RHODES UNIVERSITY

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FUZZY UNIFORM SPACES

by

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ABSTRACT

For a fuzzy uniform space, the notion of a Cauchy prefilter, a precompact fuzzy set, a complete fuzzy set and a bounded fuzzy set are defined in such a way that these notions are good extensions of the corresponding notions for a uniform space. A theory of fuzzy uniform spaces is developed which generalises the theory of uniform spaces.

KEYWORDS : Fuzzy uniformity, α–level uniformity, Cauchy filter, Cauchy prefilter, precompact fuzzy set, complete fuzzy set, compact fuzzy set, bounded fuzzy set, weak–Cauchy prefilter, weak–Cauchy filter, bounded set.

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CONTENTS

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PREFACE

Uniform spaces provide the appropriate abstract setting for the notion of completeness, which is in turn described by either Cauchy filters or Cauchy nets. Precompactness and boundedness are also uniform space concepts and it is well known that precompactness together with completeness is equivalent to compactness.

A uniformity $\mathbb{D}$ on a set X is a collection of subsets $U \subseteq X \times X$ and in 1981, Lowen published a paper entitled "Fuzzy Uniform Spaces" [Lo 3] in which a fuzzy uniformity on X is defined to be collection $\mathscr{D}$ of functions $\sigma : X \times X \longrightarrow [0,1]$. In 1982 and 1983 Lowen and Wuyts demonstrated, in [Lo 4] and [Lo 5], that a theory of completeness, precompactness and compactness of fuzzy uniform spaces $(X, \mathscr{D})$ could be developed and this theory was a generalisation of the standard theory.

In 1988 Chadwick showed that it is possible to define a notion of compactness for fuzzy sets $\mu \in I^X$ which was a "good extension" of the standard notion and that the theory of compact fuzzy sets was a generalisation of the standard theory [Ch 1].

It was suggested by Chadwick that it should be possible to define the notions of completeness and precompactness for fuzzy sets $\mu \in I^X$ which satisfied the criteria of being good extensions, which generalised the standard theory and, in particular, linked with his notion of compactness.

It was decided to use prefilters instead of fuzzy nets and the first task was to obtain a working definition of a $\mathscr{D}$–Cauchy prefilter and in order to check that the notion which seemed reasonable was a good extension of the definition of a Cauchy filter, it became necessary to find a suitable characterisation of Cauchy filters.

Prime prefilters are much more user friendly than prefilters in general and this is why some effort is made to characterise the fuzzy uniform space notions in terms of prime prefilters and indeed, in terms of subsets of the prime prefilters. When this was achieved, the theory of complete, precompact and bounded fuzzy sets was constructed while simultaneously developing the necessary theory of Cauchy prefilters and the theory obtained is indeed a generalisation of standard theory.

In order to provide examples and counterexamples it was desirable to be able to construct fuzzy uniformities from families of uniformities and the investigation into this process resulted in a rather striking theorem which is presented in Chapter 3. In essence it says that a fuzzy uniformity is completely determined by its α–level uniformities and so a family of uniformities obeying certain conditions can be used to determine a unique fuzzy uniformity. This proved to be extremely useful.

All of the necessary technical results concerning filters, prefilters and fuzzy uniformities are presented first, after which follows the theory of Cauchy prefilters and then the theory of precompact, bounded and complete fuzzy sets.

CHAPTER 1

FUNDAMENTALS

If X is a set then a subset $A \subseteq X$ has **characteristic function** $1_A \in \{0,1\}^X$ defined by :

$$1_A(x) = \left.\begin{array}{ll} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{array}\right\}$$

where the elementhood relation "$\in$" has the property that for each $x \in X$, either $x \in A$ or $x \notin A$. If the set X represents some collection of objects in the real world and A is defined by some condition, then in many instances the statement : "$x \in A$" cannot, with absolute certainty, be declared to be true or false.

With this in mind the notion of a **fuzzy set** was introduced by Zadeh [Za 1]. A fuzzy set is simply an element of I^X where $I = [0,1]$ and if $\mu \in I^X$ then we can think of $\mu(x)$ as being the "degree to which x belongs to μ". If $(\nu_j : j \in J)$ is a family of fuzzy sets and $x \in X$ we define :

$$(\sup \nu_j)(x) = (\vee\, \nu_j)(x) = \sup_{j \in J} \nu_j(x) \quad \text{and}$$

$$(\inf \nu_j)(x) = (\wedge\, \nu_j)(x) = \inf_{j \in J} \nu_j(x).$$

If $\mu,\nu \in I^X$ we define :

$$\begin{array}{ll} \mu \leq \nu \Leftrightarrow \forall x \in X & \mu(x) \leq \nu(x), \\ \mu'(x) = 1 - \mu(x) & \text{for all } x \in X, \\ 0(x) = 0 & \text{for all } x \in X, \\ 1(x) = 1_X(x) = 1 & \text{for all } x \in X. \end{array}$$

If X and Y are sets, $f : X \longrightarrow Y$, $\mu \in I^X$ and $\nu \in I^Y$ we define $f[\mu]$ and $f^{-1}[\nu]$ as follows :

For $y \in Y$ $f[\mu](y) = \sup_{x \in f^{-1}[\{y\}]} \mu(x)$

(with the convention that $\sup \emptyset = 0$),

$f^{-1}[\nu] = \nu \circ f$.

It is easy to check that these definitions reduce to the usual ones in the case where $\mu = 1_A$ with $A \subseteq X$ and $\nu = 1_B$ with $B \subseteq Y$. The following theorem summarises the properties of $f[\ ]$ and $f^{-1}[\ ]$. The proofs are routine or can be found in [Wo 2], [Wa 1], [Wa 2] and [Ch 3].

1.1 THEOREM

Let X,Y,Z be sets, $f : X \longrightarrow Y$, $g : Y \longrightarrow Z$, $\mu \in I^X$, $\nu \in I^Y$, $\lambda \in I^Z$, $(\mu_j : j \in J) \in (I^X)^J$ and $(\nu_j : j \in J) \in (I^Y)^J$.
Then

(a) $(g \circ f)[\mu] = g[f[\mu]]$,

(b) $(g \circ f)^{-1}[\lambda] = f^{-1}[g^{-1}[\lambda]]$,

(c) $f^{-1}[\sup_{j \in J} \nu_j] = \sup_{j \in J} f^{-1}[\nu_j]$,

(d) $f^{-1}[\inf_{j \in J} \nu_j] = \inf_{j \in J} f^{-1}[\nu_j]$,

(e) $f^{-1}[\nu'] = (f^{-1}[\nu])'$,

(f) $\nu_j \leq \nu_k \Rightarrow f^{-1}[\nu_j] \leq f^{-1}[\nu_k]$,

(g) $f[\sup_{j \in J} \mu_j] = \sup_{j \in J} f[\mu_j]$,

(h) $f[\inf_{j \in J} \mu_j] \leq \inf_{j \in J} f[\mu_j]$,

(i) $f[\mu'] \geq f[\mu]'$,

(j) $\mu_j \leq \mu_k \Rightarrow f[\mu_j] \leq f[\mu_k]$,

(k) $f[f^{-1}[\nu]] \leq \nu$ with equality if f is surjective,

(l) $f^{-1}[f[\mu]] \geq \mu$ with equality if f is injective,

(m) $f[f^{-1}[\nu] \wedge \mu] = \nu \wedge f[\mu]$.

Since suprema and infima will appear often in the sequel we note some easily proved facts in the following lemma. We shall frequently use these facts without specifically mentioning this lemma.

1.2 LEMMA

(a) If $f : X \times Y \to I$ then :

$$\sup_{(x,y)\in X\times Y} f(x,y) = \sup_{x\in X} \sup_{y\in Y} f(x,y),$$

$$\inf_{(x,y)\in X\times Y} f(x,y) = \inf_{x\in X} \inf_{y\in Y} f(x,y),$$

$$\sup_{x\in X} \inf_{y\in Y} f(x,y) \leq \inf_{y\in Y} \sup_{x\in X} f(x,y).$$

(b) If $X,Y \subseteq I$ then: $\sup X \wedge \sup Y = \sup_{x\in X} \sup_{y\in Y} x \wedge y,$

$\inf X \vee \inf Y = \inf_{x\in X} \inf_{y\in Y} x \vee y.$

(c) If $\nu, \mu \in I^X$ then : $\sup(\nu \wedge \mu) \leq \sup \nu \wedge \sup \mu$.

(d) If $\nu \in I^X$ and $A,B \subseteq X$ then :

$$\sup_{x\in A} \nu(x) \wedge \sup_{y\in B} \nu(y) = \sup_{x\in A} \sup_{y\in B} \nu(x) \wedge \nu(y).$$

It should be noted that here and for the entire thesis, $I = [0,1]$ and properties of I are needed for many of the proofs. We shall adopt the notation :
$I_0 = I\backslash\{0\}$ and $I_1 = I\backslash\{1\}$.

In [Ch 3] Chang defines a subset $\tau \subseteq I^X$ to be a **fuzzy topology** on X if :

(i) $0, 1 \in \tau$,

(ii) $\mu, \nu \in \tau \Rightarrow \mu \wedge \nu \in \tau$,

(iii) $(\mu_j : j \in J) \in \tau^J \Rightarrow \sup_{j\in J} \mu_j \in \tau$.

For the basic facts about fuzzy topologies the reader is referred to :
[Ch 3], [Wo 1] and [Wo 2].

In [Lo 9] Lowen defines a subset $\tau \subseteq I^X$ to be a fuzzy topology on X if (i), (ii), (iii) hold as well as :

(iv) $\forall \alpha \in I \ \alpha 1_X \in \tau$,

and there has been some debate about the merits of these definitions [Lo 12].

In [Wa 1], [Wa 2], [Wa 3] and [Lo 2] it is shown that a fuzzy topology can be defined using a **closure operator** or by a **fuzzy neighbourhood system**. The reader is referred to these papers as well as [Wu 1], [Wu 2], [Lo 1], [Lo 6], [Lo 7], [Lo 10] for details.

The fuzzy topologies that we shall encounter will all be generated by a fuzzy uniformity [Lo 3] which we shall define in Chapter 3 and such topologies are described by a closure operator or by a fuzzy neighbourhood system and (iv) will automatically hold, so they are all fuzzy topologies in the sense of Lowen.

CHAPTER 2

FILTERS AND PREFILTERS

A non–empty collection $\mathscr{F} \subseteq I^X$ is called a **prefilter** (on X) iff

(1) $0 \notin \mathscr{F}$,

(2) $\nu \in \mathscr{F}, \mu \in \mathscr{F} \Rightarrow \nu \wedge \mu \in \mathscr{F}$,

(3) $\nu \in \mathscr{F}, \nu \leq \mu \Rightarrow \mu \in \mathscr{F}$

and $\mathscr{F}$ is called a **prefilter base** (on X) iff

(1) $0 \notin \mathscr{F}$,

(2) $\nu \in \mathscr{F}, \mu \in \mathscr{F} \Rightarrow \exists\ \lambda \in \mathscr{F} : \lambda \leq \nu \wedge \mu$.

If $\mathscr{F}$ is a prefilter base then on X then :

$$<\mathscr{F}> = \{\mu \in I^X : \exists\ \nu \in \mathscr{F}, \nu \leq \mu\}$$

is called the prefilter **generated by** $\mathscr{F}$. These are natural generalisations of the notions filter and filter base from set theory and general topology and we shall in this chapter develop the theory of filters and prefilters necessary for our investigations. If $\mathscr{F}$ is a prefilter then it is automatically a prefilter base and for a prefilter (base) $\mathscr{F}$ we define the **characteristic of** $\mathscr{F}$, denoted $c(\mathscr{F})$ by :

$$c(\mathscr{F}) = \inf_{\nu \in \mathscr{F}} \sup \nu.$$

If $\nu \in \mathscr{F}$ then $(\sup \nu)\ 1_X \in \mathscr{F}$ and an easy consequence of this is :

$$c(\mathscr{F}) = \inf\{\alpha \in I : \alpha 1_X \in \mathscr{F}\}.$$

2.1 LEMMA

If $\mathscr{F}$ is a prefilter base then $c(\mathscr{F}) = c(<\mathscr{F}>)$.

PROOF

Since $\mathcal{F} \subseteq <\mathcal{F}>$ we have $c(<\mathcal{F}>) \leq c(\mathcal{F})$ immediately. For the reverse inequality suppose that $c(<\mathcal{F}>) < c(\mathcal{F})$. Choose $\alpha \in I$ such that $c(<\mathcal{F}>) < \alpha < c(\mathcal{F})$. Then there exists $\lambda \in <\mathcal{F}>$ such that $\sup \lambda < \alpha$ and there exists $\nu \in \mathcal{F}$ such that $\nu \leq \lambda$. Consequently $\sup \nu < \alpha < c(\mathcal{F})$ which contradicts the definitions of $c(\mathcal{F})$.

∎

If $\mathcal{F}$ and $\mathcal{G}$ are prefilters bases we say that they are **compatible**, denoted $\mathcal{F} \sim \mathcal{G}$ iff $\nu \wedge \mu \neq 0$ for each $\nu \in \mathcal{F}$ and $\mu \in \mathcal{G}$.

If $\mathcal{F} \sim \mathcal{G}$ we can define

$$\mathcal{F} \vee \mathcal{G} = <\{\nu \wedge \mu : \nu \in \mathcal{F}, \mu \in \mathcal{G}\}>.$$

2.2 LEMMA

> If $\mathcal{F}$ and $\mathcal{G}$ are compatible prefilter bases then $\mathcal{F} \vee \mathcal{G}$ is the smallest prefilter containing both $\mathcal{F}$ and $\mathcal{G}$.

The proof is routine and we omit it here.

For prefilters $\mathcal{F}$ and $\mathcal{G}$ we define

$$c(\mathcal{F},\mathcal{G}) = \left.\begin{array}{ll} c(\mathcal{F} \vee \mathcal{G}) & \text{if } \mathcal{F} \sim \mathcal{G}, \\ = 0 & \text{otherwise.} \end{array}\right\}$$

If $0 \neq \mu \in I^X$ then $<\mu> = \{\nu \in I^X : \mu \leq \nu\}$ is a prefilter and is called a **principal** prefilter and we write $c(\mathcal{F},\mu)$ for short instead of $c(\mathcal{F},<\mu>)$. Some authors use the notation $(\mathcal{F},\mathcal{G})$ instead of $\mathcal{F} \vee \mathcal{G}$ and in terms of this notation $c(\mathcal{F},\mathcal{G}) = c((\mathcal{F},\mathcal{G}))$ whether or not $\mathcal{F} \sim \mathcal{G}$.

Of particular interest are those prefilters $\mathcal{F}$ with the property that :

2.3 PROPERTY

> $\nu \in \mathcal{F} \Leftrightarrow \forall\, \epsilon \in I_0\ \exists\, \nu_\epsilon \in \mathcal{F} : \nu_\epsilon \leq \nu + \epsilon.$

In this connection, if $\mathscr{F}$ is a prefilter base with $c(\mathscr{F}) > 0$, let

$$\hat{\mathscr{F}} = \{ \sup_{\epsilon \in I_0} (\nu_\epsilon - \epsilon) : (\nu_\epsilon : \epsilon \in I_0) \in \mathscr{F}^{I_0}\}.$$

In this, $\sup_{\epsilon \in I_0} (\nu_\epsilon - \epsilon)(x) := \sup_{\epsilon \in I_0} (\nu_\epsilon(x) - \epsilon)$, so if for each $\epsilon \in I_0$ we have $\nu_\epsilon - \epsilon \leq \nu$, then it follows that $\sup_{\epsilon \in I_0} (\nu_\epsilon - \epsilon) \leq \nu$. It is easy to show that $\hat{\mathscr{F}}$ is also a prefilter base and that $\mathscr{F} \subseteq \hat{\mathscr{F}}$.

2.4 LEMMA

$\mathscr{F} = \hat{\mathscr{F}} \Leftrightarrow \mathscr{F}$ has property 2.3.

PROOF

($\Rightarrow$) Let $\mathscr{F} = \hat{\mathscr{F}}$ and $\nu \in \mathscr{F}$. Then $\nu \in \hat{\mathscr{F}}$ and hence there is a family $(\nu_\epsilon : \epsilon \in I_0) \in \mathscr{F}^{I_0}$ such that $\nu = \sup_{\epsilon \in I_0} (\nu_\epsilon - \epsilon)$. Thus for all $\epsilon \in I_0$ we have $\nu \geq \nu_\epsilon - \epsilon$. Conversely, if $\forall \epsilon \in I_0 \, \exists \, \nu_\epsilon \in \mathscr{F} : \nu_\epsilon - \epsilon \leq \nu$, then $\lambda := \sup_{\epsilon \in I_0} (\nu_\epsilon - \epsilon) \leq \nu$ and $\lambda \in \hat{\mathscr{F}} = \mathscr{F}$. Consequently $\nu \in \mathscr{F}$.

($\Leftarrow$) Let $\mathscr{F}$ have property 2.3 and let $\nu \in \hat{\mathscr{F}}$. Then there is a family $(\nu_\epsilon : \epsilon \in I_0) \in \mathscr{F}^{I_0}$ such that $\nu = \sup_{\epsilon \in I_0} (\nu_\epsilon - \epsilon)$ and hence for each $\epsilon \in I_0$ $\nu \geq \nu_\epsilon - \epsilon$. Invoking Property 2.3 we conclude that $\nu \in \mathscr{F}$ and so we have shown that $\hat{\mathscr{F}} \subseteq \mathscr{F}$.

∎

Property 2.3 is Lowen's condition N2 in [Lo 2], one of the conditions imposed on a collection of prefilters for them to qualify as a fuzzy neighbourhood system. We shall be concerned with fuzzy uniform spaces and the neighbourhood prefilters in a fuzzy uniform space form a fuzzy neighbourhood system, a point which we shall discuss later. For the moment it suffices to say that the properties of prefilters enjoying Property 2.3 are to be found in [Lo 2], [Lo 3] and [Lo 10]. We collect these together in the following theorem.

2.5 THEOREM

Let $\mathscr{F}$ and $\mathscr{G}$ be prefilter bases with $c(\mathscr{F}) \wedge c(\mathscr{G}) > 0$. Then

(a) $\mathscr{F} \subseteq \hat{\mathscr{F}}$,

(b) $\mathscr{F} \subseteq \mathscr{G} \Rightarrow \hat{\mathscr{F}} \subseteq \hat{\mathscr{G}}$,

(c) $\tilde{\mathscr{F}} := \langle \hat{\mathscr{F}} \rangle = \hat{\mathscr{G}}$ where $\mathscr{G} = <\mathscr{F}>$,

(d) $\hat{\hat{\mathscr{F}}} \subseteq \tilde{\mathscr{F}}$,

(e) If $\mathscr{F}$ is a prefilter then $\tilde{\mathscr{F}} = \hat{\mathscr{F}}$,

(f) If $\mathscr{F}$ is a prefilter then $c(\hat{\mathscr{F}}) = c(\mathscr{F})$.

In the sequel, we shall need to be able to construct prefilters from filters and filters from prefilters. If $\mathscr{F}$ is a prefilter, $\nu \in \mathscr{F}$ and $\alpha \in I$ then we define :

$$\nu^{\alpha} = \{x \in X : \nu(x) > \alpha\},$$
$$\nu_{\alpha} = \{x \in X : \nu(x) \geq \alpha\}.$$

Some authors call ν^0 the **support of** ν.

2.6 THEOREM (PREFILTERS FROM FILTERS)

Let $\mathbb{F}$ be a filter on X and $c > 0$. Define :

$$\mathbb{F}_c = \langle \{c\, 1_F : F \in \mathbb{F}\} \rangle$$
$$\mathbb{F}^c = \{\nu \in I^X : \forall \alpha < c\ \nu^{\alpha} \in \mathbb{F}\}.$$ Then

(a) $\mathbb{F}_c$ and $\mathbb{F}^c$ are prefilters with $\mathbb{F}_c \subseteq \mathbb{F}^c$,

(b) $\hat{\mathbb{F}}_c = \mathbb{F}^c$ and $\hat{\mathbb{F}}^c = \mathbb{F}^c$,

(c) $c(\mathbb{F}_c) = c(\mathbb{F}^c) = c$.

PROOF

(a) Straightforward check using : $1_F \wedge 1_K = 1_{F \cap K}$, $(\nu \wedge \mu)^\alpha = \nu^\alpha \cap \mu^\alpha$ and $(c1_F)^\alpha = F$ for $\alpha < c$.

(b) Let $\lambda \in \widehat{\mathbb{F}^c}$. Then there is a family $(\nu_\epsilon : \epsilon \in I_0)$ with $\nu_\epsilon \in \mathbb{F}^c$ such that $\lambda = \sup_{\epsilon \in I_0} (\nu_\epsilon - \epsilon)$. We observe that if $\alpha < c$ then :

$$\lambda^\alpha = \bigcup_{\epsilon \in I_0} \nu_\epsilon^{\alpha+\epsilon}$$

$[x \in \lambda^\alpha \Leftrightarrow \lambda(x) > \alpha \Leftrightarrow \exists\ \epsilon \in I_0 : \nu_\epsilon(x) - \epsilon > \alpha \Leftrightarrow x \in \bigcup_{\epsilon \in I_0} \nu_\epsilon^{\alpha+\epsilon}]$

So if we choose ϵ such that $\alpha < \alpha+\epsilon < c$ then $\nu_\epsilon^{\alpha+\epsilon} \subseteq \lambda^\alpha$ and, since $\nu_\epsilon^{\alpha+\epsilon} \in \mathbb{F}$, it follows that $\lambda^\alpha \in \mathbb{F}$. We have shown that for all $\alpha < c$, $\lambda^\alpha \in \mathbb{F}$ and hence $\lambda \in \mathbb{F}^c$. Since λ is arbitrary we conclude that $\widehat{\mathbb{F}^c} = \mathbb{F}^c$.

Now let $\lambda \in \mathbb{F}^c$. Then for each $\epsilon \in I_0$, $F_\epsilon := \lambda^{c-c\epsilon} \in \mathbb{F}$ and so $\nu_\epsilon := c\, 1_{F_\epsilon} \in \mathbb{F}_c$ and $\nu_\epsilon - \epsilon \leq \lambda$. [If $x \in F_\epsilon$ then $\nu_\epsilon(x) - \epsilon = c - \epsilon < c - c\epsilon < \lambda(x)$ and if $x \notin F_\epsilon$ then $\nu_\epsilon(x) = 0 \leq \lambda(x) + \epsilon$]. Consequently, $\nu := \sup_{\epsilon \in I_0} (\nu_\epsilon - \epsilon) \leq \lambda$ with $\nu \in \widehat{\mathbb{F}}_c$ and hence $\lambda \in \widehat{\mathbb{F}}_c$. Since λ is arbitrary we have shown that $\mathbb{F}^c \subseteq \widehat{\mathbb{F}}_c$ and now, since $\mathbb{F}_c \subseteq \mathbb{F}^c$, we have $\widehat{\mathbb{F}}_c \subseteq \widehat{\mathbb{F}^c} = \mathbb{F}^c \subseteq \widehat{\mathbb{F}}_c$.

(c) In the light of 2.5(f) we need only show that $c(\mathbb{F}_c) = c$ and this in turn follows from the fact that $\sup c\, 1_F = c$ for each $F \in \mathbb{F}$.

■

2.7 THEOREM (FILTERS FROM PREFILTERS)

Let $\mathscr{F}$ be a prefilter on X with $c = c(\mathscr{F}) > 0$. Define :

$$\mathscr{F}_0 = \langle \{\nu^0 : \nu \in \mathscr{F}\} \rangle,$$

$$\mathscr{F}_\alpha = \langle \{\nu^\alpha : \nu \in \mathscr{F}\} \rangle \qquad \text{for } 0 \leq \alpha < c,$$

$$\mathscr{F}^\alpha = \{\nu^\beta : \nu \in \mathscr{F}, \beta < \alpha\} \qquad \text{for } 0 < \alpha \leq c.$$

(a) If $0 \leq \beta < \alpha \leq c$ then $\mathscr{F}_0$, $\mathscr{F}_\beta$, $\mathscr{F}^\alpha$ and $\mathscr{F}^c$ are filters on X with $\mathscr{F}_0 \subseteq \mathscr{F}_\beta \subseteq \mathscr{F}^\alpha \subseteq \mathscr{F}^c$.

(b) $\mathscr{F}_0 = \{F \subseteq X : 1_F \in \mathscr{F}\}$.

(c) If $\alpha \geq \beta > 0$ and $\mathbb{F}$ is a filter on X then

$\mathbb{F} = (\mathbb{F}_\alpha)_0 = (\mathbb{F}^\alpha)_0 = (\mathbb{F}^\alpha)^\beta = (\mathbb{F}^\alpha)^\alpha$.

(d) If $F \in \mathscr{F}_0$ and $c(\mathscr{F}) < \gamma \leq 1$ then $\gamma 1_F \in \mathscr{F}$.

PROOF

(a) Follows from

(i) $\nu_1^{\alpha_1} \cap \nu_2^{\alpha_2} \supseteq (\nu_1 \wedge \nu_2)^{\alpha_1 \vee \alpha_2}$ and

(ii) $A \in \mathscr{F}^\alpha,\ A \subseteq B \Rightarrow B \in \mathscr{F}^\alpha$.

To see (ii) let $A = \nu^\beta$ with $\nu \in \mathscr{F}$ and $\beta < \alpha$ and define $\mu = \nu \vee 1_B$. Then $\mu \in \mathscr{F}$ and $\mu^\beta = B$. [If $x \in B$ then $\mu(x) = 1 > \beta$ and so $x \in \mu^\beta$. If $x \in \mu^\beta$ then $\nu(x) > \beta$ or $x \in B$. Thus $x \in B \cup \nu^\beta = B \cup A = B$.]

(b) If $F \in \mathscr{F}_0$ then $F \supseteq \nu^0$ for some $\nu \in \mathscr{F}$ and hence $\nu \leq 1_{\nu^0} \leq 1_F$. Thus $1_F \in \mathscr{F}$. Conversely, if $1_F \in \mathscr{F}$ then $F = (1_F)^0 \in \mathscr{F}_0$.

(c) If $F \in \mathbb{F}$ then $\alpha 1_F \in \mathbb{F}_\alpha$ and so :

$$F = (\alpha 1_F)^0 \in (\mathbb{F}_\alpha)_0 \subseteq (\mathbb{F}^\alpha)_0 \subseteq (\mathbb{F}^\alpha)^\beta \subseteq (\mathbb{F}^\alpha)^\alpha$$

Thus we have $\mathbb{F} \subseteq (\mathbb{F}_\alpha)_0 \subseteq (\mathbb{F}^\alpha)^\beta \subseteq (\mathbb{F}^\alpha)^\alpha$. For the reverse inclusion let $\nu^\gamma \in (\mathbb{F}^\alpha)^\alpha$ with $\nu \in \mathbb{F}^\alpha$ and $\gamma < \alpha$. Then $\nu^\gamma \in \mathbb{F}$ and hence $(\mathbb{F}^\alpha)^\alpha \subseteq \mathbb{F}$.

(d) If $F \in \mathcal{F}_0$ and $c(\mathcal{F}) < \gamma$ then $\nu_1^0 \subseteq F$ and $\sup \nu_2 < \gamma$ for some $\nu_1, \nu_2 \in \mathcal{F}$. Let $\nu = \nu_1 \wedge \nu_2$ then $\nu \in \mathcal{F}$, $\nu^0 \subseteq F$ and $\sup \nu < \gamma$. Thus $\nu \leq \gamma 1_F$ and so $\gamma 1_F \in \mathcal{F}$.

■

PRIME PREFILTERS

Recall that a filter $\mathbb{F}$ is called **prime** (or **ultra**) iff

$$F \cup K \in \mathbb{F} \Longrightarrow F \in \mathbb{F} \text{ or } K \in \mathbb{F}.$$

Analagously we call a prefilter $\mathcal{F}$ **prime** iff

$$\nu \vee \mu \in \mathcal{F} \Longrightarrow \nu \in \mathcal{F} \text{ or } \mu \in \mathcal{F}.$$

It can be shown (see for example [Wi 2] or [Bo 1]) that a filter is prime iff it is maximal but this is not true for prefilters. This can be seen by considering $\mathcal{F} = \langle \mu \rangle$ for $\mu \neq 0$. We assert that :

2.8 ASSERTION

$\mathcal{F} = \langle \mu \rangle$ is prime $\Leftrightarrow \exists\ \alpha > 0 \ni x \in X : \mu = \alpha 1_x$.

($\Longrightarrow$) If $x_1, x_2 \in \mu^0$ with $x_1 \neq x_2$ let :

$$\left.\begin{aligned} \nu_1(x) &= \mu(x) \quad \text{for } x \neq x_2, \\ &= 0 \quad\quad\ \text{for } x = x_2. \end{aligned}\right\}$$

$$\nu_2 = \mu(x_2) 1_{x_2}$$

Then $\nu_1 \vee \nu_2 = \mu \in \mathcal{F}$, so $\nu_1 \in \mathcal{F}$ or $\nu_2 \in \mathcal{F}$. But this means that $\mu \leq \nu_1$ or $\mu \leq \nu_2$ which is clearly false. Thus μ^0 is a singleton and hence μ is a fuzzy point.

($\Longleftarrow$) Let $\nu_1 \vee \nu_2 \in \langle \alpha 1_x \rangle$. Then $\alpha 1_x \leq \nu_1 \vee \nu_2$ and so $\nu_1(x) \geq \alpha$ or $\nu_2(x) \geq \alpha$. Consequently $\nu_1 \in \langle \alpha 1_x \rangle$ or $\nu_2 \in \langle \alpha 1_x \rangle$.

■

We observe that for each $x \in X$ and each $\beta \leq \alpha \leq \mu(x)$ we have $\langle \mu \rangle \subseteq \langle \alpha 1_x \rangle \subseteq \langle \beta 1_x \rangle$ with both $\langle \alpha 1_x \rangle$ and $\langle \beta 1_x \rangle$ being prime and this reveals that prime prefilters are not maximal. To remind us of the difference between filters and prefilters we shall use the term **ultrafilter** instead of the synonym prime filter. The following crucial theorem will be invoked many times.

2.9 THEOREM

Let $\mathcal{F}$ be a prefilter with $c(\mathcal{F}) = c > 0$ and $\mathbb{F}$ a filter on X. Then

(a) $\mathcal{F}$ is prime $\Leftrightarrow$ $\mathcal{F}_0$ is ultra,

(b) $\mathcal{F}$ is prime $\Rightarrow$ $\mathcal{F}_0 = \mathcal{F}^c$,

(c) $\mathbb{F}$ is ultra $\Leftrightarrow$ $\mathbb{F}_c$ is prime,

(d) $\mathbb{F}$ is ultra $\Leftrightarrow$ $\mathbb{F}^c$ is prime,

(e) $\mathcal{F} \subseteq \mathcal{G}$, $\mathcal{F}$ is prime $\Rightarrow$ $\mathcal{G}$ is prime.

PROOF

(a) Let $\mathcal{F}$ be prime and $A \cup B \in \mathcal{F}_0$. Then there exists $\nu \in \mathcal{F}$ with $\nu^0 \subseteq A \cup B$ and so
$\nu \leq 1_{\nu^0} \leq 1_{A \cup B} = 1_A \vee 1_B$. It follows that $1_A \vee 1_B \in \mathcal{F}$ and hence $1_A \in \mathcal{F}$ say, in which case $A = (1_A)^0 \in \mathcal{F}_0$.

For the converse, let $\nu_1 \vee \nu_2 \in \mathcal{F}$, $F_1 = \{x : \nu_1(x) \geq \nu_2(x)\}$ and $F_2 = \{x : \nu_2(x) \geq \nu_1(x)\}$. Thus $X = F_1 \cup F_2 \in \mathcal{F}_0$ and so $F_1 \in \mathcal{F}_0$ say. This means that we can find $\nu \in \mathcal{F}$ with $\nu^0 \subseteq F_1$ and hence $\nu \leq 1_{\nu^0} \leq 1_{F_1}$ from which we deduce that $1_{F_1} \in \mathcal{F}$. It follows that $(\nu_1 \vee \nu_2) \wedge 1_{F_1} \in \mathcal{F}$. Now :

$$((\nu_1 \vee \nu_2) \wedge 1_{F_1})(x) = (\nu_1 \vee \nu_2)(x) = \left.\begin{array}{ll} \nu_1(x) & \text{if } x \in F_1, \\ = 0 & \text{if } x \notin F_1. \end{array}\right\}$$

Consequently $(\nu_1 \vee \nu_2) \wedge 1_{F_1} \leq \nu_1$ and so $\nu_1 \in \mathcal{F}$.

(b) $\mathcal{F}_0 \subseteq \mathcal{F}^c$ and $\mathcal{F}_0$ is maximal.

(c) $(\mathbb{F}_c)_0 = \mathbb{F}$ and so (c) follows from (a).

(d) $(\mathbb{F}^c)_0 = \mathbb{F}$ and hence (d) also follows straight from (a).

(e) If $\mathcal{F} \subseteq \mathcal{G}$ then $\mathcal{F}_0 \subseteq \mathcal{G}_0$ and $\mathcal{F}_0$ is maximal, so $\mathcal{F}_0 = \mathcal{G}_0$ and hence $\mathcal{G}$ is prime.

■

MINIMAL PRIME PREFILTERS

Let $\mathbb{F}$ be a filter and $\mathcal{F}$ a prefilter. We define :

$$\mathbb{P}(\mathbb{F}) = \{\mathbb{K} : \mathbb{K} \text{ is an ultrafilter}, \mathbb{F} \subseteq \mathbb{K}\},$$
$$\mathcal{P}(\mathcal{F}) = \{\mathcal{G} : \mathcal{G} \text{ is a prime prefilter}, \mathcal{F} \subseteq \mathcal{G}\}.$$

It is as easy to show that :

2.10 LEMMA

$$\mathcal{F} = \bigcap_{\mathcal{G} \in \mathcal{P}(\mathcal{F})} \mathcal{G},$$

as it is to show that :

2.11 LEMMA

$$\mathbb{F} = \bigcap_{\mathbb{K} \in \mathbb{P}(\mathbb{F})} \mathbb{K}$$ (see [Bo 1]).

A fairly simple Zorn's Lemma argument yields the fact that $\mathcal{P}(\mathcal{F})$ has minimal elements (see [Lo 1]). As an example, consider the principal prefilter $\mathcal{F} = \langle \mu \rangle$ with its chain of prime prefilters $(\langle \alpha 1_x \rangle : \alpha \leq \mu(x))$ at each $x \in X$. The prime prefilter $\langle \mu(x) 1_x \rangle$ is minimal.

For a prefilter $\mathcal{F}$ we define :

$$\mathcal{P}_m(\mathcal{F}) = \{\mathcal{G} : \mathcal{G} \in \mathcal{P}(\mathcal{F}), \mathcal{G} \text{ is minimal}\}.$$

The result above can be improved to :

2.12 LEMMA

$$\mathcal{F} = \bigcap_{\mathcal{G} \in \mathcal{P}_m(\mathcal{F})} \mathcal{G}$$

PROOF

Let $\nu \in \bigcap_{\mathcal{G} \in \mathcal{P}_m(\mathcal{F})} \mathcal{G}$ and let $\mathcal{H} \in \mathcal{P}(\mathcal{F})$. Order $\{\mathcal{G} \in \mathcal{P}(\mathcal{F}) : \mathcal{F} \subseteq \mathcal{G} \subseteq \mathcal{H}\}$ by inclusion and apply Zorn's Lemma to obtain $\mathcal{G} \in \mathcal{P}_m(\mathcal{F}) : \mathcal{F} \subseteq \mathcal{G} \subseteq \mathcal{H}$. Then $\nu \in \mathcal{G}$ and so $\nu \in \mathcal{H}$. Since $\mathcal{H}$ is arbitrary we conclude that $\nu \in \bigcap_{\mathcal{H} \in \mathcal{P}(\mathcal{F})} \mathcal{H} = \mathcal{F}$ and hence $\bigcap_{\mathcal{G} \in \mathcal{P}_m(\mathcal{F})} \mathcal{G} \subseteq \mathcal{F}$. The reverse inclusion is obvious.

■

CHARACTERISATION OF MINIMAL PRIME PREFILTERS

The minimal prime prefilters are destined to play a major role in the theory of fuzzy uniform spaces and the following characterisation of them was found by Lowen and a proof appears in [Lo 1]. Because it is of such importance to our investigation we give a (slightly modified) proof here. If $\mathcal{F}$ is a prefilter and $\mathbb{F}$ a filter we say that they are **compatible** if $\nu \wedge 1_F \neq 0$ for all $\nu \in \mathcal{F}$ and all $F \in \mathbb{F}$, in other words, if $\mathcal{F} \sim \mathbb{F}_1$. We could abuse this notation without fear of confusion and write $\mathcal{F} \sim \mathbb{F}$ as Lowen does in [Lo 1]. In that paper

$$(\mathcal{F}, \mathbb{F}) := \langle \{\nu \wedge 1_F : \nu \in \mathcal{F}, F \in \mathbb{F}\} \rangle$$

and in terms of our notation :

$$(\mathcal{F}, \mathbb{F}) = \mathcal{F} \vee \mathbb{F}_1.$$

2.13 LEMMA

If $\mathcal{F}$ is a prefilter and $\mathbb{F}$ a filter then

(a) $\mathcal{F} \sim \mathbb{F} \Rightarrow (\mathcal{F} \vee \mathbb{F}_1)_0 = \mathcal{F}_0 \vee \mathbb{F}$.

(b) If $\mathbb{F}$ is ultra then $\mathcal{F} \sim \mathbb{F} \Leftrightarrow \mathcal{F}_0 \subseteq \mathbb{F}$.

(c) If $\mathbb{F}$ is ultra then $\mathcal{F} \sim \mathbb{F} \Leftrightarrow (\mathcal{F} \vee \mathbb{F}_1)_0 = \mathbb{F}$.

PROOF

(a) $(\mathcal{F} \vee \mathbb{F}_1)_0 = \langle \{(\nu \wedge 1_F)^0 : \nu \in \mathcal{F}, F \in \mathbb{F}\} \rangle = \langle \{\nu^0 \cap F : \nu \in \mathcal{F}, F \in \mathbb{F}\} \rangle = \mathcal{F}_0 \vee \mathbb{F}.$

(b) $\mathcal{F} \sim \mathbb{F} \iff \forall \nu \in \mathcal{F} \, \forall F \in \mathbb{F} \;\; \nu \wedge 1_F \neq 0$

$\iff \forall \nu \in \mathcal{F} \, \forall F \in \mathbb{F} \;\; \nu^0 \cap F \neq \emptyset$

$\iff \forall \nu \in \mathcal{F} \;\; \nu^0 \in \mathbb{F}$ (since $\mathbb{F}$ is an ultrafilter)

$\iff \mathcal{F}_0 \subseteq \mathbb{F}.$

(c) $\mathcal{F}_0 \subseteq \mathbb{F} \iff \mathcal{F}_0 \vee \mathbb{F} = \mathbb{F}$

$\iff (\mathcal{F} \vee \mathbb{F}_1)_0 = \mathbb{F}.$

■

Note that we have used the result from set theory :

2.14 THEOREM

> If $\mathbb{F}$ is an ultrafilter then $A \in \mathbb{F} \iff \forall F \in \mathbb{F} \;\; A \cap F \neq \emptyset$.

For a proof of this see [Wi 2] or [Bo 1].

2.15 THEOREM

> If $\mathcal{F}$ is a prefilter then :
> $\mathcal{P}_m(\mathcal{F}) = \{\mathcal{F} \vee \mathbb{F}_1 : \mathbb{F} \in \mathbb{P}(\mathcal{F}_0)\}.$

PROOF

($\Rightarrow$) Let $\mathcal{G} \in \mathcal{P}_m(\mathcal{F})$ and $\mathbb{F} = \mathcal{G}_0$. Then $\mathbb{F}$ is ultra (2.9(a)), $\mathbb{F} \supseteq \mathcal{F}_0$ and $\mathcal{F} \sim \mathbb{F}$. Now $\mathbb{F}_1 = (\mathcal{G}_0)_1 = \langle \{1_{\nu^0} : \nu \in \mathcal{G}\} \rangle \subseteq \mathcal{G}$ and so $\mathcal{F} \subseteq \mathcal{F} \vee \mathbb{F}_1 \subseteq \mathcal{F} \vee \mathcal{G} = \mathcal{G}$. Furthermore $(\mathcal{F} \vee \mathbb{F}_1)_0 = \mathcal{F}_0 \vee \mathbb{F} = \mathbb{F}$ and hence $\mathcal{F} \vee \mathbb{F}_1$ is prime. Thus by the minimality of $\mathcal{G}$ we have $\mathcal{G} = \mathcal{F} \vee \mathbb{F}_1$.

($\Leftarrow$) Let $\mathbb{F}$ be an ultrafilter such that $\mathcal{F}_0 \subseteq \mathbb{F}$. Let $\mathcal{G} = \mathcal{F} \vee \mathbb{F}_1$. Then $\mathcal{G}_0 = \mathbb{F}$ is ultra and so $\mathcal{G}$ is prime (2.9(a)). Of course $\mathcal{F} \subseteq \mathcal{G}$ and so it remains to show that $\mathcal{G}$ is minimal in $\mathcal{P}(\mathcal{F})$. To this end let $\mathcal{H} \in \mathcal{P}(\mathcal{F})$ with $\mathcal{F} \subseteq \mathcal{H} \subseteq \mathcal{G}$ and let $\lambda \in \mathcal{G}$. Then there exists $\nu \in \mathcal{F}$ and $F \in \mathbb{F}$ with $\nu \wedge 1_F \leq \lambda$. Now :

$$\nu = (\nu \wedge 1_F) \vee (\nu \wedge 1_{F'}) \in \mathcal{F} \subseteq \mathcal{H} \text{ (where } F' := X\backslash F).$$

and $\mathcal{H}$ is prime so $\nu \wedge 1_F \in \mathcal{H}$ or $\nu \wedge 1_{F'} \in \mathcal{H}$. But if $\nu \wedge 1_{F'} \in \mathcal{H}$ then $\nu \wedge 1_{F'} \in \mathcal{G}$ and we have $\nu \wedge 1_F \in \mathcal{G}$ which means that $(\nu \wedge 1_{F'}) \wedge (\nu \wedge 1_F) = 0 \in \mathcal{G}$, contradicting the fact that $\mathcal{G}$ is a prefilter. We conclude that $\nu \wedge 1_F \in \mathcal{H}$ and hence that $\lambda \in \mathcal{H}$. Since λ is arbitrary, $\mathcal{G} \subseteq \mathcal{H}$ and so $\mathcal{G} = \mathcal{H}$.

■

If $\mathcal{F}$ is a prefilter and $\mathcal{G} \in \mathcal{P}_m(\mathcal{F})$ then since $\mathcal{F} \subseteq \mathcal{G}$ we have $c(\mathcal{G}) \leq c(\mathcal{F})$ and in fact $c(\mathcal{F}) = \sup_{\mathcal{G} \in \mathcal{P}_m(\mathcal{F})} c(\mathcal{G})$. The fact that $c(\mathcal{F})$ is actually attained is very convenient for us. The proof is the first example of the usefulness of Theorem 2.15.

2.16 THEOREM

> If $\mathcal{F}$ is a prefilter then :
>
> there exists $\mathcal{G} \in \mathcal{P}_m(\mathcal{F})$ such that $c(\mathcal{G}) = c(\mathcal{F})$.

PROOF

If $c = c(\mathcal{F}) > 0$, choose an ultrafilter $\mathbb{F} \supseteq \mathcal{F}^c$ and let $\mathcal{G} = \mathcal{F} \vee \mathbb{F}_1$. Then we have $\mathcal{F}_0 \subseteq \mathcal{F}^c \subseteq \mathbb{F}$ and so $\mathcal{G} \in \mathcal{P}_m(\mathcal{F})$ from which it follows that $c(\mathcal{G}) \leq c(\mathcal{F})$. For the reverse inequality let $0 < \alpha < c(\mathcal{F})$ and let $\mu \in \mathcal{G}$. Then there exists $\nu \in \mathcal{F}$ and $F \in \mathbb{F}$ with $\nu \wedge 1_F \leq \mu$. Now $\overset{\alpha}{\nu} \in \mathcal{F}^c \subseteq \mathbb{F}$ and so $\overset{\alpha}{\nu} \cap F \neq \emptyset$. Let $x \in \overset{\alpha}{\nu} \cap F$ then $\sup \mu \geq \sup \nu \wedge 1_F \geq \nu(x) > \alpha$. Since μ is arbitrary we have $c(\mathcal{G}) \geq \alpha$ and since α is arbitrary we conclude that $c(\mathcal{F}) \leq c(\mathcal{G})$.

If $\mathscr{F}$ is a prefilter with $c(\mathscr{F}) = 0$ and we choose an ultrafilter $\mathbb{F} \supseteq \mathscr{F}_0$ and let $\mathscr{G} = \mathscr{F} \vee \mathbb{F}_1$, then again $\mathscr{G} \in \mathscr{P}_m(\mathscr{F})$ and so $c(\mathscr{G}) \leq c(\mathscr{F}) = 0$ from which we conclude that $c(\mathscr{G}) = c(\mathscr{F})$.

■

From the proof of Theorem 2.16 we glean :

2.17 THEOREM

> Let $\mathscr{F}$ be a prefilter and let $c = c(\mathscr{F})$.
> If $c > 0$ and $\mathbb{F}$ is an ultrafilter with $\mathbb{F} \supseteq \mathscr{F}^c$ then
> $\mathscr{G} = \mathscr{F} \vee \mathbb{F}_1 \in \mathscr{P}_m(\mathscr{F})$ and $c(\mathscr{G}) = c(\mathscr{F})$.

For a prefilter $\mathscr{F}$ we define the **lower characteristic of $\mathscr{F}$**, denoted $\bar{c}(\mathscr{F})$ by :

$$\bar{c}(\mathscr{F}) = \inf_{\mathscr{G} \in \mathscr{P}_m(\mathscr{G})} c(\mathscr{G}).$$

2.18 LEMMA

> If $\mathscr{F}$ is a prime prefilter then $\bar{c}(\mathscr{F}) = c(\mathscr{F})$.

PROOF

If $\mathscr{F}$ is prime then $\mathscr{P}_m(\mathscr{F}) = \{\mathscr{F}\}$.

■

For prefilters $\mathscr{F}$ and $\mathscr{G}$ we have $c(\mathscr{G}) \leq c(\mathscr{F})$ if $\mathscr{F} \subseteq \mathscr{G}$ but if $\mathscr{F} \subseteq \mathscr{G}$ we cannot say that $\bar{c}(\mathscr{G}) \leq \bar{c}(\mathscr{F})$. This is one of the reasons why the lower characteristic is difficult to work with and why we take the trouble to express our theorems in terms of prime prefilters.

The following Lemma will prove very useful.

2.19 LEMMA

> If $\mathcal{F}$ is a prefilter with $\bar{c}(\mathcal{F}) > 0$ then :
> $$0 < \alpha < \bar{c}(\mathcal{F}) \Leftrightarrow \forall \nu \in \mathcal{F}\ \nu^{\alpha} \in \mathcal{F}_0.$$

PROOF

Let $\mathcal{S} = \{\mathbb{F} : \mathbb{F}$ is an ultrafilter and $\mathbb{F} \supseteq \mathcal{F}_0\} = \mathbb{P}(\mathcal{F}_0)$. Then :

$$\bar{c}(\mathcal{F}) = \inf_{\mathbb{F} \in \mathcal{S}} c(\mathcal{F} \vee \mathbb{F}_1) = \inf_{\mathbb{F} \in \mathcal{S}} \inf_{F \in \mathbb{F}} \inf_{\nu \in \mathcal{F}} \sup \nu \wedge 1_F \text{ so :}$$

$$\begin{aligned}
0 < \alpha < \bar{c}(\mathcal{F}) \quad & \Leftrightarrow \forall \mathbb{F} \in \mathcal{S}\, \forall F \in \mathbb{F}\ \forall \nu \in \mathcal{F}\, \exists x : (\nu \wedge 1_F)(x) > \alpha \\
& \Leftrightarrow \forall \nu \in \mathcal{F}\, \forall \mathbb{F} \in \mathcal{S}\, \forall F \in \mathbb{F}\ (\nu^{\alpha} \cap F) \neq \emptyset \\
& \Leftrightarrow \forall \nu \in \mathcal{F}\, \forall \mathbb{F} \in \mathcal{S}\ \nu^{\alpha} \in \mathbb{F} \qquad (2.14) \\
& \Leftrightarrow \forall \nu \in \mathcal{F}\ \nu^{\alpha} \in \mathcal{F}_0 \qquad (2.11).
\end{aligned}$$

■

2.20 COROLLARY

> If $\mathcal{F}$ is a prefilter with $\bar{c} = \bar{c}(\mathcal{F}) > 0$ then :
>
> (a) $\bar{c}(\mathcal{F}) = \inf_{\nu \in \mathcal{F}} \sup\{\alpha : \nu^{\alpha} \in \mathcal{F}_0\} = \sup\{\alpha : \mathcal{F}_{\alpha} = \mathcal{F}_0\}$.
>
> (b) If $\mathcal{F} = \hat{\mathcal{F}}$ then $(\mathcal{F}^{\bar{c}})_c \subseteq \mathcal{F}$.

PROOF

(a) Let $\beta = \inf_{\nu \in \mathcal{F}} \sup\{\alpha : \nu^{\alpha} \in \mathcal{F}_0\}$ and let $\gamma = \sup\{\alpha : \mathcal{F}_{\alpha} = \mathcal{F}_0\}$ and note that since $\mathcal{F}_0 \subseteq \mathcal{F}_{\alpha}$, $\gamma = \sup\{\alpha : \mathcal{F}_{\alpha} \subseteq \mathcal{F}_0\}$. From 2.19 we have :

$$\begin{aligned}
\alpha < \bar{c}(\mathcal{F}) \quad & \Leftrightarrow \forall \nu \in \mathcal{F}\ \alpha \leq \sup\{\delta : \nu^{\delta} \in \mathcal{F}_0\} \\
& \Leftrightarrow \alpha \leq \inf_{\nu \in \mathcal{F}} \sup\{\delta : \nu^{\delta} \in \mathcal{F}_0\} = \beta. \text{ So } \bar{c}(\mathcal{F}) = \beta.
\end{aligned}$$

Also :

$\alpha < \bar{c}(\mathscr{F}) \quad \Leftrightarrow \forall \nu \in \mathscr{F}\ \nu^{\alpha} \in \mathscr{F}_0$

$\Leftrightarrow \mathscr{F}_{\alpha} \subseteq \mathscr{F}_0.$

$\Leftrightarrow \alpha \leq \gamma.$ So $\bar{c}(\mathscr{F}) = \gamma.$

■

(b) If $\mu \in (\mathscr{F}^{\bar{c}})_c$ then $\mu = c\,1_{\nu}{}^{\alpha}$ for some $\nu \in \mathscr{F}$ and $\alpha < \bar{c}$. By 2.19 we have $\nu^{\alpha} \in \mathscr{F}_0$. We intend to show that $\mu \in \hat{\mathscr{F}}$ and so we let $\epsilon \in I_0$ be arbitrary and show that there exists $\nu_{\epsilon} \in \mathscr{F}$ with $\nu_{\epsilon} \leq \mu + \epsilon$. Define $\gamma(\epsilon) = (c + \frac{\epsilon}{2}) \wedge 1$ and $\nu_{\epsilon} = \gamma(\epsilon) 1_{\nu}{}^{\alpha}$. Then $\nu_{\epsilon} \in \mathscr{F}$ (2.7(d) or (b)) and for $x \in \nu^{\alpha}$:

$\nu_{\epsilon}(x) = \gamma(\epsilon) \leq c + \frac{\epsilon}{2} = \mu(x) + \frac{\epsilon}{2} < \mu(x) + \epsilon.$

While for $x \notin \nu^{\alpha}$:

$\nu_{\epsilon}(x) = 0 = \mu(x) < \mu(x) + \epsilon.$

Thus $\nu_{\epsilon} \leq \mu + \epsilon$.

■

Notwithstanding the remark concerning the awkwardness of the lower characteristic, we do have the following Lemmas.

2.21 LEMMA

If $\mathscr{F}$ is a prefilter with $\bar{c}(\mathscr{F}) > 0$ then $\bar{c}(\hat{\mathscr{F}}) \geq \bar{c}(\mathscr{F})$.

PROOF

Let $0 < \alpha < \bar{c}(\mathscr{F})$, $\mathscr{G} \in \mathscr{P}_m(\hat{\mathscr{F}})$ and $\lambda \in \mathscr{G}$. There exists, according to 2.15 an ultrafilter $\mathbb{F} \supseteq (\hat{\mathscr{F}})_0$ such that $\mathscr{G} = \hat{\mathscr{F}} \vee \mathbb{F}_1$ and hence $\lambda \geq \nu \wedge 1_F$ for some $\nu \in \hat{\mathscr{F}}$ and $F \in \mathbb{F}$.

Now $\nu = \sup_{\epsilon \in I_0} (\nu_\epsilon - \epsilon)$ for some family $(\nu_\epsilon : \epsilon \in I_0) \in \mathscr{F}^{I_0}$ so choose β such that $\alpha < \beta < \bar{c}(\mathscr{F})$ and let $\epsilon < \beta - \alpha$. Then we have $\nu_\epsilon \in \mathscr{F}$ and $\nu \geq \nu_\epsilon - \epsilon$. By 2.19 we have $\nu_\epsilon^\beta \in \mathscr{F}_0 \subseteq (\hat{\mathscr{F}})_0 \subseteq \mathbb{F}$ and hence $\nu_\epsilon^\beta \cap F \neq \emptyset$. Choose $x \in \nu_\epsilon^\beta \cap F$, then :

$$\sup \lambda \geq \sup \nu \wedge 1_F = \sup_{y \in F} \nu(y) \geq \nu(x) \geq \nu_\epsilon(x) - \epsilon > \beta - \epsilon > \alpha.$$

Since λ is arbitrary, $c(\mathscr{G}) \geq \alpha$, since $\mathscr{G}$ is arbitrary, $\bar{c}(\hat{\mathscr{F}}) \geq \alpha$ and since α is arbitrary, $\bar{c}(\hat{\mathscr{F}}) \geq \bar{c}(\mathscr{F})$.

■

2.22 LEMMA

If $c > 0$ then $\bar{c}(\mathbb{F}_c) = c$ and $\bar{c}(\mathbb{F}^c) \geq c$.

PROOF

We observe that $\mathscr{G} \in \mathscr{P}_m(\mathbb{F}_c)$ iff $\mathscr{G} = \mathbb{F}_c \vee \mathbb{K}_1$ for some ultrafilter $\mathbb{K} \supseteq (\mathbb{F}_c)^0 = \mathbb{F}$ (2.7) in which case

$$\begin{aligned} c(\mathscr{G}) = c(\mathbb{F}_c \vee \mathbb{K}_1) &= \inf_{F \in \mathbb{F}} \inf_{K \in \mathbb{K}} \sup(c1_F \wedge 1_K) \\ &= \inf_{F \in \mathbb{F}} \inf_{K \in \mathbb{K}} \sup c1_{(F \cap K)} \\ &= c \text{ (since in each case } F \cap K \neq \emptyset). \end{aligned}$$

This holds for every $\mathscr{G} \in \mathscr{P}_m(\mathbb{F}_c)$ and so $\bar{c}(\mathbb{F}_c) = c$.

The fact that $\bar{c}(\mathbb{F}^c) \geq c$ follows from 2.21 and 2.6.

■

IMAGES AND PRE IMAGES

If $f : X \longrightarrow Y$ and $\mathscr{F}$ is a prefilter on X we define :

$$f[\mathscr{F}] = \{f[\nu] : \nu \in \mathscr{F}\}.$$

If $\mathscr{G}$ is a prefilter on Y we define :

$$f^{-1}[\mathscr{G}] = \{f^{-1}[\nu] : \nu \in \mathscr{G}\}.$$

2.23 LEMMA

Let $f : X \to Y$, let $\mathscr{F}$ be a prefilter on X and $\mathscr{G}$ a prefilter on Y.
Then:

(a) $f[\mathscr{F}]$ is a prefilter base.

(b) If f is surjective then $f^{-1}[\mathscr{G}]$ is a prefilter base.

(c) $c(\mathscr{F}) = c(f[\mathscr{F}])$.

(d) If f is surjective then $c(f^{-1}[\mathscr{G}]) = c(\mathscr{G})$.

(e) If f is surjective and $\mu \in I^X$ then
$c(\mathscr{G}, f[\mu]) = c(f^{-1}[\mathscr{G}],\mu)$.

(f) If $\mathscr{F}$ is prime then $\langle f[\mathscr{F}]\rangle$ is prime.

PROOF

(a) A routine check. Note that $f[\nu_1 \wedge \nu_2] \leq f[\nu_1] \wedge f[\nu_2]$.

(b) Also routine. Note that $f^{-1}[\nu_1 \wedge \nu_2] = f^{-1}[\nu_1] \wedge f^{-1}[\nu_2]$.

(c)
$$c(\mathscr{F}) = \inf_{\nu \in \mathscr{F}} \sup_{x \in X} \nu(x) = \inf_{\nu \in \mathscr{F}} \sup_{y \in f[X]} \sup_{f(x) = y} \nu(x)$$
$$= \inf_{\nu \in \mathscr{F}} \sup_{y \in Y} f[\nu](y) \qquad (\text{for } y \in Y \setminus f[X] \ f[\nu](y) := 0)$$
$$= c(f[\mathscr{F}]).$$

(d)
$$c(f^{-1}[\mathscr{G}]) = \inf_{\nu \in \mathscr{G}} \sup_{x \in X} f^{-1}[\nu](x) = \inf_{\nu \in \mathscr{G}} \sup_{x \in X} \nu(f(x))$$
$$= \inf_{\nu \in \mathscr{G}} \sup_{y \in Y} \sup_{f(x) = y} \nu(x)$$
$$= \inf_{\nu \in \mathscr{G}} \sup \nu = c(\mathscr{G}).$$

(e) Similar to (d).

(f) Let $\mu_1 \vee \mu_2 \in \langle f[\mathscr{F}]\rangle$. Then there exists $\nu \in \mathscr{F}$ with $f[\nu] \leq \mu_1 \vee \mu_2$, so $\nu \leq f^{-1}[\mu_1 \vee \mu_2] = f^{-1}[\mu_1] \vee f^{-1}[\mu_2]$ and hence $f^{-1}[\mu_1] \vee f^{-1}[\mu_2] \in \mathscr{F}$ which is prime. Suppose $f^{-1}[\mu_1] \in \mathscr{F}$, then $f[f^{-1}[\mu_1]] \leq \mu_1$ which means that $\mu_1 \in \langle f[\mathscr{F}]\rangle$.

■

In [Lo 5] (Lemma 3.3) it is shown that

2.24 LEMMA

> If $f : X \to Y$ and $\mathscr{F}$ is a prefilter on X then :
> $$\mathscr{P}_m(f[\mathscr{F}]) \subseteq \{f[\mathscr{G}] : \mathscr{G} \in \mathscr{P}_m(\mathscr{F})\}.$$

From this we can deduce the following corollary.

2.25 COROLLARY

> If $f : X \to Y$ and $\mathscr{F}$ is a prefilter on X then
> $$\bar{c}(f[\mathscr{F}]) \geq \bar{c}(\mathscr{F}).$$

PROOF

$$\bar{c}(\mathscr{F}) = \inf_{\mathscr{G} \in \mathscr{P}_m(\mathscr{F})} c(\mathscr{G}) = \inf_{\mathscr{G} \in \mathscr{P}_m(\mathscr{F})} c(f[\mathscr{G}]) \text{ (2.23(c))}$$

$$\leq \inf_{\mathscr{H} \in \mathscr{P}_m(f[\mathscr{F}])} c(\mathscr{H}) \text{ (2.24).}$$

$$= \bar{c}(f[\mathscr{F}]).$$

■

In view of Theorem 2.16 it is natural to ask whether, for a prefilter $\mathscr{F}$, we can find $\mathscr{G} \in \mathscr{P}_m(\mathscr{F})$ such that $c(\mathscr{G}) = \bar{c}(\mathscr{F})$. If this were true, some of the results which follow would have simpler proofs and so the answer to this question is worth knowing. Very recently, during 1991, Chadwick has settled this issue by finding a counter–example.

2.26 THEOREM

> There is a prefilter $\mathscr{F}$ on $\mathbb{N}$ such that for no $\mathscr{G} \in \mathscr{P}_m(\mathscr{F})$ do we have $c(\mathscr{G}) = \bar{c}(\mathscr{F})$.

PROOF

$$\text{For } n \in \mathbb{N} \text{ let } \left.\begin{aligned} \nu_n(m) &= \tfrac{1}{m} \text{ if } m \leq n, \\ &= 1 \text{ if } m > n. \end{aligned}\right\}$$

Let $\mathscr{F} = <\{\nu_n : n \in \mathbb{N}\}>$. Since $\nu_n^0 = \mathbb{N}$ for each $n \in \mathbb{N}$ we have $\mathscr{F}_o = \{\mathbb{N}\}$.

Consequently :

$$\mathbb{P}_m(\mathscr{F}) = \{(\mathscr{F},\mathbb{F}) : \mathbb{F} \text{ is an ultrafilter on } \mathbb{N}\}.$$

If $\mathbb{F}$ is a fixed ultrafilter on $\mathbb{N}$ then $\mathbb{F} = <\{m\}>$ for some $m \in \mathbb{N}$ and then :

$$\begin{aligned} c(\mathscr{F},\mathbb{F}) &= \inf_{n \in \mathbb{N}} \inf_{F \in \mathbb{F}} \sup(\nu_n \wedge 1_F) \\ &= \inf_{n \in \mathbb{N}} \inf_{F \in \mathbb{F}} \sup_{k \in F} \nu_n(k) \\ &= \frac{1}{m}. \end{aligned}$$

It follows that $\bar{c}(\mathscr{F}) = 0$ while $c(\mathscr{F},\mathbb{F}) > 0$ for any fixed ultrafilter $\mathbb{F}$ on $\mathbb{N}$.

On the other hand if $\mathbb{F}$ is a free ultrafilter on $\mathbb{N}$ then for each $n \in \mathbb{N}$ we have $A_n := \{m \in \mathbb{N} : m \geq n\} \in \mathbb{F}$. Thus for each $F \in \mathbb{F}$ and $n \in \mathbb{N}$, $F \cap A_n \neq \emptyset$.

Let $k(F,n) \in F \cap A_n$ then :

$$\begin{aligned} c(\mathscr{F},\mathbb{F}) &= \inf_{n \in \mathbb{N}} \inf_{F \in \mathbb{F}} \sup_{k \in F} \nu_n(k) \\ &\geq \inf_{n \in \mathbb{N}} \inf_{F \in \mathbb{F}} \nu_n(k(F,n)) \\ &= 1. \end{aligned}$$

We have shown that $\bar{c}(\mathscr{F}) = 0$ but $c(\mathscr{G}) > 0$ for each $\mathscr{G} \in \mathscr{P}_m(\mathscr{F})$ which establishes the theorem.

∎

CHAPTER 3

UNIFORMITIES

The reader is referred to [Bo 1], [Wi 1] and [Wi 2] for the theory of uniform spaces and in particular the interplay between the notions of compactness, precompactness, completeness and boundedness.

In [Lo 3], Lowen introduces the notion of a fuzzy uniform space in which conditions on a collection $\mathscr{D} \subseteq I^{X\times X}$ are imposed for $\mathscr{D}$ to be a fuzzy uniformity. An element $\sigma \in \mathscr{D}$ is a function from $X\times X$ to I and, in the spirit of Zadeh [Za 1], $\sigma(x,y)$ can be thought of as the degree to which (x,y) belongs to σ. If $\sigma, \psi \in I^{X\times X}$ we define :

$$\sigma_s(x,y) = \sigma(y,x)$$

$$(\sigma \circ \psi)(x,y) = \sup_{z \in X} \psi(x,z) \wedge \sigma(z,y).$$

These are natural generalisations of the standard notions, for if $\sigma = 1_U$ and $\psi = 1_V$ then :

$(1_U)_s = 1_{U_s}$ since :

$$(1_U)_s(x,y) = 1 \Leftrightarrow (1_U)(y,x) = 1 \Leftrightarrow (y,x) \in U \Leftrightarrow (x,y) \in U_s \Leftrightarrow (1_{U_s})(x,y) = 1$$

and $(1_V \circ I_U) = 1_{V\circ U}$ since :

$$\begin{aligned}(1_V \circ I_U)(x,y) = 1 \quad &\Leftrightarrow \sup_z I_U(x,z) \wedge 1_V(z,y) = 1 \\ &\Leftrightarrow \exists\, z : (x,z) \in U \text{ and } (z,y) \in V \\ &\Leftrightarrow (x,y) \in V\circ U \Leftrightarrow 1_{V\circ U}(x,y) = 1.\end{aligned}$$

$\mathscr{D} \subseteq I^{X\times X}$ is called a **fuzzy uniformity** and $(X, \mathscr{D})$ a **fuzzy uniform space** if :

(1) $\mathscr{D}$ is a prefilter and $\hat{\mathscr{D}} = \mathscr{D}$,

(2) $\forall\, \sigma \in \mathscr{D}\ \forall\, x \in X\ \ \sigma(x,x) = 1$,

(3) $\forall\, \sigma \in \mathscr{D}\ \ \sigma_s \in \mathscr{D}$,

(4) $\forall\, \sigma \in \mathscr{D}\ \forall\, \epsilon > 0\ \exists\, \psi \in \mathscr{D} : \psi \circ \psi \leq \sigma + \epsilon$.

$\mathscr{B} \subseteq I^{X\times X}$ is called a **fuzzy uniform base** for $\mathscr{D}$ if $\tilde{\mathscr{B}} := \langle \hat{\mathscr{B}} \rangle = \mathscr{D}$ and in [Lo 3] it is shown that :

THEOREM 3.1

> If $\mathscr{D}$ is a fuzzy uniformity then $\mathscr{S} := \{\sigma \in \mathscr{D} : \sigma_s = \sigma\}$ is a fuzzy uniform base for $\mathscr{D}$.

The elements of $\mathscr{S}$ are called **symmetric** and Theorem 3.1 and the fact that $\hat{\mathscr{S}} = \mathscr{S}$ assures us that if $\sigma \in \mathscr{D}$ then there exists symmetric $\psi \in \mathscr{D}$ such that $\psi \leq \sigma$ and so we shall often assume without loss of generality that an arbitrarily chosen element $\sigma \in \mathscr{D}$ is symmetric. It is immediate from the definition that :

3.2 ASSERTION

> If $\sigma \in \mathscr{D}$ then $\sigma \leq \sigma \circ \sigma$.

If $n \in \mathbb{N}$ then $\sigma^n := \sigma \circ \sigma \circ \cdots \circ \sigma$ (n factors) and by induction on n we have $\sigma \leq \sigma^n$ for any $n \in \mathbb{N}$, a fact which we shall use without mention in the sequel.

THE FUZZY UNIFORM TOPOLOGY

If $\sigma \in \mathscr{D}$ and $x \in X$ we define $\sigma\langle x\rangle$ by :

$$\sigma\langle x\rangle(y) = \sigma(y,x)$$

and $$\mathscr{D}_x = \{\sigma\langle x\rangle : \sigma \in \mathscr{D}\}.$$

$\mathscr{D}_x$ is a **fuzzy neighbourhood prefilter** and $(\mathscr{D}_x : x \in X)$ is a **fuzzy neighbourhood system** in the sense of [Lo 2]. For $\mu \in I^X$ and $\sigma \in \mathscr{D}$ we define $\sigma\langle\mu\rangle$ by

$$\sigma\langle\mu\rangle(x) = \sup \mu \wedge \sigma\langle x\rangle = \sup_{y \in X} \mu(y) \wedge \sigma(y,x).$$

If $(X,\mathbb{D})$ is a uniform space, $U \in \mathbb{D}$ and $A \subseteq X$, then :

$$\begin{aligned} 1_U <1_A>(x) = 1 &\Leftrightarrow \sup_y 1_A(y) \wedge 1_U(y,x) = 1 \\ &\Leftrightarrow \exists\, y \in A : (y,x) \in U \\ &\Leftrightarrow \exists y \in A : x \in U(y) \Leftrightarrow x \in U(A). \end{aligned}$$

Thus $1_U\langle 1_A\rangle = 1_{U(A)}$ and the definition of $\sigma\langle\mu\rangle$ is a natural generalisation of the standard notion.

If $\mu \in I^X$ we define the **closure of** μ, denoted $\bar{\mu}$ by :

$$\bar{\mu} = \inf_{\sigma \in \mathscr{D}} \sigma\langle\mu\rangle.$$

This yields a **fuzzy closure operator** as defined in [Lo 3]. Specifically :

3.3 ASSERTION

> If $\alpha \in I$ and $\mu, \nu \in I^X$ then
> (a) $\overline{\alpha 1_X} = \alpha 1_X$,
> (b) $\mu \leq \bar{\mu}$,
> (c) $\overline{\mu \vee \nu} = \bar{\mu} \vee \bar{\nu}$,
> (d) $\bar{\bar{\mu}} = \bar{\mu}$.
> (e) $\overline{\mu \wedge \nu} \leq \bar{\mu} \wedge \bar{\nu}$.

The proof of this assertion can be found in [Lo 3] and we omit it here. The closure operator defines a fuzzy topology $\tau(\mathscr{D})$ associated with $\mathscr{D}$ and $\tau(\mathscr{D})$ is called the **fuzzy uniform topology** in which the $\mathscr{D}$–closed fuzzy sets are those $\mu \in I^X$ such that $\bar{\mu} = \mu$. In other words, $\tau(\mathscr{D}) = \{1-\mu : \bar{\mu} = \mu\}$.

Alternatively, a fuzzy uniform space is a **fuzzy neighbourhood space** in the sense of [Lo 2] in which :

$$\bar{\mu}(x) = \inf_{\nu \in \mathscr{D}_x} \sup \mu \wedge \nu.$$

This is precisely what we have here because

$$\bar{\mu}(x) = \inf_{\sigma \in \mathscr{D}} \sigma\langle\mu\rangle(x) = \inf_{\sigma \in \mathscr{D}} \sup \mu \wedge \sigma\langle x\rangle.$$

Thus a fuzzy uniform space $(X, \mathscr{D})$ gives rise to a fuzzy neighbourhood space $(X, (\mathscr{N}_x : x \in X))$ in which $\mathscr{N}_x = \mathscr{D}_x$, from which we obtain a closure operator, which in turn defines the fuzzy uniform topology $\tau(\mathscr{D})$.

We collect together some basic technical facts concerning fuzzy uniform spaces in the following lemma in which :

$$\sigma^{\beta} := \{(x,y) \in X \times X : \sigma(x,y) > \beta\} \quad \text{for } \sigma \in \mathscr{D},\ \beta \in I_1.$$

3.4 LEMMA

Let $(X, \mathscr{D})$ be a fuzzy uniform space with $\sigma, \psi \in \mathscr{D}$; $\nu, \lambda \in I^X$, $\epsilon \in I$, $\beta \in I_1$, $x \in X$, and $n \in \mathbb{N}$. Then

(a) $\nu \leq \sigma\langle\nu\rangle$,

(b) $(\sigma + \epsilon)\langle\nu\rangle \leq \sigma\langle\nu\rangle + \epsilon$,

(c) $\sigma\langle\psi\langle\nu\rangle\rangle = (\sigma \circ \psi)\langle\nu\rangle$,

(d) $\sup \sigma\langle\nu\rangle \wedge \mu = \sup \nu \wedge \sigma_s\langle\mu\rangle$,

(e) $\bar{\nu}(x) = c(\mathscr{D}_x, \nu)$,

(f) $\sup \bar{\nu} = \sup \nu$,

(g) $(\sigma\langle\nu\rangle)^{\beta} = \sigma^{\beta}(\nu^{\beta})$,

(h) $\sigma\langle x\rangle^{\beta} = \sigma_s^{\beta}(x)$,

(i) $(\sigma^{\beta})^n = (\sigma^n)^{\beta}$,

(j) $\bar{\nu} = \inf_{\sigma \in \mathscr{D}} \overline{\sigma\langle\nu\rangle}$.

PROOF

(a) Immediate from the definition.

(b)
$$\begin{aligned}(\sigma + \epsilon)\langle\nu\rangle(x) &= \sup_y \nu(y) \wedge (\sigma+\epsilon)(y,x) = \sup_y \nu(y) \wedge (\sigma(y,x)+\epsilon) \\ &\leq \sup_y (\nu(y)+\epsilon) \wedge (\sigma(y,x)+\epsilon) = (\sup_y \nu(y) \wedge \sigma(y,x))+\epsilon \\ &= \sigma\langle\nu\rangle(x)+\epsilon.\end{aligned}$$

(c) Straight from the definition.

(d) $$\sup_x (\sigma\langle\nu\rangle \wedge \mu)(x) = \sup_x (\sup_y \nu(y) \wedge \sigma(y,x)) \wedge \mu(x)$$
$$= \sup_y (\sup_x \mu(x) \wedge \sigma_s(x,y)) \wedge \nu(y)$$
$$= \sup_y \sigma_s\langle\mu\rangle(y) \wedge \nu(y) = \sup \nu \wedge \sigma_s\langle\mu\rangle.$$

(e) $$\bar{\nu}(x) = \inf_{\sigma \in \mathscr{D}} \sigma\langle\nu\rangle(x) = \inf_{\sigma \in \mathscr{D}} \sup \nu \wedge \sigma\langle x\rangle = c(\mathscr{D}_x,\nu).$$

(f) For each $x \in X$ $\bar{\nu}(x) = c(\mathscr{D}_x,\nu) \leq \sup \nu$ (since $\nu \in (\mathscr{D}_x,\nu)$).

(g) $$x \in (\sigma\langle\nu\rangle)^\beta \Leftrightarrow \sigma\langle\nu\rangle(x) = \sup_y \nu(y) \wedge \sigma(y,x) > \beta$$
$$\Leftrightarrow \exists\, y \in \nu^\beta : (y,x) \in \sigma^\beta$$
$$\Leftrightarrow x \in \sigma^\beta(\nu^\beta).$$

(h) $$y \in (\sigma\langle x\rangle)^\beta \Leftrightarrow \sigma\langle x\rangle(y) = \sigma(y,x) = \sigma_s(x,y) > \beta \Leftrightarrow (x,y) \in \sigma_s^\beta$$
$$\Leftrightarrow y \in \sigma_s^\beta(x).$$

(i) $$(x,y) \in (\sigma^\beta)^n \Leftrightarrow \exists\, y_1,y_2,\cdots,y_{n-1} : (x,y_1) \in \sigma^\beta, (y_1,y_2) \in \sigma^\beta,\cdots,(y_{n-1},y) \in \sigma^\beta$$
$$\Leftrightarrow \sup\{\sigma(x,y_1) \wedge \cdots \wedge \sigma(y_{n-1},y) : y_i \in X, i \in [n-1]\} > \beta$$
$$\Leftrightarrow \sigma^n(x,y) > \beta$$
$$\Leftrightarrow (x,y) \in (\sigma^n)^\beta.$$

(j) $\bar{\nu} = \inf_{\sigma \in \mathscr{D}} \sigma\langle\nu\rangle \leq \inf_{\sigma \in \mathscr{D}} \overline{\sigma\langle\nu\rangle}$. For the reverse inequality let $\sigma \in \mathscr{D}$ and $\epsilon \in I_0$. Then there exists $\psi \in \mathscr{D}$ such that $\psi \circ \psi \leq \sigma + \epsilon$ and hence :

$$\overline{\psi\langle\nu\rangle} = \inf_{\xi \in \mathscr{D}} \xi\langle\psi\langle\nu\rangle\rangle \leq \psi\langle\psi\langle\nu\rangle\rangle = (\psi \circ \psi)\langle\nu\rangle$$
$$\leq (\sigma+\epsilon)\langle\nu\rangle \leq \sigma\langle\nu\rangle+\epsilon.$$

We have shown that : $\forall\ \sigma \in \mathscr{D}\ \forall\ \epsilon \in I_0\ \exists\ \psi \in \mathscr{D} : \overline{\psi\langle\nu\rangle} \leq \sigma\langle\nu\rangle+\epsilon$ which is equivalent to : $\inf_{\psi \in \mathscr{D}} \overline{\psi\langle\nu\rangle} \leq \inf_{\sigma \in \mathscr{D}} \sigma\langle\nu\rangle = \bar{\nu}$.

■

CONVERGENCE

If $(X, \mathscr{D})$ is a fuzzy uniform space and $\mathscr{F}$ is a prefilter (on X) the **adherence of** $\mathscr{F}$ (Adh $\mathscr{F}$) and **limit of** $\mathscr{F}$ (lim $\mathscr{F}$) are elements of 1^X defined using the fuzzy uniform topology :

$$\text{Adh } \mathscr{F} = \inf_{\nu \in \mathscr{F}} \bar{\nu},$$

$$\lim \mathscr{F} = \inf_{\mathscr{G} \in \mathscr{P}_m(\mathscr{F})} \text{Adh } \mathscr{G},$$

We assemble some observations regarding these in a lemma.

3.5 LEMMA

Let $(X, \mathscr{D})$ be a fuzzy uniform space, $\mathscr{F}$ a prefilter on X and $x \in X$.
Then:

(a) $(\text{Adh } \mathscr{F})(x) = c(\mathscr{D}_x, \mathscr{F})$.

(b) $(\lim \mathscr{F})(x) = \inf_{\mathscr{G} \in \mathscr{P}_m(\mathscr{F})} c(\mathscr{D}_x, \mathscr{G})$.

(c) If $\mathscr{F}$ is prime then $\lim \mathscr{F} = \text{Adh } \mathscr{F}$.

(d) $\sup \text{Adh } \mathscr{F} \leq c(\mathscr{F})$.

(e) $\sup \lim \mathscr{F} \leq \bar{c}(\mathscr{F})$.

(f) If $\mathscr{F} \subseteq \mathscr{G}$ then $\text{Adh } \mathscr{G} \leq \text{Adh } \mathscr{F}$.

(g) $\text{Adh } \mathscr{F} = \sup_{\mathscr{G} \in \mathscr{P}_m(\mathscr{F})} \text{Adh } \mathscr{G}$.

(h) If $\mathscr{F}$ is a prefilter base then $\text{Adh} < \mathscr{F} > = \text{Adh } \mathscr{F}$.

(i) If $\mathscr{F}$ is a prefilter base and $\sigma \in \mathscr{D}$ then $\text{Adh } \hat{\mathscr{F}} = \text{Adh } \mathscr{F}$ and $\text{Adh } \sigma < \hat{\mathscr{F}} > = \text{Adh } \sigma < \mathscr{F} >$.

PROOF

(a) $$(\text{Adh } \mathscr{F})(x) = \inf_{\nu \in \mathscr{F}} \inf_{\sigma \in \mathscr{D}} \sigma<\nu>(x) = \inf_{\nu \in \mathscr{F}} \inf_{\sigma \in \mathscr{D}} \sup \nu \wedge \sigma <x>$$
$$= c(\mathscr{F}, \mathscr{D}_x).$$

(b) Immediate from (a).

(c) Immediate since $\mathscr{P}_m(\mathscr{F}) = \{\mathscr{F}\}$.

(d) If $x \in X$ then $(\text{Adh } \mathscr{F})(x) = c(\mathscr{F}, \mathscr{D}_x) \leq c(\mathscr{F})$.

(e) If $x \in X$ then :

$$(\lim \mathscr{F})(x) = \inf_{\mathscr{G} \in \mathscr{P}_m(\mathscr{F})} c(\mathscr{G}, \mathscr{D}_x) \leq \inf_{\mathscr{G} \in \mathscr{P}_m(\mathscr{F})} c(\mathscr{G}) = \bar{c}(\mathscr{F}).$$

(f) Immediate.

(g) For each $\mathscr{G} \in \mathscr{P}_m(\mathscr{F})$, $\text{Adh } \mathscr{G} \leq \text{Adh } \mathscr{F}$ and so $\sup_{\mathscr{G} \in \mathscr{P}_m(\mathscr{F})} \text{Adh } \mathscr{G} \leq \text{Adh } \mathscr{F}$.

To prove the reverse inequality let $x \in X$. If $(\text{Adh } \mathscr{F})(x) = 0$ then $\sup_{\mathscr{G} \in \mathscr{P}_m(\mathscr{F})} (\text{Adh } \mathscr{G})(x) \geq 0$. If $(\text{Adh } \mathscr{F})(x) > 0$ choose α such that $(\text{Adh } \mathscr{F})(x) > \alpha > 0$. Then $c(\mathscr{F}, \mathscr{D}_x) = c(\mathscr{F} \vee \mathscr{D}_x) > \alpha$ so choose an ultrafilter $\mathbb{F} \supseteq (\mathscr{F} \vee \mathscr{D}_x)_\alpha$. Now $\mathscr{F}_0 \subseteq \mathscr{F}_\alpha \subseteq (\mathscr{F} \vee \mathscr{D}_x)_\alpha$ (2.7(a)) and hence $\mathscr{F}_0 \subseteq \mathbb{F}$. Let $\mathscr{G} = (\mathscr{F}, \mathbb{F}) = \mathscr{F} \vee \mathbb{F}_1$. Then $\mathscr{G} \in \mathscr{P}_m(\mathscr{F})$ and :

$$\begin{aligned}
(\text{Adh } \mathscr{G})(x) &= c(\mathscr{G}, \mathscr{D}_x) = c((\mathscr{F}, \mathbb{F}), \mathscr{D}_x) = c(\mathscr{F} \vee \mathbb{F}_1 \vee \mathscr{D}_x) \\
&= \inf_{\nu \in \mathscr{F}} \inf_{F \in \mathbb{F}} \inf_{\sigma \in \mathscr{D}} \sup \nu \wedge 1_F \wedge \sigma\langle x\rangle \\
&= \inf_{\nu \in \mathscr{F}} \inf_{F \in \mathbb{F}} \inf_{\sigma \in \mathscr{D}} \sup_{y \in F} (\nu \wedge \sigma\langle x\rangle)(y).
\end{aligned}$$

If $\nu \in \mathscr{F}$, $F \in \mathbb{F}$ and $\sigma \in \mathscr{D}$ then $\nu \wedge \sigma\langle x\rangle \in \mathscr{F} \vee \mathscr{D}_x$ and so $F \cap (\nu \wedge \sigma\langle x\rangle)^\alpha \neq \emptyset$. It follows that $\sup_{y \in F} (\nu \wedge \sigma\langle x\rangle)(y) > \alpha$ and hence $(\text{Adh } \mathscr{G})(x) \geq \alpha$. Thus $(\sup_{\mathscr{G} \in \mathscr{P}_m(\mathscr{F})} \text{Adh } \mathscr{G})(x) \geq \alpha$ and, since α is arbitrary, $(\text{Adh } \mathscr{F})(x) \leq (\sup_{\mathscr{G} \in \mathscr{P}_m(\mathscr{F})} \text{Adh } \mathscr{G})(x)$. Since this holds for all $x \in X$ the result follows.

(h) Straightforward.

(i) A fairly easy exercise.

■

It should be noted that the results in Lemma 3.5 hold in a more general setting, namely, fuzzy topological spaces and the reader is referred to [Lo 1]. The proof of 3.5(g) in the context of fuzzy topological spaces is more involved than the one we present here, the additional structure available in fuzzy uniform spaces allowing us to find this simpler proof.

If $(X, \mathscr{D})$ is a fuzzy uniform space and $\mathscr{F}$ is a prefilter on X, we shall say that $\mathscr{F}$ is **$\mathscr{D}$–convergent** or simply **convergent** iff $\bar{c}(\mathscr{F}) = \sup \lim \mathscr{F}$ and in the light of Lemma 3.5(e) this is equivalent to :

$$\mathscr{F} \text{ is convergent } \Leftrightarrow \bar{c}(\mathscr{F}) \leq \sup \lim \mathscr{F}.$$

If $\mu \in I^X$ we shall say that $\mathscr{F}$ is **$\mathscr{D}$–convergent in μ** or simply **convergent in μ** iff $\bar{c}(\mathscr{F}) = \sup \mu \wedge \lim \mathscr{F}$. Equivalently :

$$\mathscr{F} \text{ is convergent in } \mu \Leftrightarrow \bar{c}(\mathscr{F}) \leq \sup \mu \wedge \lim \mathscr{F}.$$

CONTINUITY

If $(X, \mathscr{D})$ and $(Y, \mathscr{E})$ are fuzzy uniform spaces and $f : (X, \mathscr{D}) \to (Y, \mathscr{E})$ we say that **f is continuous** if the inverse image of open fuzzy sets are open. More precisely :

$$f \text{ is continuous} \Leftrightarrow \forall\, \nu \in \tau(\mathscr{E})\ f^{-1}[\nu] \in \tau(\mathscr{D}).$$

It is straightforward to check that the following are equivalent :

(i) f is continuous.

(ii) For each $\mathscr{E}$–closed λ $f^{-1}[\lambda]$ is $\mathscr{D}$–closed.

(iii) For each $\nu \in I^Y$ $\overline{f^{-1}[\nu]} \leq f^{-1}[\bar{\nu}]$.

(iv) For each $\nu \in I^X$ $f[\bar{\nu}] \leq \overline{f[\nu]}$.

The proofs are in [Ch 3] and [Wa 1]. In [Lo 1], Lowen characterises continuity using prefilters as follows :

(v) f is continuous iff for each prefilter $\mathscr{F}$ on X
$f[\text{Adh } \mathscr{F}] \leq \text{Adh } f[\mathscr{F}]$.

(vi) f is continuous iff for each prime prefilter $\mathscr{F}$ on X
$f[\lim \mathscr{F}] \leq \lim f[\mathscr{F}]$.

This idea of characterising a certain property using only prime prefilters will be used time and time again in the sequel and its usefulness will become apparent.

HAUSDORFF FUZZY UNIFORM SPACES

If a topological space is Hausdorff (T_2) then a filter $\mathbb{F}$ on X has a limit set which is either empty, in which case we say that $\mathbb{F}$ does not converge, or a singleton $\{x\}$ say, in which case we say that $\mathbb{F}$ converges to x.

To achieve a similar situation we shall adopt the following definition. We shall call a fuzzy uniform space (X, $\mathscr{D}$) **Hausdorff** (or separated or T_2) iff for each prime prefilter $\mathscr{F}$ on X, $(\text{Adh } \mathscr{F})^0$ is either empty or a singleton. In [Lo 3] we find a useful equivalent statement.

3.6 THEOREM

A fuzzy uniform space $(X, \mathscr{D})$ is Hausdorff iff $\inf_{\sigma \in \mathscr{D}} \sigma = 1_\Delta$
(where $\Delta = \{(x,x) : x \in X\}$).

3.7 COROLLARY

If a fuzzy uniform space $(X, \mathscr{D})$ is Hausdorff then
for each $x \in X$ $\text{Adh } \mathscr{D}_x = 1_x$.

PROOF

$$(\mathrm{Adh}\ \mathscr{D}_x)(y) = \inf_{\sigma \in \mathscr{D}} \overline{\sigma<x>}(y) = \inf_{\sigma \in \mathscr{D}} \overline{\sigma<1_x>}(y) = \bar{1}_x(y) \quad (3.4(j))$$

$$= \inf_{\sigma \in \mathscr{D}} \sigma<1_x>(y)$$

$$= \inf_{\sigma \in \mathscr{D}} \sigma<x>(y) = \inf_{\sigma \in \mathscr{D}} \sigma(y,x) = 1_\Delta(y,x) = 1_x(y).$$

■

UNIFORM CONTINUITY

We extend the notion of uniform continuity in uniform spaces in a natural way as follows. If $f : X \to Y$ then $(f\times f) : X\times X \to Y\times Y$ is defined by $(f\times f)(x,y) := (f(x),f(y))$. If $(X,\mathscr{D})$ and $(Y,\mathscr{E})$ are fuzzy uniform spaces then :

f is uniformly continuous $\Leftrightarrow \forall\ \psi\in \mathscr{E}\ (f\times f)^{-1}[\psi] \in \mathscr{D}$

$\Leftrightarrow \forall\ \psi\in \mathscr{E}\ \exists\ \sigma\in\mathscr{D}: (f\times f)[\sigma] \leq \psi.$

As for uniform spaces we have :

3.8 THEOREM

> A uniformly continuous function is continuous.

The proof can be found in [Lo 3].

PRODUCTS

Let J be a non–empty index set and for each $j \in J$ let $(X_j, \mathscr{D}_j)$ be a fuzzy uniform space. Let $X = \prod_{j\in J} X_j$ and for $k \in J$ we define the **kth projection p_k** by

$$p_k : \begin{array}{l} X \to X_k \\ (x_j) \to x_k \end{array}$$ and the **kth biprojection $p_k \times p_k$** by

$$p_k \times p_k : \begin{array}{l} X\times X \to X_k\times X_k \\ (x,y) \to (p_k(x),p_k(y)) \end{array}$$. The **product fuzzy uniformity**

$\mathscr{D} = \prod_{j\in J} \mathscr{D}_j$ is defined to be that fuzzy uniformity on X whose sub–basic elements are

$$\mathscr{S} = \{(p_j \times p_j)^{-1}[\sigma_j] : \sigma_j \in \mathscr{D}_j,\ j\in J\}.$$

The basic elements of $\mathscr{D}$ have form :

$$\sigma = \inf_{j \in J_0} (p_j \times p_j)^{-1}[\sigma_j] \text{ with } J_0 \in \mathscr{P}_f(J) = \{K \subseteq J : K \text{ is finite}\}.$$

In this case $\sigma(x,y) = \inf_{j \in J_0} \sigma_j(p_j(x),p_j(y)) = \min_{j \in J_0} \sigma_j(p_j(x),p_j(y))$.

We need the following facts concerning the product space $(X, \mathscr{D})$.

3.9 THEOREM

$\mathscr{D} = \prod_{j \in J} \mathscr{D}_j$ is the weakest fuzzy uniformity on $X = \prod_{j \in J} X_j$ which makes each projection $p_j : X \to X_j$ uniformly continuous.

PROOF : A straightforward check (see [Lo 3]).

■

3.10 LEMMA

If $\mathscr{B} = \{ \max_{j \in J_0} p_j^{-1}[\lambda_j] : \lambda_j \text{ is } \tau(\mathscr{D}_j)\text{–closed}, J_0 \in \mathscr{P}_f(J)\}$

then $\mathscr{B}$ is a base for the closed fuzzy sets of $\tau(\mathscr{D})$.

PROOF

Basic $\tau(\mathscr{D})$–open fuzzy sets have form :

$$\nu = \min_{j \in J_0} p_j^{-1} [\nu_j] \text{ with } J_0 \in \mathscr{P}_f(J),\ \nu_j \text{ is } \tau(\mathscr{D}_j)\text{–open.}$$

Now λ is basic $\tau(\mathscr{D})$–closed iff $1-\lambda$ is basic $\tau(\mathscr{D})$–open and so if $\min_{j \in J_0} p_j^{-1} [\nu_j] = 1-\lambda$ then $\lambda = 1 - \min_{j \in J_0} p_j^{-1} [\nu_j] = \max_{j \in J_0} p_j^{-1} [1-\nu_j]$.

■

If for each $j \in J$ $\mu_i \in I^{X_j}$ we define the product in the natural way :

$$\prod_{j \in J} \mu_j = \inf_{j \in J} p_j^{-1}[\mu_j].$$

Thus $(\prod_{j \in J} \mu_j)(x) \geq \alpha \Leftrightarrow \forall\, j \in J\ \mu_j(p_j(x)) \geq \alpha$. Note that each p_j is surjective and so :

$$\nu \leq p_j^{-1}[p_j[\nu]] \text{ and } \nu_j = p_j[p_j^{-1}[\nu_j]].$$

We collect some easily checked facts together in a lemma.

3.11 LEMMA

Let $((X_j, \mathscr{D}_j) : j \in J)$ be a family of fuzzy uniform spaces,

$X = \prod_{j \in J} X_j$, $\mathscr{D} = \prod_{j \in J} \mathscr{D}_j$, $\mu_j \in I^{X_j}$, $\nu \in I^X$ and $\mathscr{F}$ a prefilter on X.

(a) $\forall k \in J\ \bar{\nu} \leq p_k^{-1}[\,\overline{p_k[\nu]}\,]$.

(b) $\forall k \in J\ p_k[\prod_{j \in J} \mu_j] \leq \mu_k$.

(c) $\forall k \in J \sup \nu = \sup p_k[\nu]$.

(d) If $\mathscr{S}_j \subseteq I^{X_j}$ then $p_j^{-1}[\inf_{\nu_j \in \mathscr{S}_j} \nu_j] = \inf_{\nu_j \in \mathscr{S}_j} p_j^{-1}[\nu_j]$.

(e) $c(\mathscr{F}, \prod_{j \in J} \mu_j) \leq \inf_{j \in J} c(p_j[\mathscr{F}], \mu_j)$.

(f) $\forall j \in J\ c(p_j[\mathscr{F}]) = c(\mathscr{F})$.

The reader is referred to [Ch 1] for the proofs. We also glean the following useful theorem from [Ch 1] :

3.12 THEOREM

If $\mathscr{G}$ is a prefilter on $\prod_{j \in J} X_j$ then :

(a) $\text{Adh}\ \mathscr{G} \leq \prod_{j \in J} \text{Adh}\ p_j[\mathscr{G}]$.

(b) If $\mathscr{G}$ is prime then $\text{Adh}\ \mathscr{G} = \prod_{j \in J} \text{Adh}\ p_j[\mathscr{G}]$.

PROOF (outline)

(a) Follows easily from 3.11(a).

(b) To show that $\rho := \prod_{j \in J} \text{Adh}\ p_j[\mathscr{G}] \leq \text{Adh}\ \mathscr{G}$ we let $\nu \in \mathscr{G}$ and let closed $\lambda \geq \nu$. We invoke 3.10 to obtain $\lambda = \max_{j \in J_0} p_j^{-1}[\lambda_j]$ and now, since $\lambda \in \mathscr{G}$ and $\mathscr{G}$ is prime we have $p_k^{-1}[\lambda_k] \in \mathscr{G}$ for some $k \in J_0$. We then obtain $\rho \leq p_k^{-1}[\lambda_k] \leq \lambda$ and the result follows since ν and λ are arbitrary.

■

NOTE : If there is a possibility of confusing $\mathscr{D}_j$ with $\{\sigma^{\beta} : \sigma \in \mathscr{D}, \beta < j\}$ (2.7) or μ_j with $\{x : \mu(x) \geq j\}$ we shall use the more cumbersome $\mathscr{D}(j)$ and $\mu(j)$.

THE α–LEVEL UNIFORMITIES

If $(X, \mathscr{D})$ is a fuzzy uniform space then for each $\alpha \in I_0$ we define :

$$\mathscr{D}^{\alpha} = \{\sigma^{\beta} : \sigma \in \mathscr{D}, 0 \leq \beta < \alpha\}.$$

3.13 ASSERTION

$\mathscr{D}^{\alpha}$ is a uniformity on X.

PROOF

If $U \in \mathscr{D}^{\alpha}$ then $U = \sigma^{\beta}$ for some $\sigma \in \mathscr{D}$ and $\beta < \alpha$ and so $\Delta \subseteq U$ (since for $(x,x) \in \Delta$ we have $\sigma(x,x) = 1 > \beta$ which means that $(x,x) \in \sigma^{\beta}$).

Let $U \in \mathscr{D}^{\alpha}$ and $U \subseteq V$. Again $U = \sigma^{\beta}$ for some $\beta > \alpha$ and $\sigma \in \mathscr{D}$.

Let $\psi = \sigma \vee 1_V$. Then $\sigma \leq \psi$ and so $\psi \in \mathscr{D}$. We have $\psi^\beta = V$ since :

$$\begin{aligned} \psi(x,y) > \beta \quad & \Leftrightarrow \sigma(x,y) > \beta \text{ or } (x,y) \in V \\ & \Leftrightarrow (x,y) \in \sigma^\beta = U \text{ or } (x,y) \in V \\ & \Rightarrow (x,y) \in V \\ (x,y) \in V \quad & \Rightarrow 1_V(x,y) = 1 \\ & \Rightarrow \psi(x,y) = 1 > \beta \\ & \Rightarrow (x,y) \in \psi^\beta . \end{aligned}$$

It follows that $V \in \mathscr{D}^\alpha$.

Let $U_1 = \sigma_1^{\beta_1} \in \mathscr{D}^\alpha$ and $U_2 = \sigma_2^{\beta_2} \in \mathscr{D}^\alpha$.

Then $(\sigma_1 \wedge \sigma_2)^{\beta_1 \vee \beta_2} \in \mathscr{D}^\alpha$ and $(\sigma_1 \wedge \sigma_2)^{\beta_1 \vee \beta_2} \subseteq \sigma_1^{\beta_1} \cap \sigma_1^{\beta_2}$, from which we conclude that $\sigma_1^{\beta_1} \cap \sigma_1^{\beta_2} = U_1 \cap U_2 \in \mathscr{D}^\alpha$.

Let $U = \sigma^\beta \in \mathscr{D}^\alpha$. Then $U_s = (\sigma_s)^\beta$ and, since $\sigma_s \in \mathscr{D}$, we have $U_s \in \mathscr{D}^\alpha$.

Finally, let $U = \sigma^\beta \in \mathscr{D}^\alpha$ with $\sigma \in \mathscr{D}$ and $\beta < \alpha$. Let $\epsilon = \frac{\alpha - \beta}{2}$ and $\gamma = \frac{3\alpha + 2\beta}{5}$. Then $\epsilon > 0$ so there exists $\psi \in \mathscr{D}$ with $\psi \circ \psi \leq \sigma + \epsilon$. We have $\beta < \gamma < \alpha$ and so $V := \psi^\gamma \in \mathscr{D}^\alpha$.

Furthermore $V \circ V \subseteq U$ since :

$$\begin{aligned} (x,y) \in V \circ V \quad & \Leftrightarrow \exists\, z\colon (x,z) \in V \text{ and } (z,y) \in V \\ & \Leftrightarrow \sup_z \psi(x,z) \wedge \psi(z,y) = (\psi \circ \psi)(x,y) > \gamma \\ & \Rightarrow \sigma(x,y) > \gamma - \epsilon = \frac{\alpha + 9\beta}{10} > \beta \\ & \Leftrightarrow (x,y) \in \sigma^\beta = U. \end{aligned}$$

■

The uniformity $\mathscr{D}^\alpha$ will be referred to as the **α–level uniformity**.

3.14 ASSERTION

If $0 < \beta \leq \alpha \leq 1$ then $\mathscr{D}^{\beta} \subseteq \mathscr{D}^{\alpha} \subseteq \mathscr{D}^{1}$. Furthermore
$$\mathscr{D}^{\alpha} = \bigcup_{\gamma < \alpha} \mathscr{D}^{\gamma}.$$

PROOF

The first part is clear and hence $\bigcup_{\gamma < \alpha} \mathscr{D}^{\gamma} \subseteq \mathscr{D}^{\alpha}$ is also clear.

Now let $U = \sigma^{\beta} \in \mathscr{D}^{\alpha}$ with $\sigma \in \mathscr{D}$ and $\beta < \alpha$. Choose γ such that $\beta < \gamma < \alpha$. Then $U \in \mathscr{D}^{\gamma}$ and hence $U \in \bigcup_{\gamma < \alpha} \mathscr{D}^{\gamma}$.

■

We have seen that a single fuzzy uniformity $\mathscr{D}$ on X gives rise to a family $(\mathscr{D}^{\alpha} : \alpha \in I_0)$ of uniformities on X which become stronger as α increases, the strongest being $\mathscr{D}^1 = \{\sigma^{\beta} : \sigma \in \mathscr{D}, \beta < 1\}$. Investigating a uniformity or a prefilter by examining its α–levels is a very useful device used by Lowen in [Lo 2] and [Lo 10] in which our $\mathscr{D}^{\alpha}$ is denoted $i_{1-\alpha}(\mathscr{D})$. In [Lo 2] it is shown that the convergence of a prefilter can be expressed in terms of the convergence of its α–levels. The setting there was a fuzzy neighbourhood space and we restate the theorem (Proposition 7.3 in [Lo 2]) in our setting.

Let $\mathscr{D}^{\alpha}_{x} = \{U : U \text{ is a } \mathscr{D}^{\alpha}-\text{neighbourhood of } x\}$.

3.15 THEOREM

Let $(X, \mathscr{D})$ be a fuzzy uniform space, $\mathscr{F}$ a prefilter on X, $x \in X$ and $\alpha \leq \bar{c}(\mathscr{F})$. Then :

$$(\lim \mathscr{F})(x) \geq \alpha \Leftrightarrow \mathscr{F}_0 \to x \quad \text{w.r.t. } \mathscr{D}^{\alpha}.$$

PROOF

We first prove the result for a prime prefilter $\mathcal{G}$. In this case : we have $\lim \mathcal{G} = \text{Adh } \mathcal{G}$ and $\bar{c}(\mathcal{G}) = c(\mathcal{G})$. Thus :

$$
\begin{aligned}
(\text{Adh } \mathcal{G})(x) \geq \alpha \quad & \Leftrightarrow \forall \beta < \alpha \ \forall \nu \in \mathcal{G} \ \forall \sigma \in \mathcal{D} \ \ \sigma<\nu>(x) = \sup \nu \wedge \sigma<x> > \beta \\
& \Leftrightarrow \forall \sigma \in \mathcal{D} \ \forall \beta < \alpha \ \forall \nu \in \mathcal{G} \ \ \nu^{\beta} \cap \sigma<x>^{\beta} \neq \emptyset \\
& \Leftrightarrow \forall \sigma \in \mathcal{D} \ \forall \beta < \alpha \ \forall F \in \mathcal{G}_0 \ \ F \cap \sigma_s^{\beta}(x) \neq \emptyset. \ (2.9(b)). \\
& \Leftrightarrow \forall U \in \mathcal{D}_x^{\alpha} \ \forall F \in \mathcal{G}_0 \ \ F \cap U \neq \emptyset \\
& \Leftrightarrow \forall U \in \mathcal{D}_x^{\alpha} \ \ U \in \mathcal{G}_0 \ \ (2.9(a), 2.14) \\
& \Leftrightarrow \mathcal{D}_x^{\alpha} \subseteq \mathcal{G}_0 \\
& \Leftrightarrow \mathcal{G}_0 \to x \quad \text{w.r.t. } \mathcal{D}^{\alpha}.
\end{aligned}
$$

To deal with the general case we first observe that :

$$\bigcap_{\mathcal{G} \in \mathcal{P}_m(\mathcal{F})} \mathcal{G}_0 = \mathcal{F}_0.$$

To see this let $\mathcal{S} = \{\mathbb{F} : \mathbb{F} \text{ is ultra}, \mathbb{F} \supseteq \mathcal{F}_0\} = \mathbb{P}(\mathcal{F}_0)$ then :

$$
\begin{aligned}
\bigcap_{\mathcal{G} \in \mathcal{P}_m(\mathcal{F})} \mathcal{G}_0 \quad & = \bigcap_{\mathbb{F} \in \mathcal{S}} (\mathcal{F} \vee \mathbb{F}_1)_0 = \bigcap_{\mathbb{F} \in \mathcal{S}} (\mathcal{F}_0 \vee \mathbb{F}) \ \ (2.13(a)) \\
& = \mathcal{F}_0.
\end{aligned}
$$

$$
\begin{aligned}
\text{Now } (\lim \mathcal{F})(x) \geq \alpha \quad & \Leftrightarrow \forall \mathcal{G} \in \mathcal{P}_m(\mathcal{F}) \ (\text{Adh } \mathcal{G})(x) \geq \alpha \\
& \Leftrightarrow \forall \mathcal{G} \in \mathcal{P}_m(\mathcal{F}) \ \mathcal{G}_0 \to x \quad \text{w.r.t. } \mathcal{D}^{\alpha} \\
& \Leftrightarrow \forall \mathcal{G} \in \mathcal{P}_m(\mathcal{F}) \ \mathcal{D}_x^{\alpha} \subseteq \mathcal{G}_0 \\
& \Leftrightarrow \mathcal{D}_x^{\alpha} \subseteq \bigcap_{\mathcal{G} \in \mathcal{P}_m(\mathcal{F})} \mathcal{G}_0 = \mathcal{F}_0 \\
& \Leftrightarrow \mathcal{F}_0 \to x \quad \text{w.r.t. } \mathcal{D}^{\alpha}.
\end{aligned}
$$

■

Theorem 3.15 is the first example of what we might call an **"α–level theorem"** in which we express a property of an object of a fuzzy uniform space in terms of the α–level uniformities. This idea has also been used by Katsaras and Liu in [Ka 1] in connection with convex and balanced fuzzy sets and Chadwick in [Ch 1] in connection with compact fuzzy sets. As another example, we characterise the Hausdorff property in terms of the α–level uniformities.

3.16 THEOREM

A fuzzy uniformity $\mathscr{D}$ is $T_2 \Leftrightarrow \forall \alpha \in I_0$ $\mathscr{D}^\alpha$ is T_2.

PROOF

$$\mathscr{D} \text{ is } T_2 \quad \Leftrightarrow \quad \inf_{\sigma \in \mathscr{D}} \sigma = 1_\Delta \ (3.6) \text{ and for } \alpha \in I_0,$$

$$\mathscr{D}^\alpha \text{ is } T_2 \quad \Leftrightarrow \quad \bigcap_{U \in \mathscr{D}^\alpha} U = \bigcap_{\sigma \in \mathscr{D}} \bigcap_{\beta < \alpha} \sigma^\beta = \Delta.$$

Let $\mathscr{D}$ be T_2, $\alpha \in I_0$ and $(x,y) \in \bigcap_{\sigma \in \mathscr{D}} \bigcap_{\beta < \alpha} \sigma^\beta$. Then $\sigma(x,y) > \beta$ for each $\sigma \in \mathscr{D}$ and each $\beta < \alpha$. Consequently $\sigma(x,y) \geq \alpha$ for each $\sigma \in \mathscr{D}$ and hence

$$(\inf_{\sigma \in \mathscr{D}} \sigma)(x,y) = 1_\Delta(x,y) \geq \alpha > 0.$$

Thus $1_\Delta(x,y) = 1$ and so $x=y$, in other words $(x,y) \in \Delta$. We have shown that $\bigcap_{\sigma \in \mathscr{D}} \bigcap_{\beta < \alpha} \sigma^\beta \subseteq \Delta$ and, since the reverse inclusion always holds, we have equality.

Conversely, let $\mathscr{D}^\alpha$ be T_2 for each $\alpha \in I_0$ and let $x \neq y$. Then for each $\alpha \in I_0$ $(x,y) \notin \Delta = \bigcap_{\sigma \in \mathscr{D}} \bigcap_{\beta < \alpha} \sigma^\beta$. Thus for each $\alpha \in I_0$ $\sigma(x,y) \leq \beta$ for some $\sigma \in \mathscr{D}$ and $\beta < \alpha$, from which it follows that $(\inf_{\sigma \in \mathscr{D}} \sigma)(x,y) = 0$. On the other hand, if $x = y$ then $\inf_{\sigma \in \mathscr{D}} \sigma(x,y) = 1$ and hence $\inf_{\sigma \in \mathscr{D}} \sigma = 1_\Delta$.

■

In similar vein, we characterise closedness in terms of the α–levels.

3.17 THEOREM

If $(X, \mathscr{D})$ is a fuzzy uniform space and $\mu \in I^X$ then :
μ is $\mathscr{D}$–closed $\Leftrightarrow$ $\forall \alpha \in I_0$ μ_α is $\mathscr{D}^\alpha$–closed.

PROOF

Let μ be $\mathscr{D}$–closed, $\alpha \in I_0$ and let $x \in cl_\alpha(\mu_\alpha)$, the $\mathscr{D}^\alpha$–closure of μ_α. We must show that $x \in \mu_\alpha$ and so we let $\beta < \alpha$ be arbitrary and show $\mu(x) > \beta$. Now μ is $\mathscr{D}$–closed and so

$$\mu(x) = \bar{\mu}(x) = \inf_{\sigma \in \mathscr{D}} \sigma\langle\mu\rangle(x) = \inf_{\sigma} \sup \mu \wedge \sigma\langle x\rangle.$$

We therefore let $\sigma \in \mathscr{D}$ be arbitrary and show that $\sup \mu \wedge \sigma\langle x\rangle > \beta$. Since $\sigma^\beta \in \mathscr{D}^\alpha$ we have $x \in \sigma^\beta(\mu_\alpha) \subseteq \sigma^\beta(\mu^\beta)$ and hence $\sigma(y,x) > \beta$ for some $y \in \mu^\beta$. Thus :

$$\sup \mu \wedge \sigma\langle x\rangle \geq \mu(y) \wedge \sigma(y,x) > \beta.$$

For the converse we must show that $\bar{\mu} \leq \mu$ and so we let $x \in X$ be arbitrary and show that $\bar{\mu}(x) \leq \mu(x)$. To do this we let $\alpha \leq \bar{\mu}(x)$ be arbitrary and show that $\alpha \leq \mu(x)$.

Now :

$$\begin{aligned}
\alpha \leq \bar{\mu}(x) = \inf_{\sigma \in \mathscr{D}} \sup \mu \wedge \sigma\langle x\rangle \quad & \Leftrightarrow \forall \beta < \alpha \ \forall \sigma \in \mathscr{D} \ \ \beta < \sup \mu \wedge \sigma\langle x\rangle \\
& \Leftrightarrow \forall \beta < \alpha \ \forall \sigma \in \mathscr{D} \ \ \exists y \in \mu^\beta \subseteq \mu_\beta : \sigma(y,x) > \beta \\
& \Leftrightarrow \forall \beta < \alpha \ \forall \sigma \in \mathscr{D} \ \ x \in \sigma^\beta(\mu_\beta) \\
& \Leftrightarrow \forall \beta < \alpha \ \forall \ U \in \mathscr{D}^\alpha \ \ x \in U(\mu_\beta) \\
& \Leftrightarrow \forall \beta < \alpha \ \ x \in cl_\alpha(\mu_\beta) = \mu_\beta \\
& \Leftrightarrow \forall \beta < \alpha \ \ \mu(x) \geq \beta \\
& \Leftrightarrow \mu(x) \geq \alpha.
\end{aligned}$$

■

We shall see many more α–level theorems in the theory to come, in fact, we shall obtain α–level theorems for each of the properties :
Cauchy, compactness, precompactness, boundedness and completeness.

GOOD EXTENSIONS OF THE NOTIONS OF UNIFORM SPACES

If $\mathbb{D}$ is a uniformity on X, $\mathbb{D}$ is a filter on X×X and we adopt the notation of Chapter 2 and define :

$$\mathbb{D}^1 = \{\sigma \in I^{X\times X} : \forall \alpha \in I_1 \ \sigma^{\alpha} \in \mathbb{D}\}.$$

It is shown in [Lo 3] (Theorem 3.1) that $\mathbb{D}^1$ is a fuzzy uniformity and in that paper $\mathbb{D}^1$ is denoted by $\omega_u(\mathbb{D})$.

If $\mathscr{D}$ is a fuzzy uniformity on X, $\mathscr{D}$ is a prefilter on X×X and we again adopt the notation of Chapter 2 and define :

$$\mathscr{D}^1 = \{\sigma^{\beta} : \sigma \in \mathscr{D}, \ \beta \in I_1\}.$$

In Theorem 3.1 of [Lo 3] $\mathscr{D}^1$ is shown to be a uniformity on X and there $\mathscr{D}^1$ is denoted $i_u(\mathscr{D})$.

Let UNIF denote the category of uniform spaces with uniformly continuous maps and let F.UNIF denote the category of fuzzy uniform spaces with uniformly continuous maps. Let

$$\begin{aligned} E : |\text{UNIF}| &\rightarrow |\text{F.UNIF}| \\ (X,\mathbb{D}) &\rightarrow (X,\mathbb{D}^1) \end{aligned}$$

and let E leave maps unchanged. Let

$$\begin{aligned} G : |\text{F. UNIF}| &\rightarrow |\text{UNIF}| \\ (X,\mathscr{D}) &\mapsto (X,\mathscr{D}^1) \end{aligned}$$

and let G leave maps unchanged. E and G and functors and :

$$G(E(X,\mathbb{D})) = G(X,\mathbb{D}^1) = (X,(\mathbb{D}^1)^1) = (X,\mathbb{D}) \quad (2.7(c)).$$

Thus $G \circ E((X,\mathbb{D})) = (X,\mathbb{D})$ and G turns out to be a left adjoint for E as we shall see. We first need :

3.18 LEMMA

> If $(X,\mathbb{D}) \in |\text{UNIF}|$, $(Y,\mathscr{D}) \in |\text{F.UNIF}|$ and $f : (X,\mathbb{D}) \rightarrow (Y,\mathscr{D}^1)$ is uniformly continuous then $f : (X,\mathbb{D}^1) \rightarrow (Y,\mathscr{D})$ is uniformly continuous

PROOF

Let $\sigma \in \mathscr{D}$, $\psi = (f\times f)^{-1}[\sigma]$ and $\alpha \in I_1$. Then :

$$\psi^{\alpha} = \{(x,y) : \sigma(f(x),f(y)) > \alpha\} = \{(x,y) : (f\times f)(x,y) \in \sigma^{\alpha}\} = (f\times f)^{-1}[\sigma^{\alpha}].$$

Now $\sigma^{\alpha} \in \mathscr{D}^1$ and hence $(f\times f)^{-1}[\sigma^{\alpha}] = \psi^{\alpha} \in \mathbb{D}$ since f is uniformly continuous as a morphism of UNIF. We have therefore shown that $\psi \in \mathbb{D}^1$ and consequently that f is uniformly continuous as a morphism of F.UNIF.

∎

It follows from the Lemma that G is a left adjoint for E since if $(X,\mathbb{D}) \in |\text{UNIF}|$, $(Y,\mathscr{D}) \in |\text{F.UNIF}|$ and $f: (X,\mathbb{D}) \to (Y,\mathscr{D}^1)$ is a morphism of UNIF then
$f: (X,\mathbb{D}^1) \to (Y,\mathscr{D})$ is a morphism of F.UNIF and $f = f\circ 1_X$.

$$\text{F.UNIF} \underset{G}{\overset{E}{\leftrightarrows}} \text{UNIF}$$

$$(X,\mathbb{D}^1) \xrightarrow{\ f\ } (Y,\mathscr{D}) \qquad\qquad \begin{array}{ccc} (X,\mathbb{D}) & & \\ {\scriptstyle 1_X}\downarrow & \searrow^{f} & \\ (X,\mathbb{D}) & \xrightarrow[f]{} & (Y,\mathscr{D}^1) \end{array}$$

In this sense, the functor E embeds UNIF into F.UNIF and G is a forgetful functor. An element $(X,\mathscr{D}) \in |\text{F.UNIF}|$ is said to be **uniformly generated** if $(X,\mathscr{D}) = E((X,\mathbb{D}))$ for some $(X,\mathbb{D}) \in |\text{UNIF}|$. In other words if $\mathscr{D} = \mathbb{D}^1$ for some uniformity $\mathbb{D}$. These uniformly generated fuzzy uniform spaces form a copy of UNIF in F.UNIF and if E were surjective, the theory of fuzzy uniform spaces would just be the theory of uniform spaces in disguise. This is however, not the case and $\mathscr{D} \subsetneq (\mathscr{D}^1)^1$ in general, equality occuring when $\mathscr{D}$ is uniformly generated ($((\mathbb{D}^1)^1)^1 = \mathbb{D}^1$). In extending the notions of UNIF (such as Cauchy filter, complete set) to F.UNIF, we must ensure that the extended notion (Cauchy prefilter, complete fuzzy set) coincides with the usual notion in the uniformly generated case.

We intend to extend (or generalise, to use another expression) the notions : Cauchy filter, precompact set, compact set, bounded set and complete set to the fuzzy uniform space setting and we must do so in such a way that :

$$\mathbb{F} \text{ is } \mathbb{D}\text{–Cauchy} \quad \Leftrightarrow \mathbb{F}^1 \text{ is } \mathbb{D}^1\text{–Cauchy}$$
$$A \text{ is } \mathbb{D} - P \quad \Leftrightarrow 1_A \text{ is } \mathbb{D}^1 - P$$

where P is precompact, compact, bounded or complete. In this case we will say that we have a **good extension** of the uniform space notion.

When we have obtained good extensions of these notions we shall investigate the theory that they generate using the standard theory as a guide. In this way the theory of uniform spaces will be extended to obtain the theory of fuzzy uniform spaces.

Prefilters $\mathscr{F}$ with $\hat{\mathscr{F}} = \mathscr{F}$ will be called **saturated prefilters** (as in [Mo 1]) from now on. These prefilters are special because fuzzy uniformities are prefilters with this property, and it is this property which is needed in the definition in order to produce a good theory of fuzzy uniform spaces. We shall now discover that they enjoy another interesting and useful property.

3.19 THEOREM

> If $\mathscr{D}$ and $\mathscr{L}$ are saturated prefilters on X such that $\mathscr{D}^\alpha = \mathscr{L}^\alpha$ for all $\alpha \in I_0$ then $\mathscr{D} = \mathscr{L}$.

PROOF

Let $\sigma \in \mathscr{D}$ and $\epsilon \in I_0$. We seek $\psi \in \mathscr{L}$ such that $\psi \leq \sigma + \epsilon$. Then, since ϵ is arbitrary, we will have

$$\forall \epsilon \in I_0 \ \exists \ \psi_\epsilon \in \mathscr{L} : \psi_\epsilon - \epsilon \leq \sigma$$

and consequently

$$\sup_{\epsilon \in I_0} (\psi_\epsilon - \epsilon) := \xi \leq \sigma \text{ with } \xi \in \hat{\mathscr{L}} = \mathscr{L}$$

and so we will have $\sigma \in \mathscr{L}$. Choose : $0 = \alpha_0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n = 1$ with $\alpha_i - \alpha_{i-1} < \frac{\epsilon}{4}$ for each $i \in [n]$.

(i) Since $\sigma^{\alpha_0} \in \mathscr{D}^{\alpha_1} = \mathscr{L}^{\alpha_1}$ there exists $\psi_1 \in \mathscr{L}$ and $\beta_1 < \alpha_1$ with $\psi_1^{\beta_1} = \sigma^{\alpha_0}$.

(ii) Since $\sigma^{\alpha_1} \in \mathscr{D}^{\alpha_2} = \mathscr{L}^{\alpha_2}$ there exists $\psi_2 \in \mathscr{L}$ and $\beta_2 < \alpha_2$ with $\psi_2^{\beta_2} = \sigma^{\alpha_1}$. Now $\alpha_1 < \alpha_2$ and so $\psi_2^{\alpha_1} \in \mathscr{L}^{\alpha_2}$. Thus $\psi_2^{\alpha_1} \cap \psi_2^{\beta_2} = \psi_2^{(\alpha_1 \vee \beta_2)} \in \mathscr{L}^{\alpha_2}$ and we have $\psi_2^{(\alpha_1 \vee \beta_2)} \subseteq \psi_2^{\beta_2} = \sigma^{\alpha_1}$. If we let $\beta_2' = \alpha_1 \vee \beta_2$ then $\alpha_1 \leq \beta_2'$ and $\psi_2^{\beta_2'} \subseteq \sigma^{\alpha_1}$. Thus instead of (ii) we can say :

(ii)′ There exists $\psi_2 \in \mathscr{L}$ and $\alpha_1 \leq \beta_2 < \alpha_2$ with $\psi_2^{\beta_2} \subseteq \sigma^{\alpha_1}$.

Similarly :

(iii) There exists $\psi_3 \in \mathscr{L}$ and $\alpha_2 \leq \beta_3 < \alpha_3$ with $\psi_3^{\beta_3} \subseteq \sigma^{\alpha_2}$.

In general we have : $\forall i \in [n] \ \exists \ \psi_i \in \mathscr{L} \ \exists \ \alpha_{i-1} \leq \beta_i < \alpha_i$ with $\psi_i^{\beta_i} \subseteq \sigma^{\alpha_{i-1}}$.

Let $\psi := \inf_{i \in [n]} \psi_i$. Then $\psi \in \mathscr{L}$ and :

$\forall i \in [n] \ \psi^{\beta_i} \subseteq \psi_i^{\beta_i} \subseteq \sigma^{\alpha_{i-1}}$.

Thus :

(iv) $\forall i \in [n] \ \forall x \in X \ \ \psi(x) > \beta_i \Longrightarrow \sigma(x) > \alpha_{i-1} > \alpha_i - \frac{\epsilon}{4} > \beta_i - \frac{\epsilon}{4}$.

It now follows that $\psi \leq \sigma + \epsilon$ since if $x \in X$ and $\psi(x) > \beta$ then $\beta_i \leq \beta < \beta_{i+1}$ for some i and then :

$$\begin{aligned}
\psi(x) > \beta \quad &\Longrightarrow \psi(x) > \beta_i \\
&\Longrightarrow \sigma(x) > \beta_i - \tfrac{\epsilon}{4} \qquad \text{(from (iv))} \\
&\Longrightarrow \sigma(x) + \epsilon > \beta_i + \tfrac{3}{4}\epsilon > \beta_{i+1} > \beta
\end{aligned}$$

$(\beta_{i+1} - \beta_i \leq \alpha_{i+1} - \alpha_{i-1} < \frac{\epsilon}{2})$.

We have therefore shown that $\mathscr{D} \subseteq \mathscr{L}$ and it follows by symmetry that $\mathscr{L} \subseteq \mathscr{D}$.

■

An immediate corollary is :

3.20 COROLLARY

> If $\mathscr{D}$ and $\mathscr{L}$ are fuzzy uniformities on X with $\mathscr{D}^{\alpha} = \mathscr{L}^{\alpha}$ for each $\alpha \in (0,1)$ then $\mathscr{D} = \mathscr{L}$.

THE CONSTRUCTION OF FUZZY UNIFORMITIES

We intend to build fuzzy uniformities with predetermined α–level uniformities. In other words, if $(\mathbb{D}(\alpha) : \alpha \in (0,1))$ is a family of uniformities on X, we shall construct a fuzzy uniformity $\mathscr{D}$ whose α–level uniformities are precisely the $\mathbb{D}(\alpha)$'s. We know from Assertion 3.14 that the α–level uniformities $(\mathscr{D}^{\alpha} : \alpha \in (0,1))$ satisfy :

(1) $0 < \beta \leq \alpha \leq 1 \Rightarrow \mathscr{D}^{\beta} \subseteq \mathscr{D}^{\alpha}$ and $\mathscr{D}^{\alpha} = \bigcup_{\beta < \alpha} \mathscr{D}^{\beta}$ and so the family $(\mathbb{D}(\alpha) : \alpha \in (0,1))$ must satisfy :

(2) $0 < \beta \leq \alpha \leq 1 \Rightarrow \mathbb{D}(\beta) \subseteq \mathbb{D}(\alpha)$ and $\mathbb{D}(\alpha) = \bigcup_{\beta < \alpha} \mathbb{D}(\beta)$.

Equivalently, $(\mathbb{D}(\alpha) : \alpha \in (0,1))$ must satisfy :

(3) $0 < \beta \leq \alpha \leq 1 \Rightarrow \mathbb{D}(\beta) \subseteq \mathbb{D}(\alpha)$ and $\mathbb{D}(\alpha) = \sup_{\beta < \alpha} \mathbb{D}(\beta)$

where $\sup_{\beta < \alpha} \mathbb{D}(\beta)$ is the weakest uniformity stronger that each $\mathbb{D}(\beta)$ with $\beta < \alpha$.

The equivalence of (2) and (3) follows from the fact that

$\bigcup_{\beta < \alpha} \mathbb{D}(\beta) \subseteq \sup_{\beta < \alpha} \mathbb{D}(\beta)$ by definition and if $U \in \sup_{\beta < \alpha} \mathbb{D}(\beta)$ then there exists $\alpha_1 < \alpha_2 < \cdots < \alpha_n < \alpha$ and $U_i \in \mathbb{D}(\alpha_i)$ such that $\bigcap_{i \in [n]} U_i \subseteq U$.

Now $\mathbb{D}(\alpha_1) \subseteq \mathbb{D}(\alpha_2) \subseteq \cdots \subseteq \mathbb{D}(\alpha_n)$ and so each $U_i \in \mathbb{D}(\alpha_n)$. Thus $U \in \mathbb{D}(\alpha_n) \subseteq \bigcup_{\beta < \alpha} \mathbb{D}(\beta)$ and hence we have $\sup_{\beta < \alpha} \mathbb{D}(\beta) = \bigcup_{\beta < \alpha} \mathbb{D}(\beta)$.

The uniformity $\mathbb{D}(1)$ need not be specified since it is determined by $\mathbb{D}(1) = \bigcup_{\beta < 1} \mathbb{D}(\beta)$ anyway.

3.21 THEOREM

Let $(\mathbb{D}(\alpha)\colon \alpha \in (0,1))$ be a family of uniformities on X satisfying :

(a) $0 < \beta \leq \alpha \leq 1 \Rightarrow \mathbb{D}(\beta) \subseteq \mathbb{D}(\alpha)$,

(b) $\mathbb{D}(\alpha) = \bigcup_{\beta < \alpha} \mathbb{D}(\beta)$ for each $\alpha \in I_0$.

Let $\mathscr{D} = \{\sigma \in I^{X \times X} : \forall \alpha \in (0,1) \ \forall \beta < \alpha \ \sigma^{\beta} \in \mathbb{D}(\alpha)\}$.

Then $\mathscr{D}$ is the unique fuzzy uniformity on X such that

$$\mathscr{D}^{\alpha} = \mathbb{D}(\alpha) \text{ for each } \alpha \in I_0.$$

PROOF

(1) **$\mathscr{D}$ is a saturated prefilter**

Let $\sigma \in \mathscr{D}$ and $0 \leq \beta < 1$. Choose α such that $\beta < \alpha < 1$. Then $\sigma^{\beta} \in \mathbb{D}(\alpha)$ and so $\Delta \subseteq \sigma^{\beta}$ and hence $\sigma(x,x) > \beta$ for each $x \in X$. Since β is arbitrary we have :

$\forall x \in X \ \sigma(x,x) = 1$.

In particular, $\sigma \neq 0$.

Let $\sigma_1, \sigma_2 \in \mathscr{D}$ and $0 \leq \beta < \alpha < 1$. Then $\sigma_1^{\beta}, \sigma_2^{\beta} \in \mathbb{D}(\alpha)$ and so

$\sigma_1^{\beta} \cap \sigma_2^{\beta} = (\sigma_1 \wedge \sigma_2)^{\beta} \in \mathbb{D}(\alpha)$.

Thus $\sigma_1 \wedge \sigma_2 \in \mathscr{D}$.

If $\sigma \in \mathscr{D}$ then $\sigma_s \in \mathscr{D}$ since if $0 \leq \beta < \alpha < 1$ then $(\sigma_s)^{\beta} = (\sigma^{\beta})_s \in \mathbb{D}(\alpha)$.

Let $\sigma \in \mathscr{D}$, $\sigma \leq \psi$ and $0 \leq \beta < \alpha < 1$. Then $\sigma^{\beta} \subseteq \psi^{\beta}$ and, since $\sigma^{\beta} \in \mathbb{D}(\alpha)$, $\psi^{\beta} \in \mathbb{D}(\alpha)$. Consequently $\psi \in \mathscr{D}$.

Let $\sigma = \sup_{\epsilon \in I_0} (\sigma_{\epsilon} - \epsilon) \in \hat{\mathscr{D}}$ with each $\sigma_{\epsilon} \in \mathscr{D}$ and let $0 \leq \beta < \alpha < 1$.

We note that :

$$\sigma(x,y) > \beta \Leftrightarrow \exists\ \epsilon \in I_0 : \sigma_\epsilon(x,y) - \epsilon > \beta \Leftrightarrow (x,y) \in \bigcup_{\epsilon \in I_0} \sigma_\epsilon^{\epsilon+\beta} .$$

In other words : $\sigma^\beta = \bigcup_{\epsilon \in I_0} \sigma_\epsilon^{\epsilon+\beta}$.

Choose $\epsilon \in I_0$ such that $\beta < \beta+\epsilon < \alpha$. Then :

$$\sigma_\epsilon^{\epsilon+\beta} \subseteq \sigma^\beta \text{ with } \sigma_\epsilon^{\epsilon+\beta} \in \mathbb{D}(\alpha) \text{ and so } \sigma^\beta \in \mathbb{D}(\alpha).$$

Thus $\sigma \in \mathscr{D}$ and we have shown that $\hat{\mathscr{D}} \subseteq \mathscr{D}$ from which it follows that $\hat{\mathscr{D}} = \mathscr{D}$.

(2) $\forall \sigma \in \mathscr{D}\ \forall \epsilon > 0\ \exists\ \psi \in \mathscr{D} : \psi \circ \psi \leq \sigma + \epsilon$

Let $\sigma \in \mathscr{D}$, $\epsilon > 0$ and choose $\alpha_0, \alpha_1, \cdots, \alpha_n$ such that :

$0 = \alpha_0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n = 1$ and $\alpha_i - \alpha_{i-1} < \frac{\epsilon}{2}$ for each $i \in [n]$.

For $i = 0,1,2,\cdots,n-1$ we have $\sigma^{\alpha_i} \in \mathbb{D}(\alpha_{i+1})$ and so there exists $U_{\alpha_{i+1}} \in \mathbb{D}(\alpha_{i+1})$ such that $U_{\alpha_{i+1}} \circ U_{\alpha_{i+1}} \subseteq \sigma^{\alpha_i}$. Let $U'_{\alpha_1} = U_{\alpha_1}$ and $U'_{\alpha_i} = \bigcap_{j \leq i} U_{\alpha_j}$. Then, since for each $j \leq i$, $U_{\alpha_j} \in \mathbb{D}(\alpha_j) \subseteq \mathbb{D}(\alpha_i)$, we have $U'_{\alpha_i} \in \mathbb{D}(\alpha_i)$ and $U'_{\alpha_1} \supseteq U'_{\alpha_2} \supseteq \cdots \supseteq U'_{\alpha_n}$. So we can state that :

$$\forall i \in [n]\ \exists\ U_{\alpha_i} \in \mathbb{D}(\alpha_i) : U_{\alpha_i} \circ U_{\alpha_i} \subseteq \sigma^{\alpha_{i-1}} \text{ and } U_{\alpha_1} \supseteq U_{\alpha_2} \supseteq \cdots \supseteq U_{\alpha_n}.$$

Let $\psi := \sup_{i \in [n]} \alpha_i 1_{U_{\alpha_{i-1}}}$.

Then $\psi \in \mathscr{D}$ since if $0 \leq \beta < \alpha < 1$ then $\alpha_i \leq \alpha < \alpha_{i+1}$ for some i. Thus $\beta < \alpha_{i+1}$ and hence $\psi^\beta \supseteq \psi_{\alpha_{i+1}} = U_{\alpha_i} \in \mathbb{D}(\alpha_i) \subseteq \mathbb{D}(\alpha)$. It follows that $\psi^\beta \in \mathbb{D}(\alpha)$ and so $\psi \in \mathscr{D}$.

If $\sigma(x,y) > \alpha_{n-2}$, then $\sigma(x,y)+\epsilon > \alpha_{n-2} + (\alpha_n - \alpha_{n-2}) = \alpha_n = 1$ and hence we have $(\psi \circ \psi)(x,y) \leq \sigma(x,y)+\epsilon$.

If $\sigma(x,y) \leq \alpha_{n-2}$ there exists some $i \leq n-2$ such that : $\alpha_{i-1} < \sigma(x,y) \leq \alpha_i$.

Since $(x,y) \notin \sigma^{\alpha_i}$ we have $(x,y) \notin U_{\alpha_{i+1}} \circ U_{\alpha_{i+1}}$ and so for no z do we have $(x,z) \in U_{\alpha_{i+1}}$ and $(z,y) \in U_{\alpha_{i+1}}$. In other words :

$\forall\, z\ (x,z) \notin U_{\alpha_{i+1}}$ or $(z,y) \notin U_{\alpha_{i+1}}$. Thus :

$\forall\, z\ \psi(x,z) \leq \alpha_{i+1}$ or $\psi(z,y) \leq \alpha_{i+1}$. Consequently :

$$\psi \circ \psi(x,y) = \sup_z \psi(x,z) \wedge \psi(z,y) \leq \alpha_{i+1} < \alpha_{i-1} + \epsilon < \sigma(x,y) + \epsilon.$$

(3) **$\mathscr{D}^{\alpha} = \mathbb{D}(\alpha)$ for each $\alpha \in I_0$**

If $U \in \mathscr{D}^{\alpha}$ then $U = \sigma^{\beta}$ for some $\sigma \in \mathscr{D}$ and $\beta < \alpha$ and so $\sigma^{\beta} \in \mathbb{D}(\alpha)$. Thus, on the one hand we have $\mathscr{D}^{\alpha} \subseteq \mathbb{D}(\alpha)$.

On the other hand, if $U \in \mathbb{D}(\alpha)$ then $U \in \mathbb{D}(\beta)$ for some $\beta < \alpha$ since $\mathbb{D}(\alpha) = \bigcup_{\beta < \alpha} \mathbb{D}(\beta)$. Let $\sigma = \beta 1_{X \times X} \vee 1_U$. To show that $\sigma \in \mathscr{D}$ we let $\delta < 1$ and $0 \leq \gamma < \delta$ and show that $\sigma^{\gamma} \in \mathbb{D}(\delta)$. If $\gamma < \beta$ we have $\sigma^{\gamma} = X \times X$ and if $\gamma \geq \beta$ we have $\sigma^{\gamma} = U \in \mathbb{D}(\beta) \subseteq \mathbb{D}(\gamma)$. So in both cases we have $\sigma^{\gamma} \in \mathbb{D}(\gamma) \subseteq \mathbb{D}(\delta)$. Thus $U = \sigma^{\beta}$ and hence $U \in \mathscr{D}^{\alpha}$ and we have shown that $\mathbb{D}(\alpha) \subseteq \mathscr{D}^{\alpha}$.

(4) **$\mathscr{D}$ is unique**

We invoke 3.20 and claim that there is precisely one fuzzy uniformity whose α–levels are the $\mathbb{D}(\alpha)$'s.

■

A fuzzy uniformity $\mathscr{D}$ gives rise to a family $(\mathscr{D}^{\alpha} : \alpha \in (0,1))$ of uniformities and this family in turn generates a fuzzy uniformity $\mathscr{L}$ with $\mathscr{L}^{\alpha} = \mathscr{D}^{\alpha}$ for each $\alpha \in (0,1)$. Thus $\mathscr{D} = \mathscr{L}$ and hence a fuzzy uniformity is uniquely determined by its family of α–level uniformities.

We Obtain an α–level theorem for uniform continuity.

3.22 THEOREM

Let $(X, \mathscr{D})$ and $(Y, \mathscr{E})$ be fuzzy uniform spaces. The following are equivalent :

(a) $f: (X, \mathscr{D}) \to (Y, \mathscr{E})$ is uniformly continuous.

(b) $\forall \alpha \in I_0$ $f: (X, \mathscr{D}^{\alpha}) \to (Y, \mathscr{E}^{\alpha})$ is uniformly continuous.

PROOF

We note that for $\psi \in \mathscr{E}$ and $\beta \in I_0$ we have :

$$(f\times f)^{-1}[\psi^{\beta}] = ((f\times f)^{-1}[\psi])^{\beta}$$

$$[(x_1,x_2) \in (f\times f)^{-1}[\psi^{\beta}] \Leftrightarrow (f(x_1),f(x_2)) \in \psi^{\beta} \Leftrightarrow \psi(f(x_1),f(x_2)) > \beta$$

$$\Leftrightarrow (\psi \circ (f\times f))(x_1,x_2) > \beta \Leftrightarrow (x_1,x_2) \in ((f\times f)^{-1}[\psi])^{\beta}.]$$

(a) $\Longrightarrow$ (b) Let $\alpha \in I_0$ and $V \in \mathscr{E}^{\alpha}$. Then $V = \psi^{\beta}$ for some $\psi \in \mathscr{E}$ and $\beta < \alpha$. Therefore $(f\times f)^{-1}[V] = (f\times f)^{-1}[\psi^{\beta}] = ((f\times f)^{-1}[\psi])^{\beta} \in \mathscr{D}^{\alpha}$ since f is uniformly continuous and $\psi \in \mathscr{E}$.

(b) $\Longrightarrow$ (a) Let $\psi \in \mathscr{E}$, $\alpha \in I_0$ and $\beta < \alpha$. Then $\psi^{\beta} \in \mathscr{E}^{\alpha}$ and so $(f\times f)^{-1}[\psi^{\beta}] = ((f\times f)^{-1}[\psi])^{\beta} \in \mathscr{D}^{\alpha}$. Thus :

(i) $\forall \alpha \in I_0$ $\forall \beta < \alpha$ $((f\times f)^{-1}[\psi])^{\beta} \in \mathscr{D}^{\alpha}$.

Now $\mathscr{D}$ is completely determined by its α–levels, in other words :

$$\mathscr{D} = \{\sigma \in I^{X\times X} : \forall \alpha \in I_0 \ \forall \beta < \alpha \ \sigma^{\beta} \in \mathscr{D}^{\alpha}\} \text{ so :}$$

(ii) $\sigma \in \mathscr{D} \Leftrightarrow \forall \alpha \in I_0$ $\forall \beta < \alpha$ $\sigma^{\beta} \in \mathscr{D}^{\alpha}$.

It follows from (i) and (ii) that $(f\times f)^{-1}[\psi] \in \mathscr{D}$ and hence we have shown that f is uniformly continuous.

■

EXAMPLES

We use Theorem 3.21 to provide some examples of fuzzy uniformities with predetermined α–levels and we shall refer to these later on.

3.23 EXAMPLE

Let $\mathbb{D}_w$ and $\mathbb{D}_s$ be two uniformities on X with $\mathbb{D}_w \subseteq \mathbb{D}_s$.

For $0 < \alpha \leq \frac{1}{2}$ let $\mathbb{D}(\alpha) := \mathbb{D}_w$ and for $\frac{1}{2} < \alpha \leq 1$ let $\mathbb{D}(\alpha) := \mathbb{D}_s$.

We then have :

(1) $0 < \beta \leq \alpha \leq 1 \Longrightarrow \mathbb{D}(\beta) \subseteq \mathbb{D}(\alpha)$,

(2) $\mathbb{D}(\alpha) = \bigcup_{\beta < \alpha} \mathbb{D}(\beta)$ for each $\alpha \in I_0$.

We construct $\mathscr{D}$ as in 3.21 to obtain a fuzzy uniformity with just two α–level uniformities :

For $0 < \alpha \leq \frac{1}{2}$ $\mathscr{D}^\alpha = \mathbb{D}_w$,

for $\frac{1}{2} < \alpha \leq 1$ $\mathscr{D}^\alpha = \mathbb{D}_s$.

3.24 EXAMPLE

What we have done for two uniformities, we can do for a finite collection $\{\mathbb{D}_{w_1}, \mathbb{D}_{w_2}, \cdots, \mathbb{D}_{w_n}\}$ of uniformities on X with $\mathbb{D}_{w_1} \subseteq \mathbb{D}_{w_2} \subseteq \cdots \subseteq \mathbb{D}_{w_n}$. Choose $\alpha_0, \alpha_1, \cdots, \alpha_n$ such that :

$$0 = \alpha_0 < \alpha_1 < \sigma_2 \cdots < \alpha_n = 1.$$

For each $i \in [n]$ if $\alpha_{i-1} < \alpha \leq \alpha_i$ let $\mathbb{D}(\alpha) := \mathbb{D}_{w_i}$.

Again the two conditions of Theorem 3.21 are met and the corresponding fuzzy uniformity has n α–levels satisfying :

$$\mathscr{D}^\alpha = \mathbb{D}_{w_i} \text{ if } \alpha_{i-1} < \alpha \leq \alpha_i \quad \text{for each } i \in [n].$$

CHAPTER 4

CAUCHY FILTERS AND PREFILTERS

If $(X,\mathbb{D})$ is a uniform space, a filter $\mathbb{F}$ on X is called **$\mathbb{D}$–Cauchy** if it contains arbitrarily $\mathbb{D}$–small elements. More precisely :

$$\mathbb{F} \text{ is } \mathbb{D}\text{–Cauchy} \Leftrightarrow \forall U \in \mathbb{D} \ \exists F \in \mathbb{F} : F \times F \subseteq U.$$

Our aim is to find a good extension (in the sense of Chapter 3) of this notion which allows us to generalise the theory associated with Cauchy filters to the fuzzy uniform space setting. A natural try would be to define a prefilter $\mathcal{F}$ on a fuzzy uniform space $(X,\mathcal{D})$ to be $\mathcal{D}$–Cauchy if :

$$\forall \sigma \in \mathcal{D} \ \forall \epsilon > 0 \ \exists \nu \in \mathcal{F} : \nu \times \nu \leq \sigma + \epsilon.$$

This is indeed a good extension as we shall see and we shall call such a prefilter strong–Cauchy in the sequel. The definition of a Cauchy prefilter which we shall adopt was inspired by the characterisation of Cauchy filters which follows.

4.1 THEOREM

> Let $(X,\mathbb{D})$ be a uniform space and $\mathbb{F}$ an ultrafilter on X. Then:
>
> $$\mathbb{F} \text{ is } \mathbb{D}\text{–Cauchy} \Leftrightarrow \forall\ U \in \mathbb{D} \ \bigcap_{F \in \mathbb{F}} U(F) \neq \emptyset.$$

PROOF

For the forward implication, let $\mathbb{F}$ be $\mathbb{D}$–Cauchy, $U \in \mathbb{D}$ and $F \in \mathbb{F}$. Since $\mathbb{F}$ is $\mathbb{D}$–Cauchy there exists $G \in \mathbb{F}$ such that $G \times G \in U$. Since $G \in \mathbb{F}$, $G \neq \emptyset$. Also, $G \subseteq U(F)$ since if $y \in G$, choose $x \in G \cap F$ (which is non–empty since $G \cap F \in \mathbb{F}$). Then $(x,y) \in G \times G \subseteq U$ and hence $y \in U(x) \subseteq U(G \cap F) \subseteq U(F)$. Since F is arbitrary we have $G \subseteq \bigcap_{F \in \mathbb{F}} U(F)$ and hence $\bigcap_{F \in \mathbb{F}} U(F) \neq \emptyset$.

Let us define : $U' = \bigcap_{F \in \mathbb{F}} U(F)$

$$\mathbb{F}' = \langle \{U' : U \in \mathbb{D}\} \rangle .$$

To show the reverse implication, suppose that each $U' \neq \emptyset$. Then :

(a) **$\mathbb{F}'$ is a filter on X.**

Let U, V $\in \mathbb{D}$. Then $U \cap V \in \mathbb{D}$ and so $(U \cap V)' \neq \emptyset$ and $(U \cap V)' \in \mathbb{F}'$. Now :

$$\begin{aligned}
y \in (U\cap V)' &\Leftrightarrow \forall\, F \in \mathbb{F}\ \ y \in (U\cap V)(F) \\
&\Leftrightarrow \forall\, F \in \mathbb{F}\ \ \exists\, x \in F : (x,y) \in U \text{ and } (x,y) \in V \\
&\Rightarrow \forall\, F \in \mathbb{F}\ (\exists\, x \in F : (x,y) \in U) \text{ and } (\exists\, x \in F : (x,y) \in V) \\
&\Leftrightarrow \forall\, F \in \mathbb{F}\ \ y \in U(F) \text{ and } y \in V(F) \\
&\Leftrightarrow y \in (\bigcap_{F \in \mathbb{F}} U(F)) \cap (\bigcap_{F \in \mathbb{F}} V(F)) \\
&\Leftrightarrow y \in U' \cap V'
\end{aligned}$$

Thus $(U \cap V)' \subseteq U' \cap V'$ and hence $\mathbb{F}'$ is a filter on X.

(b) **If $U \in \mathbb{D}$ is symmetric then $x \in U' \Leftrightarrow U(x) \in \mathbb{F}$.**

Let $U \in \mathbb{D}$ be symmetric. Then :

$$\begin{aligned}
x \in U' &\Leftrightarrow \forall\, F \in \mathbb{F}\ x \in U(F) \\
&\Leftrightarrow \forall\, F \in \mathbb{F}\ \exists\, y \in F : (y,x) \in U \\
&\Leftrightarrow \forall\, F \in \mathbb{F}\ \exists\, y \in F : (x,y) \in U \quad \text{(U is symmetric)} \\
&\Leftrightarrow \forall\, F \in \mathbb{F}\ \exists\, y \in F : y \in U(x) \\
&\Leftrightarrow \forall\, F \in \mathbb{F}\ U(x) \cap F \neq \emptyset \\
&\Leftrightarrow U(x) \in \mathbb{F} \quad (\mathbb{F} \text{ is ultra, 2.14}).
\end{aligned}$$

(c) **If $U \in \mathbb{D}$ is symmetric then $U' \times U' \subseteq U \circ U$.**

Let $U \in \mathbb{D}$ be symmetric. Then :

$$\begin{aligned}
(x,y) \in U' \times U' &\Leftrightarrow U(x) \in \mathbb{F} \text{ and } U(y) \in \mathbb{F} \quad \text{(by (b))} \\
&\Rightarrow U(x) \cap U(y) \neq \emptyset \\
&\Leftrightarrow \exists\, z : (x,z) \in U \text{ and } (y,z) \in U \\
&\Leftrightarrow \exists\, z : (x,z) \in U \text{ and } (z,y) \in U \text{ (U is symmetric)} \\
&\Leftrightarrow (x,y) \in U \circ U.
\end{aligned}$$

(d) **$\mathbb{F}'$ is $\mathbb{D}$–Cauchy.**

If $V \in \mathbb{D}$ we can choose symmetric $U \in \mathbb{D}$ such that $U \circ U \subseteq V$ and hence by (c) we have $U' \times U' \subseteq U \circ U \subseteq V$.

(e) **If $U \in \mathbb{D}$ and $x \in U'$ then $U(x) \subseteq (U \circ U)'$.**

Let $U \in \mathbb{D}$, $x \in U'$ and $y \in U(x)$. To show that $y \in (U \circ U)'$ we let $F \in \mathbb{F}$ be arbitrary and show that $y \in (U \circ U)(F)$. Now since $x \in U'$ we have $x \in U(F)$ and so there is some $z \in \mathbb{F}$ for which $(z,x) \in U$. Since $y \in U(x)$ we have $(x,y) \in U$ and hence $(z,y) \in U \circ U$. Thus $y \in (U \circ U)(F)$.

(f) **If $U \in \mathbb{D}$ is symmetric then $(U \circ U)' \in \mathbb{F}$.**

Let $U \in \mathbb{D}$ be symmetric and choose $x \in U'$. Then, by (b) we have $U(x) \in \mathbb{F}$ and by (e) we have $U(x) \subseteq (U \circ U)'$. It follows that $(U \circ U)' \in \mathbb{F}$.

(g) **$\mathbb{F}' \subseteq \mathbb{F}$.**

If $F \in \mathbb{F}'$ then $U' \subseteq F$ for some $U \in \mathbb{D}$. Let symmetric $V \in \mathbb{D}$ be such that $V \circ V \subseteq U$. Thus $(V \circ V)' \subseteq U'$ and it follows from (f) that $U' \in \mathbb{F}$ and hence $F \in \mathbb{F}$.

Finally, it follows from (d) and (g) that $\mathbb{F}$ is $\mathbb{D}$–Cauchy.

■

It is worth noting exactly where the fact that $\mathbb{F}$ is an ultrafilter is used in the above proof because the result does not hold in general ; that is, we do **not** have :

$$\mathbb{F} \text{ is } \mathbb{D}\text{–Cauchy} \Leftrightarrow \forall\, U \in \mathbb{D} \quad \bigcap_{F \in \mathbb{F}} U(F) \neq \emptyset.$$

for an arbitrary filter $\mathbb{F}$ in a uniform space $(X,\mathbb{D})$. To see this, consider $X = [0,1]$ and $\mathbb{D}$ the usual uniformity on $[0,1]$. Let $\{q_n : n \in \mathbb{N}\}$ be an enumeration of the rationals in $[0,1]$. In other words $\mathbb{Q} \cap [0,1] = \{q_n : n \in \mathbb{N}\}$. This sequence of rationals in $[0,1]$ is certainly non–Cauchy and hence so is the filter $\mathbb{F}$ that it generates.

Explicitly,

$$\mathbb{F} = <\{T_n : n \in \mathbb{N}\}> \text{ where } T_n = \{q_m : m \geq n\} \text{ is not Cauchy.}$$

However, $\mathbb{F}$ satisfies the above condition since if $U \in \mathbb{D}$ then $U \supseteq D_\epsilon := \{(x,y) : |x-y| < \epsilon\}$ for some $\epsilon > 0$. But if $F \in \mathbb{F}$ then $F \supseteq T_n$ for some $n \in \mathbb{N}$ and T_n contains all but a finite number of rationals in $[0,1]$.

Thus $U(F) = \bigcup_{m \geq n} U(q_m) = [0,1]$.

The theorem which characterises arbitrary Cauchy filters is as follows :

4.2 THEOREM

> Let $(X,\mathbb{D})$ be a uniform space and $\mathbb{F}$ a filter on X. Then :
> $\mathbb{F}$ is $\mathbb{D}$–Cauchy $\Leftrightarrow \forall\, U \in \mathbb{D} \ \bigcap_{\mathbb{K} \in \mathbb{P}(\mathbb{F})} \ \bigcap_{K \in \mathbb{K}} U(K) \neq \emptyset$.

PROOF

Let $\mathbb{F}$ be $\mathbb{D}$–Cauchy, $U \in \mathbb{D}$, $\mathbb{K} \in \mathbb{P}(\mathbb{F})$ and $K \in \mathbb{K}$. Choose $F \in \mathbb{F}$ such that $F \times F \subseteq U$. Since $F, K \in \mathbb{K}$ we have $F \cap K \neq \emptyset$ and furthermore, $F \subseteq U(K)$. To see this, let $y \in F$ and $x \in F \cap K$. Then $(x,y) \in F \times F \subseteq U$ and hence $y \in U(K)$. Since $\mathbb{K}$ and K are arbitrary and $F \neq \emptyset$, the result follows.

For the converse suppose that for each $U \in \mathbb{D}$

$$U' := \bigcap_{\mathbb{K} \in \mathbb{P}(\mathbb{F})} \ \bigcap_{K \in \mathbb{K}} U(K) \neq \emptyset.$$

Let $\mathbb{F}' := <\{U' : U \in \mathbb{D}\}>$.

(a) **$\mathbb{F}'$ is a filter on X.**

If $U, V \in \mathbb{D}$ then $U \cap V \neq \emptyset$ and furthermore :

$$\begin{aligned}
y \in (U \cap V)' &\Leftrightarrow \forall\, \mathbb{K} \in \mathbb{P}(\mathbb{F})\ \forall\, K \in \mathbb{K}\ y \in (U \cap V)(K) \\
&\Leftrightarrow \forall\, \mathbb{K} \in \mathbb{P}(\mathbb{F})\ \forall\, K \in \mathbb{K} \exists\, x \in K : (x,y) \in U \text{ and } (x,y) \in V \\
&\Rightarrow \forall\, \mathbb{K} \in \mathbb{P}(\mathbb{F})\ \forall\, K \in \mathbb{K} (\exists\, x \in K : (x,y) \in U) \text{ and} \\
&\quad\ (\exists\, x \in K : (x,y) \in V) \\
&\Leftrightarrow \forall\, \mathbb{K} \in \mathbb{P}(\mathbb{F})\ \forall\, K \in \mathbb{K}\ y \in U(K) \text{ and } y \in V(K) \\
&\Leftrightarrow y \in U' \cap V'.
\end{aligned}$$

Thus $(U \cap V)' \subseteq U' \cap V'$ and it follows that $\mathbb{F}'$ is a filter on X.

(b) **If $U \in \mathbb{D}$ is symmetric then $x \in U' \Leftrightarrow U(x) \in \mathbb{F}$.**

Let $U \in \mathbb{D}$ be symmetric. Then :

$$\begin{aligned} x \in U' &\Leftrightarrow \forall\, \mathbb{K} \in \mathbb{P}(\mathbb{F})\, \forall\, K \in \mathbb{K}\; x \in U(K) \\ &\Leftrightarrow \forall\, \mathbb{K} \in \mathbb{P}(\mathbb{F})\, \forall\, K \in \mathbb{K}\, \exists\, y \in K : (y,x) \in U \\ &\Leftrightarrow \forall\, \mathbb{K} \in \mathbb{P}(\mathbb{F})\, \forall\, K \in \mathbb{K}\, \exists\, y \in K : (x,y) \in U \quad \text{(U is symmetric)} \\ &\Leftrightarrow \forall\, \mathbb{K} \in \mathbb{P}(\mathbb{F})\, \forall\, K \in \mathbb{K}\; \exists\, y \in K : y \in U(x) \\ &\Leftrightarrow \forall\, \mathbb{K} \in \mathbb{P}(\mathbb{F})\, \forall\, K \in \mathbb{K}\;\; U(x) \cap K \neq \emptyset \\ &\Leftrightarrow \forall\, \mathbb{K} \in \mathbb{P}(\mathbb{F})\;\; U(x) \in \mathbb{K} \quad \text{(each } \mathbb{K} \text{ is an ultrafilter, 2.14)} \\ &\Leftrightarrow U(x) \in \bigcap_{\mathbb{K} \in \mathbb{P}(\mathbb{F})} \mathbb{K} = \mathbb{F} \quad (2.11). \end{aligned}$$

(c) **If $U \in \mathbb{D}$ is symmetric then $U' \times U' \subseteq U \circ U$.**

As in 4.1.

(d) **$\mathbb{F}'$ is $\mathbb{D}$–Cauchy.**

As in 4.1.

(e) **If $U \in \mathbb{D}$ and $x \in U'$ then $U(x) \subseteq (U \circ U)'$.**

Let $U \in \mathbb{D}$, $x \in U'$ and $y \in U(x)$. To show that $y \in (U \circ U)'$ we let $\mathbb{K} \in \mathbb{P}(\mathbb{F})$ and $K \in \mathbb{K}$ be arbitrary and show that $y \in (U \circ U)(K)$. Since $x \in U'$ we have $x \in U(K)$ and so $(z,x) \in U$ from some $z \in K$. Now $y \in U(x)$ and hence $(x,y) \in U$ and so it follows that $(z,y) \in U \circ U$. Consequently $y \in (U \circ U)(K)$.

(f) **If $U \in \mathbb{D}$ is symmetric then $(U \circ U)' \in \mathbb{F}$.**

As in 4.1.

(g) **$\mathbb{F}' \subseteq \mathbb{F}$.**

As in 4.1.

Again, as in 4.1, $\mathbb{F}$ is $\mathbb{D}$–Cauchy.

∎

Now let $(X, \mathcal{D})$ be a fuzzy uniform space and $\mathcal{F}$ a prefilter on X. With Theorem 4.2 in mind we say :

$$\mathcal{F} \text{ is } \mathcal{D}\text{–Cauchy} \Leftrightarrow \forall \sigma \in \mathcal{D}\ \forall\ \alpha < \bar{c}(\mathcal{F})\ \exists\ x \in X: \alpha 1_x \in \inf_{\mathcal{G} \in \mathcal{P}_m(\mathcal{F})} \inf_{\nu \in \mathcal{G}} \sigma\langle\nu\rangle$$

$$\Leftrightarrow \forall\ \sigma \in \mathcal{D}\ \forall \alpha < \bar{c}(\mathcal{F})\ \exists\ x \in X: \alpha \leq \inf_{\mathcal{G} \in \mathcal{P}_m(\mathcal{F})} \inf_{\nu \in \mathcal{G}} \sigma\langle\nu\rangle(x)$$

$$\Leftrightarrow \forall\ \sigma \in \mathcal{D}\ \forall\ \alpha < \bar{c}(\mathcal{F})\ \ \alpha \leq \sup \inf_{\mathcal{G} \in \mathcal{P}_m(\mathcal{F})} \inf_{\nu \in \mathcal{G}} \sigma\langle\nu\rangle$$

$$\Leftrightarrow \forall\ \sigma \in \mathcal{D}\ \ \ \bar{c}(\mathcal{F}) \leq \sup \inf_{\mathcal{G} \in \mathcal{P}_m(\mathcal{F})} \inf_{\nu \in \mathcal{G}} \sigma\langle\nu\rangle$$

$$\Leftrightarrow \bar{c}(\mathcal{F}) \leq \inf_{\sigma \in \mathcal{D}} \sup \inf_{\mathcal{G} \in \mathcal{P}_m(\mathcal{F})} \inf_{\nu \in \mathcal{G}} \sigma\langle\nu\rangle.$$

We say that a prefilter base $\mathcal{F}$ is $\mathcal{D}$–Cauchy iff $\langle \mathcal{F} \rangle$ is $\mathcal{D}$–Cauchy, in which case the prefilter base must satisfy the same condition.

We see that if a prefilter $\mathcal{F}$ is prime then :

$$\mathcal{F} \text{ is } \mathcal{D}\text{–Cauchy} \Leftrightarrow c(\mathcal{F}) \leq \inf_{\sigma \in \mathcal{D}} \sup \inf_{\nu \in \mathcal{F}} \sigma\langle\nu\rangle.$$

If $(X, \mathcal{D})$ is a fuzzy uniform space, $\mathcal{F}$ is a prefilter on X and $\sigma \in \mathcal{D}$ we define :

$$\sigma \langle \mathcal{F} \rangle = \langle \{\sigma\langle\nu\rangle : \nu \in \mathcal{F}\} \rangle$$

It is easy to see that $\sigma\langle \mathcal{F} \rangle$ is a prefilter and that $\sigma \langle \mathcal{F} \rangle \subseteq \mathcal{F}$.

Before proceeding we need the following technical lemma.

4.3 LEMMA

Let $(X, \mathcal{D})$ be a fuzzy uniform space and $\mathcal{F}$ a prefilter on X. Then :

(a) $\inf_{\sigma \in \mathcal{D}} \sup \inf_{\nu \in \mathcal{F}} \sigma\langle\nu\rangle = \inf_{\sigma \in \mathcal{D}} \sup \inf_{\nu \in \mathcal{F}} (\sigma \circ \sigma)\langle\nu\rangle,$

(b) $\inf_{\sigma \in \mathcal{D}} \sup \inf_{\mathcal{G} \in \mathcal{P}_m(\mathcal{F})} \inf_{\nu \in \mathcal{G}} \sigma\langle\nu\rangle = \inf_{\sigma \in \mathcal{D}} \sup \inf_{\mathcal{G} \in \mathcal{P}_m(\mathcal{F})} \inf_{\nu \in \mathcal{G}} (\sigma \circ \sigma)\langle\nu\rangle,$

(c) $\inf_{\sigma \in \mathcal{D}} \sup \inf_{\nu \in \mathcal{F}} \sigma\langle\nu\rangle = \inf_{\sigma \in \mathcal{D}} \sup \operatorname{Adh} \sigma\langle \mathcal{F} \rangle,$

(d) $\inf_{\sigma \in \mathcal{D}} \sup \inf_{\mathcal{G} \in \mathcal{P}_m(\mathcal{F})} \inf_{\nu \in \mathcal{G}} \sigma\langle\nu\rangle = \inf_{\sigma \in \mathcal{D}} \sup \inf_{\mathcal{G} \in \mathcal{P}_m(\mathcal{F})} \operatorname{Adh} \sigma\langle \mathcal{G} \rangle.$

PROOF

(a) We note that $\inf\limits_{\sigma\in\mathcal{D}} \sup\limits_{\nu\in\mathcal{F}} \inf (\sigma\circ\sigma)\langle\nu\rangle \leq \inf\limits_{\sigma\in\mathcal{D}} \sup\limits_{\nu\in\mathcal{F}} \inf \sigma\langle\nu\rangle$

$\Leftrightarrow \forall\, \sigma\in\mathcal{D}\ \inf\limits_{\sigma\in\mathcal{D}} \sup\limits_{\nu\in\mathcal{F}} \inf (\sigma\circ\sigma)\langle\nu\rangle \leq \sup\limits_{\nu\in\mathcal{F}} \inf \sigma\langle\nu\rangle$

$\Leftrightarrow \forall\, \sigma\in\mathcal{D}\ \forall\, \epsilon>0\ \inf\limits_{\sigma\in\mathcal{D}} \sup\limits_{\sigma\in\mathcal{F}} \inf (\sigma\circ\sigma)\langle\nu\rangle \leq (\sup\limits_{\nu\in\mathcal{F}} \inf \sigma\langle\nu\rangle)+\epsilon$

$\Leftrightarrow \forall\, \sigma\in\mathcal{D}\ \forall\, \epsilon>0\ \exists\, \psi\in\mathcal{D} : \sup\limits_{\nu\in\mathcal{F}} \inf (\psi\circ\psi)\langle\nu\rangle \leq (\sup\limits_{\nu\in\mathcal{F}} \inf \sigma\langle\nu\rangle)+\epsilon.$

So we let $\sigma\in\mathcal{D}$, $\epsilon>0$ be arbitrary. Then there exists $\psi\in\mathcal{D}$ such that $\psi\circ\psi\leq\sigma+\epsilon$. Now for $\nu\in\mathcal{F}$ we have :

$$(\psi\circ\psi)\langle\nu\rangle \leq (\sigma+\epsilon)\langle\nu\rangle \leq \sigma\langle\nu\rangle+\epsilon \qquad (3.4\text{ (b)})$$

and the desired inequality follows. The reverse inequality follows from 3.2.

(b) Similar to (a)

(c)

$$\begin{aligned}
\inf_{\sigma\in\mathcal{D}} \sup_{\nu\in\mathcal{F}} \inf \sigma\langle\nu\rangle &\leq \inf_{\sigma\in\mathcal{D}} \sup_{\nu\in\mathcal{F}} \inf \overline{\sigma\langle\nu\rangle} \quad (3.3(\text{b}))\\
&= \inf_{\sigma\in\mathcal{D}} \sup \operatorname{Adh} \sigma\langle\mathcal{F}\rangle \quad (3.5(\text{h}))\\
&= \inf_{\sigma\in\mathcal{D}} \sup_{\nu\in\mathcal{F}} \inf_{\psi\in\mathcal{D}} \inf \psi(\sigma\langle\nu\rangle)\\
&= \inf_{\sigma\in\mathcal{D}} \sup_{\nu\in\mathcal{F}} \inf_{\psi\in\mathcal{D}} \inf (\psi\circ\sigma)\langle\nu\rangle \quad (3.4(\text{c}))\\
&\leq \inf_{\sigma\in\mathcal{D}} \sup_{\nu\in\mathcal{F}} \inf (\sigma\circ\sigma)\langle\nu\rangle\\
&= \inf_{\sigma\in\mathcal{D}} \sup_{\nu\in\mathcal{F}} \inf \sigma\langle\nu\rangle \quad (4.3\text{ (a)}).
\end{aligned}$$

So we have equality all the way.

(d) Similar to (c)

■

In view of Lemma 4.3 we can state that if $(X, \mathscr{D})$ is a fuzzy uniform space and $\mathscr{F}$ is a prefilter on X then :

$$\mathscr{F} \text{ is } \mathscr{D}\text{–Cauchy} \quad \Leftrightarrow \bar{c}(\mathscr{F}) \leq \inf_{\sigma \in \mathscr{D}} \sup \inf_{\mathscr{G} \in \mathscr{P}_m(\mathscr{F})} \inf_{\nu \in \mathscr{G}} \sigma\langle\nu\rangle$$

$$\Leftrightarrow \bar{c}(\mathscr{F}) \leq \inf_{\sigma \in \mathscr{D}} \sup \inf_{\mathscr{G} \in \mathscr{P}_m(\mathscr{F})} \operatorname{Adh} \sigma\langle\mathscr{G}\rangle.$$

If $\mathscr{F}$ is prime then :

$$\mathscr{F} \text{ is } \mathscr{D}\text{–Cauchy} \quad \Leftrightarrow c(\mathscr{F}) \leq \inf_{\sigma \in \mathscr{D}} \sup \inf_{\nu \in \mathscr{F}} \sigma\langle\nu\rangle$$

$$\Leftrightarrow c(\mathscr{F}) \leq \inf_{\sigma \in \mathscr{D}} \sup \operatorname{Adh} \sigma\langle\mathscr{F}\rangle.$$

Our next task should be to check that the definition is a good extension of the definition of a Cauchy filter in UNIF ; however, with a bit of hindsight we assert that it is best to investigate the α–levels first.

4.4 THEOREM

Let $(X, \mathscr{D})$ be a fuzzy uniform space and $\mathscr{F}$ a prefilter on X with $\bar{c} = \bar{c}(\mathscr{F}) > 0$ and $c = c(\mathscr{F})$. Then :

(a) $\mathscr{F}$ is $\mathscr{D}$–Cauchy $\Leftrightarrow$ $\mathscr{F}_0$ is $\mathscr{D}^{\bar{c}}$–Cauchy.

(b) If $\mathscr{F}$ is prime then $\mathscr{F}$ is $\mathscr{D}$–Cauchy $\Leftrightarrow$ $\mathscr{F}_0$ is $\mathscr{D}^{c}$–Cauchy.

PROOF

(a) $\mathscr{F}$ is $\mathscr{D}$–Cauchy

$$\Leftrightarrow \bar{c} \leq \inf_{\sigma \in \mathscr{D}} \sup \inf_{\mathscr{G} \in \mathscr{P}_m(\mathscr{F})} \inf_{\nu \in \mathscr{G}} \sigma\langle\nu\rangle$$

$$\Leftrightarrow \forall\, \alpha < \bar{c}\ \forall\, \sigma \in \mathscr{D}\ \exists\, x \in X\ \forall\, \mathscr{G} \in \mathscr{P}_m(\mathscr{F})\ \forall\, \nu \in \mathscr{G}$$

$$\sigma\langle\nu\rangle(x) = \sup \nu \wedge \sigma\langle x\rangle > \alpha$$

$$\Leftrightarrow \forall\, \alpha < \bar{c}\ \forall\, \sigma \in \mathscr{D}\ \exists\, x \in X\ \forall\, \mathscr{G} \in \mathscr{P}_m(\mathscr{F})\ \forall\, \nu \in \mathscr{G}\ \exists\, y \in \nu^{\alpha} : \sigma(y,x) > \alpha$$

$$\Leftrightarrow \forall\, \alpha < \bar{c}\ \forall\, \sigma \in \mathscr{D}\ \exists\, x \in X\ \forall\, \mathscr{G} \in \mathscr{P}_m(\mathscr{F})\ \forall\, \nu \in \mathscr{G}\ x \in \sigma^{\alpha}(\nu^{\alpha})$$

$$\Leftrightarrow \forall\, \alpha < \bar{c}\ \forall\, \sigma \in \mathscr{D} \bigcap_{\mathscr{G} \in \mathscr{P}_m(\mathscr{F})} \bigcap_{\nu \in \mathscr{G}} \sigma^{\alpha}(\nu^{\alpha}) \neq \emptyset$$

$$\Leftrightarrow \forall\, U \in \mathscr{D}^{\bar{c}} \bigcap_{\mathbb{K} \in \mathbb{P}(\mathscr{F}_0)} \bigcap_{K \in \mathbb{K}} U(K) \neq \emptyset \qquad \cdots \text{(i)}$$

$\Leftrightarrow$ $\mathscr{F}_0$ is $\mathscr{D}^{\bar{c}}$–Cauchy (4.2).

To justify (i) we must show that :

$$\forall \alpha < \bar{c}\, \forall \sigma \in \mathscr{D} \bigcap_{\mathscr{G} \in \mathscr{P}_m(\mathscr{F})} \bigcap_{\nu \in \mathscr{G}} \sigma^{\alpha}(\nu^{\alpha}) \neq \emptyset \Leftrightarrow \forall\, U \in \mathscr{D}^{\bar{c}} \bigcap_{\mathbb{K} \in \mathbb{P}(\mathscr{F}_0)} \bigcap_{K \in \mathbb{K}} U(K) \neq \emptyset.$$

($\Rightarrow$) Let $U \in \mathscr{D}^{\bar{c}}$, $\mathbb{K} \in \mathbb{P}(\mathscr{F}_0)$ and $K \in \mathbb{K}$. Then $U = \sigma^{\alpha}$ for some $\alpha < \bar{c}$, $\sigma \in \mathscr{D}$ and $\mathscr{G} := \mathscr{F} \vee \mathbb{K}_1 \in P_m(\mathscr{F})$ (2.15). Thus $1_K \in \mathbb{K}_1 \subseteq \mathscr{G}$ and hence $\sigma^{\alpha}(1_K^{\alpha}) = U(K) \neq \emptyset$.

($\Leftarrow$) Let $\alpha < \bar{c}$, $\sigma \in \mathscr{D}$, $\mathscr{G} \in \mathscr{P}_m(\mathscr{F})$ and $\nu \in \mathscr{G}$. Then $\mathscr{G} = \mathscr{F} \vee \mathbb{K}_1$ for some $\mathbb{K} \in \mathbb{P}(\mathscr{F}_0)$ and $\nu = \mu \wedge 1_K$ for some $\mu \in \mathscr{F}$ and $K \in \mathbb{K}$. Thus $\nu^{\alpha} = \mu^{\alpha} \cap K \in \mathbb{K}$ (since $\mu^{\alpha} \in \mathscr{F}_0 \subseteq \mathbb{K}$ (2.19)) and $\sigma^{\alpha} \in \mathscr{D}^{\bar{c}}$. Consequently $\sigma^{\alpha}(\nu^{\alpha}) \neq \emptyset$.

(b) If $\mathscr{F}$ is prime then $\bar{c}(\mathscr{F}) = c(\mathscr{F})$.

■

4.5 COROLLARY

> Let $(X, \mathscr{D})$ be a fuzzy uniform space and $\mathscr{F}$ a prefilter on X with $\bar{c} := \bar{c}(\mathscr{F}) > 0$. Then :
>
> $\mathscr{F}$ is $\mathscr{D}$–Cauchy $\Leftrightarrow \forall \alpha < \bar{c}$ $\mathscr{F}_0$ is $\mathscr{D}^{\alpha}$–Cauchy.

PROOF

The forward implication is clear since each $\mathscr{D}^{\alpha} \subseteq \mathscr{D}^{\bar{c}}$.

For the reverse implication we note that $\mathscr{D}^{\bar{c}} = \bigcup_{\alpha < \bar{c}} \mathscr{D}^{\alpha}$. Consequently, if $U \in \mathscr{D}^{\bar{c}}$ then $U \in \mathscr{D}^{\alpha}$ for some $\alpha < \bar{c}$ and so $\nu^0 \times \nu^0 \subseteq U$ for some $\nu \in \mathscr{F}$. It follows that $\mathscr{F}_0$ is $\mathscr{D}^{\bar{c}}$–Cauchy and hence, by 4.4 , that $\mathscr{F}$ is $\mathscr{D}$–Cauchy.

■

Theorem 4.4 or its corollary 4.5 constitute the promised α–level theorem for the Cauchy property. As a consequence of this we establish that our definition is a good extension.

4.6 COROLLARY

Let $(X,\mathbb{D})$ be a uniform space and $\mathbb{F}$ a filter on X. Then :

$$\mathbb{F} \text{ is } \mathbb{D}\text{–Cauchy} \Leftrightarrow \mathbb{F}^1 \text{ is } \mathbb{D}^1\text{–Cauchy.}$$

PROOF

We show that $\bar{c}(\mathbb{F}^1) = 1$ as follows. Let $\mathscr{G} \in \mathscr{P}_m(\mathbb{F}^1)$. Then $\mathscr{G} = \mathbb{F}^1 \vee \mathbb{K}_1$ for some ultrafilter $\mathbb{K} \supseteq \mathbb{F} = (\mathbb{F}^1)_0$ (2.7 (c)). We show that $c(\mathscr{G}) = 1$ from which it follows, since $\mathscr{G}$ is arbitrary, that $\bar{c}(\mathbb{F}^1) = 1$. Now

$$c(\mathscr{G}) = c(\mathbb{F}^1 \vee \mathbb{K}_1) = \inf_{\nu \in \mathbb{F}^1} \inf_{K \in \mathbb{K}} \sup(\nu \wedge 1_K),$$

so let $\nu \in \mathbb{F}^1$, $K \in \mathbb{K}$ and $\alpha < 1$ be arbitrary. Then $\nu^{\alpha} \in \mathbb{F} \subseteq \mathbb{K}$ and so $\nu^{\alpha} \cap K \neq \emptyset$. It follows that $\sup(\nu \wedge 1_K) > \alpha$ and hence that $c(\mathscr{G}) = 1$.

Alternatively, we could appeal to 2.22.

Recall also (2.7 (c)) that $(\mathbb{D}^1)^1 = \mathbb{D}$. Invoking 4.4 we obtain :

$$\mathbb{F}^1 \text{ is } \mathbb{D}^1\text{–Cauchy} \Leftrightarrow (\mathbb{F}^1)_0 \text{ is } (\mathbb{D}^1)^1\text{–Cauchy} \Leftrightarrow \mathbb{F} \text{ is } \mathbb{D}\text{–Cauchy.}$$

■

4.7 COROLLARY

Let $(X, \mathscr{D})$ be a fuzzy uniform space, $\mathbb{F}$ a filter on X and $\alpha > 0$. Then:

$$\mathbb{F} \text{ is } \mathscr{D}^{\alpha}\text{–Cauchy} \Leftrightarrow \mathbb{F}_{\alpha} \text{ is } \mathscr{D}\text{–Cauchy.}$$

PROOF

We employ 4.4 noting that $\bar{c}(\mathbb{F}_{\alpha}) = \alpha$ (2.22).

$$\begin{aligned} \mathbb{F}_{\alpha} \text{ is } \mathscr{D}\text{–Cauchy} &\Leftrightarrow (\mathbb{F}_{\alpha})_0 \text{ is } \mathscr{D}^{\alpha}\text{–Cauchy} \\ &\Leftrightarrow \mathbb{F} \text{ is } \mathscr{D}^{\alpha}\text{–Cauchy} \ (2.7\ (c)). \end{aligned}$$

■

STRONG CAUCHY PREFILTERS

If $(X, \mathscr{D})$ is a fuzzy uniform space and $\mathscr{F}$ is a prefilter on X then we say that :

$$\mathscr{F} \text{ is } \textbf{strong } \mathscr{D}\textbf{–Cauchy} \Leftrightarrow \forall \sigma \in \mathscr{D} \ \forall \epsilon > 0 \ \exists \ \nu \in \mathscr{F} : \nu \times \nu \leq \sigma + \epsilon.$$

The following theorem justifies the terminology.

4.8 THEOREM

> If $(X, \mathscr{D})$ is a fuzzy uniform space and $\mathscr{F}$ is a prefilter on X, then :
> $\mathscr{F}$ is strong $\mathscr{D}$–Cauchy $\Rightarrow$ $\mathscr{F}$ is $\mathscr{D}$–Cauchy.

PROOF

Let $\mathscr{F}$ be strong $\mathscr{D}$–Cauchy, let $\bar{c} = \bar{c}(\mathscr{F})$ and $U \in \mathscr{D}^{\bar{c}}$. Then $U = \sigma^{\alpha}$ for some $\sigma \in \mathscr{D}$ and $\alpha < \bar{c}$. Let $\epsilon = \bar{c} - \alpha$ and choose $\nu \in \mathscr{F}$ with $\nu \times \nu \leq \sigma + \frac{\epsilon}{2}$.

Let $\beta = \frac{\bar{c}+\alpha}{2}$. We invoke 2.19 and assert that $\nu^{\beta} \in \mathscr{F}_0$. Now :

$$\begin{aligned}(x,y) \in \nu^{\beta} \times \nu^{\beta} &\Leftrightarrow \nu(x) \wedge \nu(y) = (\nu \times \nu)(x,y) > \beta \\ &\Rightarrow \sigma(x,y) + \tfrac{\epsilon}{2} > \beta \\ &\Leftrightarrow \sigma(x,y) > \beta - \tfrac{\epsilon}{2} = \alpha,\end{aligned}$$

which means that $\nu^{\beta} \times \nu^{\beta} \subseteq \sigma^{\alpha} = U$. We have therefore shown that $\mathscr{F}_0$ is $\mathscr{D}^{\bar{c}}$–Cauchy and hence, due to 4.4, that $\mathscr{F}$ is $\mathscr{D}$–Cauchy.

∎

For certain types of prefilter the two notions coincide. For example :

4.9 THEOREM

> If $(X, \mathscr{D})$ is a fuzzy uniform space and $\mathscr{F}$ is a prime prefilter on X, then
> $\mathscr{F}$ is $\mathscr{D}$–Cauchy $\Leftrightarrow$ $\mathscr{F}$ is strong $\mathscr{D}$–Cauchy.

PROOF

Let $\mathscr{F}$ be prime and $\mathscr{D}$–Cauchy, then according to 4.4, $\mathscr{F}_0$ is $\mathscr{D}^c$–Cauchy, where $c = c(\mathscr{F}) = \bar{c}(\mathscr{F})$. Let $\sigma \in \mathscr{D}$ and $\epsilon > 0$ be given and define $\alpha = c - \frac{\epsilon}{2}$, $\beta = (c + \frac{\epsilon}{2}) \wedge 1$. Then $\sigma^{\alpha} \in \mathscr{D}^c$ and so we can find $F \in \mathscr{F}_0$ with $F \times F \subseteq \sigma^{\alpha}$. We appeal to 2.7(d) or (b) and assert that $\nu := \beta\, 1_F \in \mathscr{F}$. Let $(x,y) \in X \times X$ be arbitrary, then :

$$(\nu \times \nu)(x,y) \leq \beta \leq c + \tfrac{\epsilon}{2} = \alpha + \epsilon \leq \sigma(x,y) + \epsilon.$$

Thus $\nu \times \nu \leq \sigma + \epsilon$ and we have shown that $\mathscr{F}$ is strong $\mathscr{D}$–Cauchy.

The converse follows from 4.8.

■

To obtain another example we first need :

4.10 THEOREM

Let $(X, \mathscr{D})$ be a fuzzy uniform space, $\mathbb{F}$ a filter on X and $\alpha > 0$. Then :

$\mathbb{F}$ is $\mathscr{D}^{\alpha}$–Cauchy $\Leftrightarrow$ $\mathbb{F}_{\alpha}$ is strong $\mathscr{D}$–Cauchy.

PROOF

We first observe that if $F \subseteq X$, $\sigma \in \mathscr{D}$ and $\beta > 0$ then :

$$F \times F \subseteq \sigma_{\beta} \Leftrightarrow \beta\, 1_F \times \beta\, 1_F \leq \sigma.$$

[If $F \times F \subseteq \sigma_{\beta}$ and $(x,y) \in F \times F$ then $(\beta\, 1_F \times \beta\, 1_F)(x,y) = \beta \leq \sigma(x,y)$, while if $(x,y) \notin F \times F$ then $(\beta\, 1_F \times \beta\, 1_F)(x,y) = 0 \leq \sigma(x,y)$. Conversely, if $\beta\, 1_F \times \beta\, 1_F \leq \sigma$ and $(x,y) \in F \times F$ then $\sigma(x,y) \geq (\beta\, 1_F \times \beta\, 1_F)(x,y) = \beta$].

Let $\mathbb{F}$ be $\mathscr{D}^{\alpha}$–Cauchy, $\sigma \in \mathscr{D}$ and $\epsilon > 0$. If we let $\beta = (\alpha - \epsilon) \vee 0$, then $\beta < \alpha$ and so we can find $F \in \mathbb{F}$ with $F \times F \subseteq \sigma^{\beta} \subseteq \sigma_{\beta}$. Thus :

$$\alpha\, 1_F \times \alpha\, 1_F \leq (\beta + \epsilon) 1_F \times (\beta + \epsilon) 1_F \leq \beta\, 1_F \times \beta\, 1_F + \epsilon \leq \sigma + \epsilon.$$

For the converse, let $\mathbb{F}_\alpha$ be strong $\mathscr{D}$–Cauchy and $U \in \mathscr{D}^\alpha$. Then $U = \sigma^\beta$ for some $\sigma \in \mathscr{D}$ and $\beta < \alpha$. If we choose $\epsilon < \alpha-\beta$ then there exists $F \in \mathbb{F}$ such that $\alpha\, 1_F \times \alpha\, 1_F \leq \sigma+\epsilon$. Thus for $(x,y) \in F\times F$ we have $(\alpha\, 1_F \times \alpha\, 1_F)(x,y) = \alpha \leq \sigma(x,y)+\epsilon$. In other words, $\sigma(x,y) \geq \alpha-\epsilon > \beta$ and so $(x,y) \in \sigma^\beta$. We have therefore shown that $F\times F \subseteq \sigma^\beta = U$.

■

4.11 COROLLARY

Let $(X,\mathscr{D})$ be a fuzzy uniform space, $\mathbb{F}$ a filter on X and $\alpha > 0$. Then :

$$\mathbb{F}_\alpha \text{ is a } \mathscr{D}\text{–Cauchy} \Leftrightarrow \mathbb{F}_\alpha \text{ is strong } \mathscr{D}\text{–Cauchy.}$$

PROOF

$$\begin{aligned} \mathbb{F}_\alpha \text{ is } \mathscr{D}\text{–Cauchy} \quad &\Leftrightarrow \mathbb{F} \text{ is } \mathscr{D}^\alpha\text{–Cauchy} \quad (4.7) \\ &\Leftrightarrow \mathbb{F}_\alpha \text{ is strong } \mathscr{D}\text{–Cauchy} \quad (4.10). \end{aligned}$$

■

We have seen that for two types of prefilter : prime prefilters and prefilters of form $\mathbb{F}_\alpha$, Cauchy is equivalent to strong Cauchy ; which prompts us to ask firstly, whether the strong Cauchy definition is a good extension and secondly, if so, are the two definitions equivalent ? To settle these questions we first need to make the following observation .

4.12 LEMMA

If $\mathscr{F}$ is a strong Cauchy prefilter and $\mathscr{F} \subseteq \mathscr{G}$ then $\mathscr{G}$ is strong Cauchy.

The proof of Lemma 4.12 is obvious. We can now settle the first question.

4.13 THEOREM

> Let $(X,\mathbb{D})$ be a uniform space and $\mathbb{F}$ a filter on X. Then :
> $\mathbb{F}$ is $\mathbb{D}$–Cauchy $\Leftrightarrow$ $\mathbb{F}^1$ is strong $\mathbb{D}^1$–Cauchy.

PROOF

$\mathbb{F}$ is $\mathbb{D}$–Cauchy	$\Leftrightarrow$ $\mathbb{F}$ is $(\mathbb{D}^1)^1$–Cauchy	(2.7(c))
	$\Leftrightarrow$ $\mathbb{F}_1$ is strong $\mathbb{D}^1$–Cauchy	(4.10)
	$\Rightarrow$ $\mathbb{F}^1$ is strong $\mathbb{D}^1$–Cauchy	(2.6(a), 4.12)
	$\Rightarrow$ $\mathbb{F}^1$ is $\mathbb{D}^1$–Cauchy	(4.8)
	$\Leftrightarrow$ $\mathbb{F}$ is $\mathbb{D}$–Cauchy.	(4.6)

■

In order to deal with the second question we first establish a necessary condition for a prefilter to be strong Cauchy.

4.14 THEOREM

> Let $(X,\mathscr{D})$ be a fuzzy uniform space, $\mathscr{F}$ a prefilter on X, $c = c(\mathscr{F})$, $\bar{c} = \bar{c}(\mathscr{F}) > 0$. Then :
>
> (a) $\mathscr{F}$ is strong $\mathscr{D}$–Cauchy $\Rightarrow$ $\mathscr{F}^c$ is $\mathscr{D}^c$–Cauchy.
>
> (b) $\mathscr{F}$ is strong $\mathscr{D}$–Cauchy $\Rightarrow$ $\mathscr{F}^{\bar{c}}$ is $\mathscr{D}^{\bar{c}}$–Cauchy.

PROOF

(a) Let $\mathscr{F}$ be strong $\mathscr{D}$–Cauchy and $U = \sigma^{\alpha} \in \mathscr{D}^c$ with $\sigma \in \mathscr{D}$ and $\alpha < c$. We can find $\nu \in \mathscr{F}$ with $\nu\times\nu \leq \sigma + \frac{c-\alpha}{2}$. Let $\beta = \frac{c+\alpha}{2}$, then $\beta < c$ and so $\nu^{\beta} \in \mathscr{F}^c$. Furthermore :

$$(x,y) \in \nu^{\beta}\times\nu^{\beta} \Rightarrow \beta < \nu(x) \wedge \nu(y) = (\nu\times\nu)(x,y) \leq \sigma(x,y) + \frac{c-\alpha}{2}.$$

Thus $\sigma(x,y) > \frac{c+\alpha}{2} + \frac{\alpha-c}{2} = \alpha$ and so we have shown that $\nu^{\beta}\times\nu^{\beta} \subseteq \sigma^{\alpha}$ and hence that $\mathscr{F}^c$ is $\mathscr{D}^c$–Cauchy.

(b) $\mathscr{F}$ is strong $\mathscr{D}$–Cauchy $\Rightarrow \mathscr{F}$ is $\mathscr{D}$–Cauchy

$\Rightarrow \mathscr{F}_0$ is $\mathscr{D}^{\bar{c}}$ –Cauchy

$\Rightarrow \mathscr{F}^{\bar{c}}$ is $\mathscr{D}^{\bar{c}}$ –Cauchy.

■

Equipped with Theorem 4.14 we can show that the two definitions are not equivalent.

4.15 EXAMPLE

Let X = I and let $0 < \alpha < \frac{1}{2} < \beta < \gamma < 1$. On I we define two uniformities :

$\mathbb{D}_w = \{I \times I\}$ the weakest uniformity in I,

$\mathbb{D}_s = \langle \Delta \rangle$ the strongest uniformity on I.

We appeal to Theorem 3.21 and assert that there is a unique fuzzy uniformity $\mathscr{D}$ on I such that :

$$\mathscr{D}^{\alpha} = \mathbb{D}_w \text{ for } 0 < \alpha \leq \tfrac{1}{2},$$
$$\mathscr{D}^{\alpha} = \mathbb{D}_s \text{ for } \tfrac{1}{2} < \alpha \leq 1.$$

Let $\lambda = \alpha\, 1_\alpha \vee \beta\, 1_\beta \vee \beta\, 1_\gamma$ and let $\mathscr{F} = <\lambda>$. Then :

$c = c(\mathscr{F}) = \beta$.

$\nu \in \mathscr{F} \Leftrightarrow \lambda \leq \nu \Leftrightarrow \nu(\alpha) \geq \alpha,\ \nu(\beta) \geq \beta,\ \nu(\gamma) \geq \beta.$

Thus $\sup \nu \geq \beta$ for each $\nu \in \mathscr{F}$ and hence $c(\mathscr{F}) \geq \beta$. Since $\lambda \in \mathscr{F}$ and $\sup \lambda = \beta$ we also have $c(\mathscr{F}) \leq \beta$.

$\bar{c} = \bar{c}(\mathscr{F}) = \alpha$.

Let $\mathscr{G}_\alpha = <\alpha\, 1_\alpha>$. Then $\mathscr{G}_\alpha$ is prime (2.8) and $\mathscr{F} \subseteq \mathscr{G}_\alpha$.

Thus $\mathscr{G}_\alpha \in \mathscr{P}(\mathscr{F})$ and in fact $\mathscr{G}_\alpha \in \mathscr{P}_m(\mathscr{F})$. To see this, let $\mathscr{F} \subseteq \mathscr{G} \subseteq \mathscr{G}_\alpha$ with $\mathscr{G}$ prime. Then $\mathscr{G}$ is also generated by a fuzzy point (2.8) so let $\mathscr{G} = <\delta\, 1_x>$ say. Thus :

$$<\delta\, 1_x> \subseteq <\alpha\, 1_\alpha> \Rightarrow \forall\ \nu\ \ (\nu(x) \geq \delta \Rightarrow \nu(\alpha) \geq \alpha).$$

We conclude that $x = \alpha$ and $\delta \geq \alpha$. Since $\mathscr{F} = <\lambda> \subseteq \mathscr{G}$ we have $\lambda \in \mathscr{G}$ and hence $\lambda(\alpha) = \alpha \geq \delta$. Thus $\mathscr{G} = \mathscr{G}_\alpha$ and we have shown that $\mathscr{G}_\alpha$ is minimal.

It follows that $\bar{c}(\mathscr{F}) \leq c(\mathscr{G}_\alpha) = \alpha$.

If $\delta < \alpha$ and $\nu \in \mathscr{F}$ then $\nu(\alpha) \geq \alpha > \delta$, $\nu(\beta) \geq \beta > \delta$ and $\nu(\gamma) \geq \beta > \delta$. Consequently, $\{\alpha, \beta, \gamma\} \subseteq \nu^{\delta}$ and so $\nu^{\delta} \in \mathscr{F}_0 = <\{\alpha, \beta, \gamma\}>$. Since δ is arbitrary we conclude that $\bar{c}(\mathscr{F}) \geq \alpha$ (2.19).

$\mathscr{F}$ is $\mathscr{D}$–Cauchy.

$\mathscr{F}$ is $\mathscr{D}$–Cauchy $\quad\Leftrightarrow$ $\mathscr{F}_0$ is $\mathscr{D}^{\bar{c}}$ –Cauchy (4.4)

$\Leftrightarrow$ $\mathscr{F}_0$ is $\mathbb{D}_w$ –Cauchy ($\bar{c} = \bar{c}(\mathscr{F}) = \alpha < \frac{1}{2}$) and any filter is $\mathbb{D}_w$ –Cauchy.

$\mathscr{F}$ is not strong $\mathscr{D}$–Cauchy.

$\mathscr{F}$ is strong $\mathscr{D}$–Cauchy $\quad\Rightarrow$ $\mathscr{F}^c$ is $\mathscr{D}^c$ –Cauchy (4.14(a))

$\Rightarrow$ $\mathscr{F}^{\beta}$ is $\mathbb{D}_s$–Cauchy ($\frac{1}{2} < \beta = c = c(\mathscr{F})$).

So to prove our assertion we show that $\mathscr{F}^{\beta}$ is not $\mathbb{D}_s$–Cauchy. If $F \in \mathscr{F}^{\beta}$ then $F = \nu^{\delta}$ for some $\nu \in \mathscr{F}$ and $\delta < \beta$. Since $\nu \in \mathscr{F}$, $\nu(\beta) \geq \beta > \delta$ and $\nu(\gamma) \geq \beta > \delta$ and hence $\{\beta,\gamma\} \subseteq \nu^{\delta} = F$. We have shown therefore that $\mathscr{F}^{\beta} \subseteq <\{\beta,\gamma\}> := \mathbb{F}$.

Consequently, if $\mathscr{F}^{\beta}$ were $\mathbb{D}_s$–Cauchy, $\mathbb{F}$ would be $\mathbb{D}_s$–Cauchy. But in that case, since $\Delta \in \mathbb{D}_s$, $F \times F \subseteq \Delta$ for some $F \supseteq \{\beta,\gamma\}$ and hence we would have $\{\beta,\gamma\} \times \{\beta,\gamma\} \subseteq \Delta$. Since $\beta < \gamma$ this is impossible. We have a contradiction which establishes our assertion.

■

This example demonstrates that there are Cauchy prefilters which are not strong Cauchy so the two definitions are not equivalent. The difference between the two notions is also illustrated by their behaviour with respect to refinements. Strong Cauchy prefilters are stable under refinements as we have already noted in Lemma 4.12, but this is not true of Cauchy prefilters as we shall see.

4.16 REMARK

> If $(X, \mathscr{D})$ is a fuzzy uniform space and $\mathscr{F}$ is a $\mathscr{D}$–Cauchy prefilter on X then it is not true that every prefilter $\mathscr{G} \supseteq \mathscr{F}$ is also $\mathscr{D}$–Cauchy.

We justify this remark by considering the fuzzy uniform space $(X, \mathscr{D})$ and the prefilter $\mathscr{F}$ defined in Example 4.15.

Let $\mathscr{G} = \langle \beta 1_\beta \vee \beta 1_\gamma \rangle$.

Then $\mathscr{G} \supseteq \mathscr{F}$ and we assert that :

$\bar{c}(\mathscr{G}) = \beta$

Since $\langle \beta 1_\beta \rangle := \mathscr{G}_\beta \in \mathscr{P}_m(\mathscr{G})$, $\bar{c}(\mathscr{G}) \leq c(\mathscr{G}_\beta) = \beta$. To obtain the reverse inequality we note that $\mathscr{G}_o = \langle\{\beta,\gamma\}\rangle$ and employ Lemma 2.19.

$[\forall\, \delta < \beta\ \forall \nu \in \mathscr{G}\ \{\beta,\gamma\} \subseteq \overset{\delta}{\nu} \Rightarrow \forall \delta < \beta\ \forall \nu \in \mathscr{G}\ \overset{\delta}{\nu} \in \mathscr{G}_o]$

Now $\langle\{\beta,\gamma\}\rangle$ is not $\mathbb{D}_s$–Cauchy as in 4.15. In other words $\mathscr{G}_o$ is not $\mathscr{D}^{\bar{c}}$–Cauchy and so $\mathscr{G}$ is not $\mathscr{D}$–Cauchy (4.4).

■

We therefore have two different good extensions of the Cauchy notion. To anticipate, we shall show that the notions of compactness, precompactness, boundedness and completeness can be expressed in terms of prime prefilters and then, because of Theorem 4.9, the two notions generate the same theory.

We have seen that our definition of a Cauchy prefilter is a good extension but we require much more, namely that the standard theorems regarding Cauchy filters carry over to the broader context of fuzzy uniform spaces. The following theorems constitute an extension of the elementary theory of Cauchy filters.

4.17 THEOREM

> Let $(X,\mathscr{D})$ be a fuzzy uniform space, $\mathscr{F}$ a prefilter on X and $\mu \in I^X$. Then:
> $$\mathscr{F} \text{ is } \mathscr{D}\text{–convergent in } \mu \Longrightarrow \mathscr{F} \text{ is } \mathscr{D}\text{–Cauchy.}$$

PROOF

$\mathscr{F}$ is $\mathscr{D}$–convergent in μ

$$\Leftrightarrow \bar{c}(\mathscr{F}) \leq \sup \mu \wedge \lim \mathscr{F}$$

$$\Leftrightarrow \bar{c}(\mathscr{F}) \leq \sup \mu \wedge \Big(\inf_{\mathscr{G} \in \mathscr{P}_m(\mathscr{F})} \operatorname{Adh} \mathscr{G} \Big)$$

$$\Longrightarrow \bar{c}(\mathscr{F}) \leq \sup \inf_{\mathscr{G} \in \mathscr{P}_m(\mathscr{F})} \operatorname{Adh} \mathscr{G}.$$

Let $\sigma \in \mathscr{D}$ and $\mathscr{G} \in \mathscr{P}_m(\mathscr{F})$. Then $\sigma<\mathscr{G}> \subseteq \mathscr{G}$ and so $\operatorname{Adh} \mathscr{G} \leq \operatorname{Adh} \sigma<\mathscr{G}>$. It follows, since σ and $\mathscr{G}$ are arbitrary, that :

$$\bar{c}(\mathscr{F}) \leq \sup \inf_{\mathscr{G} \in \mathscr{P}_m(\mathscr{F})} \operatorname{Adh} \mathscr{G} \leq \sup \inf_{\mathscr{G} \in \mathscr{P}_m(\mathscr{F})} \operatorname{Adh} \sigma<\mathscr{G}>.$$

In other words, $\mathscr{F}$ is $\mathscr{D}$–Cauchy.

This can also be seen from the α–level theorems 3.15 and 4.5 :

$\mathscr{F}$ is $\mathscr{D}$–convergent in μ

$$\Leftrightarrow \bar{c}(\mathscr{F}) \leq \sup \mu \wedge \lim \mathscr{F}$$

$$\Leftrightarrow \forall \alpha < \bar{c}(\mathscr{F})\ \exists x \in \mu^{\alpha}\ \mathscr{F}_0 \longrightarrow x \text{ w.r.t. } \mathscr{D}^{\alpha}$$

$$\Longrightarrow \forall \alpha < \bar{c}(\mathscr{F})\ \mathscr{F}_0 \text{ is } \mathscr{D}^{\alpha}\text{–Cauchy}$$

$$\Leftrightarrow \mathscr{F} \text{ is } \mathscr{D}\text{–Cauchy.}$$

■

4.18 LEMMA

> Let $\mathscr{D}$ and $\mathscr{E}$ be fuzzy uniformities on X with $\mathscr{D} \subseteq \mathscr{E}$ and $\mathscr{F}$ a prefilter on X. Then :
> $$\mathscr{F} \text{ is } \mathscr{E}\text{–Cauchy} \Longrightarrow \mathscr{F} \text{ is } \mathscr{D}\text{–Cauchy.}$$

PROOF

Let $\bar{c} = \bar{c}(\mathscr{F})$ then $\mathscr{D}^{\bar{c}} \subseteq \mathscr{E}^{\bar{c}}$ and hence :

$$\begin{aligned} \mathscr{F} \text{ is } \mathscr{E}\text{–Cauchy} \quad &\Leftrightarrow \mathscr{F}_0 \text{ is } \mathscr{E}^{\bar{c}}\text{–Cauchy} \\ &\Rightarrow \mathscr{F}_0 \text{ is } \mathscr{D}^{\bar{c}}\text{–Cauchy} \\ &\Leftrightarrow \mathscr{F} \text{ is } \mathscr{D}\text{–Cauchy.} \end{aligned}$$

■

Despite the unpleasant Remark 4.16 we do have :

4.19 LEMMA

> Let $(X, \mathscr{D})$ be a fuzzy uniform space, $\mathscr{F}$ and $\mathscr{G}$ prime prefilters on X with $\mathscr{F} \subseteq \mathscr{G}$. Then : $\mathscr{F}$ is $\mathscr{D}$–Cauchy $\Rightarrow$ $\mathscr{G}$ is $\mathscr{D}$–Cauchy.

PROOF

If $\mathscr{F} \subseteq \mathscr{G}$ then $\mathscr{F}_0 \subseteq \mathscr{G}_0$, $c(\mathscr{G}) \leq c(\mathscr{F})$ and $\mathscr{D}^{c(\mathscr{G})} \subseteq \mathscr{D}^{c(\mathscr{F})}$.

Consequently :

$$\begin{aligned} \mathscr{F} \text{ is } \mathscr{D}\text{–Cauchy} \quad &\Leftrightarrow \mathscr{F}_0 \text{ is } \mathscr{D}^{c(\mathscr{F})}\text{–Cauchy} \\ &\Rightarrow \mathscr{F}_0 \text{ is } \mathscr{D}^{c(\mathscr{G})}\text{–Cauchy} \\ &\Rightarrow \mathscr{G}_0 \text{ is } \mathscr{D}^{c(\mathscr{G})}\text{–Cauchy} \\ &\Leftrightarrow \mathscr{G} \text{ is } \mathscr{D}\text{–Cauchy.} \end{aligned}$$

■

4.20 THEOREM

> Let $(X, \mathscr{D})$ and $(Y, \mathscr{E})$ be fuzzy uniform spaces, $f : X \to Y$ uniformly continuous and $\mathscr{F}$ a prime $\mathscr{D}$–Cauchy prefilter on X. Then $f[\mathscr{F}]$ is a prime $\mathscr{E}$–Cauchy prefilter base on Y.

PROOF

We have seen in 2.23(a), (f) that $f[\mathscr{F}]$ is a prime prefilter base.
Let $\mathscr{H} = \langle f[\mathscr{F}]\rangle$ and $c = c(\mathscr{F})$. Then $c(\mathscr{H}) = c$ (2.23(c), 2.1) and we show that $\mathscr{H}$ is $\mathscr{E}$–Cauchy by showing that $\mathscr{H}_0$ is $\mathscr{E}^c$–Cauchy (4.4). To this end let $\psi^{\beta} \in \mathscr{E}^c$ with $\psi \in \mathscr{E}$ and $\beta < c$.

If $\sigma := (f\times f)^{-1}[\psi]$ then $\sigma \in \mathscr{D}$ and so $\sigma^{\beta} \in \mathscr{D}^c$. Since $\mathscr{F}$ is $\mathscr{D}$–Cauchy, $\mathscr{F}_0$ is $\mathscr{D}^c$–Cauchy and hence we can find $\nu \in \mathscr{F}$ with $\nu^0 \times \nu^0 \subseteq \sigma^{\beta}$. Now ;

$f[\nu]^0 = \{y \in Y : f^{-1}[\{y\}] \cap \nu^0 \neq \emptyset\}$.

$$
\begin{aligned}
y \in f[\nu]^0 \quad &\Leftrightarrow f[\nu](y) = \sup_{x \in f^{-1}[\{y\}]} \nu(x) > 0 \\
&\Leftrightarrow \exists x \in f^{-1}[\{y\}] : \nu(x) > 0 \\
&\Leftrightarrow \nu^0 \cap f^{-1}[\{y\}] \neq \emptyset .
\end{aligned}
$$

$f[\nu]^0 \times f[\nu]^0 \subseteq \psi^{\beta}$.

$$
\begin{aligned}
(y_1,y_2) \in f[\nu]^0 \times f[\nu]^0 \quad &\Leftrightarrow f^{-1}[\{y_1\}] \cap \nu^0 \neq \emptyset \text{ and } f^{-1}[\{y_2\}] \cap \nu^0 \neq \emptyset \\
&\Leftrightarrow \exists\ x_1 \in \nu^0\ \exists\ x_2 \in \nu^0 : f(x_1) = y_1 \text{ and } f(x_2) = y_2 .
\end{aligned}
$$

Thus $\psi(y_1,y_2) = \psi(f(x_1),f(x_2)) = \psi\circ(f\times f)(x_1,x_2) = (f\times f)^{-1}[\psi](x_1,x_2) = \sigma(x_1,x_2) > \beta$.

Since $f[\nu] \in \mathscr{H}$ we have shown that $\mathscr{H}_0$ is $\mathscr{E}^c$–Cauchy.

An alternative proof makes use of 4.9. If $\psi \in \mathscr{E}$ and $\epsilon > 0$, then $(f\times f)^{-1}[\psi] \in \mathscr{D}$ and so $\nu\times\nu \leq (f\times f)^{-1}[\psi] + \epsilon$ for some $\nu \in \mathscr{F}$. It is easy to show that $f[\nu] \times f[\nu] \leq \psi + \epsilon$ and the result follows.

■

4.21 THEOREM

> Let $((X_j, \mathscr{D}(j)) : j \in J)$ be a family of fuzzy uniform spaces,
> $X = \prod_{j \in J} X_j$, $\mathscr{D} = \prod_{j \in J} \mathscr{D}(j)$ and $\mathscr{F}$ a prime prefilter on X. Then:
>
> $\mathscr{F}$ is $\mathscr{D}$–Cauchy $\Leftrightarrow \forall j \in J\ \ p_j[\mathscr{F}]$ is $\mathscr{D}(j)$–Cauchy.

PROOF

The forward implication follows immediately from 3.9 and 4.20.

For the reverse implication we first note that for each $j \in J$:

(i) $\mathscr{F}(j) := \langle p_j[\mathscr{F}] \rangle$ is prime and $c(\mathscr{F}(j)) = c(\mathscr{F}) := c$.

(ii) $\mathscr{F}(j)$ is $\mathscr{D}(j)$–Cauchy $\Rightarrow \mathscr{F}(j)_0$ is $\mathscr{D}(j)^c$–Cauchy. (4.4).

We show that $\mathscr{F}$ is $\mathscr{D}$–Cauchy by showing that $\mathscr{F}_0$ is $\mathscr{D}^c$–Cauchy. Let $\sigma^\beta \in \mathscr{D}^c$ with $\sigma \in \mathscr{D}$ and $\beta < c$. We seek $\nu \in \mathscr{F}$ such that $\nu^0 \times \nu^0 \subseteq \sigma^\beta$. Since $\sigma \in \mathscr{D}$ there is a finite subset $J_0 \in J$ with $\sigma(j) \in \mathscr{D}(j)$ for each $j \in J_0$ and $\sigma \geq \inf_{j \in J_0} (p_j \times p_j)^{-1}[\sigma(j)] := \psi$. For each $j \in J_0$ $\sigma(j)^\beta \in \mathscr{D}(j)^c$ and so :

$\forall\, j \in J_0\ \exists\ \nu(j) \in \mathscr{F}$: $(p_j[\nu(j)])^0 \times (p_j[\nu(j)])^0 \subseteq \sigma(j)^\beta$.

Let $\nu = \inf_{j \in J_0} \nu(j)$. Then $\nu \in \mathscr{F}$ and

(iii) $\forall j \in J_0\ (p_j[\nu])^0 \times (p_j[\nu])^0 \subseteq \sigma(j)^\beta$. now

(iv) $x(j) \in (p_j[\nu])^0 \Leftrightarrow \sup_{x \in p_j^{-1}[\{x(j)\}]} \nu(x) > 0 \Leftrightarrow \nu^0 \cap p_j^{-1}[\{x(j)\}] \neq \emptyset$

$$
\begin{aligned}
\text{Thus } x \in \nu^0 \quad & \Rightarrow \forall\ j\ \ x \in \nu^0 \cap p_j^{-1}[\{p_j(x)\}] \\
& \Rightarrow \forall\ j\ p_j(x) \in (p_j[\nu])^0 \ \ \text{(by (iv))}. \\
\text{So } (x,y) \in \nu^0 \times \nu^0 \quad & \Rightarrow \forall\ j \in J_0\ (p_j(x), p_j(y)) \in (p_j[\nu])^0 \times (p_j[\nu])^0 \\
& \Rightarrow \forall\ j \in J_0\ (p_j \times p_j)(x,y) \in \sigma(j)^\beta. \ \ \text{(by (iii))} \\
& \Rightarrow \forall j \in J_0\ (p_j \times p_j)^{-1}[\sigma(j)](x,y) > \beta \\
& \Rightarrow \psi(x,y) > \beta \\
& \Rightarrow (x,y) \in \psi^\beta \subseteq \sigma^\beta.
\end{aligned}
$$

In other words $\nu^0 \times \nu^0 \subseteq \sigma^\beta$.

■

CHAPTER 5

COMPACTNESS

If $(X, \mathscr{D})$ is a fuzzy uniform space and $\mu \in I^X$, we shall call μ $\mathscr{D}$–compact if it is f–compact with respect to the fuzzy uniform topology in the sense of Chadwick in [Ch 1] and [Ch 2]. That is :

$$\mu \text{ is } \mathscr{D}\text{–compact iff for each prefilter } \mathscr{F} \text{ on } X$$
$$c(\mathscr{F},\mu) \leq \sup(\mu \wedge \mathrm{Adh}\,\mathscr{F}).$$

Equivalently :

$$\mu \text{ is } \mathscr{D}\text{–compact iff for each prefilter } \mathscr{F} \text{ on } X \text{ with } \mu \in \mathscr{F}$$
$$c(\mathscr{F}) \leq \sup(\mu \wedge \mathrm{Adh}\,\mathscr{F}).$$

This definition gives rise to a good theory of compactness in fuzzy topological spaces and we shall conduct a similar investigation of the notions of precompactness, boundedness and completeness in fuzzy uniform spaces. Proposition 3.1 of [Ch 1] expresses compactness in terms of prime prefilters and, as we shall obtain analogues of this result, we state it here.

5.1 THEOREM

μ is $\mathscr{D}$–compact iff for each prime prefilter $\mathscr{F}$ on X
$c(\mathscr{F},\mu) \leq \sup(\mu \wedge \mathrm{Adh}\ \mathscr{F})$.

Equivalently :

5.2 THEOREM

μ is $\mathscr{D}$–compact iff for each prime prefilter $\mathscr{F}$ on X with $\mu \in \mathscr{F}$
$c(\mathscr{F}) \leq \sup(\mu \wedge \mathrm{Adh}\ \mathscr{F})$.

We shall call $\mu \in I^X$ **relatively $\mathscr{D}$–compact** (see [Ch 2]) iff for each prefilter $\mathscr{F}$ we have $c(\mathscr{F},\mu) \leq \sup \mathrm{Adh}\ \mathscr{F}$.

It was shown in Theorem 3.3 of [Ch 2] that compactness could be characterised in terms of prime prefilters with characteristic 1 and Theorem 3.4 of [Ch 2] is an α–level theorem. We shall obtain similar results in the sequel and so we state these theorems here in our setting.

5.3 THEOREM

If $(X, \mathscr{D})$ is a fuzzy uniform space and $\mu \in I^X$ then the following are equivalent :

(a) μ is $\mathscr{D}$–compact.

(b) For every prime prefilter $\mathscr{F}$ with $c(\mathscr{F}) = 1$ we have

$$c(\mathscr{F},\mu) \leq \sup (\mu \wedge \mathrm{Adh}\ \mathscr{F}).$$

5.4 THEOREM

If $(X, \mathscr{D})$ is a fuzzy uniform space and $\mu \in I^X$ then the following are equivalent :

(a) μ is $\mathscr{D}$–compact.

(b) $\forall\ \alpha \leq \sup \mu\ \forall\ 0 < \beta < \alpha$

μ_α is relatively compact in $(\mu_\beta, \mathscr{D}^\beta)$.

CHAPTER 6

PRECOMPACTNESS

If $(X,\mathbb{D})$ is a uniform space and $A \subseteq X$ then we say

A is $\mathbb{D}$–precompact $\Leftrightarrow \forall\, U \in \mathbb{D}\ \exists\, F \in \mathscr{P}_f(X) : A \subseteq U(F) = \bigcup_{x \in F} U(x)$.

Equivalently :

A is $\mathbb{D}$–precompact iff every ultrafilter $\mathbb{F}$ with $A \in \mathbb{F}$ is $\mathbb{D}$–Cauchy.

If $(X, \mathscr{D})$ is a fuzzy uniform space and $\mu \in I^X$ we say

μ is **$\mathscr{D}$–precompact** iff every prime prefilter $\mathscr{F}$ with $\mu \in \mathscr{F}$ is $\mathscr{D}$–Cauchy.

Let us first check that we have a good extension.

6.1 THEOREM

Let $(X,\mathbb{D})$ be a uniform space with $A \subseteq X$. Then :

A is $\mathbb{D}$–precompact $\Leftrightarrow 1_A$ is $\mathbb{D}^1$–precompact.

PROOF

Let A be $\mathbb{D}$–precompact and $\mathscr{F}$ a prime prefilter with $1_A \in \mathscr{F}$. If $c := c(\mathscr{F}) = 0$ then $\mathscr{F}$ is automatically $\mathbb{D}^1$–Cauchy so let $c > 0$. Since $1_A \in \mathscr{F}$ we have $A = (1_A)^0 \in \mathscr{F}_0$ with $\mathscr{F}_0$ an ultrafilter (2.9(a)). Thus $\mathscr{F}_0$ is $\mathbb{D}$–Cauchy. Now $\mathbb{D} = (\mathbb{D}^1)^c$ (2.7(c)) and hence by 4.4, $\mathscr{F}$ is $\mathbb{D}^1$–Cauchy.

Conversely, let 1_A be $\mathbb{D}^1$–precompact and $\mathbb{F}$ an ultrafilter with $A \in \mathbb{F}$. Then $1_A \in \mathbb{F}_1$ and, since $(\mathbb{F}_1)_0 = \mathbb{F}$, $\mathbb{F}_1$ is prime. Consequently $\mathbb{F}_1$ is $\mathbb{D}^1$–Cauchy and hence $\mathbb{F}^1$ is $\mathbb{D}^1$–Cauchy (2.6(a)). Thus, by 4.6, $\mathbb{F}$ is $\mathbb{D}$–Cauchy.

■

Before proceeding we obtain some characterisations of precompactness.

6.2 THEOREM

Let $(X, \mathscr{D})$ be a fuzzy uniform space with $\mu \in I^X$.
The following statements are equivalent :
(a) μ is $\mathscr{D}$–precompact.
(b) For every prime prefilter $\mathscr{F}$ $c(\mathscr{F},\mu) \le \inf_{\sigma \in \mathscr{D}} \sup \operatorname{Adh} \sigma\langle\mathscr{F}\rangle$.
(c) For every prime prefilter $\mathscr{F}$ $c(\mathscr{F},\mu) \le \inf_{\sigma \in \mathscr{D}} \sup \inf_{\nu \in \mathscr{F}} \sigma\langle\nu\rangle$.
(d) For every prime prefilter $\mathscr{F}$ with $\mu \in \mathscr{F}$ $c(\mathscr{F}) \le \inf_{\sigma \in \mathscr{D}} \sup \inf_{\nu \in \mathscr{F}} \sigma\langle\nu\rangle$.

PROOF

(a)$\Longrightarrow$(b) Let $\mathscr{F}$ be a prime prefilter (on X). If $c(\mathscr{F},\mu) = 0$ then

$0 = c(\mathscr{F},\mu) \le \inf_{\sigma \in \mathscr{D}} \sup \operatorname{Adh} \sigma\langle\mathscr{F}\rangle$. If $c(\mathscr{F},\mu) > 0$ then :

$$\forall \nu \in \mathscr{F} \ \sup \nu \wedge \mu > 0 \qquad \Longrightarrow \forall \nu \in \mathscr{F} \ \nu \wedge \mu \neq 0$$
$$\Longrightarrow (\mathscr{F},\mu) = \mathscr{F} \vee \langle\mu\rangle \text{ is a prefilter on X.}$$

Furthermore, $\mathscr{F} \subseteq (\mathscr{F},\mu)$ so $(\mathscr{F},\mu)$ is prime (2.9(e)) and $\mu \in (\mathscr{F},\mu)$.

Consequently $(\mathscr{F},\mu)$ is $\mathscr{D}$–Cauchy and this means that :

$$\begin{aligned} c(\mathscr{F},\mu) &\le \inf_{\sigma \in \mathscr{D}} \sup \operatorname{Adh} \sigma\langle(\mathscr{F},\mu)\rangle \\ &\le \inf_{\sigma \in \mathscr{D}} \sup \operatorname{Adh} \sigma\langle\mathscr{F}\rangle. \end{aligned}$$

(b)$\Longrightarrow$(c) Follows from 4.3(c).

(c)$\Longrightarrow$(d) Let $\mathscr{F}$ be a prime prefilter with $\mu \in \mathscr{F}$. Then $(\mathscr{F},\mu) = \mathscr{F}$ and the result follows.

(d) $\Longrightarrow$ (a) : Follows from 4.3(c).

■

Precompactness is characterised in terms of prime prefilters with characteristic 1 in the following theorem.

6.3 THEOREM

Let $(X, \mathscr{D})$ be a fuzzy uniform space with $\mu \in I^X$. The following statements are equivalent :

(a) μ is $\mathscr{D}$–precompact.

(b) For every prime prefilter $\mathscr{H}$ with $c(\mathscr{H}) = 1$

$$c(\mathscr{H},\mu) \leq \inf_{\sigma \in \mathscr{D}} \sup \operatorname{Adh} \sigma < \mathscr{H} >.$$

PROOF

(a)$\Longrightarrow$(b) Clear.

(b)$\Longrightarrow$(a) Let $\mathscr{F}$ be a prime prefilter with $\mu \in \mathscr{F}$. Let $\mathscr{H} := (\mathscr{F}_0)_1$.

In other words : $\mathscr{H} = <\{1_F : F \in \mathscr{F}_0\}>$. Then :

(i) $\mathscr{H} \subseteq (\mathscr{H},\mu) \subseteq \mathscr{F}$.

If $\lambda \in \mathscr{H}$ then $1_F \leq \lambda$ for some $F \in \mathscr{F}_0$ and $\nu^0 \subseteq F$ for some $\nu \in \mathscr{F}$.

Thus $\nu \leq 1_{\nu^0} \leq 1_F \leq \lambda$ and so $\lambda \in \mathscr{F}$. We have shown that $\mathscr{H} \subseteq \mathscr{F}$ and so it follows that $\mathscr{H} \subseteq (\mathscr{H},\mu) \subseteq (\mathscr{F},\mu) = \mathscr{F}$.

(ii) $\mathscr{H}_0 = (\mathscr{H},\mu)_0 = \mathscr{F}_0$.

It follows from (i) that $\mathscr{H}_0 \subseteq (\mathscr{H},\mu)_0 \subseteq \mathscr{F}_0$. To show the reverse inclusion let $F \in \mathscr{F}_0$. Then $1_F \in \mathscr{H}$ and $F = (1_F)^0$, from which it follows that $F \in \mathscr{H}_0$.

(iii) $\mathscr{H}$ **is prime.**

This follows from 2.9(a) since $\mathscr{H}_0 = \mathscr{F}_0$ and $\mathscr{F}$ is prime.

(iv) $c(\mathscr{H}) = 1$.

For each $F \in \mathscr{F}_0$ we have $\sup 1_F = 1$ and so :

$$c(\mathscr{H}) = \inf_{F \in \mathscr{F}_0} \sup 1_F = 1.$$

Alternatively we could appeal to 2.6(c).

From (i), (iii), (iv) and (b) we obtain :

(v) $c(\mathscr{F}) \leq c(\mathscr{H},\mu) \leq \inf_{\sigma \in \mathscr{D}} \sup \text{Adh}\, \sigma<\mathscr{H}>$.

(vi) **Adh $\sigma <\mathscr{H}> =$ Adh $\sigma <\mathscr{F}>$**

Let $x \in X$ and $\text{Adh}\, \sigma <\mathscr{H}>(x) = \inf_{\psi \in \mathscr{D}} \inf_{F \in \mathscr{F}_0} (\psi \circ \sigma)<1_F>(x) > \alpha$.

Let $\psi \in \mathscr{D}$ and $\nu \in \mathscr{F}$. Then $\nu^{\alpha} \in \mathscr{F}_{\alpha} = \mathscr{F}_0$ (2.9(b)) and so

$$(\psi \circ \sigma)<1_{\nu^{\alpha}}>(x) = \sup 1_{\nu^{\alpha}} \wedge (\psi \circ \sigma)<x> > \alpha.$$

Thus $\sup \nu \wedge (\psi \circ \sigma)<x> = (\psi \circ \sigma)<\nu>(x) > \alpha$. Since ψ and ν are arbitrary it follows that $\text{Adh}\, \sigma<\mathscr{F}>(x) > \alpha$. Finally, since x is arbitrary we have $\text{Adh}\, \sigma<\mathscr{H}> \leq \text{Adh}\, \sigma<\mathscr{F}>$ and, since $\sigma<\mathscr{H}> \subseteq \sigma<\mathscr{F}>$ we have equality.

It follows from (v) and (vi) that $\mathscr{F}$ is $\mathscr{D}$–Cauchy and hence that μ is $\mathscr{D}$–precompact.

■

Theorem 6.3 can be improved by showing that precompactness is characterised by an even smaller collection of prefilters.

6.4 THEOREM

> Let $(X, \mathscr{D})$ be a fuzzy uniform space with $\mu \in I^X$.
> The following statements are equivalent :
> (a) μ is $\mathscr{D}$–precompact.
> (b) For every prime saturated prefilter $\mathscr{H}$ with $c(\mathscr{H}) = 1$
> $$c(\mathscr{H},\mu) \leq \inf_{\sigma \in \mathscr{D}} \sup \mathrm{Adh}\, \sigma\langle\mathscr{H}\rangle.$$
> (c) Every prime saturated prefilter $\mathscr{H}$ with $\mu \in \mathscr{H}$ is $\mathscr{D}$–Cauchy.

PROOF

(a)$\Longrightarrow$(b) Clear.

(b)$\Longrightarrow$(a) Let $\mathscr{F}$ be a prime prefilter with $c(\mathscr{F}) = 1$. Let $\mathscr{H} = (\mathscr{F}_0)^1$.

In other words $\mathscr{H} = \{ \lambda \in I^X : \forall \alpha < 1 \;\; \lambda^\alpha \in \mathscr{F}_0\}$. Then :

(i) **$\mathscr{H}$ is a saturated prefilter.**

This follows from 2.6(b).

(ii) **$c(\mathscr{H}) = 1$.**

If $\lambda \in \mathscr{H}$ then $\forall \alpha < 1 \; \lambda^\alpha \in \mathscr{F}_0 \Rightarrow \forall \alpha < 1 \; \lambda^\alpha \neq \emptyset$

$\Rightarrow \forall \alpha < 1 \; \exists \; x : \lambda(x) > \alpha$

$\Rightarrow \forall \alpha < 1 \; \sup \lambda > \alpha$

$\Rightarrow \sup \lambda = 1.$

Thus $c(\mathscr{H}) = \inf_{\lambda \in \mathscr{H}} \sup \lambda = 1.$

Alternatively we could appeal to 2.6(c).

(iii) **$\mathscr{H}_0 = \mathscr{F}_0$.**

If $H \in \mathscr{H}_0$ then $\lambda^0 \subseteq H$ for some $\lambda \in \mathscr{H}$ and so, since $\lambda^0 \in \mathscr{F}_0$, $H \in \mathscr{F}_0$.

If $F \in \mathscr{F}_0$ then $\nu^0 \subseteq F$ for some $\nu \in \mathscr{F}$ and so $1_{\nu^0} \leq 1_F$. Now :

$\forall \alpha < 1 \;\; (1_{\nu^0})^\alpha = \nu^0 \in \mathscr{F}_0 \Rightarrow 1_{\nu^0} \in \mathscr{H}.$

Thus $1_F \in \mathscr{H}$ and so $F = (1_F)^0 \in \mathscr{H}_0$.

Alternatively we could appeal to 2.7(c).

(iv) **$\mathcal{H}$ is prime.**

$\mathcal{F}$ is prime, so $\mathcal{F}_0$ is ultra and hence $\mathcal{H}$ is prime (2.9(a)).

By (i), (ii), (iv) and (b) we have :

$$c(\mathcal{H},\mu) \leq \inf_{\sigma \in \mathcal{D}} \sup \text{Adh } \sigma < \mathcal{H} >.$$

(v) **$c(\mathcal{F},\mu) \leq c(\mathcal{H},\mu)$.**

Let $\alpha < c(\mathcal{F},\mu)$ and $\lambda \in \mathcal{H}$. We show that $\alpha < \sup \lambda \wedge \mu$ and then, since α and λ are arbitrary, the result follows. Since $\lambda \in \mathcal{H}$ we have $\lambda^{\alpha} \in \mathcal{F}_0$ and so $\nu^0 \subseteq \lambda^{\alpha}$ for some $\nu \in \mathcal{F}$. Now $\nu \leq 1_{\nu^0} \leq 1_{\lambda^{\alpha}}$,hence $1_{\lambda^{\alpha}} \in \mathcal{F}$ and thus $\mu \wedge 1_{\lambda^{\alpha}} \in (\mathcal{F},\mu)$. It follows that $\sup(\mu \wedge 1_{\lambda^{\alpha}}) > \alpha$ and so $\mu(x) > \alpha$ for some $x \in \lambda^{\alpha}$. Consequently :

$$\sup(\mu \wedge \lambda) \geq (\mu \wedge \lambda)(x) > \alpha.$$

We now have :

(vi) $$c(\mathcal{F},\mu) \leq c(\mathcal{H},\mu) \leq \inf_{\sigma \in \mathcal{D}} \sup \text{Adh } \sigma < \mathcal{H} >.$$

(vii) **Adh $\sigma<\mathcal{H}>$ = Adh $\sigma<\mathcal{F}>$.**

If we let $\mathcal{K} = (\mathcal{F}_0)_1$, then $\mathcal{H} = (\mathcal{F}_0)^1 = \hat{\mathcal{K}}$ (2.6(b)). Thus :

$$\begin{aligned} \text{Adh } \sigma<\mathcal{H}> &= \text{Adh } \sigma<\hat{\mathcal{K}}> \\ &= \text{Adh } \sigma<\mathcal{K}> \quad (3.5(\text{i})) \\ &= \text{Adh } \sigma<\mathcal{F}> \quad (6.3(\text{vi})). \end{aligned}$$

It follows from (vi),(vii) and 6.3(b) that μ is $\mathcal{D}$–precompact.

(a) $\Longrightarrow$ (c) Clear.

(c) $\Longrightarrow$ (a) Let $\mathscr{F}$ be a prime prefilter with $\mu \in \mathscr{F}$ and $c(\mathscr{F}) = c$.
Let $\mathscr{H} = (\mathscr{F}_0)^c = \{\lambda \in I^X : \forall \alpha < c \;\; \lambda^\alpha \in \mathscr{F}_0\}$. Then :

(i) **$\mathscr{H}$ is a saturated prefilter.**
From 2.6(b).

(ii) **$\mathscr{F} \subseteq \mathscr{H}$.**
If $\nu \in \mathscr{F}$ then $\nu^\alpha \in \mathscr{F}_0$ for each $\alpha < c$ (2.19) and so $\nu \in \mathscr{H}$.

(iii) **$c(\mathscr{H}) = c$.**
Since $\mathscr{F} \subseteq \mathscr{H}$ we have $c(\mathscr{H}) \leq c(\mathscr{F}) = c$. Furthermore :

$$
\begin{aligned}
\forall \lambda \in \mathscr{H} \, \forall \alpha < c \;\; \lambda^\alpha \in \mathscr{F}_0 \quad & \Longrightarrow \forall \lambda \in \mathscr{H} \, \forall \alpha < c \;\; \lambda^\alpha \neq \emptyset \\
& \Longrightarrow \forall \lambda \in \mathscr{H} \, \forall \; \alpha < c \;\; \sup \lambda > \alpha \\
& \Longrightarrow \forall \lambda \in \mathscr{H} \; \sup \lambda \geq c \\
& \Longrightarrow \inf_{\lambda \in \mathscr{H}} \sup \lambda = c(\mathscr{H}) \geq c.
\end{aligned}
$$

Alternatively we could appeal to 2.6(c).

(iv) **$\mathscr{F}_0 = \mathscr{H}_0$.**
From (ii) we obtain $\mathscr{F}_0 \subseteq \mathscr{H}_0$. Now let $H \in \mathscr{H}_0$. Then there exists $\lambda \in \mathscr{H}$ such that $\lambda^0 \subseteq H$ and, since $\lambda^0 \in \mathscr{F}_0$, $H \in \mathscr{F}_0$. Thus $\mathscr{H}_0 \subseteq \mathscr{F}_0 \subseteq \mathscr{H}_0$ and we have equality. Alternatively we could appeal to 2.7(c).

(v) **$\mathscr{H}$ is prime.**
Follows from (iv) and 2.9(a).

Since $\mu \in \mathscr{F}$ we have $\mu \in \mathscr{H}$ and $\mathscr{H}$ is a prime saturated prefilter. By assumption, $\mathscr{H}$ is $\mathscr{D}$–Cauchy which means that :

$$
\begin{aligned}
c(\mathscr{F}) = c = c(\mathscr{H}) \quad & \leq \inf_{\sigma \in \mathscr{D}} \sup_{\lambda \in \mathscr{H}} \inf \sigma <\lambda> \\
& \leq \inf_{\sigma \in \mathscr{D}} \sup_{\nu \in \mathscr{F}} \inf \sigma <\nu>.
\end{aligned}
$$

We have therefore shown that $\mathscr{F}$ is $\mathscr{D}$–Cauchy and so μ is $\mathscr{D}$–precompact.

■

We have an α–level theorem for precompact fuzzy sets which we now present.

6.5 THEOREM

Let $(X, \mathscr{D})$ be a fuzzy uniform space and let $\mu \in I^X$. Then
(a) μ is $\mathscr{D}$–precompact $\Leftrightarrow \forall \alpha \in I_0$ μ_α is $\mathscr{D}^\alpha$–precompact.
(b) μ is $\mathscr{D}$–precompact $\Leftrightarrow \forall \alpha \in I_0$ μ^α is $\mathscr{D}^\alpha$–precompact.

PROOF

(a) Let μ be $\mathscr{D}$–precompact, $\alpha \in I_0$ and $\mu_\alpha \in \mathbb{F}$ with $\mathbb{F}$ an ultrafilter on X. Since $\alpha 1_{\mu_\alpha} \leq \mu$ and $\alpha 1_{\mu_\alpha} \in \mathbb{F}_\alpha$ we have $\mu \in \mathbb{F}_\alpha$. Furthermore $(\mathbb{F}_\alpha)_0 = \mathbb{F}$ (2.7(c)) which means that $\mathbb{F}_\alpha$ is prime (2.9(a)). Consequently, $\mathbb{F}_\alpha$ is $\mathscr{D}$–Cauchy and hence $\mathbb{F}$ is $\mathscr{D}^\alpha$–Cauchy (4.7). Since $\mathbb{F}$ is arbitrary we conclude that μ_α is $\mathscr{D}^\alpha$–precompact.

Conversely, suppose that $\mu \in \mathscr{F}$, $\mathscr{F}$ is a prime prefilter and $c(\mathscr{F}) = c > 0$. We show that $\mathscr{F}_0$ is $\mathscr{D}^c$–Cauchy and invoke 4.4. So let $\sigma^\alpha \in \mathscr{D}^c$ with $\sigma \in \mathscr{D}$ and $\alpha < c$. Choose β such that $\alpha < \beta < c$. Then : $\mu^\beta \subseteq \mu_\beta$ and μ_β is $\mathscr{D}^\beta$–precompact. Moreover, since $\mu \in \mathscr{F}$ and $\beta < c$, $\mu^\beta \in \mathscr{F}_0$ (2.19) and $\mathscr{F}_0$ is an ultrafilter ($\mathscr{F}$ is prime, 2.9(a)). It follows that $\mathscr{F}_0$ is $\mathscr{D}^\beta$–Cauchy. Now $\sigma^\alpha \in \mathscr{D}^\beta$, so $F \times F \subset \sigma^\alpha$ for some $F \in \mathscr{F}_0$ and we have shown that $\mathscr{F}_0$ is $\mathscr{D}^c$–Cauchy. Thus, by 4.4, $\mathscr{F}$ is $\mathscr{D}$–Cauchy.

(b) Let μ be $\mathscr{D}$–precompact and $\alpha \in I_0$. Since $\mu^\alpha \subseteq \mu_\alpha$ and μ_α is $\mathscr{D}^\alpha$–precompact, μ^α is $\mathscr{D}^\alpha$–precompact. (We note that if $\alpha = 1$ then $\mu^\alpha = \emptyset$ and $\emptyset$ is of course $\mathscr{D}^1$–precompact.)

Conversely, suppose that $\mu \in \mathscr{F}$, $\mathscr{F}$ is a prime prefilter and $c(\mathscr{F}) = c > 0$. Let $\sigma^\alpha \in \mathscr{D}^c$ with $\sigma \in \mathscr{D}$ and $\alpha < c$. Let $\alpha < \beta < c$. Then $\mu^\beta \in \mathscr{F}_0$, μ^β is $\mathscr{D}^\beta$–precompact and $\mathscr{F}_0$ is an ultrafilter. Therefore $\mathscr{F}_0$ is $\mathscr{D}^\beta$–Cauchy and, since $\sigma^\alpha \in \mathscr{D}^\beta$ we can find $F \in \mathscr{F}_0$ such that $F \times F \subseteq \sigma^\alpha$. We have shown that $\mathscr{F}_0$ is $\mathscr{D}^c$–Cauchy and hence $\mathscr{F}$ is $\mathscr{D}$–Cauchy.

■

A subset Y of a uniform space $(X,\mathbb{D})$ is $\mathbb{D}$–precompact iff for each $U \in \mathbb{D}$ there corresponds a finite subset $F \subseteq X$ such that $Y \subseteq \bigcup_{y \in F} U(y)$. We would expect a fuzzy subset $\mu \in I^X$ of a fuzzy uniform space $(X,\mathscr{D})$ to be $\mathscr{D}$–precompact iff for each $\sigma \in \mathscr{D}$ and each $\epsilon > 0$ there corresponds a finite subset $F \subseteq X$ such that $\mu \leq \sup_{y \in F} \sigma <y> + \epsilon$. The next theorem asserts that this is indeed the case.

6.6 THEOREM

> Let $(X,\mathscr{D})$ be a fuzzy uniform space and let $\mu \in I^X$. Then :
> μ is $\mathscr{D}$–precompact $\Leftrightarrow \forall \sigma \in \mathscr{D} \; \forall \epsilon > 0 \; \exists F \in \mathscr{P}_f(X): \mu \leq \sup_{y \in F} \sigma <y> + \epsilon$.

PROOF

Let $\psi \in \mathscr{D}$ be symmetric, let $\epsilon \in I_0$ and suppose that :

(i) $\quad \forall K \in \mathscr{P}_f(X) \; \exists z \in X : \mu(z) > \sup_{y \in K} \psi <y> + \epsilon.$

Choose $x_1 \in X$ then since $\{x_1\} \in \mathscr{P}_f(X)$:

$$\exists x_2 \in X : \mu(x_2) > \psi(x_2,x_1) + \epsilon.$$

Now $\{x_1,x_2\} \in \mathscr{P}_f(X)$ and so :

$$\exists x_3 \in X : \mu(x_3) > \sup_{i < 3} \psi(x_3,x_i) + \epsilon.$$

Continuing in this way we obtain a sequence $(x_n : n \in \mathbb{N})$ with the property :

(ii) $\quad \forall n \in \mathbb{N} \; \forall i < n \;\; \mu(x_n) > \psi(x_n,x_i) + \epsilon.$

For $n \in \mathbb{N}$ let $T_n = \{x_m : m \geq n\}$ and $\mathbb{K} = <\{T_n : n \in \mathbb{N}\}>$. Let $\mathscr{F} = (\mathbb{K}_1,\mu) = <\{1_K \wedge \mu : K \in \mathbb{K}\}> = <\{1_{T_n} \wedge \mu : n \in \mathbb{N}\}>$ and choose $\mathscr{G} \in \mathscr{P}_m(\mathscr{F})$ such that $c(\mathscr{G}) = c(\mathscr{F})$ (2.16). Then $\mathscr{G}$ is prime, $\mu \in \mathscr{G}$ and :

$$c(\mathscr{G}) = c(\mathscr{F}) = \inf_{n \in \mathbb{N}} \sup 1_{T_n} \wedge \mu = \inf_{n \in \mathbb{N}} \sup_{i \geq n} \mu(x_i).$$

If $n \in \mathbb{N}$ let $s_n = \sup_{i \geq n} \mu(x_i)$ then $c(\mathscr{G}) = \inf_{n \in \mathbb{N}} s_n$. Choose symmetric $\sigma \in \mathscr{D}$ such that $\sigma \circ \sigma \leq \psi + \frac{\epsilon}{2}$. Then by assumption :

$$c(\mathscr{G}) \leq \sup_{\nu \in \mathscr{G}} \inf \sigma <\nu> \leq \sup \inf_{m \in \mathbb{N}} \sigma <1_{T_m}>$$

$$\Rightarrow \inf_{n \in \mathbb{N}} s_n < \sup \inf_{m \in \mathbb{N}} \sigma <1_{T_m}> + \frac{\epsilon}{2}$$

$$\Rightarrow \exists\, x \in X \ \exists\, k \in \mathbb{N} : s_k < \inf_{m \in \mathbb{N}} \sigma <1_{T_m}>(x) + \frac{\epsilon}{2}$$

$$\Rightarrow \exists\, x \in X \ \exists\, k \in \mathbb{N} \ \forall\, m \in \mathbb{N} \ s_k < \sup_{i \geq m} \sigma(x,x_i) + \frac{\epsilon}{2}.$$

For this $x \in X$ and $k \in \mathbb{N}$ we have :

(iii) $\forall\, m \in \mathbb{N} \ \exists\, i \geq m : s_k < \sigma(x,x_i) + \frac{\epsilon}{2}$.

If we choose $m > k$ and invoke (iii) we obtain :

(iv) $\exists\, i > k : s_k < \sigma(x,x_i) + \frac{\epsilon}{2}$.

If we now choose $m > i$ and invoke (iii) we obtain :

(v) $\exists\, \ell > i : s_k < \sigma(x,x_\ell) + \frac{\epsilon}{2}$.

We therefore have $k < i < \ell$ and (from (iv), (v)) :

$$s_k < (\sigma(x,x_i) + \tfrac{\epsilon}{2}) \wedge (\sigma(x,x_\ell) + \tfrac{\epsilon}{2}) = \sigma(x,x_i) \wedge \sigma(x,x_\ell) + \tfrac{\epsilon}{2}$$
$$\leq (\sigma \circ \sigma)(x_i,x_\ell) + \tfrac{\epsilon}{2}$$
$$\leq (\psi(x_i,x_\ell) + \tfrac{\epsilon}{2}) + \tfrac{\epsilon}{2} = \psi(x_i,x_\ell) + \epsilon.$$

In other words :

$$\sup_{j \geq k} \mu(x_j) < \psi(x_i,x_\ell) + \epsilon.$$

From this we obtain :

$$\mu(x_\ell) < \psi(x_i,x_\ell) + \epsilon,$$

which contradicts (ii) and so (i) is false, which establishes the forward implication.

For the reverse implication we let $\mu \in \mathscr{G}$ with $\mathscr{G}$ a prime prefilter and show that $c(\mathscr{G}) \leq \inf_{\sigma \in \mathscr{D}} \sup_{\nu \in \mathscr{G}} \inf \sigma <\nu>$. To this end we let $\sigma \in \mathscr{D}$ and $\epsilon > o$ be arbitrary and show that :

(vi) $\exists\, y \in X\ \forall\, \nu \in \mathscr{G}\ c(\mathscr{G}) \leq \sigma <\nu>(y) + \epsilon$.

By assumption there exists $F \in \mathscr{P}_f(x)$ with $\mu \leq \sup_{y \in F} \sigma<y> + \epsilon$. Thus :

$$\mu \leq \sup_{y \in F} (\sigma <y> + \epsilon) \wedge 1_X := \lambda.$$

Since $\mu \in \mathscr{G}$ we know that $\lambda \in \mathscr{G}$ and hence, since $\mathscr{G}$ is prime and F is finite there exists $y \in F$ such that $(\sigma <y>+\epsilon) \wedge 1_X \in \mathscr{G}$. Thus if $\nu \in \mathscr{G}$ is arbitrary,

$$\nu \wedge (\sigma <y>+\epsilon) \wedge 1_X = (\sigma <y>+\epsilon) \wedge \nu \in \mathscr{G}.$$

It follows therefore that

$$c(\mathscr{G}) \leq \sup (\sigma <y>+\epsilon) \wedge \nu \leq (\sup \sigma <y> \wedge \nu) + \epsilon = \sigma <\nu>(y) + \epsilon,$$

which establishes (vi) and hence also the reverse implication.

■

Equipped with our characterisations of precompactness in F.UNIF we establish the analogues of the theorems which describe the properties of precompactness in UNIF. In other words, we build a theory of precompactness in fuzzy uniform spaces which generalises the standard theory.

In UNIF, precompact sets are stable with respect to the formation of finite unions and subsets. Compact and relatively compact sets are precompact. We establish the analagous results in F.UNIF.

6.7 THEOREM

Let $(X, \mathscr{D})$ be a fuzzy uniform space with $\nu, \mu, \nu_i \in I^X$. Then :

(a)	$\nu \leq \mu$ and μ is $\mathscr{D}$–precompact	$\Rightarrow \nu$ is $\mathscr{D}$–precompact.
(b)	$\forall\, i \in [n]\ \nu_i$ is $\mathscr{D}$–precompact	$\Rightarrow \sup\limits_{i \in [n]} \nu_i$ is $\mathscr{D}$–precompact.
(c)	μ is relatively $\mathscr{D}$–compact	$\Rightarrow \mu$ is $\mathscr{D}$–precompact.
(d)	μ is $\mathscr{D}$–compact	$\Rightarrow \mu$ is $\mathscr{D}$–precompact.

PROOF

(a) If $\nu \in \mathscr{F}$ with $\mathscr{F}$ a prime filter then $\mu \in \mathscr{F}$ and so $\mathscr{F}$ is $\mathscr{D}$–Cauchy.

(b) If $\sup\limits_{i \in [n]} \nu_i \in \mathscr{F}$ with $\mathscr{F}$ a prime filter then, since $\mathscr{F}$ is prime, $\nu_i \in \mathscr{F}$ for some $i \in [n]$. Consequently $\mathscr{F}$ is $\mathscr{D}$–Cauchy.

(c) Let $\mu \in \mathscr{F}$ with $\mathscr{F}$ a prime prefilter. Then :

$$\begin{aligned}
\forall\, \sigma \in \mathscr{D}\ \sigma<\mathscr{F}> \subseteq \mathscr{F} \quad &\Rightarrow \forall\, \sigma \in \mathscr{D}\quad \text{Adh}\ \mathscr{F} \leq \text{Adh}\ \sigma<\mathscr{F}> \\
&\Rightarrow \forall\, \sigma \in \mathscr{D}\quad \sup \text{Adh}\ \mathscr{F} \leq \sup \text{Adh}\ \sigma<\mathscr{F}> \\
&\Rightarrow \sup \text{Adh}\ \mathscr{F} \leq \inf_{\sigma \in \mathscr{D}} \sup \text{Adh}\ \sigma<\mathscr{F}>.
\end{aligned}$$

Since $\mathscr{F}$ is relatively $\mathscr{D}$–compact,

$$c(\mathscr{F}) \leq \sup \text{Adh}\ \mathscr{F} \leq \inf_{\sigma \in \mathscr{D}} \sup \text{Adh}\ \sigma<\mathscr{F}>.$$

Thus $\mathscr{F}$ is $\mathscr{D}$–precompact.

(d) If μ is $\mathscr{D}$–compact then μ is relatively $\mathscr{D}$–compact ($\sup \mu \wedge \text{Adh}\ \mathscr{F} \leq \sup \text{Adh}\ \mathscr{F}$) and the result follows.

∎

In UNIF, the closure of a precompact set is precompact. We now show that the corresponding statement in F.UNIF is also true.

6.8 THEOREM

> Let $(X, \mathscr{D})$ be a fuzzy uniform space with $\mu \in I^X$. Then :
> μ is $\mathscr{D}$–precompact $\Rightarrow \bar{\mu}$ is $\mathscr{D}$–precompact.

PROOF

Let $\sigma \in \mathscr{D}$ be symmetric and let $\epsilon > 0$. Choose symmetric $\psi \in \mathscr{D}$ such that

$$\psi \circ \psi \leq \sigma + \tfrac{\epsilon}{2}.$$

Since μ is $\mathscr{D}$–precompact there exists $F \in \mathscr{P}_f(X)$ such that

$$\mu \leq \sup_{y \in F} \psi\langle y\rangle + \tfrac{\epsilon}{2}.$$

For arbitrary $z \in X$ we have :

$$\begin{aligned}
\bar{\mu}(z) &\leq \psi\langle\mu\rangle(z) \\
&= \sup_{x \in X} \mu(x) \wedge \psi(x,z) \\
&\leq \sup_{x \in X} \Big(\sup_{y \in F} \psi(x,y) + \tfrac{\epsilon}{2} \Big) \wedge \psi(x,z) \\
&\leq \sup_{y \in F} \sup_{x \in X} \psi(y,x) \wedge \psi(x,z) + \tfrac{\epsilon}{2} \\
&= \sup_{y \in F} \psi \circ \psi(y,z) + \tfrac{\epsilon}{2} \\
&\leq \sup_{y \in F} \Big(\sigma(y,z) + \tfrac{\epsilon}{2}\Big) + \tfrac{\epsilon}{2} \\
&= \sup_{y \in F} \sigma\langle y\rangle(z) + \epsilon.
\end{aligned}$$

Thus $\bar{\mu} \leq \sup_{y \in F} \sigma\langle y\rangle + \epsilon$ and hence $\bar{\mu}$ is $\mathscr{D}$–precompact.

■

Uniformly continuous images of precompact subsets of a uniform space are precompact and the next theorem says that the same thing happens in fuzzy uniform spaces.

6.9 THEOREM

> Let $(X, \mathscr{D})$ and $(Y, \mathscr{E})$ be fuzzy uniform spaces and let $f : X \to Y$ be uniformly continuous. If $\mu \in I^X$ then :
>
> $$\mu \text{ is } \mathscr{D}\text{–precompact} \Rightarrow f[\mu] \text{ is } \mathscr{E}\text{–precompact.}$$

PROOF

First note that for $\alpha \in I_0$ we have :

$f[\mu]^\alpha = f[\mu^\alpha]$.

$$
\begin{aligned}
y \in f[\mu]^\alpha \quad & \Leftrightarrow f[\mu](y) = \sup_{x \in f^{-1}[\{y\}]} \mu(x) > \alpha \\
& \Leftrightarrow \exists\ x \in f^{-1}[\{y\}] \cap \mu^\alpha \\
& \Leftrightarrow \exists\ x \in \mu^\alpha : f(x) = y \\
& \Leftrightarrow y \in f[\mu^\alpha].
\end{aligned}
$$

$$
\begin{aligned}
\text{Now } \mu \text{ is } \mathscr{D}\text{–precompact} \quad & \Leftrightarrow \forall\ \alpha \in I_0\ \mu^\alpha \text{ is } \mathscr{D}^\alpha\text{–precompact (6.5(b))} \\
& \Rightarrow \forall\ \alpha \in I_0\ f[\mu^\alpha] \text{ is } \mathscr{E}^\alpha\text{–precompact (3.22(b))} \\
& \Leftrightarrow \forall\ \alpha \in I_0\ f[\mu]^\alpha \text{ is } \mathscr{E}^\alpha\text{–precompact.} \\
& \Leftrightarrow f[\mu] \text{ is } \mathscr{E}\text{–precompact. (6.5(b))}
\end{aligned}
$$

■

In UNIF, products of precompact sets are precompact and a subset of a product space is precompact iff each projection is precompact. The corresponding statements in F.UNIF are also true.

6.10 THEOREM

> Let $((X_j, \mathscr{D}(j)) : j \in J)$ be a family of fuzzy uniform spaces with $\mu(j) \in I^{X_j}$ for each $j \in J$. Let $X = \prod_{j \in J} X_j$ and $\mathscr{D} = \prod_{j \in J} \mathscr{D}(j)$, then
>
> (a) $\forall\ j \in J\ \mu(j)$ is $\mathscr{D}(j)$–precompact $\Rightarrow \prod_{j \in J} \mu(j)$ is $\mathscr{D}$–precompact.
>
> (b) $\mu \in I^X$ is $\mathscr{D}$–precompact $\Leftrightarrow \forall\ j \in J\ p_j[\mu]$ is $\mathscr{D}(j)$–precompact.

PROOF

(a) Let $\mathscr{F}$ be a prime prefilter on X with $\mu := \prod_{j \in J} \mu(j) \in \mathscr{F}$. Let $j \in J$ and $\mathscr{F}_j := \langle p_j[\mathscr{F}] \rangle$. Then $\mathscr{F}_j$ is prime (2.23(f)) and $p_j[\mu] \leq \mu(j)$. Thus since $p_j[\mu] \in \mathscr{F}_j$, $\mu(j) \in \mathscr{F}_j$ and so $\mathscr{F}_j$ is $\mathscr{D}(j)$–Cauchy. Since j is arbitrary we have :

$\forall\, j \in J\ \mathscr{F}_j$ is $\mathscr{D}(j)$–Cauchy $\Rightarrow \mathscr{F}$ is $\mathscr{D}$–Cauchy (4.21).

Thus we have shown that μ is $\mathscr{D}$–precompact.

(b) If $\mu \in I^X$ is $\mathscr{D}$–precompact then, since each p_j is uniformly continuous, each $p_j[\mu]$ is $\mathscr{D}(j)$ precompact.

For the converse we note that :

$$\forall\, j \in J\ \mu \leq p_j^{-1}[p_j[\mu]] \Rightarrow \mu \leq \inf_{j \in J} p_j^{-1}[p_j[\mu]] = \prod_{j \in J} p_j[\mu].$$

Now $\prod_{j \in J} p_j[\mu]$ is $\mathscr{D}$–precompact by (a) and hence so is μ (6.7(a)).

■

6.11 REMARK

It is worth remarking that we do **not** have :

$$\prod_{j \in J} \mu(j) \text{ is } \mathscr{D}\text{–precompact} \Rightarrow \forall\, j \in J\ \mu(j) \text{ is } \mathscr{D}(j)\text{–precompact}.$$

To see this, suppose that for some $k \in J$ $\mu(k) = 0 = 0\, 1_{X_k}$. Then $\mu(k)$ is $\mathscr{D}$ (k)–precompact since for no prime prefilter $\mathscr{F}_k$ on X_k do we have $\mu(k) \in \mathscr{F}_k$. Consequently :

$$\prod_{j \in J} \mu(j) = \inf_{j \in J} p_j^{-1}[\mu(j)] \leq p_k^{-1}[\mu(k)] = \mu(k) \circ p_k = 0 = 0\, 1_X.$$

So $\prod_{j \in J} \mu(j)$ is $\mathscr{D}$–precompact if for any $k \in J$ $\mu(k) = 0$. We can therefore construct a counter example with some $\mu(j)$ not $\mathscr{D}(j)$–precompact ($j \neq k$) while $\mu(k) = 0$.

CHAPTER 7

BOUNDEDNESS

If $(X,\mathbb{D})$ is a uniform space and $B \subseteq X$ we say that :

$$B \text{ is } \mathbf{\mathbb{D}\text{–bounded}} \Leftrightarrow \forall\, U \in \mathbb{D}\ \exists\, K \in \mathscr{P}_f(X)\ \exists\, n \in \mathbb{N} : B \subseteq U^n(K) = \bigcup_{x \in K} U^n(x).$$

It follows immediately from the definitions that a $\mathbb{D}$–precompact set is $\mathbb{D}$–bounded and, as we compare the two notions, we might suspect that $\mathbb{D}$–boundedness can also be described in terms of filters.

If $\mathbb{F}$ is an ultrafilter on X we shall say that :

$$\mathbb{F} \text{ is } \mathbf{weak\ \mathbb{D}\text{–Cauchy}} \Leftrightarrow \forall\, U \in \mathbb{D}\ \exists\, n \in \mathbb{N}\ \exists\, F \in \mathbb{F} : F \times F \subseteq U^n.$$

Clearly, a $\mathbb{D}$–Cauchy ultrafilter is weak $\mathbb{D}$–Cauchy and we show now that weak $\mathbb{D}$–Cauchy ultrafilters are described by an intersection condition similar to the one describing $\mathbb{D}$–Cauchy ultrafilters in Theorem 4.1.

7.1 THEOREM

Let $(X,\mathbb{D})$ be a uniform space and $\mathbb{F}$ an ultrafilter on X. Then :

$$\mathbb{F} \text{ is weak } \mathbb{D}\text{–Cauchy} \Leftrightarrow \forall\, U \in \mathbb{D}\ \exists\, n \in \mathbb{N} : \bigcap_{F \in \mathbb{F}} U^n(F) \neq \emptyset.$$

PROOF

Let $\mathbb{F}$ be weak $\mathbb{D}$–Cauchy and let $U \in \mathbb{D}$. There exists $n \in \mathbb{N}$ and $G \in \mathbb{F}$ such that $G \times G \subseteq U^n$. It follows, as in 4.1, that $G \subseteq \bigcap_{F \in \mathbb{F}} U^n(F)$ and hence that

$$\bigcap_{F \in \mathbb{F}} U^n(F) \neq \emptyset.$$

For the converse, if $U \in \mathbb{D}$ let :

$$U' = \bigcup_{n \in \mathbb{N}} \bigcap_{F \in \mathbb{F}} U^n(F),$$

$\mathbb{F}' = <\{U' : U \in \mathbb{D}\}>$ then :

(a) **$\mathbb{F}'$ is a filter on X.**

As in 4.1(a) we show that $(U \cap V)' \subseteq U' \cap V'$ for $U, V \in \mathbb{D}$.

(b) **If $U \in \mathbb{D}$ is symmetric then $x \in U' \Leftrightarrow \exists\, n \in \mathbb{N} : U^n(x) \in \mathbb{F}$.**

$$\begin{aligned} x \in U' &\Leftrightarrow \exists\, n \in \mathbb{N}\ \forall\, F \in \mathbb{F}\ \ x \in U^n(F) \\ &\Leftrightarrow \exists\, n \in \mathbb{N}\ \ U^n(x) \in \mathbb{F} \quad \text{(as in 4.1(b))}. \end{aligned}$$

(c) **If $U \in \mathbb{D}$ is symmetric then $\exists\, n \in \mathbb{N} : U' \times U' \subseteq U^{2n}$.**

$$\begin{aligned} (x,y) \in U' \times U' &\Leftrightarrow \exists\, k \in \mathbb{N} : U^k(x) \in \mathbb{F} \text{ and } \exists\, m \in \mathbb{N} : U^m(y) \in \mathbb{F} \\ &\Leftrightarrow \exists\, n \in \mathbb{N} : U^n(x) \in \mathbb{F} \text{ and } U^n(y) \in \mathbb{F}\ (n = \max(k,m)) \\ &\Rightarrow \exists\, n \in \mathbb{N} : U^n(x) \cap U^n(y) \neq \emptyset \\ &\Leftrightarrow \exists\, n \in \mathbb{N} : (x,y) \in U^{2n}. \end{aligned}$$

(d) **$\mathbb{F}'$ is weak $\mathbb{D}$–Cauchy.**

Let $U \in \mathbb{D}$ and symmetric $V \in \mathbb{D}$ with $V^2 \subseteq U$. Then $V' \in \mathbb{F}'$ and by (c) there exists $n \in \mathbb{N}$ such that $V' \times V' \subseteq V^{2n} \subseteq U^n$.

(e) **$\mathbb{F}' \subseteq \mathbb{F}$.**

If $G \in \mathbb{F}'$ then $U' \subseteq G$ for some $U \in \mathbb{D}$. Choose symmetric $V \in \mathbb{D}$ with $V^2 \subseteq U$ and choose $x \in V'$. Then $V^n(x) \in \mathbb{F}$ for some $n \in \mathbb{N}$ ((b)). Let $y \in V^n(x)$ and $F \in \mathbb{F}$. Then $V^n(x) \cap F \neq \emptyset$ so there exists $z \in V^n(x) \cap F$. Thus $(x,y) \in V^n$, $(x,z) \in V^n$ and so $(z,y) \in V^{2n}$. Since y is arbitrary, $V^n(x) \subseteq V^{2n}(F)$ and since F is arbitrary we have :

$$V^n(x) \subseteq \bigcap_{F \in \mathbb{F}} V^{2n}(F) \subseteq \bigcap_{F \in \mathbb{F}} U^n(F) \subseteq U' \subseteq G.$$

Since $V^n(x) \in \mathbb{F}$ we conclude that $G \in \mathbb{F}$.

Finally, it follows from (d) and (e) that $\mathbb{F}$ is weak $\mathbb{D}$–Cauchy.

■

We show that boundedness can be described in terms of weak–Cauchy filters.

7.2 THEOREM

> Let $(X,\mathbb{D})$ be a uniform space and $B \subseteq X$. Then :
> B is $\mathbb{D}$–bounded iff every ultrafilter $\mathbb{F}$ with $B \in \mathbb{F}$ is weak $\mathbb{D}$–Cauchy.

PROOF

Let B be $\mathbb{D}$–bounded and $\mathbb{F}$ an ultrafilter on X with $B \in \mathbb{F}$. Choose symmetric $V \in \mathbb{D}$ with $V^2 \subseteq U$. Since B is $\mathbb{D}$–bounded there exists $n \in \mathbb{N}$ and $K \in \mathscr{P}_f(X)$ such that $B \subseteq V^n(K)$. Since $B \in \mathbb{F}$ we have $V^n(K) \in \mathbb{F}$ and furthermore, since K is finite and $\mathbb{F}$ is an ultrafilter there exists $x \in K$ such that $V^n(x) \in \mathbb{F}$. If $F \in \mathbb{F}$ then $V^n(F) \in \mathbb{F}$ and hence $V^n(x) \cap V^n(F) \neq \emptyset$. If $z \in V^n(x) \cap V^n(F)$ then $(x,z) \in V^n$ and $(y,z) \in V^n$ for some $y \in F$. It follows that $(x,y) \in V^{2n} \subseteq U^n$ and so $x \in U^n(F)$. Since F is arbitrary, we have shown that $x \in \bigcap_{F \in \mathbb{F}} U^n(F)$ and consequently that $\bigcap_{F \in \mathbb{F}} U^n(F) \neq \emptyset$. Thus, by 7.1, $\mathbb{F}$ is weak $\mathbb{D}$–Cauchy.

Conversely, suppose that every ultrafilter $\mathbb{F}$ with $B \in \mathbb{F}$ is weak $\mathbb{D}$–Cauchy but B is not $\mathbb{D}$–bounded. Then

(i) $\exists$ (symmetric) $U \in \mathbb{D}$ $\forall\, n \in \mathbb{N}$ $\forall\, K \in \mathscr{P}_f(X)$ $B \setminus U^n(K) \neq \emptyset$.

Let $x_1 \in B$. Then, since $\{x_1\} \in \mathscr{P}_f(X)$ there exists $x_2 \in B \setminus U(\{x_1\})$.

Again, since $\{x_1,x_2\} \in \mathscr{P}_f(X)$ there exists $x_3 \in B \setminus U^2(\{x_1,x_2\})$. In this way we obtain a sequence $(x_n : n \in \mathbb{N})$ such that :

(ii) $\forall\, n \in \mathbb{N}$ $x_{n+1} \notin U^n(\{x_1,x_2,\cdots,x_n\})$.

For $n \in \mathbb{N}$ let $T_n = \{x_m : m \geq n\}$ and let $\mathbb{F}$ be an ultrafilter such that $\mathbb{F} \supseteq \{T_n : n \in \mathbb{N}\}$.

Since each $T_n \subseteq B$ we have $B \in \mathbb{F}$ and so $\mathbb{F}$ is weak $\mathbb{D}$–Cauchy. Thus, for some $n \in \mathbb{N}$ we have $\bigcap_{F \in \mathbb{F}} U^n(F) \neq \emptyset$ and so there exists $x \in X$ such that

$$\forall\, m \in \mathbb{N} \quad x \in U^n(T_m).$$

In other words :

(iii) $\forall\, m \in \mathbb{N}\, \exists\, \ell \geq m : (x_\ell, x) \in U^n$.

It follows from (iii) that

(iv) $\exists\, \ell > 2n : (x_\ell, x) \in U^n$ and

(v) $\exists\, k > \ell : (x_k, x) \in U^n$

From (iv) and (v) we see that $(x_k, x_\ell) \in U^{2n}$ and so :

$$x_k \in U^{2n}(x_\ell) \subseteq U^{k-1}(x_\ell) \subseteq U^{k-1}(\{x_1, x_2, \cdots, x_{k-1}\})$$

which contradicts (ii).

■

We use Theorem 7.2 to prove some elementary properties of $\mathbb{D}$–bounded sets.

7.3 THEOREM

Let $(X, \mathbb{D})$ be a uniform space with $A, B, B_i \subseteq X$ and $n \in \mathbb{N}$. Then :

(a)	$A \subseteq B$, B is $\mathbb{D}$–bounded	$\Rightarrow$ A is $\mathbb{D}$–bounded.
(b)	$\forall\, i \in [n]$ B_i is $\mathbb{D}$–bounded	$\Rightarrow$ $\bigcup_{i \in [n]} B_i$ is $\mathbb{D}$–bounded.
(c)	B is $\mathbb{D}$–precompact	$\Rightarrow$ B is $\mathbb{D}$–bounded.
(d)	B is $\mathbb{D}$–bounded	$\Rightarrow$ $\bar{B}$ is $\mathbb{D}$–bounded.

PROOF

In all cases we show that an appropriate ultrafilter is weak $\mathbb{D}$–Cauchy and then we appeal to 7.2.

(a) If $\mathbb{F}$ is an ultrafilter with $A \in \mathbb{F}$ then $B \in \mathbb{F}$ and so $\mathbb{F}$ is weak $\mathbb{D}$–Cauchy.

(b) If $\mathbb{F}$ is an ultrafilter with $\bigcup_{i \in [n]} B_i \in \mathbb{F}$ then $B_k \in \mathbb{F}$ for some $k \in [n]$. Consequently $\mathbb{F}$ is weak $\mathbb{D}$–Cauchy.

(c) If $\mathbb{F}$ is an ultrafilter with $B \in \mathbb{F}$ then $\mathbb{F}$ is $\mathbb{D}$–Cauchy and hence $\mathbb{F}$ is weak $\mathbb{D}$–Cauchy.

(d) Let $\mathbb{F}$ be an ultrafilter with $\bar{B} \in \mathbb{F}$. Let $n \in \mathbb{N}$, $U \in \mathbb{D}$ and $F \in \mathbb{F}$. Choose symmetric $V \in \mathbb{D}$ with $V \subseteq U$. Then $\bar{B} \subseteq V^n(B)$ and so $V^n(B) \in \mathbb{F}$. Thus $V^n(B) \cap F \neq \emptyset$ and so there exists $x \in F$ and $y \in B$ such that $(x,y) \in V^n$. Since $y \in V^n(F)$ we have $V^n(F) \cap B \neq \emptyset$ and hence $U^n(F) \cap B \neq \emptyset$.

We have therefore :

(i) $\forall\, n \in \mathbb{N}\ \forall\, U \in \mathbb{D}\ \forall\, F \in \mathbb{F}\ \ B \cap U^n(F) \neq \emptyset$.

The set $\{B \cap U^n(F) : U \in \mathbb{D}, F \in \mathbb{F}, n \in \mathbb{N}\}$ has the finite intersection property and so we can choose an ultrafilter $\mathbb{K}$ such that :

$$\{B \cap U^n(F) : U \in \mathbb{D}, F \in \mathbb{F}, n \in \mathbb{N}\} \subseteq \mathbb{K}.$$

Since $B \in \mathbb{K}$, $\mathbb{K}$ is weak $\mathbb{D}$–Cauchy and hence

$$\forall\, U \in \mathbb{D}\ \exists\, n \in \mathbb{N} : \bigcap_{K \in \mathbb{K}} U^n(K) \neq \emptyset.$$ Thus

(ii) $\forall\, U \in \mathbb{D}\ \exists\, n \in \mathbb{N}\ \exists\, x \in X\ \forall\, K \in \mathbb{K}\ \ x \in U^n(K)$.

Let $U \in \mathbb{D}$ be symmetric. Then by virtue of (ii) there exists $n \in \mathbb{N}$ and $x \in X$ such that :

$$\forall\, F \in \mathbb{F}\ \ x \in U^n(B \cap U^n(F)).$$ Thus

$$\forall\, F \in \mathbb{F}\ \exists\, y \in B \cap U^n(F) : (y,x) \in U^n$$ and so

$$\forall\, F \in \mathbb{F}\ \exists\, y \in B : (y,x) \in U^n\ \exists\, z \in F : (z,y) \in U^n.$$

Consequently :

$$\forall\, F \in \mathbb{F}\ \exists\ \ z \in F : (x,z) \in U^{2n}$$ and so :

$$x \in \bigcap_{F \in \mathbb{F}} U^{2n}(F).$$

We have shown that for some $m \in \mathbb{N}$ $\bigcap_{F \in \mathbb{F}} U^m(F) \neq \emptyset$ and this means that $\mathbb{F}$ is weak $\mathbb{D}$–Cauchy. Thus $\bar{B}$ is $\mathbb{D}$–bounded.

■

Our next task is to extend these notions to the fuzzy uniform space setting. If $(X, \mathscr{D})$ is a fuzzy uniform space and $\mathscr{F}$ is a prime prefilter on X we shall say that :

$$\mathscr{F} \text{ is } \textbf{weak } \mathscr{D}\textbf{–Cauchy} \Leftrightarrow c(\mathscr{F}) \leq \inf_{\sigma \in \mathscr{D}} \sup_{n \in \mathbb{N}} \sup_{\nu \in \mathscr{F}} \inf \sigma^n\langle\nu\rangle.$$

It follows immediately from the definitions that a $\mathscr{D}$–Cauchy prime prefilter is necessarily weak $\mathscr{D}$–Cauchy. Before proceeding we establish the α–level theorem for weak $\mathscr{D}$–Cauchy prime prefilters.

7.4 THEOREM

Let $(X, \mathscr{D})$ be a fuzzy uniform space and let $\mathscr{F}$ be a prime prefilter on X with $c(\mathscr{F}) = c$. Then :

$$\mathscr{F} \text{ is weak } \mathscr{D}\text{–Cauchy} \Leftrightarrow \mathscr{F}_0 \text{ is weak } \mathscr{D}^c\text{–Cauchy.}$$

PROOF

$\mathscr{F}$ is weak $\mathscr{D}$–Cauchy

$\Leftrightarrow c = c(\mathscr{F}) \leq \inf_{\sigma \in \mathscr{D}} \sup_{n \in \mathbb{N}} \sup_{\nu \in \mathscr{F}} \inf \sigma^n\langle\nu\rangle$

$\Leftrightarrow \forall\ \alpha < c\ \forall\ \sigma \in \mathscr{D}\ \exists\ x \in X\ \exists\ n \in \mathbb{N}\ \forall\ \nu \in \mathscr{F}\ \alpha < \sigma^n\langle\nu\rangle(x) = \sup \nu \wedge \sigma^n\langle x\rangle$

$\Leftrightarrow \forall\ \alpha < c\ \forall\ \sigma \in \mathscr{D}\ \exists\ x \in X\ \exists\ n \in \mathbb{N}\ \forall\ \nu \in \mathscr{F}\ \exists\ y \in \nu^{\alpha} : \sigma^n(y,x) > \alpha$

$\Leftrightarrow \forall\ \alpha < c\ \forall\ \sigma \in \mathscr{D}\ \exists\ x \in X\ \exists\ n \in \mathbb{N}\ \forall\ \nu \in \mathscr{F}\ x \in (\sigma^{n})^{\alpha}(\nu^{\alpha})$

$\Leftrightarrow \forall\ \alpha < c\ \forall\ \sigma \in \mathscr{D}\ \exists\ n \in \mathbb{N} : \bigcap_{\nu \in \mathscr{F}} (\sigma^{n})^{\alpha}(\nu^{\alpha}) = \bigcap_{\nu \in \mathscr{F}} (\sigma^{\alpha})^{n}(\nu^{\alpha}) \neq \emptyset.$ (3.4(i))

$\Leftrightarrow \forall\ U \in \mathscr{D}^c\ \exists\ n \in \mathbb{N} : \bigcap_{F \in \mathscr{F}_0} U^n(F) \neq \emptyset.$ (2.19)

$\Leftrightarrow \mathscr{F}_0$ is weak $\mathscr{D}^c$–Cauchy.

■

Our definition of a weak $\mathscr{D}$–Cauchy prime prefilter is a modification of the definition of a $\mathscr{D}$–Cauchy prefilter and, since prime prefilters are $\mathscr{D}$–Cauchy iff they are strong $\mathscr{D}$–Cauchy we expect there to be a characterisation of weak $\mathscr{D}$–Cauchy prime prefilters which resembles the definition of a strong $\mathscr{D}$–Cauchy prefilter.

7.5 THEOREM

> Let $(X, \mathscr{D})$ be a fuzzy uniform space and let $\mathscr{F}$ be a prime prefilter on X. Then
> $\mathscr{F}$ is weak $\mathscr{D}$–Cauchy $\Leftrightarrow \forall\ \sigma \in \mathscr{D}\ \forall\ \epsilon \in I_0\ \exists\ \nu \in \mathscr{F}\ \exists\ n \in \mathbb{N}$: $\nu \times \nu \leq \sigma^n + \epsilon$.

PROOF

Let $\mathscr{F}$ be a weak $\mathscr{D}$–Cauchy prime prefilter, with $c(\mathscr{F}) = c$, $\sigma \in \mathscr{D}$ and $\epsilon \in I_0$ and define $\alpha = c - \frac{\epsilon}{2}$, $\beta = (c + \frac{\epsilon}{2}) \wedge 1$. Since $\overset{\alpha}{\sigma} \in \mathscr{D}^c$ and $\mathscr{F}_0$ is weak $\mathscr{D}^c$–Cauchy, there exists $n \in \mathbb{N}$ and $F \in \mathscr{F}_0$ such that $F \times F \subseteq (\overset{\alpha}{\sigma})^n$. Let $\nu = \beta\, 1_F$. Then $\nu \in \mathscr{F}$ (2.7(d)) or (b) and if $(x,y) \in F \times F$ then :

$$(\nu \times \nu)(x,y) = \nu(x) \wedge \nu(y) = \beta \leq c + \frac{\epsilon}{2} = \alpha + \epsilon \leq \sigma^n(x,y) + \epsilon.$$

If $(x,y) \notin F \times F$ then $(\nu \times \nu)(x,y) = 0$.

Thus $\nu \times \nu \leq \sigma^n + \epsilon$.

To prove the converse we show that $\mathscr{F}_0$ is weak $\mathscr{D}^c$–Cauchy. To this end let $U \in \mathscr{D}^c$. Then $\overset{\alpha}{\sigma} = U$ for some $\sigma \in \mathscr{D}$ and $\alpha < c$. Let $\epsilon = c - \alpha$. Then there exists $\nu \in \mathscr{F}$ and $n \in \mathbb{N}$ such that $\nu \times \nu \leq \sigma^n + \frac{\epsilon}{2}$. If we let $\beta = \frac{c+\alpha}{2}$ then $\overset{\beta}{\nu} \in \mathscr{F}_0 (\beta < c$, 2.19) and moreover :

$$\begin{aligned} (x,y) \in \overset{\beta}{\nu} \times \overset{\beta}{\nu} \quad & \Leftrightarrow \nu(x) \wedge \nu(y) = (\nu \times \nu)(x,y) > \beta \\ & \Rightarrow \sigma^n(x,y) + \frac{\epsilon}{2} > \beta \\ & \Leftrightarrow \sigma^n(x,y) > \beta - \frac{\epsilon}{2} = \alpha. \end{aligned}$$

Thus $\overset{\beta}{\nu} \times \overset{\beta}{\nu} \subseteq \overset{\alpha}{(\sigma^n)} = (\overset{\alpha}{\sigma})^n = U^n$ which means that $\mathscr{F}_0$ is weak $\mathscr{D}^c$–Cauchy and consequently $\mathscr{F}$ is weak $\mathscr{D}$–Cauchy (7.4).

■

We check that the definition is a good extension :

7.6. THEOREM

> Let $(X, \mathbb{D})$ be a uniform space and let $\mathbb{F}$ be an ultrafilter on X. Then
> $\mathbb{F}$ is weak $\mathbb{D}$–Cauchy $\Leftrightarrow$ $\mathbb{F}^1$ is weak $\mathbb{D}^1$-Cauchy.

PROOF

$\mathbb{F}^1$ is weak $\mathbb{D}^1$–Cauchy $\Leftrightarrow (\mathbb{F}^1)_0$ is weak $(\mathbb{D}^1)^1$ –Cauchy $(c(\mathbb{F}^1) = 1,\ 7.4)$

$\Leftrightarrow \mathbb{F}$ is weak $\mathbb{D}$–Cauchy.

See the proof of 4.6 for the details.

■

The uniformly continuous image of a prime Cauchy prefilter is again prime Cauchy and we show that the same is true of prime weak Cauchy prefilters.

7.7 THEOREM

> Let $(X, \mathscr{D})$, $(Y, \mathscr{E})$ be fuzzy uniform spaces, let f: $(X, \mathscr{D}) \to (Y, \mathscr{E})$ be uniformly continuous and let $\mathscr{F}$ be a prime weak $\mathscr{D}$–Cauchy prefilter on X. Then $<f[\mathscr{F}]>$ is a prime weak $\mathscr{E}$–Cauchy prefilter on Y.

PROOF

Let $\mathscr{G} = <f[\mathscr{F}]>$. Then $\mathscr{G}$ is a prime prefilter (2.23 (a),(f)) and $c(\mathscr{G}) = c(\mathscr{F}) := c$ (2.23(c)). We show that $\mathscr{G}_0$ is weak $\mathscr{E}^c$ –Cauchy and to this end we let $\psi^{\alpha} \in \mathscr{E}^c$ with $\psi \in \mathscr{E}$ and $\alpha < c$. If we let $\sigma = (f{\times}f)^{-1}[\psi]$ then $\sigma \in \mathscr{D}$ (f is uniformly continuous) and so $\sigma^{\alpha} \in \mathscr{D}^c$. Since $\mathscr{F}_0$ is weak $\mathscr{D}^c$ –Cauchy there exists $n \in \mathbb{N}$ such that $\bigcap_{F \in \mathscr{F}_0} (\sigma^{\alpha})^n(F) \neq \emptyset$. Let $x \in \bigcap_{F \in \mathscr{F}_0} (\sigma^{\alpha})^n(F)$. We show that $f(x) \in \bigcap_{G \in \mathscr{G}_0} (\psi^{\alpha})^n(G)$. To do this let $G \in \mathscr{G}_0$. Then, since $\mathscr{G}_0 = \mathscr{G}_{\alpha}$ (2.9(b),2.7(a)) there exists $\nu \in \mathscr{F}$ such that $f[\nu]^{\alpha} = f[\nu^{\alpha}] \subseteq G$. Since $\nu^{\alpha} \in \mathscr{F}_0$ (2.19) it follows that $x \in (\sigma^{\alpha})^n(\nu^{\alpha})$ and so $(y,x) \in (\sigma^{\alpha})^n$ for some $y \in \nu^{\alpha}$. Consequently $\sigma^n(y,x) > \alpha$. Now :

$\sigma^n(y,x) = \sup\{\sigma(y,y_1) \wedge \sigma(y_1,y_2) \wedge \cdots \wedge \sigma(y_{n-1},x) : \underline{y} = (y_1,y_2,\cdots y_{n-1}) \in X^{n-1}\} > \alpha$

$\Rightarrow \exists\ \underline{y} \in X^{n-1}$: $\sigma(y,y_1) \wedge \sigma(y_1,y_2) \wedge \cdots \wedge \sigma(y_{n-1},x) > \alpha$

$\Rightarrow \exists\ \underline{y} \in X^{n-1}$: $\psi(f(y), f(y_1)) \wedge \psi(f(y_1), f(y_2)) \wedge \cdots \wedge \psi(f(y_{n-1}), f(x)) > \alpha$

$\Rightarrow \psi^n(f(y), f(x)) > \alpha$.

Thus $(f(y), f(x)) \in (\psi^{n})^{\alpha} = (\psi^{\alpha})^{n}$ and since $y \in \nu^{\alpha}$, $f(y) \in f[\nu^{\alpha}] = f[\nu]^{\alpha}$ from which we deduce that $f(x) \in (\psi^{\alpha})^{n}[f[\nu]^{\alpha}] \subseteq (\psi^{\alpha})^{n}(G)$. Since G is arbitrary, $f(x) \in \bigcap_{G \in \mathscr{G}_0} (\psi^{\alpha})^{n}(G)$ which means that $\bigcap_{G \in \mathscr{G}_0} (\psi^{\alpha})^{n}(G) \neq \emptyset$ and hence $\mathscr{G}_0$ is weak $\mathscr{E}$–Cauchy.

■

We can now investigate weak Cauchy prefilters on product spaces.

7.8 THEOREM

> Let $((X_j, \mathscr{D}_j) : j \in J)$ be a family of fuzzy uniform spaces,
> $X = \prod_{j \in J} X_j$, $\mathscr{D} = \prod_{j \in J} \mathscr{D}_j$ and $\mathscr{F}$ a prime prefilter on X. Then :
>
> $\mathscr{F}$ is weak $\mathscr{D}$–Cauchy $\Leftrightarrow \forall\, j\ p_j[\mathscr{F}]$ is weak $\mathscr{D}_j$–Cauchy.

PROOF

The forward implication follows from 7.7 and 3.9.

To prove the reverse implication we let $c(\mathscr{F}) = c$ and show that $\mathscr{F}_0$ is weak $\mathscr{D}^c$–Cauchy. To do this let $\overset{\alpha}{\sigma} \in \mathscr{D}^c$ with $\sigma \in \mathscr{D}$ and $\alpha < c$. There exists $J_0 \in \mathscr{P}_f(J)$, $\sigma_j \in \mathscr{D}_j$ such that $\inf_{j \in J_0} (p_j \times p_j)^{-1}[\sigma_j] \leq \sigma$. Let $\psi_j = (p_j \times p_j)^{-1}[\sigma_j]$ and $\psi = \inf_{j \in J_0} \psi_j$.

We seek $F \in \mathscr{F}_0$ and $n \in \mathbb{N}$ such that $F \times F \subseteq (\overset{\alpha}{\psi})^{n}$.

We have :

$\forall\, j \in J_0$ $\overset{\alpha}{\sigma_j} \in \mathscr{D}_j$ and $\mathscr{F}_j := \langle p_j[\mathscr{F}] \rangle$ is weak $\mathscr{D}_j$–Cauchy. Thus

$\forall\, j \in J_0\ \exists\, n_j \in \mathbb{N}\ \exists\, \nu^j \in \mathscr{F}: p_j[\nu^j]^0 \times p_j[\nu^j]^0 \subseteq (\overset{\alpha}{\sigma_j})^{n_j}$

Let $\nu = \inf_{j \in [n]} \nu^j$ and $n = \max_{j \in [n]} n_j$ then $\nu \in \mathscr{F}$ and :

(i) $\quad \forall\, j \in J_0\ p_j[\nu]^0 \times p_j[\nu]^0 \subseteq (\overset{\alpha}{\sigma_j})^{n}$

Let $(x,y) \in \nu^0 \times \nu^0$ and $j \in J_0$. Then

$p_j[\nu](p_j(x)) = \sup_{p_j(z) = p_j(x)} \nu(z) \geq \nu(x) > 0$ and similarly $p_j[\nu](p_j(y)) > 0$.

Thus $(p_j(x), p_j(y)) \in (\overset{\alpha}{\sigma_j})^{n} = (\sigma_j^{n})^{\alpha}$ ((i)) and so

(ii) $\sigma_j^n(p_j(x), p_j(y)) > \alpha$. Therefore :

$$\begin{aligned}\psi_j^n(x,y) &= \sup\{\psi_j(x,y_1) \wedge \psi_j(y_1,y_2) \wedge \cdots \wedge \psi_j(y_{n-1},y) : \underline{y} = (y_1,y_2,\cdots,y_{n-1}) \in X^{n-1}\} \\ &= \sup_{\underline{y}} \sigma_j(p_j(x), p_j(y_1)) \wedge \sigma_j(p_j(y_1), p_j(y_2)) \wedge \cdots \wedge \sigma_j(p_j(y_{n-1}), p_j(y)) \\ &= \sigma_j^n(p_j(x), p_j(y)) \ (p_j \text{ is surjective}).\end{aligned}$$

Thus $\forall\ j \in J_0\ \ \psi_j^n(x,y) > \alpha$ ((ii)).

Now $\psi(x,y) = \inf_{j \in J_0} \psi_j(x,y) = \min_{j \in J_0} \psi_j(x,y) = \psi_k(x,y)$ say.

Therefore $\psi^n(x,y) = \psi_k^n(x,y) > \alpha$ which means that $(x,y) \in (\psi^n)^\alpha = (\psi^\alpha)^n$

Since (x,y) is arbitrary, $\nu^0 \times \nu^0 \subseteq (\psi^\alpha)^n \subseteq (\sigma^\alpha)^n$ and this shows that $\mathscr{F}_0$ is weak $\mathscr{D}^c$–Cauchy and hence that $\mathscr{F}$ is weak $\mathscr{D}$–Cauchy (7.4).

■

If $(X, \mathscr{D})$ is a fuzzy uniform space we shall call $\mu \in I^X$ **$\mathscr{D}$–bounded** $\Leftrightarrow \forall\ \epsilon \in I_0\ \forall\ \sigma \in \mathscr{D}\ \exists\ F \in \mathscr{P}_f(X)\ \exists\ n \in \mathbb{N} : \mu \leq \sup_{x \in F} \sigma^n\langle x\rangle + \epsilon$.

(Note that for finite $F \subseteq X$ $\sup_{x \in F} \sigma^n\langle x\rangle = \max_{x \in F} \sigma^n\langle x\rangle$).

We check that this definition is a good extension.

7.9 THEOREM

> Let $(X,\mathbb{D})$ be a uniform space and let $B \subseteq X$. Then :
>
> B is $\mathbb{D}$–bounded $\Leftrightarrow 1_B$ is $\mathbb{D}^1$–bounded.

PROOF

Let B be $\mathbb{D}$–bounded, (symmetric) $\sigma \in \mathbb{D}^1$ and $\epsilon \in I_0$. Let $U = \sigma^{1-\epsilon}$ then since $0 \leq 1-\epsilon < 1$, $U \in \mathbb{D}$. Therefore we can find $F = \{x_1,x_2,\cdots,x_m\} \subseteq X$ and $n \in \mathbb{N}$ such that $B \subseteq \bigcup_{i=1}^{m} U^n(x_i)$. In other words :

$\forall\, b \in B\ \exists\, x \in F : (x,b) \in U^n = (\sigma^{1-\epsilon})^n = (\sigma^n)^{1-\epsilon}$ (3.4(i)) so,

$\forall\, b \in B\ \exists\, x \in F : \sigma^n(x,b) > 1-\epsilon$ and hence,

$\forall\, b \in B\ \sup_{x \in F} \sigma^n\langle x\rangle(b) > 1-\epsilon = 1_B(b)-\epsilon$ which means that

$$1_B \le \sup_{x \in F} \sigma^n\langle x\rangle + \epsilon.$$

For the converse, let 1_B be $\mathbb{D}^1$ –bounded and (symmetric) $U \in \mathbb{D}$. Let $\sigma = 1_U$. Since $\sigma \in \mathbb{D}^1$ there exists $\{x_1,x_2,\cdots,x_m\} \subseteq X$ and $n \in \mathbb{N}$ such that :

$$1_B \le \sup_{i \in [m]} \sigma^n\langle x_i\rangle + \tfrac{1}{2}.$$

Now if $b \in B$ then $1_B(b) = 1 \le \max_{i \in [m]} \sigma^n\langle b,x_i\rangle + \frac{1}{2}$ and it follows that $\max_{i \in [m]} \sigma^n\langle b,x_i\rangle \ge \frac{1}{2} > 0$. Consequently $\sigma^n(b,x_k) > \frac{1}{2}$ for some $k \in [m]$. Thus $(b,x_k) \in (\sigma^n)^{1/2} = (\sigma^{1/2})^n = ((1_U)^{1/2})^n = U^n$ and therefore $b \in U^n(x_k) \subseteq \bigcup_{i=1}^{m} U^n(x_i)$.

We have shown that $B \subseteq \bigcup_{i=1}^{m} U^n(x_i)$ and this proves that B is $\mathbb{D}$–bounded.

■

The properties of $\mathscr{D}$–bounded fuzzy sets are fairly easy to establish directly from the definition and the theory we build is a generalisation of the standard theory.

Bounded sets in a uniform space are stable with respect to the formation of finite unions and subsets. Precompact sets are bounded. The same is true of bounded fuzzy sets in a fuzzy uniform space.

7.10 THEOREM

Let $(X,\mathscr{D})$ be a fuzzy uniform space with ν, μ, $\nu_i \in I^X$. Then :

(a) $\nu \le \mu$ and μ is $\mathscr{D}$–bounded $\Rightarrow$ ν is $\mathscr{D}$–bounded.

(b) $\forall\, i \in [m]$ ν_i is $\mathscr{D}$–bounded $\Rightarrow$ $\sup_{i \in [m]} \nu_i$ is $\mathscr{D}$–bounded.

(c) μ is $\mathscr{D}$–precompact $\Rightarrow$ μ is $\mathscr{D}$–bounded.

PROOF

(a) If $\mu \leq \sup_{x \in F} \sigma^n\langle x\rangle + \epsilon$ then $\nu \leq \sup_{x \in F} \sigma^n\langle x\rangle + \epsilon$.

(b) Let $\sigma \in \mathscr{D}$ and $\epsilon > 0$. For each $i \in [m]$, ν_i is $\mathscr{D}$–bounded and so :

$$\forall\, i \in [m]\ \exists\ F(i) \in \mathscr{P}_f(X)\ \exists\ n(i) \in \mathbb{N} : \nu_i \leq \sup_{x \in F(i)} \sigma^{n(i)}\langle x\rangle + \epsilon.$$

Let $F = \bigcup_{i \in [m]} F(i)$ and $n = \max_{i \in [m]} n(i)$ then :

(i) $$\sup_{i \in [m]} \nu_i \leq \sup_{x \in F} \sigma^n\langle x\rangle + \epsilon.$$

To see this, let $z \in X$ be arbitrary. Then :

$$\sup_{i \in [m]} \nu_i(z) = \max_{i \in [m]} \nu_i(z) = \nu_k(z) \text{ say, and}$$

$$\nu_k(z) \leq \sup_{x \in F(k)} \sigma^{n(k)}(z,x) \leq \sup_{x \in F} \sigma^n(z,x) + \epsilon.$$

This establishes (i) which in turn proves that $\sup_{i \in [m]} \nu_i$ is $\mathscr{D}$–bounded.

(c) This follows from 6.6 with $n = 1$.

■

In UNIF, the closure of a bounded set is bounded and we now show that this statement is also true in F.UNIF.

7.11 THEOREM

> Let $(X, \mathscr{D})$ be a fuzzy uniform space and let $\mu \in I^X$. Then :
> μ is $\mathscr{D}$–bounded $\Rightarrow \bar{\mu}$ is $\mathscr{D}$–bounded.

PROOF

Let $\mu \in I^X$ be $\mathscr{D}$–bounded, $\epsilon \in I_0$ and $\sigma \in \mathscr{D}$ be symmetric. Since μ is $\mathscr{D}$–bounded we can find $F \in \mathscr{P}_f(X)$ and $n \in \mathbb{N}$ such that :

(i) $$\mu \leq \sup_{x \in F} \sigma^n\langle x\rangle + \epsilon.$$

We assert that :

(ii) $$\bar{\mu} \leq \sup_{x \in F} \sigma^{n+1}\langle x\rangle + \epsilon.$$

To see this let $z \in X$ then :

$$\begin{aligned}
\bar{\mu}(z) &= \inf_{\psi \in \mathscr{D}} \psi\langle\mu\rangle(z) = \inf_{\psi \in \mathscr{D}} \sup_{y \in X} \mu(y) \wedge \psi(y,z) \\
&\leq \inf_{\psi \in \mathscr{D}} \sup_{y \in X} (\sup_{x \in F} \sigma^n(y,x) + \epsilon) \wedge \psi(y,z) \quad \text{(from (i))} \\
&\leq \inf_{\psi \in \mathscr{D}} \sup_{y \in X} \sup_{x \in F} \sigma^n(y,x) \wedge \psi(y,z) + \epsilon \\
&\leq \sup_{x \in F} \sup_{y \in X} \sigma(z,y) \wedge \sigma^n(y,x) + \epsilon \\
&= \sup_{x \in F} (\sigma \circ \sigma^n)(z,x) + \epsilon \\
&= \sup_{x \in F} \sigma^{n+1}\langle x\rangle(z) + \epsilon.
\end{aligned}$$

This proves (ii) and hence $\bar{\mu}$ is $\mathscr{D}$–bounded.

■

In UNIF, the uniformly continuous image of a bounded set is bounded. We obtain the analgous result in F.UNIF.

7.12 THEOREM

Let $(X, \mathscr{D})$ and $(Y, \mathscr{E})$ be fuzzy uniform spaces and $f: (X, \mathscr{D}) \to (Y, \mathscr{E})$ be uniformly continuous. Then :

$$\mu \text{ is } \mathscr{D}\text{–bounded} \Rightarrow f[\mu] \text{ is } \mathscr{E}\text{–bounded}.$$

PROOF

Let symmetric $\psi \in \mathscr{E}$ and $\epsilon \in I_0$. Since f is uniformly continuous, $\sigma := (f \times f)^{-1}[\psi] \in \mathscr{D}$. Now since μ is $\mathscr{D}$–bounded there exists $F \in \mathscr{P}_f(X)$ and $n \in \mathbb{N}$ such that :

(i) $$\mu \leq \sup_{x \in F} \sigma^n\langle x\rangle + \epsilon.$$

We assert that :

(ii) $$f[\mu] \leq \sup_{x \in F} \psi^n \langle f(x)\rangle + \epsilon.$$

To prove (ii) let $y \in Y$. Then :

$$f[\mu](y) = \sup_{z \in f^{-1}[\{y\}]} \mu(z) \leq \sup_{z \in f^{-1}[\{y\}]} \sup_{x \in F} \sigma^n(z,x) + \epsilon.$$

Now $\sigma^n(z,x) = \sup\{\sigma(z,x_1) \wedge \sigma(x_1,x_2) \wedge \cdots \wedge \sigma(x_{n-1},x): x_1,x_2,\cdots,x_{n-1} \in X\}$

$= \sup(\psi(f(z),f(x_1)) \wedge \psi(f(x_1),f(x_2)) \wedge \cdots \wedge \psi(f(x_{n-1}),f(x)): x_1,x_2,\cdots,x_{n-1} \in X\}$

$\leq \sup\{\psi(f(z),y_1) \wedge \psi(y_1,y_2) \wedge \cdots \wedge \psi(y_{n-1},f(x)): y_1,y_2,\cdots,y_{n-1} \in Y\}$

$= \psi^n(f(z),f(x))$.

Thus $$f[\mu](y) \leq \sup_{z \in f^{-1}[\{y\}]} \sup_{x \in F} \psi^n(f(z),f(x)) + \epsilon$$
$$= \sup_{x \in F} \psi^n(y,f(x)) + \epsilon$$
$$= \sup_{x \in F} \psi^n\langle f(x)\rangle(y) + \epsilon.$$

This proves (ii) and, since $\{f(x): x \in F\} \in \mathscr{P}_f(Y)$, this shows that $f[\mu]$ is $\mathscr{E}$–bounded.

■

Boundedness in F.UNIF can be described in terms of weak Cauchy prefilters and the following theorem is an extension of Theorem 7.2.

7.13 THEOREM

> Let $(X, \mathscr{D})$ be a fuzzy uniform space and let $\mu \in I^X$. Then :
> μ is $\mathscr{D}$–bounded iff every prime prefilter $\mathscr{F}$ with $\mu \in \mathscr{F}$ is weak $\mathscr{D}$–Cauchy.

PROOF

Let μ be $\mathscr{D}$–bounded, $\mathscr{F}$ a prime prefilter with $\mu \in \mathscr{F}$, $\sigma \in \mathscr{D}$ and $\epsilon \in I_0$. Then there exists $n \in \mathbb{N}$ and $K \in \mathscr{P}_f(X)$ such that :

$$\mu \leq \sup_{y \in K} \sigma^n<y> + \epsilon = \sup_{y \in K} (\sigma^n<y> + \epsilon) \wedge 1_X := \lambda.$$

Since $\mu \in \mathscr{F}$, $\lambda \in \mathscr{F}$ and since K is finite and $\mathscr{F}$ is prime we have :

$\exists\, y \in K : (\sigma^n<y> + \epsilon) \wedge 1_X \in \mathscr{F}$. Let $\nu \in \mathscr{F}$. Then :

$$(\sigma^n<y> + \epsilon) \wedge 1_X \wedge \nu = (\sigma^n<y> + \epsilon) \wedge \nu \in \mathscr{F} \text{ and so}$$

$$c(\mathscr{F}) \leq \sup(\sigma^n<y> + \epsilon) \wedge \nu \leq (\sup \sigma^n<y> \wedge \nu) + \epsilon = \sigma^n<\nu>(y) + \epsilon.$$

Since ν is arbitrary, $c(\mathscr{F}) \leq \inf_{\nu \in \mathscr{F}} \sigma^n<\nu>(y) + \epsilon \leq \sup \sup_{n \in \mathbb{N}} \inf_{\nu \in \mathscr{F}} \sigma^n<\nu> + \epsilon$. Since σ and ϵ are arbitrary, $c(\mathscr{F}) \leq \inf_{\sigma \in \mathscr{D}} \sup \sup_{n \in \mathbb{N}} \inf_{\nu \in \mathscr{F}} \sigma^n<\nu>$ and this means that $\mathscr{F}$ is weak $\mathscr{D}$–Cauchy.

For the converse let $\psi \in \mathscr{D}$ be symmetric, let $\epsilon \in I_0$ and suppose that :

(i) $$\forall\, K \in \mathscr{P}_f(X)\ \forall\, n \in \mathbb{N}\ \exists\, z \in X : \mu(z) > \sup_{y \in K} \psi^n<y> + \epsilon.$$

Choose $x_1 \in X$ then since $\{x_1\} \in \mathscr{P}_f(X)$:

$$\exists\, x_2 \in X : \mu(x_2) > \psi^2(x_2,x_1) + \epsilon.$$

Now $\{x_1,x_2\} \in \mathscr{P}_f(X)$ and so :

$$\exists\, x_3 \in X : \mu(x_3) > \sup_{i < 3} \psi^3(x_3,x_i) + \epsilon.$$

Continuing in this way we obtain a sequence $(x_n : n \in \mathbb{N})$ with the property :

(ii) $$\forall\, n \in \mathbb{N}\ \forall\, i < n\ \ \mu(x_n) > \psi^n(x_n,x_i) + \epsilon.$$

For $n \in \mathbb{N}$ let $T_n = \{x_m : m \geq n\}$ and $\mathbb{K} = <\{T_n : n \in \mathbb{N}\}>$. Let $\mathscr{F} = (\mathbb{K}_1,\mu) = <\{1_K \wedge \mu : K \in \mathbb{K}\}> = <\{1_{T_n} \wedge \mu : n \in \mathbb{N}\}>$ and choose $\mathscr{G} \in \mathscr{P}_m(\mathscr{F})$ such that $c(\mathscr{G}) = c(\mathscr{F})$ (2.16). Then $\mathscr{G}$ is prime, $\mu \in \mathscr{G}$ and :

$$c(\mathscr{G}) = c(\mathscr{F}) = \inf_{n \in \mathbb{N}} \sup 1_{T_n} \wedge \mu = \inf_{n \in \mathbb{N}} \sup_{i \geq n} \mu(x_i).$$

If $n \in \mathbb{N}$ let $s_n = \sup_{i \geq n} \mu(x_i)$ then $c(\mathscr{G}) = \inf_{n \in \mathbb{N}} s_n$. Choose symmetric $\sigma \in \mathscr{D}$ such that $\sigma \circ \sigma \leq \psi + \frac{\epsilon}{2}$. Then by assumption :

$$c(\mathscr{G}) \leq \sup_{n \in \mathbb{N}} \sup_{\nu \in \mathscr{G}} \inf \sigma^n<\nu> \leq \sup_{n \in \mathbb{N}} \sup_{m \in \mathbb{N}} \inf \sigma^n<1_{T_m}>$$

$$\Rightarrow \inf_{n \in \mathbb{N}} s_n < \sup_{n \in \mathbb{N}} \inf_{m \in \mathbb{N}} \sup \sigma^n<1_{T_m}> + \frac{\epsilon}{2}$$

$$\Rightarrow \exists\, x \in X\ \exists\, n \in \mathbb{N}\ \exists\, k \in \mathbb{N} : s_k < \inf_{m \in \mathbb{N}} \sigma^n<1_{T_m}>(x) + \frac{\epsilon}{2}$$

$$\Rightarrow \exists\, x \in X\ \exists\, n \in \mathbb{N}\ \exists\, k \in \mathbb{N}\ \forall\, m \in \mathbb{N}\ s_k < \sup_{i \geq m} \sigma^n(x,x_i) + \frac{\epsilon}{2}.$$

For this $x \in X$, $n \in \mathbb{N}$ and $k \in \mathbb{N}$ we have :

(iii) $\quad \forall\, m \in \mathbb{N}\ \exists\, i \geq m : s_k < \sigma^n(x,x_i) + \frac{\epsilon}{2}.$

If we choose $m > k \vee n$ and invoke (iii) we obtain :

(iv) $\quad \exists\, i > k \vee n : s_k < \sigma^n(x,x_i) + \frac{\epsilon}{2}.$

If we now choose $m > i$ and invoke (iii) we obtain :

(v) $\quad \exists\, \ell > i : s_k < \sigma^n(x,x_\ell) + \frac{\epsilon}{2}.$

We therefore have $k \vee n < i < \ell$ and (from (iv), (v)) :

$$s_k < (\sigma^n(x,x_i) + \tfrac{\epsilon}{2}) \wedge (\sigma^n(x,x_\ell) + \tfrac{\epsilon}{2}) = \sigma^n(x,x_i) \wedge \sigma^n(x,x_\ell) + \tfrac{\epsilon}{2}$$
$$\leq (\sigma^n \circ \sigma^n)(x_i,x_\ell) + \tfrac{\epsilon}{2}$$
$$\leq (\psi^n(x_i,x_\ell) + \tfrac{\epsilon}{2}) + \tfrac{\epsilon}{2} = \psi^n(x_i,x_\ell) + \epsilon.$$

In other words :

$$\sup_{j \geq k} \mu(x_j) < \psi^n(x_i,x_\ell) + \epsilon.$$

From this we obtain :

$$\mu(x_\ell) < \psi^n(x_i,x_\ell) + \epsilon \leq \psi^\ell(x_i,x_\ell) + \epsilon$$

which contradicts (ii) and so (i) is false, which means that μ is $\mathscr{D}$–bounded.

∎

We have an α–level theorem for $\mathscr{D}$–bounded fuzzy sets which is as follows.

7.14 THEOREM

> Let $(X, \mathscr{D})$ be a fuzzy uniform space and let $\mu \in I^X$. Then :
> μ is $\mathscr{D}$–bounded $\Leftrightarrow \forall\ \alpha \in I_0\ \ \mu_\alpha$ is $\mathscr{D}^\alpha$–bounded.

PROOF

Let μ be $\mathscr{D}$–bounded, $\alpha \in I_0$ and $\sigma^\beta \in \mathscr{D}^\alpha$ with symmetric $\sigma \in \mathscr{D}$ and $\beta < \alpha$. Let $0 < \epsilon < \alpha - \beta$. Then we can find $K \in \mathscr{P}_f(X)$ and $n \in \mathbb{N}$ such that

$$\mu \leq \sup_{y \in K} \sigma^n\langle y\rangle + \epsilon. \text{ Thus :}$$

$$\begin{aligned} x \in \mu_\alpha \quad &\Rightarrow \max_{y \in K} \sigma^n(x,y) + \epsilon \geq \alpha \\ &\Rightarrow \max_{y \in K} \sigma^n(x,y) \geq \alpha - \epsilon > \beta \\ &\Rightarrow \exists\ y \in K : \sigma^n(x,y) > \beta \\ &\Rightarrow x \in (\sigma^n)^\beta(y) \subseteq (\sigma^\beta)^n(K). \end{aligned}$$

Thus $\mu_\alpha \subseteq (\sigma^\beta)^n(K)$ and so μ_α is $\mathscr{D}^\alpha$–bounded.

For the converse let $\mathscr{G}$ be a prime prefilter with $\mu \in \mathscr{G}$ and let $c(\mathscr{G}) = c > 0$. To show that $\mathscr{G}$ is weak $\mathscr{D}$–Cauchy let $0 < \alpha < c$ and let $\sigma \in \mathscr{D}$ be symmetric. Choose $\beta < \alpha$. Then $\sigma^\beta \in \mathscr{D}^\alpha$ and so there exists $K \in \mathscr{P}_f(X)$ and $n \in \mathbb{N}$ such that :

$$\mu_\alpha \subseteq (\sigma^\beta)^n(K).$$

Now $\mu^\alpha \in \mathscr{G}_0$ (2.19) and $\mu^\alpha \subseteq \mu_\alpha \subseteq (\sigma^\beta)^n(K)$ and so $(\sigma^\beta)^n(K) \in \mathscr{G}_0$ which is an ultrafilter (2.9(a)). Consequently $(\sigma^\beta)^n(y) \in \mathscr{G}_0$ for some $y \in K$ (since K is finite).

Now :

$$\begin{aligned} (\sigma^\beta)^n(y) \in \mathscr{G}_0 \quad &\Leftrightarrow \forall\ \nu \in \mathscr{G}\ \ \nu^\beta \cap (\sigma^\beta)^n(y) \neq \emptyset \\ &\Leftrightarrow \forall\ \nu \in \mathscr{G}\ \exists\ z \in \nu^\beta : \sigma^n(z,y) > \beta \\ &\Leftrightarrow \forall\ \nu \in \mathscr{G}\ \sup \nu \wedge \sigma^n\langle y\rangle > \beta \\ &\Leftrightarrow \inf_{\nu \in \mathscr{G}} \sigma^n\langle \nu\rangle(y) > \beta. \end{aligned}$$

Consequently :

$\beta < \sup_n \sup_{\nu \in \mathscr{G}} \inf \sigma^n\langle\nu\rangle$ and since β is arbitrary

$\alpha \leq \sup_n \sup_{\nu \in \mathscr{G}} \inf \sigma^n\langle\nu\rangle$.

Since α and σ are arbitrary,

$c(\mathscr{G}) \leq \inf_{\sigma \in \mathscr{D}} \sup \sup_{n \in \mathbb{N}} \inf_{\nu \in \mathscr{G}} \sigma^n\langle\nu\rangle$.

In other words, $\mathscr{G}$ is weak $\mathscr{D}$–Cauchy and finally, since $\mathscr{G}$ is arbitrary we have shown, according to 7.13, that μ is $\mathscr{D}$–bounded.

■

Just as we did for $\mathscr{D}$–precompactness we show that $\mathscr{D}$–boundedness can be described using only certain subsets of the prime prefilters.

7.15 THEOREM

Let $(X, \mathscr{D})$ be a fuzzy uniform space with $\mu \in I^X$. The following are equivalent :

(a) μ is $\mathscr{D}$–bounded.

(b) For every prime prefilter $\mathscr{F}$, $(\mathscr{F},\mu)$ is weak $\mathscr{D}$–Cauchy.

(c) For every prime prefilter $\mathscr{F}$,

$$c(\mathscr{F},\mu) \leq \inf_{\sigma \in \mathscr{D}} \sup \sup_{n \in \mathbb{N}} \inf_{\nu \in \mathscr{F}} \sigma^n\langle\nu\rangle := r(\mathscr{F})$$

(d) For every prime prefilter $\mathscr{F}$ with $c(\mathscr{F}) = 1$, $c(\mathscr{F},\mu) \leq r(\mathscr{F})$.

(e) For every prime saturated prefilter $\mathscr{F}$ with $c(\mathscr{F}) = 1$, $c(\mathscr{F},\mu) \leq r(\mathscr{F})$

(f) Every prime saturated prefilter $\mathscr{F}$ with $\mu \in \mathscr{F}$ is weak $\mathscr{D}$–Cauchy.

PROOF

(a) $\Longrightarrow$ (b) : If $\mathscr{F}$ is prime then $(\mathscr{F},\mu)$ is also prime and $\mu \in (\mathscr{F},\mu)$. Thus, since μ is $\mathscr{D}$–bounded, $(\mathscr{F},\mu)$ is weak $\mathscr{D}$–Cauchy.

(b) $\Longrightarrow$ (c) : If $(\mathscr{F},\mu)$ is weak $\mathscr{D}$–Cauchy then :

$$c(\mathscr{F},\mu) \leq \inf_{\sigma \in \mathscr{D}} \sup \sup_{n \in \mathbb{N}} \inf_{\nu \in \mathscr{F}} \sigma^n\langle\nu \wedge \lambda\rangle \leq r(\mathscr{F}).$$

(c) $\Longrightarrow$ (d) : Obvious.

(d) $\Longrightarrow$ (a) : Let $\mathscr{F}$ be a prime prefilter with $\mu \in \mathscr{F}$ and $c(\mathscr{F}) = c$. Let :

$\mathscr{H} = (\mathscr{F}_0)_1 = <\{1_F : F \in \mathscr{F}_0\}>$.

We refer to the proof of 6.3 and assert that :

(i) $\mathscr{H} \subseteq (\mathscr{H},\mu) \subseteq \mathscr{F}$.

(ii) $\mathscr{H}_0 = (\mathscr{H},\mu)_0 = \mathscr{F}_0$.

(iii) $\mathscr{H}$ is prime.

(iv) $c(\mathscr{H}) = 1$.

By assumption then we have :

$$c(\mathscr{F}) \leq c(\mathscr{H},\mu) \leq r(\mathscr{H}) = \inf_{\sigma \in \mathscr{D}} \sup_{n \in \mathbb{N}} \sup_{\lambda \in \mathscr{H}} \inf \sigma^n<\lambda>$$

$$= \inf_{\sigma \in \mathscr{D}} \sup_{n \in \mathbb{N}} \sup_{F \in \mathscr{F}_0} \inf \sigma^n<1_F>.$$

Thus :

$\forall\ \alpha < c\ \forall\ \sigma \in \mathscr{D}\ \exists\ x \in X\ \exists\ n \in \mathbb{N}\ \forall\ F \in \mathscr{F}_0\ \ \alpha < \sigma^n<1_F>(x)$

$\Leftrightarrow\ \forall\ \alpha < c\ \forall\ \sigma \in \mathscr{D}\ \exists\ x \in X\ \exists\ n \in \mathbb{N}\ \forall\ \nu \in \mathscr{F}\ \ \alpha < \sigma^n<1_{\nu^\alpha}>(x)$. (2.19)

$\Leftrightarrow\ \forall\ \alpha < c\ \forall\ \sigma \in \mathscr{D}\ \exists\ x \in X\ \exists\ n \in \mathbb{N}\ \forall\ \nu \in \mathscr{F}\ \ \alpha < \sigma^n<\nu>(x)$

$[\alpha < \sigma^n<1_{\nu^\alpha}>(x)\ \Leftrightarrow\ \alpha < \sup 1_{\nu^\alpha} \wedge \sigma^n<x>$

$\Leftrightarrow\ \exists\ y \in \nu^\alpha : \sigma^n(y,x) > \alpha$

$\Leftrightarrow\ \sup \nu \wedge \sigma^n<x> = \sigma^n<\nu>(x) > \alpha]$

$\Leftrightarrow\ c(\mathscr{F}) \leq \inf_{\sigma \in \mathscr{D}} \sup_{n \in \mathbb{N}} \sup_{\nu \in \mathscr{F}} \inf \sigma^n<\nu>$

$\Leftrightarrow\ \mathscr{F}$ is weak $\mathscr{D}$–Cauchy.

Consequently μ is $\mathscr{D}$–bounded.

(a) $\Longrightarrow$ (e) : Obvious.

(e) $\Longrightarrow$ (a) : Let $\mathscr{F}$ be a prime prefilter with $c(\mathscr{F}) = 1$. We intend to show that $c(\mathscr{F},\mu) \leq r(\mathscr{F})$ and then apply (d). Let :

$\mathscr{H} = (\mathscr{F}_0)^1 = \{\lambda \in I^X : \forall\ \alpha < 1\ \ \lambda^\alpha \in \mathscr{F}_0\}$.

From the proof of 6.4 we glean :

(i) $\mathscr{H}$ is a saturated prefilter.

(ii) $c(\mathscr{H}) = 1$.

(iii) $\mathscr{H}_0 = \mathscr{F}_0$.

(iv) $\mathscr{H}$ is prime.

(v) $c(\mathscr{F},\mu) \leq c(\mathscr{H},\mu)$.

Thus by assumption we have :

$$c(\mathscr{F},\mu) \leq c(\mathscr{H},\mu) \leq r(\mathscr{H}) = \inf_{\sigma \in \mathscr{D}} \sup_{n \in \mathbb{N}} \sup_{\lambda \in \mathscr{H}} \inf \sigma^n<\lambda>$$

$$\Leftrightarrow \forall\ \alpha < c(\mathscr{F},\mu)\ \forall\ \sigma \in \mathscr{D}\ \exists\ x \in X\ \exists\ n \in \mathbb{N}\ \forall\ \lambda \in \mathscr{H}\ \alpha < \sigma^n<\lambda>(x)$$

$$\Rightarrow \forall\ \alpha < c(\mathscr{F},\mu)\ \forall\ \sigma \in \mathscr{D}\ \exists\ x \in X\ \exists\ n \in \mathbb{N}\ \forall\ \nu \in \mathscr{F}\ \alpha < \sigma^n<1_\nu{}^\alpha>(x)$$

[If $\alpha < c(\mathscr{F},\mu) \leq c(\mathscr{F})$ and $\nu \in \mathscr{F}$ then $\nu^\alpha \in \mathscr{F}_0$ (2.19) and $1_\nu{}^\alpha \in \mathscr{H}$]

$$\Leftrightarrow \forall\ \alpha < c(\mathscr{F},\mu)\ \forall\ \sigma \in \mathscr{D}\ \exists\ x \in X\ \exists\ n \in \mathbb{N}\ \forall\ \nu \in \mathscr{F}\ \alpha < \sigma^n<\nu>(x)$$

$$\Leftrightarrow c(\mathscr{F},\mu) \leq \inf_{\sigma \in \mathscr{D}} \sup_{n \in \mathbb{N}} \sup_{\nu \in \mathscr{F}} \inf \sigma^n<\nu> = r(\mathscr{F}).$$

(a) $\Rightarrow$ (f) : Obvious.

(f) $\Rightarrow$ (a) : Let $\mathscr{F}$ be a prime prefilter with $\mu \in \mathscr{F}$ and $c(\mathscr{F},\mu) = c$. As we did in 6.4 we let $\mathscr{H} = (\mathscr{F}_0)^c$. Then $\mathscr{H}$ is a saturated prefilter, $\mathscr{F} \subseteq \mathscr{H}$, $c(\mathscr{H}) = c$, $\mathscr{F}_0 = \mathscr{H}_0$ and $\mathscr{H}$ is prime. By assumption then, $\mathscr{H}$ is weak $\mathscr{D}$–Cauchy and so :

$$c(\mathscr{F}) = c = c(\mathscr{H}) \leq \inf_{\sigma \in \mathscr{D}} \sup_{n \in \mathbb{N}} \sup_{\lambda \in \mathscr{H}} \inf \sigma^n<\lambda> \leq \inf_{\sigma \in \mathscr{D}} \sup_{n \in \mathbb{N}} \sup_{\nu \in \mathscr{F}} \inf \sigma^n<\nu>.$$

This means that $\mathscr{F}$ is weak $\mathscr{D}$–Cauchy and hence μ is $\mathscr{D}$–bounded (7.13).

■

The behaviour of bounded fuzzy sets in products of fuzzy uniform spaces parallels the behaviour of precompact fuzzy sets.

7.16 THEOREM

Let $((X_j, \mathscr{D}_j) : j \in J)$ be a family of fuzzy uniform spaces with $\mu_j \in I^{X_j}$ for each $j \in J$. Let $X = \prod_{j \in J} X_j$ and $\mathscr{D} = \prod_{j \in J} \mathscr{D}_j$. Then

(a) $\forall\, j \in J\ \mu_j$ is $\mathscr{D}_j$–bounded $\Rightarrow \prod_{j \in J} \mu_j$ is $\mathscr{D}$–bounded.

(b) $\mu \in I^X$ is $\mathscr{D}$–bounded $\Leftrightarrow \forall\, j \in J\ p_j[\mu]$ is $\mathscr{D}_j$–bounded.

PROOF

The proof is exactly the same as the proof of 6.10 with "bounded" replacing "precompact", "weak–Cauchy" replacing "Cauchy" and using 7.8, 7.12 and 7.10(a).

■

CHAPTER 8

COMPLETENESS

If $(X,\mathbb{D})$ is a uniform space and $A \subseteq X$ then
A is **$\mathbb{D}$–complete** iff every $\mathbb{D}$–Cauchy filter $\mathbb{F}$ with $A \in \mathbb{F}$ is convergent in A.

We extend this definition to the fuzzy uniform space setting as follows. If $(X, \mathscr{D})$ is a fuzzy uniform space and $\mu \in I^X$ we say that :
μ is **$\mathscr{D}$–complete** iff every $\mathscr{D}$–Cauchy prefilter $\mathscr{F}$ with $\mu \in \mathscr{F}$ is convergent in μ.

As we did for $\mathscr{D}$–boundedness and $\mathscr{D}$–precompactness we shall show that this is a good extension, that we have an α–level theorem for $\mathscr{D}$–completeness, that $\mathscr{D}$–completeness can be characterised using only certain subsets of the $\mathscr{D}$–Cauchy prefilters and that a theory of $\mathscr{D}$–complete fuzzy sets can be developed which generalises the elementary theory of $\mathbb{D}$–complete subsets.

We first characterise $\mathscr{D}$–completeness in terms of prime prefilters.

8.1 THEOREM

Let $(X, \mathscr{D})$ be a fuzzy uniform space with $\mu \in I^X$. Then
μ is $\mathscr{D}$–complete iff every prime, $\mathscr{D}$–Cauchy prefilter $\mathscr{G}$ with $\mu \in \mathscr{G}$ is convergent in μ.

PROOF

We only need to prove the reverse implication so let $\mathscr{F}$ be a $\mathscr{D}$–Cauchy prefilter with $\mu \in \mathscr{F}$ and $\bar{c}(\mathscr{F}) = \bar{c}$. We must show that $\mathscr{F}$ is convergent in μ, in other words :

(i) $\bar{c} \leq \sup \mu \wedge \lim \mathscr{F}$.
Let $\mathscr{H} = \langle\mu\rangle \vee (\mathscr{F}_0)_{\bar{c}} = ((\mathscr{F}_0)_{\bar{c}}, \mu)$. Then :

(ii) $\mathscr{H} = <\{\mu \wedge \bar{c}\, 1_F : F \in \mathscr{F}_0\}> := \mathscr{R}$.

$$\lambda \in \mathscr{H} \Leftrightarrow \exists\, \mu_1 \geq \mu\ \exists\, F \in \mathscr{F}_0\ \exists\, \nu \geq \bar{c}1_F : \lambda \geq \mu_1 \wedge \nu \geq \mu \wedge \bar{c}1_F$$

$$\Rightarrow \lambda \in \mathscr{R}$$

$$\lambda \in \mathscr{R} \Leftrightarrow \exists\, F \in \mathscr{F}_0 : \lambda \geq \mu \wedge \bar{c}1_F$$

$$\Rightarrow \lambda \in \mathscr{H}.$$

(iii) $\mathscr{F}_0 \subseteq \mathscr{H}_0$.

$$F \in \mathscr{F}_0 \quad \Rightarrow \bar{c}1_F \in (\mathscr{F}_0)_{\bar{c}} \subseteq <\mu> \vee (\mathscr{F}_0)_{\bar{c}} = \mathscr{H}$$

$$\Rightarrow (\bar{c}1_F)^0 = F \in \mathscr{H}_0.$$

(iv) $c(\mathscr{H}) = \bar{c}$.

Since $(\mathscr{F}_0)_{\bar{c}} \subseteq \mathscr{H}$ we have $c(\mathscr{H}) \leq c((\mathscr{F}_0)_{\bar{c}}) = \bar{c}$.

To show that $\bar{c} \leq c(\mathscr{H})$ let $0 < \alpha < \bar{c}$ and $F \in \mathscr{F}_0$. Then there exists $\nu \in \mathscr{F}$ such that $\nu^0 \subseteq F$ and, since $\mu \in \mathscr{F}$, $\nu \wedge \mu \in \mathscr{F}$. Thus $\sup \nu \wedge \mu \geq c(\mathscr{F}) \geq \bar{c} > \alpha$ and so we can find $x \in X$ with $(\nu \wedge \mu)(x) > \alpha$. Since $\nu(x) > \alpha > 0$, $x \in \nu^0 \subseteq F$ and so :
$\sup (\mu \wedge \bar{c}1_F) \geq (\mu \wedge \bar{c}1_F)(x) = \mu(x) \wedge \bar{c} \geq \alpha \wedge \bar{c} = \alpha$.

Since F and α are arbitrary we conclude that :

$$\bar{c} \leq \inf_{F \in \mathscr{F}_0} \sup (\mu \wedge \bar{c}1_F) = c(\mathscr{H}).$$

Now choose an ultrafilter $\mathbb{F} \supseteq \mathscr{H}^{\bar{c}} \supseteq \mathscr{H}_0$ and let $\mathscr{G} = \mathscr{H} \vee \mathbb{F}_1$. According to 2.17, $\mathscr{G} \in \mathscr{P}_m(\mathscr{H})$ and $c(\mathscr{G}) = c(\mathscr{H}) = \bar{c} = \bar{c}(\mathscr{F})$. Furthermore :

(v) $\mathscr{G}$ **is $\mathscr{D}$–Cauchy.**

$\mathscr{F}$ is $\mathscr{D}$–Cauchy

$\Leftrightarrow \mathscr{F}_0$ is $\mathscr{D}^{\bar{c}}$ –Cauchy	(4.4)
$\Leftrightarrow (\mathscr{F}_0)_{\bar{c}}$ is strong $\mathscr{D}$–Cauchy	(4.10)
$\Rightarrow \mathscr{H}$ is strong $\mathscr{D}$–Cauchy.	$((\mathscr{F}_0)_{\bar{c}} \subseteq \mathscr{H}$, 4.12)
$\Rightarrow \mathscr{G}$ is strong $\mathscr{D}$–Cauchy	$(\mathscr{H} \subseteq \mathscr{G}$, 4.12)
$\Rightarrow \mathscr{G}$ is $\mathscr{D}$–Cauchy.	(4.8)

We therefore have a prime, $\mathscr{D}$–Cauchy prefilter $\mathscr{G}$ with $\mu \in \mathscr{G}$ and hence, by assumption $\mathscr{G}$ is convergent in μ. In other words :

(vi) $c(\mathscr{G}) = \bar{c} \leq \sup(\mu \wedge \mathrm{Adh}\ \mathscr{G})$.

Let $0 < \alpha < \bar{c}$. Then by (vi) there exists $x \in \mu^{\alpha}$ with $(\mathrm{Adh}\ \mathscr{G})(x) > \alpha$. Consequently $\mathscr{G}_0 \to x$ w.r.t. $\mathscr{D}^{\alpha}$ (3.15). Now $\mathscr{G}_0 = (\mathscr{H} \vee \mathbb{F}_1)^0 = \mathbb{F}$ (2.13(c)) and so $\mathbb{F} \to x$ w.r.t. $\mathscr{D}^{\alpha}$. This means that x is a $\mathscr{D}^{\alpha}$–adherence point of $\mathscr{F}_0$. Now :

$\mathscr{F}$ is $\mathscr{D}$–Cauchy $\Leftrightarrow \mathscr{F}_0$ is $\mathscr{D}^{\bar{c}}$–Cauchy (4.4)

$\Rightarrow \mathscr{F}_0$ is $\mathscr{D}^{\alpha}$–Cauchy ($\mathscr{D}^{\alpha} \subseteq \mathscr{D}^{\bar{c}}$).

It follows therefore that $\mathscr{F}_0 \to x$ w.r.t. $\mathscr{D}^{\alpha}$ and hence that $(\lim \mathscr{F})(x) \geq \alpha$ (3.15). We also have $\mu(x) > \alpha$ and so :

$\sup(\mu \wedge \lim \mathscr{F}) \geq \mu(x) \wedge (\lim \mathscr{F})(x) > \alpha$.

Since α is arbitrary, (i) follows.

■

The α–level theorem for $\mathscr{D}$–completeness is not as simple as the other α–level theorems.

8.2 THEOREM

Let $(X, \mathscr{D})$ be a fuzzy uniform space with $\mu \in I^X$. The following are equivalent:

(a) μ is $\mathscr{D}$–complete.

(b) $\forall\ \alpha \in (0, \sup \mu]\ \forall\ \beta < \alpha\ \forall\ \mathbb{F}$

$\mathbb{F}$ is a $\mathscr{D}^{\alpha}$–Cauchy filter with $\mu_{\alpha} \in \mathbb{F} \Rightarrow \exists\ x \in \mu^{\beta} : \mathbb{F} \to x$ w.r.t. $\mathscr{D}^{\beta}$.

(c) $\forall\ \alpha \in (0, \sup \mu]\ \forall\ \beta < \alpha\ \forall\ \mathbb{F}$

$\mathbb{F}$ is a $\mathscr{D}^{\alpha}$–Cauchy filter with $\mu_{\alpha} \in \mathbb{F} \Rightarrow \exists\ x \in \mu_{\beta} : \mathbb{F} \to x$ w.r.t. $\mathscr{D}^{\beta}$.

(d) $\forall\ \alpha \in (0, \sup \mu]\ \forall\ \beta < \alpha\ \forall\ \mathbb{F}$

$\mathbb{F}$ is a $\mathscr{D}^{\alpha}$–Cauchy ultrafilter with $\mu_{\alpha} \in \mathbb{F} \Rightarrow \exists\ x \in \mu^{\beta} : \mathbb{F} \to x$ w.r.t. $\mathscr{D}^{\beta}$.

(e) $\forall\ \alpha \in (0, \sup \mu]\ \forall\ \beta < \alpha\ \forall\ \mathbb{F}$

$\mathbb{F}$ is a $\mathscr{D}^{\alpha}$–Cauchy ultrafilter with $\mu_{\alpha} \in \mathbb{F} \Rightarrow \exists\ x \in \mu_{\beta} : \mathbb{F} \to x$ w.r.t. $\mathscr{D}^{\beta}$.

PROOF

(a) $\Rightarrow$ (b) : Let μ be $\mathscr{D}$–complete, $0 < \alpha \leq \sup \mu$, $0 \leq \beta < \alpha$ and let $\mathbb{F}$ be a $\mathscr{D}^{\alpha}$–Cauchy filter with $\mu_{\alpha} \in \mathbb{F}$. Then $\mathbb{F}_{\alpha}$ is strong $\mathscr{D}$–Cauchy (4.10) and so $\mathbb{F}_{\alpha}$ is $\mathscr{D}$–Cauchy (4.8). Since $\alpha 1_{\mu_{\alpha}} \leq \mu$ and $\mu_{\alpha} \in \mathbb{F}$, $\mu \in \mathbb{F}_{\alpha}$. We therefore have $\mu \in \mathbb{F}_{\alpha}$, $\mathbb{F}_{\alpha}$ is $\mathscr{D}$–Cauchy and μ is $\mathscr{D}$–complete and so it follows that :

$$\bar{c}(\mathbb{F}_{\alpha}) = \alpha \leq \sup(\mu \wedge \lim \mathbb{F}_{\alpha}) \quad (2.22)$$

$$\Rightarrow \quad \exists\ x \in \mu^{\beta} : (\lim \mathbb{F}_{\alpha})(x) > \beta$$

$$\Rightarrow \quad \exists\ x \in \mu^{\beta} : (\mathbb{F}_{\alpha})_{o} = \mathbb{F} \to x \text{ w.r.t. } \mathscr{D}^{\beta} \quad (3.15, 2.7(c)).$$

(b) $\Rightarrow$ (c) : This is immediate since $\mu^{\beta} \subseteq \mu_{\beta}$.

(c) $\Rightarrow$ (e) : Obvious.

(e) $\Longrightarrow$ (d) : Let $0 < \alpha < \sup \mu$, $\beta < \alpha$ and let $\mathbb{F}$ be a $\mathscr{D}^{\alpha}$–Cauchy ultrafilter with $\mu_\alpha \in \mathbb{F}$. Choose γ such that $\beta < \gamma < \alpha$. Then by (e) there exists $x \in \mu_\gamma$ such that $\mathbb{F} \longrightarrow x$ w.r.t. $\mathscr{D}^{\gamma}$. Thus $x \in \mu^{\beta}$ and $\mathbb{F} \longrightarrow x$ w.r.t. $\mathscr{D}^{\beta}$ since $\mathscr{D}^{\beta} \subseteq \mathscr{D}^{\gamma}$.

(d) $\Longrightarrow$ (a) : Let $\mathscr{F}$ be a prime $\mathscr{D}$–Cauchy prefilter with $c(\mathscr{F}) = c > 0$ and $\mu \in \mathscr{F}$. Then $\mathscr{F}_0$ is $\mathscr{D}^c$–Cauchy (4.4) and $c \leq \sup \mu$. Let $0 \leq \beta < c$ and let $\beta < \alpha < c$. Then $\mu^{\alpha} \in \mathscr{F}_0$ (2.19) and since $\mu^{\alpha} \subseteq \mu_\alpha$, $\mu_\alpha \in \mathscr{F}_0$. Also, since $\mathscr{F}$ is prime, $\mathscr{F}_0$ is an ultrafilter (2.9(a)) and since $\mathscr{D}^{\alpha} \subseteq \mathscr{D}^c$, $\mathscr{F}_0$ is $\mathscr{D}^{\alpha}$–Cauchy. Thus there exists $x \in \mu^{\beta}$ such that $\mathscr{F}_0 \longrightarrow x$ w.r.t. $\mathscr{D}^{\beta}$ and hence $(\mathrm{Adh}\ \mathscr{F})(x) \geq \beta$ (3.15). Consequently $\sup(\mu \wedge \mathrm{Adh}\ \mathscr{F}) \geq \mu(x) \wedge \mathrm{Adh}\ \mathscr{F}(x) \geq \beta$ and since β is arbitrary, $\sup(\mu \wedge \mathrm{Adh}\ \mathscr{F}) \geq c = c(\mathscr{F})$. In other words, $\mathscr{F}$ is convergent in μ and this proves that μ is $\mathscr{D}$–complete.

■

If the fuzzy uniformity is Hausdorff then the α–level theorem reduces to a statement which is reminiscent of the other α–level theorems.

8.3 THEOREM

> If $(X, \mathscr{D})$ is a Hausdorff fuzzy uniform space, and $\mu \in I^X$ then :
> μ is $\mathscr{D}$–complete $\Leftrightarrow \forall\ \alpha \in (0, \sup \mu]$ μ_α is $\mathscr{D}^{\alpha}$–complete.

PROOF

Let μ be $\mathscr{D}$–complete, $0 < \alpha \leq \sup \mu$ and $\mathbb{K}$ be a $\mathscr{D}^{\alpha}$–Cauchy filter with $\mu_\alpha \in \mathbb{K}$. Let $\mathbb{F}$ be an ultrafilter with $\mathbb{K} \subseteq \mathbb{F}$. Then $\mathbb{F}$ is also $\mathscr{D}^{\alpha}$–Cauchy and $\mu_\alpha \in \mathbb{F}$. Since $\alpha 1_{\mu_\alpha} \leq \mu$, $\mu \in \mathbb{F}_\alpha$ and since $\mathbb{F} = (\mathbb{F}_\alpha)_0$ (2.7(c)), $\mathbb{F}_\alpha$ is $\mathscr{D}$–Cauchy and prime (4.7, 2.9(a)). Consequently $\mathbb{F}_\alpha$ is convergent in μ. In other words :

$\alpha = c(\mathbb{F}_\alpha) \leq \sup \mu \wedge \mathrm{Adh}\, \mathbb{F}_\alpha$. Thus :

(i) $\forall\, \beta < \alpha\ \exists\, x_\beta \in \mu^\beta : (\mathrm{Adh}\,\mathbb{F}_\alpha)(x_\beta) > \beta.$

We must show that $\mathbb{K}$ is $\mathscr{D}^\alpha$–convergent to some point $x \in \mu_\alpha$ and this will be achieved if we show that $\mathbb{F}$ is $\mathscr{D}^\alpha$–convergent to some $x \in \mu_\alpha$.

Now since $\mathscr{D}$ is Hausdorff and $\mathbb{F}_\alpha$ is prime ($(\mathbb{F}_\alpha)_0 = \mathbb{F}$ which is ultra, 2.9(a)), $\{x_\beta : \beta < \alpha\}$ is a singleton. So let $\{x_\beta : \beta < \alpha\} = \{x\}$.

Then $\forall\, \beta < \alpha, x \in \mu^\beta \quad \Rightarrow \forall\, \beta < \alpha\ \mu(x) > \beta$
$\Rightarrow \mu(x) \geq \alpha.$

Thus $x \in \mu_\alpha$ and from (i) we have :

$\forall\, \beta < \alpha\ (\mathrm{Adh}\,\mathbb{F}_\alpha)(x) > \beta \quad \Rightarrow (\mathrm{Adh}\,\mathbb{F}_\alpha)(x) \geq \alpha$
$\Rightarrow \mathbb{F} \to x$ w.r.t. $\mathscr{D}^\alpha$ (3.15).

We have therefore found $x \in \mu_\alpha$ such that $\mathbb{F} \to x$ w.r.t. $\mathscr{D}^\alpha$ and so μ_α is $\mathscr{D}^\alpha$–complete.

For the converse let $0 < \alpha \leq \sup \mu$, $\beta < \alpha$ and let $\mathbb{F}$ be a $\mathscr{D}^\alpha$–Cauchy filter with $\mu_\alpha \in \mathbb{F}$. By assumption μ_α is $\mathscr{D}^\alpha$–complete and so there exists $x \in \mu_\alpha$ such that $\mathbb{F} \to x$ w.r.t. $\mathscr{D}^\alpha$. Since $\mu_\alpha \subseteq \mu_\beta$ and $\mathscr{D}^\beta \subseteq \mathscr{D}^\alpha$ we have $x \in \mu_\beta$ and $\mathbb{F} \to x$ w.r.t. $\mathscr{D}^\beta$ and hence μ is $\mathscr{D}$–complete (8.2(c)).

■

The following example shows that the Hausdorff condition in Theorem 8.3 is necessary.

8.4 EXAMPLE

There is a fuzzy uniform space $(X, \mathscr{D})$ and a $\mathscr{D}$–complete $\mu \in I^X$ such that for some $\alpha \in (0, \sup \mu]$, μ_α is not $\mathscr{D}^\alpha$–complete.

Let $X = \{x \in \mathbb{R}^{\mathbb{N}} : \{n \in \mathbb{N} : x_n \neq 0\} \text{ is finite}\} = \{\text{real sequences which are eventually zero}\}$. For $x = (x_n : n \in \mathbb{N}) \in X$ define :

$$\|x\| = \sum_{i=1}^{\infty} |x_i| \quad \text{and for } m \in \mathbb{N} \text{ let } \|x\|_m = \sum_{i=1}^{m} |x_i|.$$

These are seminorms on X and $\|\cdot\|_1 \leq \|\cdot\|_2 \leq \cdots \leq \|\cdot\|$.

Let $\mathbb{D}(\|\cdot\|)$ denote the uniformity associated with $\|\cdot\|$ and for $m \in \mathbb{N}$ let $\mathbb{D}(\|\cdot\|_m)$ denote the uniformity associated with $\|\cdot\|_m$.

For $\frac{1}{2} < \alpha \leq 1$ let $\mathbb{D}(\alpha) = \mathbb{D}(\|\cdot\|)$.

For each $m \in \mathbb{N}$ and each $\alpha \in (\frac{1}{2} - \frac{1}{m+1}, \frac{1}{2} - \frac{1}{m+2}]$ let $\mathbb{D}(\alpha) = \mathbb{D}(\|\cdot\|_m)$. We therefore have :

(a) $\quad 0 < \beta \leq \alpha \leq 1 \implies \mathbb{D}(\beta) \subseteq \mathbb{D}(\alpha)$,

(b) $\quad \mathbb{D}(\alpha) = \bigcup_{\beta < \alpha} \mathbb{D}(\beta)$ for each $\alpha \in I_0$.

This is due to the fact that $\|\cdot\|_1 \leq \|\cdot\|_2 \leq \cdots \leq \|\cdot\|$ and for $x \in X$, $\|x\| = \lim_{m \to \infty} \|x\|_m$.

The conditions (a) and (b) that the family $(\mathbb{D}(\alpha) : \alpha \in I_0)$ fulfills are the conditions (a) and (b) in 3.21 which we now invoke to assert that there is a fuzzy uniformity $\mathscr{D}$ on X such that $\forall\ \alpha \in I_0\ \ \mathscr{D}^{\alpha} = \mathbb{D}(\alpha)$. We note that :

$\mathscr{D}$ is not Hausdorff.

For $\alpha < \frac{1}{2}$ $\mathscr{D}^{\alpha}$ is not Hausdorff and hence $\mathscr{D}$ is not Hausdorff (3.16).

Let $\mu = \frac{1}{2} 1_X$ then :

μ is $\mathscr{D}$–complete.

$\sup \mu = \frac{1}{2}$ so let $\alpha \in (0, \frac{1}{2}]$, $\beta < \alpha$, $\mu_{\alpha} \in \mathbb{F}$ with $\mathbb{F}$ a $\mathscr{D}^{\alpha}$–Cauchy filter on X. There are two cases to consider.

(i) $\alpha = \frac{1}{2}$: Here $\mu_{\alpha} = \mu_{1/2} = X$ and $\mathscr{D}^{\alpha} = \mathscr{D}^{1/2} = \mathbb{D}(\frac{1}{2}) = \mathbb{D}(\|\cdot\|)$. Thus $\mathbb{F}$ is $\|\cdot\|$ – Cauchy. Since $\beta < \frac{1}{2}$ there exists $m \in \mathbb{N}$ such that $\beta \in (\frac{1}{2} - \frac{1}{m+1}, \frac{1}{2} - \frac{1}{m+2}]$ and then $\mathscr{D}^{\beta} = \mathbb{D}(\|\cdot\|_m)$.

Now X is $\|\cdot\|_m$ –complete and $\mathbb{F}$ is $\|\cdot\|_m$ –Cauchy ($\|\cdot\|_m \leq \|\cdot\|$) and hence $\mathbb{F}$ is $\mathscr{D}^\beta$ convergent in $\mu_\beta = X$.

(ii) $\alpha < \frac{1}{2}$: Again $\mu_\alpha = X$ but now $\mathscr{D}^\alpha = \mathbb{D}(\|\cdot\|_m)$ for some $m \in \mathbb{N}$. Since $\beta < \alpha$, $\mathscr{D}^\beta = \mathbb{D}(\|\cdot\|_n)$ for some $n \leq m$. Thus since $\|\cdot\|_n \leq \|\cdot\|_m$ and $\mathbb{F}$ is $\|\cdot\|_m$ –Cauchy, $\mathbb{F}$ is $\|\cdot\|_n$ –Cauchy. Again since X is $\|\cdot\|_n$ –complete $\mathbb{F}$ is $\mathscr{D}^\beta$ convergent in $\mu_\beta = X$.

We appeal to 8.2(c) and assert that μ is $\mathscr{D}$–complete. However :

$\mu_{1/2}$ is not $\mathscr{D}^{1/2}$ –complete.

This follows from the fact that

$$\mu_{1/2} = X,\ \mathscr{D}^{1/2} = \mathbb{D}(\tfrac{1}{2}) = \bigcup_{\beta < \frac{1}{2}} \mathbb{D}(\beta) = \bigcup_{m \in \mathbb{N}} \mathbb{D}(\|\cdot\|_m) = \mathbb{D}(\|\cdot\|) \text{ and X is not}$$

$\|\cdot\|$ –complete.

■

The characterisation of completeness in terms of prime prefilters is very useful but, as we did for precompactness and boundedness, we can do a little better.

8.5 THEOREM

Let $(X, \mathscr{D})$ be a fuzzy uniform space and let $\mu \in I^X$. The following are equivalent :

(a) μ is $\mathscr{D}$–complete.

(b) For every prime prefilter $\mathscr{F}$ $((\mathscr{F},\mu)$ is $\mathscr{D}$–Cauchy $\Rightarrow$ $(\mathscr{F},\mu)$ is convergent in $\mu)$.

(c) For every prime prefilter $\mathscr{F}$ $((\mathscr{F},\mu)$ is $\mathscr{D}$–Cauchy $\Rightarrow$ $c(\mathscr{F},\mu) \leq \sup \mu \wedge \text{Adh } \mathscr{F})$.

(d) Every prime $\mathscr{D}$–Cauchy saturated prefilter $\mathscr{F}$ with $\mu \in \mathscr{F}$ is $\mathscr{D}$–convergent in μ.

PROOF

(a) $\Longrightarrow$ (b) : Let $\mathcal{F}$ be a prime filter with $(\mathcal{F},\mu)$ $\mathcal{D}$–Cauchy. Then $(\mathcal{F},\mu)$ is also prime (2.9(e)) and $\mu \in (\mathcal{F},\mu)$. Consequently, since μ is $\mathcal{D}$–complete, $(\mathcal{F},\mu)$ is convergent in μ.

(b) $\Longrightarrow$ (c) :

$(\mathcal{F},\mu)$ is convergent in μ

$\Leftrightarrow c(\mathcal{F},\mu) \leq \sup \mu \wedge \mathrm{Adh}(\mathcal{F},\mu)$

$\Longrightarrow c(\mathcal{F},\mu) \leq \sup \mu \wedge \mathrm{Adh}\, \mathcal{F}$.

(c) $\Longrightarrow$ (a) : Let $\mathcal{F}$ be a prime $\mathcal{D}$–Cauchy prefilter with $\mu \in \mathcal{F}$. Since $\mu \in \mathcal{F}$, $(\mathcal{F},\mu) = \mathcal{F}$ and so $(\mathcal{F},\mu)$ is $\mathcal{D}$–Cauchy. Consequently :

$c(\mathcal{F},\mu) = c(\mathcal{F}) \leq \sup \mu \wedge \mathrm{Adh}\, \mathcal{F}$.

In other words, $\mathcal{F}$ is convergent in μ.

(a) $\Longrightarrow$ (d) : Obvious.

(d) $\Longrightarrow$ (a) : Let $\mathcal{F}$ be a prime $\mathcal{D}$–Cauchy prefilter with $\mu \in \mathcal{F}$ and $c(\mathcal{F}) = c$. Let $\mathcal{H} = (\mathcal{F}_0)^c$ then from the proof of 6.4 we have : $\mathcal{H}$ is a prime saturated prefilter, $c(\mathcal{H}) = c$ and $\mathcal{F} \subseteq \mathcal{H}$. Since $\mu \in \mathcal{H}$ we have :

$c = c(\mathcal{H}) \leq \sup \mu \wedge \mathrm{Adh}\, \mathcal{H} \leq \sup \mu \wedge \mathrm{Adh}\, \mathcal{F}$.

Thus $\mathcal{F}$ is convergent in μ and so we have shown that μ is $\mathcal{D}$–complete.

■

8.6 EXAMPLE

Let X be the closed unit ball of a non–reflexive Banach space E.

Let $\mathbb{D}_w$ denote the uniformity on X associated with the weak topology and let $\mathbb{D}_s$ denote the uniformity associated with the norm topology (see [Wi 2, 38.2] for details). We observe that :

$\mathbb{D}_w$ is Hausdorff, X is $\mathbb{D}_w$–closed, X is $\mathbb{D}_w$–precompact, X is $\mathbb{D}_w$–bounded but X is not $\mathbb{D}_w$–complete (otherwise X would be $\mathbb{D}_w$–compact and hence E would be reflexive).

$\mathbb{D}_s$ is Hausdorff, X is $\mathbb{D}_s$–closed, X is $\mathbb{D}_s$–bounded, X is $\mathbb{D}_s$–complete but X is not $\mathbb{D}_s$–precompact (otherwise E would be finite–dimensional and hence reflexive).

If $0 < \alpha \leq \frac{1}{2}$ let $\mathbb{D}(\alpha) = \mathbb{D}_w$.
If $\frac{1}{2} < \alpha \leq 1$ let $\mathbb{D}(\alpha) = \mathbb{D}_s$.

As we did in 3.23 we obtain a unique fuzzy uniformity $\mathscr{D}$ on X such that :
If $0 < \alpha \leq \frac{1}{2}$ $\mathscr{D}^{\alpha} = \mathbb{D}_w$
If $\frac{1}{2} < \alpha \leq 1$ let $\mathscr{D}^{\alpha} = \mathbb{D}_s$.

We make the following observations :

(a) **$\mathscr{D}$ is Hausdorff.**
Since each $\mathscr{D}^{\alpha}$ is Hausdorff, $\mathscr{D}$ is Hausdorff.

(b) **$\alpha 1_X$ is $\mathscr{D}$–bounded for each $\alpha \in I_0$.**
If $\alpha, \beta \in I_0$ and $\mu = \alpha 1_X$ then :
$\mu_\beta = X$ if $\beta \leq \alpha$,
$\mu_\beta = \emptyset$ if $\beta > \alpha$.
Since X is $\mathbb{D}_w$–bounded and $\mathbb{D}_s$–bounded it follows from 7.14 that μ is $\mathscr{D}$–bounded.

(c) **$\alpha 1_X$ is $\mathscr{D}$–precompact for $0 < \alpha \leq \frac{1}{2}$.**
If $\mu = \alpha 1_X$ with $0 < \alpha \leq \frac{1}{2}$ then :
if $\beta \leq \alpha$: $\mu_\beta = X$ and $\mathscr{D}^{\beta} = \mathbb{D}_w$,
if $\beta > \alpha$: $\mu_\beta = \emptyset$ and $\mathscr{D}^{\beta} = \mathbb{D}_s$.
Thus μ_β is $\mathscr{D}^{\beta}$–precompact for each $\beta \in I_0$ so it follows from 6.5 that μ is $\mathscr{D}$–precompact.

(d) **$\alpha 1_X$ is not $\mathscr{D}$–precompact for $\alpha > \frac{1}{2}$.**
If $\mu = \alpha 1_X$ with $\alpha > \frac{1}{2}$ then :
If $\frac{1}{2} < \beta < \alpha$: $\mu_\beta = X$ and $\mathscr{D}^{\beta} = \mathbb{D}_s$.

Since μ_β is not $\mathscr{D}^{\beta}$–precompact we conclude from 6.5 that μ is not $\mathscr{D}$–precompact.

(e) **$\alpha 1_X$ is not $\mathscr{D}$–complete for any $\alpha \in I_0$.**

If $\mu = \alpha 1_X$ with $\alpha \in I_0$ then :

if $\beta < \frac{1}{2} \wedge \alpha : \mu_\beta = X$ and $\mathscr{D}^\beta = \mathbb{D}_w$.

Since μ_β is not $\mathscr{D}^\beta$–complete it follows from 8.3 that μ is not $\mathscr{D}$–complete.

■

We managed to obtain characterisations of bounded and precompact fuzzy sets in terms of prime prefilters with characteristic one and in [Ch 2] Theorem 3.2 we find a similar result for compact fuzzy sets. With Theorems 6.4 and 7.15 in mind we make the following conjecture.

8.7 CONJECTURE

If $(X, \mathscr{D})$ is a fuzzy uniform space and $\mu \in I^X$, then the following are equivalent :

(a) μ is $\mathscr{D}$–complete.

(b) For every prime prefilter $\mathscr{F}$ with $c(\mathscr{F}) = 1$
$(\mathscr{F},\mu)$ is $\mathscr{D}$–Cauchy $\Longrightarrow$ $\mathscr{F}$ is convergent in μ.

Example 8.6(e) reveals that this is **not** true. For if $(X, \mathscr{D})$ is the fuzzy uniform space defined in 8.6 and $\mu = 1_X$ then on the one hand μ is not $\mathscr{D}$–complete. On the other hand, if $\mathscr{F}$ is any prime prefilter with $c(\mathscr{F}) = 1$ then :

$$
\begin{aligned}
\mathscr{F} \text{ is } \mathscr{D}\text{–Cauchy} \quad &\Leftrightarrow \mathscr{F}_0 \text{ is } \mathscr{D}^1\text{–Cauchy} \quad (4.4) \\
&\Leftrightarrow \mathscr{F}_0 \text{ is } \mathbb{D}_s\text{–Cauchy} \\
&\Leftrightarrow \mathscr{F}_0 \text{ is } \mathbb{D}_s\text{–convergent in X. (X is } \mathbb{D}_s\text{–complete)} \\
&\Leftrightarrow \exists\, x \in X : \mathscr{F}_0 \rightarrow x \text{ w.r.t. } \mathscr{D}^1. \\
&\Leftrightarrow \exists\, x \in X : (\lim \mathscr{F})(x) \geq 1 \quad (3.15) \\
&\Longrightarrow \sup(\mu \wedge \lim \mathscr{F}) = 1 \\
&\Leftrightarrow \mathscr{F} \text{ is } \mathscr{D}\text{–convergent in } \mu.
\end{aligned}
$$

Now since $\mu = 1_X$, $\mu \in \mathscr{F}$ for any prefilter $\mathscr{F}$ and so every $\mathscr{D}$–Cauchy prefilter $\mathscr{F} = (\mathscr{F},\mu)$ with $c(\mathscr{F}) = 1$ is convergent in μ, but μ is not $\mathscr{D}$–complete.

■

Completeness can also be described in terms of strong Cauchy prefilters as follows.

8.8 THEOREM

> If $(X, \mathscr{D})$ is a fuzzy uniform space and $\mu \in I^X$, then the following are equivalent.
>
> (a) μ is $\mathscr{D}$–complete.
>
> (b) For every strong $\mathscr{D}$–Cauchy prefilter $\mathscr{F}$ with $\mu \in \mathscr{F}$
> $$\bar{c}(\mathscr{F}) \le \sup(\mu \wedge \lim \mathscr{F}).$$
>
> (c) For every strong $\mathscr{D}$–Cauchy prefilter $\mathscr{F}$ with $\mu \in \mathscr{F}$
> $$c(\mathscr{F}) \le \sup(\mu \wedge \text{Adh } \mathscr{F}).$$

PROOF

(a) $\Longrightarrow$ (b) : Every strong $\mathscr{D}$–Cauchy prefilter is $\mathscr{D}$–Cauchy (4.8) and $\bar{c}(\mathscr{F}) \le \sup(\mu \wedge \lim \mathscr{F})$ iff $\mathscr{F}$ is convergent in μ. Thus (b) follows from (a).

(b) $\Longrightarrow$ (a) : We intend to employ 8.2 so we let $0 < \alpha \le \sup \mu$, $\beta < \alpha$, $\mathbb{F}$ be a $\mathscr{D}^{\alpha}$ –Cauchy ultrafilter with $\mu_\alpha \in \mathbb{F}$. Then $\mathbb{F}_\alpha$ is strong $\mathscr{D}$–Cauchy (4.10), $\alpha 1_{\mu_\alpha} \in \mathbb{F}_\alpha$ and, since $\alpha 1_{\mu_\alpha} \le \mu$, $\mu \in \mathbb{F}_\alpha$. We note that $\bar{c}(\mathbb{F}_\alpha) = \alpha$ (2.22) and $\mathbb{F}_\alpha$ is prime ($(\mathbb{F}_\alpha)_0 = \mathbb{F}$ which is ultra, 2.7(c), 2.9(a)). By assumption then :

$$\bar{c}(\mathbb{F}_\alpha) = \alpha \le \sup(\mu \wedge \lim \mathbb{F}_\alpha) = \sup(\mu \wedge \text{Adh } \mathbb{F}_\alpha).$$

Since $\beta < \alpha$ there exists $x \in \mu^\beta$ such that $(\text{Adh } \mathbb{F}_\alpha)(x) > \beta$. Consequently $(\mathbb{F}_\alpha)_0 = \mathbb{F} \to x$ w.r.t. $\mathscr{D}^\beta$ (3.15) and hence we have shown, according to 8.2, that μ is $\mathscr{D}$–complete.

(a) $\Longrightarrow$ (c) : Let μ be $\mathcal{D}$–complete, $\mathcal{F}$ a strong $\mathcal{D}$–Cauchy prefilter with $\mu \in \mathcal{F}$ and let $c(\mathcal{F}) = c > 0$. Let $\gamma < c$ and choose α,β such that $\gamma < \beta < \alpha < c$. Since $\mu \in \mathcal{F}$, $c = \inf_{\nu \in \mathcal{F}} \sup \nu \leq \sup \mu$ and so $\alpha < \sup \mu$. Again since $\mu \in \mathcal{F}$, $\mu^{\alpha} \in \mathcal{F}^c$ and hence $\mu_\alpha \in \mathcal{F}^c (\mu^{\alpha} \subseteq \mu_\alpha)$. Since $\mathcal{F}$ is strong $\mathcal{D}$–Cauchy, $\mathcal{F}^c$ is $\mathcal{D}^c$–Cauchy (4.14(a)) and so $\mathcal{F}^c$ is $\mathcal{D}^{\alpha}$–Cauchy ($\mathcal{D}^{\alpha} \subseteq \mathcal{D}^c$). Putting these observations together we have : $\alpha < \sup \mu$, $\mu_\alpha \in \mathcal{F}^c$ and $\mathcal{F}^c$ is $\mathcal{D}^{\alpha}$–Cauchy and hence there exists $x \in \mu^{\beta}$ such that $\mathcal{F}^c \longrightarrow x$ w.r.t. $\mathcal{D}^{\beta}$ (8.2(b)).

Let symmetric $\sigma \in \mathcal{D}$ and $\nu \in \mathcal{F}$. Then $\sigma^{\gamma} \in \mathcal{D}^{\beta}$ and so $\sigma^{\gamma}(x)$ is a $\mathcal{D}^{\beta}$–neighbourhood of x. Thus $\sigma^{\gamma}(x) \in \mathcal{F}^c$ and $\nu^{\gamma} \in \mathcal{F}^c$ from which it follows that $\sigma^{\gamma}(x) \cap \nu^{\gamma} \neq \emptyset$. If $y \in \sigma^{\gamma}(x) \cap \nu^{\gamma}$ then $(\sigma\langle x\rangle \wedge \nu)(y) > \gamma$ and hence $\sup \sigma\langle x\rangle \wedge \nu = \sigma\langle\nu\rangle(x) > \gamma$. Since σ and ν are arbitrary,

$$\sup \mu \wedge \mathrm{Adh}\, \mathcal{F} \geq \mu(x) \wedge \mathrm{Adh}\,(x) \geq \beta \wedge \inf_{\sigma \in \mathcal{D}} \inf_{\nu \in \mathcal{F}} \sigma\langle\nu\rangle(x) \geq \beta \wedge \gamma = \gamma.$$

Since γ is arbitrary,
$\sup \mu \wedge \mathrm{Adh}\, \mathcal{F} \geq c = c(\mathcal{F})$.

(c) $\Longrightarrow$ (a) : Let $\mathcal{F}$ be a prime $\mathcal{D}$–Cauchy prefilter with $\mu \in \mathcal{F}$. Then $\mathcal{F}$ is strong $\mathcal{D}$–Cauchy and hence, by assumption,

$c(\mathcal{F}) \leq \sup \mu \wedge \mathrm{Adh}\, \mathcal{F}$.

Since $\mathcal{F}$ is prime, this means that $\mathcal{F}$ is convergent in μ and this proves that μ is $\mathcal{D}$–complete.

■

Let us check that our definition of completeness is a good extension of the standard notion.

8.9 THEOREM

> Let $(X.\mathbb{D})$ be a uniform space and let $A \subseteq X$. Then
> A is $\mathbb{D}$–complete $\Leftrightarrow$ 1_A is $\mathbb{D}^1$–complete.

PROOF

Let A be $\mathbb{D}$–complete and let $\mathscr{F}$ be a prime $\mathbb{D}^1$–Cauchy prefilter with $1_A \in \mathscr{F}$ and $c = c(\mathscr{F})$. From 2.7(c) we see that $(\mathbb{D}^1)^c = \mathbb{D}$ and hence $\mathscr{F}_0$ is $\mathbb{D}$–Cauchy (4.4). Now $(1_A)^0 = A \in \mathscr{F}_0$ and, since A is $\mathbb{D}$–complete, there exists $x \in A$ such that $\mathscr{F}_0 \to x$ w.r.t. $\mathbb{D} = (\mathbb{D}^1)^c$. Consequently we have $(\text{Adh } \mathscr{F})(x) \geq c$. Furthermore, $1_A(x) = 1 \geq c$ and so

$$\sup(1_A \wedge \text{Adh } \mathscr{F}) \geq c = c(\mathscr{F}).$$

This means that $\mathscr{F}$ is $\mathbb{D}^1$–convergent in 1_A and we have shown that 1_A is $\mathbb{D}^1$–complete.

To show the converse, let $\mathbb{F}$ be a $\mathbb{D}$–Cauchy filter on X with $A \in \mathbb{F}$. Then $\mathbb{F}^1$ is $\mathbb{D}^1$–Cauchy (4.6), $1_A \in \mathbb{F}^1$ (for $\alpha \in I_0$, $(1_A)^\alpha = A \in \mathbb{F}$) and $\bar{c}(\mathbb{F}^1) = 1$ (2.23). Since 1_A is $\mathbb{D}$–complete, we have :

$$\bar{c}(\mathbb{F}^1) = 1 \leq \sup(1_A \wedge \lim \mathbb{F}^1).$$

Thus, if we choose $0 < \alpha < 1$ there exists $x \in X$ such that $1_A(x) > \alpha$ and $(\lim \mathbb{F}^1)(x) > \alpha$. It follows that $x \in A$ and $(\mathbb{F}^1)_0 = \mathbb{F} \to x$ w.r.t. $(\mathbb{D}^1)^\alpha = \mathbb{D}$ (3.15, 2.7(c)). We have therefore shown that A is $\mathbb{D}$–complete.

■

We now investigate the properties of completeness in F.UNIF bearing in mind the properties of completeness in UNIF. In other words, we extend the theory of complete subsets of a uniform space to obtain the theory of complete fuzzy sets in a fuzzy uniform space.

In UNIF, compactness is equivalent to completeness together with precompactness. We obtain the same result in F.UNIF.

8.10 THEOREM

Let $(X, \mathscr{D})$ be a fuzzy uniform space and let $\mu \in I^X$. Then :
μ is $\mathscr{D}$–compact $\Leftrightarrow$ μ is $\mathscr{D}$–complete and $\mathscr{D}$–precompact.

PROOF

Let μ be $\mathscr{D}$–compact and let $\mathscr{F}$ be a prime prefilter with $\mu \in \mathscr{F}$. Then $\mathscr{F}$ is convergent in μ (5.2). It follows immediately that μ is $\mathscr{D}$–complete. Furthermore, since convergent prefilters are Cauchy (4.17), it follows that μ is $\mathscr{D}$–precompact.

For the converse, let $\mathscr{F}$ be a prime prefilter with $\mu \in \mathscr{F}$. Since μ is $\mathscr{D}$–precompact, $\mathscr{F}$ is $\mathscr{D}$–Cauchy and hence, since μ is $\mathscr{D}$–complete, $\mathscr{F}$ is convergent in μ. Thus μ is $\mathscr{D}$–compact (5.2).

■

Closed subsets of complete subsets of a uniform space are complete and the same is true in fuzzy uniform spaces.

8.11 THEOREM

> Let $(X,\mathscr{D})$ be a fuzzy uniform space and let $\mu, \lambda \in I^X$. Then :
> μ is $\mathscr{D}$–complete and λ is $\mathscr{D}$–closed $\Rightarrow$ $\mu \wedge \lambda$ is $\mathscr{D}$–complete.

PROOF

Let $\mathscr{F}$ be a prime $\mathscr{D}$–Cauchy prefilter with $\mu \wedge \lambda \in \mathscr{F}$. Since $\mu \wedge \lambda \le \mu$, $\mu \in \mathscr{F}$ and since μ is $\mathscr{D}$–complete, $\mathscr{F}$ is convergent in μ. So we have :

$$c(\mathscr{F}) \le \sup(\mu \wedge \operatorname{Adh} \mathscr{F}).$$

Now:

$$\begin{aligned}
\sup(\mu \wedge \lambda \wedge \operatorname{Adh} \mathscr{F}) &= \sup\,(\mu \wedge \lambda \wedge \inf_{\nu \in \mathscr{F}} \bar{\nu}) \\
&= \sup\,(\mu \wedge \inf_{\nu \in \mathscr{F}} \lambda \wedge \bar{\nu}) \\
&= \sup\,(\mu \wedge \inf_{\nu \in \mathscr{F}} \bar{\lambda} \wedge \bar{\nu}) \quad (\lambda \text{ is closed}) \\
&\ge \sup\,(\mu \wedge \inf_{\nu \in \mathscr{F}} \overline{\lambda \wedge \nu}\,) \quad (3.3(e)) \\
&= \sup\,(\mu \wedge \operatorname{Adh}(\mathscr{F},\lambda)) \\
&= \sup\,(\mu \wedge \operatorname{Adh}(\mathscr{F})) \quad (\mu \wedge \lambda \le \lambda \Rightarrow \lambda \in \mathscr{F}) \\
&\ge c(\mathscr{F}).
\end{aligned}$$

This shows that $\mathscr{F}$ is convergent in $\mu \wedge \lambda$ and hence that $\mu \wedge \lambda$ is $\mathscr{D}$–complete.

■

In UNIF, finite unions of complete sets are complete. We prove the analagous theorem in F.UNIF.

8.12 THEOREM

> Let $(X, \mathscr{D})$ be a fuzzy uniform space with $\mu_i \in I^X$ for each $i \in [n]$. Then :
> $\forall\, i \in [n]$ μ_i is $\mathscr{D}$–complete $\Rightarrow \sup_{i \in [n]} \mu_i$ is $\mathscr{D}$–complete.

PROOF

Let $\mathscr{F}$ be a prime prefilter with $\mu := \sup_{i \in [n]} \mu_i \in \mathscr{F}$. Since $\mathscr{F}$ is prime there exists $j \in [n]$ such that $\mu_j \in \mathscr{F}$ and now, since μ_j is $\mathscr{D}$–complete, $\mathscr{F}$ is convergent in μ_j. Thus :

$$c(\mathscr{F}) \leq \sup(\mu_j \wedge \text{Adh } \mathscr{F}) \leq \sup(\mu \wedge \text{Adh } \mathscr{F}).$$

and this means that $\mathscr{F}$ is convergent in μ. We have therefore shown that μ is $\mathscr{D}$–complete.

■

If a uniform space is Hausdorff then complete subsets are closed. This result extends to fuzzy uniform spaces.

8.13 THEOREM

> If $(X, \mathscr{D})$ is a Hausdorff fuzzy uniform space and $\mu \in I^X$ is $\mathscr{D}$–complete then μ is $\mathscr{D}$–closed.

PROOF

We must show that $\bar{\mu} \leq \mu$ so we let $x \in X$ be arbitrary and show that $\bar{\mu}(x) \leq \mu(x)$. Now $\bar{\mu}(x) = c(\mathscr{D}_x, \mu)$ (3.4(e)) so choose $\mathscr{G} \in \mathscr{P}_m((\mathscr{D}_x, \mu))$ with $c(\mathscr{G}) = c(\mathscr{D}_x, \mu)$ (2.16). Let $\alpha < c(\mathscr{G})$ and $\sigma \in \mathscr{D}$. Since $\sigma\langle x\rangle \in \mathscr{D}_x$, $\sigma\langle x\rangle \in \mathscr{G}$ and hence :

$$\forall\, \nu \in \mathcal{G}\ \sigma\langle x\rangle \wedge \nu \in \mathcal{G} \quad \Rightarrow \forall\, \nu \in \mathcal{G}\ \sigma\langle\nu\rangle(x) = \sup \nu \wedge \sigma\langle x\rangle \geq c(\mathcal{G}) > \alpha$$

$$\Rightarrow \inf_{\nu \in \mathcal{G}} \sigma\langle\nu\rangle(x) \geq \alpha$$

$$\Rightarrow \sup \inf_{\nu \in \mathcal{G}} \sigma\langle\nu\rangle(x) \geq \alpha.$$

Since σ and α are arbitrary we obtain :

$$c(\mathcal{G}) \leq \inf_{\sigma \in \mathcal{D}} \sup \inf_{\nu \in \mathcal{G}} \sigma\langle\nu\rangle.$$

and this means that $\mathcal{G}$ is $\mathcal{D}$–Cauchy. Since μ is $\mathcal{D}$–complete and $\mu \in \mathcal{G}$ we conclude that $\mathcal{G}$ is convergent in μ. In other words :

$$\begin{aligned} c(\mathcal{G}) &\leq \sup(\mu \wedge \operatorname{Adh} \mathcal{G}) \\ &\leq \sup(\mu \wedge \operatorname{Adh} \mathcal{D}_x) \quad (\mathcal{D}_x \subseteq \mathcal{G}) \\ &= \sup(\mu \wedge 1_x) \quad (3.7) \\ &= \mu(x). \end{aligned}$$

Finally, since $c(\mathcal{G}) = c(\mathcal{D}_x,\mu) = \bar{\mu}(x)$ we have shown that $\bar{\mu}(x) \leq \mu(x)$.

∎

8.14 COROLLARY

Let $(X, \mathcal{D})$ be a Hausdorff fuzzy uniform space with $\mu, \lambda \in I^X$. Then :

(a) μ and λ are $\mathcal{D}$–complete $\Rightarrow$ $\mu \wedge \lambda$ is $\mathcal{D}$–complete.

(b) If μ is $\mathcal{D}$–complete and $\lambda \leq \mu$ then:
λ is $\mathcal{D}$–complete $\Leftrightarrow$ λ is $\mathcal{D}$–closed

(c) If $\mu \leq \lambda$, λ is $\mathcal{D}$–complete and μ is $\mathcal{D}$–precompact then
μ is relatively $\mathcal{D}$–compact.

PROOF

(a) If μ and λ are both $\mathcal{D}$–complete then they are both $\mathcal{D}$–closed (8.13) and so $\mu \wedge \lambda$ is $\mathcal{D}$–closed. Thus $\mu \wedge \lambda$ is $\mathcal{D}$–complete (8.11, $\mu \wedge \lambda \leq \mu$).

(b) If λ is $\mathcal{D}$–complete then λ is $\mathcal{D}$–closed (8.13).
Conversely if λ is $\mathcal{D}$–closed then $\lambda \wedge \mu = \lambda$ is $\mathcal{D}$–complete (8.11).

(c) We refer to [Ch 2: Theorem 4.1] where it is shown that in a fuzzy uniform space $(X, \mathscr{D})$ μ is relatively $\mathscr{D}$–compact iff $\bar{\mu}$ is $\mathscr{D}$–compact. If λ is $\mathscr{D}$–complete then λ is $\mathscr{D}$–closed (8.13) and so $\bar{\mu} \leq \lambda$. Consequently, $\bar{\mu}$ is $\mathscr{D}$–complete (8.14(b)) and $\mathscr{D}$–precompact (6.8). It follows from 8.10 that $\bar{\mu}$ is $\mathscr{D}$–compact.

■

The closure of a complete subset of a uniform space is complete. We show now that this result is also true for fuzzy uniform spaces.

8.15 THEOREM

Let $(X, \mathscr{D})$ be a fuzzy uniform space with $\mu \in I^X$. Then :

μ is $\mathscr{D}$–complete $\Rightarrow$ $\bar{\mu}$ is $\mathscr{D}$–complete.

PROOF

Let $\mathscr{F}$ be a prime $\mathscr{D}$–Cauchy prefilter with $\bar{\mu} \in \mathscr{F}$ and let :

$$\mathscr{G} = <\{\sigma<\nu> \wedge \mu : \sigma \in \mathscr{D}, \nu \in \mathscr{F}\}>.$$

We assert that :

(i) **$\mathscr{G}$ is strong $\mathscr{D}$–Cauchy.**

Let $\psi \in \mathscr{D}$, $\epsilon \in I_0$ and choose symmetric $\sigma \in \mathscr{D}$ such that $\sigma^3 \leq \psi + \frac{\epsilon}{2}$.

Since $\mathscr{F}$ is prime and $\mathscr{D}$–Cauchy, $\mathscr{F}$ is strong $\mathscr{D}$–Cauchy (4.9) and so we can choose $\nu \in \mathscr{F}$ such that $\nu \times \nu \leq \sigma + \frac{\epsilon}{2}$. Consequently, if $(x,y) \in X \times X$ then :

$$\begin{aligned}
&[(\sigma<\nu> \wedge \mu) \times (\sigma<\nu> \wedge \mu)](x,y) = \sigma<\nu>(x) \wedge \mu(x) \wedge \sigma<\nu>(y) \wedge \mu(y) \\
&\leq (\sup_{z \in X} \nu(z) \wedge \sigma(z,x)) \wedge (\sup_{w \in X} \nu(w) \wedge \sigma(w,y)) \\
&= \sup_{z \in X} \sup_{w \in X} \nu(z) \wedge \nu(w) \wedge \sigma(z,x) \wedge \sigma(w,y) \\
&\leq \sup_{z \in X} \sup_{w \in X} (\sigma(z,w) + \tfrac{\epsilon}{2}) \wedge \sigma(z,x) \wedge \sigma(w,y) \\
&\leq \sup_{z \in X} \sup_{w \in X} \sigma(x,z) \wedge \sigma(z,w) \wedge \sigma(w,y) + \tfrac{\epsilon}{2} \\
&= \sigma^3(x,y) + \tfrac{\epsilon}{2} \\
&\leq (\psi(x,y) + \tfrac{\epsilon}{2}) + \tfrac{\epsilon}{2} = \psi(x,y) + \epsilon.
\end{aligned}$$

Since (x,y) is arbitrary this shows that $\mathscr{G}$ is strong $\mathscr{D}$–Cauchy.

(ii) $c(\mathscr{G}) \geq c(\mathscr{F})$.

$$
\begin{aligned}
c(\mathscr{G}) &= \inf_{\sigma \in \mathscr{D}} \inf_{\nu \in \mathscr{F}} \sup \sigma<\nu> \wedge \mu. \\
&= \inf_{\sigma \in \mathscr{D}} \inf_{\nu \in \mathscr{F}} \sup \nu \wedge \sigma_s<\mu> \quad (3.4(d)) \\
&= \inf_{\nu \in \mathscr{F}} \inf_{\sigma \in \mathscr{D}} \sup \nu \wedge \sigma<\mu> \quad (3.1) \\
&\geq \inf_{\nu \in \mathscr{F}} \sup \inf_{\sigma \in \mathscr{D}} \nu \wedge \sigma<\mu> \\
&= \inf_{\nu \in \mathscr{F}} \sup \nu \wedge \inf_{\sigma \in \mathscr{D}} \sigma<\mu> \\
&= \inf_{\nu \in \mathscr{F}} \sup \nu \wedge \bar{\mu} \\
&= c(\mathscr{F},\bar{\mu}) \\
&= c(\mathscr{F}) \quad (\bar{\mu} \in \mathscr{F}).
\end{aligned}
$$

We therefore have a strong $\mathscr{D}$–Cauchy prefilter $\mathscr{G}$ with $\mu \in \mathscr{G}$ and we know that μ is $\mathscr{D}$–complete. We appeal to 8.8(c) and claim that $c(\mathscr{G}) \leq \sup \mu \wedge \text{Adh } \mathscr{G}$. Consequently :

$$
\begin{aligned}
c(\mathscr{F}) &\leq c(\mathscr{G}) \quad ((ii)) \\
&\leq \sup \mu \wedge \text{Adh } \mathscr{G} \\
&\leq \sup \mu \wedge \inf_{\nu \in \mathscr{F}} \inf_{\sigma \in \mathscr{D}} \overline{\sigma<\nu> \wedge \mu} \\
&\leq \sup \inf_{\nu \in \mathscr{F}} \inf_{\sigma \in \mathscr{D}} \overline{\sigma<\nu> \wedge \mu} \\
&\leq \sup \inf_{\nu \in \mathscr{F}} \inf_{\sigma \in \mathscr{D}} \overline{\sigma<\nu>} \wedge \bar{\mu} \quad (3.3(e)) \\
&= \sup \inf_{\nu \in \mathscr{F}} \bar{\mu} \wedge \bar{\nu} \quad (3.4(j)) \\
&= \sup \bar{\mu} \wedge \text{Adh } \mathscr{F}.
\end{aligned}
$$

Thus $\mathscr{F}$ is $\mathscr{D}$–convergent in $\bar{\mu}$ and hence we have shown that $\bar{\mu}$ is $\mathscr{D}$–complete.

■

In UNIF, products of complete sets are complete and we shall prove now that there is an analagous theorem in F.UNIF.

8.16 THEOREM

> Let $((X_j, \mathscr{D}(j))$: be a family of fuzzy uniform spaces with $\mu(j) \in I^{X_j}$ for each $j \in J$. Let $X = \prod_{j \in J} X_j$ and $\mathscr{D} = \prod_{j \in J} \mathscr{D}(j)$. Then :
>
> $\forall\, j \in J\ \mu(j)$ is $\mathscr{D}(j)$ –complete $\Rightarrow \prod_{j \in J} \mu(j)$ is $\mathscr{D}$–complete.

PROOF

Let $\mu = \prod_{j \in J} \mu(j)$ and let $\mathscr{G}$ be a prime $\mathscr{D}$–Cauchy prefilter on X with $\mu \in \mathscr{G}$.

For each $j \in J$ let $\mathscr{G}(j) = \langle p_j[\mathscr{G}]\rangle$ then :

(i) $\mathscr{G}(j)$ is a prime prefilter on X_j (2.23).

(ii) $\mathscr{G}(j)$ is $\mathscr{D}(j)$ –Cauchy (4.21).

(iii) $\mu(j) \in \mathscr{G}(j)$ $(p_j[\mu] \leq \mu(j)$, (3.11(b)) and $p_j[\mu] \in \mathscr{G}(j))$.

(iv) $c(\mathscr{G}(j)) \leq \sup(\mu(j) \wedge \text{Adh } \mathscr{G}(j))$ ((i),(ii),(iii) and $\mu(j)$ is $\mathscr{D}(j)$ –complete).

Consequently :

$$
\begin{aligned}
c(\mathscr{G}) &\leq \inf_{j \in J} c(\mathscr{G}(j)) \quad (3.11(e)) \\
&\leq \inf_{j \in J} \sup \mu(j) \wedge \text{Adh } \mathscr{G}(j) \quad ((iv)) \\
&= \inf_{j \in J} \sup_{x \in X} (\mu(j) \wedge \text{Adh } \mathscr{G}(j))(p_j(x)) \\
&= \inf_{j \in J} \sup p_j^{-1}[\mu(j) \wedge \text{Adh } \mathscr{G}(j)] = p \text{ say.}
\end{aligned}
$$

We intend to show that $p \leq \sup \mu \wedge \text{Adh } \mathscr{G}$ so we let $\alpha < p$ be arbitrary and show that $\alpha < \sup \mu \wedge \text{Adh } \mathscr{G}$.

Since $\alpha < p$ we have :

$\forall\, j \in J\ \exists\, x(j) \in X : (\mu(j) \wedge \text{Adh } \mathscr{G}(j))(p_j(x(j)) > \alpha$.

Let $p_j(x(j)) = x_j$ and let $x = (x_j : j \in J)$. Then

$\forall\, j \in J\ (\mu(j) \wedge \text{Adh } \mathscr{G}(j))(p_j(x)) > \alpha$ and so :

$$\alpha < \inf_{j \in J} p_j^{-1}[\mu(j)](x) \wedge \inf_{j \in J} p_j^{-1}[\mathrm{Adh}\ \mathscr{G}(j)](x)$$
$$= (\prod_{j \in J} \mu(j) \wedge \prod_{j \in J} \mathrm{Adh}\ \mathscr{G}(j))(x)$$
$$= (\mu \wedge \mathrm{Adh}\ \mathscr{G})(x) \quad (3.12(b))$$
$$\leq \sup \mu \wedge \mathrm{Adh}\ \mathscr{G}.$$

We have shown that $c(\mathscr{G}) \leq p \leq \sup \mu \wedge \mathrm{Adh}\ \mathscr{G}$ and hence μ is $\mathscr{D}$–complete (8.1).

■

We do **not** have :

$$\prod_{j \in J} \mu(j) \text{ is } \mathscr{D}\text{–complete} \Longrightarrow \forall\ j \in J\ \mu(j) \text{ is } \mathscr{D}(j)\text{–complete.}$$

This is due to the observation that if for some $k \in J$ $\mu(k) = 0$ then :
$\prod_{j \in J} \mu(j) = 0$ which is trivially $\mathscr{D}$–complete (even if for some $j \neq k$ $\mu(j)$ is not $\mathscr{D}(j)$–complete).

In UNIF it is not true that the uniformly continuous image of a complete set is complete so we do not have this in F.UNIF. Consequently we do **not** have :

$$\mu \in I^X \text{ is } \mathscr{D}\text{–complete} \Leftrightarrow \forall\ j \in J\ p_j[\mu] \text{ is } \mathscr{D}(j)\text{–complete.}$$

CHAPTER 9

CONCLUDING REMARKS

In [Lo 4] and [Lo 5], Lowen and Wuyts define the notions of compactness, precompactness and completeness for a fuzzy uniform space $(X, \mathscr{D})$, show that these definitions are good extensions and develop a theory of these notions which generalises the standard theory. We have seen that a theory of compactness, precompactness, completeness and boundedness can be established for fuzzy sets $\mu \in I^X$ where $(X, \mathscr{D})$ is a fuzzy uniform space and it behooves us to find the relationship betweem our notions and the notions of Lowen and Wuyts in the case $\mu = 1_X$.

In [Lo 4] a fuzzy uniform space $(X, \mathscr{D})$ is defined to be **compact** iff for each family $(\sigma_x : x \in X) \in \mathscr{D}^X$ and for each $\epsilon \in I_0$ there exists a finite subfamily $(\sigma_y : y \in Y)$ such that $\sup_{x \in Y} \sigma_x\langle x\rangle \geq 1-\epsilon$. Let us say that 1_X is **Lowen–compact** iff $(X, \mathscr{D})$ is compact in this sense. In [Lo 1], Lowen shows that $(X, \mathscr{D})$ is (Lowen–)compact iff for each prefilter $\mathscr{F}$ on X we have $c(\mathscr{F}) \leq \sup \operatorname{Adh} \mathscr{F}$ and hence 1_X is Lowen–compact iff 1_X is $\mathscr{D}$–compact. Consequently Lowen–compactness generalises the standard notion of compactness and $\mathscr{D}$–compactness generalises Lowen–compactness.

The situation with regard to completeness is not nearly as transparent. In [Lo 4] a prefilter $\mathscr{H}$ in a fuzzy uniform space $(X, \mathscr{D})$ is called a **hyper Cauchy** prefilter iff $\mathscr{H}$ is a strong $\mathscr{D}$–Cauchy, saturated prefilter with $c(\mathscr{H}) = 1$. It is shown in [Lo 4] that every hyper Cauchy prefilter $\mathscr{H}$ on X has a unique minimal (w.r.t. inclusion) hyper Cauchy prefilter $\mathscr{H}_m \subseteq \mathscr{H}$ and that $\mathscr{H}_m = \tilde{\mathscr{G}}$ where $\mathscr{G} = \{\sigma\langle\nu\rangle : \sigma \in \mathscr{D}, \nu \in \mathscr{H}\}$. It then follows that the neighbourhood prefilters $\mathscr{D}_x$ are minimal hyper Cauchy prefilters. The family of all minimal hyper Cauchy prefilters on X is denoted $\mathscr{M}(X)$ and then a prefilter $\mathscr{F}$ is called **Cauchy** iff $\sup_{\mathscr{H} \in \mathscr{M}(X)} \inf_{\mathscr{G} \in \mathscr{P}_m(\mathscr{F})} c(\mathscr{H}, \mathscr{G}) \geq \bar{c}(\mathscr{F})$. Let us call $\mathscr{F}$ **Lowen–Cauchy** iff $\mathscr{F}$ is Cauchy in this sense. It follows easily that convergent prefilters are Lowen–Cauchy ([Lo 4]).

9.1 THEOREM

> If $(X,\mathscr{D})$ is a fuzzy uniform space with $(X,\mathscr{D}^1)$ complete and $\mathscr{F}$ is a strong $\mathscr{D}$–Cauchy prefilter on X with $c(\mathscr{F}) = 1$, then :
> $(\text{Adh } \mathscr{F})(x) = 1$ for some $x \in X$.

PROOF

Since $\mathscr{F}$ is strong $\mathscr{D}$–Cauchy and $c(\mathscr{F}) = 1$, it follows from 4.14 that $\mathscr{F}^1$ is $\mathscr{D}^1$ –Cauchy and, since $(X,\mathscr{D}^1)$ is complete, there exists $x \in X$ such that $\mathscr{F}^1 \rightarrow x$ w.r.t. $\mathscr{D}^1$. Thus :

$$\mathscr{D}^1_x \subseteq \mathscr{F}^1 \quad \Rightarrow \forall\ \sigma \in \mathscr{D}\ \forall\ \beta < 1\ \ \overset{\beta}{\sigma}(x) \in \mathscr{F}^1$$

$$\Rightarrow \forall\ \sigma \in \mathscr{D}\ \forall\ \beta < 1\ \forall\ \nu \in \mathscr{F}\ \ \overset{\beta}{\sigma}(x) \cap \overset{\beta}{\nu} \neq \emptyset$$

$$\Rightarrow \forall\ \sigma \in \mathscr{D}\ \forall\ \beta < 1\ \forall\ \nu \in \mathscr{F}\ \exists\ y : \sigma(y,x) \wedge \nu(y) > \beta$$

$$\Rightarrow \forall\ \beta < 1\ \forall\ \sigma \in \mathscr{D}\ \forall\ \nu \in \mathscr{F}\ \sup \sigma\langle x\rangle \wedge \nu = \sigma\langle\nu\rangle(x) > \beta$$

$$\Rightarrow \forall\ \beta < 1\ \inf_{\nu \in \mathscr{F}}\ \inf_{\sigma \in \mathscr{D}} \sigma\langle\nu\rangle(x) = \inf_{\nu \in \mathscr{F}} \bar{\nu}(x) = (\text{Adh } \mathscr{F})(x) > \beta$$

$$\Rightarrow (\text{Adh } \mathscr{F})(x) = 1.$$

■

9.2 COROLLARY

> If $(X,\mathscr{D})$ is a fuzzy uniform space with $(X,\mathscr{D}^1)$ complete and $\mathscr{H}$ is a hyper Cauchy prefilter on X then $(\text{Adh } \mathscr{H})(x) = 1$ for some $x \in X$.

PROOF

Immediate.

■

This last corollary enables us to make the interesting observation that in a fuzzy uniform space $(X,\mathscr{D})$ with $(X,\mathscr{D}^1)$ complete, the set $\mathscr{M}(X)$ of minimal hyper Cauchy prefilters is precisely the set $\{\mathscr{D}_x : x \in X\}$ of neighbourhoods.

9.3 THEOREM

> If $(X,\mathscr{D})$ is a fuzzy uniform space with $(X,\mathscr{D}^1)$ complete then
> $$\mathscr{M}(X) = \{\mathscr{D}_x : x \in X\}.$$

PROOF

We have already remarked that $\{\mathscr{D}_x : x \in X\} \subseteq \mathscr{M}(X)$, so to prove the reverse inequality, let $\mathscr{H} \in \mathscr{M}(X)$. Since $\mathscr{H}$ is hyper Cauchy, 9.2 guarantees that $(\text{Adh } \mathscr{H})(x) = 1$ for some $x \in X$. We show that $\mathscr{H} = \mathscr{D}_x$ and, since $\mathscr{H}$ is minimal hyper Cauchy, it is sufficient to show that $\mathscr{D}_x \subseteq \mathscr{H}$. To this end let $\psi\langle x\rangle \in \mathscr{D}_x$ with $\psi \in \mathscr{D}$ and let $\epsilon \in I_0$. We seek $\nu_\epsilon \in \mathscr{H}$ such that $\nu_\epsilon \leq \psi\langle x\rangle + \epsilon$. It will then follow, since ϵ is arbitrary, that $\nu := \sup_{\epsilon \in I_0} (\nu_\epsilon - \epsilon) \leq \psi\langle x\rangle$ and this in turn allows us to deduce that $\psi\langle x\rangle \in \hat{\mathscr{H}} = \mathscr{H}$.

Let $\sigma \in \mathscr{D}$ be such that $\sigma \circ \sigma \leq \psi + \frac{\epsilon}{2}$. $\mathscr{H}$ is strong $\mathscr{D}$–Cauchy and so there exists $\nu_\epsilon \in \mathscr{H}$ with $\nu_\epsilon \times \nu_\epsilon \leq \sigma + \frac{\epsilon}{2}$. Now :

$$1 = (\text{Adh } \mathscr{H})(x) = \inf_{\nu \in \mathscr{H}} \inf_{\sigma \in \mathscr{D}} \sigma\langle\nu\rangle(x) \leq \sigma\langle\nu_\epsilon\rangle(x).$$

Thus $\sigma\langle\nu_\epsilon\rangle(x) = 1$ and hence if $y \in X$ we have :

$$\begin{aligned}
\nu_\epsilon(y) &= \nu_\epsilon(y) \wedge \sigma\langle\nu_\epsilon\rangle(x) \\
&= \nu_\epsilon(y) \wedge \sup_{z \in X} \nu_\epsilon(z) \wedge \sigma(z,x) \\
&= \sup_{z \in X} \nu_\epsilon(y) \wedge \nu_\epsilon(z) \wedge \sigma(z,x) \\
&= \sup_{z \in X} (\nu_\epsilon \times \nu_\epsilon)(y,z) \wedge \sigma(z,x) \\
&\leq \sup_{z \in X} (\sigma(y,z) \wedge \sigma(z,x)) + \tfrac{\epsilon}{2} \\
&= (\sigma \circ \sigma)(y,x) + \tfrac{\epsilon}{2} \leq \psi(y,x) + \epsilon = \psi\langle x\rangle(y) + \epsilon.
\end{aligned}$$

We have shown that $\nu_\epsilon \leq \psi\langle x\rangle + \epsilon$ and so we are done.

∎

9.4 LEMMA

If $(X, \mathscr{D})$ is a fuzzy uniform space and $\mathscr{H}$ is a hyper Cauchy prefilter on X then for each $\sigma \in \mathscr{D}$ there exists $x \in X$ such that $\sigma\langle x\rangle \in \mathscr{H}$.

PROOF

Let $\mathscr{H}$ be hyper Cauchy, $\sigma \in \mathscr{D}$ and $\epsilon \in I_0$. Since $\mathscr{H}$ is strong $\mathscr{D}$–Cauchy there exists $\nu_\epsilon \in \mathscr{H}$ with $\nu_\epsilon \times \nu_\epsilon \leq \sigma + \frac{\epsilon}{2}$. We have $c(\mathscr{H}) = 1$ and so $\sup \nu_\epsilon = 1$. Consequently $\nu_\epsilon(x) > 1 - \frac{\epsilon}{2}$ for some $x \in X$. Let $y \in X$. Then :

$$\sigma(y,x) + \tfrac{\epsilon}{2} \geq \nu_\epsilon(y) \wedge \nu_\epsilon(x) \geq \nu_\epsilon(y) \wedge (1 - \tfrac{\epsilon}{2}) \geq (\nu_\epsilon(y) \wedge 1) - \tfrac{\epsilon}{2} = \nu_\epsilon(y) - \tfrac{\epsilon}{2}.$$

Thus $\nu_\epsilon(y) \leq \sigma\langle x\rangle(y) + \epsilon$ holds for each $y \in X$ and so $\nu_\epsilon \leq \sigma\langle x\rangle + \epsilon$. We have shown that for each $\epsilon \in I_0$ we have $\nu_\epsilon - \epsilon \leq \sigma\langle x\rangle$ and hence

$\nu := \sup_{\epsilon \in I_0} (\nu_\epsilon - \epsilon) \leq \sigma\langle x\rangle$. Finally, $\nu \in \hat{\mathscr{H}} = \mathscr{H}$ and so $\sigma\langle x\rangle \in \mathscr{H}$.

■

We are now ready to compare the two notions of a Cauchy prefilter.

9.5 THEOREM

Let $(X, \mathscr{D})$ be a fuzzy uniform space and let $\mathscr{F}$ be a prefilter on X.
(a) $\mathscr{F}$ is Lowen–Cauchy $\Rightarrow$ $\mathscr{F}$ is $\mathscr{D}$–Cauchy.
(b) If $(X, \mathscr{D}^1)$ is complete then $\mathscr{F}$ is Lowen–Cauchy $\Leftrightarrow$ $\mathscr{F}$ is convergent.

PROOF

(a) Let $\mathscr{F}$ be Lowen–Cauchy, $\alpha < \bar{c}(\mathscr{F})$ and $\sigma \in \mathscr{D}$. Since $\mathscr{F}$ is Lowen–Cauchy we have $\alpha < \inf_{\mathscr{G} \in \mathscr{P}_m(\mathscr{F})} c(\mathscr{H}, \mathscr{G})$ for some $\mathscr{H} \in \mathscr{M}(X)$. We invoke 9.4 to assert that $\sigma\langle x\rangle \in \mathscr{H}$ for some $x \in X$. Let $\mathscr{G} \in \mathscr{P}_m(\mathscr{F})$ and $\nu \in \mathscr{G}$. Then $\sigma\langle x\rangle \wedge \nu \in (\mathscr{H}, \mathscr{G})$ and so :

$$\sigma\langle\nu\rangle(x) = \sup \nu \wedge \sigma\langle x\rangle \geq c(\mathscr{H}, \mathscr{G}) > \alpha.$$

Since $\mathscr{G}$ and ν are arbitrary we have :

$$\inf_{\mathscr{G}\in \mathscr{P}_{m}(\mathscr{F})} \inf_{\nu \in \mathscr{G}} \sigma\langle\nu\rangle(x) \geq \alpha \text{ and so :}$$

$$\sup \inf_{\mathscr{G}\in \mathscr{P}_{m}(\mathscr{F})} \inf_{\nu \in \mathscr{G}} \sigma\langle\nu\rangle \geq \alpha.$$

Finally, since σ and α are arbitrary, we conclude that

$$\inf_{\sigma \in \mathscr{D}} \sup \inf_{\mathscr{G}\in \mathscr{P}_{m}(\mathscr{F})} \inf_{\nu \in \mathscr{G}} \sigma\langle\nu\rangle \geq \bar{c}(\mathscr{F})$$

and this means that $\mathscr{F}$ is $\mathscr{D}$–Cauchy.

(b) Since a convergent prefilter is Lowen–Cauchy [Lo 4: Theorem 2.5], we must show the reverse implication. To this end let $\mathscr{F}$ be Lowen–Cauchy. Then :

$$\begin{aligned}
\bar{c}(\mathscr{F}) &\leq \sup_{\mathscr{H}\in \mathscr{M}(X)} \inf_{\mathscr{G}\in \mathscr{P}_{m}(\mathscr{F})} c(\mathscr{H},\mathscr{G}) \\
&= \sup_{x \in X} \inf_{\mathscr{G}\in \mathscr{P}_{m}(\mathscr{F})} c(\mathscr{D}_{x},\mathscr{G}) && (9.3)\\
&= \sup_{x \in X} \inf_{\mathscr{G}\in \mathscr{P}_{m}(\mathscr{F})} (\text{Adh } \mathscr{G})(x) && (3.5(a))\\
&= \sup \lim \mathscr{F}. && (3.5(b))
\end{aligned}$$

Thus $\mathscr{F}$ is convergent.

■

A $\mathscr{D}$ –Cauchy prefilter need not be Lowen–Cauchy as the following counter–example shows.

9.6 EXAMPLE

Let $(X,\mathscr{D})$ be a Hausdorff fuzzy uniform space with $(X,\mathscr{D}^{1})$ complete but $(X,\mathscr{D}^{\alpha})$ not complete for some $\alpha \in I_0$ (see 8.6 for example). It follows from 8.3 that 1_X is not $\mathscr{D}$–complete and hence there exists a prime, $\mathscr{D}$–Cauchy prefilter $\mathscr{F}$ on X which is not convergent. Referring to 9.5(b) we see that $\mathscr{F}$ is not Lowen–Cauchy.

■

9.7 COROLLARY

There exists a fuzzy uniform space $(X,\mathscr{D})$ and a $\mathscr{D}$–Cauchy prefilter $\mathscr{F}$ on X which is not Lowen–Cauchy.

In [Lo 4], a fuzzy uniform space $(X, \mathscr{D})$ is defined to be **complete** iff every (Lowen–)Cauchy prefilter is convergent. Let us call 1_X **Lowen–complete** iff $(X, \mathscr{D})$ is complete in this sense. As an immediate consequence of Theorem 9.5(a) we obtain :

9.8 COROLLARY

If $(X, \mathscr{D})$ is a fuzzy uniform space then :
1_X is $\mathscr{D}$–complete $\Rightarrow$ 1_X is Lowen–complete.

Example 9.6 also reveals that the two notions of completeness are not equivalent when $\mu = 1_X$.

9.9 COROLLARY

There exists a fuzzy uniform space $(X, \mathscr{D})$ such that 1_X is Lowen–complete but 1_X is not $\mathscr{D}$–complete.

PROOF

If $(X, \mathscr{D})$ is as in 9.6 then it follows from 9.5(b) that every Lowen–Cauchy prefilter is convergent and hence 1_X is Lowen–complete but not $\mathscr{D}$–complete.

■

In [Lo 4] a fuzzy uniform space $(X, \mathscr{D})$ is defined to be **precompact** iff for each $\sigma \in \mathscr{D}$ and each $\epsilon \in I_0$ there exists $K \in \mathscr{P}_f(X)$ such that $\sup_{x \in K} \sigma\langle x\rangle \geq 1-\epsilon$. This is shown to be equivalent to each prime, saturated prefilter $\mathscr{F}$ with $c(\mathscr{F}) = 1$ being hyper Cauchy [Lo 4: Theorem 3.1]. Let us say that 1_X is **Lowen–precompact** if 1_X is precompact in this sense.

9.10 THEOREM

If $(X, \mathscr{D})$ is a fuzzy uniform space then :
1_X is $\mathscr{D}$–precompact $\Leftrightarrow$ 1_X is Lowen–precompact.

PROOF

Let 1_X be $\mathscr{D}$–precompact and let $\mathscr{F}$ be a prime, saturated prefilter with $c(\mathscr{F}) = 1$. Then we have :

$$c(\mathscr{F},1_X) = c(\mathscr{F}) \leq \inf_{\sigma \in \mathscr{D}} \sup \text{Adh}\, \sigma<\mathscr{F}> \quad (6.4).$$

Consequently $\mathscr{F}$ is $\mathscr{D}$–Cauchy and, since $\mathscr{F}$ is prime, $\mathscr{F}$ is strong $\mathscr{D}$–Cauchy (4.9). Thus $\mathscr{F}$ is hyper Cauchy and hence we have shown that 1_X is Lowen–precompact.

Conversely, let 1_X be Lowen–precompact and let $\mathscr{F}$ be a prime Lowen–prefilter with $c(\mathscr{F}) = 1$. Then $\mathscr{F}$ is hyper Cauchy and so in particular, $\mathscr{F}$ is strong $\mathscr{D}$–Cauchy. Thus $\mathscr{F}$ is $\mathscr{D}$–Cauchy (4.8) and so :

$$c(\mathscr{F}) = c(\mathscr{F},1_X) \leq \inf_{\sigma \in \mathscr{D}} \sup \text{Adh}\, \sigma<\mathscr{F}>.$$

Thus we have shown, according to 6.4, that 1_X is $\mathscr{D}$–precompact.

■

We have seen that even though the two notions of a Cauchy prefilter differ, the notions of precompactness coincide for $\mu = 1_X$. So Lowen–precompactness generalises the standard notion and $\mathscr{D}$ –precompactness generalises Lowen–precompactness.

CONCLUSION

We have seen that it is possible to find good extensions of the basic uniform–space notions of compactness, precompactness, boundedness and completeness to fuzzy uniform spaces and furthermore, the theory generated generalises standard theory. The notions of $\mathscr{D}$ –compactness and $\mathscr{D}$ –precompactness coincide with the previously defined notions of Lowen on the whole space $\mu = 1_X$,so in this sense they are extensions of Lowen–compactness and Lowen–precompactness, but $\mathscr{D}$–completeness is not an extension of Lowen–completeness.

BIBLIOGRAPHY

[Bo 1] N. Bourbaki, *General Topology* Chapters 1–4, Springer–Verlag.

[Bu 1] M.H. Burton, The Relationship Between a Fuzzy Uniformity and its Family of α–Level Uniformities, submitted for publication.

[Bu 2] M.H. Burton, Cauchy Filters and Prefilters, submitted for publication.

[Bu 3] M.H. Burton and J.J. Chadwick, Precompactness in Fuzzy Uniform Spaces, submitted for publication.

[Bu 4] M.H. Burton, Boundedness in Uniform Spaces and Fuzzy Uniform Spaces, submitted for publication.

[Bu 5] M.H. Burton and J.J. Chadwick, Completeness in Fuzzy Uniform Spaces, submitted for publication.

[Ch 1] J.J. Chadwick, A Generalised Form of Compactness in Fuzzy Topological Spaces. (To appear in J.Math.Anal.Appl.).

[Ch 2] J.J. Chadwick, Relative Compactness and Compactness of General Subsets in an I–Topological Space, submitted for publication.

[Ch 3] C.L. Chang, Fuzzy Topological Spaces, J.Math.Anal. Appl.**24** (1968), 182–190.

[Go 1] J.A. Goguen, The Fuzzy Tychonoff Theorem, J.Math.Anal. Appl. **43** (1973), 734–742.

[He 1] H. Herrlich and G.E. Strecker, *Category Theory*, Heldermann Verlag.

[Ka 1] A.K. Katsaras and D.B. Liu, Fuzzy Vector Spaces and Fuzzy Topological Vector Spaces, J.Math.Anal. Appl. **58** (1977) 135–146.

[Ka 2] A.K. Katsaras, Fuzzy Topological Vector Spaces I, Fuzzy Sets and Systems **6** (1981), 85–95.

[Ka 3] A.K. Katsaras, Fuzzy Topological Vector Spaces II, Fuzzy Sets and Systems **12** (1984), 143–154.

[Lo 1] R. Lowen, Convergence in Fuzzy Topological Spaces, General Topology and its Applications, **10** (1979), 147–160.

[Lo 2] R. Lowen, Fuzzy Neighbourhood Spaces, Fuzzy Sets and Systems **7** (1982), 165–189.

[Lo 3] R. Lowen, Fuzzy Uniform Spaces, J.Math.Anal. Appl. **82** (1981), 370–385.

[Lo 4] R. Lowen, and P. Wuyts, Completeness, Compactness and Precompactness in Fuzzy Uniform Spaces : Part I, J.Math.Anal. Appl. **90(2)** (1982), 563–581.

[Lo 5] R. Lowen, and P. Wuyts, Completeness, Compactness and Precompactness in Fuzzy Uniform Spaces : Part II, J.Math.Anal. Appl. **92** (1983), 342–371.

[Lo 6] R. Lowen, and P. Wuyts, Initial and Final Topologies and the Fuzzy Tychonoff Theorem, J.Math.Anal. Appl. **58(1)** (1977), 11–21.

[Lo 7] E. Lowen and R. Lowen, Characterisation of Convergence in Fuzzy Topological Spaces, Internat. J.Math. & Math.Sci. **8(3)** (1985), 497–511.

[Lo 8] R. Lowen, Compactness Notions in Fuzzy Neighbourhood Spaces, Manuscripta Math. **38** (1982), 265–287.

[Lo 9] R. Lowen, Fuzzy Topological Spaces and Fuzzy Compactness, J.Math.Anal. Appl. **56** (1976), 621–633.

[Lo 10] R. Lowen, On the Existence of Natural Non–topological Fuzzy Topological Spaces, Research and Exposition in Mathematics **II**, Helderman Verlag.

[Lo 11] R. Lowen, A comparison of Different Compactness Notions in Fuzzy Topological Spaces, J.Math.Anal. Appl. **65(2)** (1978), 447–454.

[Lo 12] R. Lowen and P. Wuyts, Concerning the Constants in Fuzzy Topology, J.Math.Anal. Appl. **129** (1988), 256–268.

[Lo 13] R. Lowen, The Relationship Between Filter and Net Convergence in Fuzzy Topological Spaces, Fuzzy Math. **12** (1983), 41–52.

[Mo 1] N.N. Morsi, Nearness Concepts in Fuzzy Neighbourhood Spaces and in their Fuzzy Proximity Spaces, Fuzzy Sets and Systems **31** (1989), 83–109.

[Pu 1] Pu Pao–Ming and Liu Ying–Ming, Fuzzy Topology.I.Neighbourhood Structure of a Fuzzy Point and Moore–Smith Convergence, J.Math.Anal. Appl. **94** (1983), 1–23.

[Wa 1] R.H. Warren, Neighbourhoods, Bases and Continuity in Fuzzy Topological Spaces, Rocky Mountain J.Math. **8(3)** (1978), 459–470.

[Wa 2] R.H. Warren, Convergence in Fuzzy Topology, Rocky Mountain J.Math. **13(1)** (1983), 31–36.

[Wa 3] R.H. Warren, Fuzzy Topologies Characterised by Neighbourhood Systems, Rocky Mountain J.Math. **9(4)** (1979), 761–764.

[Wi 1] A. Wilansky, *Topology for Analysis*, Robert E. Krieger Pub.Co.

[Wi 2] S. Willard, *General Topology,* Addison–Wesley.

[Wo 1] C.K. Wong, Fuzzy Topology : Product and Quotient Theorems, J.Math.Anal. Appl. **45** (1974), 512–521.

[Wo 2] C.K. Wong, Fuzzy Points and Local Properties of Fuzzy Topology, J.Math.Anal. Appl. **46** (1974), 316–328.

[Wu 1] P. Wuyts, On the Determination of Fuzzy Topological Spaces and Fuzzy Neighbourhood Spaces by Their Level Topologies, Fuzzy Sets and Systems **12(1)**, 71–85.

[Wu 2] P. Wuyts and R. Lowen, On Separation Axioms in Fuzzy Topological Spaces, Fuzzy Neighbourhood Spaces and Fuzzy Uniform Spaces, J.Math.Anal. Appl. **93**(1) (1983), 27–41.

[Za 1] L.A. Zadeh, Fuzzy Sets, Information and Control **8** (1965), 338–353.

INDEX OF NOTATION

$:=$ means equal by definition.

$\mathbb{N} = \{1,2,3,\cdots\}$

$[n] = \{1,2,\cdots,n\}$

$I = [0,1]$, $I_0 = I \setminus \{0\}$, $I_1 = I \setminus \{1\}$

If $A \subseteq X$ then $A' = X \setminus A$.

$I^X = \{f : f \text{ is a function and } f : X \longrightarrow I\}$

$\Delta = \{(x,x) : x \in X\}$

$1_A \;:\; 1_A(x) = \left.\begin{matrix} 1 & x \in A \\ 0 & x \notin A \end{matrix}\right\}$

$1_x \;:\; 1_x = 1_{\{x\}}$

$f[\nu] : f[\nu](y) = \sup\limits_{x \in f^{-1}[\{y\}]} \nu(x) = \sup\limits_{f(x)=y} \nu(x)$ where $\sup \emptyset = 0$.

$f^{-1}[\nu] : \quad f^{-1}[\nu] = \nu \circ f$

$<\mathscr{F}> = \{\mu : \exists\ \nu \in \mathscr{F}, \nu \leq \mu\}$

$<\mu> = \{\nu : \mu \leq \nu\}$

$c(\mathscr{F}) = \inf\limits_{\nu \in \mathscr{F}} \sup \nu$

$\mathscr{F} \vee \mathscr{G} = (\mathscr{F}, \mathscr{G}) = < \{\nu \wedge \mu : \nu \in \mathscr{F}, \mu \in \mathscr{G}\} >$

$\hat{\mathscr{F}} = \{ \sup_{\epsilon \in I_0} (\nu_\epsilon - \epsilon) : (\nu_\epsilon : \epsilon \in I_0) \in \mathscr{F}^{I_0} \}$

$\mathscr{F}$ is a **saturated prefilter** iff $\hat{\mathscr{F}} = \mathscr{F}$.

$f[\mathscr{F}] = \{f[\nu] : \nu \in \mathscr{F}\}$

$f^{-1}[\mathscr{F}] = \{f^{-1}[\nu] : \nu \in \mathscr{F}\}$

$\nu^\alpha = \{x : \nu(x) > \alpha)$ so $\nu^0 = \{x : \nu(x) > 0\}$

$\nu_\alpha = \{x : \nu(x) \geq \alpha\}$

$\mathbb{F}_\alpha = <\{\alpha 1_F : F \in \mathbb{F}\}>$

$\mathbb{F}^\alpha = \{\nu : \forall \beta < \alpha, \nu^\beta \in \mathbb{F}\}$

$\mathscr{F}_\alpha = <\{\nu^\alpha : \nu \in \mathscr{F}\}>$ so $\mathscr{F}_0 = <\{\nu^0 : \nu \in \mathscr{F}\}>$

$\mathscr{F}^\alpha = \{\nu^\beta : \nu \in \mathscr{F}, \beta < \alpha\}$

$\mathbb{P}(\mathbb{F}) = \{\mathbb{K} \supseteq \mathbb{F} : \mathbb{K} \text{ is ultra}\}$

$\mathscr{P}(\mathscr{F}) = \{ \mathscr{G} \supseteq \mathscr{F} : \mathscr{G} \text{ is prime}\}$

$\mathscr{P}_m(\mathscr{F}) = \{ \mathscr{G} \supseteq \mathscr{F} : \mathscr{G} \text{ is prime and } \mathscr{G} \text{ is minimal}\}$

$\mathscr{P}_f(X) = \{Y \subseteq X : Y \text{ is finite}\}$

$\bar{c}(\mathscr{F}) = \inf_{\mathscr{G} \in \mathscr{P}_m(\mathscr{F})} c(\mathscr{G})$

$\sigma \circ \sigma : (\sigma \circ \sigma)(x,y) = \sup_{z} \sigma(x,z) \wedge \sigma(z,y)$

$\sigma^n : \sigma^n(x,y) = \sup_{\underline{z}} \sigma(x,z_1) \wedge \sigma(z_1,z_2) \wedge \cdots \wedge \sigma(z_{n-1},y)$

$\sigma\langle\nu\rangle : \quad \sigma\langle\nu\rangle(x) = \sup \nu \wedge \sigma\langle x\rangle$

$\sigma\langle x\rangle : \quad \sigma\langle x\rangle(y) = \sigma(y,x)$

$\sigma^{\beta} = \{(x,y) : \sigma(x,y) > \beta\}$

$\sigma\langle \mathscr{F}\rangle = \langle\{\sigma\langle\nu\rangle : \nu \in \mathscr{F}\}\rangle$

$\text{Adh } \mathscr{F} = \inf_{\nu \in \mathscr{F}} \bar{\nu}$

$\lim \mathscr{F} = \inf_{\mathscr{G} \in \mathscr{P}_m(\mathscr{F})} \text{Adh } \mathscr{G}$

$\prod_{j \in J} \mathscr{D}(j)$: the product fuzzy uniformity.

$\tau(\mathscr{D})$: the fuzzy uniform topology associated with $\mathscr{D}$.

$\prod_{j \in J} \mu(j) = \inf_{j \in J} p_j^{-1} [\mu(j)]$

$\mathscr{D}^{\alpha} = \{\sigma^{\beta} : \sigma \in \mathscr{D}, \beta < \alpha\}$ so $\mathscr{D}^1 = \{\sigma^{\beta} : \sigma \in \mathscr{D}, \beta < 1\}$

$\mathbb{D}^{\alpha} = \{\sigma : \forall\ \beta < \alpha, \sigma^{\beta} \in \mathbb{D}\}$ so $\mathbb{D}^1 = \{\sigma : \forall\ \beta < 1, \sigma^{\beta} \in \mathbb{D}\}$

$\mathscr{D}_x = \{\sigma\langle x\rangle : \sigma \in \mathscr{D}\}$.

$\mathscr{D}_x^{\alpha} = \{U : U \text{ is a } \mathscr{D}^{\alpha}\text{–neighbourhood of } x\}$.