A STUDY OF A CLASS OF INVARIANT OPTIMAL CONTROL PROBLEMS ON THE EUCLIDEAN GROUP $\mathsf{SE}(2)$

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Abstract

The aim of this thesis is to study a class of left-invariant optimal control problems on the matrix Lie group $\mathsf{SE}(2)$. We classify, under detached feedback equivalence, all controllable (left-invariant) control affine systems on $\mathsf{SE}(2)$. This result produces six types of control affine systems on $\mathsf{SE}(2)$. Hence, we study six associated left-invariant optimal control problems on $\mathsf{SE}(2)$. A left-invariant optimal control problem consists of minimizing a cost functional over the trajectory-control pairs of a left-invariant control system subject to appropriate boundary conditions. Each control problem is lifted from $\mathsf{SE}(2)$ to $T^*\mathsf{SE}(2) \cong \mathsf{SE}(2) \times \mathfrak{se}(2)^*$ and then reduced to a problem on $\mathfrak{se}(2)^*$. The maximum principle is used to obtain the optimal control and Hamiltonian corresponding to the normal extremals. Then we derive the (reduced) extremal equations on $\mathfrak{se}(2)^*$. These equations are explicitly integrated by trigonometric and Jacobi elliptic functions. Finally, we fully classify, under Lyapunov stability, the equilibrium states of the normal extremal equations for each of the six types under consideration.

Keywords and phrases. Matrix Lie groups, detached feedback equivalence, (left-invariant) control affine systems, the maximum principle, normal extremals, (reduced) extremal equations, Jacobi elliptic functions, Lyapunov stability.

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Contents

Chapter 1

Introduction

This chapter contains a brief introduction to geometric control theory. We introduce the notions of a control system and a class of optimal control problems for a matrix Lie group G. We cover briefly the main details concerning the class of left-invariant optimal control problems on the matrix Lie group $\mathsf{SE}(2)$. A detailed overview of the material covered in each section of the thesis is given. Lastly, we give a list of the contributions made in this thesis.

1.1 Background

Mathematical control theory is the area of application-oriented mathematics that treats the basic mathematical principles, theory and problems underlying the analysis and design of control systems. Control theory has applications in many areas including engineering, robotics, physics and biology. A control system is described by a family of (ordinary) differential equations, parametrised by control parameters, which can be used to influence the behaviour of the system. A solution of such an equation, for each admissible control, is uniquely determined by its initial condition and is called an admissible trajectory of the system. "To characterise the states reachable from a given initial point is one of the first natural problems in control theory: the controllability problem" [1]. As soon as the possibility to reach a certain state is established, we try to do it in the best way: optimality. Other concepts such as stability, observability and realisation play important roles in control theory.

A major contribution to control theory was the discovery of the maximum principle by L.S. Pontryagin and his co-workers in the late 1950's. "The maximum principle is a far reaching generalisation of Weirstrass's necessary conditions for strong minima, which provides geometric conditions for a (strong) minimum of an integral criterion, called the cost, over the trajectories of a differential control system" [6].

The significance of the Lie bracket for problems of control became clear around the year 1970 with publication of the papers of R. Brockett, H. Hermes and C. Lobry. This work led to a partnership of differential geometry and control theory, marking the birth of geometric control theory. "The maximum principle, in its original form, suffers from some serious limitations and geometric control theory forms a theoretical foundation for extensions of the maximum principle to optimal problems on arbitrary differential manifolds. This theoretical foundation comprises important results concerning the topological and differential properties of the reachable sets and is an essential complement to modern optimal control theory" [6]. Geometric control theory has been developed by many researchers, including: R. Brockett, V. Jurdjevic, A.A. Agrachev, Y.L Sachkov, H.J. Sussmann and A. Krener. It is a fast growing active field of contemporary research.

From a geometric point of view an (ordinary) differential equation is a vector field and so a control system can be viewed as a family of vector fields, parametrised by some controls. A trajectory of the control system, associated to some admissible control, is thus the flow of such a vector field. "The structure of the admissible trajectories and attainable sets is thus intimately related to the group of transformations generated by the dynamical systems involved. In turn, groups of transformations form the heart of Geometry" [1]. In this thesis, a control system is a pair $\Sigma = (\mathsf{G}, \Xi)$, where G is a matrix Lie group and $\Xi : \mathsf{G} \times \mathbb{R}^\ell \to T\mathsf{G}$ is a smooth mapping. In classical notation, Σ is given as $\dot{g} = \Xi(g, u)$, $g \in \mathsf{G}$, $u \in \mathbb{R}^\ell$. The dynamics Ξ is left-invariant in the sense that $\Xi(g, u) = g\Xi(\mathbf{1}, u)$ for any $g \in \mathsf{G}$, where $\mathbf{1}$ is the identity element in G. The image Γ of $\Xi(\mathbf{1}, \cdot\,)$ is called the trace of the system. A trajectory of a control system Σ is some absolutely continuous curve $g(\cdot\,) : [0, T] \to \mathsf{G}$, where $g(0) = g_0$, such that there exists an admissible control $u(\cdot\,)$, which satisfies the differential equation $\dot{g} = \Xi(g, u)$. The attainable set from a point $g \in \mathsf{G}$ is the set $\mathcal{A}(g)$ of all terminal points $g(t)$, $T \geq 0$, of all trajectories $g(\cdot\,)$ of Σ starting at g. We are interested in the analysis of those control systems on a matrix Lie group G which are controllable. Here, a control system Σ is called controllable if given any pair of points $g_0, g_1 \in \mathsf{G}$, the point g_1 can be reached from the point g_0 along some trajectory of Σ for some non-negative time. That is, $\mathcal{A}(g) = G$ for any $g \in \mathsf{G}$. We classify all equivalent controllable control systems on the matrix Lie group G. Two control systems $\Sigma = (\mathsf{G}, \Xi)$ and $\tilde{\Sigma} = (\mathsf{G}, \widetilde{\Xi})$ are said to be state space equivalent provided there exists a local diffeomorphism $\Phi : V \to \tilde{V}$, where V and $\tilde{V}$ are neighbourhoods of $\mathbf{1} \in \mathsf{G}$, such that $T_\mathbf{1}\Phi \cdot \Xi_u = \widetilde{\Xi}_u$ for all $u \in \mathbb{R}^\ell$. Here $T_\mathbf{1}\Phi$ is the tangent map of Φ at $\mathbf{1}$. State space equivalence turns out to be to rigid for the particular problem of interest and so we extend this idea to what we call detached feedback equivalence. Two control systems $\Sigma = (\mathsf{G}, \Xi)$ and $\tilde{\Sigma} = (\mathsf{G}, \widetilde{\Xi})$ are said to be detached feedback equivalent if there exists a local diffeomorphism $\Phi : V \to \tilde{V}$, where V and $\tilde{V}$ are neighbourhoods of $\mathbf{1} \in \mathsf{G}$, and an affine transformation $\Psi : \mathbb{R}^\ell \to \mathbb{R}^\ell$ such that $T_\mathbf{1}\Phi \cdot \Xi_u = \widetilde{\Xi}_{\Psi(u)}$ for all $u \in \mathbb{R}^\ell$. Let G be a matrix Lie group, g_0, g_1 arbitrary but fixed points in G and $T > 0$ fixed. A left-invariant optimal control problem (LiCP) on $\Sigma = (\mathsf{G}, \Xi)$ consists of finding a trajectory-control pair $(g(\cdot\,), u(\cdot\,))$ which transfers g_0 to g_1 optimally. That is, the solution minimises the cost

$$J = \int_0^T L(u(t))dt,$$

and satisfies

$$\dot{g} = g\Xi(\mathbf{1}, u)$$

subject to the boundary conditions

$$g(0) = g_0 \text{ and } g(T) = g_1.$$

We solve an optimal control problem on $\Sigma = (\mathsf{G}, \Xi)$ as follows: for each fixed $u \in \mathbb{R}^\ell$, $\Xi(\cdot\,, u) = \Xi_u$ defines a vector field on G. We define the following family of cost-extended Hamiltonians, $H_u^\lambda(\xi) = -\frac{\lambda}{2}L(u) + \xi(\Xi(g, u))$ for all $\xi \in T_g^*\mathsf{G}$, $g \in \mathsf{G}$, $u \in \mathbb{R}^\ell$. Here $\lambda = 1$ or $\lambda = 0$. Each vector field Ξ_u can be lifted canonically to its corresponding cost-extended Hamiltonian vector field $\overrightarrow{H}_u^\lambda$.

Using the identification $\xi = dL^*_{g^{-1}}(p)$, we realise $T^*\mathsf{G}$ as $\mathsf{G} \times \mathfrak{g}^*$, where $\mathfrak{g}^*$ is the dual of the Lie algebra, $\mathfrak{g}$. In these coordinates, each Hamiltonian becomes a linear functional on $\mathfrak{g}^*$ only. Thus, we obtain a family of cost-extended Hamiltonians on $\mathfrak{g}^*$. That is, each Hamiltonian vector field $\vec{H}^\lambda_u$ on $T^*\mathsf{G} \cong \mathsf{G} \times \mathfrak{g}^*$, corresponding to the Hamiltonian H^λ_u, can be (left) reduced to a Hamiltonian vector field on $\mathfrak{g}^*$. This process is known as Poisson reduction and is well known in geometric mechanics. Since $\mathfrak{g}^*$ is a vector space, we thus conveniently bypass computations with the symplectic structure on $T^*\mathsf{G}$. We then use the maximum principle to obtain a single optimal Hamiltonian function on $\mathfrak{g}^*$ and its corresponding optimal control. The optimal trajectory of an optimal control problem is then the projection of the integral curve $(g(\cdot), p(\cdot))$ of the Hamiltonian vector field $\vec{H}^\lambda_{u(\cdot)}$, which satisfies the conditions of the maximum principle. A pair of curves $(g(\cdot), p(\cdot), u(\cdot))$ on an interval $[0, T]$ is called an extremal pair if $(g(\cdot), p(\cdot))$ is an integral curve of $\vec{H}^\lambda_{u(\cdot)}$, for either $\lambda = 0$ or $\lambda = 1$, such that the conditions of the maximum principle are satisfied. The projection $(g(\cdot), p(\cdot))$ is called an extremal. The system of differential equations with solution $(g(\cdot), p(\cdot))$ are called the extremal equations, however the optimal Hamiltonian function is a linear functional on $\mathfrak{g}^*$ only and thus we study the system of differential equations on the dual space $\mathfrak{g}^*$ whose solution is the curve $p(\cdot)$. The system of differential equations, on $\mathfrak{g}^*$, are called the (reduced) extremal equations. The extremals corresponding to $\lambda = 0$ are called the abnormal extremals and the extremals corresponding to $\lambda = 1$ are called the normal extremals. The optimal Hamiltonian, along with the Poisson structure on $\mathfrak{g}^*$, is used to determine the (reduced) extremal equations. Using detached feedback equivalence, we determine all equivalence classes of controllable control systems on our Lie group G. We then choose a representative from each equivalence class. For each representative we study the associated optimal control problem. For each left-invariant optimal control problem on G we determine the the optimal control and optimal Hamiltonian function corresponding to the normal extremals. Also, we derive and analyse the normal extremal equations on $\mathfrak{g}^*$. In particular we study solvability (or integrability) of these normal extremal equations using trigonometric and/or Jacobi elliptic functions. We also fully classify, under Lyapunov stability, the equilibrium states of the normal extremal equations. The classification, under detached feedback equivalence, and the analysis of the normal extremal equations of the associated optimal control problems are the two main areas of study of this thesis.

1.2 A class of left-invariant optimal control problems on $\mathsf{SE}(2)$

1.2.1 The matrix Lie group $\mathsf{SE}(2)$

The group of orientation-preserving isometries of the Euclidean plane consists of transformations of the form $F(x) = Rx + \mathbf{v}$, where $R \in \mathbb{R}^{2\times 2}$ such that $R^\top R = \mathbf{1}$ with $\det R = 1$ and $\mathbf{v} \in \mathbb{R}^{2\times 1}$. Here $\mathbf{1}$ is the identity element in $\mathbb{R}^{2\times 2}$. This group of transformations is isomorphic to the matrix Lie group

$$\mathsf{SE}(2) = \left\{ \begin{bmatrix} 1 & 0 \\ \mathbf{v} & R_\theta \end{bmatrix} \mid \mathbf{v} \in \mathbb{R}^{2\times 1},\ R_\theta \in \mathsf{SO}(2) \right\},$$

where $\mathsf{SO}(2)$ is the special orthogonal group. $\mathsf{SE}(2)$ is a connected, solvable, unimodular matrix Lie group. The Lie algebra $\mathfrak{se}(2)$ consists of all 3×3 matrices of the form

$$\mathfrak{se}(2) = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ x_1 & 0 & -x_3 \\ x_2 & x_3 & 0 \end{bmatrix} \mid x_1, x_2, x_3 \in \mathbb{R} \right\}.$$

The matrices

$$E_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad E_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

form the standard basis for $\mathfrak{se}(2)$.

1.2.2 Equivalence of control systems on $\mathsf{SE}(2)$

A (left-invariant) control affine system on $\mathsf{SE}(2)$ is a pair $\Sigma = (\mathsf{SE}(2), \Xi)$, where $\Xi : \mathsf{SE}(2) \times \mathbb{R}^\ell \to T^*\mathsf{SE}(2) \cong \mathsf{SE}(2) \times \mathfrak{se}(2)^*$ is such that its trace $\Gamma \subset \mathfrak{se}(2)$ is an affine subspace. That is, the trace is of the form $\Gamma = \left\{ A + \sum_{i=1}^{\ell} u_i B_i \mid (u_1, \ldots, u_\ell) \in \mathbb{R}^\ell \right\}$, where $A, B_1, \ldots, B_\ell \in \mathfrak{se}(2)$. We assume $B_1, \ldots, B_\ell$ are linearly independent. We prove that $\Sigma = (\mathsf{SE}(2), \Xi)$ is controllable if and only if it is of full rank. Then, we classify, under detached feedback equivalence, all controllable control affine systems on $\mathsf{SE}(2)$. It turns out that there are six types of equivalent controllable control affine systems on $\mathsf{SE}(2)$. We therefore consider the following six types of control affine systems $\Sigma = (\mathsf{SE}(2), \Xi)$, where in each case Ξ is such that:

$$\begin{aligned}
&\text{Type I-a} : \Xi(\mathbf{1}, u) = E_1 + uE_3, && u \in \mathbb{R}, \\
&\text{Type I-b} : \Xi(\mathbf{1}, u) = \alpha E_3 + uE_1, && u \in \mathbb{R},\ \alpha > 0, \\
&\text{Type II}^0 : \Xi(\mathbf{1}, u) = E_1 + u_1 E_2 + u_2 E_3, && u = (u_1, u_2) \in \mathbb{R}^2, \\
&\text{Type II-b} : \Xi(\mathbf{1}, u) = \alpha E_3 + u_1 E_1 + u_2 E_3, && u = (u_1, u_2) \in \mathbb{R}^2,\ \alpha > 0 \\
&\text{Type III}^0 : \Xi(\mathbf{1}, u) = u_1 E_1 + u_2 E_1 + u_3 E_3, && u = (u_1, u_2, u_3) \in \mathbb{R}^3.
\end{aligned}$$

1.2.3 Control problems on $\mathsf{SE}(2)$

Given a controllable control affine system $\Sigma = (\mathsf{SE}(2), \Xi)$, let g_1 and g_2 be arbitrary but fixed points in $\mathsf{SE}(2)$ and $T > 0$ fixed. A left-invariant optimal control problem (LiCP) on a control affine system $\Sigma = (\mathsf{SE}(2), \Xi)$ consists of finding a trajectory-control pair $(g(\cdot), u(\cdot))$, which transfers g_0 to g_1 optimally. That is, it minimises the cost

$$J = \frac{1}{2} \int_0^T (c_1 u_1^2(t) + \ldots + c_\ell u_\ell^2(t)) dt, \quad c_1, \ldots, c_\ell > 0, 1 \leq \ell \leq 3,$$

and satisfies

$$\dot{g} = g\Xi(\mathbf{1}, u)$$

subject to the boundary conditions

$$g(0) = g_1 \text{ and } g(T) = g_2.$$

The Lie algebra $\mathfrak{se}(2)$ is isomorphic to the Lie algebra of left-invariant vector fields on $\mathsf{SE}(2)$ and so we view each left-invariant vector field on $\mathsf{SE}(2)$ as an element of the Lie algebra $\mathfrak{se}(2)$. Each vector field $X = A + \sum_{i=1}^{\ell} u_i B_i$ can be then be canonically lifted to its corresponding cost-extended Hamiltonian vector field $\overrightarrow{H}_u^{\lambda}$. We define the following family of Hamiltonians on $T^*\mathsf{SE}(2)$, $H_u(\xi) = \xi(A + \sum_{i=1}^{\ell} u_i B_i)$ for all $\xi \in T_g^* G$ and all $g \in \mathsf{G}$. Using the identification $T^*\mathsf{SE}(2) \cong \mathsf{SE}(2) \times \mathfrak{se}(2)^*$, we get that this family of Hamiltonians are given by $H_u(p) = p(A + \sum_{i=1}^{\ell} u_i B_i)$, which are linear functionals on $\mathfrak{se}(2)^*$ only. We then define the family of (reduced) cost-extended Hamiltonians on $\mathfrak{se}(2)^*$, $H_u^{\lambda}(p) = -\frac{\lambda}{2}\left(\sum_{i=1}^{\ell} c_i u_i^2\right) + p(A + \sum_{i=1}^{\ell} u_i B_i)$. Here $\lambda = 0$ or $\lambda = 1$. The control problem is thus lifted from the group $\mathsf{SE}(2)$ to the cotangent bundle $T^*\mathsf{SE}(2) \cong \mathsf{SE}(2) \times \mathfrak{se}(2)^*$ and then reduced to a problem on $\mathfrak{se}(2)^*$, the dual of the Lie algebra $\mathfrak{se}(2)$. The Pontryagin maximum principle is then used to reduce this family of (reduced) cost-extended Hamiltonians to a single candidate, which we call the optimal Hamiltonian. We then obtain a system of differential equations on $\mathfrak{se}(2)^*$, known as the (reduced) extremal equations. We identify the extremal curves $p(\cdot\,)$ on $\mathfrak{se}(2)^*$ with the corresponding curves $P(\cdot\,)$ on $\mathfrak{se}(2)$ using the nondegenerate paring $\langle P(t), X\rangle = p(t)X$, for all $X \in \mathfrak{se}(2)$. Therefore if

$$P(t) = \begin{bmatrix} 0 & 0 & 0 \\ P_1(t) & 0 & -P_3(t) \\ P_2(t) & P_3(t) & 0 \end{bmatrix}$$

then $P_i(t) = p(t)(E_i) = H_{E_1}(p(t))$, $i = 1, 2, 3$. The Lie algebra $\mathfrak{se}(2)$ can then be identified with $\mathbb{R}^3_{\owedge}$, where $\mathbb{R}^3_{\owedge}$ is the vector space $\mathbb{R}^3$ with the Lie bracket, $\owedge$, given by $(x_1, x_2, x_3) \owedge (y_1, y_2, y_3) = (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, 0)$.

The explicit solutions of the normal extremal equations are found for each of the control problems using trigonometric or Jacobi elliptic functions. We also use various methods, such as the energy-Casimir method, to classify the stability of all equilibrium states of the normal extremal equations for each of the six types of control problems on $\mathsf{SE}(2)$. The elliptic/trigonometric functions obtained are plotted in MATLAB, with arbitrary chosen initial conditions and constants. The results are then compared to plots of the extremal equations solved using the ODE45 solver in MATLAB. A comparison, of the extremal equations, is also made to plots of the intersection of the Hamiltonian functions and the coadjoint orbits.

The classification, under detached feedback equivalence, of all controllable control affine systems on $\mathsf{SE}(2)$ and then the analysis of the extremal equations of the resulting associated optimal control problems are the two main areas of study of this thesis.

In the conclusion, we compare the solutions to the extremal equations and the stability results of pairs of types of control problems as follows: firstly we allow $c_2 \to \infty$ in the type II-a case and compare the results to the type I-a case; secondly we allow $c_2 \to \infty$ in the type II-b case and compare the results with the type I-b case; lastly we let $c_2 \to \infty$ in the type III^0 case and compare the results with the type II^0 case.

1.3 Overview

Chapter 2 contains the necessary mathematical preliminaries. That is, it contains the definitions and facts that will be needed in chapters 3-10.

Section 2.1: Contains the definitions of the general linear group, matrix Lie groups, the tangent space, a Lie algebra, a tangent map, a Lie algebra homomorphism, the special orthogonal group $\mathsf{SO}(2)$, the (special) Euclidean group $\mathsf{SE}(2)$, left-invariant vector fields on a Lie group, the adjoint action, unimodular Lie algebra and Lie group, the derived series, solvable Lie algebra and Lie group, the lower central series, nilpotent Lie algebra and Lie group, external and internal semi-direct product.

Section 2.2: Contains the adjoint and coadjoint action of a Lie group, the orbit of an element under the action of a group, the adjoint and coadjoint orbits, invariant bilinear form.

Section 2.3: The Poisson space, Hamiltonian vector field, Hamiltonian function, Hamilton-Poisson system, Casimir functions, the Lie-Poisson bracket.

Section 2.4: We define a control system, the trace of a control system, a trajectory of a control system, the Lie algebra of a control system, homogeneous and non-homogeneous control affine systems, the attainable set, a controllable control system, full rank control systems.

Section 2.5: Introduces the notion of state space equivalence, feedback equivalence and detached feedback equivalence of control systems.

Section 2.6: We define an optimal trajectory, the cost and the Lagrangian, the left-invariant optimal control problem, the Hamiltonian of a vector field on a Lie group, the cost-extended Hamiltonian on the dual of the Lie algebra, the maximum principle, normal extremals.

Section 2.7: We introduce the Jacobi elliptic functions and elliptic integrals.

Section 2.8: We define an equilibrium state, a (nonlinear) stable equilibrium state, an asymptotically stable equilibrium state and an unstable equilibrium state. We state the energy-Casimir method and a theorem by Ortega and Ratiu.

Chapter 3: Introduces the matrix Lie group $\mathsf{SE}(2)$, its Lie algebra $\mathfrak{se}(2)$, and some of their properties. It also contains the adjoint and coadjoint orbits of the matrix Lie group $\mathsf{SE}(2)$.

Section 3.1: The (special) Euclidean group $\mathsf{SE}(2)$, the product and inverse of elements in $\mathsf{SE}(2)$. $\mathsf{SE}(2)$ is a matrix Lie group and is non-compact.

Section 3.2: The Lie algebra $\mathfrak{se}(2)$, $\mathfrak{se}(2)$ is isomorphic to $\mathbb{R}^3_{\owedge}$.

Section 3.3: The exponential map $\exp : \mathfrak{se}(2) \to \mathsf{SE}(2)$. This map is surjective but not injective.

Section 3.4: The Lie-Poisson structure on $\mathfrak{se}(2)^*$, the pairing between $\mathfrak{se}(2)^*$ and $\mathfrak{se}(2)$. We identify the extremal curves with curves in $\mathfrak{se}(2)$ and derive relations of the Lie-Poisson bracket on $\mathfrak{se}(2)^*$.

Section 3.5: $\mathsf{SE}(2)$ is the semi-direct product $\mathbb{R}^2 \rtimes \mathsf{SO}(2)$, $\mathsf{SE}(2)$ is a solvable but not completely solvable Lie group, $\mathsf{SE}(2)$ is a unimodular Lie group.

Subsection 3.6.1: The adjoint action and adjoint orbits of $\mathsf{SE}(2)$.

Subsection 3.6.2: The coadjoint action and coadjoint orbits of $\mathsf{SE}(2)$.

Subsection 3.6.3: The non-existence of a non-degenerate invariant bilinear form on $\mathfrak{se}(2)$.

Chapter 4: Covers the controllability of (left-invariant) control systems on $\mathsf{SE}(2)$. We attempt to classify, under state space equivalence, all controllable control affine systems on $\mathsf{SE}(2)$. We classify, under detached feedback equivalence, all controllable control affine systems on $\mathsf{SE}(2)$.

Section 4.1: We prove that a control system on $\mathsf{SE}(2)$ is controllable if and only if it is of full rank.

Section 4.2: We prove an algebraic criterion for the state space equivalence of two control affine systems on a 3-dimensional matrix Lie group G. It also contains the study of the state space equivalence of control affine systems on $\mathsf{SE}(2)$.

Subsection 4.2.1: We prove two control systems are state space equivalent if and only if there exists a local diffeomorphism which locally preserves trajectories corresponding to the same controls. We also prove that two three-dimensional control affine systems are state space equivalent if and only if there exists a Lie algebra isomorphism which maps elements corresponding to the same controls.

Subsection 4.2.2: We find representatives of the equivalence classes of all state space equivalent single-input control affine systems on $\mathsf{SE}(2)$.

Subsection 4.2.3: We find representatives of the equivalence classes of all state space equivalent two-input homogeneous control affine systems on $\mathsf{SE}(2)$.

Subsection 4.2.4: We determine the conditions required for two two-input non-homogeneous control affine systems on $\mathsf{SE}(2)$ to be state space equivalent.

Subsection 4.2.5: We determine the conditions required for two three-input homogeneous control affine systems on $\mathsf{SE}(2)$ to be state space equivalent.

Section 4.3: We study the detached feedback equivalence of control affine systems on $\mathsf{SE}(2)$.

Subsection 4.3.1: We find representatives of the equivalence classes of all detached feedback equivalent single-input control affine systems on $\mathsf{SE}(2)$.

Subsection 4.3.2: We find representatives of the equivalence classes of all detached feedback equivalent two-input homogeneous control affine systems on $\mathsf{SE}(2)$.

Subsection 4.3.3: We find representatives of the equivalence classes of all detached feedback equivalent two-input non-homogeneous control affine systems on $\mathsf{SE}(2)$.

Subsection 4.3.3: We find representatives of the equivalence classes of all detached feedback equivalent three-input control affine systems on $\mathsf{SE}(2)$. We also produce a table which summarises all the types of detached feedback equivalent control affine system on $\mathsf{SE}(2)$.

In **Chapters 5-10** we perform an analysis of the class of left-invariant optimal control problems on the matrix Lie group $\mathsf{SE}(2)$. We investigate six types of left-invariant optimal control problems on $\mathsf{SE}(2)$, namely: type I-a, type I-b, type II^0, type II-a, type II-b and type III^0. Each of these chapters begins with a statement of the particular optimal control problem. We then prove a theorem which gives us the optimal control, optimal Hamiltonian and the (reduced) extremal equations corresponding to the normal extremals. In each of these chapters, we solve the normal extremal equations using either standard trigonometric functions or Jacobi elliptic functions. We then fully classify, under Lyapunov stability, the equilibrium states of the normal extremal equations.

1.4 Contributions

To the best of our knowledge the following parts of the thesis are original:

- An algebraic criterion for the state space equivalence of full rank control affine systems on a 3-dimensional matrix Lie groups. Theorem: 4.2.3.
- The classification (under state space equivalence) of single-input control affine systems and two-input homogeneous control affine systems on $\mathsf{SE}(2)$. Propositions: 4.2.4, 4.2.6 and Corollaries: 4.2.5, 4.2.7.
- The set of conditions for two control systems to be state space equivalent for two-input non-homogeneous control affine systems and three-input homogeneous control affine systems. Propositions: 4.2.8, 4.2.10 and Examples: 4.2.9, 4.2.11.
- The classification (under detached feedback equivalence) of control affine systems on $\mathsf{SE}(2)$. Propositions: 4.3.1, 4.3.2, 4.3.3 and 4.3.4.
- The explicit integration of the (normal) extremal equations, via trigonometric or Jacobi elliptic functions, in the cases type I-b, type II^0, type II-a, type II-b and type III^0. Theorems: 6.2.1, 7.2.1, 8.2.1, 9.2.1, 10.2.1 and 10.2.2.
- The full classification (under Lyapunov stability) of the equilibrium states of the (normal) extremal equations for each of the six types. Theorems: 5.3.1, 5.3.2, 5.3.3, 6.3.1, 7.3.1, 7.3.2, 7.3.3, 8.3.1, 8.3.2, 8.3.3, 8.3.4, 9.3.1, 10.3.1, 10.3.2 and 10.3.3.

Chapter 2

Preliminaries

This chapter contains the definitions, propositions and theorems required throughout this thesis. The topics included are, matrix Lie groups, adjoint and coadjoint orbits, Poisson structure, control systems and controllability, equivalence of control systems, optimal control, elliptic functions and stability.

Throughout this thesis, unless otherwise specified,
all vector fields and mappings are assumed to be smooth.

2.1 Matrix Lie groups

The references used include [17], [18], [19]. $\mathbb{R}^{n\times n}$ is the algebra of all real $n \times n$ matrices. As a (real) vector space, $\mathbb{R}^{n\times n}$ will be identified with the Euclidean space $\mathbb{R}^{n^2}$. The matrix norm $\|A\| = \operatorname{tr}(A^\top A)$, $A \in \mathbb{R}^{n\times n}$, coincides with the Euclidean norm on $\mathbb{R}^{n^2}$. We use $\mathbf{1}$ to denote the identity element of a group G.

2.1.1 Definition. The (real) **general linear group** $\mathsf{GL}(n,\mathbb{R})$ consists of all $n \times n$ real invertible matrices:

$$\mathsf{GL}(n,\mathbb{R}) = \left\{X \in \mathbb{R}^{n\times n} \mid \det X \neq 0\right\}. \tag{2.1}$$

The **exponential map**, $\exp : \mathbb{R}^{n\times n} \to \mathsf{GL}(n,\mathbb{R})$, is given by

$$A \mapsto \exp A = \mathbf{1} + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \frac{A^4}{4!} + \cdots.$$

2.1.2 Definition. A group of matrices $\mathsf{G} \subset \mathsf{GL}(n,\mathbb{R})$ is a **closed** subset of $\mathsf{GL}(n,\mathbb{R})$ if the following condition is satisfied: if $(A_r)_{r>0}$ is any sequence of matrices in G and $A_r \to A$, then either $A \in \mathsf{G}$ or A is not invertible (i.e. $A \notin \mathsf{GL}(n,\mathbb{R})$).

2.1.3 Definition. A (real) **matrix Lie group** is any closed subgroup of some general linear group.

2.1.4 Definition. A matrix Lie group G is called (path) **connected** if, for any $g_1, g_2 \in \mathsf{G}$, there exists a continuous curve $\sigma : [0,1] \to \mathsf{G}$ such that $\sigma(0) = g_1$ and $\sigma(1) = g_2$.

2.1.5 Definition. Let G be a matrix Lie group and $g \in \mathsf{G}$. Then the **tangent space** to G at g is the set

$$T_g\mathsf{G} = \{\dot{\sigma}(0) \mid \sigma(\cdot\,) \text{ is a curve in } \mathsf{G},\ \sigma(0) = g\}\,. \tag{2.2}$$

2.1.6 Definition. A (real) **Lie algebra** consists of a vector space V, together with a bilinear map $[\cdot\,,\cdot\,] : V \times V \to V$, called the **Lie bracket**, such that for all $x, y, z \in V$,

$$[x, y] = -[y, x], \qquad \text{(Skew symmetry)}$$

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \qquad \text{(Jacobi identity)}$$

Let G be a matrix Lie group. Then the tangent space at identity, $T_{\mathbf{1}}\mathsf{G}$, equipped with the Lie bracket defined by the commutator of matrices,

$$[A, B] = AB - BA, \quad \text{for all } A, B \in T_{\mathbf{1}}\mathsf{G},$$

defines a Lie algebra of the Lie group G, which we denote by $\mathfrak{g}$.

2.1.7 Proposition. *For any matrix Lie group* G*, the restriction of the exponential map carries the Lie algebra* $\mathfrak{g}$ *into* G*.*

2.1.8 Definition. Suppose that $\Phi : \mathsf{G}_1 \to \mathsf{G}_2$ is a mapping between matrix Lie groups. For any curve σ on G_1, $\Phi \circ \sigma$ is a curve on G_2. Then the tangent map of Φ is

$$T_{\mathbf{1}} \cdot \Phi : T_{\mathbf{1}}\mathsf{G}_1 \to T_{\Phi(\mathbf{1})}\mathsf{G}_2, \quad \dot{\sigma}(0) \mapsto (\Phi \circ \sigma)^{\dot{}}(0).$$

2.1.9 Definition. Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be two Lie algebras with Lie brackets given by $[\cdot\,,\cdot\,]_1$ and $[\cdot\,,\cdot\,]_2$, respectively. A linear mapping $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ is called a **Lie algebra homomorphism** if

$$\phi([X, Y]_1) = [\phi(X), \phi(Y)]_2$$

for all $X, Y \in \mathfrak{g}_1$. If ϕ is also bijective, then ϕ is called a **Lie algebra isomorphism**.

2.1.10 Definition. The group $\mathsf{SO}(\mathbf{2})$, called the **special orthogonal group**, is the set of all 2×2 orthogonal matrices with determinant 1. That is,

$$\mathsf{SO}(2) = \left\{ R \in \mathbb{R}^{2\times 2} \mid R^\top R = \mathbf{1},\ \det R = 1 \right\} \tag{2.3}$$

2.1.11 Definition. The group SE($\mathbf{2}$), called the (special) **Euclidean group**, is the following set of all 3×3 matrices:

$$\mathrm{SE}(2) := \left\{ \begin{bmatrix} 1 & 0 \\ \mathbf{v} & R_\theta \end{bmatrix} \in \mathsf{GL}(3, \mathbb{R}) \mid \mathbf{v} \in \mathbb{R}^{2\times 1} \text{ and } R_\theta \in \mathsf{SO}(2) \right\}. \tag{2.4}$$

2.1.12 Definition. A vector field on G of the form

$$V(g) = gX, \quad g \in \mathsf{G}, \quad X \in \mathfrak{g},$$

is called **left-invariant**.

We denote by $\mathfrak{X}^L(\mathsf{G})$ the space of all left-invariant vector fields on a Lie group G. Also, we denote by $C^\infty(M)$ the space of all (smooth) functions on a space M.

2.1.13 Proposition. *Let* G *be a matrix Lie group,* $\mathfrak{g}$ *its Lie algebra, and let* $g \in \mathsf{G}$. *Then*

$$T_g G = g T_{\mathbf{1}} \mathsf{G} = g\,\mathfrak{g} = \{gX \mid X \in \mathfrak{g}\}\,.$$

2.1.14 Proposition. *Let* G *be a matrix Lie group,* $\mathfrak{g}$ *its Lie algebra, and let* $X, Y \in \mathfrak{g}$. *Let* $V(g) = gX$ *and* $W(g) = gY$ *be left-invariant vector fields on* G. *Then*

$$[V, W](g) = [gX, gY] = g[X, Y] = g(XY - YX), \quad g \in \mathsf{G}.$$

2.1.15 Corollary. *Left-invariant vector fields on a Lie group* G *form a Lie algebra isomorphic to the Lie algebra* $\mathfrak{g}$. *The isomorphism is defined as follows:*

$$V \in \mathfrak{X}^L(\mathsf{G}) \leftrightarrow V(\mathbf{1}) = X \in \mathfrak{g}.$$

Thus we identify the Lie algebra of left-invariant vector fields on G with the Lie algebra $\mathfrak{g}$ of G.

2.1.16 Definition. Let G be a Lie group. Given an element X of the Lie algebra $\mathfrak{g}$, the **adjoint action of X on $\mathfrak{g}$** is the endomorphism

$$\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}, \quad \mathrm{ad}_X(Y) = [X, Y]$$

for all $Y \in \mathfrak{g}$.

2.1.17 Definition. The **derived series** of a Lie algebra $\mathfrak{g}$ is given as follows:

$$\mathfrak{g}^k = [\mathfrak{g}^{k-1}, \mathfrak{g}^{k-1}] = \mathrm{span}\,\{[x, y] \mid x, y \in \mathfrak{g}^{k-1}\},$$

$k \in \mathbb{N}$.

2.1.18 Definition. The **lower central series** of a Lie algebra $\mathfrak{g}$ is the sequence of subalgebras recursively defined by

$$\mathfrak{g}_{k+1} = [\mathfrak{g}, \mathfrak{g}_k],$$

with $\mathfrak{g}_0 = \mathfrak{g}$.

2.1.19 Definition. Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. $\mathfrak{g}$ is called

- **unimodular** if $\mathrm{tr}\,(\mathrm{ad}_X) = 0$ for all $X \in \mathfrak{g}$.
- **solvable** if its derived series $\mathfrak{g}^k$ vanishes for some $k \in \mathbb{N}$.
- **completely solvable** if it is solvable and if all adjoint operators ad_X, $X \in \mathfrak{g}$, have real spectra.
- **nilpotent** if its lower central series vanishes for some $k \in \mathbb{N}$.

The Lie group G is called unimodular, solvable, completely solvable or nilpotent if its Lie algebra is unimodular, solvable, completely solvable or nilpotent, respectively.

2.1.20 Definition. Let H, Q be groups and let $\theta : \mathsf{Q} \to \mathrm{Aut}(\mathsf{H})$ be a group homomorphism. The **external semi-direct product** $\mathsf{H} \rtimes \mathsf{Q}$ is defined to be the group with underlying set

$$\{(h, q) \mid h \in \mathsf{H},\ q \in \mathsf{Q}\}$$

and group operation

$$(h, q)(h', q') = (h\theta(q)h', qq').$$

The inverse of an element (h, q) is given by

$$(h, q)^{-1} = (\theta(q^{-1})(h^{-1}), q^{-1}).$$

2.1.21 Theorem. *Let* $\mathsf{G} := \mathsf{H} \rtimes \mathsf{Q}$ *be the external semi-direct product of* H *and* Q*. Then:*

- H *is a normal subgroup of* G
- $\mathsf{HQ} = \mathsf{G}$
- $\mathsf{H} \cap \mathsf{Q} = \mathbf{1}_{\mathsf{G}}$.

This result motivates the definition of internal semi-direct products.

2.1.22 Definition. Let G be a group with subgroups H, Q. We say G is an **internal semi-direct product** of H and Q if:

- H is a normal subgroup of G
- $\mathsf{HQ} = \mathsf{G}$
- $\mathsf{H} \cap \mathsf{Q} = \mathbf{1}_{\mathsf{G}}$.

Clearly from theorem 2.1.21 an external semi-direct product is an internal semi-direct product. We now give a theorem which shows that an internal semi-direct product is an external semi-direct product.

2.1.23 Theorem. *Suppose* G *is a group with subgroups* H *and* Q *and* G *is the internal semi-direct product of* H *and* Q*. Then* $\mathsf{G} \cong \mathsf{H} \rtimes_{\theta} \mathsf{Q}$ *where* $\theta : \mathsf{Q} \to \mathrm{Aut}(\mathsf{H})$ *is given by*

$$\theta(q)(h) := qhq^{-1}, \quad q \in \mathsf{Q},\ h \in \mathsf{H}.$$

Since we have that the internal and external semi-direct products are the same we will refer to both as just the **semi-direct product** and denote it by $\mathsf{G} = \mathsf{H} \rtimes \mathsf{Q}$ where θ is assumed to be given as in theorem 2.1.23.

2.2 Adjoint and coadjoint orbits

The references used are [10], [6].

2.2.1 Definition. By an **action** of a group G on a set M we mean a map $\tau : \mathsf{G} \times M \to M$, satisfying the following conditions:

$$\tau(\mathbf{1}, x) = x \tag{2.5}$$
$$\tau(a, \tau(b, x)) = \tau(ab, x) \tag{2.6}$$

for any $x \in M$, $b \in \mathsf{G}$. Given an action τ, to every $a \in \mathsf{G}$ there corresponds a bijective transformation $\tau_a : x \mapsto \tau(a, x)$ of the set M and the map $t : a \mapsto \tau_a$ is a homomorphism of the group G into the group S_M of all permutations (bijective transformations) of the set M. Conversely, any homomorphism $t : \mathsf{G} \to S_M$ defines an action of G on M by the formula

$$\tau(a, x) = t(a)(x), \quad a \in \mathsf{G}, x \in M.$$

Usually the action of a group G on a set M is denoted as multiplication of elements of the group G by the elements of M and written

$$\tau(a, x) = ax, \quad a \in \mathsf{G}, x \in M.$$

The group actions defined above are referred to as left actions.

The **adjoint action** of a matrix Lie group G on its Lie algebra $\mathfrak{g}$ is defined as follows:

$$\mathrm{Ad} : \mathsf{G} \times \mathfrak{g} \to \mathfrak{g}, \quad \mathrm{Ad} : (g, X) \mapsto \; gXg^{-1}$$

and hence we define the following bijective transformation:

$$\mathrm{Ad}_{\,g} : \mathfrak{g} \to \mathfrak{g}, \quad \mathrm{Ad}_{\,g} : \; X \mapsto gXg^{-1}.$$

The **coadjoint action** of a matrix Lie group G on the dual of its Lie algebra $\mathfrak{g}^*$ is defined as follows:

$$\mathrm{Ad}^* : \mathsf{G} \times \mathfrak{g}^* \to \mathfrak{g}^*, \quad \mathrm{Ad}^* : (g, \mu) \; \mapsto (\mathrm{Ad}_{g^{-1}})^* \mu$$

and hence we define the following bijective transformation for all $X \in \mathfrak{g}$:

$$\mathrm{Ad}^*_g : \mu \mapsto (\mathrm{Ad}_{g^{-1}})^* \mu, \quad \text{where} \quad (\mathrm{Ad}_{g^{-1}})^* \mu(X) = \mu(g^{-1}Xg).$$

2.2.2 Definition. Given a group G acting on a set M, the **orbit** of an element $x \in M$ is given by

$$Orb(x) = \{gx \mid g \in \mathsf{G}\}.$$

2.2.3 Definition. Given a Lie group G with Lie algebra $\mathfrak{g}$, the set

$$\mathrm{Orb}(X) = \{\mathrm{Ad}_g X \mid g \in \mathsf{G}\}$$

is called the **adjoint orbit** of G through $X \in \mathfrak{g}$.

2.2.4 Definition. Given a Lie group G with Lie algebra $\mathfrak{g}$, the set

$$\operatorname{orb}(\mu) = \left\{ (\mathrm{Ad}_{g^{-1}})^*(\mu) \mid g \in \mathsf{G} \right\}$$

is called the **coadjoint orbit** of G through $\mu \in \mathfrak{g}^*$.

2.2.5 Definition. A bilinear form on the Lie algebra $\mathfrak{g}$, of a Lie group G, $\mathcal{B} : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$, is called **invariant** if

$$\mathcal{B}(\mathrm{Ad}_g X_1, \mathrm{Ad}_g X_2) \;=\; \mathcal{B}(X_1, X_2)$$

for all $g \in \mathsf{G},\ X_1, X_2 \in \mathfrak{g}$.

2.2.6 Proposition. *If the Lie algebra $\mathfrak{g}$, of a Lie group G, admits a non-degenerate invariant bilinear form, then the adjoint and coadjoint actions are equivalent.*

2.3 Poisson structure

The references used are [1], [6], [9], [10], [13].

2.3.1 Definition. A **Poisson bracket** on a vector space M is a bilinear operation $\{\cdot\,,\cdot\}$ on $C^\infty(M)$ such that:

- $(C^\infty(M), \{\cdot\,,\cdot\})$ is a Lie algebra; and
- $\{\cdot\,,\cdot\}$ is a derivation in each factor, that is,

  $$\{FG, H\} = \{F, H\}G + F\{G, H\}$$

  for all $F, G, H \in C^\infty(M)$.

A vector space M endowed with a Poisson bracket on $C^\infty(M)$, $(M, \{\cdot\,,\cdot\})$, is called a **Poisson space**.

2.3.2 Definition. Let $(M, \{\cdot\,,\cdot\})$ be a Poisson space and $H \in C^\infty(M)$. The vector field $\overrightarrow{H}$ defined by

$$\overrightarrow{H}(F) = \{F, H\} \tag{2.7}$$

for all $F \in C^\infty(M)$, is called the **Hamiltonian vector field**, with **Hamiltonian function** H. The triple $(M, \{\cdot\,,\cdot\}, H)$ is called a **Hamilton-Poisson system**.

2.3.3 Definition. Let $(M, \{\cdot\,,\cdot\})$ be a Poisson space. Among the elements of $C^\infty(M)$ are functions C such that $\{C, F\} = 0$ for all $F \in C^\infty(M)$; that is, C is constant along the flow (i.e, an integral of motion) of all Hamiltonian vector fields or, equivalently, $\overrightarrow{H}(C) = 0$. Such functions are called **Casimir functions** of the Poisson structure.

2.3.4 Proposition. *Let $(M, \{\cdot\,,\cdot\}, H)$ be a Hamilton-Poisson system. The map $F \in C^\infty(M) \mapsto \overrightarrow{F} \in \mathfrak{X}(M)$ is a Lie algebra anti-homomorphism, i.e.,*

$$[\overrightarrow{F}, \overrightarrow{G}] = -\overrightarrow{\{F, G\}}.$$

2.3.5 Proposition. *Let* $(M, \{\cdot\,,\cdot\}, H)$ *be a Hamilton-Poisson system and let* $\phi_t = \exp t\overrightarrow{H}$ *denote the flow of* $\overrightarrow{H}$*. Then for each* $F \in C^\infty(M)$ *we have:*

(i) $H \circ \phi_t = H$ *(conservation of energy).*

(ii) $\frac{d}{dt}(F \circ \phi_t) = \{F, H\} \circ \phi_t = \{F \circ \phi_t, H\}$*, or for short*

$$\dot{F} = \{F, H\},$$

for all $F \in C^\infty(M)$*.*

2.3.6 Definition. If $\mathfrak{g}$ is a Lie algebra then its dual, $\mathfrak{g}^*$, is a Poisson space with respect to the **Lie-Poisson bracket** $\{\cdot\,,\cdot\}_-$ defined by

$$\{F, G\}_-(p) = -p\,[dF(p), dG(p)] \tag{2.8}$$

for $p \in \mathfrak{g}^*$ and $F, G \in C^\infty(\mathfrak{g}^*)$. Here $dF(p), dG(p) \in (\mathfrak{g}^*)^* \cong \mathfrak{g}$.

Let G be an n-dimensional Lie group and suppose $H : \mathfrak{g}^* \to \mathbb{R}$ is a (reduced) Hamiltonian function. Let $(\mathfrak{g}^*, \{\cdot\,,\cdot\}_-)$ denote the dual space of the Lie algebra $\mathfrak{g}$ with the Poisson structure as defined by 2.8. Let $E_1^*, \ldots, E_n^*$ be the dual basis of $\mathfrak{g}^*$. Thus $E_i^*(E_j) = \delta^i_j$. Now any $p \in \mathfrak{g}^*$ can be expressed as

$$p = \sum_{i=1}^n p_i E_i^*.$$

The coordinate functions then satisfy the differential equation (by proposition 2.3.5)

$$\dot{p}_i = \{p_i, H\}_-, \; i = 1, \ldots, n. \tag{2.9}$$

2.3.7 Proposition. ([9]) *Each left-invariant vector field* X *on a matrix Lie group* G *can be canonically lifted to a its Hamiltonian vector field* $\overrightarrow{H}_X$ *through the formula*

$$H_X(p) = p(X(g))$$

for all $g \in \mathsf{G}$ *and all* $p \in \mathfrak{g}^*$*.* H_X *is called the (reduced) Hamiltonian of* X*.*

2.3.8 Proposition. *If* H_X *and* H_Y *are the Hamiltonian functions on* $\mathfrak{g}^*$*, which correspond to the left-invariant vector fields* X *and* Y *on* G*, then* $\{H_X, H_Y\}_- = -H_{[X,Y]}$*.*

2.4 Control systems and controllability

The references used are [1], [5], [6], [18], [19]. The **class of admissible controls** is

$$\mathcal{U} = \{u(\,\cdot\,) : [0, T_u] \to \mathbb{R}^\ell \mid u(\,\cdot\,) \text{ piecewise-continuous}\},$$

where $\mathbb{R}^\ell$ is called the **input space**. Throughout this thesis we fix this class of admissible controls.

2.4.1 Definition. A (left-invariant) **control system** is a pair $\Sigma = (\mathsf{G}, \Xi)$ such that:

- $\mathsf{G} \subset \mathsf{GL}(n, \mathbb{R})$ is a matrix Lie group, called the **state space**.
- $\Xi : \mathsf{G} \times \mathbb{R}^\ell \to T\mathsf{G}$ is a mapping of the form

$$(g, u) \mapsto \Xi(g, u) = g\Xi(\mathbf{1}, u).$$

$\Xi(\mathbf{1}, \cdot\,) : \mathbb{R}^\ell \to \mathfrak{g}$ is a map. $\Gamma = \operatorname{im}(\Xi(\mathbf{1}, \cdot))$, called the **trace** of the system, is given by

$$\Gamma = \{\Xi_u \mid u \in \mathbb{R}^\ell\} \subset \mathfrak{g},$$

where each left-invariant vector field, $\Xi_u = \Xi(\,\cdot\,, u) : \mathsf{G} \to T\mathsf{G}$, is viewed as an element of the Lie algebra $\mathfrak{g}$, by corollary 2.1.15.

2.4.2 Definition. A **trajectory** of a control system Σ, through some $g_0 \in \mathsf{G}$, for some admissible control $u(\,\cdot\,) \in \mathcal{U}$, is an absolutely continuous curve $g(\,\cdot\,) : [0, T] \to \mathsf{G}$, such that $g(0) = g_0$, which satisfies the equation

$$\dot{g}(t) = \Xi(g(t), u(t)),$$

almost everywhere.

2.4.3 Definition. We define the **Lie algebra of a system** Σ, Lie$\,(\Gamma)$, as the Lie algebra generated by Γ, i.e., the smallest Lie subalgebra of $\mathfrak{g}$ containing Γ.

2.4.4 Definition. A **control affine system** is a control system Σ such that the trace Γ is of the form:

$$\Gamma = \{A + \sum_{i=1}^{\ell} u_i B_i \mid (u_1, \ldots, u_\ell) \in \mathbb{R}^\ell\},$$

where $A, B_1, \ldots, B_\ell \in \mathfrak{g}$ and $B_1, \ldots, B_\ell$ are linearly independent. If $A, B_1, \ldots, B_\ell$ are linearly independent then the system is called **non-homogeneous**, otherwise it is called **homogeneous**.

2.4.5 Definition. The **attainable set** of a control system Σ, from a point $g \in \mathsf{G}$, is the set $\mathcal{A}(g)$ of all terminal points $g(t)$, $T \geq 0$, of all trajectories $g(\cdot)$ of Σ starting at g:

$$\mathcal{A}(g) = \{g(T) \mid g(\cdot\,) \text{ a trajectory of } \Sigma,\ g(0) = g,\ T \geq 0\}\,.$$

We denote by $\mathcal{A}$ the attainable set from the identity, i.e., $\mathcal{A} = \mathcal{A}(\mathbf{1})$.

2.4.6 Lemma. *Let* $\Sigma = (\mathsf{G}, \Xi)$ *be a control system and let* g *be an arbitrary point of* G. *Then*

- $\mathcal{A}(g) = \{\exp(t_n A_n) \ldots \exp(t_1 A_1) \mid A_i \in \Gamma,\ t_i > 0,\ N \geq 0\}$*;*
- $\mathcal{A}(g) = g\mathcal{A}$*;*
- $\mathcal{A}$ *is a subsemigroup of* G*;*
- $\mathcal{A}(g)$ *is a connected subset of* G.

2.4.7 Definition. A control system Σ is called **controllable** if, given any pair of points g_0 and g_1 in G, the point g_1 can be reached from the point g_0 along a trajectory of Σ for some non-negative time:

$$g_1 \in \mathcal{A}(g_0) \text{ for any } g_0, g_1 \in \mathsf{G},$$

or if,

$$\mathcal{A}(g) = G \text{ for any } g \in \mathsf{G}.$$

2.4.8 Definition. A control system Σ is said to be of **full rank** if $\operatorname{Lie}(\Gamma) = \mathfrak{g}$.

2.4.9 Theorem. (Connectedness condition) *If a control system $\Sigma = (\mathsf{G}, \Xi)$ is controllable, then the G is connected.*

2.4.10 Theorem. (Rank condition) *Let Σ be a control system.*

(i) If Σ is controllable, then Σ is of full rank.

(ii) int$\mathcal{A} \neq \emptyset$ if and only if Σ is of full rank.

2.4.11 Theorem. *Let $\Sigma = (\mathsf{G}, \Xi)$ be a control System. Then Σ is controllable if and only if G is connected and $\mathbf{1} \in \operatorname{int}\mathcal{A}$.*

2.5 Equivalence of control systems

The references used are [4], [8], [16]. Let $\Sigma = (\mathsf{G}, \Xi)$ and $\tilde{\Sigma} = (\tilde{\mathsf{G}}, \tilde{\Xi})$ be two control systems, where we assume that $\dim \mathsf{G} = \dim \tilde{\mathsf{G}}$.

2.5.1 Definition. We say that Σ and $\tilde{\Sigma}$ are (locally) **state space equivalent** if there exists a local diffeomorphism $\Phi : V \to \tilde{V}$, $\Phi(\mathbf{1}) = \tilde{\mathbf{1}}$, where V and $\tilde{V}$ are neighbourhoods of $\mathbf{1}$ and $\tilde{\mathbf{1}}$, respectively, such that

$$T_{\mathbf{1}}\Phi \cdot \Xi_u = \tilde{\Xi}_u$$

for all $u \in \mathbb{R}^\ell$. Here $T_{\mathbf{1}}\Phi$ denotes the tangent map of Φ at $\mathbf{1} \in \mathsf{G}$.

Local state space equivalence of Σ and $\tilde{\Sigma}$ means simply that Φ establishes a correspondence between vector fields defined by constant controls.

We introduce the following notation for left iterated Lie brackets:

$$\begin{aligned} \Xi_{[u_1]} &= \Xi_{u_1} \\ \Xi_{[u_1 u_2 \ldots u_k]} &= [\Xi_{u_1}, [\Xi_{u_2}, \ldots, [\Xi_{u_{k-1}}, \Xi_{u_k}] \ldots]]. \end{aligned}$$

2.5.2 Theorem. *Let $\Sigma = (\mathsf{G}, \Xi)$ and $\tilde{\Sigma} = (\tilde{\mathsf{G}}, \tilde{\Xi})$ be two control systems such that* $\dim \mathsf{G} = \dim \tilde{\mathsf{G}}$. *Assume that* $\operatorname{Lie}(\Gamma) = \mathfrak{g}$ *and* $\operatorname{Lie}(\tilde{\Gamma}) = \tilde{\mathfrak{g}}$. *Then Σ and $\tilde{\Sigma}$ are state space equivalent if and only if there exists a linear isomorphism ψ such that*

$$\psi \cdot \Xi_{[u_1 u_2 \ldots u_k]} = \tilde{\Xi}_{[u_1 u_2 \ldots u_k]} \tag{2.10}$$

for any $k \geq 1$ and any $u_1, u_2, \ldots, u_k \in \mathbb{R}^\ell$.

This is an adaption of a result obtained by A. Krener in [8]. Henceforth, we will refer to condition 2.10 as Krener's condition.

Feedback equivalence is a generalisation of state space equivalence where we allow a transformation of the controls, as well as the state space. Here we transform the controls in a way that depends on the state: thus feeding the system back into itself.

2.5.3 Definition. Two control systems $\Sigma = (\mathsf{G}, \Xi)$ and $\tilde{\Sigma} = (\tilde{\mathsf{G}}, \tilde{\Xi})$ are (locally) **feedback equivalent**, at points g and $\tilde{g}$, if there exists a local diffeomorphism $\Omega : V \times \mathbb{R}^\ell \to \tilde{V} \times \mathbb{R}^\ell$, where V and $\tilde{V}$ are neighbourhoods of g and $\tilde{g}$, respectively, of the form

$$(\tilde{g}, \tilde{u}) = \Omega(g, u) = (\Phi(g), \Psi(g, u)),$$

which transforms the first system into the second, i.e.,

$$T_g\Phi \cdot \Xi(g, u) = \tilde{\Xi}(\Phi(g), \Psi(g, u)).$$

The notion of feedback equivalence turns out to be to general for the purposes of studying the equivalence of control affine systems on $\mathsf{SE}(2)$. This is so as we do not want the transformation of the controls to be dependent on the state space. We now define the notion of detached feedback equivalence of control systems. We define this concept independently of any other notion of equivalence, although it is of interest to note that we call this equivalence "detached feedback" as it is a special case of the standard notion of feedback equivalence. This keeps the transformation of the controls independent of the state space, since for our left-invariant optimal control problem the cost function is a quadratic function of the controls only. On the other hand, the notion of state space equivalence is to rigid as we want to allow some degree of transformation of the controls.

2.5.4 Definition. Two control systems $\Sigma = (\mathsf{G}, \Xi)$ and $\tilde{\Sigma} = (\tilde{\mathsf{G}}, \tilde{\Xi})$ are called (locally) **detached feedback equivalent** if there exists a local diffeomorphism $\Phi : V \to \tilde{V}$, where V and $\tilde{V}$ are neighbourhoods of $\mathbf{1}$ and $\tilde{\mathbf{1}}$, respectively, and an affine transformation $\Psi : \mathbb{R}^\ell \to \mathbb{R}^\ell$ such that

$$T_{\mathbf{1}}\Phi \cdot \Xi_u = \tilde{\Xi}_{\psi(u)},$$

for all $u \in \mathbb{R}^\ell$.

2.6 Optimal control

The references used are [1], [5], [6], [7], [9], [10], [18], [19], [20]. Let g_0 and g_1 be arbitrary but fixed points in G. A trajectory $g(\cdot)$ of a control system Σ is said to transfer a point g_0 to g_1 **optimally** if there exists an interval $[0, T]$, on which $g(\cdot)$ is defined, such that $g(0) = g_0$ and $g(T) = g_1$ and among all trajectories which satisfy these conditions it also minimises the **cost**, given by

$$J = \int_0^T L(g(t), u(t))dt.$$

Here $L : G \times \mathbb{R}^\ell \to \mathbb{R}_{>0}$ is called the **Lagrangian**.

2.6.1 Definition. The Lagrangian $L : \mathsf{G} \times \mathbb{R}^\ell \to \mathbb{R}_{>0}$ is called **left-invariant** if

$$L(gh, u) = L(h, u) \quad \text{for all } g, h \in \mathsf{G}.$$

2.6.2 Definition. Given a controllable control affine system $\Sigma = (\mathsf{G}, \Xi)$, let g_0 and g_1 be arbitrary but fixed points in G, and $T > 0$ fixed. A **left-invariant optimal control problem (LiCP)** on $\Sigma = (\mathsf{G}, \Xi)$ consists of finding a trajectory-control pair $(g(\cdot\,), u(\cdot\,))$ which transfers g_0 to g_1 optimally. That is, it minimises the cost

$$J = \frac{1}{2}\int_0^T (c_1 u_1^2(t) + \ldots + c_\ell u_\ell^2(t))dt, \quad c_1, \ldots, c_\ell > 0, \quad 1 \leq \ell \leq 3 \tag{2.11}$$

and satisfies

$$\dot{g} = g\Xi(\mathbf{1}, u) = g(A + \sum_{i=1}^{\ell} u_i B_i), \tag{2.12}$$

subject to the boundary conditions

$$g(0) = g_0 \text{ and } g(T) = g_1. \tag{2.13}$$

Let $\Sigma = (\mathsf{G}, \Xi)$ be a control affine system. For each admissible control $u(\cdot\,) = (u_1(\cdot\,), \ldots, u_\ell(\cdot\,)) \in \mathcal{U}$, the Hamiltonian H_u of the vector field $A + \sum_{i=1}^{\ell} u_i B_i$ is given by

$$H_u(\xi) = \xi(g(A + \sum_{i=1}^{\ell} u_i B_i))$$

for all $\xi \in T_g^*\mathsf{G}$, $g \in \mathsf{G}$. Using the change of coordinates $\xi = dL_{g^{-1}}^*(p)$, we identify $T^*\mathsf{G} \cong \mathsf{G} \times \mathfrak{g}^*$, and thus we get that the corresponding Hamiltonian, in these new non-canonical coordinates, is given by

$$\begin{aligned} H_u(g, p) &= dL_{g^{-1}}^*(p)(g(A + \sum_{i=1}^{\ell} u_i B_i)) \\ &= p(dL_{g^{-1}})(g(A + \sum_{i=1}^{\ell} u_i B_i)) \\ &= p(g^{-1}g(A + \sum_{i=1}^{\ell} u_i B_i)) \\ &= p(A + \sum_{i=1}^{\ell} u_i B_i). \end{aligned}$$

Hence H_u is a linear functional on $\mathfrak{g}^*$ only.

2.6.3 Definition. Given a LiCP, the (reduced) **cost-extended Hamiltonian** on $\mathfrak{g}^*$, for each

$u(\cdot\,) = (u_1(\cdot\,), \ldots, u_\ell(\cdot\,)) \in \mathcal{U}$, is given by

$$H_u^\lambda(p) = -\frac{\lambda}{2}\left(\sum_{i=1}^{\ell} c_i u_i^2\right) + p(A + \sum_{i=1}^{\ell} u_i B_i),$$

$p \in \mathfrak{g}^*$. Here $\lambda = 0$ or $\lambda = 1$.

2.6.4 Proposition. *Suppose that* $(g(\cdot\,), p(\cdot\,))$ *is an integral curve of the Hamiltonian vector field* $\vec{H}^\lambda_{u(\cdot)}$ *for some* $u(\cdot\,) \in \mathcal{U}$, *where* $H_u^\lambda(p) = -\frac{\lambda}{2}\left(\sum_{i=1}^{\ell} c_i u_i^2\right) + p(A + \sum_{i=1}^{\ell} u_i B_i)$ *is the (reduced) cost-extended Hamiltonian on* $\mathfrak{g}^*$. *Then*

$$\dot{g} = g\left(\frac{\partial H_u^\lambda}{\partial p}(p)\right) \quad \textit{and} \quad p(t) = \mathrm{Ad}^*_{g(t)}p(0)$$

for some $p(0) \in \mathfrak{g}$. *Consequently* $p(\cdot\,)$ *is contained in the coadjoint orbit of* G *through* $p(0)$.

We now give a statement of the maximum principle, due to Pontryagin, which gives a set of necessary conditions for a trajectory to be an optimal trajectory.

2.6.5 Theorem. (The maximum principle) *Suppose that* $u(\cdot\,)$ *is an optimal control with corresponding optimal trajectory* $g(\cdot\,)$ *of an optimal control problem. Then,* $g(\cdot\,)$ *is the projection of an integral curve* $(g(\cdot\,), p(\cdot\,))$ *of the Hamiltonian vector field* $\vec{H}^\lambda_{u(\cdot)}$ *with* $\lambda = 0$ *or* $\lambda = 1$ *such that:*

(MP1) if $\lambda = 0$, *then* $(g(\cdot\,), p(\cdot\,))$ *is not identically zero on* $[0, T]$.

(MP2) $H^\lambda_{u(\cdot)}(g(\cdot\,), p(\cdot\,)) \geq H^\lambda_u(g(\cdot\,), p(\cdot\,))$ *for any* $u \in \mathbb{R}^\ell$, *and almost all* $t \in [0, T]$.

(MP3) $H^\lambda_{u(\cdot)}(g(\cdot\,), p(\cdot\,))$ *is constant for almost all* $t \in [0, T]$.

2.6.6 Definition. A pair of curves $(g(\cdot\,), p(\cdot\,), u(\cdot\,))$, on an interval $[0, T]$, is called an **extremal pair** if $(g(\cdot\,), p(\cdot\,))$ is an integral curve of $\vec{H}^\lambda_{u(\cdot)}$, for either $\lambda = 0$ or $\lambda = 1$, such that $(MP1)$ and $(MP2)$ of the maximum principle hold. The projection $(g(\cdot\,), p(\cdot\,))$ of an extremal pair is called an **extremal**. The extremals corresponding to $\lambda = 0$ are called **abnormal** and the extremals corresponding to $\lambda = 1$ are called **normal**. The system of differential equations with solution $(g(\cdot\,), p(\cdot\,))$ are called the **extremal equations**. The system of differential equations with solution $p(\cdot\,)$ are called the (reduced) extremal equations.

As the maximum principle gives a set of necessary conditions for a trajectory to be optimal, it provides us with no more than just a possible number of candidates of optimal controls and their corresponding optimal trajectories. The problem of existence of an optimal trajectory is an important issue to consider. Indeed, in general, there does not always exist an optimal trajectory. In this thesis, some of the types of optimal control problems considered can be used to model real world physical situations, were clearly then the existence of an optimal solution is not a problem. Throughout this thesis, we assume that there does indeed exist an optimal solution to each of the types of control problems we consider. In general, even if there exists an optimal trajectory for a particular control problem the maximum principle can produce more than one candidate for the optimal control. For each type of optimal control problem in this thesis the maximum principle produces only one candidate for an optimal control and its corresponding optimal trajectory. This, with the assumption that there does indeed exist an optimal trajectory for each LiCP, motivates the reasoning behind us referring to, the optimal Hamiltonian, the optimal control and the optimal trajectory of each LiCP.

2.7 Elliptic functions

The references used in this section are [11], [21].

2.7.1 Definition. Let k be a number in $(0,1)$. The **(Jacobi) elliptic functions** $\mathrm{sn}(\cdot\,,k)$, $\mathrm{cn}(\cdot\,,k)$ and $\mathrm{dn}(\cdot\,,k)$ are defined as the solution of the system of differential equations

$$\begin{aligned} \dot{x} &= yz \\ \dot{y} &= -zx \\ \dot{z} &= -k^2xy \end{aligned} \tag{2.14}$$

that satisfy the initial conditions

$$\mathrm{sn}(0,k) = x(0) = 0, \qquad \mathrm{cn}(0,k) = y(0) = 1, \qquad \mathrm{dn}(0,k) = z(0) = 1.$$

The number k is known as the **modulus**.

The equations 2.14 are real analytic in the variables t, x, y, z and the parameter k, so the basic existence theory of ordinary differential equations ensures that the Jacobi elliptic functions are smooth or even real analytic functions of t and k.

2.7.2 Definition. An **elliptic integral** is any function F which can be expressed as

$$F(x) = \int_a^x R(t, P(t))dt,$$

where R is a rational function of its two arguments and P is the square root of a polynomial of degree 3 or 4 with no repeated roots.

2.7.3 Definition. The elliptic integrals of the **first, second and third kind** respectively are given by

1. $\int \frac{dt}{\sqrt{(A_1t^2+B_1)(A_2t^2+b_2)}}$,
2. $\int \frac{t^2dt}{\sqrt{(A_1t^2+B_1)(A_2t^2+b_2)}}$,
3. $\int \frac{dt}{(1+Nt^2)\sqrt{(A_1t^2+B_1)(A_2t^2+b_2)}}, \quad N \neq 0.$

Jacobi elliptic functions can be used to evaluate any integral of the first kind, i.e, any integral of the form $\int \frac{dx}{\sqrt{X}}$, where X is a cubic or quartic. The definition 2.7.1 immediately gives the derivatives for the functions,

$$\begin{aligned}\frac{d}{dt}\mathrm{sn}(t,k) &= \mathrm{cn}(t,k)\mathrm{dn}(t,k)\\ \frac{d}{dt}\mathrm{cn}(t,k) &= -\mathrm{dn}(t,k)\mathrm{sn}(t,k)\\ \frac{d}{dt}\mathrm{dn}(t,k) &= -k^2\mathrm{sn}(t,k)\mathrm{cn}(t,k)\end{aligned}$$

There are nine other elliptic functions defined by taking the reciprocals and quotients of $\mathrm{sn}(\cdot\,,k)$ $\mathrm{cn}(\cdot\,,k)$ and $\mathrm{sn}(\cdot\,,k)$:

$$\mathrm{ns}(\cdot\,,k) := \frac{1}{\mathrm{sn}(\cdot\,,k)},\quad \mathrm{nc}(\cdot\,,k) := \frac{1}{\mathrm{cn}(\cdot\,,k)},\quad \mathrm{nd}(\cdot\,,k) := \frac{1}{\mathrm{dn}(\cdot\,,k)},$$

$$\mathrm{sc}(\cdot\,,k) := \frac{\mathrm{sn}(\cdot\,,k)}{\mathrm{cn}(\cdot\,,k)},\quad \mathrm{cd}(\cdot\,,k) := \frac{\mathrm{cn}(\cdot\,,k)}{\mathrm{dn}(\cdot\,,k)},\quad \mathrm{ds}(\cdot\,,k) := \frac{\mathrm{dn}(\cdot\,,k)}{\mathrm{sn}(\cdot\,,k)}$$

$$\mathrm{cs}(\cdot\,,k) := \frac{\mathrm{cn}(\cdot\,,k)}{\mathrm{sn}(\cdot\,,k)},\quad \mathrm{dc}(\cdot\,,k) := \frac{\mathrm{dn}(\cdot\,,k)}{\mathrm{cn}(\cdot\,,k)},\quad \mathrm{sd}(\cdot\,,k) := \frac{\mathrm{sn}(\cdot\,,k)}{\mathrm{dn}(\cdot\,,k)}.$$

The following formulas hold:

$$\int_0^x \frac{dt}{\sqrt{(a^2-t^2)(b^2-t^2)}} = \frac{1}{a}\mathrm{sn}^{-1}(\frac{x}{b},\frac{b}{a}), \qquad 0 \le x \le b < a \tag{2.15}$$

$$\int_x^b \frac{dt}{\sqrt{(a^2+t^2)(b^2-t^2)}} = \frac{1}{\sqrt{a^2+b^2}}\mathrm{cn}^{-1}(\frac{x}{b},\frac{b}{\sqrt{a^2+b^2}}), \qquad 0 \le x \le b \tag{2.16}$$

$$\int_x^b \frac{dt}{\sqrt{(a^2-t^2)(b^2-t^2)}} = \frac{1}{a}\mathrm{cd}^{-1}(\frac{x}{b},\frac{b}{a}), \qquad 0 \le x \le b < a \tag{2.17}$$

$$\int_0^x \frac{dt}{\sqrt{(a^2+t^2)(b^2-t^2)}} = \frac{1}{\sqrt{a^2+b^2}}\mathrm{sd}^{-1}(\frac{\sqrt{a^2+b^2}x}{ab},\frac{b}{\sqrt{a^2+b^2}}), \qquad 0 \le x \le b \tag{2.18}$$

$$\int_a^x \frac{dt}{\sqrt{(t^2-a^2)(t^2-b^2)}} = \frac{1}{a}\mathrm{dc}^{-1}(\frac{x}{a},\frac{b}{a}), \qquad b < a \le x \tag{2.19}$$

$$\int_x^\infty \frac{dt}{\sqrt{(t^2-a^2)(t^2-b^2)}} = \frac{1}{a}\mathrm{ns}^{-1}(\frac{x}{a},\frac{b}{a}), \qquad b < a \le x \tag{2.20}$$

$$\int_b^x \frac{dt}{\sqrt{(a^2-t^2)(t^2-b^2)}} = \frac{1}{a}\mathrm{nd}^{-1}(\frac{x}{b},\frac{\sqrt{a^2-b^2}}{a}), \qquad b \le x \le a \tag{2.21}$$

$$\int_x^a \frac{dt}{\sqrt{(a^2-t^2)(t^2-b^2)}} = \frac{1}{a}\mathrm{dn}^{-1}(\frac{x}{a},\frac{\sqrt{a^2-b^2}}{a}), \qquad b \le x \le a \tag{2.22}$$

$$\int_a^x \frac{dt}{\sqrt{(t^2-a^2)(t^2+b^2)}} = \frac{1}{\sqrt{a^2+b^2}}\mathrm{nc}^{-1}(\frac{x}{a},\frac{b}{\sqrt{a^2+b^2}}), \qquad a \le x \tag{2.23}$$

$$\int_x^\infty \frac{dt}{\sqrt{(t^2-a^2)(t^2+b^2)}} = \frac{1}{\sqrt{a^2+b^2}}\mathrm{ds}^{-1}(\frac{x}{\sqrt{a^2+b^2}},\frac{b}{\sqrt{a^2+b^2}}), \qquad a \le x \tag{2.24}$$

In this work we will need to solve differential equations of the form

$$\dot{x}^2 = C(a_1x^2 + b_1x + c_1)(a_2x^2 + b_2x + c_2), \tag{2.25}$$

where $C > 0$. We now give the general method of reducing an equation of this type into a form which is convenient for the use of elliptic integrals. Let

$$S_1 = a_1x^2 + 2b_1x + c_1 \text{ and } S_2 = a_2x^2 + 2b_2x + c_2.$$

Consider the quadratic expression $S_1 + \lambda S_2$; this is a perfect square whenever

$$D(\lambda) = (a_1 + \lambda a_2)(c_1 + \lambda c_2) - (b_1 + \lambda b_2)^2 = 0 \tag{2.26}$$

$$\iff (a_2c_2 - b_2^2)\lambda^2 + (a_1c_2 + a_2c_1 - 2b_1b_2)\lambda + a_1c_1 - b_1^2 = 0. \tag{2.27}$$

Let λ_1 and λ_2 be the roots of $D(\lambda) = 0$. Then

$$S_1 + \lambda_1 S_2 = (a_1 + \lambda_1 a_2)(x - \alpha)^2 \tag{2.28}$$

$$S_1 + \lambda_2 S_2 = (a_1 + \lambda_2 a_2)(x - \beta)^2, \tag{2.29}$$

where

$$(x-\alpha)^2 = x^2 + \frac{2b_1 + \lambda_1 2b_2}{a_1 + \lambda_1 a_2}x + \frac{c_1 + \lambda_1 c_2}{a_1 + \lambda_1 a_2} \tag{2.30}$$

$$(x-\beta)^2 = x^2 + \frac{2b_1 + \lambda_2 2b_2}{a_1 + \lambda_2 a_2}x + \frac{c_1 + \lambda_2 c_2}{a_1 + \lambda_2 a_2}\cdot \tag{2.31}$$

Solving 2.28 and 2.29 for S_1 and S_2 we have that they can be expressed in the forms

$$S_1 = A_1(x-\alpha)^2 + B_1(x-\beta)^2 \tag{2.32}$$

$$S_2 = A_2(x-\alpha)^2 + B_2(x-\beta)^2, \tag{2.33}$$

where

$$A_1 = \frac{\lambda_2(a_1 + \lambda_1 a_2)}{\lambda_2 - \lambda_1},\ B_1 = \frac{\lambda_1(a_1 + \lambda_2 a_2)}{\lambda_1 - \lambda_2},\ A_2 = \frac{a_1 + \lambda_1 a_2}{\lambda_1 - \lambda_2},\ B_2 = \frac{a_1 + \lambda_2 a_2}{\lambda_2 - \lambda_1}\cdot \tag{2.34}$$

We can now consider solving the differential equation in the form

$$\dot{x} = \sqrt{C(A_1(x-\alpha)^2 + B_1(x-\beta)^2)(A_2(x-\alpha)^2 + B_2(x-\beta)^2)}, \tag{2.35}$$

which gives us

$$t = \frac{1}{\sqrt{C}} \int \frac{dx}{\sqrt{(A_1(x-\alpha)^2 + B_1(x-\beta)^2)(A_2(x-\alpha)^2 + B_2(x-\beta)^2)}}\cdot$$

By letting

$$u = \frac{x-\alpha}{x-\beta}, \tag{2.36}$$

we get

$$t = \frac{1}{\sqrt{C}(\alpha - \beta)\sqrt{A_1 A_2}} \int \frac{du}{\sqrt{(u^2 + \frac{B_1}{A_1})(u_2 + \frac{B_2}{A_2})}}. \tag{2.37}$$

This form can then easily be manipulated to match the form of one of the Jacobi elliptic integrals 2.15, . . . , 2.24.

2.8 Stability

The references used are [2], [3], [12], [13], [14], [15]. Let E be a (real) vector space and $W \subset E$ open. Consider the differential equation

$$\dot{p} = X(p), \; X : W \to E. \tag{2.38}$$

We assume that $X \in C^1(W)$. Here $p \in W$ is a variable describing the state of the system, X is a function of p and $\dot{p} = \frac{dp}{dt}$. The state follows a curve $p(\cdot\,)$ in W, where $p(\cdot\,)$ is uniquely determined if its initial condition $p_0 = p(0)$ is specified.

2.8.1 Definition. An **equilibrium state** (of 2.38), is a state p_e, such that $X(p_e) = 0$. The unique trajectory starting at p_e is p_e itself.

2.8.2 Definition. An equilibrium state p_e is said to be (nonlinear) **stable** (or Lyapunov stable) if for every neighbourhood U of p_e there is a neighbourhood V of p_e such that trajectories $p(\cdot\,)$ initially in V never leave U. If U can be chosen so that, in addition, $\lim_{t\to\infty} p(t) = p_e$, then p_e is **asymptotically stable**.

2.8.3 Definition. An equilibrium state p_e that is not stable is called **unstable**. That is, there exists a neighbourhood $U \ni p_e$ such that for every neighbourhood V of p_e in U, there is at least one solution $p(\cdot\,)$ starting at $p(0) \in V$, which does not lie entirely in U.

2.8.4 Theorem. *Let $W \subset E$ be open and $X : W \to E$ continuously differentiable. Suppose $X(p_e) = 0$ and p_e is a stable equilibrium state of the equation*

$$\dot{p} = X(p).$$

Then no eigenvalue of $DX(p_e)$ has positive real part. Here $DX(p_e)$ is the matrix of first order partial derivatives of X at p_e.

2.8.5 Theorem. *Suppose p_e is an equilibrium state of the equation $\dot{p} = X(p)$. Consider the linearised system $DX(p_e)$ and for each eigenvalue λ of $DX(p_e)$, suppose that m_λ denotes the algebraic multiplicity of λ and d_λ the geometric multiplicity of λ. Then the equilibrium state p_e is unstable if and only if there is an eigenvalue λ with zero real part and $d_\lambda < m_\lambda$.*

The energy-Casimir method due to Holm,Marsden,Ratiu and Weinstein is a generalisation of the Lagrange-Dirichlet theorem to Hamilton-Poisson systems. It gives sufficient conditions for nonlinear stability of equilibrium states of Poisson spaces.

The algorithm of the energy-Casimir method is as follows:

Step 1: Find a Hamilton-Poisson system $(M, \{\cdot,\cdot\}, H)$ and write the equations of motion in the Hamiltonian form:

$$\dot{p} = \{p, H\}.$$

Let p_e be an equilibrium state of this system.

Step 2: Find a family of constants of motion for the Hamiltonian system. These constants of motion are generally Casimir functions on $\mathfrak{g}^*$.

Step 3: Find a constant of the motion K from the family in **Step 2** such that the first variation of the energy-Casimir function $H + K$ is zero at the given equilibrium state, i.e, $\delta(H + K))(p_e) = 0$.

Step 4: Check to see if the second variation $\delta^2(H + K)(p_e)$, the matrix of second partial derivatives of $H + K$ at p_e, is positive or negative definite.

If the second variation is positive or negative definite, then the equilibrium state is stable. If not, the test is inconclusive.

2.8.6 THEOREM. (ENERGY-CASIMIR METHOD [13]) *Let* $(M, \{\cdot,\cdot\}, H)$ *be a Hamilton-Poisson system with equilibrium state* p_e *satisfying the above steps* $1, 2, 3, 4$. *Then its equilibrium state* p_e *is nonlinear stable.*

The following theorem is an adaption of a result of Ratiu and Ortega which gives a result similar to that of the energy-Casimir method and allows us to investigate the stability of certain equilibrium states that cannot be studied using just the energy-Casimir method.

2.8.7 THEOREM. ([12]) *Let* $(M, \{\cdot,\cdot\}, H)$ *be a Hamilton-Poisson system,* p_e *and equilibrium state of the Hamiltonian vector field* $\overrightarrow{H}$, *and* $C_1, \ldots, C_k : M \to \mathbb{R}$ *conserved quantities, that is* $\{C_i, H\} = 0$, $i = \overline{1,k}$. *Assume that there exists constants* $\lambda_0, \lambda_1, \ldots, \lambda_k$ *such that*

$$d(\lambda_0 H + \lambda_1 C_1 + \ldots + \lambda_k C_k)(p_e) = 0 \tag{2.39}$$

and the quadratic form

$$d^2(\lambda_0 H + \lambda_1 C_1 + \ldots + \lambda_k C_k + \mu F)(p_e) \mid_{W \times W} (p_e) \tag{2.40}$$

is positive definite, where

$$W = \ker dH(p_e) \cap \ker dC_1(p_e) \cap \ldots \cap \ker dC_k(p_e).$$

Then p_e *is a stable equilibrium state.*

Chapter 3

The Euclidean Group $\mathsf{SE}(2)$

In this chapter we prove some general properties of the Lie group $\mathsf{SE}(2)$ and its Lie algebra $\mathfrak{se}(2)$.

3.1 The Lie group $\mathsf{SE}(2)$

The (special) Euclidean group $\mathsf{SE}(2)$ is the following group of matrices:

$$\mathsf{SE}(2) = \left\{ \begin{bmatrix} 1 & 0 \\ \mathbf{v} & R_\theta \end{bmatrix} \in \mathsf{GL}(3,\mathbb{R}) \mid \mathbf{v} \in \mathbb{R}^{2\times 1} \text{ and } R_\theta \in \mathsf{SO}(2) \right\}, \tag{3.1}$$

where $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. It will be useful to have different ways to represent an element of $\mathsf{SE}(2)$. The following are equivalent representations of an element in $\mathsf{SE}(2)$:

$$(\mathbf{v}, R_\theta) = \begin{bmatrix} 1 & 0 \\ \mathbf{v} & R_\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ v_1 & \cos\theta & -\sin\theta \\ v_2 & \sin\theta & \cos\theta \end{bmatrix} \in \mathsf{SE}(2).$$

The group product is given by the product of matrices, that is,

$$\begin{aligned}
&\begin{bmatrix} 1 & 0 & 0 \\ v_1 & \cos\theta & -\sin\theta \\ v_2 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ w_1 & \cos\psi & -\sin\psi \\ w_2 & \sin\psi & \cos\psi \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ v_1 + w_1\cos\theta - w_2\sin\theta & \cos\theta\cos\psi - \sin\theta\sin\psi & -\cos\theta\sin\psi - \cos\psi\sin\theta \\ v_2 + w_1\sin\theta + w_2\cos\theta & \cos\psi\sin\theta + \cos\theta\sin\psi & -\sin\theta\sin\psi + \cos\theta\cos\psi \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ v_1 + w_1\cos\theta - w_2\sin\theta & \cos(\theta+\psi) & -\sin(\theta+\psi) \\ v_2 + w_1\sin\theta + w_2\cos\theta & \sin(\theta+\psi) & \cos(\theta+\psi) \end{bmatrix} \quad \text{or}
\end{aligned}$$

$$\begin{bmatrix} 1 & 0 \\ \mathbf{v} & R_\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \mathbf{w} & R_\psi \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \mathbf{v} + R_\theta\mathbf{w} & R_{\theta+\psi} \end{bmatrix}.$$

The inverse of an element is given by the matrix inverse, that is,

$$\begin{bmatrix} 1 & 0 & 0 \\ v_1 & \cos\theta & -\sin\theta \\ v_2 & \sin\theta & \cos\theta \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -\cos\theta v_1 + \sin\theta v_2 & \cos\theta & \sin\theta \\ \sin\theta v_1 - \cos\theta v_2 & -\sin\theta & \cos\theta, \quad \text{or} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ \mathbf{v} & R_\theta \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ -R_\theta^{-1}\mathbf{v} & R_\theta^{-1} \end{bmatrix}$$

We study left-invariant optimal control problems on $\mathsf{SE}(2)$. $\mathsf{SE}(2)$ is a connected, non-compact, Lie group, which is solvable (but not completely solvable) and unimodular.

3.1.1 Proposition. $\mathsf{SE}(2)$ *is a matrix Lie group.*

Proof. Let $(A_r)_{r>0}$ be any sequence of elements in $\mathsf{SE}(2)$ where each A_r is of the form

$$A_r = \begin{bmatrix} 1 & 0 \\ \mathbf{v}_r & R_{\theta r} \end{bmatrix}, \; r > 0$$

and let $A_r \to A$ as $r \to \infty$, where

$$A = \begin{bmatrix} 1 & 0 \\ \mathbf{v} & R_\theta \end{bmatrix}$$

Clearly $\mathbb{R}^2$ is a closed subset of itself and so as $r \to \infty$, $v_r \to v \in \mathbb{R}^2$. Since $\mathsf{SO}(2)$ is a closed subgroup of $\mathsf{GL}(2,\mathbb{R})$ it follows that as $r \to \infty$, $R_{\theta_r} \to R_\theta \in \mathsf{SO}(2)$. Hence $\mathsf{SE}(2)$ is closed in $\mathsf{GL}(3,\mathbb{R})$ and is therefore a matrix Lie group. ☐

3.1.2 Proposition. $\mathsf{SE}(2)$ *is not compact.*

Proof. Let

$$\begin{bmatrix} 1 & 0 & 0 \\ v_1 & \cos\theta & -\sin\theta \\ v_2 & \sin\theta & \cos\theta \end{bmatrix} \in \mathsf{SE}(2).$$

The norm is given by,

$$\begin{aligned} \left\| \begin{bmatrix} 1 & 0 & 0 \\ v_1 & \cos\theta & -\sin\theta \\ v_2 & \sin\theta & \cos\theta \end{bmatrix} \right\|_F &= \sqrt{\operatorname{tr}\left(\begin{bmatrix} 1 & v_1 & v_2 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ v_1 & \cos\theta & -\sin\theta \\ v_2 & \sin\theta & \cos\theta \end{bmatrix} \right)} \\ &= \sqrt{\operatorname{tr}\begin{bmatrix} 1 + v_1^2 + v_2^2 & . & . \\ . & \cos^2\theta + \sin^2\theta & . \\ . & . & \sin^2\theta + \cos^2\theta \end{bmatrix}} \\ &= \sqrt{3 + v_1^2 + v_2^2} \end{aligned}$$

This shows that $\mathsf{SE}(2)$ is not bounded for all $v_1, v_2 \in \mathbb{R}$ and hence $\mathsf{SE}(2)$ is not compact. ☐

3.2 The Lie algebra $\mathfrak{se}(2)$

Recall that the tangent space at identity, $T_{\mathbf{1}}\mathsf{G}$, of a Lie group G, equipped with the Lie bracket defined by the commutator of matrices, is a Lie algebra, which we denote by $\mathfrak{g}$. We now proceed to calculate the Lie algebra $\mathfrak{se}(2)$ of the Lie group $\mathsf{SE}(2)$. Let $g(\cdot)$ be a curve in $\mathsf{SE}(2)$ such that $g(0) = \mathbf{1}$. Then

$$g(t) = \begin{bmatrix} 1 & 0 \\ \mathbf{v}(t) & R_\theta(t) \end{bmatrix},$$

and thus

$$\dot{g}(t) = \begin{bmatrix} 0 & 0 \\ \dot{v}(t) & \dot{R}_\theta(t) \end{bmatrix}.$$

Now letting $t = 0$ we get that

$$\dot{g}(0) = \begin{bmatrix} 0 & 0 \\ \dot{v}(0) & \dot{R}_\theta(0) \end{bmatrix} \text{ with } \dot{v}(0) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2.$$

Then

$$\dot{g}(0) = \begin{bmatrix} 0 & 0 & 0 \\ x_1 & 0 & -x_3 \\ x_2 & x_3 & 0 \end{bmatrix}, \ x_1, x_2, x_3 \in \mathbb{R}.$$

Hence

$$\mathfrak{se}(2) = \left\{ A = \begin{bmatrix} 0 & 0 & 0 \\ x_1 & 0 & -x_3 \\ x_2 & x_3 & 0 \end{bmatrix} \mid x_1, x_2, x_3 \in \mathbb{R} \right\}.$$

The **standard basis for $\mathfrak{se}(2)$** is given by:

$$E_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ E_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \ E_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

The Lie bracket commutators are then given by:

$$[E_1, E_2] = 0, \quad [E_2, E_3] = E_1 \quad \text{and} \quad [E_3, E_1] = E_2.$$

Let $\mathbb{R}^3_{\oslash}$ denote the Lie algebra $\mathbb{R}^3$ with Lie bracket given by

$$\begin{aligned} (x_1, x_2, x_3) \oslash (y_1, y_2, y_3) &= (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, 0) \\ &= (y_3 \mathbb{J}\mathbf{x} - x_3 \mathbb{J}\mathbf{y}, 0), \end{aligned}$$

for all $(x_1, x_2, x_3), (y_1, y_2, y_3) \in \mathbb{R}^3$, where $\mathbb{J} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

3.2.1 Proposition. *The map $\phi : \mathfrak{se}(2) \to \mathbb{R}^3_{\oslash}$ given by $\begin{bmatrix} 0 & 0 \\ \mathbf{x} & -x_3\mathbb{J} \end{bmatrix} \mapsto (\mathbf{x}, x_3)$ is a Lie algebra isomorphism.*

Proof. Since $\mathfrak{se}(2)$ is a real 3 dimensional vector space it follows immediately that ϕ is a linear isomorphism. We are left to show that the Lie bracket is preserved. Therefore we show that $\phi([A, B]) = \phi(A) \circledwedge \phi(B)$. Indeed, we have

$$\begin{aligned}
\phi\left(\left[\begin{bmatrix} 0 & 0 \\ \mathbf{x} & -x_3\mathbb{J} \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ \mathbf{y} & -y_3\mathbb{J} \end{bmatrix}\right]\right) &= \phi\left(\begin{bmatrix} 0 & 0 \\ \mathbf{x} & -x_3\mathbb{J} \end{bmatrix}\begin{bmatrix} 0 & 0 \\ \mathbf{y} & -y_3\mathbb{J} \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ \mathbf{y} & -y_3\mathbb{J} \end{bmatrix}\begin{bmatrix} 0 & 0 \\ \mathbf{x} & -x_3\mathbb{J} \end{bmatrix}\right) \\
&= \phi\left(\begin{bmatrix} 0 & 0 \\ -x_3\mathbb{J}\mathbf{y} & x_3y_3\mathbb{J}^2 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ -y_3\mathbb{J}\mathbf{x} & x_3y_3\mathbb{J} \end{bmatrix}\right) \\
&= \phi\left(\begin{bmatrix} 0 & 0 \\ y_3\mathbb{J}\mathbf{x} - x_3\mathbb{J}\mathbf{y} & 0 \end{bmatrix}\right) \\
&= (y_3\mathbb{J}\mathbf{x} - x_3\mathbb{J}\mathbf{y}, 0) \\
&= (x_1, x_2, x_3) \circledwedge (y_1, y_2, y_3) \\
&= \phi\left(\begin{bmatrix} 0 & 0 \\ \mathbf{x} & -x_3\mathbb{J} \end{bmatrix}\right) \circledwedge \phi\left(\begin{bmatrix} 0 & 0 \\ \mathbf{y} & -y_3\mathbb{J} \end{bmatrix}\right).
\end{aligned}$$

□

3.2.2 Proposition. *The group of Lie algebra automorphisms on* $\mathfrak{se}(2)$ *is*

$$\mathrm{Aut}(\mathfrak{se}(2)) = \left\{ \begin{bmatrix} x & y & v \\ \mp y & \pm x & w \\ 0 & 0 & \pm 1 \end{bmatrix} \mid x, y, v, w \in \mathbb{R},\ x^2 + y^2 \neq 0 \right\}.$$

Proof. We recall that $\mathfrak{se}(2) \cong \mathbb{R}^3_{\circledwedge}$. A linear map ϕ on $\mathbb{R}^3_{\circledwedge}$ is a Lie algebra isomorphism if it preserves the Lie bracket. That is,

$$\phi(A \circledwedge B) = \phi(A) \circledwedge \phi(B)$$

for all $A, B \in \mathbb{R}^3_{\circledwedge}$. Any linear map $\phi : \mathbb{R}^3_{\circledwedge} \to \mathbb{R}^3_{\circledwedge}$ can be represented by some 3×3 matrix with nonzero determinant. Let

$$\phi = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

represent a linear map on $\mathbb{R}^3_{\circledwedge}$. The fact that this map must preserve the Lie bracket on $\mathbb{R}^3_{\circledwedge}$ is equivalent to the following set of conditions:

$$\begin{aligned}
a_1 &= b_2c_3 & \qquad a_2 &= -b_1c_3 \\
b_1 &= -a_2c_3 & \qquad b_2 &= a_1c_3 \\
c_1 &= c_2 = 0.
\end{aligned}$$

Therefore, every Lie algebra isomorphism on $\mathbb{R}^3_{\circledwedge} \cong \mathfrak{se}(2)$ is of the following form

$$\mathrm{Aut}(\mathfrak{se}(2)) = \left\{ \begin{bmatrix} x & y & v \\ \mp y & \pm x & w \\ 0 & 0 & \pm 1 \end{bmatrix} \mid x, y, v, w \in \mathbb{R},\ x^2 + y^2 \neq 0 \right\}. \qquad \square$$

3.3 The exponential map

We check proposition 2.1.7 for $\mathfrak{g} = \mathfrak{se}(2)$. Let

$$A = \begin{bmatrix} 0 & 0 & 0 \\ x_1 & 0 & -x_3 \\ x_2 & x_3 & 0 \end{bmatrix} \in \mathfrak{se}(2) \text{ and } B = \begin{bmatrix} 0 & 0 & 0 \\ x_2 & x_3 & 0 \\ -x_1 & 0 & x_3 \end{bmatrix}.$$

We then have that

$$A^2 = -x_3 \begin{bmatrix} 0 & 0 & 0 \\ x_2 & x_3 & 0 \\ -x_1 & 0 & x_3 \end{bmatrix} = -x_3 B, \qquad A^3 = -{x_3}^2 \begin{bmatrix} 0 & 0 & 0 \\ x_1 & 0 & -x_3 \\ x_2 & x_3 & 0 \end{bmatrix} = -{x_3}^2 A,$$

$$A^4 = {x_3}^3 \begin{bmatrix} 0 & 0 & 0 \\ x_2 & x_3 & 0 \\ -x_1 & 0 & x_3 \end{bmatrix} = {x_3}^3 B, \qquad A^5 = {x_3}^4 A.$$

Therefore,

$$\begin{aligned}
\exp A &= \mathbf{1} + \frac{A}{1!} - \frac{x_3 B}{2!} - \frac{{x_3}^2 A}{3!} + \frac{{x_3}^3 B}{4!} + \frac{{x_3}^4 A}{5!} - \frac{{x_3}^5 B}{6!} + \dots \\
&= \mathbf{1} + \frac{A}{x_3}\left(\frac{x_3}{1!} - \frac{{x_3}^3}{3!} + \frac{{x_3}^5}{5!} - \dots\right) + \frac{B}{x_3}\left(-\frac{{x_3}^2}{2!} + \frac{{x_3}^4}{4!} - \frac{{x_3}^6}{6!} + \dots\right) \\
&= \mathbf{1} + \frac{\sin(x_3)}{x_3} A + \frac{(\cos(x_3) - 1)}{x_3} B \\
&= \mathbf{1} + \frac{\sin(x_3)}{x_3} A + \frac{(1 - \cos(x_3))}{{x_3}^2} A^2 \\
&= \mathbf{1} + \frac{\sin(x_3)}{x_3} \begin{bmatrix} 0 & 0 & 0 \\ x_1 & 0 & -x_3 \\ x_2 & x_3 & 0 \end{bmatrix} + \frac{(\cos(x_3) - 1)}{x_3} \begin{bmatrix} 0 & 0 & 0 \\ x_2 & x_3 & 0 \\ -x_1 & 0 & x_3 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ \frac{x_1 \sin(x_3) + x_2 \cos(x_3) - x_2}{x_3} & \cos(x_3) & -\sin(x_3) \\ \frac{x_2 \sin(x_3) - x_1 \cos(x_3) + x_1}{x_3} & \sin(x_3) & \cos(x_3) \end{bmatrix} \in \mathsf{SE}(2).
\end{aligned}$$

This expression above is valid for all $A \in \mathfrak{se}(2)$ such that $x_3 \neq 0$. In the case where $x_3 = 0$ we consider the limiting case as $x_3 \to 0$

$$\begin{aligned}
\lim_{x_3 \to 0}\left(\mathbf{1} + \frac{\sin(x_3)}{x_3} A + \frac{(1 - \cos(x_3))}{{x_3}^2} A^2\right) &= \mathbf{1} + \lim_{x_3 \to 0} \frac{\sin(x_3)}{x_3} A + \lim_{x_3 \to 0} \frac{(1 - \cos(x_3))}{{x_3}^2} A^2 \\
&= \mathbf{1} + A + \frac{1}{2} A^2 \\
&= \begin{bmatrix} 1 & 0 & 0 \\ x_1 & 1 & 0 \\ x_2 & 0 & 1 \end{bmatrix} \in \mathsf{SE}(2)
\end{aligned}$$

as for $x_3 = 0$, $A^2 = 0$. This limit corresponds with putting A (with $x_3 = 0$) straight into the exponential formula.

3.3.1 Proposition. *The exponential map* $\exp : \mathfrak{se}(2) \to \mathsf{SE}(2)$ *is surjective.*

Proof. Let

$$(v, R_\theta) = \begin{bmatrix} 1 & 0 & 0 \\ v_1 & \cos\theta & -\sin\theta \\ v_2 & \sin\theta & \cos\theta \end{bmatrix} \in \mathsf{SE}(2).$$

We shall show that there is an $A = \begin{bmatrix} 0 & 0 & 0 \\ x_1 & 0 & -\theta \\ x_2 & \theta & 0 \end{bmatrix} \in \mathfrak{se}(2)$ such that $\exp A = (v, R_\theta)$. We have

$$\exp \begin{bmatrix} 0 & 0 & 0 \\ x_1 & 0 & -\theta \\ x_2 & \theta & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{x_1 \sin\theta + x_2\cos\theta - x_2}{\theta} & \cos\theta & -\sin\theta \\ \frac{x_2\sin\theta - x_1\cos\theta + x_1}{\theta} & \sin\theta & \cos\theta \end{bmatrix},$$

where $\frac{x_1 \sin\theta + x_2\cos\theta - x_2}{\theta} = v_1$ and $\frac{x_2\sin\theta - x_1\cos\theta + x_1}{\theta} = v_2$. Now

$$\begin{aligned} x_1\sin\theta + x_2\cos\theta - x_2 &= v_1\theta \\ x_1\sin\theta + x_2(\cos\theta - 1) &= v_1\theta. \end{aligned} \tag{3.2}$$

Also,

$$\begin{aligned} x_2\sin\theta - x_1\cos\theta + x_1 &= v_2\theta \\ x_2\sin\theta - x_1(\cos\theta - 1) &= v_2\theta \\ x_2 &= \frac{v_2\theta + x_1(\cos\theta - 1)}{\sin\theta}. \end{aligned} \tag{3.3}$$

Substituting 3.3 into 3.2 gives

$$x_1\sin\theta + \left(\frac{v_2\theta + x_1(\cos\theta - 1)}{\sin\theta}\right)(\cos\theta - 1) = v_1\theta$$

$$x_1\sin^2\theta + v_2\theta(\cos\theta - 1) + x_1(\cos\theta - 1)^2 = v_1\theta\sin\theta$$

$$x_1\sin^2\theta + x_1\cos^2\theta - 2x_1\cos\theta + x_1 = v_1\theta\sin\theta - v_2\theta(\cos\theta - 1)$$

$$2x_1(1 - \cos\theta) = v_1\theta\sin\theta - v_2\theta(\cos\theta - 1)$$

$$x_1 = \frac{v_1\theta\sin\theta}{2(1-\cos\theta)} + \frac{v_2\theta}{2}.$$

Similarly, we get

$$x_2 = \frac{v_2\theta\sin\theta}{2(1-\cos\theta)} - \frac{v_1\theta}{2}.$$

This proves the result. □

3.3.2 Proposition. *The exponential map* $\exp : \mathfrak{se}(2) \to \mathsf{SE}(2)$ *is not injective.*

Proof. Consider the following two elements in $\mathfrak{se}(2)$,

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -2\pi \\ 0 & 2\pi & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -4\pi \\ 1 & 4\pi & 0 \end{bmatrix}.$$

Then

$$\exp \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -2\pi \\ 0 & 2\pi & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$\begin{aligned} \exp \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -4\pi \\ 1 & 4\pi & 0 \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 \\ \frac{1(0)+1(1)-1}{4\pi} & \cos 4\pi & -\sin 4\pi \\ \frac{1(0)-1(1)+1}{4\pi} & \sin 4\pi & \cos 4\pi \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

□

3.4 Poisson structure on $\mathfrak{se}(2)^*$

3.4.1 Proposition. *$(\mathfrak{se}(2)^*, \{\cdot\,,\cdot\}_-)$ is a Poisson space.*

Proof. We show that $\{\cdot\,,\cdot\}_-$ is a Poisson bracket on $\mathfrak{se}(2)^*$. Recall that the Lie-Poisson bracket, definition 2.3.6, is given by

$$\{F, G\}_-(p) = -p[dF(p), dG(p)]$$

for $p \in \mathfrak{se}(2)^*$ and $F, G \in C^\infty(\mathfrak{se}(2)^*)$. We show that the Lie-Poisson bracket on $\mathfrak{se}(2)^*$ is indeed a Poisson bracket, by definition 2.3.1. Since the Poisson bracket is defined through the Lie bracket on $\mathfrak{se}(2)$, $(\mathfrak{se}(2)^*, \{\cdot\,,\cdot\}_-)$ is indeed a Lie algebra. We now show that $\{\cdot\,,\cdot\}_-$ is a derivation in each factor. Let $F, G, H \in C^\infty(\mathfrak{se}(2)^*)$, then for all $p \in \mathfrak{se}(2)^*$

$$\begin{aligned} \{FG, H\}_-(p) = -p[d(FG)(p), dH(p)] &= -p[(dF\cdot G + F\cdot dG)(p), dH(p)] \\ &= -p\left([dF(p), dH(p)]G(p) + F(p)[dG(p), dH(p)]\right) \\ &= \{F, H\}_- G + F\{G, H\}_-. \end{aligned}$$

Therefore $(\mathfrak{se}(2)^*, \{\cdot\,,\cdot\}_-)$ is a Poisson space. □

We denote by H_{E_i} the Hamiltonians of each of the basis elements E_i, $i = 1, 2, 3$. Every element $X \in \mathfrak{se}(2)$ is of the form

$$X = \begin{bmatrix} 0 & 0 & 0 \\ x_1 & 0 & -x_3 \\ x_2 & x_3 & 0 \end{bmatrix}.$$

We will identify $\mathfrak{se}(2)^*$ with $\mathfrak{se}(2)$ via the pairing

$$\left\langle \begin{bmatrix} 0 & 0 & 0 \\ x_1 & 0 & -x_3 \\ x_2 & x_3 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ y_1 & 0 & -y_3 \\ y_2 & y_3 & 0 \end{bmatrix} \right\rangle = x_1y_1 + x_2y_2 + x_3y_3.$$

Then each extremal curve $p(\cdot)$ is identified with a curve $P(\cdot)$ in $\mathfrak{se}(2)$ via the formula

$$\langle P(t), X \rangle = p(t)X.$$

If

$$P(t) = \begin{bmatrix} 0 & 0 & 0 \\ P_1(t) & 0 & -P_3(t) \\ P_2(t) & P_3(t) & 0 \end{bmatrix},$$

then

$$P_1(t) = p(t)(E_1) = H_{E_1}(p(t)), \tag{3.4}$$

$$P_2(t) = p(t)(E_2) = H_{E_2}(p(t)) \tag{3.5}$$

$$P_3(t) = p(t)(E_3) = H_{E_3}(p(t)). \tag{3.6}$$

Hence,

$$P(t) = \begin{bmatrix} 0 & 0 & 0 \\ H_{E_1}(p(t)) & 0 & -H_{E_3}(p(t)) \\ H_{E_2}(p(t)) & H_{E_3}(p(t)) & 0 \end{bmatrix}$$

Using proposition 2.3.8 we get that

$$\begin{aligned} \{P_i, P_j\}_- &= \{H_{E_i}, H_{E_j}\}_- \\ &= -H_{[E_i,E_j]} \end{aligned}$$

From this it follows that we get the relations

$$\{P_1, P_2\}_- = 0,\ \{P_2, P_3\}_- = -P_1 \text{ and } \{P_1, P_3\}_- = P_2. \tag{3.7}$$

3.5 Further properties of $\mathsf{SE}(2)$

3.5.1 Proposition. *The Euclidean group* $\mathsf{SE}(2)$ *is the semi-direct product* $\mathbb{R}^2 \rtimes \mathsf{SO}(2)$.

Proof. Let $T = \left\{ (\mathbf{u}, 1) = \begin{bmatrix} 1 & 0 \\ \mathbf{u} & 1 \end{bmatrix} \mid \mathbf{u} = (u_1, u_2) \in \mathbb{R}^2 \right\} \subset \mathsf{SE}(2)$. We now show that T is a normal subgroup of $\mathsf{SE}(2)$. Indeed let $(\mathbf{v}, R_\theta) \in \mathsf{SE}(2)$ and let $(\mathbf{u}, 1) \in T$, then:

$$\begin{aligned} (\mathbf{v}, R_\theta)(\mathbf{u}, 1)(\mathbf{v}, R_\theta)^{-1} &= \begin{bmatrix} 1 & 0 \\ \mathbf{v} & R_\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \mathbf{u} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\mathbf{v}R_\theta^{-1} & R_\theta^{-1} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ R_\theta \mathbf{u} & 1 \end{bmatrix} \end{aligned}$$

which is clearly still in T for any $(\mathbf{v}, R_\theta) \in \mathsf{SE}(2)$ and $(\mathbf{u}, 1) \in T$. Thus T is a normal subgroup of $\mathsf{SE}(2)$.

Let $H = \{(0, R_\theta) \mid R_\theta \in \mathsf{SO}(2)\} \subset \mathsf{SE}(2)$. Now clearly $\mathsf{SE}(2) = TH$ as a set. We also have that,

$$\begin{aligned} T \cap H &= (\mathbf{u}, 1) \cap (0, R_\theta) && (3.8) \\ &= (0, 1) && (3.9) \\ &= \mathbf{1}. && (3.10) \end{aligned}$$

We now show that $T \cong \mathbb{R}^2$. Indeed let $\pi_1 : T \to \mathbb{R}^2$ such that $\pi_1 : (\mathbf{u}, 1) = \mathbf{u}$. Then for each $\mathbf{u} = (u_1, u_2) \in \mathbb{R}^2$ the element $(\mathbf{u}, 1) \in T$ is such that $\pi_1(\mathbf{u}, 1) = \mathbf{u}$ and so π_1 is surjective. Let $(\mathbf{u}, 1), (\mathbf{v}, 1) \in T$ such that $\pi_1(\mathbf{u}, 1) = \pi_1(\mathbf{v}, 1)$, then immediately we have that $\mathbf{u} = \mathbf{v}$ and so π_1 is injective.

Lastly, let $(\mathbf{u}, 1), (\mathbf{v}, 1) \in T$. Then

$$\begin{aligned} \pi_1((\mathbf{u}, 1)(\mathbf{v}, 1)) &= \pi_1(\mathbf{u} + \mathbf{v}, 1) \\ &= \mathbf{u} + \mathbf{v} \\ &= \pi_1(\mathbf{u}, 1) + \pi_1(\mathbf{v}, 1), \end{aligned}$$

and so π_1 is a group isomorphism. Therefore $T \cong \mathbb{R}^2$ as groups.

We now also show that $H \cong \mathsf{SO}(2)$. Indeed let $\pi_2 : H \to \mathsf{SO}(2)$ such that $\pi_2(0, R_\theta) = R_\theta$. Then for each $R_\theta \in \mathsf{SO}(2)$ the element $(0, R_\theta) \in H$ is such that $\pi_2(0, R_\theta) = R_\theta$ and so π_2 is surjective. Let $(0, R_\theta), (0, R_\psi) \in H$ such that $\pi_2(0, R_\theta) = \pi_2(0, R_\psi)$, then immediately we have that $R_\theta = R_\psi$ and so π_2 is injective.

Lastly, let $(0, R_\theta), (0, R_\psi) \in H$. Then

$$\begin{aligned} \pi_2((0, R_\theta)(0, R_\psi)) &= \pi_2(0, R_\theta R_\psi) \\ &= R_\theta R_\psi \\ &= \pi_2(0, R_\theta)\pi_2(0, R_\psi), \end{aligned}$$

and so π_2 is a group isomorphism. Therefore $H \cong \mathsf{SO}(2)$ as groups. Hence $\mathsf{SE}(2)$ is the semi direct product $\mathbb{R}^2 \rtimes \mathsf{SO}(2)$. □

3.5.2 PROPOSITION. $\mathsf{SE}(2)$ *is connected.*

PROOF. As a set, $\mathsf{SE}(2)$ is the direct product $\mathbb{R}^2 \times \mathsf{SO}(2)$, by proposition 3.5.1. Now, $\mathsf{SO}(2)$ is isomorphic to the unit circle $\mathbb{S}^1$. Since $\mathbb{S}^1$ is connected, it follows that $\mathsf{SO}(2)$ is connected. Then, since $\mathsf{SE}(2)$ is the direct product of two connected sets, it follows that $\mathsf{SE}(2)$ is connected. □

3.5.3 Proposition. $\mathsf{SE}(2)$ *is solvable.*

Proof. We have that the derived series, see 2.1.17, is as follows:

$$\begin{aligned}
\mathfrak{se}(2)^{(0)} &= \mathfrak{se}(2)\\
\mathfrak{se}(2)^{(1)} &= \operatorname{span}\left\{[x,y] \mid x,y \in \mathfrak{se}(2)^{(0)}\right\}\\
&= \operatorname{span}\{E_1, E_2\}\\
\mathfrak{se}(2)^{(2)} &= \operatorname{span}\left\{[x,y] \mid x,y \in \mathfrak{se}(2)^{(1)}\right\}\\
&= \{0\}\,.
\end{aligned}$$

Therefore the derived series vanishes for some $k \in \mathbb{N}$. Hence $\mathfrak{se}(2)$ is solvable. By definition it follows that $\mathsf{SE}(2)$ is solvable. □

3.5.4 Proposition. $\mathsf{SE}(2)$ *is not completely solvable.*

Proof. Let $X = x_1E_1 + x_2E_2 + x_3E_3$ be an arbitrary element of $\mathfrak{se}(2)$. Now ad_X is a linear map and so we calculate it acting on each of the basis elements of $\mathfrak{se}(2)$:

$$\begin{aligned}
\operatorname{ad}_X(E_1) &= [x_1E_1 + x_2E_2 + x_3E_3, E_1]\\
&= x_3E_2\\
\operatorname{ad}_X(E_2) &= [x_1E_1 + x_2E_2 + x_3E_3, E_2]\\
&= -x_3E_1\\
\operatorname{ad}_X(E_3) &= [x_1E_1 + x_2E_2 + x_3E_3, E_3]\\
&= x_2E_1 - x_1E_2.
\end{aligned}$$

Therefore

$$\operatorname{ad}_X = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Now in particular

$$\operatorname{ad}_{E_3} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and the eigenvalues of this operator are given by

$$\det(\operatorname{ad}_{E_3} - \lambda\mathbf{1}) = -\lambda^3 - \lambda = 0.$$

Therefore the eigenvalues are $0, \pm i$. Since the spectrum of ad_{E_3} is not real it follows that $\mathfrak{se}(2)$ is not a completely solvable Lie algebra and hence by definition $\mathsf{SE}(2)$ is not a completely solvable Lie group. □

3.5.5 Proposition. $\mathsf{SE}(2)$ *is unimodular.*

Proof. As calculated in the proof of proposition 3.5.4, for any $X \in \mathfrak{se}(2)$ the operator ad_X can be represented as follows:

$$\mathrm{ad}_X = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence $\mathrm{tr}(\mathrm{ad}_X) = 0$ and thus $\mathfrak{se}(2)$ is a unimodular Lie algebra. By definition it then follows that $\mathsf{SE}(2)$ is a unimodular Lie group. □

3.5.6 Proposition. $\mathsf{SE}(2)$ *is not nilpotent.*

Proof. Indeed, calculating the lower central series, see 2.1.18 of $\mathfrak{se}(2)$, we have that

$$\begin{aligned} \mathfrak{se}(2)_1 &= [\mathfrak{se}(2), \mathfrak{se}(2)] \\ &= \{[X, Y] \mid X, Y \in \mathfrak{se}(2)\} \\ &= \mathrm{span}\,(E_1, E_2). \\ \mathfrak{se}(2)_2 &= [\mathfrak{se}(2), \mathfrak{se}(2)_1] \\ &= \{[X, Z] \mid X \in \mathfrak{se}(2),\ Z \in \mathfrak{se}(2)_1\} \\ &= \mathrm{span}\,(E_1, E_2). \end{aligned}$$

Therefore since $\mathfrak{se}(2)_1 = \mathfrak{se}(2)_2$ it can easily be seen that $\mathfrak{se}(2)_k = \mathrm{span}\,(E_1, E_2)$ for any $k \in \mathbb{N}$. Therefore the lower central series never terminates and so $\mathfrak{se}(2)$ is not a nilpotent Lie algebra. By definition it follows that $\mathsf{SE}(2)$ is not nilpotent. □

3.6 Adjoint and coadjoint orbits of $\mathsf{SE}(2)$

In this section we calculate the adjoint and coadjoint orbits of the Euclidean group $\mathsf{SE}(2)$. We start by calculating the adjoint and coadjoint actions of the group $\mathsf{SE}(2)$ on the Lie algebra $\mathfrak{se}(2)$ and its dual space $\mathfrak{se}^*(2)$, respectively, and use these results to calculate the adjoint and coadjoint orbits of $\mathsf{SE}(2)$. We further show from the results obtained that there cannot exist a non-degenerate invariant bilinear form on the Lie algebra $\mathfrak{se}(2)$.

3.6.1 The adjoint orbits

Let $(\mathbf{v}, R_\theta) = \begin{bmatrix} 1 & 0 \\ \mathbf{v} & R_\theta \end{bmatrix} \in \mathsf{SE}(2)$ and let X $= x_1E_1 + x_2E_2 + x_3E_3 \in \mathfrak{se}(2)$. By proposition 3.2.1 we have that $\mathfrak{se}(2) \cong \mathbb{R}^3_{\circledwedge}$. The adjoint action of $(\mathbf{v}, R_\theta) \in \mathsf{SE}(2)$ on $X \in \mathfrak{se}(2)$ is then given by,

$$\mathrm{Ad}_{(\mathbf{v}, R_\theta)}X = \begin{bmatrix} 1 & 0 \\ \mathbf{v} & R_\theta \end{bmatrix} \begin{bmatrix} 0 & 0 \\ \mathbf{x} & -x_3\mathbb{J} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -R_\theta^{-1}\mathbf{v} & R_\theta^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ R_\theta\mathbf{x} & -x_3R_\theta\mathbb{J} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -R_\theta^{-1}\mathbf{v} & R_\theta^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ R_\theta \mathbf{x} \ + \ x_3 R_\theta \mathbb{J} R_\theta^{-1} \mathbf{v} & -x_3 R_\theta \mathbb{J} R_\theta^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ R_\theta \mathbf{x} \ + \ x_3 \mathbb{J} \mathbf{v} & -x_3 \mathbb{J} \end{bmatrix}.$$

In coordinates, $\mathrm{Ad}_{(\mathbf{v},R_\theta)}(\mathbf{x}, x_3) = (x_3\mathbb{J}\mathbf{v} \ + \ R_\theta \mathbf{x}, x_3)$. In proving this, we used the identity $R_\theta \mathbb{J} = \mathbb{J} R_\theta$.

Thus the adjoint orbits are given by

$$\mathrm{Orb}(X) \ = \ \{\mathrm{Ad}_{(\mathbf{v},R_\theta)} X \mid (\mathbf{v}, R_\theta) \in \mathsf{SE}(2)\}.$$

We now list and describe the different types of adjoint orbits:

Type 1: The orbits through $(\mathbf{0}, x_3) \in \mathfrak{se}(2)$, where $x_3 \neq 0$, are given by

$$\mathrm{Orb}((\mathbf{0}, x_3)) \ = \ \{(x_3\mathbb{J}\mathbf{v}, x_3) \mid \mathbf{v} \in \mathbb{R}^2\}, \ x_3 \neq 0,$$

which are the two dimensional planes through the points x_3 parallel to the x_1x_2-plane.

Type 2: The orbits through $(\mathbf{x}, 0) \in \mathfrak{se}(2)$, where $\mathbf{x} \neq 0$, are given by

$$\mathrm{Orb}((\mathbf{x}, 0)) \ = \ \{(R_\theta \mathbf{x}, 0) \mid \theta \in \mathbb{R}\}, \ \mathbf{x} \neq 0,$$

which are the one dimensional circles in the x_1x_2-plane centred at the origin and of radius equal to $\|\mathbf{x}\|$.

Type 3: The orbit through the point $(\mathbf{0}, 0) \in \mathfrak{se}(2)$ is given by

$$\mathrm{Orb}((\mathbf{0}, 0)) \ = \ \{(\mathbf{0}, 0)\},$$

which is just the origin itself.

The collection of these orbits clearly cover the whole space, as the orbits through all $x_3 \neq 0$ cover all the planes parallel to the x_1x_2-plane, while all the orbits of type 2, i.e., the circles in the x_1x_2-plane, cover the whole of the x_1x_2-plane minus the origin. We then have that the orbit through the origin is just the origin itself, and we are done.

3.6.2 The coadjoint orbits

Similarly, we now proceed to calculate the coadjoint orbits of $\mathsf{SE}(2)$. Using the standard basis (E_1^*, E_2^*, E_3^*) of the dual space, we have that elements of $\mathfrak{se}^*(2)$ are given by 3x3 block matrices of the form

$$\begin{bmatrix} 0 & \mu \\ 0 & \mu_3 \mathbb{J} \end{bmatrix} \in \mathfrak{se}^*(2), \quad \text{where} \quad \mu = [\mu_1 \ \mu_2].$$

As $\mathfrak{se}^*(2)$ is a 3-dimensional real vector space we identify it with $\mathbb{R}^3$ via the isomorphism

$$\varphi : \begin{bmatrix} 0 & \mu \\ 0 & \mu_3 \mathbb{J} \end{bmatrix} \in \mathfrak{se}^*(2) \mapsto (\mu, \mu_3) \in \mathbb{R}^3.$$

Firstly, we calculate

$$\mathrm{Ad}_{(\mathbf{v},R_\theta)^{-1}}X = \begin{bmatrix} 1 & 0 \\ -R_\theta^{-1}\mathbf{v} & R_\theta^{-1} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ \mathbf{x} & -x_3\mathbb{J} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \mathbf{v} & R_\theta \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ R_\theta^{-1}\mathbf{x} & -x_3R_\theta^{-1}\mathbb{J} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \mathbf{v} & R_\theta \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ R_\theta^{-1}\mathbf{x} \, - \, x_3R_\theta^{-1}\mathbb{J}\mathbf{v} & -x_3R_\theta^{-1}\mathbb{J}R_\theta \end{bmatrix}.$$

In coordinates, $\mathrm{Ad}_{(\mathbf{v},R_\theta)^{-1}}(\mathbf{x},x_3) = (R_\theta^{-1}\mathbf{x} \, - \, x_3R_\theta^{-1}\mathbb{J}\mathbf{v}, x_3)$.

We now define a pairing between $\mathfrak{se}^*(2)$ and $\mathfrak{se}(2)$ to be given by

$$\langle(\mu,\mu_3),(\mathbf{x},x_3)\rangle = \mu\cdot\mathbf{x} \, + \, \mu_3x_3$$

that is, the usual dot product in $\mathbb{R}^3$. This pairing is clearly symmetric, as $\mu\cdot\mathbf{x} \, + \, \mu_3x_3 \, = \, \mathbf{x}\cdot\mu \, + \, x_3\mu_3$, and is nondegenerate, as $\langle(\mu,\mu_3),(\mathbf{x},x_3)\rangle = 0 \quad \forall(\mathbf{x},x_3)\in\mathfrak{se}(2)$ implies $(\mu,\mu_3) = 0$. We now calculate the coadjoint action of $\mathsf{SE}(2)$ on the dual of the Lie algebra $\mathfrak{se}^*(2)$ via our nondegenerate pairing as follows,

$$\begin{aligned} \left\langle(\mathrm{Ad}_{(\mathbf{v},R_\theta)^{-1}})^*(\mu,\mu_3),(\mathbf{x},x_3)\right\rangle &= \left\langle(\mu,\mu_3),\mathrm{Ad}_{(\mathbf{v},R_\theta)^{-1}}(\mathbf{x},x_3)\right\rangle \\ &= \left\langle(\mu,\mu_3),(R_\theta^{-1}\mathbf{x} \, - \, x_3R_\theta^{-1}\mathbb{J}\mathbf{v},x_3)\right\rangle \\ &= \mu_3x_3 \, - \, x_3\mu\cdot R_{\theta^{-1}}\mathbb{J}\mathbf{v} \, + \, \mu\cdot R_{\theta^{-1}}\mathbf{x} \\ &= (\mu_3 \, - \, \mu\cdot R_{\theta^{-1}}\mathbb{J}\mathbf{v})x_3 \, + \, R_\theta\mu\cdot\mathbf{x} \\ &= \langle(R_\theta\mu,\mu_3 \, - \, R_\theta\mu\cdot\mathbb{J}\mathbf{v}),(\mathbf{x},x_3)\rangle \end{aligned}$$

and therefore we get that, $(\mathrm{Ad}_{(\mathbf{v},R_\theta)^{-1}})^*(\mu,\mu_3) \, = \, (R_\theta\mu, \ \mu_3 \, - \, R_\theta\mu\cdot\mathbb{J}\mathbf{v})$. Thus the coadjoint orbits are given by,

$$\mathrm{orb}(\mu) \ = \ \{(\mathrm{Ad}_{(\mathbf{v},R_\theta)^{-1}})^*\mu \mid (\mathbf{v},R_\theta)\in\mathsf{SE}(2)\}.$$

We now list and describe the different types of coadjoint orbits:

Type 1: The orbits through $(\mathbf{0},\mu_3)\in\mathfrak{se}^*(2)$ are given by

$$\mathrm{orb}((\mathbf{0},\mu_3)) \ = \ \{(\mathbf{0},\mu_3)\},$$

which is clearly just the point $(\mathbf{0},\mu_3)$ on the μ_3-axis.

Type 2: The orbits through $(\mu,0)\in\mathfrak{se}^*(2)$, such that $\mu\neq 0$, are given by

$$\mathrm{orb}((\mu,0)) \ = \ \{(R_\theta\mu,-R_\theta\mu\cdot\mathbb{J}\mathbf{v}) \mid \mathbf{v}\in\mathbb{R}^2, \ \theta\in\mathbb{R}\},$$

which are the cylinders of infinite height centred at the origin of circular radius $\|\mu\|$.

The collection of these orbits clearly cover the whole space. This is clear as all the orbits of type 2, i.e., the cylinders of infinite height centred at the origin, cover the whole space minus the μ_3-axis, and all the orbits of type 1 clearly cover the the whole of the μ_3-axis.

3.6.3 Adjoint and coadjoint action

We now proceed to show that there cannot exist a non-degenerate invariant bilinear form on $\mathfrak{se}(2)$. For a large class of matrix Lie groups, called semi-simple Lie groups, there exists a nondegenerate invariant bilinear form on their Lie algebras called the Killing form. This allows one to identify the coadjoint orbits with the adjoint orbits and in doing so reduce the control problem from one on the dual space $\mathfrak{g}^*$ to one on the Lie algebra $\mathfrak{g}$. Since no such invariant form can exist on $\mathfrak{se}(2)^*$ we cannot identify the adjoint and coadjoint orbits of $\mathsf{SE}(2)$. Therefore finding the solutions to the extremal equations, which evolve along the coadjoint adjoint orbits of $\mathsf{SE}(2)$, cannot be reduce to a problem on the Lie algebra $\mathfrak{se}(2)$.

3.6.1 PROPOSITION. *There does not exist a non-degenerate invariant bilinear form on* $\mathfrak{se}(2)$.

PROOF. From proposition 2.2.6, it follows that if there did exist a nondegenerate invariant bilinear form on $\mathfrak{se}(2)$ then the adjoint and coadjoint actions would be equivalent. It then follows, from the definitions for the adjoint and coadjoint orbits of $\mathsf{SE}(2)$, that the orbits would be equivalent. This is clearly not the case for $\mathsf{SE}(2)$, as all the coadjoint orbits of $\mathsf{SE}(2)$ are of even dimension, while the adjoint orbits of type 2 (i.e. the circles in the $x_1 - x_2$ plane) are of dimension one. Clearly then, there cannot exist an isomorphism between the adjoint and coadjoint orbits of $\mathsf{SE}(2)$, and therefore the adjoint and coadjoint actions are not equivalent. Hence there cannot exist a non-degenerate invariant bilinear form on $\mathfrak{se}(2)$.

Chapter 4

Equivalence of Control Systems on $\mathsf{SE}(2)$

In this chapter we prove an important result which shows that a control system on $\mathsf{SE}(2)$ is controllable if and only if it is of full rank. We attempt to classify, under state space equivalence, all controllable control affine systems on $\mathsf{SE}(2)$. We then classify, under detached feedback equivalence, all controllable control affine systems on $\mathsf{SE}(2)$.

4.1 Control systems on $\mathsf{SE}(2)$

By the rank theorem 2.4.10, a controllable system Σ must be of full rank. In general the converse does not hold. For the Lie group $\mathsf{SE}(2)$, however, it turns out that a control system $\Sigma = (\mathsf{SE}(2), \Xi)$ is controllable if and only if it is of full rank. The following result is taken from [19].

4.1.1 Theorem. *A control system* $\Sigma = (\mathsf{SE}(2), \Xi)$ *is controllable if and only if* $\mathrm{Lie}\,(\Gamma) = \mathfrak{se}(2)$.

Proof. We have that $\mathsf{SE}(2) = \mathbb{R}^2 \rtimes \mathsf{SO}(2)$. We note that $\mathsf{SO}(2)$ is a compact Lie group and the action of $\mathsf{SO}(2)$ on $\mathbb{R}^2$ has no nonzero fixed points.

Let Γ be the trace associated to the control system $\Sigma = (\mathsf{SE}(2), \Xi)$. $\mathcal{A}$ is the semigroup given by $\mathcal{A} = \{\exp tA \mid A \in \Gamma,\ t \geq 0\}$. Because $\mathcal{A}$ is equal to the reachable set from the identity, it follows that $\mathcal{A}$ has a nonempty interior in $\mathsf{SE}(2)$ if and only if $\mathrm{Lie}\,(\Gamma) = \mathfrak{se}(2)$.

($\Leftarrow$) Assume that $\mathrm{Lie}\,(\Gamma) = \mathfrak{se}(2)$. Then the projection $\mathcal{A}$ on $\mathsf{SO}(2)$ is equal to $\mathsf{SO}(2)$, since the latter is compact and connected. It follows from theorem 2.4.11, that it is sufficient to show $\mathbf{1} = (0, 1)$ is contained in the interior of $\mathcal{A}$. Let $(v, R_\theta) \in \mathrm{int}\mathcal{A}$. Because the projection of $\mathcal{A}$ on $\mathsf{SO}(2)$ contains $\mathsf{SO}(2)$, it follows that their exists a $\mathbf{u} = (u_1, u_2) \in \mathbb{R}^2$ such that $(u, R_\theta^{-1}) \in \mathrm{int}\mathcal{A}$. Then $(v, R_\theta)(u, R_\theta^{-1}) = (v + R_\theta u, 1)$, and this product is in the interior of $\mathcal{A}$. Let Ω be a neighbourhood of $1 \in \mathsf{SO}(2)$ such that $\{v + R_\theta u, \Omega\} \subset \mathrm{int}\mathcal{A}$. Denote $v + R_\theta u$ by s. For any $h \in \Omega$ and any positive integer n, $(s, h)^n = (s + hs + \ldots + h^{n-1}s, h^n)$ is contained in the interior of $\mathcal{A}$. If $h^n = 1$ and if $s = hw - w$ for some $w \in \mathbb{R}^2$, then $s + hs + \ldots + h^{n-1}s = 0$ and so $(0, 1)$ is contained in the interior of $\mathcal{A}$. For $\mathsf{SE}(2)$ it follows that for any $s \in \mathbb{R}^2$ and any neighbourhood Ω of $1 \in \mathsf{SO}(2)$ there exists an element $h \in \mathsf{SO}(2) \cap \Omega$ such that s is contained in the image of $h - 1$ and $h^n = 1$ for some positive integer n.

($\Rightarrow$) Follows immediately from the rank condition, theorem 2.4.10. $\square$

4.2 State space equivalent control systems

In this section we attempt to classify (under state space equivalence) all controllable control affine system on $\mathsf{SE}(2)$. Firstly, we prove a result which gives us an algebraic criterion for the state space equivalence of two full rank control affine systems on a 3-dimensional matrix Lie group. Secondly, we find the equivalence classes (under state space equivalence) of control systems for single-input and two-input homogeneous control affine systems. Although we do not find all the equivalence classes for two-input non-homogeneous and three-input homogeneous control affine systems, we do determine the conditions which are required for two control systems of each type to be state space equivalent.

4.2.1 A classification of state space equivalent control systems

The following proposition shows that state space equivalence is very natural. Let Σ and $\tilde{\Sigma}$ be two control systems.

4.2.1 Proposition. *Σ and $\tilde{\Sigma}$ are state space equivalent if and only if there exists a local diffeomorphism which locally (in neighbourhoods of $\mathbf{1}$ and $\tilde{\mathbf{1}}$) preserves trajectories corresponding to the same controls $u(\,\cdot\,) \in \mathcal{U}$, i.e.,*

$$\Phi(g^u(t)) = \tilde{g}^u(t)$$

for any $u(\,\cdot\,) \in \mathcal{U}$ and any t for which both sides exist. Here $g^u(\cdot\,)$ and $\tilde{g}^u(\cdot\,)$ denote the trajectories of Σ and $\tilde{\Sigma}$, respectively, corresponding to the admissible control $u(\,\cdot\,) \in \mathcal{U}$ and passing through $\mathbf{1}$ and $\tilde{\mathbf{1}}$, respectively.

Proof. ($\Leftarrow$) Let $\Phi : V \to \tilde{V}$ be a diffeomorphism, as given in the statement above, such that for each admissible control $u(\,\cdot\,) \in \mathcal{U}$

$$\Phi(g^u(t)) = \tilde{g}^u(t), \tag{4.1}$$

where $g^u(\cdot\,)$ and $\tilde{g}^u(\cdot\,)$ are trajectories as given in the statement above. For each $u(\,\cdot\,) \in \mathcal{U}$, since $g^u(\cdot\,)$ and $\tilde{g}^u(\cdot\,)$ are trajectories, it follows that

$$\dot{g}(t) = \Xi(g(t), u(t)) \quad \text{and} \quad \dot{\tilde{g}}(t) = \tilde{\Xi}(\tilde{g}(t), u(t)),$$

almost everywhere. So differentiating 4.1, we get that

$$\begin{aligned} \frac{d}{dt}\Phi(g^u(t))) &= \frac{d}{dt}(\tilde{g}^u(t)) \\ T_{g(t)}\Phi \cdot \Xi(g(t), u(t)) &= \tilde{\Xi}(\tilde{g}(t), u(t)). \end{aligned}$$

It follows that (for each $u \in \mathbb{R}^\ell$)

$$\begin{aligned} T_{\mathbf{1}}\Phi \cdot \Xi(\mathbf{1}, u) &= \tilde{\Xi}(\tilde{\mathbf{1}}, u) \\ T_{\mathbf{1}}\Phi \cdot \Xi_u &= \tilde{\Xi}_u. \end{aligned}$$

Hence the control systems Σ and $\tilde{\Sigma}$ are state space equivalent.

($\Rightarrow$) Conversely, assume that the control systems Σ and $\tilde{\Sigma}$ are state space equivalent. Then there exists a local diffeomorphism $\Phi : V \to \tilde{V}$, $\Phi(\mathbf{1}) = \tilde{\mathbf{1}}$, where V and $\tilde{V}$ are neighbourhoods of

$\mathbf{1}$ and $\tilde{\mathbf{1}}$, respectively, such that

$$T_{\mathbf{1}}\Phi \cdot \Xi_u = \tilde{\Xi}_u$$

for all $u \in \mathbb{R}^\ell$. Let $g^u(\cdot)$ be a trajectory of system Σ for some admissible control $u(\cdot)$ such that $g^u(0) = \mathbf{1}$ then it must satisfy the differential equation

$$\dot{g}(t) = \Xi(g(t), u(t))$$

almost everywhere. We now show that the curve $\tilde{g}^u = \Phi \circ g^u$ is a trajectory of system $\tilde{\Sigma}$. If this is the case then $\tilde{g}^u(\cdot)$ must satisfy the equation

$$\tilde{\Xi}(\tilde{g}(t), u(t))$$

almost everywhere with $\tilde{g}^u(0) = \tilde{\mathbf{1}}$. Indeed, at $t = 0$, we have that $\Phi(g^u(0)) = \Phi(\mathbf{1}) = \tilde{\mathbf{1}} = \tilde{g}^u(0)$, and so $\tilde{g}^u(\cdot)$ is curve going through the identity at $t = 0$. Also, differentiating $\tilde{g}^u(\cdot)$, we get that

$$\begin{aligned}\frac{d}{dt}\tilde{g}^u(t) &= \frac{d}{dt}\Phi(g^u(t))\\ &= T_{g(t)}\Phi \cdot \dot{g}^u(t)\\ &= T_{g(t)}\Phi \cdot \Xi(g(t), u(t))\\ &= \tilde{\Xi}(\tilde{g}(t), u(t))\end{aligned}$$

almost everywhere. This last step follows as Σ and $\tilde{\Sigma}$ are state space equivalent; that is, since $T_{\mathbf{1}}\Phi \cdot \Xi_u = \tilde{\Xi}_u$ (for all $u \in \mathbb{R}^\ell$). Therefore, $\tilde{g}^u(\cdot)$ is a trajectory of system $\tilde{\Sigma}$ and thus we have that trajectories are mapped to trajectories. □

4.2.2 LEMMA. *Let $\Sigma = (\mathsf{G}, \Xi)$ be a full rank control affine system, where $\dim \mathsf{G} = 3$.*
(i) If $\ell = 1$, then $A, B_1, [A, B_1]$ are linearly independent.
(ii)If $\ell = 2$ and A, B_1, B_2 are linearly dependent, then $B_1, B_2, [B_1, B_2]$ are linearly independent.

PROOF. (i) Let Σ be of full rank. Let $\ell = 1$, that is $\Gamma = A + \langle B_1 \rangle$, and assume $[A, B_1], A, B_1$ are linearly dependent. Therefore

$$[A, B_1] = \alpha A + \beta B_1,$$

for some $\alpha, \beta \in \mathbb{R}$. Now, $\mathrm{Lie}\,(\Gamma)$ is the smallest Lie algebra generated by Γ and so

$$[A, [A, B_1]] = [A, \alpha A + \beta B_1] = \beta[A, B_1] = \delta A + \gamma B_1$$

for some $\delta, \gamma \in \mathbb{R}$. Hence $[A, [A, B_1]], A, B_1$ are linearly dependent. Similarly $[B_1, [A, B_1]], A, B_1$ are also linearly dependent. This gives that $\mathrm{Lie}\,(\Gamma) \neq \mathfrak{g}$, which is a contradiction. Hence $A, B_1, [A, B_1]$ are linearly independent.

(ii) For $\ell = 2$, $\Gamma = A + \langle B_1, B_2 \rangle$. Similarly, for A, B_1, B_2 linearly dependent, it follows that $B_1, B_2, [B_1, B_2]$ are linearly independent. □

4.2.3 THEOREM. *Consider two control affine systems of full rank $\Sigma = (\mathsf{G}, \Xi)$ and $\tilde{\Sigma} = (\tilde{\mathsf{G}}, \tilde{\Xi})$, where* $\dim \mathsf{G} = \dim \tilde{\mathsf{G}} = 3$. *Then Σ and $\tilde{\Sigma}$ are state space equivalent if and only if there exists a Lie algebra isomorphism $\phi : \mathfrak{g} \to \tilde{\mathfrak{g}}$ such that*

$$\phi \cdot \Xi_u = \tilde{\Xi}_u$$

for all $u \in \mathbb{R}^\ell$*.*

Proof. The traces of Σ and $\tilde{\Sigma}$, respectively, are given by

$$\Gamma = \{A + \sum_{i=1}^{\ell} u_i B_i \mid (u_1, \ldots, u_\ell) \in \mathbb{R}^\ell\}$$

$$\tilde{\Gamma} = \{\tilde{A} + \sum_{i=1}^{\ell} u_i \tilde{B}_i \mid (u_1, \ldots, u_\ell) \in \mathbb{R}^\ell\}.$$

Two control systems Σ and $\tilde{\Sigma}$ are state space equivalent if we can find a linear isomorphism $\psi : \mathfrak{g} \to \tilde{\mathfrak{g}}$ such that

$$\psi \cdot \Xi_{[u_1 u_2 \ldots u_k]} = \tilde{\Xi}_{[u_1 u_2 \ldots u_k]}$$

for all $k \geq 1$ and all $u_1, \ldots, u_k \in \mathbb{R}^\ell$.

($\Leftarrow$) We assume that there exists a Lie algebra isomorphism $\phi : \mathfrak{g} \to \tilde{\mathfrak{g}}$ such that

$$\phi \cdot \Xi_{u_1} = \tilde{\Xi}_{u_1} \text{ for all } u_1 = (u_{1,i}) \in \mathbb{R}^\ell. \tag{4.2}$$

This implies that

$$\phi(A + \sum_{i=1}^{\ell} u_{1,i} B_i) = \tilde{A} + \sum_{i=1}^{\ell} u_{1,i} \tilde{B}_i$$

for all $u_1 \in \mathbb{R}^\ell$. Then for all $u_1, \ldots, u_k \in \mathbb{R}^\ell$ we have that,

$$\begin{aligned}
& \phi \cdot \Xi_{[u_1 u_2 \ldots u_k]} \\
= \quad & \phi([A + \sum_{i=1}^{\ell} u_{1,i} B_i, [\ldots [A + \sum_{i=1}^{\ell} u_{k-1,i} B_i, A + \sum_{i=1}^{\ell} u_{k,i} B_i] \ldots]]) \\
= \quad & [\phi(A + \sum_{i=1}^{\ell} u_{1,i} B_i), [\ldots [\phi(A + \sum_{i=1}^{\ell} u_{k-1,i} B_i), \phi(A + \sum_{i=1}^{\ell} u_{k,i} B_i]) \ldots]] \\
= \quad & [\tilde{A} + \sum_{i=1}^{\ell} u_{1,i} \tilde{B}_i, [\ldots [\tilde{A} + \sum_{i=1}^{\ell} u_{k-1,i} \tilde{B}_i, \tilde{A} + \sum_{i=1}^{\ell} u_{k,i} \tilde{B}_i] \ldots]] \\
= \quad & \tilde{\Xi}_{[u_1 u_2 \ldots u_k]}
\end{aligned}$$

Hence we have shown that if ϕ is a Lie algebra isomorphism satisfying 4.2 then ϕ is a linear isomorphism that satisfies Krener's conditions 2.10 and so the two systems Σ and $\tilde{\Sigma}$ are state space equivalent.

($\Rightarrow$) Conversely, assume that the two control systems Σ and $\tilde{\Sigma}$ are state space equivalent. Then there exists a linear isomorphism $\psi : \mathfrak{g} \to \tilde{\mathfrak{g}}$ which satisfies Krener's conditions, that is,

$$\psi \cdot \Xi_{[u_1 u_2 \ldots u_k]} = \tilde{\Xi}_{[u_1 u_2 \ldots u_k]}$$

for all $k \geq 1$ and all $u_1, \ldots, u_k \in \mathbb{R}^\ell$. In particular, for $k = 1$, we have that

$$\psi \cdot \Xi_{u_1} = \tilde{\Xi}_{u_1}, \text{ or}$$

$$\psi(A + \sum_{i=1}^{\ell} u_{1,i} B_i) = \tilde{A} + \sum_{i=1}^{\ell} u_{1,i} \tilde{B}, \quad \ell \leq 3.$$

Since this holds for all $u_1 \in \mathbb{R}^\ell$, it follows that

$$\psi \cdot A = \tilde{A} \quad \text{and} \quad \psi \cdot B_i = \tilde{B}_i, \quad i = 1, \ldots, \ell.$$

Now since $\operatorname{Lie}(\Gamma) = \mathfrak{g}$ and $\dim \mathsf{G} = 3$, we can assume that at most three of the elements $A, B_1, \ldots, B_\ell$ are linearly independent. When three of the elements $A, B_1, \ldots, B_l$ are linearly independent there are two cases that follow; either A, B_1, B_2 are linearly independent, or B_1, B_2, B_3 are linearly independent.

Case 1: We assume that the elements A, B_1, B_2 are linearly independent, that is they form a basis for $\mathfrak{g}$. We have that

$$\psi \cdot A = \tilde{A} \quad \text{and} \quad \psi \cdot B_i = \tilde{B}_i, \quad i = 1, 2.$$

From Krener's condition for $k = 2$ we have that (for all $u_1, u_2 \in \mathbb{R}^2$)

$$\psi \cdot [A + u_{1,1}B + u_{1,2}B_2, A + u_{2,1}B_1 + u_{2,2}B_2]$$
$$= [\tilde{A} + u_{1,1}\tilde{B}_1 + u_{1,2}\tilde{B}_2, \tilde{A} + u_{2,1}\tilde{B}_1 + u_{2,2}\tilde{B}_2]$$

and so

$$\psi((u_{2,1} - u_{1,1})[A, B_1] + (u_{2,2} - u_{1,2})[A, B_2] + (u_{1,1}u_{2,2} - u_{1,2}u_{2,1})[B_1, B_2])$$
$$= (u_{2,1} - u_{1,1})[\tilde{A}, \tilde{B}_1] + (u_{2,2} - u_{1,2})[\tilde{A}, \tilde{B}_2] + (u_{1,1}u_{2,2} - u_{1,2}u_{2,1})[\tilde{B}_1, \tilde{B}_2].$$

This equality holds for all $u_1, u_2 \in \mathbb{R}^2$. In particular if we take $u_{2,1} = u_{1,1} = 0$ and $u_{2,2} - u_{1,2} \neq 0$ we have that

$$\psi \cdot [A, B_2] = [\tilde{A}, \tilde{B}_2].$$

Again, if we take $u_{2,2} = u_{1,2} = 0$ and $u_{2,1} - u_{1,1} \neq 0$ we get that

$$\psi \cdot [A, B_1] = [\tilde{A}, \tilde{B}_1].$$

Using these two conditions and that fact that the equality holds for all $u_1, u_2 \in \mathbb{R}^2$ it follows that

$$\psi \cdot [B_1, B_2] = [\tilde{B}_1, \tilde{B}_2].$$

So

$$\psi \cdot [A, B_i] = [\tilde{A}, \tilde{B}_i] = [\psi \cdot A, \psi \cdot B_i], \ i = 1, 2 \text{ and } \psi \cdot [B_1, B_2] = [\tilde{B}_1, \tilde{B}_2] = [\psi \cdot B_1, \psi \cdot B_2].$$

Hence we have found that the linear isomorphism ψ satisfies the property of being a Lie algebra isomorphism on all of its basis elements and so ψ is indeed a Lie algebra isomorphism.

Case 2: We assume that the elements B_1, B_2, B_3 are linearly independent and therefore form a basis for $\mathfrak{g}$. For $k = 1$ of Krener's condition we have that

$$\psi \cdot B_i = \tilde{B}_i, \quad i = 1, 2, 3.$$

From Krener's condition for $k = 2$ we have that for all $u_1, u_2, u_3 \in \mathbb{R}^2$

$$\psi \cdot [u_{1,1}B + u_{1,2}B_2 + u_{1,3}B_3, u_{2,1}B_1 + u_{2,2}B_2 + u_{2,3}B_3]$$

$$= [u_{11}\tilde{B_1} + u_{12}\tilde{B_2} + u_{1,3}B_3, u_{21}\tilde{B_1} + u_{22}\tilde{B_2} + u_{2,3}B_3]$$

and so

$$\psi((u_{1,1}u_{2,2} - u_{1,2}u_{2,1})[B_1, B_2] + (u_{1,2}u_{2,3} - u_{1,3}u_{2,2})[B_2, B_3] + (u_{1,1}u_{2,3} - u_{1,3}u_{2,1})[B_1, B_3])$$

$$= (u_{1,1}u_{2,2} - u_{1,2}u_{2,1})[\tilde{B_1}, \tilde{B_2}] + (u_{1,2}u_{2,3} - u_{1,3}u_{2,2})[\tilde{B_2}, \tilde{B_3}] + (u_{1,1}u_{2,3} - u_{1,3}u_{2,1})[\tilde{B_1}, \tilde{B_3}]$$

This equality holds for all $u_1, u_2, u_3 \in \mathbb{R}^2$. In particular, if we take $u_{1,1} = u_{1,2} = u_{2,1} = 0$ and $u_{1,3} = u_{2,2} \neq 0$, we have that

$$\psi \cdot [B_2, B_3] = [\tilde{B_2}, \tilde{B_3}].$$

Similarly, for appropriate choices of u_1, u_2, u_3 we get that

$$\psi \cdot [B_1, B_2] = [\tilde{B_1}, \tilde{B_2}]$$

and

$$\psi \cdot [B_1, B_3] = [\tilde{B_1}, \tilde{B_3}].$$

So

$$\psi \cdot [B_i, B_j] = [\tilde{B_i}, \tilde{B_j}] = [\psi \cdot B_i, \psi \cdot B_j],\ i < j = 3$$

Hence we have found that the linear isomorphism ψ satisfies the property of being a Lie algebra isomorphism on all of its basis elements and so ψ is indeed a Lie algebra isomorphism.

For the case when only two of the elements $A, B_1, \ldots, B_\ell$ are linearly independent we again have two cases.

Case 1′: A, B_1, $[A, B_1]$ are linearly independent. Since Σ is of full rank, it follows that $A, B_1, [A, B_1]$ are linearly independent, by lemma 4.2.2. Hence these three elements form a basis for $\mathfrak{g}$. Now, from Krener's condition for $k = 1$, it follows that

$$\psi \cdot A = \tilde{A} \quad \text{and} \quad \psi \cdot B_1 = \tilde{B_1}.$$

Also from Krener's condition for $k = 2$, for all $(u_1, u_2 \in \mathbb{R})$

$$\psi \cdot [A + u_1B_1, A + u_2B_1] = [\tilde{A} + u_1\tilde{B_1}, \tilde{A} + u_2\tilde{B_1}]$$

and so

$$(u_2 - u_1)\psi \cdot [A, B_1] = (u_2 - u_1)[\tilde{A}, \tilde{B_1}].$$

Since this equation folds for all $u_1, u_2 \in \mathbb{R}$ it follows that

$$\psi \cdot [A, B_1] = [\tilde{A}, \tilde{B_1}] = [\psi \cdot A, \psi \cdot B_1].$$

It then follows from Krener's condition for k=3 that for all $u_1, u_2, u_3 \in \mathbb{R}$

$$\psi \cdot [A + u_1B_1, [A + u_2B_1, A + u_3B_1]$$

$$= [\tilde{A} + u_1\tilde{B}_1, [\tilde{A} + u_2\tilde{B}_1, \tilde{A} + u_3\tilde{B}_1]]$$

and so

$$(u_3 - u_2)\psi([A, [A, B_1]] + u_1[B_1, [A, B_1]])$$
$$= (u_3 - u_2)\psi([\tilde{A}, [\tilde{A}, \tilde{B_1}]] + u_1[\tilde{B_1}, [\tilde{A}, \tilde{B_1}]]).$$

Again since this holds for all $u_1, u_2, u_3 \in \mathbb{R}$ we have that

$$\psi \cdot [A, [A, B_1]] = [\psi \cdot A, \psi \cdot [A, B_1]], \text{ and } \psi \cdot [B_1, [A, B_1]] = [\psi \cdot B_1, \psi \cdot [A, B_1]].$$

Hence we have found that the linear isomorphism ψ satisfies the property of being a Lie algebra isomorphism on all of its basis elements and so ψ is indeed a Lie algebra isomorphism.

Case 2′: B_1, B_2, $[B_1, B_2]$ are linearly independent. Since Σ is of full rank it follows that $B_1, B_2, [B_1, B_2]$ are linearly independent, by lemma 4.2.2. Hence it follows that these three elements form a basis for $\mathfrak{g}$. Now, from Krener's condition for $k = 1$, it follows that

$$\psi \cdot B_i = \tilde{B_i}, \ i = 1, 2.$$

It also follows, from Krener's condition for $k = 2$, that (for all $u_1, u_2 \in \mathbb{R}^2$)

$$\psi \cdot [u_{1,1}B_1 + u_{1,2}B_2, u_{2,1}B_2 + u_{2,2}B_2] = [u_{1,1}\tilde{B_1} + u_{1,2}\tilde{B_2}, u_{2,1}\tilde{B_2} + u_{2,2}\tilde{B_2}]$$

and so

$$(u_{1,1}u_{2,2} - u_{1,2}u_{2,1})\psi \cdot [B_1, B_2] = (u_{1,1}u_{2,2} - u_{1,2}u_{2,1})[\tilde{B_1}, \tilde{B_2}].$$

Since the equation holds for all $u_1, u_2 \in \mathbb{R}^2$ it follows that

$$\psi \cdot [B_1, B_2] = [\tilde{B_1}, \tilde{B_2}] = [\psi \cdot B_1, \psi \cdot B_2].$$

It then follows from Krener's condition for k=3, that (for all $u_1, u_2, u_3 \in \mathbb{R}$)

$$\psi \cdot [u_{1,1}B_1 + u_{1,2}B_2, [u_{2,1}B_1 + u_{2,2}B_2, u_{3,1}B_1 + u_{3,2}B_2]$$

$$= [u_{1,1}\tilde{B_1} + u_{1,2}\tilde{B_2}, [u_{2,1}\tilde{B_1} + u_{2,2}\tilde{B_2}, u_{3,1}\tilde{B_1} + u_{3,2}\tilde{B_2}]$$

and so

$$(u_{2,1}u_{3,2} - u_{2,2}u_{3,1})\psi(u_{1,1}[B_1, [B_1, B_2]] + u_{1,2}[B_1, [B_1, B_2]])$$
$$= (u_{2,1}u_{3,2} - u_{2,2}u_{3,1})((u_{1,1}[\tilde{B_1}, [\tilde{B_1}, \tilde{B_2}]] + u_{1,2}[\tilde{B_1}, [\tilde{B_1}, \tilde{B_2}]]).$$

Again this holds for all $u_1, u_2, u_3 \in \mathbb{R}$. So for $u_{2,1}u_{3,2} - u_{2,2}u_{3,1} \neq 0$, $u_{1,1} \neq 0$ and $u_{1,2} = 0$ we have that

$$\psi \cdot [B_1, [B_1, B_2]] = [\tilde{B_1}, [\tilde{B_1}, \tilde{B_2}]].$$

Similarly (with $u_{2,1}u_{3,2} - u_{2,2}u_{3,1} \neq 0$, $u_{1,2} \neq 0$ and $u_{1,1} = 0$) we have that

$$\psi \cdot [B_2, [B_1, B_2]] = [\tilde{B_2}, [\tilde{B_1}, \tilde{B_2}]].$$

Therefore

$$\psi \cdot [B_1, [B_1, B_2]] = [\psi \cdot B_1, \psi \cdot [B_1, B_2]], \text{ and } \psi \cdot [B_2, [B_1, B_2]] = [\psi \cdot B_2, \psi \cdot [B_1, B_2]].$$

Hence we have found that the linear isomorphism ψ satisfies the property of being a Lie algebra isomorphism on all of its basis elements and so ψ is indeed a Lie algebra isomorphism. ☐

4.2.2 Single-input control affine systems

A single-input controllable control affine system on $\mathsf{SE}(2)$ cannot be homogeneous. This follows since no single-input homogeneous control system on $\mathsf{SE}(2)$ is of full rank.

4.2.4 Proposition. *Any controllable non-homogeneous single-input control affine system* $\Sigma = (\mathsf{SE}(2), \Xi)$ *is state space equivalent to one of the systems* $\Sigma^{\alpha,\beta} = (\mathsf{SE}(2), \Xi^{\alpha,\beta})$ *where*

$$\Xi^{\alpha,\beta}_u = E_1 + \alpha E_3 + u(E_2 + \beta E_3)$$

for all $u \in \mathbb{R}$. *Here* $\alpha \in \mathbb{R},\ \beta \geq 0$ *and* $\alpha^2 + \beta^2 \neq 0$.

Proof. Let $\Sigma = (\mathsf{SE}(2), \Xi)$ be a controllable single-input control affine system where

$$\Xi_u = A + uB = a_1 E_1 + a_2 E_2 + a_3 E_3 + u(b_1 E_1 + b_2 E_2 + b_3 E_3), \quad u \in \mathbb{R}.$$

Let $\Sigma^{\alpha,\beta} = (\mathsf{SE}(2), \Xi^{\alpha,\beta})$ be another controllable control affine system where

$$\Xi^{\alpha,\beta}_u = C + uD = E_1 + \alpha E_3 + u(E_2 + \beta E_3), \quad u \in \mathbb{R},$$

with α, β as in the statement above. We identify each of the elements $A, B, C, D \in \mathfrak{g}$ with their corresponding elements in $\mathbb{R}^3_{\circledwedge}$,

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ \alpha \end{bmatrix}, \text{ and } \begin{bmatrix} 0 \\ 1 \\ \beta \end{bmatrix},$$

respectively. We know that any Lie algebra isomorphism taking $\Xi^{\alpha,\beta}_u$ to Ξ_u (for all $u \in \mathbb{R}$) is of the form

$$\phi = \begin{bmatrix} x & y & v \\ \mp y & \pm x & w \\ 0 & 0 & \pm 1 \end{bmatrix}, \quad x, y, v, w \in \mathbb{R},\ x^2 + y^2 \neq 0.$$

We now apply this mapping to $\Xi^{\alpha,\beta}_u$ and show that for any constants $a_1, a_2, a_3, b_1, b_2, b_3$, such that $\mathrm{Lie}\,(\Gamma) = \mathfrak{se}(2)$, we can find a Lie algebra automorphism taking $\Xi^{\alpha,\beta}_u$ to Ξ_u (for all $u \in \mathbb{R}$) and for some appropriate α, β. Therefore

$$\begin{bmatrix} x & y & v \\ \mp y & \pm x & w \\ 0 & 0 & \pm 1 \end{bmatrix} \left(\begin{bmatrix} 1 \\ 0 \\ \alpha \end{bmatrix} + u \begin{bmatrix} 0 \\ 1 \\ \beta \end{bmatrix} \right) = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + u \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

Since this holds for all $u \in \mathbb{R}$ we get the following equations:

$$\begin{aligned} x + \alpha v &= a_1 && (4.3) \\ \mp y + \alpha w &= a_2 && (4.4) \\ \pm \alpha &= a_3 && (4.5) \\ y + \beta v &= b_1 && (4.6) \\ \pm x + \beta w &= b_2 && (4.7) \\ \pm \beta &= b_3. && (4.8) \end{aligned}$$

We see that $|a_3| = |\alpha|$ and $|b_3| = \beta$. Now, from the fact that ϕ must preserve the Lie bracket, we get

$$\begin{bmatrix} x & y & v \\ \mp y & \pm x & w \\ 0 & 0 & \pm 1 \end{bmatrix} \begin{bmatrix} -\alpha \\ -\beta \\ 0 \end{bmatrix} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ 0 \end{bmatrix}$$

and so we get the equations:

$$\begin{aligned} -\alpha x - \beta y &= (a_2 b_3 - a_3 b_2) && (4.9) \\ \pm \alpha y \mp \beta x &= (a_3 b_1 - a_1 b_3). && (4.10) \end{aligned}$$

Since Σ is controllable, and hence of full rank, we have that $A, B, [A, B]$ are linearly independent, by lemma 4.2.2. Using 4.9 and 4.10 we get that:

$$x = \frac{-\alpha(a_2 b_3 - a_3 b_2) - \beta(a_3 b_1 - a_1 b_3)}{\alpha^2 + \beta^2},$$

$$y = \frac{\alpha(a_3 b_1 - a_1 b_3) - \beta(a_2 b_3 - a_3 b_2)}{\alpha^2 + \beta^2}.$$

or

$$x = \frac{-\alpha(a_2 b_3 - a_3 b_2) + \beta(a_3 b_1 - a_1 b_3)}{\alpha^2 + \beta^2},$$

$$y = \frac{-\alpha(a_3 b_1 - a_1 b_3) - \beta(a_2 b_3 - a_3 b_2)}{\alpha^2 + \beta^2}.$$

Since it cannot be that $\alpha = \beta = 0$, as otherwise $\mathrm{Lie}\,(\Gamma^{\alpha,\beta}) \neq \mathfrak{se}(2)$, one has that these solutions are defined for all possible choices of constants $a_1, \ldots, b_3$. Again, since $\alpha^2 + \beta^2 \neq 0$ we can put the values of x, y into equations 4.3 and 4.4 (and/or 4.6 and 4.7) to solve for the variables v, w. There are now three cases that follow, for all allowed a_1, a_2, a_3, b_1, b_2 and all $u \in \mathbb{R}$:

Case 1: If $b_3 > 0$, then the Lie algebra automorphism taking $\Xi_u^{\alpha,\beta}$ to Ξ_u (for all $u \in \mathbb{R}$) is given by

$$\begin{bmatrix} x & y & v \\ -y & x & w \\ 0 & 0 & 1 \end{bmatrix}$$

where

$$x = \frac{-a_3(a_2 b_3 - a_3 b_2) - b_3(a_3 b_1 - a_1 b_3)}{a_3^2 + b_3^2},$$

$$y = \frac{a_3(a_3b_1 - a_1b_3) - b_3(a_2b_3 - a_3b_2)}{a_3^2 + b_3^2}$$

$$v = \frac{1}{\beta}(b_1 - y),\ w = \frac{1}{\beta}(b_2 - x),$$

and $\beta = b_3$ and $\alpha = a_3$.

Case 2: If $b_3 < 0$, then the Lie algebra automorphism taking $\Xi_u^{\alpha,\beta}$ to Ξ_u (for all $u \in \mathbb{R}$) is given by

$$\begin{bmatrix} x & y & v \\ +y & -x & w \\ 0 & 0 & -1 \end{bmatrix}$$

$$x = \frac{a_3(a_2b_3 - a_3b_2) - b_3(a_3b_1 - a_1b_3)}{a_3^2 + b_3^2},$$

$$y = \frac{a_3(a_3b_1 - a_1b_3) + b_3(a_2b_3 - a_3b_2)}{a_3^2 + b_3^2}$$

$$v = \frac{1}{\beta}(b_1 - y),\ w = \frac{1}{\beta}(b_2 + x),$$

and $\beta = -b_3$ and $\alpha = -a_3$.

Case 3: If $b_3 = 0$, then $a_3 \neq 0$ and the Lie algebra automorphism taking $\Xi_u^{\alpha,\beta}$ to Ξ_u for all $u \in \mathbb{R}$ is given by

$$\begin{bmatrix} x & y & v \\ -y & x & w \\ 0 & 0 & 1 \end{bmatrix},$$

where

$$x = \frac{-a_3(a_2b_3 - a_3b_2) - b_3(a_3b_1 - a_1b_3)}{a_3^2 + b_3^2},$$

$$y = \frac{a_3(a_3b_1 - a_1b_3) - b_3(a_2b_3 - a_3b_2)}{a_3^2 + b_3^2}$$

$$v = \frac{1}{\alpha}(a_1 - x),\ w = \frac{1}{\alpha}(a_2 + y),$$

and $\beta = 0$ and $\alpha = a_3$.

Therefore, we have shown that any controllable single-input non-homogeneous control system Σ, with trace $\Xi_u = A + uB$ for all $u \in \mathbb{R}$, is state space equivalent to one of the control systems $\Sigma^{\alpha,\beta}$, for some $\alpha \in \mathbb{R}$, $\beta \geq 0$ and $\alpha^2 + \beta^2 \neq 0$. □

4.2.5 Corollary. *Any controllable single-input non-homogeneous control affine system Σ is state space equivalent to exactly one of the systems $\Sigma^{\alpha,\beta}$, where either α, β are both nonzero or $\beta = 0$ and $\alpha > 0$.*

Proof. Let $\Sigma^{\alpha_1,\beta_1}$ and $\Sigma^{\alpha_2,\beta_2}$ be two controllable single-input control affine systems.

(i) Assume $\alpha_1, \beta_1, \alpha_2, \beta_2$ are all nonzero. Now for these systems to be state space equivalent we must be able to find a Lie algebra isomorphism which takes $\Gamma^{\alpha_1,\beta_1}$ to $\Gamma^{\alpha_2,\beta_2}$.

From the fact that any Lie algebra isomorphism on $\mathsf{SE}(2)$ is of the form

$$\begin{bmatrix} x & y & v \\ \mp y & \pm x & w \\ 0 & 0 & \pm 1 \end{bmatrix}$$

and $\beta_1, \beta_2 > 0$ we get that $\beta_1 = \beta_2$. This means that the Lie algebra isomorphism must specifically be of the form

$$\begin{bmatrix} x & y & v \\ -y & x & w \\ 0 & 0 & 1 \end{bmatrix}.$$

Hence it must be that $\alpha_1 = \alpha_2$. Hence any different pair of α, β, where both α, β are nonzero, define an equivalence class of control systems on $\mathsf{SE}(2)$.

(ii) Assume $\beta_1 = 0$, so β_2 must be zero. If $\alpha_1 = -\alpha_2$ then the Lie algebra isomorphism given by

$$\begin{bmatrix} x & y & v \\ y & -x & w \\ 0 & 0 & -1 \end{bmatrix},$$

where x, y, v, w can be calculated by the steps previous theorem, takes $\Gamma^{\alpha_1,\beta_1}$ to $\Gamma^{\alpha_2,\beta_2}$. Hence these systems are state space equivalent. It is also clear from these steps that a control system $\Sigma^{\alpha_1\beta_1}$, where both α_1, β_1 are nonzero, cannot be equivalent to a control system $\Sigma^{\alpha_2,0}$ for any $\alpha_2 > 0$. Therefore the representatives of all equivalence classes of state space equivalent control systems on $\mathsf{SE}(2)$ are exactly the systems $\Sigma^{\alpha,\beta}$ for all α, β nonzero and the systems $\Sigma^{\alpha,0}$ for all $\alpha > 0$. □

4.2.3 Two-input homogeneous control affine systems

4.2.6 Proposition. *Any controllable two-input homogeneous control affine system* $\Sigma = (\mathsf{SE}(2), \Xi)$ *is state space equivalent to one of the systems* $\Sigma^{\alpha,\beta} = (\mathsf{SE}(2), \Xi^{\alpha,\beta})$, *where*

$$\begin{aligned} \Xi^{\alpha,\beta}_u &= D + u_1 E + u_2 F \\ &= k_1 E_1 + k_2 E_2 + (k_1\alpha + k_2\beta)E_3 + u_1(E_1 + \alpha E_3) + u_2(E_2 + \beta E_3). \end{aligned}$$

Here $\alpha \in \mathbb{R}$, $\beta \geq 0$, $\alpha^2 + \beta^2 \neq 0$ *and* $k_1, k_2 \in \mathbb{R}$ *are constants which satisfy the equation* $D = k_1 E + k_2 F$.

Proof. Let $\Sigma = (\mathsf{SE}(2), \Xi)$ be a controllable two-input homogeneous control affine system where

$$\begin{aligned} \Xi_u &= A + u_1 B + u_2 C \\ &= a_1 E_1 + a_2 E_2 + a_3 E_3 + u_1(b_{1,1}E_1 + b_{1,2}E_2 + b_{1,3}E_3) + u_2(b_{2,1}E_1 + b_{2,2}E_2 + b_{2,3}E_3). \end{aligned}$$

Let $\Sigma^{\alpha,\beta} = (\mathsf{SE}(2), \widetilde{\Xi})$ be another controllable control system where

$$\begin{aligned} \Xi^{\alpha,\beta}_u &= D + u_1 E + u_2 F \\ &= k_1 E_1 + k_2 E_2 + (k_1\alpha + k_2\beta)E_3 + u_1(E_1 + \alpha E_3) + u_2(E_2 + \beta E_3), \end{aligned}$$

for α, β, k_1, k_2 satisfying the conditions given in the statement of the theorem. We identify each of these elements $A, B, C, D, E, F \in \mathfrak{se}(2)$ with their corresponding elements in $\mathbb{R}^3_{\oslash}$,

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \quad \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}, \quad \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}, \quad \begin{bmatrix} k_1 \\ k_2 \\ k_1\alpha + k_2\beta \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ \alpha \end{bmatrix}, \text{ and } \begin{bmatrix} 0 \\ 1 \\ \beta \end{bmatrix},$$

respectively. We know that any Lie algebra isomorphism taking $\Xi_u^{\alpha,\beta}$ to Ξ_u (for all $u \in \mathbb{R}$) is of the form

$$\phi = \begin{bmatrix} x & y & v \\ \mp y & \pm x & w \\ 0 & 0 & \pm 1 \end{bmatrix}, \quad x, y, v, w \in \mathbb{R}, \text{ and } x^2 + y^2 \neq 0.$$

We now show that we can find a Lie algebra automorphism taking $\Xi_u^{\alpha,\beta}$ to Ξ_u. Therefore (for all $u_1, u_2 \in \mathbb{R}$)

$$\begin{bmatrix} x & y & v \\ \mp y & \pm x & w \\ 0 & 0 & \pm 1 \end{bmatrix} \left(\begin{bmatrix} k_1 \\ k_2 \\ k_1\alpha + k_2\beta \end{bmatrix} + u_1 \begin{bmatrix} 1 \\ 0 \\ \alpha \end{bmatrix} + u_2 \begin{bmatrix} 0 \\ 1 \\ \beta \end{bmatrix} \right) = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + u_1 \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} + u_2 \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}.$$

Since this holds for all $u_1, u_2 \in \mathbb{R}$ we get the following equations:

$$\begin{aligned} k_1 x + k_2 y + v(k_1\alpha + k_2\beta) &= a_1 && (4.11) \\ \mp k_1 y \pm k_2 x + w(k_1\alpha + k_2\beta) &= a_2 && (4.12) \\ \pm(k_1\alpha + k_2\beta) &= a_3 && (4.13) \\ x + \alpha v &= b_1 && (4.14) \\ \mp y + \alpha w &= b_2 && (4.15) \\ \pm\alpha &= b_3 && (4.16) \\ y + \beta v &= c_1 && (4.17) \\ \pm x + \beta w &= c_2 && (4.18) \\ \pm\beta &= c_3. && (4.19) \end{aligned}$$

From here it can already be seen that $|b_3| = |\alpha|$ and $|c_3| = \beta$ for these two control systems to be state space equivalent. Now, from the fact that ϕ must preserve the Lie bracket, we get that

$$\begin{bmatrix} x & y & v \\ \mp y & \pm x & w \\ 0 & 0 & \pm 1 \end{bmatrix} \begin{bmatrix} -\alpha \\ -\beta \\ 0 \end{bmatrix} = \begin{bmatrix} b_2c_3 - b_3c_2 \\ b_3c_1 - b_1c_3 \\ 0 \end{bmatrix}$$

and so we get the equations:

$$\begin{aligned} -\alpha x - \beta y &= b_2c_3 - b_3c_2 && (4.20) \\ \pm\alpha y \mp \beta x &= b_3c_1 - b_1c_3. && (4.21) \end{aligned}$$

Since Σ is controllable, and hence of full rank, we have that $B, C, [B, C]$ are linearly independent, by lemma 4.2.2. Using 4.20 and 4.21 and the fact that $\alpha^2+\beta^2 \neq 0$, we get that:

$$x = \frac{-\alpha(b_2c_3 - b_3c_2) - \beta(b_3c_1 - b_1c_3)}{\alpha^2+\beta^2},$$

$$y = \frac{\alpha(b_3c_1 - b_1c_3) - \beta(b_2c_3 - b_3c_2)}{\alpha^2+\beta^2}.$$

or

$$x = \frac{-\alpha(b_2c_3 - b_3c_2) + \beta(b_3c_1 - b_1c_3)}{\alpha^2+\beta^2},$$

$$y = \frac{-\alpha(b_3c_1 - b_1c_3) - \beta(b_2c_3 - b_3c_2)}{\alpha^2+\beta^2}.$$

Again, since $\alpha^2+\beta^2 \neq 0$, we can put the values of x, y into equations 4.14 and 4.15 (and/or 4.17 and 4.18) to solve for the variables v, w. There are now two cases that follow, for all allowed $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2$:

Case 1: If $c_3 > 0$, then the Lie algebra automorphism taking $\Xi_u^{\alpha,\beta}$ to Ξ_u (for all $u \in \mathbb{R}^2$) is given by

$$\begin{bmatrix} x & y & v \\ -y & x & w \\ 0 & 0 & 1 \end{bmatrix}$$

where

$$x = \frac{-\alpha(b_2c_3 - b_3c_2) - \beta(b_3c_1 - b_1c_3)}{\alpha^2+\beta^2},$$

$$y = \frac{\alpha(b_3c_1 - b_1c_3) - \beta(b_2c_3 - b_3c_2)}{\alpha^2+\beta^2}.$$

$$v = \frac{1}{\beta}(c_1 - y),\ w = \frac{1}{\beta}(c_2 - x),$$

$\alpha = b_3$ and $\beta = c_3$.

Case 2: If $c_3 < 0$, then the Lie algebra automorphism taking $\Xi_u^{\alpha,\beta}$ to Ξ_u (for all $u \in \mathbb{R}^2$) is given by

$$\begin{bmatrix} x & y & v \\ +y & -x & w \\ 0 & 0 & -1 \end{bmatrix},$$

where

$$x = \frac{-\alpha(b_2c_3 - b_3c_2) + \beta(b_3c_1 - b_1c_3)}{\alpha^2+\beta^2},$$

$$y = \frac{-\alpha(b_3c_1 - b_1c_3) - \beta(b_2c_3 - b_3c_2)}{\alpha^2+\beta^2},$$

$$v = \frac{1}{\beta}(c_1 - y),\ w = \frac{1}{\beta}(c_2 + x),$$

$\alpha = -b_3$ and $\beta = -c_3$.

Case 3: If $c_3 = 0$, then it must be that $b_3 \neq 0$, and so the Lie algebra automorphism taking

$\Xi_u^{\alpha,\beta}$ to Ξ_u (for all $u \in \mathbb{R}^2$) is given by

$$\begin{bmatrix} x & y & v \\ -y & x & w \\ 0 & 0 & 1 \end{bmatrix},$$

where

$$x = \frac{-\alpha(b_2c_3 - b_3c_2) - \beta(b_3c_1 - b_1c_3)}{\alpha^2+\beta^2},$$

$$y = \frac{\alpha(b_3c_1 - b_1c_3) - \beta(b_2c_3 - b_3c_2)}{\alpha^2+\beta^2}.$$

$$v = \frac{1}{\alpha}(b_1 - x),\; w = \frac{1}{\alpha}(b_2 + y),$$

$\alpha = a_3$ and $\beta = 0$.

Therefore, we have shown that any two-input controllable control system Σ is state space equivalent to one of the control systems $\Sigma^{\alpha,\beta}$, for some $\alpha, \beta \in \mathbb{R},\ \beta \geq 0$ and $\alpha^2 + \beta^2 \neq 0$. □

4.2.7 Corollary. *Any controllable two-input homogeneous control affine system Σ is state space equivalent to exactly one of the systems $\Sigma^{\alpha,\beta}$ where either $\alpha \in \mathbb{R}$ and $\beta > 0$ or $\beta = 0$ and $\alpha > 0$.*

Proof. Let $\Sigma^{\alpha_1,\beta_1}$ and $\Sigma^{\alpha_2,\beta_2}$ be two controllable two-input control affine systems.

(i) Assume β_1 is nonzero. Now for these systems to be state space equivalent we must be able to find a Lie algebra isomorphism which takes $\Gamma^{\alpha_1,\beta_1}$ to $\Gamma^{\alpha_2,\beta_2}$. From the fact that any Lie algebra isomorphism on $\mathsf{SE}(2)$ is of the form

$$\begin{bmatrix} x & y & v \\ \mp y & \pm x & w \\ 0 & 0 & \pm 1 \end{bmatrix}$$

and $\beta_1, \beta_2 > 0$ we get that

$$\beta_1 = \beta_2.$$

This means that the Lie algebra isomorphism must specifically be of the form

$$\begin{bmatrix} x & y & v \\ -y & x & w \\ 0 & 0 & 1 \end{bmatrix}.$$

Hence it must be that $\alpha_1 = \alpha_2$. Hence any different pair of α, β, where $\beta > 0$, define an equivalence class of control systems on $\mathsf{SE}(2)$.

(ii) Assume $\beta_1 = 0$, so β_2 must be zero. If $\alpha_1 = -\alpha_2$ then the Lie algebra isomorphism given by

$$\begin{bmatrix} x & y & v \\ y & -x & w \\ 0 & 0 & -1 \end{bmatrix},$$

where x, y, v, w can be calculated as in the previous theorem, takes $\Gamma^{\alpha_1,\beta_1}$ to $\Gamma^{\alpha_2,\beta_2}$. Hence these systems are state space equivalent. It is also clear from these steps that a control system $\Sigma^{\alpha_1\beta_1}$ where $\beta_1 > 0$ cannot be equivalent to a control system $\Sigma^{\alpha_2,0}$.

Therefore the representatives of all equivalence classes of state space equivalent control systems on $\mathsf{SE}(2)$ are exactly the systems $\Sigma^{\alpha,\beta}$ for all α and $\beta > 0$ and the systems $\Sigma^{\alpha,0}$ for all $\alpha > 0$. $\square$

4.2.4 Two-input non-homogeneous control affine systems

4.2.8 Proposition. *A controllable two-input non-homogeneous control affine system* $\Sigma = (\mathsf{SE}(2)\Xi)$, *where*

$$\Xi_u = A + u_1 B + u_2 C, \quad u = (u_1, u_2) \in \mathbb{R}^2$$

is state space equivalent to another controllable two-input non-homogeneous control affine system $\tilde{\Sigma} = (\mathsf{SE}(2), \widetilde{\Xi})$, *where*

$$\widetilde{\Xi}_u = D + u_1 E + u_2 F, \quad u = (u_1, u_2) \in \mathbb{R}^2$$

if and only if one the following conditions hold for $\kappa = 1$ *or* $\kappa = -1$*:*

(i) $\kappa a_3 = d_3$,

(ii) $\kappa b_3 = e_3$,

(iii) $\kappa c_3 = f_3$

(iv) $\kappa(b_2c_3 - b_3c_2)d_1 + (a_3c_2 - a_2c_3)e_1 + (a_2b_3 - a_3b_2)f_1$
$= (b_3c_1 - b_1c_3)d_2 + (a_1c_3 - a_3c_1)e_2 + (a_3b_1 - a_1b_3)f_2.$

(v) $\kappa(b_3c_1 - b_1c_3)d_1 + (a_1c_3 - a_3c_1)e_1 + (a_3b_1 - a_1b_3)f_1$
$= (b_3c_2 - b_2c_3)d_2 + (a_2c_3 - a_3c_2)e_2 + (a_3b_2 - a_2b_3)f_2.$

Proof. Let $\Sigma = (\mathsf{SE}(2), \Xi)$ be a controllable non-homogeneous two-input control affine system, where

$$\begin{aligned}\Xi_u &= A + u_1B + u_2C \\ &= a_1E_1 + a_2E_2 + a_3E_3 + u_1(b_1E_1 + b_2E_2 + b_3E_3) + u_2(c_1E_1 + c_2E_2 + c_3E_3),\end{aligned}$$

and let $\tilde{\Sigma} = (\mathsf{SE}(2), \widetilde{\Xi})$ be another controllable two-input control affine system,

$$\begin{aligned}\widetilde{\Xi}_u &= D + u_1E + u_2F \\ &= d_1E_1 + d_2E_2 + d_3E_3 + u_1(e_1E_1 + e_2E_2 + e_3E_3) + u_2(f_1E_1 + f_2E_2 + f_3E_3).\end{aligned}$$

We identify each of these elements $A, B, C, D, E, F \in \mathfrak{se}(2)$ with their corresponding elements in $\mathbb{R}^3_{\otimes}$,

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}, \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}, \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} \text{ and } \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix},$$

respectively. We know that any Lie algebra isomorphism taking Ξ_u to $\widetilde{\Xi}_u$ (for all $u \in \mathbb{R}^2$) is of the form

$$\phi = \begin{bmatrix} x & y & v \\ \mp y & \pm x & w \\ 0 & 0 & \pm 1 \end{bmatrix}, \quad x, y, v, w \in \mathbb{R}, \text{ and } x^2 + y^2 \neq 0.$$

We split the two possible types of Lie algebra isomorphisms into two cases, that is a case for 1 and a case for -1.

Case 1: We have

$$\begin{bmatrix} x & y & v \\ -y & x & w \\ 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + u_1 \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} + u_2 \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \right) = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} + u_1 \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} + u_2 \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}.$$

Since this holds for all $u_1, u_2 \in \mathbb{R}$ we get the following equations:

$$a_1 x + a_2 y + a_3 v = d_1 \tag{4.22}$$
$$-a_1 y + a_2 x + a_3 w = d_2 \tag{4.23}$$
$$a_3 = d_3 \tag{4.24}$$
$$b_1 x + b_2 y + b_3 v = e_1 \tag{4.25}$$
$$-b_1 y + b_2 x + b_3 w = e_2 \tag{4.26}$$
$$b_3 = e_3 \tag{4.27}$$
$$c_1 x + c_2 y + c_3 v = f_1 \tag{4.28}$$
$$-c_1 y + c_2 x + c_3 w = f_2 \tag{4.29}$$
$$c_3 = f_3. \tag{4.30}$$

Combining equations 4.22, 4.25, 4.28 and also equations 4.23, 4.26, 4.29 we get the following systems:

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ v \end{bmatrix} = \begin{bmatrix} d_1 \\ e_1 \\ f_1 \end{bmatrix}$$

and

$$\begin{bmatrix} a_2 & -a_1 & a_3 \\ b_2 & -b_1 & b_3 \\ c_2 & -c_1 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} = \begin{bmatrix} d_2 \\ e_3 \\ f_2 \end{bmatrix},$$

where we denote these two 3×3 matrices by M_1 and M_2, respectively. Since A, B, C are linearly independent we have that $\det M_1 \neq 0$, hence M_1 is invertible. Also since $\det M_1 = \det M_2$ it follows that M_2 is invertible. We get,

$$\begin{bmatrix} x \\ y \\ v \end{bmatrix} = \frac{1}{\det M_1} \begin{bmatrix} (b_2c_3 - b_3c_2)d_1 + (a_3c_2 - a_2c_3)e_1 + (a_2b_3 - a_3b_2)f_1 \\ (b_3c_1 - b_1c_3)d_1 + (a_1c_3 - a_3c_1)e_1 + (a_3b_1 - a_1b_3)f_1 \\ (b_1c_2 - b_2c_1)d_1 + (a_2c_1 - a_1c_2)e_1 + (a_1b_2 - a_2b_1)f_1 \end{bmatrix}$$

and

$$\begin{bmatrix} x \\ y \\ w \end{bmatrix} = \frac{1}{\det M_1} \begin{bmatrix} (b_3c_1 - b_1c_3)d_2 + (a_1c_3 - a_3c_1)e_2 + (a_3b_1 - a_1b_3)f_2 \\ (b_3c_2 - b_2c_3)d_2 + (a_2c_3 - a_3c_2)e_2 + (a_3b_2 - a_2b_3)f_2 \\ (b_1c_2 - b_2c_1)d_2 + (a_2c_1 - a_1c_2)e_2 + (a_1b_2 - a_2b_1)f_2 \end{bmatrix}.$$

Since $\det M_1 \neq 0$, it follows that $v, w \in \mathbb{R}$ for any A, B, C, D, E, F. The two conditions that need to be satisfied then are:

$$\begin{aligned} & (b_2c_3 - b_3c_2)d_1 + (a_3c_2 - a_2c_3)e_1 + (a_2b_3 - a_3b_2)f_1 \\ = \quad & (b_3c_1 - b_1c_3)d_2 + (a_1c_3 - a_3c_1)e_2 + (a_3b_1 - a_1b_3)f_2. \end{aligned}$$

and

$$\begin{aligned} & (b_3c_1 - b_1c_3)d_1 + (a_1c_3 - a_3c_1)e_1 + (a_3b_1 - a_1b_3)f_1 \\ = \quad & (b_3c_2 - b_2c_3)d_2 + (a_2c_3 - a_3c_2)e_2 + (a_3b_2 - a_2b_3)f_2. \end{aligned}$$

Case 2: We have

$$\begin{bmatrix} x & y & v \\ y & -x & w \\ 0 & 0 & -1 \end{bmatrix} \left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + u_1 \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} + u_2 \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \right) = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} + u_1 \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} + u_2 \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}.$$

Since this holds for all $u_1, u_2 \in \mathbb{R}$ we get the following equations:

$$a_1x + a_2y + a_3v = d_1 \tag{4.31}$$

$$a_1y - a_2x + a_3w = d_2 \tag{4.32}$$

$$-a_3 = d_3 \tag{4.33}$$

$$b_1x + b_2y + b_3v = e_1 \tag{4.34}$$

$$b_1y - b_2x + b_3w = e_2 \tag{4.35}$$

$$-b_3 = e_3 \tag{4.36}$$

$$c_1x + c_2y + c_3v = f_1 \tag{4.37}$$

$$c_1y - c_2x + c_3w = f_2 \tag{4.38}$$

$$-c_3 = f_3. \tag{4.39}$$

Similarly, the two conditions that need to be satisfied are:

$$\begin{aligned} & (b_2c_3 - b_3c_2)d_1 + (a_3c_2 - a_2c_3)e_1 + (a_2b_3 - a_3b_2)f_1 \\ = \quad & (-b_3c_1 + b_1c_3)d_2 + (-a_1c_3 + a_3c_1)e_2 + (-a_3b_1 + a_1b_3)f_2 \end{aligned}$$

and

$$\begin{aligned} & (b_3c_1 + b_1c_3)d_1 + (a_1c_3 - a_3c_1)e_1 + (a_3b_1 - a_1b_3)f_1 \\ = \quad & (-b_3c_2 + b_2c_3)d_2 + (-a_2c_3 + a_3c_2)e_2 + (-a_3b_2 + a_2b_3)f_2. \end{aligned}$$

□

4.2.9 Example. Every controllable two-input non-homogeneous control affine system $\Sigma^{\alpha,\beta,\gamma,\delta} = (\mathsf{SE}(2), \Xi^{\alpha,\beta,\gamma,\delta})$, where

$$\Xi_u^{\alpha,\beta,\gamma,\delta} = \begin{bmatrix} \alpha \\ \beta \\ 0 \end{bmatrix} + u_1 \begin{bmatrix} -\beta \\ \alpha \\ 0 \end{bmatrix} + u_2 \begin{bmatrix} \gamma \\ \delta \\ 1 \end{bmatrix},$$

is state space equivalent to the control system $\Sigma^{1,1,1} = (\mathsf{SE}(2), \Xi^{1,1,1})$, where

$$\Xi_u^{1,1,1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + u_1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + u_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Clearly $\alpha^2 + \beta^2 \neq 0$ otherwise $\Sigma^{\alpha,\beta,\gamma,\delta}$ is not controllable. Indeed, let $\Xi^{1,1,1} = A + u_1 B + u_2 C = E_1 + u_1 E_2 + u_2 E_3$ and let $\Xi = D + u_1 E + u_2 F$ for all $u \in \mathbb{R}$. According to theorem 4.2.8, these control systems are state space equivalent if and only if $a_1 = d_3$, $a_2 = e_3$, $a_3 = f_3$,

$$\begin{aligned} & (b_2c_3 - b_3c_2)d_1 + (a_3c_2 - a_2c_3)e_1 + (a_2b_3 - a_3b_2)f_1 \\ = \; & (b_3c_1 - b_1c_3)d_2 + (a_1c_3 - a_3c_1)e_2 + (a_3b_1 - a_1b_3)f_2, \text{ and} \\ & (b_3c_1 - b_1c_3)d_1 + (a_1c_3 - a_3c_1)e_1 + (a_3b_1 - a_1b_3)f_1 \\ = \; & (b_3c_2 - b_2c_3)d_2 + (a_2c_3 - a_3c_2)e_2 + (a_3b_2 - a_2b_3)f_2. \end{aligned}$$

Therefore, we must have that $a_3 = d_3 = 0$, $b_3 = e_3 = 0$ and $c_3 = f_3 = 1$. Now, we substitute in for the other two conditions to get that $d_1 = e_2$ and $-d_2 = e_1$. If we choose $x = d_1$, $y = -d_2$, $v = f_1$ and $w = f_2$ we can see that

$$\begin{aligned} \begin{bmatrix} d_1 & -d_2 & f_1 \\ d_2 & d_1 & f_2 \\ 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + u_1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + u_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) &= \begin{bmatrix} d_1 \\ d_2 \\ 0 \end{bmatrix} + u_1 \begin{bmatrix} -d_2 \\ d_1 \\ 0 \end{bmatrix} + u_2 \begin{bmatrix} f_1 \\ f_2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} + u_1 \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} + u_2 \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} \end{aligned}$$

if and only if the conditions of the theorem are satisfied. Therefore, we have shown that we can find a Lie algebra isomorphism taking $\Xi_u^{1,1,1}$ to $\Xi_u^{\alpha,\beta,\gamma,\delta}$, for any $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that $\alpha^2 + \beta^2 \neq 0$.

4.2.5 Three-input homogeneous control affine systems

4.2.10 Proposition. *A controllable three-input homogeneous control affine system* $\Sigma = (\mathsf{SE}(2), \Xi)$, *where*

$$\Xi_u = u_1 A + u_2 B + u_3 C, \quad u = (u_1, u_2, u_3) \in \mathbb{R}^3$$

is state space equivalent to another controllable three-input control affine systems $\tilde{\Sigma} = (\mathsf{SE}(2), \widetilde{\Xi})$, *where*

$$\widetilde{\Xi}_u = u_1 D + u_2 E + u_3 F \quad u = (u_1, u_2, u_3) \in \mathbb{R}^3$$

if and only if one the following conditions hold for $\kappa = 1$ *or* $\kappa = -1$*:*

(i) $\kappa a_3 = d_3$,

(ii) $\kappa b_3 = e_3$,

(iii) $\kappa c_3 = f_3$

(iv) $\kappa(b_2c_3 - b_3c_2)d_1 + (a_3c_2 - a_2c_3)e_1 + (a_2b_3 - a_3b_2)f_1$
$= (b_3c_1 - b_1c_3)d_2 + (a_1c_3 - a_3c_1)e_2 + (a_3b_1 - a_1b_3)f_2.$

(v) $\kappa(b_3c_1 - b_1c_3)d_1 + (a_1c_3 - a_3c_1)e_1 + (a_3b_1 - a_1b_3)f_1$
$= (b_3c_2 - b_2c_3)d_2 + (a_2c_3 - a_3c_2)e_2 + (a_3b_2 - a_2b_3)f_2.$

PROOF. Let $\Sigma = (\mathsf{SE}(2), \Xi)$ be a controllable two-input control affine system, where

$$\begin{aligned}\Xi_u &= u_1A + u_2B + u_3C \\ &= u_1(a_1E_1 + a_2E_2 + a_3E_3) + u_2(b_1E_1 + b_2E_2 + b_3E_3) + u_3(c_1E_1 + c_2E_2 + c_3E_3)\end{aligned}$$

and let $\tilde{\Sigma} = (\mathsf{SE}(2), \tilde{\Xi})$ be another controllable control system, where

$$\begin{aligned}\tilde{\Xi}_u &= u_1D + u_2E + u_3F \\ &= u_1(d_1E_1 + d_2E_2 + d_3E_3) + u_2(e_1E_1 + e_2E_2 + e_3E_3) + u_3(f_1E_1 + f_2E_2 + f_3E_3).\end{aligned}$$

We identify each of these elements $A, B, C, D, E, F \in \mathfrak{se}(2)$ with their corresponding elements in $\mathbb{R}^3_{\oslash}$, that is;

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}, \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}, \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} \text{ and } \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix},$$

respectively. We know that any Lie algebra isomorphism taking Ξ_u to $\tilde{\Xi}_u$ (for all $u \in \mathbb{R}^3$) is of the form

$$\phi = \begin{bmatrix} x & y & v \\ \mp y & \pm x & w \\ 0 & 0 & \pm 1 \end{bmatrix}, \quad x, y, v, w \in \mathbb{R}, \text{ and } x^2 + y^2 \neq 0.$$

We split the two possible types of Lie algebra isomorphisms into two cases, that is a case for 1 and a case for -1.

Case 1: We have

$$\begin{bmatrix} x & y & v \\ -y & x & w \\ 0 & 0 & 1 \end{bmatrix} \left(u_1 \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + u_2 \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} + u_3 \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \right) = u_1 \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} + u_2 \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} + u_3 \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}.$$

Since this holds for all $u_1, u_2, u_3 \in \mathbb{R}$ we get the following equations:

$$a_1x + a_2y + a_3v = d_1 \tag{4.40}$$
$$-a_1y + a_2x + a_3w = d_2 \tag{4.41}$$
$$a_3 = d_3 \tag{4.42}$$
$$b_1x + b_2y + b_3v = e_1 \tag{4.43}$$
$$-b_1y + b_2x + b_3w = e_2 \tag{4.44}$$
$$b_3 = e_3 \tag{4.45}$$

$$c_1x + c_2y + c_3v = f_1 \tag{4.46}$$
$$-c_1y + c_2x + c_3w = f_2 \tag{4.47}$$
$$c_3 = f_3. \tag{4.48}$$

Combining equations 4.40, 4.43, 4.46, and equations 4.41, 4.44, 4.47, we get the following systems:

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ v \end{bmatrix} = \begin{bmatrix} d_1 \\ e_1 \\ f_1 \end{bmatrix}$$

and

$$\begin{bmatrix} a_2 & -a_1 & a_3 \\ b_2 & -b_1 & b_3 \\ c_2 & -c_1 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} = \begin{bmatrix} d_2 \\ e_3 \\ f_2 \end{bmatrix},$$

where we denote these two 3×3 matrices by M_1 and M_2, respectively. The rest of the proof now follows exactly as in the proof of proposition 4.2.8. □

4.2.11 Example. Every controllable homogeneous three-input control affine system $\Sigma^{\alpha,\beta,\gamma,\delta} = (\mathsf{SE}(2), \Xi^{\alpha,\beta,\gamma,\delta})$, where

$$\Xi_u^{\alpha,\beta,\gamma,\delta} = u_1 \begin{bmatrix} \alpha \\ \beta \\ 0 \end{bmatrix} + u_2 \begin{bmatrix} -\beta \\ \alpha \\ 0 \end{bmatrix} + u_3 \begin{bmatrix} \gamma \\ \delta \\ 1 \end{bmatrix}$$

is state space equivalent to the control system $\Sigma^{1,1,1} = (\mathsf{SE}(2), \Xi^{1,1,1})$, where

$$\Xi_u^{1,1,1} = u_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + u_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + u_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Clearly $\alpha^2 + \beta^2 \neq 0$, otherwise $\Sigma^{\alpha,\beta,\gamma,\delta}$ is not controllable. Indeed, let $\Xi^{1,1,1} = u_1A + u_2B + u_3C = u_1E_1 + u_2E_2 + u_3E_3$ and let $\Xi = u_1D + u_2E + u_3F$ (for all $u \in \mathbb{R}$). According to the theorem 4.2.10, these control systems are state space equivalent if and only if $a_1 = d_3$, $a_2 = e_3$, $a_3 = f_3$,

$$\begin{aligned} & (b_2c_3 - b_3c_2)d_1 + (a_3c_2 - a_2c_3)e_1 + (a_2b_3 - a_3b_2)f_1 \\ = \; & (b_3c_1 - b_1c_3)d_2 + (a_1c_3 - a_3c_1)e_2 + (a_3b_1 - a_1b_3)f_2, \text{ and} \\ & (b_3c_1 - b_1c_3)d_1 + (a_1c_3 - a_3c_1)e_1 + (a_3b_1 - a_1b_3)f_1 \\ = \; & (b_3c_2 - b_2c_3)d_2 + (a_2c_3 - a_3c_2)e_2 + (a_3b_2 - a_2b_3)f_2. \end{aligned}$$

Therefore, we must have that $a_3 = d_3 = 0$, $b_3 = e_3 = 0$ and $c_3 = f_3 = 1$. Now, we substitute in for the other two conditions to get that $d_1 = e_2$ and $-d_2 = e_1$. If we choose $x = d_1$, $y = -d_2$, $v = f_1$ and $w = f_2$ we can see that

$$\begin{aligned}\begin{bmatrix} d_1 & -d_2 & f_1 \\ d_2 & d_1 & f_2 \\ 0 & 0 & 1 \end{bmatrix}\left(u_1\begin{bmatrix}1\\0\\0\end{bmatrix} + u_3\begin{bmatrix}0\\1\\0\end{bmatrix} + u_3\begin{bmatrix}0\\0\\1\end{bmatrix}\right) &= u_1\begin{bmatrix}d_1\\d_2\\0\end{bmatrix} + u_2\begin{bmatrix}-d_2\\d_1\\0\end{bmatrix} + u_3\begin{bmatrix}f_1\\f_2\\1\end{bmatrix} \\ &= u_1\begin{bmatrix}d_1\\d_2\\d_3\end{bmatrix} + u_2\begin{bmatrix}e_1\\e_2\\e_3\end{bmatrix} + u_3\begin{bmatrix}f_1\\f_2\\f_3\end{bmatrix}\end{aligned}$$

if and only if the conditions of the theorem are satisfied. Therefore, we have shown that we can find a Lie algebra isomorphism taking $\Xi_u^{1,1,1}$ to $\Xi_u^{\alpha,\beta,\gamma,\delta}$ for any $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that $\alpha^2 + \beta^2 \neq 0$.

4.2.12 Remark. There do not exist any three-input non-homogeneous control affine systems $\Sigma = (\mathsf{SE}(2), \Xi)$ where $\Xi_u = A + u_1B_1 + u_2B_2 + u_3B_3$ for all $u \in \mathbb{R}^3$. This is so as B_1, B_2, B_3 are linearly independent by definition and since $\dim \mathfrak{se}(2) = 3$ it follows that $\operatorname{span}(B_1, B_2, B_3) = \mathfrak{se}(2)$. Therefore A, B_1, B_2, B_3 must be linearly dependent.

4.3 Detached feedback equivalent control systems

We find the equivalence classes of all detached feedback equivalent controllable control affine systems on $\mathsf{SE}(2)$. It is the results of this section that give us our six associated optimal control problems, which we study in chapters $5 - 10$.

4.3.1 Single-input non-homogeneous control affine systems

4.3.1 Proposition. *Any controllable single-input control affine system* $\Sigma = (\mathsf{SE}(2), \Xi)$ *is detached feedback equivalent to exactly one of the following control systems:*
(i) $\Sigma^{1,3} = (\mathsf{SE}(2), \Xi^{1,3})$ *with trace* $\Gamma^{1,3} = E_1 + \langle E_3\rangle$*;*
(ii) $\Sigma^{\alpha,1} = (\mathsf{SE}(2), \Xi^{\alpha,1})$ *with trace* $\Gamma^{\alpha,1} = \alpha E_3 + \langle E_1\rangle$*, where* $\alpha > 0$*.*

Proof. Let Σ be a control affine system with trace given by $\Gamma = A + \langle B\rangle$. We identify A, B with their corresponding elements in $\mathbb{R}^3$,

$$\begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix} \text{ and } \begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix},$$

respectively. We note that we cannot have $a_3 = b_3 = 0$ as then $\operatorname{Lie}(\Gamma) \neq \mathfrak{g}$.

Case 1: $a_3 \in \mathbb{R}$ and $b_3 \neq 0$. We now apply the affine transformation

$$\psi : \mathbb{R} \to \mathbb{R},\ u \mapsto \frac{1}{b_3}u - \frac{a_3}{b_3}$$

to the controls of Σ. Applying this transformation we get that

$$\Gamma = \begin{bmatrix} a_1 - \frac{a_3b_1}{b_3} \\ a_2 - \frac{a_3b_2}{b_3} \\ 0 \end{bmatrix} + \left\langle \begin{bmatrix} \frac{b_1}{b_3} \\ \frac{b_2}{b_3} \\ 1 \end{bmatrix} \right\rangle.$$

The Lie algebra isomorphism taking $\Gamma^{1,3}$ to Γ is then given by

$$\begin{bmatrix} a_1 - \frac{a_3 b_1}{b_3} & \frac{a_3 b_2}{b_3} - a_2 & \frac{b_1}{b_3} \\ a_2 - \frac{a_3 b_2}{b_3} & a_1 - \frac{a_3 b_1}{b_3} & \frac{b_2}{b_3} \\ 0 & 0 & 1 \end{bmatrix}.$$

This is indeed a Lie algebra isomorphism as at least one of $a_1 - \frac{a_3 b_1}{b_3}, \frac{a_3 b_2}{b_3} - a_2$ are nonzero, otherwise $\mathrm{Lie}\,(\Gamma) \neq \mathfrak{se}(2)$, and since $b_3 \neq 0$ $\frac{b_1}{b_3}, \frac{b_2}{b_3} \in \mathbb{R}$. Therefore any single-input control affine system with $a_3 \in \mathbb{R}$ and $b_3 \neq 0$ is detached feedback equivalent to the control system $\Sigma^{1,3}$.

Case 2: $a_3 \neq 0$ and $b_3 = 0$. At least one of b_1, b_2 are nonzero, otherwise $\mathrm{Lie}\,(\Gamma) \neq \mathfrak{se}(2)$. The Lie algebra isomorphism taking $\Gamma^{\alpha,3}$ to Γ is given by

$$\begin{bmatrix} b_1 & -b_2 & \frac{a_1}{\alpha} \\ b_2 & b_1 & \frac{a_2}{\alpha} \\ 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 0 \\ 0 \\ \alpha \end{bmatrix} + u \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} a_1 \\ a_2 \\ \alpha \end{bmatrix} + u \begin{bmatrix} b_1 \\ b_2 \\ 0 \end{bmatrix},$$

where $\alpha = a_3$. We now show that for each α_1, α_2 such that $|\alpha_1| \neq |\alpha_2|$, $\Sigma^{\alpha_1,1}$ is not detached feedback equivalent to $\Sigma^{\alpha_2,1}$. Indeed, since each Lie algebra isomorphism is of the form

$$\begin{bmatrix} x & y & v \\ \mp y & \pm x & w \\ 0 & 0 & \pm 1 \end{bmatrix},\ x^2 + y^2 \neq 0,$$

it follows that $\pm\alpha_1 = \alpha_2$. Thus, for each α_1, α_2 such that $|\alpha_1| \neq |\alpha_2|$, the control systems $\Sigma^{\alpha_1,1}$ and $\Sigma^{\alpha_2,1}$ are not detached feedback equivalent. If α_1, α_2 are such that $\alpha_1 = -\alpha_2$, then the Lie algebra isomorphism taking $\Gamma^{\alpha_1,1}$ to $\Gamma^{\alpha_2,1}$ is given by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

So for $\alpha > 0$, each control system $\Sigma^{\alpha,1}$ is a member of a different equivalence class of detached feedback equivalent control systems. Therefore any single-input control affine system with $a_3 \neq 0$ and $b_3 = 0$ is detached feedback equivalent to a control system $\Sigma^{\alpha,1}$ for some $\alpha > 0$. We are now just left to show that for any $\alpha > 0$, the control systems $\Sigma^{\alpha,1}$ and $\Sigma^{1,3}$ are not detached feedback equivalent. This follows almost immediately as the third component of the control term in $\Sigma^{1,3}$ is nonzero and the third component of the control term in $\Sigma^{\alpha,1}$ is zero, or $E_3 \in \langle E_3 \rangle$ and $E_3 \notin \langle E_1 \rangle$. That is, from the form of a Lie algebra isomorphism on $\mathsf{SE}(2)$, there would have to exist an affine transformation of the controls which takes the control term of $\Gamma^{\alpha,1}$, $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, to some $\begin{bmatrix} c_1 \\ c_2 \\ 1 \end{bmatrix}$ where $c_1, c_2 \in \mathbb{R}$, which is impossible. Hence these control systems are not detached feedback equivalent. Therefore we have shown that any controllable single-input control affine system on $\mathsf{SE}(2)$ is detached feedback equivalent to one and only one of the control systems $\Sigma^{1,3}$ or $\Sigma^{\alpha,1}$ for $\alpha > 0$. □

4.3.2 Two-input homogeneous control affine systems

4.3.2 PROPOSITION. *Any controllable two-input homogeneous control affine system* $\Sigma = (\mathsf{SE}(2), \Xi)$ *is detached feedback equivalent to the control systems* $\Sigma^{1,3} = (\mathsf{SE}(2), \Xi^{1,3})$ *with trace*

$$\Gamma^{1,3} = \langle E_1, E_3 \rangle .$$

PROOF. Let Σ be a control affine system with trace given by $\Gamma = \langle B_1, B_2 \rangle$. We identify B_1, B_2 with their corresponding elements in $\mathbb{R}^3_{\otimes}$,

$$\begin{bmatrix} b_{1,1} \\ b_{1,2} \\ b_{1,3} \end{bmatrix} \text{ and } \begin{bmatrix} b_{2,1} \\ b_{2,2} \\ b_{2,3} \end{bmatrix},$$

respectively. We note that we cannot have $b_{1,3} = b_{2,3} = 0$ as then $\operatorname{Lie}(\Gamma) \neq \mathfrak{se}(2)$. With out loss of generality we assume that $b_{1,3} \neq 0$. We now apply the affine transformation

$$\psi : \mathbb{R}^2 \to \mathbb{R}^2,\ (u_1, u_2) \mapsto (\frac{1}{b_{1,3}}(u_1 - b_{2,3}u_2), u_2)$$

to the controls of Σ. Applying this transformation we get that

$$\Gamma = \left\langle \begin{bmatrix} \frac{b_{1,1}}{b_{1,3}} \\ \frac{b_{1,2}}{b_{2,3}} \\ 1 \end{bmatrix}, \begin{bmatrix} b_{2,1} - \frac{b_{1,1}b_{2,3}}{b_{1,3}} \\ b_{2,2} - \frac{b_{1,2}b_{2,3}}{b_{1,3}} \\ 0 \end{bmatrix} \right\rangle .$$

Then the Lie algebra isomorphism taking $\Gamma^{1,3}$ to Γ is given by

$$\begin{bmatrix} b_{2,1} - \frac{b_{1,1}b_{2,3}}{b_{1,3}} & \frac{b_{1,2}b_{2,3}}{b_{1,3}} - b_{2,2} & \frac{b_{1,1}}{b_{1,3}} \\ b_{2,2} - \frac{b_{1,2}b_{2,3}}{b_{1,3}} & b_{2,1} - \frac{b_{1,1}b_{2,3}}{b_{1,3}} & \frac{b_{1,2}}{b_{1,3}} \\ 0 & 0 & 1 \end{bmatrix}.$$

This is indeed a Lie algebra isomorphism as at least one of $b_{2,1} - \frac{b_{1,1}b_{2,3}}{b_{1,3}}, b_{2,2} - \frac{b_{1,2}b_{2,3}}{b_{1,3}}$ are nonzero, otherwise $\operatorname{Lie}(\Gamma) \neq \mathfrak{se}(2)$, and since $b_{1,3} \neq 0$ $\frac{b_{1,1}}{b_{1,3}}, \frac{b_{1,2}}{b_{1,3}} \in \mathbb{R}$. Therefore any two-input homogeneous control affine system is detached feedback equivalent to the control system $\Sigma^{1,3}$. ☐

4.3.3 Two-input non-homogeneous control affine systems

4.3.3 PROPOSITION. *Any controllable two-input non-homogeneous control affine system* $\Sigma = (\mathsf{SE}(2), \Xi)$ *is detached feedback equivalent to exactly one of the following control systems:*

(i) $\Sigma^{1,2,3} = (\mathsf{SE}(2), \Xi^{1,2,3})$ *with trace* $\Gamma^{1,2,3} = E_1 + \langle E_2, E_3 \rangle$;

(ii) $\Sigma^{\alpha,1,2} = (\mathsf{SE}(2), \Xi^{\alpha,1,2})$ *with trace* $\Gamma^{\alpha,1,2} = \alpha E_3 + \langle E_1, E_2 \rangle$, *where* $\alpha > 0$.

PROOF. Let Σ be a control affine system with trace given by

$$\begin{aligned} \Gamma &= A + \langle B_1, B_2 \rangle \\ &= a_1E_1 + a_2E_2 + a_3E_3 + \langle b_{1,1}E_1 + b_{1,2}E_2 + b_{1,3}E_3, b_{2,1}E_1 + b_{2,2}E_2 + b_{2,3}E_3 \rangle . \end{aligned}$$

Case 1: Either $b_{1,3}$ or $b_{2,3}$ are nonzero. With out loss of generality we assume that $b_{1,3} \neq 0$.

$$\begin{aligned}
\Gamma &= (a_1 - \frac{a_3 b_{1,1}}{b_{1,3}})E_1 + (a_2 - \frac{a_3 b_{1,2}}{b_{1,3}})E_2 \\
&+ \left\langle \frac{b_{1,1}}{b_{1,3}}E_1 + \frac{b_{1,2}}{b_{1,3}}E_2 + E_3, (b_{2,1} - \frac{b_{2,3}b_{1,1}}{b_{1,3}})E_1 + (b_{2,2} - \frac{b_{2,3}b_{1,1}}{b_{1,3}})E_1 \right\rangle \\
&= a_1'E_1 + a_2'E_2 + \langle b_1'E_1 + b_2'E_2 + E_3, c_1'E_1 + c_2'E_2 \rangle
\end{aligned}$$

for some new constants $a_i', b_i', c_i',\ i = 1, 2$. Here either c_1' or c_2' are not zero as otherwise $\operatorname{Lie}(\Gamma) \neq \mathfrak{se}(2)$. Consider now the equation

$$\begin{bmatrix} c_1' & -c_2' \\ c_2' & c_1' \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -a_1' \\ -a_2' \end{bmatrix}.$$

There exists a solution for $\mathbf{v}$ as $c_1'^2 + c_2'^2 \neq 0$. Furthermore $v_2 \neq 0$. If $v_2 = 0$ then $-v_2(c_1' + c_2') = a_1' + a_2'$ which contradicts the fact that A, B_2 are linearly independent. We now show that the mapping

$$\phi = \begin{bmatrix} v_2c_2' & v_2c_1' & b_1' \\ -v_2c_1' & v_2c_2' & b_2' \\ 0 & 0 & 1 \end{bmatrix}$$

is an automorphism taking $\Gamma^{1,2,3}$ to Γ. Indeed

$$\begin{aligned}
\phi(E_1 + \langle E_2, E_3 \rangle) &= \phi \cdot E_1 + \langle \phi \cdot E_2, \phi \cdot E_3 \rangle \\
&= v_2c_1'E_1 - v_2c_1'E_2 + \langle v_2c_1'E_1 + v_2c_2'E_2, b_1'E_1 + b_2'E_2 + E_3 \rangle \\
&= a_1'E_1 + a_2'E_2 + v_1(c_1'E_1 + c_2'E_2) + \langle c_1'E_1 + c_2'E_2, b_1'E_1 + b_2'E_2 + E_3 \rangle \\
&= a_1'E_1 + a_2'E_2 + \langle c_1'E_1 + c_2'E_2, b_1'E_1 + b_2'E_2 + E_3 \rangle .
\end{aligned}$$

Therefore these two control systems are detached feedback equivalent.

Case 2: Assume $a_3 \neq 0$ and $b_{1,3} = b_{2,3} = 0$. Therefore

$$\begin{aligned}
\Gamma &= a_1E_1 + a_2E_2 + a_3E_3 + \langle b_{1,1}E_1 + b_{1,2}E_2, b_{2,1}E_1 + b_{2,2}E_2 \rangle \\
&= a_1E_1 + a_2E_2 + a_3E_3 + \langle E_1, E_2 \rangle .
\end{aligned}$$

Case 2.1: If $a_3 > 0$, then the Lie algebra isomorphism

$$\phi = \begin{bmatrix} 1 & 0 & \frac{a_1}{\alpha} \\ 0 & 1 & \frac{a_2}{\alpha} \\ 0 & 0 & 1 \end{bmatrix}$$

is an automorphism taking $\Gamma^{\alpha,1,2}$ to Γ, where $\alpha = a_3$.

Case 2.2: If $a_3 < 0$, then the Lie algebra isomorphism

$$\phi = \begin{bmatrix} -1 & 0 & \frac{a_1}{\alpha} \\ 0 & -1 & \frac{a_2}{\alpha} \\ 0 & 0 & -1 \end{bmatrix}$$

is an automorphism taking $\Gamma^{\alpha,1,2}$ to Γ, where $\alpha = -a_3$.

We are now left to show that none of the above systems are detached feedback equivalent to any of the other systems. As in proposition 4.3.1, two control systems $\Sigma^{\alpha,1,2}$ and $\Sigma^{\beta,1,2}$, such that $|\alpha| \neq |\beta|$, are not detached feedback equivalent. Assume $\Sigma^{1,2,3}$ and $\Sigma^{\alpha,1,2}$ are detached feedback equivalent, for some $\alpha > 0$. Then there must exist some affine transformation of the controls and a Lie algebra automorphism that takes $\Gamma^{1,2,3}$ to $\Gamma^{\alpha,1,2}$, for some $\alpha > 0$. Since $E_3 \in \langle E_2, E_3\rangle$ and $E_3 \notin \langle E_1, E_2\rangle$, as in proposition 4.3.1, there cannot exist a transformation of the control taking the control term of $\Gamma^{\alpha,1,2}$ to a suitable control term, such that there exists a Lie algebra isomorphism taking the transformed control system to $\Gamma^{1,2,3}$. Hence $\Sigma^{1,2,3}$ and $\Sigma^{\alpha,1,2}$ are not detached feedback equivalent, for any $\alpha > 0$. Therefore, we have found that any controllable non-homogeneous two-input control affine system Σ is detached feedback equivalent to exactly one of the systems $\Sigma^{1,2,3}$ or $\Sigma^{\alpha,1,2}$, for some $\alpha > 0$. ☐

4.3.4 Three-input control affine systems

Note that we do not have separate cases for non-homogeneous and homogeneous systems as A, B_1, B_2, B_3 will always be linearly dependent. This is so as B_1, B_2, B_3 are linearly independent by definition of a control affine system, and since $\dim(\mathfrak{se}(2)) = 3$ it follows that the elements B_1, B_2, B_3 form a basis for $\mathfrak{se}(2)$. Hence for any element $A \in \mathfrak{se}(2)$ we have that A, B_1, B_2, B_3 are linearly dependent.

4.3.4 PROPOSITION. *Any controllable three-input control affine system* $\Sigma = (\mathsf{SE}(2), \Xi)$ *is detached feedback equivalent to the control systems* $\Sigma^{1,1,1} = (\mathsf{SE}(2), \Xi^{1,1,1})$ *with trace given by*

$$\Gamma^{1,1,1} = \langle E_1, E_2, E_3\rangle .$$

PROOF. Let Σ be a three-input control affine system with trace given by $\Gamma = \langle B_1, B_2, B_3\rangle$. Now by definition of a control affine system we know that B_1, B_2, B_3 are linearly independent and so $\langle B_1, B_2, B_3\rangle = \mathfrak{se}(2)$, that is B_1, B_2, B_3 form a basis for $\mathfrak{se}(2)$. Therefore $\langle B_1, B_2, B_3\rangle = \langle E_1, E_2, E_3\rangle = \mathfrak{se}(2)$, as E_1, E_2, E_3 form a basis for $\mathfrak{se}(2)$.

Hence any three-input control affine system Σ is detached feedback equivalent to the control system $\Sigma^{1,1,1}$. ☐

Table 4.1:

Detached Feedback Equivalence		
Type	Representative	Conditions
Type I-a	$E_1 + \langle E_3\rangle$	
Type I-b	$\alpha E_3 + \langle E_1\rangle$	$\alpha > 0$
Type II^0	$\langle E_2, E_3\rangle$	
Type II-a	$E_1 + \langle E_2, E_3\rangle$	
Type II-b	$\alpha E_3 + \langle E_1, E_2\rangle$	$\alpha > 0$
Type III^0	$\langle E_1, E_2, E_3\rangle$	

We use type I-a and type I-b to denote the two different types of representatives for the class of single-input non-homogeneous control affine systems. We use type II^0, II-a and II-b to denote the different types of representatives for the classes of two-input control affine systems, where the superscript 0 indicates the control system is homogeneous. Lastly we use type III^0 to denote the representative for the class of three-input homogeneous control affine systems. Every controllable control affine system on $\mathsf{SE}(2)$ is detached feedback equivalent to one of these types of control systems. In the next six chapters, we study the left-invariant optimal control problems associated to each of the six types of control affine systems on $\mathsf{SE}(2)$.

Chapter 5

The Type I-a Problem

We consider the LiCP

$$\begin{aligned} J &= \frac{1}{2}\int_0^T c_3 u^2(t)dt \to \min, \quad c_3 > 0 && (5.1)\\ \dot{g} &= g(E_1 + uE_3), \quad g \in \mathsf{SE}(2), \quad u \in \mathbb{R} && (5.2)\\ g(0) &= g_1 \quad \text{and} \quad g(T) = g_2. && (5.3) \end{aligned}$$

Here g_1 and g_2 are arbitrary but fixed points in $\mathsf{SE}(2)$ and

$$E_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad E_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

5.1 Normal extremals

5.1.1 Proposition. *The family of (reduced) cost-extended Hamiltonian is given by*

$$H_u^\lambda(p) = -\frac{\lambda c_3}{2}u^2 + p(E_1 + uE_3). \tag{5.4}$$

Proof. By proposition 2.3.7 we obtain the family of (reduced) Hamiltonian functions, $H_u(p) = p(E_1 + uE_3)$. It then follows, by definition 2.6.3, that the family of (reduced) cost-extended Hamiltonian on $\mathfrak{se}(2)^*$ is given by

$$H_u^\lambda(p) = -\frac{\lambda c_3}{2}u^2 + p(E_1 + uE_3).$$

□

5.1.2 Theorem. *The optimal control corresponding to the normal extremals is given by*

$$u = \frac{1}{c_3}P_3,$$

where

$$\dot{P}_1 = \frac{1}{c_3} P_3 P_2 \tag{5.5}$$

$$\dot{P}_2 = -\frac{1}{c_3} P_3 P_1 \tag{5.6}$$

$$\dot{P}_3 = -P_2. \tag{5.7}$$

PROOF. The family of (reduced) cost-extended Hamiltonians, corresponding to the normal extremals, is given by $H_u(p) = -\frac{1}{2}c_3 u^2 + p(E_1 + uE_3)$. Using 3.4, 3.5, 3.6, the corresponding cost-extended Hamiltonians are given by $H_u = -\frac{1}{2}c_3 u^2 + P_1 + uP_3$. Applying the maximality condition, $(MP2)$, of the maximum principle we obtain the control corresponding to the normal extremals,

$$\frac{\partial H_u}{\partial u} = 0 \iff -c_3 u + P_3 = 0 \iff u = \frac{1}{c_3} P_3.$$

We now substitute this expression for u into the above to obtain the optimal Hamiltonian; that is,

$$\begin{aligned} H &= -\frac{1}{2c_3} P_3^2 + P_1 + \frac{1}{c_3} P_3^2 \\ &= \frac{1}{2c_3} P_3^2 + P_1. \end{aligned}$$

We then use the properties of the Poisson bracket to get that:

$$\begin{aligned} \dot{P}_i &= \{P_i, H\}_- \\ &= \{P_i, \frac{1}{2c_3} P_3^2 + P_1\}_- \\ &= \frac{1}{2c_3}(\{P_i, P_3\}_- P_3 + P_3\{P_i, P_3\}_-) + \{P_i, P_1\}_- \\ &= \frac{1}{c_3} P_3 \{P_i, P_3\}_- + \{P_i, P_1\}_-. \end{aligned}$$

From the relations 3.7 we get that:

$$\begin{aligned} \dot{P}_1 &= \frac{1}{c_3} P_3 \{P_1, P_3\}_- + \{P_1, P_1\}_- = \frac{1}{c_3} P_3 P_2, \\ \dot{P}_2 &= \frac{1}{c_3} P_3 \{P_2, P_3\}_- + \{P_2, P_1\}_- = -\frac{1}{c_3} P_3 P_1, \\ \dot{P}_3 &= \frac{1}{c_3} P_3 \{P_3, P_3\}_- + \{P_3, P_1\}_- = -P_2. \end{aligned}$$

□

5.1.3 PROPOSITION. *The function* $K = P_1^2 + P_2^2$ *is an integral of motion for the Hamilton-Poisson system* $(\mathfrak{se}(2)^*, \{\cdot,\cdot\}_-, H)$*, where* $H = \frac{1}{2c_1} P_3^2 + P_1$.

PROOF. Indeed,

$$\dot{K} = P_1 \dot{P}_1 + P_2 \dot{P}_2.$$

Substituting in equations 5.5, 5.6, 5.7 we get that

$$\dot{K} = \frac{1}{c_3}P_1P_2P_3 - \frac{1}{c_3}P_2P_1P_3 = 0.$$

Thus K is constant along the flow of the Hamiltonian vector field and is therefore an integral of motion for this system. □

5.1.4 Remark. In fact, the function $K = P_1^2 + P_2^2$ is a Casimir function for $\mathfrak{se}(2)^*$.

5.2 Solution to the normal extremal equations

5.2.1 Theorem. *The (reduced) extremal equations* 5.5, 5.6, 5.7 *can be integrated by Jacobi elliptic functions to obtain the results:*

$$P_1(t) = \frac{\alpha - \beta b \operatorname{dc}\left(\frac{(\alpha-\beta)\sqrt{A_1A_2}}{\sqrt{c_3}}bt, \frac{a}{b}\right)}{1 - b \operatorname{dc}\left(\frac{(\alpha-\beta)\sqrt{A_1A_2}}{\sqrt{c_3}}bt, \frac{a}{b}\right)} \tag{5.8}$$

$$\textit{and/or } P_1(t) = \frac{\alpha - \beta b \operatorname{ns}\left(\frac{(\alpha-\beta)\sqrt{A_1A_2}}{\sqrt{c_3}}bt, \frac{a}{b}\right)}{1 - b \operatorname{ns}\left(\frac{(\alpha-\beta)\sqrt{A_1A_2}}{\sqrt{c_3}}bt, \frac{a}{b}\right)} \tag{5.9}$$

$$P_2(t) = \pm\sqrt{K - P_1^2(t)} \tag{5.10}$$

$$P_3(t) = \pm\sqrt{2c_3(H - P_1(t))}. \tag{5.11}$$

where $a^2 = 1$ *and* $b^2 = \frac{H+\sqrt{H^2-K}}{H-\sqrt{H^2-K}}$.

Proof. We have

$$P_3^2 = 2c_3(H - P_1) \quad \text{and} \quad P_2^2 = K - P_1^2. \tag{5.12}$$

First we square equation 5.5 and then we substitute in equations 5.12 to get

$$\dot{P_1}^2 = \frac{1}{c_3^2}P_3^2P_2^2 \tag{5.13}$$

$$\dot{P_1}^2 = \frac{1}{c_3}(2H - 2P_1)(K - P_1^2). \tag{5.14}$$

We now proceed to apply the method outlined in section 2.7 to get the equation above, in the form 2.25, to match the form of equation 2.35. Let $S_1 = K - P_1^2$, and $S_2 = 2H - 2P_1$. We then consider the quadratic expression $S_1 + \lambda S_2$. This expression is a perfect square whenever

$$D(\lambda) = \lambda^2 + (K + 2H\lambda) = 0 \tag{5.15}$$

$$\iff \lambda^2 + 2H\lambda + K = 0. \tag{5.16}$$

We now solve this expression for λ to obtain

$$\lambda_1 = -\sqrt{H^2 - K} - H \tag{5.17}$$
$$\lambda_2 = \sqrt{H^2 - K} - H. \tag{5.18}$$

Substituting $\lambda_1,\ \lambda_2$ into the equation $S_1 + \lambda S_2$ gives

$$S_1 = A_1(P_1 - \alpha)^2 + B_1(P_1 - \beta)^2 \text{ and } S_2 = A_2(P_1 - \alpha)^2 + B_2(P_1 - \beta)^2 \tag{5.19}$$

where

$$A_1 = \frac{H - \sqrt{H^2 - K}}{2\sqrt{H^2 - K}} \qquad B_1 = -\frac{H + \sqrt{H^2 - K}}{2\sqrt{H^2 - K}} \tag{5.20}$$
$$A_2 = \frac{1}{2\sqrt{H^2 - K}} \qquad B_2 = -\frac{1}{2\sqrt{H^2 - K}} \tag{5.21}$$
$$\alpha = H + \sqrt{H^2 - K} \qquad \beta = H - \sqrt{H^2 - K}. \tag{5.22}$$

Now having S_1 and S_2 in the form above we can now write 5.14 as

$$\dot{P_1}^2 = \frac{1}{c_3}(A_1(P_1 - \alpha)^2 + B_1(P_1 - \beta)^2)(A_2(P_1 - \alpha)^2 + B_2(P_1 - \beta)^2).$$

Continuing to follow the steps outlined in section 2.7 we use the substitution $u = \frac{P_1 - \alpha}{P_1 - \beta}$ to obtain the following integral equation:

$$t = \frac{\sqrt{c_3}}{(\alpha - \beta)\sqrt{A_1 A_2}} \int \frac{du}{\sqrt{(u^2 + \frac{B_1}{A_1})(u_2 + \frac{B_2}{A_2})}}. \tag{5.23}$$

We require that $A_1 A_2 > 0$ under the square root sign and so we have that

$$\frac{H - \sqrt{H^2 - K}}{4(H^2 - K)} > 0 \tag{5.24}$$

which holds true for $H > \sqrt{H^2 - K}$. We choose

$$a^2 = -\frac{B_2}{A_2} = 1 \text{ and } b^2 = -\frac{B_1}{A_1}. \tag{5.25}$$

So comparing equation 5.23 with the elliptic integral 2.19 we get that

$$t = \frac{\sqrt{c_3}}{(\alpha - \beta)\sqrt{A_1 A_2}} \int_b^{\frac{P_1 - \alpha}{P_1 - \beta}} \frac{du}{\sqrt{(u^2 - a^2)(u_2 - b^2)}} \tag{5.26}$$
$$t = \frac{\sqrt{c_3}}{(\alpha - \beta)\sqrt{A_1 A_2}} \frac{1}{b} \mathrm{dc}^{-1}\left(\frac{1}{b}\frac{P_1 - \alpha}{P_1 - \beta}, \frac{a}{b}\right). \tag{5.27}$$

Rearranging now for P_1 we get that

$$P_1(t) = \frac{\alpha - \beta b \,\mathrm{dc}\left(\frac{(\alpha-\beta)\sqrt{A_1A_2}}{\sqrt{c_3}}bt, \frac{a}{b}\right)}{1 - b\,\mathrm{dc}\left(\frac{(\alpha-\beta)\sqrt{A_1A_2}}{\sqrt{c_3}}bt, \frac{a}{b}\right)}. \tag{5.28}$$

Substituting the values of a and b into the condition $a < b \leq x$ of equation 2.20 gives

$$1 < \sqrt{\frac{H + \sqrt{H^2 - K}}{H - \sqrt{H^2 - K}}}, \tag{5.29}$$

which always holds (See Appendix A: Mathematica code). Similarly, comparing equation 5.23 with the elliptic integral 2.20, we get that

$$P_1 = \frac{\alpha - \beta b \,\mathrm{ns}\left(\frac{(\alpha-\beta)\sqrt{A_1A_2}}{\sqrt{c_3}}bt, \frac{a}{b}\right)}{1 - b\,\mathrm{ns}\left(\frac{(\alpha-\beta)\sqrt{A_1A_2}}{\sqrt{c_3}}bt, \frac{a}{b}\right)}. \tag{5.30}$$

We can then use the value of P_1 obtained to solve for P_2 and P_3. □

5.3 Stability

The equilibrium states for the system of equations 5.5, 5.6, 5.7, see 2.8.1, are:

$$P_{e_1} = (M, 0, 0), \quad P_{e_2} = (0, 0, M) \quad \text{and} \quad P_0 = (0, 0, 0).$$

Here $M \in \mathbb{R}\backslash\{0\}$.

5.3.1 Theorem. *The equilibrium state* $P_{e_1} = (M, 0, 0)$, $M \in \mathbb{R}\backslash\{0\}$ *has the following behaviour:*

(i) If $M > 0$*, then it is unstable;*

(ii) If $M < 0$*, then it is nonlinearly stable.*

Proof. (i) If $M > 0$, then $P_{e_1} = (M, 0, 0)$ is unstable. Let A be the matrix corresponding to the linearised operator of system 5.5, 5.6, 5.7, as in theorem 2.8.4. Thus

$$A = \begin{bmatrix} 0 & \frac{1}{c_3}P_3 & \frac{1}{c_3}P_2 \\ -\frac{1}{c_3}P_3 & 0 & -\frac{1}{c_3}P_1 \\ 0 & -1 & 0 \end{bmatrix},$$

and then we have that

$$A(P_{e_1}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{M}{c_3} \\ 0 & -1 & 0 \end{bmatrix}.$$

The characteristic polynomial is then given by

$$\mathrm{char}_A(\lambda) = \det(A - \lambda\mathbf{1}) = -\lambda\left(\lambda^2 - \frac{M}{c_3}\right).$$

Since $M > 0$ (and $c_3 > 0$) the eigenvalues are as follows

$$\lambda_1 = 0, \quad \lambda_2 = \sqrt{\frac{M}{c_3}}, \quad \lambda_3 = -\sqrt{\frac{M}{c_3}}.$$

Now since $\mathrm{Re}(\lambda_2) > 0$ it follows from theorem 2.8.4 that our equilibrium $P_{e_1} = (M, 0, 0),\ M > 0$ is unstable.

(ii) If $M < 0$, then $P_{e_1} = (M, 0, 0)$ is nonlinearly stable. Here we use the energy-Casimir method to prove our claim. Let H_ψ be the energy-Casimir function given by

$$H_\psi(P_1, P_2, P_3) = \frac{1}{2c_3}P_3^2 + P_1 + \psi(\frac{1}{2}(P_1^2 + P_2^2)) \tag{5.31}$$

where $\psi \in C^\infty(\mathbb{R}, \mathbb{R})$. We now calculate the first variation of H_ψ:

$$\begin{aligned} \delta H_\psi &= \frac{d}{dt}(\frac{1}{2c_3}(P_3 + t\delta_3)^2 + P_1 + t\delta_1)\Big|_{t=0} \\ &+ \frac{d}{dt}\psi(\frac{1}{2}(P_1 + t\delta_1)^2 + \frac{1}{2}(P_1 + t\delta_1)^2)\Big|_{t=0} \\ &= \frac{1}{c_3}\delta_3 P_3 + \delta_1 + (\delta_1 P_1 + \delta_2 P_2)\dot{\psi}(\frac{1}{2}(P_1^2 + P_2^2)) \end{aligned} \tag{5.32}$$

and this equals zero at $P_{e_1} = (M, 0, 0)$ if and only if

$$\begin{aligned} \delta_1 + \delta_1 M\dot{\psi}(\frac{1}{2}M^2) &= 0, \\ \dot{\psi}(\frac{1}{2}M^2) &= -\frac{1}{M}. \end{aligned} \tag{5.33}$$

Now, by using 5.32, we can calculate the second variation of H_ψ:

$$\begin{aligned} \delta^2 H_\psi &= \frac{d}{dt}(\frac{\delta_3}{c_3}(P_3 + t\delta_3) + \delta_1)\Big|_{t=0} \\ &+ \frac{d}{dt}(\delta_1(P_1 + t\delta_1) + \delta_2(P_2 + t\delta_2)) \times \dot{\psi}(\frac{1}{2}(P_1 + t\delta_1)^2 + \frac{1}{2}(P_2 + t\delta_2)^2)\Big|_{t=0} \\ &= \frac{1}{c_3}\delta_3^2 + (\delta_1^2 + \delta_2^2)\dot{\psi}(\frac{1}{2}(P_1^2 + P_2^2)) + (\delta_1 P_1 + \delta_2 P_2)^2\ddot{\psi}(\frac{1}{2}(P_1^2 + P_2^2)). \end{aligned} \tag{5.34}$$

The second variation at the equilibrium of interest is

$$\delta^2 H_\psi(M, 0, 0) = (M^2\ddot{\psi}(\frac{1}{2}M^2) - \frac{1}{M})\delta_1^2 - \frac{1}{M}\delta_2^2 + \frac{1}{c_3}\delta_3^2. \tag{5.35}$$

This can be represented in matrix form as

$$\begin{bmatrix} \delta_1 & \delta_2 & \delta_3 \end{bmatrix} \begin{bmatrix} M^2\ddot{\psi}(\frac{1}{2}M^2) - \frac{1}{M} & 0 & 0 \\ 0 & -\frac{1}{M} & 0 \\ 0 & 0 & \frac{1}{c_3} \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{bmatrix}$$

and is positive definite if and only if each of the entries along the diagonal are positive. We note that the matrix can only be positive definite as $c_3 > 0$ by assumption. Therefore we get that the matrix is positive definite if and only if

$$-\frac{1}{M} > 0, \tag{5.36}$$

$$\ddot{\psi}(\frac{1}{2}M^2) > \frac{1}{M^3}. \tag{5.37}$$

Thus, having chosen that

$$\ddot{\psi}(\frac{1}{2}M^2) > \frac{1}{M^3},$$

this quadratic form is positive definite if and only if $M < 0$ and so the equilibrium states $P_{e_1} = (M, 0, 0)$ for all $M < 0$ are nonlinear stable. Consequently, $\psi(x) = -\frac{x^2}{M^3}$ is such a function that satisfies the above conditions $5.33, 5.37$ for all $M < 0$. □

5.3.2 THEOREM. *The equilibrium state $P_{e_2} = (0, 0, M)$ is stable for all $M \in \mathbb{R}\backslash\{0\}$.*

PROOF. We use theorem 2.8.7 to prove that these equilibrium states are stable. Let $L = \lambda_0 H + \lambda_1 K$, where $\lambda_0 = 0$, $\lambda_1 = 1$ and $K = P_1^2 + P_2^2$ is a Casimir function. Then $dL(0, 0, M) = [P_1\ P_2\ 0]|_{(0,0,M)} = 0$. The Hessian is then given by

$$dL^2(0, 0, M) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot$$

Now clearly $\ker dK(P_{e_2}) = \mathbb{R}^3$. We have that $dH(0, 0, M) = \begin{bmatrix} 1 & 0 & \frac{P_3}{c_3} \end{bmatrix}\Big|_{(0,0,M)} = \begin{bmatrix} 1 & 0 & \frac{M}{c_3} \end{bmatrix}$. Therefore we have that $\ker dH(P_{e_2}) = \operatorname{span}\{(-\frac{M}{c_3}, 0, 1), (0, 1, 0)\}$ and so $W = \operatorname{span}\{(-\frac{M}{c_3}, 0, 1), (0, 1, 0)\}$.The restricted Hessian is given by

$$dL^2(0, 0, M)\big|_{W\times W} = \begin{bmatrix} \frac{M^2}{c_3^2} & 0 \\ 0 & 1 \end{bmatrix},$$

which is positive definite. So we have found constants λ_0, λ_1 such that L satisfies conditions 2.39 and 2.40 of theorem 2.8.7. Hence the equilibrium state $P_{e_2} = (0, 0, M)$ is stable for all $M \in \mathbb{R}\backslash\{0\}$. □

5.3.3 THEOREM. *The equilibrium state $P_0 = (0, 0, 0)$ is unstable.*

PROOF. Let A be the matrix corresponding to the linearised operator of system 5.5, 5.6, 5.7, as in theorem 2.8.4. Thus

$$A = \begin{bmatrix} 0 & \frac{1}{c_3}P_3 & \frac{1}{c_3}P_2 \\ -\frac{1}{c_3}P_3 & 0 & -\frac{1}{c_3}P_1 \\ 0 & -1 & 0 \end{bmatrix},$$

and thus we have

$$A(P_0) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}.$$

The characteristic polynomial is then given by

$$\operatorname{char}_{A(P_0)}(\lambda) = \det(A(P_0) - \lambda \mathbf{1}).$$

Therefore

$$\det(A(P_0) - \lambda \mathbf{1}) = -\lambda^3 = 0$$

and so the only eigenvalue of $A(P_0)$ is $\lambda = 0$. The algebraic multiplicity of λ is $m_\lambda = 3$. We now determine the geometric multiplicity of λ. Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3$. The eigenvectors associated the the eigenvalue 0 are given by the system

$$(A(P_0) - 0\mathbf{1})X = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0.$$

This system reduces to the equation $x_2 = 0$. Therefore $X = \operatorname{span}(\{1, 0, 0\}, \{0, 0, 1\})$ and so the geometric multiplicity of $\lambda = 0$ is $d_\lambda = 2$. Therefore $A(P_0)$ has an eigenvalue λ with zero real part such that $d_\lambda < m_\lambda$ and so by theorem 2.8.5 it follows that $P_0 = (0, 0, 0)$ is an unstable equilibrium state. □

Table 5.1:

Stability		
Equilibrium state	Conditions	Stable/Unstable
$P_{e_1} = (M, 0, 0)$	$M > 0$ $M < 0$	Unstable Stable
$P_{e_2} = (0, 0, M)$	$M \in \mathbb{R}\backslash\{0\}$	Stable
$P_0 = (0, 0, 0)$		Unstable

Chapter 6

The Type I-b Problem

We consider the LiCP

$$\begin{aligned} J &= \frac{1}{2}\int_0^T c_1 u^2(t)dt \to \min, \quad c_1 > 0 && (6.1)\\ \dot{g} &= g(E_3 + uE_1), \quad g \in \mathsf{SE}(2), \quad u \in \mathbb{R} && (6.2)\\ g(0) &= g_1 \quad \text{and} \quad g(T) = g_2. && (6.3)\end{aligned}$$

Here g_1 and g_2 are arbitrary but fixed points in $\mathsf{SE}(2)$ and

$$E_1 = \begin{bmatrix} 0 & 0 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} \quad \text{and} \quad E_3 = \begin{bmatrix} 0 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{bmatrix}.$$

We note that although the representative given in chapter 4 is of the form $\alpha E_3 + \langle E_1\rangle$ for $\alpha > 0$, we choose to take $\alpha = 1$ throughout this chapter to simplify calculations.

6.1 Normal extremals

6.1.1 Proposition. *The family of (reduced) cost-extended Hamiltonians is given by*

$$H_u^\lambda(p) = -\frac{\lambda c_1}{2}u^2 + p(E_3 + uE_1). \tag{6.4}$$

Proof. By proposition 2.3.7 we obtain the family of (reduced) Hamiltonian functions, $H_u(p) = p(E_3 + uE_1)$. It then follows, by definition 2.6.3, that the family of (reduced) cost-extended Hamiltonian on $\mathfrak{se}(2)^*$ is given by

$$H_u^\lambda(p) = -\frac{\lambda c_1}{2}u^2 + p(E_3 + uE_1).$$

□

6.1.2 Theorem. *The optimal control corresponding to the normal extremals is given by*

$$u = \frac{1}{c_1} P_1,$$

where

$$\dot{P}_1 = P_2 \tag{6.5}$$
$$\dot{P}_2 = -P_1 \tag{6.6}$$
$$\dot{P}_3 = -\frac{1}{c_1} P_1 P_2. \tag{6.7}$$

Proof. The family of (reduced) cost-extended Hamiltonians, corresponding to the normal extremals, is given by $H_u(p) = -\frac{1}{2}c_1u^2 + p(E_3 + uE_1)$. Using 3.4, 3.5, 3.6, the corresponding cost-extended Hamiltonians are given by $H_u = -\frac{1}{2}c_1u^2 + P_3 + uP_1$. Applying the maximality condition, $(MP2)$, of the maximum principle we obtain the optimal control corresponding to the normal extremals,

$$\frac{\partial H_u}{\partial u} = 0 \iff -c_1 u + P_1 = 0 \iff u = \frac{1}{c_1} P_1.$$

We now substitute this expression for u into the above to obtain the optimal Hamiltonian; that is,

$$\begin{aligned} H(p) &= -\frac{1}{2c_1} P_1^2 + P_3 + \frac{1}{c_1} P_1^2 \\ &= \frac{1}{2c_1} P_1^2 + P_3. \end{aligned}$$

We then use the properties of the Poisson bracket to get that:

$$\begin{aligned} \dot{P}_i &= \{P_i, H\}_- \\ &= \{P_i, \frac{1}{2c_1} P_1^2 + P_3\}_- \\ &= \frac{1}{2c_1}(P_1\{P_i, P_1\}_- + \{P_i, P_1\}_- P_1) + \{P_i, P_3\}_- \\ &= \frac{1}{c_1} P_1 \{P_i, P_1\}_- + \{P_i, P_3\}_- \end{aligned}$$

From the relations 3.7 we get that:

$$\begin{aligned} \dot{P}_1 &= \frac{1}{c_1} P_1\{P_1, P_1\}_- + \{P_1, P_3\}_- = P_2, \\ \dot{P}_2 &= \frac{1}{c_1} P_1\{P_2, P_1\}_- + \{P_2, P_3\}_- = -P_1, \\ \dot{P}_3 &= \frac{1}{c_1} P_1\{P_3, P_1\}_- + \{P_3, P_3\}_- = -\frac{1}{c_1} P_1 P_2. \end{aligned}$$

□

6.1.3 Proposition. *The function* $K = P_1^2 + P_2^2$ *is an integral of motion for the Hamilton-Poisson system* $(\mathfrak{se}(2)^*, \{\cdot\,,\cdot\}_-, H)$, *where* $H = \frac{1}{2c_1}P_1^2 + P_3$.

Proof. Indeed,

$$\dot{K} = P_1\dot{P}_1 + P_2\dot{P}_2.$$

Substituting in equations 6.5, 6.6, 6.7 we get that

$$\dot{K} = P_1P_2 - P_2P_1 = 0.$$

Thus K is constant along the flow of the Hamiltonian vector field and is therefore an integral of motion for this system. □

6.1.4 Remark. In fact, the function $K = P_1^2 + P_2^2$ is a Casimir function for $\mathfrak{se}(2)^*$.

6.2 Solution to the normal extremal equations

6.2.1 Theorem. *We can integrate the (reduced) extremal equations* 6.5, 6.6, 6.7 *using standard trigonometric integrals to obtain the results:*

$$P_1(t) = \sqrt{K}\sin(t) \tag{6.8}$$

$$P_2(t) = \sqrt{K}\cos(t) \tag{6.9}$$

$$P_3(t) = \frac{\sqrt{K}}{c_1}\cos^2(t). \tag{6.10}$$

Proof. We have that the following is a Casimir function for this system,

$$P_1^2 + P_2^2 = K. \tag{6.11}$$

From here we get that $P_2 = \pm\sqrt{K - P_1^2}$. Taking $P_2 = \sqrt{K - P_1^2}$ and substituting into equation 6.6 gives

$$\int_0^{P_1} \frac{dP_1}{\sqrt{K - P_1^2}} = \int_0^t dt$$

and so we get from standard integrals that

$$t = \sin^{-1}(\frac{P_1}{\sqrt{K}}) \tag{6.12}$$

$$P_1(t) = \sqrt{K}\sin(t). \tag{6.13}$$

Substituting this equation now into equation 6.5 gives,

$$P_2(t) = \int_0^{P_2} -\sqrt{K}\sin(t)dt \tag{6.14}$$

$$P_2(t) = \sqrt{K}\cos(t). \tag{6.15}$$

We now substitute the values for P_1 and P_2 into equation 6.7 to get that

$$\dot{P}_3 = -\frac{\sqrt{K}}{c_1}\cos(t)\sin t.$$

We simply integrate now to obtain

$$P_3(t) = \frac{\sqrt{K}}{c_1}\int -\cos(t)\sin(t)dt \tag{6.16}$$

$$P_3(t) = \frac{\sqrt{K}}{c_1}\cos^2(t). \tag{6.17}$$

□

6.3 Stability

The equilibrium states for the system of equations 6.5, 6.6, 6.7, see 2.8.1, are:

$$P_{e_1} = (0,0,M) \quad M \in \mathbb{R}.$$

6.3.1 Theorem. *The equilibrium state $P_{e_1} = (0,0,M)$ is stable for all $M \in \mathbb{R}$.*

Proof. We use theorem 2.8.7 to prove that these equilibrium states are stable. We let $L = \lambda_0 H + \lambda_1 K$, where $\lambda_0 = 0$, $\lambda_1 = 1$ and $K = P_1^2 + P_2^2$ is a Casimir function. Then $dL(0,0,M) = [P_1\ P_2\ 0]|_{(0,0,M)} = 0$. The Hessian is then given by

$$dL^2(0,0,M) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot$$

Now clearly $\ker dK(P_{e_1}) = \mathbb{R}^3$. We have that $dH(0,0,M) = \left.\begin{bmatrix} \frac{P_1}{c_1} & 0 & 1 \end{bmatrix}\right|_{(0,0,M)} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$. Therefore we have that $\ker dH(P_{e_1}) = \operatorname{span}\{(1,0,0),(0,1,0)\}$ and so

$$W = \operatorname{span}\{(1,0,0),(0,1,0)\}.$$

The restricted Hessian is given by

$$dL^2(0,0,M)\big|_{W\times W} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

which is positive definite. So we have found constants λ_0, λ_1 such that L satisfies conditions 2.39 and 2.40 of theorem 2.8.7. Hence the equilibrium state $P_{e_1} = (0,0,M)$ is stable for all $M \in \mathbb{R}$. □

Chapter 7

The Type II^0 Problem

We consider the LiCP

$$\begin{aligned} J &= \frac{1}{2}\int_0^T \big(c_1 u_1^2(t) + c_3 u_3^2(t)\big)\,dt \to \min, \quad c_1, c_3 > 0 && (7.1)\\ \dot{g} &= g(u_1 E_1 + u_3 E_3), \quad g \in \mathsf{SE}(2), \quad (u_1, u_2) \in \mathbb{R}^2 && (7.2)\\ g(0) &= g_1 \quad \text{and} \quad g(T) = g_2. && (7.3) \end{aligned}$$

Here g_1 and g_2 are arbitrary but fixed points in $\mathsf{SE}(2)$ and

$$E_1 = \begin{bmatrix} 0 & 0 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad E_3 = \begin{bmatrix} 0 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0 \end{bmatrix}.$$

7.1 Normal extremals

7.1.1 Proposition. *The family of (reduced) cost-extended Hamiltonians is given by*

$$H_u^\lambda(p) = -\lambda\left(\frac{c_1}{2}u_1^2 + \frac{c_3}{2}u_3^2\right) + p(u_1 E_1 + u_3 E_3). \tag{7.4}$$

Proof. By proposition 2.3.7 we obtain the family of (reduced) Hamiltonian functions, $H_u(p) = p(u_1 E_1 + u_3 E_3)$. It then follows, by definition 2.6.3, that the family of (reduced) cost-extended Hamiltonian on $\mathfrak{se}(2)^*$ is given by

$$H_u^\lambda(p) = -\lambda\left(\frac{c_1}{2}u_1^2 + \frac{c_3}{2}u_3^2\right) + p(u_1 E_1 + u_3 E_3).$$

□

7.1.2 Theorem. *The optimal control corresponding to the normal extremals is given by*

$$u_1 = \frac{1}{c_1}P_1, \qquad u_3 = \frac{1}{c_3}P_3,$$

where

$$\dot{P}_1 = \frac{1}{c_3} P_2 P_3 \tag{7.5}$$

$$\dot{P}_2 = -\frac{1}{c_3} P_1 P_3 \tag{7.6}$$

$$\dot{P}_3 = -\frac{1}{c_1} P_1 P_2. \tag{7.7}$$

PROOF. The family of (reduced) cost-extended Hamiltonians, corresponding to the normal extremals, is given by $H_u(p) = -\frac{1}{2}(c_1 u_1^2 + c_3 u_3^2) + p(u_1 E_1 + u_3 E_3)$. Using 3.4, 3.5, 3.6 the corresponding cost-extended Hamiltonians are given by $H_u = -\frac{1}{2}(c_1 u_1^2 + c_3 u_3^2) + u_1 P_1 + u_3 P_3$. Applying the maximality condition, $(MP2)$, of the maximum principle we obtain the control corresponding to normal extremals,

$$\frac{\partial H_u}{\partial u_1} = 0 \iff -c_1 u_1 + P_1 = 0 \iff u_1 = \frac{1}{c_1} P_1,$$

$$\frac{\partial H_u}{\partial u_3} = 0 \iff -c_3 u_3 + P_3 = 0 \iff u_3 = \frac{1}{c_3} P_3.$$

We now substitute this expression for u into the above to obtain the optimal Hamiltonian; that is,

$$\begin{aligned} H(p) &= -\frac{1}{2}(\frac{1}{c_1} P_1^2 + \frac{1}{c_3} P_3^2) + \frac{1}{c_1} P_1^2 + \frac{1}{c_3} P_3^2 \\ &= \frac{1}{2c_1} P_1^2 + \frac{1}{2c_3} P_3^2. \end{aligned}$$

We then use the properties of the Poisson bracket to get that:

$$\begin{aligned} \dot{P}_i &= \{P_i, H\}_- \\ &= \{P_i, \frac{1}{2c_1} P_1^2 + \frac{1}{2c_3} P_3^2\}_- \\ &= \frac{1}{2c_1}(\{P_i, P_1\}_- P_1 + P_1 \{P_i, P_1\}_-) + \frac{1}{2c_3}(\{P_i, P_3\}_- P_3 + P_3 \{P_i, P_3\}_-) \\ &= \frac{1}{c_1} P_1 \{P_i, P_1\}_- + \frac{1}{c_3} P_3 \{P_i, P_3\}_-. \end{aligned}$$

From the relations 3.7 we get that:

$$\begin{aligned} \dot{P}_1 &= \frac{1}{c_1} P_1 \{P_1, P_1\}_- + \frac{1}{c_3} P_3 \{P_1, P_3\}_- = \frac{1}{c_3} P_2 P_3, \\ \dot{P}_2 &= \frac{1}{c_1} P_1 \{P_2, P_1\}_- + \frac{1}{c_3} P_3 \{P_2, P_3\}_- = -\frac{1}{c_3} P_1 P_3, \\ \dot{P}_3 &= \frac{1}{c_1} P_1 \{P_2, P_1\}_- + \frac{1}{c_3} P_3 \{P_3, P_3\}_- = -\frac{1}{c_1} P_1 P_2. \end{aligned}$$

□

7.1.3 PROPOSITION. *The function* $K = P_1^2 + P_2^2$ *is an integral of motion for the Hamilton-Poisson system* $(\mathfrak{se}(2)^*, \{\cdot,\cdot\}_-, H)$, *where* $H = \frac{1}{2c_1}P_1^2 + \frac{1}{2c_3}P_3^2$.

PROOF. Indeed,

$$\dot{K} = P_1\dot{P}_1 + P_2\dot{P}_2.$$

Substituting in equations 7.5, 7.6, 7.7 we get that

$$\dot{K} = \frac{1}{c_3}P_1P_2P_3 - \frac{1}{c_3}P_2P_1P_3 = 0.$$

Thus K is constant along the flow of the Hamiltonian vector field and is therefore an integral of motion for this system. □

7.1.4 REMARK. In fact, the function $K = P_1^2 + P_2^2$ is a Casimir function for $\mathfrak{se}(2)^*$.

7.2 Solution to the normal extremal equations

7.2.1 THEOREM. *The (reduced) extremal equations* 7.5, 7.6, 7.7 *can be integrated by Jacobi elliptic functions to obtain the results:*

case 1: $K > 2c_1H$.

$$P_1(t) = b \,\mathrm{sn}\left(\frac{at}{\sqrt{c_1c_3}}, \frac{b}{a}\right) \tag{7.8}$$

$$\text{or } P_1(t) = b \,\mathrm{cd}\left(\frac{at}{\sqrt{c_1c_3}}, \frac{b}{a}\right) \tag{7.9}$$

$$P_2(t) = \pm\sqrt{K - P_1^2(t)} \tag{7.10}$$

$$P_3(t) = \pm\sqrt{2c_3H - \frac{c_3}{c_1}P_1^2(t)}. \tag{7.11}$$

where $a^2 = K$ *and* $b^2 = 2c_1H$.

case 2: $K < 2c_1H$.

$$P_1(t) = b \,\mathrm{sn}\left(\frac{at}{\sqrt{c_1c_3}}, \frac{b}{a}\right) \tag{7.12}$$

$$\text{or } P_1(t) = b \,\mathrm{cd}\left(\frac{at}{\sqrt{c_1c_3}}, \frac{b}{a}\right) \tag{7.13}$$

$$P_2(t) = \pm\sqrt{K - P_1^2(t)} \tag{7.14}$$

$$P_3(t) = \pm\sqrt{2c_3H - \frac{c_3}{c_1}P_1^2(t)}. \tag{7.15}$$

where $a^2 = 2c_1H$ *and* $b^2 = K$.

PROOF. We have

$$P_3^2 = 2c_3H - \frac{c_3}{c_1}P_1^2 \quad \text{and } P_2^2 = K - P_1^2. \tag{7.16}$$

We now square equation 7.5 and substitute in equations 7.16 to get

$$\dot{P_1}^2 = \frac{1}{c_3^2} P_2^2 P_3^2 \tag{7.17}$$

$$\dot{P_1}^2 = \frac{1}{c_1 c_3}(K - P_1^2)(2c_1 H - P_1^2). \tag{7.18}$$

Therefore we get that

$$\dot{P_1} = \sqrt{\frac{1}{c_1 c_3}(K - P_1^2)(2c_1 H - P_1^2)}\cdot \tag{7.19}$$

It now follows that

$$t = \sqrt{c_1 c_3} \int_0^{P_1} \frac{dP_1}{\sqrt{(K - P_1^2)(2c_1 H - P_1^2)}}. \tag{7.20}$$

Case 1: Firstly we assume $K > 2c_1 H$ and so choose $a^2 = K$ and $b^2 = 2c_1 H$. Now comparing equation 7.20 with the elliptic integral 2.15 we get that

$$t = \frac{\sqrt{c_1 c_3}}{\sqrt{K}} \operatorname{sn}^{-1}\left(\frac{P_1}{\sqrt{2c_1 H}}, \frac{\sqrt{2c_1 H}}{\sqrt{K}}\right) \tag{7.21}$$

and so

$$P_1(t) = \sqrt{2c_1 H} \operatorname{sn}\left(\frac{\sqrt{K}t}{\sqrt{c_1 c_3}}, \frac{\sqrt{2c_1 H}}{\sqrt{K}}\right)\cdot \tag{7.22}$$

According to equation 2.15, substituting values for a^2 and b^2, we require that $0 \leq x \leq b < a$, which gives us that

$$0 \leq P_1^2 \leq P_1^2 + P_3^2 < K = P_1^2 + P_2^2.$$

This is always true as we assume that $K > 2c_1 H$ and clearly P_1^2 is always less than or equal to $P_1^2 + P_3^2$. Similarly we can compare equation 7.20 with equation 2.17, using the same values for a^2 and b^2 and satisfying the same condition ($0 \leq x \leq b < a$), to obtain

$$t = \frac{\sqrt{c_1 c_3}}{\sqrt{K}} \operatorname{cd}^{-1}\left(\frac{P_1}{\sqrt{2c_1 H}}, \frac{\sqrt{2c_1 H}}{\sqrt{K}}\right)\cdot \tag{7.23}$$

Hence we have that

$$P_1(t) = \sqrt{2c_1 H} \operatorname{cd}\left(\frac{\sqrt{K}t}{\sqrt{c_1 c_3}}, \frac{\sqrt{2c_1 H}}{\sqrt{K}}\right)\cdot \tag{7.24}$$

Case 2: We now assume that $K < 2c_1 H$ and comparing equation 7.20 again with equation 2.15, where $a^2 = 2c_1 H$ and $b^2 = K$, we get that

$$t = \frac{\sqrt{c_1 c_3}}{\sqrt{2c_1 H}} \operatorname{sn}^{-1}\left(\frac{P_1}{\sqrt{K}}, \frac{\sqrt{K}}{\sqrt{2c_1 H}}\right) \tag{7.25}$$

and so

$$P_1(t) = \sqrt{K} \operatorname{sn}\left(\frac{\sqrt{2H}t}{\sqrt{c_3}}, \frac{\sqrt{K}}{\sqrt{2c_1 H}}\right)\cdot \tag{7.26}$$

According to equation 2.15, substituting values for a^2 and b^2, we require that $0 \leq x \leq b < a$, which gives us that

$$0 \leq P_1^2 \leq K = P_1^2 + P_2^2 < 2c_1H = P_1^2 + P_3^2.$$

This is always true as we assume that $K < 2c_1H$ and clearly P_1^2 is always less than or equal to $P_1^2 + P_2^2$. Similarly we can compare equation 7.20 again with equation 2.17 using the same values for a^2 and b^2 and satisfying the same condition, $0 \leq x \leq b < a$, to obtain

$$t = \frac{\sqrt{c_1c_3}}{\sqrt{2c_1H}} \operatorname{cd}^{-1}\left(\frac{P_1}{\sqrt{K}}, \frac{\sqrt{K}}{\sqrt{2c_1H}}\right). \tag{7.27}$$

Hence we have that

$$P_1(t) = \sqrt{K} \operatorname{cd}\left(\frac{\sqrt{2H}t}{\sqrt{c_3}}, \frac{\sqrt{K}}{\sqrt{2c_1H}}\right). \tag{7.28}$$

We can now solve for P_2 and P_3 by substituting into equations 7.16. □

7.3 Stability

The equilibrium states for the system of equations 7.5, 7.6, 7.7, see 2.8.1, are:

$$P_{e_1} = (M, 0, 0), \quad P_{e_2} = (0, M, 0), \quad P_{e_3} = (0, 0, M) \quad \text{and} \quad P_0 = (0, 0, 0).$$

Here $M \in \mathbb{R}\backslash\{0\}$.

7.3.1 THEOREM. *The equilibrium state* $P_{e_1} = (M, 0, 0)$, $M \in \mathbb{R}\backslash\{0\}$ *is unstable.*

PROOF. Let A be the matrix corresponding to the linearised operator of system 7.5, 7.6, 7.7, as given in theorem 2.8.4. Thus

$$A = \begin{bmatrix} 0 & \frac{1}{c_3}P_3 & \frac{1}{c_3}P_2 \\ -\frac{1}{c_3}P_3 & 0 & -\frac{1}{c_3}P_1 \\ -\frac{1}{c_1}P_2 & -\frac{1}{c_1}P_1 & 0 \end{bmatrix},$$

and then we have that

$$A(P_{e_1}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{M}{c_3} \\ 0 & -\frac{M}{c_1} & 0 \end{bmatrix}.$$

The characteristic polynomial is then given by

$$\operatorname{char}_A(\lambda) = -\lambda\left(\lambda^2 - \frac{M^2}{c_1c_3}\right).$$

Now, for $M > 0$ (since $c_1, c_3 > 0$) the eigenvalues are as follows

$$\lambda_1 = 0, \quad \lambda_2 = \frac{M}{\sqrt{c_1c_3}}, \quad \lambda_3 = -\frac{M}{\sqrt{c_1c_3}}.$$

Therefore the Re$(\lambda_2) > 0$ and it follows from theorem 2.8.4 that our equilibrium $P_{e_1} = (M, 0, 0),\ M > 0$ is unstable. Similarly if $M < 0$, it clearly follows that Re$(\lambda_3) > 0$ and so again by theorem 2.8.4 the equilibrium is unstable. □

7.3.2 THEOREM. *The equilibrium state* $P_{e_2} = (0, M, 0),\ M \in \mathbb{R}\backslash\{0\}$ *is nonlinearly stable.*

PROOF. Here we use the energy-Casimir method to prove our claim. Let H_ψ be the energy-Casimir function given by

$$H_\psi(P_1, P_2, P_3) = \frac{1}{2c_1}P_1^2 + \frac{1}{2c_3}P_3^2 + \psi(\frac{1}{2}(P_1^2 + P_2^2)), \tag{7.29}$$

where $\psi \in C^\infty(\mathbb{R}, \mathbb{R})$.

We now calculate the first variation of H_ψ:

$$\begin{aligned}\delta H_\psi &= \frac{d}{dt}(\frac{1}{2c_1}(P_1 + t\delta_1)^2 + \frac{1}{2c_3}(P_3 + t\delta_3)^2)\Big|_{t=0} \\ &+ \frac{d}{dt}\psi(\frac{1}{2}(P_1 + t\delta_1)^2 + \frac{1}{2}(P_1 + t\delta_1)^2)\Big|_{t=0} \\ &= \frac{1}{c_1}P_1\delta_1 + \frac{1}{c_3}P_3\delta_3 + (\delta_1 P_1 + \delta_2 P_2)\dot{\psi}(\frac{1}{2}(P_1^2 + P_2^2)) \qquad (7.30)\end{aligned}$$

and this equals zero at $P_{e_2} = (0, M, 0)$ if and only if

$$\begin{aligned}\delta_2 M\dot{\psi}(\frac{1}{2}M^2) &= 0, \\ \dot{\psi}(\frac{1}{2}M^2) &= 0. \qquad (7.31)\end{aligned}$$

Now, by using 7.30, we can calculate the second variation of H_ψ:

$$\delta^2 H_\psi = \frac{1}{c_1}\delta_1^2 + \frac{1}{c_3}\delta_3^2 + (\delta_1^2 + \delta_2^2)\dot{\psi}(\frac{1}{2}(P_1^2 + P_2^2)) + (\delta_1 P_1 + \delta_2 P_2)^2\ddot{\psi}(\frac{1}{2}(P_1^2 + P_2^2)). \tag{7.32}$$

Therefore using 7.31 the second variation at the equilibrium of interest is

$$\delta^2 H_\psi(0, M, 0) = \frac{1}{c_1}\delta_1^2 + \frac{1}{c_3}\delta_3^2 + (M^2\ddot{\psi}(\frac{1}{2}M^2))\delta_2^2. \tag{7.33}$$

This can be represented in matrix form as

$$\begin{bmatrix}\delta_1 & \delta_2 & \delta_3\end{bmatrix}\begin{bmatrix}\frac{1}{c_1} & 0 & 0 \\ 0 & M^2\ddot{\psi}(\frac{1}{2}M^2) & 0 \\ 0 & 0 & \frac{1}{c_3}\end{bmatrix}\begin{bmatrix}\delta_1 \\ \delta_2 \\ \delta_3\end{bmatrix}$$

and is positive definite if and only if each of the entries along the diagonal are positive. We note that the matrix can only be positive definite as $c_1, c_3 > 0$ by assumption. Therefore the matrix is positive definite if and only if

$$\ddot{\psi}(\frac{1}{2}M^2) > 0 \tag{7.34}$$

Consequently, $\psi(x) = \frac{x^2}{2} - \frac{M^3}{6}$ is such a function that satisfies the above conditions 7.31 and 7.34. Hence this quadratic form is positive definite for all $M \in \mathbb{R}\backslash\{0\}$ and so the equilibrium states $P_{e_2} = (0, M, 0)$ are nonlinearly stable. □

7.3.3 THEOREM. *The equilibrium states* $P_0 = (0,0,0)$ *and* $P_{e_3} = (0,0,M)$ *for all* $M \in \mathbb{R}\backslash\{0\}$ *are stable.*

PROOF. We use theorem 2.8.7 to prove that these equilibrium states are stable. Let $L = \lambda_0 H + \lambda_1 K$, where $\lambda_0 = 0$, $\lambda_1 = 1$ and $K = P_1^2 + P_2^2$ is a Casimir function. Then for all $M \in \mathbb{R}$ $dL(0,0,M) = [P_1\ P_2\ 0]|_{(0,0,M)} = 0$. The Hessian is then given by

$$dL^2(0,0,M) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot$$

Now clearly $\ker dK(P_{e_2}) = \mathbb{R}^3$. We have that $dH(0,0,M) = \begin{bmatrix} \frac{P_1}{c_1} & 0 & \frac{P_3}{c_3} \end{bmatrix}\Big|_{(0,0,M)} = \begin{bmatrix} 0 & 0 & \frac{M}{c_3} \end{bmatrix}$. Therefore we have that $\ker dH(P_{e_2}) = \text{span}\,\{(1,0,0),(0,1,0)\}$ and so

$$W = \text{span}\,\{(1,0,0),(0,1,0)\}.$$

The restricted Hessian is given by

$$dL^2(0,0,M)\big|_{W\times W} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

which is positive definite. So we have found constants λ_0, λ_1 such that L satisfies conditions 2.39 and 2.40 of theorem 2.8.7. Hence the equilibrium states $P_0 = (0,0,0)$ and $P_{e_3} = (0,0,M)$ for all $M \in \mathbb{R}\backslash\{0\}$ are stable. □

Table 7.1:

Stability		
Equilibrium state	Conditions	Stable/Unstable
$P_{e_1} = (M,0,0)$	$M \in \mathbb{R}\backslash\{0\}$	Unstable
$P_{e_2} = (0,M,0)$	$M \in \mathbb{R}\backslash\{0\}$	Stable
$P_{e_3} = (0,0,M)$	$M \in \mathbb{R}\backslash\{0\}$	Stable
$P_0 = (0,0,0)$		Stable

Chapter 8

The Type II-a Problem

We consider the LiCP

$$
\begin{aligned}
J &= \frac{1}{2}\int_0^T \left(c_2u_2^2(t) + c_3u_3^2(t)\right) dt \to \min, \quad c_2,\, c_3 > 0 && (8.1)\\
\dot{g} &= g(E_1 + u_2E_2 + u_3E_3), \quad g \in \mathsf{SE}(2), \quad (u_2, u_3) \in \mathbb{R}^2 && (8.2)\\
g(0) &= g_1 \quad \text{and} \quad g(T) = g_2. && (8.3)
\end{aligned}
$$

Here g_1 and g_2 are arbitrary but fixed points in $\mathsf{SE}(2)$ and

$$
E_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad E_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.
$$

8.1 Normal extremals

8.1.1 Proposition. *The family of (reduced) cost-extended Hamiltonians is given by*

$$
H_u^\lambda(p) = -\lambda\left(\frac{c_2}{2}u_2^2 + \frac{c_3}{2}u_3^2\right) + p(E_1 + u_1E_2 + u_3E_3). \tag{8.4}
$$

Proof. By proposition 2.3.7 we obtain the family of (reduced) Hamiltonian functions, $H_u(p) = p(E_1 + u_1E_2 + u_3E_3)$. It then follows, by definition 2.6.3, that the family of (reduced) cost-extended Hamiltonian on $\mathfrak{se}(2)^*$ is given by

$$
H_u^\lambda(p) = -\lambda\left(\frac{c_2}{2}u_2^2 + \frac{c_3}{2}u_3^2\right) + p(E_1 + u_2E_2 + u_3E_3).
$$

□

8.1.2 Theorem. *The optimal control corresponding to the normal extremals is given by*

$$
u_1 = \frac{1}{c_2}P_2, \qquad u_3 = \frac{1}{c_3}P_3,
$$

where

$$\dot{P}_1 = \frac{1}{c_3} P_2 P_3 \tag{8.5}$$

$$\dot{P}_2 = -\frac{1}{c_3} P_1 P_3 \tag{8.6}$$

$$\dot{P}_3 = \frac{1}{c_2} P_1 P_2 - P_2. \tag{8.7}$$

Proof. The family of (reduced) cost-extended Hamiltonians, corresponding to the normal extremals is given by $H_u(p) = -\frac{1}{2}(c_2 u_2^2 + c_3 u_3^2) + p(E_1 + u_2 E_2 + u_3 E_3)$. Using 3.4, 3.5, 3.6 the corresponding cost-extended Hamiltonians are given by
$H_u = -\frac{1}{2}(c_2 u_2^2 + c_3 u_3^2) + P_1 + u_2 P_2 + u_3 P_3$. Applying the maximality condition, $(MP2)$, of the maximum principle we obtain the optimal control corresponding to the normal extremals,

$$\frac{\partial H_u}{\partial u_2} = 0 \iff -c_2 u_2 + P_2 = 0 \iff u_2 = \frac{1}{c_2} P_2,$$

$$\frac{\partial H_u}{\partial u_3} = 0 \iff -c_3 u_3 + P_3 = 0 \iff u_3 = \frac{1}{c_3} P_3.$$

We now substitute this expression for u into the above to obtain the optimal Hamiltonian; that is,

$$\begin{aligned} H(p) &= -\frac{1}{2}\left(\frac{1}{c_2} P_2^2 + \frac{1}{c_3} P_3^2\right) + \frac{1}{c_2} P_2^2 + \frac{1}{c_3} P_3^2 + P_1 \\ &= \frac{1}{2c_2} P_2^2 + \frac{1}{2c_3} P_3^2 + P_1. \end{aligned}$$

We then use the properties of the Poisson bracket to get that:

$$\begin{aligned} \dot{P}_i &= \{P_i, H\}_- \\ &= \{P_i, \frac{1}{2c_2} P_2^2 + \frac{1}{2c_3} P_3^2 + P_1\}_- \\ &= \frac{1}{2c_2}(\{P_i, P_2\}_- P_2 + P_2\{P_i, P_2\}_-) + \frac{1}{2c_3}(\{P_i, P_3\}_- P_3 + P_3\{P_i, P_3\}_-) + \{P_i, P_1\}_- \\ &= \frac{1}{c_2} P_2\{P_i, P_2\}_- + \frac{1}{c_3} P_3\{P_i, P_3\}_- + \{P_i, P_1\}_-. \end{aligned}$$

From the relations 3.7 we get that:

$$\begin{aligned} \dot{P}_1 &= \frac{1}{c_2} P_2\{P_i, P_2\}_- + \frac{1}{c_3} P_3\{P_i, P_3\}_- + \{P_i, P_1\}_- = \frac{1}{c_3} P_2 P_3, \\ \dot{P}_2 &= \frac{1}{c_2} P_2\{P_i, P_2\}_- + \frac{1}{c_3} P_3\{P_i, P_3\}_- + \{P_i, P_1\}_- = -\frac{1}{c_3} P_1 P_3, \\ \dot{P}_3 &= \frac{1}{c_2} P_2\{P_i, P_2\}_- + \frac{1}{c_3} P_3\{P_i, P_3\}_- + \{P_i, P_1\}_- = \frac{1}{c_2} P_1 P_2 - P_2. \end{aligned}$$

□

8.1.3 Proposition. *The function* $K = P_1^2 + P_2^2$ *is an integral of motion for the Hamilton-Poisson system* $(\mathfrak{se}(2)^*, \{\cdot, \cdot\}_-, H)$*, where* $H = \frac{1}{2c_2}P_2^2 + \frac{1}{2c_3}P_3^2 + P_1$.

Proof. Indeed,

$$\dot{K} = P_1\dot{P}_1 + P_2\dot{P}_2.$$

Substituting in equations 7.5, 7.6, 7.7 we get that

$$\dot{K} = \frac{1}{c_3}P_1P_2P_3 - \frac{1}{c_3}P_2P_1P_3 = 0.$$

Thus K is constant along the flow of the Hamiltonian vector field and is therefore an integral of motion for this system. □

8.1.4 Remark. In fact, the function $K = P_1^2 + P_2^2$ is a Casimir function for $\mathfrak{se}(2)^*$.

8.2 Solution to the normal extremal equations

8.2.1 Theorem. *The (reduced) extremal equations* 8.5, 8.6, 8.7 *can be integrated by Jacobi elliptic functions to obtain the results:*

For $H - c_2 < \sqrt{H^2 - K}$ *we have the following two cases:*

Case 1*:* $H - c_2 < -\sqrt{H^2 - K}$

$$P_1(t) = \frac{\alpha - \beta a \operatorname{dc}\left(\frac{(\alpha-\beta)\sqrt{A_1A_2}}{\sqrt{c_1c_3}}at, \frac{b}{a}\right)}{1 - a \operatorname{dc}\left(\frac{(\alpha-\beta)\sqrt{A_1A_2}}{\sqrt{c_1c_3}}at, \frac{b}{a}\right)} \tag{8.8}$$

$$\textit{and/or } P_1(t) = \frac{\alpha - \beta a \operatorname{ns}\left(\frac{(\alpha-\beta)\sqrt{A_1A_2}}{\sqrt{c_1c_3}}at, \frac{b}{a}\right)}{1 - a \operatorname{ns}\left(\frac{(\alpha-\beta)\sqrt{A_1A_2}}{\sqrt{c_1c_3}}at, \frac{b}{a}\right)} \tag{8.9}$$

$$P_2(t) = \pm\sqrt{K - P_1^2(t)} \tag{8.10}$$

$$P_3(t) = \pm\sqrt{\frac{c_3}{c_2}P_1^2(t) - 2c_3P_1(t) + 2c_3H - \frac{c_3}{c_2}K}. \tag{8.11}$$

where $a^2 = \frac{H+\sqrt{H^2+K}}{H-\sqrt{H^2-K}}$ *and* $b^2 = -\frac{H-c_2+\sqrt{H^2-K}}{c_2-H+\sqrt{H^2-K}}$.

Case 2*:* $H - c_2 > -\sqrt{H^2 - K}$

$$P_1(t) = \frac{\alpha - \beta a \operatorname{nc}\left(\frac{\sqrt{a^2+b^2}(\alpha-\beta)\sqrt{A_1A_2}}{\sqrt{c_2c_3}}t, \frac{b}{\sqrt{a^2+b^2}}\right)}{1 - a \operatorname{nc}\left(\frac{\sqrt{a^2+b^2}(\alpha-\beta)\sqrt{A_1A_2}}{\sqrt{c_2c_3}}t, \frac{b}{\sqrt{a^2+b^2}}\right)} \tag{8.12}$$

$$\textit{and/or } P_1(t) = \frac{\alpha - \beta\sqrt{a^2+b^2} \operatorname{ds}\left(\frac{\sqrt{a^2+b^2}(\alpha-\beta)\sqrt{A_1A_2}}{\sqrt{c_2c_3}}t, \frac{b}{\sqrt{a^2+b^2}}\right)}{1 - \sqrt{a^2+b^2} \operatorname{ds}\left(\frac{\sqrt{a^2+b^2}(\alpha-\beta)\sqrt{A_1A_2}}{\sqrt{c_2c_3}}t, \frac{b}{\sqrt{a^2+b^2}}\right)}. \tag{8.13}$$

$$P_2(t) = \pm\sqrt{K - P_1^2(t)} \tag{8.14}$$

$$P_3(t) = \pm\sqrt{\frac{c_3}{c_2}P_1^2(t) - 2c_3P_1(t) + 2c_3H - \frac{c_3}{c_2}K}. \tag{8.15}$$

where $a^2 = \frac{H+\sqrt{H^2+K}}{H-\sqrt{H^2-K}}$ *and* $b^2 = \frac{H-c_2+\sqrt{H^2-K}}{c_2-H+\sqrt{H^2-K}}$.

Proof. We have

$$P_3^2 = 2c_3H - \frac{c_3}{c_2}P_2^2 - 2c_3P_1 \quad \text{and} \quad P_2^2 = K - P_1^2. \tag{8.16}$$

First we square equation 8.6 and then substitute in equations 8.16 to obtain

$$\dot{P_1}^2 = \frac{1}{c_3^2}P_2^2P_3^2 \tag{8.17}$$

$$\dot{P_1}^2 = \frac{1}{c_3c_2}(K - P_1^2)(P_1^2 - 2c_2P_1 + 2c_2H - K)\cdot \tag{8.18}$$

The expression $\sqrt{H^2 - K}$, which appears throughout the remainder of this proof, is always real since the discriminant of $P_1^2 - 2c_2P_1 + 2c_2H - K$ is given by

$$\Delta = 4c_2^2 - 8c_2H + 4K.$$

This expression is a quadratic in c_2 with discriminant given by

$$\Delta = 64(H^2 - K) > 0 \text{ (since } c_2 \text{ is real)} \implies H^2 - K > 0.$$

We now proceed to apply the method outlined in section 2.7 to get the equation, in the form 2.25, to match the form of equation 2.35. Let $S_1 = P_1^2 - 2c_2P_1 + 2c_2H - K$ and $S_2 = K - P_1^2$. We then consider the quadratic expression $S_1 + \lambda S_2$. This expression is a perfect square whenever

$$D(\lambda) = (1 - \lambda)(\lambda K - K + 2c_2H) - c_2^2 = 0 \tag{8.19}$$

$$-K\lambda^2 + (2K - 2c_2H)\lambda - K - c_2^2 + 2c_2H = 0. \tag{8.20}$$

We now solve this expression for λ to obtain

$$\lambda_1 = \frac{K - c_2H + c_2\sqrt{H^2 - K}}{K} \tag{8.21}$$

$$\lambda_2 = \frac{K - c_2H - c_2\sqrt{H^2 - K}}{K}. \tag{8.22}$$

Substituting $\lambda_1,\ \lambda_2$ into equation $S_1 + \lambda S_2$ gives

$$S_1 = A_1(P_1 - \alpha)^2 + B_1(P_3 - \beta)^2 \text{ and } S_2 = A_2(P_1 - \alpha)^2 + B_2(P_3 - \beta)^2, \text{ where} \tag{8.23}$$

$$A_1 = \frac{c_2 - H + \sqrt{H^2 - K}}{2\sqrt{H^2 - K}} \qquad B_1 = \frac{H - c_2 + \sqrt{H^2 - K}}{2\sqrt{H^2 - K}} \tag{8.24}$$

$$A_2 = \frac{H - \sqrt{H^2 - K}}{2\sqrt{H^2 - K}} \qquad B_2 = -\frac{H + \sqrt{H^2 - K}}{2\sqrt{H^2 - K}} \tag{8.25}$$

$$\alpha = H + \sqrt{H^2 - K} \qquad \beta = H - \sqrt{H^2 - K}. \tag{8.26}$$

Now having S_1 and S_2 in the form above we can now write 8.18 as

$$\dot{P_1}^2 = \frac{1}{c_2 c_3}(A_1(P_1 - \alpha)^2 + B_1(P_1 - \beta)^2)(A_2(P_1 - \alpha)^2 + B_2(P_1 - \beta)^2).$$

Continuing to follow the steps outlined in section 2.7 we use the substitution $u = \frac{P_1 - \alpha}{P_1 - \beta}$ to obtain the following integral equation:

$$t = \frac{\sqrt{c_2 c_3}}{(\alpha - \beta)\sqrt{A_1 A_2}} \int \frac{du}{\sqrt{(u^2 + \frac{B_1}{A_1})(u^2 + \frac{B_2}{A_2})}}. \tag{8.27}$$

We require that $A_1 A_2 = \frac{(c_2 - H + \sqrt{H^2 - K})(H - \sqrt{H^2 - K})}{4(H^2 - K)} > 0$ under the square root sign and, since $H^2 - K > 0$, it must follow that either

$$H > \sqrt{H^2 - K} \text{ and } c_2 - H > -\sqrt{H^2 - K}, \text{ or} \tag{8.28}$$

$$H < \sqrt{H^2 - K} \text{ and } c_2 - H < -\sqrt{H^2 - K}. \tag{8.29}$$

Here case 8.29 cannot hold as we have $H - c_2 > \sqrt{H^2 - K}$ but $H - c_2 < H < \sqrt{H^2 - K}$. Therefore we have that $\sqrt{H^2 - K} < H - c_2 < \sqrt{H^2 - K}$, which is a contradiction. Therefore for case 8.28 we have the following two cases:

Case 1: $H - c_2 < -\sqrt{H^2 - K}$. In this case we take

$$a^2 = -\frac{B_2}{A_2} = \frac{H + \sqrt{H^2 - K}}{H - \sqrt{H^2 - K}},$$

$$b^2 = -\frac{B_1}{A_1} = -\frac{H - c_2 + \sqrt{H^2 - K}}{c_2 - H + \sqrt{H^2 - K}}.$$

We rewrite equation 8.27 as

$$t = \frac{\sqrt{c_2 c_3}}{(\alpha - \beta)\sqrt{A_1 A_2}} \int_a^{\frac{P_1 - \alpha}{P_1 - \beta}} \frac{du}{\sqrt{(u^2 - a^2)(u^2 - b^2)}}. \tag{8.30}$$

Comparing equation 8.30 with the elliptic integral 2.19 we get that

$$\frac{(\alpha - \beta)\sqrt{A_1 A_2}}{\sqrt{c_2 c_3}} t = \frac{1}{a} \mathrm{dc}^{-1}\left(\frac{1}{a}\frac{P_1 - \alpha}{P_1 - \beta}, \frac{b}{a}\right). \tag{8.31}$$

Therefore

$$\frac{P_1-\alpha}{P_1-\beta} = a \text{ dc}\left(\frac{a(\alpha-\beta)\sqrt{A_1A_2}}{\sqrt{c_2c_3}}t, \frac{b}{a}\right) \tag{8.32}$$

$$P_1(t) = \frac{\alpha - \beta a \text{ dc}\left(\frac{a(\alpha-\beta)\sqrt{A_1A_2}}{\sqrt{c_2c_3}}t, \frac{b}{a}\right)}{1 - a \text{ dc}\left(\frac{a(\alpha-\beta)\sqrt{A_1A_2}}{\sqrt{c_2c_3}}t, \frac{b}{a}\right)}. \tag{8.33}$$

Substituting the values of a^2 and b^2 into the condition $a < b \leq x$ of equation 2.19 gives

$$\sqrt{\frac{H+\sqrt{H^{-}K}}{H-\sqrt{H^{-}K}}} < \sqrt{-\frac{H-c_2+\sqrt{H^{-}K}}{c_2-H+\sqrt{H^{-}K}}}.$$

Similarly we can rewrite equation 8.27 as

$$t = \frac{\sqrt{c_2c_3}}{(\alpha-\beta)\sqrt{A_1A_2}} \int_{\frac{P_1-\alpha}{P_1-\beta}}^{\infty} \frac{du}{\sqrt{(u^2-a^2)(u^2-b^2)}}. \tag{8.34}$$

Comparing equation 8.34 with the elliptic integral 2.20 we get that

$$\frac{(\alpha-\beta)\sqrt{A_1A_2}}{\sqrt{c_2c_3}}t = \frac{1}{a}\text{ns}^{-1}\left(\frac{1}{a}\frac{P_1-\alpha}{P_1-\beta}, \frac{b}{a}\right). \tag{8.35}$$

Therefore

$$\frac{P_1-\alpha}{P_1-\beta} = a \text{ ns}\left(\frac{a(\alpha-\beta)\sqrt{A_1A_2}}{\sqrt{c_2c_3}}t, \frac{b}{a}\right) \tag{8.36}$$

$$P_1(t) = \frac{\alpha - \beta a \text{ ns}\left(\frac{a(\alpha-\beta)\sqrt{A_1A_2}}{\sqrt{c_2c_3}}t, \frac{b}{a}\right)}{1 - a \text{ ns}\left(\frac{a(\alpha-\beta)\sqrt{A_1A_2}}{\sqrt{c_2c_3}}t, \frac{b}{a}\right)}. \tag{8.37}$$

Case 2: $H - c_2 > -\sqrt{H^2-K}$. In this case we take

$$a^2 = -\frac{B_2}{A_2} = \frac{H+\sqrt{H^2-K}}{H-\sqrt{H^2-K}},$$

$$b^2 = \frac{B_1}{A_1} = \frac{H-c_2+\sqrt{H^2-K}}{c_2-H+\sqrt{H^2-K}}.$$

We rewrite equation 8.27 as

$$t = \frac{\sqrt{c_2c_3}}{(\alpha-\beta)\sqrt{A_1A_2}} \int_{a}^{\frac{P_1-\alpha}{P_1-\beta}} \frac{du}{\sqrt{(u^2-a^2)(u^2+b^2)}}. \tag{8.38}$$

Comparing equation 8.38 with the elliptic integral 2.23 we get that

$$\frac{(\alpha-\beta)\sqrt{A_1A_2}}{\sqrt{c_2c_3}}t = \frac{1}{\sqrt{a^2+b^2}}\mathrm{nc}^{-1}\left(\frac{1}{a}\frac{P_1-\alpha}{P_1-\beta}, \frac{b}{\sqrt{a^2+b^2}}\right) \tag{8.39}$$

$$\frac{P_1-\alpha}{P_1-\beta} = a\ \mathrm{nc}\left(\frac{\sqrt{a^2+b^2}(\alpha-\beta)\sqrt{A_1A_2}}{\sqrt{c_2c_3}}t, \frac{b}{\sqrt{a^2+b^2}}\right) \tag{8.40}$$

$$P_1(t) = \frac{\alpha-\beta a\ \mathrm{nc}\left(\frac{\sqrt{a^2+b^2}(\alpha-\beta)\sqrt{A_1A_2}}{\sqrt{c_2c_3}}t, \frac{b}{\sqrt{a^2+b^2}}\right)}{1-a\ \mathrm{nc}\left(\frac{\sqrt{a^2+b^2}(\alpha-\beta)\sqrt{A_1A_2}}{\sqrt{c_2c_3}}t, \frac{b}{\sqrt{a^2+b^2}}\right)}. \tag{8.41}$$

Similarly we can rewrite equation 8.27 as

$$t = \frac{\sqrt{c_2c_3}}{(\alpha-\beta)\sqrt{A_1A_2}}\int_{\frac{P_1-\alpha}{P_1-\beta}}^{\infty}\frac{du}{\sqrt{(u^2-a^2)(u^2+b^2)}}. \tag{8.42}$$

Comparing equation 8.42 with the elliptic integral 2.24 we get that

$$\frac{(\alpha-\beta)\sqrt{A_1A_2}}{\sqrt{c_2c_3}}t = \frac{1}{\sqrt{a^2+b^2}}\mathrm{ds}^{-1}\left(\frac{1}{\sqrt{a^2+b^2}}\frac{P_1-\alpha}{P_1-\beta}, \frac{b}{\sqrt{a^2+b^2}}\right). \tag{8.43}$$

Therefore

$$\frac{P_1-\alpha}{P_1-\beta} = \sqrt{a^2+b^2}\ \mathrm{ds}\left(\frac{\sqrt{a^2+b^2}(\alpha-\beta)\sqrt{A_1A_2}}{\sqrt{c_2c_3}}t, \frac{b}{\sqrt{a^2+b^2}}\right) \tag{8.44}$$

$$P_1(t) = \frac{\alpha-\beta\sqrt{a^2+b^2}\ \mathrm{ds}\left(\frac{\sqrt{a^2+b^2}(\alpha-\beta)\sqrt{A_1A_2}}{\sqrt{c_2c_3}}t, \frac{b}{\sqrt{a^2+b^2}}\right)}{1-\sqrt{a^2+b^2}\ \mathrm{ds}\left(\frac{\sqrt{a^2+b^2}(\alpha-\beta)\sqrt{A_1A_2}}{\sqrt{c_2c_3}}t, \frac{b}{\sqrt{a^2+b^2}}\right)}. \tag{8.45}$$

□

8.3 Stability

The equilibrium states for the system of equations 8.5, 8.6, 8.7, see 2.8.1, are:

$$P_{e_1} = (M,0,0), \quad P_{e_2} = (0,0,M), \quad P_{e_3} = (c_2,M,0) \quad \text{and} \quad P_0 = (0,0,0).$$

Here $M \in \mathbb{R}\backslash\{0\}$

8.3.1 Theorem. *The equilibrium state* $P_{e_1} = (M,0,0)$, $M \in \mathbb{R}\backslash\{0\}$ *has the following behaviour:*

- *(i) If* $c_2 > M > 0$*, then it is unstable;*
- *(ii) If* $M \in (-\infty,0)\cup(c_2,\infty)$*, then it is nonlinearly stable.*

Proof. (i) If $c_1 > M > 0$, then $P_{e_1} = (M,0,0)$ is unstable. Let A be the matrix corresponding to the linearised operator of system 8.5, 8.6, 8.7, as given in theorem 2.8.4. Thus

$$A = \begin{bmatrix} 0 & \frac{1}{c_3}P_3 & \frac{1}{c_3}P_2 \\ -\frac{1}{c_3}P_3 & 0 & -\frac{1}{c_3}P_1 \\ \frac{1}{c_2}P_2 & \frac{1}{c_2}P_1 - 1 & 0 \end{bmatrix},$$

and then we have that

$$A(P_{e_1}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{M}{c_3} \\ 0 & \frac{M}{c_2} - 1 & 0 \end{bmatrix}.$$

The characteristic polynomial is then given by

$$\mathrm{char}_A(\lambda) = -\lambda\left(\lambda^2 + \left(\frac{M^2}{c_2c_3} - \frac{M}{c_3}\right)\right).$$

Hence the eigenvalues are as follows:

$$\lambda_1 = 0, \quad \lambda_2 = +\sqrt{\frac{M}{c_3} - \frac{M^2}{c_2c_3}}, \quad \lambda_3 = -\sqrt{\frac{M}{c_3} - \frac{M^2}{c_2c_3}}\cdot$$

Now $c_2 > M > 0 \implies \frac{M}{c_3} > \frac{M^2}{c_2c_3}$. In this case $\mathrm{Re}(\lambda_2) > 0$ and so it follows from theorem 2.8.4 that our equilibrium $P_{e_1} = (M, 0, 0),\ c_2 > M > 0$ is unstable.

(ii) If $M \in (-\infty, 0) \cup (c_2, \infty)$, then it is nonlinearly stable. Here we use the energy-Casimir method to prove our claim. Let H_ψ be the energy-Casimir function given by

$$H_\psi(P_1, P_2, P_3) = \frac{1}{2c_2}P_2^2 + \frac{1}{2c_3}P_3^2 + P_1 + \psi(\frac{1}{2}(P_1^2 + P_2^2)), \tag{8.46}$$

where $\psi \in C^\infty(\mathbb{R}, \mathbb{R})$. We now calculate the first variation of H_ψ:

$$\begin{aligned} \delta H_\psi = {} & \frac{d}{dt}(\frac{1}{2c_2}(P_2 + t\delta_2)^2 + \frac{1}{2c_3}(P_3 + t\delta_3)^2 + P_1 + t\delta_1)\Big|_{t=0} \\ & + \frac{d}{dt}\psi(\frac{1}{2}(P_1 + t\delta_1)^2 + \frac{1}{2}(P_2 + t\delta_2)^2)\Big|_{t=0} \\ = {} & \frac{1}{c_2}P_2\delta_2 + \frac{1}{c_3}P_3\delta_3 + \delta_1 + (\delta_1P_1 + \delta_2P_2)\dot{\psi}(\frac{1}{2}(P_1^2 + P_2^2)) \end{aligned} \tag{8.47}$$

and this equals zero at $P_{e_1} = (M, 0, 0)$ if and only if

$$\begin{aligned} \delta_1 + \delta_1M\dot{\psi}(\frac{1}{2}M^2) &= 0, \\ \dot{\psi}(\frac{1}{2}M^2) &= -\frac{1}{M}\cdot \end{aligned} \tag{8.48}$$

Now, using 8.47, we can calculate the second variation of H_ψ:

$$\delta^2 H_\psi = \frac{1}{c_2}\delta_2^2 + \frac{1}{c_3}\delta_3^2 + (\delta_1^2+\delta_2^2)\dot{\psi}(\frac{1}{2}(P_1^2+P_2^2)) + (\delta_1 P_1 + \delta_2 P_2)^2 \ddot{\psi}(\frac{1}{2}(P_1^2+P_2^2)). \tag{8.49}$$

The second variation at the equilibrium of interest is

$$\delta^2 H_\psi(M,0,0) = (\frac{1}{c_2} - \frac{1}{M})\delta_2^2 + \frac{1}{c_3}\delta_3^2 + (M^2\ddot{\psi}(\frac{1}{2}M^2) - \frac{1}{M})\delta_1^2. \tag{8.50}$$

This can be represented in matrix form as

$$\begin{bmatrix}\delta_1 & \delta_2 & \delta_3\end{bmatrix}\begin{bmatrix} M^2\ddot{\psi}(\frac{1}{2}M^2) - \frac{1}{M} & 0 & 0 \\ 0 & (\frac{1}{c_1} - \frac{1}{M}) & 0 \\ 0 & 0 & \frac{1}{c_3}\end{bmatrix}\begin{bmatrix}\delta_1 \\ \delta_2 \\ \delta_3\end{bmatrix}$$

and is positive definite if and only if each of the entries along the diagonal are positive. We note that the matrix can only be positive definite as $c_3 > 0$ by assumption. Therefore the matrix is positive definite if and only if

$$\frac{1}{c_2} > \frac{1}{M}, \text{ and } \ddot{\psi}(\frac{1}{2}M^2) > \frac{1}{M^3}. \tag{8.51}$$

Consequently, $\psi(x) = x(x(\frac{1}{M^3} + c_1) - \frac{2}{M} - M^2 c_1)$ is such a function that satisfies the above conditions 8.48 and 8.51. Hence this quadratic form is positive definite for all $M \in (-\infty, 0) \cup (c_2, \infty)$ and so the equilibrium states $P_{e_1} = (M,0,0)$ are nonlinearly stable for all $M \in (-\infty, 0) \cup (c_2, \infty)$. □

8.3.2 THEOREM. *The equilibrium state* $P_{e_3} = (c_2, M, 0)$ *is unstable for all* $M \in \mathbb{R}\backslash\{0\}$.

PROOF. Let A be the matrix corresponding to the linearised operator of system 8.5, 8.6, 8.7, as given in theorem 2.8.4. Thus

$$A = \begin{bmatrix} 0 & \frac{1}{c_3}P_3 & \frac{1}{c_3}P_2 \\ -\frac{1}{c_3}P_3 & 0 & -\frac{1}{c_3}P_1 \\ \frac{1}{c_2}P_2 & \frac{1}{c_2}P_1 - 1 & 0 \end{bmatrix},$$

and then we have that

$$A(P_{e_3}) = \begin{bmatrix} 0 & 0 & \frac{M}{c_3} \\ 0 & 0 & -\frac{c_2}{c_3} \\ \frac{M}{c_2} & 0 & 0 \end{bmatrix}.$$

The characteristic polynomial is then given by

$$\mathrm{char}_A(\lambda) = -\lambda\left(\lambda^2 - \frac{M^2}{c_2 c_3}\right).$$

Hence the eigenvalues are as follows:

$$\lambda_1 = 0, \quad \lambda_2 = +\frac{M}{\sqrt{c_2 c_3}}, \quad \lambda_3 = -\frac{M}{\sqrt{c_2 c_3}}\cdot$$

Now if $M < 0$ then $\operatorname{Re}(\lambda_3) > 0$ and, similarly, if $M > 0$ then $\operatorname{Re}(\lambda_2) > 0$. Hence by theorem 2.8.4 it follows that $P_{e_3} = (c_2, M, 0)$ is unstable for all $M \in \mathbb{R}\backslash\{0\}$. $\square$

8.3.3 THEOREM. *The equilibrium state* $P_{e_2} = (0, 0, M)$ *is stable for all* $M \in \mathbb{R}\backslash\{0\}$.

PROOF. We use theorem 2.8.7 to prove that these equilibrium states are stable. Let $L = \lambda_0 H + \lambda_1 K$, where $\lambda_0 = 0$, $\lambda_1 = 1$ and $K = P_1^2 + P_2^2$ is a Casimir function. Then $dL(0, 0, M) = [P_1\ P_2\ 0]\big|_{(0,0,M)} = 0$. The Hessian is then given by

$$dL^2(0, 0, M) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\cdot$$

Now clearly $\ker dK(P_{e_2}) = \mathbb{R}^3$. We have that $dH(0, 0, M) = \begin{bmatrix} 1 & \frac{P_2}{c_2} & \frac{P_3}{c_3} \end{bmatrix}\Big|_{(0,0,M)} = \begin{bmatrix} 1 & 0 & \frac{M}{c_3} \end{bmatrix}$. Therefore we have that $\ker dH(P_{e_2}) = \operatorname{span}\{(-\frac{M}{c_3}, 0, 1), (0, 1, 0)\}$ and so

$$W = \operatorname{span}\{(-\frac{M}{c_3}, 0, 1), (0, 1, 0)\}.$$

The restricted Hessian is given by

$$dL^2(0, 0, M)\big|_{W\times W} = \begin{bmatrix} \frac{M^2}{c_3^2} & 0 \\ 0 & 1 \end{bmatrix},$$

which is positive definite. So we have found constants λ_0, λ_1 such that L satisfies conditions 2.39 and 2.40 of theorem 2.8.7. Hence the equilibrium state $P_{e_2} = (0, 0, M)$ is stable for all $M \in \mathbb{R}\backslash\{0\}$. $\square$

8.3.4 THEOREM. *The equilibrium state* $P_0 = (0, 0, 0)$ *is unstable.*

PROOF. Let A be the matrix corresponding to the linearised operator of system 8.5, 8.6, 8.7, as given in theorem 2.8.4. Thus

$$A = \begin{bmatrix} 0 & \frac{1}{c_3}P_3 & \frac{1}{c_3}P_2 \\ -\frac{1}{c_3}P_3 & 0 & -\frac{1}{c_3}P_1 \\ \frac{1}{c_1}P_2 & \frac{1}{c_1}P_1 - 1 & 0 \end{bmatrix},$$

and thus we have

$$A(P_0) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}.$$

The characteristic polynomial is then given by

$$\operatorname{char}_{A(P_0)}(\lambda) = \det(A(P_0) - \lambda\mathbf{1}).$$

Therefore

$$\det(A(P_0) - \lambda\mathbf{1}) = -\lambda^3 = 0$$

and so the only eigenvalue of $A(P_0)$ is $\lambda = 0$. The algebraic multiplicity of λ is $m_\lambda = 3$. We now determine the geometric multiplicity of λ. Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3$. The eigenvectors associated the the eigenvalue 0 are given by the system

$$\begin{aligned} (A(P_0) - 0\mathbf{1})X &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\ &= 0. \end{aligned}$$

This system reduces to the equation $x_2 = 0$. Therefore $X = \operatorname{span}(\{1, 0, 0\}, \{0, 0, 1\})$ and so the geometric multiplicity of $\lambda = 0$ is $d_\lambda = 2$. Therefore $A(P_0)$ has an eigenvalue λ with zero real part such that $d_\lambda < m_\lambda$ and so by theorem 2.8.5 it follows that $P_0 = (0, 0, 0)$ is an unstable equilibrium state. $\square$

Table 8.1:

Stability		
Equilibrium state	Conditions	Stable/Unstable
$P_{e_1} = (M, 0, 0)$	$c_2 > M > 0$ $M \in (-\infty, 0) \cup (c_2, \infty)$	Unstable Stable
$P_{e_2} = (0, 0, M)$	$M \in \mathbb{R}\backslash\{0\}$	Stable
$P_{e_3} = (c_2, M, 0)$	$M \in \mathbb{R}\backslash\{0\}$	Unstable
$P_0 = (0, 0, 0)$		Unstable

Chapter 9

The Type II-b Problem

We consider the LiCP

$$\begin{aligned} J &= \frac{1}{2}\int_0^T \big(c_1 u_1^2(t) + c_2 u_2^2(t)\big)\, dt \to \min, \quad c_1, c_2 > 0 && (9.1)\\ \dot{g} &= g(E_3 + u_1 E_1 + u_2 E_2), \quad g \in \mathsf{SE}(2), \quad (u_1, u_2) \in \mathbb{R}^2 && (9.2)\\ g(0) &= g_1 \quad \text{and} \quad g(T) = g_2. && (9.3) \end{aligned}$$

Here g_1 and g_2 are arbitrary but fixed points in $\mathsf{SE}(2)$ and

$$E_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad E_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

We note that although the representative given in chapter 4 is of the form $\alpha E_3 + \langle E_1, E_2 \rangle$ for $\alpha > 0$, we choose to take $\alpha = 1$ throughout this chapter to simplify calculations.

9.1 Normal extremals

9.1.1 Proposition. *The family of (reduced) cost-extended Hamiltonians is given by*

$$H_u^\lambda(p) = -\lambda\left(\frac{c_1}{2}u_1^2 + \frac{c_2}{2}u_2^2\right) + p(E_3 + u_1E_1 + u_2E_2). \tag{9.4}$$

Proof. By proposition 2.3.7 we obtain the family of (reduced) Hamiltonian functions, $H_u(p) = p(E_3 + u_1E_1 + u_2E_2)$. It then follows, by definition 2.6.3, that the family of (reduced) cost-extended Hamiltonian on $\mathfrak{se}(2)^*$ is given by

$$H_u^\lambda(p) = -\lambda\left(\frac{c_1}{2}u_1^2 + \frac{c_2}{2}u_2^2\right) + p(E_3 + u_1E_1 + u_2E_2).$$

□

9.1.2 THEOREM. *The optimal control corresponding to the normal extremals is given by*

$$u_1 = \frac{1}{c_1}P_1, \qquad u_2 = \frac{1}{c_2}P_2,$$

where

$$\dot{P}_1 = P_2 \tag{9.5}$$

$$\dot{P}_2 = -P_1 \tag{9.6}$$

$$\dot{P}_3 = (\frac{1}{c_2} - \frac{1}{c_1})P_1P_2. \tag{9.7}$$

PROOF. The family of (reduced) cost-extended Hamiltonians is given by $H_u(p) = -\frac{1}{2}(c_1u_1^2 + c_2u_2^2) + p(E_3 + u_1E_1 + u_2E_2)$. Using 3.4, 3.5, 3.6 the corresponding cost-extended Hamiltonians are given by $H_u = -\frac{1}{2}(c_1u_1^2 + c_2u_2^2) + P_3 + u_1P_1 + u_2P_2$. Applying the maximality condition $(MP2)$ of the maximum principle we obtain the optimal control corresponding to the normal extremals,

$$\frac{\partial H_u}{\partial u_1} = 0 \iff -c_1u_1 + P_1 = 0 \iff u_1 = \frac{1}{c_1}P_1,$$

$$\frac{\partial H_u}{\partial u_2} = 0 \iff -c_2u_2 + P_2 = 0 \iff u_2 = \frac{1}{c_2}P_2.$$

We now substitute this expression for u into the above to obtain the optimal Hamiltonian; that is,

$$\begin{aligned} H(p) &= -\frac{1}{2}(\frac{1}{c_1}P_1^2 + \frac{1}{c_2}P_2^2) + \frac{1}{c_1}P_1^2 + \frac{1}{c_2}P_2^2 + P_3 \\ &= \frac{1}{2c_1}P_1^2 + \frac{1}{2c_2}P_2^2 + P_3. \end{aligned}$$

We then use the properties of the Poisson bracket to get that:

$$\begin{aligned} \dot{P}_i &= \{P_i, H\}_- = \{P_i, \frac{1}{2c_1}P_1^2 + \frac{1}{2c_2}P_2^2 + P_3\}_- \\ &= \frac{1}{2c_1}(\{P_i, P_1\}_-P_1 + P_1\{P_i, P_1\}_-) + \frac{1}{2c_2}(\{P_i, P_2\}_-P_2 + P_2\{P_i, P_2\}_-) + \{P_i, P_3\}_- \\ &= \frac{1}{c_1}P_1\{P_i, P_1\}_- + \frac{1}{c_2}P_2\{P_i, P_2\}_- + \{P_i, P_3\}_-. \end{aligned}$$

From the relations 3.7 we get that:

$$\begin{aligned} \dot{P}_1 &= \frac{1}{c_1}P_1\{P_1, P_1\}_- + \frac{1}{c_2}P_2\{P_1, P_2\}_- + \{P_1, P_3\}_- = P_2, \\ \dot{P}_2 &= \frac{1}{c_1}P_1\{P_2, P_1\}_- + \frac{1}{c_2}P_2\{P_2, P_2\}_- + \{P_2, P_3\}_- = -P_1, \\ \dot{P}_3 &= \frac{1}{c_1}P_1\{P_3, P_1\}_- + \frac{1}{c_2}P_2\{P_3, P_2\}_- + \{P_3, P_3\}_- = (\frac{1}{c_2} - \frac{1}{c_1})P_1P_2. \end{aligned}$$

□

9.1.3 Proposition. *The function* $K = P_1^2 + P_2^2$ *is an integral of motion for the Hamilton-Poisson system* $(\mathfrak{se}(2)^*, \{\cdot\,,\cdot\}_-, H)$, *where* $H = \frac{1}{2c_1}P_1^2 + \frac{1}{2c_2}P_2^2 + P_3$.

Proof. Indeed,

$$\dot{K} = P_1\dot{P}_1 + P_2\dot{P}_2.$$

Substituting in equations 9.5, 9.6, 9.7 we get that

$$\dot{K} = \frac{1}{c_2}P_1P_2 - \frac{1}{c_2}P_2P_1 = 0.$$

Thus K is constant along the flow of the Hamiltonian vector field and is therefore an integral of motion for this system. □

9.1.4 Remark. In fact, the function $K = P_1^2 + P_2^2$ is a Casimir function for $\mathfrak{se}(2)^*$.

9.2 Solution to the normal extremal equations

9.2.1 Theorem. *We can integrate the (reduced) extremal equations* 9.5, 9.6, 9.7 *using standard trigonometric integrals to obtain the results*

$$P_1(t) = \sqrt{K}\sin(t) \tag{9.8}$$

$$P_2(t) = \sqrt{K}\cos(t) \tag{9.9}$$

$$P_3(t) = \sqrt{K}(\frac{c_2 - c_1}{c_1c_2})\cos^2(t). \tag{9.10}$$

Proof. We have that the following is a Casimir function for this system,

$$P_1^2 + P_2^2 = K. \tag{9.11}$$

From here we get that $P_2 = \pm\sqrt{K - P_1^2}$. Taking $P_2 = \sqrt{K - P_1^2}$ and substituting into equation 6.6 gives

$$\int_0^{P_1} \frac{dP_1}{\sqrt{K - P_1^2}} = \int_0^t dt$$

and so we get from standard integrals that

$$t = \sin^{-1}(\frac{P_1}{\sqrt{K}}) \tag{9.12}$$

$$P_1(t) = \sqrt{K}\sin(t). \tag{9.13}$$

Substituting this solution for P_1 into equation 6.5 now gives,

$$P_2 = \int_0^{P_2} -\sqrt{K}\sin(t)dt \tag{9.14}$$

$$P_2(t) = \sqrt{K}\cos(t). \tag{9.15}$$

We now substitute the values for P_1 and P_2 into equation 6.7 to get that

$$\dot{P_3} = \sqrt{K}(\frac{c_2 - c_1}{c_1 c_2}) \cos(t) \sin(t).$$

We simply integrate now to obtain

$$P_3 = \sqrt{K}(\frac{c_2 - c_1}{c_1 c_2}) \int - \cos(t) \sin(t) dt \tag{9.16}$$

$$P_3(t) = \sqrt{K}(\frac{c_2 - c_1}{c_1 c_2}) \cos^2(t). \tag{9.17}$$

□

9.3 Stability

The equilibrium states for the system of equations 9.5, 9.6, 9.7, see 2.8.1, are:

$$P_{e_1} = (0, 0, M) \quad M \in \mathbb{R}.$$

9.3.1 THEOREM. *The equilibrium state $P_{e_1} = (0, 0, M)$ is stable for all $M \in \mathbb{R}$.*

PROOF. We use theorem 2.8.7 to prove that these equilibrium states are stable. Let $L = \lambda_0 H + \lambda_1 K$, where $\lambda_0 = 0$, $\lambda_1 = 1$ and $K = P_1^2 + P_2^2$ is a Casimir function. Then for all $M \in \mathbb{R}$ $dL(0, 0, M) = [P_1\ P_2\ 0]|_{(0,0,M)} = 0$. The Hessian is then given by

$$dL^2(0, 0, M) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot$$

Now clearly $\ker dK(P_{e_1}) = \mathbb{R}^3$. We have that $dH(0, 0, M) = \begin{bmatrix} \frac{P_1}{c_1} & \frac{P_2}{c_1} & 1 \end{bmatrix}\Big|_{(0,0,M)} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$. Therefore we have that $\ker dH(P_{e_1}) = \operatorname{span}\{(1, 0, 0), (0, 1, 0)\}$ and so

$$W = \operatorname{span}\{(1, 0, 0), (0, 1, 0)\}.$$

The restricted Hessian is then given by

$$dL^2(0, 0, M)\big|_{W \times W} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

which is positive definite. So we have found constants λ_0, λ_1 such that L satisfies conditions 2.39 and 2.40 of theorem 2.8.7. Hence the equilibrium state $P_{e_1} = (0, 0, M)$ is stable for all $M \in \mathbb{R}$. □

Chapter 10

The Type III^0 Problem

We consider the LiCP

$$\begin{aligned} J &= \frac{1}{2}\int_0^T \big(c_1u_1^2(t) + c_2u_2^2(t) + c_3u_3^2(t)\big)\, dt \to \min, \quad c_1, c_2, c_3 > 0 && (10.1)\\ \dot{g} &= g(u_1E_1 + u_2E_2 + u_3E_3), \quad g \in \mathsf{SE}(2), \quad (u_1, u_2, u_3) \in \mathbb{R}^3 && (10.2)\\ g(0) &= g_1 \quad \text{and} \quad g(T) = g_2. && (10.3)\end{aligned}$$

Here g_1 and g_2 are arbitrary but fixed points in $\mathsf{SE}(2)$ and

$$E_1 = \begin{bmatrix} 0 & 0 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 1 & 0 & 0\end{bmatrix} \quad \text{and} \quad E_3 = \begin{bmatrix} 0 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{bmatrix}.$$

10.1 Normal extremals

10.1.1 Proposition. *The family of (reduced) cost-extended Hamiltonian is given by*

$$H_u^\lambda(p) = -\lambda\left(\frac{c_1}{2}u_1^2 + \frac{c_2}{2}u_2^2 + \frac{c_3}{2}u_3^2\right) + p(u_1E_1 + u_2E_2 + u_3E_3). \qquad (10.4)$$

Proof. By proposition 2.3.7 we obtain the family of (reduced) Hamiltonian functions, $H_u(p) = p(u_1E_1 + u_2E_2 + u_3E_3)$. It then follows, by definition 2.6.3, that the family of (reduced) cost-extended Hamiltonian on $\mathfrak{se}(2)^*$ is given by

$$H_u^\lambda(p) = -\lambda\left(\frac{c_1}{2}u_1^2 + \frac{c_2}{2}u_2^2 + \frac{c_3}{2}u_3^2\right) + p(u_1E_1 + u_2E_2 + u_3E_3).$$

□

10.1.2 Theorem. *The optimal control corresponding to the normal extremals is given by*

$$u_1 = \frac{1}{c_1}P_1, \qquad u_2 = \frac{1}{c_2}P_2, \quad u_3 = \frac{1}{c_3}P_3,$$

where

$$\dot{P}_1 = \frac{1}{c_3}P_2P_3 \tag{10.5}$$

$$\dot{P}_2 = -\frac{1}{c_3}P_1P_3 \tag{10.6}$$

$$\dot{P}_3 = (\frac{1}{c_2} - \frac{1}{c_1})P_1P_2. \tag{10.7}$$

Proof. The family of (reduced) cost-extended Hamiltonian is given by $H_u(p) = -\frac{1}{2}(c_1u_1^2 + c_2u_2^2 + c_3u_3^2) + p(u_1E_1 + u_2E_2 + u_3E_3)$. Using 3.4, 3.5, 3.6 the corresponding cost-extended Hamiltonians are given by $H_u = -\frac{1}{2}(c_1u_1^2 + c_2u_2^2 + c_3u_3^2) + u_1P_1 + u_2P_2 + u_3P_3$. Applying the maximality condition $(MP2)$ of the maximum principle we obtain the optimal control corresponding to the normal extremals,

$$\frac{\partial H_u}{\partial u_1} = 0 \iff -c_1u_1 + P_1 = 0 \iff u_1 = \frac{1}{c_1}P_1,$$

$$\frac{\partial H_u}{\partial u_2} = 0 \iff -c_2u_2 + P_2 = 0 \iff u_2 = \frac{1}{c_2}P_2,$$

$$\frac{\partial H_u}{\partial u_3} = 0 \iff -c_3u_3 + P_3 = 0 \iff u_3 = \frac{1}{c_3}P_3.$$

We now substitute this expression for u into the above to obtain the optimal Hamiltonian; that is,

$$\begin{aligned} H(p) &= -\frac{1}{2}(\frac{1}{c_1}P_1^2 + \frac{1}{c_2}P_2^2 + \frac{1}{c_3}P_3^2) + \frac{1}{c_1}P_1^2 + \frac{1}{c_2}P_2^2 + \frac{1}{2c_3}P_3^2 \\ &= \frac{1}{2c_1}P_1^2 + \frac{1}{2c_2}P_2^2 + \frac{1}{2c_3}P_3^2. \end{aligned}$$

We then use the properties of the Poisson bracket to get that:

$$\begin{aligned} \dot{P}_i &= \{P_i, H\}_- = \{P_i, \frac{1}{2c_1}P_1^2 + \frac{1}{2c_2}P_2^2 + \frac{1}{2c_3}P_3^2\}_- \\ &= \frac{1}{2c_1}(\{P_i, P_1\}_-P_1 + P_1\{P_i, P_1\}_-) + \frac{1}{2c_2}(\{P_i, P_2\}_-P_2 + P_2\{P_i, P_2\}_-) \\ &\quad + \frac{1}{2c_3}(\{P_i, P_3\}_-P_3 + P_3\{P_i, P_3\}_-) \\ &= \frac{1}{c_1}P_1\{P_i, P_1\}_- + \frac{1}{c_2}P_2\{P_i, P_2\}_- + \frac{1}{c_3}P_3\{P_i, P_3\}_-. \end{aligned}$$

From the relations 3.7 we get that:

$$\begin{aligned} \dot{P}_1 &= \frac{1}{c_1}P_1\{P_1, P_1\}_- + \frac{1}{c_2}P_2\{P_1, P_2\}_- + \frac{1}{c_3}P_3\{P_1, P_3\}_- = \frac{1}{c_3}P_2P_3, \\ \dot{P}_2 &= \frac{1}{c_1}P_1\{P_2, P_1\}_- + \frac{1}{c_2}P_2\{P_2, P_2\}_- + \frac{1}{c_3}P_3\{P_2, P_3\}_- = -\frac{1}{c_3}P_1P_3, \\ \dot{P}_3 &= \frac{1}{c_1}P_1\{P_3, P_1\}_- + \frac{1}{c_2}P_2\{P_3, P_2\}_- + \frac{1}{c_3}P_3\{P_3, P_3\}_- = (\frac{1}{c_2} - \frac{1}{c_1})P_1P_2. \end{aligned}$$

□

10.1.3 Proposition. *The function* $K = P_1^2 + P_2^2$ *is an integral of motion for the Hamilton-Poisson system* $(\mathfrak{se}(2)^*, \{\cdot\,,\cdot\}_-, H)$*, where* $H = \frac{1}{2c_1}P_1^2 + \frac{1}{2c_2}P_2^2 + \frac{1}{2c_3}P_2$.

Proof. Indeed,

$$\dot{K} = P_1\dot{P}_1 + P_2\dot{P}_2.$$

Substituting in equations 10.5, 10.6, 10.7 we get that

$$\dot{K} = \frac{1}{c_3}P_1P_2P_3 - \frac{1}{c_3}P_2P_1P_3 = 0.$$

Thus K is constant along the flow of the Hamiltonian vector field and is therefore an integral of motion for this system. □

10.1.4 Remark. In fact, the function $K = P_1^2 + P_2^2$ is a Casimir function for $\mathfrak{se}(2)^*$.

10.2 Solution to the normal extremal equations

10.2.1 Theorem. *For* $c_1 = c_2$ *the solution to the (reduced) extremal equations* 10.5, 10.6, 10.7 *are given by*

$$P_1(t) = \sqrt{K}\sin(\frac{P_3(0)}{c_3}t), \tag{10.8}$$

$$P_2(t) = \sqrt{K}\cos(\frac{P_3(0)}{c_3}t), \tag{10.9}$$

$$P_3(t) = P_3(0). \tag{10.10}$$

Proof. With $c_1 = c_2$ we can rewrite equation 10.5, 10.6, 10.7 as

$$\dot{P}_1 = \frac{1}{c_3}P_2P_3, \quad \dot{P}_2 = -\frac{1}{c_3}P_1P_3, \quad \dot{P}_3 = 0.$$

Therefore we have to solve the equations

$$\dot{P}_1 = \frac{P_3(0)}{c_3}P_2, \tag{10.11}$$

$$\dot{P}_2 = -\frac{P_3(0)}{c_3}P_1, \tag{10.12}$$

$$P_3 = P_3(0). \tag{10.13}$$

We have that the following is a Casimir function for this system,

$$P_1^2 + P_2^2 = K. \tag{10.14}$$

From here we get that $P_2 = \pm\sqrt{K - P_1^2}$. Taking $P_2 = \sqrt{K - P_1^2}$ and substituting into equation 10.11 gives

$$\int_0^{P_1} \frac{dP1}{\sqrt{K - P_1^2}}dt = \int_0^t \frac{P_3(0)}{c_3}dt.$$

So we get from standard integrals that

$$\begin{aligned} \frac{P_3(0)}{c_3}t &= \sin^{-1}(\frac{P_1}{\sqrt{K}}) \\ P_1(t) &= \sqrt{K}\sin(\frac{P_3(0)}{c_3}t). \end{aligned}$$

Substituting this solution for P_1 into equation 10.12 we get that

$$\begin{aligned} P_2 &= \sqrt{K}\frac{P_3(0)}{c_3}\int_0^{P_2} - \sin(\frac{P_3(0)}{c_3}t)dt \\ P_2(t) &= \sqrt{K}\cos(\frac{P_3(0)}{c_3}t). \end{aligned}$$

□

10.2.2 THEOREM. *The (reduced) extremal equations* 10.5, 10.6, 10.7 *can be integrated by Jacobi elliptic functions to obtain the results:*

Case 1: $c_1 > c_2$.

Case 1 (i): $2c_2H > K$.

$$P_1(t) = b \text{ cn}\left(\sqrt{\frac{(a^2+b^2)(c_1-c_2)}{c_1c_2c_3}}t, \frac{b}{\sqrt{a^2+b^2}}\right) \tag{10.15}$$

$$\text{and/or } P_1(t) = \frac{ab}{\sqrt{a^2+b^2}} \text{ sd}\left(\sqrt{\frac{(a^2+b^2)(c_1-c_2)}{c_1c_2c_3}}t, \frac{b}{\sqrt{a^2+b^2}}\right) \tag{10.16}$$

$$P_2(t) = \pm\sqrt{K - P_1^2(t)} \tag{10.17}$$

$$P_3(t) = \pm\sqrt{\frac{c_3}{c_1c_2}((c_1-c_2)P_1^2(t) - c_1K + 2c_1c_2H)}. \tag{10.18}$$

where $a^2 = \frac{2c_1c_2H - c_1K}{c_1-c_2}$ *and* $b^2 = K$.

Case 1 (ii): $2c_2H < K$.

$$P_1(t) = b \text{ nd}\left(\sqrt{\frac{(c_1-c_2)}{c_1c_2c_3}}at, \frac{\sqrt{a^2-b^2}}{a}\right) \tag{10.19}$$

$$\text{and/or } P_1(t) = a \text{ dn}\left(\sqrt{\frac{(c_1-c_2)}{c_1c_2c_3}}at, \frac{\sqrt{a^2-b^2}}{a}\right) \tag{10.20}$$

$$P_2(t) = \pm\sqrt{K - P_1^2(t)} \tag{10.21}$$

$$P_3(t) = \pm\sqrt{\frac{c_3}{c_1c_2}((c_1-c_2)P_1^2(t) - c_1K + 2c_1c_2H)}. \tag{10.22}$$

where $a^2 = K$ *and* $b^2 = -\frac{2c_1c_2H - c_1K}{c_1 - c_2}\cdot$

Case 2: $c_2 > c_1$.

Case 2 (i): $2c_2H > 2c_1H > K$.

$$P_1(t) = b\,\mathrm{sn}\left(\frac{\sqrt{c_2 - c_1}}{\sqrt{c_1c_2c_3}}at, \frac{b}{a}\right) \tag{10.23}$$

$$\textit{and/or } P_1(t) = b\,\mathrm{cd}\left(\frac{\sqrt{c_2 - c_1}}{\sqrt{c_1c_2c_3}}at, \frac{b}{a}\right) \tag{10.24}$$

$$P_2(t) = \pm\sqrt{K - P_1^2(t)} \tag{10.25}$$

$$P_3(t) = \pm\sqrt{\frac{c_3}{c_1c_2}((c_1 - c_2)P_1^2(t) - c_1K + 2c_1c_2H)}. \tag{10.26}$$

where $a^2 = \frac{2c_1c_2H - c_1K}{c_2 - c_1}$ *and* $b^2 = K$.

Case 2 (ii): $2c_2H > K > 2c_1H$.

$$P_1(t) = b\,\mathrm{sn}\left(\frac{\sqrt{c_2 - c_1}}{\sqrt{c_1c_2c_3}}at, \frac{b}{a}\right) \tag{10.27}$$

$$\textit{and/or } P_1(t) = b\,\mathrm{cd}\left(\frac{\sqrt{c_2 - c_1}}{\sqrt{c_1c_2c_3}}at, \frac{b}{a}\right) \tag{10.28}$$

$$P_2(t) = \pm\sqrt{K - P_1^2(t)} \tag{10.29}$$

$$P_3(t) = \pm\sqrt{\frac{c_3}{c_1c_2}((c_1 - c_2)P_1^2(t) - c_1K + 2c_1c_2H)}. \tag{10.30}$$

where $a^2 = K$ *and* $b^2 = \frac{2c_1c_2H - c_1K}{c_2 - c_1}\cdot$

Proof. **Case 1:** $c_1 > c_2$. We have

$$P_3^2 = \frac{c_3}{c_1c_2}((c_1 - c_2)P_1^2 - c_1K + 2c_1c_2H) \quad \text{and} \quad P_2^2 = K - P_1^2. \tag{10.31}$$

We now square equation 10.5 and substitute in equations 10.31 to get

$$\dot{P_1}^2 = \frac{1}{c_3^2}P_2^2P_3^2 \tag{10.32}$$

$$\dot{P_1}^2 = \frac{c_1 - c_2}{c_1c_2c_3}(P_1^2 - \frac{c_1}{(c_1 - c_2)}K + \frac{2c_1c_2}{(c_1 - c_2)}H)(K - P_1^2)\cdot \tag{10.33}$$

Therefore we get that

$$\dot{P}_1 = \sqrt{\frac{(c_1 - c_2)}{c_1c_2c_3}(P_1^2 - \frac{c_1}{(c_1 - c_2)}K + \frac{2c_1c_2}{(c_1 - c_2)}H)(K - P_1^2)}\cdot \tag{10.34}$$

It now follows that

$$t = \frac{\sqrt{c_1c_2c_3}}{\sqrt{c_1-c_2}}\int_0^{P_1}\frac{dP_1}{(P_1^2 - \frac{c_1}{(c_1-c_2)}K + \frac{2c_1c_2}{(c_1-c_2)}H)(K-P_1^2)}. \tag{10.35}$$

Case 1 (i): $2c_2H > K$. First we assume that $2c_2H > K$ and so choose $a^2 = \frac{2c_1c_2H - c_1K}{c_1-c_2}$ and $b^2 = K$. So comparing equation 10.35 with the elliptic integral 2.16 we get that

$$t = \frac{\sqrt{c_1c_2c_3}}{\sqrt{c_1-c_2}}\int_{P_1}^{b}\frac{dP_1}{\sqrt{(a^2+P_1^2)(b^2-P_1^2)}} \tag{10.36}$$

$$t = \frac{\sqrt{c_1c_2c_3}}{\sqrt{c_1-c_2}}\frac{1}{\sqrt{a^2+b^2}}\mathrm{cn}^{-1}\left(\frac{P_1}{b}, \frac{b}{\sqrt{a^2+b^2}}\right). \tag{10.37}$$

Therefore

$$P_1(t) = b\ \mathrm{cn}\left(\sqrt{\frac{(a^2+b^2)(c_1-c_2)}{c_1c_2c_3}}t, \frac{b}{\sqrt{a^2+b^2}}\right). \tag{10.38}$$

Similarly we can compare equation 10.35 with the elliptic integral 2.18 to obtain

$$P_1(t) = \frac{ab}{\sqrt{a^2+b^2}}\ \mathrm{sd}\left(\sqrt{\frac{(a^2+b^2)(c_1-c_2)}{c_1c_2c_3}}t, \frac{b}{\sqrt{a^2+b^2}}\right). \tag{10.39}$$

According to equations 2.16, 2.18 we require that $0 \le x \le b$. So substituting in values for b^2 and x^2 gives

$$0 \le P_1^2 \le K = P_1^2 + P_2^2.$$

This clearly always holds as $P_1^2 < P_1^2 + P_2^2$ for all P_1, P_2.

Case 1 (ii): $K > 2c_2H$. We now assume that $K > 2c_2H$ and so choose $a^2 = K$ and $b^2 = -\frac{2c_1c_2H - c_1K}{c_1-c_2}$. So comparing equation 10.35 with the elliptic integral 2.21 we get that

$$t = \frac{\sqrt{c_1c_2c_3}}{\sqrt{c_1-c_2}}\int_{b}^{P_1}\frac{dP_1}{\sqrt{(a^2-P_1^2)(P_1^2-b^2)}} \tag{10.40}$$

$$t = \frac{\sqrt{c_1c_2c_3}}{\sqrt{c_1-c_2}}\frac{1}{a}\mathrm{nd}^{-1}\left(\frac{P_1}{b}, \frac{\sqrt{a^2-b^2}}{a}\right). \tag{10.41}$$

Therefore

$$P_1(t) = b\ \mathrm{nd}\left(\sqrt{\frac{(c_1-c_2)}{c_1c_2c_3}}at, \frac{\sqrt{a^2-b^2}}{a}\right). \tag{10.42}$$

Similarly we can compare equation 10.35 with the elliptic integral 2.22 to obtain

$$P_1(t) = a\ \mathrm{dn}\left(\sqrt{\frac{(c_1-c_2)}{c_1c_2c_3}}at, \frac{\sqrt{a^2-b^2}}{a}\right). \tag{10.43}$$

According to the equations 2.21, 2.22 we require that $b \leq x \leq a$. So substituting in values for a^2, b^2 and x^2 gives

$$-\frac{2c_1c_2H - c_1K}{c_1 - c_2} \leq P_1^2 \leq K = P_1^2 + P_2^2.$$

Substituting in the values for H and K this condition becomes equivalent to

$$P_1^2 - \frac{c_1c_2}{(c_1 - c_2)c_3}P_3^2 \leq P_1^2 \leq P_1^2 + P_2^2,$$

which is always true as $c_1, c_2, c_3 > 0$ and $c_1 > c_2$ by assumption.

Case 2: $c_2 > c_1$. We start by rewriting equations 10.32, 10.33 as

$$\dot{P_1}^2 = \frac{1}{c_3^2}P_2^2P_3^2 \tag{10.44}$$

$$\dot{P_1}^2 = \frac{c_2 - c_1}{c_1c_2c_3}(-P_1^2 - \frac{c_1}{(c_2 - c_1)}K + \frac{2c_1c_2}{(c_2 - c_1)}H)(K - P_1^2)\cdot \tag{10.45}$$

Therefore we get that

$$\dot{P_1} = \sqrt{\frac{(c_2 - c_1)}{c_1c_2c_3}(-P_1^2 - \frac{c_1}{(c_2 - c_1)}K + \frac{2c_1c_2}{(c_2 - c_1)}H)(K - P_1^2)}\cdot \tag{10.46}$$

It now follows that

$$t = \frac{\sqrt{c_1c_2c_3}}{\sqrt{c_2 - c_1}}\int_0^{P_1} \frac{dP_1}{(-P_1^2 - \frac{c_1}{(c_2-c_1)}K + \frac{2c_1c_2}{(c_2-c_1)}H)(K - P_1^2)}\cdot \tag{10.47}$$

Case 2 (i): $2c_2H > 2c_1H > K$. First we assume that $2c_2H > 2c_1H > K$ and so choose $a^2 = \frac{2c_1c_2H - c_1K}{c_2 - c_1}$ and $b^2 = K$. So comparing equation 10.47 with the elliptic integral 2.15 we get that

$$t = \frac{\sqrt{c_1c_2c_3}}{\sqrt{c_2 - c_1}}\int_0^{P_1} \frac{dP_1}{\sqrt{(a^2 - P_1^2)(b^2 - P_1^2)}} \tag{10.48}$$

$$t = \frac{\sqrt{c_1c_2c_3}}{\sqrt{c_2 - c_1}}\frac{1}{a}\mathrm{sn}^{-1}\left(\frac{P_1}{b}, \frac{b}{a}\right)\cdot \tag{10.49}$$

Therefore

$$P_1(t) = b\,\mathrm{sn}\left(\frac{\sqrt{c_2 - c_1}}{\sqrt{c_1c_2c_3}}at, \frac{b}{a}\right)\cdot \tag{10.50}$$

Similarly we can compare equation 10.47 with the elliptic integral 2.17 to obtain

$$P_1(t) = b\,\mathrm{cd}\left(\frac{\sqrt{c_2 - c_1}}{\sqrt{c_1c_2c_3}}at, \frac{b}{a}\right)\cdot \tag{10.51}$$

According to the equations 2.15, 2.17 we require that $0 \leq x \leq b < a$. So substituting in values for a^2, b^2 and x^2 gives

$$0 \leq P_1^2 \leq K = P_1^2 + P_2^2 < \frac{2c_1c_2H - c_1K}{c_2 - c_1}\cdot \tag{10.52}$$

Clearly $P_1^2 < P_1^2 + P_2^2$. Now rearranging for $K < \frac{2c_1c_2H - c_1K}{c_2 - c_1}$ we get that this is equivalent to $2c_1H > K$.

Case 2 (ii): $2c_2H > K > 2c_1H$. We assume that $2c_2H > K > 2c_1H$ and so choose $a^2 = K$ and $b^2 = \frac{2c_1c_2H - c_1K}{c_2 - c_1}$. So comparing equation 10.47 with the elliptic integral 2.15 we get that

$$t = \frac{\sqrt{c_1c_2c_3}}{\sqrt{c_2 - c_1}} \int_0^{P_1} \frac{dP_1}{\sqrt{(a^2 - P_1^2)(b^2 - P_1^2)}} \tag{10.53}$$

$$t = \frac{\sqrt{c_1c_2c_3}}{\sqrt{c_2 - c_1}} \frac{1}{a} \mathrm{sn}^{-1}\left(\frac{P_1}{b}, \frac{b}{a}\right). \tag{10.54}$$

Therefore

$$P_1(t) = b\ \mathrm{sn}\left(\frac{\sqrt{c_2 - c_1}}{\sqrt{c_1c_2c_3}} at, \frac{b}{a}\right). \tag{10.55}$$

Similarly we can compare equation 10.47 with the elliptic integral 2.17to obtain

$$P_1(t) = b\ \mathrm{cd}\left(\frac{\sqrt{c_2 - c_1}}{\sqrt{c_1c_2c_3}} at, \frac{b}{a}\right). \tag{10.56}$$

According to the equations 2.15, 2.17 we require that $0 \le x \le b < a$. So substituting in values for a^2, b^2 and x^2 gives

$$0 \le P_1^2 \le \frac{2c_1c_2H - c_1K}{c_2 - c_1} < K. \tag{10.57}$$

Clearly $P_1^2 < P_1^2 + P_2^2$. Now rearranging for $K > \frac{2c_1c_2H - c_1K}{c_2 - c_1}$ we get that this is equivalent to $2c_1H < K$. □

10.3 Stability

The equilibrium states for the system of equations 10.5, 10.6, 10.7, see 2.8.1, are:

$$P_{e_1} = (M, 0, 0), \quad P_{e_2} = (0, M, 0), \quad P_{e_3} = (0, 0, M) \quad \text{and} \quad P_0 = (0, 0, 0).$$

Here $M \in \mathbb{R}\backslash\{0\}$.

10.3.1 THEOREM. *The equilibrium state $P_{e_1} = (M, 0, 0)$, $M \in \mathbb{R}\backslash\{0\}$ has the following behaviour:*

(i) If $c_2 > c_1$, then it is unstable;

(ii) If $c_1 > c_2$, then it is nonlinearly stable.

PROOF. (i) If $c_2 > c_1$, then $P_{e_1} = (M, 0, 0)$ is unstable. Let A be the matrix corresponding to the linearised operator of system 10.5, 10.6, 10.7, as given in theorem 2.8.4. Thus

$$A = \begin{bmatrix} 0 & -\frac{1}{c_3}P_3 & -\frac{1}{c_3}P_2 \\ \frac{1}{c_3}P_3 & 0 & \frac{1}{c_3}P_1 \\ \frac{c_2 - c_1}{c_1c_2}P_2 & \frac{c_2 - c_1}{c_1c_2}P_1 & 0 \end{bmatrix},$$

and then we have that

$$A(P_{e_1}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{M}{c_3} \\ 0 & \frac{c_2-c_1}{c_1c_2}M & 0 \end{bmatrix}.$$

The characteristic polynomial is then given by

$$\mathrm{char}_A(\lambda) = \lambda\left(\lambda^2 - \frac{c_2-c_1}{c_1c_2c_3}M^2\right) = 0.$$

Hence the eigenvalues are as follows:

$$\lambda_1 = 0, \quad \lambda_2 = +\sqrt{\frac{c_2-c_1}{c_1c_2c_3}M^2}, \quad \lambda_3 = -\sqrt{\frac{c_2-c_1}{c_1c_2c_3}M^2}.$$

Now if $c_2 > c_1$ then $\frac{c_2-c_1}{c_1c_2c_3}M^2 > 0$ and so $\mathrm{Re}(\lambda_2) > 0$ or $\mathrm{Re}(\lambda_3) > 0$, depending on whether $M > 0$ or $M < 0$. Hence by theorem 2.8.4 it follows that the equilibrium states $P_{e_1} = (M, 0, 0)$ are unstable for all $M \in \mathbb{R}\backslash\{0\}$ and $c_2 > c_1$.

(ii) If $c_1 > c_2$, then $P_{e_1} = (M, 0, 0)$ is nonlinearly stable. Here we use the energy-Casimir method to prove our claim. Let H_ψ be the energy-Casimir function given by

$$H_\psi(P_1, P_2, P_3) = \frac{1}{2c_1}P_1^2 + \frac{1}{2c_2}P_2^2 + \frac{1}{2c_3}P_3^2 + \psi(\frac{1}{2}(P_1^2 + P_2^2)), \tag{10.58}$$

where $\psi \in C^\infty(\mathbb{R}, \mathbb{R})$. We now calculate the first variation of H_ψ:

$$\begin{aligned} \delta H_\psi &= \frac{d}{dt}(\frac{1}{2c_1}(P_1 + t\delta_1)^2 + \frac{1}{2c_2}(P_2 + t\delta_2)^2 + \frac{1}{2c_3}(P_3 + t\delta_3)^2 + t\delta_2)\Big|_{t=0} \\ &+ \frac{d}{dt}\psi(\frac{1}{2}(P_1 + t\delta_1)^2 + \frac{1}{2}(P_1 + t\delta_1)^2)\Big|_{t=0} \\ &= \frac{1}{c_1}P_1\delta_1 + \frac{1}{c_2}P_2\delta_2 + \frac{1}{c_3}P_3\delta_3 + (\delta_1 P_1 + \delta_2 P_2)\dot{\psi}(\frac{1}{2}(P_1^2 + P_2^2)) \end{aligned} \tag{10.59}$$

and this equals zero at $P_{e_1} = (M, 0, 0)$ if and only if

$$\frac{M\delta_1}{c_1} + \delta_1 M\dot{\psi}(\frac{1}{2}M^2) = 0,$$

$$\dot{\psi}(\frac{1}{2}M^2) = -\frac{1}{c_1}. \tag{10.60}$$

Now, using 10.59, we can calculate the second variation of H_ψ:

$$\delta^2 H_\psi = \frac{1}{c_1}\delta_1^2 + \frac{1}{c_2}\delta_2^2 + \frac{1}{c_3}\delta_3^2 + (\delta_1^2 + \delta_2^2)\dot{\psi}(\frac{1}{2}(P_1^2 + P_2^2)) + (\delta_1 P_1 + \delta_2 P_2)^2\ddot{\psi}(\frac{1}{2}(P_1^2 + P_2^2)). \tag{10.61}$$

The second variation at the equilibrium of interest is

$$\delta^2 H_\psi(M, 0, 0) = M^2\ddot{\psi}(\frac{1}{2}M^2)\delta_1^2 + (\frac{1}{c_2} - \frac{1}{c_1})\delta_2^2 + \frac{1}{c_3}\delta_3^2. \tag{10.62}$$

This can be represented in matrix form as

$$\begin{bmatrix}\delta_1 & \delta_2 & \delta_3\end{bmatrix}\begin{bmatrix}M^2\ddot{\psi}(\frac{1}{2}M^2) & 0 & 0\\ 0 & (\frac{1}{c_2}-\frac{1}{c_1}) & 0\\ 0 & 0 & \frac{1}{c_3}\end{bmatrix}\begin{bmatrix}\delta_1\\ \delta_2\\ \delta_3\end{bmatrix}$$

and is positive definite if and only if each entry along the diagonal is positive. We note that the matrix can only be positive definite as $c_3 > 0$ by assumption. Therefore the matrix is positive definite if and only if

$$\ddot{\psi}(\frac{1}{2}M^2) > 0, \tag{10.63}$$

$$c_1 > c_2. \tag{10.64}$$

Consequently, $\psi(x) = x(x - c_1 - M^2)$ is such a function that satisfies the above conditions 10.60 and 10.63. Hence this quadratic form is positive definite for all $c_1 > c_2$ and so the equilibrium states $P_{e_1} = (M, 0, 0)$ are nonlinearly stable for all $M \in \mathbb{R}\backslash\{0\}$ and $c_1 > c_2$. □

10.3.2 THEOREM. *The equilibrium state $P_{e_2} = (0, M, 0)$, $M \in \mathbb{R}\backslash\{0\}$ has the following behaviour:*

(i) If $c_1 > c_2$, then it is unstable;

(ii) If $c_2 > c_1$, then it is nonlinearly stable.

PROOF. (i) If $c_1 > c_2$, then $P_{e_2} = (0, M, 0)$ is unstable. Let A be the matrix corresponding to the linear system of 10.5, 10.6, 10.7, as given in theorem 2.8.4. Thus

$$A(P_{e_2}) = \begin{bmatrix}0 & 0 & -\frac{M}{c_3}\\ 0 & 0 & 0\\ \frac{c_2-c_1}{c_1c_2}M & 0 & 0\end{bmatrix}.$$

The characteristic polynomial is then given by

$$\operatorname{char}_A(\lambda) = \lambda\left(\lambda^2 - \frac{c_1 - c_2}{c_1c_2c_3}M^2\right) = 0.$$

Hence the eigenvalues are as follows:

$$\lambda_1 = 0, \quad \lambda_2 = +\sqrt{\frac{c_1 - c_2}{c_1c_2c_3}M^2}, \quad \lambda_3 = -\sqrt{\frac{c_1 - c_2}{c_1c_2c_3}M^2}.$$

Now if $c_1 > c_2$ then $\frac{c_1-c_2}{c_1c_2c_3}M^2 > 0$ and so $\mathrm{Re}(\lambda_2) > 0$ or $\mathrm{Re}(\lambda_3) > 0$, depending on whether $M > 0$ or $M < 0$. Hence by theorem 2.8.4 it follows that the equilibrium states $P_{e_2} = (0, M, 0)$ are unstable for all $M \in \mathbb{R}\backslash\{0\}$ and $c_1 > c_2$.

(ii) If $c_2 > c_1$, then $P_{e_1} = (M, 0, 0)$ is nonlinearly stable. Here we use the energy-Casimir method to prove our claim. Let H_ψ be the energy-Casimir function given by

$$H_\psi(P_1, P_2, P_3) = \frac{1}{2c_1}P_1^2 + \frac{1}{2c_2}P_2^2 + \frac{1}{2c_3}P_3^2 + \psi(\frac{1}{2}(P_1^2 + P_2^2)), \tag{10.65}$$

where $\psi \in C^{\infty}(\mathbb{R}, \mathbb{R})$. We now calculate the first variation of H_{ψ}:

$$\begin{aligned}
\delta H_{\psi} &= \frac{d}{dt}\Big(\frac{1}{2c_1}(P_1 + t\delta_1)^2 + \frac{1}{2c_2}(P_2 + t\delta_2)^2 + \frac{1}{2c_3}(P_3 + t\delta_3)^2 + t\delta_2)\Big|_{t=0} \\
&+ \frac{d}{dt}\psi(\frac{1}{2}(P_1 + t\delta_1)^2 + \frac{1}{2}(P_1 + t\delta_1)^2)\Big|_{t=0} \\
&= \frac{1}{c_1}P_1\delta_1 + \frac{1}{c_2}P_2\delta_2 + \frac{1}{c_3}P_3\delta_3 + (\delta_1 P_1 + \delta_2 P_2)\dot{\psi}(\frac{1}{2}(P_1^2 + P_2^2)) \qquad (10.66)
\end{aligned}$$

and this equals zero at $P_{e_2} = (0, M, 0)$ if and only if

$$\begin{aligned}
\frac{M\delta_2}{c_2} + \delta_2 M \dot{\psi}(\frac{1}{2}M^2) &= 0, \\
\dot{\psi}(\frac{1}{2}M^2) &= -\frac{1}{c_2}\cdot \qquad (10.67)
\end{aligned}$$

Now, using 10.66, we can calculate the second variation of H_{ψ}:

$$\delta^2 H_{\psi} = \frac{1}{c_1}\delta_1^2 + \frac{1}{c_2}\delta_2^2 + \frac{1}{c_3}\delta_3^2 + (\delta_1^2 + \delta_2^2)\dot{\psi}(\frac{1}{2}(P_1^2 + P_2^2)) + (\delta_1 P_1 + \delta_2 P_2)^2 \ddot{\psi}(\frac{1}{2}(P_1^2 + P_2^2)). \qquad (10.68)$$

The second variation at the equilibrium of interest is

$$\delta^2 H_{\psi}(M, 0, 0) = (\frac{1}{c_1} - \frac{1}{c_2})\delta_1^2 + M^2 \ddot{\psi}(\frac{1}{2}M^2)\delta_2^2 + \frac{1}{c_3}\delta_3^2. \qquad (10.69)$$

This can be represented in matrix form as

$$\begin{bmatrix} \delta_1 & \delta_2 & \delta_3 \end{bmatrix} \begin{bmatrix} (\frac{1}{c_1} - \frac{1}{c_2}) & 0 & 0 \\ 0 & M^2\ddot{\psi}(\frac{1}{2}M^2) & 0 \\ 0 & 0 & \frac{1}{c_3} \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{bmatrix}$$

and is positive definite if and only if each of the entries on the diagonal are positive. We note that the matrix can only be positive definite as $c_3 > 0$ by assumption. Therefore the matrix is positive definite if and only if

$$\begin{aligned}
c_2 &> c_1, \qquad (10.70) \\
\ddot{\psi}(\frac{1}{2}M^2) &> 0. \qquad (10.71)
\end{aligned}$$

Consequently, $\psi(x) = x(x - c_1 - M^2)$ is such a function that satisfies the above conditions 10.67 and 10.71. Hence this quadratic form is positive definite for all $c_2 > c_1$ and so the equilibrium states $P_{e_1} = (M, 0, 0)$ are nonlinearly stable for all $M \in \mathbb{R}\backslash\{0\}$ and $c_2 > c_1$. □

10.3.3 Theorem. *The equilibrium states* $P_0 = (0, 0, 0)$ $P_{e_3} = (0, 0, M)$ *are stable for all* $M \in \mathbb{R}\backslash\{0\}$.

Proof. We use theorem 2.8.7 to prove that these equilibrium states are stable. Let $L = \lambda_0 H + \lambda_1 K$, where $\lambda_0 = 0$, $\lambda_1 = 1$ and $K = P_1^2 + P_2^2$ is a Casimir function. Then for all

$M \in \mathbb{R}$ $dL(0,0,M) = [P_1\ P_2\ 0]|_{(0,0,M)} = 0$. The Hessian is then given by

$$dL^2(0,0,M) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Now clearly $\ker dK(P_{e_2}) = \mathbb{R}^3$. We have that $dH(0,0,M) = \left.\begin{bmatrix} \frac{P_1}{c_1} & \frac{P_2}{c_2} & \frac{P_3}{c_3} \end{bmatrix}\right|_{(0,0,M)} = \begin{bmatrix} 0 & 0 & \frac{M}{c_3} \end{bmatrix}$. Therefore we have that $\ker dH(P_{e_2}) = \text{span}\,\{(1,0,0),(0,1,0)\}$ and so

$$W = \text{span}\,\{(1,0,0),(0,1,0)\}.$$

The restricted Hessian is then given by

$$dL^2(0,0,M)\big|_{W \times W} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

which is positive definite. So we have found constants λ_0, λ_1 such that L satisfies conditions 2.39 and 2.40 of theorem 2.8.7. Hence the equilibrium states $P_0 = (0,0,0)$ and $P_{e_3} = (0,0,M)$ for all $M \in \mathbb{R}\backslash\{0\}$ are stable. □

Table 10.1:

Stability		
Equilibrium state	Conditions	Stable/Unstable
$P_{e_1} = (M,0,0)$	$M \in \mathbb{R}\backslash\{0\}$, $c_2 > c_1$ $M \in \mathbb{R}\backslash\{0\}$, $c_1 > c_2$	Unstable Stable
$P_{e_2} = (0,M,0)$	$M \in \mathbb{R}\backslash\{0\}$, $c_1 > c_2$ $M \in \mathbb{R}\backslash\{0\}$, $c_2 > c_1$	Unstable Stable
$P_{e_3} = (0,0,M)$	$M \in \mathbb{R}\backslash\{0\}$	Stable
$P_0 = (0,0,0)$		Stable

Chapter 11

Conclusion

In this chapter we compare the (normal) extremal equations, solutions to the extremal equations and stability results of the six types of problems from chapters $5-10$. We do this by allowing $c_2 \to \infty$ in the cases type II-a, type II-b and type III^0 and then compare the results with the cases type I-a, type I-b and type II^0, respectively. We then discuss the plots produced in Appendix C, that is, the plots produced by the Matlab ODE45 solver, the intersection of the Hamiltonian functions and the coadjoint orbits and the Jacobi elliptic functions. Lastly, we give some final remarks concerning the work that was accomplished in the thesis.

11.1 Comparison of type I-a and type II-a

11.1.1 The normal extremal equations

We allow $c_2 \to \infty$ in theorem 8.1.2. The extremal equations then become

$$\begin{aligned} P_1 &= \frac{1}{c_3}P_2P_3 \\ P_2 &= -\frac{1}{c_3}P_1P_3 \\ P_3 &= \lim_{c_2\to\infty}\frac{1}{c_2}P_1P_2 - P_2 = -P_2 \end{aligned}$$

and the optimal Hamiltonian becomes $H = \frac{1}{2c_3}P_3^2 + P_1$ which corresponds with theorem 5.1.2.

Also allowing $c_2 \to \infty$ in theorem 8.2.1, **case 2:** $H - c_2 > -\sqrt{H^2-K}$ becomes obsolete. For **case 1:** $H - c - 2 < -\sqrt{H^2-K}$ we get the following changes to the conditions: $H > \sqrt{H^2-K}$ remains unchanged,

$c_2 - H > -\sqrt{H^2-K} \implies {}^{c_2\to\infty} 1 > 0$, and

$H - c_2 < -sqrtH^2 - K \implies {}^{c_2\to\infty} -1 < 0$.

These constraints are the now the constraints of theorem 5.2.1. Now for the values of a^2 and b^2 (theorem 8.2.1) as $c_2 \to \infty$ we get that:

$$a^2 = \frac{H+\sqrt{H^2-K}}{H-\sqrt{H^2-K}}, \text{ and}$$

$$b^2 = \lim_{c_2 \to \infty} -\frac{H - c_2 + \sqrt{H^2 - K}}{c_2 - H + \sqrt{H^2 - K}} = 1,$$

which are the values of a^2 and b^2 in theorem 5.2.1. For the condition $A_1A_2 > 0$ in theorem 8.2.1 we get the following

$$A_1A_2 = \lim_{c_2 \to \infty} c_2 \frac{(1 - \frac{-H+\sqrt{H^2+K}}{c_2})(H - \sqrt{H^2 - K})}{4\sqrt{H^2 - K}}.$$

We now let $\tilde{A}_1\tilde{A}_2 = \frac{1}{c_2}A_1A_2$, then $\tilde{A}_1\tilde{A}_2 = \lim_{c_2 \to \infty} \frac{(1-\frac{-H+\sqrt{H^2+K}}{c_2})(H-\sqrt{H^2-K})}{4\sqrt{H^2-K}} = \frac{(H-\sqrt{H^2-K})}{4\sqrt{H^2-K}}$ which corresponds with the condition $A_1A_2 > 0$ of theorem 5.2.1. The reason we choose to take $\tilde{A}_1\tilde{A}_2 = \frac{1}{c_2}A_1A_2$ is because now the value for P_1 becomes

$$\begin{aligned} P_1 &= \lim_{c_2 \to \infty} \frac{\alpha - \beta a \operatorname{dc}\left(\frac{(\alpha-\beta)\sqrt{c_2}\sqrt{\tilde{A}_1\tilde{A}_2 a t}}{\sqrt{c_2 c_3}}\right)}{1 - a \operatorname{dc}\left(\frac{(\alpha-\beta)\sqrt{c_2}\sqrt{\tilde{A}_1\tilde{A}_2 a t}}{\sqrt{c_2 c_3}}\right)} \\ &= \frac{\alpha - \beta a \operatorname{dc}\left(\frac{(\alpha-\beta)\sqrt{\tilde{A}_1\tilde{A}_2 a t}}{\sqrt{c_3}}\right)}{1 - a \operatorname{dc}\left(\frac{(\alpha-\beta)\sqrt{\tilde{A}_1\tilde{A}_2 a t}}{\sqrt{c_3}}\right)}. \end{aligned}$$

These correspond exactly to the solutions of the extremal equations in theorem 5.2.1. Therefore, allowing $c_2 \to \infty$, **case 1** of theorem 8.2.1 reduces to theorem 5.2.1.

11.1.2 Stability

We allow $c_2 \to \infty$ in theorems 8.3.1, 8.3.2, 8.3.3, 8.3.4 to produce the following: for theorem 8.3.1 the equilibrium states are $P_{e_1} = (M, 0, 0)$ all $M \in \mathbb{R}\backslash\{0\}$. For **case (i)** of theorem 8.3.1 we have that $P_{e_1} = (M, 0, 0)$ for all $M \in \mathbb{R}\backslash\{0\}$ is unstable for $c_2 > M$, for $c_2 \to \infty$ this becomes the statement of **case (i)** of theorem 5.3.1, i.e., $P_{e_1} = (M, 0, 0)$ is unstable for $M > 0$. For **case (ii)** of theorem 8.3.1 we have that $P_{e_1} = (M, 0, 0)$ is stable for $M \in (-\infty, 0) \cup (c_2, \infty)$, for $c_2 \to \infty$ this becomes the statement of **case (ii)** of theorem 5.3.1. Thus, for $c_2 \to \infty$, theorem 8.3.1 reduces to theorem 5.3.1.

For theorem 8.3.2 the equilibrium states are $P_{e_2} = (c_2, M, 0)$ all $M \in \mathbb{R}\backslash\{0\}$. As $c_2 \to \infty$ this case becomes obsolete. For theorem 8.3.3 the equilibrium states are $P_{e_1} = (0, 0, M)$ all $M \in \mathbb{R}\backslash\{0\}$ and are all stable. As $c_2 \to \infty$ this statement becomes exactly that of theorem 5.3.2. Thus, as $c_2 \to \infty$, theorem 8.3.3 reduces to theorem 5.3.2.

For theorem 8.3.4 the equilibrium state is $P_{e_1} = (0, 0, 0)$ and is unstable. As $c_2 \to \infty$ this statement becomes exactly that of theorem 5.3.3. Thus, as $c_2 \to \infty$, theorem 8.3.4 reduces to theorem 5.3.3.

11.2 Comparison of type I-b and type II-b

11.2.1 The normal extremal equations

We allow $c_2 \to \infty$ in theorem 9.1.2. Clearly then the extremal equations become

$$\begin{aligned}
\dot{P}_1 &= P_2 \\
\dot{P}_2 &= -P_1 \\
\dot{P}_3 &= -\frac{1}{c_1} P_1 P_2
\end{aligned}$$

and the Hamiltonian becomes $H = \frac{1}{2c_1} P_1^2 + P_3$ which corresponds with theorem 6.1.2. Also allowing $c_2 \to \infty$ in theorem 9.2.1, we get that

$$\begin{aligned}
P_1 &= \sqrt{K} \cos(\sqrt{K} t) \\
P_2 &= -\sqrt{K} \sin(\sqrt{K} t) \\
P_3 &= \lim_{c_2 \to \infty} \sqrt{K} (\frac{1}{c_1} - \frac{1}{c_2}) \sin^2(\sqrt{K} t) = \frac{\sqrt{K}}{c_1} \sin^2(\sqrt{K} t).
\end{aligned}$$

These correspond exactly to the solutions of the extremal equations in theorem 6.2.1. Therefore, allowing $c_2 \to \infty$, theorem 9.2.1 reduces to theorem 6.2.1.

11.2.2 Stability

Allowing $c_2 \to \infty$ in theorem 9.3.1 does not change the equilibrium states, which are $P_0 = (0, 0, 0)$ and $P_{e_1} = (0, 0, M)$ for all $M \in \mathbb{R} \backslash \{0\}$. All of these equilibrium states are stable by theorem 9.3.1. The extremal equations for the case type I-b have the same equilibrium states and by theorem 6.3.1 they are also all stable. Hence, allowing $c_2 \to \infty$, theorem 9.3.1 reduces to theorem 6.3.1.

11.3 Comparison of type II^0 and type III^0

11.3.1 The normal extremal equations

We allow $c_2 \to \infty$ in theorem 10.1.2. The extremal equations then become

$$\begin{aligned}
P_1 &= \frac{1}{c_3} P_2 P_3 \\
P_2 &= -\frac{1}{c_3} P_1 P_3 \\
P_3 &= \lim_{c_2 \to \infty} (\frac{1}{c_2} - \frac{1}{c_1}) P_1 P_2 = -\frac{1}{c_1} P_1 P_2
\end{aligned}$$

and the optimal Hamiltonian becomes $H = \frac{1}{2c_1} P_1^2 + \frac{1}{2c_3} P_3^2$ which corresponds with theorem 7.1.2. Also allowing $c_2 \to \infty$ in theorem 10.2.2, **case 1:** $c_1 > c_2$ becomes obsolete. For **case 2:** $c_2 > c_1$ we get the following:

for **case (i):** $2c_2H > 2c_1H > K$ we get that

$$\begin{aligned} a^2 &= \lim_{c_2\to\infty} \frac{2c_1c_2H - c_1K}{c_2 - c_1} = 2c_1H \\ b^2 &= K. \end{aligned}$$

We then get that the solutions to the extremal equations become

$$P_1 = \lim_{c_2\to\infty} b\,\mathrm{sn}\left(\frac{\sqrt{c_2-c_1}}{\sqrt{c_1c_2c_3}}at,\, \frac{b}{a}\right) = b\mathrm{sn}\left(\frac{1}{\sqrt{c_1c_3}}at, \frac{b}{a}\right),$$

which corresponds exactly to the solutions of the extremal equations in **case 2** of theorem 7.2.1. Therefore, allowing $c_2 \to \infty$, **case 2 (i)** of theorem 10.2.2 reduces to **case 2** of theorem 7.2.1.

For **case 2 (ii):** $2c_2H > K > 2c_1H$ we get that

$$\begin{aligned} a^2 &= K \\ b^2 &= \lim_{c_2\to\infty} \frac{2c_1c_2H - c_1K}{c_2 - c_1} = 2c_1H. \end{aligned}$$

We then get that the solutions to the extremal equations become

$$P_1 = \lim_{c_2\to\infty} b\,\mathrm{sn}\left(\frac{\sqrt{c_2-c_1}}{\sqrt{c_1c_2c_3}}at,\, \frac{b}{a}\right) = b\,\mathrm{sn}\left(\frac{1}{\sqrt{c_1c_3}}at, \frac{b}{a}\right),$$

which corresponds exactly to the solutions of the extremal equations in **case 1** of theorem 7.2.1. Therefore, allowing $c_2 \to \infty$, **case 2 (ii)** of theorem 10.2.2 reduces to **case 1** of theorem 7.2.1.

11.3.2 Stability

We allow $c_2 \to \infty$ in theorems 10.3.1, 10.3.2, 10.3.3 to produce the following: for theorem 10.3.1 the equilibrium states become $P_{e_1} = (0, 0, M)$ for all $M \in \mathbb{R}\backslash\{0\}$. **Case (ii)** of theorem 10.3.1, $c_1 > c_2$, becomes obsolete. For **case (i)** of theorem 10.3.1 we have that $P_{e_1} = (0, 0, M)$ for all $M \in \mathbb{R}\backslash\{0\}$ is unstable for $c_2 > c_1$, which is exactly the statement of theorem 7.3.1.

For theorem 10.3.2 the equilibrium states become $P_{e_2} = (0, M, 0)$ for all $M \in \mathbb{R}\backslash\{0\}$. **Case (ii)** of theorem 10.3.2, $c_1 > c_2$, becomes obsolete. For **case (i)** of theorem 10.3.2 we have that $P_{e_2} = (0, M, 0)$ for all $M \in \mathbb{R}\backslash\{0\}$ is nonlinearly stable for $c_2 > c_1$, which is exactly the statement of theorem 7.3.2.

For theorem 10.3.3 the equilibrium states are $P_0 = (0, 0, 0)$ and $P_{e_3} = (0, 0, M)$ for all $M \in \mathbb{R}\backslash\{0\}$. Theorem 10.3.3 gives that all of these equilibrium states are stable, which is exactly the statement of theorem 7.3.3.

11.4 The solution curves of the normal extremal equations

We produced numerical outputs of the solutions to the (normal) extremal equations using three approaches for each of the six types of chapters $5 - 10$. It is well known that the integral curves(i.e. solutions of the extremal equations) are the intersections of level sets of the Hamiltonian function with the symplectic leaves of the dual of the Lie algebra $\mathfrak{g}^*$ (i.e. the coadjoint orbits).

Interestingly, this method of intersections of the Hamiltonian functions and the coadjoint orbits helped us to see that in many cases there are two curves produced for the solution to the extremal equations. This helped to determine when the negative curve needed to be plotted as well when using the Matlab ODE45 Solver and the Jacobi elliptic functions. Plotting the solution curves to the extremal equations, using Matlab, also helped greatly in the calculations and oversights made for the elliptic/trigonometric functions in theorems 5.1.2, 6.1.2, 7.1.2, 8.1.2, 9.1.2, 10.1.2 and 10.2.1. In most cases, the output of the elliptic/trigonometric functions produced very accurate results, in particular for type I-a, type I-b, type II^0 and type II-b. For **case 1** of type II-a, the curves produced by the elliptic functions dc and ns produce accurate results. However, for **case 2** of type II-a the curve produced by the elliptic function nc was accurate, where as the curve produced by the elliptic function ds was not entirely accurate. Similarly, for type III^0, the elliptic functions sd, dn and nd did not produce completely accurate results. Each of the elliptic functions are obtained from an elliptic integral, which has a certain condition that has to be met. The errors in the solution curves mentioned above occur in these particular cases as the condition of the elliptic integral is violated for some values of P_1, P_2 or P_3.

11.5 Final remarks

In this thesis we proved an algebraic criterion for the state space equivalence of two full-rank control systems on a 3-dimensional matrix Lie group. We found the equivalence classes, under state space equivalence, of all single-input non-homogeneous and two-input homogeneous control affine systems on $\mathsf{SE}(2)$. The conditions for two control affine systems to be state space equivalent, for two-input non-homogeneous and three-input homogeneous control affine systems on $\mathsf{SE}(2)$, were determined. We classified, under detached feedback equivalence, all controllable control affine systems on $\mathsf{SE}(2)$. From this result, we obtained six associated left-invariant optimal control problems on $\mathsf{SE}(2)$. For each LiCP we determined the optimal control and optimal Hamiltonian corresponding to the normal extremals. Also, we derived and solved the normal extremal equations. We got explicit solutions to the normal extremal equations using trigonometric and/or Jacobi elliptic functions. We confirmed our results by plotting these functions in Matlab and comparing them to plots using the Matlab ODE45 Solver and intersections of the Hamiltonian functions and coadjoint orbits. Furthermore, we fully classified, under Lyapunov stability, the equilibrium states of the normal extremal equations for each of the six types.

Appendix A

Mathematica code

Given below is the Wolfram Mathematica code used throughout this thesis.

The code used in proposition 4.2.8 **and** 4.2.10**:**

a = M1 , c = M2

a = {{a1, a2, a3}, {b1, b2, b3}, {c1, c2, c3}}

{{a1, a2, a3}, {b1, b2, b3}, {c1, c2, c3}}

c = {{a2, −a1, a3}, {b2, −b1, b3}, {c2, −c1, c3}}

{{a2, −a1, a3}, {b2, −b1, b3}, {c2, −c1, c3}}

MatrixForm[*a*]

$$\begin{pmatrix} \mathrm{a1} & \mathrm{a2} & \mathrm{a3} \\ \mathrm{b1} & \mathrm{b2} & \mathrm{b3} \\ \mathrm{c1} & \mathrm{c2} & \mathrm{c3} \end{pmatrix}$$

MatrixForm[*c*]

$$\begin{pmatrix} \mathrm{a2} & -\mathrm{a1} & \mathrm{a3} \\ \mathrm{b2} & -\mathrm{b1} & \mathrm{b3} \\ \mathrm{c2} & -\mathrm{c1} & \mathrm{c3} \end{pmatrix}$$

Det[*a*]/Det[*c*]

1

(*Inverses of the matrices M1 and M2 multiplied by there determinants*)

I1 = Inverse[*a*] ∗ Det[*a*]

{{−b3c2 + b2c3, a3c2 − a2c3, −a3b2 + a2b3}, {b3c1 − b1c3, −a3c1 + a1c3, a3b1 − a1b3}, {−b2c1 + b1c2, a2c1 − a1c2, −a2b1 + a1b2}}

MatrixForm[I1]

$$\begin{pmatrix} -\mathrm{b3c2} + \mathrm{b2c3} & \mathrm{a3c2} - \mathrm{a2c3} & -\mathrm{a3b2} + \mathrm{a2b3} \\ \mathrm{b3c1} - \mathrm{b1c3} & -\mathrm{a3c1} + \mathrm{a1c3} & \mathrm{a3b1} - \mathrm{a1b3} \\ -\mathrm{b2c1} + \mathrm{b1c2} & \mathrm{a2c1} - \mathrm{a1c2} & -\mathrm{a2b1} + \mathrm{a1b2} \end{pmatrix}$$

I2 = Inverse[*c*] ∗ Det[*c*]

{{b3c1 − b1c3, −a3c1 + a1c3, a3b1 − a1b3}, {b3c2 − b2c3, −a3c2 + a2c3, a3b2 − a2b3}, {−b2c1 + b1c2, a2c1 − a1c2, −a2b1 + a1b2}}

MatrixForm[I2]

$$\begin{pmatrix} \mathrm{b3c1}-\mathrm{b1c3} & -\mathrm{a3c1}+\mathrm{a1c3} & \mathrm{a3b1}-\mathrm{a1b3} \\ \mathrm{b3c2}-\mathrm{b2c3} & -\mathrm{a3c2}+\mathrm{a2c3} & \mathrm{a3b2}-\mathrm{a2b3} \\ -\mathrm{b2c1}+\mathrm{b1c2} & \mathrm{a2c1}-\mathrm{a1c2} & -\mathrm{a2b1}+\mathrm{a1b2} \end{pmatrix}$$

The code used in theorem 5.5.1**:**

$\mathrm{sol}=\mathrm{Solve}\{\lambda\text{\textasciicircum}2+2H\lambda+K==0,\lambda\}$

$\left\{\mathrm{Solve}\left(K+2H\lambda+\lambda^2==0\right),\mathrm{Solve}\lambda\right\}$

$\mathrm{sol}=\mathrm{Solve}[\lambda\text{\textasciicircum}2+2H\lambda+K==0,\lambda]$

$\left\{\left\{\lambda\to -H-\sqrt{H^2-K}\right\},\left\{\lambda\to -H+\sqrt{H^2-K}\right\}\right\}$

$\lambda\ 1=-H-\sqrt{H^2-K}$

$-H-\sqrt{H^2-K}$

$\lambda\ 2=-H+\sqrt{H^2-K}$

$-H+\sqrt{H^2-K}$

$\mathrm{a1}=-1,\ \mathrm{b1}=0,\ \mathrm{c1}=K,$

$\mathrm{a2}=0,\ \mathrm{b2}=-1,\ \mathrm{c2}=2H$

$\mathrm{A1}=\lambda2(\mathrm{a1}+\lambda1*\mathrm{a2})/(\lambda\ 2-\lambda\ 1)$

$-\frac{-H+\sqrt{H^2-K}}{2\sqrt{H^2-K}}$

FullSimplify[A1]

$\frac{1}{2}\left(-1+\frac{H}{\sqrt{H^2-K}}\right)$

$\mathrm{B1}=\lambda1(\mathrm{a1}+\lambda2*\mathrm{a2})/(\lambda1-\lambda2)$

$\frac{-H-\sqrt{H^2-K}}{2\sqrt{H^2-K}}$

$\mathrm{A2}=(\mathrm{a1}+\lambda1*\mathrm{a2})/(\lambda1-\lambda2)$

$\frac{1}{2\sqrt{H^2-K}}$

$\mathrm{B2}=(\mathrm{a1}+\lambda2*\mathrm{a2})/(\lambda2-\lambda1)$

$-\frac{1}{2\sqrt{H^2-K}}$

$\mathrm{ForAll}[x,x\text{\textasciicircum}2-2x\alpha+\alpha\text{\textasciicircum}2==x\text{\textasciicircum}2+(2*\mathrm{b1}+\lambda1*2*\mathrm{b2})*x/(\mathrm{a1}+\lambda1*\mathrm{a2})+(\mathrm{c1}+\lambda1*\mathrm{c2})/(\mathrm{a1}+\lambda1*\mathrm{a2})]$

$\forall_x x^2-2x\alpha+\alpha^2==-2H\left(-H-\sqrt{H^2-K}\right)-K+2\left(-H-\sqrt{H^2-K}\right)x+x^2$

$\mathrm{Resolve}[\%,\alpha]$

$\alpha==H+\sqrt{H^2-K}$

$\mathrm{ForAll}[x,x\text{\textasciicircum}2-2x\beta+\beta\text{\textasciicircum}2==x\text{\textasciicircum}2+(2*\mathrm{b1}+\lambda2*2*\mathrm{b2})*x/(\mathrm{a1}+\lambda2*\mathrm{a2})+(\mathrm{c1}+\lambda2*\mathrm{c2})/(\mathrm{a1}+\lambda2*\mathrm{a2})]$

$\forall_x x^2-2x\beta+\beta^2==-2H\left(-H+\sqrt{H^2-K}\right)-K+2\left(-H+\sqrt{H^2-K}\right)x+x^2$

$\mathrm{Resolve}[\%,\beta]$

$\beta==H-\sqrt{H^2-K}$

$\mathrm{U1}=\mathrm{B1}/\mathrm{A1}$

$-\frac{-H-\sqrt{H^2-K}}{-H+\sqrt{H^2-K}}$

$\text{U2} = \text{B2}/\text{A2} = -1$

$\text{A3} = \text{A1} * \text{A2}$

$-\frac{-H+\sqrt{H^2-K}}{4(H^2-K)}$

Sqrt[A3]

$\frac{1}{2}\sqrt{-\frac{-H+\sqrt{H^2-K}}{H^2-K}}$

$\text{Resolve}\left[\text{Exists}\left[\{H,K\},(K>0)\&\&(H>\text{Sqrt}[H{}^\wedge 2-K])\&\&\left(1>\text{Sqrt}\left[\frac{\left(H+\sqrt{H^2-K}\right)}{\left(H-\sqrt{H^2-K}\right)}\right]\right)\right]\right]$

False

The code used in theorem 8.5.1**:**

$\text{sol} = \text{Solve}\{-\text{K}\lambda{}^\wedge 2+(2K-2\text{cH})\lambda+2\text{cH}-K-c*c==0,\lambda\}$

$\left\{\text{Solve}\left(-c^2+2\text{cH}-K-\text{K}\lambda^2+(-2\text{cH}+2K)\lambda==0\right),\text{Solve}\lambda\right\}$

$\text{sol} = \text{Solve}[-K*\lambda{}^\wedge 2+(2*K-2*c*H)*\lambda+2*c*H-K-c*c==0,\lambda]$

$\left\{\left\{\lambda\to\frac{-cH+K-\sqrt{c^2H^2-c^2K}}{K}\right\},\left\{\lambda\to\frac{-cH+K+\sqrt{c^2H^2-c^2K}}{K}\right\}\right\}$

$\lambda 1 = \frac{-cH+K+\sqrt{c^2H^2-c^2K}}{K}$ //FullSimplify

$\frac{-cH+\sqrt{c^2(H^2-K)}+K}{K}$

$\lambda 2 = \frac{-cH+K-\sqrt{c^2H^2-c^2K}}{K}$//FullSimplify

$\frac{-cH-\sqrt{c^2(H^2-K)}+K}{K}$

$\text{a1} = 1;\text{b1} = -c;\text{c1} = 2*c*H-K;\text{a2} = -1;\ \text{b2} = 0;\text{c2} = K;$

$\text{A1} = \lambda 2(\text{a1}+\lambda 1*\text{a2})/(\lambda 2-\lambda 1)$//FullSimplify

$\frac{c^2-cH+\sqrt{c^2(H^2-K)}}{2\sqrt{c^2(H^2-K)}}$

$\text{B1} = \lambda 1(\text{a1}+\lambda 2*\text{a2})/(\lambda 1-\lambda 2)$//FullSimplify

$\frac{c(-c+H)+\sqrt{c^2(H^2-K)}}{2\sqrt{c^2(H^2-K)}}$

$\text{A2} = (\text{a1}+\lambda 1*\text{a2})/(\lambda 1-\lambda 2)$//FullSimplify

$\frac{1}{2}\left(-1+\frac{cH}{\sqrt{c^2(H^2-K)}}\right)$

$\text{B2} = (\text{a1}+\lambda 2*\text{a2})/(\lambda 2-\lambda 1)$//FullSimplify

$\frac{1}{2}\left(-1-\frac{cH}{\sqrt{c^2(H^2-K)}}\right)$

$\text{ForAll}[x,x{}^\wedge 2-2x\alpha+\alpha{}^\wedge 2==x{}^\wedge 2+(2*\text{b1}+\lambda 1*2*\text{b2})*x/(\text{a1}+\lambda 1*\text{a2})+(\text{c1}+\lambda 1*\text{c2})/(\text{a1}+\lambda 1*\text{a2})]$

$\forall_x x^2-2x\alpha+\alpha^2 == \frac{cH+\sqrt{c^2(H^2-K)}}{1-\frac{-cH+\sqrt{c^2(H^2-K)}+K}{K}}-\frac{2cx}{1-\frac{-cH+\sqrt{c^2(H^2-K)}+K}{K}}+x^2$

$\text{res} = \text{Resolve}[\%,\alpha]$

$c\neq 0\&\&\sqrt{c^2\left(H^2-K\right)}\neq cH\&\&H+\frac{\sqrt{c^2(H^2-K)}}{c}==\alpha$

$\text{ForAll}[x,x{}^\wedge 2-2x\beta+\beta{}^\wedge 2==x{}^\wedge 2+(2*\text{b1}+\lambda 2*2*\text{b2})*x/(\text{a1}+\lambda 2*\text{a2})+(\text{c1}+\lambda 2*\text{c2})/(\text{a1}+\lambda 2*\text{a2})]$

$\forall_x x^2 - 2x\beta + \beta^2 == \frac{cH-\sqrt{c^2(H^2-K)}}{1-\frac{-cH-\sqrt{c^2(H^2-K)}+K}{K}} - \frac{2cx}{1-\frac{-cH-\sqrt{c^2(H^2-K)}+K}{K}} + x^2$

$\text{Resolve}[\%, \beta]$

$c \neq 0\&\&cH + \sqrt{c^2\left(H^2-K\right)} \neq 0\&\&\beta == \frac{cH-\sqrt{c^2(H^2-K)}}{c}$

$\text{U1} = -\text{B1}/\text{A1}//\text{FullSimplify}$

$-\frac{c(-c+H)+\sqrt{c^2(H^2-K)}}{c^2-cH+\sqrt{c^2(H^2-K)}}$

$\text{U2} = -\text{B2}/\text{A2}//\text{FullSimplify}$

$\frac{2cH^2+2H\sqrt{c^2(H^2-K)}-cK}{cK}$

$\text{A3} = \text{A1} * \text{A2}//\text{FullSimplify}$

$\frac{\left(cH-\sqrt{c^2(H^2-K)}\right)\left(c^2-cH+\sqrt{c^2(H^2-K)}\right)}{4c^2(H^2-K)}$

The code used in theorem 5.6.2 **and** 8.6.3**:**

$M = \{\{1,0,0\},\{0,1,0\},\{0,0,0\}\};$

$\text{MatrixForm}[M]$

$$\left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$\text{E1} = \{1,0,0\};$

$\text{E2} = \{0,1,0\};$

$\text{E3} = \{0,0,1\};$

$\text{E1t} = \frac{-m}{c}\text{E1} + \text{E3};$

$\text{E2t} = \text{E2};$

$(s\text{E1t} + t\text{E2t}).M.(s\text{E1t} + t\text{E2t})$

$\frac{m^2s^2}{c^2} + t^2$

$\text{Mt} = \{\{\text{a1}, \text{a2}\}, \{\text{b1}, \text{b2}\}\};$

$\{s,t\}.\text{Mt}.\{s,t\}$

$s(\text{a1}s + \text{b1}t) + t(\text{a2}s + \text{b2}t)$

$\text{Reduce}[\{\text{ForAll}[\{s,t\}, (s\text{E1t} + t\text{E2t}).M.(s\text{E1t} + t\text{E2t}) == \{s,t\}.\text{Mt}.\{s,t\}], \text{Transpose}[\text{Mt}] == \text{Mt}\}, \{\text{a1}, \text{a2}, \text{b1}, \text{b2}\}]$

$c \neq 0\&\&\text{a1} == \frac{m^2}{c^2}\&\&\text{a2} == 0\&\&\text{b1} == 0\&\&\text{b2} == 1$

$\text{MR} = \left\{\left\{\frac{m^2}{c^2}, 0\right\}, \{0,1\}\right\}$

$\left\{\left\{\frac{m^2}{c^2}, 0\right\}, \{0,1\}\right\}$

$\text{MatrixForm}[\text{MR}]$

$$\left(\begin{array}{cc} \frac{m^2}{c^2} & 0 \\ 0 & 1 \end{array}\right)$$

The code used in theorem 6.5.1**,** 7.6.3**,** 9.5.1 **and** 10.5.3**:**

```
M = {{1, 0, 0}, {0, 1, 0}, {0, 0, 0}};
MatrixForm[M]
```

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

```
E1 = {1, 0, 0};
E2 = {0, 1, 0};
E3 = {0, 0, 1};
E1t = E1;
E2t = E2;
(sE1t + tE2t).M.(sE1t + tE2t)
```

$s^2 + t^2$

```
Mt = {{a1, a2}, {b1, b2}};
{s, t}.Mt.{s, t}
s(a1s + b1t) + t(a2s + b2t)
Reduce[{ForAll[{s, t}, (sE1t + tE2t).M.(sE1t + tE2t)=={s, t}.Mt.{s, t}], Transpose[Mt] ==
Mt}, {a1, a2, b1, b2}]
a1 == 1&&a2 == 0&&b1 == 0&&b2 == 1
MR = {{1, 0}, {0, 1}}
{{1, 0}, {0, 1}}
MatrixForm[MR]
```

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Appendix B

Matlab code

Type I-a

```
fnc1A.m
function P = fnc1A(t, p, c1)
P = zeros(3,1);
P(1) = (1/c1)*p(2)*p(3);
P(2) = -(1/c1)*p(1)*p(3);
P(3) = -p(2);
end
```

The code used to produce Figure C.3 (a) - (b):

```
c1 = 2;
[t, p] = ode45(@fnc1A,[0 10],[0 4
2*sqrt(5)],[],c1);
plot(t,p(:,1),'-',t,p(:,2),'-.',t,
p(:,3),'.')
hold
plot(t,p(:,1),'-',t,p(:,2),'-.',t,
-p(:,3),'.')
xlabel('Time')
ylabel('Output')
legend('P1','P2','P3')
axis([0 10 -6 6])

plot3(p(:,1),p(:,2),p(:,3))
hold
plot3(p(:,1),p(:,2),-p(:,3))
xlabel('P1')
ylabel('P2')
zlabel('P3')
axis([-4 4 -4 4 -6 6])
```

The code used to produce Figure C.3 (c) - (d):

```
Orb1A.m
[x3, y3, z3] = meshgrid(linspace(-10,
10));

f1 = x3.^2 + y3.^2 - 16;
f2 = (z3.^2)./4 + x3 -5;

[y2, z2] = meshgrid(linspace(-10,10));
x2 = 5 - (z2.^2)./4;
% Visualize the two surfaces.
patch(isosurface(x3, y3, z3, f1, 0),
'FaceColor',[0.5 1.0 0.5],
'EdgeColor', 'none');
patch(isosurface(x3, y3, z3, f2, 0),
'FaceColor',[1.0 0.5 0.0],
'EdgeColor', 'none');
view(3); camlight;

f3 = f1 - f2;

f3s = interp3(x3, y3, z3, f3, x2,
y2, z2);

C = contours(y2, z2, f3s, [0 0]);

yL = C(1, 2:end);
zL = C(2, 2:end);
xL = interp2(y2, z2, x2, yL, zL);

% Visualize the line.
```

```
line(xL,yL,zL,'Color','k','LineWidth'
,1);
xlabel('P1')
ylabel('P2')
zlabel('P3')

%axis([-4 4 -4 4 -6 6]);
axis([-10 10 -10 10 -10 10]);
```

The code used to produce Figure C.4 (a) - (d):

```
H = 5; K = 16; c3 = 2;
alpha = H + sqrt(H.^2 - K);
beta = H - sqrt(H.^2 - K);
A1A2 = sqrt(H - sqrt(H.^2 - K))
./(4.*H.*H - 4.*K);
a = 1; b = sqrt((-H - sqrt(H.^2 - K))
./((-H + sqrt(H.^2 - K))));

Constant = ((alpha - beta).*sqrt(A1A2)
.*b)./(sqrt(c3));
Constant2 = (sqrt(a.^2 + b.^2)
.*(alpha - beta).*sqrt(A1A2))./sqrt(c3);

t = linspace(0,10,100);
[s,c,d] = ellipj(Constant*t,a./b);

%Uncomment other P1 value
%for alternate elliptic function
P1 = (alpha - b.*beta.*(d./c))
./(1 - b.*(d./c));
%P1 = (alpha - b.*beta.*(1./s))
./(1 - b.*(1./s));
P2 = sqrt(K - P1.*P1);
P3 = sqrt(2*c3.*(H - P1));

plot(t,P1,'-',t,P2,'-.',t,P3,'.')
hold
plot(t,P1,'-',t,-P2,'-.',t,P3,'.')
plot(t,P1,'-',t,P2,'-.',t,-P3,'.')
plot(t,P1,'-',t,-P2,'-.',t,-P3,'.')
xlabel('Time')
ylabel('Output')
legend('P1','P2','P3')

plot3(P1,P2,P3)
hold
plot3(P1,-P2,P3)
plot3(P1,P2,-P3)
plot3(P1,-P2,-P3)

xlabel('P1')
ylabel('P2')
zlabel('P3')
```

Type I-b

```
fnc1B.m
function P = fnc1B(t, p, c1)
P = zeros(3,1);
P(1) = p(2);
P(2) = -p(1);
P(3) = -(1/c1).*p(1).*p(2);
end
```

The code used to produce Figure C.5 (a) - (b):

```
c1 =2;
[t, p] = ode45(@fnc1B,[0 10],[0 2 1],[]
,c1);
plot(t,p(:,1),'-',t,p(:,2),'-.',t,
p(:,3),'.')
xlabel('Time')
ylabel('Output')
legend('P1','P2','P3')

plot3(p(:,1),p(:,2),p(:,3))
xlabel('P1')
ylabel('P2')
zlabel('P3')
```

The code used to produce Figure C.5 (c) - (d):

```
[x3, y3, z3] = meshgrid(linspace(-10
,10));

f1 = x3.^2 + y3.^2 - 4;
f2 = (x3.^2)./4 + z3 - 1;

[x2, y2] = meshgrid(linspace(-10,10));
z2 = 1 -(x2.^2)./4;
% Visualize the two surfaces.
patch(isosurface(x3, y3, z3, f1, 0),
'FaceColor',[0.5 1.0 0.5],
```

```
’EdgeColor’, ’none’);
patch(isosurface(x3, y3, z3, f2, 0),
’FaceColor’,[1.0 0.5 0.0],
’EdgeColor’, ’none’);
view(3); camlight;

f3 = f1 - f2;
f3s = interp3(x3, y3, z3, f3, x2,
y2, z2);

C = contours(x2, y2, f3s, [0 0]);

xL = C(1, 2:end);
yL = C(2, 2:end);
zL = interp2(x2, y2, z2, xL, yL);

% Visualize the line.
line(xL,yL,zL,’Color’,’k’,
’LineWidth’,1);
xlabel(’P1’)
ylabel(’P2’)
zlabel(’P3’)
%axis([-2 2 -2 2 0 1]);
axis([-10 10 -10 10 -10 10]);
```

The code used to produce Figure C.5 (e) -(f):

```
K = 4; c1=2;
t = linspace(0,10,100);

P1 = sqrt(K)*sin(t);
P2 = sqrt(K)*cos(t);
P3 = sqrt(K)*cos(t).*cos(t)./c1;

plot(t,P1,’-’,t,P2,’-.’,t,P3,’.’)
xlabel(’Time’)
ylabel(’Output’)
legend(’P1’,’P2’,’P3’)
hold

plot3(P1,P2,P3)
xlabel(’P1’)
ylabel(’P2’)
zlabel(’P3’)
```

Type II^0

```
fnc2ND.m
function P = fnc2ND(t, p, c1, c2)
P = zeros(3,1);
P(1) = (1/c2)*p(2)*p(3);
P(2) = -(1/c2)*p(1)*p(3);
P(3) = -(1/c1)*p(1)*p(2);
end
```

The code used to produce Figure C.6 (a) - (b):

```
c1 =2; c2 =2;
[t, p] = ode45(@fnc2ND,[0 5],[4 sqrt(14)
 1],
[],c1,c2);
plot(t,p(:,1),’-’,t,p(:,2),’-.’,t,
p(:,3),’.’)
hold
plot(t,p(:,1),’-’,t,-p(:,2),’-.’,t,
p(:,3),’.’)
xlabel(’Time’)
ylabel(’Output’)
legend(’P1’,’P2’,’P3’)

plot3(p(:,1),p(:,2),p(:,3))
hold
plot3(p(:,1),-p(:,2),p(:,3))
xlabel(’P1’)
ylabel(’P2’)
zlabel(’P3’)
```

The code used to produce Figure C.6 (c) - (d):

```
[x3, y3, z3] = meshgrid(linspace(-10,
10));

f1 = x3.^2 + y3.^2 - 30;
f2 = (x3.^2)./4 + (z3.^2)./4 - 4.25;

[y2, z2] = meshgrid(linspace(-sqrt(30)
,sqrt(30)));
x2 = sqrt(30 - (y2.^2));
% Visualize the two surfaces.
patch(isosurface(x3, y3, z3, f1, 0),
’FaceColor’, [0.5 1.0 0.5],
’EdgeColor’, ’none’);
patch(isosurface(x3, y3, z3, f2, 0),
’FaceColor’, [1.0 0.5 0.0],
```

```
'EdgeColor', 'none');
view(3); camlight;

f3 = f1 - f2;
f3s = interp3(x3, y3, z3, f3, x2,
 y2, z2);

C = contours(y2, z2, f3s, [0 0]);

yL = C(1, 2:end);
zL = C(2, 2:end);
xL = interp2(y2, z2, x2, yL, zL);

% Visualize the line.
line(xL,yL,zL,'Color','k',
'LineWidth',1);
line(xL,yL,-zL,'Color','k',
'LineWidth',1);
line(-xL,-yL,-zL,'Color','k',
'LineWidth',1);
xlabel('P1')
ylabel('P2')
zlabel('P3')

%axis([-5 5 -10 10 -5 5]);
axis([-10 10 -10 10 -10 10]);
```

The code used to produce Figure C.7 (a) - (d):

```
H = 4.25; K = 30; c1 = 2; c3 = 2;
L1 = sqrt(2.*c1.*H);
L2 = sqrt(2.*c3.*H);
M = sqrt(K);

t = linspace(0,5,100);
[s,c,d] = ellipj(M.*t./4, L1./M);

P1 = L1.*(s);
P2 = sqrt(K - P1.*P1);
P3 = sqrt(2.*c3.*H - P1.*P1);

plot(t,P1,'-',t,P2,'-.',t,P3,'.')
hold
plot(t,P1,'-',t,-P2,'-.',t,-P3,'.')
xlabel('Time')
ylabel('Output')
legend('P1','P2','P3')

plot3(P1,P2,P3)
hold
plot3(P1,P2,-P3)
plot3(P1,-P2,P3)
plot3(P1,-P2,-P3)

xlabel('P1')
ylabel('P2')
zlabel('P3')
```

The code used to produce Figure C.8 (a) - (b):

```
c1 =2; c2 =2;
[t, p] = ode45(@fnc2ND,[0 10],[4 0 1],
[],c1,c2);
plot(t,p(:,1),'-',t,p(:,2),'-.',t,
p(:,3),'.')
hold
plot(t,p(:,1),'-',t,p(:,2),'-.',t,
-p(:,3),'.')
xlabel('Time')
ylabel('Output')
legend('P1','P2','P3')

plot3(p(:,1),p(:,2),p(:,3))
hold
plot3(p(:,1),p(:,2),-p(:,3))
xlabel('P1')
ylabel('P2')
zlabel('P3')
```

The code used to produce Figure C.8 (c) - (d):

```
[x3, y3, z3] = meshgrid(linspace(-10,
10));

f1 = x3.^2 + y3.^2 - 16;
f2 = (x3.^2)./4 + (z3.^2)./4 - 4.25;

[x2, y2] = meshgrid(linspace(-4,4));
z2 = 2*sqrt(4.25 - (x2.^2)./4);
% Visualize the two surfaces.
patch(isosurface(x3, y3, z3, f1, 0),
'FaceColor',[0.5 1.0 0.5],
```

```
’EdgeColor’, ’none’);
patch(isosurface(x3, y3, z3, f2, 0),
’FaceColor’, [1.0 0.5 0.0],
’EdgeColor’, ’none’);
view(3); camlight;

f3 = f1 - f2;
f3s = interp3(x3, y3, z3, f3, x2,
 y2, z2);

C = contours(x2, y2, f3s, [0 0]);

xL = C(1, 2:end);
yL = C(2, 2:end);
zL = interp2(x2, y2, z2, xL, yL);
% Visualize the line.
line(xL,yL,zL,’Color’,’k’,’LineWidth’
,1);

xlabel(’P1’)
ylabel(’P2’)
zlabel(’P3’)

%axis([-5 5 1 5 -5 5]);
axis([-10 10 -10 10 -10 10]);
```

The code used to produce Figure C.9 (a) - (d):

```
H = 4.25; K = 16; c1 = 2; c2 = 2;
L1 = sqrt(2.*c1.*H);
L2 = sqrt(2.*c2.*H);
L3 = 2.*H./c2;
M = sqrt(K);

t = linspace(0,10,100);
[s,c,d] = ellipj(L3.*t./4, M./L1);

%Uncomment other P1 value
%for alternate elliptic function
P1 = sqrt(K).*(s);
%P1 = sqrt(K).*(c./d);
P2 = sqrt(K - P1.*P1);
P3 = sqrt(2.*c2.*H - P1.*P1);

plot(t,P1,’-’,t,P2,’-.’,t,P3,’.’)
hold
plot(t,P1,’-’,t,-P2,’-.’,t,-P3,’.’)
xlabel(’Time’)
ylabel(’Output’)
legend(’P1’,’P2’,’P3’)

plot3(P1,P2,P3)
hold
plot3(P1,-P2,P3)
plot3(P1,-P2,-P3)
plot3(P1,P2,-P3)

xlabel(’P1’)
ylabel(’P2’)
zlabel(’P3’)
```

Type II-a

```
fnc2A.m
function P = fnc2A(t, p, c2, c3)
P = zeros(3,1);
P(1) = (1/c3)*p(2)*p(3);
P(2) = -(1/c3)*p(1)*p(3);
P(3) = (1/c2)*p(1)*p(2) - p(2);
end
```

The code used to produce Figure C.10 (a)-(b):

```
c2 =5; c3 =4;
[t, p] = ode45(@fnc2A,[0 15],[1 3
 sqrt(52./5)],
[],c2,c3);
plot(t,p(:,1),’-’,t,p(:,2),’-.’,t
,p(:,3),’.’)
hold
plot(t,p(:,1),’-’,t,p(:,2),’-.’,t
-p(:,3),’.’)
xlabel(’Time’)
ylabel(’Output’)
legend(’P1’,’P2’,’P3’)

plot3(p(:,1),p(:,2),p(:,3))
hold
plot3(p(:,1),p(:,2),-p(:,3))
xlabel(’P1’)
ylabel(’P2’)
zlabel(’P3’)
```

The code used to produce Figure C.10 (c)-(d):

```
[x3, y3, z3] = meshgrid(linspace(-10
,10));

f1 = x3.^2 + y3.^2 - 10;
f2 = (y3.^2)./10 + (z3.^2)./8 + x3 -3.2;

[y2, z2] = meshgrid(linspace(-10,10));
x2 = 3.2 - (z2.^2)./8 - (y2.^2)./10;
% Visualize the two surfaces.
patch(isosurface(x3, y3, z3, f1, 0),
’FaceColor’, [0.5 1.0 0.5],
’EdgeColor’, ’none’);
patch(isosurface(x3, y3, z3, f2, 0),
’FaceColor’, [1.0 0.5 0.0],
’EdgeColor’, ’none’);
view(3); camlight;

f3 = f1 - f2;
f3s = interp3(x3, y3, z3, f3, x2,
 y2, z2);

C = contours(y2, z2, f3s, [0 0]);

yL = C(1, 2:end);
zL = C(2, 2:end);
xL = interp2(y2, z2, x2, yL, zL);

% Visualize the line.
line(xL,yL,zL,’Color’,’k’,’LineWidth’
,1);
xlabel(’P1’)
ylabel(’P2’)
zlabel(’P3’)

%axis([-4 4 -4 4 -8 8]);
axis([-10 10 -10 10 -10 10]);
```

The code used to produce Figure C.11 (a)-(d):

```
H = 3.2; K = 10; c2 = 5; c3=4;
alpha = H + sqrt(H.^2 - K);
beta = H - sqrt(H.^2 - K);
A1A2 = (H - sqrt(H.^2 - K))
./(4.*H.*H - 4.*K);
b = sqrt(-(H - c2 + sqrt(H.^2 - K))
./(c2 - H + sqrt(H.^2 - K)));
a = sqrt((H + sqrt(H.^2 - K))
./((H - sqrt(H.^2 - K))));

Constant = a.*(alpha - beta)
.*sqrt(A1A2)./sqrt(c2.*c3);
Constant2 = (sqrt(a.^2 + b.^2)
.*(alpha - beta).*sqrt(A1A2))./sqrt(c2);

t = linspace(0,15,100);
[s,c,d] = ellipj(Constant*t,b./a);

%Uncomment other P1 value
%for alternate elliptic function
P1 = (alpha - a.*beta.*(1./s))
./(1 - a.*(1./s));
%P1 = (alpha - a.*beta.*(d./c))
./(1 - a.*(d./c));
P2 = sqrt(K - P1.*P1);
P3 = sqrt(c3.*P1.*P1./c2 - 2.*c3.*P1
+ 2.*c3.*H - c3.*K./c2);

plot(t,P1,’-’,t,P2,’-.’,t,P3,’.’)
hold
plot(t,P1,’-’,t,-P2,’-.’,t,-P3,’.’)
xlabel(’Time’)
ylabel(’Output’)
legend(’P1’,’P2’,’P3’)

plot3(P1,P2,P3)
hold
plot3(P1,-P2,P3)
plot3(P1,P2,-P3)
plot3(P1,-P2,-P3)
xlabel(’P1’)
ylabel(’P2’)
zlabel(’P3’)
```

The code used to produce Figure C.12 (a) - (b):

```
c2 =5; c3 =4;
[t, p] = ode45(@fnc2A,[0 15],[1 3
 sqrt(84./5)],
[],c2,c3);
plot(t,p(:,1),’-’,t,p(:,2),’-.’,t,
```

```
p(:,3),’.’)
hold
plot(t,p(:,1),’-’,t,p(:,2),’-.’,t,
-p(:,3),’.’)
xlabel(’Time’)
ylabel(’Output’)
legend(’P1’,’P2’,’P3’)

plot3(p(:,1),p(:,2),p(:,3))
hold
plot3(p(:,1),p(:,2),-p(:,3))
xlabel(’P1’)
ylabel(’P2’)
zlabel(’P3’)
```

The code used to produce Figure C.12 (c) - (d):

```
[x3, y3, z3] = meshgrid(linspace(-10
,10));

f1 = x3.^2 + y3.^2 - 10;
f2 = (y3.^2)./10 + (z3.^2)./8 + x3 - 4;

[y2, z2] = meshgrid(linspace(-10,10));
x2 = 4 - (z2.^2)./8 - (y2.^2)./10;
% Visualize the two surfaces.
patch(isosurface(x3, y3, z3, f1, 0),
’FaceColor’, [0.5 1.0 0.5],
’EdgeColor’, ’none’);
patch(isosurface(x3, y3, z3, f2, 0),
’FaceColor’, [1.0 0.5 0.0],
’EdgeColor’, ’none’);
view(3); camlight;

f3 = f1 - f2;
f3s = interp3(x3, y3, z3, f3, x2,
 y2, z2);

C = contours(y2, z2, f3s, [0 0]);

yL = C(1, 2:end);
zL = C(2, 2:end);
xL = interp2(y2, z2, x2, yL, zL);

% Visualize the line.
line(xL,yL,zL,’Color’,’k’,
’LineWidth’,1);
xlabel(’P1’)
ylabel(’P2’)
zlabel(’P3’)

%axis([-4 4 -4 4 -8 8]);
axis([-10 10 -10 10 -10 10]);
```

The code used to produce Figure C.13 (a) - (d):

```
H = 4; K = 10; c2 = 5; c3=4;
alpha = H + sqrt(H.^2 - K);
beta = H - sqrt(H.^2 - K);
A1A2 = (H - sqrt(H.^2 - K))
./(4.*H.*H - 4.*K);
b = sqrt((H - c2 + sqrt(H.^2 - K))
./(c2 - H + sqrt(H.^2 - K)));
a = sqrt((H + sqrt(H.^2 - K))
./((H - sqrt(H.^2 - K))));

Constant = (sqrt(a.^2 + b.^2).*(alpha
- beta).*sqrt(A1A2))./sqrt(c2.*c3);

t = linspace(0,10,100);
[s,c,d] = ellipj(Constant*t,
b./(sqrt(a.*a + b.*b)));

%Uncomment other value of P1
%for alternate elliptic function
P1 = (alpha - a.*beta.*(1./c))
./(1 - a.*(1./c));
%P1 = (alpha - (sqrt(a.^2 + b.^2))
.*beta.*(d./s))
./(1 - (sqrt(a.^2 + b.^2)).*(d./s));
P2 = sqrt(K - P1.*P1);
P3 = sqrt(c3.*P1.*P1./c2 - 2.*c3.*P1
+ 2.*c3.*H - c3.*K./c2);

plot(t,P1,’-’,t,P2,’-.’,t,P3,’.’)
hold
plot(t,P1,’-’,t,-P2,’-.’,t,-P3,’.’)
xlabel(’Time’)
ylabel(’Output’)
legend(’P1’,’P2’,’P3’)

plot3(P1,P2,P3)
```

```
hold
plot3(P1,-P2,P3)
plot3(P1,-P2,-P3)
plot3(P1,P2,-P3)
xlabel('P1')
ylabel('P2')
zlabel('P3')
```

Type II-b

```
fnc2B.m
function P = fnc2B(t, p, c1, c2)
P = zeros(3,1);
P(1) = p(2);
P(2) = -p(1);
P(3) = ((1/c2) - (1/c1))*p(1)*p(2);
end
```

The code used to produce Figure C.14 (a) - (b):

```
c1 = 2; c2 = 4;
[t, p] = ode45(@fnc2B,[0 10],[0 2 0.5],
[],c1,c2);
plot(t,p(:,1),'-',t,p(:,2),'-.',t,
p(:,3),'.')
xlabel('Time')
ylabel('Output')
legend('P1','P2','P3')

plot3(p(:,1),p(:,2),p(:,3))
xlabel('P1')
ylabel('P2')
zlabel('P3')
```

The code used to produce Figure C.14 (c) - (d):

```
[x3, y3, z3] = meshgrid(linspace(-10
,10));

f1 = x3.^2 + y3.^2 - 4;
f2 = (x3.^2)./4 + (y3.^2)./8 + z3 - 1;

[x2, y2] = meshgrid(linspace(-10
,10));
z2 = 1 -(x2.^2)./4 - (y2.^2)./8;
% Visualize the two surfaces.
patch(isosurface(x3, y3, z3, f1, 0),
'FaceColor', [0.5 1.0 0.5],
'EdgeColor','none');
patch(isosurface(x3, y3, z3, f2, 0),
'FaceColor',[1.0 0.5 0.0],
'EdgeColor','none');
view(3); camlight;

f3 = f1 - f2;
f3s = interp3(x3, y3, z3, f3, x2,
y2, z2);

C = contours(x2, y2, f3s, [0 0]);

xL = C(1, 2:end);
yL = C(2, 2:end);
zL = interp2(x2, y2, z2, xL, yL);

% Visualize the line.
line(xL,yL,zL,'Color','k',
'LineWidth',1);
xlabel('P1')
ylabel('P2')
zlabel('P3')

%axis([-2 2 -2 2 0 0.5]);
axis([-10 10 -10 10 -10 10]);
```

The code used to produce Figure C.14 (e) - (f):

```
K = 4; c1=2; c2=4;
t = linspace(0,10,100);

P1 = sqrt(K)*sin(t);
P2 = sqrt(K)*cos(t);
P3 = sqrt(K)*cos(t).*cos(t)
.*(c2-c1)./(c1*c2);

plot(t,P1,'-',t,P2,'-.',t,P3,'.')
xlabel('Time')
ylabel('Output')
legend('P1','P2','P3')

plot3(P1,P2,P3)
hold
plot3(-P1,-P2,-P3)
```

```
xlabel(’P1’)
ylabel(’P2’)
zlabel(’P3’)
```

Type III^0

```
fnc3.m
function P = fnc3(t, p, c1, c2, c3)
P = zeros(3,1);
P(1) = (1/c3)*p(2)*p(3);
P(2) = -(1/c3)*p(1)*p(3);
P(3) = ((1/c2) - (1/c1))*p(1)*p(2);
end
```

The code used to produce Figure C.15 (a) - (b):

```
c1 =4; c2 = 3; c3 = 1;
[t, p] = ode45(@fnc3,[0 10],[sqrt(12)
 0 sqrt(2)],
[],c1,c2,c3);
plot(t,p(:,1),’-’,t,p(:,2),’-.’,t,
p(:,3),’.’)
hold
plot(t,p(:,1),’-’,t,p(:,2),’-.’,t,
-p(:,3),’.’)
xlabel(’Time’)
ylabel(’Output’)
legend(’P1’,’P2’,’P3’)

plot3(p(:,1),p(:,2),p(:,3))
hold
plot3(p(:,1),p(:,2),-p(:,3))
xlabel(’P1’)
ylabel(’P2’)
zlabel(’P3’)
```

The code used to produce Figure C.16 (c) - (d):

```
[x3, y3, z3] = meshgrid(linspace(-5,
5));

f1 = x3.^2 + y3.^2 - 12;
f2 = (x3.^2)./8 + (y3.^2)./6
+ (z3.^2)./2 - 2.5;

[y2, z2] = meshgrid(linspace(-sqrt(12)
,sqrt(12)));
x2 = sqrt(12 - (y2.^2));
% Visualize the two surfaces.
patch(isosurface(x3, y3, z3, f1, 0),
’FaceColor’, [0.5 1.0 0.5],
’EdgeColor’, ’none’);
patch(isosurface(x3, y3, z3, f2, 0),
’FaceColor’, [1.0 0.5 0.0],
’EdgeColor’,’none’);
view(3); camlight;

f3 = f1 - f2;
f3s = interp3(x3, y3, z3, f3, x2,
 y2, z2);

C = contours(y2, z2, f3s, [0 0]);

yL = C(1, 2:end);
zL = C(2, 2:end);
xL = interp2(y2, z2, x2, yL, zL);

% Visualize the line.
line(xL,yL,zL,’Color’,’k’,
’LineWidth’,1);
line(xL,yL,-zL,’Color’,’k’,
’LineWidth’,1);
line(-xL,-yL,-zL,’Color’,’k’,
’LineWidth’,1);
xlabel(’P1’)
ylabel(’P2’)
zlabel(’P3’)
%axis([-4 4 -4 4 -1.5 1.5]);
axis([-5 5 -5 5 -5 5]);
```

The code used to produce Figure C.16 (a) - (d):

```
H = 5./2; K = 12; c1 = 4; c2=3; c3=1;
a = sqrt((2.*c1.*c2.*H - c1.*K)
./(c1-c2));
b = sqrt(K);

Constant = sqrt((a.*a + b.*b).*(c1-c2)
./(c1.*c2.*c3));
Constant2 = b./(sqrt(a.*a + b.*b));

t = linspace(0,10,100);
```

```
[s,c,d] = ellipj(Constant*t,Constant2);

%Uncomment other value of P1
%for alternate elliptic function
P1 = b.*(c);
%P1 = ((a.*b)./(sqrt(a.^2 + b.^2)))
.*(s./d);
P2 = sqrt(K - P1.*P1);
P3 = sqrt(c3/(c1*c2)*((c1 - c2).*P1.*P1
- c1.*K + 2*c1*c2.*H));

plot(t,P1,'-',t,P2,'-.',t,P3,'.')
hold
plot(t,P1,'-',t,-P2,'-.',t,-P3,'.')
xlabel('Time')
ylabel('Output')
legend('P1','P2','P3')

plot3(P1,P2,P3)
hold
plot3(P1,-P2,P3)
plot3(P1,P2,-P3)
plot3(P1,-P2,-P3)

xlabel('P1')
ylabel('P2')
zlabel('P3')
%axis([-4 4 -4 4 -1.5 1.5])
```

The code used to produce Figure C.17 (a) - (b):

```
c1 =4; c2 = 3; c3 = 1;
[t, p] = ode45(@fnc3,[0 10],[4 0 1],
[],c1,c2,c3);
plot(t,p(:,1),'-',t,p(:,2),'-.',t,
p(:,3),'.')
hold
plot(t,-p(:,1),'-',t,p(:,2),'-.',t,
p(:,3),'.')
xlabel('Time')
ylabel('Output')
legend('P1','P2','P3')

plot3(p(:,1),p(:,2),p(:,3))
hold
plot3(-p(:,1),p(:,2),p(:,3))
xlabel('P1')
ylabel('P2')
zlabel('P3')
```

The code used to produce Figure C.17 (c) - (d):

```
[x3, y3, z3] = meshgrid(linspace(-5
,5));

f1 = x3.^2 + y3.^2 - 16;
f2 = (x3.^2)./8 + (y3.^2)./6
+ (z3.^2)./2 - 2.5;

[y2, z2] = meshgrid(linspace(-sqrt(16)
,sqrt(16)));
x2 = sqrt(16 - (y2.^2));
% Visualize the two surfaces.
patch(isosurface(x3, y3, z3, f1, 0),
'FaceColor', [0.5 1.0 0.5],
'EdgeColor','none');
patch(isosurface(x3, y3, z3, f2, 0),
'FaceColor', [1.0 0.5 0.0],
'EdgeColor','none');
view(3); camlight;

f3 = f1 - f2;
f3s = interp3(x3, y3, z3, f3, x2,
 y2, z2);

C = contours(y2, z2, f3s, [0 0]);

yL = C(1, 2:end);
zL = C(2, 2:end);
xL = interp2(y2, z2, x2, yL, zL);

% Visualize the line.
line(xL,yL,zL,'Color','k',
'LineWidth',1);
line(xL,yL,-zL,'Color','k',
'LineWidth',1);
line(-xL,-yL,-zL,'Color','k',
'LineWidth',1);
xlabel('P1')
ylabel('P2')
zlabel('P3')
%axis([-4 4 -4 4 -1 1]);
```

```
axis([-5 5 -5 5 -5 5]);
```

The code used to produce Figure C.18 (a) - (d):

```
H = 5./2; K = 16; c1 = 4; c2=3; c3=1;
b = sqrt(-(2.*c1.*c2.*H - c1.*K)
./(c1-c2));
a = sqrt(K);

Constant = a.*sqrt((c1-c2)
./(c1.*c2.*c3));
Constant2 = sqrt(a.^2 - b.^2)./a;

t = linspace(0,10,100);
[s,c,d] = ellipj(Constant*t,Constant2);

%Uncomment other value of P1
%for alternate elliptic function
P1 = a.*(d);
%P1 = b.*(1./d);
P2 = sqrt(K - P1.*P1);
P3 = sqrt((c3./(c1*c2)).*((c1-c2).*P1.*P1
- c1.*K + 2*c1*c2.*H));

plot(t,P1,'-',t,P2,'-.',t,P3,'.')
hold
plot(t,-P1,'-',t,-P2,'-.',t,-P3,'.')
xlabel('Time')
ylabel('Output')
legend('P1','P2','P3')

plot3(P1,P2,P3)
hold
plot3(P1,-P2,P3)
plot3(P1,P2,-P3)
plot3(P1,-P2,-P3)
plot3(-P1,P2,P3)
plot3(-P1,-P2,P3)
plot3(-P1,P2,-P3)
plot3(-P1,-P2,-P3)
xlabel('P1')
ylabel('P2')
zlabel('P3')
axis([-5 5 -5 5 -1.5 1.5])
```

The code used to produce Figure C.19 (a) - (b):

```
c1 =3; c2 = 4; c3 = 1;
[t, p] = ode45(@fnc3,[0 10],[sqrt(12)
 0 1],
[],c1,c2,c3);
plot(t,p(:,1),'-',t,p(:,2),'-.',t,
p(:,3),'.')
hold
plot(t,p(:,1),'-',t,p(:,2),'-.',t,
-p(:,3),'.')
xlabel('Time')
ylabel('Output')
legend('P1','P2','P3')

plot3(p(:,1),p(:,2),p(:,3))
hold
plot3(p(:,1),p(:,2),-p(:,3))
xlabel('P1')
ylabel('P2')
zlabel('P3')
```

The code used to produce Figure C.19 (c) - (d):

```
[x3, y3, z3] = meshgrid(linspace(-5
,5));

f1 = x3.^2 + y3.^2 - 12;
f2 = (x3.^2)./6 + (y3.^2)./8
+ (z3.^2)./2 - 2.5;

[y2, z2] = meshgrid(linspace(-sqrt(12)
,sqrt(12)));
x2 = sqrt(12 - (y2.^2));
% Visualize the two surfaces.
patch(isosurface(x3, y3, z3, f1, 0),
'FaceColor', [0.5 1.0 0.5],
'EdgeColor','none');
patch(isosurface(x3, y3, z3, f2, 0),
'FaceColor', [1.0 0.5 0.0],
'EdgeColor','none');
view(3); camlight;

f3 = f1 - f2;
f3s = interp3(x3, y3, z3, f3, x2,
 y2, z2);

C = contours(y2, z2, f3s, [0 0]);
```

```
yL = C(1, 2:end);
zL = C(2, 2:end);
xL = interp2(y2, z2, x2, yL, zL);

% Visualize the line.
line(xL,yL,zL,'Color','k',
'LineWidth',1);
line(xL,yL,-zL,'Color','k',
'LineWidth',1);
line(-xL,-yL,-zL,'Color','k',
'LineWidth',1);
xlabel('P1')
ylabel('P2')
zlabel('P3')
%axis([-4 4 -4 4 -1.5 1.5]);
axis([-5 5 -5 5 -5 5]);
```

The code used to produce Figure C.20 (a) - (d):

```
H = 5./2; K = 12; c1 = 3; c2=4; c3=1;
a = sqrt((2.*c1.*c2.*H - c1.*K)
./(c2-c1));
b = sqrt(K);

Constant = sqrt((a).*(c2-c1)
./(c1.*c2.*c3));
Constant2 = b./a;

t = linspace(0,10,100);
[s,c,d] = ellipj(Constant*t,Constant2);

%Uncomment other value of P1
%for alternate elliptic function
P1 = b.*(s);
%P1 = b.*(c./d);
P2 = sqrt(K - P1.*P1);
P3 = sqrt(c3/(c1*c2)*((c1 - c2).*P1.*P1
- c1.*K + 2*c1*c2.*H));

plot(t,P1,'-',t,P2,'-.',t,P3,'.')
hold
plot(t,P1,'-',t,-P2,'-.',t,-P3,'.')
xlabel('Time')
ylabel('Output')
legend('P1','P2','P3')
plot3(P1,P2,P3)
hold
plot3(P1,-P2,P3)
plot3(P1,P2,-P3)
plot3(P1,-P2,-P3)
xlabel('P1')
ylabel('P2')
zlabel('P3')
```

The code used to produce Figure C.21 (a) - (b):

```
c1 =2; c2 = 4; c3 = 1;
[t, p] = ode45(@fnc3,[0 10],[sqrt(8)
 sqrt(16) sqrt(2)],
[],c1,c2,c3);
plot(t,p(:,1),'-',t,p(:,2),'-.',t,
p(:,3),'.')
hold
plot(t,p(:,1),'-',t,-p(:,2),'-.',t,
p(:,3),'.')
xlabel('Time')
ylabel('Output')
legend('P1','P2','P3')

plot3(p(:,1),p(:,2),p(:,3))
hold
plot3(p(:,1),-p(:,2),p(:,3))
xlabel('P1')
ylabel('P2')
zlabel('P3')
```

The code used to produce Figure C.21 (c) - (d):

```
[x3, y3, z3] = meshgrid(linspace(-10
,10));

f1 = x3.^2 + y3.^2 - 24;
f2 = (x3.^2)./4 + (y3.^2)./8
+ (z3.^2)./2 - 5;

[y2, z2] = meshgrid(linspace(-sqrt(24)
,sqrt(24)));
x2 = sqrt(24 - (y2.^2));
% Visualize the two surfaces.
patch(isosurface(x3, y3, z3, f1, 0),
```

```
’FaceColor’, [0.5 1.0 0.5],
’EdgeColor’,’none’);
patch(isosurface(x3, y3, z3, f2, 0),
’FaceColor’, [1.0 0.5 0.0],
’EdgeColor’,’none’);
view(3); camlight;

f3 = f1 - f2;
f3s = interp3(x3, y3, z3, f3, x2,
y2, z2);

C = contours(y2, z2, f3s, [0 0]);

yL = C(1, 2:end);
zL = C(2, 2:end);
xL = interp2(y2, z2, x2, yL, zL);

% Visualize the line.
line(xL,yL,zL,’Color’,’k’,
’LineWidth’,1);
line(xL,yL,-zL,’Color’,’k’,
’LineWidth’,1);
line(-xL,-yL,-zL,’Color’,’k’,
’LineWidth’,1);
xlabel(’P1’)
ylabel(’P2’)
zlabel(’P3’)
%axis([-4 4 -5 5 -2 2]);
axis([-10 10 -10 10 -10 10]);
```

The code used to produce Figure C.22 (a) - (d):

```
H = 5; K = 24; c1 = 2; c2=4; c3=1;
b = sqrt((2.*c1.*c2.*H - c1.*K)
./(c2-c1));
a = sqrt(K);

Constant = sqrt((a).*(c2-c1)
./(c1.*c2.*c3));
Constant2 = b./a;

t = linspace(0,10,100);
[s,c,d] = ellipj(Constant*t,Constant2);

%Uncomment other value of P1
%for alternate elliptic function
P1 = b.*(s);
%P1 = b.*(c./d);
P2 = sqrt(K - P1.*P1);
P3 = sqrt(c3/(c1*c2)*((c1 - c2).*P1.*P1
- c1.*K + 2*c1*c2.*H));

plot(t,P1,’-’,t,P2,’-.’,t,P3,’.’)
hold
plot(t,P1,’-’,t,-P2,’-.’,t,-P3,’.’)
plot(t,-P1,’-’,t,P2,’-.’,t,P3,’.’)
xlabel(’Time’)
ylabel(’Output’)
legend(’P1’,’P2’,’P3’)

plot3(P1,P2,P3)
hold
plot3(P1,-P2,P3)
plot3(P1,P2,-P3)
plot3(P1,-P2,-P3)
xlabel(’P1’)
ylabel(’P2’)
zlabel(’P3’)
```

Appendix C

Graphs

The first page graphs are those of the adjoint orbits of $\mathsf{SE}(2)$ using arbitrary values of $(\mathbf{x}, x_3) \in \mathbb{R}^3$.

(a) The type 1 adjoint orbits,

(b) The type 2 adjoint orbits,

(c) The type 3 adjoint orbits.

The second page graphs are those of the coadjoint orbits of $\mathsf{SE}(2)$ using arbitrary values of $(\mathbf{x}, x_3) \in \mathbb{R}^3$.

(a) The type 1 coadjoint orbits, (b) The type 2 coadjoint orbits.

The rest of the graphs contain the following:

(a) The solution curves $P_1(\cdot)$, $P_2(\cdot)$, $P_3(\cdot)$ of the normal extremal equations using the MATLAB solver.

(b) The solution curves $P(\cdot)$ of the normal extremal equations plotted in $\mathbb{R}^3$ using the MATLAB solver.

(c) The intersection of the surfaces of the Hamiltonian functions equal constant and the coadjoint orbit corresponding to the Casimir function equal constant.

(d) The resulting curve from the intersection of the two above mentioned surfaces. (e) The solution curves $P_1(\cdot)$, $P_2(\cdot)$, $P_3(\cdot)$ of the normal extremal equations using trigonometric functions.

(f) The solution curves $P(\cdot)$ of the normal extremal equations plotted in $\mathbb{R}^3$ using trigonometric functions.

In many cases the figures (e)-(f) are replace by the following page of figures:

(a) The solution curves $P_1(\cdot)$, $P_2(\cdot)$, $P_3(\cdot)$ to the normal extremal equations using a Jacobi elliptic function.

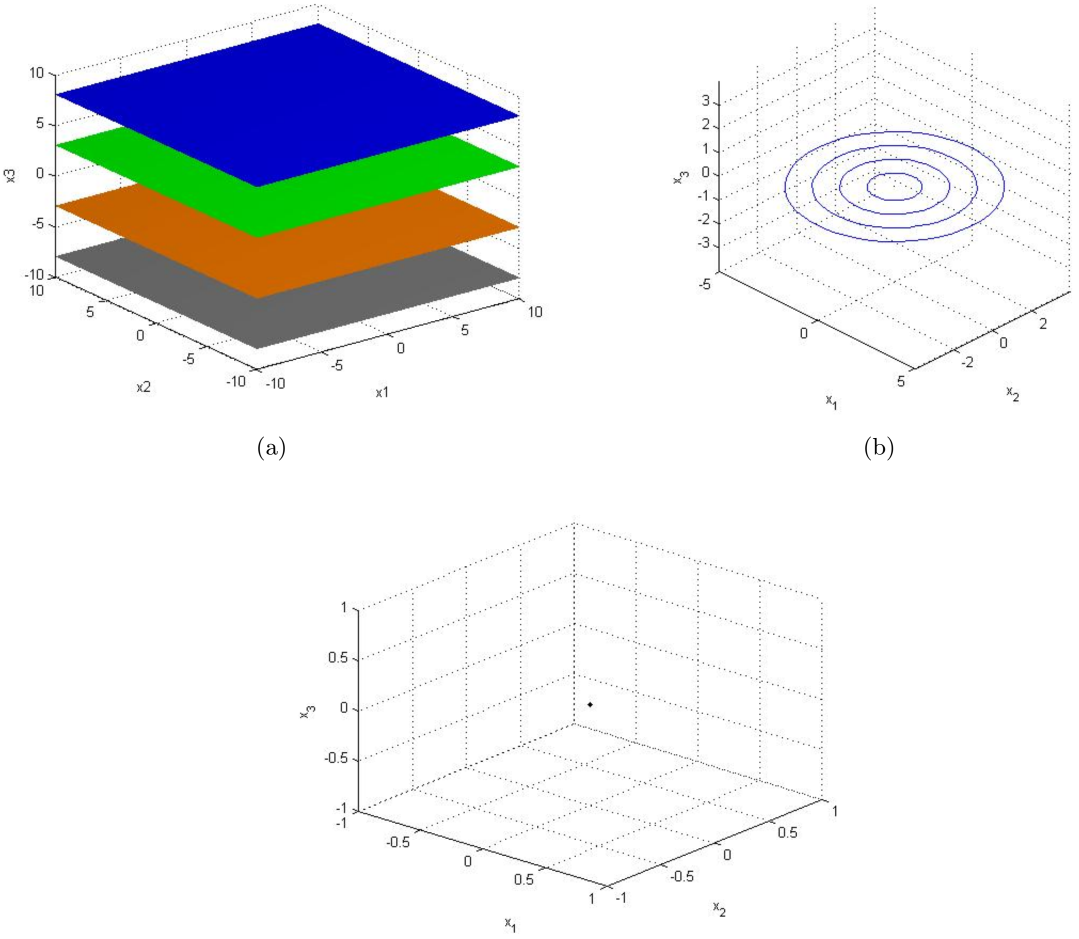
(b) The solution curves $P(\cdot)$ to the normal extremal equations plotted in $\mathbb{R}^3$ using a Jacobi elliptic function.

(c) The solution curves $P_1(\cdot)$, $P_2(\cdot)$, $P_3(\cdot)$ to the normal extremal equations using another Jacobi elliptic function.

(d) The solution curves $P(\cdot)$ to the normal extremal equations plotted in $\mathbb{R}^3$ using the other Jacobi elliptic function.

For each of the above, the result is plotted on the specified time interval $[0, 5]$, $[0, 10]$ or $[0, 15]$. For each set of graphs the following is also specified: The Case; the constant value(s) $c_1, c_2, \ldots$; the value of the Hamiltonian and Casimir functions, H and K; the initial condition and the elliptic/trigonometric function.

Adjoint orbits of $\mathsf{SE}(2)$

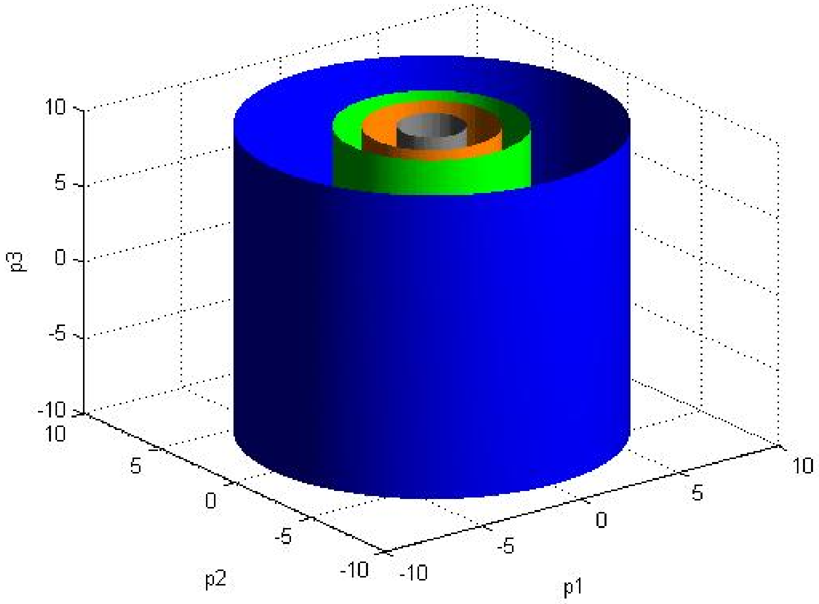


(a) (b)

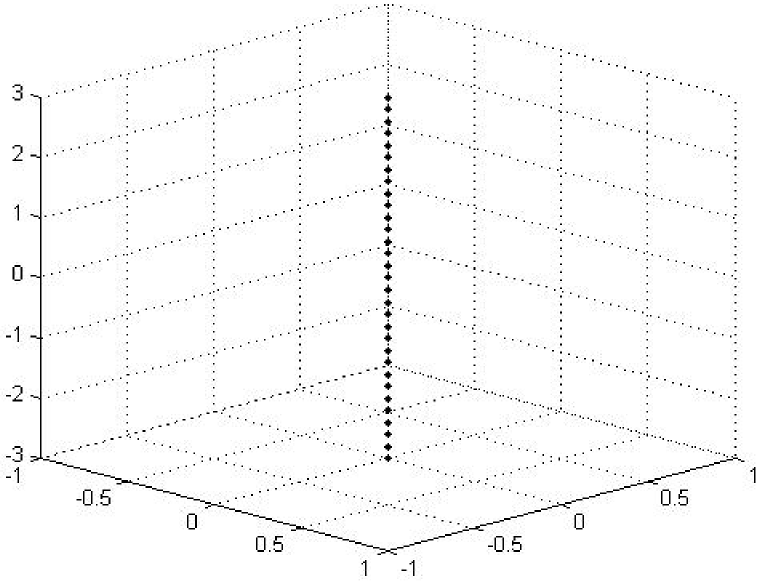
(c)

Figure C.1: Adjoint orbits: (a) type 1, (b) type 2, (c) type 3

Coadjoint orbits of $\mathsf{SE}(2)$



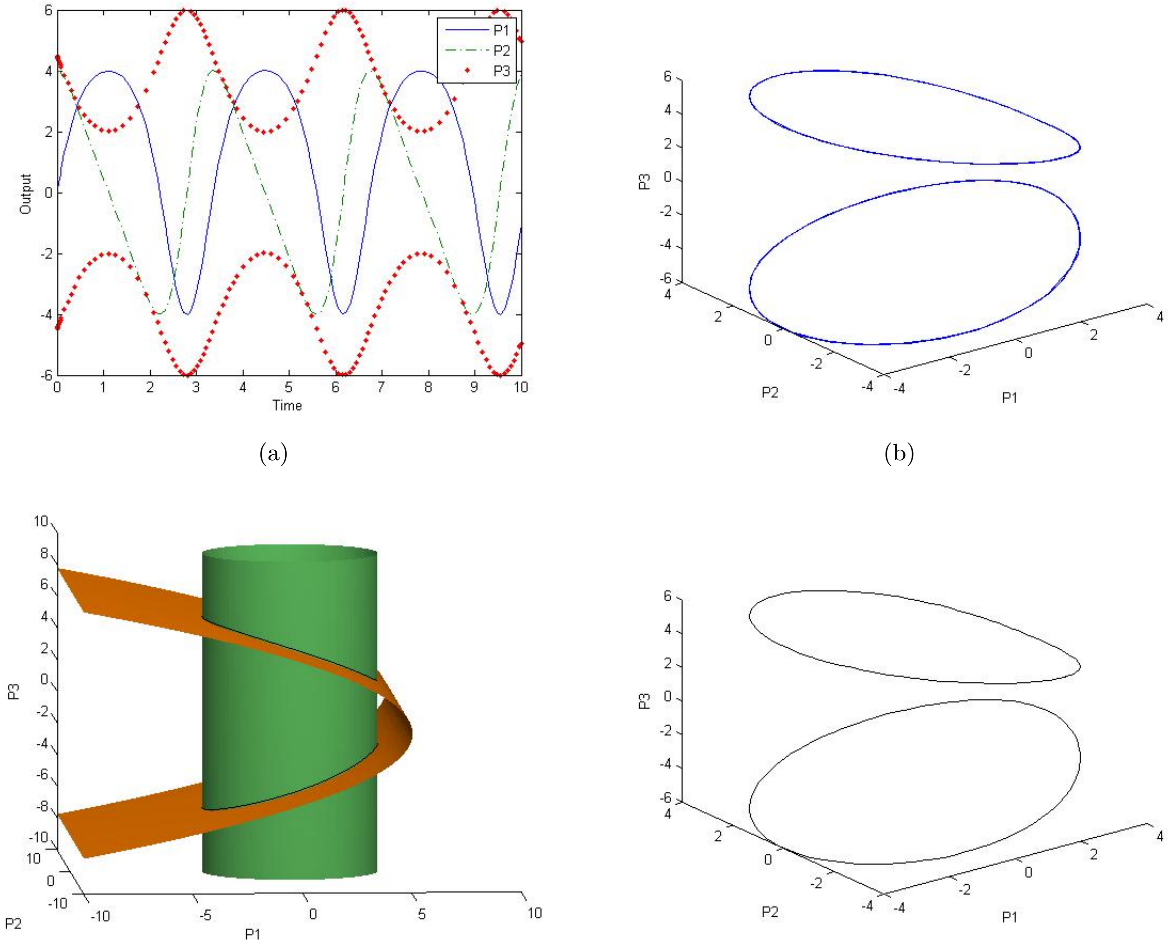
(a)



(b)

Figure C.2: Coadjoint orbits: (a) type 1, (b) type 2

Type I-a

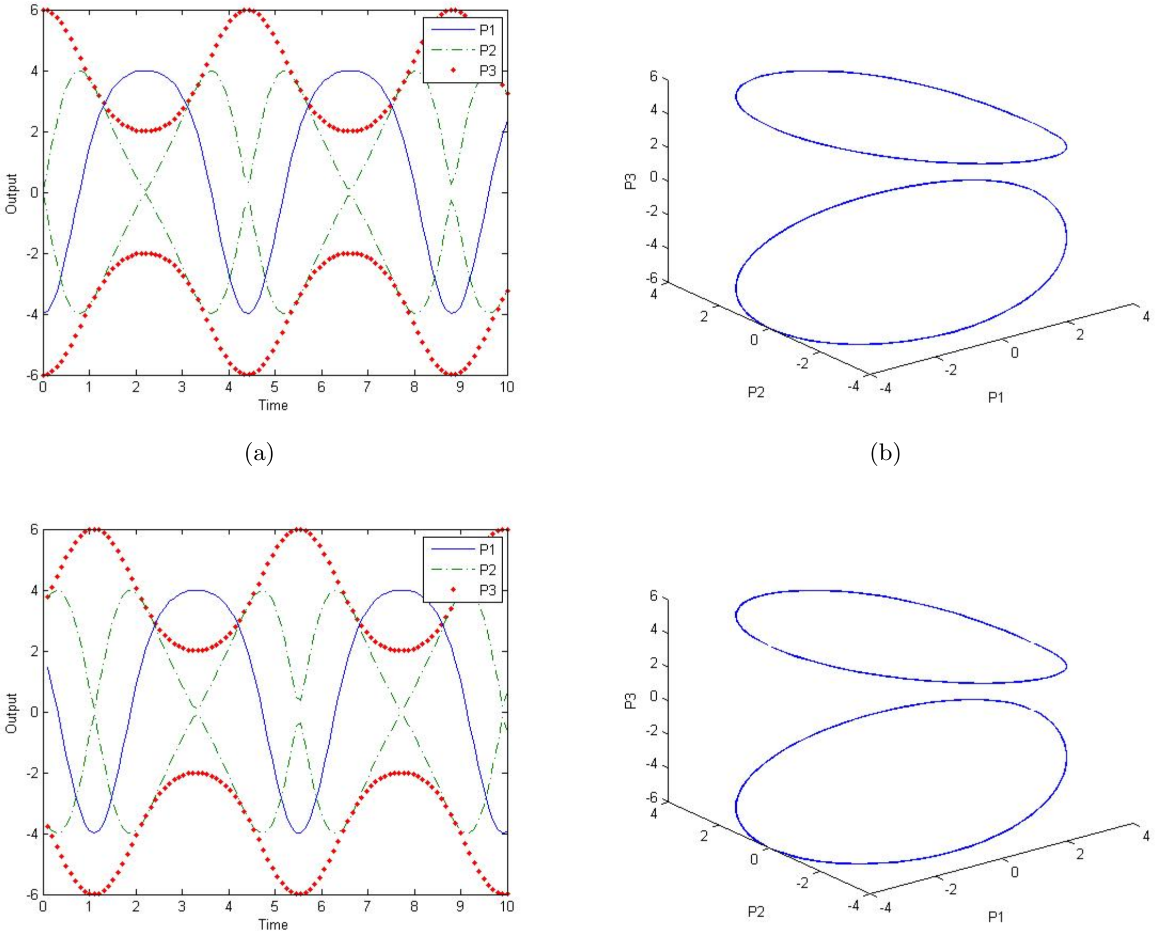


(a) (b)

(c) (d)

Figure C.3: Theorem 5.2.1: $c_1 = 2$, $P(0) = (0, 4, 2\sqrt{5})$ (a)-(b): MATLAB ODE45 solver, (c)-(d): H = 5, K = 16

Type I-a

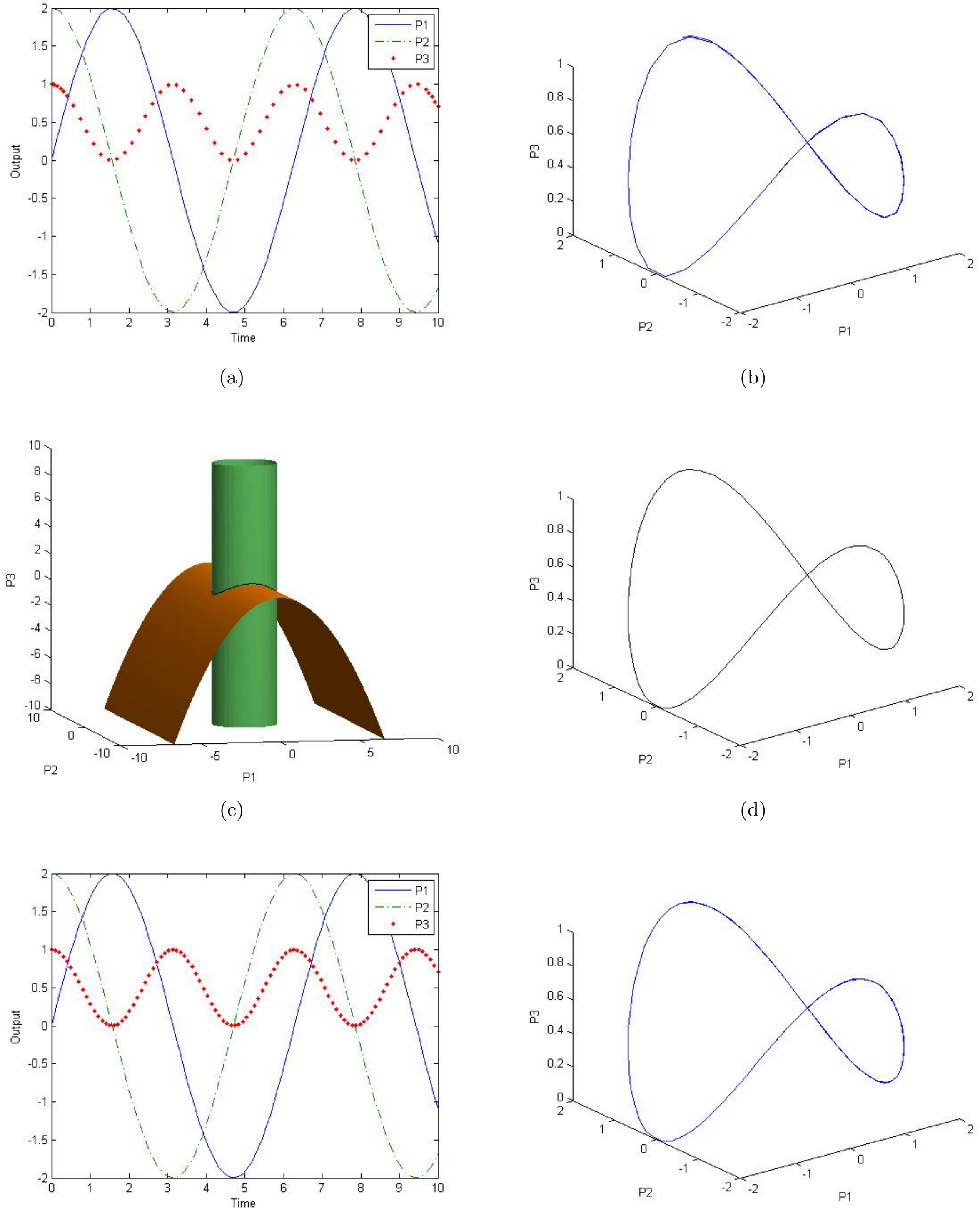


(a) (b)

(c) (d)

Figure C.4: Theorem 5.2.1: $c_1 = 2,\ H = 5,\ K = 16$ (a)-(b): dc, (c)-(d): ns

Type I-b



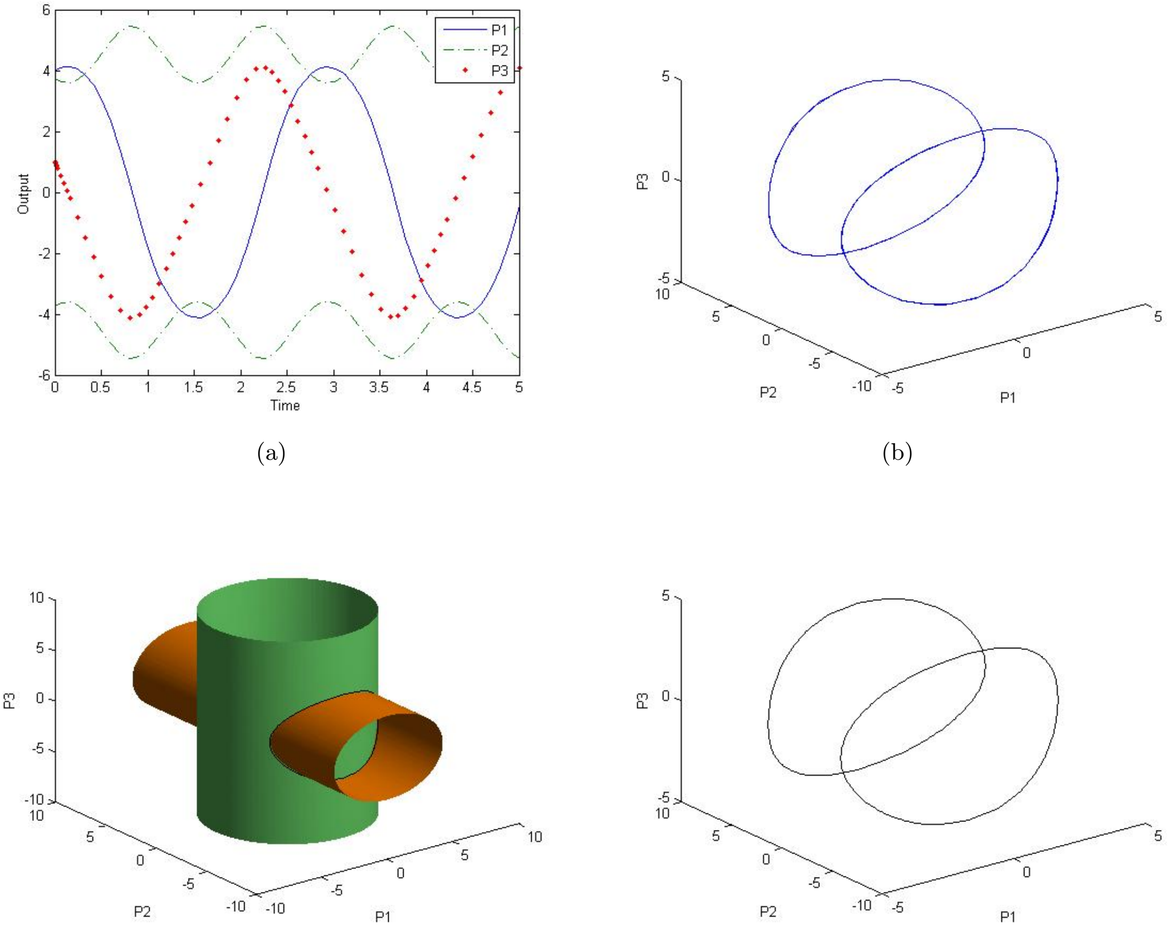
(a) (b)

(c) (d)

(e) (f)

Figure C.5: Theorem 6.2.1: $c_1 = 2$, $P(0) = (0, 2, 1)$ (a)-(b): MATLAB ODE45 solver, (c)-(d): H = 1, K=4, (e)-(f): sin, cos

Type II^0

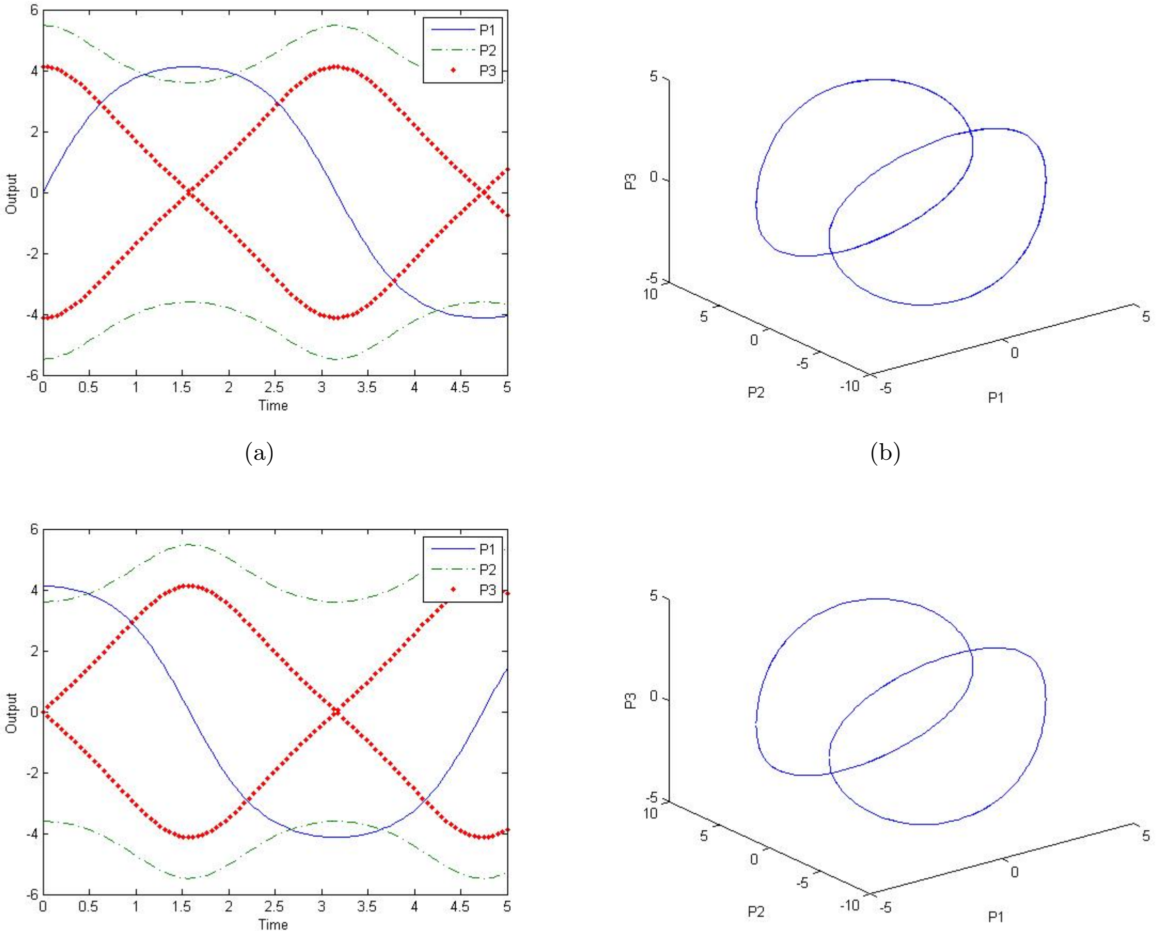


(a) (b)

(c) (d)

Figure C.6: Theorem 7.2.1 **case 1:** $c_1 = 2, c_2 = 2, \; P(0) = (4, 2, 1)$ (a)-(b): MATLAB ODE45 solver, (c)-(d): H=4.25, K=30.

Type II^0

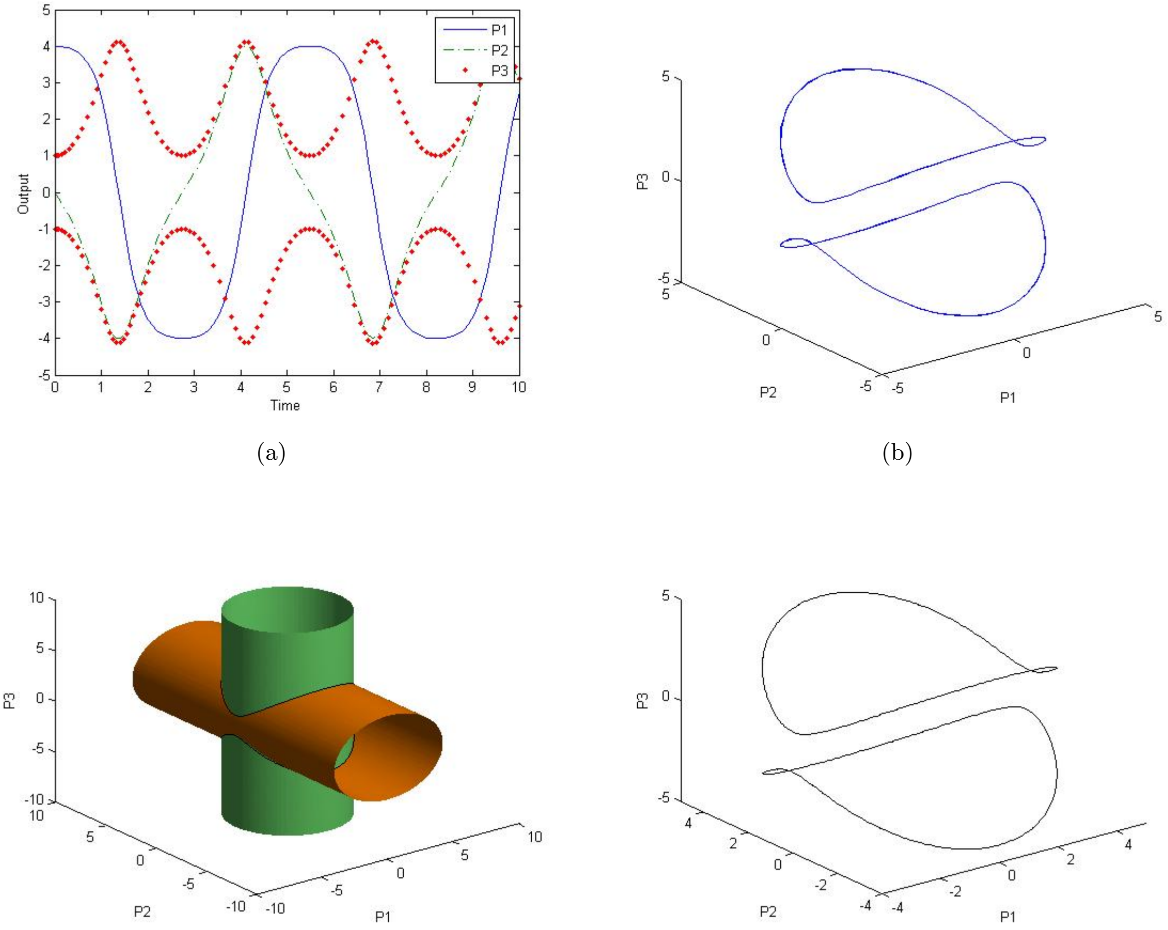


(a) (b)

(c) (d)

Figure C.7: Theorem 7.2.1 **case 1:** $c_1 = 2$, $c_2 = 2$, $H = 4.25$, $K = 30$, (a)-(b): sn, (c)-(d): cd

Type II^0

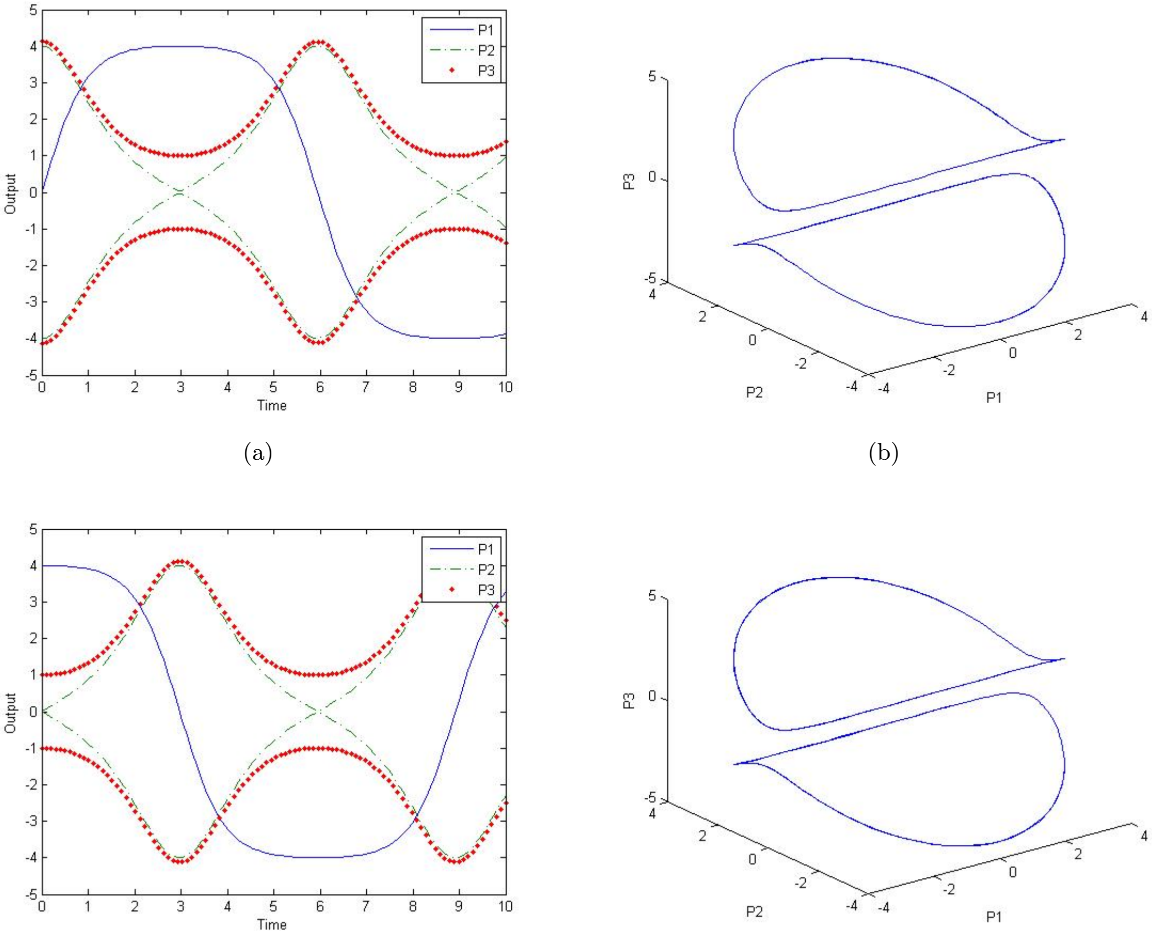


(a) (b)

(c) (d)

Figure C.8: Theorem 7.2.1 **case 2:** $c_1 = 2$, $c_2 = 2$, $P(0) = (4, 0, 1)$ (a)-(b): MATLAB ODE45 solver, (c)-(d): H =4.25, K=16.

Type II^0

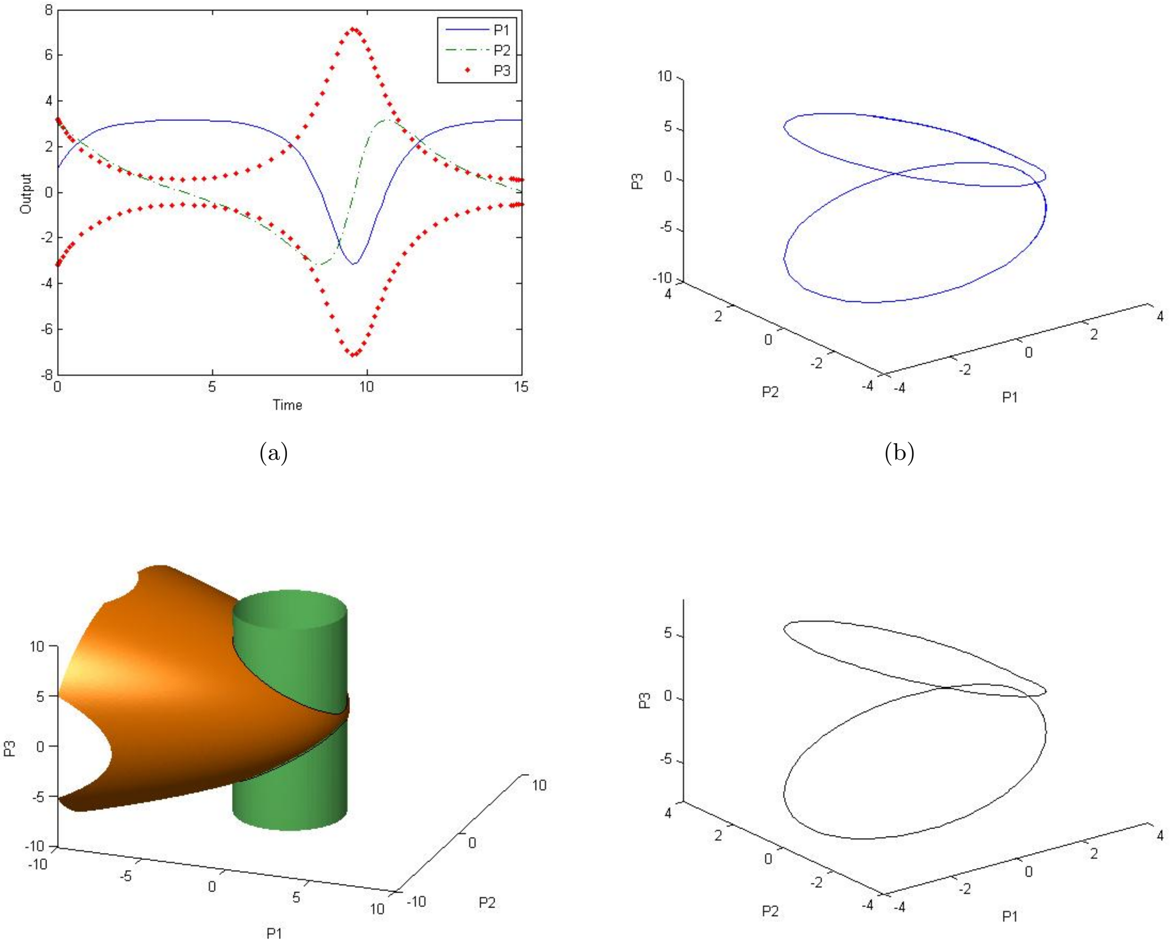


(a) (b)

(c) (d)

Figure C.9: Theorem 7.2.1 **case 2:** $c_1 = 2$, $c_2 = 2$, $H = 4.25$, $K = 16$, (a)-(b): sn, (c)-(d): cd

Type II-a

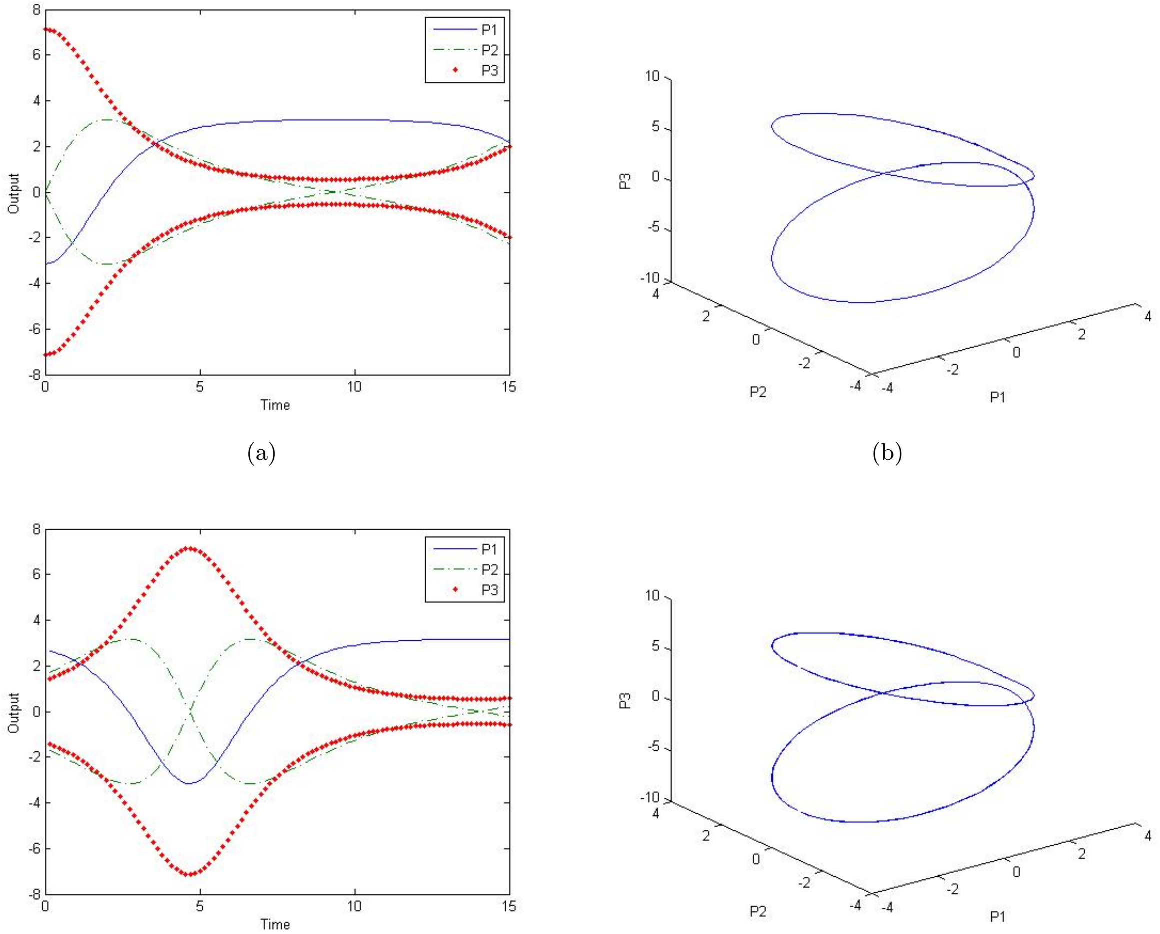


(a) (b)

(c) (d)

Figure C.10: Theorem 8.2.1 **case 1:** $c_1 = 5$, $c2 = 4$, $P(0) = (1, 3, \sqrt{\frac{52}{5}})$ (a)-(b): MATLAB ODE45 solver, (c)-(d): H=3.2, K=10

Type II-a

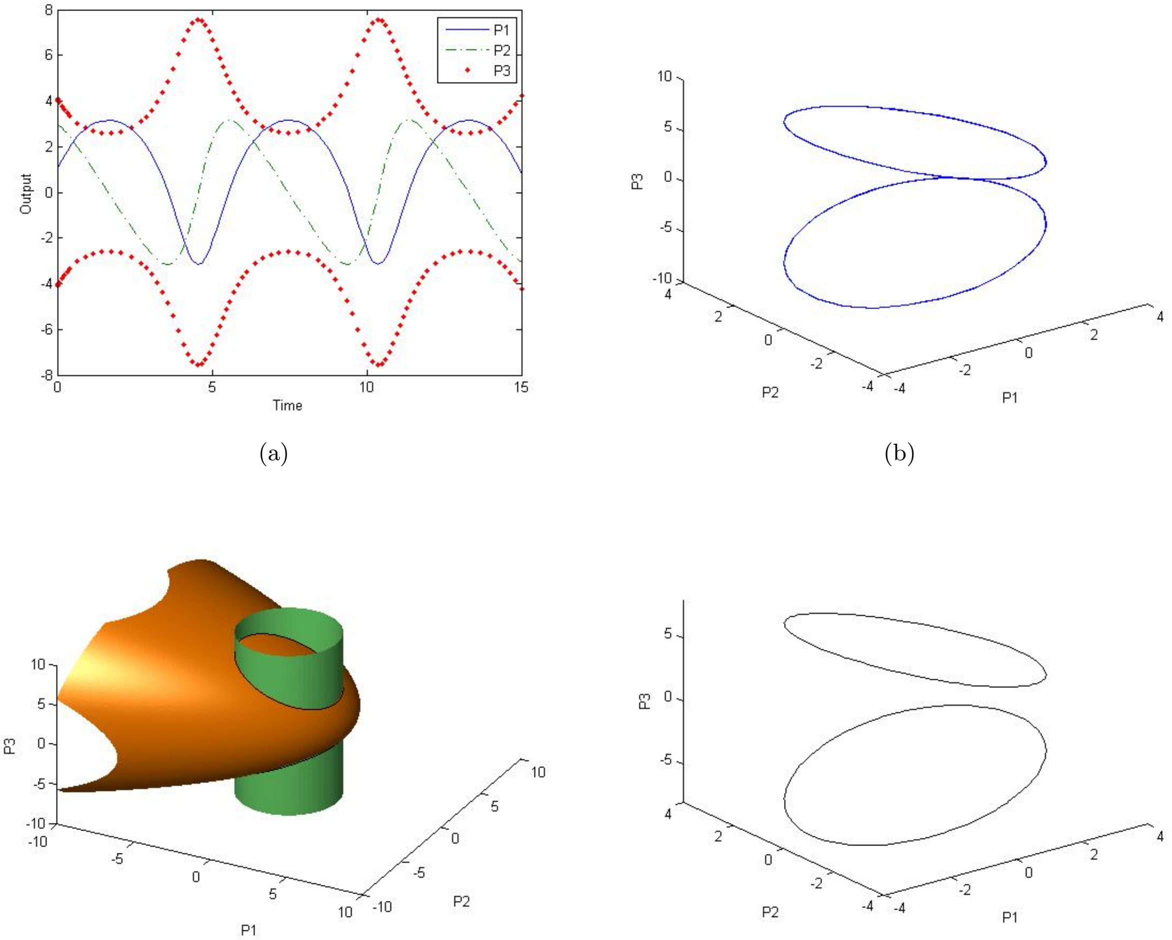


(a) (b)

(c) (d)

Figure C.11: Theorem 8.2.1 **case 1:** $c_1 = 5$, $c2 = 4$, $H = 3.2$, $K = 10$, (a)-(b): dc, (c)-(d): ns

Type II-a

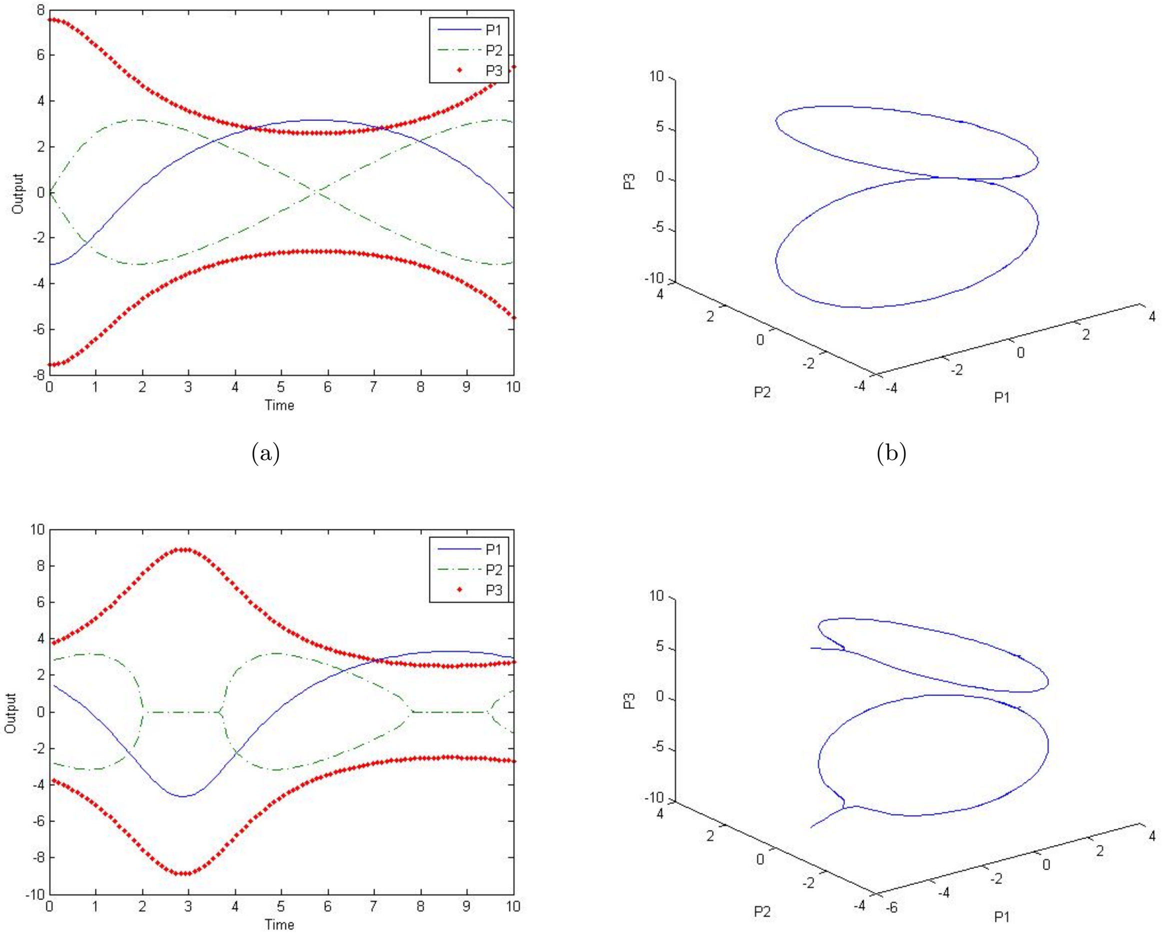


(a) (b)

(c) (d)

Figure C.12: Theorem 8.2.1 **case 2:** $c_1 = 5$, $c2 = 4$, $P(0) = (1, 3, \sqrt{\frac{84}{5}})$ (a)-(b): MATLAB ODE45 solver, (c)-(d): H=4, K=10

Type II-a

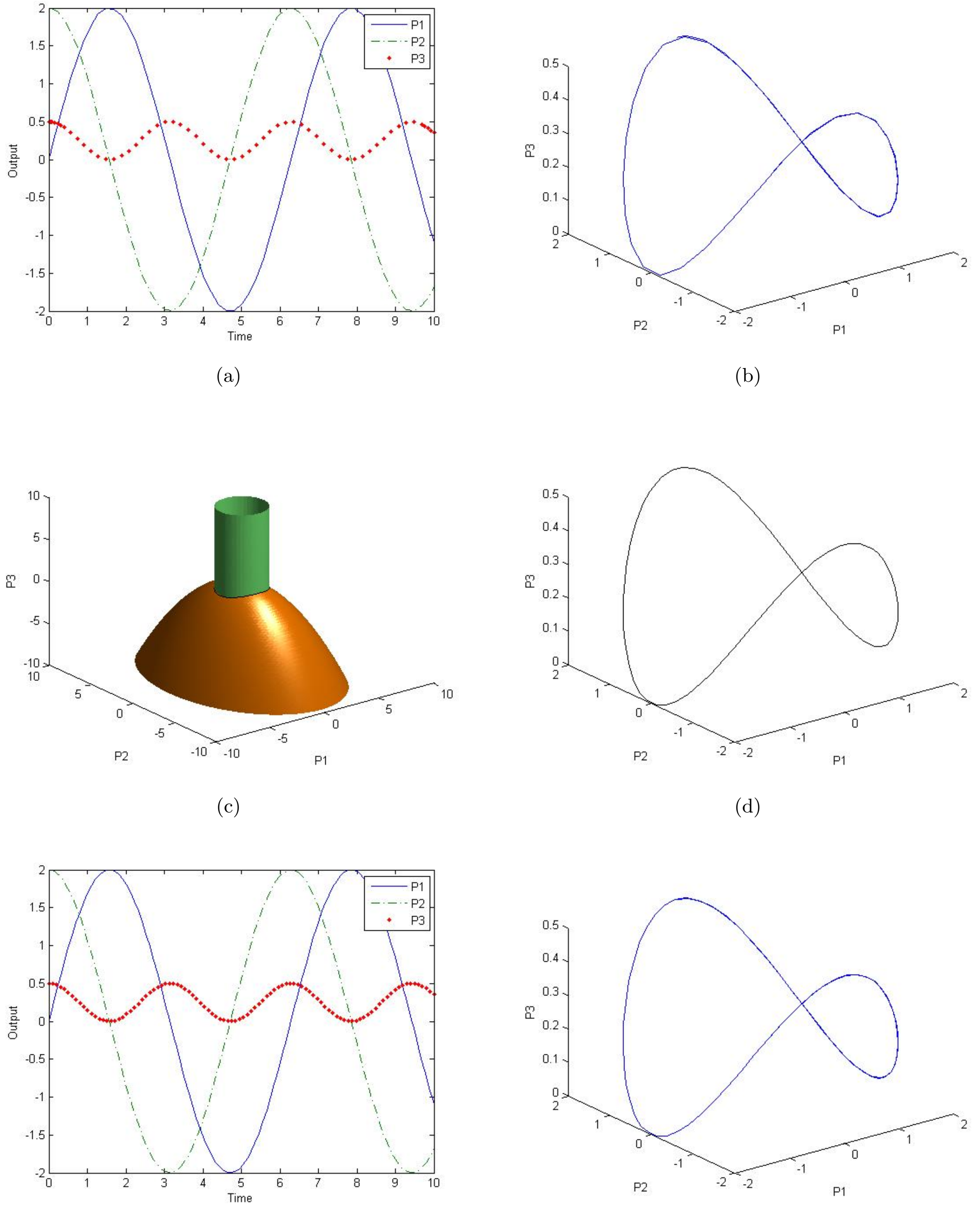


(a) (b)

(c) (d)

Figure C.13: Theorem 8.2.1 **case 2:** $c_1 = 5$, $c2 = 4$, $H = 4$, $K = 10$ (a)-(b): nc, (c)-(d): ds

Type II-b



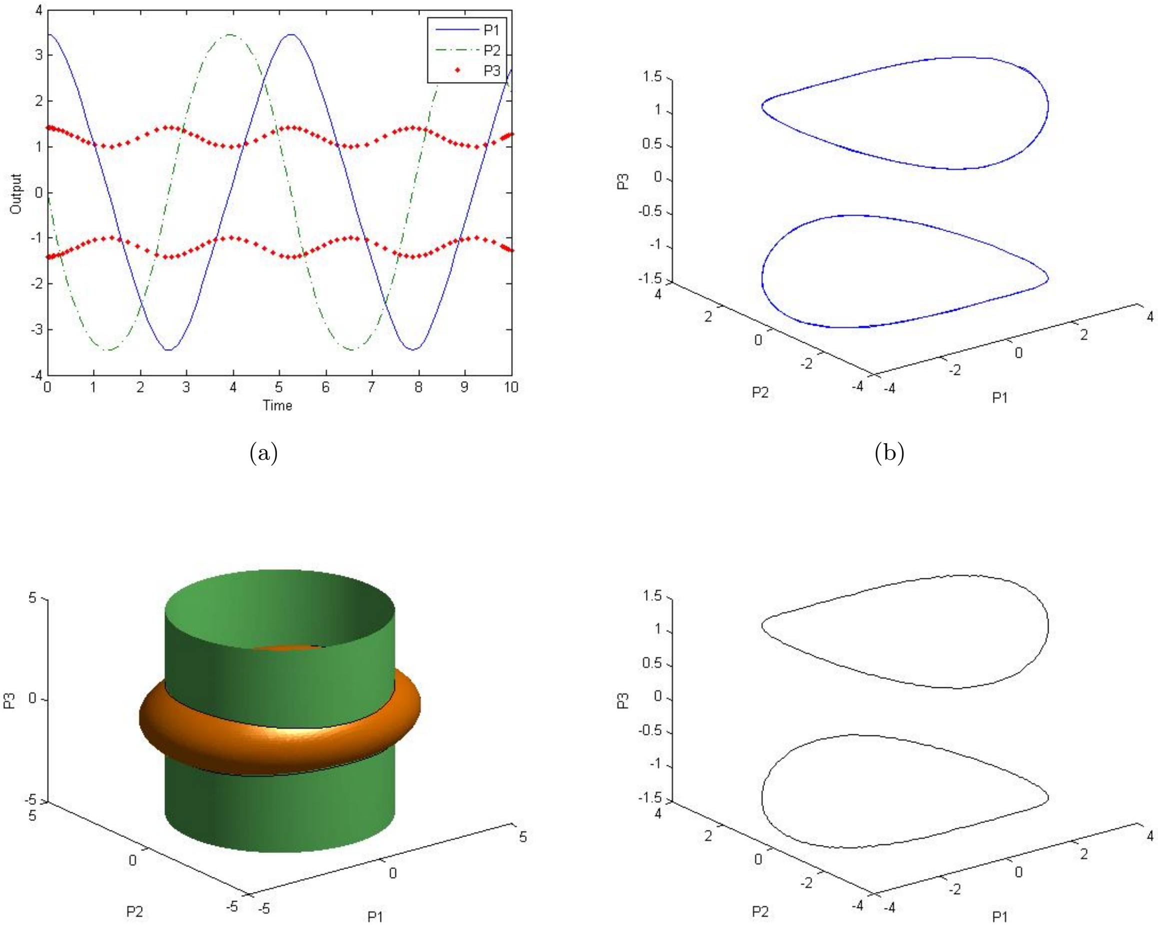
(a) (b)

(c) (d)

(e) (f)

Figure C.14: Theorem 9.2.1: $c_1 = 2$, $c2 = 4$, $P(0) = (0, 2, \frac{1}{2})$ (a)-(b): MATLAB ODE45 solver, (c)-(d): sin, cos

Type III^0

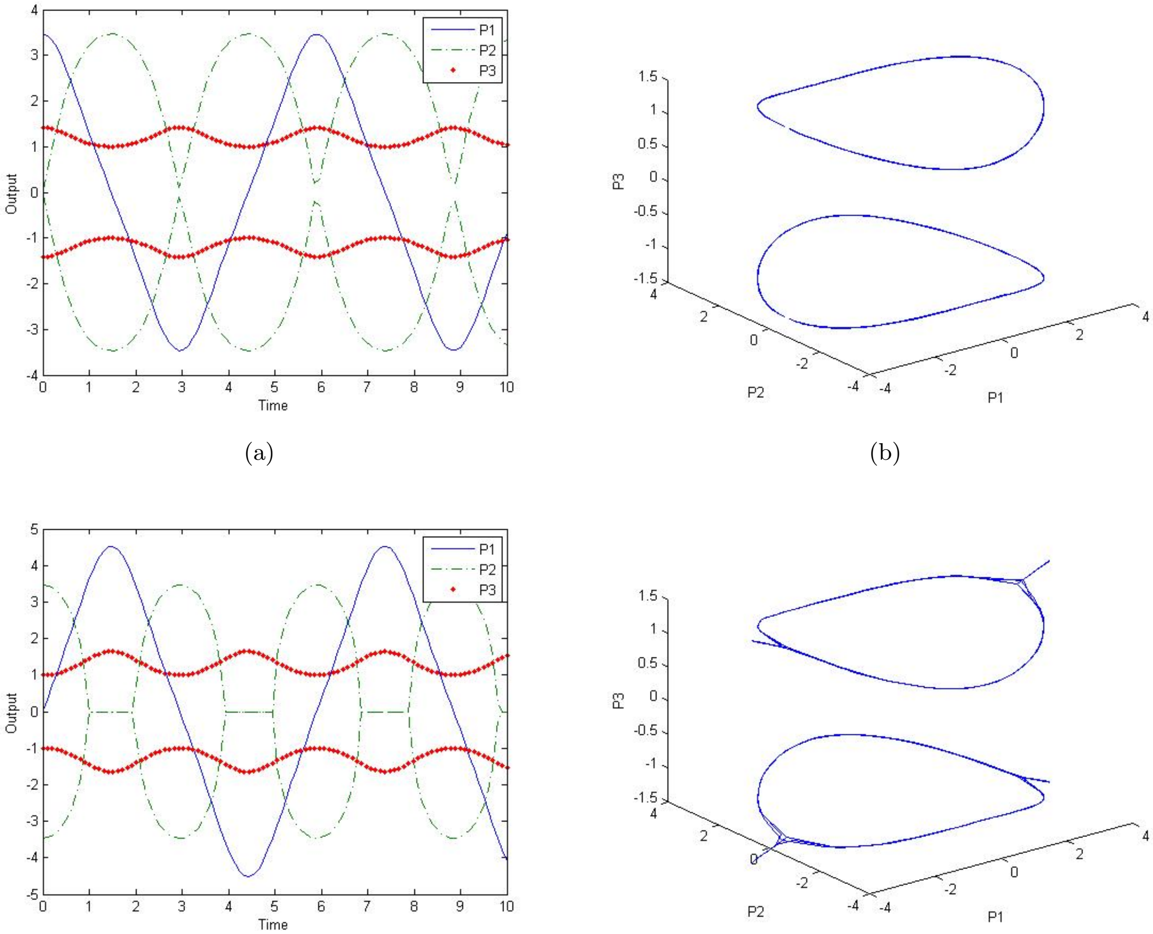


(a) (b)

(c) (d)

Figure C.15: Theorem 10.2.2 **case 1 (i):** $c_1 = 4, c2 = 3, c3 = 1,\ P(0) = (\sqrt{12}, 0, \sqrt{2})$ (a)-(b): MATLAB ODE45 solver, (c)-(d): H = 2.5, K = 12.

Type III^0

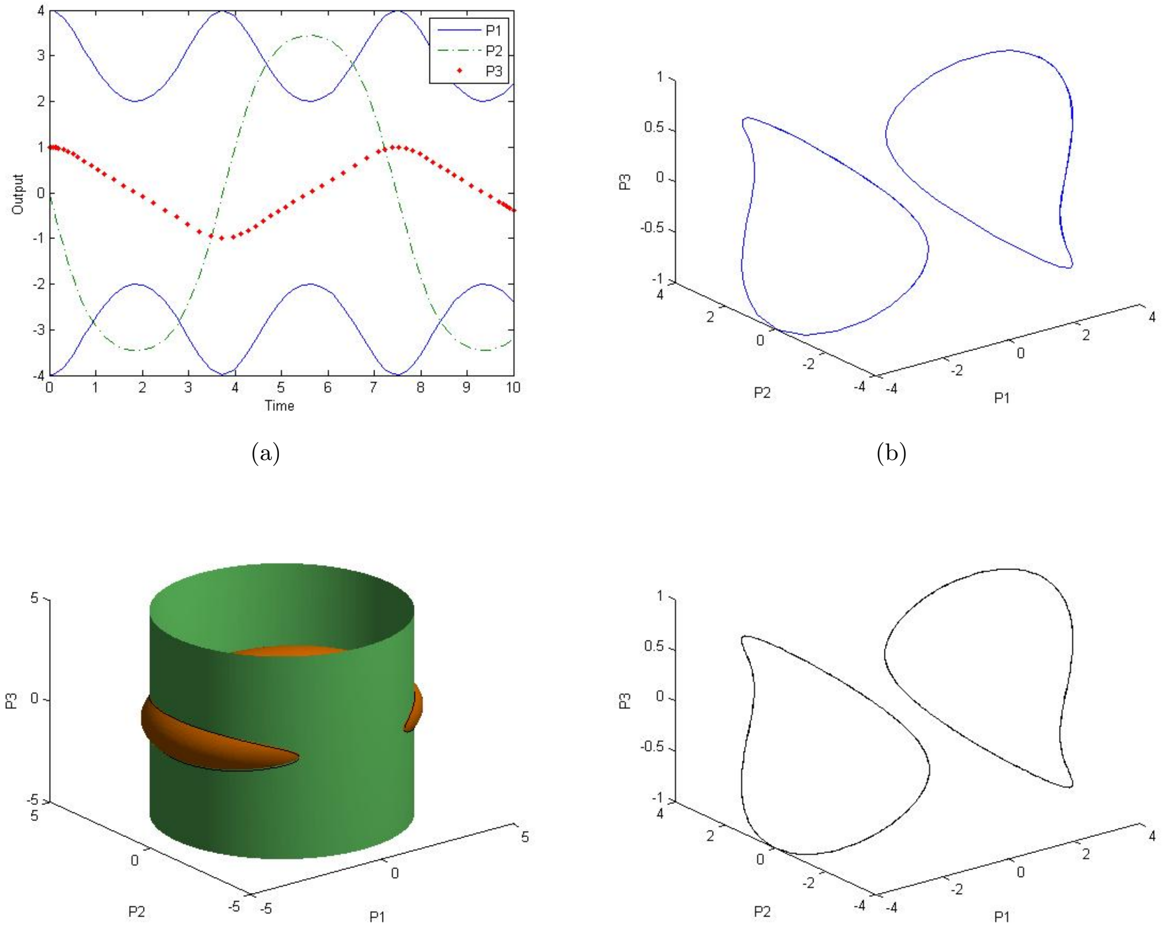


(a) (b)

(c) (d)

Figure C.16: Theorem 10.2.2 **case 1 (i):** $c_1 = 4$, $c2 = 3$, $c3 = 1$, $H = 2.5$, $K = 12$ (a)-(b): cn, (c)-(d): sd

Type III^0

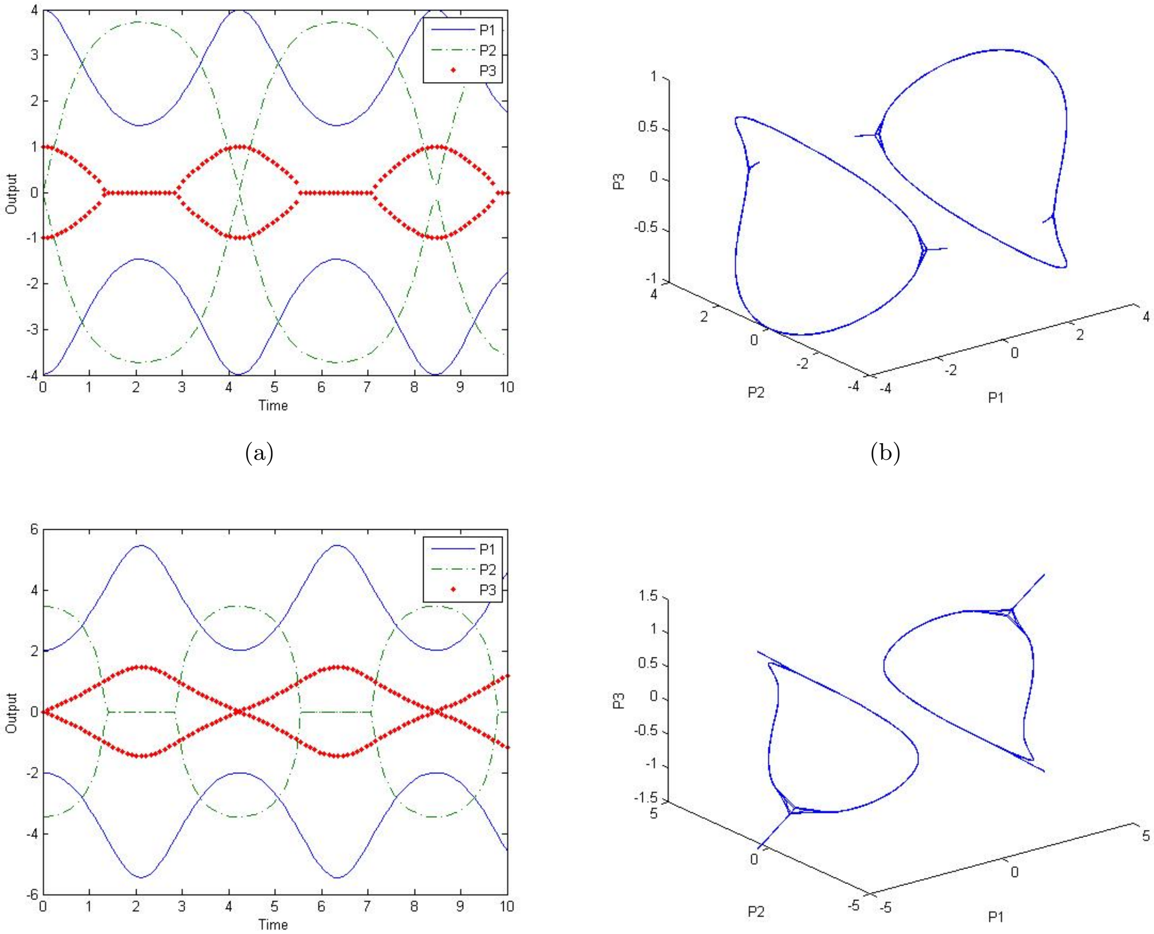


(a) (b)

(c) (d)

Figure C.17: Theorem 10.2.2 **case 1 (ii):** $c_1 = 4, c2 = 3, c3 = 1,\ P(0) = (4, 0, 1)$ (a)-(b): MATLAB ODE45 solver, (c)-(d): H = 2.5, K = 16.

Type III^0

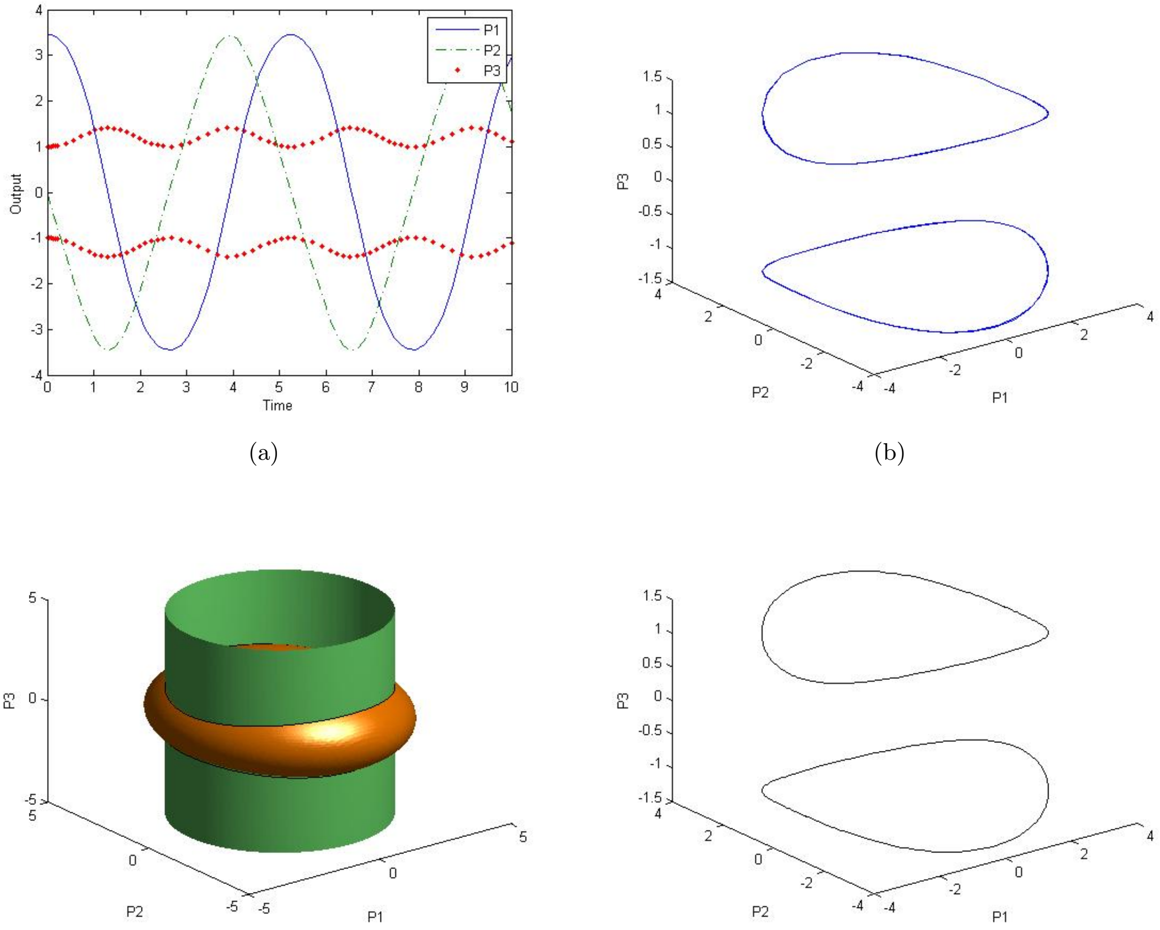


(a) (b)

(c) (d)

Figure C.18: Theorem 10.2.2 **case 1 (ii):** $c_1 = 4$, $c2 = 3$, $c3 = 1$, $H = 2.5$, $K = 16$, (a)-(b): dn, (c)-(d): nd

Type III^0

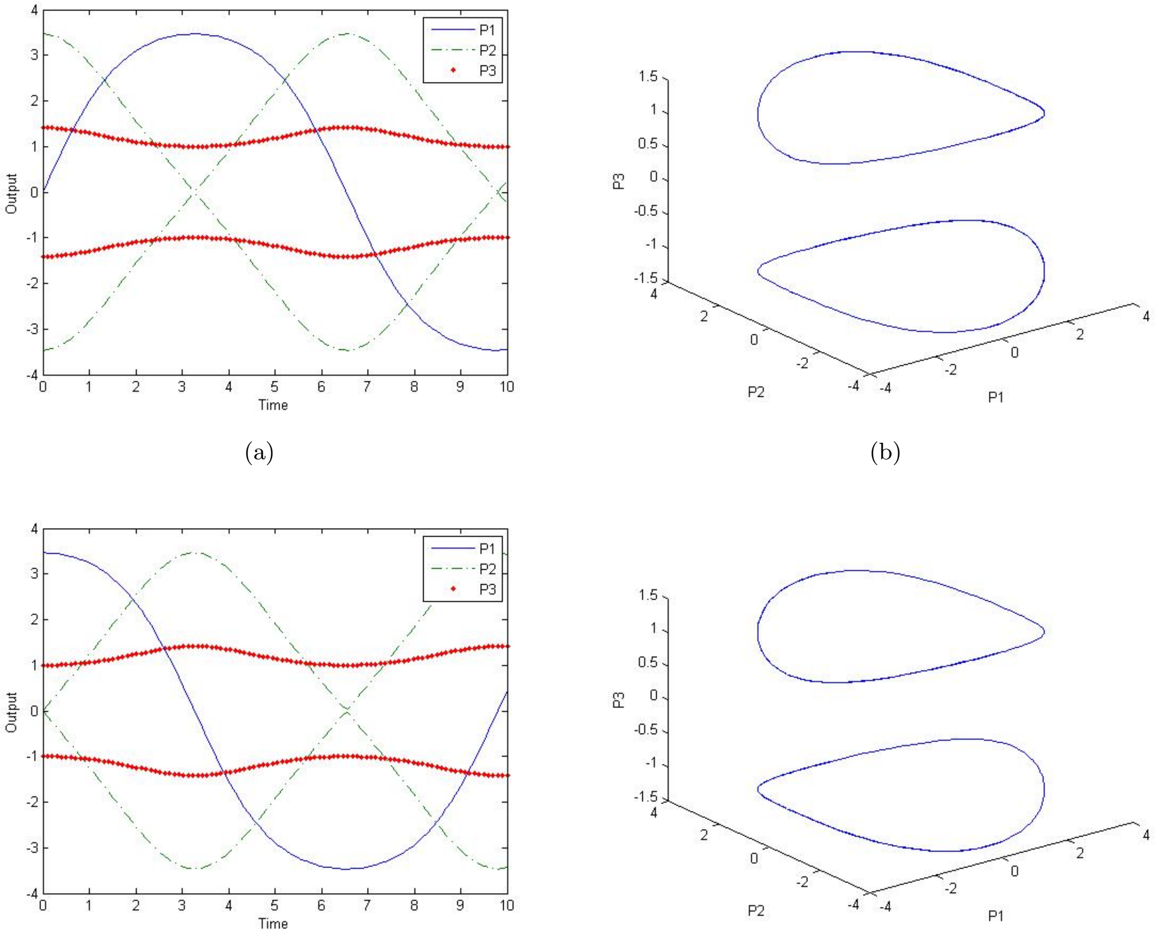


(a) (b)

(c) (d)

Figure C.19: Theorem 10.2.2 **case 2 (i):** $c_1 = 3$, $c2 = 4$, $c3 = 1$, $P(0) = (\sqrt{12}, 0, 1)$ (a)-(b): MATLAB ODE45 solver, (c)-(d): H = 2.5, K = 12.

Type III^0

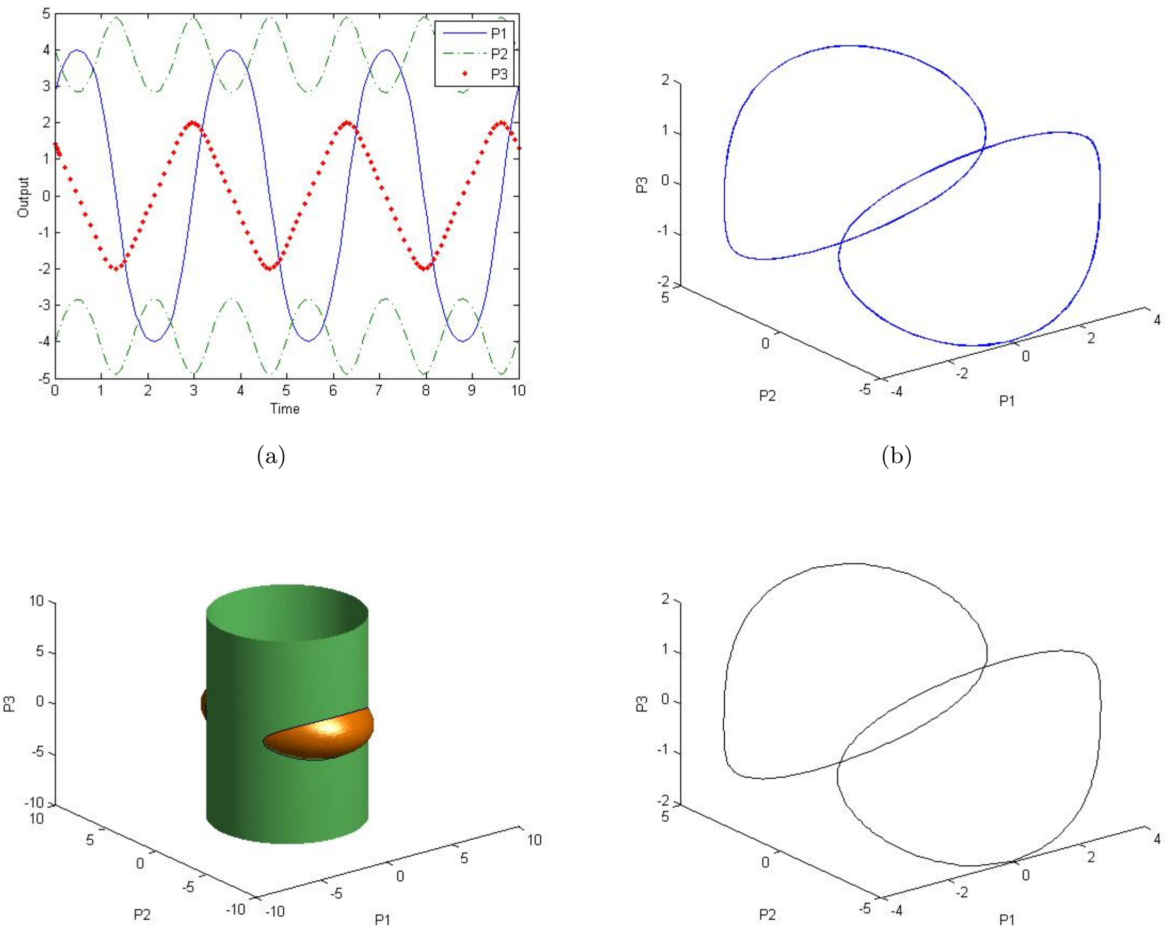


(a) (b)

(c) (d)

Figure C.20: Theorem 10.2.2 **case 2 (i):** $c_1 = 3$, $c2 = 4$, $c3 = 1$, $H = 2.5$, $K = 12$, (a)-(b): sn, (c)-(d): cd

Type III^0

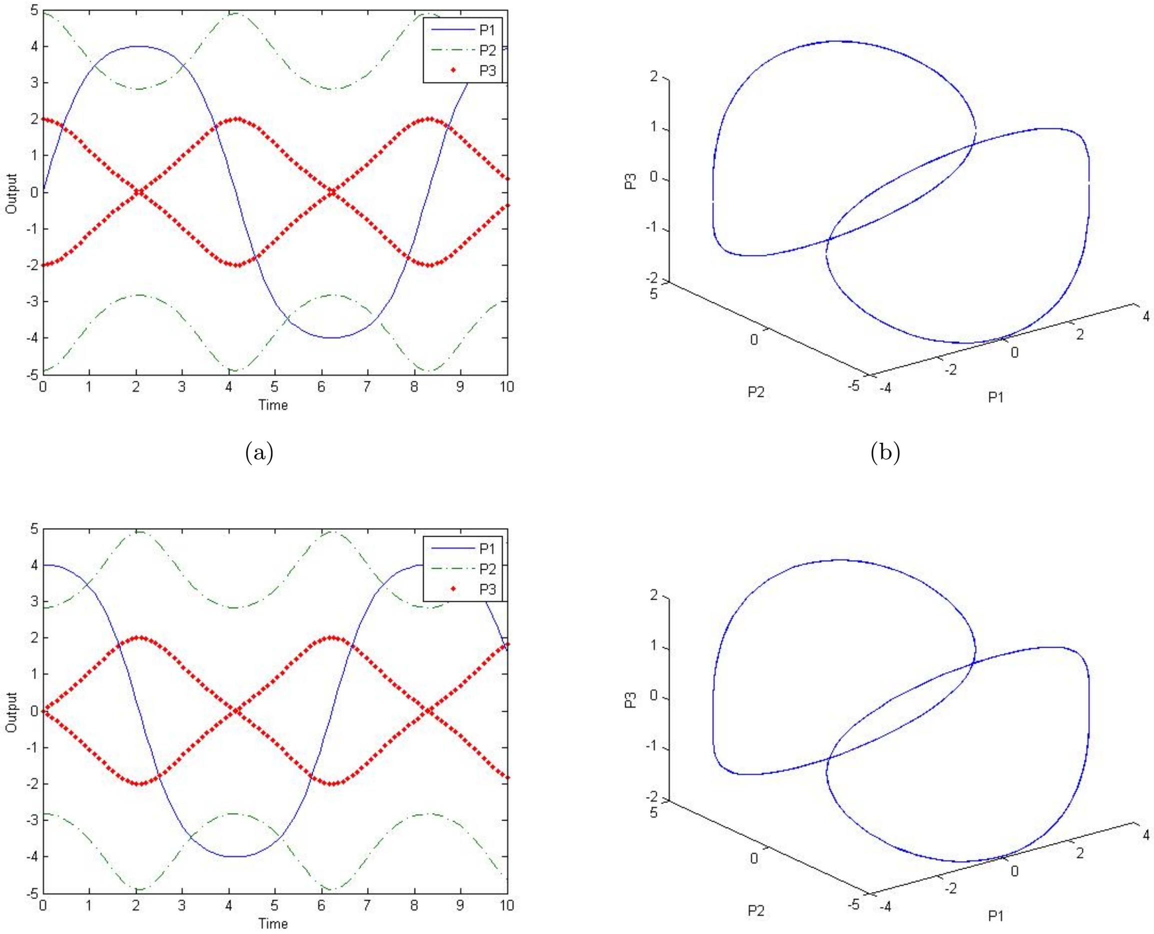


(a) (b)

(c) (d)

Figure C.21: Theorem 10.2.2 **case 2 (ii):** $c_1 = 2$, $c2 = 4$, $c3 = 1$, $P(0) = (\sqrt{8}, \sqrt{16}, \sqrt{2})$ (a)-(b): MATLAB ODE45 solver, (c)-(d): H = 5, K = 24.

Type III^0



(a) (b)

(c) (d)

Figure C.22: Theorem 10.2.2 **case 2 (ii):** $c_1 = 2$, $c2 = 4$, $c3 = 1$, $H = 5$, $K = 24$, (a)-(b): sn, (c)-(d): cd

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