mathscr{P}})^2 Z(X,Y) = K(X,Y)Z + \llbracket \mathscr{Q}([X,Y]), Z \rrbracket$.

(*ii*) $(d^{\nabla}_{\mathscr{P}})^2 \operatorname{id}_{\mathcal{D}}(X,Y,Z) = K(X,Y)Z + K(Y,Z)X + K(Z,X)Y$.

Proof. For item (i), we have

$$\begin{aligned}
(d^{\nabla}_{\mathscr{P}})^2 Z(X,Y) &= \nabla_X(d^{\nabla}_{\mathscr{P}} Z(Y)) - \nabla_Y(d^{\nabla}_{\mathscr{P}} Z(X)) - d^{\nabla}_{\mathscr{P}} Z(\llbracket X, Y\rrbracket) \\
&= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{\llbracket X,Y\rrbracket} Z \\
&= [\nabla_X, \nabla_Y]Z - \nabla_{\llbracket X,Y\rrbracket} Z - \llbracket \mathscr{Q}([X,Y]), Z\rrbracket + \llbracket \mathscr{Q}([X,Y]), Z\rrbracket \\
&= K(X,Y)Z + \llbracket \mathscr{Q}([X,Y]), Z\rrbracket.
\end{aligned}$$

Similarly, for item (ii):

$$\begin{aligned}
(d^{\nabla}_{\mathscr{P}})^2 \operatorname{id}_{\mathcal{D}}(X,Y,Z) &= \nabla_X(d^{\nabla}_{\mathscr{P}} \operatorname{id}_{\mathcal{D}}(Y,Z)) - \nabla_Y(d^{\nabla}_{\mathscr{P}} \operatorname{id}_{\mathcal{D}}(X,Z)) + \nabla_Z(d^{\nabla}_{\mathscr{P}} \operatorname{id}_{\mathcal{D}}(Y,Z)) \\
&\qquad - d^{\nabla}_{\mathscr{P}} \operatorname{id}_{\mathcal{D}}(\llbracket X,Y\rrbracket, Z) + d^{\nabla}_{\mathscr{P}} \operatorname{id}_{\mathcal{D}}(\llbracket X,Z\rrbracket, Y) - d^{\nabla}_{\mathscr{P}} \operatorname{id}_{\mathcal{D}}(\llbracket Y,Z\rrbracket, X) \\
&= \nabla_X(\nabla_Y Z - \nabla_Z Y - \llbracket Y, Z\rrbracket) - \nabla_Y(\nabla_X Z - \nabla_Z X - \llbracket X, Z\rrbracket) \\
&\qquad + \nabla_Z(\nabla_Y Z - \nabla_Z Y - \llbracket Y, Z\rrbracket) \\
&\qquad - (\nabla_{\llbracket X,Y\rrbracket} Z - \nabla_Z \llbracket X, Y\rrbracket - \llbracket \llbracket X, Y\rrbracket, Z\rrbracket) \\
&\qquad + (\nabla_{\llbracket X,Z\rrbracket} Y - \nabla_Y \llbracket X, Z\rrbracket - \llbracket \llbracket X, Z\rrbracket, Y\rrbracket) \\
&\qquad - (\nabla_{\llbracket Y,Z\rrbracket} X - \nabla_X \llbracket Y, Z\rrbracket - \llbracket \llbracket Y, Z\rrbracket, X\rrbracket).
\end{aligned}$$

Let $\sum_{\circlearrowleft(X,Y,Z)} f(X,Y,Z)$ denote the sum $f(X,Y,Z) + f(Y,Z,X) + f(Z,X,Y)$ over the cyclic permutations of (X,Y,Z). After rearranging terms in the above, we get

$$\begin{aligned}
(d^{\nabla}_{\mathscr{P}})^2 \operatorname{id}_{\mathcal{D}}(X,Y,Z) &= \sum_{\circlearrowleft(X,Y,Z)} \left([\nabla_X, \nabla_Y]Z - \nabla_{\llbracket X,Y\rrbracket} Z + \llbracket \llbracket X,Y\rrbracket, Z\rrbracket\right) \\
&= \sum_{\circlearrowleft(X,Y,Z)} \left(K(X,Y)Z + \llbracket [X,Y], Z\rrbracket\right).
\end{aligned}$$

From the Jacobi identity, we have $\sum_{\circlearrowleft(X,Y,Z)} \llbracket [X,Y], Z\rrbracket = 0$, which completes the proof. ■

The symmetries of the Riemannian curvature tensor are well known (see, e.g., [44, 55, 53]). Not all of those symmetries hold for the Schouten curvature tensor. Remarkably, however, the first Bianchi identity *does* still hold.

Proposition 2.1.2 (First Bianchi identity). *We have* $(d^{\nabla}_{\mathscr{P}})^2 \operatorname{id}_{\mathcal{D}} \equiv 0$, *i.e.,*

$$K(X,Y)Z + K(Y,Z)X + K(Z,X)Y = 0$$

for every $X, Y, Z \in \Gamma(\mathcal{D})$.

Proof. From section 1.1.4, we have that the torsion of ∇ is given by $T = d^{\nabla}_{\mathscr{P}} \operatorname{id}_{\mathcal{D}}$. Since ∇ is torsion free, it follows that $(d^{\nabla}_{\mathscr{P}})^2 \operatorname{id}_{\mathcal{D}} \equiv 0$. (The cyclic formula for K in the statement then follows from lemma 2.1.1.) ■

Remark 2.1.3. The second Bianchi identity does not hold for the Schouten tensor. Indeed, we may view K as an element of $\Omega^2(\mathcal{D}, T_1^1(\mathcal{D}))$. Furthermore, the nonholonomic connection ∇ extends to a connection of the form $\Gamma(\mathcal{D}) \times \mathcal{T}_1^1(\mathcal{D}) \to \mathcal{T}_1^1(\mathcal{D})$ (see section B.1), which we denote here by $\overline{\nabla}$. If $\varphi \in \mathcal{T}_1^1(\mathcal{D})$ and $X, U \in \Gamma(\mathcal{D})$, then $\overline{\nabla}$ is given by

$$(\overline{\nabla}_X \varphi)(U) = \nabla_X(\varphi(U)) - \varphi(\nabla_X U).$$

$\overline{\nabla}$ has an associated $\mathscr{P}$-exterior covariant derivative (defined analogously to $d_{\mathscr{P}}^{\nabla}$) of the form $d_{\mathscr{P}}^{\overline{\nabla}} : \Omega^k(\mathcal{D}, \mathcal{T}_1^1(\mathcal{D})) \to \Omega^{k+1}(\mathcal{D}, \mathcal{T}_1^1(\mathcal{D}))$. In this context, the second Bianchi identity would be given as $d_{\mathscr{P}}^{\overline{\nabla}} K \equiv 0$ (see, e.g., [56]). Instead, after a lengthy calculation, one can show that

$$d_{\mathscr{P}}^{\overline{\nabla}} K(X, Y, Z) = \nabla_{\mathcal{J}_{\mathscr{P}}(X,Y,Z)} - \pounds^{\mathscr{P}}_{\mathcal{J}_{\mathscr{Q}}(X,Y,Z)} - \sum_{\circlearrowleft(X,Y,Z)} [\nabla_X, \pounds^{\mathscr{P}}_{\mathscr{Q}([Y,Z])}]$$

for $X, Y, Z \in \Gamma(\mathcal{D})$, where $\mathcal{J}_{\mathscr{P}}$ and $\mathcal{J}_{\mathscr{Q}}$ are the Jacobiators of $[\![\cdot, \cdot]\!]$ and $\mathscr{Q}([\cdot, \cdot])$, respectively, i.e., $\mathcal{J}_{\mathscr{P}}(X, Y, Z) = \sum_{\circlearrowleft(X,Y,Z)} [\![[\![X, Y]\!], Z]\!]$ and $\mathcal{J}_{\mathscr{Q}}(X, Y, Z) = \sum_{\circlearrowleft(X,Y,Z)} \mathscr{Q}([\mathscr{Q}([X, Y]), Z])$. □

Proposition 2.1.4. $\widehat{K}$ *satisfies the following symmetries:*

(*S1*) $\widehat{K}(W, X, Y, Z) + \widehat{K}(X, W, Y, Z) = 0.$

(*S2*) $\widehat{K}(W, X, Y, Z) + \widehat{K}(W, X, Y, Z) + \widehat{K}(W, X, Y, Z) = 0.$

(*Here* $W, X, Y, Z \in \Gamma(\mathcal{D})$.)

Proof. Clearly, $K(X, X)Y = 0$, and so $\widehat{K}(X, X, Y, Z) = 0$. The symmetry (*S1*) follows by polarisation. For (*S2*), we have

$$\begin{aligned}&\widehat{K}(W, X, Y, Z) + \widehat{K}(W, X, Y, Z) + \widehat{K}(W, X, Y, Z)\\ &\quad = \mathsf{g}(K(W, X)Y + K(X, Y)W + K(Y, W)X, Z)\end{aligned}$$

which is zero by proposition 2.1.2. ■

In contrast to the $(0, 4)$ Riemannian curvature tensor, $\widehat{K}$ is generally not skew-symmetric in the final two arguments, nor is it symmetric if one swaps the first two arguments with the last two. We decompose $\widehat{K}$ into two tensors $\widehat{R}$ and $\widehat{C}$, where $\widehat{R}$ is the component of $\widehat{K}$ that is skew-symmetric in the last two arguments and $\widehat{C}$ is the component that is symmetric in the last two arguments. Specifically, we define $\widehat{R}, \widehat{C} \in \mathcal{T}_4^0(\mathcal{D})$ as

$$\widehat{R}(W, X, Y, Z) = \frac{1}{2}[\widehat{K}(W, X, Y, Z) - \widehat{K}(W, X, Z, Y)], \quad \widehat{C} = \widehat{K} - \widehat{R}.$$

(Here $W, X, Y, Z \in \Gamma(\mathcal{D})$.) In addition to (*S1*) and (*S2*), $\widehat{R}$ satisfies the following symmetries:

(*S3*) $\widehat{R}(W, X, Y, Z) + \widehat{R}(W, X, Z, Y) = 0.$

(*S4*) $\widehat{R}(W, X, Y, Z) = \widehat{R}(Y, Z, W, X)$. (This follows from (*S1*), (*S2*) and (*S3*).)

On the other hand, we have

(*S5*) $\widehat{C}(W, X, Y, Z) = \widehat{C}(W, X, Z, Y).$

Since $\widehat{R}$ satisfies all of the symmetries of the Riemannian $(0,4)$ curvature tensor, it behaves analogously to the Riemannian tensor. Thus we may define a sectional curvature, a Ricci tensor and a (Ricci) scalar curvature in terms of $\widehat{R}$ (see section 2.1.1 below). Furthermore, we can interpret $\widehat{R}$ as the tensor

$$\widehat{R} : \Gamma(\textstyle\bigwedge^2 \mathcal{D}) \times \Gamma(\textstyle\bigwedge^2 \mathcal{D}) \to \mathcal{C}^\infty(\mathsf{M}), \quad (W \wedge X, Y \wedge Z) \mapsto \widehat{R}(W, X, Z, Y).$$

(Note the order swap between Y and Z.) On the other hand, we can view $\widehat{C}$ as the tensor

$$\widehat{C} : \Gamma(\textstyle\bigwedge^2 \mathcal{D}) \times \Gamma(\textstyle\bigvee^2 \mathcal{D}) \to \mathcal{C}^\infty(\mathsf{M}), \quad (W \wedge X, Y \vee Z) \mapsto \widehat{C}(W, X, Y, Z).$$

(Due to (*S5*), the order of Y and Z here is irrelevant.)

Using $\mathbf{g}$, we can find $(1,3)$-tensors associated to $\widehat{R}$ and $\widehat{C}$. Indeed, let $R, C \in \mathcal{T}^1_3(\mathcal{D})$ be the tensors defined as

$$R(X,Y)Z = \mathbf{g}^\sharp(\widehat{R}(X,Y,Z,\,\cdot\,)) \quad \text{and} \quad C(X,Y)Z = \mathbf{g}^\sharp(\widehat{C}(X,Y,Z,\,\cdot\,)),$$

where $X, Y, Z \in \Gamma(\mathcal{D})$. Thus R and C are to $\widehat{R}$ and $\widehat{C}$, respectively, as K is to $\widehat{K}$. Furthermore, we clearly have $K = R + C$.

Proposition 2.1.5. *Let* $W, Y, X, Z \in \Gamma(\mathcal{D})$. *Then*

$$\widehat{C}(W,X,Y,Z) = \frac{1}{2}(\pounds^{\mathscr{P}}_{\mathscr{Q}([W,X])}\mathbf{g})(Y,Z).$$

Proof. From the definition of $\widehat{C}$, we have

$$\begin{aligned}
2\,\widehat{C}(W,X,Y,Z) &= \widehat{K}(W,X,Y,Z) + \widehat{K}(W,X,Z,Y) \\
&= \mathbf{g}(K(W,X)Y, Z) + \mathbf{g}(Y, K(W,X)Z) \\
&= \mathbf{g}([\nabla_W, \nabla_X]Y, Z) + \mathbf{g}(Y, [\nabla_W, \nabla_X]Z) - \mathbf{g}(\nabla_{[\![W,X]\!]}Y, Z) - \mathbf{g}(Y, \nabla_{[\![W,X]\!]}Z) \\
&\quad - \mathbf{g}([\![\mathscr{Q}([W,X]), Y]\!], Z) - \mathbf{g}(Y, [\![\mathscr{Q}([W,X]), Z]\!]).
\end{aligned}$$

The first two terms are

$$\begin{aligned}
\mathbf{g}([\nabla_W, \nabla_X]Y, Z) &= \mathbf{g}(\nabla_W \nabla_X Y, Z) - \mathbf{g}(\nabla_X \nabla_W Y, Z) \\
&= W[\mathbf{g}(\nabla_X Y, Z)] - \mathbf{g}(\nabla_X Y, \nabla_W Z) - X[\mathbf{g}(\nabla_W Y, Z)] + \mathbf{g}(\nabla_W Y, \nabla_X Z) \\
&= [W,X][\mathbf{g}(Y,Z)] - W[\mathbf{g}(Y, \nabla_X Z)] - \mathbf{g}(\nabla_X Y, \nabla_W Z) \\
&\qquad + X[\mathbf{g}(Y, \nabla_W Z)] + \mathbf{g}(\nabla_W Y, \nabla_X Z)
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{g}(Y, [\nabla_W, \nabla_X]Z) &= [W,X][\mathbf{g}(Y,Z)] - W[\mathbf{g}(\nabla_X Y, Z)] - \mathbf{g}(\nabla_W Y, \nabla_X Z) \\
&\qquad + X[\mathbf{g}(\nabla_W Y, Z)] + \mathbf{g}(\nabla_X Y, \nabla_W Z).
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\mathbf{g}([\nabla_W, \nabla_X]Y, Z) + \mathbf{g}(Y, [\nabla_W, \nabla_X]Z) \\
&\quad = 2\,[W,X][\mathbf{g}(Y,Z)] - W[\mathbf{g}(Y, \nabla_X Z) + \mathbf{g}(Z, \nabla_X Y)] + X[\mathbf{g}(\nabla_W Y, Z) + \mathbf{g}(Y, \nabla_W Z)] \\
&\quad = 2\,[W,X][\mathbf{g}(Y,Z)] - W[X[\mathbf{g}(Y,Z)]] + X[W[\mathbf{g}(Y,Z)]] \\
&\quad = [W,X][\mathbf{g}(Y,Z)].
\end{aligned}$$

Similarly, we have $-\mathbf{g}(\nabla_{[\![W,X]\!]}Y,Z)-\mathbf{g}(Y,\nabla_{[\![W,X]\!]}Z)=-[\![W,X]\!][\mathbf{g}(Y,Z)]$ and

$$\begin{aligned}-\mathbf{g}([\![\mathscr{Q}([W,X]),Y]\!],Z)-\mathbf{g}(Y,[\![\mathscr{Q}([W,X]),Z]\!]) &= -\mathscr{Q}([W,X])[\mathbf{g}(Y,Z)] \\ &\quad + (\mathscr{L}^{\mathscr{P}}_{\mathscr{Q}([W,X])}\mathbf{g})(Y,Z).\end{aligned}$$

Substituting back into the expression for $2\widehat{C}(W,X,Y,Z)$ yields the result. ■

2.1.1 Sectional curvature, the Ricci tensor and scalar curvature

Let $\mathcal{S}_q$, $q\in\mathsf{M}$ be a two-dimensional subspace of $\mathcal{D}_q$ and let (X_q,Y_q) be a basis for $\mathcal{S}_q$. We define the *sectional curvature* of $\mathcal{S}_q$, denoted $\widetilde{R}(\mathcal{S}_q)$, as

$$\widetilde{R}(\mathcal{S}_q)=\frac{\widehat{R}_q(X_q\wedge Y_q,X_q\wedge Y_q)}{\widehat{\mathbf{g}}_q(X_q\wedge Y_q,X_q\wedge Y_q)},$$

where $\widehat{\mathbf{g}}$ is the metric induced on $\bigwedge^2\mathcal{D}$ by $\mathbf{g}$, i.e.,

$$\widehat{\mathbf{g}}(W\wedge X,Y\wedge Z)=\mathbf{g}(W,Y)\mathbf{g}(X,Z)-\mathbf{g}(W,Z)\mathbf{g}(X,Y)$$

for $W,X,Y,Z\in\Gamma(\mathcal{D})$ and $\widehat{\mathbf{g}}$ is extended to the entirety of $\Gamma(\bigwedge^2\mathcal{D})\times\Gamma(\bigwedge^2\mathcal{D})$ by linearity over $\mathcal{C}^\infty(\mathsf{M})$. We shall also write $\widetilde{R}(\mathcal{S}_q)$ as $\widetilde{R}(X_q\wedge Y_q)$.

As in the Riemannian case, one can show that $\widetilde{R}$ is well defined and determines $\widehat{R}$. (We follow the corresponding proofs in [53] and [44].)

Lemma 2.1.6. *The sectional curvature of $\mathcal{S}_q$ is well defined, i.e., it does not depend on the choice of basis (X_q,Y_q).*

Proof. Let $(\widetilde{X}_q,\widetilde{Y}_q)$ be a different basis for $\mathcal{S}_q$. There exist $a,b,c,d\in\mathbb{R}$ with $ad-bc\neq 0$ such that $\widetilde{X}_q=a\,X_q+b\,Y_q$ and $\widetilde{Y}_q=c\,X_q+d\,Y_q$. Using skew-symmetry of the exterior product, we have $\widetilde{X}_q\wedge\widetilde{Y}_q=(ad-bc)X_q\wedge Y_q$, and so

$$\begin{cases}\widehat{R}_q(\widetilde{X}_q\wedge\widetilde{Y}_q,\widetilde{X}_q\wedge\widetilde{Y}_q)=(ad-bc)^2\widehat{R}_q(X_q\wedge Y_q,X_q\wedge Y_q)\\ \widehat{\mathbf{g}}_q(\widetilde{X}_q\wedge\widetilde{Y}_q,\widetilde{X}_q\wedge\widetilde{Y}_q)=(ad-bc)^2\,\widehat{\mathbf{g}}_q(X_q\wedge Y_q,X_q\wedge Y_q).\end{cases}$$

Hence

$$\widetilde{R}(\widetilde{X}_q\wedge\widetilde{Y}_q)=\frac{(ad-bc)^2\,\widehat{R}_q(X_q\wedge Y_q,X_q\wedge Y_q)}{(ad-bc)^2\,\widehat{\mathbf{g}}_q(X_q\wedge Y_q,X_q\wedge Y_q)}=\widetilde{R}(X_q\wedge Y_q).$$ ■

Proposition 2.1.7. *Let $F\in\mathcal{T}^0_4(\mathcal{D})$ satisfy (S1)–(S4), so that F can be viewed as a mapping $F_q:\bigwedge^2\mathcal{D}_q\times\bigwedge^2\mathcal{D}_q\to\mathbb{R}$, $(W_q\wedge X_q,Y_q\wedge Z_q)\mapsto F_q(W_q,X_q,Z_q,Y_q)$ for each $q\in\mathsf{M}$. If*

$$\widetilde{R}(X_q\wedge Y_q)=\frac{F_q(X_q\wedge Y_q,X_q\wedge Y_q)}{\widehat{\mathbf{g}}_q(X_q\wedge Y_q,X_q\wedge Y_q)}$$

for every nonzero $X_q\wedge Y_q\in\bigwedge^2\mathcal{D}$, then $\widehat{R}=F$.

Proof. Let $G = \widehat{R} - F$. Since G also satisfies the symmetries (*S1*)–(*S4*), it suffices to show that $G \equiv 0$ under the assumption $G_q(X_q \wedge Y_q, X_q \wedge Y_q) = 0$ for $X_q \wedge Y_q \in \bigwedge^2 \mathcal{D}_q$. Let $W_q, X_q, Y_q, Z_q \in \mathcal{D}_q$. We have

$$\begin{aligned}
0 &= G_q((X_q + Y_q) \wedge Z_q, (X_q + Y_q) \wedge Z_q) \\
&= G_q(X_q \wedge Z_q, X_q \wedge Z_q) + G_q(X_q \wedge Z_q, Y_q \wedge Z_q) + G_q(Y_q \wedge Z_q, X_q \wedge Z_q) \\
&\quad + G_q(Y_q \wedge Z_q, Y_q \wedge Z_q) \\
&= 2\, G_q(X_q \wedge Z_q, X_q \wedge Z_q),
\end{aligned}$$

and so

$$\begin{aligned}
0 &= G_q(W_q \wedge (Y_q + Z_q), X_q \wedge (Y_q + Z_q)) \\
&= G_q(W_q \wedge Y_q, X_q \wedge Y_q) + G_q(W_q \wedge Y_q, X_q \wedge Z_q) + G_q(W_q \wedge Z_q, X_q \wedge Y_q) \\
&\quad + G_q(W_q \wedge Z_q, X_q \wedge Z_q) \\
&= G_q(W_q \wedge Y_q, X_q \wedge Z_q) + G_q(W_q \wedge Z_q, X_q \wedge Y_q).
\end{aligned}$$

Accordingly, using the (first) Bianchi identity, we have

$$\begin{aligned}
0 &= G_q(W_q \wedge X_q, Y_q \wedge Z_q) + G_q(X_q \wedge Y_q, Z_q \wedge W_q) + G_q(Y_q \wedge W_q, Z_q \wedge X_q) \\
&= G_q(W_q \wedge X_q, Y_q \wedge Z_q) - G_q(X_q \wedge W_q, Z_q \wedge Y_q) - G_q(W_q \wedge Y_q, Z_q \wedge X_q) \\
&= G_q(W_q \wedge X_q, Y_q \wedge Z_q) + G_q(W_q \wedge X_q, Z_q \wedge Y_q) + G_q(W_q \wedge X_q, Z_q \wedge Y_q) \\
&= 3\, G_q(W_q \wedge X_q, Y_q \wedge Z_q).
\end{aligned}$$

Since W_q, X_q, Y_q and Z_q are arbitrary, it follows that $G_q \equiv 0$ for every $q \in \mathsf{M}$. ■

Corollary 2.1.8. *Suppose* $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ *has constant sectional curvature* $\kappa \in \mathbb{R}$ (*i.e., we have* $\widetilde{R}(\mathcal{S}_q) = \kappa$ *for every two-dimensional subspace* $\mathcal{S}_q \subseteq \mathcal{D}_q$). *Then*

$$\widehat{R}(W \wedge X, Y \wedge Z) = \kappa\, \widehat{\mathbf{g}}(W \wedge X, Y \wedge Z)$$

for every $W, X, Y, Z \in \Gamma(\mathcal{D})$.

Proof. The map $F \in \mathcal{T}^0_4(\mathcal{D})$ given by $F(W, X, Y, Z) = \kappa\, \widehat{\mathbf{g}}(W \wedge X, Y \wedge Z)$ satisfies the symmetries (*S1*)–(*S4*). Furthermore, we have $F(X, Y, X, Y) = \kappa\, \widehat{\mathbf{g}}(X \wedge Y, X \wedge Y)$ for every $X, Y \in \Gamma(\mathcal{D})$. Hence, if $\mathcal{S}_q \subseteq \mathcal{D}_q$ is spanned by $U_q \wedge V_q$, then

$$\widetilde{R}(U_q \wedge V_q) = \kappa = \frac{F_q(U_q \wedge V_q, V_q \wedge U_q)}{\widehat{\mathbf{g}}_q(U_q \wedge V_q, U_q \wedge V_q)}$$

and so $\widehat{R}(W, X, Y, Z) = F(W, X, Y, Z)$ for every $W, X, Y, Z \in \Gamma(\mathcal{D})$. ■

The *Ricci tensor* $\mathrm{Ric} \in \mathcal{T}^0_2(\mathcal{D})$ is defined as $\mathrm{Ric} = \mathrm{tr}^1_1 R$, i.e., if $X, Y \in \Gamma(\mathcal{D})$, then

$$\begin{aligned}
\mathrm{Ric}(X, Y) &= \mathrm{tr}(Z \mapsto R(Z, X)Y) \\
&= \sum_a \mathbf{g}(R(X_a, X)Y, X_a) = \sum_a \widehat{R}(X_a, X, Y, X_a),
\end{aligned}$$

where (X_a) is an orthonormal frame for $\mathcal{D}$.

Proposition 2.1.9. *The Ricci tensor is symmetric.*

Proof. Let $X, Y \in \Gamma(\mathcal{D})$ and let (X_a) be an orthonormal frame for $\mathcal{D}$. Using the (first) Bianchi identity, we have

$$\begin{aligned}\operatorname{Ric}(X,Y) &= \sum_a \widehat{R}(X_a, X, Y, X_a) = -\sum_a \left[\widehat{R}(X, Y, X_a, X_a) + \widehat{R}(Y, X_a, X, X_a)\right] \\ &= -\sum_a \widehat{R}(Y, X_a, X, X_a) = \sum_\alpha \widehat{R}(X_a, Y, X, X_a) = \operatorname{Ric}(Y, X).\end{aligned}$$

Thus Ric is symmetric. ∎

The trace of the endomorphism $\mathbf{g}^\sharp \circ \operatorname{Ric}^\flat : \Gamma(\mathcal{D}) \to \Gamma(\mathcal{D})$ is called the *scalar curvature*, denoted Scal. In terms of the orthonormal frame (X_a), we have

$$\operatorname{Scal} = \sum_a \operatorname{Ric}(X_a, X_a) = \sum_{a,b} \widehat{R}(X_b, X_a, X_a, X_b) = \sum_{a \neq b} \widetilde{R}(X_a \wedge X_b).$$

2.1.2 Contractions of C

In a similar fashion to the Ricci tensor, let $A \in \mathcal{T}_2^0(\mathcal{D})$ be defined as $A = \operatorname{tr}_1^1 C$, i.e.,

$$A(X,Y) = \sum_a \widehat{C}(X_a, X, Y, X_a), \qquad X, Y \in \Gamma(\mathcal{D}),$$

where (X_a) is an orthonormal frame for $\mathcal{D}$. In general, A is not symmetric. Thus we define two tensors A_{sym} and A_{skew} to be the symmetric and skew-symmetric parts of A, respectively. That is, $A_{sym}(X,Y) = \frac{1}{2}[A(X,Y) + A(Y,X)]$ and $A_{skew}(X,Y) = \frac{1}{2}[A(X,Y) - A(Y,X)]$, where $X, Y \in \Gamma(\mathcal{D})$. In terms of (X_a), we then have

$$\begin{aligned}A_{sym}(X,Y) &= \frac{1}{2}\sum_a \left[\widehat{C}(X_a, X, Y, X_a) + \widehat{C}(X_a, Y, X, X_a)\right], \\ A_{skew}(X,Y) &= -\frac{1}{2}\sum_a \widehat{C}(X, Y, X_a, X_a).\end{aligned}$$

(Furthermore, $A_{skew} = -\frac{1}{2}\operatorname{tr}_3^1 C$.) Both A_{sym} and A_{skew} are trace-free, so there is no analogue of the scalar curvature in this case.

2.2 The Wagner curvature tensor

Let $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ be a nonholonomic Riemannian manifold, where $\mathcal{D}$ has degree of nonholonomy $N \geq 2$. (See section 1.1.) Let $\mathcal{D}^1 \subsetneq \mathcal{D}^2 \subsetneq \cdots \subsetneq \mathcal{D}^{N-1} \subsetneq \mathcal{D}^N = T\mathsf{M}$ be the flag of $\mathcal{D}$, i.e., we have $\mathcal{D}^1 = \mathcal{D}$ and $\mathcal{D}^{i+1} = \mathcal{D}^i + [\mathcal{D}^i, \mathcal{D}^i]$ for $i \geq 1$. In addition, let $\mathcal{E}^1, \ldots, \mathcal{E}^{N-1}$ be distributions on M such that

$$\mathcal{D}^\perp = \mathcal{E}^1 \oplus \cdots \oplus \mathcal{E}^{N-1} \quad \text{and} \quad \mathcal{D}^{i+1} = \mathcal{D}^i \oplus \mathcal{E}^i \text{ for each } i = 1, \ldots, N-1. \tag{2.2.1}$$

Let $\mathscr{Q}_i : T\mathsf{M} \to \mathcal{E}^i$ denote the projection onto $\mathcal{E}^i$ and let $\mathscr{P}_i : T\mathsf{M} \to \mathcal{D}^i$ be the projection onto $\mathcal{D}^i = \mathcal{D} \oplus \mathcal{E}^1 \oplus \cdots \oplus \mathcal{E}^{i-1}$ defined as $\mathscr{P}_1 = \mathscr{P}$ and $\mathscr{P}_{i+1} = \mathscr{P} \oplus \mathscr{Q}_1 \oplus \cdots \oplus \mathscr{Q}_i$ for $i \geq 1$.

We shall see that the distributions $\mathcal{E}^1,\dots,\mathcal{E}^{N-1}$ are crucial for the definition of the Wagner curvature tensor, yet in general there is no canonical choice for these distributions (and hence they will not be preserved under nonholonomic isometries; see section 3.1.3). Consequently, the Wagner curvature tensor will not be intrinsically defined. (Because of this, Wagner [75] proposed redefining a nonholonomic Riemannian structure to include $\mathcal{E}^1,\dots,\mathcal{E}^{N-1}$.) This prompts us to make the following definition: a *Wagner structure* is a nonholonomic Riemannian structure $(\mathsf{M},\mathcal{D},\mathcal{D}^\perp,\mathbf{g})$, with degree of nonholonomy $N\ge 2$, together with distributions $\mathcal{E}^1,\dots,\mathcal{E}^{N-1}$ on M such that the equations (2.2.1) are satisfied.

If $\mathcal{D}$ is strongly nonholonomic, then we have $\mathcal{D}^2=T\mathsf{M}=\mathcal{D}\oplus\mathcal{D}^\perp$, i.e., the choice of $\mathcal{E}^1$ is canonical. Thus we have the following result.

Proposition 2.2.1. *Every nonholonomic Riemannian structure* $(\mathsf{M},\mathcal{D},\mathcal{D}^\perp,\mathbf{g})$ *with* $\mathcal{D}$ *strongly nonholonomic is a Wagner structure.*

Thus, when $\mathcal{D}$ is strongly nonholonomic, the Wagner curvature tensor is intrinsically defined. The next result also shows that, if we are dealing with a nonholonomic mechanical system with kinetic energy Lagrangian and constraints linear-in-velocities (see section 1.1.1), then its associated nonholonomic Riemannian structure is a Wagner structure.

Proposition 2.2.2. *Let* $(\mathsf{M},\widetilde{\mathbf{g}})$ *be a Riemannian manifold,* $\mathcal{D}$ *a completely nonholonomic distribution on* M *and* $\mathcal{D}^\perp$ *the orthogonal complement of* $\mathcal{D}$. *Let* $\mathcal{D}=\mathcal{D}^1\subsetneq\cdots\subsetneq\mathcal{D}^N=T\mathsf{M}$ *be the flag of* $\mathcal{D}$, *where* $N\ge 2$ *is the degree of nonholonomy of* $\mathcal{D}$, *and let* $\mathcal{E}^i$ *be the* $\widetilde{\mathbf{g}}|_{\mathcal{D}^{i+1}}$-*orthogonal complement of* $\mathcal{D}^i$ *in* $\mathcal{D}^{i+1}$, *for each* $i=1,\dots,N-1$. *Then* $(\mathsf{M},\mathcal{D},\mathcal{D}^\perp,\widetilde{\mathbf{g}}|_{\mathcal{D}})$, *together with the distributions* $\mathcal{E}^1,\dots,\mathcal{E}^{N-1}$, *is a Wagner structure.*

Proof. The second part of (2.2.1) holds by construction of $\mathcal{E}^1,\dots,\mathcal{E}^{N-1}$. For the first part, since $\mathcal{D}^i=\mathcal{D}\oplus\mathcal{E}^1\oplus\cdots\oplus\mathcal{E}^{i-1}$ and $\mathcal{D}^i\perp_{\widetilde{\mathbf{g}}}\mathcal{E}^i$, we have $\mathcal{D}\perp_{\widetilde{\mathbf{g}}}\mathcal{E}^i$. That is, $\mathcal{D}$ is orthogonal to each of $\mathcal{E}^1,\dots,\mathcal{E}^{N-1}$, and hence is orthogonal to $\mathcal{E}^1\oplus\cdots\oplus\mathcal{E}^{N-1}$. Then $\mathcal{E}^1\oplus\cdots\oplus\mathcal{E}^{N-1}$ is the orthogonal complement of $\mathcal{D}$, whence $\mathcal{D}^\perp=\mathcal{E}^1\oplus\cdots\oplus\mathcal{E}^{N-1}$. ■

For each $i=1,\dots,N-1$, define a mapping $\Delta_i:\Gamma(\bigwedge^2\mathcal{D}^i)\to\Gamma(\mathcal{E}^i)$, called the i^{th} *nonholonomy tensor of* $\mathcal{D}$, as

$$\Delta_i(X\wedge Y)=\mathscr{Q}_i([X,Y]),$$

where $X,Y\in\Gamma(\mathcal{D}^i)$.

Lemma 2.2.3. *The map* Δ_i *is tensorial and surjective, for each* $i=1,\dots,N-1$.

Proof. Let $1\le i\le N-1$, let $X,Y\in\Gamma(\mathcal{D})$ and let $f\in C^\infty(\mathsf{M})$. We have

$$\begin{aligned}\Delta_i(fX\wedge Y)&=\mathscr{Q}_i([fX,Y])=\mathscr{Q}_i(f[X,Y]-Y[f]X)\\&=f\mathscr{Q}_i([X,Y])\\&=f\Delta_i(X\wedge Y),\end{aligned}$$

and so Δ_i is tensorial. Surjectivity follows from the complete nonholonomy of $\mathcal{D}$. Indeed,

$$\mathcal{D}^i\oplus\mathcal{E}^i=\mathcal{D}^{i+1}=\mathcal{D}^i+[\mathcal{D}^i,\mathcal{D}^i]=\mathcal{D}^i\oplus\mathscr{Q}_i([\mathcal{D}^i,\mathcal{D}^i]),$$

and so $\mathcal{E}^i=\mathscr{Q}_i([\mathcal{D}^i,\mathcal{D}^i])=\Delta_i(\bigwedge^2\mathcal{D}^i)$. ■

Using $\Delta_1, \dots, \Delta_{N-1}$, the metric $\mathbf{g}$ can be extended to a Riemannian metric on M.

Theorem 2.2.4 (cf. [21]). *There exists a unique Riemannian metric* $\widetilde{\mathbf{g}}$ *on* M *satisfying the following conditions*:

(*i*) *The decomposition* $T\mathsf{M} = \mathcal{D} \oplus \mathcal{E}^1 \oplus \cdots \oplus \mathcal{E}^{N-1}$ *is orthogonal and* $\widetilde{\mathbf{g}} = \mathbf{g} \oplus \mathbf{h}^1 \oplus \cdots \oplus \mathbf{h}^{N-1}$*, where* $\mathbf{h}^i = \widetilde{\mathbf{g}}|_{\mathcal{E}^i}$ *for each* $i = 1, \dots, N-1$.

(*ii*) $\Delta_i|_{(\ker \Delta_i)^\perp} : (\ker \Delta_i)^\perp \to \mathcal{E}^i$ *is an isometry, for each* $i = 1, \dots, N-1$*, i.e.,*

$$\mathbf{h}^i(\Delta_i(W \wedge X), \Delta_i(Y \wedge Z)) = \widehat{\mathbf{g}}^i(W \wedge X, Y \wedge Z)$$

for every $W \wedge X, Y \wedge Z \in (\ker \Delta_i)^\perp$*. Here* $\widehat{\mathbf{g}}^i$ *is the metric induced on* $\bigwedge^2 \mathcal{D}^i$ *by the metric* $\mathbf{g}^i = \mathbf{g} \oplus \mathbf{h}^1 \oplus \cdots \oplus \mathbf{h}^{i-1}$ *on* $\mathcal{D}^i$*, i.e.,*

$$\widehat{\mathbf{g}}^i(W \wedge X, Y \wedge Z) = \mathbf{g}^i(W, Y)\mathbf{g}^i(X, Z) - \mathbf{g}^i(W, Z)\mathbf{g}^i(X, Y)$$

for $W, X, Y, Z \in \Gamma(\mathcal{D}^i)$ *and* $\widehat{\mathbf{g}}^i$ *is extended to the entirety of* $\Gamma(\bigwedge^2 \mathcal{D}^i) \times \Gamma(\bigwedge^2 \mathcal{D}^i)$ *by linearity over* $C^\infty(\mathsf{M})$.

Proof. Let $\mathbf{g}^1 = \mathbf{g}$ and let $\widehat{\mathbf{g}}^1$ be the corresponding metric on $\bigwedge^2 \mathcal{D}^1$. Let $(\ker \Delta_1)^\perp$ be the orthogonal complement of $\ker \Delta_1 \subseteq \bigwedge^2 \mathcal{D}^1$ with respect to $\widehat{\mathbf{g}}^1$. As Δ_1 is surjective, we can define a (positive definite) metric $\mathbf{h}^1$ on $\mathcal{E}^1$ by the requirement that the isomorphism $\Delta_1|_{(\ker \Delta_1)^\perp} : (\ker \Delta_1)^\perp \to \mathcal{E}^1$ is an isometry. Specifically, we define $\mathbf{h}^1$ as

$$\mathbf{h}^1(\Delta_1(W \wedge X), \Delta_1(Y \wedge Z)) = \widehat{\mathbf{g}}^1(W \wedge X, Y \wedge Z), \qquad W, X, Y, Z \in (\ker \Delta_1)^\perp.$$

Hence we have the metric $\mathbf{g}^2 = \mathbf{g}^1 \oplus \mathbf{h}^1$ on $\mathcal{D}^2 = \mathcal{D} \oplus \mathcal{E}^1$, which induces a metric $\widehat{\mathbf{g}}^2$ on $\bigwedge^2 \mathcal{D}^2$. Let $\mathbf{h}^2$ be the metric on $\mathcal{E}^2$ induced by $\Delta_2|_{(\ker \Delta_2)^\perp}$. Continuing in this fashion, we get the Riemannian metric $\widetilde{\mathbf{g}} = \mathbf{g} \oplus \mathbf{h}^1 \oplus \cdots \oplus \mathbf{h}^{N-1}$, defined on $T\mathsf{M} = \mathcal{D} \oplus \mathcal{E}^1 \oplus \cdots \oplus \mathcal{E}^{N-1}$. ■

Fix $1 \leq i \leq N-1$. Let $\nabla^1 = \nabla$ and let $\nabla^{i+1} : \Gamma(\mathcal{D}^{i+1}) \times \Gamma(\mathcal{D}) \to \Gamma(\mathcal{D})$ be the $\mathcal{D}^{i+1}$-restricted connection on $\mathcal{D}$ (see section B.1) specified as follows: if $Z \in \Gamma(\mathcal{D}^{i+1})$ with $X = \mathscr{P}_i(Z)$ and $A = \mathscr{Q}_i(Z)$, then

$$\nabla^{i+1}_Z U = \nabla^i_X U + K^i(\Theta_i(A))U + \llbracket A, U \rrbracket.$$

Here $\Theta_i = \Delta_i|^{-1}_{(\ker \Delta_i)^\perp}$ and $K^i : \Gamma(\bigwedge^2 \mathcal{D}^i) \times \Gamma(\mathcal{D}) \to \Gamma(\mathcal{D})$ is the curvature tensor of ∇^i defined as

$$K^i(X \wedge Y)U = [\nabla^i_X, \nabla^i_Y]U - \nabla^i_{\mathscr{P}_i([X,Y])}U - \llbracket \mathscr{Q}_i([X,Y]), U \rrbracket. \tag{2.2.2}$$

(Note that, as $\mathscr{Q}([\mathcal{D}, \mathcal{D}]) = \mathscr{Q}_1([\mathcal{D}, \mathcal{D}])$, we have $K^1 = K$.) In particular, for $i = N-1$, we have a vector bundle connection $\nabla^N : \Gamma(T\mathsf{M}) \times \Gamma(\mathcal{D}) \to \Gamma(\mathcal{D})$ on $\mathcal{D}$. Let K^N be the curvature tensor of this connection:

$$K^N(X \wedge Y)U = [\nabla^N_X, \nabla^N_Y]U - \nabla^N_{[X,Y]}U.$$

K^N is called the *Wagner curvature tensor* of $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$.

Remark 2.2.5. We have departed slightly from Wagner's original definition of the connections $\nabla^2, \dots, \nabla^N$, and consequently, of $K^2, \dots, K^N$ (see [21, 75]). Wagner defined

$$\nabla^{i+1}_Z U = \nabla^i_X U + K^i(\Delta_i^\dagger(A))U + [\![A, U]\!], \qquad Z = X + A,\ X \in \Gamma(\mathcal{D}^i),\ A \in \Gamma(\mathcal{E}^i) \tag{2.2.3}$$

where $\Delta_i^\dagger$ is the adjoint of Δ_i, i.e., $\Delta_i^\dagger = (\widehat{\mathbf{g}}^i)^\sharp \circ \Delta_i^* \circ (\mathbf{h}^i)^\flat$. In fact, it turns out that the $\Delta_i^\dagger$ in (2.2.3) can be replaced with *any* right inverse of Δ_i, and the crucial property of the ∇^i's (*viz.*, theorem 2.2.8, below) will still hold. Accordingly, there are many possible ways to define K^N. □

Before proving the key property of the $\mathcal{D}^i$-connections ∇^i (that the set of parallel tensor fields of ∇^i coincides with that of ∇^{i+1}), we first prove some basic lemmas. Specifically, we show that ∇^i is indeed a $\mathcal{D}^i$-restricted connection on $\mathcal{D}$ and that K^i is a $(1,3)$-tensor field.

Lemma 2.2.6. *∇^i is a $\mathcal{D}^i$-restricted connection on $\mathcal{D}$, for each $i = 1, \dots, N$.*

Proof. We use induction. The nonholonomic connection ∇^1 is a $\mathcal{D}$-restricted connection on $\mathcal{D}$ (theorem 1.1.5). Assume that ∇^i is a $\mathcal{D}^i$-restricted connection on $\mathcal{D}$ for some $1 \le i < N - 1$. Let $X, Y \in \Gamma(\mathcal{D}^i)$, $A, B \in \Gamma(\mathcal{E}^i)$, $U \in \Gamma(\mathcal{D})$ and $f \in \mathcal{C}^\infty(\mathsf{M})$. Clearly, ∇^{i+1} is $\mathbb{R}$-linear. Furthermore,

$$\begin{aligned}
\nabla^{i+1}_{f(X+A)} U &= \nabla^i_{fX} U + K^i(f\Theta_i(A))U + [\![fA, U]\!] \\
&= f\left(\nabla^i_X U + K^i(\Theta_i(A))U + [\![A, U]\!]\right) \\
&= f\, \nabla^{i+1}_{X+A} U,
\end{aligned}$$

which proves tensoriality in the first argument. Similarly,

$$\begin{aligned}
\nabla^{i+1}_{X+A} fU &= \nabla^i_X fU + K^i(\Theta_i(A))fU + [\![A, fU]\!] \\
&= X[f]U + f\nabla^i_X U + fK^i(\Theta_i(A))U + f[\![A, U]\!] + A[f]U \\
&= (X + A)[f]U + f\, \nabla^{i+1}_{X+A} U,
\end{aligned}$$

and so ∇^{i+1} is a derivation in its second argument. We thus have that ∇^{i+1} is a $\mathcal{D}^{i+1}$-connection on $\mathcal{D}$. ■

Lemma 2.2.7. *K^i is tensorial in all three arguments, for each $i = 1, \dots, N$.*

Proof. Let $X, Y \in \Gamma(\mathcal{D}^i)$, $U \in \Gamma(\mathcal{D})$ and $f \in \mathcal{C}^\infty(\mathsf{M})$. We have

$$\begin{aligned}
K^i(fX \wedge Y)U &= [\nabla^i_{fX}, \nabla^i_Y]U - \nabla^i_{\mathscr{P}_i([fX,Y])}U - [\![\mathscr{Q}_i([fX, Y]), U]\!] \\
&= f\nabla^i_X \nabla^i_Y U - (Y[f]\nabla^i_X U + f\nabla^i_Y \nabla^i_X U) - \nabla^i_{f\mathscr{P}_i([X,Y]) - Y[f]X} U \\
&\qquad - f[\![\mathscr{Q}_i([X, Y]), U]\!] \\
&= fK^i(X \wedge Y)U.
\end{aligned}$$

Likewise,

$$\begin{aligned}
K^i(X \wedge Y)fU &= [\nabla^i_X, \nabla^i_X](fU) - \nabla^i_{\mathscr{P}_i([X,Y])}(fU) - [\![\mathscr{Q}_i([X, Y]), fU]\!] \\
&= [X, Y][f]U + f[\nabla^i_X, \nabla^i_X]U - \mathscr{P}_i([X, Y])[f]U - f\nabla^i_{\mathscr{P}_i([X,Y])}U \\
&\qquad - f[\![\mathscr{Q}_i([X, Y]), U]\!] - \mathscr{Q}_i([X, Y])[f]U \\
&= fK^i(X \wedge Y)U.
\end{aligned}$$

Thus $K^i \in \mathcal{T}^0_2(\mathcal{D}^i) \otimes \mathcal{T}^1_1(\mathcal{D})$. ■

Theorem 2.2.8 (cf. [21]). *Let* $T \in \mathcal{T}^k_\ell(\mathcal{D})$ *and* $1 \le i \le N-1$. *We have* $\nabla^i T \equiv 0$ *if and only if* $\nabla^{i+1}T \equiv 0$.

Proof. Let $T \in \mathcal{T}^k_\ell(\mathcal{D})$. It is not difficult to see that

$$\nabla^{i+1}_Z T = \nabla^i_X T + K^i(\Theta_i(A))T + \pounds^{\mathscr{P}}_A T, \qquad Z = X + A,\ X \in \Gamma(\mathcal{D}^i),\ A \in \Gamma(\mathcal{E}^i),$$

where $K^i(\Theta_i(A))T$ is defined by extending the $(1,1)$-tensor field $K^i(\Theta_i(A))$ to an algebraic derivation of $\mathcal{T}^k_\ell(\mathcal{D})$ (see section A.3.1). Suppose that $\nabla^i T \equiv 0$ for some $1 \le i \le N-1$. Then

$$\begin{aligned} K^i(X \wedge Y)T &= \nabla^i_X \nabla^i_Y T - \nabla^i_Y \nabla^i_X T - \nabla^i_{\mathscr{P}_i([X,Y])} T - \pounds^{\mathscr{P}}_{\mathscr{Q}_i([X,Y])} T \\ &= -\pounds^{\mathscr{P}}_{\mathscr{Q}_i([X,Y])} T \end{aligned}$$

for every $X, Y \in \Gamma(\mathcal{D}^i)$. We claim that $K^i(\Theta_i(A))T + \pounds^{\mathscr{P}}_A T = 0$ for each $A \in \Gamma(\mathcal{E}^i)$. Indeed, suppose that $\Theta_i(A) = f^{jk} Y_j \wedge Z_k$ for some $f^{jk} \in \mathcal{C}^\infty(\mathsf{M})$ and $Y_j, Z_k \in \Gamma(\mathcal{D}^i)$. Then

$$\begin{aligned} K^i(\Theta_i(A))T = f^{jk} K^i(Y_j \wedge Z_k)T &= f^{jk}[\nabla^i_{Y_j}, \nabla^i_{Z_k}]T - f^{jk}\nabla_{\mathscr{P}_i([Y_j,Z_k])}T - f^{jk}\pounds^{\mathscr{P}}_{\mathscr{Q}_i([Y_j,Z_k])}T \\ &= -f^{jk}\pounds^{\mathscr{P}}_{\Delta_i(Y_j \wedge Z_k)}T = -\pounds^{\mathscr{P}}_{\Delta_i(\Theta_i(A))}T \\ &= -\pounds^{\mathscr{P}}_A T. \end{aligned}$$

Hence, if $Z \in \Gamma(\mathcal{D}^{i+1})$ with $X = \mathscr{P}_i(Z)$ and $A = \mathscr{Q}_i(Z)$, we have

$$\nabla^{i+1}_Z T = \nabla^i_X T + K^i(\Theta_i(A))T + \pounds^{\mathscr{P}}_A T = 0.$$

The converse follows from the fact that, if $X \in \Gamma(\mathcal{D}^i)$, then $\nabla^i_X T = \nabla^{i+1}_X T$. Therefore, $\nabla^i T$ vanishes identically if and only if $\nabla^{i+1} T \equiv 0$. ■

Corollary 2.2.9. *Each connection* ∇^i *is metric, i.e.,* $\nabla^i \mathbf{g} \equiv 0$ *for each* $i = 1, \dots, N$.

We have N restricted connections $\nabla^1, \dots, \nabla^N$ on $\mathcal{D}$. The first connection (a nonholonomic, or $\mathcal{D}$-restricted connection on $\mathcal{D}$) permits parallel transport only along $\mathcal{D}$-curves, whereas the last (a vector bundle connection on $\mathcal{D}$) permits parallel translation along any curve in M. In between we have the $\mathcal{D}^i$-restricted connections ∇^i (for each $i = 2, \dots, N-1$) which permit parallel translation along $\mathcal{D}^i$-curves. By corollary 2.2.9, parallel translation (with respect to any of the connections $\nabla^1, \dots, \nabla^N$) is a linear isometry.

For a $\mathcal{D}^i$-curve $\gamma : [0,1] \to \mathsf{M}$, $1 \le i \le N$, let $\Pi^{i,t}_\gamma$ denote the parallel translation along γ with respect to ∇^i. (See section B.1.1.)

Proposition 2.2.10. *Let* $1 \le i \le N-1$. *If* $\gamma : [0,1] \to \mathsf{M}$ *is a* $\mathcal{D}^i$*-curve, then* $\Pi^{i,t}_\gamma = \Pi^{i+1,t}_\gamma$.

Proof. Let $\gamma : [0,1] \to \mathsf{M}$ be a $\mathcal{D}^i$-curve, $U_0 \in \mathcal{D}_{\gamma(0)}$ and $V(t) = \Pi^{i,t}_\gamma(U_0)$, $W(t) = \Pi^{i+1,t}_\gamma(U_0)$. Let $(X^0_{a_0})$ be an orthonormal frame for $\mathcal{D}$ and $(X^i_{a_i})$ a frame for $\mathcal{E}^i$, where $1 \le a_0 \le r$ and $1 \le a_i \le \operatorname{rank}(\mathcal{E}^i)$. It follows that $(X^0_{a_0}, X^1_{a_1}, \dots, X^i_{a_i})$ is a frame for $\mathcal{D}^{i+1} = \mathcal{D} \oplus \mathcal{E}^1 \oplus \dots \oplus \mathcal{E}^i$. There exist functions $v^{a_0}, w^{a_0} \in \mathcal{C}^\infty([0,1])$ such that $V = v^{a_0}(X^0_{a_0} \circ \gamma)$ and $W = w^{a_0}(X^0_{a_0} \circ \gamma)$. Furthermore, these functions satisfy the ODEs

$$\dot{v}^{a_0} = -\Gamma^{a_0}_{b_i c_0}(\gamma)\dot{\gamma}^{b_i} v^{c_0} \quad \text{and} \quad \dot{w}^{a_0} = -\Omega^{a_0}_{b_i c_0}(\gamma)\dot{\gamma}^{b_i} w^{c_0},$$

where $\Gamma^{a_0}_{b_i c_0}, \Omega^{a_0}_{b_i c_0} \in \mathcal{C}^\infty(\mathsf{M})$ are defined by $\nabla^i_{X^j_{b_j}} X^0_{c_0} = \Gamma^{a_0}_{b_j c_0} X^0_{a_0}$ (where $0 \le j \le i$) and $\nabla^{i+1}_{X^k_{b_k}} X^0_{c_0} = \Omega^{a_0}_{b_k c_0} X^0_{a_0}$ (where $0 \le k \le i+1$). Since $\nabla^i_X U = \nabla^{i+1}_X U$ for $X \in \Gamma(\mathcal{D}^i)$ and $U \in \Gamma(\mathcal{D})$, we have $\Gamma^{a_0}_{b_j c_0} = \Omega^{a_0}_{b_j c_0}$ for $j = 0, \dots, i$. It follows that $V = W$. ■

2.2.1 Algebraic interpretation of curvature tensors

For a vector bundle connection $\widetilde{\nabla}$ on $\mathcal{D}$, the curvature tensor $(X, Y) \mapsto [\widetilde{\nabla}_X, \widetilde{\nabla}_Y] - \widetilde{\nabla}_{[X,Y]}$ can be viewed as measuring the extent to which the mapping $\Gamma(T\mathsf{M}) \to \mathsf{Der}(\mathcal{D})$, $X \mapsto \widetilde{\nabla}_X$ fails to be a homomorphism (of Lie algebras). In this section we show that a similar interpretation holds for the curvature tensors K^i, $i = 1, \ldots, N$. Section A.2 contains some necessary results (and fixes some notation) for what follows.

From proposition A.3.2 we have the vector space decomposition $\mathsf{Der}(\mathcal{D}) = \mathscr{L}^{\mathscr{P}}_{T\mathsf{M}} \oplus \mathsf{Der}_0(\mathcal{D})$. Let $\mathcal{S}$ be a distribution on M and denote $\mathscr{L}^{\mathscr{P}}_{\mathcal{S}} = \{\mathscr{L}^{\mathscr{P}}_X : X \in \Gamma(\mathcal{S})\}$. We shall say that a derivation $\delta \in \mathsf{Der}(\mathcal{D})$ is an *$\mathcal{S}$-restricted derivation* (or simply *$\mathcal{S}$-derivation*) if

$$\delta \in \mathsf{Der}_{\mathcal{S}}(\mathcal{D}) = \mathscr{L}^{\mathscr{P}}_{\mathcal{S}} \oplus \mathsf{Der}_0(\mathcal{D}).$$

Since the tangent bundle of M decomposes as $T\mathsf{M} = \mathcal{D} \oplus \mathcal{E}^1 \oplus \cdots \oplus \mathcal{E}^{N-1}$, we have a similar (vector space) decomposition $\mathscr{L}^{\mathscr{P}}_{T\mathsf{M}} = \mathscr{L}^{\mathscr{P}}_{\mathcal{D}} \oplus \mathscr{L}^{\mathscr{P}}_{\mathcal{E}^1} \oplus \cdots \oplus \mathscr{L}^{\mathscr{P}}_{\mathcal{E}^{N-1}}$. Moreover, from the decomposition $\mathcal{D}^{i+1} = \mathcal{D}^i \oplus \mathcal{E}^i = \mathcal{D} \oplus \mathcal{E}^1 \oplus \cdots \oplus \mathcal{E}^i$, we have

$$\begin{aligned}
\mathsf{Der}_{\mathcal{D}^{i+1}}(\mathcal{D}) &= \mathscr{L}^{\mathscr{P}}_{\mathcal{D}^{i+1}} \oplus \mathsf{Der}_0(\mathcal{D}) \\
&= \mathscr{L}^{\mathscr{P}}_{\mathcal{D}^i} \oplus \mathscr{L}^{\mathscr{P}}_{\mathcal{E}^i} \oplus \mathsf{Der}_0(\mathcal{D}) \\
&= \mathscr{L}^{\mathscr{P}}_{\mathcal{D}} \oplus \mathscr{L}^{\mathscr{P}}_{\mathcal{E}^1} \oplus \cdots \oplus \mathscr{L}^{\mathscr{P}}_{\mathcal{E}^i} \oplus \mathsf{Der}_0(\mathcal{D}).
\end{aligned}$$

(Consequently, $\mathsf{Der}_{\mathcal{D}}(\mathcal{D})$ is completely nonholonomic: if we define the flag $\mathcal{F}^1 \subsetneq \mathcal{F}^2 \subsetneq \cdots$ by $\mathcal{F}^1 = \mathsf{Der}_{\mathcal{D}}(\mathcal{D})$ and $\mathcal{F}^{i+1} = \mathcal{F}^i + [\mathcal{F}^i, \mathcal{F}^i]$, $i \geq 1$, then $\mathcal{F}^i = \mathsf{Der}_{\mathcal{D}^i}(\mathcal{D})$ and $\mathcal{F}^N = \mathsf{Der}(\mathcal{D})$.) We shall also use $\mathscr{P}_i$ to denote the projection $\mathsf{Der}(\mathcal{D}) \to \mathsf{Der}_{\mathcal{D}^i}(\mathcal{D})$. Likewise, let $\mathscr{Q}_i$ be the projection $\mathsf{Der}(\mathcal{D}) \to \mathscr{L}^{\mathscr{P}}_{\mathcal{E}^i}$.

Lemma 2.2.11. *Fix* $1 \leq i \leq N$ *and let* $\delta_1, \delta_2 \in \mathsf{Der}_{\mathcal{D}^i}(\mathcal{D})$, *where* $\delta_1 = \mathscr{L}^{\mathscr{P}}_X + D_X$ *and* $\delta_2 = \mathscr{L}^{\mathscr{P}}_Y + D_Y$. *Then*

$$\mathscr{P}_i([\delta_1, \delta_2]) = [\delta_1, \delta_2] - \mathscr{L}^{\mathscr{P}}_{\mathscr{Q}_i([X,Y])}.$$

(*Here* $X, Y \in \Gamma(\mathcal{D}^i)$ *and* $D_X, D_Y \in \mathsf{Der}_0(\mathcal{D})$.)

Proof. We have

$$[\delta_1, \delta_2] = [\mathscr{L}^{\mathscr{P}}_X, \mathscr{L}^{\mathscr{P}}_Y] + [\mathscr{L}^{\mathscr{P}}_X, D_Y] + [D_X, \mathscr{L}^{\mathscr{P}}_Y] + [D_X, D_Y].$$

It follows from proposition A.3.2 that $[\mathscr{L}^{\mathscr{P}}_X, D_Y], [D_X, \mathscr{L}^{\mathscr{P}}_Y], [D_X, D_Y] \in \mathsf{Der}_0(\mathcal{D})$. Consider the term $[\mathscr{L}^{\mathscr{P}}_X, \mathscr{L}^{\mathscr{P}}_Y]$. We have $[\mathscr{L}^{\mathscr{P}}_X, \mathscr{L}^{\mathscr{P}}_Y](f) = [X, Y][f]$ for every $f \in \mathcal{C}^\infty(\mathsf{M})$, and so

$$[\mathscr{L}^{\mathscr{P}}_X, \mathscr{L}^{\mathscr{P}}_Y] = \mathscr{L}^{\mathscr{P}}_{[X,Y]} + D_{X,Y},$$

for some $D_{X,Y} \in \mathsf{Der}_0(\mathcal{D})$. Moreover, as $X, Y \in \Gamma(\mathcal{D}^i)$, it follows that $[X, Y] \in \Gamma(\mathcal{D}^{i+1})$. Thus $[\mathscr{L}^{\mathscr{P}}_X, \mathscr{L}^{\mathscr{P}}_Y] = \mathscr{L}^{\mathscr{P}}_{\mathscr{P}_i([X,Y])} + \mathscr{L}^{\mathscr{P}}_{\mathscr{Q}_i([X,Y])} + D_{X,Y}$, and so

$$\mathscr{Q}_i([\mathscr{L}^{\mathscr{P}}_X, \mathscr{L}^{\mathscr{P}}_Y]) = [\mathscr{L}^{\mathscr{P}}_X, \mathscr{L}^{\mathscr{P}}_Y] - \mathscr{L}^{\mathscr{P}}_{\mathscr{Q}_i([X,Y])}.$$

Consequently, we have

$$\begin{aligned}
\mathscr{P}_i([\delta_1, \delta_2]) &= [\delta_1, \delta_2] - \mathscr{Q}_i([\delta_1, \delta_2]) \\
&= [\delta_1, \delta_2] - \mathscr{Q}_i([\mathscr{L}^{\mathscr{P}}_X, \mathscr{L}^{\mathscr{P}}_Y]) = [\delta_1, \delta_2] - \mathscr{L}^{\mathscr{P}}_{\mathscr{Q}_i([X,Y])}.
\end{aligned}$$

■

For each $\mathcal{D}^i$-restricted connection ∇^i, $i = 1, \dots, N$, we have that ∇^i_X is a $\mathcal{D}^i$-derivation, for $X \in \Gamma(\mathcal{D}^i)$. This leads us to the following result, which implies that the curvature tensor K^i measures the extent to which the mapping $\Gamma(\mathcal{D}^i) \to \mathsf{Der}_{\mathcal{D}^i}(\mathcal{D})$, $X \mapsto \nabla^i_X$ fails to be a homomorphism from $(\Gamma(\mathcal{D}^i), \mathscr{P}_i([\cdot,\cdot]))$ to $(\mathsf{Der}_{\mathcal{D}^i}(\mathcal{D}), \mathscr{P}_i([\cdot,\cdot]))$. Note that these structures are not Lie algebras, as $\mathscr{P}_i([\cdot,\cdot])$ does not generally satisfy the Jacobi identity. Instead they are so-called "almost Lie structures" [57], and so "homomorphism" refers to a homomorphism of almost Lie structures.

Theorem 2.2.12. *For each $i = 1, \dots, N$, we have*

$$K^i(X \wedge Y) = \mathscr{P}_i([\nabla^i_X, \nabla^i_Y]) - \nabla^i_{\mathscr{P}_i([X,Y])}, \qquad X, Y \in \Gamma(\mathcal{D}^i).$$

Proof. Let $X, Y \in \Gamma(\mathcal{D}^i)$ and $U \in \Gamma(\mathcal{D})$. There exist algebraic derivations $D_X, D_Y \in \mathsf{Der}_0(\mathcal{D})$ such that $\nabla^i_X = \pounds^{\mathscr{P}}_X + D_X$ and $\nabla^i_Y = \pounds^{\mathscr{P}}_Y + D_Y$. Hence, by lemma 2.2.11, we have $\mathscr{P}_i([\nabla^i_X, \nabla^i_Y]) = [\nabla^i_X, \nabla^i_Y] - \pounds^{\mathscr{P}}_{\mathscr{Q}_i([X,Y])}$. It follows that

$$\begin{aligned} \mathscr{P}_i([\nabla^i_X, \nabla^i_Y])U - \nabla^i_{\mathscr{P}_i([X,Y])}U &= [\nabla^i_X, \nabla^i_Y]U - \nabla^i_{\mathscr{P}_i([X,Y])}U - \llbracket \mathscr{Q}_i([X,Y]), U \rrbracket \\ &= K^i(X \wedge Y)U. \end{aligned}$$ ■

In particular, if we take $\llbracket \cdot,\cdot \rrbracket = \mathscr{P}_1([\cdot,\cdot])$, then the Schouten curvature tensor may be expressed as

$$K(X \wedge Y) = \llbracket \nabla_X, \nabla_Y \rrbracket - \nabla_{\llbracket X,Y \rrbracket}, \qquad X, Y \in \Gamma(\mathcal{D}).$$

2.2.2 Flat structures

By theorem 2.2.8, we have that the set of parallel tensor fields with respect to ∇ coincides with the set of parallel tensor fields with respect to ∇^{i+1}, for each $i = 1, \dots, N-1$. Accordingly, we shall simply say that such tensor fields are *parallel*.

A nonholonomic Riemannian manifold $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathsf{g})$ is called *locally flat on $\mathcal{U} \subseteq \mathsf{M}$* if there exists a parallel frame (U_a) for $\mathcal{D}$ defined on $\mathcal{U}$. If $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathsf{g})$ is locally flat on a neighbourhood about every point in M, then it is called *locally flat*. On the other hand, if $\mathcal{U} = \mathsf{M}$, i.e., there exists a global parallel frame for $\mathcal{D}$, then $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathsf{g})$ is said to be (globally) *flat*.

Keeping in mind corollary B.1.26, we shall always assume that a parallel frame for $\mathcal{D}$ is orthonormal. The next proposition characterises when the converse is true.

Proposition 2.2.13. *An orthonormal frame (U_a) for $\mathcal{D}$ is parallel if and only if $\llbracket U_a, U_b \rrbracket = 0$.*

Proof. Let (U_a) be an orthonormal frame for $\mathcal{D}$. Suppose (U_a) is parallel, i.e., $\nabla U_a \equiv 0$. Since ∇ is torsion free, it follows that $\llbracket U_a, U_b \rrbracket = \nabla_{U_a} U_b - \nabla_{U_b} U_a = 0$. Conversely, suppose $\llbracket U_a, U_b \rrbracket = 0$. Using Koszul's formula, we have $\mathsf{g}(\nabla_{U_a} U_b, U_c) = 0$, i.e., $\nabla_{U_a} U_b = 0$. Since ∇ is tensorial in its first argument, it follows that $\nabla U_b \equiv 0$, and so (U_a) is parallel. ■

Theorem 2.2.14. *Let $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathsf{g})$ be a Wagner structure. If $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathsf{g})$ is locally flat on $\mathcal{U} \subseteq \mathsf{M}$, then*

$$K^N(X \wedge Y)U = 0 \qquad \textit{for every } X, Y \in \Gamma(T\mathsf{M}) \textit{ and } U \in \Gamma(\mathcal{D}) \textit{ defined on } \mathcal{U}. \tag{2.2.4}$$

Conversely, if (2.2.4) holds for some neighbourhood $\mathcal{U} \subseteq \mathsf{M}$, then $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathsf{g})$ is locally flat on $\mathcal{U}$.

Proof. The Wagner curvature tensor K^N is exactly the curvature tensor of the vector bundle connection ∇^N, and so there exists a parallel frame for $\mathcal{D}$ if and only if K^N vanishes identically (see theorem B.1.28). Since every parallel vector field with respect to ∇^N is a parallel vector field with respect to ∇, it follows that $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ is locally flat on $\mathcal{U} \subseteq \mathsf{M}$ exactly when K^N vanishes identically on $\mathcal{U}$. ■

Corollary 2.2.15. *A Wagner structure* $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ *is locally flat if and only if* $K^N \equiv 0$.

Corollary 2.2.16. *Suppose* $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ *is a Wagner structure that is locally flat on* $\mathcal{U} \subseteq \mathsf{M}$ *and let* (U_a) *be a parallel frame for* $\mathcal{D}$*, defined on* $\mathcal{U}$*. If* $U = u^a U_a \in \Gamma(\mathcal{D})$ *is defined on* $\mathcal{U}$*, then*

$$K^i(X \wedge Y)U = \mathscr{Q}_i([X,Y])[u^a]U_a - [\![\mathscr{Q}_i([X,Y]), U]\!]$$

for every $X, Y \in \Gamma(\mathcal{D}^i)$ *defined on* $\mathcal{U}$ *and each* $i = 1, \ldots, N-1$.

Proof. Let $X, Y \in \Gamma(\mathcal{D}^i)$ and $U = u^a U_a \in \Gamma(\mathcal{D})$ be defined on $\mathcal{U}$. Then

$$\begin{aligned}
K^i(X \wedge Y)U &= u^a K^i(X \wedge Y)U_a \\
&= u^a[\nabla^i_X, \nabla^i_Y]U_a - u^a \nabla^i_{\mathscr{P}_i([X,Y])}U_a - u^a[\![\mathscr{Q}_i([X,Y]), U_a]\!] \\
&= -u^a[\![\mathscr{Q}_i([X,Y]), U_a]\!] \\
&= \mathscr{Q}_i([X,Y])[u^a]U_a - [\![\mathscr{Q}_i([X,Y]), U]\!].
\end{aligned}$$

■

2.3 Curvature of restricted Ehresmann connections

In this section we consider the curvature of the connections $\nabla^1, \ldots, \nabla^N$ from the Ehresmann connection point of view (see section B.2). In particular, we write the Schouten curvature tensor K, as well as the curvature tensors $K^1, \ldots, K^N$ in the case of a Wagner structure, in terms of horizontal lifts of vector fields. (As a corollary to the first, we then characterise the vanishing of the Schouten tensor in terms of an involutivity condition for the horizontal distribution.) We also show that the connections $\nabla^1, \ldots, \nabla^N$ are equivalently formulated as a flag of horizontal distributions on $\mathcal{D}$.

Let $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ be a nonholonomic Riemannian manifold with associated nonholonomic connection ∇. We can extend ∇ to a vector bundle connection $\mathring{\nabla}$ on $\mathcal{D}$ as follows (cf. [5, 6]):

$$\mathring{\nabla} : \Gamma(T\mathsf{M}) \times \Gamma(\mathcal{D}) \to \Gamma(\mathcal{D}), \qquad \mathring{\nabla}_Z = \nabla_{\mathscr{P}(Z)} + \pounds^{\mathscr{P}}_{\mathscr{Q}(Z)}.$$

(Note that $\mathring{\nabla}$ depends only on $\mathcal{D}$, $\mathcal{D}^\perp$ and $\mathbf{g}$, hence it is intrinsic to the nonholonomic Riemannian structure.)

Lemma 2.3.1. $\mathring{\nabla}$ *is a vector bundle connection on* $\mathcal{D}$.

Proof. Since ∇ is an affine connection, it suffices to show that $\mathring{\nabla}_{fA}X = f\,\mathring{\nabla}_A X$ and $\mathring{\nabla}_A fX = A[f]X + f\mathring{\nabla}_A X$ for every $f \in \mathcal{C}^\infty(\mathsf{M})$, $X \in \Gamma(\mathcal{D})$ and $A \in \Gamma(\mathcal{D}^\perp)$. For the first part, we have

$$\mathring{\nabla}_{fA}X = \pounds^{\mathscr{P}}_{fA}X = [\![fA, X]\!] = f[\![A, X]\!] - \mathscr{P}(X[f]A) = f\pounds^{\mathscr{P}}_A X = f\,\mathring{\nabla}_A X.$$

Likewise, for the second,

$$\begin{aligned}\overset{\circ}{\nabla}_A fX &= \pounds^{\mathscr{P}}_A(fX) = [\![A, fX]\!] \\ &= f\,[\![A, X]\!] + \mathscr{P}(A[f]X) \\ &= f\pounds^{\mathscr{P}}_A X + A[f]X = A[f]X + f\,\overset{\circ}{\nabla}_A X.\end{aligned}$$

Hence $\overset{\circ}{\nabla}$ is tensorial in its first argument, and a derivation in the second, and so is an affine connection. ■

The curvature tensor of the (vector bundle) connection $\overset{\circ}{\nabla}$ (see section B.1.3.2) is the $(1,3)$-tensor field $\overset{\circ}{R} : \Gamma(T\mathsf{M}) \times \Gamma(T\mathsf{M}) \times \Gamma(\mathcal{D}) \to \Gamma(\mathcal{D})$ given by

$$\overset{\circ}{R}(X, Y)U = [\overset{\circ}{\nabla}_X, \overset{\circ}{\nabla}_Y]U - \overset{\circ}{\nabla}_{[X,Y]}U,$$

where $X, Y \in \Gamma(T\mathsf{M})$ and $U \in \Gamma(\mathcal{D})$.

Lemma 2.3.2. *If $X, Y \in \Gamma(\mathcal{D})$, then $\overset{\circ}{R}(X, Y) = K(X, Y)$.*

Proof. Let $X, Y \in \Gamma(\mathcal{D})$. A direct calculation yields the result:

$$\begin{aligned}\overset{\circ}{R}(X, Y) &= [\overset{\circ}{\nabla}_X, \overset{\circ}{\nabla}_Y] - \overset{\circ}{\nabla}_{[X,Y]} \\ &= [\nabla_X, \nabla_Y] - \nabla_{[\![X,Y]\!]} - \overset{\circ}{\nabla}_{\mathscr{Q}([X,Y])} \\ &= [\nabla_X, \nabla_Y] - \nabla_{[\![X,Y]\!]} - \pounds^{\mathscr{P}}_{\mathscr{Q}([X,Y])} = K(X, Y).\end{aligned}$$ ■

Proposition 2.3.3. *Let $\mathcal{U}$ be an open neighbourhood in M. If $\overset{\circ}{R} \equiv 0$ on $\mathcal{U}$, then $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ is locally flat on $\mathcal{U}$. Conversely, if (U_a) is a (local) parallel frame for $\mathcal{D}$ defined on $\mathcal{U}$ such that $[U_a, \Gamma(\mathcal{D}^\perp)] \subseteq \Gamma(\mathcal{D}^\perp)$, then $\overset{\circ}{R} \equiv 0$ on $\mathcal{U}$.*

Proof. Suppose $\overset{\circ}{R} \equiv 0$ on $\mathcal{U}$. Since $\overset{\circ}{\nabla}$ is a vector bundle connection on $\mathcal{D}$, the vanishing of its curvature tensor $\overset{\circ}{R}$ implies the existence of a (local) parallel frame (U_a) for $\mathcal{D}$ on $\mathcal{U}$, where "parallel" means with respect to (the parallel transport induced by) $\overset{\circ}{\nabla}$ (see theorem B.1.28). That is, $\overset{\circ}{\nabla}_Z U_a = 0$ for every $Z \in \Gamma(T\mathsf{M})$ defined on $\mathcal{U}$. In particular, taking $Z \in \Gamma(\mathcal{D})$, it follows that $\nabla U_a \equiv 0$. Thus $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ is locally flat on $\mathcal{U}$.

Conversely, suppose there exists a (local) parallel (with respect to ∇) frame (U_a) for $\mathcal{D}$ defined on $\mathcal{U}$. Then $\overset{\circ}{\nabla}_X U_a = 0$ for every $X \in \Gamma(\mathcal{D})$ on $\mathcal{U}$. On the other hand, we have $\overset{\circ}{\nabla}_A U_a = [\![A, U_a]\!]$ for every $A \in \Gamma(\mathcal{D}^\perp)$ on $\mathcal{U}$. Accordingly, if $[U_a, \Gamma(\mathcal{D}^\perp)] \subseteq \Gamma(\mathcal{D}^\perp)$, then $\overset{\circ}{\nabla}_A U_a = 0$, i.e., (U_a) is also parallel with respect to $\overset{\circ}{\nabla}$. It follows that the curvature tensor of $\overset{\circ}{\nabla}$ vanishes on $\mathcal{U}$. ■

Remark 2.3.4. As we shall see in the following chapter (specifically, corollary 3.2.11 and section 3.2.3), the existence of an orthonormal frame (U_a) such that $[U_a, \Gamma(\mathcal{D}^\perp)] \subseteq \Gamma(\mathcal{D}^\perp)$ is sufficient for $\mathcal{D}$ to be geodesically invariant, i.e., invariant under the geodesic flow of $(\mathsf{M}, \widetilde{\mathbf{g}})$, where $\widetilde{\mathbf{g}}$ is any Riemannian extension of $\mathbf{g}$ such that $\mathcal{D}$ and $\mathcal{D}^\perp$ are orthogonal with respect to $\widetilde{\mathbf{g}}$. □

Let $\pi : \mathcal{D} \to \mathsf{M}$ be the natural projection of a $\mathcal{D}$-vector onto its base point. Associated to the nonholonomic connection ∇ and its extension $\mathring{\nabla}$ are the restricted Ehresmann connections

$$h : \pi^*\mathcal{D} \to T\mathcal{D}, \quad h(U_q, X_q) = T_qU \cdot X_q - \mathrm{vl}_{U_q} \cdot \nabla_{X_q}U$$

and

$$f : \pi^*T\mathsf{M} \to T\mathcal{D}, \quad f(U_q, Z_q) = T_qU \cdot Z_q - \mathrm{vl}_{U_q} \cdot \mathring{\nabla}_{Z_q}U,$$

respectively, where (for each $U_q \in \mathcal{D}$) $U \in \Gamma(\mathcal{D})$ is a vector field such that $U(q) = U_q$. (See section B.2, and particularly section B.2.1.) By proposition B.2.7, we have that h (resp. f) is a well-defined linear $\mathcal{D}$-connection (resp. $T\mathsf{M}$-connection) on $\mathcal{D}$ with associated covariant derivative ∇ (resp. $\mathring{\nabla}$). Furthermore, we have $f|_{\pi^*\mathcal{D}} = h$; in particular, if $X \in \Gamma(\mathcal{D})$, then $X^f = X^h$ (where X^f is the horizontal lift induced by f, and X^h the horizontal lift induced by h).

Let $\mathcal{V} = \ker T\pi$ be the vertical distribution, $\mathcal{H} = \mathrm{im}\, h$ the horizontal distribution of h and $\mathcal{F} = \mathrm{im}\, f$ the horizontal distribution of f. Clearly, we have $\mathcal{H} \subsetneq \mathcal{F}$. By proposition B.2.4 it follows that $\mathcal{V} \cap \mathcal{H} = \mathcal{V} \cap \mathcal{F} = \{0\}$, $\mathcal{V} + \mathcal{H} \subsetneq T\mathcal{D}$ and $\mathcal{V} + \mathcal{F} = T\mathcal{D}$. That is, $\mathcal{V} \oplus \mathcal{H} \subsetneq \mathcal{V} \oplus \mathcal{F} = T\mathcal{D}$. Let $\mathcal{H}^\perp$ denote the distribution on $\mathcal{D}$ given by $\mathcal{H}^\perp = \mathrm{im}(f|_{\pi^*\mathcal{D}^\perp})$, i.e.,

$$\mathcal{H}^\perp_{U_q} = \{f(U_q, X_q) : X_q \in \mathcal{D}^\perp_q\}, \qquad U_q \in \mathcal{D}.$$

Lemma 2.3.5. *We have* $\pi_*\mathcal{H}^\perp = \mathcal{D}^\perp$.

Proof. Let $X_{U_q} \in \mathcal{H}^\perp_{U_q}$, $U_q \in \mathcal{D}$. Then $X_{U_q} = f(U_q, X_q)$ for some $X_q \in \mathcal{D}^\perp_q$, and so $T_{U_q}\pi \cdot X_{U_q} = X_q$. Hence $\pi_*\mathcal{H}^\perp \subseteq \mathcal{D}^\perp$. Conversely, if $X_q \in \mathcal{D}^\perp$ and $U_q \in \mathcal{D}_q$, then $X_q = T_{U_q}\pi \cdot f(U_q, X_q) \in T_{U_q}\pi \cdot \mathcal{H}^\perp_{U_q}$. It follows that $\pi_*\mathcal{H}^\perp = \mathcal{D}^\perp$. ■

Lemma 2.3.6. *We have* $\mathcal{F} = \mathcal{H} \oplus \mathcal{H}^\perp$*. In particular,* $T\mathcal{D} = \mathcal{V} \oplus \mathcal{H} \oplus \mathcal{H}^\perp$.

Proof. Let $X_{U_q} \in \mathcal{H}_{U_q} \cap \mathcal{H}^\perp_{U_q}$, $U_q \in \mathcal{D}$. We have $X_{U_q} = h(U_q, X_q) = f(U_q, Y_q)$ for some $X_q \in \mathcal{D}_q$ and $Y_q \in \mathcal{D}^\perp_q$. Then

$$X_q = T_{U_q}\pi \cdot h(U_q, X_q) = T_{U_q}\pi \cdot f(U_q, Y_q) = Y_q.$$

Since $\mathcal{D}_q \cap \mathcal{D}^\perp_q = \{0\}$, it follows that $X_q = Y_q = 0$, and thus $X_{U_q} = 0$. Hence $\mathcal{H} \cap \mathcal{H}^\perp = \{0\}$. We clearly have $\mathcal{H} \oplus \mathcal{H}^\perp \subseteq \mathcal{F}$. For the converse, let $Z_{U_q} \in \mathcal{F}_{U_q}$. Then $Z_{U_q} = f(U_q, Z_q)$ for some $Z_q \in T_q\mathsf{M}$, and hence

$$Z_{U_q} = f(U_q, Z_q) = h(U_q, \mathscr{P}(Z_q)) + f(U_q, \mathscr{Q}(Z_q)) \in \mathcal{H}_{U_q} \oplus \mathcal{H}^\perp_{U_q}.$$

This completes the proof. ■

Lemma 2.3.7. *We have* $\mathcal{V} = (T\pi)^{-1}(0)$, $\mathcal{V} \oplus \mathcal{H} = (T\pi)^{-1}(\mathcal{D})$ *and* $\mathcal{V} \oplus \mathcal{H}^\perp = (T\pi)^{-1}(\mathcal{D}^\perp)$.

Proof. The first part is obvious. For the second part, we clearly have $\mathcal{V} \oplus \mathcal{H} \subseteq (T\pi)^{-1}(\mathcal{D})$. Let $X_{U_q} \in (T\pi)^{-1}(\mathcal{D})$, so that $T_{U_q}\pi \cdot X_{U_q} = V_q \in \mathcal{D}_q$. Then $T_{U_q}\pi \cdot (X_{U_q} - h(U_q, V_q)) = V_q - V_q = 0$, and so $X_{U_q} - h(U_q, V_q) \in \mathcal{V}_{U_q}$. It follows that $X_{U_q} \in \mathcal{V}_{U_q} \oplus \mathcal{H}_{U_q}$, whence $\mathcal{V} \oplus \mathcal{H} = (T\pi)^{-1}(\mathcal{D})$. The proof that $\mathcal{V} \oplus \mathcal{H}^\perp = (T\pi)^{-1}(\mathcal{D}^\perp)$ uses a similar argument (keeping in mind lemma 2.3.5). ■

Remark 2.3.8. As noted in [11, 27], a nonholonomic connection on $\mathcal{D}$ (i.e., a $\mathcal{D}$-restricted connection on $\mathcal{D}$) is precisely the specification of a ϕ_t-invariant complement $\mathcal{H}$ to $\mathcal{V}$ in $(T\pi)^{-1}(\mathcal{D})$ (where $\phi_t : \mathcal{D} \to \mathcal{D}$ is the dilation $U_q \mapsto e^t U_q$). Indeed, given such a complement, we have $\pi_* \mathcal{H} = \mathcal{D}$, $\mathcal{V} \cap \mathcal{H} = \{0\}$ and $(\phi_t)_* \mathcal{H} = \mathcal{H}$. Hence, by proposition B.2.5 and lemma B.2.3, there exists a unique linear $\mathcal{D}$-connection h on $\mathcal{D}$ with $\operatorname{im} h = \mathcal{H}$. □

Let $\mathscr{V} : T\mathcal{D} \to \mathcal{V}$, $\mathscr{P} : T\mathcal{D} \to \mathcal{V} \oplus \mathcal{H}$ and $\mathscr{Q} : T\mathcal{D} \to \mathcal{H}^\perp$ denote the projections corresponding to the decomposition $T\mathcal{D} = \mathcal{V} \oplus \mathcal{H} \oplus \mathcal{H}^\perp$. Let $[\![\cdot,\cdot]\!]$ be the projected Lie bracket $\mathscr{P}([\cdot,\cdot])$.

Lemma 2.3.9. *If* $X \in \Gamma(\mathcal{F})$ *is projectable, then*

$$\mathscr{P}(X) = \mathscr{P}(\pi_* X)^h \quad \textit{and} \quad \mathscr{Q}(X) = \mathscr{Q}(\pi_* X)^f.$$

Proof. Let $X \in \Gamma(\mathcal{F})$ be projectable. We have $X = (\pi_* X)^f = \mathscr{P}(\pi_* X)^h + \mathscr{Q}(\pi_* X)^f$, from which it follows that $\mathscr{P}(X) = \mathscr{P}(\pi_* X)^h$ and $\mathscr{Q}(X) = \mathscr{Q}(\pi_* X)^f$. ■

We now link the vanishing of the Schouten curvature tensor of $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ with an involutivity condition for $\mathcal{H}$. In the case of a vector bundle connection, it is well known (see, e.g., [56, 37]) that the curvature of the connection vanishes if and only if the associated horizontal distribution is involutive, or equivalently (for the case of a vector bundle connection), integrable. For a nonholonomic connection, however, the characterising condition is that the $[\![\cdot,\cdot]\!]$-bracket of two horizontal vector fields is still in $\mathcal{H}$.

Theorem 2.3.10. *We have*

$$\begin{aligned} K(X,Y)U_q &= -\operatorname{vl}_{U_q}^{-1} \cdot \left([X^h, Y^h](U_q) - [X,Y]^f(U_q)\right) \\ &= -\operatorname{vl}_{U_q}^{-1} \cdot \mathscr{V}([\![X^h, Y^h]\!])(U_q) \end{aligned}$$

for every $X, Y \in \Gamma(\mathcal{D})$ *and* $U_q \in \mathcal{D}$.

Proof. Let $X, Y \in \Gamma(\mathcal{D})$ and $\omega \in \Gamma(\mathcal{D}^*)$. By proposition B.2.13, we have $Z^f[\overline{\omega}] = \overline{\overset{\circ}{\nabla}_Z \omega}$ for every $Z \in \Gamma(T\mathsf{M})$, where for a 1-form $\eta \in \Gamma(\mathcal{D}^*)$ the function $\overline{\eta} \in \mathcal{C}^\infty(\mathcal{D})$ is given by $\overline{\eta}(U_q) = \eta_q(U_q)$. We may interpret the $(1,1)$-tensor field $K(X,Y)$ as an algebraic derivation, i.e., a derivation that vanishes on elements of $\mathcal{C}^\infty(\mathsf{M})$ (see proposition A.3.1). In particular, $K(X,Y)\omega \in \Gamma(\mathcal{D}^*)$ is given by

$$(K(X,Y)\omega)(U) = K(X,Y)(\omega(U)) - \omega(K(X,Y)U) = -\omega(K(X,Y)U),$$

for $U \in \Gamma(\mathcal{D})$. Using corollary B.2.14, we have

$$\overline{K(X,Y)\omega} = \overline{[\nabla_X, \nabla_Y]\omega} - \overline{\overset{\circ}{\nabla}_{[X,Y]}\omega} = [X^h, Y^h][\overline{\omega}] - [X,Y]^f[\overline{\omega}].$$

Since $\pi_*\left([X^h, Y^h] - [X,Y]^f\right) = [\pi_* X^h, \pi_* Y^h] - [X,Y] = 0$, we have that $[X^h, Y^h] - [X,Y]^f$ is vertical. Hence

$$\begin{aligned} [X^h, Y^h] - [X,Y]^f &= \mathscr{V}([X^h, Y^h] - [X,Y]^f) \\ &= \mathscr{V}([\![X^h, Y^h]\!] - [\![X,Y]\!]^h) + \mathscr{V}(\mathscr{Q}([X^h, Y^h]) - \mathscr{Q}([X,Y])^f) \\ &= \mathscr{V}([\![X^h, Y^h]\!]), \end{aligned}$$

and so $\overline{K(X,Y)\omega} = \mathscr{V}([\![X^h, Y^h]\!])[\overline{\omega}]$. If $U_q \in \mathcal{D}$, then

$$\begin{aligned}\omega_q(K(X,Y)U_q) &= -(K(X,Y)\omega)_q(U_q) = -(\overline{K(X,Y)\omega})(U_q)\\ &= -\mathscr{V}([X^h, Y^h])[\overline{\omega}](U_q)\\ &= -d\overline{\omega}(U_q)(\mathscr{V}([X^h, Y^h])(U_q)).\end{aligned}$$

Using lemma B.2.12, we get

$$\omega_q(K(X,Y)U_q) = -\omega_q(\mathrm{vl}_{U_q}^{-1} \cdot \mathscr{V}([X^h, Y^h])(U_q)).$$

Since ω is arbitrary, the result follows. ■

Corollary 2.3.11. $K \equiv 0$ *if and only if* $[\![\mathcal{H}, \mathcal{H}]\!] \subseteq \mathcal{H}$.

Suppose now that $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ is a Wagner structure (see the beginning of section 2.2). Let $\mathcal{D} = \mathcal{D}^1 \subsetneq \cdots \subsetneq \mathcal{D}^{N-1} \subsetneq \mathcal{D}^N = T\mathsf{M}$ be the flag of $\mathcal{D}$, where $N \geq 2$, and $\mathcal{D}^{i+1} = \mathcal{D}^i \oplus \mathcal{E}^i$ for each $i = 1, \ldots, N-1$. Associated to each connection ∇^i is a restricted Ehresmann connection, given by

$$h^i : \pi^*\mathcal{D}^i \to T\mathcal{D}, \quad h^i(U_q, X_q) = T_qU \cdot X_q - \mathrm{vl}_{U_q} \cdot \nabla^i_{X_q} U,$$

where $U \in \Gamma(\mathcal{D})$ is a smooth extension of U_q. Again by proposition B.2.7, we have that h^i is a well-defined linear $\mathcal{D}^i$-connection on $\mathcal{D}$ with associated covariant derivative ∇^i. Furthermore, we have $h^1 = h$ and $h^{i+1}|_{\pi^*\mathcal{D}^i} = h^i$ for each $i = 1, \ldots, N-1$. Let $\mathcal{H}^i = \operatorname{im} h^i$ and $\mathcal{Q}^i = \operatorname{im}(h^{i+1}|_{\pi^*\mathcal{E}^i})$. As before, we denote the vertical distribution $\ker T\pi$ by $\mathcal{V}$.

Lemma 2.3.12. *We have* $\mathcal{H}^{i+1} = \mathcal{H}^i \oplus \mathcal{Q}^i$ *for each* $i = 1, \ldots, N-1$. *Hence*

$$T\mathcal{D} = \mathcal{V} \oplus \mathcal{H}^N = \mathcal{V} \oplus \mathcal{H} \oplus \mathcal{Q}^1 \oplus \cdots \oplus \mathcal{Q}^{N-1}.$$

Proof. We first show that $\mathcal{H}^i \cap \mathcal{Q}^i = \{0\}$. Suppose otherwise, i.e., there exists $X_{U_q} \in \mathcal{H}^i_{U_q} \cap \mathcal{Q}^i_{U_q}$ for $U_q \in \mathcal{D}$. Then $X_{U_q} = h^i(U_q, X_q)$ and $X_{U_q} = h^{i+1}(U_q, Y_q)$ for some $X_q \in \mathcal{D}^i_q$ and $Y_q \in \mathcal{E}^i_q$. It follows that

$$X_q = T_{U_q}\pi \cdot h^i(U_q, X_q) = T_{U_q}\pi \cdot X_{U_q} = T_{U_q}\pi \cdot h^{i+1}(U_q, Y_q) = Y_q.$$

That is, $X_q = Y_q = 0$ since $\mathcal{D}^i_q \cap \mathcal{E}^i_q = \{0\}$, and hence $X_{U_q} = 0$. Thus $\mathcal{H}^i \cap \mathcal{Q}^i = \{0\}$, as claimed.

The inclusion $\mathcal{H}^i \oplus \mathcal{Q}^i \subseteq \mathcal{H}^{i+1}$ is obvious. Let $X_{U_q} \in \mathcal{H}^{i+1}_{U_q}$, $U_q \in \mathcal{D}$. Then $X_{U_q} = h^{i+1}(U_q, Z_q)$ for some $Z_q \in \mathcal{D}^{i+1}_q$. Since $\mathcal{D}^{i+1}_q = \mathcal{D}^i_q \oplus \mathcal{E}^i_q$, we have $Z_q = X_q + Y_q$ for some $X_q \in \mathcal{D}^i_q$ and $Y_q \in \mathcal{E}^i_q$. Hence

$$X_{U_q} = h^{i+1}(U_q, X_q) + h^{i+1}(U_q, Y_q) = \underbrace{h^i(U_q, X_q)}_{\in \mathcal{H}^i_{U_q}} + \underbrace{h^{i+1}(U_q, Y_q)}_{\in \mathcal{Q}^i_{U_q}},$$

i.e., $X_{U_q} \in \mathcal{H}^i_{U_q} \oplus \mathcal{Q}^i_{U_q}$. It follows that $\mathcal{H}^{i+1} = \mathcal{H}^i \oplus \mathcal{Q}^i$. ■

Let $\mathscr{V} : T\mathcal{D} \to \mathcal{V}$, $\mathscr{P} : T\mathcal{D} \to \mathcal{V} \oplus \mathcal{H}$ and $\mathscr{Q}_i : T\mathcal{D} \to \mathcal{Q}^i$ (for $i = 1, \ldots, N-1$) be the projections corresponding to the decomposition $T\mathcal{D} = \mathcal{V} \oplus \mathcal{H} \oplus \mathcal{Q}^1 \oplus \cdots \oplus \mathcal{Q}^{N-1}$. Similarly, let $\mathscr{P}_i = \mathscr{P} \oplus \mathscr{Q}_i \oplus \cdots \oplus \mathscr{Q}_{i-1}$ be the projection onto $\mathcal{V} \oplus \mathcal{H}^i$, for each $i = 1, \ldots, N$.

Lemma 2.3.13. *If $X \in \Gamma(\mathcal{H}^N)$ is projectable, then*

$$\mathscr{P}(X) = \mathscr{P}(\pi_* X)^h, \quad \mathscr{Q}_i(X) = \mathscr{Q}_i(\pi_* X)^{h^{i+1}} \quad \textit{and} \quad \mathscr{P}_i(X) = \mathscr{P}_i(\pi_* X)^{h^i}$$

for each $i = 1, \ldots, N-1$.

Proof. Let $X \in \Gamma(\mathcal{H}^N)$ be projectable. We have $X = (\pi_* X)^{h^N} = \mathscr{P}(\pi_* X)^h + \mathscr{Q}_1(\pi_* X)^{h^2} + \cdots + \mathscr{Q}_{N-1}(\pi_* X)^{h^N}$, from which it follows that $\mathscr{P}(X) = \mathscr{P}(\pi_* X)^h$ and $\mathscr{Q}_i(X) = \mathscr{Q}_i(\pi_* X)^{h^{i+1}}$ for each $i = 1, \ldots, N-1$. Lastly, we have

$$\begin{aligned}\mathscr{P}_i(X) &= \mathscr{P}(X) + \mathscr{Q}_1(X) + \cdots + \mathscr{Q}_{i-1}(X) \\ &= \mathscr{P}(\pi_* X)^{h^i} + \mathscr{Q}_i(\pi_* X)^{h^i} + \cdots + \mathscr{Q}_{i-1}(\pi_* X)^{h^i} = \mathscr{P}_i(\pi_* X)^{h^i}. \qquad \blacksquare\end{aligned}$$

The connections $\nabla^1, \ldots, \nabla^N$ are equivalently specified by the restricted Ehresmann connections $h^1, \ldots, h^N$, which are in turn equivalently specified by the horizontal distributions $\mathcal{H}^1, \ldots, \mathcal{H}^N$. (See section B.2.1 and proposition B.2.5.) It follows that Wagner's construction of $\nabla^1, \ldots, \nabla^N$ is equivalently formulated as the flag of horizontal distributions on $\mathcal{D}$

$$\mathcal{H}^1 \subsetneq \mathcal{H}^2 \subsetneq \cdots \subsetneq \mathcal{H}^{N-1} \subsetneq \mathcal{H}^N,$$

where, in particular, $\mathcal{H}^N$ is a full complement to $\mathcal{V}$, i.e., $\mathcal{V} \oplus \mathcal{H}^N = T\mathcal{D}$. The next result allows us to describe the flag iteratively, starting only with $\mathcal{H}^1 = \mathcal{H}$.

Theorem 2.3.14. *We have*

$$\mathcal{H}^{i+1} = \mathcal{H}^i + \{[X^{h^i}, Y^{h^i}] : X \wedge Y \in (\ker \Delta_i)^\perp\}$$

for every $i = 1, \ldots, N-1$.

Proof. Let $\mathcal{S}^1 = \mathcal{H}$ and $\mathcal{S}^{i+1} = \mathcal{S}^i + \{[X^{h^i}, Y^{h^i}] : X \wedge Y \in (\ker \Delta_i)^\perp\}$ for $i \geq 1$. We shall use induction on i to prove $\mathcal{S}^i = \mathcal{H}^i$ for each $i \geq 1$. By definition, we have $\mathcal{S}^1 = \mathcal{H}^1$. Suppose that $\mathcal{S}^i = \mathcal{H}^i$ for some $1 \leq i \leq N-1$. We claim that $\pi_* \mathcal{S}^{i+1} = \mathcal{D}^{i+1}$ and $\mathcal{V} \cap \mathcal{S}^{i+1} = \{0\}$. By uniqueness of the connection associated to $\mathcal{H}^{i+1}$ (see proposition B.2.5), it will then follow that $\mathcal{S}^{i+1} = \mathcal{H}^{i+1}$. Let $W_{U_q} + [X^{h^i}, Y^{h^i}](U_q) \in \mathcal{S}^{i+1}_{U_q}$, where $U_q \in \mathcal{D}$. Then

$$T_{U_q}\pi \cdot \big(W_{U_q} + [X^{h^i}, Y^{h^i}](U_q)\big) = T_{U_q}\pi \cdot W_{U_q} + [X, Y](q) \in \mathcal{D}^i_q + [\mathcal{D}^i, \mathcal{D}^i]_q = \mathcal{D}^{i+1}_q,$$

and so $\pi_* \mathcal{H}^{i+1} \subseteq \mathcal{D}^{i+1}$. Conversely, let $W_q \in \mathcal{D}^i_q$ and $X_q \in \mathcal{E}^i_q$; then $W_q + X_q$ is an arbitrary element of $\mathcal{D}^{i+1}_q = \mathcal{D}^i_q \oplus \mathcal{E}^i_q$. Since $\mathcal{S}^i = \mathcal{H}^i$ by the inductive hypothesis, we have $\pi_* \mathcal{S}^i = \mathcal{D}^i$. Accordingly, there exists $V_{U_q} + [Y, Z](U_q) \in \mathcal{S}^i_{U_q}$ such that $T_{U_q}\pi \cdot (V_{U_q} + [Y, Z](U_q)) = W_q$. Let $A \wedge B = \Theta_i(X) \in (\ker \Delta_i)^\perp$, where $X \in \Gamma(\mathcal{E}^i)$ is a smooth extension of X_q. (The case when $\Theta_i(X)$ is a $C^\infty(\mathsf{M})$-combination of bivector fields from $\Gamma(\mathcal{D}^i)$ is treated similarly.) Then

$$\begin{aligned} W_q + X_q &= T_{U_q}\pi \cdot \big(V_{U_q} + [Y, Z](U_q)\big) + \mathscr{Q}_i([A, B])(q) \\ &= T_{U_q}\pi \cdot \big(V_{U_q} + [Y, Z](U_q) + \mathscr{Q}_i([A, B])^{h^{i+1}}(U_q)\big) \\ &= T_{U_q}\pi \cdot \big(V_{U_q} + [Y, Z](U_q) + \mathscr{Q}_i([A^{h^i}, B^{h^i}])(U_q)\big) \\ &= T_{U_q}\pi \cdot \big(\underbrace{V_{U_q} + [Y, Z](U_q) - \mathscr{P}_i([A^{h^i}, B^{h^i}])(U_q)}_{\in\, \mathcal{S}^i_{U_q}} + [A^{h^i}, B^{h^i}](U_q)\big) \in T_{U_q}\pi \cdot \mathcal{S}^{i+1}_{U_q}. \end{aligned}$$

Hence $\pi_*\mathcal{S}^{i+1} = \mathcal{D}^{i+1}$. Let $W_{U_q} + [X^{h^i}, Y^{h^i}](U_q) \in \mathcal{V}_{U_q} \cap \mathcal{S}^{i+1}_{U_q}$, $U_q \in \mathcal{D}$. We have

$$\begin{aligned} W_{U_q} + [X^{h^i}, Y^{h^i}](U_q) &= \mathscr{V}([X^{h^i}, Y^{h^i}])(U_q) + (\mathscr{P}_i - \mathscr{V})(W_{U_q} + [X^{h^i}, Y^{h^i}](U_q)) \\ &\quad + \mathscr{Q}_i([X^{h^i}, Y^{h^i}])(U_q) \\ &= -\operatorname{vl}_{U_q} \cdot K^i(X \wedge Y)U_q + (W_{U_q} + \mathscr{P}_i([X,Y])^{h^i}(U_q)) \\ &\quad + \mathscr{Q}_i([X,Y])^{h^{i+1}}(U_q). \end{aligned}$$

Both non-vertical components must vanish; in particular, we have $\mathscr{Q}_i([X,Y])(q) = 0$, i.e., $(X \wedge Y)(q) \in \ker \Delta_{i,q}$. Since $X \wedge Y \in (\ker \Delta_i)^\perp$, it follows that $(X \wedge Y)(q) = 0$. Then

$$W_{U_q} + [X^{h^i}, Y^{h^i}](U_q) = -\operatorname{vl}_{U_q} \cdot K^i(X \wedge Y)U_q = 0.$$

Thus $\mathcal{V} \cap \mathcal{S}^{i+1} = \{0\}$, which completes the proof of the inductive case. ■

Lastly, we prove a result similar to theorem 2.3.10 for each of the curvature tensors $K^1, \ldots, K^N$. Let $\mathring{\nabla}^i$ be the $\mathcal{D}^{i+1}$-connection on $\mathcal{D}$ given by

$$\mathring{\nabla}^i_X U = \nabla^i_{\mathscr{P}_i(X)} U + \pounds^{\mathscr{P}}_{\mathscr{Q}_i(X)} U, \qquad X \in \Gamma(\mathcal{D}^{i+1}),\ U \in \Gamma(\mathcal{D}).$$

Let $f^i : \pi^*\mathcal{D}^{i+1} \to T\mathcal{D}$ be the restricted Ehresmann connection associated to $\mathring{\nabla}^i$.

Theorem 2.3.15. *Let* $1 \le i \le N$. *Then*

$$\begin{aligned} K^i(X,Y)U_q &= -\operatorname{vl}^{-1}_{U_q} \cdot ([X^{h^i}, Y^{h^i}](U_q) - [X,Y]^{f^i}(U_q)) \\ &= -\operatorname{vl}^{-1}_{U_q} \cdot \mathscr{V}(\mathscr{P}_i([X^{h^i}, Y^{h^i}]))(U_q) \end{aligned}$$

for every $X, Y \in \Gamma(\mathcal{D}^i)$ *and* $U_q \in \mathcal{D}$.

Proof. (The proof is similar to that of theorem 2.3.10; hence, we omit some details.) Let $X, Y \in \Gamma(\mathcal{D}^i)$ and $\omega \in \Gamma(\mathcal{D}^*)$. Then $\overline{K^i(X,Y)\omega} = \overline{[\nabla^i_X, \nabla^i_Y]\omega - \mathring{\nabla}^i_{[X,Y]}\omega}$. (Since $X, Y \in \Gamma(\mathcal{D}^i)$, we have $[X,Y] \in \Gamma(\mathcal{D}^{i+1})$; hence the term $\mathring{\nabla}^i_{[X,Y]}\omega$ is properly defined.) Then, by proposition B.2.13 and corollary B.2.14, we have

$$\overline{K^i(X,Y)\omega} = [X^{h^i}, Y^{h^i}][\overline{\omega}] - [X,Y]^{f^i}[\overline{\omega}].$$

Clearly, $[X^{h^i}, Y^{h^i}] - [X,Y]^{f^i}$ is vertical; thus $[X^{h^i}, Y^{h^i}] - [X,Y]^{f^i} = \mathscr{V}(\mathscr{P}_i([X^{h^i}, Y^{h^i}]))$, and so $\overline{K^i(X,Y)\omega} = \mathscr{V}(\mathscr{P}_i([X^{h^i}, Y^{h^i}]))[\overline{\omega}]$. If $U_q \in \mathcal{D}$, then

$$\omega_q(K^i(X,Y)U_q) = -\mathscr{V}(\mathscr{P}_i([X^{h^i}, Y^{h^i}]))(U_q) = -\omega_q(\operatorname{vl}^{-1}_{U_q} \cdot \mathscr{V}(\mathscr{P}_i([X^{h^i}, Y^{h^i}]))(U_q)),$$

whence $K^i(X,Y)U_q = -\operatorname{vl}^{-1}_{U_q} \cdot \mathscr{V}(\mathscr{P}_i([X^{h^i}, Y^{h^i}]))(U_q)$. ■

Corollary 2.3.16. *Let* $1 \le i \le N$. *We have* $K^i \equiv 0$ *if and only if* $\mathscr{P}_i([\mathcal{H}^i, \mathcal{H}^i]) \subseteq \mathcal{H}^i$. *In particular,* $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathsf{g})$ *is flat (i.e.,* $K^N \equiv 0$*) if and only if* $\mathcal{H}^N$ *is integrable.*

2.4 Curvature of left-invariant structures on Lie groups

In this section we show that, for a left-invariant nonholonomic Riemannian structure, the Schouten curvature tensor is left invariant, as are the related tensors R, C, etc. Likewise, the tensors associated to a left-invariant Wagner structure (formally defined below) are left invariant. Lastly, in section 2.4.1, we characterise the existence of left-invariant parallel frames. Let $(\mathsf{G}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ be a left-invariant nonholonomic Riemannian structure on a Lie group G.

Proposition 2.4.1. *The Schouten curvature tensor K is left invariant, i.e., $K = (L_g)^*K$ for every $g \in \mathsf{G}$.*

Proof. Let $g \in \mathsf{G}$ and $X, Y, Z \in \Gamma(\mathcal{D})$. Then

$$\begin{aligned}(L_g)_*(K(X,Y)Z) &= (L_g)_*(\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z) - (L_g)_*\nabla_{[\![X,Y]\!]}Z - (L_g)_*[\![\mathscr{Q}([X,Y]), Z]\!]\\ &= \nabla_{(L_g)_*X}(L_g)_*(\nabla_Y Z) - \nabla_{(L_g)_*Y}(L_g)_*(\nabla_X Z) - \nabla_{(L_g)_*[\![X,Y]\!]}(L_g)_*Z\\ &\quad - [\![(L_g)_*\mathscr{Q}([X,Y]), (L_g)_*Z]\!]\\ &= \nabla_{(L_g)_*X}\nabla_{(L_g)_*Y}(L_g)_*Z - \nabla_{(L_g)_*Y}\nabla_{(L_g)_*X}(L_g)_*Z\\ &\quad - \nabla_{[\![(L_g)_*X,(L_g)_*Y]\!]}(L_g)_*Z - [\![\mathscr{Q}([(L_g)_*X,(L_g)_*Y]), (L_g)_*Z]\!]\\ &= K((L_g)_*X, (L_g)_*Y)(L_g)_*Z.\end{aligned}$$

That is, $K = (L_g)^*K$. ■

Clearly, the process by which one obtains the tensors R, C, etc. from K preserves left invariance. Hence we have the following corollary.

Corollary 2.4.2. *We have $S = (L_g)^*S$ for every $g \in \mathsf{G}$ (i.e., S is left invariant), where S is any one of R, C, $\widehat{K}$, $\widehat{R}$, $\widehat{C}$, $\widetilde{R}$, Ric, A_{sym}, A_{skew} or Scal.*

We now define Wagner structures invariant under left translations. Let $(\mathsf{G}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ (together with distributions $\mathcal{E}^1, \dots, \mathcal{E}^{N-1}$) be a Wagner structure, with degree of nonholonomy $N \geq 2$, on a Lie group G. If the underlying nonholonomic Riemannian structure $(\mathsf{G}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ is left invariant and the distributions $\mathcal{E}^1, \dots, \mathcal{E}^{N-1}$ are also left invariant, i.e., $(L_g)_*\mathcal{E}^i = \mathcal{E}^i$ for every $g \in \mathsf{G}$ and each $i = 1, \dots, N-1$, then we call $(\mathsf{G}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ a *left-invariant Wagner structure.*

Let $(\mathsf{G}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ be a left-invariant Wagner structure. We will show that the associated tensors (the projections, the Riemannian extension of $\mathbf{g}$, the connections ∇^i and curvature tensors K^i, etc.) are all left invariant.

Lemma 2.4.3. *We have $(L_g)_*\mathscr{P}_i(X) = \mathscr{P}_i((L_g)_*X)$ and $(L_g)_*\mathscr{Q}_i(X) = \mathscr{Q}_i((L_g)_*X)$ for every $X \in \Gamma(T\mathsf{G})$ and $g \in \mathsf{G}$ and each $i = 1, \dots, N-1$.*

Proof. Fix $1 \leq i \leq N-1$, let $g \in \mathsf{G}$ and $X \in \Gamma(T\mathsf{G})$. There exist $X_0 \in \Gamma(\mathcal{D})$ and $X_j \in \Gamma(\mathcal{E}^j)$ for each $j = 1, \dots, N-1$ such that $X = X_0 + X_1 + \cdots + X_{N-1}$. Since the distributions $\mathcal{D}, \mathcal{E}^1, \dots, \mathcal{E}^{N-1}$ are left invariant, it follows that $(L_g)_*X_0 \in \Gamma(\mathcal{D})$ and $(L_g)_*X_j \in \Gamma(\mathcal{E}^j)$. Consequently, we have

$$\begin{aligned}\mathscr{Q}_i((L_g)_*X) &= \mathscr{Q}_i((L_g)_*X_0 + (L_g)_*X_1 + \cdots + (L_g)_*X_{N-1})\\ &= (L_g)_*X_i = (L_g)_*\mathscr{Q}_i(X).\end{aligned}$$

Similarly for the projection $\mathscr{P}_i$,

$$\begin{aligned}(L_g)_*\mathscr{P}_i(X) &= (L_g)_*(\mathscr{P}(X) + \mathscr{Q}_1(X) + \cdots + \mathscr{Q}_{i-1}(X)) \\ &= \mathscr{P}((L_g)_*X) + \mathscr{Q}_1((L_g)_*X) + \cdots + \mathscr{Q}_{i-1}((L_g)_*X) = \mathscr{P}_i((L_g)_*X). \qquad \blacksquare\end{aligned}$$

Lemma 2.4.4. *The tensors Δ_i and Θ_i are left invariant, i.e.,*

$$(L_g)_*\Delta_i(X \wedge Y) = \Delta_i((L_g)_*(X \wedge Y)) \quad \text{and} \quad (L_g)_*\Theta_i(A) = \Theta_i((L_g)_*A)$$

for every $g \in \mathsf{G}$, $X, Y \in \Gamma(\mathcal{D}^i)$ and $A \in \Gamma(\mathcal{E}^i)$ and each $i = 1, \dots, N-1$.

Proof. By the previous lemma, left translations preserve the projection operators $\mathscr{P}_i$ and $\mathscr{Q}_i$. Accordingly, if $g \in \mathsf{G}$ and $X, Y \in \Gamma(\mathcal{D}^i)$, then

$$\begin{aligned}(L_g)_*\Delta_i(X \wedge Y) &= (L_g)_*\mathscr{Q}_i([X,Y]) = \mathscr{Q}_i([(L_g)_*X, (L_g)_*Y]) \\ &= \Delta_i((L_g)_*X \wedge (L_g)_*Y) \\ &= \Delta_i((L_g)_*(X \wedge Y)).\end{aligned}$$

Similarly, if $g \in \mathsf{G}$ and $A \in \Gamma(\mathcal{E}^i)$, then

$$(L_g)_*A = (L_g)_*\Delta_i(\Theta_i(A)) = \Delta_i((L_g)_*\Theta_i(A)).$$

Applying Θ_i to both sides of the equation yields $\Theta_i((L_g)_*A) = (L_g)_*\Theta_i(A)$, as required. ■

Proposition 2.4.5. *The Riemannian extension $\widetilde{\mathbf{g}}$ of $\mathbf{g}$ described in theorem 2.2.4 is left invariant.*

Proof. (We follow the notation used in theorem 2.2.4.) Using induction on i, we shall prove that $\mathbf{g}^i = \mathbf{g} \oplus \mathbf{h}^1 \oplus \cdots \oplus \mathbf{h}^{i-1}$ is left invariant, for each $i = 1, \dots, N-1$. As $\widetilde{\mathbf{g}} = \mathbf{g}^{N-1}$, this will imply that $\widetilde{\mathbf{g}}$ is left invariant. Since $\mathbf{g}$ is assumed to be left invariant, the case $i = 1$ is true by assumption. Suppose that $\mathbf{g}^i$ is left invariant, for some $1 \leq i < N-1$. Since $\mathbf{g}^{i+1} = \mathbf{g}^i \oplus \mathbf{h}^i$, it suffices to show that $\mathbf{h}^i$ is left invariant. It is not difficult to see that $(\mathbf{h}^i)^\flat = \Theta_i^* \circ (\widehat{\mathbf{g}}^i)^\flat \circ \Theta_i$. Accordingly, if $g \in \mathsf{G}$ and $A, B \in \Gamma(\mathcal{E}^i)$, then

$$\begin{aligned}\mathbf{h}^i((L_g)_*A, (L_g)_*B) &= \big((\Theta_i^* \circ (\widehat{\mathbf{g}}^i)^\flat \circ \Theta_i)((L_g)_*A)\big)\big((L_g)_*B\big) \\ &= \widehat{\mathbf{g}}^i(\Theta_i((L_g)_*A), \Theta_i((L_g)_*B)) \\ &= \widehat{\mathbf{g}}^i((L_g)_*\Theta_i(A), (L_g)_*\Theta_i(B)).\end{aligned}$$

If $X_1 \wedge X_2, Y_1 \wedge Y_2 \in (\ker \Delta_i)^\perp$, then

$$\begin{aligned}\widehat{\mathbf{g}}^i((L_g)_*(X_1 \wedge X_2), (L_g)_*(Y_1 \wedge Y_2)) &= \det \begin{bmatrix} \mathbf{g}^i((L_g)_*X_1, (L_g)_*Y_1) & \mathbf{g}^i((L_g)_*X_1, (L_g)_*Y_2) \\ \mathbf{g}^i((L_g)_*X_2, (L_g)_*Y_1) & \mathbf{g}^i((L_g)_*X_2, (L_g)_*Y_2) \end{bmatrix} \\ &= \det \begin{bmatrix} \mathbf{g}^i(X_1, Y_1) \circ L_{g^{-1}} & \mathbf{g}^i(X_1, Y_2) \circ L_{g^{-1}} \\ \mathbf{g}^i(X_2, Y_1) \circ L_{g^{-1}} & \mathbf{g}^i(X_2, Y_2) \circ L_{g^{-1}} \end{bmatrix} \\ &= \widehat{\mathbf{g}}^i(X_1 \wedge X_2, Y_1 \wedge Y_2) \circ L_{g^{-1}}.\end{aligned}$$

Consequently, we have

$$\mathbf{h}^i((L_g)_*A, (L_g)_*B) = \widehat{\mathbf{g}}^i(\Theta_i(A), \Theta_i(B)) \circ L_{g^{-1}} = \mathbf{h}^i(A, B) \circ L_{g^{-1}},$$

and so $\mathbf{h}^i$ is left invariant. ■

Proposition 2.4.6. *We have* $\nabla^i = (L_g)^*\nabla^i$ *and* $K^i = (L_g)^*K^i$ *for every* $g \in \mathsf{G}$ *and each* $i = 1, \ldots, N$.

Proof. We use induction on i. The case $i = 1$ is proved in proposition 1.2.3 and proposition 2.4.1. Suppose the statement is true for some $1 \leq i \leq N-1$. Let $g \in \mathsf{G}$, $Z \in \Gamma(\mathcal{D}^{i+1})$ and $U \in \Gamma(\mathcal{D})$. If $X = \mathscr{P}_i(Z)$ and $A = \mathscr{Q}_i(Z)$, then

$$\begin{aligned}\nabla^{i+1}_{(L_g)_*Z}(L_g)_*U &= \nabla^{i}_{(L_g)_*X}(L_g)_*U + K^i(\Theta_i((L_g)_*A))(L_g)_*U + \llbracket (L_g)_*A, (L_g)_*U \rrbracket \\ &= (L_g)_*\nabla^i_X U + (L_g)_*(K^i(\Theta_i(A))U) + (L_g)_*\llbracket A, U \rrbracket \\ &= (L_g)_*\nabla^{i+1}_Z U.\end{aligned}$$

That is, ∇^{i+1} is left invariant. Then, if $X, Y \in \Gamma(\mathcal{D}^i)$ and $U \in \Gamma(\mathcal{D})$, we have

$$\begin{aligned}(L_g)_*(K^{i+1}(X \wedge Y)U) &= (L_g)_*\nabla^{i+1}_X\nabla^{i+1}_Y U - (L_g)_*\nabla^{i+1}_Y\nabla^{i+1}_X U - (L_g)_*\nabla^{i+1}_{\mathscr{P}_{i+1}([X,Y])}U \\ &\quad - (L_g)_*\llbracket \mathscr{Q}_{i+1}([X,Y]), U \rrbracket \\ &= \nabla^{i+1}_{(L_g)_*X}\nabla^{i+1}_{(L_g)_*Y}(L_g)_*U - \nabla^{i+1}_{(L_g)_*Y}\nabla^{i+1}_{(L_g)_*X}(L_g)_*U \\ &\quad - \nabla^{i+1}_{\mathscr{P}_{i+1}([(L_g)_*X,(L_g)_*Y])}(L_g)_*U \\ &\quad - \llbracket \mathscr{Q}_{i+1}([(L_g)_*X, (L_g)_*Y]), (L_g)_*U \rrbracket \\ &= K^{i+1}((L_g)_*X \wedge (L_g)_*Y)(L_g)_*U,\end{aligned}$$

and so $K^{i+1} = (L_g)^*K^{i+1}$. This completes the proof. ■

2.4.1 Left-invariant parallel frames

To conclude this chapter, we consider the existence of *left-invariant* parallel frames. Let $(\mathsf{G}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ be a left-invariant nonholonomic Riemannian structure on a Lie group. A *left-invariant parallel frame* for $\mathcal{D}$ is a left-invariant orthonormal frame (U_a) for $\mathcal{D}$ such that each U_a is parallel (with respect to the nonholonomic connection ∇). Clearly, if there exists a left-invariant parallel frame for $\mathcal{D}$, then $(\mathsf{G}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ is (globally) flat. Furthermore, the parallel translation is simply (a restriction of the tangent map of) the left translation (see proposition B.1.31).

Proposition 2.4.7. *The following statements are equivalent:*

- (i) *There exists a left-invariant parallel frame for* $\mathcal{D}$ *on* G.
- (ii) $\llbracket X, Y \rrbracket = 0$ *for every* $X, Y \in \Gamma^L(\mathcal{D})$.
- (iii) $\nabla_X Y = 0$ *for every* $X, Y \in \Gamma^L(\mathcal{D})$. (*In particular,* ∇ *is Cartan–Schouten and every left-invariant vector field is parallel.*)

Proof. Suppose (i) holds, i.e., there exists a left-invariant parallel frame (U_a) for $\mathcal{D}$. By proposition 2.2.13, we then have $\llbracket X, Y \rrbracket = x^a y^b \llbracket U_a, U_b \rrbracket = 0$ for every pair of left-invariant vector fields $X = x^a U_a$, $Y = y^a U_a$ in $\Gamma(\mathcal{D})$. If item (ii) holds, then clearly ∇ is a Cartan–Schouten connection. In fact, we have $\nabla_X Y = \frac{1}{2}\llbracket X, Y \rrbracket = 0$ for every $X, Y \in \Gamma^L(\mathcal{D})$. That (iii) implies (i) is trivial. ■

Corollary 2.4.8. *If* ∇ *is a Cartan–Schouten connection and* $\operatorname{rank}(\mathcal{D}) = 2$*, then there exists a left-invariant parallel frame for* $\mathcal{D}$*.*

Proof. By proposition 1.2.11, if ∇ is Cartan–Schouten and $\operatorname{rank}(\mathcal{D}) = 2$, then $\nabla_X Y = 0$ for every $X, Y \in \Gamma^L(\mathcal{D})$. ■

Corollary 2.4.9. *Suppose that* $(\mathsf{G}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ *is a left-invariant Wagner structure (with degree of nonholonomy* $N \geq 2$*) and there exists a left-invariant parallel frame for* $\mathcal{D}$*. Then (in the notation of section 2.2):*

(*i*) $\nabla^i_Z U = 0$ *for every* $Z \in \Gamma^L(\mathcal{D}^i)$, $U \in \Gamma^L(\mathcal{D})$ *and* $i = 1, \ldots, N$.

(*ii*) $K^i(X \wedge Y)U = -[\![\mathcal{Q}_i([X, Y]), U]\!]$ *for every* $X, Y \in \Gamma^L(\mathcal{D}^i)$, $U \in \Gamma^L(\mathcal{D})$ *and* $i = 1, \ldots, N$.

Proof. By proposition 2.4.7 we have that every left-invariant vector field is parallel. Item (*i*) now follows from the fact that $\nabla^i U \equiv 0$ if and only if $\nabla^{i+1} U \equiv 0$ (theorem 2.2.8); item (*ii*) follows from corollary 2.2.16. ■

Chapter 3

Equivalence and embeddings

There are two main topics treated in this chapter. Firstly, we consider the equivalence of nonholonomic Riemannian manifolds. We define three natural equivalence relations (of increasing strength) between nonholonomic Riemannian structures, and prove some basic properties of each equivalence. The first equivalence relation is up to a (diffeomorphic) correspondence between the nonholonomic geodesics of two structures; the second is up to a correspondence between the constraint distributions and nonholonomic connections; and the third is up to a correspondence between the constraint distributions, the complementary distributions, and the metrics. In this thesis we shall primarily be concerned with the last (and strongest) of the three equivalences. Nonholonomic Riemannian structures that are equivalent under this equivalence relation will be called "nonholonomically isometric." (Nonholonomic isometries are the natural generalisation of Riemannian isometries.)

The second topic treated in this chapter is that of "nonholonomic Riemannian submanifolds." In other words, we consider the situation when one nonholonomic Riemannian structure can be embedded inside another such structure. This is an obvious generalisation of Riemannian isometric embeddings to the case of nonholonomic Riemannian structures. As such, our approach is very much informed and inspired by the approach taken in Riemannian geometry (as set out in, for instance, [44]). We have also generalised the work of Lewis [46, 3] on "geodesic invariance." Briefly, a vector subbundle $\mathcal{S}$ of the distribution $\mathcal{D}$ of a nonholonomic Riemannian manifold $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ is *geodesically invariant* (in $\mathcal{D}$) if it is invariant under the nonholonomic geodesic flow of $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$. Geodesic invariance generalises the concept of a totally geodesic submanifold (i.e., a submanifold N of M such that every geodesic tangent to N at some point is contained entirely in N). Indeed, if $\mathcal{S}$ is a geodesically invariant integrable distribution, then its corresponding integral manifolds are totally geodesic. The reason for studying geodesic invariance is as follows. Suppose $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ is embedded inside a nonholonomic Riemannian structure $(\mathsf{M}', \mathcal{D}', \mathcal{D}'^\perp, \mathbf{g}')$. We may identify M with a submanifold of M' and $\mathcal{D}$ with a subbundle of $\mathcal{D}'|_\mathsf{M}$. The nonholonomic geodesics of $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ coincide with those of $(\mathsf{M}', \mathcal{D}', \mathcal{D}'^\perp, \mathbf{g}')$ lying in M and tangent to $\mathcal{D}$ exactly when $\mathcal{D}$ is geodesically invariant. Hence, when $\mathcal{D}$ is geodesically invariant, the geometry of $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ is essentially inherited from the ambient structure $(\mathsf{M}', \mathcal{D}', \mathcal{D}'^\perp, \mathbf{g}')$. The case when $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ can be embedded inside a Riemannian manifold such that $\mathcal{D}$ is geodesically invariant is of particular interest. In the last section of this chapter we show that, at least when $\mathcal{D}$ is strongly nonholonomic, this occurs exactly when the component C of the Schouten curvature tensor vanishes. We prove a similar result in the general case for Wagner structures.

Lastly, we mention that it is possible to generalise the concept of a Riemannian submersion to the nonholonomic Riemannian case. Indeed, given two nonholonomic Riemannian structures $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ and $(\mathsf{M}', \mathcal{D}', \mathcal{D}'^\perp, \mathbf{g}')$, we can define a *nonholonomic Riemannian submersion* to be a surjective submersion $\pi : \mathsf{M} \to \mathsf{M}'$ such that: (i) $\mathcal{D}$ and $\mathcal{D}'$ are π-related; (ii) $\mathcal{D}^\perp$ and $\mathcal{D}'^\perp$ and π-related; (iii) $T\pi$ preserves the length of horizontal vectors, where the horizontal distribution $\mathcal{H}$ is defined to be the orthogonal complement in $\mathcal{D}$ of the vertical distribution $\mathcal{V} = \ker T\pi|_{\mathcal{D}} \subsetneq \mathcal{D}$. Much of the theory of Riemannian submersions carries through to the nonholonomic Riemannian case, and their study would be of comparable interest. Nevertheless, space considerations do not permit us to consider this topic further in this thesis.

3.1 Equivalence of nonholonomic Riemannian structures

Before considering the equivalence of nonholonomic Riemannian structures, we first show that a diffeomorphism mapping one distribution to another also maps the flag of the first distribution to that of the second distribution. We also show that dilations of the metric do not affect the structure in any appreciable fashion.

Let $\mathcal{D}$ and $\mathcal{D}'$ be completely nonholonomic distributions on manifolds M and M', respectively. Let $\mathcal{D} = \mathcal{D}^1 \subsetneq \cdots \subsetneq \mathcal{D}^{N-1} \subsetneq \mathcal{D}^N = T\mathsf{M}$ and $\mathcal{D}' = \mathcal{D}'^1 \subsetneq \cdots \subsetneq \mathcal{D}'^{N'-1} \subsetneq \mathcal{D}'^{N'} = T\mathsf{M}'$ be the flags of $\mathcal{D}$ and $\mathcal{D}'$, respectively, where $N, N' \geq 2$.

Proposition 3.1.1. *If $\phi : \mathsf{M} \to \mathsf{M}'$ is a diffeomorphism such that $\phi_*\mathcal{D} = \mathcal{D}'$, then $N = N'$ and $\phi_*\mathcal{D}^i = \mathcal{D}'^i$ for every $i = 1, \ldots, N$.*

Proof. We use induction on i. The result is true by assumption for $i = 1$, i.e., $\phi_*\mathcal{D}^1 = \mathcal{D}'^1$. Suppose that $\phi_*\mathcal{D}^i = \mathcal{D}'^i$ for some $1 \leq i \leq N-1$. Then

$$\phi_*\mathcal{D}^{i+1} = \phi_*\mathcal{D}^i + \phi_*[\mathcal{D}^i, \mathcal{D}^i] = \mathcal{D}'^i + [\phi_*\mathcal{D}^i, \phi_*\mathcal{D}^i] = \mathcal{D}'^i + [\mathcal{D}'^i, \mathcal{D}'^i] = \mathcal{D}'^{i+1}.$$

It follows that $N = N'$. ■

Proposition 3.1.2. *The nonholonomic connection of a nonholonomic Riemannian manifold $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ is invariant under a (constant) rescaling of the metric $\mathbf{g}$.*

Proof. Let $\mathbf{g}' = \frac{1}{\mu^2}\mathbf{g}$ be a rescaling of the metric, where $\mu > 0$. We shall denote the nonholonomic connection of $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g}')$ by ∇'. Let $X, Y, Z \in \Gamma(\mathcal{D})$. Using the Koszul formula (theorem 1.1.5), we have

$$\begin{aligned} 2\,\mathbf{g}'(\nabla'_X Y, Z) &= \frac{1}{\mu^2} X[\mathbf{g}(Y,Z)] + \frac{1}{\mu^2} Y[\mathbf{g}(X,Z)] - \frac{1}{\mu^2} Z[\mathbf{g}(X,Y)] \\ &\quad + \frac{1}{\mu^2}\mathbf{g}([\![X,Y]\!], Z) - \frac{1}{\mu^2}\mathbf{g}([\![X,Z]\!], Y) - \frac{1}{\mu^2}\mathbf{g}([\![Y,Z]\!], X) \\ &= \frac{1}{\mu^2} 2\,\mathbf{g}(\nabla_X Y, Z) \\ &= 2\,\mathbf{g}'(\nabla_X Y, Z). \end{aligned}$$

Since Z is arbitrary, we have $\nabla'_X Y = \nabla_X Y$ for every $X, Y \in \Gamma(\mathcal{D})$, i.e., $\nabla' = \nabla$. ■

In particular, it follows that the nonholonomic geodesics of a rescaled structure are identical to those of the original structure.

3.1.1 Nonholonomic geodesic equivalence

We say that two nonholonomic Riemannian manifolds $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ and $(\mathsf{M}', \mathcal{D}', \mathcal{D}'^\perp, \mathbf{g}')$ are *NH-geodesically equivalent* if there exists a diffeomorphism $\phi : \mathsf{M} \to \mathsf{M}'$ establishing a one-to-one correspondence between the nonholonomic geodesics of the two structures, i.e., γ is a nonholonomic geodesic of $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ if and only if $\phi \circ \gamma$ is a nonholonomic geodesic of $(\mathsf{M}', \mathcal{D}', \mathcal{D}'^\perp, \mathbf{g}')$.

Lemma 3.1.3 (cf. [38]). *The following statements are equivalent:*

(i) *Two nonholonomic connections* $\nabla^1, \nabla^2 : \Gamma(\mathcal{D}) \times \Gamma(\mathcal{D}) \to \Gamma(\mathcal{D})$ *have the same geodesics.*

(ii) $\nabla^1_X X = \nabla^2_X X$ *for every* $X \in \Gamma(\mathcal{D})$.

(iii) $\langle\!\langle X : Y \rangle\!\rangle_1 = \langle\!\langle X : Y \rangle\!\rangle_2$ *for every* $X, Y \in \Gamma(\mathcal{D})$, *where* $\langle\!\langle \cdot : \cdot \rangle\!\rangle_i$ *is the symmetric bracket of* ∇^i *for* $i = 1, 2$.

Proof. Let $B(X,Y) = \nabla^1_X Y - \nabla^2_X Y$ for $X, Y \in \Gamma(\mathcal{D})$. If $f \in \mathcal{C}^\infty(\mathsf{M})$, then $B(fX, Y) = \nabla^1_{fX} Y - \nabla^2_{fX} Y = fB(X,Y)$ and

$$B(X, fY) = \nabla^1_X fY - \nabla^2_X (fY) = X[f]Y + f\nabla^1_X Y - X[f]Y - f\nabla^2_X Y = fB(X,Y).$$

That is, B is tensorial in both arguments. We will prove that ∇^1 and ∇^2 have the same geodesics if and only if $B(X,X) = 0$ for every $X \in \Gamma(\mathcal{D})$. That item ($ii$) and item ($iii$) are equivalent is immediate by polarisation.

Suppose ∇^1 and ∇^2 have the same geodesics. Let $X_q \in \mathcal{D}$ and let γ be the (unique) geodesic of ∇^1 and ∇^2 such that $\gamma(0) = q$ and $\dot\gamma(0) = X_q$. Let $Y \in \Gamma(\mathcal{D})$ be a local extension of $\dot\gamma$ along γ; we have $\nabla^1_Y Y(\gamma(t)) = \nabla^2_Y Y(\gamma(t)) = 0$ for all t. Then $B(X_q, X_q) = \nabla^1_Y Y(\gamma(0)) - \nabla^2_Y Y(\gamma(0)) = 0$, and so, by tensoriality of B, we have $B(X,X) = 0$ for every $X \in \Gamma(\mathcal{D})$. Conversely, suppose that $B(X,X) = 0$ for every $X \in \Gamma(\mathcal{D})$ and let γ be a geodesic of ∇^1. Then $\nabla^2_{\dot\gamma}\dot\gamma(t) = \nabla^1_{\dot\gamma}\dot\gamma(t) - B(\dot\gamma(t), \dot\gamma(t)) = 0$, i.e., γ is also a geodesic of ∇^2. ■

Proposition 3.1.4. $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ *and* $(\mathsf{M}', \mathcal{D}', \mathcal{D}'^\perp, \mathbf{g}')$ *are NH-geodesically equivalent if and only if there exists a diffeomorphism* $\phi : \mathsf{M} \to \mathsf{M}'$ *such that* $\phi_*\mathcal{D} = \mathcal{D}'$ *and* ∇ *and* $\phi^*\nabla'$ *have the same geodesics.*

Proof. We have that $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ and $(\mathsf{M}', \mathcal{D}', \mathcal{D}'^\perp, \mathbf{g}')$ are NH-geodesically equivalent if and only if $\nabla_{\dot\gamma}\dot\gamma(t) = (\phi^*\nabla')_{\dot\gamma}\dot\gamma(t)$ for every nonholonomic geodesic γ of $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$. Suppose that $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ and $(\mathsf{M}', \mathcal{D}', \mathcal{D}'^\perp, \mathbf{g}')$ are NH-geodesically equivalent. (The proof of the converse is immediate.) Clearly, we have $\phi_*\mathcal{D} = \mathcal{D}'$. Let $B(X,Y) = \nabla_X Y - (\phi^*\nabla')_X Y$ for $X, Y \in \Gamma(\mathcal{D})$; as in the proof of lemma 3.1.3, we have that B is tensorial in both arguments. Let $X_q \in \mathcal{D}$ and let γ be the nonholonomic geodesic such that $\gamma(0) = q$, $\dot\gamma(0) = X_q$. Let $X \in \Gamma(\mathcal{D})$ be an extension of X_q to a neighbourhood of q such that X coincides with $\dot\gamma$ along γ in the neighbourhood. Then

$$B(X_q, X_q) = \nabla_{X_q} X - (\phi^*\nabla')_{X_q} X = \nabla_{\dot\gamma}\dot\gamma(0) - (\phi^*\nabla')_{\dot\gamma}\dot\gamma(0) = 0.$$

(We have used the fact that $\nabla_X Y(q)$ depends only on the values of Y along any curve tangent to $X(q)$; see lemma B.1.5.) That is, $B(X,X) = 0$ for every $X \in \Gamma(\mathcal{D})$. By lemma 3.1.3, it follows that ∇ and $\phi^*\nabla'$ have the same geodesics. ■

3.1.2 Nonholonomic affinities

We say that two nonholonomic Riemannian manifolds $(\mathsf{M}, \mathcal{D}, \mathcal{D}^{\perp}, \mathbf{g})$ and $(\mathsf{M}', \mathcal{D}', \mathcal{D}'^{\perp}, \mathbf{g}')$ are *NH-affinely equivalent* if there exists a diffeomorphism $\phi : \mathsf{M} \to \mathsf{M}'$ such that

$$\phi_* \mathcal{D} = \mathcal{D}' \quad \text{and} \quad \nabla = \phi^* \nabla'.$$

A diffeomorphism ϕ satisfying the above properties is called an *NH-affinity.*

Proposition 3.1.5. *Let* $(\mathsf{M}, \mathcal{D}, \mathcal{D}^{\perp}, \mathbf{g})$ *and* $(\mathsf{M}', \mathcal{D}', \mathcal{D}'^{\perp}, \mathbf{g}')$ *be two nonholonomic Riemannian manifolds.*

(i) *If* $(\mathsf{M}, \mathcal{D}, \mathcal{D}^{\perp}, \mathbf{g})$ *and* $(\mathsf{M}', \mathcal{D}', \mathcal{D}'^{\perp}, \mathbf{g}')$ *are NH-affinely equivalent, then they are NH-geodesically equivalent.*

(ii) *Suppose* $(\mathsf{M}, \mathcal{D}, \mathcal{D}^{\perp}, \mathbf{g})$ *and* $(\mathsf{M}', \mathcal{D}', \mathcal{D}'^{\perp}, \mathbf{g}')$ *are NH-geodesically equivalent with respect to a diffeomorphism* $\phi : \mathsf{M} \to \mathsf{M}'$. *The two structures are NH-affinely equivalent* (*with respect to* ϕ) *if and only if* $\phi^* \nabla'$ *is torsion free* (*with respect to* $\mathscr{P}$), *i.e.,*

$$(\phi^* \nabla')_X Y - (\phi^* \nabla')_Y X = [\![X, Y]\!]$$

for every $X, Y \in \Gamma(\mathcal{D})$.

Proof. (i) Let $\phi : \mathsf{M} \to \mathsf{M}'$ be an NH-affinity between $(\mathsf{M}, \mathcal{D}, \mathcal{D}^{\perp}, \mathbf{g})$ and $(\mathsf{M}', \mathcal{D}', \mathcal{D}'^{\perp}, \mathbf{g}')$. As $\nabla = \phi^* \nabla'$, it is trivially true that ∇ and $\phi^* \nabla'$ have the same geodesics. Consequently, by proposition 3.1.4, the two structures are NH-geodesically equivalent.

(ii) Suppose that $(\mathsf{M}, \mathcal{D}, \mathcal{D}^{\perp}, \mathbf{g})$ and $(\mathsf{M}', \mathcal{D}', \mathcal{D}'^{\perp}, \mathbf{g}')$ are NH-geodesically equivalent. Then, by proposition 3.1.4, there exists a diffeomorphism $\phi : \mathsf{M} \to \mathsf{M}'$ such that $\phi_* \mathcal{D} = \mathcal{D}'$ and ∇ and $\phi^* \nabla'$ have the same geodesics. By lemma 3.1.3, this occurs exactly when

$$\nabla_X Y + \nabla_Y X = (\phi^* \nabla')_X Y + (\phi^* \nabla')_Y X$$

for every $X, Y \in \Gamma(\mathcal{D})$. Since ∇ and ∇' are torsion free, we get

$$\begin{aligned}
&\nabla_X Y + \nabla_Y X = (\phi^* \nabla')_X Y + (\phi^* \nabla')_Y X \\
\iff \quad & \nabla_X Y + \nabla_Y X = (\phi^{-1})_* (\nabla'_{\phi_* X} \phi_* Y + \nabla'_{\phi_* Y} \phi_* X) \\
\iff \quad & 2\, \nabla_X Y - [\![X, Y]\!] = 2\, (\phi^{-1})_* (\nabla'_{\phi_* X} \phi_* Y) - (\phi^{-1})_* [\![\phi_* X, \phi_* Y]\!] \\
\iff \quad & 2\, ((\phi^* \nabla')_X Y - \nabla_X Y) = (\phi^{-1})_* \mathscr{P}'(\phi_* [X, Y]) - [\![X, Y]\!].
\end{aligned}$$

The right-hand side of the last line is exactly the torsion $T(X, Y)$ of $\phi^* \nabla'$ (with respect to the projection $\mathscr{P}$):

$$\begin{aligned}
T(X, Y) &= (\phi^* \nabla')_X Y - (\phi^* \nabla')_Y X - [\![X, Y]\!] \\
&= (\phi^{-1})_* (\nabla'_{\phi_* X} \phi_* Y - \nabla'_{\phi_* Y} \phi_* X) - [\![X, Y]\!] \\
&= (\phi^{-1})_* [\![\phi_* X, \phi_* Y]\!] - [\![X, Y]\!] \\
&= (\phi^{-1})_* \mathscr{P}'(\phi_* [X, Y]) - [\![X, Y]\!].
\end{aligned}$$

Hence $\nabla_X Y = (\phi^* \nabla')_X Y$ for every $X, Y \in \Gamma(\mathcal{D})$ if and only if $\phi^* \nabla'$ is torsion free. ■

3.1.3 Nonholonomic isometries

Two nonholonomic Riemannian manifolds $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ and $(\mathsf{M}', \mathcal{D}', \mathcal{D}'^\perp, \mathbf{g}')$ are said to be *NH-isometric* if there exists a diffeomorphism $\phi : \mathsf{M} \to \mathsf{M}'$ such that

$$\phi_*\mathcal{D} = \mathcal{D}', \quad \phi_*\mathcal{D}^\perp = \mathcal{D}'^\perp \quad \text{and} \quad \mathbf{g} = \phi^*\mathbf{g}'.$$

Any map satisfying the above three properties is termed an *NH-isometry.* From [34], the collection of all NH-isometries of a nonholonomic Riemannian structure forms a finite-dimensional Lie group. In the following chapter, we shall consider the equivalence of three-dimensional nonholonomic Riemannian manifolds under NH-isometries. In particular, we classify the left-invariant structures on the (three-dimensional) simply connected Lie groups.

For the following results, let $\phi : \mathsf{M} \to \mathsf{M}'$ be an NH-isometry between nonholonomic Riemannian manifolds $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ and $(\mathsf{M}', \mathcal{D}', \mathcal{D}'^\perp, \mathbf{g}')$. We first show that ϕ preserves the projection operators of the two structures.

Lemma 3.1.6. *If $X \in \Gamma(T\mathsf{M})$, then $\phi_*\mathscr{P}(X) = \mathscr{P}'(\phi_*X)$ and $\phi_*\mathscr{Q}(X) = \mathscr{Q}'(\phi_*X)$.*

Proof. Let $X \in \Gamma(T\mathsf{M})$. There exist $X_1 \in \Gamma(\mathcal{D})$ and $X_2 \in \Gamma(\mathcal{D}^\perp)$ such that $X = X_1 + X_2$. Since $\phi_*\mathcal{D} = \mathcal{D}'$ and $\phi_*\mathcal{D}^\perp = \mathcal{D}'^\perp$, we have $\phi_*X_1 \in \Gamma(\mathcal{D}')$ and $\phi_*X_2 \in \Gamma(\mathcal{D}'^\perp)$. Consequently, we have

$$\mathscr{P}'(\phi_*X) = \mathscr{P}'(\phi_*X_1 + \phi_*X_2) = \phi_*X_1 = \phi_*\mathscr{P}(X),$$

and so $\mathscr{Q}'(\phi_*X) = \phi_*X - \mathscr{P}'(\phi_*X) = \phi_*(X - \mathscr{P}(X)) = \phi_*\mathscr{Q}(X)$. ■

As a consequence of preserving the projection operators (as well as $\mathcal{D}$ and $\mathbf{g}$), we have that every NH-isometry is an NH-affinity.

Proposition 3.1.7. *Every NH-isometry is an NH-affinity.*

Proof. We show that $\nabla = \phi^*\nabla'$. Let $X, Y, Z \in \Gamma(\mathcal{D})$. Since $\mathbf{g} = \phi^*\mathbf{g}'$, we have

$$\begin{aligned}
\mathbf{g}((\phi^*\nabla')_Z X, Y) + \mathbf{g}(X, (\phi^*\nabla')_Z Y) &= \mathbf{g}((\phi^{-1})_*(\nabla'_{\phi_*Z}\phi_*X), Y) + \mathbf{g}(X, (\phi^{-1})_*(\nabla'_{\phi_*Z}\phi_*Y)) \\
&= \mathbf{g}'(\nabla'_{\phi_*Z}\phi_*X, \phi_*Y) \circ \phi + \mathbf{g}'(\phi_*X, \nabla'_{\phi_*Z}\phi_*Y) \circ \phi \\
&= (\phi_*Z)[\mathbf{g}'(\phi_*X, \phi_*Y)] \circ \phi \\
&= (\phi_*Z)[\mathbf{g}(X, Y) \circ \phi^{-1}] \circ \phi \\
&= Z[\mathbf{g}(X, Y)].
\end{aligned}$$

That is, $(\phi^*\nabla')\mathbf{g} \equiv 0$. Likewise,

$$(\phi^*\nabla')_X Y - (\phi^*\nabla')_Y X - [\![X, Y]\!] = (\phi^{-1})_*(\nabla'_{\phi_*X}\phi_*Y - \nabla'_{\phi_*Y}\phi_*X) - [\![X, Y]\!].$$

As ϕ preserves the projection operators, we have $(T\phi)^{-1} \circ \mathscr{P}' \circ \phi = \mathscr{P}$. Hence

$$(\phi^*\nabla')_X Y - (\phi^*\nabla')_Y X - [\![X, Y]\!] = (\phi^{-1})_*(\nabla'_{\phi_*X}\phi_*Y - \nabla'_{\phi_*Y}\phi_*X - [\![\phi_*X, \phi_*Y]\!]) = 0,$$

i.e., $\phi^*\nabla'$ is torsion free. By uniqueness of the nonholonomic connection (theorem 1.1.5), it follows that $\nabla = \phi^*\nabla'$. Thus ϕ is an NH-affinity. ■

It is not difficult to see that every nonholonomic Riemannian structure that is NH-isometric to a (locally) flat structure is also (locally) flat. Indeed, the following result follows immediately from corollary B.1.23. (The converse to this result, however, does not hold: in chapter 4, and specifically section 4.2, we shall see that in three dimensions there are many non-NH-isometric flat structures.)

Proposition 3.1.8. *If* $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ *is locally flat on* $\mathcal{U} \subseteq \mathsf{M}$*, then* $(\mathsf{M}', \mathcal{D}', \mathcal{D}'^\perp, \mathbf{g}')$ *is locally flat on* $\phi(\mathcal{U})$*.*

As NH-isometries preserve the metric, nonholonomic connection and projection operators, it follows that the various curvature tensors introduced in section 2.1 are also preserved.

Proposition 3.1.9. *We have* $K = \phi^* K'$*.*

Proof. Let $X, Y, Z \in \Gamma(\mathcal{D})$. Then

$$\begin{aligned}
&\phi_*(K(X,Y)Z)\\
&\quad = \phi_*(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z) - \phi_* \nabla_{[\![X,Y]\!]} Z - \phi_* [\![\mathscr{Q}([X,Y]), Z]\!]\\
&\quad = \nabla'_{\phi_* X} \phi_*(\nabla_Y Z) - \nabla'_{\phi_* Y} \phi_*(\nabla_X Z) - \nabla'_{\phi_*[\![X,Y]\!]} \phi_* Z - [\![\phi_* \mathscr{Q}([X,Y]), \phi_* Z]\!]\\
&\quad = \nabla'_{\phi_* X} \nabla'_{\phi_* Y} \phi_* Z - \nabla'_{\phi_* Y} \nabla'_{\phi_* X} \phi_* Z - \nabla'_{[\![\phi_* X, \phi_* Y]\!]} \phi_* Z - [\![\mathscr{Q}'([\phi_* X, \phi_* Y]), \phi_* Z]\!]\\
&\quad = K'(\phi_* X, \phi_* Y)\phi_* Z.
\end{aligned}$$

That is, $K = \phi^* K'$. ■

Corollary 3.1.10. *We have* $S = \phi^* S'$*, where* S *is any one of* R*,* C*,* $\widehat{K}$*,* $\widehat{R}$*,* $\widehat{C}$*,* $\widetilde{R}$*,* Ric*,* A_{sym}*,* A_{skew} *or* Scal*.*

We conclude this section by considering (nonholonomic) isometries between Wagner structures (see section 2.2). Let $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ and $(\mathsf{M}', \mathcal{D}', \mathcal{D}'^\perp, \mathbf{g}')$ be Wagner structures, both with degree of nonholonomy $N \geq 2$. The two structures are said to be *NH-isometric as Wagner structures* if there exists a diffeomorphism $\phi : \mathsf{M} \to \mathsf{M}'$ such that

$$\phi_* \mathcal{D} = \mathcal{D}', \quad \phi_* \mathcal{E}^i = \mathcal{E}'^i \quad \text{and} \quad \mathbf{g} = \phi^* \mathbf{g}'$$

for each $i = 1, \dots, N-1$. (Since $\mathcal{D}^\perp = \mathcal{E}^1 \oplus \cdots \oplus \mathcal{E}^{N-1}$ and $\mathcal{D}'^\perp = \mathcal{E}'^1 \oplus \cdots \oplus \mathcal{E}'^{N-1}$, the middle condition clearly implies that $\phi_* \mathcal{D}^\perp = \mathcal{D}'^\perp$.) Such a map ϕ is called an *NH-isometry of Wagner structures.*

Let $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ and $(\mathsf{M}', \mathcal{D}', \mathcal{D}'^\perp, \mathbf{g}')$ be Wagner structures, both with degree of nonholonomy $N \geq 2$, and let $\phi : \mathsf{M} \to \mathsf{M}'$ be an NH-isometry of these Wagner structures. (We follow the notation introduced in section 2.2 for the projection operators, connections and curvature tensors.)

Lemma 3.1.11. *If* $X \in \Gamma(T\mathsf{M})$*, then* $\phi_* \mathscr{P}_i(X) = \mathscr{P}'_i(\phi_* X)$ *and* $\phi_* \mathscr{Q}_i(X) = \mathscr{Q}'_i(\phi_* X)$ *for each* $i = 1, \dots, N-1$*.*

Proof. Fix $1 \leq i \leq N-1$ and let $X \in \Gamma(T\mathsf{M})$. There exist $X_0 \in \Gamma(\mathcal{D})$ and $X_j \in \Gamma(\mathcal{E}^j)$ for $j = 1, \dots, N-1$ such that $X = X_0 + X_1 + \cdots + X_{N-1}$. Since $\phi_* \mathcal{D} = \mathcal{D}'$ and $\phi_* \mathcal{E}^j = \mathcal{E}'^j$, it follows that $\phi_* X_0 \in \Gamma(\mathcal{D}')$ and $\phi_* X_j \in \Gamma(\mathcal{E}'^j)$. Consequently, we have

$$\mathscr{Q}'_i(\phi_* X) = \mathscr{Q}'_i(\phi_* X_0 + \phi_* X_1 + \cdots + \phi_* X_{N-1}) = \phi_* X_i = \phi_* \mathscr{Q}_i(X)$$

and

$$\begin{aligned}\phi_*\mathscr{P}_i(X) &= \phi_*(\mathscr{P}(X) + \mathscr{Q}_1(X) + \cdots + \mathscr{Q}_{i-1}(X))\\ &= \mathscr{P}'(\phi_*X) + \mathscr{Q}'_1(\phi_*X) + \cdots + \mathscr{Q}'_{i-1}(\phi_*X) = \mathscr{P}'_i(\phi_*X).\end{aligned}$$

This completes the proof. ■

Lemma 3.1.12. *We have*

$$\phi_*\Delta_i(X \wedge Y) = \Delta'_i(\phi_*(X \wedge Y)) \quad \text{and} \quad \phi_*\Theta_i(A) = \Theta'_i(\phi_*A)$$

for every $X, Y \in \Gamma(\mathcal{D}^i)$ *and* $A \in \Gamma(\mathcal{E}^i)$ *and each* $i = 1, \dots, N-1$.

Proof. By the previous lemma, ϕ preserves the projection operators $\mathscr{P}_i$ and $\mathscr{Q}_i$. Accordingly, if $X, Y \in \Gamma(\mathcal{D}^i)$, then

$$\phi_*\Delta_i(X \wedge Y) = \phi_*\mathscr{Q}_i([X,Y]) = \mathscr{Q}'_i([\phi_*X, \phi_*Y]) = \Delta'_i(\phi_*X \wedge \phi_*Y) = \Delta'_i(\phi_*(X \wedge Y)).$$

Similarly, if $A \in \Gamma(\mathcal{E}^i)$, then $\phi_*A = \phi_*\Delta_i(\Theta_i(A)) = \Delta'_i(\phi_*\Theta_i(A))$. Applying Θ'_i to both sides of the equation yields $\Theta'_i(\phi_*A) = \phi_*\Theta_i(A)$, as required. ■

Proposition 3.1.13. *The map* ϕ *is a Riemannian isometry between* $(\mathsf{M}, \widetilde{\mathbf{g}})$ *and* $(\mathsf{M}', \widetilde{\mathbf{g}}')$, *where* $\widetilde{\mathbf{g}}$ (*resp.* $\widetilde{\mathbf{g}}'$) *is the Riemannian extension of* $\mathbf{g}$ (*resp.* $\mathbf{g}'$) *described in theorem 2.2.4.*

Proof. (We follow the notation used in theorem 2.2.4.) Using induction on i, we shall prove that $\mathbf{g}^i = \phi^*\mathbf{g}'^i$ (where $\mathbf{g}^i = \mathbf{g} \oplus \mathbf{h}^1 \oplus \cdots \oplus \mathbf{h}^{i-1}$ and $\mathbf{g}'^i = \mathbf{g}' \oplus \mathbf{h}'^1 \oplus \cdots \oplus \mathbf{h}'^{i-1}$) for each $i = 1, \dots, N-1$. As $\widetilde{\mathbf{g}} = \mathbf{g}^{N-1}$ and $\widetilde{\mathbf{g}}' = \mathbf{g}'^{N-1}$, this will imply that $\widetilde{\mathbf{g}} = \phi^*\widetilde{\mathbf{g}}'$. The case $i = 1$ is true by assumption. Suppose that $\mathbf{g}^i = \phi^*\mathbf{g}'^i$, for some $1 \le i < N-1$. Since $\mathbf{g}^{i+1} = \mathbf{g}^i \oplus \mathbf{h}^i$ and $\mathbf{g}'^{i+1} = \mathbf{g}'^i \oplus \mathbf{h}'^i$, it suffices to show that $\mathbf{h}^i = \phi^*\mathbf{h}'^i$. We have $(\mathbf{h}^i)^\flat = \Theta_i^* \circ (\widehat{\mathbf{g}}^i)^\flat \circ \Theta_i$ and $(\mathbf{h}'^i)^\flat = \Theta_i'^* \circ (\widehat{\mathbf{g}}'^i)^\flat \circ \Theta'_i$. Accordingly, if $A, B \in \Gamma(\mathcal{E}^i)$, then

$$\begin{aligned}\mathbf{h}'^i(\phi_*A, \phi_*B) &= \big((\Theta_i'^* \circ (\widehat{\mathbf{g}}'^i)^\flat \circ \Theta'_i)(\phi_*A)\big)(\phi_*B)\\ &= \widehat{\mathbf{g}}'^i(\Theta'_i(\phi_*A), \Theta'_i(\phi_*B))\\ &= \widehat{\mathbf{g}}'^i(\phi_*\Theta_i(A), \phi_*\Theta_i(B)).\end{aligned}$$

If $X_1 \wedge X_2, Y_1 \wedge Y_2 \in (\ker \Delta_i)^\perp$, then

$$\begin{aligned}\widehat{\mathbf{g}}'^i(\phi_*(X_1 \wedge X_2), \phi_*(Y_1 \wedge Y_2)) &= \det\begin{bmatrix}\mathbf{g}'^i(\phi_*X_1, \phi_*Y_1) & \mathbf{g}'^i(\phi_*X_1, \phi_*Y_2)\\ \mathbf{g}'^i(\phi_*X_2, \phi_*Y_1) & \mathbf{g}'^i(\phi_*X_2, \phi_*Y_2)\end{bmatrix}\\ &= \det\begin{bmatrix}\mathbf{g}^i(X_1, Y_1) \circ \phi^{-1} & \mathbf{g}^i(X_1, Y_2) \circ \phi^{-1}\\ \mathbf{g}^i(X_2, Y_1) \circ \phi^{-1} & \mathbf{g}^i(X_2, Y_2) \circ \phi^{-1}\end{bmatrix}\\ &= \widehat{\mathbf{g}}^i(X_1 \wedge X_2, Y_1 \wedge Y_2) \circ \phi^{-1}.\end{aligned}$$

Consequently, we have

$$\mathbf{h}'^i(\phi_*A, \phi_*B) = \widehat{\mathbf{g}}^i(\Theta_i(A), \Theta_i(B)) \circ \phi^{-1} = \mathbf{h}^i(A, B) \circ \phi^{-1},$$

and so $\mathbf{h}^i = \phi^*\mathbf{h}'^i$. ■

Proposition 3.1.14. *We have* $\nabla^i = \phi^*\nabla'^i$ *and* $K^i = \phi^* K'^i$ *for each* $i = 1, \ldots, N$.

Proof. We use induction on i. The case $i = 1$ is proved in proposition 3.1.7 and proposition 3.1.9. Suppose the statement is true for some $1 \leq i \leq N-1$. Let $Z \in \Gamma(\mathcal{D}^{i+1})$ and $U \in \Gamma(\mathcal{D})$. If $X = \mathscr{P}_i(Z)$ and $A = \mathscr{Q}_i(Z)$, then

$$\begin{aligned} \nabla^{i+1}_{\phi_* Z}\phi_* U &= \nabla^i_{\phi_* X}\phi_* U + K^i(\Theta_i(\phi_* A))\phi_* U + [\![\phi_* A, \phi * U]\!] \\ &= \phi_*\nabla^i_X U + \phi_*(K^i(\Theta_i(A))U) + \phi_*[\![A, U]\!] \\ &= \phi_*\nabla^{i+1}_Z U. \end{aligned}$$

That is, $\nabla^{i+1} = \phi^*\nabla'^{i+1}$. Then, if $X, Y \in \Gamma(\mathcal{D}^i)$ and $U \in \Gamma(\mathcal{D})$, we have

$$\begin{aligned} \phi_*(K^{i+1}(X \wedge Y)U) &= \phi_*\nabla^{i+1}_X\nabla^{i+1}_Y U - \phi_*\nabla^{i+1}_Y\nabla^{i+1}_X U - \phi_*\nabla^{i+1}_{\mathscr{P}_{i+1}([X,Y])}U \\ &\qquad - \phi_*[\![\mathscr{Q}_{i+1}([X,Y]), U]\!] \\ &= \nabla^{i+1}_{\phi_* X}\nabla^{i+1}_{\phi_* Y}(L_g)_* U - \nabla^{i+1}_{\phi_* Y}\nabla^{i+1}_{\phi_* X}\phi_* U - \nabla^{i+1}_{\mathscr{P}_{i+1}([\phi_* X, \phi_* Y])}\phi_* U \\ &\qquad - [\![\mathscr{Q}_{i+1}([\phi_* X, \phi_* Y]), \phi_* U]\!] \\ &= K^{i+1}(\phi_* X \wedge \phi_* Y)\phi_* U, \end{aligned}$$

and so $K^{i+1} = \phi^* K^{i+1}$. This completes the proof. ■

3.1.4 Structures with Cartan–Schouten connections

Let $(\mathsf{G}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ and $(\mathsf{G}', \mathcal{D}', \mathcal{D}'^\perp, \mathbf{g}')$ be two left-invariant nonholonomic Riemannian structures with Cartan–Schouten connections. It turns out that if there exists a Lie group isomorphism relating the distributions of the two structures, then they are NH-geodesically equivalent.

Proposition 3.1.15. *If* $\phi : \mathsf{G} \to \mathsf{G}'$ *is a Lie group isomorphism such that* $\phi_*\mathcal{D} = \mathcal{D}'$, *then* $(\mathsf{G}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ *and* $(\mathsf{G}', \mathcal{D}', \mathcal{D}'^\perp, \mathbf{g}')$ *are NH-geodesically equivalent* (*with respect to* ϕ).

Proof. Let $\phi : \mathsf{G} \to \mathsf{G}'$ be a Lie group isomorphism such that $\phi_*\mathcal{D} = \mathcal{D}'$. Since ϕ preserves the Lie group structure, if $g(\cdot) : t \mapsto g_0 \exp(t\,U_0)$, $g_0 \in \mathsf{G}$, $U_0 \in \mathcal{D}_\mathbf{1}$ is a nonholonomic geodesic of $(\mathsf{G}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$, then $(\phi \circ g)(t) = \phi(g_0)\exp(t\,T_\mathbf{1}\phi \cdot U_0)$, which is a nonholonomic geodesic of $(\mathsf{G}', \mathcal{D}', \mathcal{D}'^\perp, \mathbf{g}')$. The converse is proved similarly. Hence we have that $(\mathsf{G}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ and $(\mathsf{G}', \mathcal{D}', \mathcal{D}'^\perp, \mathbf{g}')$ are NH-geodesically equivalent. ■

Accordingly, there exists (up to NH-geodesic equivalence) at most one structure whose connection is a Cartan–Schouten connection, for each left-invariant completely nonholonomic distribution on a Lie group. To make a similar statement regarding NH-affine equivalence, we need to add an additional assumption: the Lie group isomorphism must also relate the complementary distributions of the two structures.

Proposition 3.1.16. *If* $\phi : \mathsf{G} \to \mathsf{G}'$ *is a Lie group isomorphism such that* $\phi_*\mathcal{D} = \mathcal{D}'$ *and* $\phi_*\mathcal{D}^\perp = \mathcal{D}'^\perp$, *then* $(\mathsf{G}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ *and* $(\mathsf{G}', \mathcal{D}', \mathcal{D}'^\perp, \mathbf{g}')$ *are NH-affinely equivalent* (*with respect to* ϕ).

Proof. Since ϕ is a Lie group isomorphism such that $\phi_*\mathcal{D} = \mathcal{D}'$, by proposition 3.1.15, we have that $(\mathsf{G}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ and $(\mathsf{G}', \mathcal{D}', \mathcal{D}'^\perp, \mathbf{g}')$ are NH-geodesically equivalent with respect to ϕ. The torsion of $\phi^*\nabla'$ (with respect to $\mathscr{P}$) is

$$T(X,Y) = (\phi^{-1})_*\mathscr{P}'(\phi_*[X,Y]) - \llbracket X,Y \rrbracket, \qquad X, Y \in \Gamma(\mathcal{D}).$$

(See the proof of proposition 3.1.5.) In fact, since $\phi_*\mathcal{D}^\perp = \mathcal{D}'^\perp$, we have

$$\begin{aligned} T(X,Y) &= (\phi^{-1})_*\mathscr{P}'(\underbrace{\phi_*\llbracket X,Y \rrbracket}_{\in \Gamma(\mathcal{D}')} + \underbrace{\phi_*\mathscr{Q}([X,Y])}_{\in \Gamma(\mathcal{D}'^\perp)}) - \llbracket X,Y \rrbracket \\ &= (\phi^{-1})_*(\phi_*\llbracket X,Y \rrbracket) - \llbracket X,Y \rrbracket \\ &= 0. \end{aligned}$$

The result now follows from proposition 3.1.5. ■

Corollary 3.1.17. *If an NH-affinity* $\phi : \mathsf{G} \to \mathsf{G}'$ *between* $(\mathsf{G}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ *and* $(\mathsf{G}', \mathcal{D}', \mathcal{D}'^\perp, \mathbf{g}')$ *is a Lie group isomorphism and* $\mathcal{D}$ *is strongly nonholonomic, then* $\phi_*\mathcal{D}^\perp = \mathcal{D}'^\perp$.

Proof. Since ∇ and ∇' are Cartan–Schouten, we have $\phi_*\mathscr{P}([X,Y]) = \mathscr{P}'(\phi_*[X,Y])$ for every $X, Y \in \Gamma^L(\mathcal{D})$. Indeed,

$$\frac{1}{2}\mathscr{P}([X,Y]) = \nabla_X Y = (\phi^*\nabla')_X Y = (\phi^{-1})_*\nabla'_{\phi_* X}\phi_* Y = \frac{1}{2}(\phi^{-1})_*\mathscr{P}'(\phi_*[X,Y]).$$

To prove $\phi_*\mathcal{D}^\perp = \mathcal{D}'^\perp$, it suffices (by left invariance) to show that $T_\mathbf{1}\phi \cdot \mathscr{P}_\mathbf{1} = \mathscr{P}'_\mathbf{1} \circ T_\mathbf{1}\phi$. Let $U \in T_\mathbf{1}\mathsf{G}$. Since $\mathcal{D}$ is strongly nonholonomic, we have $T\mathsf{G} = \mathcal{D} + [\mathcal{D}, \mathcal{D}]$. Accordingly, there exist $V \in T_\mathbf{1}\mathsf{G}$ and $X, Y \in \Gamma(\mathcal{D})$ such that $U = V + [X,Y](\mathbf{1})$. Suppose $X = x^a X_a$ and $Y = y^a X_a$ for $x^a, y^a \in \mathcal{C}^\infty(\mathsf{M})$, where (X_a) is a left-invariant frame for $\mathcal{D}$. Then

$$\begin{aligned} T_\mathbf{1}\phi \cdot \mathscr{P}_\mathbf{1}(U) &= T_\mathbf{1}\phi \cdot V + T_\mathbf{1}\phi \cdot \mathscr{P}_\mathbf{1}([X,Y](\mathbf{1})) \\ &= T_\mathbf{1}\phi \cdot V + x^a(\mathbf{1})y^b(\mathbf{1})T_\mathbf{1}\phi \cdot \mathscr{P}_\mathbf{1}([X_a, X_b](\mathbf{1})) \\ &\qquad + x^a(\mathbf{1})X_a[y^b](\mathbf{1})T_\mathbf{1}\phi \cdot X_b(\mathbf{1}) - y^b(\mathbf{1})X_b[x^a](\mathbf{1})T_\mathbf{1}\phi \cdot X_a(\mathbf{1}). \end{aligned}$$

Since X_a and X_b are left-invariant, we have $T_\mathbf{1}\phi \cdot \mathscr{P}_\mathbf{1}([X_a, X_b](\mathbf{1})) = \mathscr{P}'_\mathbf{1}(T_\mathbf{1}\phi \cdot [X_a, X_b](\mathbf{1}))$. Consequently,

$$\begin{aligned} T_\mathbf{1}\phi \cdot \mathscr{P}_\mathbf{1}(U) &= T_\mathbf{1}\phi \cdot V + \mathscr{P}'_\mathbf{1}(x^a(\mathbf{1})y^b(\mathbf{1})T_\mathbf{1}\phi \cdot [X_a, X_b](\mathbf{1})) \\ &\qquad + x^a(\mathbf{1})X_a[y^b](\mathbf{1})T_\mathbf{1}\phi \cdot X_b(\mathbf{1}) - y^b(\mathbf{1})X_b[x^a](\mathbf{1})T_\mathbf{1}\phi \cdot X_a(\mathbf{1}) \\ &= T_\mathbf{1}\phi \cdot V + \mathscr{P}'_\mathbf{1}(T_\mathbf{1}\phi \cdot [X,Y](\mathbf{1})) \\ &= \mathscr{P}'_\mathbf{1}(T_\mathbf{1}\phi \cdot U). \end{aligned}$$

That is, $T_\mathbf{1}\phi \cdot \mathscr{P}_\mathbf{1} = \mathscr{P}'_\mathbf{1} \circ T_\mathbf{1}\phi$. ■

Proposition 3.1.15 and proposition 3.1.16 go some way toward answering the question of uniqueness of left-invariant structures with Cartan–Schouten connections, at least as far as the uniqueness up to NH-geodesic equivalence or NH-affine equivalence is concerned. On the other hand, in chapter 4 (section 4.1.4 in particular) we shall find a number of examples of left-invariant nonholonomic Riemannian structures (on the same Lie group) with Cartan–Schouten connections, that are not NH-isometric. (In contrast to the question of uniqueness considered here, as we have seen in section 1.2.2, the question of existence of left-invariant structures with Cartan–Schouten connections has not been answered in much generality.)

3.2 Nonholonomic Riemannian embeddings

Let $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ and $(\mathsf{M}', \mathcal{D}', \mathcal{D}'^\perp, \mathbf{g}')$ be two nonholonomic Riemannian manifolds, where $\dim(\mathsf{M}) \leq \dim(\mathsf{M}')$. For clarity, we shall denote $[\![\cdot,\cdot]\!]_{\mathscr{P}} = \mathscr{P}([\cdot,\cdot])$ and $[\![\cdot,\cdot]\!]_{\mathscr{P}'} = \mathscr{P}'([\cdot,\cdot])$. Likewise, let $\langle\!\langle \cdot : \cdot \rangle\!\rangle_{\mathscr{P}}$ denote the symmetric bracket of ∇ and $\langle\!\langle \cdot : \cdot \rangle\!\rangle_{\mathscr{P}'}$ that of ∇'. An embedding $\iota : \mathsf{M} \to \mathsf{M}'$ (i.e., a smooth injective immersion that is a homeomorphism onto its image $\iota(\mathsf{M})$ in the subspace topology) is called a *nonholonomic Riemannian embedding* if

(i) $T_q\iota \cdot \mathcal{D}_q \subseteq \mathcal{D}'_{\iota(q)}$ for every $q \in \mathsf{M}$.

(ii) $\mathbf{g}_q = (\iota^*\mathbf{g}')_q|_{T_q\iota\cdot\mathcal{D}_q}$ for every $q \in \mathsf{M}$.

If such a map exists, then $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ is said to be a *nonholonomic Riemannian submanifold* of $(\mathsf{M}', \mathcal{D}', \mathcal{D}'^\perp, \mathbf{g}')$.

Remark 3.2.1. As in Riemannian geometry, one may define a notion of a *nonholonomic Riemannian immersion.* Since every immersion $\iota : \mathsf{M} \to \mathsf{M}'$ is locally an embedding, and the results of this section are all essentially local in nature, they are readily extended to the case of an immersion. □

Let $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ be a nonholonomic Riemannian submanifold of $(\mathsf{M}', \mathcal{D}', \mathcal{D}'^\perp, \mathbf{g}')$, with the embedding $\iota : \mathsf{M} \to \mathsf{M}'$. We shall identify $\iota(\mathsf{M})$ with M and $\mathcal{D}$ with $\bigsqcup_{q\in\mathsf{M}} T_q\iota\cdot\mathcal{D}_q$. Hence we treat $\mathcal{D}$ as a subbundle of (the pullback bundle) $\mathcal{D}'|_{\mathsf{M}} = \iota^*\mathcal{D}'$. Condition ($ii$) of the definition above may now be written more simply as $\mathbf{g} = (\iota^*\mathbf{g}')|_{\mathcal{D}}$. For convenience, we shall also write $\mathbf{g}'$ for the metric $\iota^*\mathbf{g}'$ on $\mathcal{D}'|_{\mathsf{M}}$.

Let $\mathcal{N}$ be the orthogonal complement of $\mathcal{D}$ with respect to $\mathbf{g}'$ in $\mathcal{D}'|_{\mathsf{M}}$, i.e., $\mathcal{D}'|_{\mathsf{M}} = \mathcal{D} \oplus \mathcal{N}$. We call $\mathcal{N}$ the *normal bundle.* If $X_q \in \mathcal{D}'_q$, $q \in \mathsf{M}$, then we shall write $X_q^\top \in \mathcal{D}_q$ for the tangential part of X_q and $X_q^\perp \in \mathcal{N}_q$ for the normal part.

Any (local) vector field on M' restricts to a (local) vector field on M. Conversely, every vector field on M may be smoothly extended to a vector field on M'. That is, if $Z \in \Gamma(T\mathsf{M})$ is defined on $\mathcal{U} \subseteq \mathsf{M}$, then there exists a smooth extension of Z to a neighbourhood $\mathcal{U}'$ of $\mathcal{U}$ in M', which we shall also denote Z. Let $X, Y \in \Gamma(\mathcal{D})$ be local vector fields. By the preceding argument, the expression $\nabla'_X Y$ is defined. Furthermore, by lemma B.1.4 and lemma B.1.5, it is clear that $\nabla'_X Y$ is independent of the choice of extensions of X and Y.

Lemma 3.2.2. *If* $Z \in \Gamma(T\mathsf{M})$*, then* $\mathscr{P}(Z) = \mathscr{P}'(Z)^\top$ *and* $\mathscr{Q}(Z) = \mathscr{Q}'(Z) + \mathscr{P}'(Z)^\perp$.

Proof. Let $Z \in \Gamma(T\mathsf{M})$. That $\mathscr{P}(Z) = \mathscr{P}'(Z)^\top$ is obvious. For the second part, we have

$$\mathscr{P}(Z) + \mathscr{Q}(Z) = Z = \mathscr{P}'(Z) + \mathscr{Q}'(Z) = \mathscr{P}'(Z)^\perp + \mathscr{P}'(Z)^\top + \mathscr{Q}'(Z),$$

and hence $\mathscr{Q}(Z) = \mathscr{Q}'(Z) + \mathscr{P}'(Z)^\perp$. ■

Since $\mathcal{D}'|_{\mathsf{M}} = \mathcal{D} \oplus \mathcal{N}$, we have the decomposition

$$\nabla'_X Y = (\nabla'_X Y)^\top + (\nabla'_X Y)^\perp, \qquad X, Y \in \Gamma(\mathcal{D}).$$

In fact, it turns out that the tangential part of ∇' (when ∇' is restricted to elements in $\Gamma(\mathcal{D})$) is exactly the nonholonomic connection of $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$.

Lemma 3.2.3. *We have* $\nabla_X Y = (\nabla'_X Y)^\top$ *for every* $X, Y \in \Gamma(\mathcal{D})$.

Proof. Let $X, Y, Z \in \Gamma(\mathcal{D})$. The operation $(X, Y) \mapsto (\nabla'_X Y)^\top$ is clearly an affine connection (specifically, a nonholonomic, or $\mathcal{D}$-restricted, connection on $\mathcal{D}$), i.e., tensorial in the first argument and a derivation in the second. We have $(\nabla'_X Y)^\top - (\nabla'_Y X)^\top = [\![X, Y]\!]^\top_{\mathscr{P}'} = [\![X, Y]\!]_{\mathscr{P}}$ and

$$\mathsf{g}'((\nabla'_Z X)^\top, Y) + \mathsf{g}'(X, (\nabla'_Z Y)^\top) = \mathsf{g}(\nabla'_Z X, Y) + \mathsf{g}(X, \nabla'_Z Y) = Z[\mathsf{g}(X, Y)],$$

i.e., $(X, Y) \mapsto (\nabla'_X Y)^\top$ is metric and torsion free. By uniqueness of the nonholonomic connection (theorem 1.1.5), it follows that the connections $(X, Y) \mapsto (\nabla'_X Y)^\top$ and ∇ are identical. ■

The *second fundamental form* $\mathrm{II} : \Gamma(\mathcal{D}) \times \Gamma(\mathcal{D}) \to \Gamma(\mathcal{N})$ is defined to be the normal component of ∇' (when evaluated on sections of $\mathcal{D}$), i.e.,

$$\mathrm{II}(X, Y) = \nabla'_X Y - \nabla_X Y = (\nabla'_X Y)^\perp, \qquad X, Y \in \Gamma(\mathcal{D}).$$

Accordingly, we have the decomposition $\nabla'_X Y = \nabla_X Y + \mathrm{II}(X, Y)$, called the "Gauss formula" in Riemannian geometry.

Lemma 3.2.4. *The second fundamental form is tensorial in both arguments.* (*In particular,* $\mathrm{II}(X, Y)$ *does not depend on the extensions of* X *and* Y.) *Furthermore,*

$$\mathrm{II}(X, Y) - \mathrm{II}(Y, X) = [\![X, Y]\!]^\perp_{\mathscr{P}'} \quad \text{and} \quad \mathrm{II}(X, Y) + \mathrm{II}(Y, X) = \langle\!\langle X : Y \rangle\!\rangle^\perp_{\mathscr{P}'}$$

for every $X, Y \in \Gamma(\mathcal{D})$.

Proof. Let $f \in \mathcal{C}^\infty(\mathsf{M})$ and $X, Y \in \Gamma(\mathcal{D})$. As $\nabla'_X Y$ is tensorial in X, it is clear that $\mathrm{II}(fX, Y) = f\,\mathrm{II}(X, Y)$. Furthermore, $\mathrm{II}(X, fY) = (X[f]Y)^\perp + f\,\mathrm{II}(X, Y)$. Since $Y \in \Gamma(\mathcal{D})$, it follows that $Y^\perp = 0$, and so $\mathrm{II}(X, fY) = f\,\mathrm{II}(X, Y)$. Hence II is tensorial in both arguments. Lastly, we have $\mathrm{II}(X, Y) - \mathrm{II}(Y, X) = (\nabla'_X Y - \nabla'_Y X)^\perp = [\![X, Y]\!]^\perp_{\mathscr{P}'}$ and $\mathrm{II}(X, Y) + \mathrm{II}(Y, X) = (\nabla'_X Y + \nabla'_Y X)^\perp = \langle\!\langle X : Y \rangle\!\rangle^\perp_{\mathscr{P}'}$. ■

For the special case of a Riemannian embedding (when $\mathcal{D} = T\mathsf{M}$, $\mathcal{D}^\perp = \{0\}$ and $\mathcal{D}' = T\mathsf{M}'$, $\mathcal{D}'^\perp = \{0\}$), we have $[\![X, Y]\!]^\perp_{\mathscr{P}'} = [X, Y]^\perp$. Since $X, Y \in \Gamma(\mathcal{D}) = \Gamma(T\mathsf{M})$, it follows that $[X, Y] \in \Gamma(T\mathsf{M})$, and so $[X, Y]^\perp = 0$. Hence, in this case, the second fundamental form is symmetric.

In the following two sections (sections 3.2.1 and 3.2.2) we study the relation between the nonholonomic geodesics of $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathsf{g})$ and $(\mathsf{M}', \mathcal{D}', \mathcal{D}'^\perp, \mathsf{g}')$ as well as the relation between the sub-Riemannian geodesics of the associated sub-Riemannian structures $(\mathsf{M}, \mathcal{D}, \mathsf{g})$ and $(\mathsf{M}', \mathcal{D}', \mathsf{g}')$.

3.2.1 Geodesic invariance

Let $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathsf{g})$ be a nonholonomic Riemannian submanifold of $(\mathsf{M}', \mathcal{D}', \mathcal{D}'^\perp, \mathsf{g}')$. Since $\nabla'_{\dot\gamma}\dot\gamma = \nabla_{\dot\gamma}\dot\gamma + \mathrm{II}(\dot\gamma, \dot\gamma)$ for a $\mathcal{D}$-curve γ, the proof of the following result is immediate.

Proposition 3.2.5. *A* $\mathcal{D}$*-curve* γ *in* M *is a nonholonomic geodesic of both* $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathsf{g})$ *and* $(\mathsf{M}', \mathcal{D}', \mathcal{D}'^\perp, \mathsf{g}')$ *if and only if* $\mathrm{II}(\dot\gamma, \dot\gamma)$ *vanishes identically.*

Corollary 3.2.6. *If* $\gamma : [0,1] \to \mathsf{M}$ *is a nonholonomic geodesic of* $(\mathsf{M}', \mathcal{D}', \mathcal{D}'^{\perp}, \mathbf{g}')$ *tangent to* $\mathcal{D}$*, then it is a nonholonomic geodesic of* $(\mathsf{M}, \mathcal{D}, \mathcal{D}^{\perp}, \mathbf{g})$.

Proof. Let $\gamma : [0,1] \to \mathsf{M}$ be a geodesic of ∇', i.e., $\nabla'_{\dot\gamma}\dot\gamma(t) = 0$ for every $t \in [0,1]$. Then $\mathrm{II}(\dot\gamma, \dot\gamma) = (\nabla'_{\dot\gamma}\dot\gamma)^{\perp}$ is identically zero. ■

Proposition 3.2.5 will allow us to characterise when the set of nonholonomic geodesics of $(\mathsf{M}, \mathcal{D}, \mathcal{D}^{\perp}, \mathbf{g})$ coincide with the set of nonholonomic geodesics of $(\mathsf{M}', \mathcal{D}', \mathcal{D}'^{\perp}, \mathbf{g}')$ that are tangent to $\mathcal{D}$. The characterisation involves the notion of a "geodesically invariant subbundle." A vector subbundle $\mathcal{S} \subsetneq \mathcal{D}$ is said to be *geodesically invariant in* $\mathcal{D}$ (cf. [46, 3]) if, for every nonholonomic geodesic $\gamma : [0,1] \to \mathsf{M}$ of $(\mathsf{M}, \mathcal{D}, \mathcal{D}^{\perp}, \mathbf{g})$ such that $\dot\gamma(0) \in \mathcal{S}_{\gamma(0)}$, we have $\dot\gamma(t) \in \mathcal{S}_{\gamma(t)}$ for every $t \in [0,1]$.

Proposition 3.2.7 (cf. [46, 3]). *The following statements are equivalent*:

(*i*) $\mathcal{S}$ *is geodesically invariant in* $\mathcal{D}$.

(*ii*) $\mathcal{S}$ *is preserved by the nonholonomic geodesic flow* Φ_t*, i.e.,* $\Phi_t(\mathcal{S}) = \mathcal{S}$.

(*iii*) *The restriction* $\Xi|_{\mathcal{S}}$ *of the nonholonomic geodesic spray* Ξ *of* $(\mathsf{M}, \mathcal{D}, \mathcal{D}^{\perp}, \mathbf{g})$ *is tangent to* $\mathcal{S}$*, i.e.,* $\Xi|_{\mathcal{S}} \in \Gamma(T\mathcal{S})$.

(*iv*) $\mathcal{S}$ *is invariant under parallel translation along nonholonomic geodesics with initial velocity in* $\mathcal{S}$*, i.e.,* $\Pi^t_{\gamma}(\mathcal{S}_{\gamma(0)}) = \mathcal{S}_{\gamma(t)}$ *for every nonholonomic geodesic* $\gamma : [0,1] \to \mathsf{M}$ *such that* $\dot\gamma(0) \in \mathcal{S}_{\gamma(0)}$.

(*v*) $\nabla_X X \in \Gamma(\mathcal{S})$ *for every* $X \in \Gamma(\mathcal{S})$.

(*vi*) $\langle\!\langle X : Y \rangle\!\rangle_{\mathscr{P}} \in \Gamma(\mathcal{S})$ *for every* $X, Y \in \Gamma(\mathcal{S})$.

Proof. If $\gamma : [0,1] \to \mathsf{M}$ is a nonholonomic geodesic, then $\dot\gamma(t) = \Phi_t(\dot\gamma(0))$. Hence it is clear that $\mathcal{S}$ is geodesically invariant in $\mathcal{D}$ if and only if it is preserved by Φ_t. Furthermore, $\Phi_t(\mathcal{S}) = \mathcal{S}$ is also clearly equivalent to $\Xi|_{\mathcal{S}}$ being tangent to $\mathcal{S}$. The first three items are thus equivalent. Furthermore, by polarisation it is evident that items (*v*) and (*vi*) are equivalent.

We now show that (*i*) ⇒ (*iv*) ⇒ (*v*) ⇒ (*i*). Suppose $\mathcal{S}$ is geodesically invariant in $\mathcal{D}$. Let $X_q \in \mathcal{S}$ and let $\gamma : [0,1] \to \mathsf{M}$ be the unique nonholonomic geodesic such that $\gamma(0) = q$ and $\dot\gamma(0) = X_q$. As γ is a nonholonomic geodesic, it is invariant under parallel translation along γ (see section B.1.1), i.e., $\dot\gamma(t) = \Pi^t_{\gamma}(\dot\gamma(0))$ for every $t \in [0,1]$. Thus $\Pi^t_{\gamma}(X_q) = \dot\gamma(t) \in \mathcal{S}_{\gamma(t)}$. Since Π^t_{γ} is a linear isomorphism, it follows that $\Pi^t_{\gamma}(\mathcal{S}_{\gamma(0)}) = \mathcal{S}_{\gamma(t)}$.

Suppose that $\mathcal{S}$ is invariant under parallel translation along nonholonomic geodesics with initial velocity in $\mathcal{S}$. Let $X \in \Gamma(\mathcal{S})$, $q \in \mathsf{M}$ and let $\gamma : [0,1] \to \mathsf{M}$ be the unique nonholonomic geodesic such that $\gamma(0) = q$ and $\dot\gamma(0) = X(q)$. By lemma B.1.5, the expression $\nabla_X X(q)$ depends only on the values of X along any curve tangent to $X(q)$. Consequently, using corollary B.1.12, we have

$$\nabla_X X(q) = \nabla_{\dot\gamma}(X \circ \gamma)(0) = \lim_{t \to 0} \frac{\Pi^{-t}_{\gamma}(X(\gamma(t))) - X(q)}{t} \in \mathcal{S}_q.$$

That is, $\nabla_X X \in \Gamma(\mathcal{S})$.

Suppose that $\nabla_X X \in \Gamma(\mathcal{S})$ for every $X \in \Gamma(\mathcal{S})$ and let $Y \in \Gamma(\mathcal{S})$. We have that the curve $t \mapsto Y(q) + t\,\nabla_Y Y(q)$ is a curve in $\mathcal{S}_q$, and so

$$\mathrm{vl}_{Y(q)} \cdot \nabla_Y Y(q) = \left.\frac{d}{dt}\right|_{t=0} (Y(q) + t\,\nabla_Y Y(q)) \in T_{Y(q)}\mathcal{S}.$$

(Here $\mathrm{vl}_{Y(q)} \cdot \nabla_Y Y(q)$ is the vertical lift of $\nabla_Y Y(q)$ over $Y(q)$; see section B.2.) Since we have $T_qY \cdot Y(q) \in T_{Y(q)}\mathcal{S}$, it then follows that

$$\Xi(Y(q)) = h(Y(q), Y(q)) = T_qY \cdot Y(q) - \mathrm{vl}_{Y(q)} \cdot \nabla_{Y(q)}Y \in T_{Y(q)}\mathcal{S}.$$

That is, $\Xi|_{\mathcal{S}}$ is tangent to $\mathcal{S}$, which is equivalent to the geodesic invariance of $\mathcal{S}$ in $\mathcal{D}$. ■

The concept of a geodesically invariant vector subbundle is a natural generalisation of the notion of a totally geodesic submanifold in Riemannian geometry (which can itself be generalised to nonholonomic Riemannian geometry). We say that an immersed submanifold $\mathsf{N} \subseteq \mathsf{M}$ is *totally geodesic in* M if every nonholonomic geodesic $\gamma : [0,1] \to \mathsf{M}$ such that $\gamma(0) \in \mathsf{N}$ and $\dot\gamma(0) \in T_{\gamma(0)}\mathsf{N}$ lies entirely in N.

Proposition 3.2.8. *If $\mathcal{S}$ is integrable and geodesically invariant in $\mathcal{D}$, then the integral manifolds of $\mathcal{S}$ are totally geodesic in M. Conversely, if N is totally geodesic in M, then for every nonholonomic geodesic $\gamma : [0,1] \to \mathsf{N}$ such that $\dot\gamma(0) \in T_{\gamma(0)}\mathsf{N} \cap \mathcal{D}_{\gamma(0)}$, we have $\dot\gamma(t) \in T_{\gamma(t)}\mathsf{N} \cap \mathcal{D}_{\gamma(t)}$ for every $t \in [0,1]$.*

Proof. Suppose that $\mathcal{S}$ is integrable and geodesically invariant in $\mathcal{D}$. Let $q \in \mathsf{M}$ and let $\mathsf{N} \subsetneq \mathsf{M}$ be the integral manifold of $\mathcal{S}$ through q. Then $\mathcal{S}_p = T_p\mathsf{N}$ for every $p \in \mathsf{N}$. If $\gamma : [0,1] \to \mathsf{M}$ is a nonholonomic geodesic of $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbb{g})$ such that $\gamma(0) \in \mathsf{N}$ and $\dot\gamma(0) \in T_{\gamma(0)}\mathsf{N} = \mathcal{S}_{\gamma(0)}$, then $\dot\gamma(t) \in \mathcal{S}_{\gamma(t)} = T_{\gamma(t)}\mathsf{N}$ for every $t \in [0,1]$. It follows that $\gamma(t) \in \mathsf{N}$ for every $t \in [0,1]$, i.e., N is totally geodesic in M.

Conversely, let N be totally geodesic in M and let $\mathcal{S} = T\mathsf{N} \cap \mathcal{D}$. (We note that, in general $\mathcal{S}$ is not regular, i.e., the dimension of the fibres of $\mathcal{S}$ might depend on the base point.) Let $\gamma : [0,1] \to \mathsf{M}$ be a nonholonomic geodesic of $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbb{g})$ such that $\gamma(0) \in \mathsf{N}$ and $\dot\gamma(0) \in \mathcal{S}_{\gamma(0)}$. By the assumption that N is totally geodesic, we have that γ is a $\mathcal{D}$-curve lying entirely in N. It follows that $\dot\gamma(t) \in T_{\gamma(t)}\mathsf{N} \cap \mathcal{D}_{\gamma(t)}$ for every $t \in [0,1]$. ■

We return now to characterising when the nonholonomic geodesics of the embedded structure coincide with those of the ambient structure that are tangent to $\mathcal{D}$.

Proposition 3.2.9. *The following statements are equivalent:*

(*i*) *$\mathcal{D}$ is geodesically invariant in $\mathcal{D}'|_{\mathsf{M}}$, i.e., for every nonholonomic geodesic $\gamma : [0,1] \to \mathsf{M}$ of $(\mathsf{M}', \mathcal{D}', \mathcal{D}'^\perp, \mathbb{g}')$ such that $\dot\gamma(0) \in \mathcal{D}_{\gamma(0)}$, we have $\dot\gamma(t) \in \mathcal{D}_{\gamma(t)}$ for every $t \in [0,1]$.*

(*ii*) *The set of nonholonomic geodesics of $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbb{g})$ coincides with the set of nonholonomic geodesics of $(\mathsf{M}', \mathcal{D}', \mathcal{D}'^\perp, \mathbb{g}')$ that lie in M and are tangent to $\mathcal{D}$.*

(*iii*) *The second fundamental form II is skew-symmetric.*

(*iv*) *$\langle\!\langle X : Y \rangle\!\rangle_{\mathscr{P}'} \in \Gamma(\mathcal{D})$ for every $X, Y \in \Gamma(\mathcal{D})$.*

(*v*) *$\mathrm{II}(X,Y) = \frac{1}{2}[\![X,Y]\!]^{\perp}_{\mathscr{P}'}$ for every $X, Y \in \Gamma(\mathcal{D})$.*

Proof. Suppose that $\mathcal{D}$ is geodesically invariant in $\mathcal{D}'|_{\mathsf{M}}$. Consequently, by proposition 3.2.7, we have $\nabla'_X X \in \Gamma(\mathcal{D})$ for every $X \in \Gamma(\mathcal{D})$, and so, if γ is a $\mathcal{D}$-curve in M, then $\nabla'_{\dot\gamma}\dot\gamma = \nabla_{\dot\gamma}\dot\gamma$. Hence it is clear that a $\mathcal{D}$-curve in M is a nonholonomic geodesic of $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ if and only if it is a nonholonomic geodesic of $(\mathsf{M}', \mathcal{D}', \mathcal{D}'^\perp, \mathbf{g}')$. If (*ii*) holds, then proposition 3.2.5 implies that $\mathrm{II}(X, X) = 0$ for every $X \in \Gamma(\mathcal{D})$, i.e., II is skew-symmetric. If the second fundamental form is skew-symmetric, it follows from lemma 3.2.4 that $\langle\!\langle X : Y\rangle\!\rangle^\perp_{\mathscr{P}'} = 0$, and hence $\langle\!\langle X : Y\rangle\!\rangle_{\mathscr{P}'} \in \Gamma(\mathcal{D})$ for every $X, Y \in \Gamma(\mathcal{D})$. If (*iv*) holds, then (again by lemma 3.2.4) we have $\mathrm{II}(X, Y) = \frac{1}{2}[\![X, Y]\!]^\perp_{\mathscr{P}'}$. Lastly, suppose that II is given as in item (*v*). If $X \in \Gamma(\mathcal{D})$, then

$$\nabla'_X X = \nabla_X X + \mathrm{II}(X, X) = \nabla_X X \in \Gamma(\mathcal{D}).$$

Hence, by proposition 3.2.7, $\mathcal{D}$ is geodesically invariant in $\mathcal{D}'|_{\mathsf{M}}$. ■

This leads us to further a characterisation of geodesic invariance in terms of the $\mathscr{P}$-Lie derivative of $\mathbf{g}$ (see section A.3.1).

Proposition 3.2.10. *$\mathcal{D}$ is geodesically invariant in $\mathcal{D}'|_{\mathsf{M}}$ if and only if $\mathcal{L}^{\mathscr{P}}_V \mathbf{g} \equiv 0$ for every $V \in \Gamma(\mathcal{N})$.*

Proof. Suppose $\mathcal{D}$ is geodesically invariant in $\mathcal{D}'|_{\mathsf{M}}$. Then, by proposition 3.2.9, we have that II is skew-symmetric, and so $(\nabla'_X X)^\top = \mathrm{II}(X, X) = 0$ for every $X \in \Gamma(\mathcal{D})$. Consequently, if $X \in \Gamma(\mathcal{D})$ and $V \in \Gamma(\mathcal{N})$, then

$$0 = \mathbf{g}'((\nabla'_X X)^\top, V) = \mathbf{g}'(\nabla'_X X, V) = -\mathbf{g}'(X, \nabla'_X V).$$

Using the fact that ∇' is metric and torsion free (theorem 1.1.5), we get

$$\begin{aligned} 0 &= -\mathbf{g}'(X, \nabla'_V X + [\![X, V]\!]_{\mathscr{P}'}) \\ &= -\frac{1}{2} V[\mathbf{g}(X, X)] - \mathbf{g}(X, [\![X, V]\!]_{\mathscr{P}}). \end{aligned}$$

That is, $(\mathcal{L}^{\mathscr{P}}_V \mathbf{g})(X, X) = V[\mathbf{g}(X, X)] + 2\,\mathbf{g}(X, [\![X, V]\!]_{\mathscr{P}}) = 0$. Since $\mathcal{L}^{\mathscr{P}}_V \mathbf{g}$ is symmetric, it is determined by its values on the diagonal, and hence $\mathcal{L}^{\mathscr{P}}_V \mathbf{g} \equiv 0$ for every $V \in \Gamma(\mathcal{N})$.

Conversely, suppose that $\mathcal{L}^{\mathscr{P}}_V \mathbf{g} \equiv 0$ for every $V \in \Gamma(\mathcal{N})$. A similar calculation to that above yields $\mathrm{II}(X_a, X_a) = 0$, where (X_a) is an orthonormal frame for $\mathcal{D}$. It follows that II is skew-symmetric, and so $\mathcal{D}$ is geodesically invariant in $\mathcal{D}'|_{\mathsf{M}}$. ■

Corollary 3.2.11 (cf. [67]). *If there exists an orthonormal frame (X_a) for $\mathcal{D}$ such that $[X_a, \Gamma(\mathcal{N})] \subseteq \Gamma(\mathcal{N})$, then $\mathcal{D}$ is geodesically invariant in $\mathcal{D}'|_{\mathsf{M}}$.*

Proof. Let (X_a) be an orthonormal frame for $\mathcal{D}$ such that $[X_a, \Gamma(\mathcal{N})] \subseteq \Gamma(\mathcal{N})$. If $V \in \Gamma(\mathcal{N})$, then

$$(\mathcal{L}^{\mathscr{P}}_V \mathbf{g})(X_a, X_a) = V[\mathbf{g}(X_a, X_a)] + 2\,\mathbf{g}(X_a, [\![X_a, V]\!]) = 0.$$

As $\mathcal{L}^{\mathscr{P}}_V \mathbf{g}$ is tensorial in both arguments, it follows that $\mathcal{L}^{\mathscr{P}}_V \mathbf{g} \equiv 0$ for every $V \in \Gamma(\mathcal{N})$, and so $\mathcal{D}$ is geodesically invariant in $\mathcal{D}'|_{\mathsf{M}}$. ■

Note that the condition $[X_a, \Gamma(\mathcal{N})] \subseteq \Gamma(\mathcal{N})$ may be equivalently stated as follows: the flow φ_t of X_a preserves $\mathcal{N}$, i.e., $(\varphi_t)_*\mathcal{N} = \mathcal{N}$.

3.2.2 Sub-Riemannian geodesics

Let $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ be a nonholonomic Riemannian submanifold of $(\mathsf{M}', \mathcal{D}', \mathcal{D}'^\perp, \mathbf{g}')$. Associated to $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ and $(\mathsf{M}', \mathcal{D}', \mathcal{D}'^\perp, \mathbf{g}')$ are the sub-Riemannian manifolds $(\mathsf{M}, \mathcal{D}, \mathbf{g})$ and $(\mathsf{M}', \mathcal{D}', \mathbf{g}')$, respectively (see section 1.1.5). In this section we show that the normal sub-Riemannian geodesics of $(\mathsf{M}', \mathcal{D}', \mathbf{g}')$ that are tangent to $\mathcal{D}$ are also (normal) sub-Riemannian geodesics of $(\mathsf{M}, \mathcal{D}, \mathbf{g})$. We shall use this result later to prove a sufficient condition for the nonholonomic geodesics of $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ to be normal sub-Riemannian geodesics of $(\mathsf{M}, \mathcal{D}, \mathbf{g})$.

Let d_{M} and $d_{\mathsf{M}'}$ denote the Carnot–Carathéodory metrics of $(\mathsf{M}, \mathcal{D}, \mathbf{g})$ and $(\mathsf{M}', \mathcal{D}', \mathbf{g}')$, respectively. Since the class of $\mathcal{D}'$-curves is larger than the class of $\mathcal{D}$-curves, we have $d_{\mathsf{M}'}|_{\mathsf{M}\times\mathsf{M}} \leq d_{\mathsf{M}}$.

Proposition 3.2.12. *Let $p, q \in \mathsf{M}$ and let $\gamma : [0,1] \to \mathsf{M}$ be a $\mathcal{D}$-curve such that $\gamma(0) = p$ and $\gamma(1) = q$. If γ is a length-minimising curve of $(\mathsf{M}', \mathcal{D}', \mathbf{g}')$ between p and q, then it is a length-minimising curve of $(\mathsf{M}, \mathcal{D}, \mathbf{g})$ between the same points.*

Proof. Suppose otherwise, i.e., there exists another curve $\widetilde{\gamma} : [0,1] \to \mathsf{M}$ joining p to q such that $d_{\mathsf{M}}(p,q) = \operatorname{length}(\widetilde{\gamma})$ and $\operatorname{length}(\widetilde{\gamma}) < \operatorname{length}(\gamma)$. Then

$$d_{\mathsf{M}'}(p,q) = \operatorname{length}(\gamma) > \operatorname{length}(\widetilde{\gamma}) = d_{\mathsf{M}}(p,q),$$

a contradiction, since $d_{\mathsf{M}'}|_{\mathsf{M}\times\mathsf{M}} \leq d_{\mathsf{M}}$. ■

Corollary 3.2.13. *If $\gamma : [0,1] \to \mathsf{M}$ is a normal sub-Riemannian geodesic of $(\mathsf{M}', \mathcal{D}', \mathbf{g}')$ tangent to $\mathcal{D}$, then it is a (normal) sub-Riemannian geodesic of $(\mathsf{M}, \mathcal{D}, \mathbf{g})$.*

Proof. Let $\gamma : [0,1] \to \mathsf{M}$ be a normal sub-Riemannian geodesic of $(\mathsf{M}', \mathcal{D}', \mathbf{g}')$; then γ is locally length-minimising, i.e., for every sufficiently small interval $[t_1, t_2] \subseteq [0,1]$ we have that $\gamma|_{[t_1,t_2]}$ is a length minimiser of $(\mathsf{M}', \mathcal{D}', \mathbf{g}')$ between $\gamma(t_1)$ and $\gamma(t_2)$. Furthermore, $\gamma|_{[t_1,t_2]}$ is tangent to $\mathcal{D}$ by assumption. Therefore, by proposition 3.2.12, we have that $\gamma|_{[t_1,t_2]}$ is a length minimiser of $(\mathsf{M}, \mathcal{D}, \mathbf{g})$, and hence γ is a (normal) sub-Riemannian geodesic of $(\mathsf{M}, \mathcal{D}, \mathbf{g})$. ■

3.2.3 Embeddings into a Riemannian manifold

Let $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ be a nonholonomic Riemannian manifold and let $\widetilde{\mathbf{g}}$ be an extension of $\mathbf{g}$ to a Riemannian metric on M such that $\mathcal{D} \perp_{\widetilde{\mathbf{g}}} \mathcal{D}^\perp$. Clearly $\iota = \mathrm{id}_{\mathsf{M}}$ is a nonholonomic Riemannian embedding of $(\mathsf{M}, \mathcal{D}, \mathcal{D}', \mathbf{g})$ into $(\mathsf{M}, \widetilde{\mathbf{g}})$, i.e., we have that $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ is a nonholonomic Riemannian submanifold of $(\mathsf{M}, \widetilde{\mathbf{g}})$. For this embedding, the normal bundle $\mathcal{N}$ is simply $\mathcal{D}^\perp$; hence $X_q^\top = \mathscr{P}(X_q)$ and $X_q^\perp = \mathscr{Q}(X_q)$ for $X_q \in T\mathsf{M}$. In particular, the second fundamental form is given by $\mathrm{II}(X,Y) = \mathscr{Q}(\widetilde{\nabla}_X Y)$ for $X, Y \in \Gamma(\mathcal{D})$, where $\widetilde{\nabla}$ is the Levi-Civita connection of $(\mathsf{M}, \widetilde{\mathbf{g}})$.

By definition, $\mathcal{D}$ is a vector subbundle of $T\mathsf{M}$, and so we may consider when it is geodesically invariant in $T\mathsf{M}$. In this case we shall simply say that $\mathcal{D}$ is *geodesically invariant.* The following result is an immediate consequence of proposition 3.2.9.

Proposition 3.2.14. *The set of nonholonomic geodesics of $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ coincides with the set of Riemannian geodesics of $(\mathsf{M}, \widetilde{\mathbf{g}})$ that are tangent to $\mathcal{D}$ if and only if $\mathcal{D}$ is geodesically invariant.*

In particular, when $\mathcal{D}$ is geodesically invariant, the study of the nonholonomic geodesics of $(\mathsf{M},\mathcal{D},\mathcal{D}^\perp,\mathbf{g})$ essentially reduces to the study of a subclass of Riemannian geodesics of $(\mathsf{M},\widetilde{\mathbf{g}})$. Furthermore, this implies that every nonholonomic geodesic is also a sub-Riemannian geodesic.

Proposition 3.2.15. *If $\mathcal{D}$ is geodesically invariant, then every nonholonomic geodesic of $(\mathsf{M},\mathcal{D},\mathcal{D}^\perp,\mathbf{g})$ is a (normal) sub-Riemannian geodesic of $(\mathsf{M},\mathcal{D},\mathbf{g})$.*

Proof. Suppose $\mathcal{D}$ is geodesically invariant. If γ is a nonholonomic geodesic, then by proposition 3.2.14 it is a Riemannian geodesic of $(\mathsf{M},\widetilde{\mathbf{g}})$. Hence, using corollary 3.2.13, it follows that γ is also a normal sub-Riemannian geodesic of $(\mathsf{M},\mathcal{D},\mathbf{g})$. ■

Since $\mathcal{N}=\mathcal{D}^\perp$, proposition 3.2.10 takes the form:

$\mathcal{D}$ is geodesically invariant if and only if $\mathcal{L}^{\mathscr{P}}_V\mathbf{g}\equiv 0$ for every $V\in\Gamma(\mathcal{D}^\perp)$.

In particular, this implies that whether $\mathcal{D}$ is geodesically invariant or not is a property of the original structure $(\mathsf{M},\mathcal{D},\mathcal{D}^\perp,\mathbf{g})$, and not of the extension $\widetilde{\mathbf{g}}$. Hence we cannot hope to choose a particular extension $\widetilde{\mathbf{g}}$ of $\mathbf{g}$ such that $\mathcal{D}$ is geodesically invariant with respect to the geodesics of $\widetilde{\mathbf{g}}$, unless $(\mathsf{M},\mathcal{D},\mathcal{D}^\perp,\mathbf{g})$ satisfies the above condition (in which case *any* extension of $\mathbf{g}$ for which $\mathcal{D}$ and $\mathcal{D}^\perp$ are orthogonal will suffice).

In light of proposition 2.1.5, it is clear (at least when $\mathcal{D}$ is strongly nonholonomic) that the curvature tensor C measures the geodesic invariance of $\mathcal{D}$. We have the following result.

Theorem 3.2.16. *If $\mathcal{D}$ is strongly nonholonomic, then $\mathcal{D}$ is geodesically invariant if and only if $C\equiv 0$.*

Proof. Suppose that $\mathcal{D}$ is strongly nonholonomic; then $T\mathsf{M}=\mathcal{D}^2=\mathcal{D}+[\mathcal{D},\mathcal{D}]$, whence $\mathcal{D}^\perp=\mathscr{Q}([\mathcal{D},\mathcal{D}])$. From proposition 2.1.5, we then have $C\equiv 0$ if and only if $\mathcal{L}^{\mathscr{P}}_V\mathbf{g}\equiv 0$ for every $V\in\Gamma(\mathcal{D}^\perp)$. By proposition 3.2.10, the latter condition is equivalent to the geodesic invariance of $\mathcal{D}$. ■

The assumption that $\mathcal{D}$ is strongly nonholonomic is crucial for the preceding result. However, using the Wagner curvature tensor, we can obtain a similar result to theorem 3.2.16 for Wagner structures.

Let $(\mathsf{M},\mathcal{D},\mathcal{D}^\perp,\mathbf{g})$ be a Wagner structure (see section 2.2) with degree of nonholonomy $N\geq 2$. Fix $1<k\leq N$ and let $\widetilde{\mathbf{g}}$ be an extension of $\mathbf{g}$ to a metric on $\mathcal{D}^k$ such that $\mathcal{D},\mathcal{E}^1,\ldots,\mathcal{E}^{k-1}$ are mutually orthogonal with respect to $\widetilde{\mathbf{g}}$. (As above, we shall see that the choice of a particular such extension is irrelevant.)

We have that $\iota=\mathrm{id}_{\mathsf{M}}$ is an nonholonomic Riemannian embedding of $(\mathsf{M},\mathcal{D},\mathcal{D}^\perp,\mathbf{g})$ into $(\mathsf{M},\mathcal{D}^k,\mathcal{D}^{k\perp},\widetilde{\mathbf{g}})$, where $\mathcal{D}^{k\perp}=\mathcal{E}^k\oplus\cdots\oplus\mathcal{E}^{N-1}$. Since $\mathcal{D}^k=\mathcal{D}\oplus\mathcal{E}^1\oplus\cdots\oplus\mathcal{E}^{k-1}$, it follows that $\mathcal{N}=\mathcal{E}^1\oplus\cdots\oplus\mathcal{E}^{k-1}$. Accordingly, proposition 3.2.10 takes the following form.

Proposition 3.2.17. *$\mathcal{D}$ is geodesically invariant in $\mathcal{D}^k$ if and only if $\mathcal{L}^{\mathscr{P}}_{V_i}\mathbf{g}\equiv 0$ for every $V_i\in\Gamma(\mathcal{E}^i)$ and each $i=1,\ldots,k-1$.*

From this result it is clear that the particular choice of extension $\widetilde{\mathbf{g}}$ is irrelevant, so long as the distributions $\mathcal{D},\mathcal{E}^1,\ldots,\mathcal{E}^{k-1}$ are mutually orthogonal with respect to $\widetilde{\mathbf{g}}$.

We shall now generalise theorem 3.2.16 to the case of distributions with arbitrary degree of nonholonomy (so long as the structure is a Wagner structure). Associated to each connection ∇^i is the curvature tensor $K^i \in \mathcal{T}_2^0(\mathcal{D}^i) \otimes \mathcal{T}_1^1(\mathcal{D})$ (see section 2.2). We decompose these tensors into a "Riemannian" (i.e., R-like) component and a remainder (a C-like component). Let $\widehat{K}^i \in \mathcal{T}_2^0(\mathcal{D}^i) \otimes \mathcal{T}_2^0(\mathcal{D})$ be the associated $(0,4)$-tensor

$$\widehat{K}^i(X,Y,U,V) = \mathbf{g}(K^i(X,Y)U,V), \qquad X,Y \in \Gamma(\mathcal{D}^i),\ U,V \in \Gamma(\mathcal{D}).$$

Define $\widehat{R}^i, \widehat{C}^i \in \mathcal{T}_2^0(\mathcal{D}^i) \otimes \mathcal{T}_2^0(\mathcal{D})$ as

$$\widehat{R}^i(X,Y,U,V) = \frac{1}{2}[\widehat{K}^i(X,Y,U,V) - \widehat{K}^i(X,Y,V,U)], \quad \widehat{C}^i = \widehat{K}^i - \widehat{R}^i.$$

Lastly, let R^i and C^i be the associated $(1,3)$-tensors given by $R^i(X,Y)U = \mathbf{g}^\sharp(\widehat{R}^i(X,Y,U,\,\cdot\,))$ and $C^i(X,Y)U = \mathbf{g}^\sharp(\widehat{C}^i(X,Y,U,\,\cdot\,))$. Evidently, we have $K^i = R^i + C^i$.

Theorem 3.2.18. *$\mathcal{D}$ is geodesically invariant in $\mathcal{D}^k$ if and only if $C^i \equiv 0$ for each $i = 1,\dots,k-1$.*

Proof. We have $C^i \equiv 0$ if and only if $\widehat{K}^i(X,Y,U,U) = 0$ for every $X,Y \in \Gamma(\mathcal{D}^i)$ and $U \in \Gamma(\mathcal{D})$, i.e.,

$$\mathbf{g}(\nabla^i_X \nabla^i_Y U - \nabla^i_Y \nabla^i_X U - \nabla^i_{\mathscr{P}_i([X,Y])} U - [\![\mathscr{Q}_i([X,Y]), U]\!], U) = 0.$$

Since ∇^i is metric (corollary 2.2.9), we have

$$\begin{aligned}\mathbf{g}(\nabla^i_X \nabla^i_Y U, U) &= X[\mathbf{g}(\nabla^i_Y U, U)] - \mathbf{g}(\nabla^i_Y U, \nabla^i_X U)\\ &= \frac{1}{2}X[Y[\mathbf{g}(U,U)]] - \mathbf{g}(\nabla^i_Y U, \nabla^i_X U), \text{ and}\\ \mathbf{g}(\nabla^i_Y \nabla^i_X U, U) &= \frac{1}{2}Y[X[\mathbf{g}(U,U)]] - \mathbf{g}(\nabla^i_X U, \nabla^i_Y U).\end{aligned}$$

Combining these calculations, we get $\mathbf{g}(\nabla^i_X \nabla^i_Y U - \nabla^i_Y \nabla^i_X U, U) = \frac{1}{2}[X,Y][\mathbf{g}(U,U)]$. Similarly, we have $\mathbf{g}(\nabla^i_{\mathscr{P}_i([X,Y])} U, U) = \frac{1}{2}\mathscr{P}_i([X,Y])[\mathbf{g}(U,U)]$. Accordingly,

$$\begin{aligned} C^i \equiv 0 \quad &\iff \quad \frac{1}{2}\mathscr{Q}_i([X,Y])[\mathbf{g}(U,U)] - \mathbf{g}([\![\mathscr{Q}_i([X,Y]), U]\!], U) = 0\\ &\qquad\quad \text{for every } X,Y \in \Gamma(\mathcal{D}^i),\ U \in \Gamma(\mathcal{D})\\ &\iff \quad \frac{1}{2}V[\mathbf{g}(U,U)] - \mathbf{g}([\![V,U]\!],U) = 0 \text{ for every } V \in \Gamma(\mathcal{E}^i),\ U \in \Gamma(\mathcal{D})\\ &\iff \quad (\mathcal{L}^{\mathscr{P}}_V \mathbf{g})(U,U) = 0 \text{ for every } V \in \Gamma(\mathcal{E}^i),\ U \in \Gamma(\mathcal{D})\\ &\iff \quad \mathcal{L}^{\mathscr{P}}_V \mathbf{g} \equiv 0 \text{ for every } V \in \Gamma(\mathcal{E}^i).\end{aligned}$$

(The last equivalence follows from the fact that $\mathcal{L}^{\mathscr{P}}_V \mathbf{g}$ is symmetric, hence determined by its values on the diagonal.) By proposition 3.2.17, the final condition occurs if and only if $\mathcal{D}$ is geodesically invariant in $\mathcal{D}^k$. ■

Chapter 4

Nonholonomic Riemannian structures in three dimensions

In this chapter we specialise our consideration of nonholonomic Riemannian structures to those on three-dimensional manifolds, and, in particular, to left-invariant structures on the three-dimensional simply connected Lie groups. The latter structures are the simplest prototypes of nonholonomic Riemannian structures (indeed, three is clearly the lowest dimension in which one can define a nonholonomic Riemannian structure); hence their study is a first step towards an understanding of the more general structures.

Section 4.1 addresses the equivalence, up to NH-isometry and (constant) rescaling, of three-dimensional nonholonomic Riemannian manifolds. In particular, we obtain a classification of all left-invariant nonholonomic Riemannian structures on three-dimensional simply connected Lie groups. Structures on three-dimensional manifolds with a nonholonomic distribution $\mathcal{D}$ are notable in that $\mathcal{D}$ may be described as the kernel of an intrinsic contact form. This contact form, together with the associated Reeb vector field, is central to our discussion. In section 4.1.1, we identify some basic isometric invariants of nonholonomic Riemannian structures in three dimensions. Following this section, we consider the equivalence and classification of such structures. There arise two natural cases, depending on an invariant $\vartheta \geq 0$. In the first case ($\vartheta = 0$; treated in section 4.1.2), we show that two structures are NH-isometric if and only if their associated sub-Riemannian structures are "SR-isometric." Accordingly, the classification of invariant structures reduces to a classification of (invariant) sub-Riemannian structures on three-dimensional Lie groups; these structures were recently classified by Agrachev and Barilari [1]. We present their classification in detail, adapted to our situation. The second case ($\vartheta > 0$; treated in section 4.1.3) is the generic case, and of the most interest. For these structures, we construct a canonical orthonormal frame (making use of the contact structure). The commutator relations of the canonical frame uniquely determine the nonholonomic Riemannian structure. For the left-invariant structures, the canonical frame is left invariant; consequently, any NH-isometry between these structures must be the composition of a left translation with a Lie group isomorphism. This leads to the introduction of a further three invariants ϱ_0, ϱ_1 and ϱ_2, which, together with ϑ, form a complete set of invariants for structures on the unimodular groups. For structures on (most) non-unimodular groups, we show that there are at most two non-NH-isometric structures with the same invariants ϑ, ϱ_0, ϱ_1 and ϱ_2. In section 4.1.4 we characterise (in terms of the invariants) those structures whose nonholonomic connection is Cartan–Schouten, and in section 4.1.5 we point out those (equivalence classes of) structures whose distribution is geodesically invariant.

Section 4.2 is devoted to the investigation of the flat nonholonomic Riemannian structures in three dimensions, i.e., those structures for which the parallel transport induced by the nonholonomic connection is path-independent. Specifically, we characterise (by means of the $\mathscr{P}$-exterior covariant derivative associated to the nonholonomic connection) when a structure is flat. Making use of this characterisation and the classification of three-dimensional structures obtained in section 4.1, we then characterise exactly which (equivalence classes of) structures are flat. (Remarkably, it turns out that every left-invariant nonholonomic Riemannian structure on H_3 or $\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}$ is flat.) We initially take a direct approach to characterising the flat structures; accordingly, in section 4.2.3 we relate the results obtained with the Wagner curvature tensor. In particular, we obtain another characterisation of the flat structures in terms of (the vanishing of) a contraction of the Wagner tensor.

Finally, in section 4.3 we give several examples of nonholonomic Riemannian structures in three dimensions. In section 4.3.1 we consider the equivalence class of structures on the Heisenberg group H_3 with non-trivial reduced dynamics. (Up to equivalence, there are only two structures on H_3—one with trivial reduced dynamics, i.e., a Cartan–Schouten connection—and one with non-trivial reduced dynamics.) This structure is essentially the simplest nonholonomic Riemannian structure without a Cartan–Schouten connection. We compute explicitly the canonical frame and invariants for an arbitrary representative of the equivalence class, before selecting a normal form. For the normalised structure we then give an explicit expression for the parallel transport map (the structure is flat, so the parallel transport is path-independent) and calculate explicit expressions for the nonholonomic geodesics. The other examples we consider (in section 4.3.2 and 4.3.3) are two classical problems from nonholonomic mechanics, *viz.*, the Chaplygin problem and the Suslov problem. (Both of these problems are modelled as left-invariant nonholonomic Riemannian structures on a three-dimensional Lie group.) These two problems have been well studied (see, e.g., [16, 63, 52, 24, 26]). Accordingly, we shall only discuss how they relate to our classification. In particular, we give expressions for the canonical frame and invariants, and discuss how the invariants may be used to distinguish between different qualitative cases exhibited by the (reduced) dynamics (as given in [24, 26]).

Remark 4.0.19. In appendix C we review the Bianchi–Behr classification of real three-dimensional Lie algebras and their associated simply connected Lie groups, and fix notation for the various groups and algebras. We also discuss how to distinguish between the different Lie algebras (or Lie groups) using some algebraic properties; this will be crucial for our classification. □

4.1 Equivalence and classification

In chapter 3 (and specifically, section 3.1) we discussed several natural equivalence relations between nonholonomic Riemannian structures, *viz.*, nonholonomic geodesic equivalence, nonholonomic affine equivalence and equivalence up to nonholonomic isometries. In this chapter we will be chiefly concerned with the last (and strongest) of these equivalence relations.

We have also seen in proposition 3.1.2 that a rescaling of the metric (by a constant factor) does not affect the nonholonomic connection, and hence the nonholonomic geodesics. Thus we shall consider the equivalence of (three-dimensional) nonholonomic Riemannian manifolds up to *NH-isometry and rescaling.*

4.1.1 Isometric invariants

Let $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ be a nonholonomic Riemannian manifold, where $\dim(\mathsf{M}) = 3$, and let (Y_1, Y_2) be a (local) orthonormal frame for $\mathcal{D}$. Since $\mathcal{D}$ is completely nonholonomic (or equivalently in this case, strongly nonholonomic), the one-dimensional annihilator $\mathcal{D}^\circ$ is (locally) spanned by a *contact form* ω, i.e., a 1-form such that $\omega \wedge d\omega \neq 0$. By imposing the condition $d\omega(Y_1, Y_2) = \pm 1$, we fix ω up to sign. (The value of $d\omega(Y_1, Y_2)$ is—up to sign—independent of the choice of orthonormal frame for $\mathcal{D}$.) Specified in this fashion, ω depends only on $(\mathcal{D}, \mathbf{g})$; hence it is intrinsic to three-dimensional nonholonomic Riemannian manifolds (and is preserved, up to sign, by NH-isometries).

Let $Y_0 \in \Gamma(T\mathsf{M})$ denote the Reeb vector field of ω. That is, Y_0 is the unique vector field such that $i_{Y_0}\omega = 1$ and $i_{Y_0}d\omega = 0$. As ω is unique up to sign, the same holds for Y_0. Likewise, Y_0 depends only on $(\mathcal{D}, \mathbf{g})$, and so any NH-isometry preserves Y_0 (up to sign).

There are two cases to consider, *viz.*, $Y_0 \in \Gamma(\mathcal{D}^\perp)$ and $Y_0 \notin \Gamma(\mathcal{D}^\perp)$. Clearly, these conditions are invariant under NH-isometry. Moreover, in the former case, the nonholonomic Riemannian structure reduces to a sub-Riemannian structure, i.e., $\mathcal{D}^\perp$ is determined by $(\mathcal{D}, \mathbf{g})$. It is thus natural to consider the angle between Y_0 and $\mathcal{D}^\perp$ in terms of an appropriate Riemannian metric on M (i.e., one preserved by NH-isometries). Let $\widetilde{\mathbf{g}}$ be the unique extension of $\mathbf{g}$ to a Riemannian metric on M such that (Y_0, Y_1, Y_2) is an orthonormal frame. This extension is independent of the choice of orthonormal frame for $\mathcal{D}$; furthermore, it depends only on $(\mathcal{D}, \mathbf{g})$. The angle θ between Y_0 and $\mathcal{D}^\perp$ is given by

$$\cos\theta = \frac{|\widetilde{\mathbf{g}}(Y_0, Y_3)|}{\sqrt{\widetilde{\mathbf{g}}(Y_3, Y_3)}}, \qquad 0 \leq \theta < \frac{\pi}{2},$$

where $\mathcal{D}^\perp = \operatorname{span}\{Y_3\}$. It is not difficult to see that θ is invariant under NH-isometry. Having said that, we shall find it more convenient to use the invariant $\vartheta = \tan^2\theta$. Note that $Y_0 \in \Gamma(\mathcal{D}^\perp)$ exactly when $\vartheta = 0$. Thus we have the two cases $\vartheta = 0$ and $\vartheta > 0$ to consider.

We introduce a further three isometric (curvature) invariants. The first invariant, denoted κ, is defined to be the sectional curvature of $\mathcal{D}$, i.e., $\kappa(q) = \widetilde{R}(\mathcal{D}_q)$. (Alternatively, we have $\kappa = \frac{1}{2}\operatorname{Scal}$. Furthermore, $\kappa^2 = \det(\mathbf{g}^\sharp \circ \operatorname{Ric}^\flat)$.) The second two invariants, denoted χ_1 and χ_2, are defined to be the positive eigenvalue of $\mathbf{g}^\sharp \circ A^\flat_{sym}$ and the absolute value of the Pfaffian of $\mathbf{g}^\sharp \circ A^\flat_{skew}$, respectively:

$$\chi_1 = \sqrt{-\det(\mathbf{g}^\sharp \circ A^\flat_{sym})}, \qquad \chi_2 = \sqrt{\det(\mathbf{g}^\sharp \circ A^\flat_{skew})}.$$

(Both $\mathbf{g}^\sharp \circ A^\flat_{sym}$ and $\mathbf{g}^\sharp \circ A^\flat_{skew}$ are trace free, and their determinants can be shown to always be nonpositive and nonnegative, respectively.)

Proposition 4.1.1. *$R \equiv 0$ (resp. $C \equiv 0$) if and only if $\kappa = 0$ (resp. $\chi_1 = \chi_2 = 0$).*

Proof. Let (X_0, X_1, X_2) be a (local) frame of vector fields on M such that (X_1, X_2) is an orthonormal frame for $\mathcal{D}$ and X_0 is a frame for $\mathcal{D}^\perp$. We have $[X_i, X_j] = c^k_{ij}X_k$ for some $c^k_{ij} \in \mathcal{C}^\infty(\mathsf{M})$. (Here we take the ranges $i, j, k = 0, 1, 2$.) Without loss of generality, we may assume that $c^0_{21} = 1$. We have

$$\widehat{R}(X_1, X_2, X_1, X_2) = -\kappa.$$

By the symmetries (*S1*)–(*S4*), we have that $\widehat{R}$ (and thus R) is fully determined by the value of $\widehat{R}(X_1, X_2, X_1, X_2)$. Hence $R \equiv 0$ exactly when $\kappa = 0$. Similarly, $\widehat{C}$ is fully determined by

$$\widehat{C}(X_1, X_2, X_1, X_1) = -c^1_{10}, \quad \widehat{C}(X_1, X_2, X_1, X_2) = -\frac{1}{2}(c^1_{20} + c^2_{10})$$
$$\widehat{C}(X_1, X_2, X_2, X_2) = -c^2_{20},$$

and we have

$$\chi_1 = \frac{1}{2}\sqrt{(c^1_{20} + c^2_{10})^2 + (c^1_{10} - c^2_{20})^2}, \quad \chi_2 = \frac{1}{2}|c^1_{10} + c^2_{20}|.$$

Thus $C \equiv 0$ if and only if $\chi_1 = \chi_2 = 0$. ■

Note that rescaling the metric $\mathbf{g}$ as $\frac{1}{\mu^2}\mathbf{g}$, $\mu > 0$ rescales the frame (Y_1, Y_2) as $(\mu Y_1, \mu Y_2)$, and rescales the invariants homogeneously as $\mu^2\vartheta$, $\mu^2\kappa$, $\mu^2\chi_1$ and $\mu^2\chi_2$.

For a left-invariant structure on a Lie group, it is possible to find a global left-invariant orthonormal frame (Y_1, Y_2) for $\mathcal{D}$. Furthermore, the contact form ω and Reeb vector field Y_0 are left invariant, as are the invariants ϑ, κ, χ_1 and χ_2 (hence they are constant).

Remark 4.1.2. Ehlers [22] has used Cartan's method of equivalence to determine a generating set for differential invariants of three-dimensional nonholonomic Riemannian manifolds (under NH-isometry), which he denotes K, p, q, r, s and t. Ehlers makes the claim that K, $s^2 + t^2$ and $p^2 + qr$ form a complete set of differential invariants for nonholonomic Riemannian manifolds. However, this set is not complete. Indeed, looking ahead to our classification, consider the equivalence class (4.1.21) on $\mathsf{G}_{3.2}$ with $\beta = \frac{7}{2}$. For this structure, we have $K = -\frac{191}{32}$, $s^2 + t^2 = 1$ and $p^2 + qr = \frac{81}{1024}$. Likewise, the structure (4.1.22) (on the same group) has the same invariants K, $s^2 + t^2$ and $p^2 + qr$ for $\delta = 0$, $\beta = \frac{5}{2}$. However, these two structures are not NH-isometric: for the first structure we have $\chi_2 = 0$, whereas $\chi_2 = \frac{5}{4}$ for the second. The invariants ϑ, κ, χ_1 and χ_2 we have introduced in this paper may be expressed in terms of Ehlers' generating set of functions as $\vartheta = s^2 + t^2$, $\kappa = K$, $\chi_1 = \sqrt{p^2 + \frac{1}{4}(q + r)^2}$ and $\chi_2 = \frac{1}{2}|q - r|$. A geometric interpretation for $K - (p^2 + qr) - \frac{3}{4}$ was given in [22], as a sectional curvature of $\mathcal{D}$. In this paper we have interpreted κ as a different sectional curvature of $\mathcal{D}$; we have also given a geometric interpretation of ϑ. □

To every nonholonomic Riemannian manifold $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ we can associate the *sub-Riemannian manifold* $(\mathsf{M}, \mathcal{D}, \mathbf{g})$ (see section 1.1.5). Two sub-Riemannian manifolds $(\mathsf{M}, \mathcal{D}, \mathbf{g})$ and $(\mathsf{M}', \mathcal{D}', \mathbf{g}')$ are said to be *SR-isometric* if there exists a diffeomorphism $\phi : \mathsf{M} \to \mathsf{M}'$ such that

$$\phi_* \mathcal{D} = \mathcal{D}' \quad \text{and} \quad \mathbf{g} = \phi^* \mathbf{g}'.$$

Such a map is called an *SR-isometry.* If two nonholonomic Riemannian manifolds are NH-isometric, then their associated sub-Riemannian manifolds are SR-isometric.

Remark 4.1.3. It is clear that every isometric invariant of sub-Riemannian manifolds is also an invariant of nonholonomic Riemannian manifolds. Agrachev and Barilari [1] describe two isometric invariants for sub-Riemannian structures in three dimensions, which they denote κ and χ. The κ of [1] coincides with the κ used in this paper; likewise, the invariant χ in [1] agrees with the χ_1 defined above. □

4.1.2 Case 1: $\vartheta = 0$

When $\vartheta = 0$, we have that $\mathcal{D}^\perp$ is spanned by the Reeb vector field Y_0. Thus, given a sub-Riemannian structure $(\mathsf{M}, \mathcal{D}, \mathbf{g})$, there is at most one nonholonomic Riemannian structure (up to NH-isometry) with the associated sub-Riemannian structure $(\mathsf{M}, \mathcal{D}, \mathbf{g})$. Indeed, we have the following result.

Proposition 4.1.4. *If* $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ *and* $(\mathsf{M}', \mathcal{D}', \mathcal{D}'^\perp, \mathbf{g}')$ *are two nonholonomic Riemannian manifolds such that* $\vartheta = \vartheta' = 0$*, then they are NH-isometric if and only if their associated sub-Riemannian manifolds are SR-isometric.*

Proof. Suppose $\phi : \mathsf{M} \to \mathsf{M}'$ is an SR-isometry mapping $(\mathsf{M}, \mathcal{D}, \mathbf{g})$ to $(\mathsf{M}', \mathcal{D}', \mathbf{g}')$. The (normalised) contact structure depends only on the distribution and metric, so we have $\omega = \pm\phi^*\omega'$. Accordingly, ϕ preserves the Reeb vector field (up to sign). Thus $\phi_*\mathcal{D}^\perp = \operatorname{span}\{\phi_* Y_0\} = \mathcal{D}'^\perp$, i.e., $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ and $(\mathsf{M}', \mathcal{D}', \mathcal{D}'^\perp, \mathbf{g}')$ are NH-isometric. ■

Remark 4.1.5. Even if $\vartheta = 0$, the nonholonomic Riemannian geodesics are not, in general, geodesics of the associated sub-Riemannian manifold (see section 1.1.5). □

Accordingly, in this case, the classification of the three-dimensional left-invariant nonholonomic Riemannian manifolds $(\mathsf{G}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ reduces to the classification (under SR-isometries) of three-dimensional left-invariant sub-Riemannian manifolds $(\mathsf{G}, \mathcal{D}, \mathbf{g})$. These structures have already been (locally) classified on three-dimensional Lie groups (see [1]). (It turns out that the local classification may be globalised on the simply connected Lie groups.) For completeness, we include here full details of the classification contained in [1], modified where appropriate (in particular, the authors of [1] do not use the Bianchi–Behr classification of three-dimensional Lie algebras).

Let $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ be a nonholonomic Riemannian manifold such that $\vartheta = 0$. We have $\mathcal{D}^\perp = \operatorname{span}\{Y_0\}$; let (Y_1, Y_2) be an orthonormal frame for $\mathcal{D}$ such that $d\omega(Y_1, Y_2) = 1$.

Lemma 4.1.6 ([1]). *We have* $[Y_1, Y_0], [Y_2, Y_0] \in \Gamma(\mathcal{D})$ *and* $[Y_2, Y_1] = Y_0 \mod \Gamma(\mathcal{D})$.

Proof. Since $X_1 \in \ker\omega$, $i_{Y_0}\omega = 1$ and $i_{Y_0}d\omega = 0$, we have

$$0 = d\omega(Y_0, Y_1) = Y_0[\omega(Y_1)] - Y_1[\omega(Y_0)] - \omega([Y_0, Y_1]) = \omega([Y_0, Y_1]).$$

That is, $[Y_1, Y_0] \in \ker\omega = \Gamma(\mathcal{D})$. A similar argument yields $[Y_2, Y_0] \in \Gamma(\mathcal{D})$. We have $[Y_2, Y_1] = f\, Y_0 \mod \Gamma(\mathcal{D})$ for some $f \in \mathcal{C}^\infty(\mathsf{M})$. Since $d\omega(Y_1, Y_2) = 1$, it follows that

$$1 = d\omega(Y_1, Y_2) = Y_1[\omega(Y_2)] - Y_2[\omega(Y_1)] - \omega([Y_1, Y_2]) = f.$$

Hence $[Y_2, Y_1] = Y_0 \mod \Gamma(\mathcal{D})$. ■

Accordingly, the commutator relations of (Y_0, Y_1, Y_2) take the form

$$\begin{cases} [Y_1, Y_0] = c_{10}^1 Y_1 + c_{10}^2 Y_2 \\ [Y_2, Y_0] = c_{20}^1 Y_1 + c_{20}^2 Y_2 \\ [Y_2, Y_1] = c_{21}^1 Y_1 + c_{21}^2 Y_2 + Y_0, \end{cases}$$

where $c_{ij}^k \in \mathcal{C}^\infty(\mathsf{M})$ are the structure constants of the frame. Let (ν^0, ν^1, ν^2) denote the coframe dual to (Y_0, Y_1, Y_2). We have the structure equations

$$\begin{cases} d\nu^0 = \nu^1 \wedge \nu^2 \\ d\nu^1 = c_{10}^1 \nu^0 \wedge \nu^1 + c_{20}^1 \nu^0 \wedge \nu^2 + c_{21}^1 \nu^1 \wedge \nu^2 \\ d\nu^2 = c_{10}^2 \nu^0 \wedge \nu^1 + c_{20}^2 \nu^0 \wedge \nu^2 + c_{21}^2 \nu^1 \wedge \nu^2. \end{cases}$$

Furthermore,

$$\begin{aligned} 0 = d^2\nu^0 &= d\nu^1 \wedge \nu^2 - \nu^1 \wedge d\nu^2 \\ &= c_{10}^1 \nu^0 \wedge \nu^1 \wedge \nu^2 - c_{20}^2 \nu^1 \wedge \nu^0 \wedge \nu^2 = (c_{10}^1 + c_{20}^2)\nu^0 \wedge \nu^1 \wedge \nu^2. \end{aligned}$$

That is, $c_{10}^1 + c_{20}^2 = 0$. In terms of the structure constants c_{ij}^k, the invariants κ and χ_1 take the form

$$\kappa = \frac{1}{2}(c_{10}^2 - c_{20}^1) - (c_{21}^1)^2 - (c_{21}^2)^2 - Y_1[c_{21}^2] + Y_2[c_{21}^1], \quad \chi_1 = \frac{1}{2}\sqrt{(c_{10}^2 + c_{20}^1)^2 + 4(c_{10}^1)^2}.$$

On the other hand, the invariant χ_2 (in fact, the tensor A_{skew}) vanishes.

Lemma 4.1.7. *If* $\vartheta = 0$, *then* A_{skew} *vanishes identically* (*or equivalently,* $\chi_2 = 0$).

Proof. In terms of the frame (Y_0, Y_1, Y_2), we have

$$A_{skew} = \begin{bmatrix} 0 & -\frac{1}{2}(c_{10}^1 + c_{20}^2) \\ \frac{1}{2}(c_{10}^1 + c_{20}^2) & 0 \end{bmatrix}.$$

However, as $c_{10}^1 + c_{20}^2 = 0$ (using $d^2\nu^0 = 0$ as above), it follows that $A_{skew} \equiv 0$. ■

By definition, we have $\chi_1 \geq 0$. We consider the two cases $\chi_1 = 0$ and $\chi_1 > 0$ separately.

4.1.2.1 Case (a): $\chi_1 = 0$

From $\chi_1 = 0$, we get $c_{10}^1 = 0$ and $c_{10}^2 + c_{20}^1 = 0$. We are left with the following commutator relations for (Y_0, Y_1, Y_2):

$$\begin{cases} [Y_1, Y_0] = c_{10}^2 Y_2 \\ [Y_2, Y_0] = -c_{10}^2 Y_1 \\ [Y_2, Y_1] = c_{21}^1 Y_1 + c_{21}^2 Y_2 + Y_0. \end{cases} \tag{4.1.1}$$

Similarly, the structure equations of the dual frame are

$$\begin{cases} d\nu^0 = \nu^1 \wedge \nu^2 \\ d\nu^1 = -c_{10}^2 \nu^0 \wedge \nu^2 + c_{21}^1 \nu^1 \wedge \nu^2 \\ d\nu^2 = c_{10}^2 \nu^0 \wedge \nu^1 + c_{21}^2 \nu^1 \wedge \nu^2. \end{cases} \tag{4.1.2}$$

Proposition 4.1.8 ([1]). *If* M *is simply connected and* $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ *has constant sectional curvature* (*i.e.,* κ *is constant*), *then there exists a rotation* $(\widetilde{Y}_1, \widetilde{Y}_2)$ *of* (Y_1, Y_2) *such that*

$$\begin{cases} [\widetilde{Y}_1, Y_0] = \kappa \widetilde{Y}_2 \\ [\widetilde{Y}_2, Y_0] = -\kappa \widetilde{Y}_1 \\ [\widetilde{Y}_2, \widetilde{Y}_1] = Y_0. \end{cases} \tag{4.1.3}$$

Proof. Let $(\widetilde{Y}_1, \widetilde{Y}_2) = (\cos\theta\, Y_1 - \sin\theta\, Y_2, \sin\theta\, Y_1 + \cos\theta\, Y_2)$ be a rotation of the frame (Y_1, Y_2), where $\theta \in \mathcal{C}^\infty(\mathsf{M})$. We claim that there exists θ such that the commutator relations (4.1.3) are satisfied by the frame $(Y_0, \widetilde{Y}_1, \widetilde{Y}_2)$. We have

$$[\widetilde{Y}_2, \widetilde{Y}_1] = Y_0 + (c^1_{21} - Y_1[\theta])Y_1 + (c^2_{21} - Y_2[\theta])Y_2.$$

Hence, if the requisite θ exists, then $Y_1[\theta] = c^1_{21}$ and $Y_2[\theta] = c^2_{21}$. It is not difficult to show that this is also sufficient, i.e., if there exists a smooth function θ such that $Y_1[\theta] = c^1_{21}$ and $Y_2[\theta] = c^2_{21}$, then $(Y_0, \widetilde{Y}_1, \widetilde{Y}_2)$ satisfies (4.1.3). Suppose for the moment that such a θ exists. Then

$$\begin{aligned} d\theta &= Y_0[\theta]\nu^0 + Y_1[\theta]\nu^1 + Y_2[\theta]\nu^2 \\ &= ([Y_2, Y_1][\theta] - c^1_{21}Y_1[\theta] - c^2_{21}Y_2[\theta])\nu^0 + c^1_{21}\nu^1 + c^2_{21}\nu^2 \\ &= (Y_2[c^1_{21}] - Y_1[c^2_{21}] - (c^1_{21})^2 - (c^2_{21})^2)\nu^0 + c^1_{21}\nu^1 + c^2_{21}\nu^2 \\ &= (\kappa - c^2_{10})\nu^0 + c^1_{21}\nu^1 + c^2_{21}\nu^2. \end{aligned}$$

The right-hand side is independent of θ. Accordingly, let $\varpi = (\kappa - c^2_{10})\nu^0 + c^1_{21}\nu^1 + c^2_{21}\nu^2$. We will show that ϖ is closed, i.e., $d\varpi = 0$. Since M is simply connected, it then follows that ϖ is exact: there exists $\theta \in \mathcal{C}^\infty(\mathsf{M})$ such that $\varpi = d\theta$, and hence the requisite rotation exists. We have

$$d\varpi = (c^2_{10}c^2_{21} + Y_1[c^2_{10} - \kappa] + Y_0[c^1_{21}])\nu^0 \wedge \nu^1 + (-c^2_{10}c^1_{21} + Y_2[c^2_{10} - \kappa] + Y_0[c^2_{21}])\nu^0 \wedge \nu^2.$$

Using $d^2 = 0$ on the second two structure equations (4.1.2), we get $Y_0[c^1_{21}] = -c^2_{10}c^2_{21} - Y_1[c^2_{10}]$ and $Y_0[c^2_{21}] = c^2_{10}c^1_{21} - Y_2[c^2_{10}]$. Hence

$$d\varpi = -Y_1[\kappa]\nu^0 \wedge \nu^1 - Y_2[\kappa]\nu^0 \wedge \nu^2 = d\kappa \wedge \nu^0.$$

Since κ is constant, it follows that $d\varpi = 0$. ■

Corollary 4.1.9 (cf. [1]). *Two nonholonomic Riemannian manifolds* $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ *and* $(\mathsf{M}', \mathcal{D}', \mathcal{D}'^\perp, \mathbf{g}')$ *with* $\chi_1 = \chi_1' = 0$ *and constant* κ, κ' *are NH-isometric if and only if* $\kappa = \kappa'$.

Remark 4.1.10. Proposition 4.1.8 implies that all the structures in this case (i.e., with M simply connected, $\vartheta = \chi_1 = 0$ and κ constant) are flat, since there exists an orthonormal frame (Y_1, Y_2) for $\mathcal{D}$ such that $[\![Y_2, Y_1]\!] = 0$ (see proposition 2.2.13). We discuss the flat three-dimensional structures further in section 4.2. □

Theorem 4.1.11 (cf. [1]). *Let* $(\mathsf{G}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ *be a left-invariant nonholonomic Riemannian structure on a simply connected Lie group* G. *Suppose* $\vartheta = \chi_1 = 0$.

(*i*) *If* $\kappa < 0$, *then* $(\mathsf{G}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ *is NH-isometric (up to rescaling) to the following structures:*

- *The structure on* $\widetilde{\mathsf{SL}}(2,\mathbb{R})$ *with elliptic-type distribution and metric (at identity) being* $\mathcal{K}|_{\mathcal{D}_1}$. *(Here* $\mathcal{K}$ *is the Killing form of* $\mathfrak{g}$.*)*
- *Any structure on* $\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}$.

(*ii*) *If* $\kappa = 0$, *then* $(\mathsf{G}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ *is NH-isometric (up to rescaling) to any structure on the Heisenberg group* H_3.

(*iii*) *If* $\kappa > 0$*, then* $(\mathsf{G}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ *is NH-isometric (up to rescaling) to the structure on* $\mathsf{SU}(2)$ *with metric (at identity) being* $\mathcal{K}|_{\mathcal{D}_1}$.

Proof. We proceed from the commutator relations (4.1.1) (with the c_{ij}^k constant, by left invariance) and calculate all the left-invariant structures on simply connected Lie groups with $\theta = \chi_1 = 0$. It then suffices to compare the invariant κ for these structures. We have $\kappa = c_{10}^2 - (c_{21}^1)^2 - (c_{21}^2)^2$. It is straightforward to show (e.g., using MATHEMATICA) that G is isomorphic to one of H_3, $\mathsf{SU}(2)$, $\mathsf{SL}(2,\mathbb{R})_{ell}$ or $\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}$.

- If $\mathsf{G} = \mathsf{H}_3$, then $c_{21}^1 = c_{21}^2 = c_{10}^2 = 0$ and we have $\kappa = 0$.
- If $\mathsf{G} = \mathsf{SU}(2)$, then $c_{21}^1 = c_{21}^2 = 0$, $c_{10}^2 > 0$ and we have $\kappa > 0$.
- If $\mathsf{G} = \mathsf{SL}(2,\mathbb{R})_{ell}$, then $c_{21}^1 = c_{21}^2 = 0$, $c_{10}^2 < 0$ and we have $\kappa < 0$.
- If $\mathsf{G} = \mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}$, then $c_{10}^2 = 0$ and we have $\kappa < 0$.

The result now follows from corollary 4.1.9. Lastly, for the case $\mathsf{G} = \mathsf{SU}(2)$ or $\mathsf{G} = \mathsf{SL}(2,\mathbb{R})_{ell}$, the Killing form is given (with respect to the frame $(Y_0, \widetilde{Y}_1, \widetilde{Y}_2)$) by $\mathcal{K} = \operatorname{diag}(-2\kappa^2, -2\kappa, -2\kappa)$. It follows that, in both cases, the metric is a rescaling of $\mathcal{K}|_{\mathcal{D}_1}$. ■

Remark 4.1.12. Remarkably, theorem 4.1.11 implies that there exists an NH-isometry between the non-isomorphic Lie groups $\widetilde{\mathsf{SL}}(2,\mathbb{R})$ and $\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}$. For further details see [1]. In particular, in [1] the authors find an explicit expression for a local SR-isometry between $\mathsf{SL}(2,\mathbb{R})$ and $\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{S}^1$. □

4.1.2.2 Case (b): $\chi_1 > 0$

Since $\chi_1 > 0$, we have that A_{sym} is nondegenerate. Consequently, let X_1, X_2 be orthonormal vector fields spanning $\mathcal{D}$ that are also orthogonal with respect to A_{sym}. Then (X_0, X_1, X_2), where $X_0 = Y_0$, is a well-defined (up to sign) local canonical frame on M, and the commutator relations of this frame uniquely determine (up to sign) the nonholonomic Riemannian structure. Note that a rescaling of the metric $\frac{1}{\mu^2}\mathbf{g}$, $\mu > 0$ rescales the frame as $(\mu^2 X_0, \mu X_1, \mu X_2)$.

Lemma 4.1.13 (cf. [1]). *We have*

$$\begin{cases} [X_1, X_0] = c_{10}^2 X_2 \\ [X_2, X_0] = c_{20}^1 X_1 \\ [X_2, X_1] = c_{21}^1 X_1 + c_{21}^2 X_2 + X_0, \end{cases} \tag{4.1.4}$$

where $c_{ij}^k \in \mathcal{C}^\infty(\mathsf{M})$ *(with the ranges* $i, j, k = 0, 1, 2$*). (For convenience, we use the same symbols for the structure constants of* (X_0, X_1, X_2) *as we did for* (Y_0, Y_1, Y_2)*.)*

Proof. In terms of the frame (Y_1, Y_2), we have

$$A_{sym} = \begin{bmatrix} \frac{1}{2}(c_{10}^2 + c_{20}^1) & -c_{10}^1 \\ -c_{10}^1 & -\frac{1}{2}(c_{10}^2 + c_{20}^1) \end{bmatrix}.$$

Hence, for X_1, X_2 to be orthogonal with respect to A_{sym}, we must have $c_{10}^1 = 0$ in the (X_1, X_2) frame. Hence we get the commutator relations (4.1.4). ■

In terms of the structure constants of the canonical frame, we have $\chi_1 = \frac{1}{2}|c_{10}^2 + c_{20}^1|$. Changing the frame (i.e., the sign of X_1 or X_2) if necessary, we may assume that $c_{10}^2 + c_{20}^1 > 0$.

Now suppose that $\mathsf{M} = \mathsf{G}$ is a simply connected Lie group (with Lie algebra $\mathfrak{g}$) and that the nonholonomic Riemannian structure $(\mathsf{G}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ is left-invariant. In this case the canonical frame (X_0, X_1, X_2) is left invariant, hence the structure constants c_{ij}^k are constant. Furthermore, it turns out that NH-isometries must preserve the Lie group structure.

Proposition 4.1.14. *Let* $(\mathsf{G}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ *and* $(\mathsf{G}', \mathcal{D}', \mathcal{D}'^\perp, \mathbf{g}')$ *be left-invariant nonholonomic Riemannian structures on three-dimensional simply connected Lie groups such that* $\vartheta, \vartheta' = 0$ *and* $\chi_1, \chi_1' > 0$. *If* $\phi : \mathsf{G} \to \mathsf{G}'$ *is an NH-isometry between* $(\mathsf{G}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ *and* $(\mathsf{G}', \mathcal{D}', \mathcal{D}'^\perp, \mathbf{g}')$, *then* $\phi = L_{\phi(\mathbf{1})} \circ \phi'$, *where* $L_{\phi(\mathbf{1})}$ *is a left translation and* $\phi' : \mathsf{G} \to \mathsf{G}'$ *is a Lie group isomorphism.*

Proof. Let $\phi : \mathsf{G} \to \mathsf{G}'$ be an NH-isometry between the two structures. We have $\phi_* Y_0 = \pm Y_0'$ and $\phi_* \mathscr{P}(X) = \mathscr{P}'(\phi_* X)$, $\phi_* \mathscr{Q}(X) = \mathscr{Q}'(\phi_* X)$ for every $X \in \Gamma(T\mathsf{G})$. Thus $\phi_* X_0 = \sigma_0 X_0'$, $\phi_* X_1 = \sigma_1 X_1'$ and $\phi_* X_2 = \sigma_2 X_2'$, where $\sigma_0, \sigma_1, \sigma_2 \in \{-1, 1\}$. Since X_0, X_1 and X_2 are left invariant, we have that ϕ sends every left-invariant vector field on G to a left-invariant vector field on G'. Hence $\phi' = L_{\phi(\mathbf{1})^{-1}} \circ \phi$ is a Lie group isomorphism. ■

We now proceed to classify the left-invariant structures. Appendix D lists the supporting MATHEMATICA code. For the moment, we classify *only* up to NH-isometry, and *do not* rescale. Later, in section 4.1.2.3, we shall consider the classification up to NH-isometry and rescaling.

Theorem 4.1.15 (cf. [1]). *We have the following classification (up to NH-isometry) of left-invariant nonholonomic Riemannian structures on three-dimensional simply connected Lie groups for which* $\vartheta = 0$ *and* $\chi_1 > 0$.

(*i*) *On* $\widetilde{\mathsf{SE}}(2)$ *there exists exactly one family of equivalence classes, specified as follows:*

$$\begin{cases} [X_1, X_0] = \alpha X_2 \\ [X_2, X_0] = 0 \\ [X_2, X_1] = X_0 \end{cases} \qquad \begin{cases} \kappa = \dfrac{1}{2}\alpha \\ \chi_1 = \dfrac{1}{2}\alpha. \end{cases} \tag{4.1.5}$$

Here $\alpha > 0$.

(*ii*) *On* $\mathsf{SE}(1,1)$ *there exists exactly one family of equivalence classes, specified as follows:*

$$\begin{cases} [X_1, X_0] = 0 \\ [X_2, X_0] = \alpha X_1 \\ [X_2, X_1] = X_0 \end{cases} \qquad \begin{cases} \kappa = -\dfrac{1}{2}\alpha \\ \chi_1 = \dfrac{1}{2}\alpha. \end{cases} \tag{4.1.6}$$

Here $\alpha > 0$.

(*iii*) *On* $\mathsf{SU}(2)$ *there exists exactly one family of equivalence classes, specified as follows:*

$$\begin{cases} [X_1, X_0] = \alpha_1 X_2 \\ [X_2, X_0] = -\alpha_2 X_1 \\ [X_2, X_1] = X_0 \end{cases} \qquad \begin{cases} \kappa = \dfrac{1}{2}(\alpha_1 + \alpha_2) \\ \chi_1 = \dfrac{1}{2}(\alpha_1 - \alpha_2). \end{cases} \tag{4.1.7}$$

Here $0 < \alpha_2 < \alpha_1$.

(*iv*) *On* $\widetilde{\mathsf{SL}}(2,\mathbb{R})_{ell}$ *there exists exactly one family of equivalence classes, specified as follows*:

$$\begin{cases} [X_1, X_0] = -\alpha_1 X_2 \\ [X_2, X_0] = \alpha_2 X_1 \\ [X_2, X_1] = X_0 \end{cases} \qquad \begin{cases} \kappa = -\dfrac{1}{2}(\alpha_1 + \alpha_2) \\ \chi_1 = -\dfrac{1}{2}(\alpha_1 - \alpha_2). \end{cases} \tag{4.1.8}$$

Here $0 < \alpha_1 < \alpha_2$.

(*v*) *On* $\widetilde{\mathsf{SL}}(2,\mathbb{R})_{hyp}$ *there exists exactly one family of equivalence classes, specified as follows*:

$$\begin{cases} [X_1, X_0] = \alpha_1 X_2 \\ [X_2, X_0] = \alpha_2 X_1 \\ [X_2, X_1] = X_0 \end{cases} \qquad \begin{cases} \kappa = \dfrac{1}{2}(\alpha_1 - \alpha_2) \\ \chi_1 = \dfrac{1}{2}(\alpha_1 + \alpha_2). \end{cases} \tag{4.1.9}$$

Here $\alpha_1, \alpha_2 > 0$.

(*vii*) *On* $\mathsf{G}_{3.2}$ *there exists exactly one family of equivalence classes, specified as follows*:

$$\begin{cases} [X_1, X_0] = \dfrac{\beta^2}{4} X_2 \\ [X_2, X_0] = 0 \\ [X_2, X_1] = X_0 - \beta X_2 \end{cases} \qquad \begin{cases} \kappa = -\dfrac{7\beta^2}{8} \\ \chi_1 = \dfrac{\beta^2}{8}. \end{cases} \tag{4.1.10}$$

Here $\beta \neq 0$.

(*vii*) *On* $\mathsf{G}_{3.4}^h$ *there exists exactly one family of equivalence classes, specified as follows*:

$$\begin{cases} [X_1, X_0] = 0 \\ [X_2, X_0] = -\dfrac{(h^2-1)\beta^2}{4h^2} X_1 \\ [X_2, X_1] = X_0 - \beta X_2 \end{cases} \qquad \begin{cases} \kappa = -\dfrac{(1+7h^2)\beta^2}{8h^2} \\ \chi_1 = -\dfrac{(h^2-1)\beta^2}{8h^2} \end{cases} \qquad \textit{for } 0 < h < 1 \tag{4.1.11}$$

$$\begin{cases} [X_1, X_0] = \dfrac{(h^2-1)\beta^2}{4h^2} X_2 \\ [X_2, X_0] = 0 \\ [X_2, X_1] = X_0 - \beta X_2 \end{cases} \qquad \begin{cases} \kappa = -\dfrac{(1+7h^2)\beta^2}{8h^2} \\ \chi_1 = \dfrac{(h^2-1)\beta^2}{8h^2} \end{cases} \qquad \textit{for } h > 1. \tag{4.1.12}$$

Here $\beta \neq 0$.

(*viii*) *On* $\mathsf{G}_{3.5}^h$ *there exists exactly one family of equivalence classes, specified as follows*:

$$\begin{cases} [X_1, X_0] = \dfrac{(h^2+1)\beta^2}{4h^2} X_2 \\ [X_2, X_0] = 0 \\ [X_2, X_1] = X_0 - \beta X_2 \end{cases} \qquad \begin{cases} \kappa = \dfrac{(1-7h^2)\beta^2}{8h^2} \\ \chi_1 = \dfrac{(h^2+1)\beta^2}{8h^2}. \end{cases} \tag{4.1.13}$$

Here $\beta \neq 0$.

No such structures exist on H_3 *or* $\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}$.

Proof. The structure equations of the frame (ν^0, ν^1, ν^2) dual to (X_0, X_1, X_2) are given by

$$\begin{cases} d\nu^0 = \nu^1 \wedge \nu^2 \\ d\nu^1 = c^1_{20}\nu^0 \wedge \nu^2 + c^1_{21}\nu^1 \wedge \nu^2 \\ d\nu^2 = c^2_{10}\nu^0 \wedge \nu^1 + c^2_{21}\nu^1 \wedge \nu^2. \end{cases}$$

Using $d^2 = 0$ on the second and third equation, we get the equalities $c^1_{20}c^2_{21} = 0$, $c^2_{10}c^1_{21} = 0$. As $\chi_1 = \frac{1}{2}(c^2_{10} + c^1_{20}) > 0$, we cannot have both c^2_{10} and c^1_{20} nonpositive. Accordingly, we have the three cases: (a) $c^1_{21} = c^2_{21} = 0$; (b) $c^1_{21} = c^1_{20} = 0$ and $c^2_{21} \neq 0$; (c) $c^2_{21} = c^2_{10} = 0$ and $c^1_{21} \neq 0$.
(a) $c^1_{21} = c^2_{21} = 0$. We have the commutator relations

$$\begin{cases} [X_1, X_0] = c^2_{10}X_2 \\ [X_2, X_0] = c^1_{20}X_1 \\ [X_2, X_1] = X_0. \end{cases}$$

It is not difficult to show that G is unimodular and non-nilpotent; furthermore, the Killing form $\mathcal{K}$ is degenerate if and only if $c^1_{20}c^2_{10} = 0$. Suppose $\mathcal{K}$ is degenerate. The eigenvalues of ad_U, $U = u^iX_i \in \mathfrak{g}$ are

$$0,\ -\sqrt{c^1_{20}(u^2)^2 - c^2_{10}(u^1)^2},\ \sqrt{c^1_{20}(u^2)^2 - c^2_{10}(u^1)^2}.$$

Hence:

- If $c^1_{20} = 0$, then there exists a U such that the eigenvalues of ad_U are imaginary, and hence G is isomorphic to $\widetilde{\mathsf{SE}}(2)$. Specifically, we have the family of equivalence classes (4.1.5), parametrised by $\alpha > 0$.
- If $c^1_{20} \neq 0$, then $c^2_{10} = 0$, and so the eigenvalues of ad_U are always real. Hence we get the family of equivalence classes (4.1.6) on $\mathsf{SE}(1,1)$, parametrised by $\alpha > 0$.

On the other hand, suppose $\mathcal{K}$ is nondegenerate. Let μ_1, μ_2 and μ_3 denote the leading principal minors of the matrix of $\mathcal{K}$ (with respect to the canonical frame), and λ_1, λ_2 those of $\mathcal{K}|_{\mathcal{D}_1}$. (We have $\mu_1 = 2c^1_{20}c^2_{10}$, $\mu_2 = -4c^1_{20}(c^2_{10})^2$, $\mu_3 = -8(c^1_{20}c^2_{10})^2$ and $\lambda_1 = -2c^2_{10}$, $\lambda_2 = -4c^2_{10}c^1_{20}$.) G is isomorphic to

- $\mathsf{SU}(2)$ if and only if $\mu_1 < 0$ and $\mu_2 > 0$, i.e., $c^1_{20} < 0$; in this case, we get the family of equivalence classes (4.1.7) on $\mathsf{SU}(2)$, parametrised by $\alpha_1, \alpha_2 > 0$ with $\alpha_2 < \alpha_1$.
- $\widetilde{\mathsf{SL}}(2,\mathbb{R})$ with elliptic-type distribution if and only if ($\mu_1 \geq 0$ or $\mu_2 \leq 0$) and ($\lambda_1 \neq 0$ and $\lambda_2 > 0$), i.e., $c^2_{10} < 0$; specifically, we get the family of equivalence classes (4.1.8), parametrised by $\alpha_1, \alpha_2 > 0$ with $\alpha_1 < \alpha_2$.
- $\widetilde{\mathsf{SL}}(2,\mathbb{R})$ with hyperbolic-type distribution if and only if ($\mu_1 \geq 0$ or $\mu_2 \leq 0$) and ($\lambda_1 = 0$ or $\lambda_2 \leq 0$), i.e., $c^2_{10}c^1_{20} < 0$; we get the family of equivalence classes (4.1.9), parametrised by $\alpha_1, \alpha_2 > 0$.

(b) $c^1_{21} = c^1_{20} = 0$ and $c^2_{21} \neq 0$. In this case G is non-unimodular. The eigenvalues of ad_U are

$$0,\ -\frac{1}{2}u^1\Big(c^2_{21} + \sqrt{(c^2_{21})^2 - 4c^2_{10}}\Big),\ -\frac{1}{2}u^1\Big(c^2_{21} - \sqrt{(c^2_{21})^2 - 4c^2_{10}}\Big).$$

Since $c^2_{10} > 0$, there does not exists a U such that exactly two eigenvalues are zero (i.e., G is not isomorphic to $\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}$). Hence, we have the following cases:

- If $(c^2_{21})^2 - 4c^2_{10} > 0$, then G is isomorphic to $\mathsf{G}^h_{3.4}$, $h > 1$. In particular, we have the family of equivalence classes (4.1.12), parametrised by $\beta \neq 0$.

- If $(c^2_{21})^2 - 4c^2_{10} = 0$, then we have the family of equivalence classes (4.1.10) on $\mathsf{G}_{3.2}$, parametrised by $\beta \neq 0$.

- If $(c^2_{21})^2 - 4c^2_{10} < 0$, then we have the family of equivalence classes (4.1.13) on $\mathsf{G}^h_{3.5}$, parametrised by $\beta \neq 0$.

(c) $c^2_{21} = c^2_{10} = 0$ and $c^1_{21} \neq 0$. G is non-unimodular, and the eigenvalues of ad_U are

$$0,\ \frac{1}{2}\Big(u^2c^1_{21} - |u^2|\sqrt{(c^1_{21})^2 + 4c^1_{20}}\Big),\ \frac{1}{2}\Big(u^2c^1_{21} + |u^2|\sqrt{(c^1_{21})^2 + 4c^1_{20}}\Big).$$

Since $c^1_{20} > 0$, we have that G is isomorphic to $\mathsf{G}^h_{3.4}$, $0 < h < 1$. Specifically, we have the family of equivalence classes (4.1.11), parametrised by $\beta \neq 0$. ■

4.1.2.3 Summary

In this section we summarise the results of the previous two sections, in particular theorem 4.1.11 and theorem 4.1.15. Rescaling the metric by $\mu > 0$, we can normalise the invariants κ and χ_1 so that either $(\kappa, \chi_1) = (0, 0)$, or (κ, χ_1) belongs to the (upper) semi-circle

$$\{(\kappa, \chi_1) : \kappa^2 + \chi_1^2 = 1,\ \chi_1 \geq 0\}.$$

In figure 4.1 we graph the normalised invariants. Different points on the semi-circle represent non-NH-isometric structures. It is easy to see that there are at most three non-NH-isometric structures with the same invariants κ and χ_1. (However, restricting to unimodular Lie groups, the equivalence classes are completely specified by κ and χ_1.) The classification may be summarised thusly:

Theorem 4.1.16 (cf. [1]). *Let* $(\mathsf{G}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ *and* $(\mathsf{G}', \mathcal{D}', \mathcal{D}'^\perp, \mathbf{g}')$ *be left-invariant nonholonomic Riemannian structures on three-dimensional simply connected Lie groups such that* $\vartheta = \vartheta' = 0$, $\kappa = \kappa'$ *and* $\chi_1 = \chi_1'$.

(i) *If* $\kappa = \chi_1 = 0$, *then* $(\mathsf{G}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ *is NH-isometric (up to rescaling) to any structure on the Heisenberg group* H_3.

(ii) *If* $\chi_1 \neq 0$, *or* $\chi_1 = 0$ *and* $\kappa \geq 0$, *then* $(\mathsf{G}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ *is NH-isometric (up to rescaling) to* $(\mathsf{G}', \mathcal{D}', \mathcal{D}'^\perp, \mathbf{g}')$ *if and only if* $\mathfrak{g}$ *is isomorphic to* $\mathfrak{g}'$.

(iii) *If* $\chi_1 = 0$ *and* $\kappa < 0$, *then* $(\mathsf{G}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ *is NH-isometric (up to rescaling) to the structure on* $\widetilde{\mathsf{SL}}(2, \mathbb{R})$ *with elliptic-type distribution and metric (at identity) being* $\mathcal{K}|_{\mathcal{D}_1}$.

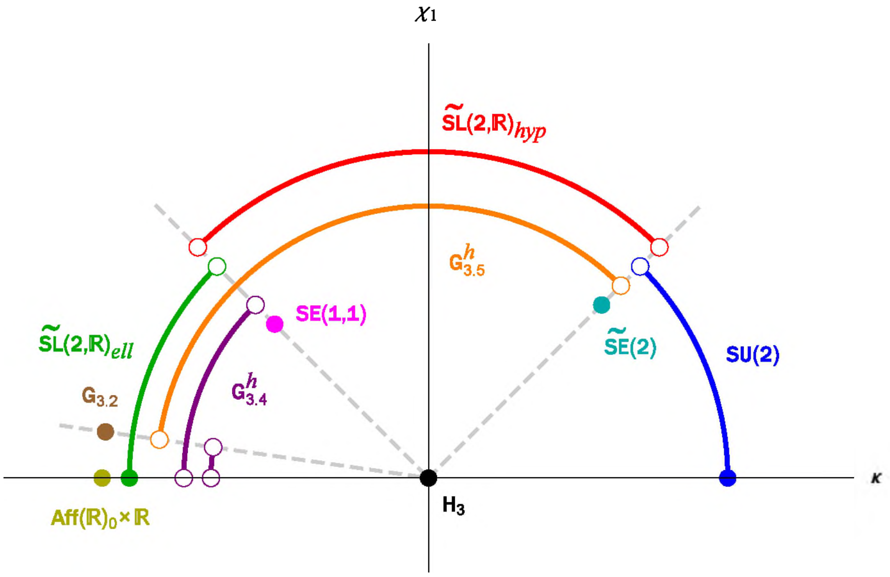

Figure 4.1: Normalised invariants for left-invariant nonholonomic Riemannian structures on three-dimensional simply connected Lie groups with vanishing ϑ (cf. [1]).

4.1.3 Case 2: $\vartheta > 0$

Let $X_0 = \mathscr{Q}(Y_0)$ and $X_1 = \mathscr{P}(Y_0)/\|\mathscr{P}(Y_0)\|$. Let X_2 be the unique unit vector field in $\mathcal{D}$ orthogonal to X_1 such that $d\omega(X_1, X_2) = 1$. A change in sign of ω results in X_0 and X_1 changing sign, while the sign of X_2 remains unchanged. Hence (X_0, X_1, X_2) is a well-defined (up to the sign of X_0, X_1) local canonical frame on M. Note that a rescaling of the metric $\frac{1}{\mu^2}\mathbf{g}$, $\mu > 0$ rescales the frame as $(\mu^2 X_0, \mu X_1, \mu X_2)$.

We now describe the commutator relations of the canonical frame and give explicit expressions for the invariants in terms of the structure constants of this frame.

Lemma 4.1.17. *We have* $i_{X_0}\omega = 1$, $[X_1, X_0] \in \Gamma(\mathcal{D})$ *and* $[X_2, X_1] = X_0 \mod \Gamma(\mathcal{D})$.

Proof. Since $X_1 \in \ker\omega$, we have $1 = i_{Y_0}\omega = i_{X_0}\omega + \|\mathscr{P}(Y_0)\| i_{X_1}\omega = i_{X_0}\omega$. On the other hand,

$$0 = d\omega(Y_0, X_1) = Y_0[\omega(X_1)] - X_1[\omega(Y_0)] - \omega([Y_0, X_1]) = \omega([X_1, X_0])$$

and so $[X_1, X_0] \in \ker\omega = \Gamma(\mathcal{D})$. Lastly, we have $[X_2, X_1] = f\,X_0 \mod \Gamma(\mathcal{D})$ for some $f \in \mathcal{C}^\infty(\mathsf{M})$. In fact,

$$1 = d\omega(X_1, X_2) = X_1[\omega(X_2)] - X_2[\omega(X_1)] - \omega([X_1, X_2]) = \omega([X_2, X_1]) = f.$$

That is, $[X_2, X_1] = X_0 \mod \Gamma(\mathcal{D})$. ∎

By lemma 4.1.17, the commutator relations of the canonical frame are given by

$$\begin{cases} [X_1, X_0] = c_{10}^1 X_1 + c_{10}^2 X_2 \\ [X_2, X_0] = c_{20}^1 X_1 + c_{20}^2 X_2 + c_{20}^0 X_0 \\ [X_2, X_1] = c_{21}^1 X_1 + c_{21}^2 X_2 + X_0, \end{cases}$$

where $c_{ij}^k \in \mathcal{C}^\infty(\mathsf{M})$ are the structure constants. (We take the ranges $i, j, k = 0, 1, 2$.) Since $\omega([X_2, Y_0]) = -d\omega(Y_0, X_2) = 0$, it follows that

$$\begin{aligned} c_{20}^0 &= \omega([X_2, X_0]) = -\omega([X_2, \mathscr{P}(Y_0)]) \\ &= -\|\mathscr{P}(Y_0)\|\omega([X_2, X_1]) \\ &= -\|\mathscr{P}(Y_0)\| < 0. \end{aligned}$$

In terms of the canonical frame, we have $\vartheta = \|\mathscr{P}(Y_0)\|^2 = (c_{20}^0)^2$. The curvature invariants κ, χ_1 and χ_2 take the form

$$\begin{aligned} \kappa &= \frac{1}{2}(c_{10}^2 - c_{20}^1) - (c_{21}^1)^2 - (c_{21}^2)^2 - X_1[c_{21}^2] + X_2[c_{21}^1], \\ \chi_1 &= \frac{1}{2}\sqrt{(c_{10}^2 + c_{20}^1)^2 + (c_{10}^1 - c_{20}^2)^2}, \quad \chi_2 = \frac{1}{2}|c_{10}^1 + c_{20}^2|. \end{aligned}$$

For the case of a left-invariant structure on a Lie group, the canonical frame is a global frame of left-invariant vector fields. It follows that the structure constants c_{ij}^k are constant. Moreover (similar to proposition 4.1.14) it turns out that NH-isometries must also preserve the Lie group structure.

Proposition 4.1.18. *Let* $(\mathsf{G}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ *and* $(\mathsf{G}', \mathcal{D}', \mathcal{D}'^\perp, \mathbf{g}')$ *be left-invariant nonholonomic Riemannian structures on three-dimensional simply connected Lie groups such that* $\vartheta, \vartheta' > 0$. *If* $\phi : \mathsf{G} \to \mathsf{G}'$ *is an NH-isometry between* $(\mathsf{G}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ *and* $(\mathsf{G}', \mathcal{D}', \mathcal{D}'^\perp, \mathbf{g}')$, *then* $\phi = L_{\phi(\mathbf{1})} \circ \phi'$, *where* $L_{\phi(\mathbf{1})}$ *is a left translation and* $\phi' : \mathsf{G} \to \mathsf{G}'$ *is a Lie group isomorphism.*

Proof. Let $\phi : \mathsf{G} \to \mathsf{G}'$ be an NH-isometry between the two structures. We have $\phi_* Y_0 = \pm Y_0'$ and $\phi_* \mathscr{P}(X) = \mathscr{P}'(\phi_* X)$, $\phi_* \mathscr{Q}(X) = \mathscr{Q}'(\phi_* X)$ for every $X \in \Gamma(T\mathsf{G})$. Thus $\phi_* X_0 = \pm X_0'$, $\phi_* X_1 = \pm X_1'$ and $\phi_* X_2 = X_2'$. Since X_0, X_1 and X_2 are left invariant, we have that ϕ sends every left-invariant vector field on G to a left-invariant vector field on G'. Hence $\phi' = L_{\phi(\mathbf{1})^{-1}} \circ \phi$ is a Lie group isomorphism. ■

It follows from proposition 4.1.18 that (when $\vartheta > 0$) every NH-isometry $\phi : \mathsf{G} \to \mathsf{G}'$ preserves the Killing form. That is, $\mathcal{K}(U, V) = \mathcal{K}'(\phi_* U, \phi_* V)$ for all left-invariant vector fields $U, V \in \Gamma(T\mathsf{G})$, where $\mathcal{K}$ (resp. $\mathcal{K}'$) is the Killing form of $\mathfrak{g}$ (resp. $\mathfrak{g}'$). Accordingly, we define scalars $\varrho_i = -\frac{1}{2}\mathcal{K}(X_i, X_i)$ for $i = 0, 1, 2$. In terms of the structure constants of the canonical frame, we have

$$\begin{gathered} \varrho_0 = -\frac{1}{2}\left[(c_{10}^1)^2 + 2c_{20}^1 c_{10}^2 + (c_{20}^2)^2\right], \quad \varrho_1 = c_{10}^2 - \frac{1}{2}(c_{21}^2)^2, \\ \varrho_2 = -\frac{1}{2}\left[(c_{20}^0)^2 + 2c_{20}^1 + (c_{21}^1)^2\right]. \end{gathered}$$

Notice that, under a rescaling of the metric $\frac{1}{\mu^2}\mathbf{g}$, $\mu > 0$, we have that ϱ_0 rescales as $\mu^4 \varrho_0$ and ϱ_1, ϱ_2 rescale as $\mu^2 \varrho_1$, $\mu^2 \varrho_2$. Furthermore, since $\phi_* X_0 = \pm X_0'$, $\phi_* X_1 = \pm X_1'$ and $\phi_* X_2 = X_2'$ for an NH-isometry $\phi : \mathsf{G} \to \mathsf{G}'$, we have that ϱ_0, ϱ_1 and ϱ_2 are isometric invariants.

The equivalence classes of left-invariant nonholonomic Riemannian structures on Lie groups with positive ϑ are better described by the invariants ϱ_0, ϱ_1 and ϱ_2 than the curvature invariants κ, χ_1 and χ_2. Accordingly, we shall prefer the ϱ_i's to the curvature invariants for these structures. In fact, for the structures on unimodular groups, ϑ, ϱ_0, ϱ_1 and ϱ_2 form a complete set of invariants. (This is not the case for the set of invariants ϑ, κ, χ_1 and χ_2.) For the structures on non-unimodular groups (with the exception of those on $\mathsf{G}_{3.5}^h$, $h = 1$), we shall find that there are at most two structures with the same invariants ϑ, ϱ_0, ϱ_1 and ϱ_2. (For structures on $\mathsf{G}_{3.5}^h$, $h = 1$ there are infinitely many structures with the same values for these invariants, but at most two with the same invariants ϑ, κ and χ_2.)

We shall distinguish between the case when G is unimodular and the case when G is non-unimodular. (As NH-isometries are left translations composed with Lie group isomorphisms, the unimodularity property is preserved under equivalence.) It is straightforward to show that G is unimodular if and only if $c_{10}^1 + c_{20}^2 = 0$, $c_{20}^0 + c_{21}^1 = 0$ and $c_{21}^2 = 0$. Suppose first that G is unimodular. In this case we have the simplified expressions

$$\varrho_0 = -(c_{10}^1)^2 - c_{20}^1 c_{10}^2, \quad \varrho_1 = c_{10}^2, \quad \varrho_2 = -c_{20}^1 - (c_{21}^1)^2.$$

Furthermore, the curvature invariants κ, χ_1 and χ_2 may be expressed as

$$\kappa = \frac{1}{2}(\varrho_1 + \varrho_2 - \vartheta), \quad \chi_1 = \frac{1}{2}\sqrt{(\varrho_1 + \varrho_2 + \vartheta)^2 - 4\varrho_0}, \quad \chi_2 = 0.$$

In appendix D (specifically, section D.1) we list the supporting Mathematica code for the proof of the following classification.

Theorem 4.1.19. *We have the following classification of left-invariant nonholonomic Riemannian structures on unimodular simply connected Lie groups, rescaled such that* $\vartheta = 1$.

(*i*) *On* H_3 *there exists exactly one equivalence class, specified as follows*:

$$\begin{cases} [X_1, X_0] = 0 \\ [X_2, X_0] = -X_0 - X_1 \\ [X_2, X_1] = X_0 + X_1 \end{cases} \qquad \begin{cases} \varrho_0 = 0 \\ \varrho_1 = 0 \\ \varrho_2 = 0. \end{cases} \tag{4.1.14}$$

(*ii*) *On* $\widetilde{\mathsf{SE}}(2)$ *there exists exactly one family of equivalence classes, specified as follows*:

$$\begin{cases} [X_1, X_0] = -\sqrt{\alpha_1\alpha_2}\,X_1 + \alpha_1 X_2 \\ [X_2, X_0] = -X_0 - (1+\alpha_2)X_1 + \sqrt{\alpha_1\alpha_2}\,X_2 \\ [X_2, X_1] = X_0 + X_1 \end{cases} \qquad \begin{cases} \varrho_0 = \alpha_1 \\ \varrho_1 = \alpha_1 \\ \varrho_2 = \alpha_2. \end{cases} \tag{4.1.15}$$

Here $\alpha_1, \alpha_2 \geq 0$ *and* $\alpha_1^2 + \alpha_2^2 \neq 0$.

(*iii*) *On* $\mathsf{SE}(1,1)$ *there exists exactly one family of equivalence classes, specified as follows*:

$$\begin{cases} [X_1, X_0] = -\sqrt{\alpha_1\alpha_2}\,X_1 - \alpha_1 X_2 \\ [X_2, X_0] = -X_0 - (1-\alpha_2)X_1 + \sqrt{\alpha_1\alpha_2}\,X_2 \\ [X_2, X_1] = X_0 + X_1 \end{cases} \qquad \begin{cases} \varrho_0 = -\alpha_1 \\ \varrho_1 = -\alpha_1 \\ \varrho_2 = -\alpha_2. \end{cases} \tag{4.1.16}$$

Here $\alpha_1, \alpha_2 \geq 0$ *and* $\alpha_1^2 + \alpha_2^2 \neq 0$.

(*iv*) *On* $\mathsf{SU}(2)$ *there exists exactly one family of equivalence classes, specified as follows*:

$$\begin{cases} [X_1, X_0] = -\delta X_1 + \alpha_1 X_2 \\ [X_2, X_0] = -X_0 - (1+\alpha_2)X_1 + \delta X_2 \\ [X_2, X_1] = X_0 + X_1 \end{cases} \qquad \begin{cases} \varrho_0 = -\delta^2 + \alpha_1(1+\alpha_2) \\ \varrho_1 = \alpha_1 \\ \varrho_2 = \alpha_2. \end{cases} \tag{4.1.17}$$

Here $\alpha_1, \alpha_2 > 0$, $\delta \geq 0$ *and* $\delta^2 - \alpha_1\alpha_2 < 0$.

(*v*) *On* $\widetilde{\mathsf{SL}}(2,\mathbb{R})_{ell}$ *there exists exactly one family of equivalence classes, specified as follows*:

$$\begin{cases} [X_1,X_0] = -\delta X_1 - \alpha_1 X_2 \\ [X_2,X_0] = -X_0 - (1-\alpha_2)X_1 + \delta X_2 \\ [X_2,X_1] = \quad X_0 + X_1 \end{cases} \qquad \begin{cases} \varrho_0 = -\delta^2 - \alpha_1(1-\alpha_2) \\ \varrho_1 = -\alpha_1 \\ \varrho_2 = -\alpha_2. \end{cases} \tag{4.1.18}$$

Here $\alpha_1, \alpha_2 > 0$, $\delta \geq 0$ *and* $\delta^2 - \alpha_1\alpha_2 < 0$.

(*vi*) *On* $\widetilde{\mathsf{SL}}(2,\mathbb{R})_{hyp}$ *there exists exactly one family of equivalence classes, specified as follows*:

$$\begin{cases} [X_1,X_0] = -\delta X_1 - \gamma_1 X_2 \\ [X_2,X_0] = -X_0 - (1-\gamma_2)X_1 + \delta X_2 \\ [X_2,X_1] = \quad X_0 + X_1 \end{cases} \qquad \begin{cases} \varrho_0 = -\delta^2 - \gamma_1(1-\gamma_2) \\ \varrho_1 = -\gamma_1 \\ \varrho_2 = -\gamma_2. \end{cases} \tag{4.1.19}$$

Here $\delta \geq 0$, $\gamma_1, \gamma_2 \in \mathbb{R}$ *and* $\delta^2 - \gamma_1\gamma_2 > 0$.

Proof. Since $\vartheta = (c_{21}^1)^2$, we rescale the frame by $\mu = \frac{1}{c_{21}^1} > 0$. We have the commutator relations

$$\begin{cases} [X_1,X_0] = c_{10}^1 X_1 + c_{10}^2 X_2 \\ [X_2,X_0] = -X_0 + c_{20}^1 X_1 - c_{10}^1 X_2 \\ [X_2,X_1] = X_0 + X_1. \end{cases}$$

(Here we have relabelled $\frac{c_{10}^1}{(c_{21}^1)^2}$ as c_{10}^1, etc.) We may assume, without loss of generality, that $c_{10}^1 \leq 0$. (Indeed, by changing the sign of ω if necessary, there exists a frame (X_0, X_1, X_2) such that c_{10}^1 is nonpositive.) We have the following four cases: (*a*) $c_{10}^1 = c_{10}^2 = 0$; (*b*) $c_{10}^1 = 0$ and $c_{10}^2 \neq 0$; (*c*) $c_{10}^1 < 0$ and $c_{10}^2 = 0$; (*d*) $c_{10}^1 < 0$ and $c_{10}^2 \neq 0$.

(*a*) $c_{10}^1 = c_{10}^2 = 0$. It is easy to show that $\mathcal{K}$ is degenerate, so G must be isomorphic to one of $\mathsf{SE}(1,1)$, H_3 or $\widetilde{\mathsf{SE}}(2)$. The nonzero eigenvalues of ad_U, $U = u^i X_i$ are $\pm|u^2|\sqrt{c_{20}^1 + 1}$. Accordingly, if $c_{20}^1 + 1 = 0$, then G is nilpotent, and hence isomorphic to the Heisenberg group H_3. Furthermore, the canonical frame has the commutator relations

$$\begin{cases} [X_1,X_0] = \quad 0 \\ [X_2,X_0] = -X_0 - X_1 \\ [X_2,X_1] = \quad X_0 + X_1. \end{cases}$$

Thus we have the equivalence class (4.1.14). On the other hand, if $c_{20}^1 + 1 > 0$, then the eigenvalues of ad_U are real for every U, and so G is completely solvable. Hence G is isomorphic to $\mathsf{SE}(1,1)$, and we have

$$\begin{cases} [X_1,X_0] = 0 \\ [X_2,X_0] = -X_0 + c_{20}^1 X_1 \\ [X_2,X_1] = X_0 + X_1. \end{cases}$$

This is the family (4.1.16) with $\alpha_1 = 0$ and $\alpha_2 = c_{20}^1 + 1 > 0$. Likewise, if $c_{20}^1 + 1 < 0$, then get have the family (4.1.15) of equivalence classes on $\widetilde{\mathsf{SE}}(2)$ with $\alpha_1 = 0$, $\alpha_2 = -(c_{20}^1 + 1) > 0$.

(*b*) $c_{10}^1 = 0$ and $c_{10}^2 \neq 0$. The Killing form $\mathcal{K}$ is nondegenerate if and only if $c_{20}^1 + 1 \neq 0$. Suppose $c_{20}^1 + 1 = 0$. The eigenvalues of ad_U are real for $c_{10}^2 < 0$ and complex for $c_{10}^2 > 0$.

These two cases yield the equivalence classes (4.1.15) and (4.1.16) on $\widetilde{\mathsf{SE}}(2)$ and $\mathsf{SE}(1,1)$, respectively, with $\alpha_1 > 0$ and $\alpha_2 = 0$. On the other hand, suppose $c^1_{20} + 1 \neq 0$. Let μ_1, μ_2 and μ_3 denote the leading principal minors of the matrix of $\mathcal{K}$ with respect to the canonical frame. (Specifically, we have $\mu_1 = 2c^2_{10}c^1_{20}$, $\mu_2 = -4(c^2_{10})^2(c^1_{20}+1)$ and $\mu_3 = -8(c^2_{10})^2(c^1_{20}+1)^2$.) Likewise, let λ_1 and λ_2 denote the leading principal minors of $\mathcal{K}|_{\mathcal{D}_1}$. (We have $\lambda_1 = -2c^2_{10}$ and $\lambda_2 = -4c^2_{10}(1 + c^1_{20})$.) G is isomorphic to:

- $\mathsf{SU}(2)$ if and only if $\mu_1 < 0$ and $\mu_2 > 0$; we get the equivalence class (4.1.17) with $\delta = 0$.
- $\widetilde{\mathsf{SL}}(2,\mathbb{R})$ with elliptic-type distribution if and only if ($\mu_1 \geq 0$ or $\mu_2 \leq 0$) and ($\lambda_1 \neq 0$ and $\lambda_2 > 0$); we get the equivalence class (4.1.18) with $\delta = 0$.
- $\widetilde{\mathsf{SL}}(2,\mathbb{R})$ with hyperbolic-type distribution if and only if ($\mu_1 \geq 0$ or $\mu_2 \leq 0$) and ($\lambda_1 = 0$ or $\lambda_2 \leq 0$); we get the equivalence class (4.1.19), with $\delta = 0$ and $\gamma_1\gamma_2 < 0$.

(*c*) $c^1_{10} < 0$ and $c^2_{10} = 0$. One can show that $\mathcal{K}$ is nondegenerate and indefinite on $\mathcal{D}$. Accordingly, G is isomorphic to $\widetilde{\mathsf{SL}}(2,\mathbb{R})$ with hyperbolic-type distribution. Moreover, we have the following commutator relations:

$$\begin{cases} [X_1, X_0] = c^1_{10}X_1 \\ [X_2, X_0] = -X_0 + c^1_{20}X_1 - c^1_{10}X_2 \\ [X_2, X_1] = X_0 + X_1. \end{cases}$$

This is exactly the family (4.1.19) of equivalence classes with $\delta = -c^1_{10} > 0$, $\gamma_1 = 0$ and $\gamma_2 = 1 + c^1_{20}$.

(*d*) $c^1_{10} < 0$ and $c^2_{10} \neq 0$. In this case $\mathcal{K}$ is nondegenerate if and only if $(c^1_{10})^2 + c^2_{10}(c^1_{20}+1) \neq 0$. Suppose $\mathcal{K}$ is degenerate. As $-c^2_{10}(c^1_{20} + 1) = (c^1_{10})^2 > 0$, we have the two cases $c^2_{10} < 0$, $c^1_{20} + 1 > 0$ and $c^2_{10} > 0$, $c^1_{20} + 1 < 0$. In the first case, the commutator relations are

$$\begin{cases} [X_1, X_0] = -\sqrt{-c^2_{10}(c^1_{20} + 1)}\, X_1 + c^2_{10}X_2 \\ [X_2, X_0] = -X_0 + c^1_{20}X_1 + \sqrt{-c^2_{10}(c^1_{20} + 1)}\, X_2 \\ [X_2, X_1] = X_0 + X_1. \end{cases}$$

It is not difficult to show that G is completely solvable, and hence isomorphic to $\mathsf{SE}(1,1)$. In fact, we have the family (4.1.16) of equivalence relations with $\alpha_1 = -c^2_{10} > 0$, $\alpha_2 = c^1_{20}+1 > 0$. Likewise, if $c^2_{10} > 0$, $c^1_{20} + 1 < 0$, then we have G isomorphic to $\widetilde{\mathsf{SE}}(2)$; specifically, the family (4.1.15) with $\alpha_1, \alpha_2 > 0$. Suppose that $\mathcal{K}$ is nondegenerate, and let μ_1, μ_2, μ_3 denote the leading principal minors of the matrix of $\mathcal{K}$ with respect to the canonical frame:

$$\mu_1 = 2[(c^1_{10})^2 + c^2_{10}c^1_{20}], \quad \mu_2 = -4c^2_{10}[(c^1_{10})^2 + c^2_{10}(c^1_{20} + 1)],$$
$$\mu_3 = -8[(c^1_{10})^2 + c^2_{10}(c^1_{20} + 1)]^2.$$

Likewise, let λ_1, λ_2 be the leading principal minors of $\mathcal{K}|_{\mathcal{D}_1}$. We have $\lambda_1 = -2c^2_{10}$ and $\lambda_2 = -4(c^1_{10})^2 - 4c^2_{10}(1 + c^1_{20})$. As before, G is isomorphic to:

- $\mathsf{SU}(2)$ if and only if $\mu_1 < 0$ and $\mu_2 > 0$;
- $\widetilde{\mathsf{SL}}(2,\mathbb{R})$ with elliptic-type distribution if and only if ($\mu_1 \geq 0$ or $\mu_2 \leq 0$) and ($\lambda_1 \neq 0$ and $\lambda_2 > 0$);

- $\widetilde{\mathsf{SL}}(2,\mathbb{R})$ with hyperbolic-type distribution if and only if ($\mu_1 \geq 0$ or $\mu_2 \leq 0$) and ($\lambda_1 = 0$ or $\lambda_2 \leq 0$).

This yields the equivalence classes (4.1.17) (with $\delta > 0$), (4.1.18) ($\delta > 0$) and (4.1.19) ($\delta > 0$, $\gamma_1 \neq 0$), respectively. ■

Corollary 4.1.20. *Let* G *be a three-dimensional unimodular simply connected Lie group. The scalars* ϑ*,* ϱ_0*,* ϱ_1 *and* ϱ_2 *form a complete set of (isometric) invariants for left-invariant nonholonomic Riemannian structures on* G *with* $\vartheta > 0$*.*

Remark 4.1.21. For the solvable groups H_3, $\widetilde{\mathsf{SE}}(2)$ and $\mathsf{SE}(1,1)$, the scalars $\vartheta > 0$, κ and χ_1 also form a complete set of isometric invariants. Indeed, for the equivalence class (4.1.14) on H_3 we have $\kappa = -\frac{1}{2}$ and $\chi_1 = \frac{1}{2}$. On the other hand, for the equivalence class (4.1.15) on $\widetilde{\mathsf{SE}}(2)$, we have $\kappa = \frac{1}{2}(\alpha_1 + \alpha_2 - 1)$ and $\chi_1 = \frac{1}{2}\sqrt{(\alpha_1+\alpha_2)^2 - 2(\alpha_1 - \alpha_2) + 1}$. Lastly, for the equivalence class (4.1.16) on $\mathsf{SE}(1,1)$, we have $\kappa = -\frac{1}{2}(\alpha_1 + \alpha_2 + 1)$ and $\chi_1 = \frac{1}{2}\sqrt{(\alpha_1+\alpha_2)^2 + 2(\alpha_1 - \alpha_2) + 1}$. (In both of the latter cases, we can solve for α_1 and α_2 uniquely in terms of κ and χ_1.) Clearly, for the semisimple groups, $\vartheta > 0$, κ and χ_1 do not form a complete set of invariants. □

Next, we suppose that G is not unimodular, i.e., at least one of $c^1_{10} + c^2_{20}$, $c^0_{20} + c^1_{21}$ or c^2_{21} is nonzero. (As before, the supporting MATHEMATICA code may be found in section D.1 of appendix D. For the non-unimodular groups in particular, many of the calculations are quite lengthy, so the use of a computer algebra system is instrumental in the following proof.)

Theorem 4.1.22. *We have the following classification of left-invariant nonholonomic Riemannian structures on non-unimodular simply connected Lie groups, rescaled such that* $\vartheta = 1$*.*

(*i*) *On* $\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}$ *there exists exactly one family of equivalence classes, specified as follows:*

$$\begin{cases} [X_1, X_0] = 0 \\ [X_2, X_0] = -X_0 - (1-\gamma)X_1 + \alpha X_2 \\ [X_2, X_1] = X_0 + (1-\gamma)X_1 - \alpha X_2 \end{cases} \qquad \begin{cases} \varrho_0 = -\frac{1}{2}\alpha^2 \\ \varrho_1 = -\frac{1}{2}\alpha^2 \\ \varrho_2 = -\frac{1}{2}\gamma^2. \end{cases} \tag{4.1.20}$$

Here $\alpha \geq 0$*,* $\gamma \in \mathbb{R}$ *and* $\alpha^2 + \gamma^2 \neq 0$*.*

(*ii*) *On* $\mathsf{G}_{3.2}$ *there exist exactly two families of equivalence classes, specified as follows:*

$$\begin{cases} [X_1, X_0] = 0 \\ [X_2, X_0] = -X_0 - \frac{1}{4}(\beta-2)^2 X_1 \\ [X_2, X_1] = X_0 + (1-\beta)X_1 \end{cases} \qquad \begin{cases} \varrho_0 = 0 \\ \varrho_1 = 0 \\ \varrho_2 = -\frac{1}{4}\beta^2 \end{cases} \tag{4.1.21}$$

$$\begin{cases} [X_1, X_0] = -\alpha X_1 + \frac{\beta^2}{4}X_2 \\ [X_2, X_0] = -X_0 - \frac{(2\alpha+\beta)^2}{\beta^2}X_1 + (\alpha+\beta)X_2 \\ [X_2, X_1] = X_0 + \frac{4\alpha+\beta}{\beta}X_1 - \beta X_2 \end{cases} \qquad \begin{cases} \varrho_0 = -\frac{1}{4}\beta^2 \\ \varrho_1 = -\frac{1}{4}\beta^2 \\ \varrho_2 = -\frac{4}{\beta^2}\alpha^2. \end{cases} \tag{4.1.22}$$

Here $\alpha \geq 0$*,* $\beta \neq 0$ *and* $(1 - \operatorname{sgn}(\alpha))\beta \geq 0$*, i.e., if* $\alpha = 0$*, then* $\beta > 0$*.*

(*iii*) *On* $\mathsf{G}_{3.4}^h$ *there exist exactly two families of equivalence classes, specified as follows*:

$$\begin{cases} [X_1, X_0] = 0 \\ [X_2, X_0] = -X_0 - \dfrac{h^2(\beta-2)^2 - \beta^2}{4h^2} X_1 \\ [X_2, X_1] = X_0 + (1-\beta) X_1 \end{cases} \qquad \begin{cases} \varrho_0 = 0 \\ \varrho_1 = 0 \\ \varrho_2 = -\dfrac{(h^2+1)\beta^2}{4h^2} \end{cases} \tag{4.1.23}$$

$$\begin{cases} [X_1, X_0] = -\alpha X_1 + \dfrac{(h^2-1)\beta^2}{4h^2} X_2 \\ [X_2, X_0] = -X_0 - \dfrac{h^2(2\alpha+\beta)^2 - \beta^2}{(h^2-1)\beta^2} X_1 + (\alpha+\beta) X_2 \\ [X_2, X_1] = X_0 + \dfrac{h^2(4\alpha+\beta) - \beta}{(h^2-1)\beta} X_1 - \beta X_2 \end{cases} \qquad \begin{cases} \varrho_0 = -\dfrac{(h^2+1)\beta^2}{4h^2} \\ \varrho_1 = -\dfrac{(h^2+1)\beta^2}{4h^2} \\ \varrho_2 = -\dfrac{4h^2(h^2+1)\alpha^2}{(h^2-1)^2\beta^2}. \end{cases} \tag{4.1.24}$$

Here $\alpha \geq 0$, $\beta \neq 0$ *and* $(1 - \operatorname{sgn}(\alpha))\beta \geq 0$.

(*iv*) *On* $\mathsf{G}_{3.5}^h$ *there exist exactly two families of equivalence classes, specified as follows*:

$$\begin{cases} [X_1, X_0] = 0 \\ [X_2, X_0] = -X_0 - \dfrac{h^2(\beta-2)^2 + \beta^2}{4h^2} X_1 \\ [X_2, X_1] = X_0 + (1-\beta) X_1 \end{cases} \qquad \begin{cases} \varrho_0 = 0 \\ \varrho_1 = 0 \\ \varrho_2 = -\dfrac{(h^2-1)\beta^2}{4h^2} \end{cases} \tag{4.1.25}$$

$$\begin{cases} [X_1, X_0] = -\alpha X_1 + \dfrac{(h^2+1)\beta^2}{4h^2} X_2 \\ [X_2, X_0] = -X_0 - \dfrac{h^2(2\alpha+\beta)^2 + \beta^2}{(h^2+1)\beta^2} X_1 + (\alpha+\beta) X_2 \\ [X_2, X_1] = X_0 + \dfrac{h^2(4\alpha+\beta) + \beta}{(h^2+1)\beta} X_1 - \beta X_2 \end{cases} \qquad \begin{cases} \varrho_0 = -\dfrac{(h^2-1)\beta^2}{4h^2} \\ \varrho_1 = -\dfrac{(h^2-1)\beta^2}{4h^2} \\ \varrho_2 = -\dfrac{4h^2(h^2-1)\alpha^2}{(h^2+1)^2\beta^2}. \end{cases} \tag{4.1.26}$$

Here $\alpha \geq 0$, $\beta \neq 0$ *and* $(1 - \operatorname{sgn}(\alpha))\beta \geq 0$.

Proof. We have $\vartheta = (c_{20}^0)^2$, and so we rescale the frame by $\mu = -\frac{1}{c_{20}^0} > 0$. The commutator relations of the canonical frame are then

$$\begin{cases} [X_1, X_0] = c_{10}^1 X_1 + c_{10}^2 X_2 \\ [X_2, X_0] = -X_0 + c_{20}^1 X_1 + c_{20}^2 X_2 \\ [X_2, X_1] = X_0 + c_{21}^1 X_1 + c_{21}^2 X_2. \end{cases}$$

(Here $\frac{c_{10}^1}{(c_{20}^0)^2}$ has been relabelled as c_{10}^1, etc.) As in the proof of theorem 4.1.19, by changing the sign of ω we may assume that $c_{10}^1 \leq 0$. If $c_{10}^1 = 0$, then we may assume, by using the same argument, that $c^2_{21} \leq 0$. Let (ν^0, ν^1, ν^2) be the coframe dual to (X_0, X_1, X_2). Then we have the structure equations

$$\begin{cases} d\nu^0 = -\nu^0 \wedge \nu^2 + \nu^1 \wedge \nu^2 \\ d\nu^1 = c^1_{10}\, \nu^0 \wedge \nu^1 + c^1_{20}\, \nu^0 \wedge \nu^2 + c^1_{21}\, \nu^1 \wedge \nu^2 \\ d\nu^2 = c^2_{10}\, \nu^0 \wedge \nu^1 + c^2_{20}\, \nu^0 \wedge \nu^2 + c^2_{21}\, \nu^1 \wedge \nu^2. \end{cases}$$

Using $d^2 = 0$, we get $c^1_{10} = -(c^2_{21} + c^2_{20})$ and

$$\begin{cases} 0 = c^2_{20}(c^1_{21} - 1) - (c^1_{20} + 1)c^2_{21} \\ 0 = c^2_{10}(c^1_{21} - 1) + (c^2_{21} + c^2_{20})c^2_{21}. \end{cases} \tag{4.1.27}$$

As $c^1_{10} \leq 0$, it follows that $c^2_{21} + c^2_{20} \geq 0$. Moreover, since G is non-unimodular, we have $c^1_{21} \neq 1$ or $c^2_{21} \neq 0$.

We have the following five cases: (a) $c^1_{21} = 1$; (b) $c^1_{21} \neq 1$, $c^2_{20} = 0$ and $c^2_{10} = 0$; (c) $c^1_{21} \neq 1$, $c^2_{20} = 0$ and $c^2_{10} \neq 0$; (d) $c^1_{21} \neq 1$, $c^2_{20} \neq 0$ and $c^2_{10} = 0$; (e) $c^1_{21} \neq 1$, $c^2_{20} \neq 0$ and $c^2_{10} \neq 0$.

(a) $c^1_{21} = 1$. Then $c^2_{21} \neq 0$ (by non-unimodularity) and (4.1.27) yields $c^1_{20} = -1$ and $c^2_{20} = -c^2_{21}$. The latter equality implies $c^1_{10} = 0$, and so we may assume (by changing the frame if necessary) that $c^2_{21} < 0$. The eigenvalues of ad_U, $U = u^i X_i$ are

$$0,\ \frac{1}{2}(u^0 - u^1)\Big(c^2_{21} - \sqrt{(c^2_{21})^2 - 4c^2_{10}}\Big),\ \frac{1}{2}(u^0 - u^1)\Big(c^2_{21} + \sqrt{(c^2_{21})^2 - 4c^2_{10}}\Big).$$

Furthermore, exactly two eigenvalues are zero for some U if and only if $c^2_{10} = 0$. Hence:

- If $(c^2_{21})^2 - 4c^2_{10} > 0$ and $c^2_{10} \neq 0$, then we have the family of equivalence classes (4.1.24) on $\mathsf{G}^h_{3.4}$, with $\alpha = 0$ and $\beta > 0$. (Using the eigenvalues of ad_U, as discussed in remark C.0.15, it is a straightforward matter to determine h in terms of the structure constants.)

- If $(c^2_{21})^2 - 4c^2_{10} > 0$ and $c^2_{10} = 0$, then we have the family of equivalence classes (4.1.20) on $\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}$, where $\alpha > 0$ and $\gamma = 0$.

- If $(c^2_{21})^2 - 4c^2_{10} = 0$, then we have the family of equivalence classes (4.1.22) on $\mathsf{G}_{3.2}$, with $\alpha = 0$ and $\beta > 0$.

- If $(c^2_{21})^2 - 4c^2_{10} < 0$, then we have the family of equivalence classes (4.1.26) on $\mathsf{G}^h_{3.5}$, with $\alpha = 0$ and $\beta > 0$.

(b) $c^1_{21} \neq 1$, $c^2_{20} = 0$ and $c^2_{10} = 0$; from (4.1.27) we get $c^2_{21} = 0$. The eigenvalues of ad_U are

$$0,\ \frac{1}{2}\Big(u^2(c^1_{21} - 1) - |u^2|\sqrt{(c^1_{21} + 1)^2 + 4c^1_{20}}\Big),\ \frac{1}{2}\Big(u^2(c^1_{21} + 1) + |u^2|\sqrt{(c^1_{21} + 1)^2 + 4c^1_{20}}\Big).$$

It is not difficult to show that exactly two eigenvalues are zero for some U if and only if $c^1_{20} + c^1_{21} = 0$. Thus we have:

- If $c^1_{20} + c^1_{21} = 0$, then we get the family of equivalence relations (4.1.20) on $\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}$, where $\alpha = 0$ and $\gamma \neq 0$.

- If $c^1_{20} + c^1_{21} \neq 0$ and $(c^1_{21} + 1)^2 + 4c^1_{20} > 0$, then we get the family (4.1.23) of equivalence classes on $\mathsf{G}^h_{3.4}$, parametrised by $\beta \neq 0$.

- If $c^1_{20} + c^1_{21} \neq 0$ and $(c^1_{21} + 1)^2 + 4c^1_{20} = 0$, then we get the family (4.1.21) of equivalence classes on $\mathsf{G}_{3.2}$, parametrised by $\beta \neq 0$.
- If $c^1_{20} + c^1_{21} \neq 0$ and $(c^1_{21} + 1)^2 + 4c^1_{20} < 0$, then we get the family (4.1.25) of equivalence classes on $\mathsf{G}^h_{3.5}$, parametrised by $\beta \neq 0$.

(c) $c^1_{21} \neq 1$, $c^2_{20} = 0$ and $c^2_{10} \neq 0$. From (4.1.27), we have

$$\begin{cases} 0 = (c^1_{20} + 1)c^2_{21} \\ 0 = c^2_{10}(c^1_{21} - 1) + (c^2_{21})^2. \end{cases}$$

The second equation implies that $c^2_{21} \neq 0$. (Indeed, if $c^2_{21} = 0$, then $c^1_{21} = 1$ or $c^2_{10} = 0$, a contradiction.) Hence $c^1_{20} = -1$. Moreover, using $c^2_{21} + c^2_{20} \geq 0$ we find $c^2_{21} > 0$. In addition, from the second equation we get $c^1_{21} = 1 - \frac{(c^2_{21})^2}{c^2_{10}}$. The nonzero eigenvalues of ad_U are

$$\frac{1}{2c^2_{10}}\Big(c^2_{10}(u^0 - u^1) - c^2_{21}u^2\Big)\Big(c^2_{21} - \sqrt{(c^2_{21})^2 - 4c^2_{10}}\Big),$$
$$\frac{1}{2c^2_{10}}\Big(c^2_{10}(u^0 - u^1) - c^2_{21}u^2\Big)\Big(c^2_{21} + \sqrt{(c^2_{21})^2 - 4c^2_{10}}\Big).$$

It is not difficult to show that there are not exactly two eigenvalues for any U (hence G is not isomorphic to $\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}$). Therefore:

- If $(c^2_{21})^2 - 4c^2_{10} > 0$, then we get the family (4.1.24) of equivalence classes on $\mathsf{G}^h_{3.4}$, with $\alpha > 0$, $\alpha + \beta = 0$.
- If $(c^2_{21})^2 - 4c^2_{10} = 0$, then we get the family (4.1.21) of equivalence classes on $\mathsf{G}_{3.2}$, with $\alpha > 0$, $\alpha + \beta = 0$.
- If $(c^2_{21})^2 - 4c^2_{10} < 0$, then we get the family (4.1.26) of equivalence classes on $\mathsf{G}^h_{3.5}$, with $\alpha > 0$, $\alpha + \beta = 0$.

(d) $c^1_{21} \neq 1$, $c^2_{20} \neq 0$ and $c^2_{10} = 0$. From (4.1.27) we get

$$\begin{cases} 0 = c^2_{20}(c^1_{21} - 1) - (c^1_{20} + 1)c^2_{21} \\ 0 = (c^2_{21} + c^2_{20})c^2_{21}. \end{cases}$$

The first equation implies that $c^2_{21} \neq 0$. (Indeed, if $c^2_{21} = 0$, then $c^1_{21} = 1$ or $c^2_{20} = 0$, a contradiction.) Accordingly, we have $c^2_{20} = -c^2_{21}$. Substituting into the first equation, we get $c^1_{20} = -c^1_{21}$. These substitutions imply that $c^1_{10} = 0$, and so (by changing the frame if necessary) we may assume that $c^2_{21} < 0$. The eigenvalues of ad_U are

$$0,\ 0,\ u^2(c^1_{21} - 1) + (u^0 - u^1)c^2_{21},$$

i.e., G is isomorphic to $\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}$. Specifically, we get the family (4.1.20) of equivalence classes, where $\alpha > 0$ and $\gamma \neq 0$.

(e) $c^1_{21} \neq 1$, $c^2_{20} \neq 0$ and $c^2_{10} \neq 0$. From (4.1.27) we see that $c^2_{21} \neq 0$, $c^1_{20} + 1 \neq 0$ and $c^2_{20} + c^2_{21} \neq 0$. (If any of these conditions are violated, it results in a contradiction.) In fact, from $c^2_{21} + c^2_{20} \geq 0$, we get $c^2_{21} + c^2_{20} > 0$. Solving for c^1_{21} and c^2_{10} in (4.1.27), we get

$$c^1_{21} = 1 + \frac{(c^1_{20} + 1)c^2_{21}}{c^2_{20}} \quad \text{and} \quad c^2_{10} = -\frac{c^2_{20}(c^2_{21} + c^2_{20})}{c^1_{20} + 1}.$$

Consider the eigenvalues of ad_U. (We do not display the eigenvalues here, as their expressions are quite lengthy.) It is not difficult to show that there does not exist a U such that ad_U has exactly two zero eigenvalues. Furthermore, there exists a U such that the eigenvalues are complex (and distinct) exactly when $(c_{20}^1 + 1)[4(c_{20}^2)^2 + 4c_{20}^2 c_{21}^2 + (c_{20}^1 + 1)(c_{21}^2)^2] > 0$; there exists a U such that there are two identical nonzero eigenvalues and one zero eigenvalue exactly when $(c_{20}^1 + 1)[4(c_{20}^2)^2 + 4c_{20}^2 c_{21}^2 + (c_{20}^1 + 1)(c_{21}^2)^2] = 0$; and there exists a U such that there are two real and distinct eigenvalues and one zero eigenvalue exactly when $(c_{20}^1 + 1)[4(c_{20}^2)^2 + 4c_{20}^2 c_{21}^2 + (c_{20}^1 + 1)(c_{21}^2)^2] < 0$. Hence:

- If $(c_{20}^1 + 1)[4(c_{20}^2)^2 + 4c_{20}^2 c_{21}^2 + (c_{20}^1 + 1)(c_{21}^2)^2] < 0$ then we get the family (4.1.24) of equivalence classes on $\mathsf{G}_{3.4}^h$, with $\alpha > 0$, $\beta \neq 0$ and $\alpha + \beta \neq 0$.
- If $(c_{20}^1 + 1)[4(c_{20}^2)^2 + 4c_{20}^2 c_{21}^2 + (c_{20}^1 + 1)(c_{21}^2)^2] = 0$ then we get the family (4.1.21) of equivalence classes on $\mathsf{G}_{3.2}$, with $\alpha > 0$, $\beta \neq 0$ and $\alpha + \beta \neq 0$.
- If $(c_{20}^1 + 1)[4(c_{20}^2)^2 + 4c_{20}^2 c_{21}^2 + (c_{20}^1 + 1)(c_{21}^2)^2] > 0$ then we get the family (4.1.26) of equivalence classes on $\mathsf{G}_{3.5}^h$, with $\alpha > 0$, $\beta \neq 0$ and $\alpha + \beta \neq 0$. ■

Corollary 4.1.23. *Let* G *be a three-dimensional non-unimodular simply connected Lie group.*

(*i*) *If* G *is isomorphic to* $\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}$, $\mathsf{G}_{3.2}$, $\mathsf{G}_{3.4}^h$ *or* $\mathsf{G}_{3.5}^h$, $h \neq 1$, *then there exist at most two non-NH-isometric structures with the same invariants* $\vartheta > 0$, ϱ_0, ϱ_1 *and* ϱ_2.

(*ii*) *If* G *is isomorphic to* $\mathsf{G}_{3.5}^h$, $h = 1$, *then there exist infinitely many non-NH-isometric structures with the same invariants* $\vartheta > 0$, $\varrho_0 = \varrho_1 = \varrho_2 = 0$, *but at most two structures with the invariants* $\vartheta > 0$, κ *and* χ_2.

Proof. (*i*) Considering the family (4.1.20) on $\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}$, we have $\alpha = \sqrt{-2\varrho_0} = \sqrt{-2\varrho_1}$ and $\gamma = \pm\sqrt{-2\varrho_2}$. Hence there exist at most two structures with the same values of ϑ, ϱ_0, ϱ_1 and ϱ_2 (corresponding to $\gamma = -\sqrt{-2\varrho_2}$ and $\gamma = \sqrt{-2\varrho_2}$). Likewise, consider the group $\mathsf{G}_{3.5}^h$, $h \neq 1$. Clearly, as $h \neq 1$, no member of the family (4.1.25) can have the same values for ϱ_0, ϱ_1 and ϱ_2 as a member of the family (4.1.26). For the family (4.1.25) we have $\beta = \pm 2h\sqrt{-\frac{\varrho_2}{h^2-1}}$ and for (4.1.26) we have $\beta = \pm 2h\sqrt{-\frac{\varrho_0}{h^2-1}}$, $\alpha = \frac{|\beta|(h^2+1)}{2h}\sqrt{-\frac{\varrho_2}{h^2-1}}$. Hence there exist at most two structures on $\mathsf{G}_{3.5}^h$, $h \neq 1$ with identical invariants ϑ, ϱ_0, ϱ_1 and ϱ_2. The argument for $\mathsf{G}_{3.2}$ and $\mathsf{G}_{3.4}^h$ is similar.

(*ii*) If $h = 1$, then clearly $\varrho_0 = \varrho_1 = \varrho_2 = 0$ for every structure on $\mathsf{G}_{3.5}^h$, and so there are uncountably many structures with the same invariants. For the family (4.1.25) we have $\kappa = -\frac{3}{4}\beta^2 - \frac{3}{2}\beta - \frac{1}{2}$ and $\chi_2 = 0$, i.e., $\beta = -1 \pm \frac{1}{\sqrt{3}}\sqrt{1 - 4\kappa}$. On the other hand, for (4.1.26) we have $\kappa = -\frac{1}{4}(2 + 3\beta^2) - \frac{3\alpha}{\beta} - \frac{3\alpha^2}{\beta^2}$ and $\chi_2 = \frac{1}{2}|\beta|$. That is, there are at most two structures with the same invariants ϑ, κ and χ_2. ■

4.1.4 Structures with Cartan–Schouten connections

We consider the structures on three-dimensional Lie groups whose nonholonomic connection is Cartan–Schouten (see section 1.2.1). Let (X_0, X_1, X_2) be a left-invariant frame on G such that (X_1, X_2) is an orthonormal frame for $\mathcal{D}$ and X_0 is a frame for $\mathcal{D}^\perp$. We have $[X_i, X_j] = c_{ij}^k X_k$, for structure constants $c_{ij}^k \in \mathbb{R}$. In terms of this frame, the nonholonomic connection is given by $\nabla_{X_1} X_1 = c^1_{21} X_2$, $\nabla_{X_1} X_2 = -c^1_{21} X_1$, $\nabla_{X_2} X_1 = c^2_{21} X_2$ and $\nabla_{X_2} X_2 = -c^2_{21} X_1$. Accordingly, ∇ is Cartan–Schouten (i.e., $\nabla_X X = 0$ for every $X \in \Gamma^L(\mathcal{D})$; see proposition 1.2.10) if and only if $c^1_{21} = c^2_{21} = 0$.

Proposition 4.1.24. *If* G *is unimodular, then* ∇ *is Cartan–Schouten if and only if* $\vartheta = 0$. *If* G *is non-unimodular and* ∇ *is Cartan–Schouten, then* $\vartheta > 0$.

Proof. If G is unimodular, then it is straightforward to show that $c^2_{20} + c^1_{10} = 0$, $c^0_{20} + c^1_{21} = 0$ and $c^0_{10} - c^2_{21} = 0$. Then $\vartheta = (c^1_{21})^2 + (c^2_{21})^2$, from which is clear that ∇ is Cartan–Schouten exactly when $\vartheta = 0$. If G is not unimodular and ∇ is Cartan–Schouten, then $c^2_{21} = c^2_{21} = 0$ and at least one of c^0_{20}, c^0_{10} is nonzero, hence $\vartheta = (c^0_{10})^2 + (c^0_{20})^2 > 0$. ■

Proposition 4.1.25. *If* G *is non-unimodular, then* ∇ *is Cartan–Schouten if and only if* $\varrho_0 = \varrho_1 = \chi_2 = 0$ *and* $\varrho_2 + \frac{1}{2}\vartheta = 2\kappa$.

Proof. Suppose that ∇ is Cartan–Schouten, so that $c^1_{21} = c^2_{21} = 0$. By proposition 4.1.24, it follows that $\vartheta > 0$. In particular, the canonical frame (X_0, X_1, X_2), as well as the invariants ϱ_0, ϱ_1 and ϱ_2, are defined. In terms of the canonical frame, we have

$$A_{skew} = \begin{bmatrix} 0 & -\frac{1}{2}(c^1_{10} + c^2_{20}) \\ \frac{1}{2}(c^1_{10} + c^2_{20}) & 0 \end{bmatrix}.$$

If (ν^0, ν^1, ν^2) denotes the coframe dual to (X_0, X_1, X_2), then

$$\begin{cases} d\nu^0 = c^0_{10}\, \nu^0 \wedge \nu^1 + c^0_{20}\, \nu^0 \wedge \nu^2 + \nu^1 \wedge \nu^2 \\ d\nu^1 = c^1_{10}\, \nu^0 \wedge \nu^1 + c^1_{20}\, \nu^0 \wedge \nu^2 \\ d\nu^2 = c^2_{10}\, \nu^0 \wedge \nu^2 + c^2_{20}\, \nu^0 \wedge \nu^2, \end{cases}$$

where $c^0_{20} < 0$. Using $d^2 = 0$ on the first equation yields $c^1_{10} + c^2_{20} = 0$, and so $A_{skew} \equiv 0$, i.e., $\chi_2 = 0$. The remaining two equations (together with $c^1_{21} = c^2_{21} = 0$ and $c^0_{20} \neq 0$) imply that $c^2_{20} = c^2_{10} = 0$. It follows that $\varrho_0 = \varrho_1 = 0$ and $\varrho_2 + \frac{1}{2}\vartheta = 2\kappa$. Conversely, suppose $\varrho_0 = \varrho_1 = \chi_2 = 0$ and $\varrho_2 + \frac{1}{2}\vartheta = 2\kappa$. The first three conditions, together with the identities obtained from the dual frame, imply that $c^2_{10} = c^2_{20} = c^2_{21} = 0$. From $\varrho_2 + \frac{1}{2}\vartheta = 2\kappa$ we then have $c^1_{12} = 0$, whence ∇ is a Cartan–Schouten connection. ■

If G is unimodular, then the classification under NH-isometry of (left-invariant) nonholonomic Riemannian structures for which ∇ is Cartan–Schouten coincides with the classification of (left-invariant) sub-Riemannian structures (proposition 4.1.24). On the other hand, it is easy to see that, on the non-unimodular three-dimensional Lie groups, the following equivalence classes of nonholonomic Riemannian structures are those whose nonholonomic connection is Cartan–Schouten:

- On $\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}$, the equivalence class (4.1.20) with $\alpha = 0$ and $\gamma = 1$.
- On $\mathsf{G}_{3.2}$, the equivalence class (4.1.21) with $\beta = 1$.
- On $\mathsf{G}^h_{3.4}$, the equivalence class (4.1.23) with $\beta = 1$.
- On $\mathsf{G}^h_{3.5}$, the equivalence class (4.1.25) with $\beta = 1$.

In particular, on a fixed (three-dimensional) non-unimodular group any two left-invariant structures whose nonholonomic connections are Cartan–Schouten connections are NH-isometric. (Hence the classification of such structures under NH-geodesic equivalence coincides with the classification under NH-isometries; see proposition 3.1.15 and remark C.0.18.) This is in contrast to the unimodular case. Indeed, again from proposition 3.1.15 and remark C.0.18, there exists, up to NH-geodesic equivalence, exactly one equivalence class of left-invariant structures on each of H_3, $\mathsf{SE}(1,1)$, $\widetilde{\mathsf{SE}}(2)$ and $\mathsf{SU}(2)$ with Cartan–Schouten connections. Likewise, there exists exactly two equivalence classes (up to NH-geodesic equivalence) on $\widetilde{\mathsf{SL}}(2,\mathbb{R})$ of left-invariant structures with Cartan–Schouten connections, according as whether the Killing form is definite or indefinite on the distribution (at identity). Hence we have a number of examples of structures with Cartan–Schouten connections that are not NH-isometric. (These are also examples of structures that are NH-geodesically equivalent, but not NH-isometric.)

4.1.5 Structures with geodesically invariant distributions

We consider the structures on three-dimensional Lie groups whose distribution is geodesically invariant (see section 3.2 and, in particular, section 3.2.3). In light of our classification of left-invariant structures on three-dimensional Lie groups, we shall list those (equivalence classes of) structures with geodesically invariant distribution. The following characterisation is an immediate consequence of theorem 3.2.16 and proposition 4.1.1.

Proposition 4.1.26. *$\mathcal{D}$ is geodesically invariant if and only if $\chi_1 = \chi_2 = 0$.*

Hence, in order to determine the (equivalence classes of) structures with geodesically invariant distribution, it is simply a matter of computing the invariants χ_1 and χ_2 for all structures. This yields the following list.

(*i*) When $\vartheta = 0$:

- Any structure on H_3 or $\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}$.
- Any structure on $\mathsf{SU}(2)$ or $\widetilde{\mathsf{SL}}(2,\mathbb{R})$ whose metric (at identity) is a rescaling of $\mathcal{K}|_{\mathcal{D}_1}$.

(*ii*) When $\vartheta > 0$:

- On $\mathsf{SE}(1,1)$, the equivalence class (4.1.16) with $\alpha_1 = 0$, $\alpha_2 = 1$.
- On $\widetilde{\mathsf{SE}}(2)$, the equivalence class (4.1.15) with $\alpha_1 = 1$, $\alpha_2 = 0$.
- On $\mathsf{SU}(2)$, the family of equivalence classes (4.1.17) with $\delta = 0$, $\alpha_1 - \alpha_2 = 1$.
- On $\widetilde{\mathsf{SL}}(2,\mathbb{R})_{ell}$, the family of equivalence classes (4.1.18) with $\delta = 0$, $\alpha_1 - \alpha_2 = -1$.
- On $\widetilde{\mathsf{SL}}(2,\mathbb{R})_{hyp}$, the family of equivalence classes (4.1.19) with $\delta = 0$, $\gamma_1 - \gamma_2 = -1$, where $-1 < \gamma_1 < 0$.
- On $\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}$, the equivalence class (4.1.20) with $\alpha = 0$, $\gamma = 1$.
- On $\mathsf{G}_{3.2}$, the equivalence class (4.1.21) with $\beta = 2$.
- On $\mathsf{G}_{3.4}^h$, the equivalence class (4.1.23) with $\beta = \frac{2h}{h+1}$ or $\beta = \frac{2h}{h-1}$.

Remarkably, there does not exist a geodesically invariant distribution on $\mathsf{G}_{3.5}^h$.

4.2 Flat structures

Having classified the left-invariant nonholonomic Riemannian structures on the three-dimensional (simply connected) Lie groups, it is of interest to determine those structures that are flat, i.e., those nonholonomic Riemannian structures whose associated parallel transport (induced by the nonholonomic connection) is path-independent. (See section 2.2.2 and section B.1.1.1.) This is the problem we consider here. We first characterise flatness in three dimensions, before using the characterisation (and the classification obtained in section 4.1) to classify the flat left-invariant structures. Our characterisation is obtained by taking a direct approach to the problem; accordingly, in section 4.2.3 we relate it with the Wagner curvature tensor. Section D.2 of appendix D lists the Mathematica code used for the calculations in this section.

4.2.1 Characterisation

Let $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ be a nonholonomic Riemannian structure on a three-dimensional manifold M. Let (X_0, X_1, X_2) be a local frame defined on a neighbourhood $\mathcal{U} \subseteq \mathsf{M}$ such that X_0 is a frame for $\mathcal{D}^\perp$ and (X_1, X_2) is an orthonormal frame for $\mathcal{D}$. Let $c_{ij}^k \in \mathcal{C}^\infty(\mathcal{U})$ denote the structure constants of this frame (where i, j, k range through 0, 1, 2). We suppose, without loss of generality, that $c_{21}^0 = 1$.

Lemma 4.2.1. *The structure* $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ *is locally flat on* $\mathcal{U}$ *if and only if there exists a function* $\theta \in \mathcal{C}^\infty(\mathcal{U})$ *such that*

$$X_1[\theta] = c_{21}^1 \quad \text{and} \quad X_2[\theta] = c_{21}^2.$$

If such a function θ *exists, then the rotated frame* $(\cos\theta X_1 - \sin\theta X_2, \sin\theta X_1 + \cos\theta X_2)$ *is parallel.*

Proof. Let (Y_1, Y_2) be any other orthonormal frame for $\mathcal{D}$ defined on $\mathcal{U}$. There exists an orthogonal transformation taking (X_1, X_2) to (Y_1, Y_2). That is, there exists $\sigma \in \{-1, 1\}$ and $\theta \in \mathcal{C}^\infty(\mathcal{U})$ such that

$$\begin{cases} Y_1 = \sigma \cos\theta X_1 - \sin\theta X_2 \\ Y_2 = \sigma \sin\theta X_1 + \cos\theta X_2. \end{cases}$$

By proposition 2.2.13, the frame (Y_1, Y_2) is parallel if and only if $\llbracket Y_2, Y_1 \rrbracket = 0$. We have $\llbracket Y_2, Y_1 \rrbracket = (\sigma c_{21}^1 - X_1[\theta])X_1 + (\sigma c_{21}^2 - X_2[\theta])X_2$, and so (Y_1, Y_2) is a parallel frame for $\mathcal{D}$ exactly when $X_1[\theta] = \sigma c_{21}^1$ and $X_2[\theta] = \sigma c_{21}^2$. By reversing the sign of θ if necessary, we may take $\sigma = 1$. ■

Lemma 4.2.2. *There exists a rotation* (Y_1, Y_2) *of* (X_1, X_2) *such that* $\llbracket Y_2, Y_1 \rrbracket = 0$ *if and only if the following equations hold*:

$$\begin{cases} (c_{10}^1 - c_{20}^2)c_{21}^1 + (c_{10}^2 + c_{20}^1)c_{21}^2 + c_{20}^0 c_{10}^1 - \frac{1}{2}c_{10}^0(c_{10}^2 + c_{20}^1) + c_{10}^0\kappa \\ \qquad = -\frac{1}{2}X_1[c_{10}^2 + c_{20}^1] + X_1[\kappa] + X_2[c_{10}^1] \\ (c_{10}^2 + c_{20}^1)c_{21}^1 - (c_{10}^1 - c_{20}^2)c_{21}^2 - c_{10}^0 c_{20}^2 + \frac{1}{2}c_{20}^0(c_{10}^2 + c_{20}^1) + c_{20}^0\kappa \\ \qquad = \frac{1}{2}X_2[c_{10}^2 + c_{20}^1] + X_2[\kappa] - X_1[c_{20}^2]. \end{cases} \tag{4.2.1}$$

Proof. Let $(Y_1, Y_2) = (\cos\theta X_1 - \sin\theta X_2, \sin\theta X_1 + \cos\theta X_2)$ be a rotation of (X_1, X_2), where $\theta \in \mathcal{C}^\infty(\mathcal{U})$. By lemma 4.2.1, we have that $[\![Y_2, Y_1]\!] = 0$ if and only if $X_1[\theta] = c^1_{21}$ and $X_2[\theta] = c^2_{21}$. We claim that there exists $\theta \in \mathcal{C}^\infty(\mathcal{U})$ satisfying the conditions $X_1[\theta] = c^1_{21}$, $X_2[\theta] = c^2_{21}$ if and only if (4.2.1) hold. If such a θ exists, then we have

$$\begin{aligned} d\theta &= X_0[\theta]\nu^0 + c^1_{21}\nu^1 + c^2_{21}\nu^2 \\ &= ([X_2, X_1][\theta] - c^1_{21}X_1[\theta] - c^2_{21}X_2[\theta])\nu^0 + c^1_{21}\nu^1 + c^2_{21}\nu^2 \\ &= (X_2[c^1_{21}] - X_1[c^2_{21}] - (c^1_{21})^2 - (c^2_{21})^2)\nu^0 + c^1_{21}\nu^1 + c^2_{21}\nu^2 \\ &= (\kappa - \frac{1}{2}(c^2_{10} - c^1_{20}))\nu^0 + c^1_{21}\nu^1 + c^2_{21}\nu^2. \end{aligned}$$

The right-hand side is independent of θ; accordingly, let $\varpi = (\kappa - \frac{1}{2}(c^2_{10} - c^1_{20}))\nu^0 + c^1_{21}\nu^1 + c^2_{21}\nu^2$. Then, $d\varpi = f_{01}\,\nu^0 \wedge \nu^1 + f_{02}\,\nu^0 \wedge \nu^2$, where

$$\begin{cases} f_{01} = c^1_{10}c^1_{21} + c^2_{10}c^2_{21} - \dfrac{1}{2}c^0_{10}(c^2_{10} - c^1_{20}) + c^0_{10}\kappa + \dfrac{1}{2}X_1[c^2_{10} - c^1_{20}] - X_1[\kappa] + X_0[c^1_{21}] \\ f_{02} = c^1_{20}c^1_{21} + c^2_{20}c^2_{21} - \dfrac{1}{2}c^0_{20}(c^2_{10} - c^1_{20}) + c^0_{20}\kappa + \dfrac{1}{2}X_2[c^2_{10} - c^1_{20}] - X_2[\kappa] + X_0[c^2_{21}]. \end{cases}$$

(Note that the $\nu^1 \wedge \nu^2$ term in $d\varpi$ vanishes.) Using $d^2 = 0$ on the structure equations $d\nu^k = \sum_{0 \le i < j \le 2} c^k_{ij}\nu^j \wedge \nu^i$ of the dual frame (ν^0, ν^1, ν^2), we get

$$\begin{cases} 0 = \quad -c^1_{10} - c^2_{20} + c^0_{10}c^1_{21} + c^0_{20}c^2_{21} + X_1[c^0_{20}] - X_2[c^0_{10}] \\ X_0[c^1_{21}] = -c^1_{20}c^0_{10} + c^1_{10}c^0_{20} - c^2_{20}c^1_{21} + c^1_{20}c^2_{21} + X_1[c^1_{20}] - X_2[c^1_{10}] \\ X_0[c^2_{21}] = -c^2_{20}c^0_{10} + c^2_{10}c^0_{20} + c^2_{10}c^1_{21} - c^1_{10}c^2_{21} + X_1[c^2_{20}] - X_2[c^2_{10}]. \end{cases}$$

Hence

$$\begin{cases} f_{01} = (c^1_{10} - c^2_{20})c^1_{21} + (c^2_{10} + c^1_{20})c^2_{21} + c^0_{20}c^1_{10} - \dfrac{1}{2}c^0_{10}(c^2_{10} + c^1_{20}) + c^0_{10}\kappa \\ \qquad + \dfrac{1}{2}X_1[c^2_{10} + c^1_{20}] - X_1[\kappa] - X_2[c^1_{10}] \\ f_{02} = (c^2_{10} + c^1_{20})c^1_{21} - (c^1_{10} - c^2_{20})c^2_{21} - c^2_{20}c^0_{10} + \dfrac{1}{2}c^0_{20}(c^2_{10} + c^1_{20}) + c^0_{20}\kappa \\ \qquad - \dfrac{1}{2}X_2[c^2_{10} + c^1_{20}] - X_2[\kappa] + X_1[c^2_{20}]. \end{cases}$$

Suppose θ exists, so that $d\varpi = d^2\theta = 0$. Then $f_{01} = f_{02} = 0$, from which we get the equations (4.2.1). Conversely, if (4.2.1) hold, then $d\varpi = 0$, i.e., ϖ is closed. Hence it is locally exact: there exists a neighbourhood $\mathcal{U}' \subseteq \mathcal{U}$ and $\theta \in \mathcal{C}^\infty(\mathcal{U}')$ such that $\varpi = d\theta$. Restricting (X_1, X_2) to $\mathcal{U}'$, it follows that the requisite rotation exists (defined on $\mathcal{U}'$). ∎

Using the two equations in lemma 4.2.2 (which depend on an orthonormal frame), we shall derive an invariant characterisation of the flat structures in three dimensions. We will require the exterior covariant derivative operator associated to ∇; see section 1.1.4. Recall the following notation: $\mathcal{D} = \ker\omega$, where ω is the normalised contact form on M (i.e., we have $d\omega(X_1, X_2) = \pm 1$). Furthermore, we have the decomposition $T\mathsf{M} = \mathcal{D} \oplus \operatorname{span}\{Y_0\}$, where Y_0 is the Reeb vector field of ω. Let $\mathscr{R} : T\mathsf{M} \to \operatorname{span}\{Y_0\}$ be the projection onto the distribution spanned by Y_0. In particular, we have

$$\mathscr{R}([X_2, X_1]) = d\omega(X_1, X_2)Y_0.$$

Theorem 4.2.3. $(\mathsf{M},\mathcal{D},\mathcal{D}^{\perp},\mathbf{g})$ *is locally flat on* $\mathcal{U}\subseteq\mathsf{M}$ *if and only if*

$$d^{\nabla}_{\mathscr{P}}F = F\circ\rho \quad \text{on}\, \mathcal{U}, \tag{4.2.2}$$

where $F=\mathbf{g}^{\sharp}\circ(\operatorname{tr}^1_1 K)^{\flat}=\mathbf{g}^{\sharp}\circ(\operatorname{Ric}^{\flat}+A^{\flat}_{sym}+A^{\flat}_{skew})$ *and* $\rho\in\Omega^2(\mathcal{D},\mathcal{D})$ *is the vector-valued 2-form given by* $\rho(X_1,X_2)=-\mathscr{P}(\mathscr{R}([X_1,X_2]))$.

Proof. We have $d^{\nabla}_{\mathscr{P}}(\mathbf{g}^{\sharp}\circ\operatorname{Ric}^{\flat})(X_1,X_2)=-X_2[\kappa]X_1+X_1[\kappa]X_2$. Indeed, since $(\mathbf{g}^{\sharp}\circ\operatorname{Ric}^{\flat})(X_a)=\operatorname{Ric}(X_a,X_1)X_1+\operatorname{Ric}(X_a,X_2)X_2$, $\operatorname{Ric}(X_a,X_a)=\kappa$ and $\operatorname{Ric}(X_a,X_b)=0$ for $a\neq b$, we have

$$\begin{aligned} &d^{\nabla}_{\mathscr{P}}(\mathbf{g}^{\sharp}\circ\operatorname{Ric}^{\flat})(X_1,X_2)\\ &\quad=\nabla_{X_1}(\mathbf{g}^{\sharp}\circ\operatorname{Ric}^{\flat})(X_2)-\nabla_{X_2}(\mathbf{g}^{\sharp}\circ\operatorname{Ric}^{\flat})(X_1)-(\mathbf{g}^{\sharp}\circ\operatorname{Ric}^{\flat})([\![X_1,X_2]\!])\\ &\quad=\nabla_{X_1}(\kappa X_2)-\nabla_{X_2}(\kappa X_1)+(c^1_{21}\kappa X_1+c^2_{21}\kappa X_2).\end{aligned}$$

Using $\nabla_{X_1}X_2=-c^1_{21}X_1$ and $\nabla_{X_2}X_1=c^2_{21}X_2$ gives

$$\begin{aligned} d^{\nabla}_{\mathscr{P}}(\mathbf{g}^{\sharp}\circ\operatorname{Ric}^{\flat})(X_1,X_2)&=X_1[\kappa]X_2-\kappa c^1_{21}X_1-X_2[\kappa]X_1-\kappa c^2_{21}X_2+(c^1_{21}\kappa X_1+c^2_{21}\kappa X_2)\\ &=-X_2[\kappa]X_1+X_1[\kappa]X_2.\end{aligned}$$

Similar calculations yield

$$\begin{aligned} d^{\nabla}_{\mathscr{P}}(\mathbf{g}^{\sharp}\circ A^{\flat})(X_1,X_2)=\Big(X_1[c^2_{20}]-\frac{1}{2}X_2[c^2_{10}+c^1_{20}]\Big)X_1+\Big(X_2[c^1_{10}]-\frac{1}{2}X_1[c^2_{10}+c^1_{20}]\Big)X_2\\ +2\,(\mathbf{g}^{\sharp}\circ A^{\flat}_{sym})([\![X_2,X_1]\!]).\end{aligned}$$

(Here $A=A_{sym}+A_{skew}$.) We also have $(\mathbf{g}^{\sharp}\circ\operatorname{Ric}^{\flat}\circ\rho)(X_1,X_2)=-c^0_{20}\kappa X_1+c^0_{10}\kappa X_2$ and

$$(\mathbf{g}^{\sharp}\circ A^{\flat}\circ\rho)(X_1,X_2)=-\Big(c^0_{10}c^1_{10}+\frac{1}{2}c^0_{20}(c^2_{10}+c^1_{20})\Big)X_1-\Big(c^0_{20}c^2_{20}+\frac{1}{2}c^0_{10}(c^2_{10}+c^1_{20})\Big)X_2.$$

Let f_{01} and f_{02} be defined as in the proof of lemma 4.2.2. The requisite rotation of (X_1,X_2) into a parallel frame for $\mathcal{D}$ exists if and only if $f_{01}=f_{02}=0$. Combining the above calculations, we see that

$$(d^{\nabla}_{\mathscr{P}}F-F\circ\rho)(X_1,X_2)=f_{02}X_1-f_{01}X_2,$$

from which the result follows immediately. (As $d^{\nabla}_{\mathscr{P}}F-F\circ\rho$ is skew-symmetric, it is fully determined by its evaluation of $X_1\wedge X_2$.) ■

Corollary 4.2.4. $(\mathsf{M},\mathcal{D},\mathcal{D}^{\perp},\mathbf{g})$ *is locally flat on* $\mathcal{U}\subseteq\mathsf{M}$ *if and only if*

$$d^{\nabla}_{\mathscr{P}}G=G\circ\rho\quad\text{on}\,\mathcal{U},$$

where $G=(\operatorname{tr}^1_1 K)^{\flat}=\operatorname{Ric}^{\flat}+A^{\flat}_{sym}+A^{\flat}_{skew}$.

Proof. Let $X,Y\in\Gamma(\mathcal{D})$. Since ∇_X and ∇_Y commute with $\mathbf{g}^{\sharp}$ (as ∇ is metric), we have

$$\begin{aligned} d^{\nabla}_{\mathscr{P}}F(X,Y)&=d^{\nabla}_{\mathscr{P}}(\mathbf{g}^{\sharp}\circ G)(X,Y)\\ &=\nabla_X(\mathbf{g}^{\sharp}\circ G)(Y)-\nabla_Y(\mathbf{g}^{\sharp}\circ G)(X)-(\mathbf{g}^{\sharp}\circ G)([\![X,Y]\!])\\ &=(\mathbf{g}^{\sharp}\circ(\nabla_X G))(Y)-(\mathbf{g}^{\sharp}\circ(\nabla_Y G))(X)-(\mathbf{g}^{\sharp}\circ G)([\![X,Y]\!])\\ &=\mathbf{g}^{\sharp}(d^{\nabla}_{\mathscr{P}}G(X,Y)).\end{aligned}$$

Hence $d^{\nabla}_{\mathscr{P}}F-F\circ\rho=\mathbf{g}^{\sharp}\circ(d^{\nabla}_{\mathscr{P}}G-G\circ\rho)$, from which the result follows immediately. ■

Corollary 4.2.5. *If* $\vartheta = 0$, *then* $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ *is locally flat on* $\mathcal{U} \subseteq \mathsf{M}$ *if and only if* $d^\nabla_{\mathscr{P}} F = 0$ (*equivalently,* $d^\nabla_{\mathscr{P}} G = 0$) *on* $\mathcal{U}$.

Proof. If $\vartheta = 0$, then $\mathscr{P}(Y_0) = 0$, and so $\rho \equiv 0$. Hence $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ is flat exactly when $d^\nabla_{\mathscr{P}} F$ (or equivalently, $d^\nabla_{\mathscr{P}} G$) vanishes. ■

Corollary 4.2.6. *If* $K \equiv 0$ *on* $\mathcal{U} \subseteq \mathsf{M}$, *then* $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ *is locally flat on* $\mathcal{U}$.

Proof. If $K \equiv 0$, then $F = \mathbf{g}^\sharp \circ (\operatorname{tr}^1_1 K)^\flat = 0$, and the condition $d^\nabla_{\mathscr{P}} F = F \circ \rho$ is trivially satisfied. ■

4.2.2 Classification

We are now in a position to classify the flat structures. Let $(\mathsf{G}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ be a left-invariant nonholonomic Riemannian structure on a three-dimensional simply connected Lie group G. We split the classification into several cases, *viz.*, $\vartheta = 0$; $\vartheta > 0$ and G is unimodular; $\vartheta > 0$ and G is non-unimodular.

Suppose that (X_0, X_1, X_2) is a left-invariant frame on G such that X_0 spans $\mathcal{D}^\perp$ and (X_1, X_2) is an orthonormal frame for $\mathcal{D}$. By left invariance, the structure constants c^k_{ij} of the frame are constant; as before, we take $c^0_{21} = 1$. With respect to (X_1, X_2), we have

$$d^\nabla_{\mathscr{P}} F(X_1, X_2) = 2\,(\mathbf{g}^\sharp \circ A^\flat_{sym})(\llbracket X_2, X_1 \rrbracket) = \begin{bmatrix} c^2_{10} + c^1_{20} & -(c^1_{10} - c^2_{20}) \\ c^2_{20} - c^1_{10} & -(c^2_{10} + c^1_{20}) \end{bmatrix} \begin{bmatrix} c^1_{21} \\ c^2_{21} \end{bmatrix} \tag{4.2.3}$$

and

$$(F \circ \rho)(X_1, X_2) = \begin{bmatrix} \kappa + \frac{1}{2}(c^2_{10} + c^1_{20}) & c^2_{20} \\ -c^1_{10} & \kappa - \frac{1}{2}(c^2_{10} + c^1_{20}) \end{bmatrix} \begin{bmatrix} -c^0_{20} \\ c^0_{10} \end{bmatrix}. \tag{4.2.4}$$

Theorem 4.2.7. *If* $\vartheta = 0$, *then* $(\mathsf{G}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ *is flat if and only if*:

(*i*) *It is a structure on a unimodular Lie group* (*with a Cartan–Schouten connection*).

(*ii*) *It is a structure on* $\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}$.

Proof. By proposition 4.1.24, we have that every nonholonomic Riemannian structure on a unimodular Lie group has a Cartan–Schouten connection, and hence, by corollary 2.4.8, is flat. Since any structure on $\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}$ is NH-isometric to a flat structure on $\widetilde{\mathsf{SL}}(2, \mathbb{R})_{ell}$, it follows by proposition 3.1.8 that any structure on $\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}$ is also flat. (However, since the NH-isometry between the structures on $\widetilde{\mathsf{SL}}(2, \mathbb{R})_{ell}$ and $\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}$ does not preserve left-invariant vector fields—see remark 4.1.12—there will not exist a *left-invariant* parallel frame on $\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}$ in this case.)

We claim that these are the only flat structures for this case. Indeed, by corollary 4.2.5 and equation (4.2.3), $(\mathsf{G}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ is flat exactly when $A^\flat_{sym}(\llbracket X_2, X_2 \rrbracket) = 0$. If $\chi_1 > 0$, then $A^\flat_{sym}$ is invertible, and $\llbracket X_2, X_1 \rrbracket = 0$ implies that G is unimodular. On the other hand, if $\chi_1 = 0$, then from theorem 4.1.11 we have that G is either unimodular or $\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}$. ■

Remark 4.2.8. The second part of the above proof (i.e., when $\chi_1 = 0$) was essentially proved by Agrachev and Barilari [1]. (Their result is replicated in proposition 4.1.8.) □

For the remainder of this section we assume that ϑ is positive. Let (X_0, X_1, X_2) be the canonical frame described in section 4.1.3.

Theorem 4.2.9. *If* $\vartheta > 0$ *and* G *is unimodular, then* $(\mathsf{G}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ *is flat if and only if* $\varrho_0 = \vartheta\varrho_1$ *and* $\varrho_2 = 0$. *Furthermore, if the structure is flat, then* $\kappa^2 = \chi_1^2$. *In particular, the following equivalence classes are flat:*

(*i*) *If* $\varrho_0 < 0$*, then the equivalence class of structures* (4.1.16) *on* $\mathsf{SE}(1,1)$ *with* $\alpha_2 = 0$.

(*ii*) *If* $\varrho_0 = 0$*, then the equivalence class of structures* (4.1.14) *on* H_3.

(*iii*) *If* $\varrho_0 > 0$*, then the equivalence class of structures* (4.1.15) *on* $\widetilde{\mathsf{SE}}(2)$ *with* $\alpha_2 = 0$.

Proof. Since G is unimodular, we have $c_{21}^2 = 0$ and $c_{21}^1 = -c_{20}^0 = \|\mathscr{P}(Y_0)\|$, whence

$$[\![X_2, X_1]\!] = d\omega(X_1, X_2)\mathscr{P}(Y_0).$$

Furthermore, we have $\chi_2 = 0$, i.e., $A_{skew} \equiv 0$. The condition (4.2.2) becomes

$$\begin{aligned} 2\,(\mathbf{g}^\sharp \circ A^\flat_{sym})([\![X_2, X_1]\!]) &= (\mathbf{g}^\sharp \circ (\mathrm{Ric}^\flat + A^\flat_{sym}))([\![X_2, X_1]\!]) \\ \iff \quad & \mathrm{Ric}^\flat([\![X_2, X_1]\!]) = A^\flat_{sym}([\![X_2, X_1]\!]) \\ \iff \quad & \mathrm{Ric}^\flat(X_1) = A^\flat_{sym}(X_1) \\ \iff \quad & A_{sym}(X_1, X_1) = \kappa \text{ and } A_{sym}(X_1, X_2) = 0. \end{aligned}$$

In terms of the structure constants, this is equivalent to the conditions $c_{10}^1 = 0$, $c_{20}^1 = -(c_{20}^0)^2$, which are in turn equivalent to $\varrho_0 = \vartheta\varrho_1$ and $\varrho_2 = 0$. Furthermore, this implies that $\kappa^2 = \chi_1^2$. In terms of the classification in theorem 4.1.19, these are the following structures:

- If $\varrho_0 < 0$, then the equivalence class of structures (4.1.16) on $\mathsf{SE}(1,1)$ with $\alpha_2 = 0$.
- If $\varrho_0 = 0$, then the equivalence class of structures (4.1.14) on H_3.
- If $\varrho_0 > 0$, then the equivalence class of structures (4.1.15) on $\widetilde{\mathsf{SE}}(2)$ with $\alpha_2 = 0$. ■

Notice that, apart from the structures with Cartan–Schouten connections (theorem 4.2.7), there are no flat structures on the semisimple groups $\mathsf{SU}(2)$ and $\widetilde{\mathsf{SL}}(2,\mathbb{R})$.

Theorem 4.2.10. *If* $\vartheta > 0$ *and* G *is non-unimodular, then* $(\mathsf{G}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ *is flat in exactly the following circumstances:*

(*i*) $\mathsf{G} = \mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}$, *i.e., every structure on* $\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}$ *is flat.*

(*ii*) *If* $\chi_2 = 0$*, then any structure on* $\mathsf{G}_{3.2}$*,* $\mathsf{G}_{3.4}^h$ *and* $\mathsf{G}_{3.5}^h$ *with a Cartan–Schouten connection (i.e.,* $\varrho_0 = \varrho_1 = \chi_2 = 0$ *and* $\varrho_2 + \frac{1}{2}\vartheta = 2\kappa$*; see proposition 4.1.25).*

(*iii*) *If* $\chi_2 > 0$*, then any structure on* $\mathsf{G}_{3.2}$*,* $\mathsf{G}_{3.4}^h$ *and* $\mathsf{G}_{3.5}^h$ *with* $\chi_2 > 0$ *that is NH-isometric (up to rescaling) to the following structures:*

(*a*) *On* $\mathsf{G}_{3.2}$*, the equivalence class of structures* (4.1.22) *with*

$$\alpha = -\frac{\beta}{8}\left(1 \pm \sqrt{1 - 4\beta^2}\right) \quad \textit{and} \quad -\frac{1}{2} \leq \beta < 0.$$

(*b*) *On* $\mathsf{G}_{3.4}^{h}$*, the equivalence class of structures* (4.1.24) *with*

$$\begin{aligned} \alpha &= -\frac{(h^2-1)\beta}{8h^2}\big(1 \pm \sqrt{1-4\beta^2}\big) \quad \text{and} \quad 0 < \beta \leq \frac{1}{2} \qquad \text{when } 0 < h < 1 \\ \alpha &= -\frac{(h^2-1)\beta}{8h^2}\big(1 \pm \sqrt{1-4\beta^2}\big) \quad \text{and} \quad -\frac{1}{2} \leq \beta < 0 \qquad \text{when } 1 < h. \end{aligned}$$

(*c*) *On* $\mathsf{G}_{3.5}^{h}$*, the equivalence class of structures* (4.1.26) *with*

$$\alpha = -\frac{(h^2+1)\beta}{8h^2}\big(1 \pm \sqrt{1-4\beta^2}\big) \quad \text{and} \quad -\frac{1}{2} \leq \beta < 0.$$

Proof. Considering the equivalence class representatives of theorem 4.1.22, a direct (but tedious) calculation, using the condition that (4.2.3) equals (4.2.4), yields the result. We have used MATHEMATICA for the computations; see appendix D, and specifically section D.1, for the code. Nevertheless, we illustrate here with the case of $\mathsf{G}_{3.2}$. Consider first the family of equivalence classes (4.1.21); we have

$$\begin{aligned} &(d^{\nabla}_{\mathscr{P}}F - F \circ \rho)(X_1, X_2) \\ &\quad = \begin{bmatrix} -\frac{1}{4}(\beta-2)^2 & 0 \\ 0 & \frac{1}{4}(\beta-2)^2 \end{bmatrix} \begin{bmatrix} 1-\beta \\ 0 \end{bmatrix} - \begin{bmatrix} -(\beta-1)^2 & 0 \\ 0 & -\frac{3}{4}\beta(\beta-\frac{4}{3}) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &\quad = \begin{bmatrix} \frac{1}{4}\beta^2(\beta-1) \\ 0 \end{bmatrix}. \end{aligned}$$

Hence such a structure is flat exactly when $\beta = 1$. From section 4.1.4, this is exactly the structure on $\mathsf{G}_{3.2}$ with a Cartan–Schouten connection. On the other hand, consider the family of equivalence classes (4.1.22):

$$(d^{\nabla}_{\mathscr{P}}F - F \circ \rho)(X_1, X_2) = \begin{bmatrix} -\frac{\alpha}{\beta^3}(16\alpha^2 + 4\alpha\beta + \beta^4) \\ \frac{1}{4\beta}(16\alpha^2 + 4\alpha\beta + \beta^4) \end{bmatrix}.$$

Thus the structure is flat if and only if $16\alpha^2 + 4\alpha\beta + \beta^4 = 0$. It is not difficult to show (we have again used MATHEMATICA) that this occurs exactly when $\alpha = -\frac{\beta}{8}(1 \pm \sqrt{1-4\beta^2})$ and $-\frac{1}{2} \leq \beta < 0$. ■

Remarkably, from theorem 4.2.7, theorem 4.2.9 and theorem 4.2.10 we have that every left-invariant nonholonomic Riemannian structure on H_3 and $\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}$ is flat.

4.2.3 Characterisation using the Wagner curvature tensor

In this section we relate the results obtained in the previous two sections with the Wagner curvature tensor (section 2.2). In particular, we obtain another invariant characterisation for a structure to be flat in terms of a contraction of the Wagner tensor.

Let $Z \in \Gamma(T\mathsf{M})$ and let $X = \mathscr{P}(Z)$ and $A = \mathscr{Q}(Z)$. The (vector bundle) connection $\nabla^2 : \Gamma(T\mathsf{M}) \times \Gamma(\mathcal{D}) \to \Gamma(\mathcal{D})$ is given by

$$\nabla^2_Z U = \nabla_X U + K(\Theta(A))U + [\![A, U]\!].$$

Here $\Theta = \Delta|^{-1}_{(\ker\Delta)^\perp}$, where $\Delta : \Gamma(\bigwedge^2 \mathcal{D}) \to \Gamma(\mathcal{D}^\perp)$, $X \wedge Y \mapsto \mathscr{Q}([X,Y])$. (In fact, we have $\ker\Delta = \{0\}$; see below.) For convenience, we shall denote $\widetilde{\nabla} = \nabla^2$. The Wagner curvature tensor $\widetilde{K} = K^2 \in \mathcal{T}^0_2(T\mathsf{M}) \otimes \mathcal{T}^1_1(\mathcal{D})$ is given by

$$\widetilde{K}(X \wedge Y)U = [\widetilde{\nabla}_X, \widetilde{\nabla}_Y]U - \widetilde{\nabla}_{[X,Y]}U, \qquad X, Y \in \Gamma(T\mathsf{M}),\ U \in \Gamma(\mathcal{D}).$$

Proposition 4.2.11. *Δ is an isomorphism, i.e., the vector bundles $\bigwedge^2 \mathcal{D}$ and $\mathcal{D}^\perp$ are isomorphic.*

Proof. Since Δ is surjective (lemma 2.2.3), it suffices to show that its kernel is trivial. We have

$$\Delta(X_1 \wedge X_2) = \mathscr{Q}([X_1, X_2]) = -X_0.$$

Since Δ is tensorial, it follows that $\Delta(X \wedge Y) = 0$ for $X, Y \in \Gamma(\mathcal{D})$ if and only if $X \wedge Y = 0$, and hence $\ker\Delta = \{0\}$. ■

Let (X_0, X_1, X_2) be a (local) frame, such that X_0 is a frame for $\mathcal{D}^\perp$ and (X_1, X_2) is an orthonormal frame for $\mathcal{D}$. Let $c^k_{ij} \in \mathcal{C}^\infty(\mathsf{M})$ be the structure constants of this frame. As usual, we may suppose that $c^0_{21} = 1$. Let f_{01} and f_{02} be the expressions given in the proof of lemma 4.2.2, i.e.,

$$\begin{cases} f_{01} = (c^1_{10} - c^2_{20})c^1_{21} + (c^2_{10} + c^1_{20})c^2_{21} + c^0_{20}c^1_{10} - \frac{1}{2}c^0_{10}(c^2_{10} + c^1_{20}) + c^0_{10}\kappa \\ \qquad + \frac{1}{2}X_1[c^2_{10} + c^1_{20}] - X_1[\kappa] - X_2[c^1_{10}] \\ f_{02} = (c^2_{10} + c^1_{20})c^1_{21} - (c^1_{10} - c^2_{20})c^2_{21} - c^2_{20}c^0_{10} + \frac{1}{2}c^0_{20}(c^2_{10} + c^1_{20}) + c^0_{20}\kappa \\ \qquad - \frac{1}{2}X_2[c^2_{10} + c^1_{20}] - X_2[\kappa] + X_1[c^2_{20}]. \end{cases}$$

It turns out that f_{01} and f_{02} are, up to sign, the only components of the Wagner curvature tensor $\widetilde{K}$.

Lemma 4.2.12. *We have*

$$\begin{cases} \widetilde{K}(X_0 \wedge X_1)X_1 = f_{01}X_2 \\ \widetilde{K}(X_0 \wedge X_1)X_2 = -f_{01}X_1 \end{cases} \quad \begin{cases} \widetilde{K}(X_0 \wedge X_2)X_1 = f_{02}X_2 \\ \widetilde{K}(X_0 \wedge X_2)X_2 = -f_{02}X_1 \end{cases} \quad \begin{cases} \widetilde{K}(X_1 \wedge X_2)X_1 = 0 \\ \widetilde{K}(X_1 \wedge X_2)X_2 = 0. \end{cases}$$

Proof. See appendix D (section D.2) for the MATHEMATICA code that calculates these expressions; while the computations are straightforward, they are tedious and quite lengthy. (Note that the above expressions yield another proof that the structure is flat exactly when $f_{01} = f_{02} = 0$.) ■

Define a tensor $\widetilde{\mathrm{Ric}} \in \mathcal{T}^0_1(T\mathsf{M}) \otimes \mathcal{T}^0_1(\mathcal{D})$ as follows:

$$\widetilde{\mathrm{Ric}}(X, U) = \mathbf{g}(\widetilde{K}(X_1 \wedge X)U, X_1) + \mathbf{g}(\widetilde{K}(X_2 \wedge X)U, X_2),$$

where $X \in \Gamma(T\mathsf{M})$ and $U \in \Gamma(\mathcal{D})$. Notice that $\widetilde{\mathrm{Ric}}$ could be considered as a trace, in the first contravariant and covariant slots, of $\widetilde{K}$. Accordingly, one can show that $\widetilde{\mathrm{Ric}}$ is well defined (i.e., it does not depend on the choice of X_1 and X_2).

Proposition 4.2.13. *We have* $\overrightarrow{\widetilde{\mathrm{Ric}}}^{\flat} \circ \Delta = d^{\nabla}_{\mathscr{P}} G - G \circ \rho$.

Proof. Using lemma 4.2.12, we get

$$\begin{aligned}\widetilde{\mathrm{Ric}}(X_0, X_1) &= \mathbf{g}(\widetilde{K}(X_1 \wedge X_0)X_1, X_1) + \mathbf{g}(\widetilde{K}(X_2 \wedge X_0)X_1, X_2)\\ &= \mathbf{g}(-f_{01}X_2, X_1) + \mathbf{g}(-f_{02}X_2, X_2)\\ &= -f_{02}.\end{aligned}$$

Similarly, $\widetilde{\mathrm{Ric}}(X_0, X_2) = f_{01}$, and so $\overrightarrow{\widetilde{\mathrm{Ric}}}^{\flat}(X_0) = -f_{02}\nu^1 + f_{01}\nu^2$, where (ν^0, ν^1, ν^2) is the coframe dual to (X_0, X_1, X_2). Then

$$(\overrightarrow{\widetilde{\mathrm{Ric}}}^{\flat} \circ \Delta)(X_2 \wedge X_1) = \overrightarrow{\widetilde{\mathrm{Ric}}}^{\flat}(X_0) = f_{02}\nu^1 - f_{01}\nu^2 = (d^{\nabla}_{\mathscr{P}} G - G \circ \rho)(X_2 \wedge X_1).$$

Since both $\overrightarrow{\widetilde{\mathrm{Ric}}}^{\flat} \circ \Delta$ and $d^{\nabla}_{\mathscr{P}} G - G \circ \rho$ are determined by their evaluation on $X_2 \wedge X_1$, the proof is complete. ∎

Theorem 4.2.14. $(\mathsf{M}, \mathcal{D}, \mathcal{D}^{\perp}, \mathbf{g})$ *is locally flat on* $\mathcal{U} \subseteq \mathsf{M}$ *if and only if* $\widetilde{\mathrm{Ric}}$ *vanishes identically on* $\mathcal{U}$*.*

Proof. From lemma 4.2.12, if $X, Y \in \Gamma(\mathcal{D})$, then

$$\widetilde{\mathrm{Ric}}(X, Y) = \mathbf{g}(\widetilde{K}(X_1 \wedge X)Y, X_1) + \mathbf{g}(\widetilde{K}(X_2 \wedge X)Y, X_2) = 0.$$

That is, $\widetilde{\mathrm{Ric}}|_{\mathcal{D}} \equiv 0$; it follows that $\widetilde{\mathrm{Ric}}$ vanishes if and only if $\widetilde{\mathrm{Ric}} \circ \Delta$ vanishes. From proposition 4.2.13 it is thus clear that $\widetilde{\mathrm{Ric}}$ vanishes identically on $\mathcal{U}$ exactly when $d^{\nabla}_{\mathscr{P}} G - G \circ \rho$ does so, and by corollary 4.2.4 this is equivalent to the local flatness of $(\mathsf{M}, \mathcal{D}, \mathcal{D}^{\perp}, \mathbf{g})$ on $\mathcal{U}$. ∎

4.3 Some examples

In this section we consider three examples of left-invariant nonholonomic Riemannian structures on three-dimensional Lie groups. We first consider the equivalence class of structures on the Heisenberg group H_3 whose reduced dynamics are non-trivial. As we have seen (theorem 4.1.16 and theorem 4.1.19), there exist (up to NH-isometry and rescaling) exactly two left-invariant nonholonomic Riemannian structures on H_3, corresponding to whether $\vartheta = 0$ or $\vartheta > 0$. Since H_3 is unimodular, the structure with $\vartheta = 0$ has a Cartan–Schouten connection (see proposition 4.1.24), and hence the reduced dynamics are trivial; it is the other case (*viz.*, with $\vartheta > 0$) that we shall consider here. For convenience, we shall refer to this example as the *Heisenberg problem.*

The second two examples are (generalisations of) classical problems from nonholonomic mechanics, *viz.*, the Chaplygin problem and the Suslov problem. These problems are typically modelled on the Lie groups $\mathsf{SE}(2)$ and $\mathsf{SO}(3)$, respectively. However, we shall consider them on the universal covering groups $\widetilde{\mathsf{SE}}(2)$ and $\mathsf{SU}(2)$. As mentioned in the introduction to this chapter, these problems have been well studied. Accordingly, rather than repeating work that has already been done, we aim to link our classification and invariants with some known results. In particular, we discuss how some of the invariants may be used in distinguishing between the different qualitative cases mentioned in [24, 26]. In order to facilitate a comparison between our work and what was done in those papers, for the Chaplygin and Suslov problems we shall work with a (left-invariant) Riemannian structure $(\mathsf{G}, \widetilde{\mathbf{g}})$ endowed with a (left-invariant) distribution $\mathcal{D}$. As we have seen in section 1.1.1, this type of structure is typically used in nonholonomic mechanics, which is where the Chaplygin and Suslov problems originate. The associated nonholonomic Riemannian structure is given by $(\mathsf{G}, \mathcal{D}, \mathcal{D}^\perp, \widetilde{\mathbf{g}}|_\mathcal{D})$, where $\mathcal{D}^\perp$ is the orthogonal complement of $\mathcal{D}$ with respect to $\widetilde{\mathbf{g}}$. We shall also follow some of the notational and basis conventions used in [24, 26].

4.3.1 The Heisenberg problem

We begin by proving some results that hold for all left-invariant nonholonomic Riemannian structures on three-dimensional unimodular Lie groups with $\vartheta > 0$. Let $(\mathsf{G}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ be such a structure. We shall assume that the structure has been rescaled so that $\vartheta = 1$. Let $g(\cdot) : \mathbb{R} \to \mathsf{G}$ be a unit-speed geodesic of the associated nonholonomic connection and let $U(\cdot) : \mathbb{R} \to \mathcal{D}_\mathbf{1}$ be the curve in $\mathcal{D}_\mathbf{1}$ such that $\dot{g}(t) = T_\mathbf{1} L_{g(t)} \cdot U(t)$ for all t. (By proposition 1.2.6, every left-invariant structure is geodesically complete; hence we may assume the domain of $g(\cdot)$ and $U(\cdot)$ is $\mathbb{R}$.) Suppose

$$U(t) = u^1(t) X_1 + u^2(t) X_2,$$

where (X_0, X_1, X_2) is the canonical frame for $(\mathsf{G}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ (see section 4.1.3) and suppose we have the initial conditions $u^1(0) = u^1_0$, $u^2(0) = u^2_0$. Since $g(\cdot)$ has unit speed, it follows that $(u^1)^2 + (u^2)^2 = 1$.

Proposition 4.3.1. $\mathcal{S} = \operatorname{span}\{X_2\}$ *is the only left-invariant vector subbundle that is geodesically invariant in* $\mathcal{D}$*.*

Proof. Let $\mathcal{S} = \operatorname{span}\{aX_1 + bX_2\}$, where $a, b \in \mathbb{R}$ are not both zero. By proposition 3.2.7, we have that $\mathcal{S}$ is geodesically invariant in $\mathcal{D}$ if and only if $\nabla_{aX_1+bX_2}(aX_1 + bX_2) \in \Gamma(\mathcal{S})$. If $c^k_{ij} \in \mathbb{R}$ are the structure constants of (X_0, X_1, X_2) (see lemma 4.1.17 and the succeeding discussion), then

$$\nabla_{aX_1+bX_2}(aX_1 + bX_2) = -ab\, c^1_{21} X_1 + a^2 c^1_{21} X_2.$$

If $a = 0$, then $\mathcal{S} = \operatorname{span}\{X_2\}$ and $\mathcal{S}$ is clearly geodesically invariant in $\mathcal{D}$. Suppose $a \neq 0$. Then

$$\begin{aligned}
&\nabla_{aX_1+bX_2}(aX_1 + bX_2) \in \Gamma(\mathcal{S}) \\
&\quad\iff \text{there exists } f \in \mathcal{C}^\infty(\mathsf{G}) \text{ such that } \nabla_{aX_1+bX_2}(aX_1 + bX_2) = f(aX_1 + bX_2) \\
&\quad\iff \text{there exists } f \in \mathcal{C}^\infty(\mathsf{G}) \text{ such that } -a(f + b\, c^1_{21})X_1 - (bf - a^2 c^1_{21})X_2 = 0 \\
&\quad\iff f = -b\, c^2_{21} \text{ and } bf = a^2 c^2_{21} \\
&\quad\iff (a^2 + b^2) c^2_{21} = 0.
\end{aligned}$$

Since $\vartheta = (c^1_{21})^2 > 0$, we can never have $(a^2 + b^2)c^2_{21} = 0$, and so this case cannot hold. ■

$\mathcal{S}$ is clearly integrable, and hence (by proposition 3.2.8) the integral manifolds of $\mathcal{S}$ are totally geodesic in G. In other words, we have the following corollary:

Corollary 4.3.2. *The integral curves* $t \mapsto g_0 \exp(tX_2)$*,* $g_0 \in \mathsf{G}$ *of* X_2 *are nonholonomic geodesics.*

We now find the reduced nonholonomic geodesics by integrating the reduced equations of motion (see equation (1.2.1)). From the commutator relations given in theorem 4.1.19, the reduced equations of motion (for any unimodular group) are given by

$$\begin{cases} \dot{u}^1 = u^1 u^2 \\ \dot{u}^2 = -(u^1)^2. \end{cases} \tag{4.3.1}$$

Proposition 4.3.3. *If $u_0^1 \neq 0$, then $U(t) = \bar{U}(t + t_0)$ for every t, where $t_0 = \operatorname{sech}^{-1}(|u_0^1|)$ and $\bar{U}(t) = \bar{u}^1(t)X_1 + \bar{u}^2(t)X_2$ is given by*

$$\begin{cases} \bar{u}^1(t) = \operatorname{sgn}(u_0^1)\operatorname{sech} t \\ \bar{u}^2(t) = -\tanh t. \end{cases}$$

If $u_0^1 = 0$, then $U(t) = U(0)$ for all t.

Proof. Suppose $u_0^1 \neq 0$. (The points $(0, u_0^2)$ are evidently equilibrium points of (4.3.1); hence, if $u_0^1 = 0$, then $U(\cdot)$ is constant.) Let $\bar{U}(\cdot)$ be a reduced nonholonomic geodesic (i.e., $\dot{\bar{U}}(t) = -\nabla_{\bar{U}(t)}\bar{U}(t)$), where $\bar{U}(t) = \bar{u}^1(t)X_1 + \bar{u}^2(t)X_2$ and $(\bar{u}^1)^2 + (\bar{u}^2)^2 = 1$. Starting from the equations of motion (4.3.1) in $\bar{u}^1$ and $\bar{u}^2$, standard integration techniques (ignoring translations in time) yield

$$\begin{cases} \bar{u}^1(t) = \sigma \operatorname{sech} t \\ \bar{u}^2(t) = -\tanh t \end{cases}$$

for some $\sigma \in \{-1, 1\}$. We claim that there exists $t_0 \in \mathbb{R}$ and $\sigma \in \{-1, 1\}$ such that $U(t) = \bar{U}(t + t_0)$. Indeed, let $\sigma = \operatorname{sgn}(u_0^1)$. We have $-1 < u^2(t), \bar{u}^2(t) < 1$, $\lim_{t\to-\infty} \bar{u}^2(t) = 1$ and $\lim_{t\to\infty} \bar{u}^2(t) = -1$. Therefore, since $\bar{u}^2$ is continuous, there exists $t_0 \in \mathbb{R}$ such that $\bar{u}^2(t + t_0) = u^2(t)$. Then

$$(u^1(t))^2 = 1 - (u^2(t))^2 = 1 - (\bar{u}^2(t + t_0))^2 = (\bar{u}^1(t + t_0))^2$$

and since $\operatorname{sgn}(u^1(0)) = \sigma = \operatorname{sgn}(\bar{u}^2(t_0))$, we have $\bar{u}^1(t + t_0) = u^1(t)$. Therefore, as $t \mapsto U(t)$ and $t \mapsto \bar{U}(t + t_0)$ are both solutions to the same system of ODEs passing through the same point at $t = 0$, they both solve the same Cauchy problem, and hence are identical. Lastly, using $u_0^1 = u^1(0) = \bar{u}^1(t_0)$, we get that $t_0 = \operatorname{sech}^{-1}(|u_0^1|)$. ■

We now specialise to the case $\mathsf{G} = \mathsf{H}_3$. The (three-dimensional) Heisenberg group H_3 (see appendix C) is the connected, simply connected and nilpotent matrix Lie group given by

$$\mathsf{H}_3 = \left\{ \varphi(x, y, z) = \begin{bmatrix} 1 & y & x \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix} : x, y, z \in \mathbb{R} \right\}.$$

Note that the map $\varphi : \mathbb{R}^3 \to \mathsf{H}_3$ is a (global) diffeomorphism from $\mathbb{R}^3$ to H_3; hence we shall work in coordinates (x, y, z). The Lie algebra $\mathfrak{h}_3$ of H_3 is

$$\mathfrak{h}_3 = \left\{ xE_1 + yE_2 + zE_3 = \begin{bmatrix} 0 & y & x \\ 0 & 0 & z \\ 0 & 0 & 0 \end{bmatrix} : x, y, z \in \mathbb{R} \right\},$$

together with the matrix commutator $[X, Y] = XY - YX$. In terms of the standard basis (E_1, E_2, E_3), the only nonzero commutator is $[E_2, E_3] = E_1$.

The equivalence class of structures on H_3 with $\vartheta > 0$ are the simplest non-trivial (i.e., whose nonholonomic connection is not a Cartan–Schouten connection) examples of nonholonomic Riemannian structures. In this section we shall illustrate how to explicitly calculate the canonical frame associated to an arbitrary member of this equivalence class.

Lemma 4.3.4. *Up to Lie algebra automorphism,* $\operatorname{span}\{E_2, E_3\}$ *is the only completely nonholonomic two-dimensional subspace of* $\mathfrak{h}_3$*.*

Proof. Let $\mathfrak{s} = \operatorname{span}\{u^1E_1 + u^2E_2 + u^3E_3, v^1E_1 + v^2E_2 + v^3E_3\}$ be a completely nonholonomic two-dimensional subspace of $\mathfrak{h}_3$. By complete nonholonomy, we have $u^2v^3 - v^2u^3 \neq 0$. Then

$$\psi = \begin{bmatrix} u^2v^3 - v^2u^3 & u^1 & v^1 \\ 0 & u^2 & v^2 \\ 0 & u^3 & v^3 \end{bmatrix}$$

is a Lie algebra automorphism of $\mathfrak{h}_3$ (written with respect to the standard basis) such that $\psi \cdot \operatorname{span}\{E_2, E_3\} = \mathfrak{s}$. ■

Let $(\mathsf{H}_3, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ be a left-invariant nonholonomic Riemannian structure on the Heisenberg group. By lemma 4.3.4, we may assume that $\mathcal{D}_\mathbf{1} = \operatorname{span}\{E_2, E_3\}$. Suppose that

$$\mathcal{D}_\mathbf{1}^\perp = \operatorname{span}\{E_1 + y^2E_2 + y^3E_3\} \quad \text{and} \quad \mathbf{g}_\mathbf{1} = \frac{1}{\mu^2}\begin{bmatrix} \alpha_1 & \beta \\ \beta & \alpha_2 \end{bmatrix},$$

where $y^2, y^3 \in \mathbb{R}$, $\alpha_1, \alpha_2, \mu > 0$, $\alpha_1\alpha_2 - \beta^2 > 0$ and $\mathbf{g}_\mathbf{1}$ is written with respect to the basis (E_2, E_3) for $\mathcal{D}_\mathbf{1}$. Let (Y_1, Y_2, Y_3) be a left-invariant frame on H_3, where $Y_3 = E_1 + y^2E_2 + y^3E_3$ and (Y_1, Y_2) is the orthonormal frame for $\mathcal{D}$ obtained by applying the Gram–Schmidt procedure to (E_2, E_3), i.e.,

$$Y_1 = \frac{\mu}{\sqrt{\alpha_1}}E_2 \quad \text{and} \quad Y_2 = \frac{\mu}{\sqrt{\alpha_2}\sqrt{\alpha_1\alpha_2 - \beta^2}}\left(-\beta E_2 + \sqrt{\alpha_1\alpha_2}E_3\right).$$

Let (ν^1, ν^2, ν^3) be the coframe dual to (Y_1, Y_2, Y_3) and let $\omega = \omega_1\nu^1 + \omega_2\nu^2 + \omega_3\nu^3$ be the normalised contact form. Since $\mathcal{D} = \ker\omega$ and $d\omega(Y_1, Y_2) = \sigma \in \{-1, 1\}$, we have

$$\omega = \frac{\sigma\sqrt{\alpha_1\alpha_2 - \beta^2}}{\mu^2}\nu^3.$$

Let Y_0 be the Reeb vector field. Using $i_{Y_0}\omega = 1$ and $i_{Y_0}d\omega = 0$, we get

$$Y_0 = -\frac{\sigma\mu(\beta y^3 - \alpha_1 y^2)}{\sqrt{\alpha_1}\sqrt{\alpha_1\alpha_2 - \beta^2}}Y_1 - \frac{\sigma\mu y_3}{\sqrt{\alpha_1}}Y_2 - \frac{\sigma\mu^2}{\sqrt{\alpha_1\alpha_2 - \beta^2}}Y_3.$$

In particular, we can now calculate the vector fields $X_0 = \mathscr{Q}(Y_0)$ and $X_1 = \mathscr{P}(Y_0)/\|\mathscr{P}(Y_0)\|$ of the canonical frame:

$$X_0 = -\frac{\sigma\mu^2}{\sqrt{\alpha_1\alpha_2 - \beta^2}}Y_3 \quad \text{and} \quad X_1 = -\frac{\sigma(\beta y^3 + \alpha_1 y^2)}{\sqrt{f\alpha_1}}Y_1 - \frac{\sigma y_3\sqrt{\alpha_1\alpha_2 - \beta^2}}{\sqrt{f\alpha_1}}Y_2.$$

Here $f = \alpha_1(y^2)^2 + 2\beta y^2 y^3 + \alpha_2(y^3)^2$; as $\vartheta > 0$, it follows that $f > 0$. The final vector field X_2 in the canonical frame is now readily calculated as the (unique) unit vector field orthogonal to X_1 such that $d\omega(X_1, X_2) = 1$. We get

$$X_2 = -\frac{y^3\sqrt{\alpha_1\alpha_2 - \beta^2}}{\sqrt{f\alpha_1}}\, Y_1 + \frac{\beta y^3 + \alpha_1 y^2}{\sqrt{f\alpha_1}}\, Y_2.$$

(Notice that X_0 and X_1 both depend on σ, whereas X_2 does not.) In particular, in terms of the standard basis (E_1, E_2, E_3) for $\mathfrak{h}_3$, the canonical frame is given by

$$\begin{cases} X_0 = \dfrac{\sigma\mu^2}{\sqrt{\alpha_1\alpha_2 - \beta^2}}\, E_1 + \dfrac{\sigma\mu^2 y^2}{\sqrt{\alpha_1\alpha_2 - \beta^2}}\, E_2 + \dfrac{\sigma\mu^2 y^3}{\sqrt{\alpha_1\alpha_2 - \beta^2}}\, E_3 \\ X_1 = -\dfrac{\sigma\mu y^2}{\sqrt{f}}\, E_2 - \dfrac{\sigma\mu y^3}{\sqrt{f}}\, E_3 \\ X_2 = -\dfrac{\mu(\beta y^2 + \alpha_2 y^3)}{\sqrt{f}\sqrt{\alpha_1\alpha_2 - \beta^2}}\, E_2 + \dfrac{\mu(\beta y^3 + \alpha_1 y^2)}{\sqrt{f}\sqrt{\alpha_1\alpha_2 - \beta^2}}\, E_3. \end{cases}$$

Normalising so that $\vartheta = 1$, we take $\mu = \frac{\sqrt{\alpha_1\alpha_2 - \beta^2}}{\sqrt{f}}$. The resultant commutator relations for the canonical frame then take the form given in theorem 4.1.19, *viz.*,

$$\begin{cases} [X_1, X_0] = 0 \\ [X_2, X_0] = -X_0 - X_1 \\ [X_2, X_1] = \quad X_0 + X_1. \end{cases}$$

The invariants take the expressions $\vartheta = 1$, $\kappa = -\frac{1}{2}$, $\chi_1 = \frac{1}{2}$, $\chi_2 = 0$ and $\varrho_0 = \varrho_1 = \varrho_2 = 0$. The nonholonomic connection is given by

$$\nabla_{X_1} X_1 = X_2, \quad \nabla_{X_1} X_2 = -X_1, \quad \nabla_{X_2} X_1 = 0, \quad \nabla_{X_2} X_2 = 0.$$

At this stage we may choose σ, α_1, α_2, β and y^2, y^3 for convenience, in order to select a normal form for the equivalence class of structures. We shall work with the normal form $(\mathsf{H}_3, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ given by

$$\mathcal{D}_{\mathbf{1}} = \operatorname{span}\{E_2, E_3\}, \qquad \mathcal{D}_{\mathbf{1}}^\perp = \operatorname{span}\{E_1 + E_2\}, \qquad \mathbf{g}_{\mathbf{1}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The canonical frame (X_0, X_1, X_2) on H_3 is then given by $X_0 = -E_1 - E_2$, $X_1 = E_2$, $X_2 = E_3$. Let $\widetilde{\nabla}$ be the vector bundle connection on $\mathcal{D}$ described in section 4.2.3. We have

$$\widetilde{\nabla}_{X_0} X_1 = -X_2 \quad \text{and} \quad \widetilde{\nabla}_{X_0} X_2 = X_1.$$

(By definition, $\widetilde{\nabla}_{X_1} = \nabla_{X_1}$ and $\widetilde{\nabla}_{X_2} = \nabla_{X_2}$.) We calculate an explicit expression for every parallel vector field on H_3. In particular, this will allow us to find a parallel frame for $\mathcal{D}$.

Proposition 4.3.5. *Every normalised parallel vector field $U \in \Gamma(\mathcal{D})$ is of the form*

$$U = \cos(\theta_0 - y)X_1 + \sin(\theta_0 - y)X_2,$$

where $U(\mathbf{1}) = (\cos\theta_0, \sin\theta_0)$ in terms of the basis (E_2, E_3).

Proof. Let $U = u^1 X_1 + u^2 X_2 \in \Gamma(\mathcal{D})$. Since $\widetilde{\nabla} U \equiv 0$, we have

$$\begin{aligned} \nabla^2_{X_0} U = 0 \quad &\Longleftrightarrow \quad (X_0[u^1] + u^2) X_1 + (X_0[u^2] - u^1) X_2 = 0 \\ &\Longleftrightarrow \quad u^1 = X_0[u^2] \text{ and } u^2 = -X_0[u^1]. \end{aligned}$$

Likewise, from $\widetilde{\nabla}_{X_1} U = 0$ and $\widetilde{\nabla}_{X_2} U = 0$, we get

$$u^1 = -X_1[u^2], \quad u^2 = X_1[u^1], \quad X_2[u^1] = X_2[u^2] = 0.$$

We shall write these differential equations in coordinates in $\mathbb{R}^3$. Let ∂_x, ∂_y, ∂_z denote the coordinate vector fields on H_3. Using φ^{-1}, we push the canonical frame to $\mathbb{R}^3$. The result is

$$(\varphi^{-1})_* X_0 = -\partial_x - \partial_y, \quad (\varphi^{-1})_* X_1 = \partial_y, \quad (\varphi^{-1})_* X_2 = y\,\partial_x + \partial_z.$$

Identifying $u^1 \circ \varphi$ and $u^2 \circ \varphi$ with u^1 and u^2, respectively, the differential equations for $(\varphi^{-1})_* U$ are given by

$$\begin{cases} u^1 = -\dfrac{\partial u^2}{\partial x} - \dfrac{\partial u^2}{\partial y} \\ u^2 = \dfrac{\partial u^1}{\partial x} + \dfrac{\partial u^1}{\partial y} \end{cases} \qquad \begin{cases} u^1 = -\dfrac{\partial u^2}{\partial y} \\ u^2 = \dfrac{\partial u^1}{\partial y} \end{cases} \qquad \begin{cases} y\,\dfrac{\partial u^1}{\partial x} + \dfrac{\partial u^1}{\partial z} = 0 \\ y\,\dfrac{\partial u^2}{\partial x} + \dfrac{\partial u^2}{\partial z} = 0. \end{cases}$$

It is a straightforward matter to solve these PDEs (we have used Mathematica), yielding the solutions

$$\begin{cases} u^1(x, y, z) = \cos(\theta_0 - y) \\ u^2(x, y, z) = \sin(\theta_0 - y), \end{cases}$$

where $u^1(0, 0, 0) = \cos\theta_0$, $u^2(0, 0, 0) = \sin\theta_0$. Applying φ to pushforward $(\varphi^{-1})_* U$ to H_3 completes the result. ■

In particular, let (U_1, U_2) be the parallel frame for $\mathcal{D}$ given by

$$\begin{bmatrix} U_1(g) \\ U_2(g) \end{bmatrix} = \begin{bmatrix} \cos y & -\sin y \\ \sin y & \cos y \end{bmatrix} \begin{bmatrix} X_1(g) \\ X_2(g) \end{bmatrix}, \quad g = (x, y, z) \in \mathsf{H}_3.$$

Note that $U_1(\mathbf{1}) = E_2$ and $U_2(\mathbf{1}) = E_3$. Furthermore, we have $[U_2, U_1] = X_0$. (However, U_1 and U_2 are not left invariant.)

If $\Pi_{\mathbf{1}}^{g} : \mathcal{D}_{\mathbf{1}} \to \mathcal{D}_g$ is the parallel transport map from identity to $g = (x, y, z) \in \mathsf{H}_3$, then in the basis $(X_1(\mathbf{1}), X_2(\mathbf{1})) = (E_2, E_3)$ for $\mathcal{D}_{\mathbf{1}}$ and $(X_1(g), X_2(g)) = (T_{\mathbf{1}} L_g \cdot E_2, T_{\mathbf{1}} L_g \cdot E_3)$ for $\mathcal{D}_g$, we have

$$\Pi_{\mathbf{1}}^{g} = \begin{bmatrix} \cos y & \sin y \\ -\sin y & \cos y \end{bmatrix}.$$

(By proposition B.1.22, it suffices to consider the parallel transport map starting from identity; every other parallel transport map is then obtained by a composition with a left translation.)

Lastly for this section, we find explicit expressions for the nonholonomic geodesics of $(\mathsf{H}_3, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$. By corollary 1.2.4, it suffices to consider the nonholonomic geodesics starting from identity; every other nonholonomic geodesic may then be obtained via a suitable left translation. In figure 4.2 we have graphed some typical nonholonomic geodesics starting from identity (over the time interval $[-3, 3]$), as well as the (restricted) exponential image $\{\exp_{\mathbf{1}}(tU) : t \in [-3, 3],\ U \in \mathcal{D}_{\mathbf{1}}\}$ (see section 1.1.2).

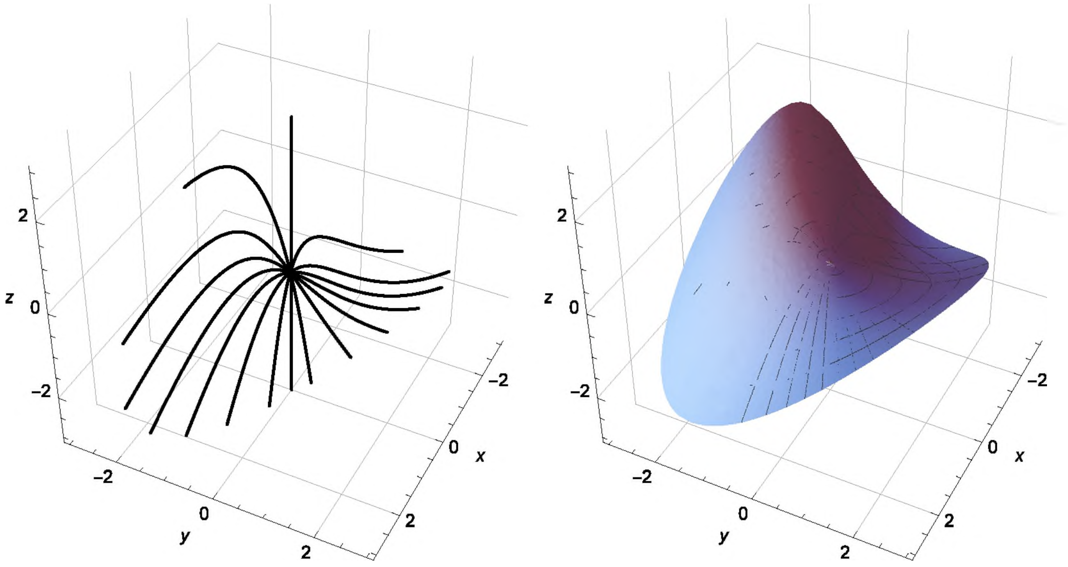

Figure 4.2: Typical nonholonomic geodesics from identity (graphed for $t \in [-3, 3]$) and the exponential image $\{\exp_{\mathbf{1}}(tU) : t \in [-3, 3],\ U \in \mathcal{D}_{\mathbf{1}}\}$ for the Heisenberg problem.

Proposition 4.3.6. *Let $g(\cdot)$ be the unit-speed nonholonomic geodesic of* $(\mathsf{H}_3, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ *starting from identity and with initial velocity $U_0 = u_0^1 X_1 + u_0^2 X_2 \in \mathcal{D}_{\mathbf{1}}$.*

- (*i*) *If $u_0^1 \neq 0$, then $g(t) = \bar{g}(t_0)^{-1}\bar{g}(t + t_0)$ for every t, where $t_0 = \operatorname{sech}^{-1}(|u_0^1|)$ and $\bar{g}(\cdot) = (\bar{x}(\cdot), \bar{y}(\cdot), \bar{z}(\cdot)) : \mathbb{R} \to \mathsf{H}_3$ is given by*

$$\begin{cases} \bar{x}(t) = -\operatorname{sgn}(u_0^1) \displaystyle\int_0^t \operatorname{gd} s \tanh s \, ds \\ \bar{y}(t) = \operatorname{sgn}(u_0^1) \operatorname{gd} t \\ \bar{z}(t) = -\ln(\cosh t). \end{cases}$$

  (*Here* $\operatorname{gd} t = 2\tanh^{-1}(\tanh(t/2))$ *is the* Gudermannian function.)

- (*ii*) *If $u_0^1 = 0$, then $g(t) = (0, 0, u_0^2 t)$.*

Proof. (*i*) Suppose $u_0^1 \neq 0$. Let $U(\cdot) = u^1(\cdot)X_1 + u^2(\cdot)X_2$ be a curve in $\mathcal{D}_{\mathbf{1}}$ such that $U(0) = U_0$ and suppose that $\dot{g}(t) = T_{\mathbf{1}}L_{g(t)} \cdot U(t)$. By proposition 4.3.3, we have $U(t) = \bar{U}(t + t_0)$ for every t, where $\bar{U}(\cdot)$ is given in proposition 4.3.3 and $t_0 = \operatorname{sech}^{-1}(|u_0^1|)$. Let $\bar{g}(\cdot) = (\bar{x}(\cdot), \bar{y}(\cdot), \bar{z}(\cdot))$ be the nonholonomic geodesic starting from identity such that $\dot{\bar{g}}(t) = T_{\mathbf{1}}L_{\bar{g}(t)} \cdot \bar{U}(t)$. It follows that $\bar{g}(\cdot)$ satisfies the following system of ODEs:

$$\begin{cases} \dot{\bar{x}} = -\bar{y} \tanh t \\ \dot{\bar{y}} = \operatorname{sgn}(u_0^1) \operatorname{sech} t \\ \dot{\bar{z}} = -\tanh t. \end{cases}$$

Integrating these equations, together with the initial condition $\bar{g}(0) = \mathbf{1}$, yields the expressions in the statement of the proposition. We have $\dot{g}(t) = T_{\mathbf{1}}L_{g(t)} \cdot U(t) = T_{\mathbf{1}}L_{g(t)} \cdot \bar{U}(t + t_0)$. Likewise, if $h(\cdot)$ denotes the curve $t \mapsto \bar{g}(t_0)^{-1}\bar{g}(t+t_0)$, then

$$\begin{aligned}\dot{h}(t) &= T_{\bar{g}(t+t_0)}L_{\bar{g}(t_0)^{-1}} \cdot T_{\mathbf{1}}L_{\bar{g}(t+t_0)} \cdot \bar{U}(t+t_0)\\ &= T_{\mathbf{1}}L_{\bar{g}(t_0)^{-1}\bar{g}(t+t_0)} \cdot \bar{U}(t+t_0)\\ &= T_{\mathbf{1}}L_{h(t)} \cdot \bar{U}(t+t_0).\end{aligned}$$

Furthermore, we have $h(0) = \mathbf{1} = g(0)$. Therefore $t \mapsto h(t)$ and $t \mapsto g(t)$ both solve the same Cauchy problem, and hence are identical.

(*ii*) Suppose $u_0^1 = 0$. Again by proposition 4.3.3, we have $\dot{g}(t) = T_{\mathbf{1}}L_{g(t)} \cdot U_0$. Hence $g(\cdot)$ is the one-parameter subgroup $g(t) = \exp(t\,U_0) = (0, 0, u_0^2 t)$. ■

Note that the nonholonomic geodesics of the form $t \mapsto (0, 0, u_0^2 t)$ are exactly the reparametrisations of the integral curves of X_2 (cf. corollary 4.3.2).

4.3.2 The Chaplygin problem

The Chaplygin problem models the motion of a planar rigid body equipped with a blade, along which the body slides, that prohibits motion in directions orthogonal to the blade. The problem may be described by means of a left-invariant Riemannian metric on the Euclidean group $\mathsf{SE}(2)$, together with a left-invariant completely nonholonomic distribution. It is known that the dynamics exhibit three qualitatively different cases (of increasing analytical complexity) [24]: the Chaplygin "skate" (when the centre of mass is at the point of contact); the Chaplygin sleigh (the classical statement of the problem); and a generalisation of the Chaplygin sleigh, called the hydrodynamic Chaplygin sleigh (first introduced in [24]). For further details on the Chaplygin problem, see, e.g., [16, 52, 8, 24].

We consider the problem on the universal covering group of $\mathsf{SE}(2)$, *viz.*,

$$\widetilde{\mathsf{SE}}(2) = \left\{ \begin{bmatrix} 1 & 0 & 0 & 0\\ x^1 & \cos x^3 & -\sin x^3 & 0\\ x^2 & \sin x^3 & \cos x^3 & 0\\ 0 & 0 & 0 & e^{x^3}\end{bmatrix} : x^1, x^2, x^3 \in \mathbb{R} \right\},$$

which has the Lie algebra

$$\mathfrak{se}(2) = \left\{ u^1E_1 + u^2E_2 + u^3E_3 = \begin{bmatrix} 0 & 0 & 0 & 0\\ u^1 & 0 & -u^3 & 0\\ u^2 & u^3 & 0 & 0\\ 0 & 0 & 0 & u^3\end{bmatrix} : u^1, u^2, u^3 \in \mathbb{R} \right\}.$$

The nonzero commutator relations are $[E_2, E_3] = E_1$, $[E_3, E_1] = E_2$. For compatibility with the notation used for the body's inertia tensor in [24], we shall work in the (ordered) basis (E_3, E_1, E_2). We may assume, without loss of generality, that the constraint distribution $\mathcal{D}$ is specified by $\mathcal{D}_{\mathbf{1}} = \operatorname{span}\{E_1, E_3\}$. Let $\tilde{\mathbf{g}}$ be the left-invariant Riemannian metric on $\widetilde{\mathsf{SE}}(2)$ specified with respect to (E_3, E_1, E_2) by

$$\tilde{\mathbf{g}}_{\mathbf{1}} = \begin{bmatrix} J & -L_2 & L_1\\ -L_2 & M & Z\\ L_1 & Z & N\end{bmatrix}.$$

The associated nonholonomic Riemannian structure is given by $(\widetilde{\mathsf{SE}}(2), \mathcal{D}, \mathcal{D}^\perp, \widetilde{\mathbf{g}}|_{\mathcal{D}})$, where $\mathcal{D}^\perp$ is the $\widetilde{\mathbf{g}}$-orthogonal complement of $\mathcal{D}$. The first invariant ϑ is given by

$$\vartheta = \frac{JZ^2 + 2ZL_1L_2 + ML_1^2}{(JM - L_2^2)^2}.$$

We have the following cases:

(*i*) $\vartheta = 0$. This occurs exactly when $L_1 = Z = 0$, and corresponds to the case with the simplest qualitative behaviour (the Chaplygin skate). Indeed, the reduced equations of motion are trivial, hence the nonholonomic connection is Cartan–Schouten; furthermore, we have $\kappa = \chi_1 = \frac{1}{2}\frac{M}{JM-L_2^2}$.

(*ii*) $\vartheta > 0$. At least one of L_1, Z is nonzero. The canonical frame (X_0, X_1, X_2) (rescaled so that $\vartheta = 1$) is readily calculated:

$$\begin{cases} X_0 = \mu(ML_1 + ZL_2)E_3 + \mu(JZ + L_1L_2)E_1 - \mu(JM - L_2^2)E_2 \\ X_1 = -\mu(ML_1 + ZL_2)E_3 - \mu(JZ + L_1L_2)E_1 \\ X_2 = \mu\big(Z\sqrt{JM - L_2^2}\big)E_3 - \mu\big(L_1\sqrt{JM - L_2^2}\big)E_1. \end{cases}$$

Here $\mu = \frac{\sqrt{JM-L_2^2}}{JZ^2+2ZL_1L_2+ML_1^2}$. The values of the parameters α_1, α_2 in the equivalence class representative (4.1.15) may likewise be calculated:

$$\alpha_1 = \mu^2(ML_1 + ZL_2)^2, \quad \alpha_2 = \mu^2 Z^2 (JM - L_2^2).$$

The structure $(\widetilde{\mathsf{SE}}(2), \widetilde{\mathbf{g}})$ together with $\mathcal{D}$ describes the Chaplygin sleigh exactly when $L_1 \neq 0$, $Z = 0$ (cf. [24]). This occurs exactly when $\alpha_2 = 0$ (or equivalently, when $\kappa^2 = \chi_1^2$; see remark 4.1.21). For $\alpha_2 > 0$ ($\kappa^2 \neq \chi_1^2$), the structure describes the hydrodynamic Chaplygin sleigh.

Lastly, we note that, in light of the characterisation of flat structures in section 4.2, the flat structures of the Chaplygin problem correspond exactly to the case of a Chaplygin skate (when $\vartheta = 0$; see theorem 4.2.7) and the Chaplygin sleigh (when $\vartheta > 0$; see theorem 4.2.9).

4.3.3 The Suslov problem

The Suslov problem describes the motion of a rigid body in $\mathbb{R}^3$ about a fixed point subject to the (nonholonomic) constraint $W \bullet \Omega = 0$. (Here Ω is the angular velocity of the body, W is a fixed vector in the body frame and $\bullet$ denotes the standard dot product of $\mathbb{R}^3$.) For more details on the Suslov problem, see, e.g., [63, 26].

The Suslov problem is typically modelled by means of a left-invariant Riemannian metric on the orthogonal group $\mathsf{SO}(3)$, together with a left-invariant completely nonholonomic distribution. However, we shall consider the problem on the universal covering group

$$\mathsf{SU}(2) = \{x \in \mathbb{C}^{2\times 2} : xx^\dagger = \mathbf{1},\ \det x = 1\}.$$

Its Lie algebra

$$\mathfrak{su}(2) = \left\{ u^1E_1 + u^2E_2 + u^3E_3 = \begin{bmatrix} \frac{i}{2}u^1 & \frac{1}{2}(iu^3 + u^2) \\ \frac{1}{2}(iu^3 - u^2) & -\frac{i}{2}u^1 \end{bmatrix} : u^1, u^2, u^3 \in \mathbb{R} \right\}$$

has commutator relations $[E_2, E_3] = E_1$, $[E_3, E_1] = E_2$, $[E_1, E_2] = E_3$. We may assume, without loss of generality, that $W = E_3$; it follows that $\mathcal{D}_\mathbf{1} = \operatorname{span}\{E_1, E_2\}$. Let $\widetilde{\mathbf{g}}$ be a left-invariant Riemannian metric on $\mathsf{SU}(2)$. By rotating the basis (E_1, E_2) (to obtain a basis (F_1, F_2, F_3) for $\mathfrak{su}(2)$, where $F_3 = E_3$), we may assume that $\widetilde{\mathbf{g}}_\mathbf{1}$ takes the form

$$\widetilde{\mathbf{g}}_\mathbf{1} = \begin{bmatrix} I_{11} & 0 & I_{13} \\ 0 & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix}.$$

(We follow the notation for the components of $\widetilde{\mathbf{g}}_\mathbf{1}$ used in [26].) Let $\mathcal{D}^\perp$ be the orthogonal complement of $\mathcal{D}$ with respect to $\widetilde{\mathbf{g}}$; then $(\mathsf{SU}(2), \mathcal{D}, \mathcal{D}^\perp, \widetilde{\mathbf{g}}|_\mathcal{D})$ is the associated nonholonomic Riemannian structure. The invariant ϑ is given by

$$\vartheta = \frac{I_{11}I_{23}^2 + I_{22}I_{13}^2}{I_{11}^2 I_{22}^2}.$$

We have the following cases:

(*i*) $\vartheta = 0$. This occurs exactly when $I_{13} = I_{23} = 0$, and corresponds to the simplest qualitative case (when the nonholonomic connection is Cartan–Schouten). The invariants are given by

$$\kappa = \frac{1}{2}\left(\frac{1}{I_{11}} + \frac{1}{I_{22}}\right), \quad \chi_1 = \frac{1}{2}\frac{|I_{11} - I_{22}|}{I_{11}I_{22}}.$$

In particular, $\widetilde{\mathbf{g}}_\mathbf{1}|_{\mathcal{D}_1}$ is a rescaling of $\mathcal{K}|_{\mathcal{D}_1}$ exactly when $I_{11} = I_{22}$. (This is also the only case when the structure is flat; see theorem 4.2.7 and theorem 4.2.9.)

(*ii*) $\vartheta > 0$. At least one of I_{13}, I_{23} is nonzero. The canonical frame (rescaled so that $\vartheta = 1$) is given by

$$\begin{cases} X_0 = \mu I_{13} F_1 + \mu \dfrac{I_{11}I_{23}}{I_{22}} F_2 - \mu I_{11} F_3 \\ X_1 = -\mu I_{13} F_1 - \mu \dfrac{I_{11}I_{23}}{I_{22}} F_2 \\ X_2 = \mu \sqrt{\dfrac{I_{11}}{I_{22}}} I_{23} F_1 - \mu \sqrt{\dfrac{I_{11}}{I_{22}}} I_{13} F_2. \end{cases}$$

Here $\mu = \frac{\sqrt{I_{11}I_{22}^3}}{I_{11}I_{23}^2 + I_{22}I_{13}^2}$. The parameters α_1, α_2, δ in the equivalence class representative (4.1.17) are:

$$\alpha_1 = \mu^2 \frac{I_{11}^2 I_{23}^2 + I_{22}^2 I_{13}^2}{I_{22}^2}, \quad \alpha_2 = \mu^2 \frac{I_{11}(I_{13}^2 + I_{23}^2)}{I_{22}},$$
$$\delta = \mu^2 \frac{|(I_{11} - I_{22}) I_{13} I_{23}| \sqrt{I_{11}}}{I_{22}^{3/2}}.$$

We have $\delta = 0$ (equivalently, $\varrho_0 = \varrho_1(1 + \varrho_2)$) exactly when $I_{11} = I_{22}$, $I_{13} = 0$ or $I_{23} = 0$. Thus δ may be used to distinguish between the different qualitative cases for the Suslov problem discussed in [26].

Conclusion

In this thesis we have considered nonholonomic Riemannian structures on manifolds, and especially the (prototypical) case of left-invariant nonholonomic Riemannian structures on Lie groups. Three broad topics concerning nonholonomic Riemannian structures were investigated, *viz.*, curvature (particularly, the Schouten and Wagner curvature tensors); equivalence (particularly, up to nonholonomic isometry) and embeddings (i.e., nonholonomic Riemannian submanifolds); and nonholonomic Riemannian structures in three dimensions (particularly, the equivalence, classification and flatness of such structures). We summarise the main contributions of the thesis below.

Chapter 1 served mainly to establish the necessary preliminaries for an understanding of the remainder of the thesis. We also compared nonholonomic Riemannian geometry with sub-Riemannian geometry (the first is essentially concerned with the "straightest" curves, and the second, the "shortest" curves) and discussed the relation between nonholonomic Riemannian geometry and nonholonomic mechanics. In the second part of the chapter we considered left-invariant nonholonomic Riemannian structures on Lie groups, and characterised those with trivial reduced dynamics (i.e., when the nonholonomic connection is Cartan–Schouten). The existence of structures with Cartan–Schouten connections is an important question; we were able to prove a positive existence result for structures with rank two distribution.

In chapter 2 we investigated the curvature of nonholonomic Riemannian manifolds, particularly the Schouten and Wagner curvature tensors. The former tensor is canonically (or intrinsically) associated to every nonholonomic Riemannian structure, but (its vanishing) does not characterise the flat structures. The latter tensor corrects the deficiency of the Schouten tensor regarding the characterisation of the flat structures, but in general, is not intrinsic. (Accordingly, it can only be used to characterise flatness when some additional conditions are met.) It is clear that the topic of curvature in nonholonomic Riemannian geometry requires considerable further study (we discuss several open problems below). Indeed, the only recent paper treating this topic is [21], which essentially only provides a modern introduction to Wagner's construction. A notable contribution of this chapter is simply to provide a more comprehensive overview, as well as complete proofs, of what is currently known with regards to curvature in nonholonomic Riemannian geometry. Looking beyond that, the (restricted) Ehresmann approach to curvature (for which we made an initial effort in section 2.3) should prove fruitful in elucidating many aspects of Wagner's construction.

Chapter 3 treated the equivalence of nonholonomic Riemannian structures. There are several (natural) choices of equivalence relation; our main consideration was equivalence up to nonholonomic isometry. (In particular, nonholonomic isometries preserve the nonholonomic geodesics, the nonholonomic connection, and the Schouten curvature tensor.) This work was chiefly to establish some basic results for the classification in chapter 4. In the second part of the chapter, we considered a generalisation of nonholonomic isometries to the case when one nonholonomic Riemannian structure $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ is embedded in a “larger” structure $(\mathsf{M}', \mathcal{D}', \mathcal{D}'^\perp, \mathbf{g}')$, i.e., the first structure is viewed as a nonholonomic Riemannian submanifold of the second structure. (We have not considered nonholonomic Riemannian submersions in this thesis; nevertheless, their study would be of comparable interest.) The main contribution of this part of the chapter is characterising when the embedding is compatible with the geometry of the enveloping structure. This essentially reduces to characterising when $\mathcal{D}$ is invariant under the nonholonomic geodesic flow of $(\mathsf{M}', \mathcal{D}', \mathcal{D}'^\perp, \mathbf{g}')$, i.e., when $\mathcal{D}$ is geodesically invariant. $(\mathsf{M}, \mathcal{D}, \mathcal{D}^\perp, \mathbf{g})$ is naturally embedded inside a Riemannian manifold $(\mathsf{M}, \widetilde{\mathbf{g}})$, where $\widetilde{\mathbf{g}}$ is any Riemannian extension of $\mathbf{g}$ such that $\mathcal{D}$ and $\mathcal{D}^\perp$ are orthogonal. Studying the conditions under which $\mathcal{D}$ is geodesically invariant for this embedding is of the most interest. We showed, at least when $\mathcal{D}$ is strongly nonholonomic, that this is characterised by the vanishing of a component of the Schouten curvature tensor. A more general result was also obtained using the curvature tensors involved in Wagner’s construction.

Lastly, chapter 4 specialised to the case of nonholonomic Riemannian structures on three-dimensional manifolds. (Three is obviously the lowest dimension in which one can define a nonholonomic Riemannian manifold.) Many of the results in this chapter can be viewed as an application of theory developed in earlier chapters. The main contribution of the chapter is the classification of left-invariant structures on the (three-dimensional) simply connected Lie groups. The equivalence classes were also described in terms of isometric invariants (as well as the commutator relations of a canonical frame). In most cases, we exhibited a complete set of invariants. The classification of the left-invariant structures also resulted in a list of examples of nonholonomic Riemannian structures with Cartan–Schouten connections, as well as structures with a geodesically invariant distribution. (We also characterised these types of structures in three dimensions.) Furthermore, the classification also led to a classification of the flat structures; in order to obtain the latter, we first characterised flatness (of the three-dimensional structures), making use of an exterior covariant derivative operator induced by the nonholonomic connection, a contraction of the Schouten tensor, and the contact structure. We also showed how this characterisation relates to the Wagner curvature tensor, resulting in a further characterisation of flatness in three dimensions in terms of the vanishing of a contraction of the Wagner tensor. In the last part of the chapter we considered three examples of (left-invariant) nonholonomic Riemannian structures. The Heisenberg problem (on H_3) has not, to our knowledge, been investigated in the literature. Since it is the simplest example of a non-trivial nonholonomic Riemannian manifold, an in-depth study of this problem seems warranted; finding the nonholonomic geodesics (as we have done) is a first step toward that end. By contrast, the other two examples considered (relating to the Chaplygin problem and Suslov problem) have been extensively studied; our classification and invariants have given some new tools for studying these problems.

In closing, we discuss below several open problems that we have identified as being of particular interest for future research. The first problem involves the study of (left-invariant) nonholonomic Riemannian structures with Cartan–Schouten connections, whereas the others involve the curvature of nonholonomic Riemannian structures.

(*i*) Left-invariant nonholonomic Riemannian structures with Cartan–Schouten connections are a distinguished class of structures. A deeper understanding of the properties of these structures should elucidate general features of nonholonomic Riemannian manifolds. In particular, a better understanding of the existence and uniqueness of such structures (see section 1.2.2 and section 3.1.4) would be a first step.

(*ii*) In Riemannian geometry, Euclidean space is the first-order approximation of a Riemannian manifold. It is not entirely clear what the analogue of Euclidean space would be for a nonholonomic Riemannian manifold. (Evidently, flatness is not the only property such a structure would possess.) It seems likely that the answer to this problem would involve the nilpotentisation of the structure (as in sub-Riemannian geometry; see, e.g., [7, 50]).

(*iii*) The Wagner curvature tensor is not intrinsic. Does there exist a curvature tensor that characterises the flat nonholonomic Riemannian structures without use of the additional assumptions requires for Wagner's construction?

(*iv*) In Riemannian geometry, there are a number of enlightening geometric interpretations of the Riemannian curvature tensor, the Ricci tensor, etc. It would be desirable to find similar interpretations of the Schouten and Wagner curvature tensors and related constructions. (For instance, in chapter 3 we showed—at least, when the distribution is strongly nonholonomic—that the component C of the Schouten tensor measures the extent to which the distribution is geodesically invariant.) Wagner himself made some progress in this regard, though only in the three-dimensional case. In [77] he characterised flatness (in three dimensions) by the vanishing of a "curvature vector," and in the later work [76] gave a geometric interpretation of this curvature vector (as well as parallel transport in directions transversal to the distribution).

Appendix A

Tensor fields and tensor derivations

This appendix serves primarily to establish the conventions we use with regard to tensor fields on a distribution $\mathcal{D}$ (i.e., sections of $\bigotimes^k \mathcal{D} \otimes \bigotimes^\ell \mathcal{D}^*$) and their contractions, and tensor derivations. In section A.3 we also introduce two new objects, both dependent on a projection $\mathscr{P}$ onto a vector subbundle of a manifold. The first object is a generalisation of the Lie derivative, which we have called the *$\mathscr{P}$-Lie derivative.* The second object is a generalisation of the exterior derivative, and is called the *$\mathscr{P}$-exterior derivative.*

A.1 Tensors and tensor fields

We briefly outline the conventions we employ for tensors, tensor fields, and their contractions. Let V be a (real) vector space. A *(k, ℓ)-tensor on* V is a multilinear map of the form

$$\underbrace{\mathsf{V}^* \times \cdots \times \mathsf{V}^*}_{k \text{ times}} \times \underbrace{\mathsf{V} \times \cdots \times \mathsf{V}}_{\ell \text{ times}} \to \mathbb{R}.$$

The space of all (k, ℓ)-tensors on V is denoted $T^k_\ell(\mathsf{V}) = \bigotimes^k \mathsf{V} \otimes \bigotimes^\ell \mathsf{V}^*$. A $(0, \ell)$-tensor is also called a *covariant ℓ-tensor on* V, whereas a $(k, 0)$-tensor is also called a *contravariant k-tensor on* V.

Let $\mathcal{D}$ be a vector bundle over a manifold M. The bundle over M of *(k, ℓ)-tensors on* $\mathcal{D}$ is defined as

$$T^k_\ell(\mathcal{D}) = \bigsqcup_{q \in \mathsf{M}} T^k_\ell(\mathcal{D}_q).$$

Smooth sections of this bundle are called *(k, ℓ)-tensor fields.* (As before, smooth sections of $T^0_\ell(\mathcal{D})$ are also called *covariant ℓ-tensor fields*, and smooth sections of $T^k_0(\mathcal{D})$, *contravariant k-tensor fields.*) For convenience, "tensor fields" are often referred to simply as "tensors." Let $\mathcal{T}^k_\ell(\mathcal{D}) = \Gamma(T^k_\ell(\mathcal{D}))$ denote the space of (k, ℓ)-tensor fields. Evidently, we have $\mathcal{T}^1_0(\mathcal{D}) = \Gamma(\mathcal{D})$ and $\mathcal{T}^0_1(\mathcal{D}) = \Gamma(\mathcal{D}^*)$. By convention, we take $\mathcal{T}^0_0(\mathcal{D}) = \mathcal{C}^\infty(\mathsf{M})$.

The k^{th}-exterior power of $\mathcal{D}$ (i.e., the bundle of skew-symmetric $(0, k)$-tensors on $\mathcal{D}$) is denoted $\bigwedge^k \mathcal{D}$; likewise, the k^{th}-symmetric power of $\mathcal{D}$ is denoted $\bigvee^k \mathcal{D}$. Accordingly, the space of differential k-forms on $\mathcal{D}$ is given by $\Omega^k(\mathcal{D}) = \Gamma(\bigwedge^k \mathcal{D}^*)$. (Evidently, we have $\Omega^1(\mathcal{D}) = \Gamma(\mathcal{D}^*)$ and $\Omega^0(\mathcal{D}) = \mathcal{C}^\infty(\mathsf{M})$.) If $\mathcal{S}$ is a vector bundle over M, then the space of all $\mathcal{S}$-valued differential k-forms on $\mathcal{D}$ is denoted $\Omega^k(\mathcal{D}, \mathcal{S}) = \Omega^k(\mathcal{D}) \otimes \Gamma(\mathcal{S})$.

A.1.1 Tensor contractions

Let $k, \ell \geq 1$. For each $1 \leq i \leq k$ and $1 \leq j \leq \ell$ there exists a unique $\mathcal{C}^\infty(\mathsf{M})$-linear map $\operatorname{tr}^i_j : \mathcal{T}^{k+1}_{\ell+1}(\mathcal{D}) \to \mathcal{T}^k_\ell(\mathcal{D})$, called the *trace* (or *contraction*) on the i^{th} contravariant slot and j^{th} covariant slot, such that

(*i*) $\operatorname{tr}^1_1(\omega \otimes X) = \omega(X)$ for every $\omega \in \Gamma(\mathcal{D}^*)$ and $X \in \Gamma(\mathcal{D})$.

(*ii*) If $T \in \mathcal{T}^{k+1}_{\ell+1}(\mathcal{D})$, then $(\operatorname{tr}^i_j T)(\omega^1, \ldots, \omega^k, X_1, \ldots, X_\ell)$ is the trace of the $(1,1)$-tensor field
$$(\eta, Y) \mapsto T(\omega^1, \ldots, \eta, \ldots, \omega^k, X_1, \ldots, Y, \ldots, X_\ell),$$
where η appears in the i^{th} contravariant slot and Y in the j^{th} covariant slot. (Here $\omega^1, \ldots, \omega^k \in \Gamma(\mathcal{D}^*)$ and $X_1, \ldots, X_\ell \in \Gamma(\mathcal{D})$.)

Note that condition (*i*) says that the trace tr^1_1 of a $(1,1)$-tensor field is the usual trace of an endomorphism $\mathcal{D} \to \mathcal{D}$. Accordingly, in this case we usually just write tr instead of tr^1_1.

A.2 Tensor derivations

In this section we introduce derivations of tensor fields on $\mathcal{D}$ and state some necessary results. We follow O'Neill's [53] presentation; in particular, we omit proofs, as they are quite similar to those given by O'Neill.

A *derivation* of $\mathcal{T}^k_\ell(\mathcal{D})$ is a collection of $\mathbb{R}$-linear maps $\delta^k_\ell : \mathcal{T}^k_\ell(\mathcal{D}) \to \mathcal{T}^k_\ell(\mathcal{D})$ (for every $k, \ell \geq 0$), all denoted by δ when convenient, such that:

(*i*) $\delta(T \otimes S) = \delta(T) \otimes S + T \otimes \delta(S)$ for every $S, T \in \mathcal{T}^k_\ell(\mathcal{D})$.

(*ii*) δ commutes with contractions, i.e., $\delta(\operatorname{tr}^i_j T) = \operatorname{tr}^i_j \delta(T)$ for every $T \in \mathcal{T}^k_\ell(\mathcal{D})$.

Note that, if $f, g \in \mathcal{C}^\infty(\mathsf{M})$, then $fg = f \otimes g$, and so (*i*) becomes $\delta(fg) = \delta(f)g + f\delta(g)$. Likewise, for a function $f \in \mathcal{C}^\infty(\mathsf{M})$ and a tensor field $T \in \mathcal{T}^k_\ell(\mathcal{D})$, we have
$$\delta(fT) = \delta(f \otimes T) = \delta(f)T + f\delta(T).$$

The space of all derivations of $\mathcal{T}^k_\ell(\mathcal{D})$, together with the commutator $[\delta_1, \delta_2] = \delta_1 \circ \delta_2 - \delta_2 \circ \delta_1$, forms a Lie algebra, denoted $\mathsf{Der}(\mathcal{D})$.

Lemma A.2.1. *Let* $\delta \in \mathsf{Der}(\mathcal{D})$.

(*i*) *There exists a unique* $Z \in \Gamma(T\mathsf{M})$ *such that* $\delta^0_0 = \pounds_Z$.

(*ii*) *If* $\delta_1, \delta_2 \in \mathsf{Der}(\mathcal{D})$ *are derivations such that* $\delta_1|^0_0 = \pounds_{Z_1}$, $\delta_2|^0_0 = \pounds_{Z_2}$ *for* $Z_1, Z_2 \in \Gamma(T\mathsf{M})$, *then* $[\delta_1, \delta_2]^0_0 = \pounds_{[Z_1, Z_2]}$.

Note that, despite item (*i*) in the above lemma, the Lie derivative is *not* a member of $\mathsf{Der}(\mathcal{D})$. However, given a projection $\mathscr{P}$ onto $\mathcal{D}$, we can define an analogue of the Lie derivative which *is* in $\mathsf{Der}(\mathcal{D})$, and which we call the "$\mathscr{P}$-Lie derivative." This is discussed further in section A.3.1.

Proposition A.2.2. *Let* $\delta \in \mathsf{Der}(\mathcal{D})$. *If* $T \in \mathcal{T}^k_\ell(\mathcal{D})$, *then*

$$\begin{aligned}\delta(T)(\omega^1,\dots,\omega^k,X_1,\dots,X_\ell) = \delta(T(\omega^1,\dots,\omega^k,X_1,\dots,X_\ell))& \\ -\sum_{i=1}^{k} T(\omega^1,\dots,\delta(\omega^i),\dots,\omega^k,X_1,\dots,X_\ell)& \\ -\sum_{j=1}^{\ell} T(\omega^1,\dots,\omega^k,X_1,\dots,\delta(X_j),\dots,X_\ell)&\end{aligned}$$

for every $\omega^1,\dots,\omega^k \in \Gamma(\mathcal{D}^*)$ *and* $X_1,\dots,X_\ell \in \Gamma(\mathcal{D})$.

Corollary A.2.3. *If* $\omega \in \Gamma(\mathcal{D}^*)$, *then* $\delta(\omega)(X) = \delta(\omega(X)) - \omega(\delta(X))$ *for every* $X \in \Gamma(\mathcal{D})$.

Corollary A.2.4. *If* $\delta_1, \delta_2 \in \mathsf{Der}(\mathcal{D})$ *satisfy* $\delta_1(f) = \delta_2(f)$ *and* $\delta_1(W) = \delta_2(W)$ *for every* $f \in C^\infty(\mathsf{M})$ *and* $W \in \Gamma(\mathcal{D})$, *then* $\delta_1 = \delta_2$.

Corollary A.2.4 states that derivations of $\mathcal{T}^k_\ell(\mathcal{D})$ are completely specified by their action on $C^\infty(\mathsf{M})$ and $\Gamma(\mathcal{D})$. Accordingly, given a map that acts as a derivation of $C^\infty(\mathsf{M})$ and $\Gamma(\mathcal{D})$, it may be uniquely extended to a derivation of $\mathcal{T}^k_\ell(\mathcal{D})$.

Proposition A.2.5. *If* $Z \in \Gamma(T\mathsf{M})$ *and* $D : \Gamma(\mathcal{D}) \to \Gamma(\mathcal{D})$ *is an* $\mathbb{R}$*-linear map such that*

$$D(fX) = Z[f]X + fD(X) \qquad \text{for every } f \in C^\infty(\mathsf{M}) \text{ and } X \in \Gamma(\mathcal{D}),$$

then there exists a unique $\delta \in \mathsf{Der}(\mathcal{D})$ *such that* $\delta^0_0 = \pounds_Z$ *and* $\delta^1_0 = D$.

A.3 Generalisations of the Lie derivative and exterior derivative

In this section we consider a generalisation, denoted $\pounds^{\mathscr{P}}_Z$, $Z \in \Gamma(T\mathsf{M})$ and $d_{\mathscr{P}}$, of the Lie derivative $\pounds_Z$ and the exterior derivative d, respectively. These generalisations depend on a projection $\mathscr{P} : T\mathsf{M} \to \mathcal{D}$ onto a vector subbundle $\mathcal{D}$ of $T\mathsf{M}$. (Naturally, when $\mathcal{D} = T\mathsf{M}$, we have $\pounds^{\mathscr{P}}_Z = \pounds_Z$ and $d_{\mathscr{P}} = d$.) For both of these objects, we prove only what is required for the thesis.

Throughout this section, we assume that $\mathcal{D}$ is a vector subbundle of $T\mathsf{M}$ for which there exists a projection $\mathscr{P} : T\mathsf{M} \to \mathcal{D}$. Let $[\![\cdot,\cdot]\!]$ denote the projected Lie bracket $\mathscr{P}([\cdot,\cdot])$ and let $\mathscr{Q} = \mathrm{id}_{T\mathsf{M}} - \mathscr{P}$ be the complementary projection.

A.3.1 The $\mathscr{P}$-Lie derivative

If $Z \in \Gamma(T\mathsf{M})$, then we define the derivation $\pounds^{\mathscr{P}}_Z \in \mathsf{Der}(\mathcal{D})$ by the requirement that

$$\pounds^{\mathscr{P}}_Z f = Z[f] \quad \text{and} \quad \pounds^{\mathscr{P}}_Z X = [\![Z, X]\!],$$

where $f \in C^\infty(\mathsf{M})$ and $X \in \Gamma(\mathcal{D})$. By proposition A.2.5, there exists a unique extension of $\pounds^{\mathscr{P}}_Z$ to a derivation of $\mathcal{T}^k_\ell(\mathcal{D})$, which we also denote $\pounds^{\mathscr{P}}_Z$. We refer to $\pounds^{\mathscr{P}}_Z$ as the *$\mathscr{P}$-Lie derivative* along Z.

Proposition A.3.1. *Let* $\delta \in \mathsf{Der}(\mathcal{D})$. *Then:*

(*i*) $\delta_0^0 = 0$ *if and only if* δ_0^1 *is* $\mathcal{C}^\infty(\mathsf{M})$*-linear, i.e.,* $\delta_0^1 \in \mathcal{T}_1^1(\mathcal{D})$.

(*ii*) *There exists a unique vector field* $Z \in \Gamma(T\mathsf{M})$ *and a unique* $D_Z \in \mathsf{Der}(\mathcal{D})$ *such that* $\delta = \pounds_Z^{\mathscr{P}} + D_Z$, *where* $D_Z|_0^0 = 0$ *and* $D_Z|_0^1 \in \mathcal{T}_1^1(\mathcal{D})$.

Proof. If $f \in \mathcal{C}^\infty(\mathsf{M})$ and $X \in \Gamma(\mathcal{D})$, then $\delta(fX) = \delta(f)X + f\delta(X)$. Hence $\delta_0^0(f) = 0$ exactly when δ_0^1 is $\mathcal{C}^\infty(\mathsf{M})$-linear. On the other hand, let $Z \in \Gamma(T\mathsf{M})$ be the (unique) vector field such that $\pounds_Z^{\mathscr{P}} = \delta_0^0$ and let $D_Z = \delta - \pounds_Z^{\mathscr{P}}$. We have $D_Z(f) = \delta(f) - Z[f] = 0$ for every $f \in \mathcal{C}^\infty(\mathsf{M})$, i.e., $D_Z|_0^1 \in \mathcal{T}_1^1(\mathcal{D})$. Hence δ decomposes as $\delta = \pounds_Z^{\mathscr{P}} + D_Z$. ■

We shall denote the set of all derivations $\delta \in \mathsf{Der}(\mathcal{D})$ for which $\delta_0^0 = 0$ by by $\mathsf{Der}_0(\mathcal{D})$; such derivations are said to be *algebraic*. (By proposition A.3.1, we have $\mathsf{Der}_0(\mathcal{D}) \cong \mathcal{T}_1^1(\mathcal{D})$.) Likewise, let $\pounds_{T\mathsf{M}}^{\mathscr{P}} = \{\pounds_Z^{\mathscr{P}} : Z \in \Gamma(T\mathsf{M})\}$.

Proposition A.3.2. *We have the (vector space) decomposition* $\mathsf{Der}(\mathcal{D}) = \pounds_{T\mathsf{M}}^{\mathscr{P}} \oplus \mathsf{Der}_0(\mathcal{D})$. *Furthermore,* $[\mathsf{Der}(\mathcal{D}), \mathsf{Der}_0(\mathcal{D})] \subseteq \mathsf{Der}_0(\mathcal{D})$.

Proof. The decomposition of $\mathsf{Der}(\mathcal{D})$ follows from proposition A.3.1. Let $\delta = \pounds_Z^{\mathscr{P}} + D_Z$ be an element of $\mathsf{Der}(\mathcal{D})$, where $Z \in \Gamma(T\mathsf{M})$ and $D_Z \in \mathsf{Der}_0(\mathcal{D})$. Let $D \in \mathsf{Der}_0(\mathcal{D})$. If $f \in \mathcal{C}^\infty(\mathsf{M})$, then

$$[\delta, D](f) = (\pounds_Z^{\mathscr{P}} \circ D)(f) + (D_Z \circ D)(f) - (D \circ \pounds_Z^{\mathscr{P}})(f) - (D \circ D_Z)(f) = 0.$$

That is, $[\delta, D] \in \mathsf{Der}_0(\mathcal{D})$, and so we have $[\mathsf{Der}(\mathcal{D}), \mathsf{Der}_0(\mathcal{D})] \subseteq \mathsf{Der}_0(\mathcal{D})$. ■

A.3.2 The $\mathscr{P}$-exterior derivative

The *$\mathscr{P}$-exterior derivative* is the operator $d_{\mathscr{P}} : \Omega^k(\mathcal{D}) \to \Omega^{k+1}(\mathcal{D})$, defined as follows:

(*i*) If $f \in \Omega^0(\mathsf{M}) = \mathcal{C}^\infty(\mathsf{M})$, then $d_{\mathscr{P}} f(X) = X[f]$ for every $X \in \Gamma(\mathcal{D})$.

(*ii*) If $\omega \in \Omega^k(\mathcal{D})$, $k \geq 1$ then

$$\begin{aligned} d_{\mathscr{P}}\omega(X_0, \ldots, X_k) &= \sum_{0 \leq i \leq k} (-1)^i (\pounds_{X_i}^{\mathscr{P}} \omega)(X_0, \ldots, \widehat{X}_i, \ldots, X_k) \\ &= \sum_{0 \leq i \leq k} (-1)^i X_i[\omega(X_0, \ldots, \widehat{X}_i, \ldots, X_k)] \\ &\quad + \sum_{0 \leq i < j \leq k} (-1)^{i+j} \omega([\![X_i, X_j]\!], X_0, \ldots, \widehat{X}_i, \ldots, \widehat{X}_j, \ldots, X_k) \end{aligned}$$

for every $X_0, \ldots, X_k \in \Gamma(\mathcal{D})$. (The hat indicates the omission of that element.)

In the particular case of a 1-form ω, we have

$$d_{\mathscr{P}}\omega(X, Y) = X[\omega(Y)] - Y[\omega(X)] - \omega([\![X, Y]\!])$$

for every $X, Y \in \Gamma(\mathcal{D})$. Many properties of the usual exterior derivative generalise to $d_{\mathscr{P}}$. In particular, we have the following result.

Lemma A.3.3. *If* $f \in \mathcal{C}^\infty(\mathsf{M})$ *and* $\omega \in \Omega^k(\mathcal{D})$*, then*

$$d_{\mathscr{P}}(f\omega) = d_{\mathscr{P}} f \wedge \omega + f d_{\mathscr{P}}\omega.$$

Proof. Let $f \in \mathcal{C}^\infty(\mathsf{M})$, $\omega \in \Omega^k(\mathcal{D})$ and $X_0, \dots, X_k \in \Gamma(\mathcal{D})$. Then

$$\begin{aligned} d_{\mathscr{P}}(f\eta)(X_0, \dots, X_k) &= \sum_{i=0}^{k} (-1)^i (\pounds^{\mathscr{P}}_{X_i} f\eta)(X_0, \dots, \widehat{X}_i, \dots, X_k) \\ &= \sum_{i=0}^{k} (-1)^i (X_i[f]\eta + f\pounds^{\mathscr{P}}_{X_i}\eta)(X_0, \dots, \widehat{X}_i, \dots, X_k) \\ &= \sum_{i=0}^{k} (-1)^i X_i[f]\eta(X_0, \dots, \widehat{X}_i, \dots, X_k) + f d_{\mathscr{P}}\eta(X_0, \dots, X_k). \end{aligned}$$

Since

$$\begin{aligned} (d_{\mathscr{P}} f \wedge \eta)(X_0, \dots, X_k) &= \frac{1}{k!} \sum_{\sigma \in \mathsf{S}_{k+1}} (\operatorname{sgn} \sigma)(d_{\mathscr{P}} f \otimes \eta)(X_{\sigma(0)}, \dots, X_{\sigma(k)}) \\ &= \frac{1}{k!} \sum_{\sigma \in \mathsf{S}_{k+1}} (\operatorname{sgn} \sigma) X_{\sigma(0)}[f]\eta(X_{\sigma(1)}, \dots, X_{\sigma(k)}) \\ &= \sum_{i=0}^{k} (-1)^i X_i[f]\eta(X_0, \dots, \widehat{X}_i, \dots, X_k), \end{aligned}$$

(where S_{k+1} is the symmetric group on $k+1$ elements) it follows that

$$d_{\mathscr{P}}(f\eta)(X_0, \dots, X_k) = (d_{\mathscr{P}} f \wedge \eta)(X_0, \dots, X_k) + f d_{\mathscr{P}}\eta(X_0, \dots, X_k).$$

That is, $d_{\mathscr{P}}(f\eta) = d_{\mathscr{P}} f \wedge \eta + f d_{\mathscr{P}}\eta$. ■

One crucial property of the exterior derivative, *viz.*, that $d^2 = 0$, no longer holds in general for the $\mathscr{P}$-exterior derivative. Indeed, if $f \in \mathcal{C}^\infty(\mathsf{M})$ and $X, Y \in \Gamma(\mathcal{D})$, then

$$\begin{aligned} d^2_{\mathscr{P}} f(X, Y) &= X[d_{\mathscr{P}} f(Y)] - Y[d_{\mathscr{P}} f(X)] - d_{\mathscr{P}} f([\![X, Y]\!]) \\ &= [X, Y][f] - [\![X, Y]\!][f] \\ &= \mathscr{Q}([X, Y])[f]. \end{aligned}$$

Hence $d^2_{\mathscr{P}} f = 0$ if and only if $\mathcal{D}$ is integrable.

Appendix B

Restricted connections

In this appendix we consider "restricted connections," i.e., connections whose associated parallel transport is restricted to a subclass of curves (specifically, curves tangent to a given vector subbundle of the tangent bundle). We consider two approaches to restricted connections. In the first, which we term the "Koszul connection approach," we view a restricted connection as a covariant derivative operator ∇. This approach is considered in section B.1. We start from the definition of a restricted connection given in [38]. (The authors of [38] introduced the idea of a restricted connection and stated some basic results, but did not publish any further development of the theory.) After considering some basic properties of these restricted connections, we show how to define a notion of parallel translation associated to the connection, discuss parallel frames and parallel tensor fields, and discuss pullbacks of these connections by diffeomorphisms. We conclude the section by briefly discussing some particular types of restricted connections, *viz.*, metric connections, vector bundle connections and left-invariant connections (on Lie groups).

In section B.2 we consider restricted connections from a more geometric point of view: essentially as a horizontal distribution complementary to the vertical distribution. (However, the horizontal distribution will not, in general, form a full complement.) We shall refer to such connections as "restricted Ehresmann connections." Our approach is largely based upon the (quite general) theory developed in [12]. However, unlike in section B.1, we shall only consider what is necessary for an understanding of the results in this thesis. In particular, we do not consider how to define a parallel translation in this formalism (although this is covered in [12]). (We do, however, link the geometric approach of this section with the Koszul connection approach in section B.1.)

Although many of the results of this appendix are essentially new (they are not found in the literature), they are a straightforward generalisation of well-known theory. Nevertheless, in most cases we have included full proofs.

B.1 The Koszul connection approach

Let M be an n-dimensional (connected) manifold and let $\mathcal{E}$ and $\mathcal{D}$ be rank m and r, respectively, distributions on M. In order to simplify the discussion of parallel transport, we shall assume that $\mathcal{E}$ is nonintegrable and completely nonholonomic (see the beginning of section 1.1). In particular, under this assumption the Chow–Rashevskii theorem (theorem 1.1.3) guarantees that any two points in M can be joined by an $\mathcal{E}$-curve.

Remark B.1.1. Although we assume that $\mathcal{D}$ is a distribution on M, most of the results in this section easily generalise to the case when $\mathcal{D}$ is an arbitrary vector bundle over M. □

Remark B.1.2. As usual, we use the summation convention throughout this appendix. However, we use slightly different ranges for the indices. Unless stated otherwise, we shall assume that i ranges through $1, \ldots, m$ and a, b, c range through $1, \ldots, r$. If these indices are themselves indexed (e.g., $a_1, a_2, \ldots$), then they range through the same values (e.g., $a_1, a_2, \ldots$ range through $1, \ldots, r$). □

An *$\mathcal{E}$-restricted (Koszul) connection ∇ on $\mathcal{D}$* (cf. [38]) (or simply *$\mathcal{E}$-connection on $\mathcal{D}$*) is an $\mathbb{R}$-linear mapping

$$\nabla : \Gamma(\mathcal{E}) \times \Gamma(\mathcal{D}) \to \Gamma(\mathcal{D}), \quad (X, W) \mapsto \nabla_X W$$

that is

(i) tensorial in its first argument: $\nabla_{fX} W = f \nabla_X W$ for every $f \in \mathcal{C}^\infty(\mathsf{M})$, $X \in \Gamma(\mathcal{E})$ and $W \in \Gamma(\mathcal{D})$.

(ii) a derivation in its second argument: $\nabla_X f W = X[f] W + f \nabla_X W$ for every $f \in \mathcal{C}^\infty(\mathsf{M})$, $X \in \Gamma(\mathcal{E})$ and $W \in \Gamma(\mathcal{D})$.

$\nabla_X W$ is called the *covariant derivative* of W along X.

Let ∇ be an $\mathcal{E}$-connection on $\mathcal{D}$. Let (U_a) be a local frame for $\mathcal{D}$ and (X_i) a local frame (defined in the same neighbourhood of (U_a)) for $\mathcal{E}$. Let $\Gamma^b_{ia} \in \mathcal{C}^\infty(\mathsf{M})$ be the functions (called the *connection coefficients of* ∇ with respect to the two frames) defined by $\nabla_{X_i} U_a = \Gamma^b_{ia} U_b$.

Remark B.1.3. If $\mathcal{E} = T\mathsf{M}$, then we shall see that parallel translation (of $\mathcal{D}$-vectors) is defined along any curve in M (as opposed to only $\mathcal{E}$-curves). In this case, ∇ is called a *vector bundle connection on $\mathcal{D}$*. On the other hand, if $\mathcal{D} = \mathcal{E}$, then ∇ is called a *nonholonomic connection* on $\mathcal{E}$, and parallel translation (of $\mathcal{E}$-vectors) is only along $\mathcal{E}$-curves. □

Lemma B.1.4. *Let $X \in \Gamma(\mathcal{E})$ and $W \in \Gamma(\mathcal{D})$. The value of $\nabla_X W(q)$, $q \in \mathsf{M}$ depends only on the value of X at q.*

Proof. The result follows from the tensoriality of $(X, W) \mapsto \nabla_X W$ in X. ■

Accordingly, for every $X_q \in \mathcal{E}$ we have the mapping $\nabla_{X_q} : \Gamma(\mathcal{D}) \to \mathcal{D}_q$ defined by $\nabla_{X_q} W = \nabla_X W(q)$, where $X \in \Gamma(\mathcal{E})$ is any vector field such that $X(q) = X_q$.

Lemma B.1.5. *Let $X \in \Gamma(\mathcal{E})$ and $W \in \Gamma(\mathcal{D})$. The expression $\nabla_X W(q)$ depends only on the values of W along any $\mathcal{E}$-curve tangent to $X(q)$. That is, if γ is an $\mathcal{E}$-curve with $\gamma(0) = q$, $\dot{\gamma}(0) = X(q)$ and $W_1, W_2 \in \Gamma(\mathcal{D})$ satisfy $W_1(\gamma(t)) = W_2(\gamma(t))$ for all t, then*

$$(\nabla_X W_1)(q) = (\nabla_X W_2)(q).$$

Proof. We have $X = x^i X_i$ and $W = w^a U_a$ for functions $x^i, w^a \in \mathcal{C}^\infty(\mathsf{M})$. Hence

$$\begin{aligned}(\nabla_X W)(q) &= X[w^a](q) U_a(q) + x^i(q) w^a(q) \nabla_{X_i} U_a(q) \\ &= X[w^a](q) U_a(q) + x^i(q) w^a(q) \Gamma^b_{ia}(q) U_b(q).\end{aligned}$$

The term $X[w^a](q)$ depends only on the values of w^a along any curve tangent to $X(q)$; the other terms depend on $W(q)$. It follows that $\nabla_X W(q)$ depends only on the values of W along such curves. ■

Covariant differentiation (along a vector field) can be extended, in a unique fashion, to arbitrary tensor fields. In order to do so, we make use of the theory of (tensor field) derivations treated in appendix A. Let $X \in \Gamma(\mathcal{E})$ and define

$$\nabla_X f = X[f], \qquad \text{for every } f \in \mathcal{C}^\infty(\mathsf{M}).$$

Then ∇_X is a derivation of $\mathcal{C}^\infty(\mathsf{M})$ and $\Gamma(\mathcal{D})$, and hence, by proposition A.2.5, there exists a unique extension of ∇_X to a derivation of $\mathcal{T}^k_\ell(\mathcal{D})$. Indeed, if $T \in \mathcal{T}^k_\ell(\mathcal{D})$, then

$$\begin{aligned}(\nabla_X T)(\omega^1, \dots, \omega^k, W_1, \dots, W_\ell) = X[T(\omega^1, \dots, \omega^k, W_1, \dots, W_\ell)] \\ - \sum_{i=1}^{k} T(\omega^1, \dots, \nabla_X \omega^i, \dots, \omega^k, W_1, \dots, W_\ell) \\ - \sum_{j=1}^{\ell} T(\omega^1, \dots, \omega^k, W_1, \dots, \nabla_X W_j, \dots, W_\ell)\end{aligned}$$

for every $\omega^1, \dots, \omega^k \in \Gamma(\mathcal{D}^*)$ and $W_1, \dots, W_\ell \in \Gamma(\mathcal{D})$. The *total covariant derivative* of T, denoted ∇T, is the tensor field in $\mathcal{T}^k_\ell(\mathcal{D}) \otimes \mathcal{T}^0_1(\mathcal{E})$ defined as

$$(\nabla T)(\omega^1, \dots, \omega^k, W_1, \dots, W_\ell, X) = (\nabla_X T)(\omega^1, \dots, \omega^k, W_1, \dots, W_\ell).$$

Lemma B.1.4 and lemma B.1.5 also generalise to covariant differentiation of tensor fields.

Lemma B.1.6. *Let $X \in \Gamma(\mathcal{E})$ and $T \in \mathcal{T}^k_\ell(\mathcal{D})$. The value of $(\nabla_X T)(q)$, $q \in \mathsf{M}$ depends only on the value of X at q and only on the values of T along any $\mathcal{E}$-curve tangent to $X(q)$.*

Proof. The first part again follows by tensoriality of ∇ in its first argument. Let (ν^a) be the frame for $\mathcal{D}^*$ dual to (U_a). Then $(B^{a_1 \cdots a_\ell}_{b_1 \cdots b_k}) = (U_{b_1} \otimes \cdots \otimes U_{b_k} \otimes \nu^{a_1} \otimes \cdots \otimes \nu^{a_\ell})$ is a frame for $\mathcal{T}^k_\ell(\mathcal{D})$. For brevity, we shall abbreviate $B^{a_1 \cdots a_\ell}_{b_1 \cdots b_k}$ by $B^{\boldsymbol{a}}_{\boldsymbol{b}}$. If $T = T^{\boldsymbol{b}}_{\boldsymbol{a}} B^{\boldsymbol{a}}_{\boldsymbol{b}} \in \mathcal{T}^k_\ell(\mathcal{D})$, then

$$\begin{aligned}(\nabla_X T)(q) &= (\nabla_X T^{\boldsymbol{b}}_{\boldsymbol{a}} B^{\boldsymbol{a}}_{\boldsymbol{b}})(q) \\ &= X[T^{\boldsymbol{b}}_{\boldsymbol{a}}](q) B^{\boldsymbol{a}}_{\boldsymbol{b}}(q) + T^{\boldsymbol{b}}_{\boldsymbol{a}}(q) \nabla_X B^{\boldsymbol{a}}_{\boldsymbol{b}}(q).\end{aligned}$$

The first term $X[T^{\boldsymbol{b}}_{\boldsymbol{a}}](q)$ depends only on the values of X along any curve tangent to $X(q)$. Consider the second term $\nabla_X B^{\boldsymbol{a}}_{\boldsymbol{b}}(q)$. Using the derivation properties of ∇_X, we have

$$\begin{aligned}\nabla_X B^{\boldsymbol{a}}_{\boldsymbol{b}} &= \sum_{i=1}^{k} U_{b_1} \otimes \cdots \otimes \nabla_X U_{b_i} \otimes \cdots \otimes U_{b_k} \otimes \nu^{a_1} \otimes \cdots \otimes \nu^{a_\ell} \\ &+ \sum_{j=1}^{\ell} U_{b_1} \otimes \cdots \otimes U_{b_k} \otimes \nu^{a_1} \otimes \cdots \otimes \nabla_X \nu^{a_j} \otimes \cdots \otimes \nu^{a_\ell}.\end{aligned}$$

We claim that $\nabla_X \omega(q)$, $\omega \in \Gamma(\mathcal{D}^*)$ depends only on the values of ω along any curve tangent to $X(q)$. (It then follows that the entire expression $\nabla_X B^{\boldsymbol{a}}_{\boldsymbol{b}}$ will only depend on the values along such a curve, which will complete the proof.) Indeed, since $X = x^i X_i$ and $\omega = \omega_a \nu^a$ for some $x^i, \omega_a \in \mathcal{C}^\infty(\mathsf{M})$, we have

$$(\nabla_X \omega)(q) = X[\omega_a](q) \nu^a(q) + x^i(q) \omega_a(q) \nabla_{X_i} \nu^a(q).$$

The expression $X[\omega_a](q)$ depends only on the values of ω^a along any curve tangent to $X(q)$. It follows that $\nabla_X \omega(q)$ (and $\nabla_X T(q)$ in turn) has a likewise dependence. ■

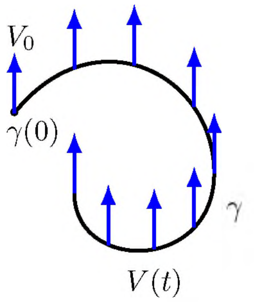

Figure B.1: Parallel translation of $V_0 \in \mathcal{D}_{\gamma(0)}$ along an $\mathcal{E}$-curve γ

B.1.1 Parallel translation

A section of $\mathcal{D}$ along an $\mathcal{E}$-curve $\gamma : [0,1] \to \mathsf{M}$ is a mapping $V : [0,1] \to \mathcal{D}$ such that $V(t) \in \mathcal{D}_{\gamma(t)}$ for every $t \in [0,1]$. The space of sections of $\mathcal{D}$ along γ is denoted $\Gamma(\gamma^*\mathcal{D})$. Clearly, if $W \in \Gamma(\mathcal{D})$, then $W \circ \gamma$ is a section along γ.

A section $V \in \Gamma(\gamma^*\mathcal{D})$ along an $\mathcal{E}$-curve γ is said to be *parallel along* γ with respect to ∇ if $\nabla_{\dot\gamma} V(t) = 0$ for all t. If $W \in \Gamma(\mathcal{D})$ is parallel along every $\mathcal{E}$-curve (specifically, $W \circ \gamma$ is parallel along γ for every $\mathcal{E}$-curve γ), then it is called *parallel.*

Lemma B.1.7. *A section $W \in \Gamma(\mathcal{D})$ is parallel if and only if $\nabla W \equiv 0$.*

Proof. Suppose $W \in \Gamma(\mathcal{D})$ is parallel and let $X_q \in \mathcal{E}_q$. There exists an $\mathcal{E}$-curve γ in M such that $\gamma(0) = q$ and $\dot\gamma(0) = X_q$. Accordingly, we have $\nabla_{X_q} W = \nabla_{\dot\gamma}(W \circ \gamma)(0) = 0$. Since q and X_q are arbitrary, it follows that $\nabla W \equiv 0$. Conversely, if ∇W vanishes identically, then for any $\mathcal{E}$-curve γ in M, we have $\nabla_{\dot\gamma}(W \circ \gamma) = 0$. That is, W is parallel. ■

Any element $V_0 \in \mathcal{D}_{\gamma(0)}$ along an $\mathcal{E}$-curve γ can be uniquely extended to a parallel section of $\mathcal{D}$ along the entirety of γ; this section is called the *parallel translate* of V_0. (See figure B.1.)

Theorem B.1.8. *Let $\gamma : [0,1] \to \mathsf{M}$ be an $\mathcal{E}$-curve in M and let $V_0 \in \mathcal{D}_{\gamma(0)}$. There exists a unique parallel section $V \in \Gamma(\gamma^*\mathcal{D})$, defined on all of $[0,1]$, such that $V(0) = V_0$.*

Proof. Any section $V \in \Gamma(\gamma^*\mathcal{D})$ can be written as $V = v^a(U_a \circ \gamma)$ for functions $v^a \in \mathcal{C}^\infty([0,1])$. Hence

$$\begin{aligned}\nabla_{\dot\gamma} V(t) &= \dot v^a(t) U_a(\gamma(t)) + v^a(t) \nabla_{\dot\gamma}(U_a \circ \gamma)(t) \\ &= \left[\dot v^b(t) + \Gamma^b_{ia}(\gamma(t))\dot\gamma^i(t) v^a(t)\right] U_b(\gamma(t)).\end{aligned}$$

Accordingly, V is parallel if and only if $v^1, \dots, v^r$ satisfies the system of first-order linear ODEs $\dot v^b = -\Gamma^b_{ia}(\gamma)\dot\gamma^i v^a$. Given initial conditions $v^1(0), \dots, v^r(0)$, such a system has a unique solution, and since the equations are linear, the solution is defined on the entirety of $[0,1]$. ■

Let $\gamma : [0,1] \to \mathsf{M}$ be an $\mathcal{E}$-curve. Using parallel translation along γ, we may define an operator (referred to as *parallel translation*)

$$\Pi^t_\gamma : \mathcal{D}_{\gamma(0)} \to \mathcal{D}_{\gamma(t)}, \qquad t \in [0,1]$$

specified by setting $\Pi^t_\gamma(V_0) = V(t)$, where V is the parallel translate of $V_0 \in \mathcal{D}_{\gamma(0)}$ along γ. Accordingly, if $W \in \Gamma(\mathcal{D})$ is parallel, then it is invariant under parallel translation, i.e., $W(\gamma(t)) = \Pi^t_\gamma(W(\gamma(0)))$ for every $\mathcal{E}$-curve $\gamma : [0,1] \to \mathsf{M}$.

Lemma B.1.9. *The parallel translation Π^t_γ does not depend on the parametrisation of γ.*

Proof. Let $\gamma : [0,1] \to \mathsf{M}$ be an $\mathcal{E}$-curve and let $\widetilde{\gamma} = \gamma \circ \phi : [0,1] \to \mathsf{M}$ be a reparametrisation of γ, where $\phi : [0,1] \to [0,1]$ is a diffeomorphism. Let $V_0 \in \mathcal{D}_{\gamma(0)}$, $\widetilde{V}_0 \in \mathcal{D}_{\widetilde{\gamma}(0)}$ and $V(t) = \Pi^t_\gamma(V_0)$, $\widetilde{V}(t) = \Pi^t_{\widetilde{\gamma}}(\widetilde{V}_0)$. We claim that $\widetilde{V} = V \circ \phi$. Indeed, there exist functions $v^a, \widetilde{v}^a \in \mathcal{C}^\infty(\mathsf{M})$ such that $V = v^a(U_a \circ \gamma)$ and $\widetilde{V} = \widetilde{v}^a(U_a \circ \widetilde{\gamma})$. Furthermore, $\dot{v}^a = -\Gamma^a_{ib}(\gamma)\dot{\gamma}^i v^b$ and $\dot{\widetilde{v}}^a = -\Gamma^a_{ib}(\widetilde{\gamma})\dot{\widetilde{\gamma}}^i \widetilde{v}^a$. We have

$$\frac{d}{dt}(v^a \circ \phi) = \dot{v}^a(\phi)\dot{\phi} = -\Gamma^a_{ib}(\gamma(\phi))\dot{\gamma}^i(\phi)v^a(\phi)\dot{\phi} = -\Gamma^a_{ib}(\widetilde{\gamma})\dot{\widetilde{\gamma}}^i(\phi)\widetilde{v}^a$$

and $v^a(\phi(0)) = \widetilde{v}^a(0)$. Thus $t \mapsto (V \circ \phi)(t)$ and $t \mapsto \widetilde{V}(t)$ both solve the same Cauchy problem, and hence are identical. ■

Proposition B.1.10. *Parallel translation is a linear isomorphism. Furthermore, the inverse $(\Pi^1_\gamma)^{-1} : \mathcal{D}_{\gamma(1)} \to \mathcal{D}_{\gamma(0)}$ is exactly the parallel translation $\Pi^{-1}_\gamma = \Pi^1_{\widetilde{\gamma}}$, where $\widetilde{\gamma} : [0,1] \to \mathsf{M}$, $t \mapsto \gamma(1-t)$.*

Proof. That parallel translation is a linear isomorphism follows immediately from the fact that the parallel translate $V \in \Gamma(\gamma^*\mathcal{D})$ of $V_0 \in \mathcal{D}_{\gamma(0)}$ is unique (theorem B.1.8). Let $V_0 \in \mathcal{D}_{\gamma(1)} = \mathcal{D}_{\widetilde{\gamma}(0)}$. We have $\Pi^1_\gamma(V_0) = V(1)$ and $\Pi^1_{\widetilde{\gamma}}(V(1)) = \widetilde{V}(1)$. We claim that $V(t) = \widetilde{V}(1-t)$. Indeed, there are functions $v^a, \widetilde{v}^a \in \mathcal{C}^\infty(\mathsf{M})$ such that $V = v^a(U_a \circ \gamma)$ and $\widetilde{V} = \widetilde{v}^a(U_a \circ \widetilde{\gamma})$. Furthermore, the v^a's satisfy the ODEs

$$\dot{v}^a(t) = -\Gamma^a_{ib}(\gamma(t))\dot{\gamma}^i(t)v^b(t),$$

and the $\widetilde{v}^a$'s satisfy

$$\dot{\widetilde{v}}^a(t) = -\Gamma^a_{ib}(\widetilde{\gamma}(t))\dot{\widetilde{\gamma}}^i(t)\widetilde{v}^b(t) = \Gamma^a_{ib}(\gamma(1-t))\dot{\gamma}^i(1-t)\widetilde{v}^b(t).$$

It is easy to show that $w^a : t \mapsto v^a(1-t)$ are solutions to the second system of ODEs. Furthermore, we have $w^a(0) = \widetilde{v}^a(0)$. It follows that $t \mapsto V(t)$ and $t \mapsto \widetilde{V}(1-t)$ both solve the same Cauchy problem, and hence are identical. ■

Lemma B.1.11. *If $\gamma : [0,1] \to \mathsf{M}$ is an $\mathcal{E}$-curve and $V \in \Gamma(\gamma^*\mathcal{D})$, then*

$$\Pi^{-t}_\gamma(\nabla_{\dot{\gamma}}V(t)) = \frac{d}{dt}\Pi^{-t}_\gamma(V(t)).$$

Proof. Let (E_a) be a basis for $\mathcal{D}_{\gamma(0)}$ and define a frame (V_a) for $\gamma^*\mathcal{D}$ (i.e., a frame for $\mathcal{D}$ along γ) by $V_a(t) = \Pi^t_\gamma(E_a)$. By definition, we have $\nabla_{\dot{\gamma}}V_a(t) = 0$ for all $t \in [0,1]$. There exist functions $v^a \in \mathcal{C}^\infty([0,1])$ such that $V = v^a V_a$. Consequently, we have

$$\begin{aligned}
\Pi^{-t}_\gamma(\nabla_{\dot{\gamma}}V(t)) &= \Pi^{-t}_\gamma\left(\dot{\gamma}[v^a](t)V_a(t) + v^a(t)\nabla_{\dot{\gamma}}V_a(t)\right)\\
&= \Pi^{-t}_\gamma\left(\dot{v}^a(t)V_a(t)\right)\\
&= \dot{v}^a(t)\Pi^{-t}_\gamma\left(V_a(t)\right) = \dot{v}^a(t)V_a(0).
\end{aligned}$$

Likewise,

$$\frac{d}{dt}\Pi^{-t}_\gamma(V(t)) = \frac{d}{dt}\Pi^{-t}_\gamma(v^a(t)V_a(t)) = \dot{v}^a(t)\Pi^{-t}_\gamma(V_a(t)) = \dot{v}^a(t)V_a(0).$$ ■

Corollary B.1.12. *If $\gamma : [0,1] \to \mathsf{M}$ is an $\mathcal{E}$-curve and $V \in \Gamma(\gamma^*\mathcal{D})$, then*

$$\nabla_{\dot\gamma} V(0) = \lim_{t\to 0} \frac{\Pi_\gamma^{-t}(V(t)) - V(0)}{t}.$$

Corollary B.1.12 implies that, given a notion of parallel transport of $\mathcal{D}$-vectors along $\mathcal{E}$-curves, we can define an $\mathcal{E}$-connection on $\mathcal{D}$. Therefore parallel transport is an equivalent means of defining a connection.

B.1.1.1 Parallel frames

A *parallel frame* (U_a) for $\mathcal{D}$ is a frame for $\mathcal{D}$ such that each vector field of the frame is parallel. The existence of parallel frames is, of course, not guaranteed. In fact, their existence places severe restrictions on the connection ∇. The connection ∇ is called *locally flat on $\mathcal{U} \subseteq \mathsf{M}$* if there exists a parallel frame (U_a) for $\mathcal{D}$ defined on $\mathcal{U}$. If ∇ is locally flat on a neighbourhood about every point in M, then we say that it is *locally flat*. If there exists a global parallel frame for $\mathcal{D}$, then ∇ is said to be (globally) *flat*.

Lemma B.1.13. *If (U_a) is parallel, then the connection coefficients Γ_{ia}^b vanish identically.*

Theorem B.1.14. *∇ is locally flat on $\mathcal{U} \subseteq \mathsf{M}$ if and only if for any two points $q_0, q_1 \in \mathcal{U}$ and for any $\mathcal{E}$-curve $\gamma : [0,1] \to \mathcal{U}$ such that $\gamma(0) = q_0$ and $\gamma(1) = q_1$, the parallel translation $\Pi_\gamma^1 : \mathcal{D}_{q_0} \to \mathcal{D}_{q_1}$ does not depend on the choice of γ. (In this case, we shall denote Π_γ^1 by $\Pi_{q_0}^{q_1}$.)*

Proof. Suppose there exists a parallel frame (U_a) for $\mathcal{D}$, defined on $\mathcal{U} \subseteq \mathsf{M}$. Let $q_0, q_1 \in \mathcal{U}$ and let $\gamma_1, \gamma_2 : [0,1] \to \mathsf{M}$ be $\mathcal{E}$-curves joining q_0 to q_1. Let $V_0 \in \mathcal{D}_{q_0}$ and let V_1, V_2 denote the parallel translates of V_0 along γ_1 and γ_2, respectively. We have $\Pi_{\gamma_1}^1(V_0) = V_1(1)$ and $\Pi_{\gamma_2}^1(V_0) = V_2(1)$. Since $V_1 = v_1^a(U_a \circ \gamma_1)$ and $V_2 = v_2^a(U_a \circ \gamma_2)$ for functions $v_1^a, v_2^a \in \mathcal{C}^\infty([0,1])$, $\nabla_{\dot\gamma_1} V_1 = \nabla_{\dot\gamma_2} V_2 = 0$, and (U_a) is a parallel frame, it follows that

$$\begin{cases} 0 = \nabla_{\dot\gamma_1} V_1 = \dot v_1^a (U_a \circ \gamma_1) + v_1^a \nabla_{\dot\gamma_1}(U_a \circ \gamma_1) = \dot v_1^a(U_a \circ \gamma_1) \\ 0 = \nabla_{\dot\gamma_2} V_2 = \dot v_2^a (U_a \circ \gamma_2) + v_2^a \nabla_{\dot\gamma_2}(U_a \circ \gamma_2) = \dot v_2^a(U_a \circ \gamma_2). \end{cases}$$

That is, $\dot v_1^a = \dot v_2^a = 0$. As V_1 and V_2 both have the same initial conditions, they solve the same Cauchy problem, and hence are identical. Therefore $\Pi_{\gamma_1}^1 = \Pi_{\gamma_2}^1$. In particular, the parallel translation from $\mathcal{D}_{q_0}$ to $\mathcal{D}_{q_1}$ does not depend on the path taken.

Conversely, suppose the parallel translation along $\mathcal{E}$-curves in $\mathcal{U}$ does not depend on the curve. Let $q_0 \in \mathcal{U}$ and let (E_a) be a basis for $\mathcal{D}_{q_0}$. Define sections $U_1, \ldots, U_r$ of $\mathcal{D}$ on $\mathcal{U}$ by

$$U_a(q) = \Pi_{q_0}^q(E_a) = V_a(1)$$

where V_a is the parallel translate of E_a. It follows that (U_a) forms a frame for $\mathcal{D}$ on $\mathcal{U}$. Furthermore, for any $\mathcal{E}$-curve $\gamma : [0,1] \to \mathsf{M}$ with $\gamma(0) = q_0$, we have $\nabla_{\dot\gamma}(U_a \circ \gamma)(t) = \nabla_{\dot\gamma} V_a(t) = 0$. That is, each U_a is parallel. ■

Corollary B.1.15. *If ∇ is locally flat on $\mathcal{U} \subseteq \mathsf{M}$, then $\mathcal{D}|_{\mathcal{U}}$ is trivial, i.e., $\mathcal{D}|_{\mathcal{U}} = \mathcal{U} \times \mathbb{R}^r$.*

Proof. The typical fibre of $\mathcal{D} \to \mathsf{M}$ is (isomorphic to) $\mathbb{R}^r$. Fix a point $q_0 \in \mathcal{U}$. The map $\Phi : \mathcal{D}|_{\mathcal{U}} \to \mathcal{U} \times \mathcal{D}_{q_0} \cong \mathcal{U} \times \mathbb{R}^r$, $U_q \mapsto (q, \Pi_q^{q_0}(U_q))$ is a trivialisation map. ■

B.1.1.2 Parallel tensor fields

We can generalise the idea of a parallel section of $\mathcal{D}$ to arbitrary tensor fields in $\mathcal{T}^k_\ell(\mathcal{D})$. We first discuss the notion of parallel sections of $\mathcal{D}^*$. Let $\gamma : [0,1] \to \mathsf{M}$ be an $\mathcal{E}$-curve. A section of $\mathcal{D}^*$ along a curve is a mapping $\sigma : [0,1] \to \mathcal{D}^*$ such that $\sigma_t = \sigma(t) \in \mathcal{D}^*_{\gamma(t)}$ for every $t \in [0,1]$. The space of such sections is denoted $\Gamma(\gamma^*\mathcal{D}^*)$. As for sections of $\mathcal{D}$, if $\omega \in \Gamma(\mathcal{D}^*)$, then $\omega \circ \gamma$ is an element of $\Gamma(\gamma^*\mathcal{D}^*)$.

A section $\sigma \in \Gamma(\gamma^*\mathcal{D}^*)$ is said to be *parallel along* γ with respect to ∇ if $\nabla_{\dot\gamma}\sigma(t) = 0$ for all t. If $\omega \in \Gamma(\mathcal{D}^*)$ is parallel along every $\mathcal{E}$-curve (specifically, $\omega \circ \gamma$ is parallel along γ, for every $\mathcal{E}$-curve γ), then it is said to be *parallel*. The proof of the following result is similar to that of lemma B.1.7.

Lemma B.1.16. *A section* $\omega \in \Gamma(\mathcal{D}^*)$ *is parallel if and only if* $\nabla\omega \equiv 0$.

Let $\gamma : [0,1] \to \mathsf{M}$ be an $\mathcal{E}$-curve. The parallel translate $\sigma \in \Gamma(\gamma^*\mathcal{D}^*)$ of an element $\sigma_0 \in \mathcal{D}^*_{\gamma(0)}$ is defined as

$$\sigma_t = (\Pi_\gamma^{-t})^*(\sigma_0),$$

where $(\Pi_\gamma^{-t})^*$ is the map dual to Π_γ^{-t}. The following result confirms this to be the correct definition.

Lemma B.1.17. *If* $\gamma : [0,1] \to \mathsf{M}$ *is an* $\mathcal{E}$*-curve and* σ *is the parallel translate of* $\sigma_0 \in \mathcal{D}^*_{\gamma(0)}$, *then* σ *is parallel along* γ.

Proof. Let $V \in \Gamma(\gamma^*\mathcal{D})$. We have

$$\begin{aligned}((\nabla_{\dot\gamma}\sigma)(V))(t) &= \nabla_{\dot\gamma}(\sigma(V))(t) - \sigma_t(\nabla_{\dot\gamma}V(t))\\ &= \dot\gamma[\sigma(V)](t) - \sigma_0(\Pi_\gamma^{-t}(\nabla_{\dot\gamma}V(t)))\\ &= \frac{d}{dt}\sigma_0(\Pi_\gamma^{-t}(V(t))) - \sigma_0(\Pi_\gamma^{-t}(\nabla_{\dot\gamma}V(t))).\end{aligned}$$

By lemma B.1.11, the second term is

$$\sigma_0(\Pi_\gamma^{-t}(\nabla_{\dot\gamma}V(t))) = \sigma_0\Big(\frac{d}{dt}\Pi_\gamma^{-t}(V(t))\Big) = \frac{d}{dt}\sigma_0(\Pi_\gamma^{-t}(V(t))).$$

It follows that $(\nabla_{\dot\gamma}\sigma)(V) = 0$. Since V is arbitrary, we thus have $\nabla_{\dot\gamma}\sigma = 0$. ■

There is also an analogue of lemma B.1.11:

Lemma B.1.18. *If* $\gamma : [0,1] \to \mathsf{M}$ *is an* $\mathcal{E}$*-curve and* $\sigma \in \Gamma(\gamma^*\mathcal{D}^*)$, *then*

$$(\Pi_\gamma^t)^*(\nabla_{\dot\gamma}\sigma(t)) = \frac{d}{dt}\,(\Pi_\gamma^t)^*(\sigma_t).$$

Proof. Let $V_0 \in \mathcal{D}_{\gamma(0)}$. We have

$$\begin{aligned}((\Pi_\gamma^t)^*((\nabla_{\dot\gamma}\sigma)_t))(V_0) &= (\nabla_{\dot\gamma}\sigma)_t(\Pi_\gamma^t(V_0))\\ &= \dot\gamma[\sigma(\Pi_\gamma^t(V_0))](t) - \sigma_t(\nabla_{\dot\gamma}\Pi_\gamma^t(V_0))\\ &= \frac{d}{dt}\sigma_t(\Pi_\gamma^t(V_0))\\ &= \frac{d}{dt}((\Pi_\gamma^t)^*(\sigma_t))(V_0).\end{aligned}$$

As V_0 is arbitrary, the result follows. ■

We can now generalise the idea of parallelness to (k, ℓ)-tensor fields. Let $\gamma : [0, 1] \to \mathsf{M}$ be an $\mathcal{E}$-curve. A section of $T^k_\ell(\mathcal{D})$ along γ is a mapping $A : [0, 1] \to T^k_\ell(\mathcal{D})$ such that $A_t = A(t) \in T^k_\ell(\mathcal{D}_{\gamma(t)})$ for every $t \in [0, 1]$. The space of such sections is denoted $\Gamma(\gamma^* T^k_\ell(\mathcal{D}))$. If $T \in \mathcal{T}^k_\ell(\mathcal{D})$, then $T \circ \gamma$ is a section of $T^k_\ell(\mathcal{D})$ along γ.

A section $A \in \Gamma(\gamma^* T^k_\ell(\mathcal{D}))$ is said to be *parallel along* γ with respect to ∇ if $\nabla_{\dot\gamma} A(t) = 0$ for all t. If $T \in \mathcal{T}^k_\ell(\mathcal{D})$ is parallel along every $\mathcal{E}$-curve (specifically, $T \circ \gamma$ is parallel along γ, for every $\mathcal{E}$-curve γ), then it is said to be *parallel.* The following lemma is proved similarly to lemma B.1.7.

Lemma B.1.19. *A tensor field* $T \in \mathcal{T}^k_\ell(\mathcal{D})$ *is parallel if and only if* $\nabla T \equiv 0$.

Let $\gamma : [0, 1] \to \mathsf{M}$ be an $\mathcal{E}$-curve. We can define the parallel translate $A \in \Gamma(\gamma^* T^k_\ell(\mathcal{D}))$ of $A_0 \in T^k_\ell(\mathcal{D}_{\gamma(0)})$ as follows:

$$\begin{aligned} &A_t(\sigma^1_t, \ldots, \sigma^k_t, V_1(t), \ldots, V_\ell(t)) \\ &\qquad = A_0((\Pi^t_\gamma)^*(\sigma^1_t), \ldots, (\Pi^t_\gamma)^*(\sigma^k_t), \Pi^{-t}_\gamma(V_1(t)), \ldots, \Pi^{-t}_\gamma(V_\ell(t))). \end{aligned}$$

Here $\sigma^1, \ldots, \sigma^k \in \Gamma(\gamma^* \mathcal{D}^*)$ and $V_1, \ldots, V_\ell \in \Gamma(\gamma^* \mathcal{D})$. Accordingly, if $T \in \mathcal{T}^k_\ell(\mathcal{D})$ is parallel, then we may interpret this to mean that T is invariant under parallel transport. That is, for every $q \in \mathsf{M}$, we have

$$T_{\gamma(t)}((\Pi^{-t}_\gamma)^*(\varepsilon^1), \ldots, (\Pi^{-t}_\gamma)^*(\varepsilon^k), \Pi^t_\gamma(E_1), \ldots, \Pi^t_\gamma(E_\ell)) = T_q(\varepsilon^1, \ldots, \varepsilon^k, E_1, \ldots, E_\ell)$$

for every $\mathcal{E}$-curve $\gamma : [0, 1] \to \mathsf{M}$ such that $\gamma(0) = q$ and every $\varepsilon^1, \ldots, \varepsilon^k \in \mathcal{D}^*_q$, $E_1, \ldots, E_\ell \in \mathcal{D}_q$.

One can again confirm that this is the correct definition of parallel translation of a tensor, i.e., we have the following result. (The proof is lengthy but similar to that of lemma B.1.17.)

Lemma B.1.20. *If* $\gamma : [0, 1] \to \mathsf{M}$ *is an* $\mathcal{E}$*-curve and* A *is the parallel translate of a tensor* $A_0 \in T^k_\ell(\mathcal{D}_{\gamma(0)})$*, then* A *is parallel along* γ*.*

B.1.2 Pullback connections

Let ∇ be an $\mathcal{E}$-restricted connection on $\mathcal{D}$. Let $\phi : \mathsf{M}' \to \mathsf{M}$ be a diffeomorphism and let $\mathcal{E}' = (\phi^{-1})_* \mathcal{E}$, $\mathcal{D}' = (\phi^{-1})_* \mathcal{D}$. The *pullback connection* $\phi^* \nabla$ is the $\mathcal{E}'$-restricted connection on $\mathcal{D}'$ defined as

$$(\phi^* \nabla)_X Y = (\phi^{-1})_* (\nabla_{\phi_* X} \phi_* Y), \qquad X \in \Gamma(\mathcal{E}'),\ Y \in \Gamma(\mathcal{D}').$$

Lemma B.1.21. *The pullback connection* $\phi^* \nabla$ *is an* $\mathcal{E}'$*-connection on* $\mathcal{D}'$*.*

Proof. Let $f \in \mathcal{C}^\infty(\mathsf{M}')$, $X \in \Gamma(\mathcal{E}')$ and $W \in \Gamma(\mathcal{D}')$. We have

$$(\phi^* \nabla)_{fX} W = (\phi^{-1})_* \nabla_{\phi_*(fX)} \phi_* W = (\phi^{-1})_* ((f \circ \phi^{-1}) \nabla_{\phi_* X} \phi_* W) = f (\phi^* \nabla)_X W,$$

i.e., $\phi^* \nabla$ is tensorial in its first argument. Moreover,

$$\begin{aligned} (\phi^* \nabla)_X (fW) &= (\phi^{-1})_* \nabla_{\phi_* X} \phi_* (fW) \\ &= (\phi^{-1})_* \nabla_{\phi_* X} (f \circ \phi^{-1}) \phi_* W \\ &= (\phi^{-1})_* ((\phi_* X)[f \circ \phi^{-1}] \phi_* W + (f \circ \phi^{-1}) \nabla_{\phi_* X} \phi_* W) \\ &= (\phi^{-1})_* ((X[f] \circ \phi^{-1}) \phi_* W) + (f \circ \phi^{-1}) \nabla_{\phi_* X} \phi_* W) \\ &= X[f] W + f (\phi^* \nabla)_X W, \end{aligned}$$

and so $\phi^*\nabla$ is a derivation in its second argument. It follows that the pullback connection is an $\mathcal{E}'$-restricted connection on $\mathcal{D}'$. ■

Proposition B.1.22. *We have:*

(*i*) *$W \in \Gamma(\mathcal{D}')$ is parallel with respect to $\phi^*\nabla$ if and only if $\phi_* W \in \Gamma(\mathcal{D})$ is parallel with respect to ∇.*

(*ii*) *If $\gamma : [0,1] \to \mathsf{M}'$ is an $\mathcal{E}'$-curve, then $\phi \circ \gamma$ is an $\mathcal{E}$-curve, and*

$$T_{\gamma(t)}\phi \cdot \Pi^t_\gamma = \Pi^t_{\phi\circ\gamma} \circ T_{\gamma(0)}\phi\big|_{\mathcal{D}'_{\gamma(0)}}$$

for every $t \in [0,1]$.

Proof. For item (*i*), let $W \in \Gamma(\mathcal{D}')$. Then

$$\begin{aligned}(\phi^*\nabla)W \equiv 0 \quad &\iff \quad (\phi^*\nabla)_X W = 0 \text{ for every } X \in \Gamma(\mathcal{E}') \\ &\iff \quad \nabla_{\phi_* X}\phi_* W = 0 \text{ for every } X \in \Gamma(\mathcal{E}') \\ &\iff \quad \nabla\phi_* W \equiv 0.\end{aligned}$$

That is, W is parallel if and only if $\phi_* W$ is parallel. For item (*ii*), let $\gamma : [0,1] \to \mathsf{M}'$ be an $\mathcal{E}'$-curve. As $\frac{d}{dt}\phi(\gamma(t)) = T_{\gamma(t)}\phi \cdot \dot\gamma(t) \in T_{\gamma(t)}\phi \cdot \mathcal{E}'_{\gamma(t)} = \mathcal{E}_{\gamma(t)}$ for every $t \in [0,1]$, we have that $\phi \circ \gamma$ is an $\mathcal{E}$-curve. Let $V_0 \in \mathcal{D}'_{\gamma(0)}$ and let $V(t) = \Pi^t_\gamma(V_0)$ be the parallel translate of V_0 along γ and with respect to $\phi^*\nabla$. Let

$$\widetilde{V} : [0,1] \to \mathcal{D}', \quad t \mapsto (T_{\gamma(t)}\phi)^{-1} \cdot \Pi^t_{\phi\circ\gamma}(T_{\gamma(t)}\phi \cdot V_0).$$

We have

$$\begin{tikzcd} \mathcal{D}'_{\gamma(0)} \arrow[r, "T_{\gamma(0)}\phi"] & \mathcal{D}_{(\phi\circ\gamma)(0)} \arrow[r, "\Pi^t_{\phi\circ\gamma}"] & \mathcal{D}_{(\phi\circ\gamma)(t)} \arrow[r, "(T_{\gamma(0)}\phi)^{-1}"] & \mathcal{D}'_{\gamma(t)}, \end{tikzcd}$$

i.e., $\widetilde{V}(t) \in \mathcal{D}'_{\gamma(t)}$ for every $t \in [0,1]$. Furthermore, as $\Pi^0_{\phi\circ\gamma} = \mathrm{id}_{\mathcal{D}_{(\phi\circ\gamma)(0)}}$, it follows that $\widetilde{V}(0) = V_0$. Thus $\widetilde{V}$ is a section of $\mathcal{D}$ along γ starting from V_0. We have that $\phi_*\widetilde{V}(t) = T_{\gamma(t)}\phi \cdot \widetilde{V}(t) = \Pi^t_{\phi\circ\gamma}(T_{\gamma(0)}\phi \cdot V_0)$ is parallel (with respect to ∇) along $\phi\circ\gamma$, and hence

$$(\phi^*\nabla)_{\dot\gamma}\widetilde{V}(t) = (\phi^{-1})_*\nabla_{\phi_*\dot\gamma}\phi_*\widetilde{V}(t) = 0.$$

Therefore $\widetilde{V}$ is parallel along γ (with respect to $\phi^*\nabla$). By uniqueness of the parallel translate, it follows that $\widetilde{V} = V$. Since V_0 is arbitrary, the result follows. ■

Corollary B.1.23. *$\phi^*\nabla$ is locally flat on $\mathcal{U} \subseteq \mathsf{M}'$ if and only if ∇ is locally flat on $\phi(\mathcal{U}) \subseteq \mathsf{M}$.*

Similar results hold for parallel tensor fields. (The proof of the following result is analogous to that of proposition B.1.22, only more involved, and hence we omit it.)

Proposition B.1.24. *$T \in \mathcal{T}^k_\ell(\mathcal{D})$ is parallel (with respect to ∇) if and only if $\phi^*T \in \mathcal{T}^k_\ell(\mathcal{D}')$ is parallel (with respect to $\phi^*\nabla$).*

B.1.3 Special classes of restricted connections

In this section we collect some pertinent results for special types of connections, *viz.*, metric connections, vector bundle connections and left-invariant connections on Lie groups.

B.1.3.1 Metric connections

An $\mathcal{E}$-connection ∇ on $\mathcal{D}$ is said to be *metric* if there exists a parallel (positive definite) fibre metric $\mathbf{g}$ on $\mathcal{D}$. That is, if there exists a Riemannian metric $\mathbf{g}$ on $\mathcal{D}$ such that $\nabla \mathbf{g} \equiv 0$, or equivalently,

$$X[\mathbf{g}(W_1, W_2)] = \mathbf{g}(\nabla_X W_1, W_2) + \mathbf{g}(W_1, \nabla_X W_2),$$

for every $X \in \Gamma(\mathcal{E})$ and $W_1, W_2 \in \Gamma(\mathcal{D})$.

Let $\mathbf{g}$ be a (positive definite) fibre metric on $\mathcal{D}$ and ∇ an $\mathcal{E}$-connection on $\mathcal{D}$ such that $\nabla \mathbf{g} \equiv 0$. Let $\|\cdot\|$ be the norm on $\mathcal{D}$ induced by $\mathbf{g}$.

Proposition B.1.25. *If $W \in \Gamma(\mathcal{D})$ is parallel, then so is the normalised vector field $W/\|W\|$.*

Proof. Let $f = 1/\|W\|$. If $X \in \Gamma(\mathcal{E})$, then $\nabla_X fW = X[f]W + f\nabla_X W = X[f]W$. However, we have

$$X[f] = -\frac{1}{2}\frac{X[\mathbf{g}(W, W)]}{\mathbf{g}(W, W)^{3/2}} = -\frac{\mathbf{g}(\nabla_X W, W)}{\mathbf{g}(W, W)^{3/2}} = 0.$$

That is, $\nabla_X fW = 0$ for every $X \in \Gamma(\mathcal{E})$, and so fW is parallel. ■

Corollary B.1.26. *If there exists a (local) parallel frame for $\mathcal{D}$, then there exists an orthonormal parallel frame for $\mathcal{D}$ (defined on the same neighbourhood).*

Proof. Let (U_a) be a parallel frame for $\mathcal{D}$, defined on $\mathcal{U} \subseteq \mathsf{M}$. Following the Gram–Schmidt process, define the frame (W_a) on $\mathcal{U}$ as

$$W_1 = U_1, \quad W_{k+1} = U_{k+1} - \sum_{i=1}^{k} \frac{\mathbf{g}(U_i, W_i)}{\mathbf{g}(W_i, W_i)} W_i \quad \text{for } k = 2, \dots, r.$$

Then (W_a) is orthogonal, and hence $(W_a/\|W_a\|)$ is orthonormal. Accordingly, it suffices to show that (W_a) is parallel. We use induction on a. Clearly, the vector field W_a, $a = 1$ is parallel. Suppose that $W_1, \dots, W_k$ are parallel for some $1 \le k < r$. Then, if $X \in \Gamma(\mathcal{E})$, we have

$$\begin{aligned}
\nabla_X W_{k+1} &= \nabla_X U_{k+1} - \sum_{i=1}^{k} \nabla_X \left(\frac{\mathbf{g}(U_i, W_i)}{\mathbf{g}(W_i, W_i)} W_i \right) \\
&= -\sum_{i=1}^{k} \left(X\Big[\frac{\mathbf{g}(U_i, W_i)}{\mathbf{g}(W_i, W_i)}\Big] W_i + \frac{\mathbf{g}(U_i, W_i)}{\mathbf{g}(W_i, W_i)} \nabla_X W_i \right) \\
&= -\sum_{i=1}^{k} \left(\frac{X[\mathbf{g}(U_i, W_i)]}{\mathbf{g}(W_i, W_i)} + \mathbf{g}(U_i, W_i) X\Big[\frac{1}{\mathbf{g}(W_i, W_i)}\Big] \right) W_i \\
&= -\sum_{i=1}^{k} \left(\frac{X[\mathbf{g}(U_i, W_i)]}{\mathbf{g}(W_i, W_i)} - \mathbf{g}(U_i, W_i) \frac{X[\mathbf{g}(W_i, W_i)]}{\mathbf{g}(W_i, W_i)^2} \right) W_i.
\end{aligned}$$

Since ∇ is metric, it follows that $X[\mathbf{g}(U_i, W_i)] = \mathbf{g}(\nabla_X U_i, W_i) + \mathbf{g}(U_i, \nabla_X W_i) = 0$ and (similarly) $X[\mathbf{g}(W_i, W_i)] = 0$ for each $i = 1, \ldots, k$. Hence $\nabla_X W_{k+1} = 0$, and so $\nabla W_{k+1} \equiv 0$. By induction we then have that (W_a) is parallel. ■

Accordingly, for a metric connection we may always assume that a parallel frame (if one exists) is orthonormal.

Proposition B.1.27. *The parallel transport* Π_γ^t *along an* $\mathcal{E}$*-curve* $\gamma : [0,1] \to \mathsf{M}$ *is a linear isometry.*

Proof. The result follows immediately from the fact that, if $\mathbf{g}$ is parallel, then it is invariant under parallel translation. ■

B.1.3.2 Vector bundle connections on $\mathcal{D}$

Suppose that $\mathcal{E} = T\mathsf{M}$, i.e., parallel translation is defined along any curve in M. In this case, an $\mathcal{E}$-connection ∇ on $\mathcal{D}$ is called a *connection on the vector bundle* $\mathcal{D}$ (or simply a *vector bundle connection on* $\mathcal{D}$). Vector bundle connections have been extensively studied (see, e.g., [56, 37]) and are well understood. In particular, there is a well-known definition of a curvature tensor for a vector bundle connection, whose vanishing characterises the flatness of the connection.

Let ∇ be a vector bundle connection on $\mathcal{D}$. The *curvature tensor* $R \in \mathcal{T}_2^0(T\mathsf{M}) \otimes \mathcal{T}_1^1(\mathcal{D})$ of ∇ is defined as

$$\begin{aligned} R(X,Y)U &= \nabla_X \nabla_Y U - \nabla_Y \nabla_X U - \nabla_{[X,Y]} U \\ &= [\nabla_X, \nabla_Y] U - \nabla_{[X,Y]} U, \end{aligned}$$

where $X, Y \in \Gamma(T\mathsf{M})$ and $U \in \Gamma(\mathcal{D})$. ($R$ is sometimes also called the *Riemannian curvature tensor* of ∇, as it originates in Riemannian geometry.) Calling R the "curvature" is justified by the following result. (The following theorem is a standard result for connections on vector bundles; see, e.g., [56]. Nevertheless, we provide here a proof, closely following that given in [44] for the Levi-Civita connection.)

Theorem B.1.28. ∇ *is flat on* $\mathcal{U} \subseteq \mathsf{M}$ *if and only if* $R(X,Y)U = 0$ *for every* $X, Y \in \Gamma(T\mathsf{M})$ *and* $U \in \Gamma(\mathcal{D})$ *defined on* $\mathcal{U}$*.*

Proof. Suppose ∇ is flat on $\mathcal{U} \subseteq \mathsf{M}$, i.e., there exists a parallel frame (U_a) for $\mathcal{D}$ on $\mathcal{U}$. Then $K(X,Y)U_a = [\nabla_X, \nabla_Y]U_a - \nabla_{[X,Y]}U_a = 0$ for every $X, Y \in \Gamma(T\mathsf{M})$ defined on $\mathcal{U}$. It follows (from tensoriality of K) that $K \equiv 0$ on $\mathcal{U}$.

Conversely, suppose $K \equiv 0$ on some neighbourhood $\mathcal{U}$ in M. Let $q \in \mathcal{U}$ and let (q^i) be local coordinates about q, defined in an open subset of $\mathcal{U}$, such that (E_a), $E_a = \partial_a(q)$ is a basis for $\mathcal{D}_q$. Extend (E_a) to a basis (E_a, E_λ) for $T_q\mathsf{M}$ (where λ ranges through $r+1, \ldots, n$). We may assume, without loss of generality, that the image of the coordinate chart is a cube $C_\epsilon = \{q : |q^i| < \epsilon\}$.

We shall construct a parallel frame for $\mathcal{D}$ by parallel translating the vectors $E_1, \ldots, E_r$. Begin by parallel translating each vector E_a along the q^1-axis. Next, from each point on the q^1-axis, parallel translate along the coordinate line parallel to the q^2-axis. Then successively parallel translate along coordinate lines parallel to the q^3- through q^n-axes. (See figure B.2.)

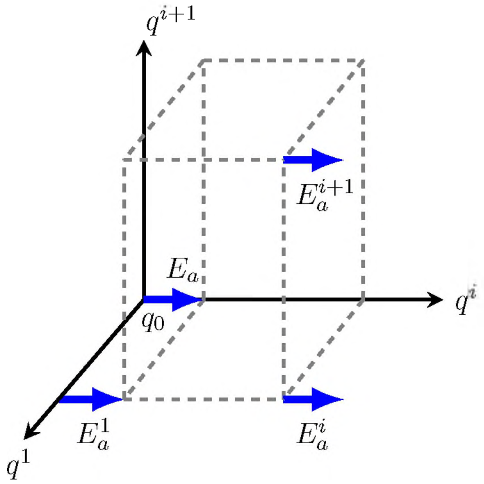

Figure B.2: Construction of a parallel orthonormal frame for $\mathcal{D}$

The result is r vector fields $U_1, \ldots, U_r$, defined in C_ϵ. We claim that this frame is parallel, i.e., $\nabla U_a \equiv 0$. By tensoriality of ∇ in the first argument, it suffices to show that $\nabla_{\partial_i} U_a = 0$.

Let $1 \leq a \leq r$ be fixed. By construction, we have

- $\nabla_{\partial_1} U_a = 0$ on the q^1-axis;
- $\nabla_{\partial_2} U_a = 0$ on the (q^1, q^2)-plane;

  $\vdots$
- $\nabla_{\partial_k} U_a = 0$ on the subset $M_k = \{q \in C_\epsilon : q^{k+1} = \cdots = q^n = 0\}$;

  $\vdots$
- $\nabla_{\partial_n} U_a = 0$ on C_ϵ.

Using induction on k, we shall prove that

$$\nabla_{\partial_1} U_a = \cdots = \nabla_{\partial_k} U_a = 0 \text{ on } M_k. \tag{B.1.1}$$

This is true by construction for $k = 1$ and for $k = n$ it means that (U_a) is parallel on the entire cube C_ϵ. Suppose (B.1.1) holds for some $1 \leq k \leq n-1$. By construction, we have $\nabla_{\partial_{k+1}} U_a = 0$ on M_{k+1}. Similarly, by the inductive hypothesis we have $\nabla_{\partial_i} U_a = 0$ for $1 \leq i \leq k$ on the hyperplane $M_k = \{q \in M_{k+1} : q^{k+1} = 0\}$. It remains to show that $\nabla_{\partial_i} U_a = 0$ on the set $\{q \in M_{k+1} : q^{k+1} \neq 0\}$ for each $i = 1, \ldots, k$.

Since $[\partial_{k+1}, \partial_i] = 0$, $\nabla_{\partial_{k+1}} U_a = 0$ on M_{k+1} and $K \equiv 0$, we have

$$\begin{aligned} 0 &= K(\partial_{k+1}, \partial_i) U_a \\ &= \nabla_{\partial_{k+1}}(\nabla_{\partial_i} U_a) - \nabla_{\partial_i}(\nabla_{\partial_{k+1}} U_a) - \nabla_{[\partial_{k+1}, \partial_i]} U_a \\ &= \nabla_{\partial_{k+1}}(\nabla_{\partial_i} U_a) \text{ on } M_{k+1}. \end{aligned}$$

That is, $\nabla_{\partial_i} U_a$ is parallel along curves parallel to the q^{k+1}-axis starting on M_k, and thus is equal to the parallel transport of the zero vector. Hence it is zero on every such curve, and since every point of M_{k+1} is on one of these curves, it follows that $\nabla_{\partial_i} U_a = 0$ on all of M_{k+1}. That is, the frame (U_a) is parallel. ■

B.1.3.3 Left-invariant connections

Suppose that $\mathsf{M} = \mathsf{G}$ is a Lie group and $\mathcal{D}$, $\mathcal{E}$ and ∇ are *left invariant*, i.e., $(L_g)_*\mathcal{D} = \mathcal{D}$, $(L_g)_*\mathcal{E} = \mathcal{E}$ and $\nabla = (L_g)^*\nabla$ for every $g \in \mathsf{G}$. In particular, since ∇ is equal to the pullback connection $(L_g)^*\nabla$, all of the results of section B.1.2 apply.

Proposition B.1.29 (cf. [31]). *There is a one-to-one correspondence between the set of left-invariant $\mathcal{E}$-connections on $\mathcal{D}$ and the set of bilinear maps*

$$\mathcal{E}_{\mathbf{1}} \times \mathcal{D}_{\mathbf{1}} \to \mathcal{D}_{\mathbf{1}}, \quad (X, W) \mapsto \nabla_{X^L} W^L(\mathbf{1}). \tag{B.1.2}$$

Here $X^L \in \Gamma(\mathcal{E})$ and $W^L \in \Gamma(\mathcal{D})$ are the left-invariant vector fields given by $X^L(g) = T_{\mathbf{1}}L_g \cdot X$ and $W^L(g) = T_{\mathbf{1}}L_g \cdot W$.

Proof. Let (E_a) be a basis for $\mathcal{D}_{\mathbf{1}}$ and (F_i) a basis for $\mathcal{E}_{\mathbf{1}}$. Then (E_a^L) and (F_i^L) are left-invariant frame for $\mathcal{D}$ and $\mathcal{E}$, respectively. Given a bilinear map $\varphi : \mathcal{E}_{\mathbf{1}} \times \mathcal{D}_{\mathbf{1}} \to \mathcal{D}_{\mathbf{1}}$, let ∇ be specified by

$$\nabla_{F_i^L} E_a^L = \varphi(F_i, E_a)^L.$$

It is straightforward to show that ∇ is an $\mathcal{E}$-connection on $\mathcal{D}$. We claim that ∇ is left invariant. Let $X \in \Gamma(\mathcal{E})$ and $W \in \Gamma(\mathcal{D})$. We have $X = x^i F_i^L$ and $W = w^a E_a^L$ for $x^i, w^a \in C^\infty(\mathsf{G})$. Accordingly,

$$\begin{aligned}
\nabla_{(L_g)_*X}(L_g)_*W &= (x^i \circ L_g^{-1})\nabla_{(L_g)_*F_i^L}(L_g)_*W \\
&= (x^i \circ L_g^{-1})\nabla_{F_i^L}(L_g)_*W \\
&= (x^i \circ L_g^{-1})\left[F_i^L[w^q \circ L_g^{-1}]E_a^L + (w^a \circ L_g^{-1})\nabla_{F_i^L}E_a^L\right] \\
&= (x^i \circ L_g^{-1})\left[(F_i^L[w^q] \circ L_g^{-1})E_a^L + (w^a \circ L_g^{-1})\nabla_{F_i^L}E_a^L\right] \\
&= (L_g)_*\left(x^i\left[F_i^L[w^a]E_a^L + w^a\nabla_{F_i^L}E_a^L\right]\right) = (L_g)_*\nabla_X W,
\end{aligned}$$

and hence ∇ is left invariant. Conversely, if ∇ is a left-invariant $\mathcal{E}$-connection on $\mathcal{D}$, then $\nabla_{F_i^L}E_a^L$ is left invariant, and (B.1.2) prescribes the associated bilinear map. ■

Proposition B.1.30. *If there exists a left-invariant parallel frame for $\mathcal{D}$, then every left-invariant vector field in $\Gamma(\mathcal{D})$ is parallel.*

Proof. This is immediate from the fact that every left-invariant vector field in $\Gamma(\mathcal{D})$ is an $\mathbb{R}$-linear combination of elements of a left-invariant frame for $\mathcal{D}$, and that ∇ is $\mathbb{R}$-linear in its second argument. ■

Proposition B.1.31. *Suppose there exists a left-invariant parallel frame for $\mathcal{D}$. The parallel translation $\Pi_{g_0}^{g_1} : \mathcal{D}_{g_0} \to \mathcal{D}_{g_1}$ is exactly (the restriction of the tangent map of) the left translation $T_{g_0}L_{g_1 g_0^{-1}}\big|_{\mathcal{D}_{g_0}}$.*

Proof. Let (U_a) be a left-invariant parallel frame for $\mathcal{D}$ and let $\gamma : [0,1] \to \mathsf{G}$ be an $\mathcal{E}$-curve such that $\gamma(0) = g_0$ and $\gamma(1) = g_1$. Fix $1 \leq a \leq r$ and let $V(t) = \Pi_\gamma^1(U_a(g_0))$. There exist functions $v^b \in \mathcal{C}^\infty([0,1])$ such that $V = v^b(U_b \circ \gamma)$, where $\dot{v}^b = -\Gamma^b_{ic}(\gamma)\dot{\gamma}^i v^c = 0$. Thus $V(t) = U_a(\gamma(t))$, and so

$$\Pi^{g_1}_{g_0}(U_a(g_0)) = V(1) = U_a(\gamma(1)) = U_a(g_1) = T_{g_0}L_{g_1 g_0^{-1}} \cdot U_a(g_0).$$

Since a is arbitrary and the parallel translation maps are linear, it follows that $\Pi^{g_1}_{g_0}$ is exactly the restriction to $\mathcal{D}_{g_0}$ of $T_{g_0}L_{g_1 g_0^{-1}}$. ■

B.2 The Ehresmann connection approach

Let M be an n-dimensional (connected) manifold and let $\mathcal{E}$ and $\mathcal{D}$ be rank m and r, respectively, vector subbundles of $T\mathsf{M}$. By $\tau_\mathsf{M} : T\mathsf{M} \to \mathsf{M}$ we denote the canonical projection of a tangent vector onto its base point. Let $\pi = \tau_\mathsf{M}|_\mathcal{D} : \mathcal{D} \to \mathsf{M}$. We may view $\mathcal{D}$ as a submanifold of $T\mathsf{M}$; then $\tau_\mathcal{D} : T\mathcal{D} \to \mathcal{D}$ denotes the corresponding canonical projection. Consider the pullback bundle $\pi^*\mathcal{E}$:

$$\pi^*\mathcal{E} = \{(U_q, X_q) \in \mathcal{D} \times \mathcal{E} : \pi(U_q) = \tau_\mathsf{M}(X_q)\} \subsetneq \mathcal{D} \times \mathcal{E}.$$

We can view $\pi^*\mathcal{E}$ as a vector bundle over $\mathcal{D}$ and over $\mathcal{E}$, with projections

$$\widetilde{\pi}_1 : \pi^*\mathcal{E} \ni (U_q, X_q) \mapsto U_q \in \mathcal{D} \quad \text{and} \quad \widetilde{\pi}_2 : \pi^*\mathcal{E} \ni (U_q, X_q) \mapsto X_q \in \mathcal{E},$$

respectively. The fibres of $\widetilde{\pi}_1$ and $\widetilde{\pi}_2$ are given by

$$(\widetilde{\pi}_1)^{-1}(U_q) = \{U_q\} \times \mathcal{E}_q \quad \text{and} \quad (\widetilde{\pi}_2)^{-1}(X_q) = \mathcal{D}_q \times \{X_q\},$$

respectively, where $U_q \in \mathcal{D}$ and $X_q \in \mathcal{E}$. Clearly, we have $\tau_\mathsf{M} \circ \widetilde{\pi}_2 = \pi \circ \widetilde{\pi}_1$.

An *$\mathcal{E}$-restricted (Ehresmann) connection on $\mathcal{D}$* (cf. [12]) (or *$\mathcal{E}$-connection on $\mathcal{D}$*) is a map $h : \pi^*\mathcal{E} \to T\mathcal{D}$ that is

(*i*) a linear bundle map from $\widetilde{\pi}_1$ to $\tau_\mathcal{D}$ covering $\mathrm{id}_\mathcal{D}$, i.e., $\tau_\mathcal{D} \circ h = \widetilde{\pi}_1$;

(*ii*) a bundle map from $\widetilde{\pi}_2$ to $T\pi$ covering the inclusion $\iota : \mathcal{E} \to T\mathsf{M}$, i.e., $T\pi \cdot h = \iota \circ \widetilde{\pi}_2$.

That is, the following two diagrams commute:

$$\begin{tikzcd}
\pi^*\mathcal{E} \arrow[r, "h"] \arrow[d, "\widetilde{\pi}_1"'] & T\mathcal{D} \arrow[dl, "\tau_\mathcal{D}"] \\
\mathcal{D} &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\pi^*\mathcal{E} \arrow[r, "h"] \arrow[d, "\widetilde{\pi}_2"'] & T\mathcal{D} \arrow[d, "T\pi"] \\
\mathcal{E} \arrow[r, "\iota"'] & T\mathsf{M}
\end{tikzcd}$$

The linearity condition in (*i*) stipulates that h is fibrewise $\mathbb{R}$-linear in its second argument, i.e., $h(U_q, \alpha X_q + \beta Y_q) = \alpha\, h(U_q, X_q) + \beta\, h(U_q, Y_q)$ for every $U_q \in \mathcal{D}_q$, $X_q, Y_q \in \mathcal{E}_q$ and $\alpha, \beta \in \mathbb{R}$. In section B.2.1 we clarify the relation between the above definition and the definition of a restricted connection given in section B.1. In particular, to avoid ambiguity, in this section we shall refer to the operator ∇ of B.1 as an *$\mathcal{E}$-restricted Koszul connection on $\mathcal{D}$*.

We say that h is a *linear connection* if $T_{U_q}\phi_t \cdot h(U_q, X_q) = h(\phi_t(U_q), X_q)$ for every $(U_q, X_q) \in \pi^*\mathcal{E}$, where $\phi_t : \mathcal{D} \to \mathcal{D}$ denotes the dilation $\phi_t(U_q) = e^t U_q$. (That is, ϕ_t is the flow of the canonical dilation vector field on $\mathcal{D}$.)

Let $\mathcal{V} = \ker T\pi$ be the *vertical distribution* and $\mathcal{H} = \operatorname{im} h$ the *horizontal distribution.* (Since both $T\pi$ and ι have constant rank, it follows that both $\mathcal{V}$ and $\mathcal{H}$ are vector subbundles of $T\mathcal{D}$.) Vectors in $\mathcal{V}$ are called *vertical*, whereas those in $\mathcal{H}$ are called *horizontal.* Likewise, vector fields in $\Gamma(\mathcal{V})$ are called *vertical vector fields*, whereas those in $\Gamma(\mathcal{H})$ are *horizontal vector fields.*

Lemma B.2.1. *The fibres of* $\mathcal{V}$ *are the tangent spaces of the fibres of* $\mathcal{D}$*, i.e.,* $\mathcal{V}_{U_q} = T_{U_q}\mathcal{D}_q$ *for every* $U_q \in \mathcal{D}_q$*.*

Proof. Let $U_q, V_q \in \mathcal{D}_q$. Then $\frac{d}{dt}\big|_{t=0}(U_q + tV_q) \in T_{U_q}\mathcal{D}_q$. But

$$T_{U_q}\pi \cdot \left.\frac{d}{dt}\right|_{t=0}(U_q + tV_q) = \left.\frac{d}{dt}\right|_{t=0}\pi(U_q + tV_q) = 0,$$

since $\pi(U_q + tV_q) = q$ is constant in t. Thus we have $T_{U_q}\mathcal{D}_q \subseteq \mathcal{V}_{U_q}$. Equality follows from the fact that both $T_{U_q}\mathcal{D}_q$ and $\mathcal{V}_{U_q}$ are r-dimensional. ■

The map $\mathrm{vl} : \pi^*\mathcal{D} \to \mathcal{V}$ given by

$$\mathrm{vl}(U_q, V_q) = \left.\frac{d}{dt}\right|_{t=0}(U_q + tV_q), \qquad (U_q, V_q) \in \pi^*\mathcal{D}$$

is evidently a vector bundle isomorphism. If $U_q \in \mathcal{D}$, then the linear isomorphism

$$\mathrm{vl}_{U_q} = \mathrm{vl}(U_q, \,\cdot\,) : \mathcal{D}_q \to \mathcal{V}_{U_q}$$

is called the *vertical lift over* U_q. If $X \in \Gamma(\mathcal{D})$, then we define the *vertical lift of* X to be the vertical vector field $X^v \in \Gamma(\mathcal{V})$ given by

$$X^v(U_q) = \mathrm{vl}(U_q, X(q)), \qquad U_q \in \mathcal{D}_q.$$

Lemma B.2.2. *We have* $\pi_*\mathcal{H} = \mathcal{E}$*, i.e.,* $T_{U_q}\pi \cdot \mathcal{H}_{U_q} = \mathcal{E}_q$ *for every* $U_q \in \mathcal{D}_q$*.*

Proof. Let $X_{U_q} \in \mathcal{H}_{U_q}$. Then there exists $X_q \in \mathcal{E}_q$ such that $X_{U_q} = h(U_q, X_q)$. Consequently, we have

$$T_{U_q}\pi \cdot X_{U_q} = T_{U_q}\pi \cdot h(U_q, X_q) = (\iota \circ \widetilde{\pi}_2)(U_q, X_q) = X_q \in \mathcal{E}_q,$$

and so $T_{U_q}\pi \cdot \mathcal{H}_{U_q} \subseteq \mathcal{E}_q$. Conversely, let $X_q \in \mathcal{E}_q$. Then $h(U_q, X_q) \in \mathcal{H}_{U_q}$ for every $U_q \in \mathcal{D}_q$, and $X_q = T_{U_q}\pi \cdot h(U_q, X_q) \in T_{U_q}\pi \cdot \mathcal{H}_{U_q}$. It follows that $T_{U_q}\pi \cdot \mathcal{H}_{U_q} = \mathcal{E}_q$. ■

Lemma B.2.3. *An* $\mathcal{E}$*-connection* h *on* $\mathcal{D}$ *is linear if and only if* $(\phi_t)_*\mathcal{H} = \mathcal{H}$*.*

Proof. Suppose h is linear. Then by definition, if $(U_q, X_q) \in \pi^*\mathcal{E}$, we have $T_{U_q}\phi_t \cdot h(U_q, X_q) = h(\phi_t(U_q), X_q) \in \mathcal{H}_{\phi_t(U_q)}$, and so $(\phi_t)_*\mathcal{H} \subseteq \mathcal{H}$. Equality follows from the fact that ϕ_t is a diffeomorphism. Conversely, suppose $(\phi_t)_*\mathcal{H} = \mathcal{H}$. If $(U_q, X_q) \in \pi^*\mathcal{E}$, then $T_{U_q}\phi_t \cdot h(U_q, X_q) = h(\phi_t(U_q), Y_q)$ for some $Y_q \in \mathcal{E}_q$. But

$$Y_q = T_{\phi_t(U_q)}\pi \cdot h(\phi_t(U_q), Y_q) = T_{\phi_t(U_q)}\pi \cdot T_{U_q}\phi_t \cdot h(U_q, X_q) = T_{U_q}(\phi_t \circ \pi) \cdot h(U_q, X_q).$$

Clearly $\phi_t \circ \pi = \pi$, and so $Y_q = X_q$. It follows that h is linear. ■

Proposition B.2.4 (cf. [12]). *We have:*

(*i*) $\mathcal{V} \cap \mathcal{H} = \{0\}$.

(*ii*) $\mathcal{V} + \mathcal{H} \subseteq T\mathcal{D}$ *with equality if and only if* $\mathcal{E} = T\mathsf{M}$.

Proof. (*i*) If $X_{U_q} \in \mathcal{V}_{U_q} \cap \mathcal{H}_{U_q}$, then $X_{U_q} = h(U_q, Y_q)$ for some $Y_q \in \mathcal{E}_q$. Hence

$$0 = T_{U_q}\pi \cdot X_{U_q} = T_{U_q}\pi \cdot h(U_q, Y_q) = (\iota \circ \widetilde{\pi}_2)(U_q, Y_q) = Y_q.$$

Since h is linear in its second argument, it follows that $h(U_q, 0) = 0$. Hence $X_{U_q} \in \mathcal{V}_{U_q} \cap \mathcal{H}_{U_q}$ if and only if $X_{U_q} = 0$.

(*ii*) Clearly $\mathcal{V} + \mathcal{H} \subseteq T\mathcal{D}$. Suppose that $\mathcal{V} + \mathcal{H} = T\mathcal{D}$ and let $Z_q \in T_q\mathsf{M}$. Since $T\pi$ is surjective, there exists $X_{U_q} \in T_{U_q}\mathcal{D}$ such that $T_{U_q}\pi \cdot X_{U_q} = Z_q$. The $\mathcal{H}_{U_q}$-component of X_{U_q} is given by $h(U_q, X_q)$ for some $X_q \in \mathcal{E}_q$. Accordingly,

$$Z_q = T_{U_q}\pi \cdot h(U_q, X_q) = (\iota \circ \widetilde{\pi}_2)(U_q, X_q) = \iota(X_q) \in \mathcal{E}_q.$$

Since Z_q is arbitrary, this is a contradiction unless $\mathcal{E} = T\mathsf{M}$. ■

Proposition B.2.5 (cf. [12]). *Let* $\mathcal{E}$ *and* $\mathcal{D}$ *be distributions on* M *and* $\pi = \tau_{\mathsf{M}}|_{\mathcal{D}} : \mathcal{D} \to \mathsf{M}$. *If* $\mathcal{H}$ *is a distribution on* $\mathcal{D}$ *such that*

$$\pi_*\mathcal{H} = \mathcal{E} \quad \text{and} \quad \mathcal{V} \cap \mathcal{H} = \{0\}, \tag{B.2.1}$$

then there exists a unique (*not necessarily linear*) $\mathcal{E}$*-connection* h *on* $\mathcal{D}$ *such that* $\mathcal{H} = \operatorname{im} h$.

Proof. For each $U_q \in \mathcal{D}$, we can construct a map $h_{U_q} : \mathcal{E}_q \to T_{U_q}\mathcal{D}$ by the requirement that

$$\{h_{U_q}(X_q)\} = \mathcal{H}_{U_q} \cap (T_{U_q}\pi)^{-1}(X_q) \tag{B.2.2}$$

for every $X_q \in \mathcal{E}_q$. We first show that (B.2.2) determines a unique point. Since $T_{U_q}\pi \cdot \mathcal{H}_{U_q} = \mathcal{E}_q$, we have $\mathcal{H}_{U_q} \subseteq (T_{U_q}\pi)^{-1}(X_q)$. Thus the intersection $\mathcal{H}_{U_q} \cap (T_{U_q}\pi)^{-1}(X_q)$ is non-empty. Suppose there exist two points $X_{U_q}, Y_{U_q} \in \mathcal{H}_{U_q} \cap (T_{U_q}\pi)^{-1}(X_q)$. Then $T_{U_q}\pi \cdot (X_{U_q} - Y_{U_q}) = 0$, i.e., $X_{U_q} - Y_{U_q}$ is vertical. Since $X_{U_q} - Y_{U_q} \in \mathcal{H}_{U_q}$ and $\mathcal{V} \cap \mathcal{H} = \{0\}$, this holds if and only if $X_{U_q} = Y_{U_q}$. Thus (B.2.2) specifies a unique point $h_{U_q}(X_q)$, as claimed. Furthermore, if $X_q, Y_q \in \mathcal{E}_q$ and $\alpha, \beta \in \mathbb{R}$, then

$$\begin{aligned}
\{h_{U_q}(\alpha X_q + \beta Y_q)\} &= \mathcal{H}_{U_q} \cap (T_{U_q}\pi)^{-1}(\alpha X_q + \beta Y_q) \\
&= \alpha \left[\mathcal{H}_{U_q} \cap (T_{U_q}\pi)^{-1}(X_q)\right] + \beta \left[\mathcal{H}_{U_q} \cap (T_{U_q}\pi)^{-1}(Y_q)\right] \\
&= \{\alpha\, h_{U_q}(X_q) + \beta\, h_{U_q}(Y_q)\}.
\end{aligned}$$

That is, h_{U_q} is $\mathbb{R}$-linear. Define a map $h : \pi^*\mathcal{E} \to T\mathcal{D}$ as $h(U_q, X_q) = h_{U_q}(X_q)$. Then clearly $\tau_{\mathcal{D}} \circ h = \widetilde{\pi}_1$ and $T\pi \cdot h = \iota \circ \widetilde{\pi}_2$. Since $h(U_q, \cdot) = h_{U_q}$ is $\mathbb{R}$-linear, we have that h is an $\mathcal{E}$-connection on $\mathcal{D}$; moreover, by definition, $\operatorname{im} h = \mathcal{H}$.

It remains to show that h is unique. Suppose there exists another $\mathcal{E}$-connection $\widetilde{h}$ on $\mathcal{D}$ such that $\operatorname{im} \widetilde{h} = \mathcal{H}$. Then for each $U_q \in \mathcal{D}_q$ there exists $X_q, Y_q \in \mathcal{E}_q$ such that $\widetilde{h}(U_q, X_q) = h(U_q, Y_q)$. Applying $T_{U_q}\pi$ to both sides, we get $X_q = Y_q$. That is, $\widetilde{h} = h$. ■

B.2.1 Associated covariant derivative

Suppose that h is a linear $\mathcal{E}$-connection on $\mathcal{D}$. Let $X_q \in \mathcal{E}$ and $Y_{U_q} \in T_{U_q}\mathcal{D}$, where $U_q \in \mathcal{D}_q$ and Y_{U_q} satisfies $T_{U_q}\pi \cdot Y_{U_q} = X_q$. Then

$$T_{U_q}\pi \cdot [Y_{U_q} - h(U_q, X_q)] = X_q - (\iota \circ \widetilde{\pi}_2)(U_q, X_q) = 0.$$

That is, $Y_{U_q} - h(U_q, X_q) \in \mathcal{V}_{U_q}$. Hence we can define an operator $\nabla : \Gamma(\mathcal{E}) \times \Gamma(\mathcal{D}) \to \Gamma(\mathcal{D})$ by

$$\nabla_X U(q) = \mathrm{vl}_{U(q)}^{-1} \cdot \big[T_q U \cdot X(q) - h(U(q), X(q))\big]. \tag{B.2.3}$$

∇ is called the *covariant derivative associated to* h, and (as the following lemma demonstrates) is exactly an $\mathcal{E}$-restricted Koszul connection on $\mathcal{D}$ (see section B.1).

Lemma B.2.6. *∇ is an $\mathcal{E}$-restricted Koszul connection on $\mathcal{D}$.*

Proof. Let $X \in \Gamma(\mathcal{E})$, $U \in \Gamma(\mathcal{D})$ and $f \in \mathcal{C}^\infty(\mathsf{M})$. Since h is fibrewise $\mathbb{R}$-linear in its second argument, we have

$$\begin{aligned}
\nabla_{fX} U(q) &= \mathrm{vl}_{U(q)}^{-1} \cdot \big[T_q U \cdot f(q)X(q) - h(U(q), f(q)X(q))\big] \\
&= \mathrm{vl}_{U(q)}^{-1} \cdot f(q)\big[T_q U \cdot X(q) - h(U(q), X(q))\big] \\
&= f(q)(\nabla_X U)(q),
\end{aligned}$$

i.e., ∇ is $\mathcal{C}^\infty(\mathsf{M})$-linear in the first argument. Similarly, it is a derivation in its second argument. Indeed, since

$$T_q(fU) \cdot X(q) = X[f](q)\, \mathrm{vl}_{U(q)} \cdot U(q) + f(q) T_q U \cdot X(q)$$

we have

$$\begin{aligned}
\nabla_X fU(q) &= \mathrm{vl}_{U(q)}^{-1} \cdot \big[T_q(fU) \cdot X(q) - h(f(q)U(q), X(q))\big] \\
&= X[f](q)U(q) + f(q)\, \mathrm{vl}_{U(q)}^{-1} \cdot \big[T_q U \cdot X(q) - h(U(q), X(q))\big] \\
&= X[f](q)U(q) + f(q)\nabla_X U(q).
\end{aligned}$$

Therefore ∇ is an $\mathcal{E}$-restricted Koszul connection on $\mathcal{D}$. ■

Conversely, given an $\mathcal{E}$-restricted Koszul connection ∇ on $\mathcal{D}$, there exists a unique linear $\mathcal{E}$-restricted Ehresmann connection h whose associated covariant derivative is exactly ∇.

Proposition B.2.7 (cf. [12]). *Let $\nabla : \Gamma(\mathcal{E}) \times \Gamma(\mathcal{D}) \to \Gamma(\mathcal{D})$ be an $\mathcal{E}$-restricted Koszul connection. Then ∇ is the covariant derivative associated to the unique $\mathcal{E}$-connection h on $\mathcal{D}$ given by*

$$h(U_q, X_q) = T_q U \cdot X_q - \mathrm{vl}_{U_q} \cdot \nabla_{X_q} U, \qquad (U_q, X_q) \in \pi^*\mathcal{E}. \tag{B.2.4}$$

Here $U \in \Gamma(\mathcal{D})$ satisfies $U(q) = U_q$. (The definition of h does not depend on the choice of U.)

Proof. Let $(U_q, X_q) \in \pi^*\mathcal{E}$. It is straightforward to check that the map $\Gamma(\mathcal{D}) \to T_{U_q}\mathcal{D}$, $U \mapsto T_q U \cdot X_q - \mathrm{vl}_{U_q} \cdot \nabla_{X_q} U$ is $\mathcal{C}^\infty(\mathsf{M})$-linear, hence depends only on the value of U at q.

Define h using (B.2.4). Since ∇ is $\mathbb{R}$-linear, h is clearly (fibrewise) $\mathbb{R}$-linear in its second argument. We have

$$\tau_{\mathcal{D}}(h(U_q, X_q)) = \tau_{\mathcal{D}}(T_qU \cdot X_q - \mathrm{vl}_{U_q} \cdot \nabla_{X_q}U) = U_q = \widetilde{\pi}_1(U_q, X_q).$$

Likewise,

$$\begin{aligned} T_{U_q}\pi \cdot h(U_q, X_q) &= T_{U_q}\pi \cdot (T_qU \cdot X_q - \mathrm{vl}_{U_q} \cdot \nabla_{X_q}U) = T_q(\pi \circ U) \cdot X_q - 0 \\ &= X_q = (\iota \circ \widetilde{\pi}_2)(U_q, X_q). \end{aligned}$$

Thus h is an $\mathcal{E}$-connection on $\mathcal{D}$. It is also linear, since

$$\begin{aligned} T_{U_q}\phi_t \cdot h(U_q, X_q) &= T_{U_q}\phi_t \cdot [T_qU \cdot X_q - \mathrm{vl}_{U_q} \cdot \nabla_{X_q}U] \\ &= T_q(\phi_t \circ U) \cdot X_q - T_{U_q}\phi_t \cdot \left.\frac{d}{ds}\right|_{s=0} (U_q + s\,\nabla_{X_q}U) \\ &= T_q(\phi_t \circ U) \cdot X_q - \left.\frac{d}{ds}\right|_{s=0} (\phi_t(U_q) + s\,\phi_t(\nabla_{X_q}U)) \\ &= T_q(\phi_t \circ U) \cdot X_q - \mathrm{vl}_{\phi_t(U_q)} \cdot \phi_t(\nabla_{X_q}U) \\ &= T_q(\phi_t \circ U) \cdot X_q - \mathrm{vl}_{\phi_t(U_q)} \cdot \nabla_{X_q}(\phi_t \circ U) \\ &= h(\phi_t(U_q), X_q). \end{aligned}$$

Lastly, considering equations (B.2.3) and (B.2.4), it is clear that ∇ is the covariant derivative associated to h. ■

B.2.2 Horizontal lifts

If $(U_q, X_q) \in \pi^*\mathcal{E}$, then we call $h(U_q, X_q) \in \mathcal{H}_{U_q}$ the *horizontal lift of* X_q *over* U_q (or, explicitly, the *h-lift of* X_q *over* U_q). Similarly, if $X \in \Gamma(\mathcal{E})$, we define the *horizontal lift* (*h-lift*) *of* X to be the section $X^h \in \Gamma(\mathcal{H})$ given by

$$X^h(U_q) = h(U_q, X(q)), \qquad U_q \in \mathcal{D}.$$

A vector field $Z \in \Gamma(T\mathcal{D})$ is called *projectable* if there exists $X \in \Gamma(T\mathsf{M})$ such that $T\pi \cdot Z = X \circ \pi$, i.e., if Z and X are π-related. We shall write X as π_*Z. The Lie bracket of two projectable vector fields is also projectable. In fact, we have $\pi_*[Z_1, Z_1] = [\pi_*Z_1, \pi_*Z_2]$ for every pair of projectable vector fields $Z_1, Z_2 \in \Gamma(T\mathsf{M})$. Furthermore, if X is projectable, then $X[f \circ \pi] = (\pi_*X)[f] \circ \pi$ for every $f \in C^\infty(\mathsf{M})$.

Lemma B.2.8. *Projectable horizontal vector fields are horizontal lifts of vector fields in* $\Gamma(\mathcal{E})$*, i.e., if* $X \in \Gamma(\mathcal{H})$ *is projectable, then* $X = (\pi_*X)^h$*.*

Proof. Let $X \in \Gamma(\mathcal{H})$ be projectable and let $U_q \in \mathcal{D}$. We have $X(U_q) = h(U_q, Y_q)$ for some $Y_q \in \mathcal{E}_q$. In fact,

$$T_{U_q}\pi \cdot X(U_q) = T_{U_q}\pi \cdot h(U_q, Y_q) = (\iota \circ \widetilde{\pi}_2)(U_q, Y_q) = Y_q.$$

Consequently, we have $(\pi_*X)^h(U_q) = h(U_q, (\pi_*X)(q)) = h(U_q, T_{U_q}\pi \cdot X(U_q)) = X(U_q)$. Hence $X = (\pi_*X)^h$. ■

Lemma B.2.9. *If* $X, Y \in \Gamma(\mathcal{E})$ *and* $f \in \mathcal{C}^\infty(\mathsf{M})$*, then*

(i) $(X + Y)^h = X^h + Y^h$.

(ii) $(fX)^h = (f \circ \pi)X^h$.

(iii) $\pi_* X^h = X$.

Proof. Let $U_q \in \mathcal{D}$. For (*i*), since h is fibrewise $\mathbb{R}$-linear in its second argument, we have

$$\begin{aligned}(X + Y)^h(U_q) &= h(U_q, X(q) + Y(q))\\ &= h(U_q, X(q)) + h(U_q, Y(q)) = X^h(U_q) + Y^h(U_q).\end{aligned}$$

Likewise, $(fX)^h(U_q) = h(U_q, f(q)X(q)) = f(q)X^h(U_q)$, which proves (*ii*). Lastly, for (*iii*), we have

$$T_{U_q}\pi \cdot X^h(U_q) = T_{U_q}\pi \cdot h(U_q, X(q)) = (\iota \circ \widetilde{\pi}_2)(U_q, X(q)) = (\iota \circ X)(q) = (\iota \circ X \circ \pi)(U_q).$$

It follows that $\pi_* X^h = X$ (where we make the usual identification of $\iota \circ X$ with X). ■

Lemma B.2.10. *Let* $X \in \Gamma(\mathcal{E})$*. Then* $X^h(U_q) = 0$ *for some* $U_q \in \mathcal{D}$ *if and only if* $X(q) = 0$.

Proof. If $X(q) = 0$, then $X^h(U_q) = h(U_q, X(q)) = h(U_q, 0) = 0$. On the other hand, if $X^h(U_q) = 0$, then $h(U_q, X(q)) = 0$. Since $\ker h(U_q, \cdot\,) = \{0\}$, it follows that $X(q) = 0$. ■

Lemma B.2.11. *If* h *is linear, then* $(\phi_t)_* X^h = X^h$ *for every* $X \in \Gamma(\mathcal{E})$*. (Here* ϕ_t *is the flow of the dilation vector field on* $\mathcal{D}$*.)*

Proof. Let $U_q \in \mathcal{D}$. Then

$$T_{U_q}\phi_t \cdot X^h(U_q) = T_{U_q}\phi_t \cdot h(U_q, X(q)) = h(\phi_t(U_q), X(q)) = (X^h \circ \phi_t)(U_q).$$

That is, $(\phi_t)_* X^h = X^h$. ■

The next proposition relates the horizontal lift of a vector field $X \in \Gamma(\mathcal{E})$ and the $\mathcal{E}$-derivation ∇_X. We first prove a technical lemma. If $\omega \in \Gamma(\mathcal{D}^*)$, then we shall denote by $\overline{\omega} \in \mathcal{C}^\infty(\mathcal{D})$ the function given by $\overline{\omega}(U_q) = \omega_q(U_q)$.

Lemma B.2.12. *If* $W \in \Gamma(\mathcal{V})$ *and* $\omega \in \Gamma(\mathcal{D}^*)$*, then*

$$W[\overline{\omega}](U_q) = \omega_q(\mathrm{vl}_{U_q}^{-1} \cdot W(U_q))$$

for every $U_q \in \mathcal{D}$.

Proof. Let (U_a) be a (local) frame for $\mathcal{D}$ and let (ν^a) be the dual frame for $\mathcal{D}^*$. There exist functions $\omega_a \in \mathcal{C}^\infty(\mathsf{M})$ such that $\omega = \omega_a \nu^a$; furthermore, we have $\overline{\omega} = (\omega_a \circ \pi)\overline{\nu}^a$. Let $U_q \in \mathcal{D}$ and $V_q = \mathrm{vl}_{U_q}^{-1} \cdot W(U_q) \in \mathcal{D}_q$; we have $V_q = v_q^a U_a(q)$ for some $v_q^a \in \mathbb{R}$. Then

$$\begin{aligned}W[\overline{\omega}](U_q) &= d\overline{\omega}(U_q)(W(U_q))\\ &= \frac{d}{dt}\bigg|_{t=0} \overline{\omega}(U_q + t\,V_q)\\ &= \frac{d}{dt}\bigg|_{t=0} \omega_a(q)\nu_q^a(U_q + t\,V_q)\\ &= \omega_a(q)v_q^a = \omega_q(V_q) = \omega_q(\mathrm{vl}_{U_q}^{-1} \cdot W(U_q)).\end{aligned}$$

■

Proposition B.2.13 (cf. [82]). *If* $X \in \Gamma(\mathcal{E})$ *and* $\omega \in \Gamma(\mathcal{D}^*)$*, then* $X^h[\overline{\omega}] = \overline{\nabla_X \omega}$*.*

Proof. Let $U_q \in \mathcal{D}$ and let $U \in \Gamma(\mathcal{D})$ be a smooth extension of U_q. We have

$$X^h[\overline{\omega}](U_q) = d\overline{\omega}(U_q)(X^h(U_q)).$$

Likewise, recalling the expression (B.2.3) for $\nabla_X U(q)$, and making use of lemma B.2.12, we have

$$\begin{aligned}(\overline{\nabla_X\omega})(U_q) = (\nabla_X\omega)_q(U_q) &= X[\omega(U)](q) - \omega(\nabla_X U)(q) \\ &= X[\omega(U)](q) - \omega_q\big(\mathrm{vl}_{U_q}^{-1} \cdot [T_q U \cdot X(q) - X^h(U_q)]\big) \\ &= X[\omega(U)](q) - d\overline{\omega}(U_q)\big(T_q U \cdot X(q) - X^h(U_q)\big).\end{aligned}$$

Let γ be a curve in M such that $\gamma(0) = q$ and $\dot{\gamma}(0) = U_q$. The first part of this expression is then

$$X[\omega(U)](q) = \frac{d}{dt}\bigg|_{t=0} \overline{\omega}(U(\gamma(t))) = d\overline{\omega}(U_q)\Big(\frac{d}{dt}\bigg|_{t=0} U(\gamma(t))\Big) = d\overline{\omega}(U_q)(T_q U \cdot X(q)).$$

Therefore

$$(\overline{\nabla_X\omega})(U_q) = d\bar{\omega}(U_q)(T_q U \cdot X(q)) - d\overline{\omega}(U_q)\big(T_q U \cdot X(q) - X^h(U_q)\big) = X^h[\overline{\omega}](U_q).$$

Since U_q is arbitrary, it follows that $\overline{\nabla_X\omega} = X^h[\overline{\omega}]$. ■

Corollary B.2.14 (cf. [82]). *If* $X, Y \in \Gamma(\mathcal{E})$ *and* $\omega \in \Gamma(\mathcal{D}^*)$*, then*

$$[X^h, Y^h][\overline{\omega}] = \overline{[\nabla_X, \nabla_Y]\omega}.$$

Proof. Using proposition B.2.13, we have

$$[X^h, Y^h][\overline{\omega}] = X^h[\overline{\nabla_Y\omega}] - Y^h[\overline{\nabla_X\omega}] = \overline{\nabla_X\nabla_Y\omega} - \overline{\nabla_Y\nabla_X\omega} = \overline{[\nabla_X, \nabla_Y]\omega}.$$ ■

Appendix C

The real three-dimensional Lie algebras

The classification of real three-dimensional Lie algebras is well known. Our preference is for the Bianchi–Behr enumeration [48, 40, 51]. In terms of an appropriate ordered basis (E_1, E_2, E_3), the commutator relations are given by

$$\begin{cases} [E_2, E_3] = n_1 E_1 - a E_2 \\ [E_3, E_1] = a E_1 + n_2 E_2 \\ [E_1, E_2] = n_3 E_3. \end{cases}$$

The coefficients a, n_1, n_2 and n_3 for each type of algebra may be found in table C.1, together with the (unique) simply connected Lie group corresponding to each algebra. In addition, the algebraic properties (unimodular, nilpotent, etc.) are listed alongside each algebra. A Lie algebra $\mathfrak{g}$ is

- *unimodular* if $\operatorname{tr}(\operatorname{ad}_X) = 0$ for every $X \in \mathfrak{g}$.
- *nilpotent* if the eigenvalues of ad_X are all zero for every $X \in \mathfrak{g}$.
- *completely solvable* if the eigenvalues of ad_X are all real for every $X \in \mathfrak{g}$.
- *solvable* if the Lie algebra $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent.
- *semisimple* if the Killing form $\mathcal{K} : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$, $\mathcal{K}(X, Y) = \operatorname{tr}(\operatorname{ad}_X \circ \operatorname{ad}_Y)$ is nondegenerate for every $X, Y \in \mathfrak{g}$.

Nilpotency implies complete solvability, which in turn implies solvability. Furthermore, a semisimple algebra is always unimodular and cannot be solvable. A Lie group is called unimodular, nilpotent, etc. if its Lie algebra has the corresponding property.

Remark C.0.15. The algebras $\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1$, $\mathfrak{g}_{3.2}$ and $\mathfrak{g}_{3.4}^h$ are all non-unimodular and completely solvable. They may be distinguished by considering the eigenvalues of ad_X for $X \in \mathfrak{g}$. Indeed, a non-unimodular completely solvable three-dimensional Lie algebra $\mathfrak{g}$ is isomorphic to

- $\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1$ when ad_X has exactly two zero eigenvalues (and one nonzero eigenvalue) for some $X \in \mathfrak{g}$.
- $\mathfrak{g}_{3.2}$ when ad_X has exactly two identical nonzero eigenvalues and one zero eigenvalue for some $X \in \mathfrak{g}$.
- $\mathfrak{g}_{3.4}^h$ when ad_X has exactly two real and distinct nonzero eigenvalues and one zero eigenvalue for some $X \in \mathfrak{g}$.

On the other hand, the eigenvalues of ad_X can be used to determine the parameter h in the infinite families $\mathfrak{g}_{3.4}^h$ and $\mathfrak{g}_{3.5}^h$. Indeed, in terms of (E_1, E_2, E_3), the eigenvalues of ad_X, $X = x^i E_i$ are $\{0, x^3(h-1), x^3(h+1)\}$ and $\{0, x^3(h-\sqrt{-1}), x^3(h+\sqrt{-1})\}$ on $\mathfrak{g}_{3.4}^h$ and $\mathfrak{g}_{3.5}^h$, respectively. The scalars

$$\left(\frac{x^3(h-1)+x^3(h+1)}{x^3(h-1)-x^3(h+1)}\right)^2 = h^2 \quad \text{and} \quad \left(\frac{x^3(h-\sqrt{-1})+x^3(h+\sqrt{-1})}{x^3(h-\sqrt{-1})-x^3(h+\sqrt{-1})}\right)^2 = -h^2$$

(i.e., the ratio between the sum and difference of the nonzero eigenvalues, ignoring sign) are invariant under automorphism, and determine h. □

Remark C.0.16. The two semisimple algebras $\mathfrak{g}_{3.6}$ and $\mathfrak{g}_{3.7}$ may be distinguished by inspecting the Killing form: for $\mathfrak{g}_{3.7}$ it is definite, whereas for $\mathfrak{g}_{3.6}$ it is indefinite. □

Remark C.0.17. The Abelian group $\mathbb{R}^3$ and the group $\mathsf{G}_{3.3}$ do not admit completely nonholonomic left-invariant distributions. (Accordingly, they do not admit nonholonomic Riemannian structures, and hence do not appear in the classification in chapter 4.) □

Remark C.0.18. Apart from $\widetilde{\mathsf{SL}}(2,\mathbb{R})$ (the universal cover of $\mathsf{SL}(2,\mathbb{R})$) there exists, up to Lie algebra automorphism, at most one completely nonholonomic left-invariant distribution on each three-dimensional simply connected Lie group. On $\widetilde{\mathsf{SL}}(2,\mathbb{R})$ there exist exactly two such distributions up to automorphism, according as whether the Killing form restricted to the distribution (at identity) is definite or indefinite. Following [1], if the Killing form is definite on a given distribution, we shall say that the distribution is of *elliptic type*, and denote the group as $\widetilde{\mathsf{SL}}(2,\mathbb{R})_{ell}$. On the other hand, when the Killing form is indefinite on the distribution, we shall say that it is of *hyperbolic type*, and write $\widetilde{\mathsf{SL}}(2,\mathbb{R})_{hyp}$ for the group. □

$\mathfrak{g}_{3.3}$	V	1	0	0	0	$\mathsf{G}_{3.3}$		●	●	
$\mathfrak{g}_{3.4}^{0}$	VI_0	0	1	-1	0	$\mathsf{SE}(1,1)$	●	●	●	
$\mathfrak{g}_{3.4}^{h}$	VI_h	$h{>}0$ $h{\neq}1$	1	-1	0	$\mathsf{G}_{3.4}^{h}$		●	●	
$\mathfrak{g}_{3.5}^{0}$	VII_0	0	1	1	0	$\widetilde{\mathsf{SE}}(2)$	●		●	
$\mathfrak{g}_{3.5}^{h}$	VII_h	$h{>}0$	1	1	0	$\mathsf{G}_{3.5}^{h}$			●	
$\mathfrak{g}_{3.6}$	VIII	0	1	1	-1	$\widetilde{\mathsf{SL}}(2,\mathbb{R})$	●			●
$\mathfrak{g}_{3.7}$	IX	0	1	1	1	$\mathsf{SU}(2)$	●			●

Table C.1: Bianchi–Behr classification of real three-dimensional Lie algebras

Type	Bianchi	a	n_1	n_2	n_3	Simply connected Lie group	Unimodular	Nilpotent	Compl. solvable	Solvable	Semisimple
$3\mathfrak{g}_1$	I	0	0	0	0	$\mathbb{R}^3$	●	●	●	●	
$\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1$	III	1	1	-1	0	$\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}$			●	●	
$\mathfrak{g}_{3.1}$	II	0	1	0	0	H_3	●	●	●	●	
$\mathfrak{g}_{3.2}$	IV	1	1	0	0	$\mathsf{G}_{3.2}$			●	●	

Appendix D

Mathematica code

In this appendix we list the MATHEMATICA [81] code that was developed for many of the calculations performed in this thesis. Text in bold is the actual MATHEMATICA code. Non-bold text (apart from the explanatory comments) is output of the preceding code. (However, for brevity we have omitted the output in most cases.) Each section of this appendix corresponds to a different MATHEMATICA notebook.

D.1 Classification of nonholonomic Riemannian structures in three dimensions

Levi-Civita connection for an extension of the metric on $\mathcal{D}$ to a Riemannian metric preserving the decomposition $T\mathsf{M} = \mathcal{D} \oplus \mathcal{D}^{\perp}$. (Using the Levi-Civita connection for such a metric is an easy way to calculate the nonholonomic connection.) Defines rules so that it is tensorial in the first argument, a derivation in the second, etc.

```
Clear[Con]
Con[X_ + Y_, A_]:=Con[X, A] + Con[Y, A];
Con[a_X_{i_}, A_]:=aCon[X_i, A];

Con[A_, X_ + Y_]:=Con[A, X] + Con[A, Y];
Con[X_{i_}, a_X_{j_}]:=X_i[a]X_j + aCon[X_i, X_j];

Con[0, A_]:=0;
Con[A_, 0]:=0;

Con[X_{i_}, X_{j_}]:=Sum[ε_{i,j,k}X_k, {k, 0, 2}];
```

Nonholonomic connection $\nabla : \Gamma(\mathcal{D}) \times \Gamma(\mathcal{D}) \to \Gamma(\mathcal{D})$. Defines rules so that it is tensorial in the first argument, a derivation in the second, etc.

```
Clear[ConNH]
ConNH[X_ + Y_, A_]:=ConNH[X, A] + ConNH[Y, A];
ConNH[a_X_{i_}, A_]:=aConNH[X_i, A];

ConNH[A_, X_ + Y_]:=ConNH[A, X] + ConNH[A, Y];
ConNH[X_{i_}, a_X_{j_}]:=X_i[a]X_j + aConNH[X_i, X_j];

ConNH[c_{i_,j_,k_}A_, B_]:=c_{i,j,k}ConNH[A, B];

ConNH[0, A_]:=0;
ConNH[A_, 0]:=0;

ConNH[X_{i_}, X_{j_}]:=Sum[ε_{i,j,k}X_k, {k, 1, 2}];
```

Rules for directional derivatives along the elements of the frame (X_0, X_1, X_2) and for using the dual frame (ν^0, ν^1, ν^2).

```
X_{m_}[-A_]:= - X_m[A];
X_{m_}[A_ + B_]:=X_m[A] + X_m[B];
X_{m_}[a_]/;NumberQ[a]:=0;

ν_{i_}[A_]:=⟨X_i, A⟩;
```

Connection coefficients and structure constants.

```
ε_{i_,j_,k_}:=1/2(c_{k,i,j} + c_{i,j,k} - c_{j,k,i})
      /.{c_{0,1,0} → 0, c_{1,2,0} → -1};
c_{i_,j_,k_}/;!OrderedQ[{i, j}]:= - c_{j,i,k};
c_{i_,j_,k_}/;i == j:=0;
```

Projection onto $\mathcal{D}$ (i.e., $\mathscr{P} : T\mathsf{M} \to \mathcal{D}$) and onto $\mathcal{D}^{\perp}$ (i.e., $\mathcal{Q} : T\mathsf{M} \to \mathcal{D}^{\perp}$).

```
Clear[P, Q]
P[X_ + Y_ + Z_]:=P[X] + P[Y] + P[Z];
P[X_ + Y_]:=P[X] + P[Y];
P[a_X_{i_}]:=aP[X_i];
P[c_{i_,j_,k_}A_]:=c_{i,j,k}P[A];

P[X_{i_}]/;i == 1||i == 2:=X_i;
P[X_{i_}]/;i == 0:=0;

P[0]:=0;

Q[X_]:=X - P[X];
```

Metric on $\mathcal{D}$; defines rules so that it is bilinear, symmetric, etc.

```
Clear[AngleBracket]
⟨A_, X_ + Y_⟩:=⟨A, X⟩ + ⟨A, Y⟩;
⟨A_, X_ + Y_ + Z_⟩:=⟨A, X⟩ + ⟨A, Y⟩ + ⟨A, Z⟩;
⟨A_, a_X_{i_}⟩:=a⟨A, X_{i}⟩;

⟨X_ + Y_, A_⟩:=⟨X, A⟩ + ⟨Y, A⟩;
⟨X_ + Y_ + Z_, A_⟩:=⟨X, A⟩ + ⟨Y, A⟩ + ⟨Z, A⟩;
⟨a_X_{i_}, A_⟩:=a⟨X_{i}, A⟩;

⟨X_{i_}, X_{j_}⟩/;i == j:=1;
⟨X_{i_}, X_{j_}⟩/;i ≠ j:=0;

⟨0, A_⟩:=0;
⟨A_, 0⟩:=0;

⟨A_, B_⟩/;!OrderedQ[{A, B}]:=⟨B, A⟩;
```

Lie bracket; defines rules so that it is bilinear, a derivation in each argument, etc.

```
Clear[lie]
lie[A_, X_ + Y_]:=lie[A, X] + lie[A, Y];
lie[A_, X_ + Y_ + Z_]:=lie[A, X] + lie[A, Y] + lie[A, Z];
lie[A_, g_X_{i_}]:=glie[A, X_{i}] + A[g]X_{i};

lie[X_ + Y_, A_]:=lie[X, A] + lie[Y, A];
lie[X_ + Y_ + Z_, A_]:=lie[X, A] + lie[Y, A]
    +lie[Z, A];
lie[f_X_{i_}, A_]:=flie[X_{i}, A] − A[f]X_{i};

lie[X_{i_}, X_{j_}]/;i == j:=0;
lie[0, A_]:=0;
lie[A_, 0]:=0;

lie[A_, B_]/;!OrderedQ[{A, B}]:= − lie[B, A];

lie[X_{0}, X_{1}]:= − (c_{1,0,1}X_{1} + c_{1,0,2}X_{2});
lie[X_{0}, X_{2}]:= − (c_{2,0,0}X_{0} + c_{2,0,1}X_{1} + c_{2,0,2}X_{2});
lie[X_{1}, X_{2}]:= − (X_{0} + c_{2,1,1}X_{1} + c_{2,1,2}X_{2});
```

The adjoint map $\mathrm{ad}_X = [X, \cdot]$, $X \in \mathfrak{g}$.

```
ad[1, 0, 0]:=With[{a = lie[X_{0}, X_{1}], b = lie[X_{0}, X_{2}]},
 {
   {0, 0, 0},
   If[PossibleZeroQ[a], {0, 0, 0}, a],
   If[PossibleZeroQ[b], {0, 0, 0}, b]
 }/.Thread[{X_{0}, X_{1}, X_{2}} → IdentityMatrix[3]]
    //Transpose//Simplify];
ad[0, 1, 0]:=With[{a = lie[X_{1}, X_{0}], b = lie[X_{1}, X_{2}]},
 {
   If[PossibleZeroQ[a], {0, 0, 0}, a],
   {0, 0, 0},
   If[PossibleZeroQ[b], {0, 0, 0}, b]
 }/.Thread[{X_{0}, X_{1}, X_{2}} → IdentityMatrix[3]]
    //Transpose//Simplify];
ad[0, 0, 1]:=With[{a = lie[X_{2}, X_{0}], b = lie[X_{2}, X_{1}]},
 {
   If[PossibleZeroQ[a], {0, 0, 0}, a],
   If[PossibleZeroQ[b],
   {0, 0, 0}, b], {0, 0, 0}
 }/.Thread[{X_{0}, X_{1}, X_{2}} → IdentityMatrix[3]]
    //Transpose//Simplify];
ad[a_, b_, c_]:=aad[1, 0, 0] + bad[0, 1, 0] + cad[0, 0, 1];
```

Curvature tensors (K, $\widehat{K}$, $\widehat{R}$, $\widehat{C}$, etc., as well as their contractions).

```
K[A_, B_, C_]:=ConNH[A, ConNH[B, C]]
    −ConNH[B, ConNH[A, C]]
    −ConNH[P[lie[A, B]], C]
    −P[lie[Q[lie[A, B]], C]];
Khat[A_, B_, C_, D_]:=⟨K[A, B, C], D⟩;

Rhat[A_, B_, C_, D_]:=
    1/2(Khat[A, B, C, D] − Khat[A, B, D, C]);
Chat[A_, B_, C_, D_]:=
    Khat[A, B, C, D] − Rhat[A, B, C, D];

Rtilde[X_, Y_]:= Rhat[X,Y,Y,X] / (⟨X,X⟩⟨Y,Y⟩−⟨X,Y⟩^2);

Ric[A_, B_]:=Sum[Rhat[X_{ℓ}, A, B, X_{ℓ}], {ℓ, 1, 2}];
Scal = Sum[Ric[X_{ℓ}, X_{ℓ}], {ℓ, 1, 2}];

RicC[A_, B_]:=Sum[Chat[X_{ℓ}, A, B, X_{ℓ}], {ℓ, 1, 2}];
RicCSym[A_, B_]:=1/2(RicC[A, B] + RicC[B, A]);
RicCSkw[A_, B_]:=1/2(RicC[A, B] − RicC[B, A]);

𝒢sharp = Inverse[( ⟨X_{1}, X_{1}⟩  ⟨X_{1}, X_{2}⟩
                   ⟨X_{2}, X_{1}⟩  ⟨X_{2}, X_{2}⟩ )]
    //Simplify;
Ricflat = ( Ric[X_{1}, X_{1}]  Ric[X_{1}, X_{2}]
            Ric[X_{2}, X_{1}]  Ric[X_{2}, X_{2}] )
    //Simplify;

R = 𝒢sharp.Ricflat^T//FullSimplify[#, X_[_] == 0]&;
R//Simplify//MatrixForm

RicCSymflat = ( RicCSym[X_{1}, X_{1}]  RicCSym[X_{1}, X_{2}]
                RicCSym[X_{2}, X_{1}]  RicCSym[X_{2}, X_{2}] )
    //Simplify;

RSym = 𝒢sharp.RicCSymflat^T//FullSimplify;
RSym//Simplify//MatrixForm

RicCSkwflat = ( RicCSkw[X_{1}, X_{1}]  RicCSkw[X_{1}, X_{2}]
                RicCSkw[X_{2}, X_{1}]  RicCSkw[X_{2}, X_{2}] )
    //Simplify;

RSkw = 𝒢sharp.RicCSkwflat^T//FullSimplify;
RSkw//Simplify//MatrixForm
```

Scalar invariants κ, χ_1, χ_2 and ϑ.

```
κ = 1/2Tr[R];
χ_{1} = √(−Det[RSym]);
χ_{2} = √(Det[RSkw]);
ϑ = (c_{0,2,0})^2;
```

Killing form $\mathcal{K}$ and its restriction to the distribution $\mathcal{K}|_{\mathcal{D}_1}$.

```
Kil[A_, B_]:=Tr[(ad@@A).(ad@@B)];
KilForm = {
   {Kil[{1, 0, 0}, {1, 0, 0}], Kil[{1, 0, 0}, {0, 1, 0}],
    Kil[{1, 0, 0}, {0, 0, 1}]}
   {Kil[{0, 1, 0}, {1, 0, 0}], Kil[{0, 1, 0}, {0, 1, 0}],
    Kil[{0, 1, 0}, {0, 0, 1}]}
   {Kil[{0, 0, 1}, {1, 0, 0}], Kil[{0, 0, 1}, {0, 1, 0}],
    Kil[{0, 0, 1}, {0, 0, 1}]}
 }//FullSimplify;
KilFormD = {
   {Kil[{0, 1, 0}, {0, 1, 0}], Kil[{0, 1, 0}, {0, 0, 1}]}
```

```
   {Kil[{0,0,1},{0,1,0}],Kil[{0,0,1},{0,0,1}]}
 }//FullSimplify;
```

The other three scalar invariants: ϱ_0, ϱ_1 and ϱ_2.

```
ϱ_0 = -1/2 Kil[{1,0,0},{1,0,0}]
ϱ_1 = -1/2 Kil[{0,1,0},{0,1,0}]
ϱ_2 = -1/2 Kil[{0,0,1},{0,0,1}]
```

Calculate the principal minors of the Killing form $\mathcal{K}$ and the Killing form along the distribution $\mathcal{K}|_{\mathcal{D}_1}$. We use these in the classification code below for distinguishing structures on the semisimple groups.

```
PrincipalMinors[Mat_]:=With[{P = Minors[Mat,#]},
      P_[[1,1]]]&/@Range[1,Length[Mat]];

{m1,m2,m3} = PrincipalMinors[KilForm];
{k1,k2} = PrincipalMinors[KilFormD];
```

Classification

1 $Y_0 \notin \mathcal{D}^\perp$

```
{lie[X_1,X_0],lie[X_2,X_0],lie[X_2,X_1]}
    //Simplify//Column
```

$$\begin{array}{l} -X_1 c_{0,1,1} - X_2 c_{0,1,2} \\ -X_0 c_{0,2,0} - X_1 c_{0,2,1} - X_2 c_{0,2,2} \\ X_0 - X_1 c_{1,2,1} - X_2 c_{1,2,2} \end{array}$$

Equations coming from the Jacobi identity for the Lie bracket:

```
{-c_{0,1,1} - c_{0,2,2} - c_{0,2,0}c_{1,2,2} == 0,
-c_{0,1,1}c_{0,2,0} + c_{0,2,2}c_{1,2,1} - c_{0,2,1}c_{1,2,2} == 0,
c_{0,1,2}(-c_{0,2,0} - c_{1,2,1}) + c_{0,1,1}c_{1,2,2} == 0
}//FullSimplify
```

1.1 Unimodular case: $c_{01}^1 + c_{02}^2 = 0$, $c_{02}^0 + c_{12}^1 = 0$ and $c_{12}^2 = 0$

```
{lie[X_1,X_0],lie[X_2,X_0],lie[X_2,X_1]}
    /.{c_{0,2,2} → -c_{0,1,1}, c_{0,2,0} → -c_{1,2,1}, c_{1,2,2} → 0}
    //Simplify//Column
```

$$\begin{array}{l} -X_1 c_{0,1,1} - X_2 c_{0,1,2} \\ X_2 c_{0,1,1} - X_1 c_{0,2,1} + X_0 c_{1,2,1} \\ X_0 - X_1 c_{1,2,1} \end{array}$$

```
{lie[μX_1,μ^2X_0],lie[μX_2,μ^2X_0],lie[μX_2,μX_1]}
    /.A_[_] → 0
    /.{c_{0,2,2} → -c_{0,1,1}, c_{0,2,0} → -c_{1,2,1}, c_{1,2,2} → 0}
    /.Thread[{X_0,X_1,X_2} → {1/μ^2 X_0, 1/μ X_1, 1/μ X_2}]
    //Simplify;
%/.μ → -1/c_{1,2,1}//Collect[#,X_i_]&//Column
```

$$\begin{array}{l} -\frac{X_1 c_{0,1,1}}{c_{1,2,1}^2} - \frac{X_2 c_{0,1,2}}{c_{1,2,1}^2} \\ -X_0 + \frac{X_2 c_{0,1,1}}{c_{1,2,1}^2} - \frac{X_1 c_{0,2,1}}{c_{1,2,1}^2} \\ X_0 + X_1 \end{array}$$

```
{lie[X_1,X_0],lie[X_2,X_0],lie[X_2,X_1]}
    /.{c_{0,2,2} → -c_{0,1,1}, c_{0,2,0} → -c_{1,2,1}, c_{1,2,2} → 0}
    /.{c_{1,2,1} → -1}//Simplify//Column
```

$$\begin{array}{l} -X_1 c_{0,1,1} - X_2 c_{0,1,2} \\ -X_0 + X_2 c_{0,1,1} - X_1 c_{0,2,1} \\ X_0 + X_1 \end{array}$$

CheckAlg routine

Checks algebraic properties of the algebra generated by the canonical frame (using the eigenvalues of the adjoint operator and also using the Killing form) in order to distinguish the different three-dimensional (unimodular) Lie algebras. Prints out messages indicating if the given algebra has the various properties.

```
CheckAlgU[Subs_,Assumps_,Assumps2_]:=Module[{
    adeigs,uni,nil,compsolv,Kil,KilDet,KilMins,
    Kilre,KilreEigs,KilEigs,KilreMins,nondeg,
    def,defre},

adeigs = Eigenvalues[ad[a_0,a_1,a_2]]/.Subs
    //FullSimplify[#,Join[Assumps,Assumps2]]&;

(* Print[adeigs];*)

uni = TimeConstrained[Reduce[
  ForAll[{a_0,a_1,a_2},{a_0,a_1,a_2} ∈ Reals,.
    Tr[ad[a_0,a_1,a_2]/.Subs] == 0]&&And@@Assumps,
    {c_{0,1,1},c_{0,1,2},c_{0,2,0},c_{0,2,1},c_{0,2,2},c_{1,2,1},c_{1,2,2}},Reals]
      //FullSimplify[#,Join[Assumps,Assumps2]]&,
 5,"Timeout"];
Print["Unimodular?\n",
    If[uni,"Yes, ","No, ","Not sure, "],uni];

nil = TimeConstrained[Reduce[
  ForAll[{a_0,a_1,a_2},And@@Thread[adeigs^2 == 0]]
    &&And@@Assumps,
    {c_{0,1,1},c_{0,1,2},c_{0,2,0},c_{0,2,1},c_{0,2,2},c_{1,2,1},c_{1,2,2}}]
      //FullSimplify[#,Join[Assumps,Assumps2]]&,
 5,"Timeout"];
Print["Nilpotent?\n",
    If[nil,"Yes, ","No, ","Not sure, "],nil];

compsolv = TimeConstrained[Reduce[
  ForAll[{a_0,a_1,a_2},{a_0,a_1,a_2} ∈ Reals,
    And@@Thread[Im[adeigs] == 0]]&&And@@Assumps,
    {c_{0,1,1},c_{0,1,2},c_{0,2,0},c_{0,2,1},c_{0,2,2},c_{1,2,1},c_{1,2,2}}]
      //FullSimplify[#,Join[Assumps,Assumps2]]&,
 2,"Timeout"];
Print["Completely solvable?\n",
    If[compsolv,"Yes, ","No, ","Not sure, "],
        compsolv];

Kil = KilForm/.Subs
    //FullSimplify[#,Join[Assumps,Assumps2]]&;
KilEigs = Eigenvalues[Kil]
    //Simplify[#,Join[Assumps,Assumps2]]&;

KilDet = Det[Kil]
    //FullSimplify[#,Join[Assumps,Assumps2]]&;
nondeg = TimeConstrained[Reduce[
  KilDet ≠ 0&&And@@Assumps,{},Reals]
    //FullSimplify[#,Join[Assumps,Assumps2]]&,
 5,"Timeout"];
Print["Killing form nondegenerate?\n",
    If[nondeg,"Yes, ","No, ","Not sure, "],nondeg];

KilMins = Flatten[PrincipalMinors[Kil]]
    //FullSimplify[#,Join[Assumps,Assumps2]]&;

def = TimeConstrained[
  (KilMins_[[1]] > 0&&KilMins_[[2]] > 0&&KilMins_[[3]] > 0)
 ||(KilMins_[[1]] < 0&&KilMins_[[2]] > 0&&KilMins_[[3]] < 0)
```

```
    //FullSimplify[#, Join[Assumps, Assumps2]]&,
 5, "Timeout"];
Print["Killing form definite?\n",
    If[def, "Yes, ", "No, ", "Not sure, "], def];

Kilre = KilFormD./Subs
    //FullSimplify[#, Join[Assumps, Assumps2]]&;
KilreEigs = Eigenvalues[Kilre]
    //Simplify[#, Join[Assumps, Assumps2]]&;

KilreMins = Flatten[PrincipalMinors[Kilre]]
    //FullSimplify[#, Join[Assumps, Assumps2]]&;

defre = TimeConstrained[
  (KilreMins[[1]] > 0&&KilreMins[[2]] > 0)
 ||(KilreMins[[1]] < 0&&KilreMins[[2]] > 0)
    //FullSimplify[#, Join[Assumps, Assumps2]]&,
 5, "Timeout"];
Print["Killing form definite on ⟨X1, X2⟩?\n",
    If[defre, "Yes, ", "No, ", "Not sure, "], defre];
];
```

1.1.1 $c^1_{01} = 0$ and $c^2_{01} = 0$

```
Det[KilForm]/.{c1,2,1 → −1, c1,2,2 → 0, c0,2,0 → 1,
c0,2,2 → 0, c0,1,1 → 0, c0,1,2 → 0}//Simplify
```

```
0
```

```
ad[a0, a1, a2]/.{c1,2,1 → −1, c1,2,2 → 0, c0,2,0 → 1,
c0,2,2 → 0, c0,1,1 → 0, c0,1,2 → 0}
    //Eigenvalues
    //FullSimplify[#, ai_ ∈ Reals]&
```

```
{0, −Abs[a2]√(1 − c0,2,1), Abs[a2]√(1 − c0,2,1)}
```

1.1.1.1 $c^1_{02} < 1$

For this case, we get a one-parameter family of equivalence classes on SE(1, 1)

```
{lie[X1, X0], lie[X2, X0], lie[X2, X1]}
    /.{c1,2,1 → −1, c1,2,2 → 0, c0,2,0 → 1, c0,2,2 → 0,
c0,1,1 → 0, c0,1,2 → 0}//Simplify;
%/.c0,2,1 → 1 − α//Column
```

```
0
−X0 − (1 − α)X1
X0 + X1
```

```
CheckAlgU[{c0,1,0 → 0, c0,1,1 → 0, c0,1,2 → 0, c0,2,0 → 1,
c0,2,1 → 1 − α, c0,2,2 → 0, c1,2,0 → −1, c1,2,1 → −1,
c1,2,2 → 0}, {α > 0}, {}]
```

```
CheckAlgU[{c1,2,1 → −1, c1,2,2 → 0, c0,2,0 → 1, c0,2,2 → 0,
c0,1,1 → 0, c0,1,2 → 0}, {c0,2,1 < 1}, {}]
```

```
Unimodular?
Yes, True
Nilpotent?
No, False
Completely solvable?
Yes, True
Killing form nondegenerate?
No, False
Killing form definite?
No, False
Killing form definite on <X_1,X_2>?
No, False
```

1.1.1.2 $c^1_{02} = 1$

For this case, we get a single equivalence class on H_3.

```
{lie[X1, X0], lie[X2, X0], lie[X2, X1]}
    /.{c1,2,1 → −1, c1,2,2 → 0, c0,2,0 → 1, c0,2,2 → 0,
c0,1,1 → 0, c0,1,2 → 0, c0,2,1 → 1}//Simplify;
%//Column
```

```
0
−X0 − X1
X0 + X1
```

```
CheckAlgU[{c1,2,1 → −1, c1,2,2 → 0, c0,2,0 → 1, c0,2,2 → 0,
c0,1,1 → 0, c0,1,2 → 0, c0,2,1 → 1}, {}, {}]
```

```
Unimodular?
Yes, True
Nilpotent?
Yes, True
Completely solvable?
Yes, True
Killing form nondegenerate?
No, False
Killing form definite?
No, False
Killing form definite on <X_1,X_2>?
No, False
```

1.1.1.3 $c^1_{02} > 1$

For this case, we get a one-parameter family of equivalence classes on $\widetilde{\mathsf{SE}}(2)$.

```
{lie[X1, X0], lie[X2, X0], lie[X2, X1]}
    /.{c1,2,1 → −1, c1,2,2 → 0, c0,2,0 → 1, c0,2,2 → 0,
c0,1,1 → 0, c0,1,2 → 0}
    //Simplify;
%/.c0,2,1 → 1 + α//Column
```

```
0
−X0 − (1 + α)X1
X0 + X1
```

```
CheckAlgU[{c0,1,0 → 0, c0,1,1 → 0, c0,1,2 → 0, c0,2,0 → 1,
c0,2,1 → 1 + α, c0,2,2 → 0, c1,2,0 → −1, c1,2,1 → −1,
c1,2,2 → 0}, {α > 0}, {}]
```

```
CheckAlgU[{c1,2,1 → −1, c1,2,2 → 0, c0,2,0 → 1, c0,2,2 → 0,
c0,1,1 → 0, c0,1,2 → 0}, {c0,2,1 > 1}, {}]
```

```
Unimodular?
Yes, True
Nilpotent?
No, False
Completely solvable?
No, False
Killing form nondegenerate?
No, False
Killing form definite?
No, False
Killing form definite on <X_1,X_2>?
No, False
```

For brevity, we suppress the output for the remainder of the (unimodular-case) classification code.

1.1.2 $c_{01}^1 = 0$ and $c_{01}^2 \neq 0$

```
Det[KilForm] == 0
    /.{c_{1,2,1} → −1, c_{1,2,2} → 0, c_{0,2,0} → 1, c_{0,2,2} → 0,
c_{0,1,1} → 0}//Simplify[#, c_{0,1,2} ≠ 0]&

ad[a_0, a_1, a_2]/.{c_{1,2,1} → −1, c_{1,2,2} → 0, c_{0,2,0} → 1,
c_{0,2,2} → 0, c_{0,1,1} → 0}
    //Eigenvalues
    //FullSimplify[#, a_i_ ∈ Reals&&c_{0,1,2} ≠ 0]&
ForAll[{a_0, a_1, a_2}, {a_0, a_1, a_2} ∈ Reals,
(−a_2^2(c_{0,2,1} − 1) + c_{0,1,2}(2a_0a_1 + a_1^2 + a_0^2c_{0,2,1}) ≥ 0)
    &&
(−a_2^2(c_{0,2,1} − 1) + c_{0,1,2}(2a_0a_1 + a_1^2 + a_0^2c_{0,2,1}) ≥ 0)]
    //Resolve//FullSimplify[#, c_{0,1,2} ≠ 0]&
```

1.1.2.1 $c_{02}^1 = 1$ $(m_3 = 0)$

1.1.2.1.1 $c_{01}^2 > 0$

For this case, we get a one-parameter family of equivalence classes on SE(1, 1).

```
{lie[X_1, X_0], lie[X_2, X_0], lie[X_2, X_1]}
    /.{c_{1,2,1} → −1, c_{1,2,2} → 0, c_{0,2,0} → 1, c_{0,2,2} → 0,.
c_{0,1,1} → 0, c_{0,2,1} → 1}//Simplify;
%/.c_{0,1,2} → α//Column

CheckAlgU[{c_{0,1,0} → 0, c_{0,1,1} → 0, c_{0,1,2} → α, ..
c_{0,2,0} → 1, c_{0,2,1} → 1, c_{0,2,2} → 0, c_{1,2,0} → −1,
c_{1,2,1} → −1, c_{1,2,2} → 0}, {α > 0}, {}]

CheckAlgU[{c_{1,2,1} → −1, c_{1,2,2} → 0, c_{0,2,0} → 1, ..
c_{0,2,2} → 0, c_{0,1,1} → 0, c_{0,2,1} → 1}, {c_{0,1,2} > 0}, {}]
```

1.1.2.1.2 $c_{01}^2 < 0$

For this case, we get a one-parameter family of equivalence classes on $\widetilde{\mathsf{SE}}(2)$.

```
{lie[X_1, X_0], lie[X_2, X_0], lie[X_2, X_1]}
    /.{c_{1,2,1} → −1, c_{1,2,2} → 0, c_{0,2,0} → 1, c_{0,2,2} → 0,.
c_{0,1,1} → 0, c_{0,2,1} → 1}//Simplify;
%/.c_{0,1,2} → −α//Column

CheckAlgU[{c_{0,1,0} → 0, c_{0,1,1} → 0, c_{0,1,2} → −α, ..
c_{0,2,0} → 1, c_{0,2,1} → 1, c_{0,2,2} → 0, c_{1,2,0} → −1,
c_{1,2,1} → −1, c_{1,2,2} → 0}, {α > 0}, {}]

CheckAlgU[{c_{1,2,1} → −1, c_{1,2,2} → 0, c_{0,2,0} → 1, ..
c_{0,2,2} → 0, c_{0,1,1} → 0, c_{0,2,1} → 1}, {c_{0,1,2} < 0}, {}]
```

1.1.2.2 $c_{02}^1 \neq 1$ $(m_3 \neq 0)$

```
m3 < 0/.{c_{1,2,1} → −1, c_{1,2,2} → 0, c_{0,2,0} → 1,.
c_{0,2,2} → 0, c_{0,1,1} → 0}
    //Simplify[#, c_{0,1,2} ≠ 0&&c_{0,2,1} ≠ 1]&

{m1, m2, m3}/.{c_{1,2,1} → −1, c_{1,2,2} → 0, c_{0,2,0} → 1,.
c_{0,2,2} → 0, c_{0,1,1} → 0}
    //Simplify[#, c_{0,1,2} ≠ 0&&c_{0,2,1} ≠ 1]&
{k1, k2}/.{c_{1,2,1} → −1, c_{1,2,2} → 0, c_{0,2,0} → 1,.
c_{0,2,2} → 0, c_{0,1,1} → 0}
    //Simplify[#, c_{0,1,2} ≠ 0&&c_{0,2,1} ≠ 1]&
```

```
(* KilForm def *)
(m1 < 0&&m2 > 0)/.{c_{1,2,1} → −1, c_{1,2,2} → 0,
c_{0,2,0} → 1, c_{0,2,2} → 0, c_{0,1,1} → 0}
    //Simplify[#, c_{0,1,2} ≠ 0&&c_{0,2,1} ≠ 1]&;
Reduce[%, {}, Reals]//
FullSimplify[#, c_{0,1,2} ≠ 0&&c_{0,2,1} ≠ 1]&

(* KilForm indef *)
(m1 ≥ 0||m2 ≤ 0)/.
{c_{1,2,1} → −1, c_{1,2,2} → 0, c_{0,2,0} → 1, c_{0,2,2} → 0,
c_{0,1,1} → 0}
    //Simplify[#, c_{0,1,2} ≠ 0&&c_{0,2,1} ≠ 1]&;
Reduce[%, {}, Reals]
    //FullSimplify[#, c_{0,1,2} ≠ 0&&c_{0,2,1} ≠ 1]&
```

1.1.2.2.1 $m_1 < 0$ and $m_2 > 0$ ($\mathcal{K}$ neg. def.)

For this case, we get a two-parameter family of equivalence classes on SU(2).

```
{lie[X_1, X_0], lie[X_2, X_0], lie[X_2, X_1]}
    /.{c_{1,2,1} → −1, c_{1,2,2} → 0, c_{0,2,0} → 1, c_{0,2,2} → 0,
c_{0,1,1} → 0}//Simplify;
%/.c_{0,1,2} → −α_1/.c_{0,2,1} → 1 + α_2//Column

Assumps = {c_{0,1,2} ≠ 0, c_{0,2,1} ≠ 1};
Assumps2 = Join[Assumps, {c_{0,1,2} < 0&&c_{0,2,1} > 1}];
Reduce[−c_{0,1,2} > 0&&And@@Assumps]
    //FullSimplify[#, Assumps2]&
Reduce[c_{0,2,1} − 1 > 0&&And@@Assumps]
    //FullSimplify[#, Assumps2]&

CheckAlgU[{c_{0,1,0} → 0, c_{0,1,1} → 0, c_{0,1,2} → −α_1,
c_{0,2,0} → 1, c_{0,2,1} → 1 + α_2, c_{0,2,2} → 0, c_{1,2,0} → −1,
c_{1,2,1} → −1, c_{1,2,2} → 0}, {α_1 > 0, α_2 > 0}, {}]

CheckAlgU[{c_{1,2,1} → −1, c_{1,2,2} → 0, c_{0,2,0} → 1,
c_{0,2,2} → 0, c_{0,1,1} → 0}, {c_{0,1,2} ≠ 0, c_{0,2,1} ≠ 1},
{c_{0,1,2} < 0, c_{0,2,1} > 1}]
```

1.1.2.2.2 $m_1 \geq 0$ or $m_2 \leq 0$ ($\mathcal{K}$ indef.)

```
(* KilFormD def *)
(m1 ≥ 0||m2 ≤ 0)&&(k1 ≠ 0&&k2 > 0)
    /.{c_{1,2,1} → −1, c_{1,2,2} → 0, c_{0,2,0} → 1,
c_{0,2,2} → 0, c_{0,1,1} → 0}
    //FullSimplify[#, c_{0,1,2} ≠ 0&&c_{0,2,1} ≠ 1]&;
Reduce[%, {}, Reals]
    //FullSimplify[#, c_{0,1,2} ≠ 0&&c_{0,2,1} ≠ 1]&
(* KilFormD indef *)
(m1 ≥ 0||m2 ≤ 0)&&(k1 == 0||k2 ≤ 0)
    /.{c_{1,2,1} → −1, c_{1,2,2} → 0, c_{0,2,0} → 1,
c_{0,2,2} → 0, c_{0,1,1} → 0}
    //FullSimplify[#, c_{0,1,2} ≠ 0&&c_{0,2,1} ≠ 1]&;
Reduce[%, {}, Reals]
    //FullSimplify[#, c_{0,1,2} ≠ 0&&c_{0,2,1} ≠ 1]&
```

1.1.2.2.2.1 $k_1 \neq 0$ and $k_2 > 0$ ($\mathcal{K}_{\mathcal{D}_1}$ def.)

For this case, we get a two-parameter family of equivalence classes on $\widetilde{\mathsf{SL}}(2,\mathbb{R})_{ell}$.

```
{lie[X_1, X_0], lie[X_2, X_0], lie[X_2, X_1]}
    /.{c_{1,2,1} → −1, c_{1,2,2} → 0, c_{0,2,0} → 1,
c_{0,2,2} → 0, c_{0,1,1} → 0}//Simplify;
%/.c_{0,1,2} → α_1/.c_{0,2,1} → 1 − α_2//Column
```

```
Assumps = {c_{0,1,2} ≠ 0, c_{0,2,1} ≠ 1};
Assumps2 = Join[Assumps, {c_{0,1,2} > 0, c_{0,2,1} < 1}];
Reduce[c_{0,1,2} > 0&&And@@Assumps]
    //FullSimplify[#, Assumps2]&
Reduce[1 − c_{0,2,1} > 0&&And@@Assumps]
    //FullSimplify[#, Assumps2]&
```

```
CheckAlgU[{c_{0,1,0} → 0, c_{0,1,1} → 0, c_{0,1,2} → α_1,
c_{0,2,0} → 1, c_{0,2,1} → 1 − α_2, c_{0,2,2} → 0,
c_{1,2,0} → −1, c_{1,2,1} → −1, c_{1,2,2} → 0},
{α_1 > 0, α_2 > 0}, {}]
```

```
CheckAlgU[{c_{1,2,1} → −1, c_{1,2,2} → 0, c_{0,2,0} → 1,
c_{0,2,2} → 0, c_{0,1,1} → 0}, {c_{0,1,2} ≠ 0, c_{0,2,1} ≠ 1},
{c_{0,1,2} > 0, c_{0,2,1} < 1}]
```

1.1.2.2.2.2 $k_1 = 0$ or $k_2 \leq 0$ ($\mathcal{K}_{\mathcal{D}_1}$ indef.)

For this case, we get a two-parameter family of equivalence classes on $\widetilde{\mathsf{SL}}(2, \mathbb{R})_{hyp}$.

Case 1: $c_{0,1,2} < 0$ and $c_{0,2,1} \leq 1$

```
{lie[X_1, X_0], lie[X_2, X_0], lie[X_2, X_1]}
    /.{c_{1,2,1} → −1, c_{1,2,2} → 0, c_{0,2,0} → 1,
c_{0,2,2} → 0, c_{0,1,1} → 0}//Simplify;
%/.c_{0,1,2} → −α_1/.c_{0,2,1} → 1 − α_2//Column
```

```
Assumps = {c_{0,1,2} ≠ 0, c_{0,2,1} ≠ 1};
Assumps2 = Join[Assumps, {c_{0,1,2} < 0&&c_{0,2,1} ≤ 1}];
Reduce[−c_{0,1,2} > 0&&And@@Assumps]
    //FullSimplify[#, Assumps2]&
Reduce[1 − c_{0,2,1} > 0&&And@@Assumps]
    //FullSimplify[#, Assumps2]&
```

```
CheckAlgU[{c_{0,1,0} → 0, c_{0,1,1} → 0, c_{0,1,2} → −α_1,
c_{0,2,0} → 1, c_{0,2,1} → 1 − α_2, c_{0,2,2} → 0,
c_{1,2,0} → −1, c_{1,2,1} → −1, c_{1,2,2} → 0},
{α_1 > 0, α_2 > 0}, {}]
```

```
CheckAlgU[{c_{1,2,1} → −1, c_{1,2,2} → 0, c_{0,2,0} → 1,
c_{0,2,2} → 0, c_{0,1,1} → 0}, {c_{0,1,2} ≠ 0, c_{0,2,1} ≠ 1},
{c_{0,1,2} < 0, c_{0,2,1} ≤ 1}]
```

Case 2: $c_{0,1,2} > 0$ and $c_{0,2,1} \geq 1$

```
{lie[X_1, X_0], lie[X_2, X_0], lie[X_2, X_1]}
    /.{c_{1,2,1} → −1, c_{1,2,2} → 0, c_{0,2,0} → 1,
c_{0,2,2} → 0, c_{0,1,1} → 0}//Simplify;
%/.c_{0,1,2} → α_1/.c_{0,2,1} → 1 + α_2//Column
```

```
Assumps = {c_{0,1,2} ≠ 0, c_{0,2,1} ≠ 1};
Assumps2 = Join[Assumps, {c_{0,1,2} > 0, c_{0,2,1} ≥ 1}];
Reduce[c_{0,1,2} > 0&&And@@Assumps]
    //FullSimplify[#, Assumps2]&
Reduce[c_{0,2,1} − 1 > 0&&And@@Assumps]
    //FullSimplify[#, Assumps2]&
```

```
CheckAlgU[{c_{0,1,0} → 0, c_{0,1,1} → 0, c_{0,1,2} → α_1,
c_{0,2,0} → 1, c_{0,2,1} → 1 + α_2, c_{0,2,2} → 0,
c_{1,2,0} → −1, c_{1,2,1} → −1, c_{1,2,2} → 0},
{α_1 > 0, α_2 > 0}, {}]
```

```
CheckAlgU[{c_{1,2,1} → −1, c_{1,2,2} → 0, c_{0,2,0} → 1,
c_{0,2,2} → 0, c_{0,1,1} → 0}, {c_{0,1,2} ≠ 0, c_{0,2,1} ≠ 1},
{c_{0,1,2} > 0, c_{0,2,1} ≥ 1}]
```

1.1.3 $c^1_{01} > 0$ and $c^2_{01} = 0$

For this case, we get a two-parameter family of equivalence classes on $\widetilde{\mathsf{SL}}(2, \mathbb{R})_{hyp}$.

```
Det[KilForm] ≠ 0
    /.{c_{1,2,1} → −1, c_{1,2,2} → 0, c_{0,2,0} → 1,
c_{0,2,2} → −c_{0,1,1}, c_{0,1,2} → 0}
    //Simplify[#, c_{0,1,1} > 0]&
m3 < 0/.{c_{1,2,1} → −1, c_{1,2,2} → 0, c_{0,2,0} → 1,
c_{0,2,2} → −c_{0,1,1}, c_{0,1,2} → 0}
    //Simplify[#, c_{0,1,1} > 0]&
```

```
{m1, m2, m3}/.{c_{1,2,1} → −1, c_{1,2,2} → 0,
c_{0,2,0} → 1, c_{0,2,2} → −c_{0,1,1}, c_{0,1,2} → 0}
    //Simplify[#, c_{0,1,1} > 0]&
(* KilForm def *)
(m1 < 0&&m2 > 0)/.{c_{1,2,1} → −1, c_{1,2,2} → 0, c_{0,2,0} → 1,
c_{0,2,2} → −c_{0,1,1}, c_{0,1,2} → 0}
    //Simplify[#, c_{0,1,1} > 0]&;
Reduce[%, {}, Reals]
    //FullSimplify[#, c_{0,1,1} > 0]&
```

```
(* KilForm indef *)
(m1 ≥ 0||m2 ≤ 0)
    /.{c_{1,2,1} → −1, c_{1,2,2} → 0, c_{0,2,0} → 1,
c_{0,2,2} → −c_{0,1,1}, c_{0,1,2} → 0}
    //Simplify[#, c_{0,1,1} > 0]&;
Reduce[%, {}, Reals]
    //FullSimplify[#, c_{0,1,1} > 0]&
```

```
(* KilFormD def *)
(m1 ≥ 0||m2 ≤ 0)&&(k1 ≠ 0&&k2 > 0)
    /.{c_{1,2,1} → −1, c_{1,2,2} → 0, c_{0,2,0} → 1,
c_{0,2,2} → −c_{0,1,1}, c_{0,1,2} → 0}
    //Simplify[#, c_{0,1,1} > 0]&;
Reduce[%, {}, Reals]
    //Simplify[#, c_{0,1,1} > 0]&
```

```
(* KilFormD indef *)
(m1 ≥ 0||m2 ≤ 0)&&(k1 == 0||k2 ≤ 0)
    /.{c_{1,2,1} → −1, c_{1,2,2} → 0, c_{0,2,0} → 1,
c_{0,2,2} → −c_{0,1,1}, c_{0,1,2} → 0}
    //Simplify[#, c_{0,1,1} > 0]&;
Reduce[%, {}, Reals]
    //Simplify[#, c_{0,1,1} > 0]&
```

```
{lie[X_1, X_0], lie[X_2, X_0], lie[X_2, X_1]}
    /.{c_{1,2,1} → −1, c_{1,2,2} → 0, c_{0,2,0} → 1,
c_{0,2,2} → −c_{0,1,1}, c_{0,1,2} → 0}//Simplify;
%/.c_{0,1,1} → α/.c_{0,2,1} → −γ + 1//Column
```

```
Assumps = {c_{0,1,1} > 0};
Reduce[c_{0,1,1} > 0&&And@@Assumps]
    //FullSimplify[#, Assumps]&
Reduce[c_{0,2,1} ∈ Reals&&And@@Assumps]
    //FullSimplify[#, Assumps]&
```

```
CheckAlgU[{c_{0,1,0} → 0, c_{0,1,1} → α, c_{0,1,2} → 0,
c_{0,2,0} → 1, c_{0,2,1} → 1 − γ, c_{0,2,2} → −α,
c_{1,2,0} → −1, c_{1,2,1} → −1, c_{1,2,2} → 0},
{α > 0, γ ∈ Reals}, {}]
```

```
CheckAlgU[{c_{1,2,1} → −1, c_{1,2,2} → 0, c_{0,2,0} → 1,
c_{0,2,2} → −c_{0,1,1}, c_{0,1,2} → 0}, {c_{0,1,1} > 0}, {}]
```

1.1.4 $c_{01}^1 > 0$ and $c_{01}^2 \neq 0$

Det[KilForm] $== 0$**/.**$\{c_{1,2,1} \to -1, c_{1,2,2} \to 0, c_{0,2,0} \to 1,$
$c_{0,2,2} \to -c_{0,1,1}\}$
//Simplify[#, $c_{0,1,1} > 0 \&\& c_{0,1,2} \neq 0$**]&**
Solve[%, $c_{0,1,1}$**]//Simplify**

1.1.4.1 $(c_{01}^1)^2 + c_{01}^2(c_{02}^1 - 1) = 0$ $(m_3 = 0)$

1.1.4.1.1 $c_{01}^2 > 0$ and $c_{02}^1 < 1$

For this case, we get a two-parameter family of equivalence classes on SE(1, 1).

{lie[X_1, X_0**], lie[**X_2, X_0**], lie[**X_2, X_1**]}**
/.$\{c_{1,2,1} \to -1, c_{1,2,2} \to 0, c_{0,2,0} \to 1,$
$c_{0,2,2} \to -\sqrt{-c_{0,1,2}(c_{0,2,1} - 1)},$
$c_{0,1,1} \to \sqrt{-c_{0,1,2}(c_{0,2,1} - 1)}\}$
//Simplify;
%/.$c_{0,1,2} \to \alpha_1$**/.**$c_{0,2,1} \to 1 - \alpha_2$
//Simplify[#, $\alpha_1 > 0 \&\& \alpha_2 > 0$**]&//Column**

CheckAlgU[$\{c_{0,1,0} \to 0, c_{0,1,1} \to \sqrt{\alpha_1\alpha_2}, c_{0,1,2} \to \alpha_1,$
$c_{0,2,0} \to 1, c_{0,2,1} \to (1 - \alpha_2), c_{0,2,2} \to -\sqrt{\alpha_1\alpha_2},$
$c_{1,2,0} \to -1, c_{1,2,1} \to -1, c_{1,2,2} \to 0\},$
$\{\alpha_1 > 0, \alpha_2 > 0\}, \{\}$**]**

CheckAlgU[$\{c_{1,2,1} \to -1, c_{1,2,2} \to 0, c_{0,2,0} \to 1,$
$c_{0,2,2} \to -\sqrt{-c_{0,1,2}(c_{0,2,1} - 1)},$
$c_{0,1,1} \to \sqrt{-c_{0,1,2}(c_{0,2,1} - 1)}\},$
$\{c_{0,1,2} > 0, c_{0,2,1} < 1\}, \{\}$**]**

1.1.4.1.2 $c_{01}^2 < 0$ and $c_{02}^1 > 1$

For this case, we get a two-parameter family of equivalence classes on $\widetilde{\text{SE}}(2)$.

{lie[X_1, X_0**], lie[**X_2, X_0**], lie[**X_2, X_1**]}**
/.$\{c_{1,2,1} \to -1, c_{1,2,2} \to 0, c_{0,2,0} \to 1,$
$c_{0,2,2} \to -\sqrt{-c_{0,1,2}(c_{0,2,1} - 1)},$
$c_{0,1,1} \to \sqrt{-c_{0,1,2}(c_{0,2,1} - 1)}\}$
//Simplify;
%/.$c_{0,1,2} \to -\alpha_1$**/.**$c_{0,2,1} \to 1 + \alpha_2$
//Simplify[#, $\alpha_1 > 0 \&\& \alpha_2 > 0$**]&//Column**

CheckAlgU[$\{c_{0,1,0} \to 0, c_{0,1,1} \to \sqrt{\alpha_1\alpha_2}, c_{0,1,2} \to -\alpha_1,$
$c_{0,2,0} \to 1, c_{0,2,1} \to (1 + \alpha_2), c_{0,2,2} \to -\sqrt{\alpha_1\alpha_2},$
$c_{1,2,0} \to -1, c_{1,2,1} \to -1, c_{1,2,2} \to 0\},$
$\{\alpha_1 > 0, \alpha_2 > 0\}, \{\}$**]**

CheckAlgU[$\{c_{1,2,1} \to -1, c_{1,2,2} \to 0, c_{0,2,0} \to 1,$
$c_{0,2,2} \to -\sqrt{-c_{0,1,2}(c_{0,2,1} - 1)},$
$c_{0,1,1} \to \sqrt{-c_{0,1,2}(c_{0,2,1} - 1)}\},$
$\{c_{0,1,2} < 0, c_{0,2,1} > 1\}, \{\}$**]**

1.1.4.2 $(c_{01}^1)^2 + c_{01}^2(c_{02}^1 - 1) \neq 0$ $(m_3 \neq 0)$

m3 < 0**/.**$\{c_{1,2,1} \to -1, c_{1,2,2} \to 0, c_{0,2,0} \to 1,$
$c_{0,2,2} \to -c_{0,1,1}\}$
//Simplify[#, $c_{0,1,1} > 0 \&\& c_{0,1,2} \neq 0$
$\&\& c_{0,1,1}^2 + c_{0,1,2}(c_{0,2,1} - 1) \neq 0$**]&**

{m1, m2, m3}/.$\{c_{1,2,1} \to -1, c_{1,2,2} \to 0, c_{0,2,0} \to 1,$
$c_{0,2,2} \to -c_{0,1,1}\}$
//Simplify[#, $c_{0,1,1} > 0 \&\& c_{0,1,2} \neq 0$
$\&\& c_{0,1,1}^2 + c_{0,1,2}(c_{0,2,1} - 1) \neq 0$**]&**
{k1, k2}/.$\{c_{1,2,1} \to -1, c_{1,2,2} \to 0, c_{0,2,0} \to 1,$
$c_{0,2,2} \to -c_{0,1,1}\}$
//Simplify[#, $c_{0,1,1} > 0 \&\& c_{0,1,2} \neq 0$
$\&\& c_{0,1,1}^2 + c_{0,1,2}(c_{0,2,1} - 1) \neq 0$**]&**

(* KilForm def *)
(m1 $< 0 \&\&$ **m2** > 0**)/.**$\{c_{1,2,1} \to -1, c_{1,2,2} \to 0,$
$c_{0,2,0} \to 1, c_{0,2,2} \to -c_{0,1,1}\}$
//Simplify[#, $c_{0,1,1} > 0 \&\& c_{0,1,2} \neq 0$
$\&\& c_{0,1,1}^2 + c_{0,1,2}(c_{0,2,1} - 1) \neq 0$**]&;**
Reduce[%, {}, Reals]
//FullSimplify[#, $c_{0,1,1} > 0 \&\& c_{0,1,2} \neq 0$
$\&\& c_{0,1,1}^2 + c_{0,1,2}(c_{0,2,1} - 1) \neq 0$**]&**

(* KilForm indef *)
(m1 $\geq 0 \|$**m2** ≤ 0**)/.**$\{c_{1,2,1} \to -1, c_{1,2,2} \to 0,$
$c_{0,2,0} \to 1, c_{0,2,2} \to -c_{0,1,1}\}$
//Simplify[#, $c_{0,1,1} > 0 \&\& c_{0,1,2} \neq 0$
$\&\& c_{0,1,1}^2 + c_{0,1,2}(c_{0,2,1} - 1) \neq 0$**]&;**
Reduce[%, {}, Reals]
//FullSimplify[#, $c_{0,1,1} > 0 \&\& c_{0,1,2} \neq 0$
$\&\& c_{0,1,1}^2 + c_{0,1,2}(c_{0,2,1} - 1) \neq 0$**]&**

1.1.4.2.1 $m_1 < 0$ and $m_2 > 0$ ($\mathcal{K}$ neg. def.)

For this case, we get a two-parameter family of equivalence classes on SU(2).

{lie[X_1, X_0**], lie[**X_2, X_0**], lie[**X_2, X_1**]}**
/.$\{c_{1,2,1} \to -1, c_{1,2,2} \to 0, c_{0,2,0} \to 1,$
$c_{0,2,2} \to -c_{0,1,1}\}$
//Simplify;
%/.$c_{0,1,1} \to \alpha_1$**/.**$c_{0,1,2} \to -\alpha_2$**/.**$c_{0,2,1} \to 1 + \alpha_3$
//Column

Assumps $= \{c_{0,1,1} > 0, c_{0,1,2} \neq 0\}$**;**
Assumps2 = Join[Assumps, $\{c_{0,1,1}^2 + c_{0,1,2}(c_{0,2,1} - 1) \neq 0,$
$c_{0,1,2} < 0, \frac{c_{0,1,1}^2}{c_{0,1,2}} + c_{0,2,1} > 1\}$**];**
Reduce[$c_{0,1,1} > 0 \&\&$**And@@Assumps]**
//FullSimplify[#, Assumps2]&
Reduce[$-c_{0,1,2} > 0 \&\&$**And@@Assumps]**
//FullSimplify[#, Assumps2]&
Reduce[$c_{0,2,1} - 1 > -\frac{c_{0,1,1}^2}{c_{0,1,2}} > 0 \&\&$**And@@Assumps]**
//FullSimplify[#, Assumps2]&

CheckAlgU[$\{c_{0,1,0} \to 0, c_{0,1,1} \to \alpha_1, c_{0,1,2} \to -\alpha_2,$
$c_{0,2,0} \to 1, c_{0,2,1} \to 1 + \alpha_3, c_{0,2,2} \to -\alpha_1,$
$c_{1,2,0} \to -1, c_{1,2,1} \to -1, c_{1,2,2} \to 0\},$
$\{\alpha_1 > 0, \alpha_2 > 0, \alpha_3 > 0\}, \{\alpha_1^2 < \alpha_2\alpha_3\}$**]**

CheckAlgU[$\{c_{1,2,1} \to -1, c_{1,2,2} \to 0, c_{0,2,0} \to 1,$
$c_{0,2,2} \to -c_{0,1,1}\}, \{c_{0,1,1} > 0, c_{0,1,2} \neq 0\},$
$\{c_{0,1,1}^2 + c_{0,1,2}(c_{0,2,1} - 1) \neq 0, c_{0,1,2} < 0,$
$\frac{c_{0,1,1}^2}{c_{0,1,2}} + c_{0,2,1} > 1\}$**]**

1.1.4.2.2 $m_1 \geq 0$ or $m_2 \leq 0$ ($\mathcal{K}$ indef.)

(* KilFormD def *)
(m1 $\geq 0 \|$**m2** ≤ 0**)&&(k1** $\neq 0 \&\&$**k2** > 0**)**
/.$\{c_{1,2,1} \to -1, c_{1,2,2} \to 0, c_{0,2,0} \to 1,$
$c_{0,2,2} \to -c_{0,1,1}\}$

$\texttt{//Simplify}[\#, c_{0,1,1} > 0\&\&c_{0,1,2} \neq 0$
$\&\&c^2_{0,1,1} + c_{0,1,2}(c_{0,2,1} - 1) \neq 0]\&;$
$\texttt{Reduce}[\%, \{\}, \texttt{Reals}]$
$\texttt{//FullSimplify}[\#, c_{0,1,1} > 0\&\&c_{0,1,2} \neq 0$
$\&\&c^2_{0,1,1} + c_{0,1,2}(c_{0,2,1} - 1) \neq 0]\&$

$\texttt{(* KilFormD indef *)}$
$(\texttt{m1} \geq 0||\texttt{m2} \leq 0)\&\&(\texttt{k1} == 0||\texttt{k2} \leq 0)$
$/.\{c_{1,2,1} \to -1, c_{1,2,2} \to 0, c_{0,2,0} \to 1,$
$c_{0,2,2} \to -c_{0,1,1}\}$
$\texttt{//Simplify}[\#, c_{0,1,1} > 0\&\&c_{0,1,2} \neq 0$
$\&\&c^2_{0,1,1} + c_{0,1,2}(c_{0,2,1} - 1) \neq 0]\&;$
$\texttt{Reduce}[\%, \{\}, \texttt{Reals}]$
$\texttt{//FullSimplify}[\#, c_{0,1,1} > 0\&\&c_{0,1,2} \neq 0$
$\&\&c^2_{0,1,1} + c_{0,1,2}(c_{0,2,1} - 1) \neq 0]\&$

1.1.4.2.2.1 $k_1 \neq 0$ and $k_2 > 0$ ($\mathcal{K}_{\mathcal{D}_1}$ def.)

For this case, we get a two-parameter family of equivalence classes on $\widetilde{\mathrm{SL}}(2,\mathbb{R})_{ell}$.

$\{\texttt{lie}[X_1, X_0], \texttt{lie}[X_2, X_0], \texttt{lie}[X_2, X_1]\}$
$/.\{c_{1,2,1} \to -1, c_{1,2,2} \to 0, c_{0,2,0} \to 1,$
$c_{0,2,2} \to -c_{0,1,1}\}$
$\texttt{//Simplify};$
$\%/.c_{0,1,1} \to \alpha_1/.c_{0,1,2} \to \alpha_2/.c_{0,2,1} \to 1 - \alpha_3$
$\texttt{//Column}$

$\texttt{Assumps} = \{c_{0,1,1} > 0, c_{0,1,2} \neq 0\};$
$\texttt{Assumps2} = \texttt{Join}[\texttt{Assumps},$
$\{c^2_{0,1,1} + c_{0,1,2}(c_{0,2,1} - 1) \neq 0,$
$c_{0,1,2} > 0, \frac{c^2_{0,1,1}}{c_{0,1,2}} + c_{0,2,1} < 1\}];$
$\texttt{Reduce}[c_{0,1,1} > 0\&\&\texttt{And@@Assumps}]$
$\texttt{//FullSimplify}[\#, \texttt{Assumps2}]\&$
$\texttt{Reduce}[c_{0,1,2} > 0\&\&\texttt{And@@Assumps}]$
$\texttt{//FullSimplify}[\#, \texttt{Assumps2}]\&$
$\texttt{Reduce}[1 - c_{0,2,1} > 0 > -\frac{c^2_{0,1,1}}{c_{0,1,2}}\&\&\texttt{And@@Assumps}]$
$\texttt{//FullSimplify}[\#, \texttt{Assumps2}]\&$

$\texttt{CheckAlgU}[\{c_{0,1,0} \to 0, c_{0,1,1} \to \alpha_1, c_{0,1,2} \to \alpha_2,$
$c_{0,2,0} \to 1, c_{0,2,1} \to 1 - \alpha_3, c_{0,2,2} \to -\alpha_1,$
$c_{1,2,0} \to -1, c_{1,2,1} \to -1, c_{1,2,2} \to 0\},$
$\{\alpha_1 > 0, \alpha_2 > 0, \alpha_3 > 0\}, \{\alpha_2\alpha_3 > \alpha_1^2\}]$

$\texttt{CheckAlgU}[\{c_{1,2,1} \to -1, c_{1,2,2} \to 0, c_{0,2,0} \to 1,$
$c_{0,2,2} \to -c_{0,1,1}\}, \{c_{0,1,1} > 0, c_{0,1,2} \neq 0\},$
$\{c^2_{0,1,1} + c_{0,1,2}(c_{0,2,1} - 1) \neq 0, c_{0,1,2} > 0,$
$\frac{c^2_{0,1,1}}{c_{0,1,2}} + c_{0,2,1} < 1\}]$

1.1.4.2.2.2 $k_1 = 0$ or $k_2 \leq 0$ ($\mathcal{K}_{\mathcal{D}_1}$ indef.)

For this case, we get a two-parameter family of equivalence classes on $\widetilde{\mathrm{SL}}(2,\mathbb{R})_{hyp}$.

Case 1: $c_{0,1,2} < 0$ and $\frac{c^2_{0,1,1}}{c_{0,1,2}} + c_{0,2,1} \leq 1$

$\{\texttt{lie}[X_1, X_0], \texttt{lie}[X_2, X_0], \texttt{lie}[X_2, X_1]\}$
$/.\{c_{1,2,1} \to -1, c_{1,2,2} \to 0, c_{0,2,0} \to 1,$
$c_{0,2,2} \to -c_{0,1,1}\}$
$\texttt{//Simplify};$
$\%/.c_{0,1,1} \to \alpha_1/.c_{0,1,2} \to -\alpha_2/.c_{0,2,1} \to -\gamma + 1$
$\texttt{//Column}$

$\texttt{Assumps} = \{c_{0,1,1} > 0, c_{0,1,2} \neq 0\};$
$\texttt{Assumps2} = \texttt{Join}[\texttt{Assumps}, \{c^2_{0,1,1} + c_{0,1,2}(c_{0,2,1} - 1) \neq 0,$
$c_{0,1,2} < 0, \frac{c^2_{0,1,1}}{c_{0,1,2}} + c_{0,2,1} \leq 1\}];$
$\texttt{Reduce}[c_{0,1,1} > 0\&\&\texttt{And@@Assumps}]$
$\texttt{//FullSimplify}[\#, \texttt{Assumps2}]\&$
$\texttt{Reduce}[-c_{0,1,2} > 0\&\&\texttt{And@@Assumps}]$
$\texttt{//FullSimplify}[\#, \texttt{Assumps2}]\&$
$\texttt{Reduce}[1 - c_{0,2,1} \geq \frac{c^2_{0,1,1}}{c_{0,1,2}}\&\&\texttt{And@@Assumps}]$
$\texttt{//FullSimplify}[\#, \texttt{Assumps2}]\&$

$\texttt{CheckAlgU}[\{c_{0,1,0} \to 0, c_{0,1,1} \to \alpha_1, c_{0,1,2} \to -\alpha_2,$
$c_{0,2,0} \to 1, c_{0,2,1} \to 1 - \gamma, c_{0,2,2} \to -\alpha_1,$
$c_{1,2,0} \to -1, c_{1,2,1} \to -1, c_{1,2,2} \to 0\},$
$\{\alpha_1 > 0, \alpha_2 > 0, \gamma \in \texttt{Reals}\}, \{\alpha_1^2 + \gamma\alpha_2 > 0\}]$

$\texttt{CheckAlgU}[\{c_{1,2,1} \to -1, c_{1,2,2} \to 0, c_{0,2,0} \to 1,$
$c_{0,2,2} \to -c_{0,1,1}\}, \{c_{0,1,1} > 0, c_{0,1,2} \neq 0\},$
$\{c^2_{0,1,1} + c_{0,1,2}(c_{0,2,1} - 1) \neq 0, c_{0,1,2} < 0,$
$\frac{c^2_{0,1,1}}{c_{0,1,2}} + c_{0,2,1} \leq 1\}]$

Case 2: $c_{0,1,2} > 0$ and $\frac{c^2_{0,1,1}}{c_{0,1,2}} + c_{0,2,1} \geq 1$

$\{\texttt{lie}[X_1, X_0], \texttt{lie}[X_2, X_0], \texttt{lie}[X_2, X_1]\}$
$/.\{c_{1,2,1} \to -1, c_{1,2,2} \to 0, c_{0,2,0} \to 1,$
$c_{0,2,2} \to -c_{0,1,1}\}$
$\texttt{//Simplify};$
$\%/.c_{0,1,1} \to \alpha_1/.c_{0,1,2} \to \alpha_2/.c_{0,2,1} \to -\gamma + 1$
$\texttt{//Column}$

$\texttt{Assumps} = \{c_{0,1,1} > 0, c_{0,1,2} \neq 0\};$
$\texttt{Assumps2} = \texttt{Join}[\texttt{Assumps}, \{c^2_{0,1,1} + c_{0,1,2}(c_{0,2,1} - 1) \neq 0,$
$c_{0,1,2} > 0, \frac{c^2_{0,1,1}}{c_{0,1,2}} + c_{0,2,1} \geq 1\}];$
$\texttt{Reduce}[c_{0,1,1} > 0\&\&\texttt{And@@Assumps}]$
$\texttt{//FullSimplify}[\#, \texttt{Assumps2}]\&$
$\texttt{Reduce}[c_{0,1,2} > 0\&\&\texttt{And@@Assumps}]$
$\texttt{//FullSimplify}[\#, \texttt{Assumps2}]\&$
$\texttt{Reduce}[c_{0,2,1} - 1 \geq -\frac{c^2_{0,1,1}}{c_{0,1,2}}\&\&\texttt{And@@Assumps}]$
$\texttt{//FullSimplify}[\#, \texttt{Assumps2}]\&$

$\texttt{CheckAlgU}[\{c_{0,1,0} \to 0, c_{0,1,1} \to \alpha_1, c_{0,1,2} \to \alpha_2,$
$c_{0,2,0} \to 1, c_{0,2,1} \to 1 - \gamma, c_{0,2,2} \to -\alpha_1,$
$c_{1,2,0} \to -1, c_{1,2,1} \to -1, c_{1,2,2} \to 0\},$
$\{\alpha_1 > 0, \alpha_2 > 0, \gamma \in \texttt{Reals}\}, \{\alpha_1^2 - \gamma\alpha_2 > 0\}]$

$\texttt{CheckAlgU}[\{c_{1,2,1} \to -1, c_{1,2,2} \to 0, c_{0,2,0} \to 1,$
$c_{0,2,2} \to -c_{0,1,1}\}, \{c_{0,1,1} > 0, c_{0,1,2} \neq 0\},$
$\{c^2_{0,1,1} + c_{0,1,2}(c_{0,2,1} - 1) \neq 0, c_{0,1,2} > 0,$
$\frac{c^2_{0,1,1}}{c_{0,1,2}} + c_{0,2,1} \geq 1\}]$

Equivalence Classes

SE(1, 1)

On SE(1, 1), we have a two-parameter family of equivalence classes.

$\texttt{Subs} = \{c_{0,1,0} \to 0, c_{0,1,1} \to \sqrt{\alpha_1\alpha_2}, c_{0,1,2} \to \alpha_1,$
$c_{0,2,0} \to 1, c_{0,2,1} \to (1 - \alpha_2), c_{0,2,2} \to -\sqrt{\alpha_1\alpha_2},$
$c_{1,2,0} \to -1, c_{1,2,1} \to -1, c_{1,2,2} \to 0\};$

{lie$[X_1, X_0]$**, lie**$[X_2, X_0]$**, lie**$[X_2, X_1]$**}/.Subs**
//Simplify;
%//Simplify[#, $\alpha_1 \geq 0 \&\& \alpha_2 \geq 0$**]&//Column**

{$\varrho_0, \varrho_1, \varrho_2$**}/.Subs//Simplify[#,** $\alpha_1 \geq 0 \&\& \alpha_2 \geq 0$**]&**
{$\kappa, \chi_1, \vartheta$**}/.Subs//FullSimplify[#,** $\alpha_1 \geq 0 \&\& \alpha_2 \geq 0$**]&**

CheckAlgU[Subs, $\{\alpha_1 \geq 0, \alpha_2 \geq 0\}, \{\alpha_1 \neq 0 || \alpha_2 \neq 0\}$**]**

H_3

On H_3, we have a single equivalence class.

Subs = $\{c_{1,2,1} \to -1, c_{1,2,2} \to 0, c_{0,2,0} \to 1, c_{0,2,2} \to 0, c_{0,1,1} \to 0, c_{0,1,2} \to 0, c_{0,2,1} \to 1\}$;

{lie$[X_1, X_0]$**, lie**$[X_2, X_0]$**, lie**$[X_2, X_1]$**}/.Subs**
//Simplify;
%//Column

{$\kappa, \chi_1, \chi_2, \vartheta$**}/.Subs//Simplify**
{$\varrho_0, \varrho_1, \varrho_2$**}/.Subs//Simplify**

CheckAlgU[$\{c_{1,2,1} \to -1, c_{1,2,2} \to 0, c_{0,2,0} \to 1, c_{0,2,2} \to 0, c_{0,1,1} \to 0, c_{0,1,2} \to 0, c_{0,2,1} \to 1\}, \{\}, \{\}$**]**

$\widetilde{\mathsf{SE}}(2)$

On $\widetilde{\mathsf{SE}}(2)$, we have a two-parameter family of equivalence classes.

Subs = $\{c_{0,1,0} \to 0, c_{0,1,1} \to \sqrt{\alpha_1\alpha_2}, c_{0,1,2} \to -\alpha_1, c_{0,2,0} \to 1, c_{0,2,1} \to (1+\alpha_2), c_{0,2,2} \to -\sqrt{\alpha_1\alpha_2}, c_{1,2,0} \to -1, c_{1,2,1} \to -1, c_{1,2,2} \to 0\}$;

{lie$[X_1, X_0]$**, lie**$[X_2, X_0]$**, lie**$[X_2, X_1]$**}/.Subs**
//Simplify;
%//Simplify[#, $\alpha_1 \geq 0 \&\& \alpha_2 \geq 0$**]&//Column**

{$\kappa, \chi_1, \chi_2, \vartheta$**}/.Subs**
//FullSimplify[#, $\alpha_1 \geq 0 \&\& \alpha_2 \geq 0$**]&**
{$\varrho_0, \varrho_1, \varrho_2$**}/.Subs**
//FullSimplify[#, $\alpha_1 \geq 0 \&\& \alpha_2 \geq 0$**]&**

CheckAlgU[Subs, $\{\alpha_1 \geq 0 \&\& \alpha_2 \geq 0\}, \{\alpha_1 \neq 0 || \alpha_2 \neq 0\}$**]**

$\mathsf{SU}(2)$

On $\mathsf{SU}(2)$, we have a three-parameter family of equivalence classes.

Subs = $\{c_{0,1,0} \to 0, c_{0,1,1} \to \delta, c_{0,1,2} \to -\alpha_1, c_{0,2,0} \to 1, c_{0,2,1} \to 1+\alpha_2, c_{0,2,2} \to -\delta, c_{1,2,0} \to -1, c_{1,2,1} \to -1, c_{1,2,2} \to 0\}$;

{lie$[X_1, X_0]$**, lie**$[X_2, X_0]$**, lie**$[X_2, X_1]$**}/.Subs**
//Simplify;
%//Column

{m1, m2, m3}/.Subs
//FullSimplify[#, $\delta \geq 0 \&\& \alpha_1 > 0 \&\& \alpha_2 > 0$
$\&\& \delta^2 - \alpha_1\alpha_2 < 0$**]&**
{m1 < 0**, m2** > 0**, m3** < 0**}/.Subs**
//FullSimplify[#, $\delta \geq 0 \&\& \alpha_1 > 0 \&\& \alpha_2 > 0$
$\&\& \delta^2 - \alpha_1\alpha_2 < 0$**]&**

{$\kappa, \chi_1, \chi_2, \vartheta$**}/.Subs**
//FullSimplify[#, $\delta \geq 0 \&\& \alpha_1 > 0 \&\& \alpha_2 > 0$
$\&\& \delta^2 - \alpha_1\alpha_2 < 0$**]&**
{$\varrho_0, \varrho_1, \varrho_2$**}/.Subs**
//FullSimplify[#, $\delta \geq 0 \&\& \alpha_1 > 0 \&\& \alpha_2 > 0$
$\&\& \delta^2 - \alpha_1\alpha_2 < 0$**]&**

CheckAlgU[Subs, $\{\delta \geq 0, \alpha_1 > 0, \alpha_2 > 0\}, \{\delta^2 - \alpha_1\alpha_2 < 0\}$**]**

$\widetilde{\mathsf{SL}}(2, \mathbb{R})_{ell}$

On $\widetilde{\mathsf{SL}}(2, \mathbb{R})_{ell}$, we have a three-parameter family of equivalence classes.

Subs = $\{c_{0,1,0} \to 0, c_{0,1,1} \to \delta, c_{0,1,2} \to \alpha_1, c_{0,2,0} \to 1, c_{0,2,1} \to 1-\alpha_2, c_{0,2,2} \to -\delta, c_{1,2,0} \to -1, c_{1,2,1} \to -1, c_{1,2,2} \to 0\}$;

{lie$[X_1, X_0]$**, lie**$[X_2, X_0]$**, lie**$[X_2, X_1]$**}/.Subs**
//Simplify;
%//Column

{$\kappa, \chi_1, \chi_2, \vartheta$**}/.Subs/.**$\alpha_2 \to 1 + \alpha_1$
//FullSimplify[#, $\delta \geq 0 \&\& \alpha_1 > 0 \&\& \alpha_2 > 0$
$\&\& \delta^2 - \alpha_1\alpha_2 < 0$**]&**
{$\varrho_0, \varrho_1, \varrho_2$**}/.Subs**
//FullSimplify[#, $\delta \geq 0 \&\& \alpha_1 > 0 \&\& \alpha_2 > 0$
$\&\& \delta^2 - \alpha_1\alpha_2 < 0$**]&**

CheckAlgU[Subs, $\{\delta \geq 0, \alpha_1 > 0, \alpha_2 > 0\}, \{\delta^2 - \alpha_1\alpha_2 < 0\}$**]**

$\widetilde{\mathsf{SL}}(2, \mathbb{R})_{hyp}$

On $\widetilde{\mathsf{SL}}(2, \mathbb{R})_{hyp}$, we have a three-parameter family of equivalence classes.

Subs = $\{c_{0,1,0} \to 0, c_{0,1,1} \to \delta, c_{0,1,2} \to \gamma_1, c_{0,2,0} \to 1, c_{0,2,1} \to 1-\gamma_2, c_{0,2,2} \to -\delta, c_{1,2,0} \to -1, c_{1,2,1} \to -1, c_{1,2,2} \to 0\}$;

{lie$[X_1, X_0]$**, lie**$[X_2, X_0]$**, lie**$[X_2, X_1]$**}/.Subs**
//Simplify;
%//Column

{κ, χ_1, χ_2**}/.Subs/.**$\gamma_2 \to \gamma_1 + 1$
//FullSimplify[#, $\delta \geq 0 \&\& \{\gamma_1, \gamma_2\} \in$ **Reals**
$\&\& \delta^2 - \gamma_1\gamma_2 > 0$**]&**
{$\varrho_0, \varrho_1, \varrho_2$**}/.Subs**
//FullSimplify[#, $\delta \geq 0 \&\& \{\gamma_1, \gamma_2\} \in$ **Reals**
$\&\& \delta^2 - \gamma_1\gamma_2 > 0$**]&**

CheckAlgU[Subs, $\{\delta \geq 0, \{\gamma_1, \gamma_2\} \in$ **Reals**$\}, \{\delta^2 - \gamma_1\gamma_2 > 0\}$**]**

1.2 Non-unimodular case: $c_{01}^1 + c_{02}^2 \neq 0$ or $c_{02}^0 + c_{12}^1 \neq 0$ or $c_{12}^2 \neq 0$

{lie$[X_1, X_0]$**, lie**$[X_2, X_0]$**, lie**$[X_2, X_1]$**}**
//Simplify//Column

$-X_1 c_{0,1,1} - X_2 c_{0,1,2}$
$-X_0 c_{0,2,0} - X_1 c_{0,2,1} - X_2 c_{0,2,2}$
$X_0 - X_1 c_{1,2,1} - X_2 c_{1,2,2}$

{lie$[\mu X_1, \mu^2 X_0]$**, lie**$[\mu X_2, \mu^2 X_0]$**, lie**$[\mu X_2, \mu X_1]$**}**
/.A_[_] $\to 0$
/.Thread[$\{X_0, X_1, X_2\} \to \{\frac{1}{\mu^2} X_0, \frac{1}{\mu} X_1, \frac{1}{\mu} X_2\}$**]**

```
    //Simplify;
%/.μ → 1/c_{0,2,0}//Collect[#, {X_0, X_1, X_2}]&//Column
```

$$\begin{array}{l} -\frac{X_1 c_{0,1,1}}{c_{0,2,0}^2} - \frac{X_2 c_{0,1,2}}{c_{0,2,0}^2} \\ -X_0 - \frac{X_1 c_{0,2,1}}{c_{0,2,0}^2} - \frac{X_2 c_{0,2,2}}{c_{0,2,0}^2} \\ X_0 - \frac{X_1 c_{1,2,1}}{c_{0,2,0}} - \frac{X_2 c_{1,2,2}}{c_{0,2,0}} \end{array}$$

```
{lie[X_1, X_0], lie[X_2, X_0], lie[X_2, X_1]}/.{c_{0,2,0} → 1}
    //Simplify//Column
```

$$\begin{array}{l} -X_1 c_{0,1,1} - X_2 c_{0,1,2} \\ -X_0 - X_1 c_{0,2,1} - X_2 c_{0,2,2} \\ X_0 - X_1 c_{1,2,1} - X_2 c_{1,2,2} \end{array}$$

```
And@@{-c_{0,1,1} - c_{0,2,2} - c_{0,2,0}c_{1,2,2} == 0,
-c_{0,1,1}c_{0,2,0} + c_{0,2,2}c_{1,2,1} - c_{0,2,1}c_{1,2,2} == 0,
c_{0,1,2}(-c_{0,2,0} - c_{1,2,1}) + c_{0,1,1}c_{1,2,2} == 0}
    /.{c_{0,2,0} → 1}//FullSimplify
And@@{-c_{0,1,1} - c_{0,2,2} - c_{0,2,0}c_{1,2,2} == 0,
-c_{0,1,1}c_{0,2,0} + c_{0,2,2}c_{1,2,1} - c_{0,2,1}c_{1,2,2} == 0,
c_{0,1,2}(-c_{0,2,0} - c_{1,2,1}) + c_{0,1,1}c_{1,2,2} == 0}
    /.{c_{0,2,0} → 1}/.{c_{0,1,1} → -c_{0,2,2} - c_{1,2,2}}
    //FullSimplify
```

$$\begin{array}{l} c_{0,1,1} + c_{0,2,2} + c_{1,2,2} == 0\&\& \\ c_{0,1,1} + c_{0,2,1}c_{1,2,2} == c_{0,2,2}c_{1,2,1}\&\& \\ c_{0,1,2}(1 + c_{1,2,1}) == c_{0,1,1}c_{1,2,2} \end{array}$$

$$\begin{array}{l} c_{0,2,2}(1 + c_{1,2,1}) == (c_{0,2,1} - 1)c_{1,2,2}\&\& \\ c_{0,1,2}(1 + c_{1,2,1}) + c_{1,2,2}(c_{0,2,2} + c_{1,2,2}) == 0 \end{array}$$

```
Or@@Thread[{c_{0,1,1} + c_{0,2,2}, c_{0,2,0} + c_{1,2,1}, c_{1,2,2}} ≠ 0]
  /.{c_{0,2,0} → 1}/.{c_{0,1,1} → -c_{0,2,2} - c_{1,2,2}}
    //FullSimplify
```

$$1 + c_{1,2,1} \neq 0 \| c_{1,2,2} \neq 0$$

Since $c_{10}^1 \geq 0$, we have:

```
-c_{0,2,2} - c_{1,2,2} ≥ 0//Simplify
```

$$c_{0,2,2} + c_{1,2,2} \leq 0$$

CheckAlg routine

Checks algebraic properties of the algebra generated by the canonical frame (using the eigenvalues of the adjoint operator and also using the Killing form) in order to distinguish the different three-dimensional (non-unimodular) Lie algebras. Prints out messages indicating if the given algebra has the various properties. Some of the checks are done in different ways, since sometimes one way will work and another approach will time out; hence there are some repeated checks.

```
CheckAlgN[Subs_, Assumps_, Assumps2_]:=Module[{
    adeigs, uni, compsolv, affrr, eigseq},

adeigs = Eigenvalues[ad[a_0, a_1, a_2]/.Subs]
    //FullSimplify[#, Join[Assumps, Assumps2]]&;

Print[adeigs];

uni = TimeConstrained[Reduce[
  ForAll[{a_0, a_1, a_2}, {a_0, a_1, a_2} ∈ Reals,
    Tr[ad[{a_0, a_1, a_2}]] == 0]&&And@@Assumps,
    {c_{0,1,1}, c_{0,1,2}, c_{0,2,0}, c_{0,2,1}, c_{0,2,2}, c_{1,2,1}, c_{1,2,2}}, Reals]
      //FullSimplify[#, Join[Assumps, Assumps2]]&,
 5, "Timeout"];
Print["Unimodular?\n",
    If[uni, "Yes, ", "No, ", "Not sure, "], uni];

compsolv = TimeConstrained[Reduce[
  ForAll[{a_0, a_1, a_2}, {a_0, a_1, a_2} ∈ Reals,
    Thread[adeigs ∈ Reals]]&&And@@Assumps,
    {c_{0,1,1}, c_{0,1,2}, c_{0,2,0}, c_{0,2,1}, c_{0,2,2}, c_{1,2,1}, c_{1,2,2}}]
      //FullSimplify[#, Join[Assumps, Assumps2]]&,
 5, "Timeout"];
Print["Completely solvable?\n",
    If[compsolv, "Yes, ", "No, ", "Not sure, "],
      compsolv];

compsolv = TimeConstrained[Reduce[
  ForAll[{a_0, a_1, a_2}, {a_0, a_1, a_2} ∈ Reals,
    And@@Thread[Im[adeigs] == 0]]&&And@@Assumps,
    {c_{0,1,1}, c_{0,1,2}, c_{0,2,0}, c_{0,2,1}, c_{0,2,2}, c_{1,2,1}, c_{1,2,2}}]
      //FullSimplify[#, Join[Assumps, Assumps2]]&,
 5, "Timeout"];
Print["Completely solvable?\n",
    If[compsolv, "Yes, ", "No, ", "Not sure, "],
      compsolv];

compsolv = TimeConstrained[Reduce[
  !Exists[{a_0, a_1, a_2}, {a_0, a_1, a_2} ∈ Reals,
    Or@@Thread[Im[adeigs] ≠ 0]]&&And@@Assumps,
    {c_{0,1,1}, c_{0,1,2}, c_{0,2,0}, c_{0,2,1}, c_{0,2,2}, c_{1,2,1}, c_{1,2,2}}]
      //FullSimplify[#, Join[Assumps, Assumps2]]&,
 5, "Timeout"];
Print["Completely solvable?\n",
    If[compsolv, "Yes, ", "No, ", "Not sure, "],
      compsolv];

affrr = TimeConstrained[Resolve[
  (ForAll[{a_0, a_1, a_2}, {a_0, a_1, a_2} ∈ Reals,
    adeigs[[2]] == 0]
    &&
   Exists[{a_0, a_1, a_2}, {a_0, a_1, a_2} ∈ Reals,
    adeigs[[3]] ≠ 0])
 || (ForAll[{a_0, a_1, a_2}, {a_0, a_1, a_2} ∈ Reals,
    adeigs[[3]] == 0]
    &&
   Exists[{a_0, a_1, a_2}, {a_0, a_1, a_2} ∈ Reals,
    adeigs[[2]] ≠ 0])
 && And@@Assumps,
    {c_{0,1,1}, c_{0,1,2}, c_{0,2,0}, c_{0,2,1}, c_{0,2,2}, c_{1,2,1}, c_{1,2,2}}]
      //FullSimplify[#, Join[Assumps, Assumps2]]&,
 5, "Timeout"];
Print["Aff(R)_0 x R?\n",
    If[affrr, "Yes, ", "No, ", "Not sure, "], affrr];

eigseq = TimeConstrained[Reduce[
  ForAll[{a_0, a_1, a_2}, {a_0, a_1, a_2} ∈ Reals,
    adeigs[[2]] == adeigs[[3]]]&&And@@Assumps,
    {c_{0,1,1}, c_{0,1,2}, c_{0,2,0}, c_{0,2,1}, c_{0,2,2}, c_{1,2,1}, c_{1,2,2}}]
      //FullSimplify[#, Join[Assumps, Assumps2]]&,
 5, "Timeout"];
Print["Eigenvalues equal?\n",
    If[eigseq, "Yes, ", "No, ", "Not sure, "], eigseq];

eigseq = TimeConstrained[Reduce[
  !Exists[{a_0, a_1, a_2}, {a_0, a_1, a_2} ∈ Reals,
    adeigs[[2]] ≠ adeigs[[3]]]&&And@@Assumps,
    {c_{0,1,1}, c_{0,1,2}, c_{0,2,0}, c_{0,2,1}, c_{0,2,2}, c_{1,2,1}, c_{1,2,2}}]
      //FullSimplify[#, Join[Assumps, Assumps2]]&,
```

```
5, "Timeout"];
Print["Eigenvalues equal?\n",
    If[eigseq, "Yes, ", "No, ", "Not sure, "], eigseq];
];
```

1.2.1 $c^1_{12} = -1$

$$\begin{aligned}
&\texttt{Or@@Thread}[\{c_{0,1,1}+c_{0,2,2}, c_{0,2,0}+c_{1,2,1}, c_{1,2,2}\} \neq 0]\\
&\quad /.\, c_{0,2,0} \to 1/.\{c_{0,1,1} \to -c_{1,2,2}-c_{0,2,2}\}\\
&\quad /.\, c_{1,2,1} \to -1\\
&\quad \texttt{//FullSimplify}
\end{aligned}$$

$c_{1,2,2} \neq 0$

This implies that $c^2_{12} \neq 0$.

$$\begin{aligned}
&c_{0,1,1}+c_{0,2,2}+c_{1,2,2} == 0\\
&\&\&c_{0,1,1}+c_{0,2,1}c_{1,2,2} == c_{0,2,2}c_{1,2,1}\\
&\&\&c_{0,1,2}(1+c_{1,2,1}) == c_{0,1,1}c_{1,2,2}\\
&\quad /.\{c_{0,2,0} \to 1\}/.\{c_{0,1,1} \to -c_{1,2,2}-c_{0,2,2}\}\\
&\quad /.\, c_{1,2,1} \to -1\\
&\quad \texttt{//FullSimplify}[\#, c_{1,2,2} \neq 0]\&
\end{aligned}$$

$c_{0,2,1} == 1\&\&c_{0,2,2}+c_{1,2,2} == 0$

This implies that $c^1_{10} = 0$.

$$\begin{aligned}
&\texttt{CheckAlgN}[\{c_{0,2,0} \to 1, c_{0,1,1} \to 0, c_{1,2,1} \to -1, c_{0,2,1} \to 1,\\
&c_{0,2,2} \to -c_{1,2,2}\}, \{c_{1,2,2} \neq 0, c_{\texttt{u\_,v\_,w\_}} \in \texttt{Reals}\}, \{\}]
\end{aligned}$$

$$\begin{aligned}
&\texttt{Resolve}[\texttt{ForAll}[\{a_0,a_1,a_2\}, \{a_0,a_1,a_2\} \in \texttt{Reals},\\
&\quad -\tfrac{1}{2}(a_0+a_1)(\sqrt{4c_{0,1,2}+c^2_{1,2,2}}-c_{1,2,2}) == 0\\
&\quad \|\ \tfrac{1}{2}(a_0+a_1)(\sqrt{4c_{0,1,2}+c^2_{1,2,2}}+c_{1,2,2}) == 0]]\\
&\qquad \texttt{//FullSimplify}[\#, c_{1,2,2} \neq 0\&\&c_{1,2,2} \in \texttt{Reals}]\&\\
&\texttt{Resolve}[\texttt{ForAll}[\{a_0,a_1,a_2\}, \{a_0,a_1,a_2\} \in \texttt{Reals},\\
&\qquad -\tfrac{1}{2}(a_0+a_1)(\sqrt{4c_{0,1,2}+c^2_{1,2,2}}-c_{1,2,2})\\
&\quad == \tfrac{1}{2}(a_0+a_1)(\sqrt{4c_{0,1,2}+c^2_{1,2,2}}+c_{1,2,2})]]\\
&\qquad \texttt{//FullSimplify}[\#, c_{1,2,2} \neq 0\&\&c_{1,2,2} \in \texttt{Reals}]\&
\end{aligned}$$

$c_{0,1,2} == 0$

$c_{1,2,2} == -2i\sqrt{c_{0,1,2}}\|c_{1,2,2} == 2i\sqrt{c_{0,1,2}}$

Since $c^1_{10} = 0$, we may assume $c^2_{21} < 0$ (by changing the frame if necessary).

1.2.1.1 $(c^2_{12})^2 + 4c^2_{01} > 0$ and $c^2_{01} \neq 0$

In this case, we get a one-parameter family of equivalence classes on $\mathsf{G}^h_{3.4}$.

$$\begin{aligned}
&\{\texttt{lie}[X_1,X_0], \texttt{lie}[X_2,X_0], \texttt{lie}[X_2,X_1]\}\\
&\quad /.\{c_{0,2,0} \to 1, c_{0,1,1} \to 0, c_{1,2,1} \to -1, c_{0,2,1} \to 1,\\
&c_{0,2,2} \to -c_{1,2,2}\}\\
&\quad \texttt{//Simplify};\\
&\%/.\,c_{1,2,2} \to \alpha/.\,c_{0,1,2} \to \beta/.\,\beta \to -\tfrac{(h^2-1)\alpha^2}{4h^2}\texttt{//}\\
&\texttt{Column}
\end{aligned}$$

$$\begin{aligned}
&\tfrac{(h^2-1)\alpha^2 X_2}{4h^2}\\
&-X_0 - X_1 + \alpha X_2\\
&X_0 + X_1 - \alpha X_2
\end{aligned}$$

$$\begin{aligned}
&\texttt{Assumps} = \{c_{1,2,2} > 0, c_{0,1,2} \neq 0, c_{\texttt{u\_,v\_,w\_}} \in \texttt{Reals}\};\\
&\texttt{Assumps2} = \texttt{Join}[\texttt{Assumps}, \{4c_{0,1,2}+c^2_{1,2,2} > 0\}];\\
&\texttt{Reduce}[c_{1,2,2} > 0\&\&\texttt{And@@Assumps}]\\
&\quad \texttt{//FullSimplify}[\#, \texttt{Assumps2}]\&\\
&\texttt{Reduce}[c_{0,1,2} \neq 0\&\&\texttt{And@@Assumps}]\\
&\quad \texttt{//FullSimplify}[\#, \texttt{Assumps2}]\&
\end{aligned}$$

```
True

True
```

$$\begin{aligned}
&\texttt{CheckAlgN}[\{c_{0,1,0} \to 0, c_{0,1,1} \to 0, c_{0,1,2} \to \beta, c_{0,2,0} \to 1,\\
&c_{0,2,1} \to 1, c_{0,2,2} \to -\alpha, c_{1,2,0} \to -1, c_{1,2,1} \to -1,\\
&c_{1,2,2} \to \alpha\}/.\,\beta \to -\tfrac{(h^2-1)\alpha^2}{4h^2}, \{\alpha > 0, \beta \neq 0, \beta \in \texttt{Reals},\\
&h > 0, h \neq 1\}, \{\alpha^2 + 4\beta > 0\}]
\end{aligned}$$

$$\begin{aligned}
&\texttt{CheckAlgN}[\{c_{0,2,0} \to 1, c_{0,1,1} \to 0, c_{1,2,1} \to -1, c_{0,2,1} \to 1,\\
&c_{0,2,2} \to -c_{1,2,2}\}, \{c_{1,2,2} > 0, c_{0,1,2} \neq 0, c_{\texttt{u\_,v\_,w\_}} \in \texttt{Reals}\},\\
&\{4c_{0,1,2}+c^2_{1,2,2} > 0, c_{0,1,2} < 0 \Rightarrow\\
&\quad (c_{1,2,2} > 2\sqrt{-c_{0,1,2}}\|c_{1,2,2} < -2\sqrt{-c_{0,1,2}})\}]
\end{aligned}$$

$\{0, -\frac{1}{2}(a_0-a_1)(c_{1,2,2}+\sqrt{4c_{0,1,2}+c^2_{1,2,2}}), \frac{1}{2}(a_0-a_1)(-c_{1,2,2}+\sqrt{4c_{0,1,2}+c^2_{1,2,2}})\}$

```
Unimodular?
No, False
Completely solvable?
Yes, True
Completely solvable?
Yes, True
Completely solvable?
Yes, True
Aff(R)_0 x R?
Not sure, Timeout
Eigenvalues equal?
No, False
Eigenvalues equal?
No, False
```

Find h:

$$\begin{aligned}
&\texttt{Subs} = \{c_{0,1,0} \to 0, c_{0,1,1} \to 0, c_{0,1,2} \to \beta, c_{0,2,0} \to 1,\\
&c_{0,2,1} \to 1, c_{0,2,2} \to -\alpha, c_{1,2,0} \to -1, c_{1,2,1} \to -1,\\
&c_{1,2,2} \to \alpha\};
\end{aligned}$$

$$\begin{aligned}
&\{\texttt{a0}, \texttt{a1}, \texttt{a2}\} = \texttt{Eigenvalues}[\texttt{ad}[a_0,a_1,a_2]]/.\texttt{Subs}\\
&\quad \texttt{//FullSimplify}[\#, \alpha > 0\&\&\beta \neq 0\\
&\qquad \&\&\{a_{i\_}, \beta\} \in \texttt{Reals}]\&;\\
&(\tfrac{\texttt{a1+a2}}{\texttt{a1-a2}})^2\\
&\quad \texttt{//FullSimplify}[\#, \alpha > 0\&\&\beta \neq 0\\
&\qquad \&\&\{a_{i\_}, \beta\} \in \texttt{Reals}]\&\\
&\texttt{Solve}[\% == h^2, h]\\
&\quad \texttt{//FullSimplify}[\#, \alpha > 0\&\&\beta \neq 0\\
&\qquad \&\&\{a_{i\_}, \beta\} \in \texttt{Reals}]\&
\end{aligned}$$

$\frac{\alpha^2}{\alpha^2+4\beta}$

$\{\{h \to -\frac{\alpha}{\sqrt{\alpha^2+4\beta}}\}, \{h \to \frac{\alpha}{\sqrt{\alpha^2+4\beta}}\}\}$

1.2.1.2 $(c^2_{12})^2 + 4c^2_{01} > 0$ and $c^2_{01} = 0$

In this case, we get a one-parameter family of equivalence classes on $\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}$.

$$\begin{aligned}
&\{\texttt{lie}[X_1,X_0], \texttt{lie}[X_2,X_0], \texttt{lie}[X_2,X_1]\}\\
&\quad /.\{c_{0,2,0} \to 1, c_{0,1,1} \to 0, c_{1,2,1} \to -1, c_{0,2,1} \to 1,\\
&c_{0,2,2} \to -c_{1,2,2}, c_{0,1,2} \to 0\}\\
&\quad \texttt{//Simplify};\\
&\%/.\,c_{1,2,2} \to \alpha\texttt{//Column}
\end{aligned}$$

$$\begin{aligned}
&0\\
&-X_0 - X_1 + \alpha X_2\\
&X_0 + X_1 - \alpha X_2
\end{aligned}$$

$\mathbf{CheckAlgN}[\{c_{0,2,0}\to 1, c_{0,1,1}\to 0, c_{1,2,1}\to -1, c_{0,2,1}\to 1,$
$c_{0,2,2}\to -c_{1,2,2}, c_{0,1,2}\to 0\},$
$\{c_{1,2,2}>0, c_{\mathtt{u\_},\mathtt{v\_},\mathtt{w\_}}\in \mathbf{Reals}\}, \{\}]$

$\{0, 0, (-a_0+a_1)c_{1,2,2}\}$

```
Unimodular?
No, False
Completely solvable?
Yes, True
Completely solvable?
Yes, True
Completely solvable?
Yes, True
Aff(R)_0 x R?
Yes, True
Eigenvalues equal?
No, False
Eigenvalues equal?
No, False
```

1.2.1.3 $(c_{12}^2)^2 + 4c_{01}^2 = 0$

In this case, we get a one-parameter family of equivalence classes on $\mathsf{G}_{3.2}$.

$\{\mathtt{lie}[X_1, X_0], \mathtt{lie}[X_2, X_0], \mathtt{lie}[X_2, X_1]\}$
$\quad /.\{c_{0,2,0}\to 1, c_{0,1,1}\to 0, c_{1,2,1}\to -1, c_{0,2,1}\to 1,$
$c_{0,2,2}\to -c_{1,2,2}, c_{0,1,2}\to -\frac{1}{4}(c_{1,2,2})^2\}$
$\quad \mathbf{//Simplify;}$
$\%/.c_{1,2,2}\to \alpha \mathbf{//Column}$

$\frac{\alpha^2 X_2}{4}$
$-X_0 - X_1 + \alpha X_2$
$X_0 + X_1 - \alpha X_2$

$\mathbf{CheckAlgN}[\{c_{0,1,0}\to 0, c_{0,1,1}\to 0, c_{0,1,2}\to -\frac{1}{4}\alpha^2,$
$c_{0,2,0}\to 1, c_{0,2,1}\to 1, c_{0,2,2}\to -\alpha, c_{1,2,0}\to -1,$
$c_{1,2,1}\to -1, c_{1,2,2}\to \alpha\}, \{\alpha>0\}, \{\}]$

$\mathbf{CheckAlgN}[\{c_{0,2,0}\to 1, c_{0,1,1}\to 0, c_{1,2,1}\to -1, c_{0,2,1}\to 1,$
$c_{0,2,2}\to -c_{1,2,2}, c_{0,1,2}\to -\frac{1}{4}(c_{1,2,2})^2\},$
$\{c_{1,2,2}>0, c_{\mathtt{u\_},\mathtt{v\_},\mathtt{w\_}}\in \mathbf{Reals}\}, \{\}]$

$\{0, \frac{1}{2}(-a_0+a_1)c_{1,2,2}, \frac{1}{2}(-a_0+a_1)c_{1,2,2}\}$

```
Unimodular?
No, False
Completely solvable?
Yes, True
Completely solvable?
Yes, True
Completely solvable?
Yes, True
Aff(R)_0 x R?
No, False
Eigenvalues equal?
Yes, True
Eigenvalues equal?
Yes, True
```

1.2.1.4 $(c_{12}^2)^2 + 4c_{01}^2 < 0$

In this case we get a one-parameter family of equivalence classes on $\mathsf{G}_{3.5}^h$.

$\{\mathtt{lie}[X_1, X_0], \mathtt{lie}[X_2, X_0], \mathtt{lie}[X_2, X_1]\}$
$\quad /.\{c_{0,2,0}\to 1, c_{0,1,1}\to 0, c_{1,2,1}\to -1,$
$c_{0,2,1}\to 1, c_{0,2,2}\to -c_{1,2,2}\}$
$\quad \mathbf{//Simplify;}$
$\%/.c_{1,2,2}\to \alpha_1/.c_{0,1,2}\to -\alpha_2$
$\quad /.\alpha_2\to \frac{(1+h^2)\alpha_1^2}{4h^2}/.\alpha_1\to \alpha$
$\quad \mathbf{//Column}$

$\frac{(1+h^2)\alpha^2 X_2}{4h^2}$
$-X_0 - X_1 + \alpha X_2$
$X_0 + X_1 - \alpha X_2$

$\mathbf{Assumps} = \{c_{1,2,2}>0, c_{0,1,2}\neq 0, c_{\mathtt{u\_},\mathtt{v\_},\mathtt{w\_}}\in \mathbf{Reals}\};$
$\mathbf{Assumps2} = \mathbf{Join}[\mathbf{Assumps}, \{4c_{0,1,2}+c_{1,2,2}^2<0\}];$
$\mathbf{Reduce}[c_{1,2,2}>0\&\&\mathbf{And@@Assumps}]$
$\quad \mathbf{//FullSimplify}[\#, \mathbf{Assumps2}]\&$
$\mathbf{Reduce}[-c_{0,1,2}>0\&\&\mathbf{And@@Assumps}]$
$\quad \mathbf{//FullSimplify}[\#, \mathbf{Assumps2}]\&$

```
True
```

```
True
```

$\mathbf{CheckAlgN}[\{c_{0,1,0}\to 0, c_{0,1,1}\to 0, c_{0,1,2}\to -\frac{(1+h^2)\alpha^2}{4h^2},$
$c_{0,2,0}\to 1, c_{0,2,1}\to 1, c_{0,2,2}\to -\alpha, c_{1,2,0}\to -1,$
$c_{1,2,1}\to -1, c_{1,2,2}\to \alpha\}, \{\alpha>0, h>0\}, \{\}]$

$\mathbf{CheckAlgN}[\{c_{0,2,0}\to 1, c_{0,1,1}\to 0, c_{1,2,1}\to -1, c_{0,2,1}\to 1,$
$c_{0,2,2}\to -c_{1,2,2}\}, \{c_{1,2,2}>0, c_{0,1,2}\neq 0, c_{\mathtt{u\_},\mathtt{v\_},\mathtt{w\_}}\in \mathbf{Reals}\},$
$\{4c_{0,1,2}+c_{1,2,2}^2<0, -2\sqrt{-c_{0,1,2}}<c_{1,2,2}<2\sqrt{-c_{0,1,2}}\}]$

$\{0, -\frac{1}{2}(a_0-a_1)(c_{1,2,2}+\sqrt{4c_{0,1,2}+c_{1,2,2}^2}),$
$\frac{1}{2}(a_0-a_1)(-c_{1,2,2}+\sqrt{4c_{0,1,2}+c_{1,2,2}^2})\}$

```
Unimodular?
No, False
Completely solvable?
No, False
Completely solvable?
No, False
Completely solvable?
No, False
Aff(R)_0 x R?
No, False
Eigenvalues equal?
No, False
Eigenvalues equal?
No, False
```

Find h:

$\mathbf{Subs} = \{c_{0,1,0}\to 0, c_{0,1,1}\to 0, c_{0,1,2}\to -\alpha_2, c_{0,2,0}\to 1,$
$c_{0,2,1}\to 1, c_{0,2,2}\to -\alpha_1, c_{1,2,0}\to -1, c_{1,2,1}\to -1,$
$c_{1,2,2}\to \alpha_1\};$

$\{\mathbf{a0}, \mathbf{a1}, \mathbf{a2}\} = \mathbf{Eigenvalues}[\mathbf{ad}[a_0, a_1, a_2]]/.\mathbf{Subs}$
$\quad \mathbf{//FullSimplify}[\#, \alpha>0\&\&\beta\neq 0$
$\qquad \&\&\{a_{i\_}, \beta\}\in \mathbf{Reals}]\&;$
$(\frac{\mathbf{a1}+\mathbf{a2}}{\mathbf{a1}-\mathbf{a2}})^2$
$\quad \mathbf{//FullSimplify}[\#, \alpha>0\&\&\beta\neq 0$
$\qquad \&\&\{a_{i\_}, \beta\}\in \mathbf{Reals}]\&$
$\mathbf{Solve}[\% == -h^2, h]$
$\quad \mathbf{//FullSimplify}[\#, \alpha>0\&\&\beta\neq 0$
$\qquad \&\&\{a_{i\_}, \beta\}\in \mathbf{Reals}]\&$

$\frac{\alpha_1^2}{\alpha_1^2-4\alpha_2}$

$$\{\{h\to -\frac{\alpha_1}{\sqrt{-\alpha_1^2+4\alpha_2}}\},\{h\to \frac{\alpha_1}{\sqrt{-\alpha_1^2+4\alpha_2}}\}\}$$

For brevity, we suppress (most of) the output for the remainder of the (non-unimodular-case) classification code.

1.2.2 $c^1_{12} \neq -1$ and $c^2_{02} = 0$ and $c^2_{01} = 0$

$$\begin{array}{l}
\texttt{And@@}\{-c_{0,1,1}-c_{0,2,2}-c_{0,2,0}c_{1,2,2}==0,\\
-c_{0,1,1}c_{0,2,0}+c_{0,2,2}c_{1,2,1}-c_{0,2,1}c_{1,2,2}==0,\\
c_{0,1,2}(-c_{0,2,0}-c_{1,2,1})+c_{0,1,1}c_{1,2,2}==0\}\\
\quad /.\{c_{0,2,0}\to 1\}/.\{c_{0,1,1}\to -c_{0,2,2}-c_{1,2,2}\}\\
\quad /.c_{0,2,2}\to 0/.c_{0,1,2}\to 0\\
\quad \texttt{//FullSimplify}
\end{array}$$

$$c_{1,2,2}==0$$

$$\begin{array}{l}
\texttt{CheckAlgN}[\{c_{0,2,0}\to 1,c_{0,1,1}\to 0,c_{0,2,2}\to 0,c_{0,1,2}\to 0,\\
c_{1,2,2}\to 0\},\{\},\{c_{1,2,1}\neq -1,c_{\texttt{u\_,v\_,w\_}}\in \texttt{Reals}\}]
\end{array}$$

$$\begin{array}{l}
\texttt{ForAll}[\{a_0,a_1,a_2\},\{a_0,a_1,a_2\}\in \texttt{Reals},\\
\quad \frac{1}{2}(-\sqrt{a_2^2(-4c_{0,2,1}+(c_{1,2,1}-1)^2)}-a_2(1+c_{1,2,1}))\\
\qquad ==0\\
\ \ ||\ \frac{1}{2}(+\sqrt{a_2^2(-4c_{0,2,1}+(c_{1,2,1}-1)^2)}-a_2(1+c_{1,2,1}))\\
\qquad ==0]\\
\quad \texttt{//Resolve}\\
\quad \texttt{//FullSimplify}[\#,c_{1,2,1}\neq 1\&\&c_{\texttt{u\_,v\_,w\_}}\in \texttt{Reals}]\&\\
\texttt{ForAll}[\{a_0,a_1,a_2\},\{a_0,a_1,a_2\}\in \texttt{Reals},\\
\quad \frac{1}{2}(-\sqrt{a_2^2(-4c_{0,2,1}+(c_{1,2,1}-1)^2)}-a_2(1+c_{1,2,1}))\\
==\ \frac{1}{2}(+\sqrt{a_2^2(-4c_{0,2,1}+(c_{1,2,1}-1)^2)}-a_2(1+c_{1,2,1}))]\\
\quad \texttt{//Resolve}\\
\quad \texttt{//FullSimplify}[\#,c_{1,2,1}\neq 1\&\&c_{\texttt{u\_,v\_,w\_}}\in \texttt{Reals}]\&
\end{array}$$

$$c_{0,2,1}+c_{1,2,1}==0$$

$$2\sqrt{c_{0,2,1}}+c_{1,2,1}==1\|1+2\sqrt{c_{0,2,1}}==c_{1,2,1}$$

$$\texttt{Solve}[-4c_{0,2,1}+(1+c_{1,2,1})^2==0,c_{0,2,1}]$$

$$\{\{c_{0,2,1}\to \tfrac{1}{4}(1+c_{1,2,1})^2\}\}$$

$$\begin{array}{l}
\{\frac{1}{2}(-\sqrt{a_2^2(-4c_{0,2,1}+(c_{1,2,1}-1)^2)}-a_2(1+c_{1,2,1})),\\
\ \frac{1}{2}(+\sqrt{a_2^2(-4c_{0,2,1}+(c_{1,2,1}-1)^2)}-a_2(1+c_{1,2,1}))\}\\
\quad /.\{c_{0,2,1}\to -c_{1,2,1}\}\\
\quad \texttt{//FullSimplify}[\#,c_{1,2,1}\neq -1\&\&c_{\texttt{u\_,v\_,w\_}}\}\in \texttt{Reals}]\&\\
\{\frac{1}{2}(-\sqrt{a_2^2(-4c_{0,2,1}+(c_{1,2,1}-1)^2)}-a_2(1+c_{1,2,1})),\\
\ \frac{1}{2}(+\sqrt{a_2^2(-4c_{0,2,1}+(c_{1,2,1}-1)^2)}-a_2(1+c_{1,2,1}))\}\\
\quad /.\{c_{1,2,1}\to 1+\sigma 2\sqrt{c_{0,2,1}}\}\\
\quad \texttt{//FullSimplify}[\#,c_{1,2,1}\neq -1\\
\qquad \&\&\{a_{\texttt{i\_}},c_{\texttt{u\_,v\_,w\_}}\}\in \texttt{Reals}\&\&\sigma^2==1]\&
\end{array}$$

$$\{\tfrac{1}{2}(-\sqrt{a_2^2}|c_{1,2,1}+1|-a_2(c_{1,2,1}+1)),\tfrac{1}{2}(\sqrt{a_2^2}|c_{1,2,1}+1|-a_2(c_{1,2,1}+1))\}$$

$$\{-a_2(1+\sigma\sqrt{c_{0,2,1}}),-a_2(1+\sigma\sqrt{c_{0,2,1}})\}$$

Changing the frame can't help us:

$$\begin{array}{l}
\{c_{0,1,1},c_{2,0,2},c_{1,2,2}\}\\
\quad /.\{c_{0,2,0}\to 1,c_{0,1,1}\to 0,c_{0,2,2}\to 0,c_{0,1,2}\to 0,\\
c_{1,2,2}\to 0\}\\
\quad \texttt{//Simplify}
\end{array}$$

$$\{0,0,0\}$$

1.2.2.1 $c^1_{02} + c^1_{12} = 0$

In this case, we get a one-parameter family of equivalence classes on $\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}$.

$$\begin{array}{l}
\{\texttt{lie}[X_1,X_0],\texttt{lie}[X_2,X_0],\texttt{lie}[X_2,X_1]\}\\
\quad /.\{c_{0,2,0}\to 1,c_{0,1,1}\to 0,c_{0,2,2}\to 0,c_{0,1,2}\to 0,\\
c_{1,2,2}\to 0,c_{0,2,1}\to -c_{1,2,1}\}\\
\quad \texttt{//Simplify;}\\
\%/.c_{1,2,1}\to \beta-1\texttt{//Column}
\end{array}$$

$$\begin{array}{l}
\texttt{CheckAlgN}[\{c_{0,1,0}\to 0,c_{0,1,1}\to 0,c_{0,1,2}\to 0,c_{0,2,0}\to 1,\\
c_{0,2,1}\to 1-\beta,c_{0,2,2}\to 0,c_{1,2,0}\to -1,c_{1,2,1}\to \beta-1,\\
c_{1,2,2}\to 0\},\{\beta\neq 0,\beta\in \texttt{Reals}\},\{\}]
\end{array}$$

$$\begin{array}{l}
\texttt{CheckAlgN}[\{c_{0,2,0}\to 1,c_{0,1,1}\to 0,c_{0,2,2}\to 0,c_{0,1,2}\to 0,\\
c_{1,2,2}\to 0,c_{0,2,1}\to -c_{1,2,1}\},\{\},\{c_{1,2,1}\neq -1,\\
c_{\texttt{u\_,v\_,w\_}}\in \texttt{Reals}\}]
\end{array}$$

1.2.2.2 $c^1_{02}+c^1_{12} \neq 0$ and $(c^1_{12}-1)^2-4c^1_{02} > 0$

In this case, we get a one-parameter family of equivalence classes on $\mathrm{G}^h_{3.4}$.

$$\begin{array}{l}
\{\texttt{lie}[X_1,X_0],\texttt{lie}[X_2,X_0],\texttt{lie}[X_2,X_1]\}\\
\quad /.\{c_{0,2,0}\to 1,c_{0,1,1}\to 0,c_{0,2,2}\to 0,c_{0,1,2}\to 0,\\
c_{1,2,2}\to 0\}\\
\quad \texttt{//Simplify;}\\
\%/.c_{0,2,1}\to \gamma/.c_{1,2,1}\to \beta-1/.\gamma\to \frac{h^2(2-\beta)^2-\beta^2}{4h^2}\\
\quad \texttt{//Column}
\end{array}$$

$$\begin{array}{l}
\texttt{Assumps}=\{c_{1,2,1}\neq -1,c_{\texttt{u\_,v\_,w\_}}\in \texttt{Reals},\\
c_{0,2,1}+c_{1,2,1}\neq 0\};\\
\texttt{Assumps2}=\texttt{Join}[\texttt{Assumps},\{(c_{1,2,1}-1)^2-4c_{0,2,1}>0,\\
\quad c_{0,2,1}\geq 0\Rightarrow\\
\qquad (c_{1,2,1}-1>2\sqrt{c_{0,2,1}}||c_{1,2,1}-1<-2\sqrt{c_{0,2,1}})\}];\\
\texttt{Reduce}[c_{1,2,1}+1\neq 0\&\&\texttt{And@@Assumps}]\\
\quad \texttt{//FullSimplify}[\#,\texttt{Assumps2}]\&\\
\texttt{Reduce}[c_{0,2,1}\in \texttt{Reals}\&\&\texttt{And@@Assumps}]\\
\quad \texttt{//FullSimplify}[\#,\texttt{Assumps2}]\&\\
\texttt{Reduce}[c_{1,2,1}+c_{0,2,1}\neq 0\&\&\texttt{And@@Assumps}]\\
\quad \texttt{//FullSimplify}[\#,\texttt{Assumps2}]\&
\end{array}$$

$$\begin{array}{l}
\texttt{CheckAlgN}[\{c_{0,1,0}\to 0,c_{0,1,1}\to 0,c_{0,1,2}\to 0,c_{0,2,0}\to 1,\\
c_{0,2,1}\to \gamma,c_{0,2,2}\to 0,c_{1,2,0}\to -1,c_{1,2,1}\to \beta-1,\\
c_{1,2,2}\to 0\},\{\beta\neq 0,\{\beta,\gamma\}\in \texttt{Reals}\},\\
\{\beta+\gamma-1\neq 0,(\beta-2)^2-4\gamma>0\}]
\end{array}$$

$$\begin{array}{l}
\texttt{CheckAlgN}[\{c_{0,2,0}\to 1,c_{0,1,1}\to 0,c_{0,2,2}\to 0,c_{0,1,2}\to 0,\\
c_{1,2,2}\to 0\},\{\},\{c_{1,2,1}\neq -1,c_{\texttt{u\_,v\_,w\_}}\in \texttt{Reals},\\
c_{0,2,1}+c_{1,2,1}\neq 0,(c_{1,2,1}-1)^2-4c_{0,2,1}>0,\\
c_{0,2,1}\geq 0\Rightarrow\\
\quad (c_{1,2,1}-1>2\sqrt{c_{0,2,1}}||c_{1,2,1}-1<-2\sqrt{c_{0,2,1}})\}]
\end{array}$$

Find h:

$$\begin{array}{l}
\texttt{Subs}=\{c_{0,1,0}\to 0,c_{0,1,1}\to 0,c_{0,1,2}\to 0,c_{0,2,0}\to 1,\\
c_{0,2,1}\to \gamma,c_{0,2,2}\to 0,c_{1,2,0}\to -1,c_{1,2,1}\to \beta-1,\\
c_{1,2,2}\to 0\};
\end{array}$$

$$\begin{array}{l}
\{\mathbf{a0},\mathbf{a1},\mathbf{a2}\}=\texttt{Eigenvalues}[\mathrm{ad}[a_0,a_1,a_2]]/.\texttt{Subs}\\
\quad \texttt{//FullSimplify}[\#,\alpha>0\&\&\beta\neq 0\\
\qquad \&\&\{a_{\texttt{i\_}},\beta\}\in \texttt{Reals}]\&\\
(\frac{\mathbf{a1}+\mathbf{a2}}{\mathbf{a1}-\mathbf{a2}})^2\texttt{//FullSimplify}[\#,(\beta-2)^2-4\gamma>0\\
\quad \&\&\alpha>0\&\&\beta\neq 0\&\&\{a_{\texttt{i\_}},\beta\}\in \texttt{Reals}]\&\\
\texttt{Reduce}[\%==h^2,h]\\
\quad \texttt{//FullSimplify}[\#,(\beta-2)^2-4\gamma>0\&\&\alpha>0\\
\qquad \&\&\beta\neq 0\&\&\{a_{\texttt{i\_}},\beta\}\in \texttt{Reals}]\&
\end{array}$$

1.2.2.3 $c^1_{02}+c^1_{12} \neq 0$ and $(c^1_{12}-1)^2-4c^1_{02} = 0$

In this case, we get a one-parameter family of equivalence classes on $\mathrm{G}_{3.2}$.

$$\begin{array}{l}
\{\texttt{lie}[X_1,X_0],\texttt{lie}[X_2,X_0],\texttt{lie}[X_2,X_1]\}\\
\quad /.\{c_{0,2,0}\to 1,c_{0,1,1}\to 0,c_{0,2,2}\to 0,c_{0,1,2}\to 0,\\
c_{1,2,2}\to 0,c_{0,2,1}\to \frac{1}{4}(1-c_{1,2,1})^2\}\texttt{//Simplify};\\
\%/.c_{1,2,1}\to \beta-1\texttt{//Column}
\end{array}$$

$$\begin{array}{l}
\texttt{Assumps}=\{c_{1,2,1}\neq -1,c_{\texttt{u\_},\texttt{v\_},\texttt{w\_}}\in \texttt{Reals},c_{0,2,1}\neq c_{1,2,1}\};\\
\texttt{Reduce}[c_{1,2,1}+1\neq 0\&\&\texttt{And@@Assumps}]\\
\quad \texttt{//FullSimplify}[\#,\texttt{Assumps}]\&
\end{array}$$

$$\begin{array}{l}
\texttt{CheckAlgN}[\{c_{0,1,0}\to 0,c_{0,1,1}\to 0,c_{0,1,2}\to 0,c_{0,2,0}\to 1,\\
c_{0,2,1}\to \frac{1}{4}(\beta-2)^2,c_{0,2,2}\to 0,c_{1,2,0}\to -1,\\
c_{1,2,1}\to \beta-1,c_{1,2,2}\to 0\},\{\beta\neq 0,\beta\in \texttt{Reals}\},\{\}]
\end{array}$$

$$\begin{array}{l}
\texttt{CheckAlgN}[\{c_{0,2,0}\to 1,c_{0,1,1}\to 0,c_{0,2,2}\to 0,c_{0,1,2}\to 0,\\
c_{1,2,2}\to 0\},\{c_{1,2,1}\neq -1,c_{\texttt{u\_},\texttt{v\_},\texttt{w\_}}\in \texttt{Reals},c_{0,2,1}\neq c_{1,2,1}\},\\
\{(1-c_{1,2,1})^2-4c_{0,2,1}==0\}]
\end{array}$$

1.2.2.4 $c^1_{02}+c^1_{12} \neq 0$ and $(c^1_{12}-1)^2-4c^1_{02} < 0$

In this case, we get a one-parameter family of equivalence classes on $\mathrm{G}^h_{3.5}$.

$$\begin{array}{l}
\{\texttt{lie}[X_1,X_0],\texttt{lie}[X_2,X_0],\texttt{lie}[X_2,X_1]\}\\
\quad /.\{c_{0,2,0}\to 1,c_{0,1,1}\to 0,c_{0,2,2}\to 0,c_{0,1,2}\to 0,\\
c_{1,2,2}\to 0\}\\
\quad \texttt{//Simplify};\\
\%/.c_{0,2,1}\to \alpha/.c_{1,2,1}\to \beta-1/.\alpha\to \frac{\beta^2+h^2(2-\beta)^2}{4h^2}\\
\quad \texttt{//Column}
\end{array}$$

$$\begin{array}{l}
\texttt{Assumps}=\{c_{1,2,1}\neq -1,c_{\texttt{u\_},\texttt{v\_},\texttt{w\_}}\in \texttt{Reals},\\
c_{0,2,1}+c_{1,2,1}\neq 0\};\\
\texttt{Assumps2}=\texttt{Join}[\texttt{Assumps},\{(1-c_{1,2,1})^2-4c_{0,2,1}<0\}];\\
\texttt{Reduce}[c_{0,2,1}>0\&\&\texttt{And@@Assumps}]\\
\quad \texttt{//FullSimplify}[\#,\texttt{Assumps2}]\&\\
\texttt{Reduce}[c_{1,2,1}+1\neq 0\&\&\texttt{And@@Assumps}]\\
\quad \texttt{//FullSimplify}[\#,\texttt{Assumps2}]\&
\end{array}$$

$$\begin{array}{l}
\texttt{CheckAlgN}[\{c_{0,1,0}\to 0,c_{0,1,1}\to 0,c_{0,1,2}\to 0,c_{0,2,0}\to 1,\\
c_{0,2,1}\to \alpha,c_{0,2,2}\to 0,c_{1,2,0}\to -1,c_{1,2,1}\to \beta-1,\\
c_{1,2,2}\to 0\},\{\beta\neq 0,\beta\in \texttt{Reals},h>0\},\\
\{\alpha+\beta-1\neq 0,(\beta-2)^2-4\alpha<0\}]
\end{array}$$

$$\begin{array}{l}
\texttt{CheckAlgN}[\{c_{0,2,0}\to 1,c_{0,1,1}\to 0,c_{0,2,2}\to 0,c_{0,1,2}\to 0,\\
c_{1,2,2}\to 0\},\{\},\{c_{1,2,1}\neq -1,c_{\texttt{u\_},\texttt{v\_},\texttt{w\_}}\in \texttt{Reals},\\
c_{0,2,1}+c_{1,2,1}\neq 0,(1-c_{1,2,1})^2-4c_{0,2,1}<0,\\
-2\sqrt{c_{0,2,1}}<1-c_{1,2,1}<2\sqrt{c_{0,2,1}}\}]
\end{array}$$

Find h:

$$\begin{array}{l}
\texttt{Subs}=\{c_{0,1,0}\to 0,c_{0,1,1}\to 0,c_{0,1,2}\to 0,c_{0,2,0}\to 1,\\
c_{0,2,1}\to \alpha,c_{0,2,2}\to 0,c_{1,2,0}\to -1,c_{1,2,1}\to \beta-1,\\
c_{1,2,2}\to 0\};
\end{array}$$

$$\begin{array}{l}
\{\texttt{a0},\texttt{a1},\texttt{a2}\}=\texttt{Eigenvalues}[\texttt{ad}[a_0,a_1,a_2]]/.\texttt{Subs}\\
\quad \texttt{//FullSimplify}[\#,\beta\neq 0\&\&\{a_{\texttt{i\_}},\beta\}\in \texttt{Reals}]\&\\
(\frac{\texttt{a1}+\texttt{a2}}{\texttt{a1}-\texttt{a2}})^2\texttt{//FullSimplify}[\#,\beta\neq 0\&\&\{a_{\texttt{i\_}},\beta\}\in \texttt{Reals}]\&\\
\texttt{Reduce}[\%==-h^2,h,\texttt{Reals}]\\
\quad \texttt{//FullSimplify}[\#,-4\alpha+(2-\beta)^2<0\\
\qquad \&\&\beta\neq 0\&\&\{a_{\texttt{i\_}},\beta\}\in \texttt{Reals}]\&
\end{array}$$

1.2.3 $c^1_{12} \neq -1$ and $c^2_{02} = 0$ and $c^2_{01} \neq 0$

$$\begin{array}{l}
\texttt{And@@}\{-c_{0,1,1}-c_{0,2,2}-c_{0,2,0}c_{1,2,2}==0,\\
-c_{0,1,1}c_{0,2,0}+c_{0,2,2}c_{1,2,1}-c_{0,2,1}c_{1,2,2}==0,\\
c_{0,1,2}(-c_{0,2,0}-c_{1,2,1})+c_{0,1,1}c_{1,2,2}==0\}\\
\quad /.\{c_{0,2,0}\to 1\}/.\{c_{0,1,1}\to -c_{1,2,2}-c_{0,2,2}\}\\
\quad /.c_{0,2,2}\to 0\\
\quad \texttt{//FullSimplify}[\#,c_{1,2,1}\neq -1\&\&c_{0,1,2}\neq 0]\&
\end{array}$$

$(c_{0,2,1}-1)c_{1,2,2}==0\&\&c_{0,1,2}(1+c_{1,2,1})+c^2_{1,2,2}==0$

The second equation implies that $c^2_{12} \neq 0$, whence $c^1_{02} = 1$. In fact, from $c^1_{10} \leq 0$, we have

$$\begin{array}{l}
c_{0,1,1}\geq 0/.\{c_{0,2,0}\to 1\}/.\{c_{0,1,1}\to -c_{1,2,2}-c_{0,2,2}\}\\
\quad /.c_{0,2,2}\to 0\texttt{//FullSimplify}[\#,c_{1,2,2}\neq 0]\&
\end{array}$$

$c_{1,2,2}\leq 0$

That is, $c^2_{12} < 0$. Now using $c^1_{02} = 1$:

$$\begin{array}{l}
c_{0,1,2}(1+c_{1,2,1})+c^2_{1,2,2}==0/.c_{0,2,1}\to 1\\
\quad \texttt{//Simplify}[\#,c_{0,1,2}\neq 0]\&\\
\texttt{Solve}[\%,c_{1,2,1}]\texttt{//Simplify}
\end{array}$$

$c_{0,1,2}(1+c_{1,2,1})+c^2_{1,2,2}==0$

$\{\{c_{1,2,1}\to -1-\frac{c^2_{1,2,2}}{c_{0,1,2}}\}\}$

$$\begin{array}{l}
\texttt{CheckAlgN}[\{c_{0,2,0}\to 1,c_{0,1,1}\to -c_{1,2,2},c_{0,2,2}\to 0,\\
c_{0,2,1}\to 1,c_{1,2,1}\to -1-\frac{c^2_{1,2,2}}{c_{0,1,2}}\},\{c_{1,2,1}\neq -1,\\
c_{0,1,2}\neq 0,c_{1,2,2}<0\},\{\}]
\end{array}$$

$$\begin{array}{l}
\texttt{eig1}=-\frac{((a_0-a_1)c_{0,1,2}-a_2c_{1,2,2})(\sqrt{4c_{0,1,2}+c^2_{1,2,2}}+c_{1,2,2})}{2c_{0,1,2}};\\
\texttt{eig2}=+\frac{((a_0-a_1)c_{0,1,2}-a_2c_{1,2,2})(\sqrt{4c_{0,1,2}+c^2_{1,2,2}}-c_{1,2,2})}{2c_{0,1,2}};
\end{array}$$

$$\begin{array}{l}
\texttt{Reduce}[(\texttt{eig1}==0\&\&\texttt{eig2}\neq 0)\\
\quad ||(\texttt{eig1}\neq 0\&\&\texttt{eig2}==0),\{\},\texttt{Reals}]\\
\quad\quad \texttt{//FullSimplify}[\#,\{c_{1,2,1}\neq -1,c_{0,1,2}\neq 0,\\
\qquad\quad c_{1,2,2}<0,4c_{0,1,2}+c^2_{1,2,2}>0\}]\&\\
\texttt{Reduce}[\texttt{eig1}==\texttt{eig2},\{\},\texttt{Reals}]\\
\quad\quad \texttt{//FullSimplify}[\#,\{c_{1,2,1}\neq -1,c_{0,1,2}\neq 0,\\
\qquad\quad c_{1,2,2}<0,4c_{0,1,2}+c^2_{1,2,2}>0\}]\&
\end{array}$$

`False`

$(c_{0,1,2}\geq 0\|2\sqrt{-c_{0,1,2}}<c_{1,2,2}\|2\sqrt{-c_{0,1,2}}+c_{1,2,2}<0)$
$\&\&a_0==a_1+\frac{a_2c_{1,2,2}}{c_{0,1,2}}$

$$\begin{array}{l}
\texttt{Reduce}[\texttt{ForAll}[\{a_0,a_1,a_2\},\{a_0,a_1,a_2\}\in \texttt{Reals},\\
\quad \texttt{eig1}==\texttt{eig2}]\&\&\texttt{And@@}\{c_{1,2,1}\neq -1,c_{0,1,2}\neq 0,\\
\quad c_{1,2,2}<0\},\{\},\texttt{Reals}]\\
\quad\quad \texttt{//FullSimplify}[\#,\{c_{1,2,1}\neq -1,c_{0,1,2}\neq 0,\\
\qquad\quad c_{1,2,2}<0\}]\&
\end{array}$$

$4c_{0,1,2}+c^2_{1,2,2}==0$

1.2.3.1 $(c^2_{12})^2+4c^2_{01} > 0$

In this case, we get a one-parameter family of equivalence classes on $\mathrm{G}^h_{3.4}$.

$$\begin{array}{l}
\{\texttt{lie}[X_1,X_0],\texttt{lie}[X_2,X_0],\texttt{lie}[X_2,X_1]\}\\
\quad /.\{c_{0,2,0}\to 1,c_{0,1,1}\to -c_{1,2,2},c_{0,2,2}\to 0,c_{0,2,1}\to 1,\\
c_{1,2,1}\to -1-\frac{c^2_{1,2,2}}{c_{0,1,2}}\}
\end{array}$$

$\quad\mathbf{//Simplify};$
$\%/.c_{1,2,2} \to -\alpha/.c_{0,1,2} \to \beta/.\beta \to -\frac{(h^2-1)\alpha^2}{4h^2}$
$\quad\mathbf{//Column}$

$\mathbf{Assumps} = \{c_{1,2,1} \neq -1, c_{0,1,2} \neq 0, c_{1,2,2} < 0,$
$c_{0,1,2} \in \mathbf{Reals}\};$
$\mathbf{Assumps2} = \mathbf{Join}[\mathbf{Assumps}, \{c_{1,2,2}^2 + 4c_{0,1,2} > 0,$
$\quad c_{0,1,2} < 0 \Rightarrow$
$\qquad (c_{1,2,2} < -2\sqrt{-c_{0,1,2}} || c_{1,2,2} > 2\sqrt{-c_{0,1,2}})\}];$
$\mathbf{Reduce}[c_{1,2,2} < 0\&\&\mathbf{And@@Assumps}]$
$\quad\mathbf{//FullSimplify}[\#, \mathbf{Assumps2}]\&$
$\mathbf{Reduce}[c_{0,1,2} \neq 0\&\&\mathbf{And@@Assumps}]$
$\quad\mathbf{//FullSimplify}[\#, \mathbf{Assumps2}]\&$

$\mathbf{CheckAlgN}[\{c_{0,1,0} \to 0, c_{0,1,1} \to \alpha, c_{0,1,2} \to \beta, c_{0,2,0} \to 1,$
$c_{0,2,1} \to 1, c_{0,2,2} \to 0, c_{1,2,0} \to -1, c_{1,2,1} \to -(1 + \frac{\alpha^2}{\beta}),$
$c_{1,2,2} \to -\alpha\}, \{\alpha > 0, \beta \neq 0, \beta \in \mathbf{Reals}\}, \{\alpha^2 + 4\beta > 0\}]$

$\mathbf{CheckAlgN}[\{c_{0,2,0} \to 1, c_{0,1,1} \to -c_{1,2,2}, c_{0,2,2} \to 0,$
$c_{0,2,1} \to 1, c_{1,2,1} \to -1 - \frac{c_{1,2,2}^2}{c_{0,1,2}}\}, \{c_{1,2,1} \neq -1, c_{0,1,2} \neq 0,$
$c_{1,2,2} < 0, c_{0,1,2} \in \mathbf{Reals}\}, \{c_{1,2,2}^2 + 4c_{0,1,2} > 0,$
$c_{0,1,2} < 0 \Rightarrow (c_{1,2,2} < -2\sqrt{-c_{0,1,2}} || c_{1,2,2} > 2\sqrt{-c_{0,1,2}})\}]$

Find h:

$\mathbf{Subs} = \{c_{0,1,0} \to 0, c_{0,1,1} \to \alpha, c_{0,1,2} \to \beta, c_{0,2,0} \to 1,$
$c_{0,2,1} \to 1, c_{0,2,2} \to 0, c_{1,2,0} \to -1, c_{1,2,1} \to -(1 + \frac{\alpha^2}{\beta}),$
$c_{1,2,2} \to -\alpha\};$

$\{\mathbf{a0}, \mathbf{a1}, \mathbf{a2}\} = \mathbf{Eigenvalues}[\mathbf{ad}[a_0, a_1, a_2]]/.\mathbf{Subs}$
$\quad\mathbf{//FullSimplify}[\#, \alpha > 0\&\&\beta \neq 0$
$\qquad\&\&\{a_{\mathbf{i\_}}, \beta\} \in \mathbf{Reals}]\&$
$(\frac{\mathbf{a1}+\mathbf{a2}}{\mathbf{a1}-\mathbf{a2}})^2\mathbf{//FullSimplify}[\#, \alpha^2 + 4\beta > 0$
$\quad\&\&\alpha > 0\&\&\beta \neq 0\&\&\{a_{\mathbf{i\_}}, \beta\} \in \mathbf{Reals}]\&$
$\mathbf{Reduce}[\% == h^2, h]$
$\quad\mathbf{//FullSimplify}[\#, \alpha^2 + 4\beta > 0\&\&\alpha > 0\&\&\beta \neq 0$
$\qquad\&\&\{a_{\mathbf{i\_}}, \beta\} \in \mathbf{Reals}]\&$

1.2.3.2 $(c_{12}^2)^2 + 4c_{01}^2 = 0$

In this case, we get a one-parameter family of equivalence classes on $\mathsf{G}_{3.2}$.

$\{\mathbf{lie}[X_1, X_0], \mathbf{lie}[X_2, X_0], \mathbf{lie}[X_2, X_1]\}$
$\quad/.\{c_{0,2,0} \to 1, c_{0,1,1} \to -c_{1,2,2}, c_{0,2,2} \to 0, c_{0,2,1} \to 1,$
$c_{1,2,1} \to -1 - \frac{c_{1,2,2}^2}{c_{0,1,2}}\}/.c_{0,1,2} \to -\frac{1}{4}c_{1,2,2}^2$
$\quad\mathbf{//Simplify};$
$\%/.c_{1,2,2} \to -\alpha\mathbf{//Simplify}[\#, \alpha > 0]\&$
$\quad\mathbf{//Collect}[\#, X_{\mathbf{i\_}}, \mathbf{Simplify}]\&\mathbf{//Column}$

$\mathbf{Assumps} = \{c_{1,2,1} \neq -1, c_{0,1,2} \neq 0, c_{1,2,2} < 0,$
$c_{0,1,2} \in \mathbf{Reals}\};$
$\mathbf{Reduce}[-c_{1,2,2} > 0\&\&\mathbf{And@@Assumps}]$
$\quad\mathbf{//FullSimplify}[\#, \mathbf{Assumps}]\&$

$\mathbf{CheckAlgN}[\{c_{0,1,0} \to 0, c_{0,1,1} \to \alpha, c_{0,1,2} \to -\frac{1}{4}\alpha^2,$
$c_{0,2,0} \to 1, c_{0,2,1} \to 1, c_{0,2,2} \to 0, c_{1,2,0} \to -1, c_{1,2,1} \to 3,$
$c_{1,2,2} \to -\alpha\}, \{\alpha > 0\}, \{\}]$

$\mathbf{CheckAlgN}[\{c_{0,2,0} \to 1, c_{0,1,1} \to -c_{1,2,2}, c_{0,2,2} \to 0,$
$c_{0,2,1} \to 1, c_{1,2,1} \to -1 - \frac{c_{1,2,2}^2}{c_{0,1,2}}\}, \{c_{1,2,1} \neq -1,$
$c_{0,1,2} \neq 0, c_{1,2,2} < 0, c_{0,1,2} \in \mathbf{Reals}\},$
$\{c_{1,2,2}^2 + 4c_{0,1,2} == 0\}]$

1.2.3.3 $(c_{12}^2)^2 + 4c_{01}^2 < 0$

In this case, we get a one-parameter family of equivalence classes on $\mathsf{G}_{3.5}^h$.

$\{\mathbf{lie}[X_1, X_0], \mathbf{lie}[X_2, X_0], \mathbf{lie}[X_2, X_1]\}$
$\quad/.\{c_{0,2,0} \to 1, c_{0,1,1} \to -c_{1,2,2}, c_{0,2,2} \to 0, c_{0,2,1} \to 1,$
$c_{1,2,1} \to -1 - \frac{c_{1,2,2}^2}{c_{0,1,2}}\}\mathbf{//Simplify};$
$\%/.c_{1,2,2} \to -\alpha_1/.c_{0,1,2} \to -\alpha_2/.\alpha_2 \to \frac{(1+h^2)\alpha_1^2}{4h^2}$
$\quad/.\alpha_1 \to \alpha\mathbf{//Column}$

$\mathbf{Assumps} = \{c_{1,2,1} \neq -1, c_{0,1,2} \neq 0, c_{1,2,2} < 0,$
$c_{0,1,2} \in \mathbf{Reals}\};$
$\mathbf{Assumps2} = \mathbf{Join}[\mathbf{Assumps}, \{c_{1,2,2}^2 + 4c_{0,1,2} < 0\}];$
$\mathbf{Reduce}[-c_{1,2,2} > 0\&\&\mathbf{And@@Assumps}]$
$\quad\mathbf{//FullSimplify}[\#, \mathbf{Assumps2}]\&$
$\mathbf{Reduce}[-c_{0,1,2} > 0\&\&\mathbf{And@@Assumps}]$
$\quad\mathbf{//FullSimplify}[\#, \mathbf{Assumps2}]\&$

$\mathbf{CheckAlgN}[\{c_{0,1,0} \to 0, c_{0,1,1} \to \alpha_1, c_{0,1,2} \to -\alpha_2,$
$c_{0,2,0} \to 1, c_{0,2,1} \to 1, c_{0,2,2} \to 0, c_{1,2,0} \to -1,$
$c_{1,2,1} \to -(1 - \frac{\alpha_1^2}{\alpha_2}), c_{1,2,2} \to -\alpha_1\}, \{\alpha_1 > 0, \alpha_2 > 0,$
$h > 0\}, \{\alpha_1^2 - 4\alpha_2 < 0\}]$

$\mathbf{CheckAlgN}[\{c_{0,2,0} \to 1, c_{0,1,1} \to -c_{1,2,2}, c_{0,2,2} \to 0,$
$c_{0,2,1} \to 1, c_{1,2,1} \to -1 - \frac{c_{1,2,2}^2}{c_{0,1,2}}\}, \{c_{1,2,1} \neq -1,$
$c_{0,1,2} \neq 0, c_{1,2,2} < 0, c_{0,1,2} \in \mathbf{Reals}\},$
$\{c_{1,2,2}^2 + 4c_{0,1,2} < 0, -2\sqrt{-c_{0,1,2}} < c_{1,2,2} < 2\sqrt{-c_{0,1,2}}\}]$

Find h:

$\mathbf{Subs} = \{c_{0,1,0} \to 0, c_{0,1,1} \to \alpha_1, c_{0,1,2} \to -\alpha_2, c_{0,2,0} \to 1,$
$c_{0,2,1} \to 1, c_{0,2,2} \to 0, c_{1,2,0} \to -1, c_{1,2,1} \to -(1 - \frac{\alpha_1^2}{\alpha_2}),$
$c_{1,2,2} \to -\alpha_1\};$

$\{\mathbf{a0}, \mathbf{a1}, \mathbf{a2}\} = \mathbf{Eigenvalues}[\mathbf{ad}[a_0, a_1, a_2]]/.\mathbf{Subs}$
$\quad\mathbf{//FullSimplify}[\#, \alpha_1 > 0\&\&\alpha_2 > 0$
$\quad\&\&a_{\mathbf{i\_}} \in \mathbf{Reals}]\&$
$(\frac{\mathbf{a1}+\mathbf{a2}}{\mathbf{a1}-\mathbf{a2}})^2\mathbf{//FullSimplify}[\#, \alpha_1^2 - 4\alpha_2 < 0\&\&\alpha_1 > 0$
$\quad\&\&\alpha_2 > 0\&\&a_{\mathbf{i\_}} \in \mathbf{Reals}]\&$
$\mathbf{Reduce}[\% == -h^2, h, \mathbf{Reals}]$
$\quad\mathbf{//FullSimplify}[\#, \alpha_1^2 - 4\alpha_2 < 0\&\&\alpha_1 > 0$
$\qquad\&\&\alpha_2 > 0\&\&a_{\mathbf{i\_}} \in \mathbf{Reals}]\&$

1.2.4 $c_{12}^1 \neq -1$ and $c_{02}^2 \neq 0$ and $c_{01}^2 = 0$

In this case, we get a two-parameter family of equivalence classes on $\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}$.

$\mathbf{And@@}\{-c_{0,1,1} - c_{0,2,2} - c_{0,2,0}c_{1,2,2} == 0,$
$-c_{0,1,1}c_{0,2,0} + c_{0,2,2}c_{1,2,1} - c_{0,2,1}c_{1,2,2} == 0,$
$c_{0,1,2}(-c_{0,2,0} - c_{1,2,1}) + c_{0,1,1}c_{1,2,2} == 0\}$
$\quad/.\{c_{0,2,0} \to 1\}/.\{c_{0,1,1} \to -c_{1,2,2} - c_{0,2,2}\}$
$\quad/.c_{0,1,2} \to 0$
$\quad\mathbf{//FullSimplify}[\#, c_{1,2,1} \neq -1\&\&c_{0,2,2} \neq 0]\&$

$c_{0,2,2}(1 + c_{1,2,1}) == (c_{0,2,1} - 1)c_{1,2,2}\&\&$
$c_{1,2,2}(c_{0,2,2} + c_{1,2,2}) == 0$

The first equation implies that $c_{12}^2 \neq 0$, whence $c_{02}^2 = -c_{12}^2$.

$c_{0,2,2}(1 + c_{1,2,1}) == (c_{0,2,1} - 1)c_{1,2,2}/.c_{0,2,2} \to -c_{1,2,2}$
$\quad\mathbf{//Simplify}[\#, c_{0,2,2} \neq 0]\&$
$\mathbf{Solve}[\%, c_{1,2,1}]\mathbf{//Simplify}$

$(c_{0,2,1} + c_{1,2,1})c_{1,2,2} == 0$

$$\{\{c_{1,2,1}\to -c_{0,2,1}\}\}$$

Since $c^1_{10}=0$, we may assume $c^2_{21}<0$ (by changing the frame if necessary).

$$\begin{array}{l}
\{\mathtt{lie}[X_1,X_0],\mathtt{lie}[X_2,X_0],\mathtt{lie}[X_2,X_1]\}\\
\quad /.\{c_{0,2,0}\to 1,c_{0,1,1}\to 0,c_{0,1,2}\to 0,c_{0,2,2}\to -c_{1,2,2},\\
c_{0,2,1}\to -c_{1,2,1}\}\\
\quad //\mathtt{Simplify};\\
\%/.c_{1,2,2}\to \alpha/.c_{1,2,1}\to \beta-1//\mathtt{Column}
\end{array}$$

$$\begin{array}{l}
\mathtt{CheckAlgN}[\{c_{0,1,0}\to 0,c_{0,1,1}\to 0,c_{0,1,2}\to 0,c_{0,2,0}\to 1,\\
c_{0,2,1}\to 1-\beta,c_{0,2,2}\to -\alpha,c_{1,2,0}\to -1,c_{1,2,1}\to \beta-1,\\
c_{1,2,2}\to \alpha\},\{\alpha>0,\beta\neq 0,\beta\in \mathtt{Reals}\},\{\}]
\end{array}$$

$$\begin{array}{l}
\mathtt{CheckAlgN}[\{c_{0,2,0}\to 1,c_{0,1,1}\to 0,c_{0,1,2}\to 0,\\
c_{0,2,2}\to -c_{1,2,2},c_{0,2,1}\to -c_{1,2,1}\},\{c_{1,2,1}\neq -1,\\
c_{1,2,2}>0,c_{1,2,1}\in \mathtt{Reals}\},\{\}]
\end{array}$$

1.2.5 $c^1_{12}\neq -1$ and $c^2_{02}\neq 0$ and $c^2_{01}\neq 0$

$$\begin{array}{l}
\mathtt{And@@}\{-c_{0,1,1}-c_{0,2,2}-c_{0,2,0}c_{1,2,2}==0,\\
-c_{0,1,1}c_{0,2,0}+c_{0,2,2}c_{1,2,1}-c_{0,2,1}c_{1,2,2}==0,\\
c_{0,1,2}(-c_{0,2,0}-c_{1,2,1})+c_{0,1,1}c_{1,2,2}==0\}\\
\quad /.\{c_{0,2,0}\to 1\}/.\{c_{0,1,1}\to -c_{1,2,2}-c_{0,2,2}\}\\
\quad //\mathtt{FullSimplify}[\#,c_{1,2,1}\neq -1\&\&c_{0,2,2}\neq 0\\
\qquad \&\&c_{0,1,2}\neq 0]\&
\end{array}$$

$$\begin{array}{l}
c_{0,2,2}(1+c_{1,2,1})==(c_{0,2,1}-1)c_{1,2,2}\&\&\\
c_{0,1,2}(1+c_{1,2,1})+c_{1,2,2}(c_{0,2,2}+c_{1,2,2})==0
\end{array}$$

We see that $c^2_{12}\neq 0$, $c^1_{02}\neq 1$ and $c^2_{12}+c^2_{02}\neq 0$. In fact, from $c^1_{10}\geq 0$, we get:

$$\begin{array}{l}
c_{0,1,1}\geq 0/.\{c_{0,1,1}\to -c_{1,2,2}-c_{0,2,2}\}\\
\quad //\mathtt{FullSimplify}[\#,c_{1,2,1}\neq -1\&\&c_{0,2,2}\neq 0\\
\quad \&\&c_{0,1,2}\neq 0\&\&c_{1,2,2}\neq 0\&\&c_{0,2,1}\neq 1\\
\qquad \&\&c_{1,2,2}+c_{0,2,2}\neq 0]\&
\end{array}$$

$$c_{0,2,2}+c_{1,2,2}\leq 0$$

That is, $c^2_{12}+c^2_{02}<0$. Thus:

$$\begin{array}{l}
\mathtt{Solve}[c_{0,2,2}(1+c_{1,2,1})==(c_{0,2,1}-1)c_{1,2,2},c_{1,2,1}]\\
\quad //\mathtt{FullSimplify}[\#,c_{1,2,1}\neq -1\&\&c_{0,2,2}\neq 0\\
\qquad \&\&c_{0,1,2}\neq 0\&\&c_{1,2,2}\neq 0\&\&c_{0,2,1}\neq 1\\
\qquad \&\&c_{1,2,2}+c_{0,2,2}\neq 0]\&\\
\mathtt{Solve}[c_{0,1,2}(1+c_{1,2,1})+c_{1,2,2}(c_{0,2,2}+c_{1,2,2})==0\\
\ \ /.c_{1,2,1}\to \frac{(c_{0,2,1}-1)c_{1,2,2}}{c_{0,2,2}}-1,c_{0,1,2}]\\
\quad //\mathtt{FullSimplify}[\#,c_{1,2,1}\neq -1\&\&c_{0,2,2}\neq 0\\
\qquad \&\&c_{0,1,2}\neq 0\&\&c_{1,2,2}\neq 0\&\&c_{0,2,1}\neq 1\\
\qquad \&\&c_{1,2,2}+c_{0,2,2}\neq 0]\&
\end{array}$$

$$\{\{c_{1,2,1}\to \frac{(c_{0,2,1}-1)c_{1,2,2}}{c_{0,2,2}}-1\}\}$$

$$\{\{c_{0,1,2}\to -\frac{c_{0,2,2}(c_{0,2,2}+c_{1,2,2})}{c_{0,2,1}-1}\}\}$$

$$\begin{array}{l}
\{\{c_{0,2,0}\to 1,c_{0,1,1}\to -c_{1,2,2}-c_{0,2,2}\},\\
\ \{c_{1,2,1}\neq -1,c_{0,2,2}\neq 0,c_{0,1,2}\neq 0,c_{1,2,2}\neq 0\},\\
\ \{c_{0,2,1}\neq 1,c_{1,2,2}+c_{0,2,2}<0\}\}\\
\quad /.c_{1,2,1}\to \frac{(c_{0,2,1}-1)c_{1,2,2}}{c_{0,2,2}}-1\\
\quad /.c_{0,1,2}\to -\frac{c_{0,2,2}(c_{0,2,2}+c_{1,2,2})}{c_{0,2,1}-1}\\
\quad //\mathtt{FullSimplify}
\end{array}$$

$$\begin{array}{l}
\{\{c_{0,2,0}\to 1,c_{0,1,1}\to -c_{0,2,2}-c_{1,2,2}\},\\
\{\frac{(c_{0,2,1}-1)c_{1,2,2}}{c_{0,2,2}}\neq 0,c_{0,2,2}\neq 0,\frac{c_{0,2,2}(c_{0,2,2}+c_{1,2,2})}{c_{0,2,1}-1}\neq 0,\\
c_{1,2,2}\neq 0\},\\
\{c_{0,2,1}\neq 1,c_{0,2,2}+c_{1,2,2}<0\}\}
\end{array}$$

$$\begin{array}{l}
\mathtt{ad}[a_0,a_1,a_2]/.\{c_{0,2,0}\to 1,c_{0,1,1}\to -c_{1,2,2}-c_{0,2,2},\\
c_{1,2,1}\to \frac{(c_{0,2,1}-1)c_{1,2,2}}{c_{0,2,2}}-1,\\
c_{0,1,2}\to -\frac{c_{0,2,2}(c_{0,2,2}+c_{1,2,2})}{c_{0,2,1}-1}\}\\
\quad //\mathtt{Eigenvalues}\\
\quad //\mathtt{FullSimplify}[\#,\mathtt{And@@}\{c_{1,2,1}\neq -1,c_{0,2,2}\neq 0,\\
\qquad c_{0,1,2}\neq 0,c_{1,2,2}\neq 0,c_{0,2,1}\neq 1,\\
\qquad c_{1,2,2}+c_{0,2,2}<0\}]\&
\end{array}$$

$$\begin{array}{l}
\mathbf{eig1}=-\Big((a_2(c_{0,2,1}-1)+(a_0-a_1)c_{0,2,2})\\
\times((c_{0,2,1}-1)c_{1,2,2}+\sqrt{c_{0,2,1}-1}\\
\times\sqrt{-4c^2_{0,2,2}-4c_{0,2,2}c_{1,2,2}+(c_{0,2,1}-1)c^2_{1,2,2}})\Big)/\\
(2(c_{0,2,1}-1)c_{0,2,2});\\
\mathbf{eig2}=\Big((a_2(c_{0,2,1}-1)+(a_0-a_1)c_{0,2,2})\\
\times(-(c_{0,2,1}-1)c_{1,2,2}+\sqrt{c_{0,2,1}-1}\\
\times\sqrt{-4c^2_{0,2,2}-4c_{0,2,2}c_{1,2,2}+(c_{0,2,1}-1)c^2_{1,2,2}})\Big)/\\
(2(c_{0,2,1}-1)c_{0,2,2});
\end{array}$$

$$\begin{array}{l}
\mathtt{Reduce}[(\mathbf{eig1}==0\&\&\mathbf{eig2}\neq 0)\\
\quad ||(\mathbf{eig1}\neq 0\&\&\mathbf{eig2}==0),\{\},\mathtt{Reals}]\\
\quad //\mathtt{FullSimplify}[\#,\mathtt{And@@}\{c_{1,2,1}\neq -1,c_{0,2,2}\neq 0,\\
\qquad c_{0,1,2}\neq 0,c_{1,2,2}\neq 0,c_{0,2,1}\neq 1,\\
\qquad c_{1,2,2}+c_{0,2,2}<0\}]\&
\end{array}$$

```
False
```

Hence: $\mathbf{eig1}=\mathbf{0}$ if and only if $\mathbf{eig2}=\mathbf{0}$ (hence we're not on $\mathsf{Aff}(\mathbb{R})_0\times\mathbb{R}$); and there exists A such that ad_A has 1 zero, 2 nonzero and identical eigenvalues exactly when (the bit under the square roots) = 0.

1.2.5.1 Eigenvalues real and distinct

In this case, we get a two-parameter family of equivalence classes on $\mathrm{G}^h_{3.4}$.

$$\begin{array}{l}
\{\mathtt{lie}[X_1,X_0],\mathtt{lie}[X_2,X_0],\mathtt{lie}[X_2,X_1]\}\\
\quad /.\{c_{0,2,0}\to 1,c_{0,1,1}\to -c_{1,2,2}-c_{0,2,2},\\
c_{1,2,1}\to \frac{(c_{0,2,1}-1)c_{1,2,2}}{c_{0,2,2}}-1,\\
c_{0,1,2}\to -\frac{c_{0,2,2}(c_{0,2,2}+c_{1,2,2})}{c_{0,2,1}-1}\}\\
\quad //\mathtt{Simplify};\\
\%/.c_{1,2,2}\to \beta_1/.c_{0,2,2}\to \beta_2/.c_{0,2,1}\to \beta_3+1\\
\quad /.\beta_3\to \frac{4h^2\beta_2(\beta_1+\beta_2)}{(h^2-1)\beta_1^2}\\
\quad //\mathtt{FullSimplify}[\#,\beta_1\neq 0\&\&\beta_2\neq 0\\
\quad \&\&\beta_{\mathtt{i\_}}\in \mathtt{Reals}]\&;\\
\%/.\mathtt{Flatten@Solve}[-\beta_1-\beta_2==\alpha\&\&\beta_1==\beta,\{\beta_1,\beta_2\}]\\
\quad //\mathtt{Collect}[\#,X_{\mathtt{i\_}},\mathtt{FullSimplify}]\&//\mathtt{Column}
\end{array}$$

$$\begin{array}{l}
\mathtt{Reduce}[(\alpha+\beta\neq 0/.\alpha\to -(c_{1,2,2}+c_{0,2,2})/.\beta\to c_{1,2,2})\\
\quad \&\&\mathtt{And@@Assumps}]\\
\quad //\mathtt{FullSimplify}[\#,\mathtt{Assumps}]\&
\end{array}$$

$$\begin{array}{l}
\mathtt{Assumps}=\{c_{1,2,1}\neq -1,c_{0,2,2}\neq 0,c_{0,1,2}\neq 0,c_{1,2,2}\neq 0,\\
c_{0,2,1}\neq 1,c_{1,2,2}+c_{0,2,2}<0,c_{0,2,1}\in \mathtt{Reals}\};\\
\mathtt{Assumps2}=\mathtt{Join}[\mathtt{Assumps},\{(c_{0,2,1}-1)\\
\ \ \times(-4c^2_{0,2,2}-4c_{0,2,2}c_{1,2,2}+(c_{0,2,1}-1)c^2_{1,2,2})>0\}];\\
\mathtt{Reduce}[c_{1,2,2}\neq 0\&\&\mathtt{And@@Assumps}]\\
\quad //\mathtt{FullSimplify}[\#,\mathtt{Assumps2}]\&\\
\mathtt{Reduce}[c_{0,2,2}\neq 0\&\&\mathtt{And@@Assumps}]
\end{array}$$

$$\begin{array}{l}
\quad\texttt{//FullSimplify}[\#,\texttt{Assumps2}]\&\\
\texttt{Reduce}[c_{0,2,1}-1\neq 0\&\&\texttt{And@@Assumps}]\\
\quad\texttt{//FullSimplify}[\#,\texttt{Assumps2}]\&
\end{array}$$

$$\begin{array}{l}
\texttt{Reduce}[-(c_{1,2,2}+c_{0,2,2})>0\&\&\texttt{And@@Assumps}]\\
\quad\texttt{//FullSimplify}[\#,\texttt{Assumps2}]\&
\end{array}$$

$$\begin{array}{l}
\texttt{CheckAlgN}[\{c_{0,1,0}\to 0,c_{0,1,1}\to-(\beta_1+\beta_2),\\
c_{0,1,2}\to-\frac{\beta_2(\beta_1+\beta_2)}{\beta_3},c_{0,2,0}\to 1,c_{0,2,1}\to\beta_3+1,\\
c_{0,2,2}\to\beta_2,c_{1,2,0}\to-1,c_{1,2,1}\to-(1-\frac{\beta_1\beta_3}{\beta_2}),\\
c_{1,2,2}\to\beta_1\},\{\beta_1\neq 0,\beta_2\neq 0,\beta_3\neq 0,\\
\{\beta_1,\beta_2,\beta_3\}\in\texttt{Reals}\},\{\beta_3(-4\beta_1\beta_2-4\beta_2^2+\beta_1^2\beta_3)>0\}]
\end{array}$$

$$\begin{array}{l}
\texttt{CheckAlgN}[\{c_{0,2,0}\to 1,c_{0,1,1}\to-c_{1,2,2}-c_{0,2,2},\\
c_{1,2,1}\to\frac{(c_{0,2,1}-1)c_{1,2,2}}{c_{0,2,2}}-1,\\
c_{0,1,2}\to-\frac{c_{0,2,2}(c_{0,2,2}+c_{1,2,2})}{c_{0,2,1}-1}\},\\
\{c_{1,2,1}\neq-1,c_{0,2,2}\neq 0,c_{0,1,2}\neq 0,c_{1,2,2}\neq 0,c_{0,2,1}\neq 1,\\
c_{1,2,2}+c_{0,2,2}<0,c_{0,2,1}\in\texttt{Reals}\},\{(c_{0,2,1}-1)\\
\quad\times(-4c_{0,2,2}^2-4c_{0,2,2}c_{1,2,2}+(c_{0,2,1}-1)c_{1,2,2}^2)>0\}]
\end{array}$$

Find h:

$$\begin{array}{l}
\texttt{Subs}=\{c_{0,1,0}\to 0,c_{0,1,1}\to-(\beta_1+\beta_2),\\
c_{0,1,2}\to-\frac{\beta_2(\beta_1+\beta_2)}{\beta_3},c_{0,2,0}\to 1,c_{0,2,1}\to\beta_3+1,\\
c_{0,2,2}\to\beta_2,c_{1,2,0}\to-1,c_{1,2,1}\to-(1-\frac{\beta_1\beta_3}{\beta_2}),\\
c_{1,2,2}\to\beta_1\};
\end{array}$$

$$\begin{array}{l}
\{\texttt{a0},\texttt{a1},\texttt{a2}\}=\texttt{Eigenvalues}[\texttt{ad}[a_0,a_1,a_2]]\texttt{/.Subs}\\
\quad\texttt{//FullSimplify}[\#,\beta_1\neq 0\&\&\beta_2\neq 0\&\&\beta_3\neq 0\\
\qquad\&\&\{a_{\texttt{i\_}},\beta_{\texttt{j\_}}\}\in\texttt{Reals}]\&\\
(\frac{\texttt{a1+a2}}{\texttt{a1-a2}})^2\texttt{//FullSimplify}[\#,\beta_3(\beta_1^2\beta_3-4\beta_1\beta_2-4\beta_2^2)>0\\
\quad\&\&\beta_1\neq 0\&\&\beta_2\neq 0\&\&\beta_3\neq 0\\
\quad\&\&\{a_{\texttt{i\_}},\beta_{\texttt{j\_}}\}\in\texttt{Reals}]\&\\
\texttt{Reduce}[\%==h^2,h]\\
\quad\texttt{//FullSimplify}[\#,\beta_3(-4\beta_1\beta_2-4\beta_2^2+\beta_1^2\beta_3)>0\\
\qquad\&\&\beta_1\neq 0\&\&\beta_2\neq 0\&\&\beta_3\neq 0\\
\qquad\&\&\{a_{\texttt{i\_}},\beta_{\texttt{j\_}}\}\in\texttt{Reals}]\&
\end{array}$$

1.2.5.2 Eigenvalues equal

In this case, we get a two-parameter family of equivalence classes on $\mathrm{G}_{3.2}$.

$$\begin{array}{l}
\{\texttt{lie}[X_1,X_0],\texttt{lie}[X_2,X_0],\texttt{lie}[X_2,X_1]\}\\
\quad\texttt{/.}\{c_{0,2,0}\to 1,c_{0,1,1}\to-c_{1,2,2}-c_{0,2,2},\\
c_{1,2,1}\to\frac{(c_{0,2,1}-1)c_{1,2,2}}{c_{0,2,2}}-1,\\
c_{0,1,2}\to-\frac{c_{0,2,2}(c_{0,2,2}+c_{1,2,2})}{c_{0,2,1}-1}\}\\
\quad\texttt{/.}c_{0,2,1}\to\frac{(2c_{0,2,2}+c_{1,2,2})^2}{c_{1,2,2}^2}\\
\quad\texttt{//Simplify};\\
\%\texttt{/.}c_{1,2,2}\to\beta_1\texttt{/.}c_{0,2,2}\to\beta_2\\
\quad\texttt{//FullSimplify}[\#,\beta_1\neq 0\&\&\beta_2\neq 0\\
\quad\&\&\beta_{\texttt{i\_}}\in\texttt{Reals}]\&;\\
\%\texttt{/.Flatten@Solve}[\beta_1+\beta_2==-\alpha\&\&\beta_1==\beta,\{\beta_1,\beta_2\}]\\
\quad\texttt{//Collect}[\#,X_{\texttt{i\_}},\texttt{FullSimplify}]\&\texttt{//Column}
\end{array}$$

$$\begin{array}{l}
\texttt{Reduce}[(\alpha+\beta\neq 0\texttt{/.}\alpha\to-(c_{1,2,2}+c_{0,2,2})\texttt{/.}\beta\to c_{1,2,2})\\
\quad\&\&\texttt{And@@Assumps}]\\
\quad\texttt{//FullSimplify}[\#,\texttt{Assumps}]\&
\end{array}$$

$$\begin{array}{l}
\texttt{Assumps}=\{c_{1,2,1}\neq-1,c_{0,2,2}\neq 0,c_{0,1,2}\neq 0,c_{1,2,2}\neq 0,\\
c_{0,2,1}\neq 1,c_{1,2,2}+c_{0,2,2}<0,c_{0,2,1}\in\texttt{Reals}\};\\
\texttt{Reduce}[c_{1,2,2}\neq 0\&\&\texttt{And@@Assumps}]\\
\quad\texttt{//FullSimplify}[\#,\texttt{Assumps}]\&\\
\texttt{Reduce}[c_{0,2,2}\neq 0\&\&\texttt{And@@Assumps}]\\
\quad\texttt{//FullSimplify}[\#,\texttt{Assumps}]\&
\end{array}$$

$$\begin{array}{l}
\texttt{Reduce}[-(c_{1,2,2}+c_{0,2,2})>0\&\&\texttt{And@@Assumps}]\\
\quad\texttt{//FullSimplify}[\#,\texttt{Assumps}]\&
\end{array}$$

$$\begin{array}{l}
\texttt{CheckAlgN}[\{c_{0,1,0}\to 0,c_{0,1,1}\to\alpha,c_{0,1,2}\to-\frac{\beta^2}{4},\\
c_{0,2,0}\to 1,c_{0,2,1}\to\frac{(\beta-2\alpha)^2}{\beta^2},c_{0,2,2}\to\beta-\alpha,\\
c_{1,2,0}\to-1,c_{1,2,1}\to\frac{4\alpha}{\beta}-1,c_{1,2,2}\to-\beta\},\\
\{\alpha>0,\beta\neq 0,\beta\in\texttt{Reals}\},\{\}]
\end{array}$$

$$\begin{array}{l}
\texttt{CheckAlgN}[\{c_{0,2,0}\to 1,c_{0,1,1}\to-c_{0,2,2}-c_{1,2,2},\\
c_{1,2,1}\to 3+\frac{4c_{0,2,2}}{c_{1,2,2}},c_{0,1,2}\to-\frac{1}{4}c_{1,2,2}^2,\\
c_{0,2,1}\to\frac{(2c_{0,2,2}+c_{1,2,2})^2}{c_{1,2,2}^2}\},\{c_{1,2,1}\neq-1,\\
c_{0,2,2}\neq 0,c_{0,1,2}\neq 0,c_{1,2,2}\neq 0,c_{0,2,1}\neq 1,\\
c_{1,2,2}+c_{0,2,2}<0,c_{\texttt{u\_},\texttt{v\_},\texttt{w\_}}\in\texttt{Reals}\},\{\}]
\end{array}$$

1.2.5.3 Eigenvalues complex and distinct

In this case, we get a two-parameter family of equivalence classes on $\mathrm{G}_{3.5}^h$.

$$\begin{array}{l}
\{\texttt{lie}[X_1,X_0],\texttt{lie}[X_2,X_0],\texttt{lie}[X_2,X_1]\}\\
\quad\texttt{/.}\{c_{0,2,0}\to 1,c_{0,1,1}\to-c_{1,2,2}-c_{0,2,2},\\
c_{1,2,1}\to\frac{(c_{0,2,1}-1)c_{1,2,2}}{c_{0,2,2}}-1,\\
c_{0,1,2}\to-\frac{c_{0,2,2}(c_{0,2,2}+c_{1,2,2})}{c_{0,2,1}-1}\}\\
\quad\texttt{//Simplify};\\
\%\texttt{/.}c_{1,2,2}\to\beta_1\texttt{/.}c_{0,2,2}\to\beta_2\texttt{/.}c_{0,2,1}\to\beta_3+1\\
\quad\texttt{/.}\beta_3\to\frac{4h^2\beta_2(\beta_1+\beta_2)}{(1+h^2)\beta_1^2}\\
\quad\texttt{//FullSimplify}[\#,\beta_1\neq 0\&\&\beta_2\neq 0\\
\qquad\&\&\beta_{\texttt{i\_}}\in\texttt{Reals}]\&;\\
\%\texttt{/.Flatten@Solve}[\beta_1+\beta_2==-\alpha\&\&\beta_1==\beta,\{\beta_1,\beta_2\}]\\
\quad\texttt{//Collect}[\#,X_{\texttt{i\_}},\texttt{Simplify}]\&\texttt{//Column}
\end{array}$$

$$\begin{array}{l}
\texttt{Reduce}[(\alpha+\beta\neq 0\texttt{/.}\alpha\to-(c_{1,2,2}+c_{0,2,2})\texttt{/.}\beta\to c_{1,2,2})\\
\quad\&\&\texttt{And@@Assumps}]\\
\quad\texttt{//FullSimplify}[\#,\texttt{Assumps2}]\&
\end{array}$$

$$\begin{array}{l}
\texttt{Assumps}=\{c_{1,2,1}\neq-1,c_{0,2,2}\neq 0,c_{0,1,2}\neq 0,c_{1,2,2}\neq 0,\\
c_{0,2,1}\neq 1,c_{1,2,2}+c_{0,2,2}<0,c_{0,2,1}\in\texttt{Reals}\};\\
\texttt{Assumps2}=\texttt{Join}[\texttt{Assumps},\{(c_{0,2,1}-1)\\
\quad\times(-4c_{0,2,2}^2-4c_{0,2,2}c_{1,2,2}+(c_{0,2,1}-1)c_{1,2,2}^2)<0\}];\\
\texttt{Reduce}[c_{1,2,2}\neq 0\&\&\texttt{And@@Assumps}]\\
\quad\texttt{//FullSimplify}[\#,\texttt{Assumps2}]\&\\
\texttt{Reduce}[c_{0,2,2}\neq 0\&\&\texttt{And@@Assumps}]\\
\quad\texttt{//FullSimplify}[\#,\texttt{Assumps2}]\&\\
\texttt{Reduce}[c_{0,2,1}-1\neq 0\&\&\texttt{And@@Assumps}]\\
\quad\texttt{//FullSimplify}[\#,\texttt{Assumps2}]\&
\end{array}$$

$$\begin{array}{l}
\texttt{Reduce}[-(c_{1,2,2}+c_{0,2,2})>0\&\&\texttt{And@@Assumps}]\\
\quad\texttt{//FullSimplify}[\#,\texttt{Assumps2}]\&
\end{array}$$

$$\begin{array}{l}
\texttt{CheckAlgN}[\{c_{0,1,0}\to 0,c_{0,1,1}\to\alpha,c_{0,1,2}\to-\frac{(1+h^2)\beta^2}{4h^2},\\
c_{0,2,0}\to-1,c_{0,2,1}\to\frac{\beta^2+h^2(-2\alpha+\beta)^2}{(1+h^2)\beta^2},c_{0,2,2}\to-(\alpha-\beta),\\
c_{1,2,0}\to-1,c_{1,2,1}\to-\frac{h^2(4\alpha-\beta)-\beta}{(1+h^2)\beta},c_{1,2,2}\to\beta\},\\
\{\alpha>0,\beta\neq 0,\beta\in\texttt{Reals},h>0\},\{\}]
\end{array}$$

$$\begin{array}{l}
\texttt{CheckAlgN}[\{c_{0,2,0}\to 1,c_{0,1,1}\to-c_{1,2,2}-c_{0,2,2},\\
c_{1,2,1}\to\frac{(c_{0,2,1}-1)c_{1,2,2}}{c_{0,2,2}}-1,
\end{array}$$

$$\begin{array}{l}
c_{0,1,2} \to -\frac{c_{0,2,2}(c_{0,2,2}+c_{1,2,2})}{c_{0,2,1}-1}\}, \{c_{1,2,1} \neq -1, \\
c_{0,2,2} \neq 0, c_{0,1,2} \neq 0, c_{1,2,2} \neq 0, c_{0,2,1} \neq 1, \\
c_{1,2,2} + c_{0,2,2} < 0, c_{0,2,1} \in \mathbf{Reals}\}, \{(c_{0,2,1} - 1) \\
\quad \times(-4c_{0,2,2}^2 - 4c_{0,2,2}c_{1,2,2} + (c_{0,2,1} - 1)c_{1,2,2}^2) < 0\}]
\end{array}$$

Find h:

$$\begin{array}{l}
\mathbf{Subs} = \{c_{0,1,0} \to 0, c_{0,1,1} \to -(\beta_1 + \beta_2), \\
c_{0,1,2} \to -\frac{\beta_2(\beta_1+\beta_2)}{\beta_3}, c_{0,2,0} \to 1, c_{0,2,1} \to 1 + \beta_3, \\
c_{0,2,2} \to \beta_2, c_{1,2,0} \to -1, c_{1,2,1} \to -(1 - \frac{\beta_1\beta_3}{\beta_2}), \\
c_{1,2,2} \to \beta_1\};
\end{array}$$

$$\begin{array}{l}
\{c_{1,2,1} \neq 1, c_{0,2,2} \neq 0, c_{0,1,2} \neq 0, c_{1,2,2} \neq 0, c_{0,2,1} \neq 1, \\
c_{1,2,2} > c_{0,2,2}, c_{0,2,1} \in \mathbf{Reals}\}\mathbf{/.Subs} \\
\quad \mathbf{//FullSimplify}[\#, \beta_3(4\beta_1\beta_2 - 4\beta_2^2 + \beta_1^2\beta_3) < 0 \\
\qquad \&\&\beta_1 + \beta_2 < 0\&\&\beta_1 \neq 0\&\&\beta_2 \neq 0\&\&\beta_3 \neq 0 \\
\qquad \&\&\{a_{\mathtt{i\_}}, \beta_{\mathtt{j\_}}\} \in \mathbf{Reals}]\&
\end{array}$$

$\{2\beta_2 \neq \beta_1\beta_3, \mathtt{True}, \mathtt{True}, \mathtt{True}, \mathtt{True}, \beta_1 > \beta_2, \mathtt{True}\}$

$$\begin{array}{l}
\{\mathbf{a0}, \mathbf{a1}, \mathbf{a2}\} = \mathbf{Eigenvalues}[\mathbf{ad}[a_0, a_1, a_2]]\mathbf{/.Subs} \\
\quad \mathbf{//FullSimplify}[\#, \beta_1 \neq 0\&\&\beta_2 \neq 0\&\&\beta_3 \neq 0 \\
\quad \&\&\{a_{\mathtt{i\_}}, \beta_{\mathtt{j\_}}\} \in \mathbf{Reals}]\& \\
(\frac{\mathbf{a1+a2}}{\mathbf{a1-a2}})^2\mathbf{//FullSimplify}[\#, \beta_3(4\beta_1\beta_2 - 4\beta_2^2 + \beta_1^2\beta_3) < 0 \\
\quad \&\&\beta_1 \neq 0\&\&\beta_2 \neq 0\&\&\beta_3 \neq 0 \\
\quad \&\&\{a_{\mathtt{i\_}}, \beta_{\mathtt{j\_}}\} \in \mathbf{Reals}]\& \\
\mathbf{Reduce}[\% == -h^2, h, \mathbf{Reals}] \\
\quad \mathbf{//FullSimplify}[\#, h > 0\&\&2\beta_2 \neq \beta_1\beta_3\&\&\beta_1 > \beta_2 \\
\qquad \&\&\beta_3(4\beta_1\beta_2 - 4\beta_2^2 + \beta_1^2\beta_3) < 0\&\&\beta_1 + \beta_2 < 0 \\
\qquad \&\&\beta_1 \neq 0\&\&\beta_2 \neq 0\&\&\beta_3 \neq 0 \\
\qquad \&\&\{a_{\mathtt{i\_}}, \beta_{\mathtt{j\_}}\} \in \mathbf{Reals}]\&
\end{array}$$

Equivalence Classes

Aff($\mathbb{R}$)$_0$ × $\mathbb{R}$

On Aff($\mathbb{R}$)$_0$ × $\mathbb{R}$, we have a two-parameter family of equivalence classes.

$$\begin{array}{l}
\mathbf{Subs} = \{c_{0,1,0} \to 0, c_{0,1,1} \to 0, c_{0,1,2} \to 0, c_{0,2,0} \to 1, \\
c_{0,2,1} \to 1 - \gamma, c_{0,2,2} \to -\delta, c_{1,2,0} \to -1, c_{1,2,1} \to \gamma - 1, \\
c_{1,2,2} \to \delta\};
\end{array}$$

$$\begin{array}{l}
\{\mathbf{lie}[X_1, X_0], \mathbf{lie}[X_2, X_0], \mathbf{lie}[X_2, X_1]\}\mathbf{/.Subs} \\
\quad \mathbf{//Simplify;} \\
\mathbf{\%//Simplify}[\#, \delta \geq 0\&\&\gamma \in \mathbf{Reals}]\&\mathbf{//Column}
\end{array}$$

$$\begin{array}{l}
\{\varrho_0, \varrho_1, \varrho_2\}\mathbf{/.Subs} \\
\quad \mathbf{//FullSimplify}[\#, \delta \geq 0\&\&\gamma \in \mathbf{Reals}]\& \\
\{\kappa, \chi_1, \chi_2, \vartheta\}\mathbf{/.Subs} \\
\quad \mathbf{//FullSimplify}[\#, \delta \geq 0\&\&\gamma \in \mathbf{Reals}]\&
\end{array}$$

$\mathbf{CheckAlgN}[\mathbf{Subs}, \{\delta \geq 0, \gamma \in \mathbf{Reals}\}, \{\delta \neq 0 || \gamma \neq 0\}]$

Check which of the equivalence classes are flat (using the characterising equations found in chapter 4).

$$\begin{array}{l}
\{-c_{0,1,0}c_{0,2,2} + c_{0,2,1}c_{1,2,1} + c_{0,1,2}(c_{0,2,0} + c_{1,2,1}) \\
+(-c_{0,1,1} + c_{0,2,2})c_{1,2,2} + c_{0,2,0}(c_{1,2,1}^2 + c_{1,2,2}^2), \\
c_{0,2,2}c_{1,2,1} - c_{0,1,1}(c_{0,2,0} + c_{1,2,1}) - (c_{0,1,2} + c_{0,2,1})c_{1,2,2} \\
+c_{0,1,0}(c_{0,2,1} - c_{1,2,1}^2 - c_{1,2,2}^2)\} \\
\quad \mathbf{/.Subs//Simplify}
\end{array}$$

$\{0, 0\}$

That is, all structures (for this case, i.e., $\vartheta > 0$) on Aff($\mathbb{R}$)$_0$ × $\mathbb{R}$ are flat.

G$_{3.2}$, case 1

On G$_{3.2}$, we have a one-parameter family and a two-parameter family of equivalence classes. This section deals with the first family.

$$\begin{array}{l}
\mathbf{Subs} = \{c_{0,1,0} \to 0, c_{0,1,1} \to 0, c_{0,1,2} \to 0, c_{0,2,0} \to 1, \\
c_{0,2,1} \to \frac{1}{4}(\beta - 2)^2, c_{0,2,2} \to 0, c_{1,2,0} \to -1, \\
c_{1,2,1} \to \beta - 1, c_{1,2,2} \to 0\};
\end{array}$$

$$\begin{array}{l}
\{\mathbf{lie}[X_1, X_0], \mathbf{lie}[X_2, X_0], \mathbf{lie}[X_2, X_1]\}\mathbf{/.Subs} \\
\quad \mathbf{//Simplify;} \\
\mathbf{\%//Simplify}[\#, \alpha > 0]\&\mathbf{//Collect}[\#, X_{\mathtt{i\_}}]\&\mathbf{//Column}
\end{array}$$

$$\begin{array}{l}
\{\varrho_0, \varrho_1, \varrho_2\}\mathbf{/.Subs} \\
\quad \mathbf{//FullSimplify}[\#, \beta \neq 0\&\&\beta \in \mathbf{Reals}]\& \\
\{\kappa, \chi_1, \chi_2, \vartheta\}\mathbf{/.Subs} \\
\quad \mathbf{//FullSimplify}[\#, \beta \neq 0\&\&\beta \in \mathbf{Reals}]\&
\end{array}$$

$\mathbf{CheckAlgN}[\mathbf{Subs}, \{\beta \neq 0, \beta \in \mathbf{Reals}\}, \{\}]$

Check which of the equivalence classes are flat (using the characterising equations found in chapter 4).

$$\begin{array}{l}
\{-c_{0,1,0}c_{0,2,2} + c_{0,2,1}c_{1,2,1} + c_{0,1,2}(c_{0,2,0} + c_{1,2,1}) \\
+(-c_{0,1,1} + c_{0,2,2})c_{1,2,2} + c_{0,2,0}(c_{1,2,1}^2 + c_{1,2,2}^2), \\
c_{0,2,2}c_{1,2,1} - c_{0,1,1}(c_{0,2,0} + c_{1,2,1}) - (c_{0,1,2} + c_{0,2,1})c_{1,2,2} \\
+c_{0,1,0}(c_{0,2,1} - c_{1,2,1}^2 - c_{1,2,2}^2)\} \\
\quad \mathbf{/.Subs//Simplify}
\end{array}$$

$\{\frac{1}{4}(\beta - 1)\beta^2, 0\}$

That is, all structures (for this case, i.e., $\vartheta > 0$ and considering the one-parameter family of equivalence classes) on G$_{3.2}$ with $\beta = 1$ are flat.

G$_{3.2}$, case 2

On G$_{3.2}$, we have a one-parameter family and a two-parameter family of equivalence classes. This section deals with the second family.

$$\begin{array}{l}
\mathbf{Subs} = \{c_{0,1,0} \to 0, c_{0,1,1} \to \delta, c_{0,1,2} \to -\frac{\beta^2}{4}, c_{0,2,0} \to 1, \\
c_{0,2,1} \to \frac{(2\delta+\beta)^2}{\beta^2}, c_{0,2,2} \to -(\delta + \beta), c_{1,2,0} \to -1, \\
c_{1,2,1} \to -(1 + \frac{4\delta}{\beta}), c_{1,2,2} \to \beta\};
\end{array}$$

$$\begin{array}{l}
\{\mathbf{lie}[X_1, X_0], \mathbf{lie}[X_2, X_0], \mathbf{lie}[X_2, X_1]\}\mathbf{/.Subs} \\
\quad \mathbf{//Simplify;} \\
\mathbf{\%//FullSimplify}[\#, \delta \geq 0\&\&\beta \neq 0\&\&\beta \in \mathbf{Reals}]\& \\
\quad \mathbf{//Collect}[\#, X_{\mathtt{i\_}}]\&\mathbf{//Column}
\end{array}$$

$$\begin{array}{l}
\{\kappa, \chi_1, \chi_2, \vartheta\}\mathbf{/.Subs} \\
\quad \mathbf{//FullSimplify}[\#, \delta \geq 0\&\&\beta \neq 0\&\&\beta \in \mathbf{Reals}]\& \\
\{\varrho_0, \varrho_1, \varrho_2\}\mathbf{/.Subs} \\
\quad \mathbf{//FullSimplify}[\#, \delta \geq 0\&\&\beta \neq 0\&\&\beta \in \mathbf{Reals}]\&
\end{array}$$

$\mathbf{CheckAlgN}[\mathbf{Subs}, \{\delta \geq 0, \beta \neq 0, \beta \in \mathbf{Reals}\}, \{\}]$

Check which of the equivalence classes are flat (using the characterising equations found in chapter 4).

$$\begin{array}{l}
\{-c_{0,1,0}c_{0,2,2} + c_{0,2,1}c_{1,2,1} + c_{0,1,2}(c_{0,2,0} + c_{1,2,1}) \\
+(-c_{0,1,1} + c_{0,2,2})c_{1,2,2} + c_{0,2,0}(c_{1,2,1}^2 + c_{1,2,2}^2), \\
c_{0,2,2}c_{1,2,1} - c_{0,1,1}(c_{0,2,0} + c_{1,2,1}) - (c_{0,1,2} + c_{0,2,1})c_{1,2,2} \\
+c_{0,1,0}(c_{0,2,1} - c_{1,2,1}^2 - c_{1,2,2}^2)\} \\
\quad \mathbf{/.Subs//Simplify}
\end{array}$$

$\{-\frac{\delta(\beta^4+4\beta\delta+16\delta^2)}{\beta^3}, \frac{\beta^3}{4} + \delta + \frac{4\delta^2}{\beta}\}$

$$\begin{aligned}&\texttt{Reduce}[\beta^4+4\beta\delta+16\delta^2==0\&\&\delta\geq 0\\ &\quad\&\&\beta\neq 0\&\&\beta\in\texttt{Reals},\{\beta,\delta\},\texttt{Reals}]\\ &\qquad\texttt{//FullSimplify}[\#,\delta\geq 0\&\&\beta\neq 0\&\&\beta\in\texttt{Reals}]\&\end{aligned}$$

$$\begin{aligned}&(\beta==-\tfrac{1}{2}\&\&16\delta==1)\|(-\tfrac{1}{2}<\beta<0\&\&\\ &(\beta+\sqrt{\beta^2-4\beta^4}+8\delta==0\|\sqrt{\beta^2-4\beta^4}==\beta+8\delta))\end{aligned}$$

That is, all structures (for this case, i.e., $\vartheta > 0$ and considering the two-parameter family of equivalence classes) on $\mathrm{G}_{3.2}$ satisfying the conditions above are flat.

$\mathrm{G}^h_{3.4}$, case 1

On $\mathrm{G}^h_{3.4}$, we have a one-parameter family and a two-parameter family of equivalence classes. This section deals with the first family.

$$\begin{aligned}&\texttt{Subs}=\{c_{0,1,0}\to 0,c_{0,1,1}\to 0,c_{0,1,2}\to 0,c_{0,2,0}\to 1,\\ &c_{0,2,1}\to\tfrac{(h^2(\beta-2)^2-\beta^2)}{4h^2},c_{0,2,2}\to 0,c_{1,2,0}\to -1,\\ &c_{1,2,1}\to -(1-\beta),c_{1,2,2}\to 0\};\end{aligned}$$

$$\begin{aligned}&\{\texttt{lie}[X_1,X_0],\texttt{lie}[X_2,X_0],\texttt{lie}[X_2,X_1]\}\texttt{/.Subs}\\ &\qquad\texttt{//Simplify;}\\ &\texttt{\%//Simplify//Column}\end{aligned}$$

$$\begin{aligned}&\{\kappa,\chi_1,\chi_2,\vartheta\}\texttt{/.Subs}\\ &\qquad\texttt{//FullSimplify}[\#,\beta\neq 0\&\&\beta\in\texttt{Reals}\&\&h>0\\ &\qquad\qquad\&\&h\neq 1]\&\\ &\{\varrho_0,\varrho_1,\varrho_2\}\texttt{/.Subs}\\ &\qquad\texttt{//FullSimplify}[\#,\beta\neq 0\&\&\beta\in\texttt{Reals}\&\&h>0\\ &\qquad\qquad\&\&h\neq 1]\&\end{aligned}$$

$$\texttt{CheckAlgN}[\texttt{Subs},\{\beta\neq 0,\beta\in\texttt{Reals},h>0,h\neq 1\},\{\}]$$

Check which of the equivalence classes are flat (using the characterising equations found in chapter 4).

$$\begin{aligned}&\{-c_{0,1,0}c_{0,2,2}+c_{0,2,1}c_{1,2,1}+c_{0,1,2}(c_{0,2,0}+c_{1,2,1})\\ &+(-c_{0,1,1}+c_{0,2,2})c_{1,2,2}+c_{0,2,0}(c^2_{1,2,1}+c^2_{1,2,2}),\\ &c_{0,2,2}c_{1,2,1}-c_{0,1,1}(c_{0,2,0}+c_{1,2,1})-(c_{0,1,2}+c_{0,2,1})c_{1,2,2}\\ &+c_{0,1,0}(c_{0,2,1}-c^2_{1,2,1}-c^2_{1,2,2})\}\\ &\qquad\texttt{/.Subs//Simplify}\end{aligned}$$

$$\{\tfrac{(h^2-1)(\beta-1)\beta^2}{4h^2},0\}$$

That is, all structures (for this case, i.e., $\vartheta > 0$ and considering the one-parameter family of equivalence classes) on $\mathrm{G}^h_{3.4}$ with $\beta = 1$ are flat.

$\mathrm{G}^h_{3.4}$, case 2

On $\mathrm{G}^h_{3.4}$, we have a one-parameter family and a two-parameter family of equivalence classes. This section deals with the second family.

$$\begin{aligned}&\texttt{Subs}=\{c_{0,1,0}\to 0,c_{0,1,1}\to\delta,c_{0,1,2}\to -\tfrac{(h^2-1)\beta^2}{4h^2},\\ &c_{0,2,0}\to 1,c_{0,2,1}\to -(\tfrac{\beta^2-h^2(2\delta+\beta)^2}{(h^2-1)\beta^2}),c_{0,2,2}\to -(\delta+\beta),\\ &c_{1,2,0}\to -1,c_{1,2,1}\to -\tfrac{-\beta+h^2(4\delta+\beta)}{(h^2-1)\beta},c_{1,2,2}\to\beta\};\end{aligned}$$

$$\begin{aligned}&\{\texttt{lie}[X_1,X_0],\texttt{lie}[X_2,X_0],\texttt{lie}[X_2,X_1]\}\texttt{/.Subs}\\ &\qquad\texttt{//Simplify;}\\ &\texttt{\%//Simplify}[\#,\delta\geq 0\&\&\beta\neq 0\&\&\beta\in\texttt{Reals}]\&\\ &\qquad\texttt{//Collect}[\#,X_{\texttt{i\_}}]\&\texttt{//Column}\end{aligned}$$

$$\begin{aligned}&\{\kappa,\chi_1,\chi_2,\vartheta\}\texttt{/.Subs}\\ &\qquad\texttt{//FullSimplify}[\#,\delta\geq 0\&\&\beta\neq 0\&\&\beta\in\texttt{Reals}\\ &\qquad\qquad\&\&h>0\&\&h\neq 1]\&\\ &\{\varrho_0,\varrho_1,\varrho_2\}\texttt{/.Subs}\\ &\qquad\texttt{//FullSimplify}[\#,\delta\geq 0\&\&\beta\neq 0\&\&\beta\in\texttt{Reals}\\ &\qquad\qquad\&\&h>0\&\&h\neq 1]\&\end{aligned}$$

$$\texttt{CheckAlgN}[\texttt{Subs},\{\delta\geq 0,\beta\neq 0,\beta\in\texttt{Reals},h>0\},\{\}]$$

Check which of the equivalence classes are flat (using the characterising equations found in chapter 4).

$$\begin{aligned}&\{-c_{0,1,0}c_{0,2,2}+c_{0,2,1}c_{1,2,1}+c_{0,1,2}(c_{0,2,0}+c_{1,2,1})\\ &+(-c_{0,1,1}+c_{0,2,2})c_{1,2,2}+c_{0,2,0}(c^2_{1,2,1}+c^2_{1,2,2}),\\ &c_{0,2,2}c_{1,2,1}-c_{0,1,1}(c_{0,2,0}+c_{1,2,1})-(c_{0,1,2}+c_{0,2,1})c_{1,2,2}\\ &+c_{0,1,0}(c_{0,2,1}-c^2_{1,2,1}-c^2_{1,2,2})\}\\ &\qquad\texttt{/.Subs//Simplify}\end{aligned}$$

$$\begin{aligned}&\{-\tfrac{\delta((h^2-1)^2\beta^4+4h^2(h^2-1)\beta\delta+16h^4\delta^2)}{(h^2-1)^2\beta^3},\\ &\tfrac{(h^2-1)^2\beta^4+4h^2(h^2-1)\beta\delta+16h^4\delta^2}{4h^2(h^2-1)\beta}\}\end{aligned}$$

$$\begin{aligned}&\texttt{Reduce}[-\tfrac{\delta((h^2-1)^2\beta^4+4h^2(h^2-1)\beta\delta+16h^4\delta^2)}{(h^2-1)^2\beta^3}==0\\ &\quad\&\&\tfrac{(h^2-1)^2\beta^4+4h^2(h^2-1)\beta\delta+16h^4\delta^2}{4h^2(h^2-1)\beta}==0\\ &\quad\&\&\delta\geq 0\&\&\beta\neq 0\&\&\beta\in\texttt{Reals}\&\&0<h<1,\\ &\quad\{\beta,\delta\},\texttt{Reals}]\\ &\qquad\texttt{//FullSimplify}[\#,\delta\geq 0\&\&\beta\neq 0\&\&\beta\in\texttt{Reals}\\ &\qquad\qquad\&\&0<h<1]\&\\ &\texttt{Reduce}[-\tfrac{\delta((h^2-1)^2\beta^4+4h^2(h^2-1)\beta\delta+16h^4\delta^2)}{(h^2-1)^2\beta^3}==0\\ &\quad\&\&\tfrac{(h^2-1)^2\beta^4+4h^2(h^2-1)\beta\delta+16h^4\delta^2}{4h^2(h^2-1)\beta}==0\\ &\quad\&\&\delta\geq 0\&\&\beta\neq 0\&\&\beta\in\texttt{Reals}\&\&1<h,\{\beta,\delta\},\texttt{Reals}]\\ &\qquad\texttt{//FullSimplify}[\#,\delta\geq 0\&\&\beta\neq 0\&\&\beta\in\texttt{Reals}\\ &\qquad\qquad\&\&1<h]\&\end{aligned}$$

$$\begin{aligned}&(\beta==-\tfrac{1}{2}\&\&1+16h^2\delta==h^2)\|\\ &(-\tfrac{1}{2}<\beta<0\&\&(\delta==-\tfrac{(h^2-1)(\beta+\sqrt{\beta^2-4\beta^4})}{8h^2}\|\\ &\delta==\tfrac{(h^2-1)(-\beta+\sqrt{\beta^2-4\beta^4})}{8h^2}))\end{aligned}$$

$$\begin{aligned}&(0<\beta<\tfrac{1}{2}\&\&(\delta==\tfrac{(h^2-1)(-\beta+\sqrt{\beta^2-4\beta^4})}{8h^2}\|\\ &\delta==-\tfrac{(h^2-1)(\beta+\sqrt{\beta^2-4\beta^4})}{8h^2}))\|\\ &(2\beta==1\&\&h^2(1+16\delta)==1)\end{aligned}$$

That is, all structures (for this case, i.e., $\vartheta > 0$ and considering the two-parameter family of equivalence classes) on $\mathrm{G}^h_{3.4}$ satisfying the conditions above (in the first case, with $0 < h < 1$, and in the second, with $1 < h$) are flat.

$\mathrm{G}^h_{3.5}$, case 1

On $\mathrm{G}^h_{3.5}$, we have a one-parameter family and a two-parameter family of equivalence classes. This section deals with the first family.

$$\begin{aligned}&\texttt{Subs}=\{c_{0,1,0}\to 0,c_{0,1,1}\to 0,c_{0,1,2}\to 0,c_{0,2,0}\to 1,\\ &c_{0,2,1}\to\tfrac{h^2(\beta-2)^2+\beta^2}{4h^2},c_{0,2,2}\to 0,c_{1,2,0}\to -1,\\ &c_{1,2,1}\to\beta-1,c_{1,2,2}\to 0\};\end{aligned}$$

$$\begin{aligned}&\{\texttt{lie}[X_1,X_0],\texttt{lie}[X_2,X_0],\texttt{lie}[X_2,X_1]\}\texttt{/.Subs}\\ &\qquad\texttt{//Simplify;}\\ &\texttt{\%//Simplify//Column}\end{aligned}$$

$$\begin{aligned}&\{\varrho_0,\varrho_1,\varrho_2\}\texttt{/.Subs}\\ &\qquad\texttt{//FullSimplify}[\#,\beta\neq 0\&\&\beta\in\texttt{Reals}\&\&h>0]\&\\ &\{\kappa,\chi_1,\chi_2\}\texttt{/.Subs}\\ &\qquad\texttt{//FullSimplify}[\#,\beta\neq 0\&\&\beta\in\texttt{Reals}\&\&h>0]\&\end{aligned}$$

$$\texttt{CheckAlgN[Subs,}\,\{\beta\neq 0,\beta\in\texttt{Reals},h>0\},\{\}]$$

Check which of the equivalence classes are flat (using the characterising equations found in chapter 4).

$$\begin{aligned}&\{-c_{0,1,0}c_{0,2,2}+c_{0,2,1}c_{1,2,1}+c_{0,1,2}(c_{0,2,0}+c_{1,2,1})\\&+(-c_{0,1,1}+c_{0,2,2})c_{1,2,2}+c_{0,2,0}(c^2_{1,2,1}+c^2_{1,2,2}),\\&c_{0,2,2}c_{1,2,1}-c_{0,1,1}(c_{0,2,0}+c_{1,2,1})-(c_{0,1,2}+c_{0,2,1})c_{1,2,2}\\&+c_{0,1,0}(c_{0,2,1}-c^2_{1,2,1}-c^2_{1,2,2})\}\\&\qquad\texttt{/.Subs//Simplify}\end{aligned}$$

$$\{\tfrac{(1+h^2)(\beta-1)\beta^2}{4h^2},0\}$$

That is, all structures (for this case, i.e., $\vartheta>0$ and considering the one-parameter family of equivalence classes) on $\mathsf{G}^h_{3.5}$ with $\beta=1$ are flat.

$\mathsf{G}^h_{3.5}$, case 2

On $\mathsf{G}^h_{3.5}$, we have a one-parameter family and a two-parameter family of equivalence classes. This section deals with the second family.

$$\begin{aligned}&\texttt{Subs}=\{c_{0,1,0}\to 0,c_{0,1,1}\to\delta,c_{0,1,2}\to-\tfrac{(1+h^2)\beta^2}{4h^2},\\&c_{0,2,0}\to 1,c_{0,2,1}\to\tfrac{\beta^2+h^2(2\delta+\beta)^2}{(1+h^2)\beta^2},c_{0,2,2}\to-(\delta+\beta),\\&c_{1,2,0}\to -1,c_{1,2,1}\to-\tfrac{\beta+h^2(4\delta+\beta)}{(1+h^2)\beta},c_{1,2,2}\to\beta\};\end{aligned}$$

$$\begin{aligned}&\{\texttt{lie}[X_1,X_0],\texttt{lie}[X_2,X_0],\texttt{lie}[X_2,X_1]\}\texttt{/.Subs}\\&\qquad\texttt{//Simplify;}\\&\texttt{\%//Simplify}[\#,\delta\geq 0\&\&\beta\neq 0\&\&\beta\in\texttt{Reals}]\&\\&\qquad\texttt{//Collect}[\#,X_{\texttt{i\_}}]\&\texttt{//Column}\end{aligned}$$

$$\begin{aligned}&\{\kappa,\chi_1,\chi_2\}\texttt{/.Subs}\\&\qquad\texttt{//FullSimplify}[\#,\delta\geq 0\&\&\beta\neq 0\&\&\beta\in\texttt{Reals}\\&\qquad\qquad\&\&h>0]\&\\&\{\varrho_0,\varrho_1,\varrho_2\}\texttt{/.Subs}\\&\qquad\texttt{//FullSimplify}[\#,\delta\geq 0\&\&\beta\neq 0\&\&\beta\in\texttt{Reals}\\&\qquad\qquad\&\&h>0]\&\end{aligned}$$

$$\texttt{CheckAlgN[Subs,}\,\{\delta\geq 0,\beta\neq 0,\beta\in\texttt{Reals},h>0\},\{\}]$$

Check which of the equivalence classes are flat (using the characterising equations found in chapter 4).

$$\begin{aligned}&\{-c_{0,1,0}c_{0,2,2}+c_{0,2,1}c_{1,2,1}+c_{0,1,2}(c_{0,2,0}+c_{1,2,1})\\&+(-c_{0,1,1}+c_{0,2,2})c_{1,2,2}+c_{0,2,0}(c^2_{1,2,1}+c^2_{1,2,2}),\\&c_{0,2,2}c_{1,2,1}-c_{0,1,1}(c_{0,2,0}+c_{1,2,1})-(c_{0,1,2}+c_{0,2,1})c_{1,2,2}\\&+c_{0,1,0}(c_{0,2,1}-c^2_{1,2,1}-c^2_{1,2,2})\}\\&\qquad\texttt{/.Subs//Simplify}\end{aligned}$$

$$\begin{aligned}&\{-\tfrac{\delta((1+h^2)^2\beta^4+4h^2(1+h^2)\beta\delta+16h^4\delta^2)}{(1+h^2)^2\beta^3},\\&\tfrac{(1+h^2)^2\beta^4+4h^2(1+h^2)\beta\delta+16h^4\delta^2}{4h^2(1+h^2)\beta}\}\end{aligned}$$

$$\begin{aligned}&\texttt{Reduce}[-\tfrac{\delta((1+h^2)^2\beta^4+4h^2(1+h^2)\beta\delta+16h^4\delta^2)}{(1+h^2)^2\beta^3}==0\\&\quad\&\&\tfrac{(1+h^2)^2\beta^4+4h^2(1+h^2)\beta\delta+16h^4\delta^2}{4h^2(1+h^2)\beta}==0\\&\quad\&\&\delta\geq 0\&\&\beta\neq 0\&\&\beta\in\texttt{Reals}\\&\quad\&\&0<h,\{\beta,\delta\},\texttt{Reals}]\\&\qquad\texttt{//FullSimplify}[\#,\delta\geq 0\&\&\beta\neq 0\&\&\beta\in\texttt{Reals}\\&\qquad\qquad\&\&0<h]\&\end{aligned}$$

$$\begin{aligned}&(\beta==-\tfrac{1}{2}\&\&h^2(16\delta-1)==1)\|\\&(-\tfrac{1}{2}<\beta<0\&\&(\delta==-\tfrac{(1+h^2)(\beta+\sqrt{\beta^2-4\beta^4})}{8h^2}\|\\&\beta+h^2\beta+8h^2\delta==(1+h^2)\sqrt{1-4\beta^2}\texttt{Abs}[\beta]))\end{aligned}$$

That is, all structures (for this case, i.e., $\vartheta>0$ and considering the two-parameter family of equivalence classes) on $\mathsf{G}^h_{3.5}$ satisfying the conditions above are flat.

2 $Y_0\in\mathcal{D}^\perp$

$$\begin{aligned}&\{\texttt{lie}[X_1,X_0],\texttt{lie}[X_2,X_0],\texttt{lie}[X_2,X_1]\}\\&\qquad\texttt{/.}\{c_{0,2,0}\to 0,c_{0,2,2}\to -c_{0,1,1}\}\\&\qquad\texttt{//Simplify//Column}\end{aligned}$$

We can reuse the **CheckAlg** routines from the previous section (**CheckAlgU** for a uniodular algebra, and **CheckAlgN** for a non-unimodular algebra).

2.1 $\chi_1>0$

$$\begin{aligned}&\{\texttt{lie}[X_1,X_0],\texttt{lie}[X_2,X_0],\texttt{lie}[X_2,X_1]\}\\&\qquad\texttt{/.}\{c_{0,2,0}\to 0,c_{0,2,2}\to -c_{0,1,1}\}\texttt{/.}\{c_{0,1,1}\to 0\}\\&\qquad\texttt{//Simplify//Column}\end{aligned}$$

2.1.1 $c^1_{21}=c^2_{21}=0$

$$\begin{aligned}&\{\texttt{lie}[X_1,X_0],\texttt{lie}[X_2,X_0],\texttt{lie}[X_2,X_1]\}\\&\qquad\texttt{/.}\{c_{0,2,0}\to 0,c_{0,2,2}\to -c_{0,1,1}\}\texttt{/.}c_{0,1,1}\to 0\\&\qquad\texttt{/.}\{c_{1,2,1}\to 0,c_{1,2,2}\to 0\}\\&\qquad\texttt{//Simplify//Column}\end{aligned}$$

$$\begin{aligned}&\{\texttt{m1}<0\&\&\texttt{m2}>0,\texttt{m1}\geq 0\|\texttt{m2}\leq 0,\texttt{m3}==0\}\\&\qquad\texttt{/.}\{c_{0,2,0}\to 0,c_{0,2,2}\to -c_{0,1,1}\}\texttt{/.}c_{0,1,1}\to 0\\&\qquad\texttt{/.}\{c_{1,2,1}\to 0,c_{1,2,2}\to 0\}\\&\qquad\texttt{//FullSimplify}[\#,c_{0,1,2}+c_{0,2,1}<0]\&\texttt{//Column}\end{aligned}$$

$$\begin{aligned}&\{\texttt{k1}\neq 0\&\&\texttt{k2}>0,\texttt{k1}==0\|\texttt{k2}\leq 0\}\\&\qquad\texttt{/.}\{c_{0,2,0}\to 0,c_{0,2,2}\to -c_{0,1,1}\}\texttt{/.}c_{0,1,1}\to 0\\&\qquad\texttt{/.}\{c_{1,2,1}\to 0,c_{1,2,2}\to 0\}\\&\qquad\texttt{//FullSimplify}[\#,c_{0,1,2}+c_{0,2,1}<0]\&\texttt{//Column}\end{aligned}$$

2.1.1.1 $m_3=0$ ($\mathcal{K}$ deg.)

2.1.1.1.1 $c^1_{20}=0$

In this case, we get a one-parameter family of equivalence classes on $\widetilde{\mathsf{SE}}(2)$.

$$\begin{aligned}&\{\texttt{lie}[X_1,X_0],\texttt{lie}[X_2,X_0],\texttt{lie}[X_2,X_1]\}\\&\qquad\texttt{/.}\{c_{0,2,0}\to 0,c_{0,2,2}\to 0,c_{0,1,1}\to 0,c_{1,2,1}\to 0,\\&c_{1,2,2}\to 0\}\texttt{/.}c_{0,2,1}\to 0\\&\qquad\texttt{//Simplify;}\\&\texttt{\%/.}c_{0,1,2}\to-\alpha\texttt{//Column}\end{aligned}$$

$$\begin{aligned}&\{\kappa,\tfrac{1}{2}(c_{1,0,2}+c_{2,0,1})\}\texttt{/.}\{c_{0,2,0}\to 0,c_{0,2,2}\to 0,c_{0,1,1}\to 0,\\&c_{1,2,1}\to 0,c_{1,2,2}\to 0,c_{0,2,1}\to 0\}\texttt{/.}c_{0,1,2}\to-\alpha\\&\qquad\texttt{//Simplify}[\#,\alpha>0]\&\end{aligned}$$

$$\begin{aligned}&\texttt{CheckAlgU}[\{c_{0,2,0}\to 0,c_{0,2,2}\to 0,c_{0,1,1}\to 0,c_{1,2,1}\to 0,\\&c_{1,2,2}\to 0,c_{0,2,1}\to 0,c_{0,1,2}\to-\alpha\},\{\alpha>0\},\{\}]\end{aligned}$$

$$\begin{aligned}&\texttt{CheckAlgU}[\{c_{0,2,0}\to 0,c_{0,2,2}\to 0,c_{0,1,1}\to 0,c_{1,2,1}\to 0,\\&c_{1,2,2}\to 0,c_{0,2,1}\to 0\},\{c_{\texttt{u\_},\texttt{v\_},\texttt{w\_}}\in\texttt{Reals},c_{0,1,2}<0\},\{\}]\end{aligned}$$

2.1.1.1.2 $c^1_{20} \neq 0$

In this case, we get a one-parameter family of equivalence classes on SE(1, 1).

$\{\texttt{lie}[X_1,X_0],\texttt{lie}[X_2,X_0],\texttt{lie}[X_2,X_1]\}$
$\quad/.\{c_{0,2,0}\to 0,c_{0,2,2}\to 0,c_{0,1,1}\to 0,c_{1,2,1}\to 0,$
$c_{1,2,2}\to 0\}/.c_{0,1,2}\to 0$
$\quad//\texttt{Simplify};$
$\%/.c_{0,2,1}\to -\alpha//\texttt{Column}$

$\{\kappa,\frac{1}{2}(c_{1,0,2}+c_{2,0,1})\}/.\{c_{0,2,0}\to 0,c_{0,2,2}\to 0,c_{0,1,1}\to 0,$
$c_{1,2,1}\to 0,c_{1,2,2}\to 0,c_{0,1,2}\to 0\}/.c_{0,2,1}\to -\alpha$
$\quad//\texttt{Simplify}[\#,\alpha>0]\&$

$\texttt{CheckAlgU}[\{c_{0,2,0}\to 0,c_{0,2,2}\to 0,c_{0,1,1}\to 0,c_{1,2,1}\to 0,$
$c_{1,2,2}\to 0,c_{0,1,2}\to 0,c_{0,2,1}\to -\alpha\},\{\alpha>0\},\{\}]$

$\texttt{CheckAlgU}[\{c_{0,2,0}\to 0,c_{0,2,2}\to 0,c_{0,1,1}\to 0,c_{1,2,1}\to 0,$
$c_{1,2,2}\to 0,c_{0,1,2}\to 0\},\{c_{\texttt{u\_},\texttt{v\_},\texttt{w\_}}\in\texttt{Reals},c_{0,2,1}<0\},\{\}]$

2.1.1.2 $m_1 < 0$ and $m_2 > 0$ ($\mathcal{K}$ neg. def.)

In this case, we get a two-parameter family of equivalence classes on SU(2).

$\{\texttt{lie}[X_1,X_0],\texttt{lie}[X_2,X_0],\texttt{lie}[X_2,X_1]\}$
$\quad/.\{c_{0,2,0}\to 0,c_{0,2,2}\to 0,c_{0,1,1}\to 0,c_{1,2,1}\to 0,$
$c_{1,2,2}\to 0\}$
$\quad//\texttt{Simplify};$
$\%/.c_{0,2,1}\to \alpha_2/.c_{0,1,2}\to -\alpha_1//\texttt{Column}$

$-c_{0,1,2}>0//\texttt{FullSimplify}[\#,c_{0,1,2}+c_{0,2,1}<0$
$\quad\&\&c_{0,2,1}>0]\&$
$c_{0,2,1}<-c_{0,1,2}//\texttt{FullSimplify}[\#,c_{0,1,2}+c_{0,2,1}<0$
$\quad\&\&c_{0,2,1}>0]\&$

$\{\kappa,\frac{1}{2}(c_{1,0,2}+c_{2,0,1})\}/.\{c_{0,2,0}\to 0,c_{0,2,2}\to 0,c_{0,1,1}\to 0,$
$c_{1,2,1}\to 0,c_{1,2,2}\to 0\}/.c_{0,2,1}\to \alpha_2/.c_{0,1,2}\to -\alpha_1$
$\quad//\texttt{FullSimplify}[\#,\alpha_1>0\&\&\alpha_2>0\&\&\alpha_1>\alpha_2]\&$

$\texttt{CheckAlgU}[\{c_{0,2,0}\to 0,c_{0,2,2}\to 0,c_{0,1,1}\to 0,c_{1,2,1}\to 0,$
$c_{1,2,2}\to 0,c_{0,2,1}\to \alpha_2,c_{0,1,2}\to -\alpha_1\},\{\alpha_1>0,\alpha_2>0\},$
$\{\alpha_1>\alpha_2\}]$

$\texttt{CheckAlgU}[\{c_{0,2,0}\to 0,c_{0,2,2}\to 0,c_{0,1,1}\to 0,c_{1,2,1}\to 0,$
$c_{1,2,2}\to 0\},\{c_{\texttt{u\_},\texttt{v\_},\texttt{w\_}}\in\texttt{Reals},c_{0,2,1}>0\},$
$\{c_{0,1,2}+c_{0,2,1}<0\}]$

2.1.1.3 $m_3 < 0$ and ($m_1 \geq 0$ or $m_2 \leq 0$) ($\mathcal{K}$ nondeg. but indef.)

$\texttt{m3}<0\&\&(\texttt{m1}\geq 0\|\texttt{m2}\leq 0)\&\&(\texttt{k1}\neq 0\&\&\texttt{k2}>0)$
$\quad/.\{c_{0,2,0}\to 0,c_{0,2,2}\to -c_{0,1,1}\}/.c_{0,1,1}\to 0$
$\quad/.\{c_{1,2,1}\to 0,c_{1,2,2}\to 0\}$
$\quad//\texttt{FullSimplify}[\#,c_{0,1,2}+c_{0,2,1}<0]\&$
$\texttt{m3}<0\&\&(\texttt{m1}\geq 0\|\texttt{m2}\leq 0)\&\&(\texttt{k1}==0\|\texttt{k2}\leq 0)$
$\quad/.\{c_{0,2,0}\to 0,c_{0,2,2}\to -c_{0,1,1}\}/.c_{0,1,1}\to 0$
$\quad/.\{c_{1,2,1}\to 0,c_{1,2,2}\to 0\}$
$\quad//\texttt{FullSimplify}[\#,c_{0,1,2}+c_{0,2,1}<0]\&$

2.1.1.3.1 $k_1 \neq 0$ and $k_2 > 0$ ($\mathcal{K}_{\mathcal{D}_1}$ def.)

In this case, we get a two-parameter family of equivalence classes on $\widetilde{\mathsf{SL}}(2,\mathbb{R})_{ell}$.

$\{\texttt{lie}[X_1,X_0],\texttt{lie}[X_2,X_0],\texttt{lie}[X_2,X_1]\}$
$\quad/.\{c_{0,2,0}\to 0,c_{0,2,2}\to 0,c_{0,1,1}\to 0,c_{1,2,1}\to 0,$
$c_{1,2,2}\to 0\}$
$\quad//\texttt{Simplify};$
$\%/.c_{0,1,2}\to \alpha_1/.c_{0,2,1}\to -\alpha_2//\texttt{Column}$

$-c_{0,2,1}>0//\texttt{FullSimplify}[\#,c_{0,1,2}+c_{0,2,1}<0$
$\quad\&\&c_{0,1,2}>0]\&$
$c_{0,2,1}<-c_{0,1,2}//\texttt{FullSimplify}[\#,c_{0,1,2}+c_{0,2,1}<0$
$\quad\&\&c_{0,1,2}>0]\&$

$\{\kappa,\frac{1}{2}(c_{1,0,2}+c_{2,0,1})\}/.\{c_{0,2,0}\to 0,c_{0,2,2}\to 0,c_{0,1,1}\to 0,$
$c_{1,2,1}\to 0,c_{1,2,2}\to 0\}/.c_{0,1,2}\to \alpha_1/.c_{0,2,1}\to -\alpha_2$
$\quad//\texttt{Simplify}[\#,\alpha_1>0\&\&\alpha_2>0\&\&\alpha_1<\alpha_2]\&$

$\texttt{CheckAlgU}[\{c_{0,2,0}\to 0,c_{0,2,2}\to 0,c_{0,1,1}\to 0,c_{1,2,1}\to 0,$
$c_{1,2,2}\to 0,c_{0,1,2}\to \alpha_1,c_{0,2,1}\to -\alpha_2\},\{\alpha_1>0,\alpha_2>0\},$
$\{\alpha_1<\alpha_2\}]$

$\texttt{CheckAlgU}[\{c_{0,2,0}\to 0,c_{0,2,2}\to 0,c_{0,1,1}\to 0,c_{1,2,1}\to 0,$
$c_{1,2,2}\to 0\},\{c_{\texttt{u\_},\texttt{v\_},\texttt{w\_}}\in\texttt{Reals},c_{0,1,2}>0\},$
$\{c_{1,0,2}+c_{2,0,1}>0\}]$

2.1.1.3.2 $k_1 = 0$ or $k_2 \leq 0$ ($\mathcal{K}_{\mathcal{D}_1}$ indef.)

In this case, we get a two-parameter family of equivalence classes on $\widetilde{\mathsf{SL}}(2,\mathbb{R})_{hyp}$.

$\{\texttt{lie}[X_1,X_0],\texttt{lie}[X_2,X_0],\texttt{lie}[X_2,X_1]\}$
$\quad/.\{c_{0,2,0}\to 0,c_{0,2,2}\to 0,c_{0,1,1}\to 0,$
$c_{1,2,1}\to 0,c_{1,2,2}\to 0\}$
$\quad//\texttt{Simplify};$
$\%/.c_{0,1,2}\to -\alpha_1/.c_{0,2,1}\to -\alpha_2//\texttt{Column}$

$-c_{0,1,2}>0//\texttt{FullSimplify}[\#,c_{0,1,2}+c_{0,2,1}<0$
$\quad\&\&c_{0,1,2}c_{0,2,1}>0]\&$
$-c_{0,2,1}>0//\texttt{FullSimplify}[\#,c_{0,1,2}+c_{0,2,1}<0$
$\quad\&\&c_{0,1,2}c_{0,2,1}>0]\&$

$\{\kappa,\frac{1}{2}(c_{1,0,2}+c_{2,0,1})\}/.\{c_{0,2,0}\to 0,c_{0,2,2}\to 0,c_{0,1,1}\to 0,$
$c_{1,2,1}\to 0,c_{1,2,2}\to 0\}/.c_{0,1,2}\to -\alpha_1/.c_{0,2,1}\to -\alpha_2$
$\quad//\texttt{Simplify}[\#,\alpha_1>0\&\&\alpha_2>0]\&$

$\texttt{CheckAlgU}[\{c_{0,2,0}\to 0,c_{0,2,2}\to 0,c_{0,1,1}\to 0,c_{1,2,1}\to 0,$
$c_{1,2,2}\to 0,c_{0,1,2}\to -\alpha_1,c_{0,2,1}\to -\alpha_2\},\{\alpha_1>0,$
$\alpha_2>0\},\{\}]$

$\texttt{CheckAlgU}[\{c_{0,2,0}\to 0,c_{0,2,2}\to 0,c_{0,1,1}\to 0,c_{1,2,1}\to 0,$
$c_{1,2,2}\to 0\},\{c_{\texttt{u\_},\texttt{v\_},\texttt{w\_}}\in\texttt{Reals}\},$
$\{c_{0,1,2}c_{0,2,1}>0,c_{1,0,2}+c_{2,0,1}>0\}]$

2.1.2 $c^1_{20} = 0$ and $c^1_{21} = 0$ and $c^2_{21} \neq 0$

$\frac{1}{2}(c_{1,0,2}+c_{2,0,1})>0/.\{c_{0,2,0}\to 0,c_{0,2,2}\to 0,c_{0,1,1}\to 0,$
$c_{0,2,1}\to 0,c_{1,2,1}\to 0\}$
$\quad//\texttt{Simplify}$

$\texttt{Resolve}[\texttt{Exists}[\{a_0,a_1,a_2\},\{a_0,a_1,a_2\}\in\texttt{Reals},$
$\quad(\frac{1}{2}a_1(c_{1,2,2}-\sqrt{4c_{0,1,2}+c^2_{1,2,2}})==0$
$\&\&\ \frac{1}{2}a_1(c_{1,2,2}+\sqrt{4c_{0,1,2}+c^2_{1,2,2}})\neq 0)\|$
$\quad(\frac{1}{2}a_1(c_{1,2,2}-\sqrt{4c_{0,1,2}+c^2_{1,2,2}})\neq 0$
$\&\&\ \frac{1}{2}a_1(c_{1,2,2}+\sqrt{4c_{0,1,2}+c^2_{1,2,2}})==0)]]$
$\quad//\texttt{Simplify}[\#,c_{\texttt{u\_},\texttt{v\_},\texttt{w\_}}\in\texttt{Reals}\&\&c_{0,1,2}<0]\&$

2.1.2.1 $(c^2_{12})^2 + 4c^2_{01} > 0$

In this case, we get a one-parameter family of equivalence classes on $G^h_{3.4}$, $1 < h$.

$$\begin{array}{l}
\{\mathtt{lie}[X_1,X_0],\mathtt{lie}[X_2,X_0],\mathtt{lie}[X_2,X_1]\}\\
\quad /.\{c_{0,2,0}\to 0,c_{0,2,2}\to 0,c_{0,1,1}\to 0,c_{0,2,1}\to 0,\\
c_{1,2,1}\to 0\}\\
\quad \mathtt{//Simplify;}\\
\%/.c_{0,1,2}\to -\alpha/.c_{1,2,2}\to\beta/.\alpha\to\frac{(h^2-1)\beta^2}{4h^2}\\
\quad \mathtt{//Column}
\end{array}$$

$$\begin{array}{l}
\{\kappa,\frac{1}{2}(c_{1,0,2}+c_{2,0,1})\}/.\{c_{0,2,0}\to 0,c_{0,2,2}\to 0,\\
c_{0,1,1}\to 0,c_{0,2,1}\to 0,c_{1,2,1}\to 0\}/.c_{0,1,2}\to -\alpha\\
\quad /.c_{1,2,2}\to\beta/.\alpha\to\frac{(h^2-1)\beta^2}{4h^2}\\
\quad \mathtt{//Simplify}[\#,0<h\&\&\beta\neq 0\&\&\beta\in\mathtt{Reals}]\&\\
\mathtt{Reduce}[\frac{(h^2-1)\beta^2}{8h^2}>0,h,\mathtt{Reals}]\\
\quad \mathtt{//Simplify}[\#,0<h\&\&\beta\neq 0\&\&\beta\in\mathtt{Reals}]\&
\end{array}$$

$$\begin{array}{l}
\mathtt{CheckAlgN}[\{c_{0,2,0}\to 0,c_{0,2,2}\to 0,c_{0,1,1}\to 0,c_{0,2,1}\to 0,\\
c_{1,2,1}\to 0,c_{0,1,2}\to -\alpha,c_{1,2,2}\to\beta\}/.\alpha\to\frac{(h^2-1)\beta^2}{4h^2},\\
\{\beta\neq 0,h>1,\{h,\beta\}\in\mathtt{Reals}\},\{\}]
\end{array}$$

$$\begin{array}{l}
\mathtt{CheckAlgN}[\{c_{0,2,0}\to 0,c_{0,2,2}\to 0,c_{0,1,1}\to 0,c_{0,2,1}\to 0,\\
c_{1,2,1}\to 0\},\{c_{\mathtt{u\_,v\_,w\_}}\in\mathtt{Reals},c_{1,2,2}\neq 0,c_{0,1,2}<0\},\\
\{c^2_{1,2,2}+4c_{0,1,2}>0\}]
\end{array}$$

Find h:

$$\begin{array}{l}
\mathtt{Subs}=\{c_{0,2,0}\to 0,c_{0,2,2}\to 0,c_{0,1,1}\to 0,c_{0,2,1}\to 0,\\
c_{1,2,1}\to 0,c_{0,1,2}\to -\alpha,c_{1,2,2}\to\beta\};
\end{array}$$

$$\begin{array}{l}
\{\mathtt{a0},\mathtt{a1},\mathtt{a2}\}=\mathtt{Eigenvalues}[\mathtt{ad}[a_0,a_1,a_2]]/.\mathtt{Subs}\\
\quad \mathtt{//FullSimplify}[\#,\alpha>0\&\&\beta\neq 0\&\&\beta\in\mathtt{Reals}]\&\\
(\frac{\mathtt{a1+a2}}{\mathtt{a1-a2}})^2\mathtt{//FullSimplify}[\#,\alpha>0\&\&\beta\neq 0\\
\quad \&\&\beta\in\mathtt{Reals}]\&\\
\mathtt{Solve}[\%==h^2,h]\\
\quad \mathtt{//FullSimplify}[\#,\alpha>0\&\&\beta\neq 0\&\&\beta\in\mathtt{Reals}]\&
\end{array}$$

2.1.2.2. $(c^2_{12})^2 + 4c^2_{01} = 0$

In this case, we get a one-parameter family of equivalence classes on $G_{3.2}$.

$$\begin{array}{l}
\{\mathtt{lie}[X_1,X_0],\mathtt{lie}[X_2,X_0],\mathtt{lie}[X_2,X_1]\}\\
\quad /.\{c_{0,2,0}\to 0,c_{0,2,2}\to 0,c_{0,1,1}\to 0,c_{0,2,1}\to 0,\\
c_{1,2,1}\to 0\}/.c_{0,1,2}\to -\frac{1}{4}c^2_{1,2,2}\\
\quad \mathtt{//Simplify;}\\
\%/.c_{1,2,2}\to\beta\mathtt{//Column}
\end{array}$$

$$\begin{array}{l}
\{\kappa,\frac{1}{2}(c_{1,0,2}+c_{2,0,1})\}/.\{c_{0,2,0}\to 0,c_{0,2,2}\to 0,c_{0,1,1}\to 0,\\
c_{0,2,1}\to 0,c_{1,2,1}\to 0\}/.c_{0,1,2}\to -\frac{1}{4}c^2_{1,2,2}/.c_{1,2,2}\to\beta\\
\quad \mathtt{//Simplify}[\#,\beta\neq 0\&\&\beta\in\mathtt{Reals}]\&
\end{array}$$

$$\begin{array}{l}
\mathtt{CheckAlgN}[\{c_{0,2,0}\to 0,c_{0,2,2}\to 0,c_{0,1,1}\to 0,c_{0,2,1}\to 0,\\
c_{1,2,1}\to 0,c_{0,1,2}\to -\frac{1}{4}\beta^2,c_{1,2,2}\to\beta\},\{\beta\neq 0,\\
\beta\in\mathtt{Reals}\},\{\}]
\end{array}$$

$$\begin{array}{l}
\mathtt{CheckAlgN}[\{c_{0,2,0}\to 0,c_{0,2,2}\to 0,c_{0,1,1}\to 0,c_{0,2,1}\to 0,\\
c_{1,2,1}\to 0\},\{c_{\mathtt{u\_,v\_,w\_}}\in\mathtt{Reals},c_{1,2,2}\neq 0,c_{0,1,2}<0\},\\
\{c^2_{1,2,2}+4c_{0,1,2}==0\}]
\end{array}$$

2.1.2.3. $(c^2_{12})^2 + 4c^2_{01} < 0$

In this case, we get a one-parameter family of equivalence classes on $G^h_{3.5}$.

$$\begin{array}{l}
\{\mathtt{lie}[X_1,X_0],\mathtt{lie}[X_2,X_0],\mathtt{lie}[X_2,X_1]\}\\
\quad /.\{c_{0,2,0}\to 0,c_{0,2,2}\to 0,c_{0,1,1}\to 0,\\
c_{0,2,1}\to 0,c_{1,2,1}\to 0\}\\
\quad \mathtt{//Simplify;}\\
\%/.c_{0,1,2}\to -\alpha/.c_{1,2,2}\to\beta/.\alpha\to\frac{(1+h^2)\beta^2}{4h^2}\\
\quad \mathtt{//Column}
\end{array}$$

$$\begin{array}{l}
\{\kappa,\frac{1}{2}(c_{1,0,2}+c_{2,0,1})\}/.\{c_{0,2,0}\to 0,c_{0,2,2}\to 0,c_{0,1,1}\to 0,\\
c_{0,2,1}\to 0,c_{1,2,1}\to 0\}/.c_{0,1,2}\to -\alpha/.c_{1,2,2}\to\beta\\
\quad /.\alpha\to\frac{(1+h^2)\beta^2}{4h^2}\\
\quad \mathtt{//Simplify}[\#,0<h\&\&\beta\neq 0\&\&\beta\in\mathtt{Reals}]\&
\end{array}$$

$$\begin{array}{l}
\mathtt{CheckAlgN}[\{c_{0,2,0}\to 0,c_{0,2,2}\to 0,c_{0,1,1}\to 0,c_{0,2,1}\to 0,\\
c_{1,2,1}\to 0,c_{0,1,2}\to -\alpha,c_{1,2,2}\to\beta,\alpha\to\frac{(1+h^2)\beta^2}{4h^2}\},\\
\{\beta\neq 0,\{\beta,h\}\in\mathtt{Reals}\},\{\}]
\end{array}$$

$$\begin{array}{l}
\mathtt{CheckAlgN}[\{c_{0,2,0}\to 0,c_{0,2,2}\to 0,c_{0,1,1}\to 0,c_{0,2,1}\to 0,\\
c_{1,2,1}\to 0\},\{c_{\mathtt{u\_,v\_,w\_}}\in\mathtt{Reals},c_{1,2,2}\neq 0,c_{0,1,2}<0\},\\
\{c^2_{1,2,2}+4c_{0,1,2}<0\}]
\end{array}$$

Find h:

$$\begin{array}{l}
\mathtt{Subs}=\{c_{0,2,0}\to 0,c_{0,2,2}\to 0,c_{0,1,1}\to 0,c_{0,2,1}\to 0,\\
c_{1,2,1}\to 0,c_{0,1,2}\to -\alpha,c_{1,2,2}\to\beta\};
\end{array}$$

$$\begin{array}{l}
\{\mathtt{a0},\mathtt{a1},\mathtt{a2}\}=\mathtt{Eigenvalues}[\mathtt{ad}[a_0,a_1,a_2]]/.\mathtt{Subs}\\
\quad \mathtt{//FullSimplify}[\#,\alpha>0\&\&\beta\neq 0\&\&\beta\in\mathtt{Reals}]\&\\
(\frac{\mathtt{a1+a2}}{\mathtt{a1-a2}})^2\mathtt{//FullSimplify}[\#,\alpha>0\&\&\beta\neq 0\\
\quad \&\&\beta\in\mathtt{Reals}]\&\\
\mathtt{Solve}[\%==-h^2,h]\\
\quad \mathtt{//FullSimplify}[\#,\alpha>0\&\&\beta\neq 0\&\&\beta\in\mathtt{Reals}]\&
\end{array}$$

2.1.3 $c^2_{10} = 0$ and $c^2_{21} = 0$ and $c^1_{21} \neq 0$

In this case, we get a one-parameter family of equivalence classes on $G^h_{3.4}$, $0 < h < 1$.

$$\begin{array}{l}
\frac{1}{2}(c_{1,0,2}+c_{2,0,1})>0/.\{c_{0,2,0}\to 0,c_{0,2,2}\to 0,c_{0,1,1}\to 0,\\
c_{0,1,2}\to 0,c_{1,2,2}\to 0\}\\
\quad \mathtt{//Simplify}
\end{array}$$

$$\begin{array}{l}
\mathtt{Resolve}[\mathtt{Exists}[\{a_0,a_1,a_2\},\{a_0,a_1,a_2\}\in\mathtt{Reals},\\
\quad (\frac{1}{2}(-a_2c_{1,2,1}-\sqrt{a_2^2(-4c_{0,2,1}+c^2_{1,2,1})})==0\\
\&\&\ \frac{1}{2}(-a_2c_{1,2,1}+\sqrt{a_2^2(-4c_{0,2,1}+c^2_{1,2,1})})\neq 0)\|\\
\quad (\frac{1}{2}(-a_2c_{1,2,1}-\sqrt{a_2^2(-4c_{0,2,1}+c^2_{1,2,1})})\neq 0\\
\&\&\ \frac{1}{2}(-a_2c_{1,2,1}+\sqrt{a_2^2(-4c_{0,2,1}+c^2_{1,2,1})})==0)]]\\
\quad \mathtt{//Simplify}[\#,c_{\mathtt{u\_,v\_,w\_}}\in\mathtt{Reals}\&\&c_{0,2,1}<0]\&
\end{array}$$

$$\begin{array}{l}
\{\mathtt{lie}[X_1,X_0],\mathtt{lie}[X_2,X_0],\mathtt{lie}[X_2,X_1]\}\\
\quad /.\{c_{0,2,0}\to 0,c_{0,2,2}\to 0,c_{0,1,1}\to 0,\\
c_{0,1,2}\to 0,c_{1,2,2}\to 0\}\\
\quad \mathtt{//Simplify;}\\
\%/.c_{0,2,1}\to -\alpha/.c_{1,2,1}\to\beta/.\alpha\to -\frac{(h^2-1)\beta^2}{4h^2}\\
\quad \mathtt{//Column}
\end{array}$$

$$\begin{array}{l}
\{\kappa,\frac{1}{2}(c_{1,0,2}+c_{2,0,1})\}/.\{c_{0,2,0}\to 0,c_{0,2,2}\to 0,c_{0,1,1}\to 0,\\
c_{0,1,2}\to 0,c_{1,2,2}\to 0\}/.c_{0,2,1}\to -\alpha/.c_{1,2,1}\to\beta\\
\quad /.\alpha\to -\frac{(h^2-1)\beta^2}{4h^2}\\
\quad \mathtt{//Simplify}[\#,0<h\&\&\beta\neq 0\&\&\beta\in\mathtt{Reals}]\&\\
\mathtt{Reduce}[-\frac{(h^2-1)\beta^2}{8h^2}>0,\{h\},\mathtt{Reals}]\\
\quad \mathtt{//Simplify}[\#,0<h\&\&\beta\neq 0\&\&\beta\in\mathtt{Reals}]\&
\end{array}$$

```
CheckAlgN[{c_{0,2,0} → 0, c_{0,2,2} → 0, c_{0,1,1} → 0,
c_{0,1,2} → 0, c_{1,2,2} → 0, c_{0,2,1} → −α, c_{1,2,1} → β}
/.α → −((h^2−1)β^2)/(4h^2), {0 < h < 1, β ≠ 0,
β ∈ Reals}, {}]
```

```
CheckAlgN[{c_{0,2,0} → 0, c_{0,2,2} → 0, c_{0,1,1} → 0, c_{0,1,2} → 0,
c_{1,2,2} → 0}, {c_{u_,v_,w_} ∈ Reals, c_{1,2,1} ≠ 0, c_{0,2,1} < 0}, {}]
```

Find h:

```
Subs = {c_{0,2,0} → 0, c_{0,2,2} → 0, c_{0,1,1} → 0, c_{0,1,2} → 0,
c_{1,2,2} → 0, c_{0,2,1} → −α, c_{1,2,1} → β};
```

```
{a0, a1, a2} = Eigenvalues[ad[a_0, a_1, a_2]]/.Subs
     //FullSimplify[#, α > 0&&β ≠ 0&&β ∈ Reals]&
((a1+a2)/(a1−a2))^2//FullSimplify[#, α > 0&&β ≠ 0
     &&β ∈ Reals]&
Solve[% == h^2, h]
     //FullSimplify[#, α > 0&&β ≠ 0&&β ∈ Reals]&
```

2.2 $\chi_1 = 0$

```
{κ, χ_1}/.{c_{0,2,0} → 0, c_{0,2,2} → 0, c_{0,1,1} → 0,
c_{0,2,1} → −c_{0,1,2}}
     //Simplify
{lie[X_1, X_0], lie[X_2, X_0], lie[X_2, X_1]}
     /.c_{0,2,0} → 0, c_{0,2,2} → 0, c_{0,1,1} → 0, c_{0,2,1} → −c_{0,1,2}}
     //Simplify//Column
```

H_3

```
{lie[X_1, X_0], lie[X_2, X_0], lie[X_2, X_1]}
     /.{c_{0,2,0} → 0, c_{0,2,2} → 0, c_{0,1,1} → 0,
c_{0,2,1} → 0, c_{1,2,1} → 0, c_{1,2,2} → 0, c_{0,1,2} → 0}
     //Simplify//Column
```

```
{κ, χ_1}/.{c_{0,2,0} → 0, c_{0,2,2} → 0, c_{0,1,1} → 0, c_{0,2,1} → 0,
c_{1,2,1} → 0, c_{1,2,2} → 0, c_{0,1,2} → 0}
```

```
CheckAlgU[{c_{0,2,0} → 0, c_{0,2,2} → 0, c_{0,1,1} → 0, c_{0,2,1} → 0,
c_{1,2,1} → 0, c_{1,2,2} → 0, c_{0,1,2} → 0}, {}, {}]
```

SU(2)

```
{lie[X_1, X_0], lie[X_2, X_0], lie[X_2, X_1]}
     /.{c_{0,2,0} → 0, c_{0,2,2} → 0, c_{0,1,1} → 0,
c_{0,2,1} → −c_{0,1,2}, c_{1,2,1} → 0, c_{1,2,2} → 0}
     //Simplify//Column
```

```
m1 < 0&&m2 > 0&&m3 < 0
     /.{c_{0,2,0} → 0, c_{0,2,2} → 0, c_{0,1,1} → 0,
c_{0,2,1} → −c_{0,1,2}, c_{1,2,1} → 0, c_{1,2,2} → 0}
   //FullSimplify
```

```
{κ, χ_1}/.{c_{0,2,0} → 0, c_{0,2,2} → 0, c_{0,1,1} → 0,
c_{0,2,1} → −c_{0,1,2}, c_{1,2,1} → 0, c_{1,2,2} → 0}
     //Simplify
```

```
KilForm/.{c_{0,2,0} → 0, c_{0,2,2} → 0, c_{0,1,1} → 0,
c_{0,2,1} → −c_{0,1,2}, c_{1,2,1} → 0, c_{1,2,2} → 0}
     /.c_{0,1,2} → −"κ"
     //Simplify//MatrixForm
```

```
CheckAlgU[{c_{0,2,0} → 0, c_{0,2,2} → 0, c_{0,1,1} → 0,
c_{0,2,1} → −c_{0,1,2}, c_{1,2,1} → 0, c_{1,2,2} → 0},
{c_{0,1,2} < 0, c_{u_,v_,w_} ∈ Reals}, {}]
```

$\widetilde{\mathsf{SL}}(2,\mathbb{R})$

```
{lie[X_1, X_0], lie[X_2, X_0], lie[X_2, X_1]}
     /.{c_{0,2,0} → 0, c_{0,2,2} → 0, c_{0,1,1} → 0, c_{0,2,1} → −c_{0,1,2},
c_{1,2,1} → 0, c_{1,2,2} → 0}
     //Simplify//Column
```

```
m3 < 0&&(m1 ≥ 0||m2 ≤ 0)&&(k1 ≠ 0&&k2 > 0)
     /.{c_{0,2,0} → 0, c_{0,2,2} → 0, c_{0,1,1} → 0,
c_{0,2,1} → −c_{0,1,2}, c_{1,2,1} → 0, c_{1,2,2} → 0}
     //FullSimplify
```

```
{κ, χ_1}/.{c_{0,2,0} → 0, c_{0,2,2} → 0, c_{0,1,1} → 0,
c_{0,2,1} → −c_{0,1,2}, c_{1,2,1} → 0, c_{1,2,2} → 0}
     //Simplify
KilForm/.{c_{0,2,0} → 0, c_{0,2,2} → 0, c_{0,1,1} → 0,
c_{0,2,1} → −c_{0,1,2}, c_{1,2,1} → 0, c_{1,2,2} → 0}
     /.c_{0,1,2} → −"κ"
     //Simplify//MatrixForm
```

```
CheckAlgU[{c_{0,2,0} → 0, c_{0,2,2} → 0, c_{0,1,1} → 0,
c_{0,2,1} → −c_{0,1,2}, c_{1,2,1} → 0, c_{1,2,2} → 0},
{c_{0,1,2} > 0, c_{u_,v_,w_} ∈ Reals}, {}]
```

$\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}$

```
{lie[X_1, X_0], lie[X_2, X_0], lie[X_2, X_1]}
     /.{c_{0,2,0} → 0, c_{0,2,2} → 0, c_{0,1,1} → 0, c_{0,2,1} → −c_{0,1,2}}
     //Simplify//Column
```

From the structures equations of the dual frame (ν^0, ν^1, ν^2) and $d^2 = 0$, we have $c_{0,1,2}c_{1,2,2} = 0$ and $-c_{0,1,2}c_{1,2,1} = 0$.

```
Reduce[c_{0,1,2}c_{1,2,2} == 0&& − c_{0,1,2}c_{1,2,1} == 0, {}, Reals]
     //FullSimplify[#, c_{1,2,1} ≠ 0||c_{1,2,2} ≠ 0]&
```

```
{κ, χ_1}/.{c_{0,2,0} → 0, c_{0,2,2} → 0, c_{0,1,1} → 0, c_{0,2,1} → 0,
c_{0,1,2} → 0}
     //Simplify
```

```
CheckAlgN[{c_{0,2,0} → 0, c_{0,2,2} → 0, c_{0,1,1} → 0, c_{0,2,1} → 0,
c_{0,1,2} → 0}, {c_{u_,v_,w_} ∈ Reals}, {c_{1,2,1} ≠ 0||c_{1,2,2} ≠ 0}]
```

D.2 Nonholonomic Riemannian structures in three dimensions

In this notebook we have used a (third-party) MATHEMATICA package "Exterior Differential Calculus and Symbolic Matrix Algebra" [9]. This package will need to be available in order to execute several sections of the code.

Much of the code at the beginning of this notebook is identical or extremely similar to that at the beginning of the notebook in section D.1. As such, we have elided many of the comments for brevity.

Load the EDC package:

```
<<"path\to\EDC\matrixEDC.m"
$EDCversion

FuncMatchList = {c_{u_,v_,w_}};
FuncQ[f_]:=AnyTrue[FuncMatchList, MatchQ[f, #]&];

Clear[Con]
Con[X_ + Y_, A_]:=Con[X, A] + Con[Y, A];
Con[a_X_{i_}, A_]:=aCon[X_i, A];

Con[A_, X_ + Y_]:=Con[A, X] + Con[A, Y];
Con[X_{i_}, a_?NumericQX_{j_}]:=aCon[X_i, X_j];
Con[X_{i_}, a_X_{j_}]:=X_i[a]X_j + aCon[X_i, X_j];

Con[0, A_]:=0;

Clear[ConNH]
ConNH[X_ + Y_, A_]:=ConNH[X, A] + ConNH[Y, A];
ConNH[a_X_{i_}, A_]:=aConNH[X_i, A];

ConNH[A_, X_ + Y_]:=ConNH[A, X] + ConNH[A, Y];
ConNH[X_, a_?NumericQY_]:=aConNH[X, Y];
ConNH[X_{i_}, a_X_{j_}]:=X_i[a]X_j + aConNH[X_i, X_j];

ConNH[0, A_]:=0;
ConNH[A_, 0]:=0;

Γ_{i_,j_,k_}:=1/2(c_{k,i,j} + c_{i,j,k} − c_{j,k,i});
c_{i_,j_,k_}/;!OrderedQ[{i, j}]:= − c_{j,i,k};
c_{i_,i_,k_}:=0;


X_{i_}[a_?NumericQ]:=0;
X_{i_}[a_?NumericQf_]:=aX_i[f];
X_{i_}[f_ + g_]:=X_i[f] + X_i[g];

X_{i_}[f_^2]:=2fX_i[f];
X_{i_}[f_g_]:=X_i[f]g + fX_i[g];
X_{i_}[1/f_]:= − 1/f^2 X_i[f];

ν_{i_}[X_{j_}]/;i == j:=1;
ν_{i_}[X_{j_}]/;i ≠ j:=0;

ν_{i_}[f_ + g_]:=ν_i[f] + ν_i[g];

Clear[P, Q]
P[X_ + Y_]:=P[X] + P[Y];
P[a_?NumericQX_]:=aP[X];
P[a_X_{i_}]:=aP[X_i];
P[f_?FuncQX_]:=fP[X];

P[X_0]:=0;
P[X_1]:=X_1;
P[X_2]:=X_2;
```

```
Clear[AngleBracket]
SetAttributes[AngleBracket, Orderless]

⟨A_, X_ + Y_⟩:=⟨A, X⟩ + ⟨A, Y⟩;
⟨A_, f_?FuncQX_⟩:=f⟨A, X⟩;
⟨A_, a_?NumericQX_⟩:=a⟨A, X⟩;
⟨A_, f_X_{i_}⟩:=f⟨A, X_i⟩;

⟨X_{i_}, X_{j_}⟩/;i == j&&(i == 1‖i == 2):=1;
⟨X_{i_}, X_{j_}⟩/;i ≠ j:=0;

Clear[lie]
lie[0, A_]:=0;
lie[A_, 0]:=0;

lie[A_, X_ + Y_]:=lie[A, X] + lie[A, Y];
lie[A_, a_?NumericQX_]:=alie[A, X];
lie[A_, f_?FuncQX_]:=flie[A, X] + A[f]X;
lie[A_, g_X_{i_}]:=glie[A, X_i] + A[g]X_i;

lie[X_ + Y_, A_]:=lie[X, A] + lie[Y, A];
lie[X_ + Y_ + Z_, A_]:=lie[X, A] + lie[Y, A] + lie[Z, A];
lie[a_?NumericQX_, A_]:=alie[X, A];
lie[f_?FuncQX_, A_]:=flie[X, A] − A[f]X;
lie[f_X_{i_}, A_]:=flie[X_i, A] − A[f]X_i;

ad[1, 0, 0]:=With[{a = lie[X_0, X_1], b = lie[X_0, X_2]},
 {
   {0, 0, 0},
   If[PossibleZeroQ[a], {0, 0, 0}, a],
   If[PossibleZeroQ[b], {0, 0, 0}, b]
 }/.Thread[{X_0, X_1, X_2} → IdentityMatrix[3]]
     //Transpose//Simplify];
ad[0, 1, 0]:=With[{a = lie[X_1, X_0], b = lie[X_1, X_2]},
 {
   If[PossibleZeroQ[a], {0, 0, 0}, a],
   {0, 0, 0},
   If[PossibleZeroQ[b], {0, 0, 0}, b]
 }/.Thread[{X_0, X_1, X_2} → IdentityMatrix[3]]
     //Transpose//Simplify];
ad[0, 0, 1]:=With[{a = lie[X_2, X_0], b = lie[X_2, X_1]},
 {
   If[PossibleZeroQ[a], {0, 0, 0}, a],
   If[PossibleZeroQ[b],
   {0, 0, 0}, b], {0, 0, 0}
 }/.Thread[{X_0, X_1, X_2} → IdentityMatrix[3]]
     //Transpose//Simplify];
ad[a_, b_, c_]:=aad[1, 0, 0] + bad[0, 1, 0] + cad[0, 0, 1];

K[A_, B_, C_]:=ConNH[A, ConNH[B, C]]
     −ConNH[B, ConNH[A, C]]
     −ConNH[P[lie[A, B]], C]
     −P[lie[Q[lie[A, B]], C]];
Khat[A_, B_, C_, D_]:=⟨K[A, B, C], D⟩;

Rhat[A_, B_, C_, D_]:=
     1/2(Khat[A, B, C, D] − Khat[A, B, D, C]);
Chat[A_, B_, C_, D_]:=
     Khat[A, B, C, D] − Rhat[A, B, C, D];

Rtilde[X_, Y_]:=Rhat[X,Y,Y,X]/(⟨X,X⟩⟨Y,Y⟩−⟨X,Y⟩^2);
```

```
Ric[A_,B_]:=Sum[Rhat[X_ℓ,A,B,X_ℓ],{ℓ,1,2}];
Scal = Sum[Ric[X_ℓ,X_ℓ],{ℓ,1,2}];
```

```
RicC[A_,B_]:=Sum[Chat[X_ℓ,A,B,X_ℓ],{ℓ,1,2}];
RicCSym[A_,B_]:=1/2(RicC[A,B] + RicC[B,A]);
RicCSkw[A_,B_]:=1/2(RicC[A,B] − RicC[B,A]);
```

```
𝒢sharp = Inverse[(⟨X_1,X_1⟩  ⟨X_1,X_2⟩
                  ⟨X_2,X_1⟩  ⟨X_2,X_2⟩)]
    //Simplify;
RicflatM = (Ric[X_1,X_1]  Ric[X_1,X_2]
            Ric[X_2,X_1]  Ric[X_2,X_2])
    //Simplify;
```

```
R = 𝒢sharp.RicflatM^T//FullSimplify[#,X_[_] == 0]&;
R//Simplify//MatrixForm
```

```
RicCSymflatM = (RicCSym[X_1,X_1]  RicCSym[X_1,X_2]
                RicCSym[X_2,X_1]  RicCSym[X_2,X_2])
    //Simplify;
```

```
RSym = 𝒢sharp.RicCSymflatM^T//FullSimplify;
RSym//Simplify//MatrixForm
```

```
RicCSkwflatM = (RicCSkw[X_1,X_1]  RicCSkw[X_1,X_2]
                RicCSkw[X_2,X_1]  RicCSkw[X_2,X_2])
    //Simplify;
```

```
RSkw = 𝒢sharp.RicCSkwflatM^T//FullSimplify;
RSkw//Simplify//MatrixForm
```

```
κ = 1/2Tr[R];
χ_1 = √(−Det[RSym]);
χ_2 = √(Det[RSkw]);
ϑ = (c_{0,1,0})^2 + (c_{0,2,0})^2;
```

```
Kil[A_,B_]:=Tr[(ad@@A).(ad@@B)];
KilForm = {
   {Kil[{1,0,0},{1,0,0}],Kil[{1,0,0},{0,1,0}],
    Kil[{1,0,0},{0,0,1}]}
   {Kil[{0,1,0},{1,0,0}],Kil[{0,1,0},{0,1,0}],
    Kil[{0,1,0},{0,0,1}]}
   {Kil[{0,0,1},{1,0,0}],Kil[{0,0,1},{0,1,0}],
    Kil[{0,0,1},{0,0,1}]}
 }//FullSimplify;
KilFormD = {
   {Kil[{0,1,0},{0,1,0}],Kil[{0,1,0},{0,0,1}]}
   {Kil[{0,0,1},{0,1,0}],Kil[{0,0,1},{0,0,1}]}
 }//FullSimplify;
```

```
ϱ_0 = 1/2(−c^2_{0,1,1} − 2c_{0,1,2}c_{0,2,1} − c^2_{0,2,2});
ϱ_1 = −c_{0,1,2} − 1/2c^2_{1,2,2};
ϱ_2 = −1/2c^2_{0,2,0} + c_{0,2,1} − 1/2c^2_{1,2,1};
```

```
PrincipalMinors[Mat_]:=With[{P = Minors[Mat,#]},
     P_[[1,1]]]&/@Range[1,Length[Mat]];
```

Structure equations of the dual frame (ν^0, ν^1, ν^2)

```
dν_i_[X_,Y_]:=X[ν_i[Y]] − Y[ν_i[X]] − ν_i[lie[X,Y]];
```

```
{dν_0[X_0,X_1],dν_0[X_0,X_2],dν_0[X_1,X_2]}
{dν_1[X_0,X_1],dν_1[X_0,X_2],dν_1[X_1,X_2]}
{dν_2[X_0,X_1],dν_2[X_0,X_2],dν_2[X_1,X_2]}
```

Using $d^2 = 0$ on the structure equations, we get the following equations that must be satisfied by the structure constants:

```
Thread[{
   (c_{1,0,1} + c_{2,0,2})c_{2,1,0} − c_{1,0,0}c_{2,1,1} − c_{2,0,0}c_{2,1,2}
     +X_0[c_{2,1,0}] − X_1[c_{2,0,0}] + X_2[c_{1,0,0}],
   −c_{1,0,1}c_{2,0,0} + c_{1,0,0}c_{2,0,1} + c_{2,0,2}c_{2,1,1} − c_{2,0,1}c_{2,1,2}
     +X_0[c_{2,1,1}] − X_1[c_{2,0,1}] + X_2[c_{1,0,1}],
   c_{1,0,0}c_{2,0,2} − c_{1,0,2}(c_{2,0,0} + c_{2,1,1}) + c_{1,0,1}c_{2,1,2}
     +X_0[c_{2,1,2}] − X_1[c_{2,0,2}] + X_2[c_{1,0,2}]
 } == 0]//Simplify//Column
```

Reeb vector field

```
Rb = Sum[r_i X_i,{i,0,2}]
     /.{r_1 → r_0c_{0,2,0},r_2 → −r_0c_{0,1,0}}/.r_0 → σ;
Ω = σν_0;
```

```
Ω[Rb]/.(σF_)[A_] → σF[A]//Simplify[#,σ^2 == 1]&
Ω[lie[X_2,X_1]]/.(σF_)[A_] → σF[A]//Simplify
```

```
With[{A = a_0X_0 + a_1X_1 + a_2X_2},
   Rb[Ω[A]] − A[Ω[Rb]] − Ω[lie[Rb,A]]]
     //.{(σF_)[A_] → σF[A],F_[σA_] → σF[A],
(F_ + G_)[A_] → F[A] + G[A],(a_i_X_j_)[A_] → a_iX_j[A],
F_[σ] → 0,F_[0] → 0}
     //Simplify
```

Hence the scalar invariant ϑ is given by:

```
⟨P[Rb],P[Rb]⟩//Simplify[#,σ^2 == 1]&
```

Wagner curvature tensor

We use the wedge symbol for bivectors in the following code, and hence it is not compatible with the use of the wedge symbol by the EDC package.

```
Clear[Wedge]
```

```
Δ[X_ ∧ Y_]:=Q[lie[X,Y]];
```

```
ΔInv[X_0]:= − (X_1 ∧ X_2);
ΔInv[f_X_0]:=fΔInv[X_0];
ΔInv[0]:=0;
```

```
K[X_ ∧ Y_][U_]:=K[X,Y,U];
K[−(X_ ∧ Y_)][U_]:= − K[X,Y,U];
K[f_?FuncQX_ ∧ Y_][U_]:=fK[X ∧ Y][U];
K[−f_?FuncQX_ ∧ Y_][U_]:= − fK[X ∧ Y][U];
K[0][U_]:=0;
```

```
ConW[Z_,U_]:=ConNH[P[Z],U] + K[ΔInv[Q[Z]]][U]
     +P[lie[Q[Z],U]];
KW[X_ ∧ Y_][U_]:=ConW[X,ConW[Y,U]] − ConW[Y,ConW[X,U]]
     −ConW[lie[X,Y],U];
```

In the following code, **ConW** is the (vector bundle) connection $\widetilde{\nabla}$ used in section 4.2.3, and **KW** is the Wagner curvature tensor (denoted $\widetilde{K}$ in section 4.2.3).

```
KWhat[X_ ∧ Y_][U_ ⊗ V_]:=⟨KW[X ∧ Y][U],V⟩;
RWhat[X_ ∧ Y_][U_ ∧ V_]:=
     1/2(KWhat[X ∧ Y][U ⊗ V] − KWhat[X ∧ Y][V ⊗ U]);
CWhat[X_ ∧ Y_][U_ ∨ V_]:=
     1/2(KWhat[X ∧ Y][U ⊗ V] + KWhat[X ∧ Y][V ⊗ U]);
```

$$\texttt{RicW[A\_,B\_]:=Sum[RWhat}[X_a \wedge A][B \wedge X_a], \{a, 1, 2\}];$$

$$\begin{aligned}
&\texttt{Subs} = \{(-X_{\texttt{i\_}})[\texttt{A\_}] \rightarrow -X_i[A],\\
&(\texttt{f\_} + \texttt{g\_} + \texttt{h\_})[\texttt{A\_}] \rightarrow f[A] + g[A] + h[A],\\
&(X_{\texttt{i\_}} c_{\texttt{u\_},\texttt{v\_},\texttt{w\_}})[\texttt{A\_}] \rightarrow c_{u,v,w} X_i[A]\};
\end{aligned}$$

$$\begin{aligned}
&\texttt{KW}[X_0 \wedge X_1][X_1]\texttt{/.}(-\texttt{A\_})[\texttt{f\_}] \rightarrow -A[f]\\
&\quad \texttt{/.}X_2[X_1[\texttt{f\_}]] \rightarrow X_0[f] + c_{2,1,1}X_1[f] + c_{2,1,2}X_2[f]\\
&\qquad +X_1[X_2[f]]\\
&\quad \texttt{//Collect[\#,}X_{\texttt{i\_}}\texttt{,Simplify]\&}\\
&\texttt{KW}[X_0 \wedge X_1][X_2]\texttt{/.}(-\texttt{A\_})[\texttt{f\_}] \rightarrow -A[f]\\
&\quad \texttt{/.}X_2[X_1[\texttt{f\_}]] \rightarrow X_0[f] + c_{2,1,1}X_1[f] + c_{2,1,2}X_2[f]\\
&\qquad +X_1[X_2[f]]\\
&\quad \texttt{//Collect[\#,}X_{\texttt{i\_}}\texttt{,Simplify]\&}
\end{aligned}$$

$$\begin{aligned}
&\texttt{KW}[X_0 \wedge X_2][X_1]\texttt{/.}(-\texttt{A\_})[\texttt{f\_}] \rightarrow -A[f]\\
&\quad \texttt{/.}X_2[X_1[\texttt{f\_}]] \rightarrow X_0[f] + c_{2,1,1}X_1[f] + c_{2,1,2}X_2[f]\\
&\qquad +X_1[X_2[f]]\\
&\quad \texttt{//Collect[\#,}X_{\texttt{i\_}}\texttt{,Simplify]\&}\\
&\texttt{KW}[X_0 \wedge X_2][X_2]\texttt{/.}(-\texttt{A\_})[\texttt{f\_}] \rightarrow -A[f]\\
&\quad \texttt{/.}X_2[X_1[\texttt{f\_}]] \rightarrow X_0[f] + c_{2,1,1}X_1[f] + c_{2,1,2}X_2[f]\\
&\qquad +X_1[X_2[f]]\\
&\quad \texttt{//Collect[\#,}X_{\texttt{i\_}}\texttt{,Simplify]\&}
\end{aligned}$$

$$\begin{aligned}
&\texttt{KW}[X_1 \wedge X_2][X_1]\texttt{/.}(-\texttt{A\_})[\texttt{f\_}] \rightarrow -A[f]\\
&\quad \texttt{/.}X_2[X_1[\texttt{f\_}]] \rightarrow X_0[f] + c_{2,1,1}X_1[f] + c_{2,1,2}X_2[f]\\
&\qquad +X_1[X_2[f]]\\
&\quad \texttt{//Collect[\#,}X_{\texttt{i\_}}\texttt{,Simplify]\&}\\
&\texttt{KW}[X_1 \wedge X_2][X_2]\texttt{/.}(-\texttt{A\_})[\texttt{f\_}] \rightarrow -A[f]\\
&\quad \texttt{/.}X_2[X_1[\texttt{f\_}]] \rightarrow X_0[f] + c_{2,1,1}X_1[f] + c_{2,1,2}X_2[f]\\
&\qquad +X_1[X_2[f]]\\
&\quad \texttt{//Collect[\#,}X_{\texttt{i\_}}\texttt{,Simplify]\&}
\end{aligned}$$

Calculations for lemma 4.2.12:

$$\begin{aligned}
&\texttt{f01} = (c_{1,0,1} - c_{2,0,2})c_{2,1,1} + (c_{1,0,2} + c_{2,0,1})c_{2,1,2}\\
&\quad +c_{1,0,1}c_{2,0,0} - \tfrac{1}{2}c_{1,0,0}(c_{1,0,2} + c_{2,0,1}) + c_{1,0,0}\kappa\\
&\quad +X_1[\tfrac{1}{2}(c_{1,0,2} + c_{2,0,1})] - X_1[\kappa] - X_2[c_{1,0,1}];\\
&\texttt{f02} = (c_{1,0,2} + c_{2,0,1})c_{2,1,1} - (c_{1,0,1} - c_{2,0,2})c_{2,1,2}\\
&\quad -c_{2,0,2}c_{1,0,0} + \tfrac{1}{2}c_{2,0,0}(c_{1,0,2} + c_{2,0,1}) + c_{2,0,0}\kappa\\
&\quad -X_2[\tfrac{1}{2}(c_{1,0,2} + c_{2,0,1})] - X_2[\kappa] + X_1[c_{2,0,2}];
\end{aligned}$$

From the structure equations of the dual frame, we have the identities:

$$\begin{aligned}
&X_0[c_{1,2,1}] \rightarrow -(c_{0,1,1}c_{0,2,0} - c_{0,2,2}c_{1,2,1}\\
&\quad -c_{0,2,1}(c_{0,1,0} - c_{1,2,2}) - X_1[c_{0,2,1}] + X_2[c_{0,1,1}])\\
&X_0[c_{1,2,2}] \rightarrow -(-c_{0,1,0}c_{0,2,2} - c_{0,1,2}(-c_{0,2,0} - c_{1,2,1})\\
&\quad -c_{0,1,1}c_{1,2,2} - X_1[c_{0,2,2}] + X_2[c_{0,1,2}])
\end{aligned}$$

$$\begin{aligned}
&\{\texttt{KW}[X_0 \wedge X_1][X_1] - \texttt{f01}X_2,\\
&\ \texttt{KW}[X_0 \wedge X_1][X_2] + \texttt{f01}X_1,\\
&\ \texttt{KW}[X_0 \wedge X_2][X_1] - \texttt{f02}X_2,\\
&\ \texttt{KW}[X_0 \wedge X_2][X_2] + \texttt{f02}X_1\\
&\}\texttt{/.}(-\texttt{A\_})[\texttt{f\_}] \rightarrow -A[f]\texttt{//Simplify;}\\
&\texttt{\%/.}X_2[X_1[\texttt{f\_}]] \rightarrow X_0[f] + c_{2,1,1}X_1[f] + c_{2,1,2}X_2[f]\\
&\quad +X_1[X_2[f]]\\
&\quad \texttt{//Simplify;}\\
&\texttt{\%/.}\{X_0[c_{1,2,1}] \rightarrow -(c_{0,1,1}c_{0,2,0} - c_{0,2,2}c_{1,2,1}\\
&-c_{0,2,1}(c_{0,1,0} - c_{1,2,2}) - X_1[c_{0,2,1}] + X_2[c_{0,1,1}]),\\
&X_0[c_{1,2,2}] \rightarrow -(-c_{0,1,0}c_{0,2,2} - c_{0,1,2}(-c_{0,2,0} - c_{1,2,1})\\
&-c_{0,1,1}c_{1,2,2} - X_1[c_{0,2,2}] + X_2[c_{0,1,2}])\}\\
&\quad \texttt{//Simplify}
\end{aligned}$$

$\mathscr{P}$-exterior covariant derivative of Ric, A_{sym} and A_{skew}

$\mathscr{P}$-exterior covariant derivative of a $(1,1)$-tensor field A:

$$\begin{aligned}
&\texttt{Pexcoder[A\_][X\_,Y\_]:=}\\
&\quad \texttt{ConNH}[X, A[Y]] - \texttt{ConNH}[Y, A[X]] - A[P\texttt{@lie}[X, Y]];
\end{aligned}$$

$\mathscr{P}$-exterior covariant derivative of Ric:

$$\begin{aligned}
&\texttt{gsharpRicflat[A\_]:=Ric}[A, X_1]X_1 + \texttt{Ric}[A, X_2]X_2;\\
&\texttt{Pexcoder[gsharpRicflat]}[X_1, X_2]\texttt{//Simplify;}\\
&\texttt{\%//Collect[\#,}X_{\texttt{i\_}}\texttt{,FullSimplify]\&}
\end{aligned}$$

$$\texttt{PexcoderRic} = \begin{pmatrix} -X_2[\kappa] \\ +X_1[\kappa] \end{pmatrix};$$

$\mathscr{P}$-exterior covariant derivative of A_{sym}:

$$\begin{aligned}
&\texttt{gsharpRicCSymflat[A\_]:=}\\
&\quad \texttt{RicCSym}[A, X_1]X_1 + \texttt{RicCSym}[A, X_2]X_2;\\
&\texttt{Pexcoder[gsharpRicCSymflat]}[X_1, X_2]\texttt{//Simplify;}\\
&\texttt{\%/.}(-X_0)[\texttt{A\_}] \rightarrow -X_0[A]\texttt{//Simplify;}\\
&\texttt{\%/.}X_2[X_1[\texttt{A\_}]] \rightarrow X_0[A] - c_{1,2,1}X_1[A] - c_{1,2,2}X_2[A]\\
&\quad +X_1[X_2[A]]\texttt{//Simplify;}\\
&\texttt{\%//Collect[\#,}\{X_1, X_2\}\texttt{,Simplify]\&}
\end{aligned}$$

$$\begin{aligned}
&\texttt{PexcoderRicCSym} = 2\{c_{2,1,1}, c_{2,1,2}\}\texttt{.RSym}\\
&\quad -\tfrac{1}{2}\begin{pmatrix} X_2[c_{1,0,2} + c_{2,0,1}] + X_1[c_{1,0,1} - c_{2,0,2}] \\ X_1[c_{1,0,2} + c_{2,0,1}] - X_2[c_{1,0,1} - c_{2,0,2}] \end{pmatrix};
\end{aligned}$$

$\mathscr{P}$-exterior covariant derivative of A_{skew}:

$$\begin{aligned}
&\texttt{gsharpRicCSkwflat[A\_]:=}\\
&\quad \texttt{RicCSkw}[A, X_1]X_1 + \texttt{RicCSkw}[A, X_2]X_2;\\
&\texttt{Pexcoder[gsharpRicCSkwflat]}[X_1, X_2]\texttt{//Simplify;}\\
&\texttt{\%/.}(-X_0)[\texttt{A\_}] \rightarrow -X_0[A]\texttt{//Simplify;}\\
&\texttt{\%/.}X_2[X_1[\texttt{A\_}]] \rightarrow X_0[A] - c_{1,2,1}X_1[A] - c_{1,2,2}X_2[A]\\
&\quad +X_1[X_2[A]]\texttt{//Simplify;}\\
&\texttt{\%//Collect[\#,}\{X_1, X_2\}\texttt{,Simplify]\&}
\end{aligned}$$

$$\texttt{PexcoderRicCSkw} = \begin{pmatrix} \tfrac{1}{2}X_1[c_{1,0,1} + c_{2,0,2}] \\ \tfrac{1}{2}X_2[c_{1,0,1} + c_{2,0,2}] \end{pmatrix};$$

Rotated frame when $\vartheta = \chi_1 = 0$

$$\begin{aligned}
&\texttt{Format[c100,StandardForm]:=}\gamma_{1,0,0};\\
&\texttt{Format[c200,StandardForm]:=}\gamma_{2,0,0};\\
&\texttt{Format[c210,StandardForm]:=}\gamma_{2,1,0};\\
&\texttt{Format[c101,StandardForm]:=}\gamma_{1,0,1};\\
&\texttt{Format[c201,StandardForm]:=}\gamma_{2,0,1};\\
&\texttt{Format[c211,StandardForm]:=}\gamma_{2,1,1};\\
&\texttt{Format[c102,StandardForm]:=}\gamma_{1,0,2};\\
&\texttt{Format[c202,StandardForm]:=}\gamma_{2,0,2};\\
&\texttt{Format[c212,StandardForm]:=}\gamma_{2,1,2};
\end{aligned}$$

$$\begin{aligned}
&\texttt{Subs} = \{\texttt{c100} \rightarrow 0, \texttt{c200} \rightarrow 0, \texttt{c210} \rightarrow 0, \texttt{c101} \rightarrow 0,\\
&\texttt{c201} \rightarrow -\texttt{c102}, \texttt{c202} \rightarrow 0\};
\end{aligned}$$

$$\texttt{DeclareForms}[\{0\}, \{1, \nu_0, \nu_1, \nu_2\}];$$

$$\begin{aligned}
&d[\nu_0] = \texttt{c100}\nu_0 \wedge \nu_1 + \texttt{c200}\nu_0 \wedge \nu_2 + \texttt{c210}\nu_1 \wedge \nu_2\texttt{/.Subs}\\
&\quad \texttt{//Simplify}\\
&d[\nu_1] = \texttt{c101}\nu_0 \wedge \nu_1 + \texttt{c201}\nu_0 \wedge \nu_2 + \texttt{c211}\nu_1 \wedge \nu_2\texttt{/.Subs}\\
&\quad \texttt{//Simplify}\\
&d[\nu_2] = \texttt{c102}\nu_0 \wedge \nu_1 + \texttt{c202}\nu_0 \wedge \nu_2 + \texttt{c212}\nu_1 \wedge \nu_2\texttt{/.Subs}\\
&\quad \texttt{//Simplify}
\end{aligned}$$

$$\begin{aligned}
&d[d[\nu_0]]\texttt{/.}d[\texttt{A\_}] \rightarrow X_0[A]\nu_0 + X_1[A]\nu_1 + X_2[A]\nu_2\texttt{//reWrite}\\
&\quad \texttt{//FullSimplify}
\end{aligned}$$

$d[d[\nu_1]]/.d[\mathtt{A\_}] \to X_0[A]\nu_0 + X_1[A]\nu_1 + X_2[A]\nu_2$**//reWrite**
//FullSimplify
$d[d[\nu_2]]/.d[\mathtt{A\_}] \to X_0[A]\nu_0 + X_1[A]\nu_1 + X_2[A]\nu_2$**//reWrite**
//FullSimplify

JacSubs $= \{$
$X_0[\gamma_{2,1,1}] \to -\gamma_{1,0,2}\gamma_{2,1,2} - X_1[\gamma_{1,0,2}],$
$X_0[\gamma_{2,1,2}] \to +\gamma_{1,0,2}\gamma_{2,1,1} - X_2[\gamma_{1,0,2}]$
$\}$;

$\varpi = (K - \mathbf{c102})\nu_0 + \mathbf{c211}\nu_1 + \mathbf{c212}\nu_2;$
$d[\varpi]/.d[\mathtt{A\_}] \to X_0[A]\nu_0 + X_1[A]\nu_1 + X_2[A]\nu_2$**/.Subs**
//reWrite//Simplify

$\{\gamma_{1,0,2}\gamma_{2,1,2} + X_0[\gamma_{2,1,1}] + X_1[\gamma_{1,0,2}] - X_1[K],$
$-\gamma_{1,0,2} + \gamma_{2,1,1}^2 + \gamma_{2,1,2}^2 + K + X_1[\gamma_{2,1,2}] - X_2[\gamma_{2,1,1}],$
$-\gamma_{1,0,2}\gamma_{2,1,1} + X_0[\gamma_{2,1,2}] + X_2[\gamma_{1,0,2}] - X_2[K]$
}/.JacSubs//Simplify;
%//FullSimplify//Column

Parallel Frame

Format[c100, StandardForm]:=$\gamma_{1,0,0}$;
Format[c200, StandardForm]:=$\gamma_{2,0,0}$;
Format[c210, StandardForm]:=$\gamma_{2,1,0}$;
Format[c101, StandardForm]:=$\gamma_{1,0,1}$;
Format[c201, StandardForm]:=$\gamma_{2,0,1}$;
Format[c211, StandardForm]:=$\gamma_{2,1,1}$;
Format[c102, StandardForm]:=$\gamma_{1,0,2}$;
Format[c202, StandardForm]:=$\gamma_{2,0,2}$;
Format[c212, StandardForm]:=$\gamma_{2,1,2}$;

DeclareForms$[\{0\}, \{1, \nu_0, \nu_1, \nu_2\}]$;

$d[\nu_0] = \mathbf{c100}\nu_0 \wedge \nu_1 + \mathbf{c200}\nu_0 \wedge \nu_2 + \mathbf{c210}\nu_1 \wedge \nu_2$
//Simplify
$d[\nu_1] = \mathbf{c101}\nu_0 \wedge \nu_1 + \mathbf{c201}\nu_0 \wedge \nu_2 + \mathbf{c211}\nu_1 \wedge \nu_2$
//Simplify
$d[\nu_2] = \mathbf{c102}\nu_0 \wedge \nu_1 + \mathbf{c202}\nu_0 \wedge \nu_2 + \mathbf{c212}\nu_1 \wedge \nu_2$
//Simplify

$d[d[\nu_0]]/.d[\mathtt{A\_}] \to X_0[A]\nu_0 + X_1[A]\nu_1 + X_2[A]\nu_2$
//reWrite//FullSimplify
$d[d[\nu_1]]/.d[\mathtt{A\_}] \to X_0[A]\nu_0 + X_1[A]\nu_1 + X_2[A]\nu_2$
//reWrite//FullSimplify
$d[d[\nu_2]]/.d[\mathtt{A\_}] \to X_0[A]\nu_0 + X_1[A]\nu_1 + X_2[A]\nu_2$
//reWrite//FullSimplify

Solve$[(\gamma_{1,0,1} + \gamma_{2,0,2})\gamma_{2,1,0} - \gamma_{1,0,0}\gamma_{2,1,1} - \gamma_{2,0,0}\gamma_{2,1,2}$
$+X_0[\gamma_{2,1,0}] - X_1[\gamma_{2,0,0}] + X_2[\gamma_{1,0,0}] == 0, X_0[\gamma_{2,1,0}]]$
//FullSimplify
Solve$[-\gamma_{1,0,1}\gamma_{2,0,0} + \gamma_{1,0,0}\gamma_{2,0,1} + \gamma_{2,0,2}\gamma_{2,1,1}$
$-\gamma_{2,0,1}\gamma_{2,1,2} + X_0[\gamma_{2,1,1}] - X_1[\gamma_{2,0,1}]$
$+X_2[\gamma_{1,0,1}] == 0, X_0[\gamma_{2,1,1}]]$
//FullSimplify
Solve$[\gamma_{1,0,0}\gamma_{2,0,2} - \gamma_{1,0,2}(\gamma_{2,0,0} + \gamma_{2,1,1}) + \gamma_{1,0,1}\gamma_{2,1,2}$
$+X_0[\gamma_{2,1,2}] - X_1[\gamma_{2,0,2}] + X_2[\gamma_{1,0,2}] == 0, X_0[\gamma_{2,1,2}]]$
//FullSimplify

$-\gamma_{1,0,1}\gamma_{2,1,0} + \gamma_{1,0,0}\gamma_{2,1,1} - \gamma_{2,0,2}\gamma_{2,1,0} + \gamma_{2,0,0}\gamma_{2,1,2}$
$+X_1[\gamma_{2,0,0}] - X_2[\gamma_{1,0,0}]$
$-\gamma_{2,0,1}\gamma_{1,0,0} + \gamma_{1,0,1}\gamma_{2,0,0} - \gamma_{2,0,2}\gamma_{2,1,1} + \gamma_{2,0,1}\gamma_{2,1,2}$
$+X_1[\gamma_{2,0,1}] - X_2[\gamma_{1,0,1}]$
$-\gamma_{2,0,2}\gamma_{1,0,0} + \gamma_{1,0,2}\gamma_{2,0,0} + \gamma_{1,0,2}\gamma_{2,1,1} - \gamma_{1,0,1}\gamma_{2,1,2}$
$+X_1[\gamma_{2,0,2}] - X_2[\gamma_{1,0,2}]$

JacSubs $= \{$
$X_0[\gamma_{2,1,1}] \to \gamma_{1,0,1}\gamma_{2,0,0} - \gamma_{2,0,2}\gamma_{2,1,1}$
$\gamma_{2,0,1}(-\gamma_{1,0,0} + \gamma_{2,1,2}) + X_1[\gamma_{2,0,1}] - X_2[\gamma_{1,0,1}],$
$X_0[\gamma_{2,1,2}] \to -\gamma_{1,0,0}\gamma_{2,0,2} + \gamma_{1,0,2}(\gamma_{2,0,0} + \gamma_{2,1,1})$
$-\gamma_{1,0,1}\gamma_{2,1,2} + X_1[\gamma_{2,0,2}] - X_2[\gamma_{1,0,2}]$
$\}$;

$d[(\gamma\kappa - \frac{1}{2}(\mathbf{c102} - \mathbf{c201}))\nu_0 + \mathbf{c211}\nu_1 + \mathbf{c212}\nu_2]$
$/.d[\mathtt{A\_}] \to X_0[A]\nu_0 + X_1[A]\nu_1 + X_2[A]\nu_2/.\mathbf{c210} \to 1$
//reWrite//FullSimplify

$\mathbf{f01} = (\gamma_{1,0,1} - \gamma_{2,0,2})\gamma_{2,1,1} + (\gamma_{1,0,2} + \gamma_{2,0,1})\gamma_{2,1,2}$
$+\gamma_{1,0,1}\gamma_{2,0,0} - \frac{1}{2}\gamma_{1,0,0}(\gamma_{1,0,2} + \gamma_{2,0,1}) + \gamma_{1,0,0}\gamma\kappa$
$+X_1[\frac{1}{2}(\gamma_{1,0,2} + \gamma_{2,0,1})] - X_1[\gamma\kappa] - X_2[\gamma_{1,0,1}];$
$\mathbf{f02} = (\gamma_{1,0,2} + \gamma_{2,0,1})\gamma_{2,1,1} - (\gamma_{1,0,1} - \gamma_{2,0,2})\gamma_{2,1,2}$
$-\gamma_{2,0,2}\gamma_{1,0,0} + \frac{1}{2}\gamma_{2,0,0}(\gamma_{1,0,2} + \gamma_{2,0,1}) + \gamma_{2,0,0}\gamma\kappa$
$-X_2[\frac{1}{2}(\gamma_{1,0,2} + \gamma_{2,0,1})] - X_2[\gamma\kappa] + X_1[\gamma_{2,0,2}];$

$\{\frac{1}{2}(2\gamma_{1,0,1}\gamma_{2,1,1} + 2\gamma_{1,0,2}\gamma_{2,1,2}$
$+\gamma_{1,0,0}(-\gamma_{1,0,2} + \gamma_{2,0,1} + 2\gamma\kappa) + 2X_0[\gamma_{2,1,1}]$
$+X_1[\gamma_{1,0,2}] - X_1[\gamma_{2,0,1}] - 2X_1[\gamma\kappa]),$
$\frac{1}{2}(-\gamma_{1,0,2} + \gamma_{2,0,1} + 2(\gamma_{2,1,1}^2 + \gamma_{2,1,2}^2 + \gamma\kappa) + 2X_1[\gamma_{2,1,2}]$
$02X_2[\gamma_{2,1,1}]),$
$\frac{1}{2}(-\gamma_{1,0,2}\gamma_{2,0,0} + \gamma_{2,0,0}\gamma_{2,0,1} + 2\gamma_{2,0,1}\gamma_{2,1,1}$
$+2\gamma_{2,0,2}\gamma_{2,1,2} + 2\gamma_{2,0,0}\gamma\kappa + 2X_0[\gamma_{2,1,2}] + X_2[\gamma_{1,0,2}]$
$-X_2[\gamma_{2,0,1}] - 2X_2[\gamma\kappa])$
}/.JacSubs//Simplify;
$\% - \{\mathbf{f01}, 0, \mathbf{f02}\}/.\gamma\kappa \to \frac{1}{2}(\gamma_{1,0,2} - \gamma_{2,0,1}$
$-2(\gamma_{2,1,1}^2 + \gamma_{2,1,2}^2 + X_1[\gamma_{2,1,2}] - X_2[\gamma_{2,1,1}]))$
//FullSimplify//Column

$\gamma\mathrm{R} = (R/.c \to \gamma/.\gamma_{\mathtt{u\_,v\_,w\_}} \to \mathbf{If}[u < v, -\gamma_{v,u,w}, \gamma_{u,v,w}])$
$/.\frac{1}{2}(\gamma_{1,0,2} - \gamma_{2,0,1} - 2(\gamma_{2,1,1}^2 + \gamma_{2,1,2}^2$
$+X_1[\gamma_{2,1,2}] - X_2[\gamma_{2,1,1}])) \to \gamma\kappa$
//Simplify;
$\gamma\mathrm{RSym} = (\mathrm{RSym}/.c \to \gamma$
$/.\gamma_{\mathtt{u\_,v\_,w\_}} \to \mathbf{If}[u < v, -\gamma_{v,u,w}, \gamma_{u,v,w}])$
//Simplify;
$\gamma\mathrm{RSkw} = (\mathrm{RSkw}/.c \to \gamma$
$/.\gamma_{\mathtt{u\_,v\_,w\_}} \to \mathbf{If}[u < v, -\gamma_{v,u,w}, \gamma_{u,v,w}])$
//Simplify;

γ**PexcoderRic** $=$ (**PexcoderRic/.**$c \to \gamma$
$/.\gamma_{\mathtt{u\_,v\_,w\_}} \to \mathbf{If}[u < v, -\gamma_{v,u,w}, \gamma_{u,v,w}])$
//Simplify;
γ**PexcoderRicCSym** $=$ (**PexcoderRicCSym/.**$c \to \gamma$
$/.\gamma_{\mathtt{u\_,v\_,w\_}} \to \mathbf{If}[u < v, -\gamma_{v,u,w}, \gamma_{u,v,w}])$
//Simplify;
γ**PexcoderRicCSkw** $=$ (**PexcoderRicCSkw/.**$c \to \gamma$
$/.\gamma_{\mathtt{u\_,v\_,w\_}} \to \mathbf{If}[u < v, -\gamma_{v,u,w}, \gamma_{u,v,w}])$
//Simplify;

$\begin{pmatrix} +\mathbf{f02} \\ -\mathbf{f01} \end{pmatrix} - ((\gamma\mathbf{PexcoderRic} + \gamma\mathbf{PexcoderRicCSym}$
$+\gamma\mathbf{PexcoderRicCSkw})$
$-(\gamma\mathbf{R} + \gamma\mathbf{RSym} + \gamma\mathbf{RSkw})^{\mathsf{T}}.\{-\gamma_{2,0,0}, \gamma_{1,0,0}\})$
$/.\gamma\kappa \to \frac{1}{2}(\gamma_{1,0,2} - \gamma_{2,0,1}) - \gamma_{2,1,1}^2 - \gamma_{2,1,2}^2$
$-X_1[\gamma_{2,1,2}] + X_2[\gamma_{2,1,1}]$
//FullSimplify

Rescaling the metric

$\{Y_0, Y_1, Y_2\}/.\{Y_0 \to \mu^2 X_0, Y_1 \to \mu X_1, Y_2 \to \mu X_2\}$
$\%/.\{X_0 \to \frac{1}{\mu^2}Y_0, X_1 \to \frac{1}{\mu}Y_1, X_2 \to \frac{1}{\mu}Y_2\}$**//Simplify**

$\{\mathbf{lie}[Y_1, Y_0], \mathbf{lie}[Y_2, Y_0], \mathbf{lie}[Y_2, Y_1]\}$
$/.\{Y_0 \to \mu^2 X_0, Y_1 \to \mu X_1, Y_2 \to \mu X_2\}$

```
    //FullSimplify;
%/.{X_0 → (1/μ^2) Y_0, X_1 → (1/μ) Y_1, X_2 → (1/μ) Y_2}
    /.Y_i_[μ] → 0
    /.(Y_i_/μ)[μ^2] → 0/.(Y_i_/μ)[μ] → 0//Simplify;
%//Collect[#, {Y_0, Y_1, Y_2}]&//Column
```

```
{k_{1,0,0}, k_{1,0,1}, k_{1,0,2}} = {−μc_{0,1,0}, −μ^2 c_{0,1,1}, −μ^2 c_{0,1,2}};
{k_{0,1,0}, k_{0,1,1}, k_{0,1,2}} = −{k_{1,0,0}, k_{1,0,1}, k_{1,0,2}};
{k_{1,0,0}, k_{1,0,1}, k_{1,0,2}}//Column
```

```
{k_{2,0,0}, k_{2,0,1}, k_{2,0,2}} = {−μc_{0,2,0}, −μ^2 c_{0,2,1}, −μ^2 c_{0,2,2}};
{k_{0,2,0}, k_{0,2,1}, k_{0,2,2}} = −{k_{2,0,0}, k_{2,0,1}, k_{2,0,2}};
{k_{2,0,0}, k_{2,0,1}, k_{2,0,2}}//Column
```

```
{k_{2,1,1}, k_{2,1,2}} = {−μc_{1,2,1}, −μc_{1,2,2}};
{k_{1,2,1}, k_{1,2,2}} = −{k_{2,1,1}, k_{2,1,2}};
{k_{2,1,1}, k_{2,1,2}}//Column
```

Calculate how the scalar invariants are affected by dilations of the metric:

```
{(ϑ/.{X_0→μ^2 X_0, X_1→μX_1, X_2→μX_2}/.c→k)/ϑ,
 (κ/.{X_0→μ^2 X_0, X_1→μX_1, X_2→μX_2}/.c→k)/κ,
 (χ_1/.{X_0→μ^2 X_0, X_1→μX_1, X_2→μX_2}/.c→k)/χ_1,
 (χ_2/.{X_0→μ^2 X_0, X_1→μX_1, X_2→μX_2}/.c→k)/χ_2
}/.(μX_i_)[f_] → μX_i[f]/.X_i_[μ] → 0
    //Simplify[#, c_{u_,v_,w_} ∈ Reals&&μ > 0]&
```

```
{(ϱ_0/.{X_0→μ^2 X_0, X_1→μX_1, X_2→μX_2}/.c→k)/ϱ_0,
 (ϱ_1/.{X_0→μ^2 X_0, X_1→μX_1, X_2→μX_2}/.c→k)/ϱ_1,
 (ϱ_2/.{X_0→μ^2 X_0, X_1→μX_1, X_2→μX_2}/.c→k)/ϱ_2
}//Simplify
```

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