INVARIANT CONTROL SYSTEMS AND SUB-RIEMANNIAN STRUCTURES ON LIE GROUPS: EQUIVALENCE AND ISOMETRIES

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RORY BIGGS

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Abstract

In this thesis we consider invariant optimal control problems and invariant sub-Riemannian structures on Lie groups. Primarily, we are concerned with the equivalence and classification of problems (resp. structures). In the first chapter, both the class of invariant optimal control problems and the class of invariant sub-Riemannian structures are organised as categories. The latter category is shown to be functorially equivalent to a subcategory of the former category. Via the Pontryagin Maximum Principle, we associate to each invariant optimal control problem (resp. invariant sub-Riemannian structure) a quadratic Hamilton-Poisson system on the associated Lie-Poisson space. These Hamiltonian systems are also organised as a category and hence the Pontryagin lift is realised as a contravariant functor. Basic properties of these categories are investigated.

The rest of this thesis is concerned with the classification (and investigation) of certain subclasses of structures and systems. In the second chapter, the homogeneous positive semidefinite quadratic Hamilton-Poisson systems on three-dimensional Lie-Poisson spaces are classified up to compatibility with a linear isomorphism; a list of 23 normal forms is exhibited. In the third chapter, we classify the invariant sub-Riemannian structures in three dimensions and calculate the isometry group for each normal form. By comparing our results with known results, we show that most isometries (in three dimensions) are in fact the composition of a left translation and a Lie group isomorphism. In the fourth and last chapter of this thesis, we classify the sub-Riemannian and Riemannian structures on the $(2n+1)$-dimensional Heisenberg groups. Furthermore, we find the isometry group and geodesics of each normal form.

Key words and phrases: invariant control system, sub-Riemannian geometry, Lie groups, feedback equivalence, isometry, quadratic Hamilton-Poisson systems.

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Contents

Introduction

Geometric control theory began in the late 1960s with the study of (nonlinear) control systems by using concepts and methods from differential geometry (cf. [70, 114]). A (smooth) control system may roughly be described a family of vector fields on a manifold, smoothly parametrized by a set of controls; an integral curve corresponding to some admissible control curve is called a trajectory of the system. The first basic question one asks of a control system is whether or not any two points can be connected by a trajectory: this is known as the controllability problem. Once one has established that two points can be connected by a trajectory, one may wish to find a trajectory that minimizes some (practical) cost function: this is known as the optimality problem. A key result in optimal control theory is the Pontryagin Maximum Principle which provides necessary conditions for optimal trajectories (see, e.g., [14]).

A significant class of control systems rich in symmetry are those evolving on Lie groups and invariant under left translations; for such systems, the left translation of any trajectory is a trajectory. This class of systems was first considered in 1972 by Brockett [45] and by Jurdjevic and Sussmann [73]; it forms a natural geometric framework for various (variational) problems in mathematical physics, mechanics, elasticity, and dynamical systems (cf. [14, 42, 70, 71]). In the last few decades substantial work on applied nonlinear control has drawn attention to left-invariant control affine systems evolving on matrix Lie groups of low dimension (see, e.g., [72, 103, 108, 109] and the references therein). We give two motivating examples of (optimal control) problems; such problems initially stimulated the growth of this research area.

The plate-ball problem ([69]). The plate-ball problem consists of the following kinematic situation: a ball rolls without slipping between two horizontal plates; the lower plate is fixed and the ball is rolled through the horizontal movement of the upper plate. The problem is to transfer the ball from a given initial position and orientation to a prescribed final position and orientation along a path which minimizes $\int_0^T \|v(t)\| dt$. Here $v(t)$ denotes the velocity of the moving plate and T is the time of transfer.

This problem can be regarded as invariant optimal control problem on the five-dimensional Lie group

$$\mathbb{R}^2 \times \mathsf{SO}\,(3) = \left\{ \begin{bmatrix} e^{x_1} & 0 & 0 \\ 0 & e^{x_2} & 0 \\ 0 & 0 & R \end{bmatrix} \,:\, x_1, x_2 \in \mathbb{R},\ R \in \mathsf{SO}\,(3) \right\}$$

specified by the dynamical law

$$\dot{g} = g \left(u_1 \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} + u_2 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \right)$$

boundary conditions $g(0) = g_0$, $g(T) = g_1$, and cost functional $\int_0^T u_1^2 + u_2^2 \, dt \to \min$.

Control of Serret-Frenet systems ([70, p. 25], see also [107, 109]). A differentiable curve $\gamma(t)$ in the Euclidean plane $\mathbb{E}^2$, parametrized by arc length, can be lifted to the group of motions of $\mathbb{E}^2$ by means of a positively oriented orthonormal moving frame v_1, v_2 defined by

$$\dot{\gamma}(t) = v_1(t), \qquad \dot{v}_1(t) = \kappa(t) v_2(t), \qquad \dot{v}_2(t) = -\kappa(t) v_1(t) \tag{0.1}$$

where $\kappa(t)$ is the signed curvature of $\gamma(t)$. The moving frame can be expressed by a rotation matrix $R(t)$ whose columns consist of the coordinates of v_1 and v_2 relative to a fixed orthonormal frame $e_1, e_2 \in \mathbb{E}^2$. Identifying vectors with their coordinate vectors, we have $R(t)e_i = v_i$. The curve $\gamma(t)$, along with its moving frame, can be represented as a curve $g(t)$ on the group of motions of $\mathbb{E}^2$, namely

$$\mathsf{SE}\,(2) = \left\{ \begin{bmatrix} 1 & 0 \;\; 0 \\ \gamma_1 & \\ \gamma_2 & R \end{bmatrix} \, : \, \gamma_1, \gamma_2 \in \mathbb{R}, R \in \mathsf{SO}\,(2) \right\}.$$

Interpreting the curvature $\kappa(t)$ as a control function, the Serret-Frenet differential system (0.1) can then written as an (inhomogeneous) left-invariant control system

$$\dot{g} = g \left(\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \kappa(t) \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \right), \qquad g \in \mathsf{SE}\,(2).$$

In this way, many classic variational problems in geometry become problems in optimal control. For example, the problem of finding a curve $\gamma(t)$ that will satisfy the given boundary conditions $\gamma(0) = a$, $\dot{\gamma}(0) = \dot{a}$, $\gamma(T) = b$, $\dot{\gamma}(T) = \dot{b}$, and will minimize $\int_0^T \kappa^2(t) \, dt$ goes back to Euler; its solutions are known as the elastica.

On the other hand, a sub-Riemannian structure on a smooth manifold consists of a non-integrable distribution with a Riemannian metric on this distribution. Sub-Riemannian structures can be thought of as generalizing Riemannian structures, in the sense that if the distribution is the entire tangent bundle, then we recover a Riemannian structure. When the distribution is bracket generating, then any two points on the manifold can be joined by a curve tangent to the distribution ([47, 94]). A fundamental problem in sub-Riemannian geometry is to determine the minimizing geodesics of a structure, i.e., to find those curves tangent to the distribution that have minimal length (with respect to the given metric).

It is perhaps not surprising that there is some significant overlap between the theory of smooth optimal control problems and geodesics of various structures in differential geometry (cf. [12, 13]). Indeed, (locally) a sub-Riemannian structure can be regarded as a smooth control system, linear in the controls, together with a quadratic cost. Moreover, the Pontryagin Maximum Principle is often used to obtain first order necessary conditions for minimising geodesics of sub-Riemannian structures. “Even in the classical case of Riemannian geometry, the maximum principle approach to finding geodesics leads to a final result much simpler and shorter than the traditional method of using the Levi-Civita connection” ([13]).

A sub-Riemannian structure on a Lie group is said to be left-invariant if the both the distribution and the metric are invariant under left translations (i.e., each left translation by a group element is an isometry of the structure). Such structures are the basic models of sub-Riemannian manifolds and as such serve to elucidate general features of sub-Riemannian geometry (cf. [10]). We note that the former of the two optimal control problems mentioned above (the plate-ball problem) may be reinterpreted as determining a minimising geodesic for a left-invariant sub-Riemannian structure on $\mathbb{R}^2 \times \mathsf{SO}(3)$; the latter problem however cannot be reinterpreted in this way (as the system is affine but not linear in the controls). Recently, left-invariant sub-Riemannian structures (especially on low-dimensional Lie groups) have received a fair amount of attention (see e.g., [3, 10, 17, 18, 44, 46, 89, 92, 111], also [62, 75, 117]).

In this thesis we consider invariant optimal control problems and left-invariant sub-Riemannian structures on Lie groups, as well as the associated Hamilton-Poisson systems (obtained via the Pontryagin Maximum Principle). Primarily, we are interested in the equivalence and interrelations between systems (resp. structures) in each of these classes, as well as the functorial interrelations between these classes themselves. Furthermore, we are also interested in the classification of various subclasses of systems (resp. structures).

Chapter 1 is concerned with formalising (as categories) invariant optimal control problems, left-invariant sub-Riemannian structures, and Hamilton-Poisson systems. The class of invariant optimal control problems (with affine dynamics and quadratic cost) is organised as a category of cost-extended systems. Morphisms in this category are Lie group homomorphisms preserving the underlying left-invariant control affine system as well as the quadratic cost. By means of the Pontryagin Maximum Principle, we associate to each cost-extended system a quadratic Hamilton-Poisson system on the associated Lie-Poisson space; this in turn yields a contravariant functor from the category of cost-extended systems to the category of Hamilton-Poisson systems. Left-invariant sub-Riemannian structures are also organised into a category (with morphisms being Lie group homomorphisms preserving the distribution and the metric) which is shown to be functorially equivalent to a full subcategory of the category of cost-extended systems. Lastly, a few examples illustrating some of the results of the chapter are discussed.

The rest of this thesis is concerned with the classification (and investigation) of certain subclasses of structures (and systems). Generally, there are two strategies followed when investigating (various) structures on Lie groups or Lie algebras. The first strategy is to make a systematic study of structures on low-dimensional Lie groups (or algebras). The second strategy is to investigate structures on a (sufficiently) regular family of groups (usually parametrized by dimension); frequently occurring cases are subclasses of the nilpotent Lie groups. We shall consider structures on the three-dimensional groups and algebras, as well as on the family of Heisenberg groups.

Chapter 2 is concerned with the classification of a class of quadratic Hamilton-Poisson systems in three dimensions. Specifically, we consider the homogeneous quadratic Hamilton-Poisson systems on three-dimensional Lie-Poisson spaces. Such systems are of interest in their own right (apart from arising in invariant optimal control), as class of Hamiltonian dynamical systems. For each Lie-Poisson space, we classify the systems up to linear equivalence (i.e., the associated Hamiltonian vector fields being compatible with a linear isomorphism). Thereafter we determine whether any systems on distinct (non-isomorphic) Lie-Poisson spaces are linearly equivalent and so arrive at a general classification; a list of 23 normal forms is exhibited.

Chapter 3 is concerned with the classification of left-invariant sub-Riemannian structures in three dimensions, as well as the description of isometries between such structures. For each simply connected three-dimensional Lie group, we classify the left-invariant sub-Riemannian structures up to isometric group isomorphism. For each normal form we determine the associated quadratic Hamilton-Poisson system (and its normal form as presented in Chapter 2), as well as the subgroup of isometries fixing the identity. By comparing our results to the classification of Agrachev and Barilari [10], we are able to show that most isometries are in fact the composition of a left translation and a group isomorphism.

Chapter 4 is concerned with sub-Riemannian and Riemannian structures on the $(2n+1)$-dimensional Heisenberg groups. We classify these structures up to isometry. The associated isometry group for each normal form is determined. By use of the isometries, simple expressions for the geodesics are obtained. Finally, the similarities between the results for the Riemannian and sub-Riemannian structures are (partially) explained by a result in the first chapter.

We append (Appendix A) a classification of the three-dimensional Lie groups and Lie algebras, as well as matrix representations of these groups and automorphism groups of the Lie algebras. Appendix B contains some tables relevant to Chapters 2 and 3. Mathematica was used to facilitate some computations in Chapters 2 and 3; sample code for some typical cases is given in Appendix C.

Much of the material presented in this thesis expands on the following publications:

[38] Biggs R., Remsing C.C., Cost-extended control systems on Lie groups, *Mediterr. J. Math.* **11** (2014), 193–215.

[33] Biggs R., Remsing C.C., A classification of quadratic Hamilton-Poisson systems in three dimensions, in Fifteenth International Conference on Geometry, Integrability and Quantization, Editors I.M. Mladenov, A. Ludu, A. Yoshioka, Bulgarian Academy of Sciences, Varna, Bulgaria, 2013, 67–78.

[30] Biggs R., Nagy P.T., A classification of sub-Riemannian structures on the Heisenberg groups, *Acta Polytech. Hungar.* **10** (2013), 41–52.

Among others, the following publications (although not part of the presentation made in this thesis) form part of its conceptualization:

[31] Biggs R., Remsing C.C., A category of control systems, *An. Ştiinţ. Univ. "Ovidius" Constanţa Ser. Mat.* **20** (2012), 355–367.

[37] Biggs R., Remsing C.C., Control systems on three-dimensional Lie groups: equivalence and controllability, *J. Dyn. Control Syst.* **20** (2014), 307–339.

Several papers, expanding on the results of this thesis, are in preparation.

Chapter 1

Invariant systems and structures on Lie groups

In this chapter we consider invariant optimal control problems for which the underlying invariant system is affine in the controls and for which the cost function is quadratic in the controls. This class of problems has received considerable attention in the last few decades. Various physical problems have been modelled in this manner, for instance optimal path planning for airplanes, motion planning for wheeled mobile robots, spacecraft attitude control, and the control of underactuated underwater vehicles ([83, 103, 124]); also, the control of quantum systems and the dynamic formation of DNA ([52, 61]). Many problems (as well as sub-Riemannian structures) on various low-dimensional matrix Lie groups have been considered by a number of authors (see, e.g., [28, 29, 43, 69, 72, 74, 93, 96, 102, 108, 109]).

We wish to develop a language for comparing such invariant optimal control problems; in particular, we want to introduce a form of equivalence. Category theory (see, e.g., [2, 86]) provides a convenient (and elegant) framework for these purposes. Indeed, for various problems in systems and control theory such a framework has proved useful (see, e.g., [31, 97, 115, 116]). For invariant sub-Riemannian structures, our formulation of equivalence turns out to be equivalent to structures being related by an isometric Lie group isomorphism (i.e., an isometry which is also a Lie group isomorphism).

In the first section we formally define what we mean by a left-invariant control affine system. We briefly discuss equivalence of these control systems (our equivalence of optimal control problems involves the equivalence of the underlying control systems). We then formally introduce the class of invariant optimal control problems to be considered and state the Pontryagin Maximum Principle (adapted to invariant optimal control problems).

To each invariant optimal control problem we associate a cost-extended system. In the second section we introduce the category of cost-extended systems and investigate some basic properties. In particular, we show that morphisms in the category behave well with respect to optimal (and extremal) trajectories and characterise equivalence in this category.

By means of the Pontryagin Maximum Principle, we associate to each cost-extended system a quadratic Hamilton-Poisson systems on the associated Lie-Poisson space. In the third section we introduce a category of Hamilton-Poisson systems and realize the Pontryagin lift as a contravariant functor from the category of cost-extended systems to this category. In particular, we find that if two cost-extended systems are equivalent, then their associated Hamilton-Poisson systems are equivalent.

In the fourth section we consider invariant sub-Riemannian structures. We organise these structures as a category and show that this category is functorially equivalent to a full subcategory of the cost-extended systems. We briefly compare our formalism of equivalence (of sub-Riemannian structures) to that of being isometric. Lastly, we show that by promoting some central vector fields of a sub-Riemannian structure to an orthogonal complement of the distribution, one obtains a sub-Riemannian (or Riemannian) structure with very similar geodesics.

In the fifth and last section we provide examples illustrating some of the main results of this chapter (as well as one counter example).

Note. Much of the material (regarding cost-extended systems) presented in this chapter appears in [38] (which is an extension of [32]).

1.1 Invariant control affine systems

A k-input left-invariant control affine system $\Sigma = (\mathsf{G}, \Xi)$ on a (real, finite-dimensional, connected) Lie group G consists of a family of left-invariant vector fields Ξ_u on G, affinely parametrized by controls $u \in \mathbb{R}^k$. Such a system is written as

$$\dot{g} = \Xi_u(g) = \Xi(g, u) = g(A + u_1 B_1 + \cdots + u_k B_k), \qquad g \in \mathsf{G},\ u \in \mathbb{R}^k.$$

Here $A, B_1, \ldots, B_k$ are elements of the Lie algebra $\mathfrak{g}$ with $B_1, \ldots, B_k$ linearly independent. The "product" gA denotes the left translation $T_\mathbf{1} L_g \cdot A$ of $A \in \mathfrak{g}$ by the tangent map of $L_g : \mathsf{G} \to \mathsf{G}$, $h \mapsto gh$. (When G is a matrix Lie group, this product is simply a matrix multiplication.) Note that the dynamics $\Xi : \mathsf{G} \times \mathbb{R}^k \to T\mathsf{G}$ are invariant under left translations, i.e., $\Xi(g, u) = g\,\Xi(\mathbf{1}, u)$. Σ is completely determined by the specification of its state space G and its parametrization map $\Xi(\mathbf{1}, \cdot)$. When G is fixed, we specify $\Sigma = (\mathsf{G}, \Xi)$ by simply writing

$$\Sigma\ :\ A + u_1 B_1 + \cdots + u_k B_k.$$

The *trace* Γ of a system Σ is the affine subspace $\Gamma = A + \Gamma^0 = A + \langle B_1, \ldots, B_k \rangle$ of $\mathfrak{g}$. (Here $\Gamma^0 = \langle B_1, \ldots, B_k \rangle$ is the subspace of $\mathfrak{g}$ spanned by $B_1, \ldots, B_k$.) A system Σ is called *homogeneous* if $A \in \Gamma^0$, and *inhomogeneous* otherwise; Σ is said to be *drift free* if $A = 0$. Also, Σ is said to have *full rank* if its trace generates the whole Lie algebra, i.e., $\mathsf{Lie}(\Gamma) = \mathfrak{g}$.

Remark 1.1. We have the following characterization of the full-rank condition for systems on G when $\dim \mathsf{G} = 3$. No homogeneous single-input system has full rank. An inhomogeneous single-input system has full rank if and only if A, B_1, and $[A, B_1]$ are linearly independent. A homogeneous two-input system has full rank if and only if B_1, B_2, and $[B_1, B_2]$ are linearly independent. Any inhomogeneous two-input or (homogeneous) three-input system has full rank.

The admissible controls are piecewise continuous maps $u(\cdot) : [0, T] \to \mathbb{R}^k$. A *trajectory* for an admissible control $u(\cdot)$ is an absolutely continuous curve $g(\cdot) : [0, T] \to \mathsf{G}$ such that

$\dot{g}(t) = g(t)\,\Xi\,(\mathbf{1}, u(t))$ for almost every $t \in [0, T]$; the pair $(g(\cdot), u(\cdot))$ is referred to as a *controlled trajectory*. We say that a system Σ is *controllable* if for any $g_0, g_1 \in \mathsf{G}$, there exists a trajectory $g(\cdot) : [0, T] \to \mathsf{G}$ such that $g(0) = g_0$ and $g(T) = g_1$. If Σ is controllable, then it has full rank. For more details about invariant control systems see, e.g., [14, 70, 73, 110].

Remark 1.2. If $(g(\cdot), u(\cdot))$ is a controlled trajectory, then its left translation $(h\,g(\cdot), u(\cdot))$ is a controlled trajectory.

Note. Left-invariant control systems on Lie groups were organised as a category in [31]; some basic properties were investigated.

1.1.1 Equivalence of systems

The most natural equivalence relation for control systems is equivalence up to coordinate changes in the state space. This is called *state space equivalence* (see [68]). State space equivalence is well understood. It establishes a one-to-one correspondence between the trajectories of the equivalent systems. However, this equivalence relation is very strong. In the (general) analytic case, Krener characterized local state space equivalence in terms of the existence of a linear isomorphism preserving iterated Lie brackets of the system's vector fields ([77], see also [14, 113, 114]).

Another fundamental equivalence relation for control systems is that of *feedback equivalence*. Two feedback equivalent control systems have the same set of trajectories (up to a diffeomorphism in the state space) which are parametrized differently by admissible controls. Feedback equivalence has been extensively studied in the last few decades (see [105] and the references therein).

We briefly consider these equivalences in the context of left-invariant control systems ([41]). Characterizations of state space equivalence and (detached) feedback equivalence are obtained in terms of Lie group isomorphisms.

State space equivalence

Two systems Σ and Σ' are called *state space equivalent* if there exists a diffeomorphism $\phi : \mathsf{G} \to \mathsf{G}'$ such that for each control value $u \in \mathbb{R}^k$ the vector fields Ξ_u and Ξ'_u are ϕ-related, i.e., $T_g\phi \cdot \Xi\,(g, u) = \Xi'\,(\phi(g), u)$ for $g \in \mathsf{G}$ and $u \in \mathbb{R}^k$. We have the following simple algebraic characterization of this equivalence.

Proposition 1.3 ([41], see also [77]). *Two full-rank systems Σ and Σ' are state space equivalent if and only if there exists a Lie group isomorphism $\phi : \mathsf{G} \to \mathsf{G}'$ such that $T_{\mathbf{1}}\phi \cdot \Xi\,(\mathbf{1}, u) = \Xi'\,(\mathbf{1}, u)$ for all $u \in \mathbb{R}^k$.*

Proof sketch. Suppose that Σ and Σ' are state space equivalent equivalent. By composition with a left translation, we may assume $\phi(\mathbf{1}) = \mathbf{1}$. As the elements $\Xi_u(\mathbf{1})$, $u \in \mathbb{R}^k$ generate $\mathfrak{g}$ and the push-forward $\phi_*\Xi_u$ of left-invariant vector fields Ξ_u are left invariant, it follows that ϕ is a Lie group isomorphism satisfying the requisite property (cf. [31]). Conversely, suppose that $\phi : \mathsf{G} \to \mathsf{G}'$ is a Lie group isomorphism as prescribed. Then $T_g\phi \cdot \Xi(g, u) = T_{\mathbf{1}}(\phi \circ L_g) \cdot \Xi(\mathbf{1}, u) = T_{\mathbf{1}}(L_{\phi(g)} \circ \phi) \cdot \Xi(\mathbf{1}, u) = \Xi'(\phi(g), u)$. □

State space equivalence is quite a strong equivalence relation. Hence, there are so many equivalence classes that any general classification appears to be very difficult if not impossible. However, there is a chance for some reasonable classification in low dimensions. We give an example to illustrate this point.

Example 1.4 ([4])**.** Any two-input inhomogeneous full-rank control affine system on the Euclidean group $\mathsf{SE}(2)$ is state space equivalent to exactly one of the following systems

$$\begin{aligned}
\Sigma_{1,\alpha\beta\gamma} &: \alpha E_3 + u_1(E_1 + \gamma_1 E_2) + u_2(\beta E_2) \\
\Sigma_{2,\alpha\beta\gamma} &: \beta E_1 + \gamma_1 E_2 + \gamma_2 E_3 + u_1(\alpha E_3) + u_2 E_2 \\
\Sigma_{3,\alpha\beta\gamma} &: \beta E_1 + \gamma_1 E_2 + \gamma_2 E_3 + u_1(E_2 + \gamma_3 E_3) + u_2(\alpha E_3).
\end{aligned}$$

Here $\alpha > 0, \beta \neq 0$ and $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{R}$, with different values of these parameters yielding distinct (non-equivalent) class representatives.

Note. A complete classification (under state space equivalence) of systems on $\mathsf{SE}(2)$ appears in [4], whereas a classification of systems on $\mathsf{SE}(1,1)$ appears in [24]. For a classification of systems on $\mathsf{SO}(2,1)_0$, see [40].

Detached feedback equivalence

We specialize feedback equivalence in the context of invariant systems by requiring that the feedback transformations are compatible with the Lie group structure. Two systems Σ and Σ' are called *detached feedback equivalent* if there exist diffeomorphisms $\phi : \mathsf{G} \to \mathsf{G}'$ and $\varphi : \mathbb{R}^k \to \mathbb{R}^k$ such that for each control value $u \in \mathbb{R}^k$ the vector fields Ξ_u and $\Xi'_{\varphi(u)}$ are ϕ-related, i.e., $T_g\phi \cdot \Xi(g,u) = \Xi'(\phi(g), \varphi(u))$ for $g \in \mathsf{G}$ and $u \in \mathbb{R}^k$. We have the following simple algebraic characterization of this equivalence in terms of the traces $\Gamma = \operatorname{im} \Xi(\mathbf{1}, \cdot)$ and $\Gamma' = \operatorname{im} \Xi'(\mathbf{1}, \cdot)$ of Σ and Σ'.

Proposition 1.5 ([41])**.** *Two full-rank systems Σ and Σ' are detached feedback equivalent if and only if there exists a Lie group isomorphism $\phi : \mathsf{G} \to \mathsf{G}'$ such that $T_{\mathbf{1}}\phi \cdot \Gamma = \Gamma'$.*

Proof sketch. Suppose Σ and Σ' are detached feedback equivalent. By composing ϕ with an appropriate left translation, we may assume $\phi(\mathbf{1}) = \mathbf{1}'$. Hence $T_{\mathbf{1}}\phi \cdot \Xi(\mathbf{1}, u) = \Xi'(\mathbf{1}', \varphi(u))$ and so $T_{\mathbf{1}}\phi \cdot \Gamma = \Gamma'$. Moreover, as the elements $\Xi_u(\mathbf{1})$, $u \in \mathbb{R}^k$ generate $\mathfrak{g}$ and the push-forward of the left-invariant vector fields Ξ_u are left invariant, it follows that ϕ is a group isomorphism (cf. [31]). On the other hand, suppose there exists a group isomorphism $\phi : \mathsf{G} \to \mathsf{G}'$ such that $T_{\mathbf{1}}\phi \cdot \Gamma = \Gamma'$. Then there exists a unique affine isomorphism $\varphi : \mathbb{R}^k \to \mathbb{R}^{k'}$ such that $T_{\mathbf{1}}\phi \cdot \Xi(\mathbf{1}, u) = \Xi'(\mathbf{1}', \varphi(u))$. As with state space equivalence, by left-invariance and the fact that ϕ is a Lie group isomorphism, it then follows that $T_g\phi \cdot \Xi(g, u) = \Xi'(\phi(g), \varphi(u))$. □

Detached feedback equivalence is notably weaker than state space equivalence. To illustrate this point, we give a classification, under detached feedback equivalence, of the same class of systems considered in Example 1.4.

Example 1.6 ([36,37])**.** Any two-input inhomogeneous full-rank control affine system on $\mathsf{SE}(2)$ is detached feedback equivalent to exactly one of the following systems

$$\begin{aligned}\Sigma_1 &: E_1 + u_1 E_2 + u_2 E_3 \\ \Sigma_{2,\alpha} &: \alpha E_3 + u_1 E_1 + u_2 E_2.\end{aligned}$$

Here $\alpha > 0$ parametrizes a family of class representatives, each different value corresponding to a distinct non-equivalent representative.

Note. Equivalence and controllability of left-invariant control affine systems on three-dimensional matrix Lie groups was investigated in [37] (see also [34–36]). On higher dimensional Lie groups, left-invariant control affine systems have been classified on the six-dimensional orthogonal group $\mathsf{SO}(4)$ [6] and the four-dimensional oscillator group [39].

1.1.2 Optimal control and Pontryagin Maximum Principle

We shall consider the class of left-invariant optimal control problems on Lie groups with fixed terminal time, affine dynamics, and affine quadratic cost. Formally, such problems are written as

$$\dot{g} = g\,(A + u_1 B_1 + \cdots + u_k B_k)\,, \quad g \in \mathsf{G},\ u \in \mathbb{R}^k \tag{1.1}$$

$$g(0) = g_0, \quad g(T) = g_1 \tag{1.2}$$

$$\mathcal{J} = \int_0^T \chi(u(t))\,dt \longrightarrow \min, \qquad \chi(u) = (u - \upsilon)^\top Q\,(u - \upsilon). \tag{1.3}$$

Here G is a (real, finite-dimensional) connected Lie group with Lie algebra $\mathfrak{g}$, A, $B_1, \ldots, B_k \in \mathfrak{g}$ (with $B_1, \ldots, B_k$ linearly independent), $u = (u_1, \ldots, u_k) \in \mathbb{R}^k$, $\upsilon \in \mathbb{R}^k$, and Q is a positive definite $k \times k$ matrix. To each such problem, we associate a *cost-extended system* (Σ, χ). Here Σ is the control system (1.1) and the cost function $\chi : \mathbb{R}^k \to \mathbb{R}$ has the form $\chi(u) = (u - \upsilon)^\top Q\,(u - \upsilon)$. Each cost-extended system corresponds to a family of invariant optimal control problems; by specification of the boundary data $\mathcal{B}(g_0, g_1, T)$, the associated problem is uniquely determined.

The Pontryagin Maximum Principle provides necessary conditions for optimality which are naturally expressed in the language of the geometry of the cotangent bundle $T^*\mathsf{G}$ of G (see [14, 57, 70]). The cotangent bundle $T^*\mathsf{G}$ can be trivialized (from the left) such that $T^*\mathsf{G} = \mathsf{G} \times \mathfrak{g}^*$; here $\mathfrak{g}^*$ is the dual of the Lie algebra $\mathfrak{g}$. More precisely, each element $(g, p) \in \mathsf{G} \times \mathfrak{g}^*$ is identified with $(T_g L_{g^{-1}})^* \cdot p \in T_g^*\mathsf{G}$. To an optimal control problem (1.1)–(1.2)–(1.3) we associate, for each real number λ and each control parameter $u \in \mathbb{R}^k$ a Hamiltonian function on $T^*\mathsf{G} = \mathsf{G} \times \mathfrak{g}^*$:

$$\begin{aligned}H_u^\lambda(\xi) &= \lambda\chi(u) + \xi(\Xi_u(g)) \\ &= \lambda\chi(u) + p(\Xi_u(\mathbf{1})), \qquad \xi = (g, p) \in T^*\mathsf{G}.\end{aligned} \tag{1.4}$$

We denote by $\vec{H}_u^\lambda$ the corresponding Hamiltonian vector field (with respect to the symplectic structure on $T^*\mathsf{G}$).

Lemma 1.7 ([70, Chapter 12], see also [14, Chapter 18]). *A curve* $\xi(\cdot) = (g(\cdot), p(\cdot))$ *is an integral curve of (the time varying Hamiltonian vector field)* $\vec{H}^{\lambda}_{u(t)}$ *if and only if*

$$\dot{g}(t) = \Xi(g(t), u(t)) \qquad \qquad \dot{p}(t) = \mathrm{ad}^* \, \Xi_{u(t)}(\mathbf{1}) \cdot p(t).$$

Here $(\mathrm{ad}^* \, A \cdot p)(B) = p([A, B])$ *for* $A, B \in \mathfrak{g}$ *and* $p \in \mathfrak{g}^*$.

In terms of the above Hamiltonians, the Maximum Principle can be stated as follows.

Maximum Principle. *Suppose the controlled trajectory* $(\bar{g}(\cdot), \bar{u}(\cdot))$ *defined over the interval* $[0, T]$ *is a solution for the optimal control problem* (1.1)–(1.2)–(1.3). *Then, there exists a curve* $\xi(\cdot) : [0, T] \to T^*\mathsf{G}$ *with* $\xi(t) \in T^*_{\bar{g}(t)}\mathsf{G}$, $t \in [0, T]$, *and there exists a real number* $\lambda \leq 0$, *such that the following conditions hold for almost every* $t \in [0, T]$ *:*

$$(\lambda, \xi(t)) \not\equiv (0, 0) \tag{1.5}$$

$$\dot{\xi}(t) = \vec{H}^{\lambda}_{\bar{u}(t)}(\xi(t)) \tag{1.6}$$

$$H^{\lambda}_{\bar{u}(t)}(\xi(t)) = \max_{u} H^{\lambda}_{u}(\xi(t)) = \textit{constant}. \tag{1.7}$$

Any optimal trajectory, $g(\cdot) : [0, T] \to \mathsf{G}$ is the projection of an integral curve $\xi(\cdot)$ of the (time-varying) Hamiltonian vector field $\vec{H}^{\lambda}_{\bar{u}(t)}$. A trajectory-control pair $(\xi(\cdot), u(\cdot))$ is said to be an extremal pair if it satisfies the conditions (1.5), (1.6), and (1.7). We call the projection $(g(\cdot), u(\cdot))$ of an extremal pair an extremal controlled trajectory (here $g(t) = \pi(\xi(t))$ where $\pi : T^*\mathsf{G} \to \mathsf{G}$ is the canonical projection). An extremal controlled trajectory is called normal if $\lambda < 0$ and abnormal if $\lambda = 0$.

1.2 Cost-extended systems

1.2.1 Definition and basic properties

We define now the concrete category $\mathbf{LiCAS}^{\bowtie}$ of cost-extended left-invariant control affine systems. An *object* is a cost-extended system (Σ, χ), where the system $\Sigma = (\mathsf{G}, \Xi)$ is a (k-input) left-invariant control affine system and the cost $\chi : \mathbb{R}^k \to \mathbb{R}$, $u \mapsto (u - v)^\top Q \, (u - v)$ is a positive definite affine quadratic form. A *morphism* $\Phi = (\phi, \varphi) : (\Sigma, \chi) \to (\Sigma', \chi')$ is a mapping

$$\Phi : \mathsf{G} \times \mathbb{R}^k \to \mathsf{G}' \times \mathbb{R}^{k'}, \quad (g, u) \mapsto (\phi(g), \varphi(u))$$

where the *state component* $\phi : \mathsf{G} \to \mathsf{G}'$ is a Lie group homomorphism and the *feedback component* $\varphi : \mathbb{R}^k \to \mathbb{R}^{k'}$ is an affine isomorphism (in particular, $k = k'$), such that the diagrams

$$\begin{CD} \mathsf{G} \times \mathbb{R}^k @>{\Phi}>> \mathsf{G}' \times \mathbb{R}^{k'} \\ @V{\Xi}VV @VV{\Xi'}V \\ T\mathsf{G} @>>{T\phi}> T\mathsf{G}' \end{CD} \qquad \qquad \begin{CD} \mathbb{R}^k @>{\varphi}>> \mathbb{R}^{k'} \\ @V{\chi}VV @VV{\chi'}V \\ \mathbb{R} @>>{\delta_r}> \mathbb{R} \end{CD}$$

commute, or equivalently

$$T_g\phi \cdot \Xi_u(g) = \Xi'_{\varphi(u)}(\phi(g)) \qquad \text{and} \qquad \chi' \circ \varphi = r\chi$$

for some $r > 0$. Here δ_r denotes the dilation by r.

Remark 1.8. If $\Phi = (\phi, \varphi) : (\Sigma, \chi) \to (\Sigma', \chi')$ and $\Phi' = (\phi', \varphi') : (\Sigma', \chi') \to (\Sigma'', \chi'')$ are morphisms, then $\Phi' \circ \Phi = (\phi' \circ \phi, \varphi' \circ \varphi) : (\Sigma, \chi) \to (\Sigma'', \chi'')$ is indeed a morphism as

$$T_g(\phi' \circ \phi) \cdot \Xi_u(g) = T_{\phi(g)}\phi' \cdot T_g\phi \cdot \Xi_u(g) = T_{\phi(g)}\phi' \cdot \Xi'_{\varphi(u)}(\phi(g)) = \Xi''_{(\varphi' \circ \varphi)(u)}((\phi' \circ \phi)(g))$$

and $\chi'' \circ (\varphi' \circ \varphi) = r'\chi' \circ \varphi = rr'\chi$.

Lemma 1.9. *Let* $\Phi = (\phi, \varphi) : (\Sigma, \chi) \to (\Sigma', \chi')$ *be a* $\mathbf{LiCAS}^{\bowtie}$*-morphism.*

1. *The constant* r *is unique.*
2. *The state component* $\phi : \mathsf{G} \to \mathsf{G}'$ *of a* $\mathbf{LiCAS}^{\bowtie}$*-morphism maps controlled trajectories to controlled trajectories.*
3. *The feedback component* $\varphi : \mathbb{R}^k \to \mathbb{R}^k$ *uniquely determined by* $T_{\mathbf{1}}\phi \cdot \Xi(\mathbf{1}, u) = \Xi'(\mathbf{1}, \varphi(u))$.

Proof. (1). If $\chi' \circ \varphi = r\chi$, then $\frac{(\chi' \circ \varphi)(u)}{\chi(u)} = r$ for $u \neq v$.

(2). Suppose $\Phi = (\phi, \varphi) : (\Sigma, \chi) \to (\Sigma', \chi')$ is a morphism and let $(g(\cdot), u(\cdot))$ be a controlled trajectory of Σ. Then $\frac{d}{dt}\phi(g(t)) = T_{g(t)}\phi \cdot \Xi(g(t), u(t)) = \Xi'(\phi(g(t)), \varphi(u(t)))$, i.e., $(\phi(g(\cdot)), \varphi(u(\cdot)))$ is a controlled trajectory of Σ'.

(3). The map $\Xi'(\mathbf{1}, \cdot) : \mathbb{R}^k \to \Gamma$ is an affine isomorphism and hence has inverse $\Xi'^{-1} : \Gamma \to \mathbb{R}^k$. Consequently, if $T_{\mathbf{1}}\phi \cdot \Xi(\mathbf{1}, u) = \Xi'(\mathbf{1}, \varphi(u))$ for $u \in \mathbb{R}^k$, then $\varphi = \Xi^{-1} \circ T_{\mathbf{1}}\phi \circ \Xi(\mathbf{1}, \cdot)$. □

We can characterise morphisms at the level of Lie algebras as follows.

Lemma 1.10. $\Phi = (\phi, \varphi) : (\Sigma = (\mathsf{G}, \Xi), \chi) \to (\Sigma' = (\mathsf{G}', \Xi'), \chi')$ *is a morphism if and only if* $\phi : \mathsf{G} \to \mathsf{G}'$ *is a Lie group homomorphism,* $\varphi : \mathbb{R}^k \to \mathbb{R}^{k'}$ *is an affine isomorphism, and for some* $r > 0$ *the diagrams*

$$\begin{array}{ccc} \mathbb{R}^k & \xrightarrow{\varphi} & \mathbb{R}^{k'} \\ {\scriptstyle \Xi(\mathbf{1},\cdot)}\downarrow & & \downarrow{\scriptstyle \Xi'(\mathbf{1},\cdot)} \\ \mathfrak{g} & \xrightarrow[T_{\mathbf{1}}\phi]{} & \mathfrak{g}' \end{array} \qquad\qquad \begin{array}{ccc} \mathbb{R}^k & \xrightarrow{\varphi} & \mathbb{R}^{k'} \\ {\scriptstyle \chi}\downarrow & & \downarrow{\scriptstyle \chi'} \\ \mathbb{R} & \xrightarrow[\delta_r]{} & \mathbb{R} \end{array}$$

commute, or equivalently,

$$T_{\mathbf{1}}\phi \cdot \Xi(\mathbf{1}, u) = \Xi'(\mathbf{1}, \varphi(u)) \qquad \text{and} \qquad \chi' \circ \varphi = r\chi.$$

Proof. Necessity follows by definition. Suppose the maps ϕ and φ, and a constant $r > 0$, satisfy the conditions of the proposition. As ϕ is a Lie group morphism, we have $(\phi \circ L_h)(g) = (L_{\phi(h)} \circ \phi)(g)$ for $g, h \in \mathsf{G}$. Hence $T_g\phi \cdot \Xi(g, u) = T_{\mathbf{1}}L_{\phi(g)} \cdot T_{\mathbf{1}}\phi \cdot \Xi(\mathbf{1}, u)$. Therefore

$$T_g\phi \cdot \Xi(g, u) = T_{\mathbf{1}}L_{\phi(g)} \cdot T_{\mathbf{1}}\phi \cdot \Xi(\mathbf{1}, u) = T_{\mathbf{1}}L_{\phi(g)} \cdot \Xi'(\mathbf{1}, \varphi(u)) = \Xi'(\phi(g), \varphi(u)).$$ □

Let $(g(\cdot), u(\cdot))$ be a controlled trajectory, defined over an interval $[0, T]$, of a cost-extended system (Σ, χ). We say $(g(\cdot), u(\cdot))$ is a *virtually optimal controlled trajectory* or *VOCT* of (Σ, χ) if it is a solution for the associated optimal control problem (with boundary data $\mathcal{B}(g(0), g(T), T)$). On the other hand, we say $(g(\cdot), u(\cdot))$ is a normal (resp. abnormal) *extremal controlled trajectory* or *ECT* of (Σ, χ) if it satisfies the necessary conditions of the Pontryagin Maximum Principle (see Section 1.1.2).

Lemma 1.11. *If $\big(g(\cdot), u(\cdot)\big)$ is a VOCT (resp. ECT), then its left translation $\big(h\, g(\cdot), u(\cdot)\big)$ by $h \in \mathsf{G}$ is a VOCT (resp. ECT).*

Proof. Suppose $(g(\cdot), u(\cdot))$ is a VOCT, but the controlled trajectory $\big(h\, g(\cdot), u(\cdot)\big)$ is not a VOCT. Then there exists a controlled trajectory $(g'(\cdot), u'(\cdot))$ such that $g'(0) = h\, g(0)$, $g'(T) = h\, g(T)$ and $\mathcal{J}(u'(\cdot)) = \int_0^T \chi(u'(t)) < \int_0^T \chi(u(t)) = \mathcal{J}(u(\cdot))$. Accordingly, $(h^{-1} g'(\cdot), u(\cdot))$ is a controlled trajectory such that $h^{-1}g'(0) = g(0)$, $h^{-1}g'(T) = g(T)$, and $\mathcal{J}(u'(\cdot)) < \mathcal{J}(u(\cdot))$. However, this contradicts the fact that $(g(\cdot), u(\cdot))$ is a VOCT. Hence the left translation $\big(h\, g(\cdot), u(\cdot)\big)$ of a VOCT $\big(g(\cdot), u(\cdot)\big)$ must be a VOCT.

Suppose $(g(\cdot), u(\cdot))$ is an ECT. Then there exists $p(\cdot) : [0, T] \to \mathfrak{g}*$ such that $(\xi(\cdot), u(\cdot))$, $\xi(\cdot) = (g(\cdot), p(\cdot))$ satisfies (1.5)-(1.6)-(1.7). We claim that $(\xi'(\cdot), u(\cdot))$, $\xi'(\cdot) = (h\, g(\cdot), p(\cdot))$ also satisfies (1.5)-(1.6)-(1.7) and so $(h\, g(\cdot), u(\cdot))$ is an ECT. The condition $(\lambda, \xi(t)) \not\equiv (0, 0)$ (1.5) is equivalent to $(\lambda, p(\cdot)) \not\equiv 0$ and thus holds. As $\frac{d}{dt} h\, g(t) = h\Xi(g(t), u(t)) = \Xi(h\, g(t), u(t))$, it follows by Lemma 1.7 that (1.6) holds. Finally, as the Hamiltonian H_u^λ (1.4) is G-invariant, it follows that (1.7) holds. □

We now investigate the compatibility of VOCTs and ECTs with morphisms. First, we show that if the image under a morphism of a controlled trajectory is a VOCT (resp. ECT), then that controlled trajectory is a VOCT (resp. ECT).

Theorem 1.12. *Let $\Phi = (\phi, \varphi) : (\Sigma, \chi) \to (\Sigma', \chi')$ be a $\mathbf{LiCAS}^{\bowtie}$-morphism and let $\big(g(\cdot), u(\cdot)\big)$ be a controlled trajectory of Σ.*

1. *If $\big(\phi \circ g(\cdot), \varphi \circ u(\cdot)\big)$ is a VOCT of (Σ', χ'), then $\big(g(\cdot), u(\cdot)\big)$ is a VOCT of (Σ, χ).*

2. *If $(\phi \circ g(\cdot), \varphi \circ u(\cdot))$ is a normal ECT of (Σ', χ'), then $(g(\cdot), u(\cdot))$ is a normal ECT of (Σ', χ').*

3. *If Φ is an epimorphism and $(\phi \circ g(\cdot), \varphi \circ u(\cdot))$ is an abnormal ECT of (Σ', χ'), then $(g(\cdot), u(\cdot))$ is an abnormal ECT of (Σ', χ').*

Proof. (1). Suppose $(g(\cdot), u(\cdot))$ is a controlled trajectory of (Σ, χ) such that its image $\big(\phi \circ g(\cdot), \varphi \circ u(\cdot)\big)$ is a VOCT of (Σ', χ'), but $(g(\cdot), u(\cdot))$ is not a VOCT of (Σ, χ). Then there exists another controlled trajectory $(h(\cdot), v(\cdot))$ of (Σ, χ) such that $h(0) = g(0)$, $h(T) = g(T)$, and

$$\mathcal{J}(v(\cdot)) = \int_0^T \chi(v(t))dt < \int_0^T \chi(u(t))dt = \mathcal{J}(u(\cdot)).$$

Hence $\big(\phi \circ h(\cdot), \varphi \circ v(\cdot)\big)$ is a controlled trajectory of (Σ', χ') such that (for some $r > 0$)

$$\begin{aligned}\mathcal{J}'(\varphi \circ v(\cdot)) &= \int_0^T (\chi' \circ \varphi)(v(t))\, dt = r \int_0^T \chi(v(t))\, dt \\ &< r \int_0^T \chi(u(t))\, dt = \int_0^T (\chi' \circ \varphi)(u(t))\, dt = \mathcal{J}'(\varphi \circ u(\cdot)).\end{aligned}$$

This contradicts the fact that $(\phi \circ g(\cdot), \varphi \circ u(\cdot))$ is a VOCT of (Σ', χ').

(2 and 3). The Hamiltonian functions (1.4) associated to (Σ, χ) and (Σ', χ') are given by

$$H_u(g, p) = p(\Xi_u(\mathbf{1})) + \lambda\chi(u) \quad \text{and} \quad H'_{u'}(g', p') = p'(\Xi'_{u'}(\mathbf{1})) + \lambda\chi'(u)$$

respectively (for a fixed λ). Let $g'(\cdot) = \phi \circ g(\cdot)$ and $u'(\cdot) = \varphi \circ u(\cdot)$. We suppose $(g'(\cdot), u'(\cdot))$ is an ECT of (Σ', χ'). Then there exists $p'(\cdot) : [0, T] \to \mathfrak{g}^*$ such that $(\xi'(\cdot), u'(\cdot))$, $\xi'(t) = (g'(t), p'(t))$ satisfies (1.5)-(1.6)-(1.7). As $\dot{\xi}'(t) = \vec{H}'_{u(t)}(\xi'(t))$ (1.6) we have (see Lemma 1.7)

$$\dot{g}'(t) = \Xi'(g'(t), u'(t)) \qquad\qquad \dot{p}'(t) = \mathrm{ad}^*\, \Xi'_{u'(t)}(\mathbf{1}) \cdot p'(t).$$

Here $(\mathrm{ad}^*\, A \cdot p)(B) = p([A, B])$ for $A, B \in \mathfrak{g}'$ and $p \in \mathfrak{g}'^*$. Let $p(\cdot) = \frac{1}{r}\,(T_{\mathbf{1}}\phi)^* \cdot p'(\cdot)$ and $\xi(t) = (g(t), p(t))$. (Here $r > 0$ is the unique constant associated to Φ).

We show that $(\xi(\cdot), u(\cdot))$ satisfies (1.6)-(1.7). By assumption we have that $\dot{g}(t) = \Xi(g(t), u(t))$. Thus, in order to satisfy (1.6), we are left to show that $\dot{p}(t) = \mathrm{ad}^*\, \Xi_{u(t)}(\mathbf{1}) \cdot p(t)$. Indeed, for $A \in \mathfrak{g}^*$ we have

$$\begin{aligned}\dot{p}(t) \cdot A &= \tfrac{1}{r}\left((T_{\mathbf{1}}\phi)^* \cdot \dot{p}'(t)\right) \cdot A \\ &= \tfrac{1}{r}\left(\dot{p}'(t)\right) \cdot (T_{\mathbf{1}}\phi \cdot A) \\ &= \tfrac{1}{r}\left(\mathrm{ad}^*\, \Xi'_{u'(t)}(\mathbf{1}) \cdot p'(t)\right) \cdot (T_{\mathbf{1}}\phi \cdot A) \\ &= \tfrac{1}{r}\, p'(t) \cdot \left[\Xi'_{u'(t)}(\mathbf{1}), T_{\mathbf{1}}\phi \cdot A\right] \\ &= \tfrac{1}{r}\, p'(t) \cdot \left[T_{\mathbf{1}}\phi \cdot \Xi_{u(t)}(\mathbf{1}), T_{\mathbf{1}}\phi \cdot A\right] \\ &= \tfrac{1}{r}\, p'(t) \cdot \left(T_{\mathbf{1}}\phi \cdot \left[\Xi_{u(t)}(\mathbf{1}), A\right]\right) \\ &= \left(\mathrm{ad}^*\, \Xi_{u(t)}(\mathbf{1}) \cdot p(t)\right) \cdot A.\end{aligned}$$

In order to show that $(\xi(\cdot), u(\cdot))$ satisfies (1.7), we first show that $H_u(\xi(t)) = \frac{1}{r} H'_{\varphi(u)}(\xi'(t))$ for $u \in \mathbb{R}^k$. Indeed,

$$\begin{aligned}H_u(\xi(t)) &= p(t) \cdot \Xi_u(\mathbf{1}) + \lambda\chi(u) \\ &= (\tfrac{1}{r}(T_{\mathbf{1}}\phi)^* \cdot p'(t)) \cdot \Xi_u(\mathbf{1}) + \lambda\chi(u) \\ &= \tfrac{1}{r} p'(t) \cdot \Xi'_{\varphi(u)}(\mathbf{1}) + \tfrac{1}{r}\lambda\chi'(\varphi(u)) \\ &= \tfrac{1}{r} H'_{\varphi(u)}(\xi'(t)).\end{aligned}$$

Hence, it follows that

$$\begin{aligned} H_{u(t)}(\xi(t)) &= \tfrac{1}{r} H'_{u'(t)}(\xi'(t)) \\ &= \max_{u' \in \mathbb{R}^{k'}} \tfrac{1}{r} H'_{u'}\left(\xi'(t)\right) \qquad\qquad (= \text{constant}) \\ &= \max_{u \in \mathbb{R}^{k}} \tfrac{1}{r} H'_{\varphi(u)}\left(\xi'(t)\right) \\ &= \max_{u \in \mathbb{R}^{k}} H_u\left(\xi(t)\right) = \text{constant} \end{aligned}$$

It is left to show that (1.5) holds when $(g'(\cdot), u'(\cdot))$ is a normal or when $(g'(\cdot), u'(\cdot))$ is abnormal and Φ is an epimorphism. If $(g'(\cdot), u'(\cdot))$ is normal, then $\lambda < 0$ and so $(\lambda, \xi(\cdot)) \not\equiv 0$. Suppose Φ is an epimorphism and $(g'(\cdot), u'(\cdot))$ is abnormal. As the state component ϕ is surjective we have that $(T_{\mathbf{1}}\phi)^*$ is injective. Hence, as $(\lambda, \xi'(\cdot)) \not\equiv 0$, i.e., $p'(\cdot) \not\equiv 0$, it follows that $p(\cdot) = \frac{1}{r}(T_{\mathbf{1}}\phi)^* \cdot p'(t) \not\equiv 0$, i.e., $(\lambda, \xi(\cdot)) \not\equiv 0$. □

Next we show that, under a $\mathbf{LiCAS}^{\bowtie}$-epimorphism, any VOCT (resp. ECT) is the image of some VOCT (resp. ECT).

Theorem 1.13. *Let $\Phi = (\phi, \varphi) : (\Sigma, \chi) \to (\Sigma', \chi')$ be a $\mathbf{LiCAS}^{\bowtie}$-epimorphism.*

(i) If $(g'(\cdot), u'(\cdot))$ is a VOCT of (Σ', χ'), then there exists a VOCT $(g(\cdot), u(\cdot))$ of (Σ, χ) such that $(g'(\cdot), u'(\cdot)) = (\phi \circ g(\cdot), \varphi \circ u(\cdot))$.

(ii) If $(g'(\cdot), u'(\cdot))$ is an ECT of (Σ', χ'), then there exists an ECT $(g(\cdot), u(\cdot))$ of (Σ, χ) such that $(g'(\cdot), u'(\cdot)) = (\phi \circ g(\cdot), \varphi \circ u(\cdot))$.

Proof. Suppose that $(g'(\cdot), u'(\cdot))$ is a VOCT (resp. ECT) of (Σ', χ'). As ϕ is surjective, there exists $g \in \mathsf{G}$ such that $\phi(g) = g'(0)$. Accordingly, there exists a controlled trajectory $(g(\cdot), \varphi^{-1} \circ u'(\cdot))$ of Σ such that $g(0) = g$. We claim that the image of $(g(\cdot), \varphi^{-1} \circ u'(\cdot))$ under Φ is $(g'(\cdot), u'(\cdot))$. Now

$$\frac{d}{dt}\phi(g(t)) = T_{g(t)}\phi \cdot \Xi(g(t), \varphi^{-1}(u'(t))) = \Xi'(\phi(g(t)), (\varphi \circ \varphi^{-1})(u'(t))) = \Xi'(\phi(g(t)), u'(t)).$$

Hence, as $\phi(g(\cdot))$ and $g'(\cdot)$ solve the same Cauchy problem, they are equal. By Theorem 1.12, it then follows that $(g(\cdot), \varphi^{-1} \circ u(\cdot))$ is a VOCT (resp. ECT) of (Σ, χ). □

Accordingly, we have that a $\mathbf{LiCAS}^{\bowtie}$-isomorphism establishes a one-to-one correspondence between VOCTs (resp. ECTs).

Corollary 1.14. *Suppose $\Phi = (\phi, \varphi) : (\Sigma, \chi) \to (\Sigma', \chi')$ is a $\mathbf{LiCAS}^{\bowtie}$-isomorphism.*

(i) $(g(\cdot), u(\cdot))$ is a VOCT if and only if $(\phi \circ g(\cdot), \varphi \circ u(\cdot))$ is a VOCT.

(ii) $(g(\cdot), u(\cdot))$ is an ECT if and only if $(\phi \circ g(\cdot), \varphi \circ u(\cdot))$ is an ECT.

Lastly, we show that when the state component $\phi : \mathsf{G} \to \mathsf{G}'$ is a Lie group covering homomorphism (i.e., ϕ is surjective and $T_{\mathbf{1}}\phi$ is a linear isomorphism), then ECTs project and lift to ECTs. We note that the proof is formally very similar to that of Theorem 1.12.

Theorem 1.15. *Suppose $\Phi = (\phi, \varphi) : (\Sigma, \chi) \to (\Sigma', \chi')$ is a* **LiCAS**$^{\bowtie}$*-epimorphism with state component ϕ a covering homomorphism. Then a controlled trajectory $(g(\cdot), u(\cdot))$ is an ECT of (Σ, χ) if and only if $(\phi \circ g(\cdot), \varphi \circ u(\cdot))$ is an ECT of (Σ', χ').*

Proof. If $(g(\cdot), u(\cdot))$ is a controlled trajectory and $(\phi \circ g(\cdot), \varphi \circ u(\cdot))$ is an ECT, then $(g(\cdot), u(\cdot))$ is an ECT by Theorem 1.12. Conversely, suppose $(g(\cdot), u(\cdot))$ is an ECT. The Hamiltonian functions (1.4) associated to (Σ, χ) and (Σ', χ') are given by

$$H_u(g, p) = p(\Xi_u(\mathbf{1})) + \lambda \chi(u) \quad \text{and} \quad H'_{u'}(g', p') = p'(\Xi'_{u'}(\mathbf{1})) + \lambda \chi'(u)$$

respectively (for a fixed λ). Accordingly, there exists $p(\cdot) : [0, T] \to \mathfrak{g}^*$ such that $(\xi(\cdot), u(\cdot))$, $\xi(t) = (g(t), p(t))$ satisfies (1.5)-(1.6)-(1.7). As $\dot{\xi}(t) = \vec{H}^{\lambda}_{u(t)}(\xi(t))$ (1.6) we have (see Lemma 1.7)

$$\dot{g}(t) = \Xi(g(t), u(t)) \qquad\qquad \dot{p}(t) = \mathrm{ad}^* \, \Xi_{u(t)}(\mathbf{1}) \cdot p(t).$$

Here $(\mathrm{ad}^* A \cdot p)(B) = p([A, B])$ for $A, B \in \mathfrak{g}$ and $p \in \mathfrak{g}^*$. Let $g'(\cdot) = \phi \circ g(\cdot)$ and $u'(\cdot) = \varphi \circ u(\cdot)$. As ϕ is a covering homomorphism, $T_{\mathbf{1}}\phi$ is a Lie algebra isomorphism and so $((T_{\mathbf{1}}\phi)^{-1})^* : \mathfrak{g}^* \to \mathfrak{g}'^*$ is well defined. Let $p'(\cdot) = r((T_{\mathbf{1}}\phi)^{-1})^* \cdot p(\cdot)$. (Here $r > 0$ is the unique constant associated to Φ). We claim that $(\xi'(\cdot), u'(\cdot))$, $\xi'(\cdot) = (g(\cdot), p(\cdot))$ satisfies (1.5)-(1.6)-(1.7).

As $T_g\phi \cdot \Xi(g(t), u(t)) = \Xi'(g'(t), u'(t))$, we have that $\dot{g}'(t) = \Xi'(g'(t), u'(t))$. Thus, in order to satisfy (1.6), we are left to show that $\dot{p}'(t) = \mathrm{ad}^* \, \Xi'_{u'(t)}(\mathbf{1}) \cdot p'(t)$. Indeed, for $A \in \mathfrak{g}'^*$ we have

$$\begin{aligned}
\dot{p}'(t) \cdot A &= r \left(((T_{\mathbf{1}}\phi)^{-1})^* \cdot \dot{p}(t)\right) \cdot A \\
&= r \left(\dot{p}(t)\right) \cdot ((T_{\mathbf{1}}\phi)^{-1} \cdot A) \\
&= r \left(\mathrm{ad}^* \, \Xi_{u(t)}(\mathbf{1}) \cdot p(t)\right) \cdot (T_{\mathbf{1}}\phi \cdot A) \\
&= r \, p(t) \cdot \left[\Xi_{u(t)}(\mathbf{1}), (T_{\mathbf{1}}\phi)^{-1} \cdot A\right] \\
&= r \, p(t) \cdot \left[(T_{\mathbf{1}}\phi)^{-1} \cdot \Xi_{u'(t)}(\mathbf{1}), (T_{\mathbf{1}}\phi)^{-1} \cdot A\right] \\
&= r \, p(t) \cdot \left((T_{\mathbf{1}}\phi)^{-1} \cdot \left[\Xi_{u(t)}(\mathbf{1}), A\right]\right) \\
&= \left(\mathrm{ad}^* \, \Xi_{u'(t)}(\mathbf{1}) \cdot p'(t)\right) \cdot A.
\end{aligned}$$

In order to show that $(\xi(\cdot), u(\cdot))$ satisfies (1.7), we first show that $H'_{\varphi(u)}(\xi'(t)) = rH_u(\xi(t))$ for $u \in \mathbb{R}^k$. Indeed,

$$\begin{aligned}
H'_{\varphi(u)}(\xi'(t)) &= p'(t) \cdot \Xi'_{\varphi(u)}(\mathbf{1}) + \lambda \chi'(\varphi(u)) \\
&= r(((T_{\mathbf{1}}\phi)^{-1})^* \cdot p(t)) \cdot \Xi'_{\varphi(u)}(\mathbf{1}) + \lambda \chi'(\varphi(u)) \\
&= rp(t) \cdot \Xi_u(\mathbf{1}) + \lambda r \chi(u) \\
&= rH_u(\xi(t)).
\end{aligned}$$

Hence, it follows that

$$\begin{aligned} H'_{u'(t)}(\xi'(t)) &= rH_{u(t)}(\xi(t)) \\ &= \max_{u\in\mathbb{R}^k} rH_u\left(\xi(t)\right) \qquad\qquad (= \text{constant}) \\ &= \max_{u'\in\mathbb{R}^{k'}} rH_{\varphi^{-1}(u')}\left(\xi(t)\right) \\ &= \max_{u'\in\mathbb{R}^{k'}} H'_{u'}\left(\xi'(t)\right) = \text{constant} \end{aligned}$$

It is left to show that (1.5) holds. We have $(\lambda, p(\cdot)) \not\equiv 0$. Hence, as $((T_{\mathbf{1}}\phi)^{-1})^*$ is bijective, we have $(\lambda, p'(\cdot)) \equiv (\lambda, r((T_{\mathbf{1}}\phi)^{-1})^* \cdot p(\cdot)) \not\equiv 0$. □

Remark 1.16. The above result is not true for VOCTs.

1.2.2 Equivalence

We say that two cost-extended systems (Σ, χ) and (Σ', χ') are *cost equivalent* (shortly *C*-equivalent) if they are isomorphic in $\mathbf{LiCAS}^{\bowtie}$. We have the following basic results.

Proposition 1.17. *$(\Sigma = (\mathsf{G}, \Xi), \chi)$ and $(\Sigma' = (\mathsf{G}', \Xi'), \chi')$ are C-equivalent if and only if there exist a Lie group isomorphism $\phi : \mathsf{G} \to \mathsf{G}'$ and an affine isomorphism $\varphi : \mathbb{R}^k \to \mathbb{R}^{k'}$ such that $T_{\mathbf{1}}\phi \cdot \Xi(\mathbf{1}, u) = \Xi'(\mathbf{1}, \varphi(u))$ and $\chi' \circ \varphi = r\chi$ for some $r > 0$.*

Proof. $(\Sigma = (\mathsf{G}, \Xi), \chi)$ and $(\Sigma' = (\mathsf{G}', \Xi'), \chi')$ are C-equivalent, i.e., we have a $\mathbf{LiCAS}^{\bowtie}$-isomorphism $(\phi, \varphi) : \mathsf{G} \times \mathbb{R}^k \to \mathsf{G}' \times \mathbb{R}^{k'}$, $(g, u) \mapsto (\phi(g), \varphi(u))$. It follows that Lie group homomorphism ϕ is bijective and hence a Lie group isomorphism. Hence, by Lemma 1.10 we have that ϕ and φ satisfy the required properties. Conversely, suppose there exist a Lie group isomorphism $\phi : \mathsf{G} \to \mathsf{G}'$ and an affine isomorphism $\varphi : \mathbb{R}^k \to \mathbb{R}^{k'}$ such that $T_{\mathbf{1}}\phi \cdot \Xi(\mathbf{1}, u) = \Xi'(\mathbf{1}, \varphi(u))$ and $\chi' \circ \varphi = r\chi$ for some $r > 0$. By Lemma 1.10 we have that (ϕ, φ) is a morphism. We have that (ϕ, φ) is an isomorphism as it has inverse $(\phi^{-1}, \varphi^{-1})$. □

Corollary 1.18. *If (Σ, χ) and (Σ', χ') are C-equivalent, then Σ and Σ' are detached feedback equivalent.*

Proof. Follows by Propositions 1.5 and 1.17. □

Corollary 1.19. *Let Σ and Σ' be two k-input full-rank systems, and let $\chi : \mathbb{R}^k \to \mathbb{R}$ be any admissible cost.*

(i) If Σ and Σ' are state space equivalent, then (Σ, χ) and (Σ', χ) are C-equivalent.

(ii) If Σ and Σ' are detached feedback equivalent, then $(\Sigma, \chi \circ \varphi)$ and (Σ', χ) are C-equivalent. (Here (ϕ, φ) defines the detached feedback equivalence.)

Proposition 1.20. *Any cost-extended system $(\Sigma = (\mathsf{G}, \Xi), \chi)$ is C-equivalent to a system $(\Sigma' = (\mathsf{G}, \Xi'), \chi')$ with $\Gamma' = \Gamma$ and $\chi'(u) = u^\top u$.*

Proof. Let $\chi(u) = (u-v)^\top Q\,(u-v)$. As Q is symmetric and positive-definite, there exists (see e.g., [90]) a non-singular real matrix R such that $R^\top Q R = I$. Let $\varphi : \mathbb{R}^k \to \mathbb{R}^k,\ u \mapsto Ru + v$ and let $\Xi' : \mathsf{G} \times \mathbb{R}^k \to T\mathsf{G},\ \Xi'(\mathbf{1}, u) = \Xi(\mathbf{1}, \varphi(u))$. Then $T_{\mathbf{1}}\,\mathrm{id}_{\mathsf{G}} \cdot \Xi'(\mathbf{1}, u) = \Xi(\mathbf{1}, \varphi(u))$ and $\chi(\varphi(u)) = u^\top R^\top Q R u = u^\top u$. The result then follows by Proposition 1.17. ☐

For a k-input system $\Sigma = (\mathsf{G}, \Xi)$, let $\mathcal{T}_\Sigma$ denote the group of *feedback transformations leaving Σ invariant.* More precisely,

$$\mathcal{T}_\Sigma = \{\varphi \in \mathsf{Aff}\,(\mathbb{R}^k) \,:\, \exists \psi \in d\,\mathsf{Aut}\,(\mathsf{G}),\ \psi \cdot \Xi(\mathbf{1}, u) = \Xi(\mathbf{1}, \varphi(u))\}.$$

Likewise, for an admissible cost $\chi : \mathbb{R}^k \to \mathbb{R}$, let $\mathcal{T}_\chi$ denote the group of *feedback transformations leaving χ invariant.* More precisely,

$$\mathcal{T}_\chi = \{\varphi \in \mathsf{Aff}\,(\mathbb{R}^k) \,:\, \chi \circ \varphi = r\chi \text{ for some } r > 0\}.$$

Here $\mathsf{Aff}\,(\mathbb{R}^k)$ is the group of affine automorphisms of $\mathbb{R}^k$, $\mathsf{Aut}(\mathsf{G})$ is the group of Lie group automorphisms, and $d\,\mathsf{Aut}(\mathsf{G}) = \{T_{\mathbf{1}}\phi \,:\, \phi \in \mathsf{Aut}(\mathsf{G})\}$.

Proposition 1.21. *(Σ, χ) and (Σ, χ') are C-equivalent if and only if there exists $\varphi \in \mathcal{T}_\Sigma$ such that $\chi' = r\chi \circ \varphi$ for some $r > 0$.*

Proof. Suppose there exists a $\mathbf{LiCAS}^{\bowtie}$-isomorphism $\Phi = \phi \times \bar\varphi : (\Sigma, \chi) \to (\Sigma, \chi')$. Then $T_{\mathbf{1}}\phi \in d\,\mathsf{Aut}\,(\mathsf{G})$ and $T_{\mathbf{1}}\phi \cdot \Xi(\mathbf{1}, u) = \Xi(\mathbf{1}, \bar\varphi(u))$. Hence $\bar\varphi \in \mathcal{T}_\Sigma$. Also, as Φ is a $\mathbf{LiCAS}^{\bowtie}$-isomorphism, there exists a constant $r > 0$ such that $\chi' \circ \bar\varphi = r\chi$. Therefore, $\varphi = \bar\varphi^{-1} \in \mathcal{T}_\Sigma$ and $\chi' = r\chi \circ \varphi$.

Conversely, suppose there exists some $\bar\varphi^{-1} \in \mathcal{T}_\Sigma$ such that $\chi' = r\chi \circ \bar\varphi^{-1}$ for some $r > 0$, i.e., $\chi' \circ \bar\varphi = r\chi$. As $\bar\varphi \in \mathcal{T}_\Sigma$, there exists an automorphism $\phi : \mathsf{G} \to \mathsf{G}$ such that $T_{\mathbf{1}}\phi \cdot \Xi(\mathbf{1}, u) = \Xi(\mathbf{1}, \bar\varphi(u))$. The result then follows by Proposition 1.17. ☐

Proposition 1.22. *$(\Sigma = (\mathsf{G}, \Xi), \chi)$ and $(\Sigma' = (\mathsf{G}', \Xi'), \chi)$ are C-equivalent if and only if there exists a Lie group isomorphism $\phi : \mathsf{G} \to \mathsf{G}'$ and $\varphi \in \mathcal{T}_\chi$ such that $\Xi'(\mathbf{1}, u) = T_{\mathbf{1}}\phi \cdot \Xi(\mathbf{1}, \varphi(u))$.*

Proof. Suppose we have a $\mathbf{LiCAS}^{\bowtie}$-isomorphism $\Phi = \phi \times \bar\varphi : (\Sigma, \chi) \to (\Sigma', \chi)$. Then $T_{\mathbf{1}}\phi \cdot \Xi(\mathbf{1}, u) = \Xi'(\mathbf{1}, \bar\varphi(u))$ and $\chi \circ \bar\varphi = r\chi$ for some $r > 0$. Hence $\bar\varphi \in \mathcal{T}_\chi$ and so $\varphi = \bar\varphi^{-1} \in \mathcal{T}_\chi$. Also, $T_{\mathbf{1}}\phi \cdot \Xi(\mathbf{1}, \varphi(u)) = \Xi'(\mathbf{1}, u)$.

Conversely, suppose that $\phi : \mathsf{G} \to \mathsf{G}'$ is a Lie group isomorphism, $\bar\varphi^{-1} \in \mathcal{T}_\chi$, and $\Xi'(\mathbf{1}, u) = T_{\mathbf{1}}\phi \cdot \Xi(\mathbf{1}, \bar\varphi^{-1}(u))$. Then $T_{\mathbf{1}}\phi \cdot \Xi(\mathbf{1}, u) = \Xi'(\mathbf{1}, \bar\varphi(u))$ and $\chi \circ \bar\varphi = r\chi$ for some $r > 0$ (as $\bar\varphi \in \mathcal{T}_\chi$). The result then follows by proposition 1.17. ☐

1.3 Pontryagin lift and Hamilton-Poisson systems

To any cost-extended system (Σ, χ) on a Lie group G we associate a (lifted) Hamilton-Poisson system on the associated Lie-Poisson space $\mathfrak{g}^*_-$, via the Pontryagin Maximum Principle (cf. [14,70,110]). Indeed, the maximum condition (1.7) eliminates the parameter u from the family of Hamiltonians (H_u); as a result, we obtain a smooth G-invariant function H on $T^*\mathsf{G} = \mathsf{G} \times \mathfrak{g}^*$.

This Hamilton-Poisson system on $T^*\mathsf{G}$ can be reduced (cf. [78]) to a Hamilton-Poisson system on the (minus) Lie-Poisson space $\mathfrak{g}^*_-$, with Poisson bracket given by

$$\{F, G\} = -p \cdot [dF(p), dG(p)].$$

Here $F, G \in C^\infty(\mathfrak{g}^*)$ and $dF(p), dG(p)$ are elements of the double dual $\mathfrak{g}^{**}$ which is canonically identified with the Lie algebra $\mathfrak{g}$. We shall realize the Pontryagin lift as a contravariant functor (cf. [57]) between the category of cost-extended systems and an appropriate category of quadratic Hamilton-Poisson systems.

A *quadratic Hamilton-Poisson system* $(\mathfrak{g}^*_-, H_{A,\mathcal{Q}})$ is specified by

$$H_{A,\mathcal{Q}} : \mathfrak{g}^* \to \mathbb{R}, \quad p \mapsto p \cdot A + \mathcal{Q}(p).$$

Here $A \in \mathfrak{g}$ and $\mathcal{Q}$ is a quadratic form on $\mathfrak{g}^*$. If $A = 0$, then the system is called *homogeneous*; otherwise, it is called *inhomogeneous*. When $\mathfrak{g}^*_-$ is fixed, a system $(\mathfrak{g}^*_-, H_{A,\mathcal{Q}})$ is identified with its Hamiltonian $H_{A,\mathcal{Q}}$. To each function $H \in C^\infty(\mathfrak{g}^*)$, we associate a *Hamiltonian vector field* $\vec{H}$ on $\mathfrak{g}^*$ specified by $\vec{H}[F] = \{F, H\}$. A function $C \in C^\infty(\mathfrak{g}^*)$ is a *Casimir function* if $\{C, F\} = 0$ for all $F \in C^\infty(\mathfrak{g}^*)$, or, equivalently $\vec{C} = 0$. A linear map $\psi : \mathfrak{g}^* \to \mathfrak{h}^*$ is a *linear Poisson morphism* if $\{F, G\} \circ \psi = \{F \circ \psi, G \circ \psi\}$ for all $F, G \in C^\infty(\mathfrak{h}^*)$. Linear Poisson morphisms are exactly the dual maps of Lie algebra homomorphisms; indeed, this leads to a contravariant functor from the category of finite-dimensional Lie algebras to the category of Poisson manifolds (see, e.g., [79, 88]).

Let $(E_1, \ldots, E_n)$ be an ordered basis for the Lie algebra $\mathfrak{g}$ and let $(E_1^*, \ldots, E_n^*)$ denote the corresponding dual basis for $\mathfrak{g}^*$. We write elements $B \in \mathfrak{g}$ as column vectors and elements $p \in \mathfrak{g}^*$ as row vectors. Whenever convenient, linear maps will be identified with their matrices. If we write elements $u \in \mathbb{R}^k$ as column vectors as well, then we can express $\Xi_u(\mathbf{1}) = A + u_1 B_1 + \ldots + u_k B_k$ as $\Xi_u(\mathbf{1}) = A + \mathbf{B}\, u$, where $\mathbf{B} = \begin{bmatrix} B_1 & \cdots & B_k \end{bmatrix}$ is a $n \times k$ matrix. The equations of motion for the integral curve $p(\cdot)$ of the Hamiltonian vector field $\vec{H}$ corresponding to $H \in C^\infty(\mathfrak{g}^*)$ then take the form $\dot{p}_i = -p \cdot [E_i, dH(p)]$.

Let (Σ, χ) be a cost-extended system with

$$\Xi_u(\mathbf{1}) = A + \mathbf{B}\, u, \qquad \chi(u) = (u - \upsilon)^\top Q (u - \upsilon).$$

An application of the Pontryagin Maximum Principle yields the following result.

Theorem 1.23 (cf. [70, 78])**.** *Any normal ECT $(g(\cdot), u(\cdot))$ of (Σ, χ) is given by*

$$\dot{g}(t) = \Xi(g(t), u(t)), \qquad u(t) = Q^{-1}\, \mathbf{B}^\top\, p(t)^\top + \upsilon$$

where $p(\cdot) : [0, T] \to \mathfrak{g}^*$ is an integral curve for the Hamilton-Poisson system on $\mathfrak{g}^*_-$ specified by

$$H(p) = p\,(A + \mathbf{B}\,\upsilon) + \tfrac{1}{2}\, p\, \mathbf{B}\, Q^{-1}\, \mathbf{B}^\top\, p^\top. \tag{1.8}$$

Proof. The Hamiltonian (1.4) is given by $H_u(g, p) = p\, A + p\, \mathbf{B}\, u + \lambda\, (u - \upsilon)^\top\, Q\, (u - \upsilon)$. Now, $\frac{\partial H_u}{\partial u}(g, p) = p\, \mathbf{B} + 2\lambda (u - \upsilon)^\top\, Q$. Hence the maximum $\max_u H_u(g, p)$ occurs at $u^\top =$

$-\frac{1}{2\lambda} p\,\mathbf{B}\,Q^{-1} + v^\top$ (for a normal ECT we have $\lambda < 0$). Therefore the maximized Hamiltonian is given by

$$\begin{aligned} H(g,p) &= \max_{u \in \mathbb{R}^k} H_u(g,p) \\ &= p\,A + p\,\mathbf{B}\,(-\tfrac{1}{2\lambda} Q^{-1}\,\mathbf{B}^\top p^\top + v) + \tfrac{\lambda}{4\lambda^2}\, p\,\mathbf{B}\,Q^{-1}\,Q\,Q^{-1}\,\mathbf{B}^\top\,p^\top \\ &= p\,(A + \mathbf{B}\,v) - \tfrac{1}{4\lambda}\,p\,\mathbf{B}\,Q^{-1}\,\mathbf{B}^\top\,p^\top. \end{aligned}$$

As the maximized Hamiltonian H is defined and smooth, it follows that if $(\xi(\cdot), u(\cdot))$ satisfies (1.5)-(1.6)-(1.7) , then $\dot{\xi}(t) = \vec{H}(\xi(t))$ and $u(\cdot) = -\frac{1}{2\lambda}\,Q^{-1}\,\mathbf{B}^\top\,p(\cdot) + v$; moreover, if $\xi(\cdot) = (g(\cdot), p(\cdot))$ is an integral curve of $\vec{H}$, then the pair $(\xi(\cdot), \tilde{u}(\cdot))$, $\tilde{u}(\cdot) = -\frac{1}{2\lambda}\,Q^{-1}\,\mathbf{B}^\top\,p(\cdot) + v$ satisfies (1.5)-(1.6)-(1.7) (see [14, Chapter 12]). Accordingly the normal ECTs are the projection $(g(\cdot), p(\cdot))$ of the pairs $(\xi(\cdot), u(\cdot))$ where $u(\cdot) = -\frac{1}{2\lambda}\,Q^{-1}\,\mathbf{B}^\top\,p(\cdot) + v$ and $\xi(\cdot) = (g(\cdot), p(\cdot))$ is an integral curve of the Hamiltonian vector field $\vec{H}$ on $T^*\mathsf{G} = \mathsf{G} \times \mathfrak{g}^*$. However, as the Hamiltonian H is G-invariant, the integral curves $(g(\cdot), p(\cdot))$ of H are given by (see, e.g., [70, Chapter 12])

$$\dot{g}(t) = g(t)\,dH(p(t)) \qquad \text{and} \qquad \dot{p}(t) = \mathrm{ad}^*\,dH(p(t)) \cdot p(t)$$

where H is viewed as a function on $\mathfrak{g}^*$ and $dH(p) \in \mathfrak{g}^{**}$ is canonically identified with an element in $\mathfrak{g}$. Accordingly we get $\dot{g}(t) = \Xi(g(t), u(t))$ where $u(t) = -\frac{1}{2\lambda} Q^{-1}\,\mathbf{B}^\top\,p(t)^\top + v$ and $p(t)$ is an integral curve of the Hamiltonian system $(\mathfrak{g}^*_-, H)$. Finally, as the pair $(\lambda, \xi(\cdot))$ can be multiplied by any positive number [14, Chapter 12], we take $\lambda = -\frac{1}{2}$ for convenience. □

Remark 1.24. $\mathbf{B}\,Q^{-1}\,\mathbf{B}^\top$ is a positive semidefinite matrix (of rank k).

Accordingly, the study of extremal controls of a cost-extended system may be reduced, essentially, to the study of the associated Hamilton-Poisson system (1.8). We define the concrete category of quadratic Hamilton-Poisson systems, $\mathbf{HP}^\square$, as follows. An *object* is a pair $(\mathfrak{g}^*_-, H)$, where $H = H_{A,\mathcal{Q}} : \mathfrak{g}^*_- \to \mathbb{R}$, $p \mapsto p(A) + \mathcal{Q}(p)$. Here $A \in \mathfrak{g}$ and $\mathcal{Q}$ is a positive semidefinite quadratic form on $\mathfrak{g}^*_-$. (When $\mathfrak{g}^*_-$ is fixed, $(\mathfrak{g}^*_-, H)$ is identified with H.) A *morphism* $\psi : (\mathfrak{g}^*_-, H) \to ((\mathfrak{g}')^*_-, H')$ is a linear mapping $\psi : \mathfrak{g}^* \to (\mathfrak{g}')^*$ such that $\vec{H}$ and $\vec{H}'$ are compatible with ψ, i.e., $T_p\psi \cdot \vec{H}(p) = \vec{H}'(\psi(p))$ for $p \in \mathfrak{g}^*$. If $\psi : (\mathfrak{g}^*_-, H) \to ((\mathfrak{g}')^*_-, H')$ and $\psi' : ((\mathfrak{g}')^*_-, H') \to ((\mathfrak{g}'')^*_-, H'')$ are morphisms, then $\psi' \circ \psi : (\mathfrak{g}^*_-, H) \to ((\mathfrak{g}'')^*_-, H'')$ is indeed a morphism as $T_p(\psi' \circ \psi) \cdot \vec{H}(p) = T_{\psi(p)}\psi' \cdot \vec{H}'(\psi(p)) = \vec{H}''((\psi' \circ \psi)(p))$. We will say that two Hamilton-Poisson systems $(\mathfrak{g}^*_-, H)$ and $((\mathfrak{g}')^*_-, H')$ are *linearly equivalent* or *L-equivalent* if they are isomorphic in $\mathbf{HP}^\square$.

Proposition 1.25. *The following pairs of Hamilton-Poisson systems (on $\mathfrak{g}^*_-$, specified by their Hamiltonians) are L-equivalent:*

1. $H_{A,Q} \circ \psi$ *and* $H_{A,Q}$*, where* $\psi : \mathfrak{g}^*_- \to \mathfrak{g}^*_-$ *is a linear Lie-Poisson automorphism;*
2. $H_{A,Q}$ *and* $H_{A,rQ}$*, where* $r > 0$*;*
3. $H_{A,Q}$ *and* $H_{A,Q} + C$*, where* C *is a Casimir function.*

Proof. (1). As ψ is a (linear) Poisson isomorphism, it follows that $T\psi \circ \overrightarrow{H_{A,Q} \circ \psi} = \vec{H}_{A,Q} \circ \psi$ (see e.g., [88]), i.e., $\psi_* \overrightarrow{H_{A,Q} \circ \psi} = \vec{H}_{A,Q}$.

(2). Let $p(\cdot)$ be an integral curve of $\vec{H}_{A,Q}$, i.e., suppose $\dot{p}_i(t) = -p(t) \cdot [E_i, A + p(t)^\top Q]$. Let $\bar{p}(\cdot) = \frac{1}{r}p(\cdot)$. Then $\dot{\bar{p}}_i(t) = -\frac{1}{r}p(t) \cdot [E_i, A + p(t)^\top Q] = -\bar{p}(t) \cdot [E_i, A + \bar{p}(t)^\top rQ]$. Thus $\bar{p}(\cdot)$ is an integral curve of $\vec{H}_{A,rQ}$. Thus $\vec{H}_{A,Q}$ and $\vec{H}_{A,rQ}$ are compatible with the dilation $\delta_{1/r} : p \mapsto \frac{1}{r}p$ (as $\delta_{1/r}$ maps the flow of $\vec{H}_{A,Q}$ to the flow of $\vec{H}_{A,rQ}$).

(3). We have $\{H_{A,Q} + C, F\} = \{H_{A,Q}, F\}$ for $F \in C^\infty(\mathfrak{g}^*)$. Thus $\overrightarrow{H_{A,Q} + C} = \vec{H}_{A,Q}$, i.e., these vector fields are compatible with the identity map. □

We now realize the Pontryagin lift as a contravariant functor from $\mathbf{LiCAS}^{\bowtie}$ to $\mathbf{HP}^{\square}$.

Lemma 1.26. *Let (Σ, χ) and (Σ', χ') be cost-extended systems with $\Xi_u(\mathbf{1}) = A + \mathbf{B}\,u$ and $\Xi'_{u'}(\mathbf{1}) = A' + \mathbf{B}'\,u'$, respectively. If $\Phi = (\phi, \varphi) : (\Sigma, \chi) \to (\Sigma', \chi')$, $\varphi(u) = Ru + \varphi_0$ is a $\mathbf{LiCAS}^{\bowtie}$-morphism, then*

$$T_{\mathbf{1}}\phi\, A = A' + \mathbf{B}'\,\varphi_0 \qquad\qquad Rv + \varphi_0 = v'$$
$$T_{\mathbf{1}}\phi\, \mathbf{B} = \mathbf{B}' R \qquad\qquad R\,Q^{-1}\,R^\top = r\,(Q')^{-1}.$$

Here r is the unique positive constant associated to the morphism Φ.

Proof. As $T_{\mathbf{1}}\phi \cdot \Xi(\mathbf{1}, u) = \Xi'(\mathbf{1}, \varphi(u))$, we have $T_{\mathbf{1}}\phi\, A + T_{\mathbf{1}}\phi\, \mathbf{B}\, u = A' + \mathbf{B}'\varphi_0 + \mathbf{B}'\, R\, u$. Equating the coefficients of u yields $T_{\mathbf{1}}\phi\, A = A' + \mathbf{B}'\,\varphi_0$ and $T_{\mathbf{1}}\phi\, \mathbf{B} = \mathbf{B}' R$. Also, as $r\chi = \chi' \circ \varphi$ for some $r > 0$, we have

$$\begin{aligned} r(u - v)^\top\, Q\,(u - v) &= (\varphi(u) - v)^\top\, Q'\,(\varphi(u) - v) \\ &= (u - R^{-1}(v' - \varphi_0))^\top\, R^\top\, Q'\, R\,(u - R^{-1}(v' - \varphi_0)). \end{aligned}$$

Consequently, as Q and $R^\top\, Q'\, R$ are symmetric and positive definite, it follows that $rQ = R^\top Q' R$ and $Rv + \varphi_0 = v'$. □

Theorem 1.27. *The assignment*

$$\mathfrak{P}(\Sigma, \chi) = (\mathfrak{g}^*_-, H_{(\Sigma,\chi)})$$
$$\mathfrak{P}\Big((\Sigma, \chi) \xrightarrow{\Phi = (\phi,\varphi)} (\Sigma', \chi')\Big) = ((\mathfrak{g}')^*_-, H_{(\Sigma',\chi')}) \xrightarrow{\frac{1}{r}\,(T_{\mathbf{1}}\phi)^*} (\mathfrak{g}^*_-, H_{(\Sigma,\chi)})$$

defines a contravariant functor $\mathfrak{P} : \mathbf{LiCAS}^{\bowtie} \to \mathbf{HP}^{\square}$. Here $H_{(\Sigma,\chi)}$ denotes the Hamiltonian associated to (Σ, χ) as given in (1.8).

Proof. Let (Σ, χ) and (Σ', χ') be cost-extended systems with $\Xi(\mathbf{1}, u) = A + \mathbf{B}\, u$ and $\Xi'(\mathbf{1}, u') = A' + \mathbf{B}'\, u'$, respectively. The associated Hamilton-Poisson systems on $\mathfrak{g}^*_-$ and $(\mathfrak{g}')^*_-$, are respectively given by

$$\begin{aligned} H_{(\Sigma,\chi)}(p) &= p\,(A + \mathbf{B}\,v) + \tfrac{1}{2}\,p\,\mathbf{B}\,Q^{-1}\,\mathbf{B}^\top\,p^\top \\ H_{(\Sigma',\chi')}(p) &= p\,(A' + \mathbf{B}'\,v') + \tfrac{1}{2}\,p\,\mathbf{B}'\,(Q')^{-1}\,(\mathbf{B}')^\top\,p^\top. \end{aligned}$$

Let $\Phi = (\phi, \varphi) : (\Sigma, \chi) \to (\Sigma', \chi')$, $\varphi(u) = Ru + \varphi_0$ be a $\mathbf{LiCAS}^{\bowtie}$-morphism. Then, by Lemma 1.26, we have

$$\begin{aligned}(H_{(\Sigma,\chi)} \circ (T_\mathbf{1}\phi)^*)(p) &= p\,(T_\mathbf{1}\phi\,(A + \mathbf{B}\,\upsilon)) + \tfrac{1}{2}\,p\,T_\mathbf{1}\phi\,\mathbf{B}\,Q^{-1}\,\mathbf{B}^\top\,T_\mathbf{1}\phi^\top\,p^\top \\ &= p\,(A' + \mathbf{B}'\,(R\,\upsilon + \varphi_0) + \tfrac{1}{2}\,p\,\mathbf{B}'\,R\,Q^{-1}\,R^\top\,\mathbf{B}'^\top\,p^\top \\ &= p\,(A' + \mathbf{B}'\,\upsilon') + \tfrac{r}{2}\,p\,\mathbf{B}'\,(Q')^{-1}\,\mathbf{B}'^\top\,p^\top.\end{aligned}$$

Thus the vector fields associated with $H_{(\Sigma',\chi')}$ and $H_{(\Sigma,\chi)} \circ (T_\mathbf{1}\phi)^*$, respectively, are compatible with the dilation $\delta_{1/r}$ (Proposition 1.25). Furthermore, the vector fields associated with $H_{(\Sigma,\chi)} \circ (T_\mathbf{1}\phi)^*$ and $H_{(\Sigma,\chi)}$, respectively, are compatible with the linear Poisson morphism $(T_\mathbf{1}\phi)^*$. Consequently $\frac{1}{r}(T_\mathbf{1}\phi)^* : ((\mathfrak{g}')^*,\, H_{(\Sigma',\chi')}) \to (\mathfrak{g}^*,\, H_{(\Sigma,\chi)})$ is a $\mathbf{HP}^{\square}$-morphism.

For $\mathbf{LiCAS}^{\bowtie}$ morphisms $\Phi : (\Sigma, \chi) \to (\Sigma', \chi')$ and $\Phi' : (\Sigma', \chi') \to (\Sigma'', \chi'')$ we have

$$\mathfrak{P}(\Phi' \circ \Phi) = \tfrac{1}{rr'}(T_\mathbf{1}(\phi' \circ \phi))^* = \tfrac{1}{r}(T_\mathbf{1}\phi)^*\,\tfrac{1}{r'}(T_\mathbf{1}\phi')^* = \mathfrak{P}(\Phi) \circ \mathfrak{P}(\Phi')$$

For the identity morphism $\mathrm{id}_{(\Sigma,\chi)} = (\mathrm{id}_\mathsf{G}, \mathrm{id}_{\mathbb{R}^k}) : (\Sigma, \chi) \to (\Sigma, \chi)$ we have $\mathfrak{P}(\mathrm{id}_{(\Sigma,\chi)}) = \mathrm{id}_{\mathfrak{g}^*} = \mathrm{id}_{\mathfrak{P}(\Sigma,\chi)}$. □

Corollary 1.28. *If (Σ, χ) and (Σ', χ') are C-equivalent, then $\mathfrak{P}(\Sigma, \chi)$ and $\mathfrak{P}(\Sigma', \chi')$ are L-equivalent.*

Remark 1.29. The converse of the above statement is not true in general. In fact, one can construct cost-extended systems with different numbers of inputs, but equivalent Hamiltonians ([38]).

1.4 Invariant sub-Riemannian structures

1.4.1 Formalism

A *left-invariant sub-Riemannian manifold* is a triplet $(\mathsf{G}, \mathcal{D}, \mathbf{g})$, where G is a (real, finite-dimensional) connected Lie group, $\mathcal{D}$ is a smooth nonintegrable left-invariant distribution on G, and $\mathbf{g}$ is a left-invariant Riemannian metric on $\mathcal{D}$. More precisely, $\mathcal{D}(\mathbf{1})$ is a linear subspace of the Lie algebra $\mathfrak{g}$ of G and $\mathcal{D}(g) = g\mathcal{D}(\mathbf{1})$; the metric $\mathbf{g_1}$ is a positive definite symmetric bilinear form on $\mathfrak{g}$ and $\mathbf{g}_g(gA, gB) = \mathbf{g_1}(A, B)$ for $A, B \in \mathfrak{g}$, $g \in \mathsf{G}$. Again, the product gA is given by $T_\mathbf{1}L_g \cdot A$, where $L_g : h \mapsto gh$ is the left translation by g and $T_\mathbf{1}L_g$ is the tangent map of L_g at identity. When $\mathcal{D} = T\mathsf{G}$ (i.e., $\mathcal{D}(\mathbf{1}) = \mathfrak{g}$) then we have a left-invariant Riemannian manifold which we simply denote $(\mathsf{G}, \mathbf{g})$. Note that the structure $(\mathcal{D}, \mathbf{g})$ on G is completely determined by the subspace $\mathcal{D}(\mathbf{1}) \subseteq \mathfrak{g}$ and the scalar product $\mathbf{g_1}$ on $\mathcal{D}(\mathbf{1})$. We say that a list of k smooth vector fields $(X_1, \ldots, X_k)$ is an *orthonormal frame* for $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ if $\mathcal{D}(g) = \mathrm{span}(X_1(g), \ldots, X_k(g))$ and $\mathbf{g}(X_i, X_j) = \delta_{ij}$. We note that any left-invariant sub-Riemannian structure on a Lie group admits a (global) orthonormal frame of left-invariant vector fields.

An absolutely continuous curve $g(\cdot) : [0, T] \to \mathsf{G}$ is called a *$\mathcal{D}$-curve* if $\dot{g}(t) \in \mathcal{D}(g(t))$ for almost all $t \in [0, T]$. We shall assume that $\mathcal{D}$ satisfies the bracket generating condition, i.e., $\mathcal{D}(\mathbf{1})$ generates $\mathfrak{g}$; this condition is necessary and sufficient for any two points in G to be connected by a $\mathcal{D}$-curve (see, e.g., [47, 94]). The *length* of a horizontal curve $g(\cdot)$ is given by

$$\ell(g(\cdot)) = \int_0^T \sqrt{\mathbf{g}(\dot{g}(t), \dot{g}(t))}\, dt.$$

A sub-Riemannian manifold $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ is endowed with a natural metric space structure, namely the *Carnot-Carathéodory distance* ([47, 94]):

$$d_{cc}(g, h) = \inf\{\ell(g(\cdot)) \,:\, g(\cdot) \text{ is a horizontal curve joining } g \text{ and } h\}.$$

A standard argument (see, e.g., [100], cf. [11]) shows that the length minimization problem

$$\begin{gathered} \dot{g}(t) \in \mathcal{D}(g(t)), \qquad g(0) = g_0, \quad g(T) = g_1, \\ \int_0^T \sqrt{\mathbf{g}(\dot{g}(t), \dot{g}(t))} \longrightarrow \min \end{gathered} \tag{1.9}$$

is equivalent to the energy minimization problem, or invariant optimal control problem

$$\begin{gathered} \dot{g} = \Xi_u(g), \; u \in \mathbb{R}^k \qquad g(0) = g_0, \; g(T) = g_1 \\ \int_0^T \chi(u(t))\, dt \longrightarrow \min. \end{gathered} \tag{1.10}$$

Here $\Xi_u(\mathbf{1}) = u_1 B_1 + \cdots + u_k B_k$ where $B_1, \ldots, B_k$ are some linearly independent elements of $\mathfrak{g}$ such that $\langle B_1, \ldots, B_k \rangle = \mathcal{D}(\mathbf{1})$; $\chi(u(t)) = u(t)^\top Q\, u(t) = \mathbf{g_1}(\Xi_{u(t)}(\mathbf{1}), \Xi_{u(t)}(\mathbf{1}))$ for some $k \times k$ positive definite (symmetric) matrix Q. Indeed, energy minimizers are exactly those length minimizers which have constant speed

Accordingly, to solve the length minimising problem (1.9), it suffices to solve a corresponding optimal control problem (1.10). We shall call the projection $g(\cdot)$ of a normal (resp. abnormal) ECT $(g(\cdot), u(\cdot))$ a *normal (resp. abnormal) geodesic.* Likewise, the projection $g(\cdot)$ of a VOCT $(g(\cdot), u(\cdot))$ will be referred to as a *minimising geodesic.* Note that a minimising geodesic realizes the Carnot-Carathéodory distance between its endpoints. We find it convenient to restate a modified form of Theorem 1.23 in this context.

Theorem 1.30. *Let $(X_1, \ldots, X_k)$ be an orthonormal frame of left-invariant vector fields for $(\mathsf{G}, \mathcal{D}, \mathbf{g})$. The normal geodesics $g(\cdot)$ of $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ are given by*

$$\begin{gathered} \dot{g} = g(u_1 X_1(\mathbf{1}) + \cdots + u_k X_k(\mathbf{1})) \\ u_i(t) = p(t) \cdot X_i(\mathbf{1}), \quad \dot{p}(t) = \vec{H}(p(t)) \end{gathered}$$

where $(\mathfrak{g}^*_-, H)$ is the Hamilton-Poisson system specified by $H(p) = \frac{1}{2} \sum_{i=1}^k p(X_i(\mathbf{1}))^2$.

1.4.2 Functorial relation to $\mathbf{LiCAS}^{\bowtie}$

We define the concrete category **LiSR** of left-invariant sub-Riemannian manifolds as follows. An object is a left-invariant sub-Riemannian (or Riemannian) manifold $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ with bracket generating-distribution $\mathcal{D}$. A morphism $\phi : (\mathsf{G}, \mathcal{D}, \mathbf{g}) \to (\mathsf{G}', \mathcal{D}', \mathbf{g}')$ is a Lie group homomorphism $\phi : \mathsf{G} \to \mathsf{G}'$ such that

1. $\mathcal{D}(\mathbf{1}) \cap \ker T_{\mathbf{1}}\phi = \{0\}$,
2. $T_g\phi \cdot \mathcal{D}(g) = \mathcal{D}'(\phi(g))$ for $g \in \mathsf{G}$,
3. and $r\mathbf{g}_g(gA, gB) = \mathbf{g}'_{\phi(g)}(T\phi \cdot gA, T\phi \cdot gB)$ for $g \in \mathsf{G}$, $A, B \in \mathcal{D}(\mathbf{1})$ and some $r > 0$.

Note that the first two conditions imply that $\dim \mathcal{D}(\mathbf{1}) = \dim \mathcal{D}'(\mathbf{1})$; we also have that for a given morphism ϕ, the constant $r > 0$ is uniquely defined. Furthermore, if $\phi : (\mathsf{G}, \mathcal{D}, \mathbf{g}) \to (\mathsf{G}', \mathcal{D}', \mathbf{g}')$ and $\phi' : (\mathsf{G}', \mathcal{D}', \mathbf{g}') \to (\mathsf{G}'', \mathcal{D}'', \mathbf{g}'')$ are morphisms, then $\phi' \circ \phi : (\mathsf{G}, \mathcal{D}, \mathbf{g}) \to (\mathsf{G}'', \mathcal{D}'', \mathbf{g}'')$ is indeed a morphism.

Analogous to Lemma 1.10, we can characterise morphisms as follows.

Lemma 1.31. *A Lie group homomorphism $\phi : \mathsf{G} \to \mathsf{G}'$ defines a morphism $\phi : (\mathsf{G}, \mathcal{D}, \mathbf{g}) \to (\mathsf{G}', \mathcal{D}', \mathbf{g}')$ if and only if $\mathcal{D}(\mathbf{1}) \cap \ker T_{\mathbf{1}}\phi = \{0\}$,*

$$T_{\mathbf{1}}\phi \cdot \mathcal{D}(\mathbf{1}) = \mathcal{D}'(\mathbf{1}') \quad \textit{and} \quad r\,\mathbf{g}_{\mathbf{1}}(A, B) = \mathbf{g}'_{\mathbf{1}'}(T_{\mathbf{1}}\phi \cdot A, T_{\mathbf{1}}\phi \cdot B) \quad A, B \in \mathfrak{g}.$$

Proof. Necessity follows by definition. Suppose the map ϕ and a constant $r > 0$ satisfy the conditions of the proposition. As ϕ is a Lie group morphism, we have $(\phi \circ L_h)(g) = (L_{\phi(h)} \circ \phi)(g)$ for $g, h \in \mathsf{G}$. Hence $T_g\phi \cdot \mathcal{D}(g) = T_g\phi \cdot g\mathcal{D}(\mathbf{1}) = T_{\mathbf{1}}L_{\phi(g)} \cdot T_{\mathbf{1}}\phi \cdot \mathcal{D}(\mathbf{1}) = T_{\mathbf{1}}L_{\phi(g)} \cdot \mathcal{D}'(\mathbf{1}') = \mathcal{D}'(\phi(g))$ for $g \in \mathsf{G}$. Likewise, we have $r\mathbf{g}_g(gA, gB) = r\mathbf{g}_{\mathbf{1}}(A, B) = \mathbf{g}'_{\mathbf{1}'}(T_{\mathbf{1}}\phi \cdot A, T_{\mathbf{1}}\phi \cdot B) = \mathbf{g}'_{\phi(g)}(T_{\mathbf{1}}L_{\phi(g)} \cdot (T_{\mathbf{1}}\phi \cdot A), T_{\mathbf{1}}L_{\phi(g)} \cdot (T_{\mathbf{1}}\phi \cdot B)) = \mathbf{g}'_{\phi(g)}(T_g\phi \cdot gA, T_g\phi \cdot gB)$ for $g \in \mathsf{G}$, $A, B \in \mathcal{D}(\mathbf{1})$. □

We show that **LiSR** is equivalent (as a category) to a full subcategory of $\mathbf{LiCAS}^{\bowtie}$. Let $\mathbf{LiCAS}_0^{\bowtie}$ be the full subcategory of $\mathbf{LiCAS}^{\bowtie}$ with objects (Σ, χ) being full-rank homogeneous systems (i.e., $\Xi(\mathbf{1}, 0) = 0$ and $\mathsf{Lie}(\Gamma) = \mathfrak{g}$) with homogeneous cost (i.e., $\chi(0) = 0$). Let (Σ, χ) be a $\mathbf{LiCAS}_0^{\bowtie}$ object. We denote by $\mathcal{D}_\Sigma$ the associated left-invariant distribution specified by $\mathcal{D}_\Sigma(\mathbf{1}) = \Gamma = \operatorname{im}\Xi(\mathbf{1}, \cdot)$. We denote by $\mathbf{g}^{(\Sigma,\chi)}$ the unique left-invariant metric (on $\mathcal{D}_\Sigma$) satisfying $\mathbf{g}_{\mathbf{1}}^{(\Sigma,\chi)}\big(\Xi(\mathbf{1}, u), \Xi(\mathbf{1}, u)\big) = \chi(u)$ for $u \in \mathbb{R}^k$.

Theorem 1.32. *The assignment*

$$\mathfrak{F}(\Sigma, \chi) = (\mathsf{G},\ \mathcal{D}_\Sigma,\ \mathbf{g}^{(\Sigma,\chi)})$$
$$\mathfrak{F}\Big((\Sigma, \chi) \xrightarrow{\Phi=(\phi,\varphi)} (\Sigma', \chi')\Big) = (\mathsf{G},\ \mathcal{D}_\Sigma,\ \mathbf{g}^{(\Sigma,\chi)}) \xrightarrow{\ \phi\ } (\mathsf{G}',\ \mathcal{D}_{\Sigma'},\ \mathbf{g}^{(\Sigma',\chi')})$$

defines a covariant functor $\mathfrak{F} : \mathbf{LiCAS}_0^{\bowtie} \to \mathbf{LiSR}$.

Proof. Let $\Phi = (\phi, \varphi) : (\Sigma, \chi) \to (\Sigma', \chi')$ be a $\mathbf{LiCAS}_0^{\bowtie}$-morphism. Furthermore, let $(\mathsf{G}, \mathcal{D}, \mathbf{g}) = \mathfrak{F}(\Sigma, \chi)$ and $(\mathsf{G}', \mathcal{D}', \mathbf{g}') = \mathfrak{F}(\Sigma', \chi')$. We have $T_{\mathbf{1}}\phi \cdot \Xi(\mathbf{1}, u) = \Xi'(\mathbf{1}', \varphi(u))$ and $\chi' \circ \varphi = r\chi$. Hence $T_{\mathbf{1}}\phi \cdot \mathcal{D}(\mathbf{1}) = T_{\mathbf{1}}\phi \cdot \operatorname{im}\Xi(\mathbf{1}, \cdot) = \operatorname{im}\Xi'(\mathbf{1}', \cdot) = \mathcal{D}'(\mathbf{1}')$. Let $A, B \in \mathcal{D}(\mathbf{1})$. There exists $u \in \mathbb{R}^k$ such that $A = \Xi(1, u)$. Therefore

$$\begin{aligned}
\mathbf{g}'_{\mathbf{1}'}(T_{\mathbf{1}}\phi \cdot A, T_{\mathbf{1}}\phi \cdot A) &= \mathbf{g}'_{\mathbf{1}'}(\Xi'(\mathbf{1}, \varphi(u)), \Xi'(\mathbf{1}, \varphi(u))) \\
&= (\chi' \circ \varphi)(u) \\
&= r\chi(u) \\
&= r\mathbf{g}_{\mathbf{1}}(\Xi(\mathbf{1}, u), \Xi(\mathbf{1}, u)) \\
&= r\mathbf{g}_{\mathbf{1}}(A, A).
\end{aligned}$$

Thus $\mathbf{g}'_{\mathbf{1}'}(T_{\mathbf{1}}\phi \cdot A, T_{\mathbf{1}}\phi \cdot B) = r\mathbf{g}_{\mathbf{1}}(A, B)$. Consequently, by Lemma 1.31, we have that $\mathcal{F}(\Phi) : (\mathsf{G}, \mathcal{D}, \mathbf{g}) \to (\mathsf{G}', \mathcal{D}', \mathbf{g}')$ is indeed a morphism.

For $\mathbf{LiCAS}_0^{\bowtie}$ morphisms $\Phi : (\Sigma, \chi) \to (\Sigma', \chi')$ and $\Phi' : (\Sigma', \chi') \to (\Sigma'', \chi'')$ we have $\mathfrak{F}(\Phi' \circ \Phi) = \phi' \circ \phi = \mathfrak{F}(\Phi') \circ \mathfrak{F}(\Phi)$. For the identity morphism $\mathrm{id}_{(\Sigma,\chi)} = (\mathrm{id}_{\mathsf{G}}, \mathrm{id}_{\mathbb{R}^k}) : (\Sigma, \chi) \to (\Sigma, \chi)$ we have $\mathfrak{F}(\mathrm{id}_{(\Sigma,\chi)}) = \mathrm{id}_{\mathsf{G}} = \mathrm{id}_{\mathfrak{F}(\Sigma,\chi)}$. □

Lemma 1.33. *$\mathfrak{F}$ is surjective on objects, i.e., for any* **LiSR***-object $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ there exists a* $\mathbf{LiCAS}_0^{\bowtie}$*-object (Σ, χ) such that $\mathfrak{F}(\Sigma, \chi) = (\mathsf{G}, \mathcal{D}, \mathbf{g})$.*

Proof. Let $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ be a **LiSR**-object and let $(A_1, A_2, \ldots, A_k)$ be an orthonormal basis for $(\mathcal{D}(\mathbf{1}), \mathbf{g}_{\mathbf{1}})$. Define a cost-extended system (Σ, χ) on G by $\Xi(\mathbf{1}, u) = u_1 A_1 + u_2 A_2 + \cdots + u_k A_k$ and $\chi(u) = u^\top u$. Then $\mathcal{F}(\Sigma, \chi) = (\mathsf{G}, \mathcal{D}, \mathbf{g})$. □

Theorem 1.34. *$\mathfrak{F}$ is an equivalence of categories.*

Proof. By the preceding lemma we have that $\mathfrak{F}$ is isomorphism-dense. Hence it suffices to show that $\mathfrak{F}$ is full and faithful.

Let (Σ, χ) and (Σ', χ') be a pair of $\mathbf{LiCAS}_0^{\bowtie}$-objects and let $(\mathsf{G}, \mathcal{D}, \mathbf{g}) = \mathfrak{F}(\Sigma, \chi)$ and $(\mathsf{G}', \mathcal{D}', \mathbf{g}') = \mathfrak{F}(\Sigma', \chi')$. Suppose $\phi : \mathfrak{F}(\Sigma, \chi) \to \mathfrak{F}(\Sigma', \chi')$ is a **LiSR**-morphism. Then $T_{\mathbf{1}}\phi \cdot \mathrm{im}\,\Xi(\mathbf{1}, \cdot) = \mathrm{im}\,\Xi'(\mathbf{1}, \cdot)$. Hence, there exists a unique linear isomorphism $\varphi : \mathbb{R}^k \to \mathbb{R}^k$ such that $T_{\mathbf{1}}\phi \cdot \Xi(\mathbf{1}, u) = \Xi'(\mathbf{1}, \varphi(u))$. We claim that $\Phi = (\phi, \varphi)$ is a $\mathbf{LiCAS}_0^{\bowtie}$-morphism. Indeed, $T_{\mathbf{1}}\phi \cdot \Xi(\mathbf{1}, u) = \Xi'(\mathbf{1}, \varphi(u))$ by definition and

$$\begin{aligned}(\chi' \circ \varphi)(u) &= \mathbf{g}'_{\mathbf{1}'}(\Xi'(\mathbf{1}, \varphi(u)), \Xi'(\mathbf{1}, \varphi(u))) \\ &= \mathbf{g}'_{\mathbf{1}'}(T_{\mathbf{1}}\phi \cdot \Xi(\mathbf{1}, u), T_{\mathbf{1}}\phi \cdot \Xi(\mathbf{1}, u)) \\ &= r\mathbf{g}_{\mathbf{1}}(\Xi(\mathbf{1}, u), \Xi(\mathbf{1}, u)) \\ &= r\chi(u).\end{aligned}$$

Thus $\mathfrak{F}(\Phi) = \phi$ and so $\mathfrak{F}$ is full.

Suppose $\Phi = (\phi, \varphi), \Phi' = (\phi', \varphi') : (\Sigma, \chi) \to (\Sigma', \chi')$ are two $\mathbf{LiCAS}_0^{\bowtie}$-morphisms such that $\mathfrak{F}(\Phi) = \phi = \phi' = \mathfrak{F}(\Phi')$. As $T_{\mathbf{1}}\phi \cdot \Xi(\mathbf{1}, u) = T_{\mathbf{1}}\phi' \cdot \Xi(\mathbf{1}, u)$ we have $\Xi'(\mathbf{1}', \varphi(u)) = \Xi'(\mathbf{1}', \varphi'(u))$ and so $\varphi = \varphi'$ (see Lemma 1.9). Thus $\Phi = \Phi'$. Consequently, $\mathfrak{F}$ is faithful. □

Corollary 1.35. *Two* $\mathbf{LiCAS}_0^{\bowtie}$*-objects (Σ, χ) and (Σ', χ') are C-equivalent if and only if $\mathfrak{F}(\Sigma, \chi)$ and $\mathfrak{F}(\Sigma', \chi')$ are* **LiSR***-isomorphic.*

Accordingly, if $\phi : (\mathsf{G}, \mathcal{D}, \mathbf{g}) \to (\mathsf{G}', \mathcal{D}', \mathbf{g}')$ is a **LiSR**-morphism, then there exists a $\mathbf{LiCAS}_0^{\bowtie}$-morphism $\Phi : (\Sigma, \chi) \to (\Sigma', \chi')$ such that $\phi = \mathfrak{F}(\Phi)$ (and $(\mathsf{G}, \mathcal{D}, \mathbf{g}) = \mathfrak{F}(\Sigma, \chi)$, $(\mathsf{G}', \mathcal{D}', \mathbf{g}') = \mathfrak{F}(\Sigma', \chi')$). We can thus restate results concerning the compatibility of $\mathbf{LiCAS}_0^{\bowtie}$-morphisms with VOCTs and ECTs in terms of the compatibility of **LiSR**-morphisms with geodesics.

Corollary 1.36. *Let $\phi : (\mathsf{G}, \mathcal{D}, \mathbf{g}) \to (\mathsf{G}', \mathcal{D}', \mathbf{g}')$ be a* **LiSR***-morphism and let $g(\cdot)$ be a $\mathcal{D}$-curve.*

- *Suppose ϕ is a* **LiSR***-morphism.*

- *If $\phi \circ g(\cdot)$ is a minimising geodesic, then $g(\cdot)$ is a minimizing geodesic.*
- *If $\phi \circ g(\cdot)$ is a normal geodesic, then $g(\cdot)$ is a normal geodesic.*

- *Suppose ϕ is a* **LiSR**-*epimorphism.*
  - *If $\phi \circ g(\cdot)$ is an abnormal geodesic, then $g(\cdot)$ is an abnormal geodesic.*
  - *If ϕ is a Lie group covering homomorphism, then $g(\cdot)$ is a normal (resp. abnormal) geodesic if and only if $\phi \circ g(\cdot)$ is a normal (resp. abnormal) geodesic.*
- *Suppose ϕ is a* **LiSR**-*isomorphism.*
  - *$g(\cdot)$ is a minimising geodesic if and only if $\phi \circ g(\cdot)$ is a minimising geodesic.*
  - *$g(\cdot)$ is a normal (resp. abnormal) geodesic if and only if $\phi \circ g(\cdot)$ is a normal (resp. abnormal) geodesic.*

Remark 1.37. Let $\phi : (\mathsf{G}, \mathcal{D}, \mathbf{g}) \to (\mathsf{G}', \mathcal{D}', \mathbf{g}')$ be a **LiSR**-epimorphism. In this case we can restate the above result in terms of lifted curves. We have that $T_g\phi|_{\mathcal{D}(g)} : \mathcal{D}(g) \to \mathcal{D}'(\phi(g))$ is a linear isomorphism for $g \in \mathsf{G}$. Hence, for every $\mathcal{D}'$-curve $g'(\cdot)$ and $g_0 \in \mathsf{G}$ such that $\phi(g_0) = g'(0)$ there exists a unique $\mathcal{D}$-curve $g(\cdot)$ such that $g(0) = g_0$ and $\phi \circ g(\cdot) = g'(\cdot)$. We call $g(\cdot)$ the $\mathcal{D}$-lift through g_0 of $g'(\cdot)$. In these terms, the $\mathcal{D}$-lift of a (minimising, normal, or abnormal) geodesic of $(\mathsf{G}', \mathcal{D}', \mathbf{g}')$ is a (resp. minimising, normal, or abnormal) geodesic of $(\mathsf{G}, \mathcal{D}, \mathbf{g})$.

Remark 1.38. Analogous to Lemma 1.11, we have that the left translation $h\,g(\cdot)$ of any (minimising, normal, or abnormal) geodesic is a (resp. minimising, normal, or abnormal) geodesic.

Analogous to Theorem 1.27, we have a contravariant functor from **LiSR** to $\mathbf{HP}^{\square}$.

Theorem 1.39. *There exists a unique contravariant functor $\mathfrak{H} : \mathbf{LiSR} \to \mathbf{HP}^{\square}$ such that the diagram*

$$\begin{array}{ccc} \mathbf{LiCAS}_0^{\bowtie} & \xrightarrow{\ \mathfrak{F}\ } & \mathbf{LiSR} \\ \downarrow{\scriptstyle \mathfrak{P}} & \swarrow{\scriptstyle \mathfrak{H}} & \\ \mathbf{HP}^{\square} & & \end{array}$$

commutes.

Proof. As $\mathfrak{F}$ is an equivalence of categories and $\mathfrak{F}$ surjective on objects, there exists a functor $\mathfrak{G} : \mathbf{LiSR} \to \mathbf{LiCAS}_0^{\bowtie}$ such that $\mathfrak{G}$ is an equivalence and $\mathfrak{F} \circ \mathfrak{G} : \mathbf{LiSR} \to \mathbf{LiSR}$ is the identity functor (see, e.g., [86]). Let $\mathfrak{H} = \mathfrak{P} \circ \mathfrak{G}$. Then $\mathfrak{H} : \mathbf{LiSR} \to \mathbf{HP}^{\square}$ is a contravariant functor.

We claim that $\mathfrak{H} \circ \mathfrak{F} = \mathfrak{P}$, i.e., $\mathfrak{P} \circ \mathfrak{G} \circ \mathfrak{F} = \mathfrak{P}$. Let (Σ, χ) be a $\mathbf{LiCAS}_0^{\bowtie}$-object and let $(\Sigma', \chi') = (\mathfrak{G} \circ \mathfrak{F})(\Sigma, \chi)$. We have that $\mathfrak{F}(\Sigma', \chi') = (\mathfrak{F} \circ \mathfrak{G} \circ \mathfrak{F})(\Sigma, \chi) = \mathfrak{F}(\Sigma, \chi)$. Therefore, as $\mathfrak{G} \circ \mathfrak{F}$ is full, there exists a $\mathbf{LiCAS}_0^{\bowtie}$-isomorphism $\phi \times \varphi : (\Sigma', \chi') \to (\Sigma, \chi)$ such that $\mathfrak{F}(\phi \times \varphi) = \mathrm{id}_{\mathfrak{F}(\Sigma, \chi)}$. It follows that $\phi = \mathrm{id}_{\mathsf{G}}$ and $r = 1$. Consequently, if (Σ, χ) is given by $\Xi_u(\mathbf{1}) = \mathbf{B}u$ and $\chi(u) = u^\top Q u$, then (Σ', χ') is given by $\Xi'_{u'}(\mathbf{1}) = \mathbf{B}\varphi u$ and $\chi'(u') = u'^\top \varphi^\top Q \varphi u$ (here the linear map $\varphi : \mathbb{R}^k \to \mathbb{R}^k$ is identified with its matrix). Consequently the Hamiltonian systems (1.8) associated to these systems are $\mathfrak{P}(\Sigma, \chi) = (\mathfrak{g}^*_-, H)$, $H = \frac{1}{2} p \mathbf{B} Q^{-1} \mathbf{B}^\top p^\top$ and $\mathfrak{P}(\Sigma', \chi') =$

$(\mathfrak{g}_-^*, H')$, $H' = \frac{1}{2} p \mathbf{B} \varphi (\varphi^{-1} Q^{-1} \varphi^{\top -1}) \varphi^\top \mathbf{B}^\top p^\top = H$. That is to say, $\mathfrak{P}(\Sigma', \chi') = \mathfrak{P}(\Sigma, \chi)$, i.e., $(\mathfrak{P} \circ \mathfrak{G} \circ \mathfrak{F})(\Sigma, \chi) = \mathfrak{P}(\Sigma, \chi)$ for any $\mathbf{LiCAS}_0^{\bowtie}$-object (Σ, χ).

Now let $\phi \times \varphi : (\Sigma, \chi) \to (\Sigma', \chi')$ be a $\mathbf{LiCAS}_0^{\bowtie}$-morphism between some $\mathbf{LiCAS}_0^{\bowtie}$-objects (Σ, χ) and (Σ', χ'); further, let $\phi' \times \varphi' = (\mathfrak{G} \circ \mathfrak{F})(\phi \times \varphi)$. We have that $\mathfrak{F}(\phi \times \varphi) = \mathfrak{F}(\phi' \times \varphi')$ and so $\phi = \phi'$ and $r = r'$. Consequently,

$$\begin{aligned}
(\mathfrak{P} \circ \mathfrak{G} \circ \mathfrak{F}) \Big((\Sigma, \chi) \xrightarrow{\phi \times \varphi} (\Sigma', \chi') \Big) &= (\mathfrak{P} \circ \mathfrak{G} \circ \mathfrak{F})(\Sigma, \chi) \xrightarrow{\frac{1}{r'}(T_1 \phi')^*} (\mathfrak{P} \circ \mathfrak{G} \circ \mathfrak{F})(\Sigma', \chi') \\
&= \mathfrak{P}(\Sigma, \chi) \xrightarrow{\frac{1}{r}(T_1 \phi)^*} \mathfrak{P}(\Sigma', \chi') \\
&= \mathfrak{P} \Big((\Sigma, \chi) \xrightarrow{\phi \times \varphi} (\Sigma', \chi') \Big).
\end{aligned}$$

Hence, we have that $\mathfrak{P} \circ \mathfrak{G} \circ \mathfrak{F} = \mathfrak{P}$ as claimed.

Suppose $\mathfrak{H}' : \mathbf{LiSR} \to \mathbf{HP}^{\square}$ is another contravariant functor such that $\mathfrak{H}' \circ \mathfrak{F} = \mathfrak{P}$. Then $\mathfrak{H}' \circ \mathfrak{F} \circ \mathfrak{G} = \mathfrak{P} \circ \mathfrak{G}$ and so $\mathfrak{H}' = \mathfrak{H}$. □

Corollary 1.40 (Compare with Theorems 1.27 and 1.30). *The contravariant functor* $\mathfrak{H} : \mathbf{LiSR} \to \mathbf{HP}^{\square}$ *is given by*

$$\mathfrak{H}(\mathsf{G}, \mathcal{D}, \mathbf{g}) = (\mathfrak{g}_-^*, H_{(\mathsf{G}, \mathcal{D}, \mathbf{g})}), \qquad H_{(\mathsf{G}, \mathcal{D}, \mathbf{g})} = \tfrac{1}{2} \sum_{i=1}^{k} p(X_i(\mathbf{1}))^2$$

$$\mathfrak{H}\Big((\mathsf{G}, \mathcal{D}, \mathbf{g}) \xrightarrow{\phi} (\mathsf{G}', \mathcal{D}', \mathbf{g}') \Big) = ((\mathfrak{g}')_-^*, H_{(\mathsf{G}', \mathcal{D}', \mathbf{g}')}) \xrightarrow{\frac{1}{r}(T_1 \phi)^*} (\mathfrak{g}_-^*, H_{(\mathsf{G}, \mathcal{D}, \mathbf{g})}).$$

Here $(X_1, \ldots, X_k)$ *is any left-invariant orthonormal frame for* $(\mathsf{G}, \mathcal{D}, \mathbf{g})$.

1.4.3 Isometries and $\mathfrak{L}$-isometries

An *isometry* between two left-invariant sub-Riemannian (or Riemannian) manifolds $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ and $(\mathsf{G}', \mathcal{D}', \mathbf{g}')$ is a diffeomorphism $\phi : \mathsf{G} \to \mathsf{G}'$ such that

$$\phi_* \mathcal{D} = \mathcal{D}' \qquad \text{and} \qquad \mathbf{g} = \phi^* \mathbf{g}'$$

i.e., $T_g \phi \cdot \mathcal{D}(g) = \mathcal{D}'(\phi(g))$ and $\mathbf{g}_g(gA, gB) = \mathbf{g}'_{\phi(g)}(T_g \phi \cdot gA, T_g \phi \cdot gB)$ for $g \in \mathsf{G}$ and $A, B \in \mathfrak{g}$. If the isometry ϕ is a Lie group isomorphism, we shall say it is an $\mathfrak{L}$-*isometry*. If there exists an isometry (resp. $\mathfrak{L}$-isometry) between two structures, then we say they are *isometric* (resp. $\mathfrak{L}$-isometric). Note that, by definition, any left translation $L_g : \mathsf{G} \to \mathsf{G}$, $h \mapsto gh$ is an isometry between a left-invariant structure $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ and itself.

We can characterize isomorphisms in $\mathbf{LiSR}$ in terms of $\mathfrak{L}$-isometries.

Proposition 1.41. *Two invariant sub-Riemannian structures* $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ *and* $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ *are* **LiSR**-*isomorphic if and only if they are* $\mathfrak{L}$-*isometric up to rescaling, i.e., there exists a Lie group isomorphism* $\phi : \mathsf{G} \to \mathsf{G}'$ *such that* $\phi_* \mathcal{D} = \mathcal{D}'$ *and* $r\mathbf{g} = \phi^* \mathbf{g}'$ *for some* $r > 0$.

Corollary 1.42. *Two* $\mathbf{LiCAS}_0^{\bowtie}$-*objects* (Σ, χ) *and* (Σ', χ') *are* C-*equivalent if and only if* $\mathfrak{F}(\Sigma, \chi)$ *and* $\mathfrak{F}(\Sigma', \chi')$ *are* $\mathfrak{L}$-*isometric up to rescaling.*

Clearly every $\mathfrak{L}$-isometry is a isometry. At least in some contexts, every isometry is the composition of a left translation and a $\mathfrak{L}$-isometry.

Theorem 1.43 (cf. [80, 81, 125]). *Let* $(\mathsf{G}, \mathbf{g})$ *and* $(\mathsf{G}', \mathbf{g}')$ *be two invariant Riemannian structures on simply connected nilpotent Lie groups* G *and* G'*, respectively. A diffeomorphism* $\phi : \mathsf{G} \to \mathsf{G}'$ *is an isometry between* $(\mathsf{G}, \mathbf{g})$ *and* $(\mathsf{G}', \mathbf{g}')$ *if and only if* ϕ *is the composition* $\phi = L_{\phi(\mathbf{1})} \circ \phi'$ *of a left translation* $L_{\phi(\mathbf{1})}$ *on* G' *and a Lie group isomorphism* $\phi' = L_{\phi(\mathbf{1})^{-1}} \circ \phi : \mathsf{G} \to \mathsf{G}'$ *such that* $\mathbf{g}_{\mathbf{1}}(A, B) = \mathbf{g}'_{\mathbf{1}'}(T_{\mathbf{1}}\phi' \cdot A, T_{\mathbf{1}}\phi' \cdot B)$.

Corollary 1.44. *Two invariant Riemannian structures on simply connected nilpotent Lie groups are isometric if and only if they are* $\mathfrak{L}$*-isometric.*

A k-step *Carnot* group G is a simply connected nilpotent Lie group whose Lie algebra $\mathfrak{g}$ has stratification $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_k$ with $[\mathfrak{g}_1, \mathfrak{g}_j] = \mathfrak{g}_{1+j}$, $j = 1, \ldots, k-1$ and $[\mathfrak{g}_j, \mathfrak{g}_k] = \{0\}$, $j = 1, \ldots, k$. Let $\mathcal{D}$ be the left-invariant distribution on G specified by $\mathcal{D}(\mathbf{1}) = \mathfrak{g}_1$. Once we fix a left-invariant metric $\mathbf{g}$ on $\mathcal{D}$ we have a left-invariant structure on G (note that $\mathcal{D}$ is bracket generating). We shall refer to such a structure as a *sub-Riemannian Carnot group.* Analogous to the above theorem, we have the following characterization of the isometries between such structures.

Theorem 1.45 (cf. [63, 75], see also [48, 82]). *Let* $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ *and* $(\mathsf{G}', \mathcal{D}', \mathbf{g}')$ *be two sub-Riemannian Carnot groups. A diffeomorphism* $\phi : \mathsf{G} \to \mathsf{G}'$ *is an isometry between* $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ *and* $(\mathsf{G}', \mathcal{D}', \mathbf{g}')$ *if and only if* ϕ *is the composition* $\phi = L_{\phi(\mathbf{1})} \circ \phi'$ *of a left translation* $L_{\phi(\mathbf{1})}$ *on* G' *and a Lie group isomorphism* $\phi' = L_{\phi(\mathbf{1})^{-1}} \circ \phi : \mathsf{G} \to \mathsf{G}'$ *such that* $T_{\mathbf{1}}\phi' \cdot \mathcal{D}(\mathbf{1}) = \mathcal{D}'(\mathbf{1}')$ *and* $\mathbf{g}_{\mathbf{1}}(A, B) = \mathbf{g}'_{\mathbf{1}'}(T_{\mathbf{1}}\phi' \cdot A, T_{\mathbf{1}}\phi' \cdot B)$.

Corollary 1.46. *Two sub-Riemannian Carnot groups are isometric if and only if they are* $\mathfrak{L}$*-isometric.*

Remark 1.47. Among the three-dimensional simply connected Lie groups, only the Abelian $\mathbb{R}^3$ and Heisenberg H_3 groups are nilpotent. Among the four-dimensional simply connected Lie groups, only the Abelian $\mathbb{R}^4$, trivially extended Heisenberg $\mathsf{H}_3 \times \mathbb{R}$ and Engel $\mathsf{G}_{4.1}$ groups are nilpotent (see, e.g, [95, 98, 101]). Except for the Abelian groups, all these groups admit sub-Riemannian Carnot groups.

Remark 1.48. If two sub-Riemannian structures are $\mathfrak{L}$-isometric, then their corresponding quadratic Hamilton-Poisson systems are L-equivalent (Theorem 1.39). However, this does not hold for isometries in general. A counterexample is given by the isometric sub-Riemannian structures on the three-dimensional Lie groups $\mathsf{Aff}(\mathbb{R}) \times \mathbb{R}$ and $\widetilde{\mathsf{SL}}(2, \mathbb{R})$. Although the structures are isomorphic (Theorem 3.7, [10]), their associated Hamilton-Poisson systems are not L-equivalent (see Theorem 2.15 and Propositions 3.9 and 3.16).

1.4.4 Central expansions

Let G be a Lie group with Lie algebra $\mathfrak{g}$ and let N a closed central subgroup of G with Lie algebra $\mathfrak{n}$. Further, let $q : \mathsf{G} \to \mathsf{G}/\mathsf{N}$ be the canonical quotient map. We say that an invariant sub-Riemannian (or Riemannian) structure $(\mathsf{G}, \tilde{\mathcal{D}}, \tilde{\mathbf{g}})$ is a *central expansion* of a sub-Riemannian structure $(\mathsf{G}, \mathcal{D}, \mathbf{g})$, with respect to N, if

1. $(\mathsf{G}, \tilde{\mathcal{D}}, \tilde{\mathbf{g}})$ tames $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ (i.e., $\mathcal{D}(\mathbf{1}) \subset \tilde{\mathcal{D}}(\mathbf{1})$ and $\tilde{\mathbf{g}}\big|_{\mathcal{D}} = \mathbf{g}$);
2. the $\tilde{\mathbf{g}}_{\mathbf{1}}$-orthogonal complement of $\tilde{\mathcal{D}}(\mathbf{1}) \cap \mathfrak{n}$ is $\mathcal{D}(\mathbf{1})$, i.e.,
$$\mathcal{D}(\mathbf{1}) = (\tilde{\mathcal{D}}(\mathbf{1}) \cap \mathfrak{n})^{\perp} = \{A \in \tilde{\mathcal{D}}(\mathbf{1}) \,:\, \tilde{\mathbf{g}}_{\mathbf{1}}(A, B) = 0 \text{ for } B \in \tilde{\mathcal{D}}(\mathbf{1}) \cap \mathfrak{n}\}.$$

As the following Lemma shows, any central expansion $(\mathsf{G}, \tilde{\mathcal{D}}, \tilde{\mathbf{g}})$ of $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ with respect to N can be constructed by promoting central vector fields.

Lemma 1.49. $(\mathsf{G}, \tilde{\mathcal{D}}, \tilde{\mathbf{g}})$ *is a central expansion of* $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ *with respect to* N *if and only if* $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ *admits a left-invariant orthonormal frame* $(X_1, \ldots, X_k)$ *such that*
$$\operatorname{span}(X_1(\mathbf{1}), \ldots, X_k(\mathbf{1})) \cap \mathfrak{n} = \{0\}$$
and $(\mathsf{G}, \tilde{\mathcal{D}}, \tilde{\mathbf{g}})$ *admits a corresponding left-invariant orthonormal frame* $(X_1, \ldots, X_k, Z_1, \ldots, Z_m)$ *such that* $Z_1(\mathbf{1}), \ldots, Z_m(\mathbf{1}) \in \mathfrak{n} \subseteq Z(\mathfrak{g})$.

Proof. Suppose $(\mathsf{G}, \tilde{\mathcal{D}}, \tilde{\mathbf{g}})$ is a central expansion of $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ with respect to N. As $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ is an invariant sub-Riemannian structure, it admits a left-invariant orthonormal frame $(X_1, \ldots, X_k)$. Moreover, we have $\mathcal{D}(\mathbf{1}) = (\tilde{\mathcal{D}}(\mathbf{1}) \cap \mathfrak{n})^{\perp}$ and so $\operatorname{span}(X_1(\mathbf{1}), \ldots, X_k(\mathbf{1})) \cap \mathfrak{n} = \{0\}$. As $(\mathsf{G}, \tilde{\mathcal{D}}, \tilde{\mathbf{g}})$ tames $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ the vectors $(X_1, \ldots, X_k)$ are orthonormal for $(\mathsf{G}, \tilde{\mathcal{D}}, \tilde{\mathbf{g}})$. As $\tilde{\mathcal{D}}(\mathbf{1})$ decomposes as the orthogonal direct sum $\tilde{\mathcal{D}}(\mathbf{1}) = \mathcal{D}(\mathbf{1}) \oplus \tilde{\mathcal{D}}(\mathbf{1}) \cap \mathfrak{n}$ and $X_1(\mathbf{1}), \ldots, X_k(\mathbf{1})$ span $\mathcal{D}(\mathbf{1})$, there exists left-invariant vector fields $Z_1, \ldots, Z_m$ with $Z_1(\mathbf{1}), \ldots, Z_m(\mathbf{1}) \in \mathfrak{n}$ such that $(X_1, \ldots, X_k, Z_1, \ldots, Z_m)$ is an orthonormal frame for $(\mathsf{G}, \tilde{\mathcal{D}}, \tilde{\mathbf{g}})$.

Conversely, suppose $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ admits a left-invariant orthonormal frame $(X_1, \ldots, X_k)$ with $\operatorname{span}(X_1(\mathbf{1}), \ldots, X_k(\mathbf{1})) \cap \mathfrak{n} = \{0\}$ and $(\mathsf{G}, \tilde{\mathcal{D}}, \tilde{\mathbf{g}})$ admits a corresponding left-invariant orthonormal frame $(X_1, \ldots, X_k, Z_1, \ldots, Z_m)$ with $Z_1(\mathbf{1}), \ldots, Z_m(\mathbf{1}) \in \mathfrak{n} \subseteq Z(\mathfrak{g})$. Then $\mathcal{D}(\mathbf{1}) = \operatorname{span}(X_1, \ldots, X_k) \subset \tilde{\mathcal{D}}(\mathbf{1})$ and $\mathbf{g}_g(a_1 X_1(g) + \ldots + a_k X_k(g), b_1 X_1(g) + \ldots + b_k X_k(g)) = a_1 b_1 + \ldots + a_k b_k = \tilde{\mathbf{g}}_g(a_1 X_1(g) + \ldots + a_k X_k(g), b_1 X_1(g) + \ldots + b_k X_k(g))$ so $\tilde{\mathbf{g}}\big|_{\mathcal{D}} = \mathbf{g}$, i.e., $(\mathsf{G}, \tilde{\mathcal{D}}, \tilde{\mathbf{g}})$ tames $(\mathsf{G}, \mathcal{D}, \mathbf{g})$. Moreover, we have that $\tilde{\mathcal{D}}(\mathbf{1})$ decomposes as an orthonormal direct sum $\tilde{\mathcal{D}}(\mathbf{1}) = \operatorname{span}(X_1, \ldots, X_k) \oplus \operatorname{span}(Z_1, \ldots, Z_m) = \mathcal{D}(\mathbf{1}) \oplus \tilde{\mathcal{D}}(\mathbf{1}) \cap \mathfrak{n}$. □

We show that the normal geodesics of a central expansion of a structure are related to the normal geodesics of that structure. Let $(\mathsf{G}, \tilde{\mathcal{D}}, \tilde{\mathbf{g}})$ be a central expansion of $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ with respect to N. Further, let $\mathcal{G}$ (resp. $\tilde{\mathcal{G}}$) denote the set of normal geodesics of $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ (resp. $(\mathsf{G}, \tilde{\mathcal{D}}, \tilde{\mathbf{g}})$).

Theorem 1.50. *The images of the sets* $\mathcal{G}$ *and* $\tilde{\mathcal{G}}$ *under* $q : \mathsf{G} \to \mathsf{G}/\mathsf{N}$ *are identical, i.e.,* $q(\mathcal{G}) = q(\tilde{\mathcal{G}})$.

Proof. Let $(X_1, \ldots, X_k)$ be a left-invariant orthonormal frame for $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ and let $(X_1, \ldots, X_k, X_{k+1}, \ldots, X_n)$ be a left-invariant orthonormal frame for $(\mathsf{G}, \tilde{\mathcal{D}}, \tilde{\mathbf{g}})$ with $X_{k+1}(\mathbf{1}), \ldots, X_n(\mathbf{1}) \in \mathfrak{n}$ (see Lemma 1.49). The Hamiltonian systems on $\mathfrak{g}^*_-$ associated to $(\mathsf{G}, \tilde{\mathcal{D}}, \tilde{\mathbf{g}})$ and $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ are given by (see Theorem 1.30)
$$\tilde{H}(p) = \tfrac{1}{2} \sum_{i=1}^{n} p(X_i(\mathbf{1}))^2 \qquad \text{and} \qquad H(p) = \tfrac{1}{2} \sum_{i=1}^{k} p(X_i(\mathbf{1}))^2$$

respectively. We claim that $C(p) = \frac{1}{2}\sum_{i=k+1}^{n} p(X_i(\mathbf{1}))^2$ is a Casimir function. Indeed, $\{F, C\}(p) = -p([dF(p), \sum_{i=k+1}^{n} X_i(\mathbf{1})]) = -p(0) = 0$ as $\sum_{i=k+1}^{n} X_i(\mathbf{1}) \in \mathrm{Z}(\mathfrak{g})$. Consequently, as $\tilde{H} = H + C$, it follows that the Hamiltonian vector fields on $\mathfrak{g}_-^*$ associated to $\tilde{H}$ and H are identical. We denote this vector field $\vec{H}$.

The normal geodesics for $(\mathsf{G}, \tilde{\mathcal{D}}, \tilde{\mathbf{g}})$ and $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ are given by (see Theorem 1.30)

$$\frac{d}{dt}\tilde{g}(t) = \tilde{u}_1(t)X_1(\tilde{g}(t)) + \cdots + \tilde{u}_n(t)X_n(\tilde{g}(t)), \quad \tilde{u}_i(t) = p(t)\cdot X_i(\mathbf{1}), \quad \frac{d}{dt}p(t) = \vec{H}(p(t))$$

and

$$\frac{d}{dt}g(t) = u_1(t)X_1(g(t)) + \cdots + u_k(t)X_k(g(t)), \quad u_i(t) = p(t)\cdot X_i(\mathbf{1}), \quad \frac{d}{dt}p(t) = \vec{H}(p(t))$$

respectively. Note that $\tilde{u}_i(\cdot) = u_i(\cdot)$ for $i = 1, \ldots, k$. We claim that $T_g q \cdot X_i(g) = 0$ for $i = k+1, \ldots, n$. Indeed, the flow of X_i is $\varphi_t(g) = g\exp(tX_i(\mathbf{1}))$ and as $X_i(\mathbf{1}) \in \mathfrak{n}$ we have that $\exp(tX_i(\mathbf{1})) \in \mathsf{N}$; consequently $q(\varphi_t(g)) = q(g)\,q(\exp(tX_i(\mathbf{1})) = q(g)$ and so $T_g q \cdot X_i(g) = 0$. Therefore, for the normal geodesics $\tilde{g}(\cdot)$ and $g(\cdot)$ corresponding to a fixed integral curve $p(\cdot)$ of $\vec{H}$, we have

$$\begin{aligned}\frac{d}{dt}\,q(\tilde{g}(t)) &= T_{\tilde{g}(t)}q\cdot \tilde{u}_1(t)X_1(\tilde{g}(t)) + \cdots + T_{\tilde{g}(t)}q\cdot \tilde{u}_n(t)X_n(\tilde{g}(t)) \\ &= u_1(t)T_{\tilde{g}(t)}q\cdot X_1(\tilde{g}(t)) + \cdots + u_k(t)T_{\tilde{g}(t)}q\cdot X_k(\tilde{g}(t)) + 0 + \cdots + 0\end{aligned}$$

and likewise

$$\frac{d}{dt}\,q(g(t)) = u_1(t)T_{g(t)}q\cdot X_1(g(t)) + \cdots + u_k(t)T_{g(t)}q\cdot X_k(g(t))$$

Hence, if $q(\tilde{g}(0)) = q(g(0))$, then the curves $t \mapsto q(\tilde{g}(t))$ and $t \mapsto q(g(t))$ solve the same Cauchy problem and hence are identical. It therefore follows (by varying through all initial conditions and integral curves $p(\cdot)$ of $\vec{H}$) that $q(\tilde{\mathcal{G}}) = q(\mathcal{G})$. □

Corollary 1.51. *Let $g(\cdot)$ be a $\mathcal{D}$-curve; $g(\cdot)$ is normal geodesic of $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ if and only if there exists normal geodesic $\tilde{g}(\cdot)$ of the central expansion $(\mathsf{G}, \tilde{\mathcal{D}}, \tilde{\mathbf{g}})$ such that $g(0) = \tilde{g}(0)$ and $q \circ g(\cdot) = q \circ \tilde{g}(\cdot)$.*

Proof. Suppose $g(\cdot)$ is a normal geodesic of $(\mathsf{G}, \mathcal{D}, \mathbf{g})$. As $q(\tilde{\mathcal{G}}) = q(\mathcal{G})$, there exists a normal geodesic $\tilde{g}(\cdot)$ of $(\mathsf{G}, \tilde{\mathcal{D}}, \tilde{\mathbf{g}})$ such that $q(g(t)) = q(\tilde{g}(t))$. By use of a suitable left-translation (by an element $h \in \ker q = \mathsf{N}$), we may assume $\tilde{g}(0) = g(0)$ (see Remark 1.38).

Conversely, suppose $\tilde{g}(\cdot)$ is a normal geodesic of $(\mathsf{G}, \tilde{\mathcal{D}}, \tilde{\mathbf{g}})$ such that $g(0) = \tilde{g}(0)$ and $q \circ g(\cdot) = q \circ \tilde{g}(\cdot)$. As $q(\tilde{\mathcal{G}}) = q(\mathcal{G})$, there exists a normal geodesic $g'(\cdot)$ of $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ such that $q \circ g'(\cdot) = q \circ \tilde{g}(\cdot)$. Again, by use of a suitable left-translation (by an element $h \in \ker q = \mathsf{N}$), we may assume $g'(0) = \tilde{g}(0)$. We claim that $g(\cdot) = g'(\cdot)$. Indeed, as $q \circ g'(\cdot) = q \circ g(\cdot)$ we have that

$$T_{\mathbf{1}}q\cdot g(t)^{-1}\,\dot{g}(t) = T_{q(g(t))}L_{q(g(t))^{-1}}\cdot T_{g(t)}q\cdot\dot{g}(t) = T_{q(g'(t))}L_{q(g'(t))^{-1}}\cdot T_{g'(t)}q\cdot\dot{g}'(t) = T_{\mathbf{1}}q\cdot g'(t)^{-1}\,\dot{g}'(t).$$

Consequently, as $T_{\mathbf{1}}q\big|_{\mathcal{D}(\mathbf{1})} : \mathcal{D}(\mathbf{1}) \to \mathfrak{g}/\mathfrak{n}$ is injective (since $\mathcal{D}(\mathbf{1}) \cap \mathfrak{n} = \{0\}$), we have $g(t)^{-1}\,\dot{g}(t) = g'(t)^{-1}\,\dot{g}'(t)$. Therefore the curves $g(\cdot)$ and $g'(\cdot)$ solve the same Cauchy problem and hence are identical. □

Remark 1.52. For any $\tilde{\mathcal{D}}$-curve $\tilde{g}(\cdot)$, there exists a unique $\mathcal{D}$-curve $g(\cdot)$ such that $g(0) = \tilde{g}(0)$ and $q \circ g(\cdot) = q \circ \tilde{g}(\cdot)$. We call $g(\cdot)$ the $\mathcal{D}$-projection of $\tilde{g}(\cdot)$. Note that the $\mathcal{D}$-projection of $\tilde{g}(\cdot)$ is the $\mathcal{D}$-lift through $\tilde{g}(0)$ of the $\mathcal{D}'$-curve $q \circ \tilde{g}(\cdot)$ where $\mathcal{D}'(\mathbf{1}') = T_{\mathbf{1}}q \cdot \mathcal{D}(\mathbf{1}) \subseteq \mathfrak{g}/\mathfrak{n}$ (see Remark 1.37). In these terms, the normal geodesics of $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ are exactly the $\mathcal{D}$-projections of the normal geodesics of the normal geodesics of $(\mathsf{G}, \tilde{\mathcal{D}}, \tilde{\mathbf{g}})$.

Corollary 1.53. *If* $(\mathsf{G}, \tilde{\mathcal{D}}_1, \tilde{\mathbf{g}}^1)$ *and* $(\mathsf{G}, \tilde{\mathcal{D}}_2, \tilde{\mathbf{g}}^2)$ *are two central expansions of* $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ *with respect to* N, *then* $q(\tilde{\mathcal{G}}^1) = q(\tilde{\mathcal{G}}^2)$. *Here* $\tilde{\mathcal{G}}^1$ *and* $\tilde{\mathcal{G}}^2$ *denote the respective sets of normal geodesics.*

Remark 1.54. Theorem 1.50 does not hold for abnormal geodesics. A counterexample shall be provided in Section 1.5.2.

Remark 1.55. Central expansions are stable under **LiSR**-isomorphisms, i.e., if $\phi : (\mathsf{G}, \mathcal{D}, \mathbf{g}) \to (\mathsf{G}', \mathcal{D}', \mathbf{g}')$ is a **LiSR**-isomorphism and $(\mathsf{G}, \tilde{\mathcal{D}}, \tilde{\mathbf{g}})$ is a central expansion of $(\mathsf{G}, \mathcal{D}, \mathbf{g})$, then $(\mathsf{G}', \phi_* \tilde{\mathcal{D}}, \phi_* \tilde{\mathbf{g}})$ is a central expansion of $(\mathsf{G}', \mathcal{D}', \mathbf{g}')$. However, this is not true for isometries in general. For example the sub-Riemannian structure on $\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}$ (see Proposition 3.9) admits a Riemannian central expansion whereas the isometric structure on $\widetilde{\mathsf{SL}}\,(2, \mathbb{R})$ (see Proposition 3.16 and Theorem 3.7) does not.

1.5 Examples

In the first subsection, we illustrate the main results of this chapter by giving simple but demonstrative examples involving the three-dimensional Heisenberg group. In the second subsection we give a counterexample showing that Theorem 1.50 does not hold for abnormal geodesics.

1.5.1 Structures on the three-dimensional Heisenberg group

The three-dimensional Heisenberg group

$$\mathsf{H}_3 = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} = m(z, x, y) \; : \; z, x, y \in \mathbb{R} \right\}$$

has Lie algebra

$$\mathfrak{h}_3 = \left\{ \begin{bmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{bmatrix} = zE_1 + xE_2 + yE_3 \; : \; z, x, y \in \mathbb{R} \right\}.$$

The center of $\mathfrak{h}_3$ is spanned by E_1 and correspondingly the center of H_3 is $\{m(z, 0, 0) \, : \, z \in \mathbb{R}\}$. The group of automorphisms of $\mathfrak{h}_3$ is given by

$$\mathsf{Aut}(\mathfrak{h}_3) = \left\{ \begin{bmatrix} x_2 y_3 - y_2 x_3 & x_1 & y_1 \\ 0 & x_2 & y_2 \\ 0 & x_3 & y_3 \end{bmatrix} \; : \; x_1, x_2, x_3, y_1, y_2, y_3 \in \mathbb{R}, \; x_2 y_3 - y_2 x_3 \neq 0 \right\}.$$

As H_3 is simply connected, $d\,\mathsf{Aut}(\mathsf{H}_3) = \mathsf{Aut}(\mathfrak{h}_3)$.

Proposition 1.56. *Any controllable cost-extended system on* H_3 *is* C*-equivalent to exactly one of the cost-extended systems*

$$\left(\Sigma^{(2,0)}, \chi_1^{(2,0)}\right) : \quad \begin{cases} \Sigma^{(2,0)} \;:\; u_1 E_2 + u_2 E_3 \\ \chi_1^{(2,0)}(u) = u_1^2 + u_2^2 \end{cases}$$

$$\left(\Sigma^{(2,0)}, \chi_2^{(2,0)}\right) : \quad \begin{cases} \Sigma^{(2,0)} \;:\; u_1 E_2 + u_2 E_3 \\ \chi_2^{(2,0)}(u) = (u_1 - 1)^2 + u_2^2 \end{cases}$$

$$\left(\Sigma^{(2,1)}, \chi_\alpha^{(2,1)}\right) : \quad \begin{cases} \Sigma^{(2,1)} \;:\; E_1 + u_1 E_2 + u_2 E_3 \\ \chi_\alpha^{(2,1)}(u) = (u_1 - \alpha)^2 + u_2^2 \end{cases}$$

$$\left(\Sigma^{(3,0)}, \chi_{\boldsymbol{\alpha}}^{(3,0)}\right) : \quad \begin{cases} \Sigma^{(3,0)} \;:\; u_1 E_1 + u_2 E_2 + u_3 E_3, \\ \chi_{\boldsymbol{\alpha}}^{(3,0)}(u) = (u_1 - \alpha_1)^2 + (u_2 - \alpha_2)^2 + u_3^2. \end{cases}$$

Here $\alpha, \alpha_1, \alpha_2 \geq 0$ *parametrize families of (non-equivalent) class representatives.*

Proof. H_3 is simply connected and completely solvable. Hence, a control system Σ with trace $\Gamma = A + \Gamma^0$ on H_3 is controllable if and only if $\mathsf{Lie}(\Gamma^0) = \mathfrak{h}_3$ (cf. [110]). Consequently, it is fairly easy to show that any controllable system on H_3 is detached feedback equivalent to exactly one of $\Sigma^{(2,0)}$, $\Sigma^{(2,1)}$, or $\Sigma^{(3,0)}$ ([35]). Accordingly, any controllable cost-extended system (Σ, χ) is C-equivalent to $(\Sigma^{(2,0)}, \chi')$, $(\Sigma^{(2,1)}, \chi')$, or $(\Sigma^{(3,0)}, \chi')$ for some cost χ' (see Corollary 1.19).

Suppose (Σ, χ) is C-equivalent to $(\Sigma^{(2,0)}, \chi')$. For $\psi \in d\,\mathsf{Aut}(\mathsf{H}_3)$ we have $\psi \cdot \Xi^{(2,0)}(\mathbf{1}, u) = u_1(x_1 E_1 + x_2 E_2 + x_3 E_3) + u_2(y_1 E_1 + y_2 E_2 + y_3 E_3)$. Hence, if $\psi \cdot \Xi^{(2,0)}(\mathbf{1}, u) = \Xi^{(2,0)}(\mathbf{1}, \varphi(u))$ for some affine isomorphism φ and all $u \in \mathbb{R}^k$, then φ is a linear isomorphism, $x_1 = y_1 = 0$, and $\varphi(u) = (x_2 u_1 + y_2 u_2, x_3 u_1 + y_3 u_2)$. Accordingly $\mathcal{T}_{\Sigma^{(2,0)}} = \mathsf{GL}\,(2, \mathbb{R})$. Let $\chi'(u) = (u - v)^\top Q (u - v)$ for some $v \in \mathbb{R}^2$ and positive definite matrix $Q = \begin{bmatrix} a_1 & b \\ b & a_2 \end{bmatrix}$. We have

$$\varphi' = \begin{bmatrix} \frac{1}{\sqrt{a_1}} & -\frac{b}{a_1 \sqrt{-\frac{b^2}{a_1} + a_2}} \\ 0 & \frac{1}{\sqrt{-\frac{b^2}{a_1} + a_2}} \end{bmatrix} \in \mathcal{T}_{\Sigma^{(2,0)}} \quad \text{and} \quad \varphi'^\top Q \, \varphi' = \operatorname{diag}(1, 1).$$

Hence $(\Sigma^{(2,0)}, \chi')$ is C-equivalent to $(\Sigma^{(2,0)}, \chi'')$, with $\chi''(u) = (\chi' \circ \varphi')(u) = (u_1 - \rho \cos\theta)^2 + (u_2 - \rho \sin\theta)^2$ for some $\rho \geq 0$ and $\theta \in \mathbb{R}$ (see Proposition 1.21). If $\rho = 0$, then $(\Sigma^{(2,0)}, \chi'') = (\Sigma^{(2,0)}, \chi_1^{(2,0)})$. Suppose $\rho > 0$. Then, $\varphi'' = \begin{bmatrix} \rho\cos\theta & -\rho\sin\theta \\ \rho\sin\theta & \rho\cos\theta \end{bmatrix}$ is an element of $\mathcal{T}_{\Sigma^{(2,0)}}$ such that

$$\varphi''^{-1} \begin{bmatrix} \rho\cos\theta \\ \rho\sin\theta \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \varphi''^\top \varphi'' = \begin{bmatrix} \rho^2 & 0 \\ 0 & \rho^2 \end{bmatrix}.$$

Thus $(\Sigma^{(2,0)}, \chi'')$ is C-equivalent to $(\Sigma^{(2,0)}, \chi_2^{(2,0)})$, as $\chi_2^{(2,0)} = \frac{1}{\rho^2} \chi'' \circ \varphi'' = (u_1 - 1)^2 + u_2^2$ (again by Proposition 1.21).

We claim that $(\Sigma^{(2,0)}, \chi_1^{(2,0)})$ and $(\Sigma^{(2,0)}, \chi_2^{(2,0)})$ are not C-equivalent. Suppose that they are C-equivalent. Then there exists $\varphi = \begin{bmatrix} y_1 & y_2 \\ z_1 & z_2 \end{bmatrix} \in \mathcal{T}_{\Sigma^{(2,0)}}$ such that $\chi_1^{(2,0)} = r\chi_2^{(2,0)} \circ \varphi$ for some $r > 0$ (Proposition 1.21). In particular, we have $\varphi^{-1} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, a contradiction.

A similar argument holds for the inhomogeneous two-input case. We have $\mathcal{T}_{\Sigma^{(2,1)}} = \mathsf{SL}(2, \mathbb{R})$; more details may be found in [32, Example 4]. Likewise, in the three-input case any cost-extended system is equivalent to exactly one cost-extended system $(\Sigma^{(3,0)}, \chi_{\boldsymbol{\alpha}}^{(3,0)})$ with $\alpha_1, \alpha_2 \geq 0$. $\square$

Corollary 1.57 (cf. Proposition 3.10; also Theorems 4.6 and 4.8)**.** *Any invariant sub-Riemannian structure on* H_3 *is isometric up to rescaling to the structure* $(\mathsf{H}_3, \mathcal{D}^{(1)}, \mathbf{g}^{(1)})$ *given by*

$$\mathcal{D}^{(1)}(\mathbf{1}) = \langle E_2, E_3 \rangle, \qquad \mathbf{g}_{\mathbf{1}}^{(1)} = \operatorname{diag}(1, 1)$$

i.e., with orthonormal frame (E_2^L, E_3^L)*. On the other hand, any invariant Riemannian structure on* H_3 *is isometric up to rescaling to the structure* $(\mathsf{H}_3, \mathbf{g}^{(0)})$ *given by*

$$\mathbf{g}_{\mathbf{1}}^{(0)} = \operatorname{diag}(1, 1, 1)$$

i.e., with orthonormal frame (E_1^L, E_2^L, E_3^L)*. Here* E_1^L*,* E_2^L*, and* E_3^L *denote the left-invariant vector fields specified by* $E_i^L(\mathbf{1}) = E_i$*.*

Proof. For any invariant sub-Riemannian structure $(\mathsf{H}_3, \mathcal{D}, \mathbf{g})$ there exists a controllable cost-extended system (Σ, χ) with $\chi(0) = 0$ and $\Xi(\mathbf{1}, 0) = 0$ such that $\mathcal{F}(\Sigma, \chi) = (\mathsf{H}_3, \mathcal{D}, \mathbf{g})$ (Lemma 1.33). If (Σ, χ) is C-equivalent to some cost-extended system (Σ', χ') with $\Xi'(\mathbf{1}, 0) = 0$, then $\chi'(0) = 0$. Consequently (Σ, χ) is C-equivalent to $(\Sigma^{(2,0)}, \chi_1^{(2,0)})$. It therefore follows, by Corollary 1.42, that $(\mathsf{H}_3, \mathcal{D}, \mathbf{g}) = \mathfrak{F}(\Sigma, \chi)$ is isometric up to rescaling to $\mathfrak{F}(\Sigma^{(2,0)}, \chi_1^{(2,0)}) = (\mathsf{H}_3, \mathcal{D}^{(1)}, \mathbf{g}^{(1)})$. Likewise, any invariant Riemannian structure is equivalent to $\mathfrak{F}(\Sigma^{(3,0)}, \chi_{\mathbf{0}}^{(3,0)}) = (\mathsf{H}_3, \mathbf{g}^{(0)})$. $\square$

A classification of the cost-extended systems or sub-Riemannian structures also yields some information regarding the equivalence of the corresponding quadratic Hamilton-Poisson systems. For instance, we have the following result.

Corollary 1.58. *Any quadratic Hamilton-Poisson systems* $((\mathfrak{h}_3)^*_-, H)$ *with Hamiltonian* $H(p) = \mathcal{Q}(p)$ *being a positive-definite quadratic form is* L*-equivalent to the system on* $(\mathfrak{h}_3)^*_-$ *with Hamiltonian* $H'(p) = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2)$*.*

Proof. Let $((\mathfrak{h}_3)^*_-, H)$ be a quadratic Hamilton-Poisson system with Hamiltonian $H(p) = p\,Q\,p^\top$ being a positive-definite quadratic form. For the cost-extended system (Σ, χ) on H_3 specified by

$$(\Sigma, \chi) \; : \qquad \begin{cases} \Sigma \; : \; u_1 E_1 + u_2 E_2 + u_3 E_3 \\ \chi(u) = \frac{1}{2} u^\top Q^{-1} u \end{cases}$$

we have that $\mathfrak{P}(\Sigma, \chi) = ((\mathfrak{h}_3)^*_-, H)$. Furthermore, we have that (Σ, χ) is C-equivalent to $(\Sigma^{(3,0)}, \chi_{\mathbf{0}}^{(3,0)})$ (we have $\boldsymbol{\alpha} = 0$ as $\Xi(\mathbf{1}, 0) = \Xi^{(3,0)}(\mathbf{1}, 0) = 0$ and $\chi(0) = 0$). Consequently, by Corollary 1.28, we have that $((\mathfrak{h}_3)^*_-, H)$ is L-equivalent to $\mathfrak{P}(\chi_{\mathbf{0}}^{(3,0)}) = ((\mathfrak{h}_3)^*_-, H')$. $\square$

Next, we give some applications of the properties of epimorphisms in $\mathbf{LiCAS}^{\bowtie}$ (Theorem 1.12) and **LiSR** (Corollary 1.36). Let $q : \mathsf{H}_3 \to \mathbb{R}^2 \cong \mathsf{H}_3/Z(\mathsf{H}_3)$ be the Lie group epimorphism given by

$$q : \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \mapsto (x, y).$$

Further, let $\bar{\Sigma}$ be the invariant control system on the Abelian group $\mathbb{R}^2$ specified by

$$\bar{\Sigma} : u_1 T_{\mathbf{1}}q \cdot E_2 + u_2 T_{\mathbf{1}}q \cdot E_3.$$

The quotient map $q : \mathsf{H}_3 \to \mathbb{R}^2$ induces the following $\mathbf{LiCAS}^{\bowtie}$-epimorphisms

1. $(q, \mathrm{id}_{\mathbb{R}^2}) : (\Sigma^{(2,0)}, \chi_1^{(2,0)}) \to (\bar{\Sigma}, \chi_1^{(2,0)})$
2. $(q, \mathrm{id}_{\mathbb{R}^2}) : (\Sigma^{(2,0)}, \chi_2^{(2,0)}) \to (\bar{\Sigma}, \chi_2^{(2,0)})$
3. $(q, \mathrm{id}_{\mathbb{R}^2}) : (\Sigma^{(2,1)}, \chi_\alpha^{(2,1)}) \to (\bar{\Sigma}, \chi_\alpha^{(2,1)})$

and the following **LiSR**-epimorphism

4. $q : (\mathsf{H}_3, \mathcal{D}^{(1)}, \mathbf{g}^{(1)}) \to (\mathbb{R}^2, \bar{\mathbf{g}}) = \mathfrak{F}(\bar{\Sigma}^{(2,0)}, \chi_1^{(2,0)})$.

Accordingly, a subclass of the ECTs of $(\Sigma^{(2,0)}, \chi_1^{(2,0)})$, $(\Sigma^{(2,0)}, \chi_2^{(2,0)})$, and $(\Sigma^{(2,1)}, \chi_\alpha^{(2,1)})$, respectively, are controlled trajectories with image (under $(q, \mathrm{id}_{\mathbb{R}^2})$) being ECTs of the Abelian structures $(\bar{\Sigma}, \chi_1^{(2,0)})$, $(\bar{\Sigma}, \chi_2^{(2,0)})$, and $(\bar{\Sigma}, \chi_\alpha^{(2,1)})$, respectively. Likewise for the sub-Riemannian structure $(\mathsf{H}_3, \mathcal{D}^{(1)}, \mathbf{g}^{(1)})$ a subclass of the normal geodesics are the lifts of the geodesics (being straight lines) of the Euclidean space $(\mathbb{R}^2, \bar{\mathbf{g}})$ (see Remark 1.37). We provide details for the latter situation. The (normal) geodesics of $(\mathbb{R}^2, \bar{\mathbf{g}})$ are simply $(x(t), y(t)) = (a_1 + b_1 t, a_2 + b_2 t)$, $a_1, a_2, b_1, b_2 \in \mathbb{R}$. Accordingly,

Proposition 1.59. *The $\mathcal{D}^{(1)}$-curves*

$$g(t) = \begin{bmatrix} 1 & a_1 + b_1 t & c_0 + a_1 b_2 t + \frac{1}{2} b_1 b_2 t^2 \\ 0 & 1 & a_2 + b_2 t \\ 0 & 0 & 1 \end{bmatrix}, \qquad c_0, a_1, a_2, b_1, b_2 \in \mathbb{R}$$

are some normal geodesics of $(\mathsf{H}_3, \mathcal{D}^{(1)}, \mathbf{g}^{(1)})$.

Proof. Suppose $g(t) = m(z(t), x(t), y(t))$ is a $\mathcal{D}^{(1)}$-curve such that $q(g(t)) = (a_1 + b_1 t, a_2 + b_2 t)$, i.e., $x(t) = a_1 + b_1 t$ and $y(t) = a_2 + b_2 t$. As $g(t)$ is a $\mathcal{D}^{(1)}$-curve we have that $\frac{d}{dt} m(z(t), x(t), y(t)) \in \mathcal{D}^{(2,0)}(m(z(t), x(t), y(t)))$ or equivalently

$$m(z(t), x(t), y(t))^{-1} \frac{d}{dt} m(z(t), x(t), y(t)) \in \mathcal{D}^{(1)}(\mathbf{1})$$

$$\begin{bmatrix} 0 & \dot{x}(t) & -x(t)\dot{y}(t) + \dot{z}(t) \\ 0 & 0 & \dot{y}(t) \\ 0 & 0 & 0 \end{bmatrix} \in \langle E_2, E_3 \rangle .$$

Therefore, $\dot{z}(t) = x(t)\,\dot{y}(t) = a_1 b_2 + b_1 b_2 t$ and so $z(t) = c_0 + a_1 b_2 t + \frac{1}{2} b_1 b_2 t^2$, $c_0 \in \mathbb{R}$. By Corollary 1.36, it follows that $g(t)$ is a normal geodesic. ☐

Lastly, we claim that the Riemannian structures on H_3 are simply central expansions of the sub-Riemannian structures on H_3.

Proposition 1.60. *The Riemannian structure* $(\mathsf{H}_3, \mathbf{g}^{(0)})$ *is a central expansion of the sub-Riemannian structure* $(\mathsf{H}_3, \mathcal{D}^{(1)}, \mathbf{g}^{(1)})$ *with respect to* $Z(\mathsf{H}_3)$*.*

Accordingly, the image under q of the set of normal geodesics of $(\mathsf{H}_3, \mathbf{g}^{(0)})$ and the image under q of the set of normal geodesics of $(\mathsf{H}_3, \mathcal{D}^{(1)}, \mathbf{g}^{(1)})$ are identical. More details, and explicit calculation of the geodesics for structures on H_{2n+1}, will be given in Chapter 4.

1.5.2 Abnormal geodesics on $\mathsf{SO}(3) \times \mathbb{R}$

The four-dimensional trivial expansion of the orthogonal group

$$\mathsf{SO}(3) \times \mathbb{R} = \left\{ \begin{bmatrix} & & & 0 \\ & g & & 0 \\ & & & 0 \\ 0 & 0 & 0 & e^w \end{bmatrix} : g^\top g = \mathbf{1},\, \det g = 1,\, w \in \mathbb{R} \right\}$$

has Lie algebra

$$\mathfrak{so}(3) \oplus \mathbb{R} = \left\{ \begin{bmatrix} 0 & x & -y & 0 \\ -x & 0 & z & 0 \\ y & -z & 0 & 0 \\ 0 & 0 & 0 & w \end{bmatrix} = xE_1 + yE_2 + zE_3 + wE_4 \,:\, x, y, z, w \in \mathbb{R} \right\}.$$

The nonzero commutators of $\mathfrak{so}(3) \oplus \mathbb{R}$ are $[E_2, E_3] = E_1$, $[E_3, E_1] = E_2$, $[E_1, E_2] = E_3$. The center of $\mathfrak{so}(3) \oplus \mathbb{R}$ is $\operatorname{span}(E_4)$. Correspondingly, the center of $\mathsf{SO}(3) \times \mathbb{R}$ is $\{\mathbf{1}\} \times \mathbb{R}$. The quotient $q : \mathsf{SO}(3) \times \mathbb{R} \to \mathsf{SO}(3) \cong (\mathsf{SO}(3) \times \mathbb{R})/Z(\mathsf{SO}(3) \times \mathbb{R})$ may simply be realized as $q : (g, w) \mapsto g$.

We give an example of an invariant sub-Riemannian structure on $\mathsf{SO}(3) \times \mathbb{R}$ admitting central expansion such that images under $q : \mathsf{SO}(3) \times \mathbb{R} \to \mathsf{SO}(3)$ of the respective classes of abnormal geodesics are not identical. By left translation, it is enough to show this for the abnormal geodesics through identity (i.e., $g(0) = \mathbf{1}$).

Consider the sub-Riemannian structure $(\mathsf{SO}(3) \times \mathbb{R}, \mathcal{D}, \mathbf{g})$ with left-invariant orthonormal frame (X_1, X_2) given by $X_1(\mathbf{1}) = E_1 + E_4$ and $X_2(\mathbf{1}) = E_1 + E_2 + 2E_4$.

Proposition 1.61. *The abnormal geodesics of* $(\mathsf{SO}(3) \times \mathbb{R}, \mathcal{D}, \mathbf{g})$ *through the identity are the curves on the subgroup*

$$\exp(\mathbb{R}X_2(\mathbf{1})) = \left\{ \begin{bmatrix} \cos(\sqrt{2}\,t) & \frac{1}{\sqrt{2}}\sin(\sqrt{2}\,t) & -\frac{1}{\sqrt{2}}\sin(\sqrt{2}\,t) & 0 \\ -\frac{1}{\sqrt{2}}\sin(\sqrt{2}\,t) & \frac{1}{2}\left(1 + \cos(\sqrt{2}\,t)\right) & \frac{1}{2}\left(1 - \cos(\sqrt{2}\,t)\right) & 0 \\ \frac{1}{\sqrt{2}}\sin(\sqrt{2}\,t) & \frac{1}{2}\left(1 - \cos(\sqrt{2}\,t)\right) & \frac{1}{2}\left(1 + \cos(\sqrt{2}\,t)\right) & 0 \\ 0 & 0 & 0 & e^{2t} \end{bmatrix} : t \in \mathbb{R} \right\}$$

with Lie algebra $\operatorname{span}(X_2(\mathbf{1}))$*.*

Proof. Consider the optimal control problem (1.10) associated to $(\mathsf{SO}(3) \times \mathbb{R}, \mathcal{D}, \mathbf{g})$ via the orthonormal frame (X_1, X_2), i.e., $\Xi(\mathbf{1}, u) = u_1 X_1(\mathbf{1}) + u_2 X_2(\mathbf{1})$ and $\chi(u) = u_1^2 + u_2^2$. Suppose $(\xi(\cdot), u(\cdot))$, $\xi(t) = (g(t), p(t)) \in (\mathsf{SO}(3) \times \mathbb{R}) \times (\mathfrak{so}(3) \oplus \mathbb{R})^*$ is an abnormal extremal. Then (1.7) implies $p(t) \cdot X_1(\mathbf{1}) = p(t) \cdot X_2(\mathbf{1}) = 0$, i.e., $p_1(t) + p_4(t) = 0$ and $p_1(t) + p_2(t) + 2p_4(t) = 0$ (here $p = p_1 E_1^* + p_2 E_2^* + p_3 E_3^*$). Consequently (see Lemma 1.7)

$$0 = \frac{d}{dt}(p_1(t) + p_4(t)) = -p(t) \cdot [X_1(\cdot), \Xi_{u(t)}(\mathbf{1})] = u_2(t) p_3(t)$$
$$0 = \frac{d}{dt}(p_1(t) + p_2(t) + 2p_4(t)) = -p(t) \cdot [X_2(\mathbf{1}), \Xi_{u(t)}(\mathbf{1})] = -u_1(t) p_3(t).$$

If $p_3(\cdot) \neq 0$ then $u_1(\cdot) = u_2(\cdot) = 0$ and $g(\cdot)$ is constant, i.e., $g(t) = g(0)$. Suppose $p_3(\cdot) = 0$. Then

$$0 = \frac{d}{dt} p_3(t) = -p(t) \cdot [E_3, \Xi_{u(t)}(\mathbf{1})] = -u_2 p_1(t) + (u_1 + u_2) p_2(t) = -u_1 p_4(t).$$

By (1.5) we have that $p_4(\cdot) \neq 0$ and so $u_1(\cdot) = 0$. Hence, $\dot{g}(t) = u_2(t) X_2(g(t))$ and so $g(t)$ is a curve evolving on $\exp(\mathbb{R} X_2(\mathbf{1}))$.

On the other hand, for any curve $g(t)$ evolving on $\exp(\mathbb{R} X_2(\mathbf{1}))$ there exists $u_2(\cdot)$ such that $\dot{g}(t) = u_2(t) X_2(g(t))$. We have that $(\xi(\cdot), u(\cdot))$, $\xi(t) = (g(t), p(t))$, $p(t) = (-1, -1, 0, 1)$, $u_1(t) = 0$ satisfies (1.5)–(1.6)–(1.7). □

Remark 1.62 ([84, Section 9.5])**.** The integral curves of X_2 are strictly abnormal geodesics and are locally minimizing.

On the other hand, let $(\mathsf{SO}(3) \times \mathbb{R}, \tilde{\mathcal{D}}, \tilde{\mathbf{g}})$ be the central expansion of $(\mathsf{SO}(3) \times \mathbb{R}, \mathcal{D}, \mathbf{g})$ with respect to $Z(\mathsf{SO}(3) \times \mathbb{R})$ having left-invariant orthonormal frame (X_1, X_2, X_3) with $X_3(\mathbf{1}) = E_4$ (see Lemma 1.49).

Proposition 1.63. *The abnormal geodesics of* $(\mathsf{SO}(3) \times \mathbb{R}, \tilde{\mathcal{D}}, \tilde{\mathbf{g}})$ *through the identity are the curves on the center* $Z(\mathsf{SO}(3) \times \mathbb{R}, \tilde{\mathcal{D}}, \tilde{\mathbf{g}}) = \{\mathbf{1}\} \times \mathbb{R}$.

Proof. Consider the optimal control problem (1.10) associated to $(\mathsf{SO}(3) \times \mathbb{R}, \tilde{\mathcal{D}}, \tilde{\mathbf{g}})$ via the orthonormal frame (X_1, X_2, X_3), i.e., $\Xi(\mathbf{1}, u) = u_1 X_1(\mathbf{1}) + u_2 X_2(\mathbf{1}) + u_2 X_3(\mathbf{1})$ and $\chi(u) = u_1^2 + u_2^2 + u_3^2$. Suppose $(\xi(\cdot), u(\cdot))$, $\xi(t) = (g(t), p(t)) \in (\mathsf{SO}(3) \times \mathbb{R}) \times (\mathfrak{so}(3) \oplus \mathbb{R})^*$ is an abnormal extremal. Then (1.7) implies $p(t) \cdot X_1(\mathbf{1}) = p(t) \cdot X_2(\mathbf{1}) = p(t) \cdot X_3(\mathbf{1}) = 0$, i.e., $p_1(t) = p_2(t) = p_4(t) = 0$. Consequently (see Lemma 1.7)

$$0 = \frac{d}{dt}(p_1(t) + p_4(t)) = -p(t) \cdot [X_1(\cdot), \Xi_{u(t)}(\mathbf{1})] = u_2(t) p_3(t)$$
$$0 = \frac{d}{dt}(p_1(t) + p_2(t) + 2p_4(t)) = -p(t) \cdot [X_2(\mathbf{1}), \Xi_{u(t)}(\mathbf{1})] = -u_1(t) p_3(t)$$

By (1.5) we have that $p_3(\cdot) \neq 0$ and so $u_1(\cdot) = u_2(\cdot) = 0$. Consequently $\dot{g}(t) = u_3(t) X_3(g(t))$ and so $g(\cdot)$ evolves on $Z(\mathsf{SO}(3) \times \mathbb{R})$.

On the other hand, for any curve $g(t)$ evolving on $\exp(\mathbb{R} X_3(\mathbf{1}))$ there exists $u_3(\cdot)$ such that $\dot{g}(t) = u_3(t) X_3(g(t))$. We have that $(\xi(\cdot), u(\cdot))$, $\xi(t) = (g(t), p(t))$, $p(t) = (0, 0, 1, 0)$, $u_1(t) = u_2(t) = 0$ satisfies (1.5)–(1.6)–(1.7).

□

Hence the image under q of the abnormal geodesics (through identity) of $(\mathsf{SO}\,(3) \times \mathbb{R}, \mathcal{D}, \mathbf{g})$ are simply curves tangent to $\exp(\mathbb{R}\, T_{\mathbf{1}} q \cdot X_2(\mathbf{1})) = q(\exp(\mathbb{R} X_2(\mathbf{1})))$. On the other hand, the abnormal geodesics (through identity) of $(\mathsf{SO}\,(3) \times \mathbb{R}, \tilde{\mathcal{D}}, \tilde{\mathbf{g}})$ have trivial image under q. Consequently, we have that $q(\mathcal{G}) \neq q(\tilde{\mathcal{G}})$ when $\mathcal{G}$ and $\tilde{\mathcal{G}}$ denote the respective collections of abnormal geodesics.

Chapter 2

Quadratic Hamilton-Poisson systems in three dimensions

The dual space of a Lie algebra admits a natural Poisson structure, namely the Lie-Poisson structure. Quadratic and homogeneous Hamiltonian systems on these structures form a natural setting for a variety of interesting dynamical systems; prevalent examples are Euler's classic equations for the rigid body as well as its extensions and generalizations (see, e.g., [1, 65–67, 70, 88, 120]).

As discussed in the first chapter (Section 1.3), quadratic Hamilton-Poisson systems (on Lie-Poisson spaces) arise naturally in the study of invariant optimal control problems as well as invariant sub-Riemannian structures on Lie groups. In this vein, a number of quadratic Hamilton-Poisson systems on lower-dimensional Lie-Poisson spaces have been considered (see, e.g., [5, 9, 29, 51, 93, 104, 109]).

The equivalence of quadratic Hamilton-Poisson systems on Lie-Poisson spaces has been investigated by a few authors. Specifically, normal forms have been computed for a special class of quadratic systems ([119–121]). Orthogonal equivalence of systems on $\mathfrak{so}(3)^*_-$ has also been considered ([53]) whereas linear equivalence (of both homogeneous and inhomogeneous quadratic) systems was considered in [8]. On the other hand, we have classifications of homogeneous quadratic systems on $\mathfrak{se}(2)^*_-$ ([7]) and $\mathfrak{se}(1,1)^*_-$ ([26]).

In this chapter, we classify a significant subclass of quadratic Hamilton-Poisson systems on three-dimensional Lie-Poisson spaces. More precisely, we consider those systems that are both homogeneous and for which the underlying quadratic form is positive semidefinite (in particular, the Hamiltonian associated to any $\mathbf{LiCAS}_0^{\bowtie}$-object or invariant sub-Riemannian structure belongs to this class). This class covers a number of systems recently considered by several authors (see, e.g., [7, 8, 20–23, 26, 121]). In our opinion, the most significant consequence of our classification is that a number of these systems (on distinct Lie-Poisson spaces), which have been treated independently, are in fact linearly equivalent. Moreover, the reduction to normal form should dramatically simplify the computational complexity in investigating stability and integration.

Our classification of systems (under L-equivalence, see Section 1.3) is carried out in two parts. First, we classify systems within the context of each three-dimensional Lie-Poisson space (by making use of the Bianchi-Behr classification of three-dimensional Lie algebras, see Appendix A.1). Subsequently, we consider equivalences of systems on non-isomorphic Lie-Poisson spaces. Finally, an exhaustive and non-redundant list of normal forms is exhibited. Moreover, by identifying some simple invariants, we arrive at a taxonomy of systems.

Some of the computations involved in this chapter are quite demanding and hence Mathematica was used to facilitate these computations (typical code is given in Appendix C.1); these details shall often be omitted from the proofs. We note that as the same classification procedure is carried out for each Lie algebra, some of the details are repetitive. Furthermore, we note that we shall make no explicit mention of the case of trivial dynamics (i.e., $\vec{H}(p) = 0$).

Note. Some of the material presented in this chapter appears in [33]. A substantially expanded version of this paper, dealing with the classification, and the systematic investigation of stability and integration, of positive semidefinite homogeneous quadratic Hamilton-Poisson systems on three-dimensional Lie-Poisson spaces is in preparation.

Preliminaries

For the purposes of this chapter we find it convenient to change notation slightly. Given a basis (E_1, E_2, E_3) for a Lie algebra $\mathfrak{g}$ (as in Appendix A.1), an element $p = p_1 E_1^* + p_2 E_2^* + p_3 E_3^*$ expressed in the dual basis (E_1^*, E_2^*, E_3^*) will be written as a column matrix $p = (p_i)_{1 \leq i \leq 3}$. A system H_Q (on $\mathfrak{g}^*_-$) is then represented as $H_Q(p) = p^\top Q\, p$, where Q is a positive semidefinite 3×3 matrix. The equations of motion of a Hamiltonian H (on each of the respective associated Lie-Poisson spaces) take the form

$$\dot{p}_i = -p([E_i, dH(p)]), \quad i = 1, \ldots, n.$$

For the sake of convenience, all linear maps will be identified with their corresponding matrices. Accordingly, as linear Poisson automorphisms are exactly the dual maps of Lie algebra automorphism, the group of (matrices of) linear Poisson automorphisms can be obtained from the group of Lie algebra automorphisms (Appendix A.3) by simply taking the transpose.

An exhaustive list of Casimir functions, for low-dimensional Lie algebras, was obtained by Patera et al. [98]; For each three-dimensional Lie-Poisson space $\mathfrak{g}^*_-$ (associated to a three-dimensional Lie algebra $\mathfrak{g}$) we exhibit its Casimir function:

$$\begin{aligned}
&\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1 : C(p) = p_3 && \mathfrak{g}_{3.1} : C(p) = p_1 \\
&\mathfrak{g}_{3.2} : C(p) = p_1\, \mathrm{e}^{\frac{p_2}{p_1}} && \mathfrak{g}_{3.3} : C(p) = \frac{p_2}{p_1} \\
&\mathfrak{g}_{3.4}^0 : C(p) = p_1^2 - p_2^2 && \underset{1 \neq \alpha > 0}{\mathfrak{g}_{3.4}^\alpha} : C(p) = \frac{\frac{1}{2}p_1 + \frac{1}{2}p_2}{(\pm\frac{1}{2}p_1 \mp \frac{1}{2}p_2)^{\frac{\alpha-1}{\alpha+1}}} \\
&\mathfrak{g}_{3.5}^0 : C(p) = p_1^2 + p_2^2 && \underset{\alpha > 0}{\mathfrak{g}_{3.5}^\alpha} : C(p) = (p_1^2 + p_2^2)\left(\frac{p_1 - ip_2}{p_1 + ip_2}\right)^{i\alpha} \\
&\mathfrak{g}_{3.6} : p_1^2 + p_2^2 - p_3^2 && \mathfrak{g}_{3.7} : p_1^2 + p_2^2 + p_3^2.
\end{aligned}$$

On the trivial Lie-Poisson space $(3\mathfrak{g}_1)^*_-$, every function is a Casimir function. Note that only $3\mathfrak{g}_1$, $\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1$, $\mathfrak{g}_{3.1}$, $\mathfrak{g}_{3.4}^0$, $\mathfrak{g}_{3.5}^0$, $\mathfrak{g}_{3.6}$, and $\mathfrak{g}_{3.7}$ admit globally defined Casimir functions.

2.1 Systems on solvable Lie algebras

For each solvable Lie algebra $\mathfrak{g}$, we classify all Hamilton-Poisson systems $(\mathfrak{g}^*_-, H_Q)$, where Q is a positive semidefinite quadratic form on $\mathfrak{g}^*$; an exhaustive and non-redundant list of class representatives is obtained. The classification procedure is as follows. We use the three basic types of equivalence (Proposition 1.25) to simplify the form of Q as much as possible (note however that some Lie-Poisson spaces admit no polynomial Casimir.) This type of normalization is often sufficient for arriving at class representatives. However, in some cases further normalization may be required; in such cases we find, explicitly, linear isomorphisms conjugating the Hamiltonian vector fields in question.

Lastly, we verify that no two of the class representatives obtained are equivalent, as follows. Suppose two representatives are equivalent. Then the associated Hamiltonian vector fields $\vec{H}$ and $\vec{H}'$ are compatible with a linear isomorphism $\psi = (\psi_{ij})_{1\leq i,j\leq 3}$, i.e., $\psi \cdot \vec{H} = \vec{H}' \circ \psi$. This is shown to lead to a contradiction.

2.1.1 Type $\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1$

The nonzero commutators of $\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1$ are $[E_1, E_2] = E_1$; $C(p) = p_3$ is a Casimir function.

Proposition 2.1. *On* $(\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1)^*_-$, *any system is equivalent to exactly one of the systems*

$$H_1(p) = p_1^2 \qquad H_2(p) = p_2^2 \qquad H_3(p) = p_1^2 + p_2^2$$
$$H_4(p) = (p_1 + p_3)^2 \qquad H_5(p) = p_2^2 + (p_1 + p_3)^2.$$

Proof. Let $H_Q(p) = p^\top Q\, p$ and let

$$Q = \begin{bmatrix} a_1 & b_1 & b_2 \\ b_1 & a_2 & b_3 \\ b_2 & b_3 & a_3 \end{bmatrix}.$$

Suppose $a_2 > 0$. Then

$$\psi = \begin{bmatrix} x & 0 & 0 \\ -\frac{xb_1}{a_2} & 1 & -\frac{vb_3}{a_2} \\ 0 & 0 & v \end{bmatrix}$$

is a linear Poisson automorphism (for any nonzero real numbers x and v) such that

$$\tfrac{1}{a_2}\, \psi^\top Q\, \psi = \begin{bmatrix} \frac{x^2(a_1a_2 - b_1^2)}{a_2^2} & 0 & \frac{vx(a_2b_2 - b_1b_3)}{a_2^2} \\ 0 & 1 & 0 \\ \frac{vx(a_2b_2 - b_1b_3)}{a_2^2} & 0 & \frac{v^2(a_2a_3 - b_3^2)}{a_2^2} \end{bmatrix}.$$

If $a_1a_2 - b_1^2 = 0$, then H is equivalent to H_2, as $\frac{1}{a_2}(H \circ \psi) = H_2(p) + \frac{v^2(a_2a_3 - b_3^2)}{a_2^2} C(p)$. Suppose $a_1a_2 - b_1^2 \neq 0$ and let $x = \sqrt{\frac{a_2^2}{a_1a_2 - b_1^2}}$. If $a_2b_2 - b_1b_3 = 0$, then H is equivalent to H_3.

If $a_2b_2 - b_1b_3 \neq 0$, then there exists $v, w \in \mathbb{R}$ such that $(H \circ \psi)(p) = (p_1 + p_3)^2 + p_2^2 + wC(p)$; hence H is equivalent to H_5.

Suppose $a_2 = 0$. Then $\psi = \operatorname{diag}(x, 1, v)$ is a linear Poisson automorphism (for any nonzero numbers x and v) such that

$$\psi^\top Q\, \psi = \begin{bmatrix} x^2 a_1 & 0 & vxb_2 \\ 0 & 0 & 0 \\ vxb_2 & 0 & v^2 a_3 \end{bmatrix}.$$

If $a_1 \neq 0$ and $b_2 \neq 0$, then H is equivalent to H_4. If $a_1 \neq 0$ and $b_2 = 0$, then H is equivalent to H_1. (If $a_1 = 0$, then H is equivalent to the trivial system $H_0(p) = 0$.)

It is straightforward to show that no two of these five systems are equivalent. We provide details only for one case: we show that $H_3(p) = p_1^2 + p_2^2$ and $H_4(p) = (p_1 + p_3)^2$ are not equivalent. Suppose H_4 and H_5 are equivalent, i.e., suppose there exists a linear isomorphism $\psi : \mathfrak{g}_{2.1} \oplus \mathfrak{g}_1 \to \mathfrak{g}_{2.1} \oplus \mathfrak{g}_1$ such that $\psi \cdot \vec{H}_4 = \vec{H}_5 \circ \psi$. Let ψ have matrix $(\psi_{ij})_{1 \leq i,j \leq 3}$. Then

$$\psi \cdot \vec{H}_4(p) = \begin{bmatrix} 2p_1\,(\psi_{12}p_1 - \psi_{11}p_2) \\ 2p_1\,(\psi_{22}p_1 - \psi_{21}p_2) \\ 2p_1\,(\psi_{32}p_1 - \psi_{31}p_2) \end{bmatrix} \qquad (\vec{H}_5 \circ \psi)(p) = \begin{bmatrix} 0 \\ *_1 \\ 0 \end{bmatrix}$$

$$*_1 = 2\,(\psi_{11}p_1 + \psi_{12}p_2 + \psi_{13}p_3)\,((\psi_{11} + \psi_{31})p_1 + (\psi_{12} + \psi_{32})p_2 + (\psi_{13} + \psi_{33})p_3)$$

Hence, by equating coefficients, we get $\psi_{11} = \psi_{12} = \psi_{31} = \psi_{32} = 0$. This contradicts that ψ is an isomorphism. □

2.1.2 Type $\mathfrak{g}_{3.1}$

The Heisenberg Lie algebra $\mathfrak{g}_{3.1}$ has nonzero commutators $[E_2, E_3] = E_1$; $C(p) = p_1$ is a Casimir function.

Proposition 2.2 ([32]). *On $(\mathfrak{g}_{3.1})^*_-$, any system is equivalent to exactly one of the systems*

$$H_1(p) = p_3^2 \qquad\qquad H_2(p) = p_2^2 + p_3^2.$$

Proof. Let $H_Q(p) = p^\top Q\, p$, where

$$Q = \begin{bmatrix} a_1 & b_1 & b_2 \\ b_1 & a_2 & b_3 \\ b_2 & b_3 & a_3 \end{bmatrix}.$$

Suppose $a_3 \neq 0$. Then

$$\psi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{b_2}{a_3} & -\frac{b_3}{a_3} & 1 \end{bmatrix}$$

is a linear Poisson automorphism such that

$$\psi^\top Q\, \psi = \begin{bmatrix} a_1 - \frac{b_2^2}{a_3} & b_1 - \frac{b_2 b_3}{a_3} & 0 \\ b_1 - \frac{b_2 b_3}{a_3} & a_2 - \frac{b_3^2}{a_3} & 0 \\ 0 & 0 & a_3 \end{bmatrix} = \begin{bmatrix} a_1' & b_1' & 0 \\ b_1' & a_2' & 0 \\ 0 & 0 & a_3 \end{bmatrix}.$$

If $a_2' = 0$, then H is equivalent to H_1. Suppose $a_2' \neq 0$. Then

$$\psi' = \begin{bmatrix} \frac{1}{\sqrt{a_3}\sqrt{a_2'}} & 0 & 0 \\ -\frac{b_1'}{\sqrt{a_3}\left(a_2'\right)^{3/2}} & \frac{1}{\sqrt{a_2'}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{a_3}} \end{bmatrix}$$

is a linear Poisson automorphism such that

$$\psi'^\top \psi^\top Q \psi \psi' = \begin{bmatrix} \frac{a_1'a_2' - \left(b_1'\right)^2}{a_3\left(a_2'\right)^2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus H is equivalent to H_2.

Now suppose $a_3 = 0$ and assume $a_2 \neq 0$ (if $a_2 = a_3 = 0$, then H is equivalent to the trivial system $H_0(p) = 0$). Then

$$\psi = \begin{bmatrix} -\frac{1}{\sqrt{a_2}} & 0 & 0 \\ \frac{b_1}{a_2^{3/2}} & 0 & \frac{1}{\sqrt{a_2}} \\ 0 & 1 & 0 \end{bmatrix}$$

is a linear Poisson automorphism such that

$$\psi^\top Q \psi = \begin{bmatrix} \frac{a_1a_2 - b_1^2}{a_2^2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Hence H is equivalent to H_1

It remains to be shown that H_1 and H_2 are not equivalent. Assume that they are equivalent, i.e., suppose there exists a linear isomorphism ψ such that $\psi \cdot H_1 = H_2 \circ \psi$. Let ψ have matrix $(\psi_{ij})_{1 \leq i,j \leq 3}$. Then

$$\begin{bmatrix} -2\psi_{12}p_1p_3 \\ -2\psi_{22}p_1p_3 \\ -2\psi_{32}p_1p_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -2\left(\psi_{11}p_1 + \psi_{12}p_2 + \psi_{13}p_3\right)\left(\psi_{31}p_1 + \psi_{32}p_2 + \psi_{33}p_3\right) \\ 2\left(\psi_{11}p_1 + \psi_{12}p_2 + \psi_{13}p_3\right)\left(\psi_{21}p_1 + \psi_{22}p_2 + \psi_{23}p_3\right) \end{bmatrix}$$

must hold for all $p_1, p_2, p_3 \in \mathbb{R}$. A straightforward argument shows that ψ is not an isomorphism, a contradiction. □

2.1.3 Type $\mathfrak{g}_{3.2}$

The nonzero commutators of $\mathfrak{g}_{3.2}$ are $[E_2, E_3] = E_1 - E_2$ and $[E_3, E_1] = E_1$; $(\mathfrak{g}_{3.2})^*_-$ does not admit any polynomial Casimir functions.

Proposition 2.3. *On* $(\mathfrak{g}_{3.2})^*_-$, *any system is equivalent to exactly one of the systems*

$$H_1(p) = p_1^2 \qquad H_2(p) = p_2^2 \qquad H_3(p) = p_3^2$$
$$H_4(p) = p_1^2 + p_2^2 \qquad H_5(p) = p_1^2 + p_3^2 \qquad H_{6,\beta}(p) = \beta\, p_1^2 + p_2^2 + p_3^2.$$

Here $\beta \geq 0$ *parametrizes a family of non-equivalent class representatives.*

Proof. Let $H_Q(p) = p^\top Q\, p$, where

$$Q = \begin{bmatrix} a_1 & b_1 & b_2 \\ b_1 & a_2 & b_3 \\ b_2 & b_3 & a_3 \end{bmatrix}.$$

Suppose $a_3 \neq 0$. Then

$$\psi = \begin{bmatrix} a_3 & 0 & 0 \\ 0 & a_3 & 0 \\ -b_2 & -b_3 & 1 \end{bmatrix}$$

is a linear Poisson automorphism such that

$$\tfrac{1}{a_3}\, \psi^\top Q\, \psi = \begin{bmatrix} a_1 a_3 - b_2^2 & a_3 b_1 - b_2 b_3 & 0 \\ a_3 b_1 - b_2 b_3 & a_2 a_3 - b_3^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_1' & b_1' & 0 \\ b_1' & a_2' & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

If $a_2' = 0$, then it is easy to show that H is equivalent to either H_3 or H_5. Suppose $a_2' \neq 0$. Then

$$\psi' = \begin{bmatrix} \frac{1}{\sqrt{a_2'}} & 0 & 0 \\ -\frac{b_1'}{(a_2')^{3/2}} & \frac{1}{\sqrt{a_2'}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is a linear Poisson automorphism such that

$$\tfrac{1}{a_3}\, \psi'^\top\, \psi^\top Q\, \psi\, \psi' = \begin{bmatrix} \frac{a_1' a_2' - (b_1')^2}{(a_2')^2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus H is equivalent to $H_{6,\beta}$ with $\beta \geq 0$.

Suppose $a_3 = 0$. If $a_2 = 0$, then H is equivalent to H_1. Suppose $a_2 \neq 0$. Then

$$\psi = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{b_1}{a_2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is a linear Poisson automorphism such that

$$\tfrac{1}{a_2} \psi^\top Q\, \psi = \begin{bmatrix} \frac{a_1 a_2 - b_1^2}{a_2^2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence H is equivalent to the intermediate system $H'_\beta = \beta\, p_1^2 + p_2^2$ with $\beta \geq 0$. If $\beta = 0$, then H'_β is equivalent to H_2. If $0 < \beta < \frac{1}{4}$, then H'_β is equivalent to H_2. Indeed,

$$\psi = \begin{bmatrix} 4\beta & -2 & 0 \\ 2\beta & \sqrt{1-4\beta}-1 & 0 \\ 0 & 0 & -4\beta \end{bmatrix}$$

is a linear isomorphism such that

$$\psi \cdot \vec{H}'_\beta(p) = \begin{bmatrix} 0 \\ 0 \\ 8\beta\left(\beta p_1^2 - p_1 p_2 + p_2^2\right) \end{bmatrix} = (\vec{H}_2 \circ \psi)(p).$$

If $\beta = \frac{1}{4}$, then

$$\psi = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

is a linear isomorphism such that

$$\psi \cdot \vec{H}'_\beta(p) = \begin{bmatrix} 0 \\ 0 \\ -2\left(p_1 - 2p_2\right)^2 \end{bmatrix} = (\vec{H}_1 \circ \psi)(p).$$

Thus $H'_{\frac{1}{4}}$ is equivalent to H_1. If $\beta > \frac{1}{4}$, then H'_β is equivalent to H_4. Indeed,

$$\psi = \begin{bmatrix} 4\beta & -2 & 0 \\ 2\beta & -1-\sqrt{12\beta-3} & 0 \\ 0 & 0 & 12\beta \end{bmatrix}$$

is a linear isomorphism such that

$$\psi \cdot \vec{H}'_\beta(p) = \begin{bmatrix} 0 \\ 0 \\ -24\beta\left(\beta p_1^2 - p_1 p_2 + p_2^2\right) \end{bmatrix} = (\vec{H}_4 \circ \psi)(p).$$

Tedious but straightforward computations show that none of the class representatives are equivalent. In particular, $H_{6,\beta}$ and $H_{6,\beta'}$ are equivalent only if $\beta = \beta'$ (see Appendix C.1 for sample Mathematica code to verify this fact). □

2.1.4 Type $\mathfrak{g}_{3.3}$

The nonzero commutators of $\mathfrak{g}_{3.3}$ are $[E_2, E_3] = -E_2$ and $[E_3, E_1] = E_1$; $(\mathfrak{g}_{3.3})^*_-$ does not admit any polynomial Casimir functions.

Proposition 2.4. *On $(\mathfrak{g}_{3.3})^*_-$, any system is equivalent to exactly one of the systems*

$$H_1(p) = p_1^2 \qquad H_2(p) = p_3^2 \qquad H_3(p) = p_1^2 + p_3^2$$
$$H_4(p) = p_1^2 + p_2^2 \qquad H_5(p) = p_1^2 + p_2^2 + p_3^2.$$

Proof. Let $H_Q(p) = p^\top Q\, p$, where

$$Q = \begin{bmatrix} a_1 & b_1 & b_2 \\ b_1 & a_2 & b_3 \\ b_2 & b_3 & a_3 \end{bmatrix}.$$

Suppose $a_3 \neq 0$. Then

$$\psi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{b_2}{a_3} & -\frac{b_3}{a_3} & 1 \end{bmatrix}$$

is a linear Poisson automorphism such that

$$\tfrac{1}{a_3}\, \psi^\top Q\, \psi = \tfrac{1}{a_3} \begin{bmatrix} a_1 - \frac{b_2^2}{a_3} & b_1 - \frac{b_2 b_3}{a_3} & 0 \\ b_1 - \frac{b_2 b_3}{a_3} & a_2 - \frac{b_3^2}{a_3} & 0 \\ 0 & 0 & a_3 \end{bmatrix} = \begin{bmatrix} a_1' & b_1' & 0 \\ b_1' & a_2' & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

If $a_2' = 0$, then it is easy to show that H is equivalent to either H_2 or H_3. Suppose $a_2' \neq 0$. Then

$$\psi' = \begin{bmatrix} x & 0 & 0 \\ -\frac{x b_1'}{a_2'} & v & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is a linear Poisson automorphism (for all nonzero x and v) such that

$$\tfrac{1}{a_3}\, \psi'^\top \psi^\top Q\, \psi\, \psi' = \begin{bmatrix} x^2(a_1' - \frac{(b_1')^2}{a_2'}) & 0 & 0 \\ 0 & v^2 a_2' & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus H is equivalent to H_5 or the intermediate system $H'(p) = p_2^2 + p_3^2$. However,

$$\psi'' = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is an automorphism such that $H' \circ \psi'' = H_3$.

Likewise, if $a_3 = 0$, then H is equivalent to H_1 or H_4 (or the trivial system $H_0(p) = 0$). It is straightforward to verify that none of the representatives are equivalent. □

2.1.5 Type $\mathfrak{g}_{3.4}^0$

The semi-Euclidean Lie algebra $\mathfrak{g}_{3.4}^0$ has nonzero commutators $[E_2, E_3] = E_1$ and $[E_3, E_1] = -E_2$; $C(p) = p_1^2 - p_2^2$ is a Casimir function.

Proposition 2.5 ([26]). *On $(\mathfrak{g}_{3.4}^0)^*_-$, any system is equivalent to exactly one of the systems*

$$H_1(p) = p_1^2 \qquad H_2(p) = p_3^2 \qquad H_3(p) = p_1^2 + p_3^2$$
$$H_4(p) = (p_1 + p_2)^2 \qquad H_5(p) = (p_1 + p_2)^2 + p_3^2.$$

Proof. Let $H_Q(p) = p^\top Q\, p$, where

$$Q = \begin{bmatrix} a_1 & b_1 & b_2 \\ b_1 & a_2 & b_3 \\ b_2 & b_3 & a_3 \end{bmatrix}.$$

Suppose $a_3 \neq 0$. Then

$$\psi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{b_2}{a_3} & -\frac{b_3}{a_3} & 1 \end{bmatrix}$$

is a linear Poisson automorphism such that

$$\tfrac{1}{a_3}\,\psi^\top Q\,\psi = \begin{bmatrix} \frac{a_1a_3-b_2^2}{a_3^2} & \frac{a_3b_1-b_2b_3}{a_3^2} & 0 \\ \frac{a_3b_1-b_2b_3}{a_3^2} & \frac{a_2a_3-b_3^2}{a_3^2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_1' & b_1' & 0 \\ b_1' & a_2' & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Suppose $b_1' \neq 0$ and suppose that $a_1' \neq a_2'$ or $(a_1')^2 \neq (b_1')^2$. Then

$$\psi' = \begin{bmatrix} 1 & -\frac{a_1'+a_2'+\sqrt{\left(a_1'+a_2'\right)^2-4\left(b_1'\right)^2}}{2b_1'} & 0 \\ -\frac{a_1'+a_2'+\sqrt{\left(a_1'+a_2'\right)^2-4\left(b_1'\right)^2}}{2b_1'} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is a linear Poisson automorphism such that

$$\tfrac{1}{a_3}\,\psi'^\top\,\psi^\top Q\,\psi\,\psi' = \begin{bmatrix} a_1'' & 0 & 0 \\ 0 & a_2'' & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

for some $a_1'' \geq 0$, $a_2'' \geq 0$. (If $b_1' = 0$, then we have the same situation.) If $a_1'' = a_2'' = 0$, then H is equivalent to H_2. Suppose $a_1'' > 0$ or $a_2'' > 0$. Then H is equivalent to $H' = (a_1''+a_2'')p_1^2+p_3^2$, as $H'(p) - a_2''C(p) = a_1''p_1^2 + a_2''p_2^2 + p_3^2$. Furthermore $\psi'' = \mathrm{diag}(\frac{1}{\sqrt{a_1''+a_2''}}, \frac{1}{\sqrt{a_1''+a_2''}}, 1)$ is a linear Poisson automorphism such that $(H' \circ \psi'')(p) = p_1^2 + p_3^2$. Thus H is equivalent to H_3.

Suppose that $a_1' = a_2'$ and $(a_1')^2 = (b_1')^2$. Then

$$\tfrac{1}{a_3}\,\psi^\top Q\,\psi = \begin{bmatrix} a_1' & ka_1' & 0 \\ ka_1' & a_1' & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

with $k = \pm 1$. If $a_1' = 0$, then H is equivalent to H_2. Suppose $a_1' > 0$. Then $\psi' = \mathrm{diag}(\frac{1}{\sqrt{a_1'}}, \frac{1}{\sqrt{a_1'}}, 1)$ is a linear Poisson automorphism such that

$$\tfrac{1}{a_3}\,\psi'^\top\,\psi^\top Q\,\psi\,\psi' = \begin{bmatrix} 1 & k & 0 \\ k & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

If $k = 1$, then H is equivalent to H_5. If $k = -1$, then $\psi'' = \mathrm{diag}(1, -1, -1)$ is a linear Poisson automorphism such that

$$\tfrac{1}{a_3}\, \psi''^{\top}\, \psi'^{\top}\, \psi^{\top}\, Q\, \psi\, \psi'\, \psi'' = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Hence if $k = -1$, then H is again equivalent to H_5.

Similarly, if $a_3 = 0$, then H is equivalent to H_1 or H_4 (or the trivial system $H_0(p) = 0$). Straightforward computations again show that none of the class representatives are equivalent. □

2.1.6 Type $\mathfrak{g}_{3.4}^{\alpha}$

The nonzero commutators of $\mathfrak{g}_{3.4}^{\alpha}$ are $[E_2, E_3] = E_1 - \alpha\, E_2$ and $[E_3, E_1] = \alpha\, E_1 - E_2$; $(\mathfrak{g}_{3.4}^{\alpha})_{-}^{*}$ does not admit any polynomial Casimir functions. The classification of systems on $(\mathfrak{g}_{3.4}^{\alpha})_{-}^{*}$ depends on the value of α. We find it convenient to classify the entire collection $\{((\mathfrak{g}_{3.4}^{\alpha})_{-}^{*}, H) : \alpha > 0,\ \alpha \neq 1\}$ of systems, rather than classifying for each fixed value for α. Moreover, we will use class representatives not strictly belonging to this collection.

Proposition 2.6. *Any system* $((\mathfrak{g}_{3.4}^{\alpha})_{-}^{*}, H)$, $\alpha > 0$, $\alpha \neq 1$ *is equivalent to exactly one of the systems*

$$\begin{array}{ll} ((\mathfrak{g}_{3.4}^{\alpha})_{-}^{*}, p_3^2) & ((\mathfrak{g}_{3.4}^{\alpha})_{-}^{*}, (p_1 + p_2)^2 + p_3^2) \\ ((\mathfrak{g}_{3.4}^{\alpha})_{-}^{*}, (p_1 - p_2)^2 + p_3^2) & ((\mathfrak{g}_{3.4}^{\alpha})_{-}^{*}, \beta\, p_1^2 + p_2^2 + p_3^2) \\ ((\mathfrak{g}_{3.4}^{0})_{-}^{*}, p_1^2) & ((\mathfrak{g}_{3.3})_{-}^{*}, p_1^2 + p_2^2) \\ ((\mathfrak{g}_{3.3})_{-}^{*}, p_1^2). & \end{array}$$

Here $0 \leq \beta \leq 1$, $\alpha > 0$, $\alpha \neq 1$ *parametrize families of non-equivalent systems.*

Proof. Consider the system $((\mathfrak{g}_{3.4}^{\alpha})_{-}^{*}, H_Q)$, $\alpha > 0$, $\alpha \neq 1$. By arguments similar to those in proof of Proposition 2.5, we have that $((\mathfrak{g}_{3.4}^{\alpha})_{-}^{*}, H_Q)$ is equivalent to one of the systems

$$\begin{array}{lll} H_1(p) = p_3^2 & H_3 = (p_1 + p_2)^2 & H_5(p) = (p_1 + p_2)^2 + p_3^2 \\ H_{2,\beta}(p) = p_1^2 + \beta\, p_2^2 & H_4 = (p_1 - p_2)^2 & H_6(p) = (p_1 - p_2)^2 + p_3^2 \\ H_{7,\beta}(p) = \beta\, p_1^2 + p_2^2 + p_3^2 & & \end{array}$$

on $(\mathfrak{g}_{3.4}^{\alpha})_{-}^{*}$. Here $0 \leq \beta \leq 1$ parametrizes families of representatives. It turns out that no further reduction of H_1, H_5, H_6, or $H_{7,\beta}$ is possible.

The systems $((\mathfrak{g}_{3.4}^{\alpha})_{-}^{*}, H_3)$ and $((\mathfrak{g}_{3.4}^{\alpha})_{-}^{*}, H_4)$ are both equivalent to the system $((\mathfrak{g}_{3.3})_{-}^{*}, p_1^2)$, for any $\alpha > 0$, $\alpha \neq 1$. Indeed, the required linear isomorphisms have matrices

$$\psi = \begin{bmatrix} 1-\alpha & 1-\alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1+\alpha \end{bmatrix} \quad \text{and} \quad \psi' = \begin{bmatrix} -1-\alpha & 1+\alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1+\alpha \end{bmatrix}$$

respectively. More precisely, $\psi, \psi' : (\mathfrak{g}_{3.4}^{\alpha})_-^* \to (\mathfrak{g}_{3.3})_-^*$ and, in coordinates,

$$(\psi \cdot \vec{H}_3 \circ \psi^{-1})(p) = \begin{bmatrix} 0 \\ 0 \\ -2p_1^2 \end{bmatrix} = (\psi' \cdot \vec{H}_4 \circ \psi'^{-1})(p)$$

which is the Hamiltonian vector field associated to $((\mathfrak{g}_{3.3})_-^*, p_1^2)$.

Next we consider the (two-parameter) family $((\mathfrak{g}_{3.4}^{\alpha})_-^*, H_{2,\beta})$. It turns out that each member of this family is equivalent to one of three systems. Let $\kappa_\alpha^{\pm} = -1 + 2\alpha^2 \pm 2\alpha\sqrt{\alpha^2 - 1}$. If $\alpha > 1$, then $0 < \kappa_\alpha^- < 1 < \kappa_\alpha^+$. The following three conditions partition the family $((\mathfrak{g}_{3.4}^{\alpha})_-^*, H_{2,\beta})$ into its equivalence classes

$$0 < \alpha < 1 \ \wedge \ 0 \leq \beta \leq 1 \qquad \text{or} \qquad \alpha > 1 \ \wedge \ 0 \leq \beta < \kappa_\alpha^- \tag{2.1}$$

$$\alpha > 1 \ \wedge \ \beta = \kappa_\alpha^- \tag{2.2}$$

$$\alpha > 1 \ \wedge \ \kappa_\alpha^- < \beta \leq 1. \tag{2.3}$$

Let $\vec{H}$ be the Hamiltonian vector field associated to $((\mathfrak{g}_{3.4}^{\alpha})_-^*, H_{2,\beta})$. Suppose (2.1) holds. Then $\psi : (\mathfrak{g}_{3.4}^{\alpha})_-^* \to (\mathfrak{g}_{3.4}^{0})_-^*$,

$$\begin{aligned}\psi &= \begin{bmatrix} 2\alpha & -1-\beta-\sqrt{1+\beta\,(2-4\alpha^2+\beta)} & 0 \\ -2\alpha & 1+\beta-\sqrt{1+\beta\,(2-4\alpha^2+\beta)} & 0 \\ 0 & 0 & 4\alpha \end{bmatrix} \\ &= \begin{bmatrix} 2\alpha & -1-\beta-\sqrt{(\kappa_\alpha^- - \beta)(\kappa_\alpha^+ - \beta)} & 0 \\ -2\alpha & 1+\beta-\sqrt{(\kappa_\alpha^- - \beta)(\kappa_\alpha^+ - \beta)} & 0 \\ 0 & 0 & 4\alpha \end{bmatrix}\end{aligned}$$

is a linear isomorphism (with $\det\psi = -16\alpha^2\sqrt{(\kappa_\alpha^- - \beta)(\kappa_\alpha^+ - \beta)}$) such that

$$\psi \cdot \vec{H} \circ \psi^{-1} = \begin{bmatrix} 0 \\ 0 \\ 2p_1p_2 \end{bmatrix}.$$

Thus, if (2.1) is satisfied, then $((\mathfrak{g}_{3.4}^{\alpha})_-^*, H_{2,\beta})$ is equivalent to $((\mathfrak{g}_{3.4}^{0})_-^*, p_1^2)$. Next suppose (2.2) holds. Then $\psi : (\mathfrak{g}_{3.4}^{\alpha})_-^* \to (\mathfrak{g}_{3.3})_-^*$,

$$\psi = \begin{bmatrix} -\alpha & \alpha^2 - \alpha\sqrt{\alpha^2-1} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \alpha \end{bmatrix}$$

is a linear isomorphism (with $\det\psi = -\alpha^2$) such that

$$\psi \cdot \vec{H} \circ \psi^{-1} = \begin{bmatrix} 0 \\ 0 \\ -2p_1^2 \end{bmatrix}.$$

Hence $((\mathfrak{g}_{3.4}^{\alpha})^*_-, p_1^2 + \kappa_\alpha^- p_2^2)$ is equivalent to $((\mathfrak{g}_{3.3})^*_-, p_1^2)$. Lastly, suppose (2.3) holds. Then $\psi : (\mathfrak{g}_{3.4}^{\alpha})^*_- \to (\mathfrak{g}_{3.3})^*_-$,

$$\psi = \begin{bmatrix} 2\alpha & -1-\beta & 0 \\ 0 & \sqrt{(\beta-\kappa_\alpha^-)(\kappa_\alpha^+-\beta)} & 0 \\ 0 & 0 & 4\alpha \end{bmatrix}$$

is a linear isomorphism (with $\det\psi = 8\alpha^2\sqrt{(\beta-\kappa_\alpha^-)(\kappa_\alpha^+-\beta)})$ such that

$$\psi \cdot \vec{H} \circ \psi^{-1} = \begin{bmatrix} 0 \\ 0 \\ -2\left(p_1^2+p_2^2\right) \end{bmatrix}.$$

Therefore, if (2.3) holds, then $((\mathfrak{g}_{3.4}^{\alpha})^*_-, H_{2,\beta})$ is equivalent to $((\mathfrak{g}_{3.3})^*_-, p_1^2+p_2^2)$.

Finally, straightforward but tedious computations again show that none of the class representatives are equivalent. ☐

2.1.7 Type $\mathfrak{g}_{3.5}^0$

The nonzero commutators of the Euclidean Lie algebra $\mathfrak{g}_{3.5}^0$ are $[E_2, E_3] = E_1$ and $[E_3, E_1] = E_2$; $C(p) = p_1^2 + p_2^2$ is a Casimir function.

Proposition 2.7 (cf. [7]). *On $(\mathfrak{g}_{3.5}^0)^*_-$, any system is equivalent to exactly one of the systems*

$$H_1(p) = p_2^2 \qquad H_2(p) = p_3^2 \qquad H_3(p) = p_2^2 + p_3^2.$$

Proof. Let $H_Q(p) = p^\top Q\, p$, where

$$Q = \begin{bmatrix} a_1 & b_1 & b_2 \\ b_1 & a_2 & b_3 \\ b_2 & b_3 & a_3 \end{bmatrix}.$$

Suppose $a_3 \neq 0$. We have that

$$\psi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{b_2}{a_3} & -\frac{b_3}{a_3} & 1 \end{bmatrix}$$

is a linear Poisson automorphism such that

$$\tfrac{1}{a_3}\,\psi^\top Q\,\psi = \begin{bmatrix} \frac{a_1a_3-b_2^2}{a_3^2} & \frac{a_3b_1-b_2b_3}{a_3^2} & 0 \\ \frac{a_3b_1-b_2b_3}{a_3^2} & \frac{a_2a_3-b_3^2}{a_3^2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_1' & b_1' & 0 \\ b_1' & a_2' & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Suppose $b_1' \neq 0$. Then

$$\psi' = \begin{bmatrix} 1 & -\frac{-a_1'+a_2'+\sqrt{\left(-a_1'+a_2'\right)^2+4\left(b_1'\right)^2}}{2b_1'} & 0 \\ \frac{-a_1'+a_2'+\sqrt{\left(-a_1'+a_2'\right)^2+4\left(b_1'\right)^2}}{2b_1'} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is a linear Poisson automorphism such that

$$\tfrac{1}{a_3}\,\psi'^\top\,\psi^\top\,Q\,\psi\,\psi' = \begin{bmatrix} a_1'' & 0 & 0 \\ 0 & a_2'' & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

for some $a_1'' \geq 0$, $a_2'' \geq 0$. (If $b_1' = 0$, then we have the same situation.) If $a_1'' = a_2''$, then H is equivalent to H_2. Suppose that $a_1'' \neq a_2''$ and that $a_1'' > 0$ or $a_2'' > 0$. We may assume $a_1'' < a_2''$. If not, the linear Poisson automorphism

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

serves to swap the values. It follows that H is equivalent to $H' = (a_2'' - a_1'')p_2^2 + p_3^2$, as $H'(p) + a_1''C(p) = a_1''p_1^2 + a_2''p_2^2 + p_3^2$. Furthermore $\psi'' = \operatorname{diag}(\frac{1}{\sqrt{a_2''-a_1''}}, \frac{1}{\sqrt{a_2''-a_1''}}, 1)$ is a linear Poisson automorphism such that $(H' \circ \psi'')(p) = p_2^2 + p_3^2$. Thus H is equivalent to H_3.

Similarly, if $a_3 = 0$, then H is equivalent to H_1 (or the trivial system $H_0(p) = 0$). Straightforward computations show that none of the class representatives are equivalent. □

2.1.8 Type $\mathfrak{g}_{3.5}^{\alpha}$

The nonzero commutators of $\mathfrak{g}_{3.5}^{\alpha}$ are $[E_2, E_3] = E_1 - \alpha\,E_2$ and $[E_3, E_1] = \alpha\,E_1 + E_2$; $(\mathfrak{g}_{3.5}^{\alpha})_-^*$ does not admit any polynomial Casimir functions. We again find it convenient to classify the entire collection $\{((\mathfrak{g}_{3.5}^{\alpha})_-^*, H) \,:\, \alpha > 0\}$ of systems, rather than classifying for each fixed value for α; we shall use class representatives not strictly belonging to this collection.

Proposition 2.8. *Any system* $((\mathfrak{g}_{3.5}^{\alpha})_-^*, H)$, $\alpha > 0$ *is equivalent to exactly one of the systems*

$$((\mathfrak{g}_{3.5}^{\alpha})_-^*, p_3^2) \qquad ((\mathfrak{g}_{3.5}^{\alpha})_-^*, \beta\,p_1^2 + p_2^2 + p_3^2) \qquad ((\mathfrak{g}_{3.5}^{0})_-^*, p_2^2)$$
$$((\mathfrak{g}_{3.3})_-^*, p_1^2 + p_2^2) \qquad ((\mathfrak{g}_{3.3})_-^*, p_1^2).$$

Here $0 \leq \beta \leq 1$, $\alpha > 0$ *parametrize families of non-equivalent systems.*

Proof. Consider the system $((\mathfrak{g}_{3.5}^{\alpha})_-^*, H_Q)$, $\alpha > 0$. By arguments similar to those in the proof of Proposition 2.7, we have that $((\mathfrak{g}_{3.5}^{\alpha})_-^*, H_Q)$ is equivalent to one of the systems

$$H_1(p) = p_3^2 \qquad H_{2,\beta}(p) = \beta\,p_1^2 + p_2^2 \qquad H_{3,\beta}(p) = \beta\,p_1^2 + p_2^2 + p_3^2$$

on $(\mathfrak{g}_{3.5}^{\alpha})_-^*$. Here $0 \leq \beta \leq 1$ parametrizes families of representatives.

We consider the (two-parameter) family $((\mathfrak{g}_{3.5}^{\alpha})_-^*, H_{2,\beta})$, $\alpha > 0$, $0 \leq \beta \leq 1$. It turns out that each member of this family is equivalent to one of three systems. Let $\kappa_\alpha^\pm = 1 + 2\alpha^2 \pm 2\alpha\sqrt{\alpha^2+1}$; note that $0 < \kappa_\alpha^- < 1 < \kappa_\alpha^+$. The family is partitioned into its equivalence classes by the conditions $0 \leq \beta < \kappa_\alpha^-$, $\beta = \kappa_\alpha^-$, and $\kappa_\alpha^- < \beta \leq 1$. Let $\vec{H}$ be the Hamiltonian vector field associated to $((\mathfrak{g}_{3.5}^{\alpha})_-^*, H_{2,\beta})$. Suppose $0 \leq \beta < \kappa_\alpha^-$. Then $\psi : (\mathfrak{g}_{3.5}^{\alpha})_-^* \to (\mathfrak{g}_{3.5}^{0})_-^*$,

$$\psi = \begin{bmatrix} 2\alpha\beta & \beta - 1 - \sqrt{(\kappa_\alpha^- - \beta)(\kappa_\alpha^+ - \beta)} & 0 \\ -\frac{1}{2} & -\frac{\alpha}{\beta - 1 - \sqrt{(\kappa_\alpha^- - \beta)(\kappa_\alpha^+ - \beta)}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is a linear isomorphism (with $\det\psi = -\sqrt{(\kappa_\alpha^- - \beta)(\kappa_\alpha^+ - \beta)}$) such that

$$\psi \cdot \vec{H} \circ \psi^{-1} = \begin{bmatrix} 0 \\ 0 \\ 2p_1p_2 \end{bmatrix}.$$

Thus $((\mathfrak{g}_{3.5}^{\alpha})_-^*, H_{2,\beta})$ is equivalent to $((\mathfrak{g}_{3.5}^{0})_-^*, p_2^2)$. Next, suppose $\beta = \kappa_\alpha^-$. Then $\psi : (\mathfrak{g}_{3.5}^{\alpha})_-^* \to (\mathfrak{g}_{3.3})_-^*$,

$$\psi = \begin{bmatrix} 1 & -\alpha - \sqrt{1+\alpha^2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{\alpha} + 2\alpha + 2\sqrt{1+\alpha^2} \end{bmatrix}$$

is a linear isomorphism (with $\det\psi = \frac{1}{\alpha} + 2\alpha + 2\sqrt{1+\alpha^2}$) such that

$$\psi \cdot \vec{H} \circ \psi^{-1} = \begin{bmatrix} 0 \\ 0 \\ -2p_1^2 \end{bmatrix}.$$

Hence $((\mathfrak{g}_{3.5}^{\alpha})_-^*, \kappa_\alpha^- p_1^2 + p_2^2)$ is equivalent to $((\mathfrak{g}_{3.3})_-^*, p_1^2)$. Lastly, suppose $\kappa_\alpha^- < \beta \leq 1$. Then $\psi : (\mathfrak{g}_{3.5}^{\alpha})_-^* \to (\mathfrak{g}_{3.3})_-^*$,

$$\psi = \begin{bmatrix} 0 & -\sqrt{(\beta - \kappa_\alpha^-)(\kappa_\alpha^+ - \beta)} & 0 \\ 2\alpha\beta & \beta - 1 & 0 \\ 0 & 0 & 4\alpha\beta \end{bmatrix}$$

is a linear isomorphism (with $\det\psi = 8\alpha^2\beta^2\sqrt{(\beta - \kappa_\alpha^-)(\kappa_\alpha^+ - \beta)}$) such that

$$\psi \cdot \vec{H} \circ \psi^{-1} = \begin{bmatrix} 0 \\ 0 \\ -2\left(p_1^2 + p_2^2\right) \end{bmatrix}.$$

Therefore $((\mathfrak{g}_{3.5}^{\alpha})_-^*, H_{2,\beta})$ is equivalent to $((\mathfrak{g}_{3.3})_-^*, p_1^2 + p_2^2)$.

Finally, straightforward but tedious computations again show that none of the class representatives are equivalent. □

2.2 Systems on semisimple Lie algebras

For each of the two simple Lie algebras $\mathfrak{g}_{3.6}$ and $\mathfrak{g}_{3.7}$, we classify the positive semidefinite quadratic Hamilton-Poisson systems on the associated Lie-Poisson space. The procedure is essentially the same as in the solvable case. However, for $\mathfrak{g}_{3.6}$ the argument becomes more involved. More precisely, simple composition with linear Poisson automorphisms is not feasible; we use a Cholesky factorization of the quadratic form and perform the normalization on the Cholesky factor.

2.2.1 Type $\mathfrak{g}_{3.6}$

The pseudo-orthogonal Lie algebra $\mathfrak{g}_{3.6}$ has nonzero commutators $[E_2, E_3] = E_1$, $[E_3, E_1] = E_2$, $[E_1, E_2] = -E_3$; $C(p) = p_1^2 + p_2^2 - p_3^2$ is a Casimir function. The group $\mathsf{SO}(2,1)$ is generated by

$$\rho_1(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh t & \sinh t \\ 0 & \sinh t & \cosh t \end{bmatrix} \qquad \rho_2(t) = \begin{bmatrix} \cosh t & 0 & \sinh t \\ 0 & 1 & 0 \\ \sinh t & 0 & \cosh t \end{bmatrix}$$

$$\rho_3(t) = \begin{bmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \eta(t) = \begin{bmatrix} 1 - \frac{1}{2}t^2 & t & \frac{1}{2}t^2 \\ -t & 1 & t \\ -\frac{1}{2}t^2 & t & 1 + \frac{1}{2}t^2 \end{bmatrix}$$

$$\varsigma = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

The following lemma proves helpful in classifying systems on $\mathfrak{so}(2,1)^*_-$.

Lemma 2.9 (cf. [40]). *For any matrix*

$$R = \begin{bmatrix} r_1 & 0 & 0 \\ 0 & r_2 & 0 \\ r_3 & r_4 & 0 \end{bmatrix}, \qquad r_i \in \mathbb{R},\ r_1, r_2 > 0$$

there exists $\psi \in \mathsf{SO}(2,1)$ *and* $s > 0$ *such that* $s\,\psi\,R$ *equals*

$$\begin{bmatrix} x & 0 & 0 \\ y & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ x & 1 & 0 \\ y & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ k & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ y & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ x & 1 & 0 \\ y & 1 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} k & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Here $x, y, \in \mathbb{R}$, *and* $k \in \{-1, 1\}$.

Proof. The group $\mathsf{SO}(2,1)$ acts transitively on each level set $\mathcal{H}_\delta = \{p \in \mathfrak{so}(2,1)^* \backslash \{0\} : C(p) = \delta\}$. ($\mathcal{H}_\delta$ is a hyperboloid of two sheets when $\delta < 0$, a hyperboloid of one sheet when $\delta > 0$, and a punctured cone when $\delta = 0$.) Therefore, for any nonzero $p \in \mathfrak{so}(2,1)^*$, there exists $\psi \in \mathsf{SO}(2,1)$ such that $\psi \cdot p$ equals δE_2^*, δE_3^* or $E_2 + E_3$ with $\delta > 0$.

Assume $r_2^2 - r_4^2 > 0$. There exists $\rho_1(t)$ and $s > 0$ such that

$$s\rho_1(t)\,R = \begin{bmatrix} a_1 & 0 & 0 \\ a_2 & 1 & 0 \\ a_3 & 0 & 0 \end{bmatrix}$$

for some $a_1, a_2, a_3 \in \mathbb{R}$. Suppose $a_1^2 - a_3^2 > 0$. Then there exists $\rho_2(t)$ such that

$$\rho_2(t) \begin{bmatrix} a_1 & 0 & 0 \\ a_2 & 1 & 0 \\ a_3 & 0 & 0 \end{bmatrix} = \begin{bmatrix} x & 0 & 0 \\ y & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Suppose $a_1^2 - a_3^2 < 0$. Likewise, there then there exists $\rho_2(t)$ such that

$$\rho_2(t) \begin{bmatrix} a_1 & 0 & 0 \\ a_2 & 1 & 0 \\ a_3 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ x & 1 & 0 \\ y & 0 & 0 \end{bmatrix}.$$

Suppose $a_1^2 = a_3^2$. (Note the situation $a_1 = a_3 = 0$ is impossible.) There exists $\psi = \rho_2(t)$, $\psi = \varsigma\, \rho_2(t)$, $\psi = \rho_3(\pi)\, \rho_2(t)\, \rho_3(\pi)$, or $\psi = \varsigma\, \rho_3(\pi)\, \rho_2(t)\, \rho_3(\pi)$ such that

$$\psi \begin{bmatrix} a_1 & 0 & 0 \\ a_2 & 1 & 0 \\ a_3 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ k & 0 & 0 \end{bmatrix}.$$

Assume $r_2^2 - r_4^2 < 0$. There exists $\rho_1(t)$ and $s > 0$ such that

$$s\rho_1(t)\, R = \begin{bmatrix} a_1 & 0 & 0 \\ a_2 & 0 & 0 \\ a_3 & 1 & 0 \end{bmatrix}$$

for some $a_1, a_2, a_3 \in \mathbb{R}$. Accordingly, there exists $\rho_3(t)$ such that

$$\rho_3(t) \begin{bmatrix} a_1 & 0 & 0 \\ a_2 & 0 & 0 \\ a_3 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ y & 1 & 0 \end{bmatrix}$$

Assume $r_2^2 = r_4^2$. Then there exists an automorphism $\psi = \mathbf{1}$ or $\psi = \varsigma$ such that

$$\tfrac{1}{r_2}\psi\, R = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & 1 & 0 \\ a_3 & 1 & 0 \end{bmatrix}$$

for some $a_1, a_3 \in \mathbb{R}$. Suppose $a_3 \neq 0$. Then

$$\rho_3(\tfrac{\pi}{2})\, \eta(\tfrac{a_1}{a_3})\, \rho_3(-\tfrac{\pi}{2}) \begin{bmatrix} a_1 & 0 & 0 \\ 0 & 1 & 0 \\ a_3 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ x & 1 & 0 \\ y & 1 & 0 \end{bmatrix}.$$

Suppose $a_3 = 0$. Then there exists $\rho_1(t)$ such that

$$\tfrac{1}{|a_1|}\, \rho_1(t) \begin{bmatrix} a_1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} k & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

□

Proposition 2.10. *On $(\mathfrak{g}_{3.6})^*_-$, any system is equivalent to exactly one of the systems*

$$H_1(p) = p_1^2 \qquad H_2(p) = p_3^2 \qquad H_3(p) = p_1^2 + p_3^2$$
$$H_4(p) = (p_2 + p_3)^2 \qquad H_5(p) = p_2^2 + (p_1 + p_3)^2.$$

Proof. Let $H_Q(p) = p^\top Q\, p$. Symmetric matrices are diagonalizable by orthogonal matrices; hence there exists $\theta \in \mathbb{R}$ such that

$$Q' = \rho_3(\theta)^\top Q\, \rho_3(\theta) = \begin{bmatrix} a_1 & 0 & b_2 \\ 0 & a_2 & b_3 \\ b_2 & b_3 & a_3 \end{bmatrix}$$

for some $a_i, b_i \in \mathbb{R}$.

If $a_1 = 0$ or $a_2 = 0$, then

$$Q' = \begin{bmatrix} 0 & 0 & 0 \\ 0 & a_2 & b_3 \\ 0 & b_3 & a_3 \end{bmatrix} \quad \text{or} \quad \rho_3(\tfrac{\pi}{2})^\top Q'\, \rho_3(\tfrac{\pi}{2}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & a_1 & b_2 \\ 0 & b_2 & a_3 \end{bmatrix}$$

respectively. We deal with this case below.

Assume $a_1, a_2 \neq 0$. Let K be the matrix of the underlying quadratic form of the Casimir function $C(p) = p_1^2 + p_2^2 - p_3^2$. We claim that $Q' + xK$ has a Cholesky decomposition $Q' + xK = R^\top R$, where

$$R = \begin{bmatrix} r_1 & 0 & r_3 \\ 0 & r_2 & r_4 \\ 0 & 0 & r_5 \end{bmatrix}$$

with $r_1, r_2 \neq 0$ and $r_5 = 0$ for some $x \geq 0$. Indeed, $r_1 = \sqrt{x + a_1}$, $r_2 = -\sqrt{x + a_2}$, $r_3 = \frac{b_2}{\sqrt{x + a_1}}$, $r_4 = -\frac{b_3}{\sqrt{x+a_2}}$, and $r_5 = \frac{\sqrt{\det(Q'+xK)}}{\sqrt{(x+a_1)(x+a_2)}}$ is a solution to $Q' + xK = R^\top R$. (If $x \geq 0$, then $x + a_1 > 0$ and $x + a_2 > 0$.) Now $\det(Q' + 0K) \geq 0$ and $\det(Q' + xK)$ is a cubic in x with leading term $-x^3$. Thus there exists $x \geq 0$ such that $\det(Q' + xK) = 0$.

Note that $\psi^\top (Q' + xK)\psi = (R\psi)^\top R\psi = \psi^\top R^\top (\psi^\top R^\top)^\top$. (Also, $\psi \in \mathsf{SO}(2,1)$ if and only if $\psi^\top \in \mathsf{SO}(2,1)$.) Hence, by the preceding lemma, H_Q is equivalent to a system $H_{\mathcal{R}_i}(p) = p^\top R_i\, p$ with quadratic form

$$R_1 = \begin{bmatrix} a_1 & b_1 & 0 \\ b_1 & a_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad R_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & a_2 & b_3 \\ 0 & b_3 & a_3 \end{bmatrix}$$

$$R_3 = \begin{bmatrix} 1 & x & k \\ x & 1 + x^2 & kx \\ k & kx & 1 \end{bmatrix} \qquad R_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

(Here $k = \pm 1$, $x, a_1, a_2, a_3, b_1, b_3 \in \mathbb{R}$, and each matrix R_i is PSD.)

Consider the case $H_{\mathcal{R}_1}$. There exists $\theta \in \mathbb{R}$ such that $\rho_2(\theta)^\top Q_1\, \rho_2(\theta) = \operatorname{diag}(a_1', a_2', 0)$ for some $a_1', a_2' \geq 0$. If $a_1' = 0$ or $a_2' = 0$, then H_{Q_1} is equivalent to H_1. (For the case $a_1 = 0$, $a_2' \neq 0$, apply an automorphism $\rho_3(\frac{\pi}{2})$.) Suppose $a_1', a_2' > 0$. Then there exists an automorphism ψ ($\psi = \mathbf{1}$ or $\psi = \rho_3(\frac{\pi}{2})$) and constant $r = \max\{a_1', a_2'\} > 0$ such that $\frac{1}{r}\psi^\top \operatorname{diag}(a_1', a_2', 0)\, \psi = \operatorname{diag}(1, \alpha, 0)$ with $0 < \alpha \leq 1$. Hence H_Q is equivalent to the (intermediate) system $H'(p) = p_1^2 + \alpha\, p_2^2$. Suppose $\alpha = 1$. Then $\psi = \operatorname{diag}(1, -1, -1)$ is an automorphism such that $\psi \cdot \vec{H}' = H_2 \circ \psi$, i.e., H_Q is equivalent to H_2. Suppose $0 < \alpha < 1$. Then

$\psi = \operatorname{diag}(\frac{1}{\sqrt{2}}\sqrt{1-\alpha},\ \sqrt{(1-\alpha)\alpha},\ \frac{1}{\sqrt{2}}\sqrt{\alpha})$ is a linear isomorphism such that $\psi \cdot \vec{H}' = H_3 \circ \psi$. Thus $H_{\mathcal{Q}}$ is equivalent to H_3.

Next, consider the case $H_{\mathcal{R}_2}$. If $a_2 = 0$ or $a_3 = 0$, then $H_{\mathcal{R}_2}$ is equivalent to H_1 or H_2 (or the trivial system $H_0(p) = 0$). Suppose $a_2, a_3 > 0$. If $b_3 = 0$, then $H_{\mathcal{R}_2}$ is equivalent to the (intermediate system) $H'(p) = p_2^2 + \alpha p_3^2$, $\alpha > 0$. As R_2 is PSD, we have $(a_2+a_3)^2 \geq 4b_3^2$. Suppose $b_3 \neq 0$ and $(a_2+a_3)^2 > 4b_3^2$. Then

$$\psi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}}\sqrt{1 + \frac{a_2+a_3}{\sqrt{(a_2+a_3)^2-4b_3^2}}} & -\frac{\operatorname{sgn}(b_3)\sqrt{a_2+a_3-\sqrt{(a_2+a_3)^2-4b_3^2}}}{\sqrt{2}\left((a_2+a_3)^2-4b_3^2\right)^{1/4}} \\ 0 & -\frac{\operatorname{sgn}(b_3)\sqrt{a_2+a_3-\sqrt{(a_2+a_3)^2-4b_3^2}}}{\sqrt{2}\left((a_2+a_3)^2-4b_3^2\right)^{1/4}} & \frac{1}{\sqrt{2}}\sqrt{1 + \frac{a_2+a_3}{\sqrt{(a_2+a_3)^2-4b_3^2}}} \end{bmatrix}$$

is an automorphism such that

$$\psi^\top R_2\, \psi = \operatorname{diag}\left(0,\ \tfrac{1}{2}\Big(a_2 - a_3 + \sqrt{(a_2+a_3)^2 - 4b_3^2}\Big),\ \tfrac{1}{2}\Big(-a_2 + a_3 + \sqrt{(a_2+a_3)^2 - 4b_3^2}\Big)\right).$$

Thus $s\,\psi^\top R_2\, \psi = \operatorname{diag}(0, 1, \alpha)$ for some $s, \alpha > 0$, i.e., $H_{\mathcal{R}_2}$ is equivalent to the (intermediate system) $H'(p) = p_2^2 + \alpha p_3^2$. We have that

$$\psi = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}}\sqrt{1+\alpha} & 0 \\ 0 & 0 & \sqrt{\alpha(1+\alpha)} \\ -\frac{1}{\sqrt{2}}\sqrt{\alpha} & 0 & 0 \end{bmatrix}$$

is a linear isomorphism such that $\psi \cdot \vec{H}' = H_3 \circ \psi$. Now suppose $b_3 \neq 0$ and $(a_2+a_3)^2 = 4b_3^2$. It follows that $a_2 a_3 = b_3^2$. Therefore

$$R_2' = \tfrac{1}{a_2} R_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & \alpha \\ 0 & \alpha & \alpha^2 \end{bmatrix}$$

for some $\alpha \neq 0$. If $\alpha^2 > 1$ (resp. $\alpha^2 < 1$), then

$$\psi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{|\alpha|}{\sqrt{\alpha^2-1}} & -\frac{\operatorname{sgn}(\alpha)}{\sqrt{\alpha^2-1}} \\ 0 & -\frac{\operatorname{sgn}(\alpha)}{\sqrt{\alpha^2-1}} & \frac{|\alpha|}{\sqrt{\alpha^2-1}} \end{bmatrix} \qquad \left(\text{resp.}\quad \psi = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{1}{\sqrt{1-\alpha^2}} & 0 & -\frac{\alpha}{\sqrt{1-\alpha^2}} \\ \frac{\alpha}{\sqrt{1-\alpha^2}} & 0 & \frac{1}{\sqrt{1-\alpha^2}} \end{bmatrix}\right)$$

is an automorphism such that $\frac{1}{|1-\alpha^2|} H_{\mathcal{R}_2'} \circ \psi$ equals H_2 (resp. H_1). If $\alpha = \pm 1$, then $\psi = \mathbf{1}$ or $\psi = \varsigma$ is an automorphism such that $H_{\mathcal{R}_2'} \circ \psi = H_4$.

Now consider the case $H_{\mathcal{R}_3}$. We have that

$$\psi = \begin{bmatrix} 1 - x^2 & 2x & k\left(3 + x^2\right) \\ kx & k\left(1 + x^2\right) & x \\ x^2 & -2x & -k\left(2 + x^2\right) \end{bmatrix}$$

is a linear isomorphism (with $\det\psi = -2\left(1+x^2\right)^2$) such that $\psi \cdot \vec{H}_{\mathcal{R}_3} = \vec{H}_5 \circ \psi$.

For the last case $H_{\mathcal{R}_4}$, we have $H_{\mathcal{R}_4} \circ \rho_3(\frac{\pi}{2}) = H_5$. It is fairly straightforward to verify that no two representatives are equivalent. □

Remark 2.11. Let $H_\mathcal{Q}$ be a system on $(\mathfrak{g}_{3.6})^*_-$ with $\mathcal{Q}$ positive definite. There exists $\psi \in \mathsf{SO}(2,1)$ such that $\psi^\top Q \psi$ is diagonal ([118]). Accordingly, $H_\mathcal{Q}$ is equivalent to H_1, H_2, or H_3.

2.2.2 Type $\mathfrak{g}_{3.7}$

The pseudo-orthogonal Lie algebra $\mathfrak{g}_{3.7}$ has nonzero commutators $[E_2, E_3] = E_1$, $[E_3, E_1] = E_2$, $[E_1, E_2] = E_3$; $C(p) = p_1^2 + p_2^2 + p_3^2$ is a Casimir function.

Proposition 2.12 ([8,38]). *On $(\mathfrak{g}_{3.7})^*_-$, any system is equivalent to exactly one of the systems*

$$H_1(p) = p_1^2 \qquad\qquad H_2(p) = p_1^2 + \tfrac{1}{2}p_2^2.$$

Proof. Let $H_\mathcal{Q}(p) = p^\top Q p$. We may assume that Q is positive definite. Indeed, if Q is not positive definite, then H is equivalent to a system $H + \lambda C$ for which the quadratic form is positive definite (for some sufficiently large λ). There exists $\psi \in \mathsf{SO}(3)$ such that $\psi^\top Q \psi = \operatorname{diag}(a_1, a_2, a_3)$, where $a_1 \geq a_2 \geq a_3 > 0$. Hence $(H_\mathcal{Q} \circ \psi)(p) - a_3 C(p) = p \operatorname{diag}(a_1 - a_3, a_2 - a_3, 0)\, p^\top$ with $a_1 - a_3 \geq a_2 - a_3 \geq 0$. If $a_1 - a_3 = 0$, then $H_\mathcal{Q}$ is equivalent to the trivial system. If $a_1 - a_3 > 0$, then $(H_\mathcal{Q} \circ \psi)(p) - a_3 C(p) = (a_1 - a_3)\, p \operatorname{diag}(1, \frac{a_2 - a_3}{a_1 - a_3}, 0)\, p^\top$ and so $H_\mathcal{Q}$ is equivalent to (the intermediate system)

$$H_{3,\alpha}(p) = p_1^2 + \alpha\, p_2^2, \qquad \alpha = \tfrac{a_2 - a_3}{a_1 - a_3},\ 0 \leq \alpha \leq 1.$$

If $\alpha = 0$, then $H_{3,\alpha} = H_1$. If $\alpha = 1$, then $\vec{H}_{3,\alpha}$ and $\vec{H}_1$ are compatible with the linear isomorphism

$$\psi = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

If $0 < \alpha < 1$, then $\vec{H}_{3,\alpha}$ and $\vec{H}_2$ are compatible with the linear isomorphism

$$\psi = \begin{bmatrix} -\sqrt{2}\sqrt{1-\alpha} & 0 & 0 \\ 0 & 2\sqrt{\alpha(1-\alpha)} & 0 \\ 0 & 0 & -\sqrt{2}\sqrt{\alpha} \end{bmatrix}.$$

It is easy to show that H_1 and H_2 are not equivalent. □

Remark 2.13. The above result holds true in the context of all homogeneous quadratic Hamilton-Poisson systems on $(\mathfrak{g}_{3.7})^*_-$. This is due to the fact that for any quadratic form $\mathcal{Q}$ on $\mathfrak{g}^*_{3.7}$, there exists a scalar multiple of the Casimir function C such that the sum $\mathcal{Q} + \lambda C$ is positive definite.

2.3 General classification and remarks

We now proceed to classify systems in the context of all three-dimensional Lie-Poisson spaces. First we determine if any systems on different Lie-Poisson systems are equivalent; a summary appears in Tables 2.1 and 2.2. The main classification result then follows.

Table 2.1: Equivalence of ruled systems (systems in the same column are equivalent)

	R(3)	R(4)	R(5)
$\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1$		p_1^2	$(p_1+p_3)^2$
$\mathfrak{g}_{3.1}$			p_3^2
$\mathfrak{g}_{3.2}$	$p_1^2+p_2^2$	p_1^2	p_2^2
$\mathfrak{g}_{3.3}$	$p_1^2+p_2^2$	p_1^2	
$\mathfrak{g}_{3.4}^0$		$(p_1+p_2)^2$	p_1^2
$\mathfrak{g}_{3.5}^0$			p_2^2

Table 2.2: Equivalence of non-ruled systems (systems in the same column are equivalent)

	P(6)	P(8)	Np(3)	Np(7)
$\mathfrak{g}_{3.1}$		$p_2^2+p_3^2$		
$\mathfrak{g}_{3.4}^0$	p_3^2		$(p_1+p_2)^2+p_3^2$	$p_1^2+p_3^2$
$\mathfrak{g}_{3.5}^0$		p_3^2		$p_2^2+p_3^2$
$\mathfrak{g}_{3.6}$	p_1^2	p_3^2	$p_2^2+(p_1+p_3)^2$	$p_1^2+p_3^2$
$\mathfrak{g}_{3.7}$		p_1^2		$p_1^2+\frac{1}{2}p_2^2$

Proposition 2.14. *In each of the following cases, any two systems are equivalent:*

1. $((\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1)_-^*, (p_1+p_3)^2)$, $((\mathfrak{g}_{3.1})_-^*, p_3^2)$, $((\mathfrak{g}_{3.2})_-^*, p_2^2)$, $(\mathfrak{g}_{3.4}^0)_-^*, p_1^2)$, $((\mathfrak{g}_{3.5}^0)_-^*, p_2^2)$.
2. $((\mathfrak{g}_{3.1})_-^*, p_2^2+p_3^2)$, $((\mathfrak{g}_{3.5}^0)_-^*, p_3^2)$, $((\mathfrak{g}_{3.6})_-^*, p_3^2)$, $((\mathfrak{g}_{3.7})_-^*, p_1^2)$.
3. $((\mathfrak{g}_{3.4}^0)_-^*, p_1^2+p_3^2)$, $((\mathfrak{g}_{3.5}^0)_-^*, p_2^2+p_3^2)$, $((\mathfrak{g}_{3.6})_-^*, p_1^2+p_3^2)$, $((\mathfrak{g}_{3.7})_-^*, p_1^2+\frac{1}{2}p_2^2)$.
4. $((\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1)_-^*, p_1^2)$, $((\mathfrak{g}_{3.2})_-^*, p_1^2)$, $((\mathfrak{g}_{3.3})_-^*, p_1^2)$, $((\mathfrak{g}_{3.4}^0)_-^*, (p_1+p_2)^2)$.
5. $((\mathfrak{g}_{3.4}^0)_-^*, p_3^2)$, $((\mathfrak{g}_{3.6})_-^*, p_1^2)$.
6. $((\mathfrak{g}_{3.4}^0)_-^*, (p_1+p_2)^2+p_3^2)$, $((\mathfrak{g}_{3.6})_-^*, p_2^2+(p_1+p_3)^2)$.
7. $((\mathfrak{g}_{3.2})_-^*, p_1^2+p_2^2)$, $((\mathfrak{g}_{3.3})_-^*, p_1^2+p_2^2)$.

Proof. We prove only item 1; the other items follow very similarly. We claim that each of the systems is equivalent to $((\mathfrak{g}_{3.1})_-^*, p_3^2)$. Indeed,

$$\psi_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad \psi_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}, \quad \psi_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad \psi_4 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$$

are linear isomorphisms with codomain $(\mathfrak{g}_{3.1})_-^*$ such that $\psi_i \cdot \vec{H}_i \circ \psi_i^{-1} = \vec{H}$. Here $\vec{H}$ is the vector field associated with $((\mathfrak{g}_{3.1})_-^*, p_3^2)$; $\vec{H}_1$, $\vec{H}_2$, $\vec{H}_3$, and $\vec{H}_4$ are the vector fields associated with $((\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1)_-^*, (p_1+p_3)^2)$, $((\mathfrak{g}_{3.2})_-^*, p_2^2)$, $(\mathfrak{g}_{3.4}^0)_-^*, p_1^2)$, and $((\mathfrak{g}_{3.5}^0)_-^*, p_2^2)$, respectively. □

We say that a system $(\mathfrak{g}^*_-, H)$ is *ruled*, if for each integral curve of $\vec{H}$ there exists a line containing its trace. Likewise, $(\mathfrak{g}^*_-, H)$ is called *planar* if it is not ruled and for each integral curve of $\vec{H}$ there exists a plane containing its trace. Otherwise, $(\mathfrak{g}^*_-, H)$ is called *non-planar*. The ruled, planar, and non-planar properties are each invariant under equivalence, i.e., if two systems are equivalent, then they must belong to the same class.

Theorem 2.15. *Suppose* $(\mathfrak{g}^*_-, H)$, $H(p) = \mathcal{Q}(p)$ *is a Hamilton-Poisson system on a three-dimensional Lie-Poisson space (and suppose* $\mathcal{Q}$ *is positive semidefinite).*

1. *If* $(\mathfrak{g}^*_-, H)$ *is ruled, then it is equivalent to exactly one of the systems*

$$\mathsf{R}(1) : ((\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1)^*_-, p_2^2)$$
$$\mathsf{R}(2) : ((\mathfrak{g}_{3.3})^*_-, p_3^2) \qquad \mathsf{R}(3) : ((\mathfrak{g}_{3.3})^*_-, p_1^2 + p_2^2)$$
$$\mathsf{R}(4) : ((\mathfrak{g}^0_{3.4})^*_-, (p_1 + p_2)^2) \qquad \mathsf{R}(5) : ((\mathfrak{g}^0_{3.5})^*_-, p_2^2).$$

2. *If* $(\mathfrak{g}^*_-, H)$ *is planar, then it is equivalent to exactly one of the systems*

$$\mathsf{P}(1) : ((\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1)^*_-, p_1^2 + p_2^2) \qquad \mathsf{P}(2) : ((\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1)^*_-, p_2^2 + (p_1 + p_3)^2)$$
$$\mathsf{P}(3) : ((\mathfrak{g}_{3.2})^*_-, p_3^2)$$
$$\mathsf{P}(4) : ((\mathfrak{g}_{3.3})^*_-, p_1^2 + p_3^2) \qquad \mathsf{P}(5) : ((\mathfrak{g}_{3.3})^*_-, p_1^2 + p_2^2 + p_3^2)$$
$$\mathsf{P}(6) : ((\mathfrak{g}^0_{3.4})^*_-, p_3^2) \qquad \mathsf{P}(7) : ((\mathfrak{g}^\alpha_{3.4})^*_-, p_3^2)$$
$$\mathsf{P}(8) : ((\mathfrak{g}^0_{3.5})^*_-, p_3^2) \qquad \mathsf{P}(9) : ((\mathfrak{g}^\alpha_{3.5})^*_-, p_3^2)$$
$$\mathsf{P}(10) : ((\mathfrak{g}_{3.6})^*_-, (p_2 + p_3)^2).$$

3. *If* $(\mathfrak{g}^*_-, H)$ *is non-planar, then it is equivalent to exactly one of the systems*

$$\mathsf{Np}(1) : ((\mathfrak{g}_{3.2})^*_-, p_1^2 + p_3^2) \qquad \mathsf{Np}(2) : ((\mathfrak{g}_{3.2})^*_-, \delta p_1^2 + p_2^2 + p_3^2)$$
$$\mathsf{Np}(3) : ((\mathfrak{g}^0_{3.4})^*_-, (p_1 + p_2)^2 + p_3^2) \qquad \mathsf{Np}(4) : ((\mathfrak{g}^\alpha_{3.4})^*_-, (p_1 + p_2)^2 + p_3^2)$$
$$\mathsf{Np}(5) : ((\mathfrak{g}^\alpha_{3.4})^*_-, (p_1 - p_2)^2 + p_3^2) \qquad \mathsf{Np}(6) : ((\mathfrak{g}^\alpha_{3.4})^*_-, \beta p_1^2 + p_2^2 + p_3^2)$$
$$\mathsf{Np}(7) : ((\mathfrak{g}^0_{3.5})^*_-, p_2^2 + p_3^2) \qquad \mathsf{Np}(8) : ((\mathfrak{g}^\alpha_{3.5})^*_-, \beta p_1^2 + p_2^2 + p_3^2).$$

Here $\alpha > 0$, $0 \leq \beta \leq 1$, *and* $\delta \geq 0$ *parametrize families of class representatives, each different value corresponding to a distinct non-equivalent representative. (For* $(\mathfrak{g}^\alpha_{3.4})^*_-$, *we have* $\alpha \neq 1$.*)*

The associated Hamiltonian vector fields (and their equilibrium states) are tabulated in appendix B.1 (Tables B.1, B.2, B.3, and B.4).

Proof. By the preceding propositions, we have that the system $(\mathfrak{g}^*_-, H)$ is indeed equivalent to one of the given normal forms. Some computationally taxing calculations show that no two normal forms are equivalent. Indeed, for the majority of pairs this has already been established in the foregoing propositions. (Some further invariants distinguishing between the normal forms will be discussed below.)

It is not difficult to establish the class of the system (ruled, planar or non-planar) by investigating the Hamiltonian and Casimir function, as any integral curve must evolve on the intersection of some level sets of the Hamiltonian and Casimir. Moreover, in many cases simple inspection of the equations of motion shows that a system is ruled or planar (as the evolution along certain coordinates is constant). Nonetheless, we can determine the class of a system $(\mathfrak{g}^*_-, H)$ as follows. In coordinates $p = (p_1, p_2, p_3)$ we can interpret an integral curve of $\vec{H}$ as a space curve and hence calculate its curvature and torsion. We have that the system is ruled if and only if every integral curve has zero curvature; the system is planar if and only if there exists integral curves with nonzero curvature and every integral curve with nonzero curvature has zero torsion; the non-planar system are those for which there exist integral curves having nonzero curvature and nonzero torsion. Although the curvature and torsion are dependent on the choice of coordinates (i.e., choice of induced inner product), their being zero does not depend on the choice of coordinates. We give details for $\mathsf{P}(10)$ as a typical case. Suppose $p(\cdot)$ is an integral curve of the system $\mathsf{P}(10)$, i.e., $\dot{p} = (2(p_2+p_3)^2, -p_1(p_2+p_3), 2p_1(p_2+p_3))$. Then we have

$$\begin{aligned}\ddot{p} &= (4(p_2+p_3)(\dot{p}_2+\dot{p}_3), -2\dot{p}_1(p_2+p_3) - 2p_1(\dot{p}_2+\dot{p}_3), 2\dot{p}_1(p_2+p_3) + 2p_1\,(\dot{p}_2+\dot{p}_3))\\ &= (0, -4(p_2+p_3)^3, 4(p_2+p_3)^3).\end{aligned}$$

Consequently, $\kappa = \frac{\|\ddot{p}\times\dot{p}\|}{\|\dot{p}\|^3} = \frac{\sqrt{2}\sqrt{(p_2+p_3)^{10}}}{\left((p_2+p_3)^2\left(2p_1^2+(p_2+p_3)^2\right)\right)^{3/2}} \neq 0$ and so $\mathsf{P}(10)$ is not ruled. However, $\dddot{p}(t) = (0,0,0)$ and so $p(\cdot)$ has zero torsion; therefore $\mathsf{P}(10)$ is planar. □

Remark 2.16. In [120] it was shown that a number of quadratic Hamilton-Poisson systems are equivalent to the free rigid body dynamics

$$\begin{cases}\dot{p}_1 = (\lambda_3 - \lambda_2)p_2p_3\\ \dot{p}_1 = (\lambda_1 - \lambda_3)p_2p_3\\ \dot{p}_1 = (\lambda_2 - \lambda_1)p_2p_3\end{cases} \qquad \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}.$$

The above system may be realised as $((\mathfrak{g}_{3.7})^*_-, \lambda_1 p_1^2+\lambda_2 p_2^2+\lambda_3 p_3^2)$; we may assume $\lambda_1, \lambda_2, \lambda_3 > 0$ by adding a multiple of the Casimir. Note however, that by the above theorem, any (non-trivial) system $((\mathfrak{g}_{3.7})^*_-, \lambda_1 p_1^2 + \lambda_2 p_2^2 + \lambda_3 p_3^2)$ is equivalent to $\mathsf{P}(8)$ or $\mathsf{Np}(7)$.

Remark 2.17. We remark on some interesting features inferred from the above theorem and preceding propositions.

- Any system on $(\mathfrak{g}_{3.1})^*_-$ or $(\mathfrak{g}_{3.7})^*_-$ is equivalent to one on $(\mathfrak{g}^0_{3.5})^*_-$.
- Any system on $(\mathfrak{g}_{3.1})^*_-$, $(\mathfrak{g}_{2.1}\oplus\mathfrak{g}_1)^*_-$ or $(\mathfrak{g}_{3.3})^*_-$ is a planar (or ruled) one. (This is follows immediately from the fact that the coadjoint orbits for each of these spaces are contained in planes.)
- Every system on $(\mathfrak{g}_{3.1})^*_-$, $(\mathfrak{g}^0_{3.4})^*_-$, $(\mathfrak{g}^0_{3.5})^*_-$, or $(\mathfrak{g}_{3.7})^*_-$ may be realized on more than one Lie-Poisson space. (For $(\mathfrak{g}_{3.6})^*_-$, the only exceptions are those systems equivalent to $\mathsf{P}(10)$.)

Remark 2.18. It will be shown in the next chapter that the Hamilton-Poisson system associated to any invariant sub-Riemannian structure on a three-dimensional Lie group is equivalent to $\mathsf{P}(2)$, $\mathsf{P}(8)$, $\mathsf{Np}(2)_{\delta=0}$, $\mathsf{Np}(6)_{\beta=0}$, $\mathsf{Np}(7)$, or $\mathsf{Np}(8)$. In particular, for structures on unimodular groups, the associated Hamilton-Poisson system is equivalent to $\mathsf{P}\,(8)$ or $\mathsf{Np}\,(7)$.

Remark 2.19. It turns out that among those spaces that admit global Casimir functions (i.e., $(\mathfrak{g}_{2.1}\oplus\mathfrak{g}_1)^*_-$, $(\mathfrak{g}_{3.1})^*_-$, $(\mathfrak{g}^0_{3.4})^*_-$, $(\mathfrak{g}^0_{3.5})^*_-$, $(\mathfrak{g}_{3.6})^*_-$, and $(\mathfrak{g}_{3.7})^*_-$) explicit expressions for the integral curves can be obtained in terms of elementary or Jacobi elliptic functions. Indeed, the integral curves of $\mathsf{R}(1)$, $\mathsf{R}(4)$, $\mathsf{R}(5)$, $\mathsf{P}(1)$, $\mathsf{P}(2)$, $\mathsf{P}(6)$, $\mathsf{P}(8)$, $\mathsf{P}(10)$ and $\mathsf{Np}(3)$ can be found in terms of elementary functions, whereas the integral curves of $\mathsf{Np}(7)$ can be found in terms of Jacobi elliptic functions (cf. [7–9]).

We note that the set of equilibrium points for each of the normal forms is the finite union of some lines and planes (see Table B.4). Accordingly, as the set of equilibria of equivalent systems are related by a linear isomorphism, the same property holds true for all homogeneous systems on three-dimensional Lie-Poisson spaces. Suppose the set of equilibrium points for a system is the union of i lines and j planes. We refer to the pair of numbers (i, j) as the *equilibrium index* of a system.

Proposition 2.20. *If two systems are equivalent, they have the same equilibrium index.*

Often the equilibrium index of a system, together with its class (i.e., ruled, planar, or non-planar), is enough to determine the normal form of a system on a given Lie-Poisson space. Accordingly, this gives us a taxonomy of systems for each Lie-Poisson space, see Appendix B.1.

Chapter 3

Sub-Riemannian structures on three-dimensional Lie groups

The classification of sub-Riemannian structures (up to local isometry) in three dimensions has been considered by a number of authors. In particular, Strichartz [112] classified the symmetric sub-Riemannian structures, Falbel and Gorodski [56] classified the homogeneous sub-Riemannian structures, and Agrachev and Barilari [10] classified the left-invariant sub-Riemannian structures.

For some classes of structures, it is known that the isometries and $\mathfrak{L}$-isometries are closely related. More precisely, for invariant nilpotent Riemannian manifolds and sub-Riemannian Carnot groups, it is known that any isometry is the composition of a left translation and a Lie group isomorphism (see Theorems 1.43 and 1.45).

In this chapter, we aim to explore the relation between isometries and $\mathfrak{L}$-isometries for invariant sub-Riemannian structures in three dimensions; the results of Agrachev and Barilari [10] are used in our comparison. We shall carry out the following program for each simply connected three-dimensional Lie group G:

1. We classify the sub-Riemannian structures on G up to $\mathfrak{L}$-isometry and rescaling (i.e., **LiSR**-isomorphism). A list of normal forms $(\mathsf{G}, \mathcal{D}_i, \mathbf{g}^i)$ is exhibited; for each solvable group it turns out there is only one structure. A similar classification is stated, but not proved, in [123, p. 52].

2. We calculate the associate group of $\mathfrak{L}$-isometries (for each normal form).

3. We determine the normal form of the associated quadratic Hamilton-Poisson system $\mathfrak{H}(\mathsf{G}, \mathcal{D}_i, \mathbf{g}^i)$ (see Corollary 1.40 and Theorem 2.15).

4. To every (invariant) sub-Riemannian structure in three-dimensions, one can associate a contact structure which is preserved (up to a change of sign) by isometries (cf. [10]).

    (a) We calculate the Reeb vector field associated to this structure and hence determine the scalar invariants (χ, κ) for the structure as given in [10]. (As a typical case, accompaning Mathematica code for these calculations is given in the case of $\widetilde{\mathsf{SE}}(2)$ in Appendix C.2.)

(b) By promoting the associated Reeb vector field to an orthogonal complement of the distribution, one obtains a Riemannian structure; the isometries of the sub-Riemannian structure are a subgroup of isometries of this Riemannian structure. By exploiting this property, we determine the subgroup of linearized isotropies of the identity, which amounts to describing the full isometry group. (As a typical case, accompaning Mathematica code for these calculations is given in the case of $\mathsf{G}_{3.2}$ in Appendix C.3.)

After carrying out this program, we find (by comparing our results to those of [10]) that two invariant structures on the same Lie group are isometric if and only if they are $\mathfrak{L}$-isometric. Moreover, it turns out that most isometries are in fact the composition of a left translation and a $\mathfrak{L}$-isometry (by virtue of Theorem 1.45, this was already known for isometries between structures on the Heisenberg group.)

The Abelian group $\mathbb{R}^3$ and the group $\mathsf{G}_{3.3}$ admit no invariant bracket generating distributions (cf. [35]); hence, these cases are ruled out from the start. For the classification of three-dimensional Lie groups and the corresponding automorphism groups, refer to Appendix A. We note that since the same program is carried out for each three-dimensional Lie group, some of the details are somewhat repetitive. A list of normal forms (up to $\mathfrak{L}$-isometry and rescaling) of invariant structures, the corresponding linearized isotropy subgroups, and the corresponding normal forms of the associated Hamilton-Poisson systems, are exhibited in Appendix B.2, Tables B.8 and B.9.

3.1 Preliminaries

3.1.1 Elements of invariant Riemannian geometry

Let $(\mathsf{G}, \mathbf{g})$ be a left-invariant Riemannian structure and let ∇ denote the associated Riemannian (or Levi-Civita) connection. For left-invariant vector fields Y, Z, and W, we have ([91])

$$\mathbf{g}(\nabla_Y Z, W) = \tfrac{1}{2}(\mathbf{g}([Y,Z],W) - \mathbf{g}([Z,W],Y) + \mathbf{g}([W,Y],Z)).$$

Accordingly, if (X_1, X_2, X_3) is a left-invariant orthonormal frame for $(\mathsf{G}, \mathbf{g})$, then

$$\nabla_Y Z = \mathbf{g}(\nabla_Y Z, X_1)X_1 + \mathbf{g}(\nabla_Y Z, X_2)X_2 + \mathbf{g}(\nabla_Y Z, X_3)X_3.$$

The curvature tensor R for $(\mathsf{G}, \mathbf{g})$ is given by $R_{YZ} = \nabla_{[Y,Z]} - \nabla_Y \nabla_Z + \nabla_Z \nabla_Y$ for vector fields Y, Z; its covariant derivative ∇R is given by

$$\nabla R(Y, Z_1, Z_2, Z_3) = \nabla_Y\, R(Z_1, Z_2, Z_3) - R(\nabla_Y Z_1, Z_2, Z_3) - R(Z_1, \nabla_Y Z_2, Z_3) - R(Z_1, Z_2, \nabla_Y Z_3)$$

for vector fields Y, Z_1, Z_2, Z_3.

Isometries are compatible with the Riemannian connection. That is, if $\phi : (\mathsf{G}, \mathbf{g}) \to (\mathsf{G}, \mathbf{g})$ is an isometry, then

$$\phi_* \nabla_Y Z = \nabla_{\phi_* Y} \phi_* Z$$

for vector fields Y and Z (see, e.g., [100]). Accordingly, $\phi_* R(Y,Z,W) = R(\phi_* Y, \phi_* Z, \phi_* W)$ and $\phi_* \nabla R(Y, Z_1, Z_2, Z_3) = \nabla R(\phi_* Y, \phi_* Z_1, \phi_* Z_2, \phi_* Z_3)$ for any isometry ϕ. In particular, if $\phi(\mathbf{1}) = \mathbf{1}$, then

$$T_{\mathbf{1}}\phi \cdot R(Y(\mathbf{1}), Z(\mathbf{1}), W(\mathbf{1})) = R(T_{\mathbf{1}}\phi \cdot Y(\mathbf{1}), T_{\mathbf{1}}\phi \cdot Z(\mathbf{1}), T_{\mathbf{1}}\phi \cdot W(\mathbf{1})) \tag{3.1}$$

and

$$\begin{aligned} T_{\mathbf{1}}\phi \cdot \nabla R(Y(\mathbf{1}), Z_1(\mathbf{1}), Z_2(\mathbf{1}), Z_3(\mathbf{1})) \\ = \nabla R(T_{\mathbf{1}}\phi \cdot Y(\mathbf{1}), T_{\mathbf{1}}\phi \cdot Z_1(\mathbf{1}), T_{\mathbf{1}}\phi \cdot Z_2(\mathbf{1}), T_{\mathbf{1}}\phi \cdot Z_3(\mathbf{1})) \end{aligned} \tag{3.2}$$

The above two conditions turn out to often fully determine the linearization of the isometries fixing identity of a invariant Riemannian structure in three dimensions. As isometries are uniquely determined by there tangent maps at a single point (see, e.g., [100]), this amounts to determining the subgroup of isometries fixing the identity.

3.1.2 Contact structure

Let (Y_1, Y_2) be an orthonormal frame for an invariant sub-Riemannian structure $(\mathsf{G}, \mathcal{D}, \mathbf{g})$. There exists a unique contact one form ω on G such that ([10])

$$\ker \omega = \mathcal{D} = \operatorname{span}(Y_1, Y_2) \qquad \text{and} \qquad d\omega(Y_1, Y_2) = 1. \tag{3.3}$$

(A one form ω on a three-dimensional group G is contact if $d\omega \wedge \omega$ is a non-vanishing volume form.) Moreover, any other choice of orthonormal frame (Y_1, Y_2) yields the same one form, up to a change of sign. We claim that pull back of ω by any isometry is $\pm\omega$.

Lemma 3.1. *If $\phi : (\mathsf{G}, \mathcal{D}, \mathbf{g}) \to (\mathsf{G}, \mathcal{D}, \mathbf{g})$ is an isometry, then $\phi^*\omega = \pm\omega$.*

Proof. Let $\phi : (\mathsf{G}, \mathcal{D}, \mathbf{g}) \to (\mathsf{G}, \mathcal{D}, \mathbf{g})$ be an isometry, let (Y_1, Y_2) be an orthonormal frame for $(\mathsf{G}, \mathcal{D}, \mathbf{g})$, and let ω be the contact one form given by (3.3) with respect to the orthonormal frame $(\phi_* Y_1, \phi_* Y_2)$ for $(\mathsf{G}, \mathcal{D}, \mathbf{g})$. Further, let $\omega' = \phi^*\omega$, i.e., $\omega'_g(X(g), Y(g)) = \omega_{\phi(g)}(T_g\phi \cdot X(g), T_g\phi \cdot Y(g))$. We have that $\omega'(Y_i) = \phi^*\omega(Y_i) = \omega(\phi_* Y_i) = 0$ and $i_{Y_i} d\omega' = i_{Y_i}\phi^* d\omega = i_{\phi_* Y_i} d\omega = 0$ for $i = 1, 2$. Hence ω' is the contact one form given by (3.3) with respect to the orthonormal frame (Y_1, Y_2). Consequently, as any choice of orthonormal frame (Y_1, Y_2) yields the same contact one form up to a change of sign, we have $\omega' = \pm\omega$. □

The Reeb vector field associated to the contact one form ω is the unique vector field Y_0 such that $\omega(Y_0) = 1$ and $i_{Y_0} d\omega = 0$. Note that the Reeb vector field is uniquely determined for a structure $(\mathsf{G}, \mathcal{D}, \mathbf{g})$, up to a change of sign (as ω is the same, up to a change in sign, for any orthonormal frame). The Reeb vector field is likewise preserved by isometries (up to a change in sign).

Lemma 3.2. *If $\phi : (\mathsf{G}, \mathcal{D}, \mathbf{g}) \to (\mathsf{G}, \mathcal{D}, \mathbf{g})$ is an isometry, then $\phi_* Y_0 = \pm Y_0$.*

Proof. By Lemma 3.1, we have $i_{\phi_* Y_0} d\omega = i_{Y_0}\phi^* d\omega = \pm i_{Y_0} d\omega = 0$ and $\omega(\phi_* Y_0) = \phi^*\omega(Y_0) = \pm\omega(Y_0) = \pm 1$. Thus either $\phi_* Y_0$ or $-\phi_* Y_0$ is identical to Y_0. □

Remark 3.3. As left translations are (orientation preserving) isometries for any invariant sub-Riemannian structure, it follows that the Reeb vector field is left invariant.

We note that rescaling the metric $\mathbf{g}$ by a constant $\lambda > 0$ rescales the associated Reeb vector field by $\frac{1}{\lambda}$.

Lemma 3.4. *If Y_0 is the Reeb vector field for $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ with respect to an orthonormal frame (Y_1, Y_2), then $\frac{1}{\lambda} Y_0$ is the Reeb vector field for $(\mathsf{G}, \mathcal{D}, \lambda\mathbf{g})$ with respect to the orthonormal frame $(\frac{1}{\sqrt{\lambda}} Y_1, \frac{1}{\sqrt{\lambda}} Y_2)$.*

Proof. Let ω be the contact one form for $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ with respect to (Y_1, Y_2). We have that $d\omega(\frac{1}{\sqrt{\lambda}} Y_1, \frac{1}{\sqrt{\lambda}} Y_2) = \frac{1}{\lambda}$ and so the contact one form for $(\mathsf{G}, \mathcal{D}, \lambda\mathbf{g})$ with respect to $(\frac{1}{\sqrt{\lambda}} Y_1, \frac{1}{\sqrt{\lambda}} Y_2)$ is $\lambda\omega$. Consequently, $(\lambda\omega)(Y_0) = \lambda$ and so $\frac{1}{\lambda} Y_0$ is the Reeb vector field for $(\mathsf{G}, \mathcal{D}, \lambda\mathbf{g})$ with respect to $(\frac{1}{\sqrt{\lambda}} Y_1, \frac{1}{\sqrt{\lambda}} Y_2)$. □

We find it convenient to calculate the contact one form and Reeb vector field in three-dimensional Cartesian space. Let $m : \mathbb{R}^3 \to \mathsf{G}$, $\boldsymbol{x} = (x_1, x_2, x_3) \mapsto m(x_1, x_2, x_3)$ be a parametrization of G. For the simply connected solvable Lie groups, m can be taken to be a diffeomorphism (see Appendix A.2). However, for our purposes a diffeomorphism between some neighbourhoods of $0 \in \mathbb{R}^3$ and $\mathbf{1} \in \mathsf{G}$ is sufficient.

The pull back of a left-invariant vector field A^L, $A^L(\mathbf{1}) = A \in \mathfrak{g}$ can be explicitly calculated by $m^* A^L(\boldsymbol{x}) = Tm^{-1} \cdot A^L(m(\boldsymbol{x})) = Tm^{-1} \cdot m(\boldsymbol{x})\, A$. We shall denote by (X_1, X_2, X_3) the pull back $m^* E_i^L$ of the (Maurer-Cartan) frame (E_1^L, E_2^L, E_3^L) of left-invariant vector fields corresponding to the given basis (E_1, E_2, E_3) for $\mathfrak{g}$. We shall denote by (ν_1, ν_2, ν_3) the (left-invariant) coframe dual to (X_1, X_2, X_3).

Note. We make use of the "Differential Forms" Mathematica package (Version 3.1, February 2007) by Frank Zizza for the exterior algebra of differential forms in n-dimensional Cartesian space (`http://library.wolfram.com/infocenter/MathSource/482/`).

3.1.3 Isometries and characteristic expansion

We define the *characteristic Riemannian expansion* $(\mathsf{G}, \tilde{\mathbf{g}})$ of a sub-Riemannian structure $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ as the structure obtained by promoting the Reeb vector field to an orthogonal complement of the distribution. That is, if (Y_1, Y_2) is an orthonormal frame for $(\mathsf{G}, \mathcal{D}, \mathbf{g})$, then its characteristic Riemannian expansion is the Riemannian structure admitting orthonormal frames $(\pm Y_0, Y_1, Y_2)$. Note that $(\mathsf{G}, \tilde{\mathbf{g}})$ does not depend on the choice of orthonormal frame (Y_1, Y_2). We show that the isometries of $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ are exactly those isometries of $(\mathsf{G}, \tilde{\mathbf{g}})$ preserving $\mathcal{D}$.

Proposition 3.5. *Let $(\mathsf{G}, \tilde{\mathbf{g}})$ be the characteristic Riemannian expansion of a sub-Riemannian structure $(\mathsf{G}, \mathcal{D}, \mathbf{g})$. A diffeomorphism $\phi : \mathsf{G} \to \mathsf{G}$ is an isometry of $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ if and only if it is an isometry of $(\mathsf{G}, \tilde{\mathbf{g}})$ such that $\phi_* \mathcal{D} = \mathcal{D}$.*

Proof. Suppose ϕ is an isometry of a sub-Riemannian structure $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ with orthonormal frame (Y_1, Y_2). We have that $(\phi_* Y_1, \phi_* Y_2)$ is an orthonormal frame for $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ and so

$(\pm Y_0, \phi_* Y_1, \phi_* Y_2)$ is an orthonormal frame for $(\mathsf{G}, \tilde{\mathbf{g}})$. That is, ϕ pushes forward the orthonormal frame (Y_0, Y_1, Y_2) of $(\mathsf{G}, \tilde{\mathbf{g}})$ to an orthonormal frame $(\pm Y_0, \phi_* Y_1, \phi_* Y_2)$ of $(\mathsf{G}, \tilde{\mathbf{g}})$. Thus ϕ is an isometry of $(\mathsf{G}, \tilde{\mathbf{g}})$ such that $\phi_* \mathcal{D} = \mathcal{D}$.

Conversely, suppose ϕ is an isometry of $(\mathsf{G}, \tilde{\mathbf{g}})$ such that $\phi_* \mathcal{D} = \mathcal{D}$. Let (Y_1, Y_2) be an orthonormal frame for $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ and (Y_0, Y_1, Y_2) be an orthonormal frame for $(\mathsf{G}, \tilde{\mathbf{g}})$. We have that $(\phi_* Y_0, \phi_* Y_1, \phi_* Y_2)$ is an orthonormal frame for $(\mathsf{G}, \tilde{\mathbf{g}})$. Moreover, as $\phi_* \mathcal{D} = \mathcal{D}$, we have that $(\phi_* Y_1, \phi_* Y_2)$ is an orthonormal frame for $\mathcal{D}$ with respect to $\mathbf{g}$ (as $\tilde{\mathbf{g}}\big|_{\mathcal{D}} = \mathbf{g}$). Hence ϕ is an isometry of $(\mathsf{G}, \mathcal{D}, \mathbf{g})$. □

We shall denote the group of isometries of a structure $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ by $\mathsf{Iso}(\mathsf{G}, \mathcal{D}, \mathbf{g})$. The subgroup of isotropies fixing an element $g \in \mathsf{G}$ will be denoted by $\mathsf{Iso}_g(\mathsf{G}, \mathcal{D}, \mathbf{g})$. By left-invariance, we have that $\mathsf{Iso}(\mathsf{G}, \mathcal{D}, \mathbf{g})$ is generated by the left translations $L_g : h \mapsto gh,\ g \in \mathsf{G}$ and the isotropy subgroup of identity. Indeed, any isometry $\phi \in \mathsf{Iso}(\mathsf{G}, \mathcal{D}, \mathbf{g})$ can be written as $\phi = L_{\phi(\mathbf{1})} \circ \phi'$, where $\phi' \in \mathsf{Iso}_{\mathbf{1}}(\mathsf{G}.\mathcal{D}, \mathbf{g})$. Moreover, any isotropy subgroup $\mathsf{Iso}_g(\mathsf{G}, \mathcal{D}, \mathbf{g})$ is conjugate to $\mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathcal{D}, \mathbf{g})$; indeed $\mathsf{Iso}_g(\mathsf{G}, \mathcal{D}, \mathbf{g}) = L_g \circ \mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathcal{D}, \mathbf{g}) \circ L_{g^{-1}}$.

As any isometry of $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ is an isometry of its characteristic Riemannian expansion, it is uniquely determined by its tangent map at a point. Accordingly, we shall denote by

$$d\,\mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathcal{D}, \mathbf{g}) = \{T_{\mathbf{1}}\phi \,:\, \phi \in \mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathcal{D}, \mathbf{g})\}$$

the corresponding linearized isotropy group. Any element $T_{\mathbf{1}}\phi \in d\,\mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathcal{D}, \mathbf{g})$ must satisfy (3.1) and (3.2) for the characteristic Riemannian expansions of $(\mathsf{G}, \mathcal{D}, \mathbf{g})$; we shall use this fact in determining $d\,\mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathcal{D}, \mathbf{g})$ for each invariant sub-Riemannian structure (on a three-dimensional Lie group).

On the other hand, we denote the group of $\mathfrak{L}$-isometries of a structure $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ by $\mathfrak{L}\text{-}\mathsf{Iso}(\mathsf{G}, \mathcal{D}, \mathbf{g})$. We have that $\mathfrak{L}\text{-}\mathsf{Iso}(\mathsf{G}, \mathcal{D}, \mathbf{g}) = \mathsf{Iso}(\mathsf{G}, \mathcal{D}, \mathbf{g}) \cap \mathsf{Aut}(\mathsf{G})$ and that $\mathfrak{L}\text{-}\mathsf{Iso}(\mathsf{G}, \mathcal{D}, \mathbf{g})$ is a subgroup of $\mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathcal{D}, \mathbf{g})$. We denote by $d\,\mathfrak{L}\text{-}\mathsf{Iso}(\mathsf{G}, \mathcal{D}, \mathbf{g})$ the subgroup of linearized $\mathfrak{L}$-isometries, i.e.,

$$\begin{aligned} d\,\mathfrak{L}\text{-}\mathsf{Iso}(\mathsf{G}, \mathcal{D}, \mathbf{g}) &= \{T_{\mathbf{1}}\phi \,:\, \phi \in \mathfrak{L}\text{-}\mathsf{Iso}(\mathsf{G}, \mathcal{D}, \mathbf{g})\} \\ &= \{\psi \in d\,\mathsf{Aut}(\mathsf{G}) \,:\, \psi \cdot \mathcal{D}(\mathbf{1}) = \mathcal{D}(\mathbf{1}),\, \psi^* \mathbf{g}_{\mathbf{1}} = \mathbf{g}_{\mathbf{1}}\}\,. \end{aligned}$$

We shall show that for almost all structures in three dimensions $d\,\mathfrak{L}\text{-}\mathsf{Iso}(\mathsf{G}, \mathcal{D}, \mathbf{g}) = d\,\mathsf{Iso}(\mathsf{G}, \mathcal{D}, \mathbf{g})$.

3.1.4 Classification and scalar invariants

We restate here adaptations of the classification results for invariant sub-Riemannian structures in three dimensions obtained in [10], specialized to our purposes.

Let (Y_1, Y_2) be a left-invariant orthonormal frame for a sub-Riemannian structure $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ and let Y_0 be the associated (left-invariant) Reeb vector field as specified in Section 3.1.2. The Lie algebra of the vector fields Y_0, Y_1, Y_2 then takes the form

$$\begin{aligned} [Y_1, Y_0] &= c_{01}^1 Y_1 + c_{01}^2 Y_2 \\ [Y_2, Y_0] &= c_{02}^1 Y_1 + c_{02}^2 Y_2 \\ [Y_2, Y_1] &= c_{12}^1 Y_1 + c_{12}^2 Y_2 + Y_0. \end{aligned}$$

In terms of the structure constants c_{ij}^k, we have the following scalar invariants

$$\chi = \tfrac{1}{2}\sqrt{(c_{02}^1 + c_{01}^2)^2 - 4c_{01}^1 c_{02}^2}$$
$$\kappa = -(c_{12}^1)^2 - (c_{12}^2)^2 + \tfrac{1}{2}(c_{01}^2 - c_{02}^1).$$

The constants χ and κ are scalar invariants in the following sense.

Theorem 3.6 ([10]). *If* $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ *and* $(\mathsf{G}', \mathcal{D}', \mathbf{g}')$ *are isometric, then* $\chi = \chi'$ *and* $\kappa = \kappa'$.

The invariants κ and χ are homogeneous (with degree two) with respect to a dilation of the orthonormal frame (Y_1, Y_2). In other words, if $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ has scalar invariants (χ, κ), then $(\mathsf{G}, \mathcal{D}, \frac{1}{\lambda^2}\mathbf{g})$, $\lambda > 0$ has scalar invariants $(\lambda^2\kappa, \lambda^2\chi)$. Hence by rescaling the metric, we may assume that either

$$\chi = \kappa = 0 \quad \text{or} \quad \chi^2 + \kappa^2 = 1$$

For a given structure, the scalar invariants (satisfying the above equation) for a suitably rescaled structure will be referred to as the *normalized scalar invariants*.

In terms of the scalar invariants χ and κ, we have the following classification result.

Theorem 3.7 ([10]). *Let* $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ *and* $(\mathsf{G}', \mathcal{D}', \mathbf{g}')$ *be two left-invariant sub-Riemannian structures with matching scalar invariants* χ *and* κ.

1. *If* $\chi = \kappa = 0$, *then* $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ *is locally isometric to the Heisenberg group.*

2. *If* $\chi \neq 0$ *or* $\chi = 0$ *and* $\kappa \geq 0$, *then* $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ *and* $(\mathsf{G}', \mathcal{D}', \mathbf{g}')$ *are locally isometric if and only if their Lie algebras are isomorphic.*

3. *If* $\chi = 0$, $\kappa = -1$, *and* G *is simply connected, then* $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ *is isometric to* $\tilde{\mathsf{A}} = \widetilde{\mathsf{SL}}(2)$ *with elliptic type Killing metric.*

In Figure 3.1 we graph the normalized scalar invariants χ and κ for the respective normal forms of sub-Riemannian structures (obtained in the next two sections).

3.1.5 Metric subspaces

For invariant sub-Riemannian structures on a simply connected Lie group G, the classification up to $\mathfrak{L}$-isometry and rescaling of the metric can be accomplished by classifying metric subspaces of the associated Lie algebra up to Lie algebra isomorphism (and dilation of the metric).

We say that a pair $(\Gamma, \boldsymbol{\mu})$ is a *metric subspace* of a Lie algebra $\mathfrak{g}$ if Γ is a full-rank subspace of $\mathfrak{g}$ and $\boldsymbol{\mu}$ is an inner product on Γ. Two metric subspaces $(\Gamma, \boldsymbol{\mu})$ and $(\Gamma', \boldsymbol{\mu}')$ are said to be equivalent if there exists a Lie algebra automorphism $\psi \in \mathsf{Aut}(\mathfrak{g})$ such that $\psi \cdot \Gamma = \Gamma'$ and $r\boldsymbol{\mu}(A, B) = \boldsymbol{\mu}'(\psi \cdot A, \psi \cdot B)$ for $A, B \in \Gamma$ and some $r > 0$. We will likewise say that two full-rank subspaces Γ and Γ' are equivalent if there exists a Lie algebra automorphism $\psi \in \mathsf{Aut}(\mathfrak{g})$ such that $\psi \cdot \Gamma = \Gamma'$.

Lemma 3.8. *The sub-Riemannian structures* $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ *and* $(\mathsf{G}, \mathcal{D}', \mathbf{g}')$ *on a simply connected Lie group* G *are* **LiSR**-*isomorphic (i.e.,* $\mathfrak{L}$*-isometric up to rescaling) if and only if the associated metric subspaces* $(\mathcal{D}(\mathbf{1}), \mathbf{g_1})$ *and* $(\mathcal{D}'(\mathbf{1}), \mathbf{g}'_\mathbf{1})$ *of* $\mathfrak{g}$ *are equivalent.*

Proof. This follows immediately from Lemma 1.31 (as for any **LiSR**-isomorphism ϕ we have $\ker\phi = \{0\}$). □

We shall denote by $\mathsf{Aut}_\Gamma(\mathfrak{g})$ the subgroup of Lie algebra automorphisms preserving Γ, i.e., $\mathsf{Aut}_\Gamma(\mathfrak{g}) = \{\psi \in \mathsf{Aut}(\Gamma) \,:\, \psi \cdot \Gamma = \Gamma\}$. We denote by $\mathsf{Aut}(\mathfrak{g})\big|_\Gamma$ the restriction to Γ of the subgroup of automorphisms preserving Γ, i.e,

$$\mathsf{Aut}(\mathfrak{g})\big|_\Gamma = \left\{\psi \in \mathsf{GL}\,(\Gamma) \,:\, \exists\, \tilde{\psi} \in \mathsf{Aut}_\Gamma(\mathfrak{g}),\, \tilde{\psi}\big|_\Gamma = \psi\right\}.$$

Accordingly, the classification of the metric subspaces of a Lie algebra can be accomplished by first classifying the full-rank subspaces up to isomorphism, then calculating the transformations $\mathsf{Aut}(\mathfrak{g})\big|_\Gamma$ preserving a given a normal form Γ, and then reducing the inner products on Γ by these transformations.

We note that each automorphism is identified with its matrix (with respect to the standard basis as given in Appendix A). Moreover, given a subspace $\Gamma = \langle A_1, A_2\rangle$ implicitly with basis (A_1, A_2), each linear transformation $\psi \in \mathsf{Aut}(\mathfrak{g})\big|_\Gamma$ and each quadratic form on Γ will also be identified with its matrix.

3.2 Structures on solvable groups

We consider first those structures on solvable Lie groups. Table B.9 in Appendix B.2 contains a summary (listing the normal forms of sub-Riemannian structures, the corresponding linearized isotropy subgroups, and the corresponding normal forms of the associated Hamilton-Poisson systems).

3.2.1 Type $\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1$

The nonzero commutators of $\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1$ are $[E_1, E_2] = E_1$; the corresponding simply connected Lie group $\mathsf{Aff}\,(\mathbb{R})_0 \times \mathbb{R}$ is diffeomorphic to $\mathbb{R}^3$.

Proposition 3.9. *Any sub-Riemannian structure on* $\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}$ *is* $\mathfrak{L}$*-isometric up to rescaling to the structure* $(\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}, \mathcal{D}, \mathbf{g})$ *given by* $\mathcal{D}(\mathbf{1}) = \langle E_1 + E_3, E_2\rangle$ *and* $\mathbf{g_1} = \mathrm{diag}(1, 1)$*, i.e., with orthonormal frame* $(E_1^L + E_3^L, E_2^L)$*.*

1. *The group of linearized* $\mathfrak{L}$*-isometries is given by*

$$d\,\mathfrak{L}\text{-}\mathsf{Iso}(\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}, \mathcal{D}, \mathbf{g}) = \left\{ \begin{bmatrix} \sigma & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sigma \end{bmatrix} \,:\, \sigma = \pm 1 \right\}.$$

2. *The associated Hamiltonian system on* $(\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1)^*_-$ *is given by* $H(p) = \frac{1}{2}((p_1 + p_3)^2 + p_2^2)$ *and is* L*-equivalent to* $\mathsf{P}(2)$*.*

3. *The Reeb vector field corresponding to* $(\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}, \mathcal{D}, \mathbf{g})$ *is* $\pm E_3^L$. *The normalized scalar invariants are given by* $\chi = 0$ *and* $\kappa = -1$.
4. *The subgroup of linearized isotropies is given by*

$$d\,\mathsf{Iso}_{\mathbf{1}}(\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}, \mathcal{D}, \mathbf{g}) = \left\{ \begin{bmatrix} \sigma\cos\theta & -\sigma\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ -\sigma + \sigma\cos\theta & -\sigma\sin\theta & \sigma \end{bmatrix} \,:\, \theta \in \mathbb{R},\ \sigma = \pm 1 \right\} \cong \mathsf{O}(2)$$

Proof. It is a simple matter to show that any full-rank subspace of $\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1$ is equivalent to the full-rank subspace $\Gamma = \langle E_1 + E_3, E_2 \rangle$ ([36]). The subgroup of automorphisms preserving Γ is given by

$$\mathsf{Aut}_\Gamma(\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1) = \left\{ \begin{bmatrix} x & y & 0 \\ 0 & 1 & 0 \\ 0 & y & x \end{bmatrix} \,:\, x, y \in \mathbb{R},\ x \neq 0 \right\}.$$

The restriction to Γ of the subgroup of automorphisms preserving Γ is

$$\mathsf{Aut}(\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1)\big|_\Gamma = \left\{ \begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix} \,:\, x, y \in \mathbb{R},\ x \neq 0 \right\}.$$

Let $\boldsymbol{\mu} = \begin{bmatrix} a_1 & b \\ b & a_2 \end{bmatrix}$ be positive definite quadratic form on Γ (here the matrix for $\boldsymbol{\mu}$ is written with respect to the basis $(E_1 + E_3, E_2)$ for Γ). We have that

$$\psi = \begin{bmatrix} \frac{\sqrt{a_1 a_2 - b^2}}{a_1} & -\frac{b}{a_1} \\ 0 & 1 \end{bmatrix} \in \mathsf{Aut}(\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1)\big|_\Gamma$$

and $\boldsymbol{\mu} \circ \psi = \psi^\top \boldsymbol{\mu} \psi = (a_2 - \frac{b^2}{a_1})\,\mathrm{diag}(1,1)$. Therefore any metric subspace of $\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1$ is equivalent to the one associated with $(\mathcal{D}, \mathbf{g})$. Consequently, by Lemma 3.8, the result follows.

(1 and 2). The group of linearized $\mathfrak{L}$-isometries $d\,\mathfrak{L}\text{-}\mathsf{Iso}(\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R})$ consists of those automorphisms

$$\psi \in \mathsf{Aut}_\Gamma(\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1) = \left\{ \begin{bmatrix} x & y & 0 \\ 0 & 1 & 0 \\ 0 & y & x \end{bmatrix} \,:\, x, y \in \mathbb{R},\ x \neq 0 \right\}$$

satisfying $\mathbf{g_1}(A, B) = \mathbf{g_1}(\psi \cdot A, \psi \cdot B)$ for $A, B \in \mathcal{D}(\mathbf{1})$, i.e., $\psi\big|_{\mathcal{D}(\mathbf{1})} \in \mathsf{O}(2)$. Thus an automorphism $\psi \in \mathsf{Aut}_\Gamma(\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1)$ is an element of $d\,\mathfrak{L}\text{-}\mathsf{Iso}(\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R})$ if and only if $\begin{bmatrix} x^2 & xy \\ xy & 1 + y^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, i.e., $x = \pm 1$ and $y = 0$. The associated Hamiltonian system is $((\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1)^*_-, \frac{1}{2}((p_1 + p_3)^2 + p_2^2))$; it is a simple matter to see that this system is L-equivalent to $\mathsf{P}(2)$ (see Proposition 1.25, item 2).

(3). The pull back (X_1, X_2, X_3) of the frame (E_1^L, E_2^L, E_3^L) to $\mathbb{R}^3$ and its corresponding dual frame is given by

$$\begin{cases} X_1 = \mathrm{e}^{-x_2}\,\partial_{x_1} \\ X_2 = \partial_{x_2} \\ X_3 = \partial_{x_3}. \end{cases} \qquad \begin{cases} \nu_1 = \mathrm{e}^{x_2}\,dx_1 \\ \nu_2 = dx_2 \\ \nu_3 = dx_3. \end{cases}$$

Let $Y_1 = \lambda(X_1 + X_3) = \lambda\, \mathrm{e}^{-x_2}\, \partial_{x_1} + \lambda\partial_{x_3}$ and $Y_2 = \lambda X_2 = \lambda\partial_{x_2}$; we have that (Y_1, Y_2) is an orthonormal frame for $(\mathsf{Aff}\,(\mathbb{R}) \times \mathbb{R}, \mathcal{D}, \frac{1}{\lambda^2}\mathbf{g})$ (or rather, of the pull back by the parametrisation $m : \mathbb{R}^3 \to \mathsf{Aff}\,(\mathbb{R})_0 \times \mathbb{R}$ of this structure). The (left-invariant) contact one-form ω can be expressed as $\omega = a_1\nu_1 + a_2\nu_2 + a_3\nu_3$ for some $a_1, a_2, a_3 \in \mathbb{R}$. The condition $Y_1, Y_2 \in \ker\omega$ implies $a_2 = 0$ and $a_3 = -a_1$. The exterior derivative $d\omega = -a_1\, \mathrm{e}^{x_2}\, dx_1 \wedge dx_2$ and so $d\omega(Y_1, Y_2) = -a_1\lambda^2$. Thus we have $\omega = -\frac{1}{\lambda^2}\nu_1 + \frac{1}{\lambda^2}\nu_2 = -\frac{1}{\lambda^2}\, \mathrm{e}^{x_2}\, dx_1 + \frac{1}{\lambda^2}\, dx_3$ and $d\omega = \frac{1}{\lambda^2}\, \nu_1 \wedge \nu_2 = \frac{\mathrm{e}^{x_2}}{\lambda^2}\, dx_1 \wedge dx_2$. The (left-invariant) Reeb vector field Y_0 can be expressed as $Y_0 = a_1X_1 + a_2X_2 + a_3X_3$ for some $a_1, a_2, a_3 \in \mathbb{R}$. We have $i_{Y_0}d\omega = -\frac{a_2}{\lambda^2}\, \mathrm{e}^{x_2}\, dx_1 + \frac{a_1}{\lambda^2}\, dx_2$ and so $a_1 = a_2 = 0$, i.e., $Y_0 = a_3X_3$. Hence, as $\omega(Y_0) = \frac{a_3}{\lambda^2}$, we have $Y_0 = \lambda^2X_3$. Therefore the Reeb vector field corresponding to $(\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}, \mathcal{D}, \mathbf{g})$ is E_3^L (with respect to the orientation defined by the frame $(E_1^L + E_3^L, E_2^L)$). Accordingly, we have

$$\begin{aligned}
[Y_1, Y_0] &= 0Y_1 + 0Y_2\\
[Y_2, Y_0] &= 0Y_1 + 0Y_2\\
[Y_2, Y_1] &= -\lambda Y_1 + 0Y_2 + Y_0
\end{aligned}$$

and so $\chi = 0$ and $\kappa = -\lambda^2$. For $\lambda = 1$ we obtain normalized scalar invariants $\chi = 0$ and $\kappa = -1$.

(4). The Riemannian characteristic expansion $(\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}, \tilde{\mathbf{g}})$ of $(\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}, \mathcal{D}, \mathbf{g})$ has orthonormal frame $(E_1^L + E_3^L, E_2^L, E_3^L)$. Consequently, we have

$$\tilde{\mathbf{g}}_\mathbf{1} = \begin{bmatrix} 2 & 0 & -1\\ 0 & 1 & 0\\ -1 & 0 & 1 \end{bmatrix}$$

with respect to the basis (E_1, E_2, E_3). Suppose $\psi \in d\,\mathsf{Iso}_\mathbf{1}(\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}, \mathcal{D}, \mathbf{g})$ and let $\phi \in \mathsf{Iso}_\mathbf{1}(\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}, \mathcal{D}, \mathbf{g})$ be the unique isometry such that $T_\mathbf{1}\phi = \psi$. As ψ preserves $\mathcal{D}(\mathbf{1})$ and $\tilde{\mathbf{g}}_\mathbf{1}$, it follows that

$$\psi = \begin{bmatrix} \sigma_1\cos\theta & -\sigma_1\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & \sigma_2 \end{bmatrix}$$

with respect to the basis $(E_1 + E_3, E_2, E_3)$ for some $\theta \in \mathbb{R}$ and $\sigma_1, \sigma_2 \in \{-1, 1\}$; equivalently,

$$\psi = \begin{bmatrix} \sigma_1\cos\theta & -\sigma_1\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ -\sigma_2 + \sigma_1\cos\theta & -\sigma_1\sin\theta & \sigma_2 \end{bmatrix}$$

with respect to the basis (E_1, E_2, E_3). Furthermore, as ϕ is an isometry of $(\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}, \tilde{\mathbf{g}})$ (Proposition 3.5), we have that ψ preserves the associated curvature tensor R and its covariant derivative ∇R (see (3.1) and (3.2)). It turns out that ψ preserves R, but from preservation of ∇R we get

$$\begin{gathered}
\psi \cdot \nabla R(E_2, E_2, E_3, E_1) = \nabla R(\psi \cdot E_2, \psi \cdot E_2, \psi \cdot E_3, \psi \cdot E_1)\\
-\sigma_1 \sin\theta E_1 + \cos\theta E_2 - \sigma_1 \sin\theta E_3 = -\sigma_2 \sin\theta E_1 + \sigma_1\sigma_2 \cos\theta E_2 - \sigma_2 \sin\theta E_3.
\end{gathered}$$

Hence $\sigma_1 = \sigma_2$ and so it follows that any $\psi \in d\,\mathsf{Iso}_\mathbf{1}(\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}, \mathcal{D}, \mathbf{g})$ is an element of the group

$$\mathsf{H} = \left\{ \begin{bmatrix} \sigma\cos\theta & -\sigma\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ -\sigma + \sigma\cos\theta & -\sigma\sin\theta & \sigma \end{bmatrix} : \theta \in \mathbb{R},\ \sigma = \pm 1 \right\} \cong \mathsf{O}\,(2).$$

The sub-Riemannian structure $(\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}, \mathcal{D}, \mathbf{g})$ turns out to be isometric to the structure $(\tilde{\mathsf{A}}, \mathcal{D}_1, \mathbf{g}^{1,\alpha})$, $\alpha = 1$ (see Proposition 3.16) as both have normalized scalar invariants $(\chi, \kappa) = (0, -1)$ (see Theorem 3.7). Accordingly the linearized isotropy groups $d\,\mathsf{Iso}_\mathbf{1}(\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}, \mathcal{D}, \mathbf{g})$ and $d\,\mathsf{Iso}_\mathbf{1}(\tilde{\mathsf{A}}, \mathcal{D}_1, \mathbf{g}^{1,\alpha})$, $\alpha = 1$ are isomorphic. Moreover, it will be shown that $d\,\mathsf{Iso}_\mathbf{1}(\tilde{\mathsf{A}}, \mathcal{D}_1, \mathbf{g}^{1,1}) \cong \mathsf{O}(2)$ (Proposition 3.16). It follows that $d\,\mathsf{Iso}_\mathbf{1}(\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}, \mathcal{D}, \mathbf{g})$ is a subgroup of $\mathsf{H} \cong \mathsf{O}(2)$ which is isomorphic to $\mathsf{O}(2)$. Consequently, $d\,\mathsf{Iso}_\mathbf{1}(\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}, \mathcal{D}, \mathbf{g}) = \mathsf{H}$ (as the only subgroup $\mathsf{O}\,(2)$ isomorphic to $\mathsf{O}\,(2)$ is $\mathsf{O}\,(2)$ itself). □

Note. A description of the isometry between the isometric structures on $\mathsf{Aff}\,(\mathbb{R})_0 \times \mathbb{R}$ and $\tilde{\mathsf{A}} = \widetilde{\mathsf{SL}}\,(2, \mathbb{R})$ can be found in [10].

3.2.2 Type $\mathfrak{g}_{3.1}$

The Heisenberg Lie algebra $\mathfrak{g}_{3.1}$ has nonzero commutators $[E_2, E_3] = E_1$; the corresponding simply connected Lie group H_3 is diffeomorphic to $\mathbb{R}^3$.

Proposition 3.10 (cf. Theorems 4.6 and 4.8; also Proposition 1.57). *Any sub-Riemannian structure on H_3 is $\mathfrak{L}$-isometric to the structure $(\mathsf{H}_3, \mathcal{D}, \mathbf{g})$ given by $\mathcal{D}(\mathbf{1}) = \langle E_2, E_3 \rangle$ and $\mathbf{g}_\mathbf{1} = \mathrm{diag}(1, 1)$, i.e., with orthonormal frame (E_2^L, E_3^L).*

1. *The group of linearized $\mathfrak{L}$-isometries is given by*

$$d\,\mathfrak{L}\text{-}\mathsf{Iso}(\mathsf{H}_3, \mathcal{D}, \mathbf{g}) = \left\{ \begin{bmatrix} \det g & 0 \quad 0 \\ 0 & \\ 0 & g \end{bmatrix} : g \in \mathsf{O}\,(2) \right\}.$$

2. *The associated Hamiltonian system on $(\mathfrak{h}_3)^*_-$ is given by $H(p) = \frac{1}{2}(p_2^2 + p_3^2)$ and is L-equivalent to $\mathsf{P}(8)$.*
3. *The Reeb vector field corresponding to $(\mathsf{H}_3, \mathcal{D}, \mathbf{g})$ is $\pm E_1^L$. The normalized scalar invariants are given by $\chi = 0$ and $\kappa = 0$.*
4. *The subgroup of linearized isotropies $d\,\mathsf{Iso}_\mathbf{1}(\mathsf{H}_3, \mathcal{D}, \mathbf{g})$ is identical to $d\,\mathfrak{L}\text{-}\mathsf{Iso}(\mathsf{H}_3, \mathcal{D}, \mathbf{g})$.*

Proof. Any full-rank subspace $\Gamma \subset \mathfrak{h}_3$ is the image under some automorphisms of the full-rank subspace $\Gamma = \langle E_2, E_3 \rangle$ ([35]). The subgroup of automorphisms preserving Γ is given by

$$\mathsf{Aut}_\Gamma(\mathfrak{h}_3) = \left\{ \begin{bmatrix} wy - vz & 0 & 0 \\ 0 & y & v \\ 0 & z & w \end{bmatrix} : y, z, v, w \in \mathbb{R},\ wy - vz \neq 0 \right\}.$$

The restriction to Γ of the subgroup of automorphisms preserving Γ is $\mathsf{Aut}(\mathfrak{h}_3)\big|_\Gamma = \mathsf{GL}\,(\Gamma)$. Hence, for any positive definite quadratic form $\boldsymbol{\mu}$ on Γ there exists $\psi \in \mathsf{Aut}(\mathfrak{h}_3)$ such that

$\psi^\top \boldsymbol{\mu} \psi = \mathrm{diag}(1,1)$ (note that there is no need to rescale the quadratic form). Therefore any metric subspace is equivalent to the one associated with $(\mathcal{D}, \mathbf{g})$. Consequently, by Lemma 3.8 the result follows.

(1 and 2). A direct computation yields the subgroup of linearized $\mathfrak{L}$-isometries and the associated Hamiltonian system; by Proposition 2.14 we have that $((\mathfrak{h}_3)^*_-, H)$ is L-equivalent to $\mathsf{P}(8)$.

(3). The pull back (X_1, X_2, X_3) of the frame (E_1^L, E_2^L, E_3^L) to $\mathbb{R}^3$ and its corresponding dual frame is given by

$$\begin{cases} X_1 = \partial_{x_1} \\ X_2 = \partial_{x_2} \\ X_3 = x_2 \partial_{x_1} + \partial_{x_3}. \end{cases} \qquad \begin{cases} \nu_1 = dx_1 - x_2\, dx_3 \\ \nu_2 = dx_2 \\ \nu_3 = dx_3. \end{cases}$$

Let $Y_1 = \lambda X_2 = \lambda \partial_{x_2}$ and $Y_2 = \lambda X_3 = \lambda(x_2 \partial_{x_1} + \partial_{x_3})$. The contact one-form ω is given by $\omega = -\frac{1}{\lambda^2}\nu_1 = -\frac{1}{\lambda^2}(dx_1 - x_2\, dx_3)$ and has exterior derivative $d\omega = \frac{1}{\lambda^2}\,\nu_2 \wedge \nu_3 = \frac{1}{\lambda^2}\, dx_2 \wedge dx_3$. Accordingly the corresponding Reeb vector field is $Y_0 = -\lambda^2 X_1 = -\lambda^2 \partial_{x_1}$. Hence we get

$$\begin{aligned} [Y_1, Y_0] &= 0 Y_1 + 0 Y_2 \\ [Y_2, Y_0] &= 0 Y_1 + 0 Y_2 \\ [Y_2, Y_1] &= 0 Y_1 + 0 Y_2 + Y_0 \end{aligned}$$

and so $\chi = 0$ and $\kappa = 0$.

(4). The structure $(\mathsf{H}_3, \mathcal{D}, \mathbf{g})$ is a sub-Riemannian Carnot group and so the result follows by Theorem 1.45; see also Theorem 4.8. □

3.2.3 Type $\mathfrak{g}_{3.2}$

The nonzero commutators of $\mathfrak{g}_{3.2}$ are $[E_2, E_3] = E_1 - E_2$ and $[E_3, E_1] = E_1$; the corresponding simply connected Lie group is diffeomorphic to $\mathbb{R}^3$.

Proposition 3.11. *Any sub-Riemannian structure on the group $\mathsf{G}_{3.2}$ is $\mathfrak{L}$-isometric up to rescaling to the structure $(\mathsf{G}_{3.2}, \mathcal{D}, \mathbf{g})$ given by $\mathcal{D}(\mathbf{1}) = \langle E_2, E_3 \rangle$ and $\mathbf{g_1} = \mathrm{diag}(1,1)$, i.e., with orthonormal frame (E_2^L, E_3^L).*

1. *The group of linearized $\mathfrak{L}$-isometries is given by*

$$d\,\mathfrak{L}\text{-}\mathsf{Iso}(\mathsf{G}_{3.2}, \mathcal{D}, \mathbf{g}) = \left\{ \begin{bmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & 1 \end{bmatrix} \,:\, \sigma = \pm 1 \right\}$$

2. *The associated Hamiltonian system on $(\mathfrak{g}_{3.2})^*_-$ is given by $H(p) = \frac{1}{2}(p_2^2 + p_3^2)$ and is L-equivalent to $\mathsf{Np}(2)_{\delta=0}$.*
3. *The Reeb vector field corresponding to $(\mathsf{G}_{3.2}, \mathcal{D}, \mathbf{g})$ is $\pm(E_1^L + E_2^L)$. The normalized scalar invariants are given by $\chi = \frac{1}{5\sqrt{2}}$ and $\kappa = -\frac{7}{5\sqrt{2}}$.*
4. *The subgroup of linearized isotropies $d\,\mathsf{Iso}_\mathbf{1}(\mathsf{G}_{3.2}, \mathcal{D}, \mathbf{g})$ is identical to $d\,\mathfrak{L}\text{-}\mathsf{Iso}(\mathsf{G}_{3.2}, \mathcal{D}, \mathbf{g})$.*

Proof. Any full-rank subspace of $\mathfrak{g}_{3.2}$ is the image under some automorphism of the full-rank subspace $\Gamma = \langle E_2, E_3 \rangle$ ([35]). The subgroup of automorphisms preserving Γ is given by

$$\mathsf{Aut}_\Gamma(\mathfrak{g}_{3.2}) = \left\{ \begin{bmatrix} u & 0 & 0 \\ 0 & u & z \\ 0 & 0 & 1 \end{bmatrix} \; : \; u, z \in \mathbb{R}, \; u \neq 0 \right\}.$$

The restriction to Γ of the subgroup of automorphisms preserving Γ is

$$\mathsf{Aut}(\mathfrak{g}_{3.2})\big|_\Gamma = \left\{ \begin{bmatrix} u & z \\ 0 & 1 \end{bmatrix} \; : \; u, z \in \mathbb{R}, \; u \neq 0 \right\}.$$

As in the proof of Proposition 3.9, it then follows that any metric subspace of $\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1$ is equivalent to the one associated with $(\mathcal{D}, \mathbf{g})$. Consequently, by Lemma 3.8, the result follows.

(1 and 2). A direct computation yields the subgroup of linearized $\mathfrak{L}$-isometries and the associated Hamiltonian system; $((\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1)^*_-, H)$ is L-equivalent to $\mathsf{Np}(2)_{\delta=0}$ (see Proposition 1.25, item 2).

(3). The pull back (X_1, X_2, X_3) of the frame (E_1^L, E_2^L, E_3^L) to $\mathbb{R}^3$ and its corresponding dual frame is given by

$$\begin{cases} X_1 = \mathrm{e}^{x_3}\, \partial_{x_1} \\ X_2 = -x_3\, \mathrm{e}^{x_3}\, \partial_{x_1} + \mathrm{e}^{x_3}\, \partial_{x_2} \\ X_3 = \partial_{x_3} \end{cases} \qquad \begin{cases} \nu_1 = \mathrm{e}^{-x_3}\, dx_1 + x_3\, \mathrm{e}^{-x_3}\, dx_2 \\ \nu_2 = \mathrm{e}^{-x_3}\, dx_2 \\ \nu_3 = dx_3. \end{cases}$$

Let $Y_1 = \lambda X_2 = \lambda(-x_3\, \mathrm{e}^{x_3}\, \partial_{x_1} + \mathrm{e}^{x_3}\, \partial_{x_2})$ and $Y_2 = \lambda X_3 = \lambda \partial_{x_3}$. The contact one-form ω is given by $\omega = -\frac{1}{\lambda^2}\nu_1 = -\frac{1}{\lambda^2}(\mathrm{e}^{-x_3}\, dx_1 + x_3\, \mathrm{e}^{-x_3}\, dx_2)$ and has exterior derivative $d\omega = \frac{1}{\lambda^2}\nu_2 \wedge \nu_3 + \frac{1}{\lambda^2}\nu_3 \wedge \nu_1 = -\frac{1}{\lambda^2}\mathrm{e}^{-x_3}\, dx_1 \wedge dx_3 - \frac{1}{\lambda^2}(x_3 - 1)\,\mathrm{e}^{-x_3}\, dx_2 \wedge dx_3$. Accordingly the corresponding Reeb vector field is $Y_0 = -\lambda^2(X_1 + X_2) = \lambda^2(x_3 - 1)\,\mathrm{e}^{x_3}\, \partial_{x_1} - \lambda^2\, \mathrm{e}^{x_3}\, \partial_{x_2}$. Hence we get

$$\begin{aligned} [Y_1, Y_0] &= 0Y_1 + 0Y_2 \\ [Y_2, Y_0] &= -\lambda^2 Y_1 + 0Y_2 \\ [Y_2, Y_1] &= 2\lambda Y_1 + 0Y_2 + Y_0 \end{aligned}$$

and so $\chi = \frac{1}{2}\lambda^2$ and $\kappa = -\frac{7}{2}\lambda^2$. For $\lambda = \sqrt{\frac{\sqrt{2}}{5}}$ we obtain normalized scalar invariants $\chi = \frac{1}{5\sqrt{2}}$ and $\kappa = -\frac{7}{5\sqrt{2}}$.

(4). Accompanying Mathematica code is given in Appendix C.3. The Riemannian characteristic expansion $(\mathsf{G}_{3.2}, \tilde{\mathbf{g}})$ of $(\mathsf{G}_{3.2}, \mathcal{D}, \mathbf{g})$ has orthonormal frame $(E_2^L, E_3^L, E_1^L + E_2^L)$. Consequently, we have

$$\tilde{\mathbf{g}}_\mathbf{1} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

with respect to the basis (E_1, E_2, E_3). Suppose $\psi \in d\,\mathsf{Iso}_\mathbf{1}(\mathsf{G}_{3.2}, \mathcal{D}, \mathbf{g})$ and let $\phi \in \mathsf{Iso}_\mathbf{1}(\mathsf{G}_{3.2}, \mathcal{D}, \mathbf{g})$ be the unique isometry such that $T_\mathbf{1}\phi = \psi$. As ψ preserves $\mathcal{D}(\mathbf{1})$ and $\tilde{\mathbf{g}}_\mathbf{1}$, it follows that

$$\psi = \begin{bmatrix} \sigma_1 \cos\theta & -\sigma_1 \sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & \sigma_2 \end{bmatrix}$$

with respect to the basis $(E_2, E_3, E_1 + E_3)$ for some $\theta \in \mathbb{R}$ and $\sigma_1, \sigma_2 \in \{-1, 1\}$; equivalently,

$$\psi = \begin{bmatrix} \sigma_2 & 0 & 0 \\ \sigma_2 - \sigma_1 \cos\theta & \sigma_1 \cos\theta & -\sigma_1 \sin\theta \\ -\sin\theta & \sin\theta & \cos\theta \end{bmatrix}$$

with respect to the basis (E_1, E_2, E_3). Furthermore, as ϕ is an isometry of $(\mathsf{G}_{3.2}, \tilde{\mathbf{g}})$ (Proposition 3.5), we have that ψ preserves the associated curvature tensor R and its covariant derivative ∇R (see (3.1) and (3.2)). Hence,

$$\begin{aligned} \psi \cdot R(E_1, E_2, E_3) &= R(\psi \cdot E_1, \psi \cdot E_2, \psi \cdot E_3) \\ 0 &= \sigma_2 \sin(2\theta) E_2 + 2\sigma_1\sigma_2 \sin\theta^2 E_3 \end{aligned}$$

and so $\sin\theta = 0$. Similarly,

$$\begin{aligned} \psi \cdot R(E_3, E_2, E_3) &= R(\psi \cdot E_3, \psi \cdot E_2, \psi \cdot E_3) \\ -2\sigma_2 E_1 - 2(\sigma_2 + 2\sigma_1 \cos\theta) E_2 - 4 \sin\theta E_3 &= -2\sigma_1 \cos\theta E_1 - 6\sigma_1 \cos\theta E_2 - 4 \sin\theta E_3 \end{aligned}$$

and so $\cos\theta = \sigma_1\sigma_2$. Consequently we have that $\psi = \operatorname{diag}(\sigma_2, \sigma_2, \sigma_1\sigma_2)$. Hence

$$\begin{aligned} \psi \cdot \nabla R(E_1, E_2, E_3, E_1) &= \nabla R(\psi \cdot E_1, \psi \cdot E_2, \psi \cdot E_3, \psi \cdot E_1) \\ -4\sigma_2 E_1 - 8\sigma_2 E_2 &= -4\sigma_1 E_1 - 8\sigma_1 E_2 \end{aligned}$$

which implies that $\sigma_1 = \sigma_2$ and so $\psi = \operatorname{diag}(\sigma_2, \sigma_2, 1)$. Thus $\psi \in d\,\mathfrak{L}\text{-}\mathsf{Iso}(\mathsf{G}_{3.2}, \mathcal{D}, \mathbf{g})$ and therefore $d\,\mathsf{Iso}_{\mathbf{1}}(\mathsf{G}_{3.2}, \mathcal{D}, \mathbf{g}) = d\,\mathfrak{L}\text{-}\mathsf{Iso}(\mathsf{G}_{3.2}, \mathcal{D}, \mathbf{g})$. □

3.2.4 Type $\mathfrak{g}_{3.4}^0$

The semi-Euclidean Lie algebra $\mathfrak{g}_{3.4}^0$ has nonzero commutators $[E_2, E_3] = E_1$ and $[E_3, E_1] = -E_2$; the corresponding simply connected Lie group $\mathsf{SE}(1,1)$ is diffeomorphic to $\mathbb{R}^3$.

Proposition 3.12 (cf. [25])**.** *Any sub-Riemannian structure on* $\mathsf{SE}(1,1)$ *is* $\mathfrak{L}$*-isometric up to rescaling to the structure* $(\mathsf{SE}(1,1), \mathcal{D}, \mathbf{g})$ *given by* $\mathcal{D}(\mathbf{1}) = \langle E_2, E_3 \rangle$ *and* $\mathbf{g}_{\mathbf{1}} = \operatorname{diag}(1,1)$*, i.e., with orthonormal frame* (E_2^L, E_3^L)*.*

1. *The group of linearized* $\mathfrak{L}$*-isometries is given by*

$$d\,\mathfrak{L}\text{-}\mathsf{Iso}(\mathsf{SE}(1,1), \mathcal{D}, \mathbf{g}) = \left\{ \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_1\sigma_2 & 0 \\ 0 & 0 & \sigma_2 \end{bmatrix} \, : \, \sigma_1, \sigma_2 = \pm 1 \right\}.$$

2. *The associated Hamiltonian system on* $\mathfrak{se}(1,1)^*_-$ *is given by* $H(p) = \frac{1}{2}(p_2^2 + p_3^2)$ *and is* L*-equivalent to* $\mathsf{Np}(7)$*.*
3. *The Reeb vector field corresponding to* $(\mathsf{SE}(1,1), \mathcal{D}, \mathbf{g})$ *is* $\pm E_1^L$*. The normalized scalar invariants are given by* $\chi = \frac{1}{\sqrt{2}}$ *and* $\kappa = -\frac{1}{\sqrt{2}}$*.*
4. *The subgroup* $d\,\mathsf{Iso}_{\mathbf{1}}(\mathsf{SE}(1,1), \mathcal{D}, \mathbf{g})$ *is identical to* $d\,\mathfrak{L}\text{-}\mathsf{Iso}(\mathsf{SE}(1,1), \mathcal{D}, \mathbf{g})$*.*

Proof. Any full-rank subspace of $\mathfrak{se}(1,1)$ is the image under some automorphism of the full-rank subspace $\Gamma = \langle E_2, E_3 \rangle$ ([24, 36]). The subgroup of automorphisms preserving Γ is given by

$$\mathsf{Aut}_\Gamma(\mathfrak{se}(1,1)) = \left\{ \begin{bmatrix} x & 0 & 0 \\ 0 & \sigma x & v \\ 0 & 0 & \sigma \end{bmatrix} \ : \ x, v \in \mathbb{R},\ x \neq 0,\ \sigma = \pm 1 \right\}.$$

The restriction to Γ of the subgroup of automorphisms preserving Γ is

$$\mathsf{Aut}(\mathfrak{se}(1,1))\big|_\Gamma = \left\{ \begin{bmatrix} \sigma x & v \\ 0 & \sigma \end{bmatrix} \ : \ x, v \in \mathbb{R},\ x \neq 0,\ \sigma = \pm 1 \right\}.$$

As in the proof of Proposition 3.9, it then follows that any metric subspace of $\mathfrak{se}(1,1)$ is equivalent to the one associated with $(\mathcal{D}, \mathbf{g})$. Consequently, by Lemma 3.8, the result follows.

(1 and 2). A direct computation yields the subgroup of linearized $\mathfrak{L}$-isometries and the associated Hamiltonian system. It is straightforward to show that $(\mathfrak{se}(1,1)^*_-, H)$ is L-equivalent to $(\mathfrak{se}(1,1)^*_-, p_1^2 + p_3^2)$ (see Proposition 1.25, items 2 and 3). The system $(\mathfrak{se}(1,1)^*_-, p_1^2 + p_3^2)$ in turn is L-equivalent to $\mathsf{Np}(7)$ by Proposition 2.14.

(3). The pull back (X_1, X_2, X_3) of the frame (E_1^L, E_2^L, E_3^L) to $\mathbb{R}^3$ and its corresponding dual frame is given by

$$\begin{cases} X_1 = \cosh x_3\, \partial_{x_1} - \sinh x_3\, \partial_{x_2} \\ X_2 = -\sinh x_3\, \partial_{x_1} + \cosh x_3\, \partial_{x_2} \\ X_3 = \partial_{x_3} \end{cases} \qquad \begin{cases} \nu_1 = \cosh x_3\, dx_1 + \sinh x_3\, dx_2 \\ \nu_2 = \sinh x_3\, dx_1 + \cosh x_3\, dx_2 \\ \nu_3 = dx_3. \end{cases}$$

Let $Y_1 = \lambda X_2 = \lambda(-\sinh x_3\, \partial_{x_1} + \cosh x_3\, \partial_{x_2})$ and $Y_2 = \lambda X_3 = \lambda \partial_{x_3}$. The contact one-form ω is given by $\omega = -\frac{1}{\lambda^2}\nu_1 = -\frac{1}{\lambda^2}(\cosh x_3\, dx_1 + \sinh x_3\, dx_2)$ and has exterior derivative $d\omega = \frac{1}{\lambda^2}\nu_2 \wedge \nu_3 = \frac{1}{\lambda^2}\sinh x_3\, dx_1 \wedge dx_3 + \frac{1}{\lambda^2}\cosh x_3\, dx_2 \wedge dx_3$. Accordingly the corresponding Reeb vector field is $Y_0 = -\lambda^2 X_1 = -\lambda^2 \cosh x_3 \partial_{x_1} + \lambda^2 \sinh x_3 \partial_{x_2}$. Hence we get

$$\begin{aligned} [Y_1, Y_0] &= 0Y_1 + 0Y_2 \\ [Y_2, Y_0] &= \lambda^2 Y_1 + 0Y_2 \\ [Y_2, Y_1] &= 0Y_1 + 0Y_2 + Y_0 \end{aligned}$$

and so $\chi = \frac{\sqrt{\lambda^4}}{2}$ and $\kappa = -\frac{\lambda^2}{2}$. For $\lambda = 2^{1/4}$ we obtain normalized scalar invariants $\chi = \frac{1}{\sqrt{2}}$ and $\kappa = -\frac{1}{\sqrt{2}}$.

(4). The Riemannian characteristic expansion $(\mathsf{SE}(1,1), \tilde{\mathbf{g}})$ of $(\mathsf{SE}(1,1), \mathcal{D}, \mathbf{g})$ has orthonormal frame (E_2^L, E_3^L, E_1^L). Consequently, we have $\tilde{\mathbf{g}}_\mathbf{1} = \operatorname{diag}(1,1,1)$ with respect to the basis (E_1, E_2, E_3). Suppose $\psi \in d\,\mathsf{Iso}_\mathbf{1}(\mathsf{SE}(1,1), \mathcal{D}, \mathbf{g})$ and let $\phi \in \mathsf{Iso}_\mathbf{1}(\mathsf{SE}(1,1), \mathcal{D}, \mathbf{g})$ be the unique isometry such that $T_\mathbf{1}\phi = \psi$. As ψ preserves $\mathcal{D}(\mathbf{1})$ and $\tilde{\mathbf{g}}_\mathbf{1}$, it follows that

$$\psi = \begin{bmatrix} \sigma_1 \cos\theta & -\sigma_1 \sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & \sigma_2 \end{bmatrix}$$

with respect to the basis (E_2, E_3, E_1) for some $\theta \in \mathbb{R}$ and $\sigma_1, \sigma_2 \in \{-1, 1\}$; equivalently,

$$\psi = \begin{bmatrix} \sigma_2 & 0 & 0 \\ 0 & \sigma_1 \cos\theta & -\sigma_1 \sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

with respect to the basis (E_1, E_2, E_3). Furthermore, as ϕ is an isometry of $(\mathsf{SE}\,(1,1), \tilde{\mathbf{g}})$ (Proposition 3.5), we have that ψ preserves the associated curvature tensor R and its covariant derivative ∇R (see (3.1) and (3.2)). Hence,

$$\begin{aligned} \psi \cdot R(E_2, E_1, E_1) &= R(\psi \cdot E_2, \psi \cdot E_1, \psi \cdot E_1) \\ -\sigma_1 \cos\theta E_2 - \sin\theta E_3 &= -\sigma_1 \cos\theta E_2 + \sin\theta E_3 \end{aligned}$$

and so $\sin\theta = 0$ and $\cos\theta = \pm 1$. Therefore, we have $\psi = \mathrm{diag}(\sigma_1, \sigma_2, \sigma_3)$ for some $\sigma_1, \sigma_2, \sigma_3 \in \{-1, 1\}$. Hence

$$\begin{aligned} \psi \cdot \nabla R(E_2, E_2, E_3, E_1) &= \nabla R(\psi \cdot E_2, \psi \cdot E_2, \psi \cdot E_3, \psi \cdot E_1) \\ 2\sigma_2 E_2 &= 2\sigma_1 \sigma_3 E_2 \end{aligned}$$

which implies that $\sigma_2 = \sigma_1 \sigma_3$ and so $\psi = \mathrm{diag}(\sigma_1, \sigma_1\sigma_3, \sigma_3)$. Thus $\psi \in d\,\mathfrak{L}\text{-}\mathsf{Iso}(\mathsf{SE}\,(1,1), \mathcal{D}, \mathbf{g})$ and therefore $d\,\mathsf{Iso}_{\mathbf{1}}(\mathsf{SE}\,(1,1), \mathcal{D}, \mathbf{g}) = d\,\mathfrak{L}\text{-}\mathsf{Iso}(\mathsf{SE}\,(1,1), \mathcal{D}, \mathbf{g})$. ☐

3.2.5 Type $\mathfrak{g}_{3.4}^{\alpha}$

The nonzero commutators of $\mathfrak{g}_{3.4}^{\alpha}$ are $[E_2, E_3] = E_1 - \alpha E_2$ and $[E_3, E_1] = \alpha E_1 - E_2$; the corresponding simply connected Lie group $\mathsf{G}_{3.4}^{\alpha}$ is diffeomorphic to $\mathbb{R}^3$.

Proposition 3.13. *Any sub-Riemannian structure on the group* $\mathsf{G}_{3.4}^{\alpha}$ *is* $\mathfrak{L}$*-isometric up to rescaling to the structure* $(\mathsf{G}_{3.4}^{\alpha}, \mathcal{D}, \mathbf{g})$ *given by* $\mathcal{D}(\mathbf{1}) = \langle E_2, E_3 \rangle$ *and* $\mathbf{g}_{\mathbf{1}} = \mathrm{diag}(1,1)$*, i.e., with orthonormal frame* (E_2^L, E_3^L).

1. *The group of linearized* $\mathfrak{L}$*-isometries is given by*

$$d\,\mathfrak{L}\text{-}\mathsf{Iso}(\mathsf{G}_{3.4}^{\alpha}, \mathcal{D}, \mathbf{g}) = \left\{ \begin{bmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & 1 \end{bmatrix} \; : \; \sigma = \pm 1 \right\}.$$

2. *The associated Hamiltonian system on* $(\mathfrak{g}_{3.4}^{\alpha})_{-}^{*}$ *is given by* $H(p) = \frac{1}{2}(p_2^2 + p_3^2)$ *and is* L*-equivalent to* $\mathsf{Np}(6)_{\beta=0}$.
3. *The Reeb vector field corresponding to* $(\mathsf{G}_{3.4}^{\alpha}, \mathcal{D}, \mathbf{g})$ *is* $\pm(E_1^L + \alpha E_2^L)$. *The normalized scalar invariants are given by* $\chi = \frac{|\alpha^2 - 1|}{\sqrt{2 + 12\alpha^2 + 50\alpha^4}}$ *and* $\kappa = \frac{-1 - 7\alpha^2}{\sqrt{2 + 12\alpha^2 + 50\alpha^4}}$.
4. *The subgroup of linearized isotropies* $d\,\mathsf{Iso}_{\mathbf{1}}(\mathsf{G}_{3.4}^{\alpha}, \mathcal{D}, \mathbf{g})$ *is identical to* $d\,\mathfrak{L}\text{-}\mathsf{Iso}(\mathsf{G}_{3.4}^{\alpha}, \mathcal{D}, \mathbf{g})$.

Proof. Any full-rank subspace of $\mathfrak{g}_{3.4}^{\alpha}$ is the image under some automorphism of the full-rank subspace $\Gamma = \langle E_2, E_3 \rangle$ ([36]). The subgroup of automorphisms preserving Γ is given by

$$\mathsf{Aut}_{\Gamma}(\mathfrak{g}_{3.4}^{\alpha}) = \left\{ \begin{bmatrix} x & 0 & 0 \\ 0 & x & v \\ 0 & 0 & 1 \end{bmatrix} \; : \; x, v \in \mathbb{R},\, x \neq 0 \right\}.$$

The restriction to Γ of the subgroup of automorphisms preserving Γ is

$$\mathsf{Aut}(\mathfrak{g}_{3.4}^{\alpha})\big|_{\Gamma} = \left\{ \begin{bmatrix} x & v \\ 0 & 1 \end{bmatrix} \,:\, x, v \in \mathbb{R},\, x \neq 0 \right\}.$$

As in the proof of Proposition 3.9, it then follows that any metric subspace of $\mathfrak{g}_{3.4}^{\alpha}$ is equivalent to the one associated with $(\mathcal{D}, \mathbf{g})$. Consequently, by Lemma 3.8, the result follows.

(1 and 2). A direct computation yields the subgroup of linearized $\mathfrak{L}$-isometries and the associated Hamiltonian system; $(\mathfrak{g}_{3.4}^{\alpha}, H)$ is L-equivalent to $\mathsf{Np}(6)_{\beta=0}$ (see Proposition 1.25, item 2).

(3). The pull back (X_1, X_2, X_3) of the frame (E_1^L, E_2^L, E_3^L) to $\mathbb{R}^3$ and its corresponding dual frame is given by

$$\begin{cases} X_1 = \mathrm{e}^{\alpha x_3} \cosh x_3\, \partial_{x_1} - \mathrm{e}^{\alpha x_3} \sinh x_3\, \partial_{x_2} \\ X_2 = -\,\mathrm{e}^{\alpha x_3} \sinh x_3\, \partial_{x_1} + \mathrm{e}^{\alpha x_3} \cosh x_3\, \partial_{x_2} \\ X_3 = \partial_{x_3} \end{cases} \qquad \begin{cases} \nu_1 = \mathrm{e}^{-\alpha x_3} \cosh x_3\, dx_1 + \mathrm{e}^{-\alpha x_3} \sinh x_3\, dx_2 \\ \nu_2 = \mathrm{e}^{-\alpha x_3} \sinh x_3\, dx_1 + \mathrm{e}^{-\alpha x_3} \cosh x_3\, dx_2 \\ \nu_3 = dx_3. \end{cases}$$

Let $Y_1 = \lambda X_2 = \lambda(-\,\mathrm{e}^{\alpha x_3} \sinh x_3\, \partial_{x_1} + \mathrm{e}^{\alpha x_3} \cosh x_3\, \partial_{x_2})$ and $Y_2 = \lambda X_3 = \lambda \partial_{x_3}$. The contact one-form ω is given by $\omega = -\frac{1}{\lambda^2}\nu_1 = -\frac{1}{\lambda^2}(\mathrm{e}^{-\alpha x_3} \cosh x_3\, dx_1 + \mathrm{e}^{-\alpha x_3} \sinh x_3\, dx_2)$ and has exterior derivative $d\omega = \frac{1}{\lambda^2}\,\nu_2 \wedge \nu_3 + \frac{\alpha}{\lambda^2}\,\nu_3 \wedge \nu_1 = \frac{1}{\lambda^2}\,\mathrm{e}^{-\alpha x_3}(\sinh x_3 - \alpha \cosh x_3)\, dx_1 \wedge dx_3 + \frac{1}{\lambda^2}\,\mathrm{e}^{-\alpha x_3}(\cosh x_3 - \alpha \sinh x_3)\, dx_2 \wedge dx_3$. Accordingly the corresponding Reeb vector field is $Y_0 = -\lambda^2 X_1 - \alpha\lambda^2 X_2 = -\lambda^2\,\mathrm{e}^{\alpha x_3}(\cosh x_3 - \alpha \sinh x_3)\partial_{x_1} + \lambda^2\,\mathrm{e}^{\alpha x_3}(\sinh x_3 - \alpha \cosh x_3)\partial_{x_2}$. Hence we get

$$\begin{aligned} [Y_1, Y_0] &= 0Y_1 + 0Y_2 \\ [Y_2, Y_0] &= -\lambda^2 \left(\alpha^2 - 1\right) Y_1 + 0Y_2 \\ [Y_2, Y_1] &= 2\alpha\lambda Y_1 + 0Y_2 + Y_0 \end{aligned}$$

and so $\chi = \frac{1}{2}\lambda^2 \left|\alpha^2 - 1\right|$ and $\kappa = -\frac{1}{2}\lambda^2 \left(1 + 7\alpha^2\right)$. For $\lambda = \frac{2^{1/4}}{(1+6\alpha^2+25\alpha^4)^{1/4}}$ we obtain normalized scalar invariants $\chi = \frac{|\alpha^2-1|}{\sqrt{2+12\alpha^2+50\alpha^4}}$ and $\kappa = \frac{-1-7\alpha^2}{\sqrt{2+12\alpha^2+50\alpha^4}}$.

(4). The Riemannian characteristic expansion $(\mathsf{G}_{3.4}^{\alpha}, \tilde{\mathbf{g}})$ of $(\mathsf{G}_{3.4}^{\alpha}, \mathcal{D}, \mathbf{g})$ has orthonormal frame $(E_2^L, E_3^L, E_1^L + \alpha E_2^L)$. Consequently, we have

$$\tilde{\mathbf{g}}_{\mathbf{1}} = \begin{bmatrix} 1 + \alpha^2 & -\alpha & 0 \\ -\alpha & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

with respect to the basis (E_1, E_2, E_3). Suppose $\psi \in d\,\mathsf{Iso}_{\mathbf{1}}(\mathsf{G}_{3.4}^{\alpha}, \mathcal{D}, \mathbf{g})$ and let $\phi \in \mathsf{Iso}_{\mathbf{1}}(\mathsf{G}_{3.4}^{\alpha}, \mathcal{D}, \mathbf{g})$ be the unique isometry such that $T_{\mathbf{1}}\phi = \psi$. As ψ preserves $\mathcal{D}(\mathbf{1})$ and $\tilde{\mathbf{g}}_{\mathbf{1}}$, it follows that

$$\psi = \begin{bmatrix} \sigma_1 \cos\theta & -\sigma_1 \sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & \sigma_2 \end{bmatrix}$$

with respect to the basis $(E_2, E_3, E_1 + \alpha E_2)$ for some $\theta \in \mathbb{R}$ and $\sigma_1, \sigma_2 \in \{-1, 1\}$; equivalently,

$$\psi = \begin{bmatrix} \sigma_2 & 0 & 0 \\ \sigma_2\alpha - \sigma_1\alpha\cos\theta & \sigma_1\cos\theta & -\sigma_1\sin\theta \\ -\alpha\sin\theta & \sin\theta & \cos\theta \end{bmatrix}$$

with respect to the basis (E_1, E_2, E_3). Furthermore, as ϕ is an isometry of $(\mathsf{G}^{\alpha}_{3.4}, \tilde{\mathbf{g}})$ (Proposition 3.5), we have that ψ preserves the associated curvature tensor R and its covariant derivative ∇R (see (3.1) and (3.2)). Hence, we have

$$\begin{aligned} \psi \cdot R(E_3, E_2, E_2) - R(\psi \cdot E_3, \psi \cdot E_2, \psi \cdot E_2) = 0 \\ 2\alpha(\alpha^2 - 1)\sigma_1 \sin\theta E_1 + 2\alpha^2(\alpha^2 - 1)\sigma_1 \sin\theta E_2 = 0 \end{aligned}$$

and

$$\begin{aligned} \psi \cdot R(E_3, E_2, E_3) - R(\psi \cdot E_3, \psi \cdot E_2, \psi \cdot E_3) = 0 \\ -2\alpha(\alpha^2 - 1)(\sigma_2 - \sigma_1\cos\theta)E_1 - 2\alpha^2(\alpha^2 - 1)(\sigma_2 - \sigma_1\cos\theta)E_2 = 0. \end{aligned}$$

Thus $\sin\theta = 0$ and $\cos\theta = \sigma_1\sigma_2$. Therefore, we have $\psi = \operatorname{diag}(\sigma_2, \sigma_2, \sigma_1\sigma_2)$. Hence

$$\begin{aligned} \psi \cdot \nabla R(E_1, E_1, E_2, E_2) - \nabla R(\psi \cdot E_1, \psi \cdot E_1, \psi \cdot E_2, \psi \cdot E_2) = 0 \\ \tfrac{1}{2}\alpha^3 \left(2 + 3\alpha^2\right) \sigma_2 \left(-\sigma_1 + \sigma_2^3\right) = 0 \end{aligned}$$

which implies that $\sigma_1 = \sigma_2$ and so $\psi = \operatorname{diag}(\sigma_1, \sigma_1, 1)$. Thus $\psi \in d\,\mathfrak{L}\text{-}\mathsf{Iso}(\mathsf{G}^{\alpha}_{3.4}, \mathcal{D}, \mathbf{g})$ and therefore $d\,\mathsf{Iso}_{\mathbf{1}}(\mathsf{G}^{\alpha}_{3.4}, \mathcal{D}, \mathbf{g}) = d\,\mathfrak{L}\text{-}\mathsf{Iso}(\mathsf{G}^{\alpha}_{3.4}, \mathcal{D}, \mathbf{g})$. □

3.2.6 Type $\mathfrak{g}^0_{3.5}$

The nonzero commutators of the Euclidean Lie algebra $\mathfrak{g}^0_{3.5}$ are $[E_2, E_3] = E_1$ and $[E_3, E_1] = E_2$; the corresponding simply connected Lie group $\widetilde{\mathsf{SE}}\,(2)$ is diffeomorphic to $\mathbb{R}^3$.

Proposition 3.14. *Any sub-Riemannian structure on $\widetilde{\mathsf{SE}}\,(2)$ is $\mathfrak{L}$-isometric up to rescaling to the structure $(\widetilde{\mathsf{SE}}\,(2), \mathcal{D}, \mathbf{g})$ given by $\mathcal{D}(\mathbf{1}) = \langle E_2, E_3\rangle$ and $\mathbf{g}_{\mathbf{1}} = \operatorname{diag}(1, 1)$, i.e., with orthonormal frame (E_2^L, E_3^L).*

1. *The group of linearized $\mathfrak{L}$-isometries is given by*

$$d\,\mathfrak{L}\text{-}\mathsf{Iso}(\widetilde{\mathsf{SE}}\,(2), \mathcal{D}, \mathbf{g}) = \left\{ \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_1\sigma_2 & 0 \\ 0 & 0 & \sigma_2 \end{bmatrix} : \sigma_1, \sigma_2 = \pm 1 \right\}.$$

2. *The associated Hamiltonian system on $\widetilde{\mathfrak{se}}\,(2)^*_-$ is given by $H(p) = \frac{1}{2}(p_2^2 + p_3^2)$ and is L-equivalent to $\mathsf{Np}(7)$.*
3. *The Reeb vector field corresponding to $(\widetilde{\mathsf{SE}}\,(2), \mathcal{D}, \mathbf{g})$ is $\pm E_1^L$. The normalized scalar invariants are given by $\chi = \frac{1}{\sqrt{2}}$ and $\kappa = \frac{1}{\sqrt{2}}$.*
4. *The subgroup of linearized isotropies $d\,\mathsf{Iso}_{\mathbf{1}}(\widetilde{\mathsf{SE}}\,(2), \mathcal{D}, \mathbf{g})$ is identical to $d\,\mathfrak{L}\text{-}\mathsf{Iso}(\widetilde{\mathsf{SE}}\,(2), \mathcal{D}, \mathbf{g})$.*

Proof. Any full-rank subspace of $\widetilde{\mathfrak{se}}\,(2)$ is the image under some automorphism of the full-rank subspace $\Gamma = \langle E_2, E_3 \rangle$ ([36]). The subgroup of automorphisms preserving Γ is given by

$$\mathsf{Aut}_\Gamma(\widetilde{\mathfrak{se}}\,(2)) = \left\{ \begin{bmatrix} x & 0 & 0 \\ 0 & \sigma x & v \\ 0 & 0 & \sigma \end{bmatrix} \; : \; x, v \in \mathbb{R},\; x \neq 0,\; \sigma = \pm 1 \right\}.$$

The restriction to Γ of the subgroup of automorphisms preserving Γ is

$$\mathsf{Aut}(\widetilde{\mathfrak{se}}\,(2))\big|_\Gamma = \left\{ \begin{bmatrix} \sigma x & v \\ 0 & \sigma \end{bmatrix} \; : \; x, v \in \mathbb{R},\; x \neq 0,\; \sigma = \pm 1 \right\}.$$

As in the proof of Proposition 3.9, it then follows that any metric subspace of $\widetilde{\mathfrak{se}}\,(2)$ is equivalent to the one associated with $(\mathcal{D}, \mathbf{g})$. Consequently, by Lemma 3.8, the result follows.

(1 and 2). A direct computation yields the subgroup of linearized $\mathfrak{L}$-isometries and the associated Hamiltonian system; $(\widetilde{\mathfrak{se}}\,(2)^*_-, H)$ is L-equivalent to $\mathsf{Np}(7)$ (see Proposition 1.25, item 2).

(3). Accompanying Mathematica code is given in Appendix C.2. The pull back (X_1, X_2, X_3) of the frame (E_1^L, E_2^L, E_3^L) to $\mathbb{R}^3$ and its corresponding dual frame is given by

$$\begin{cases} X_1 = \cos x_3\, \partial_{x_1} + \sin x_3\, \partial_{x_2} \\ X_2 = -\sin x_3\, \partial_{x_1} + \cos x_3\, \partial_{x_2} \\ X_3 = \partial_{x_3} \end{cases} \qquad \begin{cases} \nu_1 = \cos x_3\, dx_1 + \sin x_3\, dx_2 \\ \nu_2 = -\sin x_3\, dx_1 + \cos x_3\, dx_2 \\ \nu_3 = dx_3. \end{cases}$$

Let $Y_1 = \lambda X_2 = \lambda(-\sin x_3\, \partial_{x_1} + \cos x_3\, \partial_{x_2})$ and $Y_2 = \lambda X_3 = \lambda \partial_{x_3}$. The contact one-form ω is given by $\omega = -\frac{1}{\lambda^2}\nu_1 = -\frac{1}{\lambda^2}(\cos x_3\, dx_1 + \sin x_3\, dx_2)$ and has exterior derivative $d\omega = \frac{1}{\lambda^2}\, \nu_2 \wedge \nu_3 = -\frac{1}{\lambda^2} \sin x_3\, dx_1 \wedge dx_3 + \frac{1}{\lambda^2} \cos x_3\, dx_2 \wedge dx_3$. Accordingly the corresponding Reeb vector field is $Y_0 = -\lambda^2 X_1 = -\lambda^2 \cos x_3 \partial_{x_1} - \lambda^2 \sin x_3 \partial_{x_2}$. Hence we get

$$\begin{aligned} [Y_1, Y_0] &= 0Y_1 + 0Y_2 \\ [Y_2, Y_0] &= -\lambda^2 Y_1 + 0Y_2 \\ [Y_2, Y_1] &= 0Y_1 + 0Y_2 + Y_0 \end{aligned}$$

and so $\chi = \frac{\lambda^2}{2}$ and $\kappa = \frac{\lambda^2}{2}$. For $\lambda = 2^{1/4}$ we obtain normalized scalar invariants $\chi = \frac{1}{\sqrt{2}}$ and $\kappa = \frac{1}{\sqrt{2}}$.

(4). We note that the isotropy group $\mathsf{Iso}_\mathbf{1}(\widetilde{\mathsf{SE}}\,(2), \mathcal{D}, \mathbf{g})$ of a structure and the isotropy subgroup $\mathsf{Iso}_\mathbf{1}(\widetilde{\mathsf{SE}}\,(2), \mathcal{D}, \lambda\mathbf{g})$, $\lambda > 0$ of any rescaled structure are identical. We find it more convenient to take a Riemannian characteristic expansion of $(\widetilde{\mathsf{SE}}\,(2), \mathcal{D}, \frac{1}{4}\mathbf{g})$ than of $(\widetilde{\mathsf{SE}}\,(2), \mathcal{D}, \mathbf{g})$; in the former case it is enough to impose the conditions (3.1) and (3.2) to determine the linearized isotropy subgroup whereas in the latter case the condition $\phi_* \mathcal{D} = \mathcal{D}$ needs to be imposed as well.

The Riemannian characteristic expansion $(\widetilde{\mathsf{SE}}\,(2), \tilde{\mathbf{g}})$ of $(\widetilde{\mathsf{SE}}\,(2), \mathcal{D}, \frac{1}{4}\mathbf{g})$ has orthonormal frame $(2E_2^L, 2E_3^L, 4E_1^L)$. Consequently, we have

$$\tilde{\mathbf{g}}_\mathbf{1} = \begin{bmatrix} \frac{1}{16} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$$

with respect to (E_1, E_2, E_3). Suppose $\psi \in d\,\mathsf{Iso}_{\mathbf{1}}(\widetilde{\mathsf{SE}}\,(2), \mathcal{D}, \mathbf{g})$ and let $\phi \in \mathsf{Iso}_{\mathbf{1}}(\widetilde{\mathsf{SE}}\,(2), \mathcal{D}, \frac{1}{4}\mathbf{g})$ be the unique isometry such that $T_{\mathbf{1}}\phi = \psi$. As ψ preserves $\mathcal{D}(\mathbf{1})$ and $\tilde{\mathbf{g}}_{\mathbf{1}}$, it follows that

$$\psi = \begin{bmatrix} \sigma_1 \cos\theta & -\sigma_1 \sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & \sigma_2 \end{bmatrix}$$

with respect to the basis $(2E_2, 2E_3, 4E_1)$ for some $\theta \in \mathbb{R}$ and $\sigma_1, \sigma_2 \in \{-1, 1\}$; equivalently,

$$\psi = \begin{bmatrix} \sigma_2 & 0 & 0 \\ 0 & \sigma_1 \cos\theta & -\sigma_1 \sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

with respect to the basis (E_1, E_2, E_3). Furthermore, as ϕ is an isometry of $(\widetilde{\mathsf{SE}}\,(2), \tilde{\mathbf{g}})$ (Proposition 3.5), we have that ψ preserves the associated curvature tensor R and its covariant derivative ∇R (see (3.1) and (3.2)). Hence,

$$\begin{aligned} \psi \cdot R(E_1, E_2, E_1) - R(\psi \cdot E_1, \psi \cdot E_2, \psi \cdot E_1) &= 0 \\ \tfrac{3}{4} \sin\theta E_3 &= 0 \end{aligned}$$

and so $\sin\theta = 0$ and $\cos\theta = \pm 1$. Therefore, we have $\psi = \operatorname{diag}(\sigma_1, \sigma_2, \sigma_3)$ for some $\sigma_1, \sigma_2, \sigma_3 \in \{-1, 1\}$. Hence

$$\begin{aligned} \psi \cdot \nabla R(E_1, E_1, E_2, E_1) - \nabla R(\psi \cdot E_1, \psi \cdot E_1, \psi \cdot E_2, \psi \cdot E_1) &= 0 \\ -\tfrac{9}{32}\left(\sigma_1^3 \sigma_2 - \sigma_3\right) &= 0 \end{aligned}$$

which implies that $\sigma_3 = \sigma_1 \sigma_2$ and so $\psi = \operatorname{diag}(\sigma_1, \sigma_2, \sigma_1 \sigma_2)$. Thus $\psi \in d\,\mathfrak{L}\text{-}\mathsf{Iso}(\widetilde{\mathsf{SE}}\,(2), \mathcal{D}, \mathbf{g})$ and therefore $d\,\mathsf{Iso}_{\mathbf{1}}(\widetilde{\mathsf{SE}}\,(2), \mathcal{D}, \frac{1}{4}\mathbf{g}) = d\,\mathsf{Iso}_{\mathbf{1}}(\widetilde{\mathsf{SE}}\,(2), \mathcal{D}, \mathbf{g}) = d\,\mathfrak{L}\text{-}\mathsf{Iso}(\widetilde{\mathsf{SE}}\,(2), \mathcal{D}, \mathbf{g})$. □

3.2.7 Type $\mathfrak{g}_{3.5}^{\alpha}$

The nonzero commutators of $\mathfrak{g}_{3.5}^{\alpha}$ are $[E_2, E_3] = E_1 - \alpha\, E_2$ and $[E_3, E_1] = \alpha\, E_1 + E_2$; the corresponding simply connected Lie group $\mathsf{G}_{3.5}^{\alpha}$ is diffeomorphic to $\mathbb{R}^3$.

Proposition 3.15. *Any sub-Riemannian structure on the group $\mathsf{G}_{3.5}^{\alpha}$ is $\mathfrak{L}$-isometric up to rescaling to the structure $(\mathsf{G}_{3.5}^{\alpha}, \mathcal{D}, \mathbf{g})$ given by $\mathcal{D}(\mathbf{1}) = \langle E_2, E_3 \rangle$ and $\mathbf{g}_{\mathbf{1}} = \operatorname{diag}(1, 1)$, i.e., with orthonormal frame (E_2^L, E_3^L).*

1. *The group of linearized $\mathfrak{L}$-isometries is given by*

$$d\,\mathfrak{L}\text{-}\mathsf{Iso}(\mathsf{G}_{3.5}^{\alpha}, \mathcal{D}, \mathbf{g}) = \left\{ \begin{bmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & 1 \end{bmatrix} \; : \; \sigma = \pm 1 \right\}.$$

2. *The associated Hamiltonian system on $(\mathfrak{g}_{3.5}^{\alpha})_{-}^{*}$ is given by $H(p) = \frac{1}{2}(p_2^2 + p_3^2)$ and is L-equivalent to $\mathsf{Np}(8)_{\beta=0}$.*

3. *The Reeb vector field corresponding to* $(\mathsf{G}_{3.5}^{\alpha}, \mathcal{D}, \mathbf{g})$ *is* $\pm(E_1^L + \alpha E_2^L)$. *The normalized scalar invariants are given by* $\chi = \frac{1+\alpha^2}{\sqrt{2-12\alpha^2+50\alpha^4}}$ *and* $\kappa = \frac{1-7\alpha^2}{\sqrt{2-12\alpha^2+50\alpha^4}}$.
4. *The subgroup of linearized isotropies* $d\,\mathsf{Iso}_{\mathbf{1}}(\mathsf{G}_{3.5}^{\alpha}, \mathcal{D}, \mathbf{g})$ *is identical to* $d\,\mathfrak{L}\text{-}\mathsf{Iso}(\mathsf{G}_{3.5}^{\alpha}, \mathcal{D}, \mathbf{g})$.

Proof. Any full-rank subspace of $\mathfrak{g}_{3.5}^{\alpha}$ is the image under some automorphism of the full-rank subspace $\Gamma = \langle E_2, E_3 \rangle$ ([36]). The subgroup of automorphisms preserving Γ is given by

$$\mathsf{Aut}_{\Gamma}(\mathfrak{g}_{3.5}^{\alpha}) = \left\{ \begin{bmatrix} x & 0 & 0 \\ 0 & x & v \\ 0 & 0 & 1 \end{bmatrix} \; : \; x, v \in \mathbb{R},\, x \neq 0 \right\}.$$

The restriction to Γ of the subgroup of automorphisms preserving Γ is

$$\mathsf{Aut}(\mathfrak{g}_{3.5}^{\alpha})\big|_{\Gamma} = \left\{ \begin{bmatrix} x & v \\ 0 & 1 \end{bmatrix} \; : \; x, v \in \mathbb{R},\, x \neq 0 \right\}.$$

As in the proof of Proposition 3.9, it then follows that any metric subspace of $\mathfrak{g}_{3.5}^{\alpha}$ is equivalent to the one associated with $(\mathcal{D}, \mathbf{g})$. Consequently, by Lemma 3.8, the result follows.

(1 and 2). A direct computation yields the subgroup of linearized $\mathfrak{L}$-isometries and the associated Hamiltonian system; $((\mathfrak{g}_{3.5}^{\alpha})_{-}^{*}, H)$ is L-equivalent to $\mathsf{Np}(8)_{\beta=0}$ (see Proposition 1.25, item 2).

(3). The pull back (X_1, X_2, X_3) of the frame (E_1^L, E_2^L, E_3^L) to $\mathbb{R}^3$ and its corresponding dual frame is given by

$$\begin{cases} X_1 = \mathrm{e}^{\alpha x_3} \cos x_3\, \partial_{x_1} + \mathrm{e}^{\alpha x_3} \sin x_3\, \partial_{x_2} \\ X_2 = -\,\mathrm{e}^{\alpha x_3} \sin x_3\, \partial_{x_1} + \mathrm{e}^{\alpha x_3} \cos x_3\, \partial_{x_2} \\ X_3 = \partial_{x_3} \end{cases} \qquad \begin{cases} \nu_1 = \mathrm{e}^{-\alpha x_3} \cos x_3\, dx_1 + \mathrm{e}^{-\alpha x_3} \sin x_3\, dx_2 \\ \nu_2 = -\,\mathrm{e}^{-\alpha x_3} \sin x_3\, dx_1 + \mathrm{e}^{-\alpha x_3} \cos x_3\, dx_2 \\ \nu_3 = dx_3. \end{cases}$$

Let $Y_1 = \lambda X_2 = \lambda(-\,\mathrm{e}^{\alpha x_3} \sin x_3\, \partial_{x_1} + \mathrm{e}^{\alpha x_3} \cos x_3\, \partial_{x_2})$ and $Y_2 = \lambda X_3 = \lambda \partial_{x_3}$. The contact one-form ω is given by $\omega = -\frac{1}{\lambda^2}\nu_1 = -\frac{1}{\lambda^2}(\mathrm{e}^{-\alpha x_3} \cos x_3\, dx_1 + \mathrm{e}^{-\alpha x_3} \sin x_3\, dx_2)$ and has exterior derivative $d\omega = \frac{1}{\lambda^2}\, \nu_2 \wedge \nu_3 + \frac{\alpha}{\lambda^2}\, \nu_3 \wedge \nu_1 = -\frac{1}{\lambda^2}\, \mathrm{e}^{-\alpha x_3}(\sin x_3 + \alpha \cos x_3)\, dx_1 \wedge dx_3 + \frac{1}{\lambda^2}\, \mathrm{e}^{-\alpha x_3}(\cos x_3 - \alpha \sin x_3)\, dx_2 \wedge dx_3$. Accordingly the corresponding Reeb vector field is $Y_0 = -\lambda^2 X_1 - \alpha\lambda^2 X_2 = -\lambda^2\, \mathrm{e}^{\alpha x_3}(\cos x_3 - \alpha \sin x_3)\partial_{x_1} + \lambda^2\, \mathrm{e}^{\alpha x_3}(\sin x_3 + \alpha \cos x_3)\partial_{x_2}$. Hence we get

$$\begin{aligned} [Y_1, Y_0] &= 0Y_1 + 0Y_2 \\ [Y_2, Y_0] &= -\lambda^2\left(1 + \alpha^2\right) Y_1 + 0Y_2 \\ [Y_2, Y_1] &= 2\alpha\lambda Y_1 + 0Y_2 + Y_0 \end{aligned}$$

and so $\chi = \frac{1}{2}\lambda^2\left(1 + \alpha^2\right)$ and $\kappa = \frac{1}{2}\lambda^2\left(1 - 7\alpha^2\right)$. For $\lambda = \frac{2^{1/4}}{(1-6\alpha^2+25\alpha^4)^{1/4}}$ we obtain normalized scalar invariants $\chi = \frac{1+\alpha^2}{\sqrt{2-12\alpha^2+50\alpha^4}}$ and $\kappa = \frac{1-7\alpha^2}{\sqrt{2-12\alpha^2+50\alpha^4}}$.

(4). The Riemannian characteristic expansion $(\mathsf{G}_{3.5}^{\alpha}, \tilde{\mathbf{g}})$ of $(\mathsf{G}_{3.5}^{\alpha}, \mathcal{D}, \mathbf{g})$ has orthonormal frame $(E_2^L, E_3^L, E_1^L + \alpha E_2^L)$. Consequently, we have

$$\tilde{\mathbf{g}}_{\mathbf{1}} = \begin{bmatrix} 1 + \alpha^2 & -\alpha & 0 \\ -\alpha & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

with respect to the basis (E_1, E_2, E_3). Suppose $\psi \in d\,\mathsf{Iso}_{\mathbf{1}}(\mathsf{G}_{3.5}^{\alpha}, \mathcal{D}, \mathbf{g})$ and let $\phi \in \mathsf{Iso}_{\mathbf{1}}(\mathsf{G}_{3.5}^{\alpha}, \mathcal{D}, \mathbf{g})$ be the unique isometry such that $T_{\mathbf{1}}\phi = \psi$. As ψ preserves $\mathcal{D}(\mathbf{1})$ and $\tilde{\mathbf{g}}_{\mathbf{1}}$, it follows that

$$\psi = \begin{bmatrix} \sigma_1 \cos\theta & -\sigma_1 \sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & \sigma_2 \end{bmatrix}$$

with respect to the basis $(E_2, E_3, E_1 + \alpha E_2)$ for some $\theta \in \mathbb{R}$ and $\sigma_1, \sigma_2 \in \{-1, 1\}$; equivalently,

$$\psi = \begin{bmatrix} \sigma_2 & 0 & 0 \\ \sigma_2\alpha - \sigma_1\alpha\cos\theta & \sigma_1 \cos\theta & -\sigma_1 \sin\theta \\ -\alpha\sin\theta & \sin\theta & \cos\theta \end{bmatrix}$$

with respect to the basis (E_1, E_2, E_3). Furthermore, as ϕ is an isometry of $(\mathsf{G}_{3.5}^{\alpha}, \tilde{\mathbf{g}})$ (Proposition 3.5), we have that ψ preserves the associated curvature tensor R and its covariant derivative ∇R (see (3.1) and (3.2)). Hence, we have

$$\begin{aligned} \psi \cdot R(E_3, E_2, E_2) - R(\psi \cdot E_3, \psi \cdot E_2, \psi \cdot E_2) = 0 \\ 2\alpha(1+\alpha^2)\sigma_1 \sin\theta E_1 + 2\alpha^2(1+\alpha^2)\sigma_1 \sin\theta E_2 = 0 \end{aligned}$$

and

$$\begin{aligned} \psi \cdot R(E_3, E_2, E_3) - R(\psi \cdot E_3, \psi \cdot E_2, \psi \cdot E_3) = 0 \\ -2\alpha(1+\alpha^2)(\sigma_2 - \sigma_1\cos\theta)E_1 - 2\alpha^2(1+\alpha^2)(\sigma_2 - \sigma_1\cos\theta)E_2 = 0. \end{aligned}$$

Thus $\sin\theta = 0$ and $\cos\theta = \sigma_1\sigma_2$. Therefore, we have $\psi = \operatorname{diag}(\sigma_2, \sigma_2, \sigma_1\sigma_2)$. Hence

$$\begin{aligned} \psi \cdot \nabla R(E_1, E_1, E_2, E_1) - \nabla R(\psi \cdot E_1, \psi \cdot E_1, \psi \cdot E_2, \psi \cdot E_1) = 0 \\ \tfrac{1}{2}\alpha^4 \left(5 + 2\alpha^2\right) \sigma_2 \left(\sigma_1 - \sigma_2^3\right) E_3 = 0 \end{aligned}$$

which implies that $\sigma_1 = \sigma_2$ and so $\psi = \operatorname{diag}(\sigma_1, \sigma_1, 1)$. Thus $\psi \in d\,\mathfrak{L}\text{-}\mathsf{Iso}(\mathsf{G}_{3.5}^{\alpha}, \mathcal{D}, \mathbf{g})$ and therefore $d\,\mathsf{Iso}_{\mathbf{1}}(\mathsf{G}_{3.5}^{\alpha}, \mathcal{D}, \mathbf{g}) = d\,\mathfrak{L}\text{-}\mathsf{Iso}(\mathsf{G}_{3.5}^{\alpha}, \mathcal{D}, \mathbf{g})$. □

3.3 Structures on semisimple groups

Next we treat those structures on semisimple groups. Table B.8 in Appendix B.2 contains a summary (listing the normal forms of sub-Riemannian structures, the corresponding linearized isotropy subgroups, and the corresponding normal forms of the associated Hamilton-Poisson systems).

3.3.1 Type $\mathfrak{g}_{3.6}$

The pseudo-orthogonal Lie algebra $\mathfrak{g}_{3.6}$ has nonzero commutators $[E_2, E_3] = E_1$, $[E_3, E_1] = E_2$, $[E_1, E_2] = -E_3$; the corresponding simply connected Lie group $\tilde{\mathsf{A}}$ is diffeomorphic to $\mathbb{R}^3$.

Proposition 3.16. *Any left-invariant sub-Riemannian structure on* $\tilde{\mathsf{A}} \cong \widetilde{\mathsf{SL}}(2)$ *is* $\mathfrak{L}$*-isometric up to rescaling to exactly one of the structures*

$$\begin{array}{lllll} (\tilde{\mathsf{A}}, \mathcal{D}_1, \mathbf{g}^{1,\alpha}) & : & \mathcal{D}_1(\mathbf{1}) = \langle E_1, E_2 \rangle, & \mathbf{g}_{\mathbf{1}}^{1,\alpha} = \operatorname{diag}(\alpha, 1), & 0 < \alpha \leq 1 \\ (\tilde{\mathsf{A}}, \mathcal{D}_2, \mathbf{g}^{2,\alpha}) & : & \mathcal{D}_2(\mathbf{1}) = \langle E_2, E_3 \rangle, & \mathbf{g}_{\mathbf{1}}^{2,\alpha} = \operatorname{diag}(\alpha, 1), & 0 < \alpha \end{array}$$

i.e., with orthonormal frame $(\frac{1}{\sqrt{\alpha}}E_1^L, E_2^L)$ *or* $(\frac{1}{\sqrt{\alpha}}E_2^L, E_3^L)$ *respectively.*

1. *The respective groups of linearized* $\mathfrak{L}$*-isometries are given by*

$$d\,\mathfrak{L}\text{-}\mathsf{Iso}(\tilde{\mathsf{A}}, \mathcal{D}_1, \mathbf{g}^{1,\alpha}) = \left\{ \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_1\sigma_2 \end{bmatrix} : \sigma_1, \sigma_2 = \pm 1 \right\} \qquad \text{if} \quad 0 < \alpha < 1$$

$$d\,\mathfrak{L}\text{-}\mathsf{Iso}(\tilde{\mathsf{A}}, \mathcal{D}_1, \mathbf{g}^{1,\alpha}) = \left\{ \begin{bmatrix} \multicolumn{2}{c}{\multirow{2}{*}{g}} & 0 \\ & & 0 \\ 0 & 0 & \det g \end{bmatrix} : g \in \mathsf{O}(2) \right\} \qquad \text{if} \quad \alpha = 1$$

$$d\,\mathfrak{L}\text{-}\mathsf{Iso}(\tilde{\mathsf{A}}, \mathcal{D}_2, \mathbf{g}^{2,\alpha}) = \left\{ \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_1\sigma_2 \end{bmatrix} : \sigma_1, \sigma_2 = \pm 1 \right\}.$$

2. *The respective associated Hamiltonian systems (and their normal forms) are given by*

$$\mathfrak{H}(\tilde{\mathsf{A}}, \mathcal{D}_1, \mathbf{g}^{1,\alpha}) = \left((\mathfrak{g}_{3.6})^*_-, \tfrac{1}{2}(\tfrac{1}{\alpha}p_1^2 + p_2^2)\right) \qquad \text{normal form: } \begin{cases} \mathsf{Np}(7) & \text{if } 0 < \alpha < 1 \\ \mathsf{P}(8) & \text{if } \alpha = 1 \end{cases}$$

$$\mathfrak{H}(\tilde{\mathsf{A}}, \mathcal{D}_2, \mathbf{g}^{2,\alpha}) = \left((\mathfrak{g}_{3.6})^*_-, \tfrac{1}{2}(\tfrac{1}{\alpha}p_2^2 + p_3^2)\right) \qquad \text{normal form: } \mathsf{Np}(7).$$

3. *The respective Reeb vector fields and normalized scalar invariants are given by*

$$\begin{array}{llll} (\tilde{\mathsf{A}}, \mathcal{D}_1, \mathbf{g}^{1,\alpha}) : & \pm\frac{1}{\sqrt{\alpha}}E_3^L & \chi = \dfrac{1-\alpha}{\sqrt{2}\sqrt{1+\alpha^2}}, & \kappa = -\dfrac{1+\alpha}{\sqrt{2}\sqrt{1+\alpha^2}} \\ (\tilde{\mathsf{A}}, \mathcal{D}_2, \mathbf{g}^{2,\alpha}) : & \pm\frac{1}{\sqrt{\alpha}}E_1^L & \chi = \dfrac{1+\alpha}{\sqrt{2}\sqrt{1+\alpha^2}}, & \kappa = -\dfrac{1-\alpha}{\sqrt{2}\sqrt{1+\alpha^2}}. \end{array}$$

4. *Each respective group of linearized isotropies* $d\,\mathsf{Iso}_{\mathbf{1}}(\tilde{\mathsf{A}}, \mathcal{D}, \mathbf{g})$ *is identical to the corresponding group of linearized* $\mathfrak{L}$*-isometries* $d\,\mathfrak{L}\text{-}\mathsf{Iso}(\tilde{\mathsf{A}}, \mathcal{D}, \mathbf{g})$.

Proof. Any full-rank subspace of $\mathfrak{g}_{3.6}$ is equivalent to exactly one of the full-rank subspaces $\Gamma_1 = \langle E_1, E_2 \rangle$ and $\Gamma_2 = \langle E_2, E_3 \rangle$ ([34]). We consider first the subspace Γ_1. The automorphisms $\psi \in \mathsf{Aut}(\mathfrak{g}_{3.6})$ are exactly those linear transformations ψ which have $\det \psi = 1$ and preserve the Lorentzian product $A \odot B = a_1b_1 + a_2b_2 - a_3b_3$ (here $A = a_1E_1 + a_2E_2 + a_3E_3$ and $B = b_1E_1 + b_2E_2 + b_3E_3$), i.e., $(\psi \cdot A) \odot (\psi \cdot B) = A \odot B$ for $A, B \in \mathfrak{g}_{3.6}$. Hence an automorphism preserves $\Gamma_1 = \langle E_1, E_2 \rangle$ if and only if it preserves its orthogonal complement $\Gamma_1^\perp = \langle E_3 \rangle$. Consequently, as ψ preserves both $\langle E_1, E_2 \rangle$ and $\langle E_3 \rangle$, the subgroup of automorphism preserving Γ_1 is given by (cf. [40])

$$\mathsf{Aut}_{\Gamma_1}(\mathfrak{g}_{3.6}) = \left\{ \begin{bmatrix} \multicolumn{2}{c}{\multirow{2}{*}{g}} & 0 \\ & & 0 \\ 0 & 0 & \det g \end{bmatrix} : g \in \mathsf{O}(2) \right\}.$$

The restriction to Γ_1 of the subgroup of automorphisms preserving Γ_1 is $\mathsf{Aut}(\mathfrak{g}_{3.6})\big|_{\Gamma_1} = \mathsf{O}\,(2)$. Let $\boldsymbol{\mu}$ be a positive definite quadratic form on Γ_1. There exists an orthogonal transformation $\psi \in \mathsf{Aut}(\mathfrak{g}_{3.6})\big|_{\Gamma_1}$ such that $\boldsymbol{\mu} \circ \psi = \psi^\top \boldsymbol{\mu} \psi = \mathrm{diag}(\lambda_1, \lambda_2) = \lambda_2\,\mathrm{diag}(\frac{\lambda_1}{\lambda_2}, 1)$ with $\lambda_2 \geq \lambda_1 > 0$. It then follows that any metric subspace $(\mathfrak{g}_{3.6}, \Gamma, \boldsymbol{\mu})$ with Γ equivalent to Γ_1 is equivalent to the metric subspace associated with $(\mathcal{D}_1, \mathbf{g}^{1,\alpha})$, $0 < \alpha \leq 1$. Moreover, the metric subspaces $(\Gamma_1, \boldsymbol{\mu}^\alpha)$, $\boldsymbol{\mu}^\alpha = \mathrm{diag}(\alpha, 1)$, $0 < \alpha \leq 1$ and $(\Gamma_1, \boldsymbol{\mu}^{\alpha'})$, $0 < \alpha' \leq 1$ are equivalent only if $\alpha = \alpha'$. Indeed, if $\psi^\top \boldsymbol{\mu}^\alpha \psi = r\boldsymbol{\mu}^{\alpha'}$ for some $r > 0$ and $\psi \in \mathsf{Aut}(\mathfrak{g}_{3.6})\big|_{\Gamma_1} = \mathsf{O}\,(2)$ then the eigenvalues of $\boldsymbol{\mu}^\alpha$ must coincide with those of $r\boldsymbol{\mu}^{\alpha'}$, i.e., $\{\alpha, 1\} = \{r\alpha', r\}$ and so $\alpha = \alpha'$.

We now consider the subspace $\Gamma_2 = \langle E_2, E_3\rangle$. An automorphism preserves Γ_2 if and only if it preserves its orthogonal complement $\Gamma_2^\perp = \langle E_1\rangle$. Consequently, the subgroup of automorphism preserving Γ_2 is given by (cf. [40])

$$\mathsf{Aut}_{\Gamma_1}(\mathfrak{g}_{3.6}) = \left\{ \begin{bmatrix} \det g & 0 & 0 \\ 0 & & \\ 0 & \multicolumn{2}{c}{g} \end{bmatrix} \; : \; g \in \mathsf{O}\,(1,1) \right\}$$

where $\mathsf{O}\,(1,1) = \{g \in \mathbb{R}^{2\times 2} \; : \; g^\top J' g = J'\}$, $J' = \mathrm{diag}(1, -1)$. Accordingly, the restriction to Γ_2 of the subgroup of automorphisms preserving Γ_2 is $\mathsf{Aut}(\mathfrak{g}_{3.6})\big|_{\Gamma_2} = \mathsf{O}\,(1,1)$. Let $\boldsymbol{\mu}$ be a positive definite quadratic form on Γ_2. There exists $\psi \in \mathsf{O}\,(1,1)$ such that $\boldsymbol{\mu} \circ \psi = \psi^\top \boldsymbol{\mu} \psi = \mathrm{diag}(\lambda_1, \lambda_2) = \lambda_2\,\mathrm{diag}(\frac{\lambda_1}{\lambda_2}, 1)$ with $\lambda_1, \lambda_2 > 0$ (see e.g., [118]). It then follows that any metric subspace $(\mathfrak{g}_{3.6}, \Gamma, \boldsymbol{\mu})$ with Γ equivalent to Γ_2 is equivalent to the one associated with $(\mathcal{D}_2, \mathbf{g}^{2,\alpha})$, $\alpha > 0$. Moreover, the metric subspaces $(\Gamma_2, \boldsymbol{\mu}^\alpha)$, $\boldsymbol{\mu}^\alpha = \mathrm{diag}(\alpha, 1)$, $0 < \alpha$ and $(\Gamma_2, \boldsymbol{\mu}^{\alpha'})$, $0 < \alpha'$ are equivalent only if $\alpha = \alpha'$. Indeed, if $\psi^\top \boldsymbol{\mu}^\alpha \psi = r\boldsymbol{\mu}^{\alpha'}$ for some $r > 0$ and $\psi \in \mathsf{Aut}(\mathfrak{g}_{3.6})\big|_{\Gamma_2} = \mathsf{O}\,(1,1)$ then $\psi^{-1} J'\,\mathrm{diag}(\alpha, 1)\psi = rJ'\,\mathrm{diag}(\alpha', 1)$ and so the eigenvalues of $J'\,\mathrm{diag}(\alpha, 1)$ must coincide with those of $rJ'\,\mathrm{diag}(\alpha', 1)$, i.e., $\{\alpha, -1\} = \{r\alpha', -r\}$ and so $\alpha = \alpha'$. Consequently, as any metric subspace is equivalent to exactly one of the metric subspaces associated with $(\mathcal{D}_1, \mathbf{g}^{1,\alpha})$ and $(\mathcal{D}_2, \mathbf{g}^{2,\alpha})$, the result follows (by Lemma 3.8).

(1 and 2). It is straightforward to calculate the subgroups of linearized $\mathfrak{L}$-isometries (given $\mathsf{Aut}(\mathfrak{g}_{3.6})\big|_{\Gamma_1}$ and $\mathsf{Aut}(\mathfrak{g}_{3.6})\big|_{\Gamma_2}$). We have $\mathfrak{H}(\tilde{\mathsf{A}}, \mathcal{D}_1, \mathbf{g}^{1,\alpha}) = ((\mathfrak{g}_{3.6})^*_-, H)$, $H(p) = \frac{1}{2}(\frac{1}{\alpha}p_1^2 + p_2^2)$. When $\alpha = 1$, then $((\mathfrak{g}_{3.6})^*_-, H)$ is L-equivalent to $((\mathfrak{g}_{3.6})^*_-, p_3^2)$ (see Proposition 1.25, items 2 and 3); the system $((\mathfrak{g}_{3.6})^*_-, p_3^2)$ in turn is L-equivalent to $\mathsf{P}(8)$ by Proposition 2.14. When $0 < \alpha < 1$, then the corresponding Hamiltonian vector field $\vec{H} = (p_2p_3, -\frac{1}{\alpha}p_1p_3, \frac{\alpha-1}{\alpha}p_1p_2)$ has equilibria $(\mu, 0, 0)$, $(0, \mu, 0)$, $(0, 0, \mu)$, $\mu \in \mathbb{R}$ and thus equilibrium index $(3, 0)$ (see Proposition 2.20). Consequently, we have that $((\mathfrak{g}_{3.6})^*_-, H)$ is L-equivalent to $\mathsf{Np}(7)$ if $0 < \alpha < 1$ (see Table B.5). On the other hand, we have $\mathfrak{H}(\tilde{\mathsf{A}}, \mathcal{D}_2, \mathbf{g}^{2,\alpha}) = ((\mathfrak{g}_{3.6})^*_-, H)$, $H(p) = \frac{1}{2}(\frac{1}{\alpha}p_2^2 + p_3^2)$. The corresponding Hamiltonian vector field $\vec{H} = (\frac{1+\alpha}{\alpha}p_2p_3, -p_1p_3, \frac{1}{\alpha}p_1p_2)$ has equilibria $(\mu, 0, 0)$, $(0, \mu, 0)$, $(0, 0, \mu)$, $\mu \in \mathbb{R}$ and thus equilibrium index $(3, 0)$. Consequently $((\mathfrak{g}_{3.6})^*_-, H)$ is L-equivalent to $\mathsf{Np}(7)$ (see Table B.5)

(3). Locally (i.e., between some neighbourhoods of the origin and identity), the pull back

(X_1, X_2, X_3) of the frame (E_1^L, E_2^L, E_3^L) to $\mathbb{R}^3$ and its corresponding dual frame is given by

$$\begin{cases} X_1 = \operatorname{sech} x_2 \cos x_3\, \partial_{x_1} + \sin x_3 \partial_{x_2} + \tanh x_2 \cos x_3 \partial_{x_3} \\ X_2 = -\operatorname{sech} x_2 \sin x_3\, \partial_{x_1} + \cos x_3 \partial_{x_2} - \tanh x_2 \sin x_3 \partial_{x_3} \\ X_3 = \partial_{x_3} \end{cases}$$
$$\begin{cases} \nu_1 = \cosh x_2 \cos x_3\, dx_1 + \sin x_3\, dx_2 \\ \nu_2 = -\cosh x_2 \sin x_3\, dx_1 + \cos x_3\, dx_2 \\ \nu_3 = -\sinh x_2\, dx_1 + dx_3. \end{cases}$$

As we need only realize the frame (X_1, X_2, X_3) locally, it sufficed to consider (locally) the pull back of the corresponding frame (E_1^L, E_2^L, E_3^L) for $\mathsf{SO}(2,1)_0$ by $m : \mathbb{R}^3 \to \mathsf{SO}(2,1)_0$, $(x_1, x_2, x_3) \mapsto \exp(x_1 E_1)\exp(x_2 E_2)\exp(x_3 E_3)$. (One can show that $\tilde{\mathsf{A}}$ is diffeomorphic to $\mathbb{R}^3$ by means of its Cartan decomposition, see e.g., [59]; hence it is possible to realize a frame (X_1, X_2, X_3) globally with respect to some diffeomorphism $\tilde{m} : \mathbb{R}^3 \to \tilde{\mathcal{A}}$. However, we find the above frame preferable.)

We consider first $(\tilde{\mathsf{A}}, \mathcal{D}_1, \mathbf{g}^{1,\alpha})$. Let $Y_1 = \frac{\lambda}{\sqrt{\alpha}} X_1 = \frac{\lambda}{\sqrt{\alpha}} \cos x_3 \operatorname{sech} x_2 \partial_{x_1} + \frac{\lambda}{\sqrt{\alpha}} \sin x_3 \partial_{x_2} + \frac{\lambda}{\sqrt{\alpha}} \cos x_3 \tanh x_2 \partial_{x_3}$ and $Y_2 = \lambda X_2 = -\lambda \operatorname{sech} x_2 \sin x_3 \partial_{x_1} + \lambda \cos x_3 \partial_{x_2} - \lambda \sin x_3 \tanh x_2 \partial_{x_3}$. The contact one-form ω is given by $\omega = \frac{\sqrt{\alpha}}{\lambda^2} \nu_3 = \frac{\sqrt{\alpha}}{\lambda^2}(-\sinh x_2\, dx_1 + dx_3)$ and has exterior derivative $d\omega = \frac{\sqrt{\alpha}}{\lambda^2}\, \nu_1 \wedge \nu_2 = \frac{\sqrt{\alpha}}{\lambda^2} \cosh x_2\, dx_1 \wedge dx_2$. Accordingly the corresponding Reeb vector field is $Y_0 = \frac{\lambda^2}{\sqrt{\alpha}} X_3 = \frac{\lambda^2}{\sqrt{\alpha}} \partial_{x_3}$. Hence we get

$$\begin{aligned} [Y_1, Y_0] &= 0Y_1 + -\tfrac{\lambda^2}{\alpha} Y_2 \\ [Y_2, Y_0] &= \lambda^2 Y_1 + 0Y_2 \\ [Y_2, Y_1] &= 0Y_1 + 0Y_2 + Y_0 \end{aligned}$$

and so $\chi = \frac{\lambda^2}{2\alpha}(1-\alpha)$ and $\kappa = -\frac{\lambda^2}{2\alpha}(1+\alpha)$. For $\lambda = \frac{2^{1/4}}{\left(1+\frac{1}{\alpha^2}\right)^{1/4}}$ we obtain normalized scalar invariants $\chi = \frac{|1-\alpha|}{\sqrt{2}\sqrt{1+\alpha^2}}$ and $\kappa = -\frac{1+\alpha}{\sqrt{2}\sqrt{1+\alpha^2}}$.

Next we consider $(\tilde{\mathsf{A}}, \mathcal{D}_2, \mathbf{g}^{2,\alpha})$. Let $Y_1 = \frac{\lambda}{\sqrt{\alpha}} X_2 = -\frac{\lambda}{\sqrt{\alpha}} \operatorname{sech} x_2 \sin x_3\, \partial_{x_1} + \frac{\lambda}{\sqrt{\alpha}} \cos x_3\, \partial_{x_2} - \frac{\lambda}{\sqrt{\alpha}} \sin x_3 \tanh x_2\, \partial_{x_3}$ and $Y_2 = \lambda X_3 = \lambda \partial_{x_3}$. The contact one-form ω is given by $\omega = -\frac{\sqrt{\alpha}}{\lambda^2} \nu_1 = -\frac{\sqrt{\alpha}}{\lambda^2}(\cos x_3 \cosh x_2\, dx_1 + \sin x_3\, dx_2)$ and has exterior derivative $d\omega = \frac{\sqrt{\alpha}}{\lambda^2}\, \nu_2 \wedge \nu_3 = \frac{\sqrt{\alpha}}{\lambda^2} \cos x_3 \sinh x_2\, dx_1 \wedge dx_2 + \frac{\sqrt{\alpha}}{\lambda^2} \cos x_3\, dx_2 \wedge dx_3 - \frac{\sqrt{\alpha}}{\lambda^2} \cosh x_2 \sin x_3\, dx_1 \wedge dx_3$. Accordingly the corresponding Reeb vector field is $Y_0 = -\frac{\lambda^2}{\sqrt{\alpha}} X_1 = -\frac{\lambda^2}{\sqrt{\alpha}} \cos x_3 \operatorname{sech} x_2\, \partial_{x_1} - \frac{\lambda^2}{\sqrt{\alpha}} \sin x_3\, \partial_{x_3} - \frac{\lambda^2}{\sqrt{\alpha}} \cos x_3 \tanh x_2\, \partial_{x_3}$. Hence we get

$$\begin{aligned} [Y_1, Y_0] &= 0Y_1 - \tfrac{\lambda^2}{\alpha} Y_2 \\ [Y_2, Y_0] &= -\lambda^2 Y_1 + 0Y_2 \\ [Y_2, Y_1] &= 0Y_1 + 0Y_2 + Y_0 \end{aligned}$$

and so $\chi = \frac{(1+\alpha)\lambda^2}{2\alpha}$ and $\kappa = \frac{(-1+\alpha)\lambda^2}{2\alpha}$. For $\lambda = \frac{2^{1/4}}{\left(1+\frac{1}{\alpha^2}\right)^{1/4}}$ we obtain normalized scalar invariants $\chi = \frac{1+\alpha}{\sqrt{2}\sqrt{1+\alpha^2}}$ and $\kappa = -\frac{1-\alpha}{\sqrt{2}\sqrt{1+\alpha^2}}$.

(4). We consider first $(\tilde{\mathsf{A}}, \mathcal{D}_1, \mathbf{g}^{1,\alpha})$. The Riemannian characteristic expansion $(\tilde{\mathsf{A}}, \tilde{\mathbf{g}})$ of $(\tilde{\mathsf{A}}, \mathcal{D}_1, \mathbf{g}^{1,\alpha})$ has orthonormal frame $(\frac{1}{\sqrt{\alpha}}E_1^L, E_2^L, \frac{1}{\sqrt{\alpha}}E_3^L)$. Consequently, we have $\tilde{\mathbf{g}}_\mathbf{1} = \operatorname{diag}(\alpha, 1, \alpha)$ with respect to the basis (E_1, E_2, E_3). Suppose $\psi \in d\,\mathsf{Iso}_\mathbf{1}(\tilde{\mathsf{A}}, \mathcal{D}_1, \mathbf{g}^{1,\alpha})$ and let $\phi \in \mathsf{Iso}_\mathbf{1}(\tilde{\mathsf{A}}, \mathcal{D}_1, \mathbf{g}^{1,\alpha})$ be the unique isometry such that $T_\mathbf{1}\phi = \psi$. As ψ preserves $\mathcal{D}(\mathbf{1})$ and $\tilde{\mathbf{g}}_\mathbf{1}$, it follows that

$$\psi = \begin{bmatrix} \sigma_1 \cos\theta & -\sigma_1 \sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & \sigma_2 \end{bmatrix}$$

with respect to the basis $(\frac{1}{\sqrt{\alpha}}E_1, E_2, \frac{1}{\sqrt{\alpha}}E_3)$ for some $\theta \in \mathbb{R}$ and $\sigma_1, \sigma_2 \in \{-1, 1\}$; equivalently,

$$\psi = \begin{bmatrix} \sigma_1 \cos\theta & -\frac{\sigma_1 \sin\theta}{\sqrt{\alpha}} & 0 \\ \sqrt{\alpha}\sin\theta & \cos\theta & 0 \\ 0 & 0 & \sigma_2 \end{bmatrix}$$

with respect to the basis (E_1, E_2, E_3). Furthermore, as ϕ is an isometry of $(\tilde{\mathsf{A}}, \tilde{\mathbf{g}})$ (Proposition 3.5), we have that ψ preserves the associated curvature tensor R and its covariant derivative ∇R (see (3.1) and (3.2)). First, suppose $\alpha = 1$. Then we have

$$\begin{aligned} \psi \cdot \nabla R(E_1, E_2, E_1, E_1) - \nabla R(\psi \cdot E_1, \psi \cdot E_2, \psi \cdot E_1, \psi \cdot E_1) &= 0 \\ (\sigma_1 - \sigma_2)E_3 &= 0 \end{aligned}$$

which implies that $\sigma_1 = \sigma_2$. Thus $\psi \in d\,\mathfrak{L}\text{-}\mathsf{Iso}(\tilde{\mathsf{A}}, \mathcal{D}_1, \mathbf{g}^{1,\alpha})$ and therefore $d\,\mathsf{Iso}_\mathbf{1}(\tilde{\mathsf{A}}, \mathcal{D}_1, \mathbf{g}^{1,\alpha}) = d\,\mathfrak{L}\text{-}\mathsf{Iso}(\tilde{\mathsf{A}}, \mathcal{D}_1, \mathbf{g}^{1,\alpha})$, when $\alpha = 1$.

On the other hand, suppose $0 < \alpha < 1$. We have

$$\begin{aligned} \psi \cdot R(E_3, E_2, E_3) - R(\psi \cdot E_3, \psi \cdot E_2, \psi \cdot E_3) &= 0 \\ \tfrac{(-4-4\alpha+8\alpha^2)\sigma_1}{4\alpha^{3/2}} \sin\theta E_1 &= 0. \end{aligned}$$

and so $\sin\theta = 0$ and $\cos\theta = \pm 1$. Thus $\psi = \operatorname{diag}(\sigma_1, \sigma_2, \sigma_3)$ for some $\sigma_1, \sigma_2, \sigma_3 \in \{-1, 1\}$. Hence, we get

$$\begin{aligned} \psi \cdot \nabla R(E_1, E_2, E_1, E_1) - \nabla R(\psi \cdot E_1, \psi \cdot E_2, \psi \cdot E_1, \psi \cdot E_1) &= 0 \\ \frac{(1-2\alpha)^2(1+\alpha)\left(\sigma_1^3\sigma_2 - \sigma_3\right)}{2\alpha^2} E_3 &= 0 \end{aligned}$$

which implies that $\sigma_3 = \sigma_1\sigma_2$ and so $\psi = \operatorname{diag}(\sigma_1, \sigma_2, \sigma_1\sigma_2)$. Thus $\psi \in d\,\mathfrak{L}\text{-}\mathsf{Iso}(\tilde{\mathsf{A}}, \mathcal{D}_1, \mathbf{g}^{1,\alpha})$ and therefore $d\,\mathsf{Iso}_\mathbf{1}(\tilde{\mathsf{A}}, \mathcal{D}_1, \mathbf{g}^{1,\alpha}) = d\,\mathfrak{L}\text{-}\mathsf{Iso}(\tilde{\mathsf{A}}, \mathcal{D}_1, \mathbf{g}^{1,\alpha})$ for $0 < \alpha < 1$.

Next, we consider $(\tilde{\mathsf{A}}, \mathcal{D}_2, \mathbf{g}^{2,\alpha})$. The Riemannian characteristic expansion $(\tilde{\mathsf{A}}, \tilde{\mathbf{g}})$ of the structure $(\tilde{\mathsf{A}}, \mathcal{D}_2, \mathbf{g}^{2,\alpha})$ has orthonormal frame $(\frac{1}{\sqrt{\alpha}}E_2^L, E_3^L, \frac{1}{\sqrt{\alpha}}E_1^L)$. Consequently, we have $\tilde{\mathbf{g}}_\mathbf{1} = \operatorname{diag}(\alpha, \alpha, 1)$ with respect to the basis (E_1, E_2, E_3). Suppose $\psi \in d\,\mathsf{Iso}_\mathbf{1}(\tilde{\mathsf{A}}, \mathcal{D}_2, \mathbf{g}^{2,\alpha})$ and let $\phi \in \mathsf{Iso}_{\mathbf{1}}(\tilde{\mathsf{A}}, \mathcal{D}_2, \mathbf{g}^{2,\alpha})$ be the unique isometry such that $T_{\mathbf{1}}\phi = \psi$. As ψ preserves $\mathcal{D}(\mathbf{1})$ and $\tilde{\mathbf{g}}_{\mathbf{1}}$, it follows that

$$\psi = \begin{bmatrix} \sigma_1 \cos\theta & -\sigma_1 \sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & \sigma_2 \end{bmatrix}$$

with respect to the basis $(\frac{1}{\sqrt{\alpha}}E_2, E_3, \frac{1}{\sqrt{\alpha}}E_1)$ for some $\theta \in \mathbb{R}$ and $\sigma_1, \sigma_2 \in \{-1, 1\}$; equivalently,

$$\psi = \begin{bmatrix} \sigma_2 & 0 & 0 \\ 0 & \sigma_1 \cos\theta & -\frac{\sigma_1 \sin\theta}{\sqrt{\alpha}} \\ 0 & \sqrt{\alpha}\sin\theta & \cos\theta \end{bmatrix}$$

with respect to the basis (E_1, E_2, E_3). Furthermore, as ϕ is an isometry of $(\tilde{\mathsf{A}}, \tilde{\mathbf{g}})$ (Proposition 3.5), we have that ψ preserves the associated curvature tensor R and its covariant derivative ∇R (see (3.1) and (3.2)). We have

$$\begin{aligned} \psi \cdot R(E_1, E_2, E_1) - R(\psi \cdot E_1, \psi \cdot E_2, \psi \cdot E_1) &= 0 \\ -\tfrac{1+\alpha}{\sqrt{\alpha}} \sin\theta E_3 &= 0. \end{aligned}$$

and so $\sin\theta = 0$ and $\cos\theta = \pm 1$. Thus $\psi = \operatorname{diag}(\sigma_1, \sigma_2, \sigma_3)$ for some $\sigma_1, \sigma_2, \sigma_3 \in \{-1, 1\}$. Hence, we get

$$\begin{aligned} \psi \cdot \nabla R(E_1, E_2, E_1, E_1) - \nabla R(\psi \cdot E_1, \psi \cdot E_2, \psi \cdot E_1, \psi \cdot E_1) &= 0 \\ \frac{(1+\alpha)\left(\sigma_1^3\sigma_2 - \sigma_3\right)}{2\alpha} E_3 &= 0 \end{aligned}$$

which implies that $\sigma_3 = \sigma_1\sigma_2$ and so $\psi = \operatorname{diag}(\sigma_1, \sigma_2, \sigma_1\sigma_2)$. Thus $\psi \in d\,\mathfrak{L}\text{-}\mathsf{Iso}(\tilde{\mathsf{A}}, \mathcal{D}_2, \mathbf{g}^{2,\alpha})$ and therefore $d\,\mathsf{Iso}_{\mathbf{1}}(\tilde{\mathsf{A}}, \mathcal{D}_2, \mathbf{g}^{2,\alpha}) = d\,\mathfrak{L}\text{-}\mathsf{Iso}(\tilde{\mathsf{A}}, \mathcal{D}_2, \mathbf{g}^{2,\alpha})$. □

3.3.2 Type $\mathfrak{g}_{3.7}$

The pseudo-orthogonal Lie algebra $\mathfrak{g}_{3.7}$ has nonzero commutators $[E_2, E_3] = E_1$, $[E_3, E_1] = E_2$, $[E_1, E_2] = E_3$; the corresponding simply connected Lie group $\mathsf{SU}(2)$ is diffeomorphic to the three-sphere $\mathbb{S}^3$.

Proposition 3.17. *Any left-invariant sub-Riemannian structure on* $\mathsf{SU}(2)$ *is* $\mathfrak{L}$*-isometric up to rescaling to exactly one of the structures* $(\mathcal{D}, \mathbf{g}^\alpha)$, $0 < \alpha \leq 1$ *given by* $\mathcal{D}_1(\mathbf{1}) = \langle E_2, E_3 \rangle$ *and* $\mathbf{g}^\alpha_{\mathbf{1}} = \operatorname{diag}(\alpha, 1)$, *i.e., with orthonormal frame* $(\frac{1}{\sqrt{\alpha}}E_2^L, E_3^L)$.

1. *The respective groups of linearized* $\mathfrak{L}$*-isometries are given by*

$$d\,\mathfrak{L}\text{-}\mathsf{Iso}(\mathsf{SU}(2), \mathcal{D}, \mathbf{g}^\alpha) = \left\{ \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_1\sigma_2 \end{bmatrix} : \sigma_1, \sigma_2 = \pm 1 \right\} \qquad \text{if} \quad 0 < \alpha < 1$$

$$d\,\mathfrak{L}\text{-}\mathsf{Iso}(\mathsf{SU}(2), \mathcal{D}, \mathbf{g}^\alpha) = \left\{ \begin{bmatrix} \det g & 0 & 0 \\ 0 & \multicolumn{2}{c}{\multirow{2}{*}{g}} \\ 0 & & \end{bmatrix} : g \in \mathsf{O}(2) \right\} \qquad \text{if} \quad \alpha = 1.$$

2. *The respective associated Hamiltonian systems (and their normal forms) are given by*

$$\mathfrak{H}(\tilde{\mathsf{A}}, \mathcal{D}, \mathbf{g}^\alpha) = \left((\mathfrak{g}_{3.7})^*_-, \tfrac{1}{2}(\tfrac{1}{\alpha}p_2^2 + p_3^2)\right) \qquad \textit{normal form:} \quad \begin{cases} \mathsf{Np}(7) & \textit{if } 0 < \alpha < 1 \\ \mathsf{P}(8) & \textit{if } \alpha = 1. \end{cases}$$

3. *The Reeb vector field corresponding to* $(\mathsf{SU}(2), \mathcal{D}, \mathbf{g}^\alpha)$ *is* $\pm\frac{1}{\sqrt{\alpha}}E_1^L$. *The normalized scalar invariants are given by* $\chi = \frac{1-\alpha}{\sqrt{2}\sqrt{1+\alpha^2}}$ *and* $\kappa = \frac{1+\alpha}{\sqrt{2}\sqrt{1+\alpha^2}}$.

4. *The group of linearized isotropies* $d\,\mathsf{Iso}_\mathbf{1}(\mathsf{SU}(2), \mathcal{D}, \mathbf{g}^\alpha)$ *is identical to* $d\,\mathfrak{L}\text{-}\mathsf{Iso}(\mathsf{SU}(2), \mathcal{D}, \mathbf{g}^\alpha)$.

Proof. Any full-rank subspace of $\mathfrak{g}_{3.7}$ is equivalent to the full-rank subspace $\Gamma = \langle E_2, E_3\rangle$ ([34]). The automorphisms $\psi \in \mathsf{Aut}(\mathfrak{g}_{3.7})$ are exactly those linear transformations ψ which have $\det\psi = 1$ and preserve the dot product $A\bullet B = a_1b_1+a_2b_2+a_3b_3$ (here $A = a_1E_1+a_2E_2+a_3E_3$ and $B = b_1E_1 + b_2E_2 + b_3E_3$), i.e., $(\psi \cdot A) \bullet (\psi \cdot B) = A \bullet B$ for $A, B \in \mathfrak{g}_{3.7}$. Hence an automorphism preserves $\Gamma = \langle E_2, E_3\rangle$ if and only if it preserves its orthogonal complement $\Gamma^\perp = \langle E_1\rangle$. Consequently, the subgroup of automorphism preserving Γ is given by

$$\mathsf{Aut}_\Gamma(\mathfrak{g}_{3.7}) = \left\{ \begin{bmatrix} \det g & 0 & 0 \\ 0 & & \\ 0 & & g \end{bmatrix} \; : \; g \in \mathsf{O}(2) \right\}.$$

The restriction to Γ of the subgroup of automorphisms preserving Γ is $\mathsf{Aut}(\mathfrak{g}_{3.7})\big|_\Gamma = \mathsf{O}(2)$. Let $\boldsymbol{\mu}$ be a positive definite quadratic form on Γ. There exists an orthogonal transformation $\psi \in \mathsf{Aut}(\mathfrak{g}_{3.7})\big|_\Gamma$ such that $\boldsymbol{\mu}\circ\psi = \psi^\top\boldsymbol{\mu}\psi = \mathrm{diag}(\lambda_1, \lambda_2) = \lambda_2\,\mathrm{diag}(\frac{\lambda_1}{\lambda_2}, 1)$ with $\lambda_2 \geq \lambda_1 > 0$. It follows that any metric subspace of $\mathfrak{g}_{3.7}$ is equivalent to the one associated with $(\mathcal{D}_1, \mathbf{g}^{1,\alpha})$ for some $0 < \alpha \leq 1$. Moreover, the metric subspaces $(\Gamma, \boldsymbol{\mu}^\alpha)$, $\boldsymbol{\mu}^\alpha = \mathrm{diag}(\alpha, 1)$, $0 < \alpha \leq 1$ and $(\Gamma_1, \boldsymbol{\mu}^{\alpha'})$, $0 < \alpha' \leq 1$ are equivalent only if $\alpha = \alpha'$. Indeed, if $\psi^\top\boldsymbol{\mu}^\alpha\psi = r\boldsymbol{\mu}^{\alpha'}$ for some $r > 0$ and $\psi \in \mathsf{Aut}(\mathfrak{g}_{3.7})\big|_\Gamma = \mathsf{O}(2)$ then the eigenvalues of $\boldsymbol{\mu}^\alpha$ must coincide with those of $r\boldsymbol{\mu}^{\alpha'}$, i.e., $\{\alpha, 1\} = \{r\alpha', r\}$ and so $\alpha = \alpha'$. Consequently, by Lemma 3.8, we get the result.

(1 and 2). It is straightforward to calculate the subgroups of linearized $\mathfrak{L}$-isometries (given $\mathsf{Aut}(\mathfrak{g}_{3.7})\big|_\Gamma$). We have $\mathfrak{H}(\mathsf{SU}(2), \mathcal{D}, \mathbf{g}^\alpha) = ((\mathfrak{g}_{3.7})^*_-, H)$, $H(p) = \frac{1}{2}(\frac{1}{\alpha}p_2^2+p_3^2)$. When $\alpha = 1$, then $((\mathfrak{g}_{3.7})^*_-, H)$ is L-equivalent to $((\mathfrak{g}_{3.7})^*_-, p_1^2)$ (see Proposition 1.25, items 2 and 3); the system $((\mathfrak{g}_{3.7})^*_-, p_1^2)$ in turn is L-equivalent to $\mathsf{P}(8)$ by Proposition 2.14. On the other hand, when $0 < \alpha < 1$, then the corresponding Hamiltonian vector field $\vec{H} = (p_2p_3, -\frac{1}{\alpha}p_1p_3, \frac{\alpha-1}{\alpha}p_1p_2)$ has equilibria $(\mu, 0, 0)$, $(0, \mu, 0)$, $(0, 0, \mu)$, $\mu \in \mathbb{R}$ and thus equilibrium index $(3, 0)$ (see Proposition 2.20). Consequently $((\mathfrak{g}_{3.7})^*_-, H)$ is L-equivalent to $\mathsf{Np}(7)$ if $0 < \alpha < 1$ (see Table B.5).

(3). Locally (i.e., between some neighbourhoods of the origin and identity), the pull back (X_1, X_2, X_3) of the frame (E_1^L, E_2^L, E_3^L) to $\mathbb{R}^3$ and its corresponding dual frame is given by

$$\begin{cases} X_1 = \cos x_3 \sec x_2\,\partial_{x_1} + \sin x_3\,\partial_{x_2} - \cos x_3 \tan x_2\,\partial_{x_3} \\ X_2 = -\sec x_2 \sin x_3\,\partial_{x_1} + \cos x_3\,\partial_{x_2} + \sin x_3 \tan x_2\,\partial_{x_3} \\ X_3 = \partial_{x_3} \end{cases}$$

$$\begin{cases} \nu_1 = \cos x_2 \cos x_3\,dx_1 + \sin x_3\,dx_2 \\ \nu_2 = -\cos x_2 \sin x_3\,dx_1 + \cos x_3\,dx_2 \\ \nu_3 = \sin x_2\,dx_1 + dx_3 \end{cases}$$

More precisely, (X_1, X_2, X_3) is (locally) the pull back of (E_1^L, E_2^L, E_3^L) under $m : \mathbb{R}^3 \to \mathsf{SU}(2)$, $(x_1, x_2, x_3) \mapsto \exp(x_1 E_1)\exp(x_2 E_2)\exp(x_3 E_3)$.

Let $Y_1 = \frac{\lambda}{\sqrt{\alpha}} X_2 = -\frac{\lambda}{\sqrt{\alpha}} \sec x_2 \sin x_3\, \partial_{x_1} + \frac{\lambda}{\sqrt{\alpha}} \cos x_3\, \partial_{x_2} + \frac{\lambda}{\sqrt{\alpha}} \sin x_3 \tan x_2\, \partial_{x_3}$ and $Y_2 = \lambda X_3 = \lambda \partial_{x_3}$. The contact one-form ω is given by $\omega = -\frac{\sqrt{\alpha}}{\lambda^2}\nu_1 = -\frac{\sqrt{\alpha}}{\lambda^2}(\cos x_2 \cos x_3\, dx_1 + \sin x_3\, dx_2)$ and has exterior derivative $d\omega = \frac{\sqrt{\alpha}}{\lambda^2}\, \nu_2 \wedge \nu_3 = \frac{\sqrt{\alpha}}{\lambda^2} \cos x_3\, dx_2 \wedge dx_3 - \frac{\sqrt{\alpha}}{\lambda^2} \cos x_3 \sin x_2\, dx_1 \wedge dx_2 - \frac{\sqrt{\alpha}}{\lambda^2} \cos x_2 \sin x_3\, dx_1 \wedge dx_3$. Accordingly the corresponding Reeb vector field is $Y_0 = -\frac{\lambda^2}{\sqrt{\alpha}} X_1 = -\frac{\lambda^2}{\sqrt{\alpha}} \cos x_3 \sec x_2\, \partial_{x_1} - \frac{\lambda^2}{\sqrt{\alpha}} \sin x_3\, \partial_{x_2} + \frac{\lambda^2}{\sqrt{\alpha}} \cos x_3 \tan x_2\, \partial_{x_3}$. Hence we get

$$\begin{aligned}
[Y_1, Y_0] &= 0Y_1 + \tfrac{\lambda^2}{\alpha} Y_2 \\
[Y_2, Y_0] &= -\lambda^2 Y_1 + 0Y_2 \\
[Y_2, Y_1] &= 0Y_1 + 0Y_2 + Y_0
\end{aligned}$$

and so $\chi = \frac{\lambda^2}{2\alpha}(1-\alpha)$ and $\kappa = \frac{\lambda^2}{2\alpha}(1+\alpha)$. For $\lambda = \sqrt{\frac{\sqrt{2}\alpha}{\sqrt{1+\alpha^2}}}$ we obtain normalized scalar invariants $\chi = \frac{1-\alpha}{\sqrt{2}\sqrt{1+\alpha^2}}$ and $\kappa = \frac{1+\alpha}{\sqrt{2}\sqrt{1+\alpha^2}}$.

(4). As in the proof of Proposition 3.14, we find it preferable to take a central expansion not of $(\mathsf{SU}(2), \mathcal{D}, \mathbf{g}^\alpha)$ but rather of the rescaled structure $(\mathsf{SU}(2), \mathcal{D}, \frac{1}{4}\mathbf{g}^\alpha)$ (note however that $d\,\mathsf{Iso}_{\mathbf{1}}(\mathsf{SU}(2), \mathcal{D}, \frac{1}{4}\mathbf{g}^\alpha) = d\,\mathsf{Iso}_{\mathbf{1}}(\mathsf{SU}(2), \mathcal{D}, \mathbf{g}^\alpha)$).

The Riemannian characteristic expansion $(\mathsf{SU}(2), \tilde{\mathbf{g}})$ of $(\mathsf{SU}(2), \mathcal{D}, \frac{1}{4}\mathbf{g}^\alpha)$ has orthonormal frame $(\frac{2}{\sqrt{\alpha}} E_2^L, 2E_3^L, \frac{4}{\sqrt{\alpha}} E_1^L)$. Consequently, we have $\tilde{\mathbf{g}}_{\mathbf{1}} = \operatorname{diag}(\frac{\alpha}{16}, \frac{\alpha}{4}, \frac{1}{4})$ with respect to the basis (E_1, E_2, E_3). Suppose $\psi \in d\,\mathsf{Iso}_{\mathbf{1}}(\mathsf{SU}(2), \mathcal{D}, \frac{1}{4}\mathbf{g}^\alpha)$ and let $\phi \in \mathsf{Iso}_{\mathbf{1}}(\mathsf{SU}(2), \mathcal{D}, \frac{1}{4}\mathbf{g}^\alpha)$ be the unique isometry such that $T_{\mathbf{1}}\phi = \psi$. As ψ preserves $\mathcal{D}(\mathbf{1})$ and $\tilde{\mathbf{g}}_{\mathbf{1}}$, it follows that

$$\psi = \begin{bmatrix} \sigma_1 \cos\theta & -\sigma_1 \sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & \sigma_2 \end{bmatrix}$$

with respect to the basis $(\frac{2}{\sqrt{\alpha}} E_2, 2E_3, \frac{4}{\sqrt{\alpha}} E_1)$ for some $\theta \in \mathbb{R}$ and $\sigma_1, \sigma_2 \in \{-1, 1\}$; equivalently,

$$\psi = \begin{bmatrix} \sigma_2 & 0 & 0 \\ 0 & \sigma_1 \cos\theta & -\frac{\sigma_1 \sin\theta}{\sqrt{\alpha}} \\ 0 & \sqrt{\alpha}\sin\theta & \cos\theta \end{bmatrix}$$

with respect to the basis (E_1, E_2, E_3). Furthermore, as ϕ is an isometry of $(\mathsf{SU}(2), \tilde{\mathbf{g}})$ (Proposition 3.5), we have that ψ preserves the associated curvature tensor R and its covariant derivative ∇R (see (3.1) and (3.2)). First, suppose $\alpha = 1$. Then we have

$$\begin{aligned}
\psi \cdot \nabla R(E_3, E_2, E_3, E_3) - \nabla R(\psi \cdot E_3, \psi \cdot E_2, \psi \cdot E_3, \psi \cdot E_3) &= 0 \\
-\tfrac{3}{8}(\sigma_1 - \sigma_2)E_3 &= 0
\end{aligned}$$

which implies that $\sigma_1 = \sigma_2$. Thus it follows that $\psi \in d\,\mathfrak{L}\text{-}\mathsf{Iso}(\mathsf{SU}(2), \mathcal{D}, \mathbf{g}^\alpha)$ and therefore $d\,\mathsf{Iso}_{\mathbf{1}}(\mathsf{SU}(2), \mathcal{D}, \frac{1}{4}\mathbf{g}^\alpha) = d\,\mathsf{Iso}_{\mathbf{1}}(\mathsf{SU}(2), \mathcal{D}, \mathbf{g}^\alpha) = d\,\mathfrak{L}\text{-}\mathsf{Iso}(\tilde{\mathsf{A}}, \mathcal{D}_1, \mathbf{g}^{1,\alpha})$, when $\alpha = 1$.

On the other hand, suppose $0 < \alpha < 1$. We have

$$\begin{aligned} \psi \cdot R(E_1, E_2, E_1) - R(\psi \cdot E_1, \psi \cdot E_2, \psi \cdot E_1) &= 0 \\ \tfrac{-4+\alpha+3\alpha^2}{4\sqrt{\alpha}} \sin\theta E_3 &= 0. \end{aligned}$$

and so $\sin\theta = 0$ and $\cos\theta = \pm 1$. Thus $\psi = \operatorname{diag}(\sigma_1, \sigma_2, \sigma_3)$ for some $\sigma_1, \sigma_2, \sigma_3 \in \{-1, 1\}$. Hence, we get

$$\begin{aligned} \psi \cdot \nabla R(E_3, E_2, E_3, E_3) - \nabla R(\psi \cdot E_3, \psi \cdot E_2, \psi \cdot E_3, \psi \cdot E_3) &= 0 \\ \tfrac{3(4-5\alpha)^2\left(\sigma_1 - \sigma_2\sigma_3^3\right)}{8\alpha^2} &= 0 \end{aligned}$$

which implies that $\sigma_3 = \sigma_1\sigma_2$ and so $\psi = \operatorname{diag}(\sigma_1, \sigma_2, \sigma_1\sigma_2)$. Thus $\psi \in d\,\mathfrak{L}\text{-}\mathsf{Iso}(\mathsf{SU}(2), \mathcal{D}, \mathbf{g}^\alpha)$ and therefore $d\,\mathsf{Iso}_1(\mathsf{SU}(2), \mathcal{D}, \frac{1}{4}\mathbf{g}^\alpha) = d\,\mathsf{Iso}_1(\mathsf{SU}(2), \mathcal{D}, \mathbf{g}^\alpha) = d\,\mathfrak{L}\text{-}\mathsf{Iso}(\tilde{\mathsf{A}}, \mathcal{D}_1, \mathbf{g}^{1,\alpha})$, when $0 < \alpha < 1$. □

3.4 Description of isometries

In Figure 3.1, we graph the normalized invariants χ and κ for each normal form (up to $\mathfrak{L}$-isometry and rescaling) obtained in Sections 3.2 and 3.3. By Theorem 3.7, the only two normal forms isometric up to rescaling are the two structures with normalized scalar invariants $(\chi, \kappa) = (0, -1)$ on $\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}$ and $\tilde{\mathsf{A}}$ respectively. Accordingly, we find that two structures on the same Lie group are isometric if and only if they are $\mathfrak{L}$-isometric.

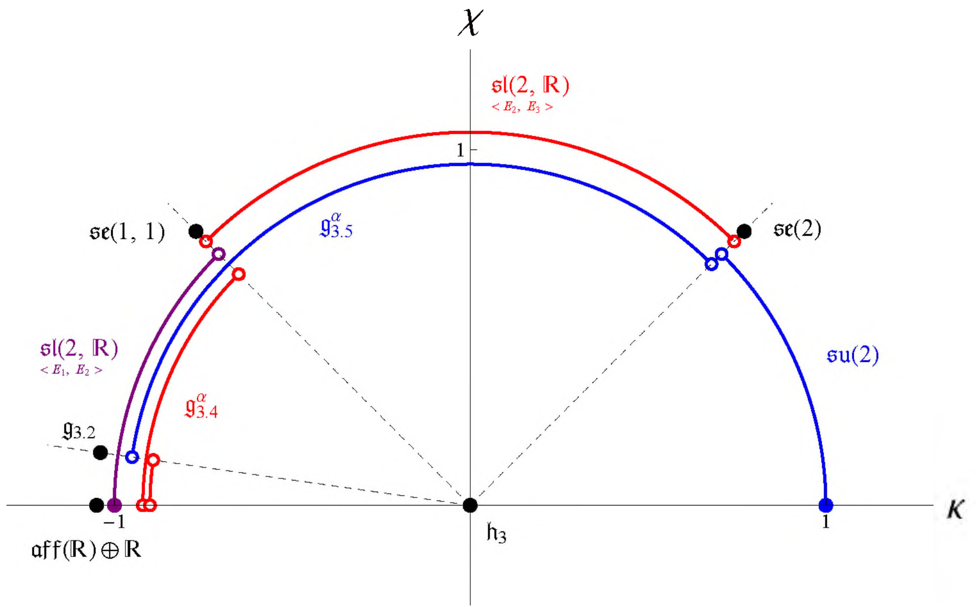


Figure 3.1: Scalar invariants χ and κ for normal forms of sub-Riemannian structures

Lemma 3.18. *Let* $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ *be a left-invariant sub-Riemannian structure (on a three-dimensional Lie group) with scalar invariants* χ *and* κ. *If* $\chi^2 + \kappa^2 \neq 0$, *then* $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ *is not isometric to any rescaled structure* $(\mathsf{G}, \mathcal{D}, \lambda\mathbf{g})$, $\lambda > 0$, $\lambda \neq 1$.

Proof. We have that $(\mathsf{G}, \mathcal{D}, \lambda\mathbf{g})$ has scalar invariants $(\frac{1}{\lambda}\chi, \frac{1}{\lambda}\kappa)$. If $\chi^2 + \kappa^2 \neq 0$, then $(\chi, \kappa) \neq (\frac{1}{\lambda}\chi, \frac{1}{\lambda}\kappa)$ and hence the result follows from Theorem 3.6. □

Theorem 3.19. *Two invariant structures* $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ *and* $(\mathsf{G}, \mathcal{D}', \mathbf{g}')$ *on the same simply connected three-dimensional Lie group* G *are isometric if and only if they are* $\mathfrak{L}$*-isometric.*

Proof. Clearly, if two structures on G are $\mathfrak{L}$-isometric, then they are isometric. To prove the converse, it suffices to show that no two normal forms (of structures on G, with respect to $\mathfrak{L}$-isometries) are isometric. Indeed, suppose $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ and $(\mathsf{G}, \mathcal{D}', \mathbf{g}')$ are isometric; let $(\mathsf{G}, \mathcal{D}_1, \mathbf{g}^1)$ and $(\mathsf{G}, \mathcal{D}_2, \mathbf{g}^2)$ be their respective normal forms (i.e., $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ is $\mathfrak{L}$-isometric to $(\mathsf{G}, \mathcal{D}_1, \mathbf{g}^1)$ and $(\mathsf{G}, \mathcal{D}', \mathbf{g}')$ is $\mathfrak{L}$-isometric to $(\mathsf{G}, \mathcal{D}_2, \mathbf{g}^2)$). If $(\mathsf{G}, \mathcal{D}_1, \mathbf{g}^1)$ and $(\mathsf{G}, \mathcal{D}_2, \mathbf{g}^2)$ are not isometric, then $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ and $(\mathsf{G}, \mathcal{D}', \mathbf{g}')$ cannot be isometric. Hence, if no two normal forms are isometric, then $(\mathsf{G}, \mathcal{D}_1, \mathbf{g}^1)$ and $(\mathsf{G}, \mathcal{D}_2, \mathbf{g}^2)$ must be identical and so $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ and $(\mathsf{G}, \mathcal{D}', \mathbf{g}')$ must be $\mathfrak{L}$-isometric.

For the Heisenberg group (Proposition 3.10), there is exactly one normal form with respect to $\mathfrak{L}$-isometries (i.e., all structures on H_3 are $\mathfrak{L}$-isometric). Similarly, for each other solvable group G (Section 3.2), every sub-Riemannian structure on G is $\mathfrak{L}$-isometric up to rescaling to exactly one structure on G; for these structures we have that $\chi^2 + \kappa^2 \neq 0$ and so by the preceding Lemma no two rescaled versions of the normal form are isometric. For the simple Lie groups (Section 3.3) we have one or two one-parameter families of normal forms. However, each permissible value of the parameter gives rise to distinct normalized scalar invariants χ and κ and so no two normal forms are isometric up to rescaling. Moreover, as $\chi^2 + \kappa^2 \neq 0$, no two rescaled versions of the same normal form are isometric (again by the preceding Lemma). Thus it follows that no two rescaled normal forms are isometric. □

We have shown that the isotropy subgroup of the identity $\mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathcal{D}, \mathbf{g})$ is a subgroup of the automorphism group for each normal form (except the one on $\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}$). In summary, we have the following result.

Theorem 3.20. *Let* $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ *be an invariant structure on a three-dimensional simply connected Lie group* G *with Lie algebra* $\mathfrak{g}$ *not isomorphic to* $\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1$.

1. *The isotropy subgroup* $\mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathcal{D}, \mathbf{g})$ *is a subgroup of* $\mathsf{Aut}(\mathsf{G})$.
2. *The isometry group* $\mathsf{Iso}(\mathsf{G}, \mathcal{D}, \mathbf{g})$ *decomposes as a semi-direct product* $L_{\mathsf{G}} \rtimes \mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathcal{D}, \mathbf{g})$ *of the left translations* $L_{\mathsf{G}} = \{L_g \,:\, g \in \mathsf{G}\}$ *and the isotropy subgroup* $\mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathcal{D}, \mathbf{g})$.

Proof. (1). We have already shown that $d\,\mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathcal{D}, \mathbf{g}) = d\,\mathfrak{L}\text{-}\mathsf{Iso}(\mathsf{G}, \mathcal{D}, \mathbf{g})$ for every group (except $\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}$) from which it follows that $\mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathcal{D}, \mathbf{g})$ is a subgroup of $\mathsf{Aut}(\mathsf{G})$.

(2). Clearly $L_{\mathsf{G}} \cap \mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathcal{D}, \mathbf{g}) = \{\mathrm{id}_{\mathsf{G}}\}$. Also, for any $\phi \in \mathsf{Iso}(\mathsf{G}, \mathcal{D}, \mathbf{g})$ we have that $L_{\phi(\mathbf{1})^{-1}} \circ \phi \in \mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathcal{D}, \mathbf{g})$ and so $L_{\mathsf{G}}\, \mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathcal{D}, \mathbf{g}) = \mathsf{Iso}(\mathsf{G}, \mathcal{D}, \mathbf{g})$. It only remains to be shown that L_{G} is normal in $\mathsf{Iso}(\mathsf{G}, \mathcal{D}, \mathbf{g})$. Let $L_g \circ \phi \in \mathsf{Iso}(\mathsf{G}, \mathcal{D}, \mathbf{g})$, $\phi \in \mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathcal{D}, \mathbf{g})$. We have $(L_g \circ \phi)^{-1} L_h (L_g \circ \phi) = L_{\phi^{-1}(g^{-1}hg)} \in L_{\mathsf{G}}$ as ϕ is an automorphism. □

Consequently we have the following simple description of the isometries between structures on three-dimensional Lie groups.

Theorem 3.21. *Let* $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ *and* $(\mathsf{G}', \mathcal{D}', \mathbf{g}')$ *be two invariant structures on some three-dimensional simply connected Lie groups; suppose neither* $\mathfrak{g}$ *nor* $\mathfrak{g}'$ *is isomorphic to* $\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1$*. If* $\phi : \mathsf{G} \to \mathsf{G}'$ *is an isometry between* $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ *and* $(\mathsf{G}', \mathcal{D}', \mathbf{g}')$*, then* $\phi = L_{\phi(\mathbf{1})} \circ \phi'$ *is the composition of a left translation* $L_{\phi(\mathbf{1})}$ *on* G' *and a Lie group isomorphism* $\phi' : \mathsf{G} \to \mathsf{G}'$*.*

Proof. Let $\phi : \mathsf{G} \to \mathsf{G}'$ be an isometry between $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ and $(\mathsf{G}', \mathcal{D}', \mathbf{g}')$. First, we claim that $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ and $(\mathsf{G}', \mathcal{D}', \mathbf{g}')$ are $\mathfrak{L}$-isometric. Indeed, if the scalar invariants satisfy $\chi \neq 0$ or $\chi = 0$ and $\kappa \geq 0$ then the Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$ must be isomorphic (Theorem 3.7) and so the Lie groups G and G' must be isomorphic; hence we may assume $\mathsf{G} = \mathsf{G}'$ and so, by Theorem 3.19, $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ and $(\mathsf{G}', \mathcal{D}', \mathbf{g}')$ are $\mathfrak{L}$-isometric. On the other hand, if $\chi = 0$ and $\kappa < 0$, then we may assume $\mathsf{G} = \mathsf{G}' = \tilde{\mathcal{A}}$ (as the only groups that admit invariant structures with $\chi = 0$ and $\kappa < 0$ are $\tilde{\mathsf{A}}$ and $\mathsf{Aff}\,(\mathbb{R})_0 \times \mathbb{R}$, see Figure 3.1, but $\mathsf{Aff}\,(\mathbb{R})_0 \times \mathbb{R}$ has been ruled out); again it follows that $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ and $(\mathsf{G}', \mathcal{D}', \mathbf{g}')$ are $\mathfrak{L}$-isometric by Theorem 3.19.

Let $\varphi : \mathsf{G} \to \mathsf{G}'$ be an $\mathfrak{L}$-isometry between $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ and $(\mathsf{G}', \mathcal{D}', \mathbf{g}')$ and let $\phi' = L_{\phi(\mathbf{1})^{-1}} \circ \phi$. We have that $\varphi^{-1} \circ \phi' \in \mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathcal{D}, \mathbf{g})$. By Theorem 3.20, $\varphi^{-1} \circ \phi'$ is a Lie group automorphism, and so (as φ is a Lie group isomorphism) it follows that ϕ' is a Lie group isomorphism. As $\phi = L_{\phi(\mathbf{1})} \circ \phi'$, the result follows. □

Remark 3.22. A symmetric (invariant) sub-Riemannian structure is essentially one for which there exists an isotropy $\phi \in \mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathcal{D}, \mathbf{g})$ such that $T_g\phi\big|_{\mathcal{D}(g)} = -\,\mathrm{id}_{\mathcal{D}(g)}$. By inspection of Tables B.8 and B.9, we have that any invariant sub-Riemannian structure on a simply connected unimodular Lie group, or on $\mathsf{Aff}\,(\mathbb{R})_0 \times \mathbb{R}$, is a symmetric structure. Moreover, as the structure on $\mathsf{Aff}\,(\mathbb{R})_0 \times \mathbb{R}$ is isometric to a structure on $\tilde{\mathsf{A}}$, we have that any symmetric invariant sub-Riemannian structure (on a simply connected three-dimensional Lie group) is isometric to exactly one of our structures on a unimodular group. This is consistent with the classification by Strichartz [112] of three-dimensional symmetric sub-Riemannian structures.

Chapter 4

Sub-Riemannian and Riemannian structures on the Heisenberg groups

Among the invariant sub-Riemannian (and Riemannian) structures on Lie groups, those on the Heisenberg groups are arguably the simplest and serve as prototypes. In this chapter we classify these structures up to isometry; a parametrized family of normal forms is exhibited. We then determine the isometry group for each normal form and hence find the geodesics.

A standard computation yields the automorphism group of H_{2n+1}, a subgroup of which is a symplectic group. By use of the automorphisms, we normalize the distributions on H_{2n+1}. Equivalence class representatives are then constructed by successively applying automorphisms, that preserve the normalized distribution, to the metric. (The Riemannian case is treated similarly.) Central to our argument is Williamson's theorem, which states that any positive definite symmetric matrix can be diagonalized, in a certain way, by symplectic matrices.

Once normal forms have been determined, the isometry group of each normal form is determined; this amounts to finding the isotropy subgroup at identity. Next, explicit expressions for the geodesics of each structure are calculated; by use of the isometry group, these expressions are brought to a very simple form. By inspection of the geodesics, some totally geodesic subgroups are identified. It is a simple matter to show that the Riemannian structures are central extensions of the sub-Riemannian structures; this explains the similarity between the respective geodesics. (In fact, the similarity between these two classes of geodesics is what instigated the investigation into central expansions in Section 1.4.4.)

As prototypical structures, sub-Riemannian and Riemannian structures on the Heisenberg groups have been considered by quite a few authors. Vershik and Gershkovich [122] describe the geodesics and wave front in the three-dimensional case. Tan and Yang [117] find explicit expressions minimizing sub-Riemannian geodesics on H-type groups as well as describing the isometry groups; for sub-Riemannian structures on the Heisenberg group, this covers only the case of maximal symmetry. Ambrosio and Rigot [16] also determine minimising geodesics in case of maximal symmetry. Monroy-Pérez and Anzaldo-Meneses [93] consider a class of invariant optimal control problems on H_{2n+1} (which turns out to cover all sub-Riemannian structures on H_{2n+1} up to isometry); they find expressions for the geodesics and determine the conjugate locus. Beals et al. [27] also consider a class of structures on H_{2n+1} (which turns out to cover all sub-Riemannian structures on H_{2n+1} up to isometry); in particular, they describe the minimising geodesics for these structures. On the other hand, Riemannian structures on Heisenberg groups (and generalizations) are well understood (see [55,80,81,125]). For a classical derivation of the Riemannian geodesics (in the case of maximal symmetry), see [87].

We believe that we make the following novel contributions: the classification of both the sub-Riemannian and Riemannian structures on the Heisenberg groups, up to isometry; explicit calculation of the isometry groups (to our knowledge only the case of maximal symmetry has been covered before); normal forms for geodesics (i.e., expressions for geodesics simplified by use of isometries); exposition of some similarities between the Riemannian and sub-Riemannian cases (and more specifically, that the Riemannian structures are central expansions of the sub-Riemannian structures). Moreover, it turns out that in the case of maximal symmetry, a structure on H_{2n+1} is reducible (in a certain sense) to a structure on H_3 (see Remarks 4.14 and 4.17).

Note. The first part of this chapter, regarding the classification of structures on H_{2n+1}, appears in [30].

4.1 The Heisenberg groups

The $(2n+1)$-dimensional Heisenberg group may be realized as a matrix Lie group

$$\mathsf{H}_{2n+1} = \left\{ \begin{bmatrix} 1 & x_1 & x_2 & \cdots & x_n & z \\ 0 & 1 & 0 & & 0 & y_1 \\ 0 & 0 & 1 & & 0 & y_2 \\ \vdots & & & \ddots & & \vdots \\ 0 & & \cdots & & 1 & y_n \\ 0 & & \cdots & & 0 & 1 \end{bmatrix} = m(z,x,y) \;:\; x_i, y_i, z \in \mathbb{R} \right\}.$$

The diffeomorphism $m : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathsf{H}_{2n+1}$ is used simply as convenient notation. H_{2n+1} is a two-step Carnot group with one-dimensional centre $\{m(z,0,0) \;:\; z \in \mathbb{R}\}$; its Lie algebra

$$\mathfrak{h}_{2n+1} = \left\{ \begin{bmatrix} 0 & x_1 & x_2 & \cdots & x_n & z \\ 0 & 0 & 0 & & 0 & y_1 \\ 0 & 0 & 0 & & 0 & y_2 \\ \vdots & & & \ddots & & \vdots \\ 0 & & \cdots & & 0 & y_n \\ 0 & & \cdots & & 0 & 0 \end{bmatrix} = zZ + \sum_{i=1}^{n} (x_i X_i + y_i Y_i) \;:\; x_i, y_i, z \in \mathbb{R} \right\}$$

has non-zero commutators $[X_i, Y_j] = \delta_{ij} Z$. (If $\mathfrak{g}_1 = \operatorname{span}(X_1, Y_1, \ldots, X_n, Y_n)$ and $\mathfrak{g}_2 = \operatorname{span}(Z)$, then $\mathfrak{h}_{2n+1} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ with $[\mathfrak{g}_1, \mathfrak{g}_1] = \mathfrak{g}_2$ and $[\mathfrak{g}_1, \mathfrak{g}_2] = \{0\}$.)

The automorphisms of $\mathfrak{h}_{2n+1}$ are exactly those linear isomorphisms that preserve the center $\mathfrak{z}$ of $\mathfrak{h}_{2n+1}$ and for which the induced map on $\mathfrak{h}_{2n+1}/\mathfrak{z}$ preserves an appropriate symplectic structure (cf. [60]). More precisely, let ω be the skew-symmetric bilinear form on $\mathfrak{h}_{2n+1}$ specified by

$$[A, B] = \omega(A, B) Z, \quad A, B \in \mathfrak{h}_{2n+1}.$$

Note that $\omega(X_i, Y_j) = \delta_{ij}$ and that ω is zero on the remaining pairs of basis vectors. Accordingly, we get the following characterization of automorphisms.

Lemma 4.1. *A linear isomorphism $\psi : \mathfrak{h}_{2n+1} \to \mathfrak{h}_{2n+1}$ is a Lie algebra automorphism if and only if*

$$\psi \cdot Z = cZ \qquad \text{and} \qquad \omega(\psi \cdot A, \psi \cdot B) = c\,\omega(A, B)$$

for some $c \neq 0$.

Proof. Suppose ψ is an automorphism. As ψ preserves the center of $\mathfrak{h}_{2n+1}$ we have $\psi \cdot Z = cZ$ for some $c \neq 0$. For $A, B \in \mathfrak{h}_{2n+1}$, we have $[\psi \cdot A, \psi \cdot B] = \psi \cdot [A, B]$ and so $\omega(\psi \cdot A, \psi \cdot B)Z = \psi \cdot \omega(A, B)Z$, i.e., $\omega(\psi \cdot A, \psi \cdot B) = c\,\omega(A, B)$. Conversely, suppose ψ is a linear isomorphism such that the given conditions hold. For $A, B \in \mathfrak{h}_{2n+1}$, we have

$$[\psi \cdot A, \psi \cdot B] = \omega(\psi \cdot A, \psi \cdot B)Z = c\,\omega(A, B)Z = \psi \cdot \omega(A, B)Z = \psi \cdot [A, B]. \qquad \square$$

We now proceed to give a matrix representation for the group of automorphisms. Throughout, we shall make use the ordered basis

$$(Z, X_1, Y_1, X_2, Y_2, \ldots, X_n, Y_n)$$

for $\mathfrak{h}_{2n+1}$; linear maps will be identified with their corresponding matrices. Furthermore, the left-invariant vector fields corresponding to Z, X_i, and Y_i will be denoted by Z^L, X_i^L, and Y_i^L, respectively. The bilinear form ω takes the form

$$\omega = \begin{bmatrix} 0 & 0 \\ 0 & J \end{bmatrix}, \quad \text{where} \quad J = \begin{bmatrix} 0 & 1 & & & 0 \\ -1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ 0 & & & -1 & 0 \end{bmatrix}.$$

We note that the linear involution

$$\varsigma = \begin{bmatrix} -1 & 0 & & \cdots & & 0 \\ 0 & 0 & 1 & & & 0 \\ & 1 & 0 & & & \\ \vdots & & & \ddots & & \\ & & & & 0 & 1 \\ 0 & 0 & & & 1 & 0 \end{bmatrix} \tag{4.1}$$

is an automorphism (indeed $\varsigma \cdot Z = (-1)Z$ and $\varsigma^\top \omega \varsigma = (-1)\omega$).

Proposition 4.2 (cf. [106]). *The group of automorphisms $\mathsf{Aut}(\mathfrak{h}_{2n+1})$ is given by*

$$\left\{ \begin{bmatrix} r^2 & v \\ 0 & rg \end{bmatrix}, \varsigma \begin{bmatrix} r^2 & v \\ 0 & rg \end{bmatrix} \; : \; r > 0,\, v \in \mathbb{R}^{1 \times 2n},\, g \in \mathsf{Sp}\,(n, \mathbb{R}) \right\}$$

where

$$\mathsf{Sp}\,(n, \mathbb{R}) = \left\{ g \in \mathbb{R}^{2n \times 2n} \; : \; g^\top J g = J \right\}$$

is the $n(2n+1)$-dimensional symplectic group over $\mathbb{R}$.

Proof. If $\psi = \begin{bmatrix} r^2 & v \\ 0 & rg \end{bmatrix}$, then $\psi \cdot Z = r^2 Z$ and

$$\begin{aligned} \psi^\top \omega \, \psi &= \begin{bmatrix} r^2 & 0 \\ v^\top & rg^\top \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & J \end{bmatrix} \begin{bmatrix} r^2 & v \\ 0 & rg \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & r^2 g^\top J g \end{bmatrix} = r^2 \omega. \end{aligned}$$

Thus $\begin{bmatrix} r^2 & v \\ 0 & rg \end{bmatrix}$ and $\varsigma \begin{bmatrix} r^2 & v \\ 0 & rg \end{bmatrix}$ are indeed automorphisms.

Suppose ψ is an automorphism. We have $\psi \cdot Z = cZ$ for some $c \neq 0$. We may assume $c > 0$ (if $c < 0$, then $\varsigma\psi$ has this property). Thus $\psi = \begin{bmatrix} r^2 & v \\ 0 & h \end{bmatrix}$ for some $r > 0$, $v \in \mathbb{R}^{2n}$ and $h \in \mathsf{GL}\,(2n, \mathbb{R})$. As $\psi^\top \omega \psi = r^2 \omega$, it follows that $h^\top J h = r^2 J$. For $g = \frac{1}{r} h$, we get $g^\top J \, g = J$. Thus $\psi = \begin{bmatrix} r^2 & v \\ 0 & rg \end{bmatrix}$ for some $r > 0$, $v \in \mathbb{R}^{2n}$ and $g \in \mathsf{Sp}\,(n, \mathbb{R})$. Finally note that if $\varsigma\psi = \begin{bmatrix} r^2 & v \\ 0 & rg \end{bmatrix}$, then $\psi = \varsigma \begin{bmatrix} r^2 & v \\ 0 & rg \end{bmatrix}$. □

Remark 4.3. The group of automorphisms $\mathsf{Aut}(\mathsf{H}_{2n+1})$ decomposes as a semidirect product

$$\mathsf{Aut}(\mathfrak{h}_{2n+1}) \cong \mathbb{R}^{2n} \rtimes \mathbb{R} \rtimes \mathsf{Sp}(n, \mathbb{R}) \rtimes \{\mathbf{1}, \varsigma\}$$

of the following subgroups:

1. the subgroup of inner automorphisms $\mathsf{Int}(\mathsf{H}_3) = \left\{ \begin{bmatrix} 1 & v \\ 0 & I_{2n} \end{bmatrix} \; : \; v \in \mathbb{R}^{2n} \right\} \cong \mathbb{R}^{2n}$;
2. the dilation subgroup $\left\{ \begin{bmatrix} r^2 & 0 \\ 0 & rI_{2n} \end{bmatrix} \; : \; r > 0 \right\} \cong \mathbb{R}$;
3. the symplectic subgroup $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & g \end{bmatrix} \; : \; g \in \mathsf{Sp}\,(n, \mathbb{R}) \right\} \cong \mathsf{Sp}\,(n, \mathbb{R})$;
4. and the two-element subgroup $\{\mathbf{1}, \varsigma\}$.

4.2 Classification of structures

We consider the sub-Riemannian case first; we start by normalizing the distribution.

Lemma 4.4. *If $\mathcal{D}$ is a bracket-generating left-invariant distribution on H_{2n+1}, then there exists an (inner) automorphism $\phi \in \mathsf{Aut}(\mathsf{H}_{2n+1})$ such that $\phi_* \mathcal{D} = \bar{\mathcal{D}}$, where $\bar{\mathcal{D}}$ is the left-invariant distribution specified by $\bar{\mathcal{D}}(\mathbf{1}) = \mathrm{span}(X_1, Y_1, X_2, Y_2, \ldots, X_n, Y_n)$.*

Proof. It suffices to show that there exists an (inner) automorphism $\psi \in \mathsf{Aut}(\mathfrak{h}_{2n+1})$ such that $\psi \cdot \mathcal{D}(\mathbf{1}) = \bar{\mathcal{D}}(\mathbf{1})$. For any subspace $\mathfrak{s} \subseteq \mathfrak{h}_{2n+1}$, we have $\mathsf{Lie}(\mathfrak{s}) \leq \mathrm{span}(\mathfrak{s}, Z)$. Therefore, if $\mathsf{Lie}(\mathfrak{s}) = \mathfrak{h}_{2n+1}$ and $\mathfrak{s} \neq \mathfrak{h}_{2n+1}$, then $\mathfrak{s}$ has codimension one, $Z \notin \mathfrak{s}$ and so $\mathfrak{s}$ takes the form

$$\mathfrak{s} = \mathrm{span}(X_1 + v_1 Z, Y_1 + v_2 Z, \ldots, X_n + v_{2n-1} Z, Y_n + v_{2n} Z).$$

Accordingly,

$$\psi = \begin{bmatrix} 1 & -v \\ 0 & I_{2n} \end{bmatrix}, \quad v = \begin{bmatrix} v_1 & v_2 & \cdots & v_{2n} \end{bmatrix}$$

is an inner automorphism such that $\psi \cdot \mathfrak{s} = \operatorname{span}(X_1, Y_1, X_2, Y_2, \ldots, X_n, Y_n)$. ☐

We now proceed to normalise the sub-Riemannian metric and so obtain a classification of the sub-Riemannian structures. We shall make use of the fact that positive definite matrices are diagonalizable by symplectic matrices (see e.g., [54], Chapter 8.3: "Symplectic Spectrum and Williamson's Theorem"). More precisely,

Lemma 4.5. *If $M \in \mathbb{R}^{2n\times 2n}$ is positive definite, then there exists $g \in \mathsf{Sp}\,(n, \mathbb{R})$ such that*

$$g^\top M\, g = \operatorname{diag}(\lambda_1, \lambda_1, \lambda_2, \lambda_2, \ldots, \lambda_n, \lambda_n)$$

where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0$; $\pm i\lambda_j$ are exactly the eigenvalues of JM. Moreover, the symplectic spectrum $\operatorname{Spec}(M) = (\lambda_1, \ldots, \lambda_n)$ is a symplectic invariant, i.e., $\operatorname{Spec}(g^\top M\, g) = \operatorname{Spec}(M)$ for $g \in \mathsf{Sp}\,(n, \mathbb{R})$.

Theorem 4.6. *Any left-invariant sub-Riemannian structure on H_{2n+1} is isometric to exactly one of the structures $(\mathcal{D}, \mathbf{g}^\lambda)$ specified by*

$$\begin{cases} \mathcal{D}(\mathbf{1}) = \operatorname{span}(X_1, Y_1, X_2, Y_2, \ldots, X_n, Y_n) \\ \quad \mathbf{g}^\lambda_{\mathbf{1}} = \Lambda = \operatorname{diag}(\lambda_1, \lambda_1, \lambda_2, \lambda_2, \ldots, \lambda_n, \lambda_n). \end{cases} \tag{4.2}$$

i.e., with orthonormal frame

$$\left(\tfrac{1}{\sqrt{\lambda_1}} X_1^L, \tfrac{1}{\sqrt{\lambda_1}} Y_1^L, \tfrac{1}{\sqrt{\lambda_2}} X_2^L, \tfrac{1}{\sqrt{\lambda_2}} Y_2^L, \ldots, \tfrac{1}{\sqrt{\lambda_n}} X_n^L, \tfrac{1}{\sqrt{\lambda_n}} Y_n^L\right).$$

Here $1 = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0$ parametrize a family of (non-isometric) class representatives.

Proof. By Lemma 4.4, any sub-Riemannian structure on H_{2n+1} is isometric to one on the distribution specified by $\mathcal{D}(\mathbf{1}) = \operatorname{span}(X_1, Y_1, X_2, Y_2, \ldots, X_n, Y_n)$. As sub-Riemannian Carnot groups, any two such structures are isometric if and only if they are $\mathfrak{L}$-isometric (see Theorem 1.43). Consequently, it remains only to classify the positive definite quadratic forms on $\mathcal{D}(\mathbf{1})$ up to Lie group automorphism.

Let $\boldsymbol{\mu}$ be a positive definite quadratic form on $\mathcal{D}(\mathbf{1})$. By Lemma 4.5 there exists $g \in \mathsf{Sp}\,(n, \mathbb{R})$ such that

$$g^\top \boldsymbol{\mu}\, g = \operatorname{diag}(\lambda_1, \lambda_1, \lambda_2, \lambda_2, \ldots, \lambda_n, \lambda_n)$$

where $(\lambda_1, \ldots, \lambda_n) = \operatorname{Spec}(\boldsymbol{\mu})$. Hence $(\frac{1}{\sqrt{\lambda_1}} g)^\top \boldsymbol{\mu}\, (\frac{1}{\sqrt{\lambda_1}} g) = \operatorname{diag}(1, 1, \frac{\lambda_2}{\lambda_1}, \frac{\lambda_2}{\lambda_1}, \ldots, \frac{\lambda_n}{\lambda_1}, \frac{\lambda_n}{\lambda_1})$. Therefore

$$\psi = \begin{bmatrix} \frac{1}{\lambda_1} & 0 \\ 0 & \frac{1}{\sqrt{\lambda_1}} g \end{bmatrix}$$

is a Lie algebra automorphism such that $\psi \cdot \mathcal{D}(\mathbf{1}) = \mathcal{D}(\mathbf{1})$ and $\psi^* \boldsymbol{\mu} = \operatorname{diag}(1, 1, \frac{\lambda_2}{\lambda_1}, \frac{\lambda_2}{\lambda_1}, \ldots, \frac{\lambda_n}{\lambda_1}, \frac{\lambda_n}{\lambda_1})$. Consequently, as H_{2n+1} is simply connected, by Lemma 1.31 (and relabelling $\frac{\lambda_i}{\lambda_1}$ as λ_i) we get the given normal forms.

It remains to be shown that no two class representatives are equivalent. Suppose $(\mathcal{D}, \mathbf{g}^{\lambda})$ and $(\mathcal{D}, \mathbf{g}^{\lambda'})$ are isometric (and so $\mathfrak{L}$-isometric) structures of the form given. Then there exists a Lie algebra automorphism

$$\psi = \begin{bmatrix} r^2 & v \\ 0 & rg \end{bmatrix} \quad \text{or} \quad \psi = \varsigma \begin{bmatrix} r^2 & v \\ 0 & rg \end{bmatrix}$$

such that $\psi \cdot \mathcal{D}(\mathbf{1}) = \mathcal{D}(\mathbf{1})$ and $\mathbf{g}_{\mathbf{1}}^{\lambda}(A, B) = \mathbf{g}_{\mathbf{1}}^{\lambda'}(\psi \cdot A, \psi \cdot B)$ for $A, B \in \mathcal{D}(\mathbf{1})$. The former condition implies $v = 0$ and so the latter implies $\Lambda = r^2 g^\top \Lambda' g$, where $\Lambda = \operatorname{diag}(\lambda_1, \lambda_1, \ldots, \lambda_n, \lambda_n)$ and $\Lambda' = \operatorname{diag}(\lambda'_1, \lambda'_1, \ldots, \lambda'_n, \lambda'_n)$. Thus, by Lemma 4.5, we have $\operatorname{Spec}(\Lambda) = r^2 \operatorname{Spec}(\Lambda')$. However, for both $\operatorname{Spec}(\Lambda)$ and $\operatorname{Spec}(\Lambda')$ the largest value is one; so $r = 1$. Consequently $\Lambda = \Lambda'$. That is to say, $(\mathcal{D}, \mathbf{g}^{\lambda})$ and $(\mathcal{D}, \mathbf{g}^{\lambda'})$ are isometric only if $\lambda = \lambda'$. □

Next, we consider the Riemannian case; the classification is very similar to the sub-Riemannian case.

Theorem 4.7. *Any left-invariant Riemannian structure on* H_{2n+1} *is isometric to exactly one of the structures* $\mathbf{g}^{\lambda}$ *specified by*

$$\mathbf{g}_{\mathbf{1}}^{\lambda} = \begin{bmatrix} 1 & 0 \\ 0 & \Lambda \end{bmatrix}, \qquad \Lambda = \operatorname{diag}(\lambda_1, \lambda_1, \lambda_2, \lambda_2, \ldots, \lambda_n, \lambda_n) \tag{4.3}$$

i.e., with orthonormal frame

$$\left(Z^L, \tfrac{1}{\sqrt{\lambda_1}} X_1^L, \tfrac{1}{\sqrt{\lambda_1}} Y_1^L, \tfrac{1}{\sqrt{\lambda_2}} X_2^L, \tfrac{1}{\sqrt{\lambda_2}} Y_2^L, \ldots, \tfrac{1}{\sqrt{\lambda_n}} X_n^L, \tfrac{1}{\sqrt{\lambda_n}} Y_n^L\right).$$

Here $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0$ *parametrize a family of (non-equivalent) class representatives.*

Proof. As H_{2n+1} is simply connected and nilpotent, two Riemannian structures $\mathbf{g}$ and $\mathbf{g}'$ on H_{2n+1} are isometric if and only if there exists $\psi \in \mathsf{Aut}(\mathfrak{h}_{2n+1})$ such that $\mathbf{g}_{\mathbf{1}}(A, B) = \mathbf{g}'(\psi \cdot A, \psi \cdot B)$ for $A, B \in \mathfrak{h}_{2n+1}$ (see Theorem 1.43). Hence it suffices to classify the positive definite quadratic forms on $\mathfrak{h}_{2n+1}$ up to Lie algebra automorphism.

Let $\boldsymbol{\mu}$ be a positive definite quadratic form on $\mathfrak{h}_{2n+1}$. We have

$$\boldsymbol{\mu} = \begin{bmatrix} \frac{1}{r^4} & v \\ v^\top & Q \end{bmatrix}$$

for some $r > 0$, $v \in \mathbb{R}^{2n}$ and $Q \in \mathbb{R}^{2n \times 2n}$. Hence

$$\psi = \begin{bmatrix} r^2 & -r^5 v \\ 0 & r I_{2n} \end{bmatrix} \in \mathsf{Aut}(\mathfrak{h}_{2n+1}) \quad \text{and} \quad \psi^\top \boldsymbol{\mu}\, \psi = \begin{bmatrix} 1 & 0 \\ 0 & Q' \end{bmatrix}$$

for some positive definite matrix Q'. Accordingly, there exists an automorphism $\psi' = \begin{bmatrix} 1 & 0 \\ 0 & g \end{bmatrix}$, $g \in \mathsf{Sp}\,(n, \mathbb{R})$ such that

$$(\psi \circ \psi')^\top \boldsymbol{\mu}\, (\psi \circ \psi') = \begin{bmatrix} 1 & 0 \\ 0 & \Lambda \end{bmatrix}$$

where $\Lambda = \operatorname{diag}(\lambda_1, \lambda_1, \ldots, \lambda_n, \lambda_n)$ (and $(\lambda_1, \ldots, \lambda_n) = \operatorname{Spec}(Q'))$.

As in the sub-Riemannian case, it is a simple matter to show that none of these structures are isometric. Suppose $\mathbf{g}^\lambda$ and $\mathbf{g}^{\lambda'}$ are isometric (and so $\mathfrak{L}$-isometric) structures of the form given. Then there exists a Lie algebra automorphism

$$\psi = \begin{bmatrix} r^2 & v \\ 0 & rg \end{bmatrix} \quad \text{or} \quad \psi = \varsigma \begin{bmatrix} r^2 & v \\ 0 & rg \end{bmatrix}$$

such that $\mathbf{g}_\mathbf{1}^\lambda(A,B) = \mathbf{g}_\mathbf{1}^{\lambda'}(\psi \cdot A, \psi \cdot B)$ for $A, B \in \mathfrak{h}_{2n+1}$. In either case we have $1 = r^4$, $0 = r^2 v$ and $\Lambda = r^2 g^\top \Lambda' g$. Hence $v = 0$, $r = 1$ and so, by Lemma 4.5, $\operatorname{Spec}(\Lambda) = \operatorname{Spec}(\Lambda')$. Consequently $\Lambda = \Lambda'$, i.e., $\mathbf{g}^\lambda$ and $\mathbf{g}^{\lambda'}$ are isometric only if $\lambda = \lambda'$. $\square$

4.3 Isometry groups

We calculate the group of isometries for each of the normal forms given in Section 4.2. Again, we denote the group of isometries of a structure $(\mathsf{H}_{2n+1}, \mathcal{D}, \mathbf{g})$ by $\mathsf{Iso}(\mathsf{H}_{2n+1}, \mathcal{D}, \mathbf{g})$; the subgroup of isotropies fixing the identity element is denoted $\mathsf{Iso}_\mathbf{1}(\mathsf{H}_{2n+1}, \mathcal{D}, \mathbf{g})$. By Theorems 1.43 and 1.45, the isotropy subgroup is given by

$$\mathsf{Iso}_\mathbf{1}(\mathsf{H}_{2n+1}, \mathcal{D}, \mathbf{g}) = \left\{ \phi \in \mathsf{Aut}(\mathsf{H}_{2n+1}) \ : \ \begin{array}{c} T_\mathbf{1}\phi \cdot \mathcal{D}(\mathbf{1}) = \mathcal{D}(\mathbf{1}) \\ \mathbf{g}_\mathbf{1}(A,B) = \mathbf{g}_\mathbf{1}(T_\mathbf{1}\phi \cdot A, T_\mathbf{1}\phi \cdot B) \end{array} \right\}.$$

The isometry group $\mathsf{Iso}(\mathsf{H}_{2n+1}, \mathcal{D}, \mathbf{g})$ decomposes as a semidirect product of the subgroup $L_{\mathsf{H}_{2n+1}} = \{L_g \ : \ g \in \mathsf{H}_{2n+1}\}$ of left translations and the isotropy subgroup $\mathsf{Iso}_\mathbf{1}(\mathsf{H}_{2n+1}, \mathcal{D}, \mathbf{g})$ of the identity (compare with Theorem 3.20).

As H_{2n+1} is simply connected, there is a one-to-one correspondence between the automorphisms of H_{2n+1} and the automorphisms of its Lie algebra $\mathfrak{h}_{2n+1}$. Moreover, as H_{2n+1} is nilpotent, the exponential map $\exp : \mathfrak{h}_{2n+1} \to \mathsf{H}_{2n+1}$ is a diffeomorphism (with inverse $\log : \mathsf{H}_{2n+1} \to \mathfrak{h}_{2n+1}$, see e.g., [59]) and so $\mathsf{Aut}(\mathsf{H}_{2n+1}) = \{\exp \circ \psi \circ \log \ : \ \psi \in \mathsf{Aut}(\mathfrak{h}_{2n+1})\}$. Accordingly, we shall denote by $d\,\mathsf{Iso}_\mathbf{1}(\mathsf{H}_{2n+1}, \mathcal{D}, \mathbf{g})$ the group $\{T_\mathbf{1}\phi \ : \ \phi \in \mathsf{Iso}_\mathbf{1}(\mathsf{H}_{2n+1}, \mathcal{D}, \mathbf{g})\}$ of linearized isotropies; we have $\mathsf{Iso}_\mathbf{1}(\mathsf{H}_{2n+1}, \mathcal{D}, \mathbf{g}) \cong d\,\mathsf{Iso}_\mathbf{1}(\mathsf{H}_{2n+1}, \mathcal{D}, \mathbf{g})$ and $\mathsf{Iso}_\mathbf{1}(\mathsf{H}_{2n+1}, \mathcal{D}, \mathbf{g}) = \{\exp \circ \psi \circ \log \ : \ \psi \in d\,\mathsf{Iso}_\mathbf{1}(\mathsf{H}_{2n+1}, \mathcal{D}, \mathbf{g})\}$. Furthermore,

$$d\,\mathsf{Iso}_\mathbf{1}(\mathsf{H}_{2n+1}, \mathcal{D}, \mathbf{g}) = \left\{ \psi \in \mathsf{Aut}(\mathfrak{h}_{2n+1}) \ : \ \begin{array}{c} \psi \cdot \mathcal{D}(\mathbf{1}) = \mathcal{D}(\mathbf{1}) \\ \mathbf{g}_\mathbf{1}(A,B) = \mathbf{g}_\mathbf{1}(\psi \cdot A, \psi \cdot B) \end{array} \right\}.$$

For a sub-Riemannian manifold (4.2) (resp. Riemannian manifold (4.3)) on H_{2n+1}, let $\eta_1 > \eta_2 > \cdots > \eta_k > 0$ denote the distinct values in the list $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $m_1, \ldots, m_k$ denote the corresponding multiplicities. We refer to the pair (η, m) as the *metric data* for the structure.

Theorem 4.8. *The group of linearized isotropies for the sub-Riemannian structure* (4.2) *(resp. Riemannian structure* (4.3)*) with metric data* (η, m) *is given by*

$$d\,\mathsf{Iso}_\mathbf{1}(\mathsf{H}_{2n+1}, \mathcal{D}, \mathbf{g}^\lambda) = \left\{ \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & g_1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & g_k \end{bmatrix}, \ \varsigma \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & g_1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & g_k \end{bmatrix} \ : \ g_i \in \mathsf{U}(m_i) \right\}.$$

Here the unitary group $\mathsf{U}(m_i) = \mathsf{Sp}(m_i, \mathbb{R}) \cap \mathsf{O}(2m_i)$ *where the orthogonal group* $\mathsf{O}(n) = \{g \in \mathbb{R}^{n\times n} : g^\top g = \mathbf{1}\}$*;* ς *is the involutive automorphism* (4.1).

Remark 4.9. In the following representations of the complex group, symplectic group, and orthogonal group

$$\begin{aligned}\mathsf{GL}(n, \mathbb{C}) &= \{g \in \mathbb{R}^{2n\times 2n} : g^{-1}Jg = J\}\\ \mathsf{Sp}(n, \mathbb{R}) &= \{g \in \mathbb{R}^{2n\times 2n} : g^\top Jg = J\}\\ \mathsf{O}(2n) &= \{g \in \mathbb{R}^{2n\times 2n} : g^\top = g^{-1}\}\end{aligned}$$

we have (see, e.g., [19, p. 225])

$$\mathsf{U}(n) = \mathsf{GL}(n, \mathbb{C}) \cap \mathsf{Sp}(n, \mathbb{R}) = \mathsf{GL}(n, \mathbb{C}) \cap \mathsf{O}(2n) = \mathsf{Sp}(n, \mathbb{R}) \cap \mathsf{O}(2n).$$

Proof. Suppose that $\psi \in d\,\mathsf{Iso}_{\mathbf{1}}(\mathsf{H}_{2n+1}, \mathcal{D}, \mathbf{g}^\lambda)$, i.e., $\psi \in \mathsf{Aut}(\mathfrak{h}_{2n+1})$, $\psi \cdot \mathcal{D}(\mathbf{1}) = \mathcal{D}(\mathbf{1})$, and $\mathbf{g}^\lambda_{\mathbf{1}}(A, B) = \mathbf{g}^\lambda_{\mathbf{1}}(\psi \cdot A, \psi \cdot B)$. As ψ is a Lie algebra automorphism, we have

$$\psi = \begin{bmatrix} r^2 & v \\ 0 & rg \end{bmatrix} \quad \text{or} \quad \psi = \varsigma \begin{bmatrix} r^2 & v \\ 0 & rg \end{bmatrix}$$

for some $r > 0$, $v \in \mathbb{R}^{1\times 2n}$ and $g \in \mathsf{Sp}(n, \mathbb{R})$. We need only consider the former case as $\varsigma \cdot \mathcal{D}(\mathbf{1}) = \mathcal{D}(\mathbf{1})$, and $\mathbf{g}^\lambda_{\mathbf{1}}(A, B) = \mathbf{g}^\lambda_{\mathbf{1}}(\varsigma \cdot A, \varsigma \cdot B)$, i.e., $\varsigma \in d\,\mathsf{Iso}_{\mathbf{1}}(\mathsf{H}_{2n+1}, \mathcal{D}, \mathbf{g}^\lambda)$.

For the sub-Riemannian case, we have $v = 0$ as $\psi \cdot \mathcal{D}(\mathbf{1}) = \mathcal{D}(\mathbf{1})$ and $r = 1$, $g^\top \Lambda g = \Lambda$ as $\mathbf{g}_{\mathbf{1}}(A, B) = \mathbf{g}_{\mathbf{1}}(\psi \cdot A, \psi \cdot B)$ for $A, B \in \mathcal{D}(\mathbf{1})$ (compare with the last paragraph of the proof of Theorem 4.6). Likewise, in the Riemannian case we have $r = 1$, $v = 0$ and $g^\top \Lambda g = \Lambda$ as $\mathbf{g}_{\mathbf{1}}(A, B) = \mathbf{g}_{\mathbf{1}}(\psi \cdot A, \psi \cdot B)$ for $A, B \in \mathfrak{h}_{2n+1}$ (compare with the last paragraph of the proof of Theorem 4.7).

In either case, we have

$$\psi = \begin{bmatrix} 1 & 0 \\ 0 & g \end{bmatrix}, \quad g^\top Jg = J, \quad \text{and} \quad g^\top \Lambda g = \Lambda.$$

From $g^\top Jg = J$ it follows that $g^\top = -Jg^{-1}J$. Hence from $g^\top \Lambda g = \Lambda$ it follows that $-Jg^{-1}J\Lambda g = \Lambda$. Consequently we have $J\Lambda g = gJ\Lambda$. As $\Lambda = -J\Lambda J$ we also have $\Lambda^2 = -J\Lambda J\Lambda$. Thus $\Lambda^2 g = g\Lambda^2$. Hence $\Lambda^2 g\boldsymbol{x} = \eta_i^2 g\boldsymbol{x}$ whenever $\boldsymbol{x}$ is an eigenvector of Λ^2 associated to the eigenvalue η_i^2. It follows that g (resp. $g^\top$) preserves each eigenspace of Λ^2. That is, $g\,\mathfrak{a}_i = \mathfrak{a}_i$ and $g^\top\,\mathfrak{a}_i = \mathfrak{a}_i$, $i = 1, \ldots, k$ where $\mathfrak{a}_i$ denotes the eigenspace of Λ^2 corresponding to μ_i^2. Therefore g takes block diagonal form

$$g = \begin{bmatrix} g_1 & & 0 \\ & \ddots & \\ 0 & & g_k \end{bmatrix}, \qquad g_i \in \mathsf{GL}(2m_i, \mathbb{R}).$$

Moreover, as $g^\top Jg = J$ and $g^\top \Lambda g = \Lambda$, we have $g_i \in \mathsf{Sp}(m_i, \mathbb{R}) \cap \mathsf{O}(2m_i) = \mathsf{U}(m_i)$.

On the other hand, suppose

$$\psi = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & g_1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & g_k \end{bmatrix}$$

is a linear map such that $g_i \in \mathsf{Sp}(m_i, \mathbb{R}) \cap \mathsf{O}(2m_i) = \mathsf{U}(m_i)$. As $g_i \in \mathsf{Sp}(m_i, \mathbb{R})$ we have that ψ is a Lie algebra automorphism. In the sub-Riemannian case we have that $\psi \cdot \mathcal{D}(\mathbf{1}) = \mathcal{D}(\mathbf{1})$ and $\mathbf{g_1}(A, B) = \mathbf{g_1}(\psi \cdot A, \psi \cdot B)$ for $A, B \in \mathcal{D}(\mathbf{1})$ as $g_i \in \mathsf{O}(2m_i)$. In the Riemannian case we likewise have $\mathbf{g_1}(A, B) = \mathbf{g_1}(\psi \cdot A, \psi \cdot B)$ for $A, B \in \mathfrak{h}_{2n+1}$. Thus the given maps are indeed linearized isotropies. □

Remark 4.10. We have that

$$\psi = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & g_1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & g_n \end{bmatrix}, \qquad g_i \in \mathsf{SO}(2) \tag{4.4}$$

is always a linearized isotropy of identity irrespective of the metric data. Hence, the isotropy subgroup must be at least n-dimensional.

Corollary 4.11. *For a sub-Riemannian or Riemannian manifold* $(\mathsf{H}_{2n+1}, \mathcal{D}, \mathbf{g}^\lambda)$ *with metric data* (η, m)*, we have*

$$\mathsf{Iso}_\mathbf{1}(\mathsf{H}_{2n+1}, \mathcal{D}, \mathbf{g}^\lambda) \cong (\mathsf{U}(m_1) \times \cdots \times \mathsf{U}(m_k)) \rtimes \{1, \varsigma\}$$

and so

$$n \leq \dim \mathsf{Iso}_\mathbf{1}(\mathsf{H}_{2n+1}, \mathcal{D}, \mathbf{g}^\lambda) = \sum_{i=1}^{k} m_i^2 \leq n^2.$$

The minimal dimension $\dim \mathsf{Iso}_\mathbf{1}(\mathsf{H}_{2n+1}, \mathcal{D}, \mathbf{g}^\lambda) = n$ *is attained when all values* $\lambda_1, \ldots, \lambda_n$ *are distinct (i.e.,* $(m_1, \ldots, m_k) = (1, 1, \ldots, 1)$*); the maximal dimension* $\dim \mathsf{Iso}_\mathbf{1}(\mathsf{H}_{2n+1}, \mathcal{D}, \mathbf{g}^\lambda) = n^2$ *is attained when the values* $\lambda_1, \ldots, \lambda_n$ *are all identical (i.e.,* $m_1 = n,\ k = 1$*).*

4.4 Geodesics

We determine (normal forms for) the geodesics of the sub-Riemannian and Riemannian structures on H_{2n+1}. In the sub-Riemannian case, the distribution is strongly bracket generating and hence there are no abnormal geodesics (see, e.g., [47, 94]). Henceforth, we shall refer to normal geodesics simply as geodesics. By inspection of the normal forms for the geodesics, we identify a number of totally geodesic subgroups.

Theorem 4.12. *The unit speed geodesics for the sub-Riemannian structure* (4.2) *are, up to composition with an isometry, given by* *(i)* $g(t) = m(z(t), x(t), y(t))$, *where*

$$\begin{aligned} z(t) &= \tfrac{1}{4}\sum_{i=1}^{n} \tfrac{c_i^2}{c_0^2}\left(\tfrac{2c_0}{\lambda_i}t - \sin(\tfrac{2c_0}{\lambda_i}t)\right) \\ x_i(t) &= \tfrac{c_i}{c_0}\sin(\tfrac{c_0}{\lambda_i}t) \\ y_i(t) &= \tfrac{c_i}{c_0}(1 - \cos(\tfrac{c_0}{\lambda_i}t)); \end{aligned}$$

(ii) $g(t) = m(0, x(t), 0)$, *where* $x_i(t) = \frac{c_i}{\lambda_i}t$.

Here $c_1, \ldots, c_n \geq 0$, $\sum_{i=1}^{n} \frac{c_i^2}{\lambda_i} = 1$ *and* $c_0 > 0$ *parametrize a family of geodesics.*

Proof. The sub-Riemannian structure (4.2) has orthonormal frame

$$(Z^L, \tfrac{1}{\sqrt{\lambda_1}}X_1^L, \tfrac{1}{\sqrt{\lambda_1}}Y_1^L, \tfrac{1}{\sqrt{\lambda_2}}X_2^L, \tfrac{1}{\sqrt{\lambda_2}}Y_2^L, \ldots, \tfrac{1}{\sqrt{\lambda_n}}X_n^L, \tfrac{1}{\sqrt{\lambda_n}}Y_n^L).$$

Accordingly, by Theorem 1.30, the geodesics are given by

$$\begin{aligned} \dot{z} &= \sum_{i=1}^{n} \tfrac{1}{\lambda_i}x_i p_{Y_i} & \dot{p}_Z &= 0 \\ \dot{x}_i &= \tfrac{1}{\lambda_i}p_{X_i} & \dot{p}_{X_i} &= -\tfrac{1}{\lambda_i}p_Z p_{Y_i} \\ \dot{y}_i &= \tfrac{1}{\lambda_i}p_{Y_i} & \dot{p}_{Y_i} &= \tfrac{1}{\lambda_i}p_Z p_{X_i}. \end{aligned}$$

Here $p = p_Z Z^* + \sum_{i=1}^{n}(p_{X_i}X_i^* + p_{Y_i}Y_i^*)$ where $(Z^*, X_1^*, Y_1^*, \ldots, X_n^*, Y_n^*)$ is the basis for $(\mathfrak{h}_{2n+1})^*_-$ dual to $(Z, X_1, Y_1, \ldots, X_n, Y_n)$; furthermore, $H(p) = \sum_{i=1}^{n} \frac{1}{\lambda_i}(p_{X_i}^2 + p_{Y_i}^2)$ and the extremal controls are $u_{X_i} = \frac{1}{\sqrt{\lambda_i}}p_{X_i}$ and $u_{Y_i} = \frac{1}{\sqrt{\lambda_i}}p_{Y_i}$.

We note that the action of an isotropy $\phi \in \mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathcal{D}, \mathbf{g}^\lambda)$ on a geodesic corresponds to the action $(g, p) \mapsto (\phi(g), (T_{\mathbf{1}}\phi)^* \cdot p)$ on an extremal curve $\xi(\cdot) = (g(\cdot), p(\cdot))$ (see the proof of Theorem 1.12). Hence, by application of the isometry ς (see (4.1)), we may assume $p_Z \geq 0$. By application of a left translation, we may assume $g_0 = \mathbf{1}$, i.e., $z(0) = x_i(0) = y_i(0) = 0$. By application of an isotropy of the form (4.4), we may assume $\dot{y}_i(0) = 0$ and $\dot{x}_i(0) \geq 0$, i.e., $p_{Y_i}(0) = 0$ and $p_{X_i}(0) \geq 0$. Let $a_i = p_{X_i}(0) \geq 0$ and $a_0 = p_Z \geq 0$. The corresponding integral curves (satisfying these initial conditions) are given by

$$\begin{aligned} z(t) &= \tfrac{1}{4}\sum_{i=1}^{n} \tfrac{a_i^2}{a_0^2}\left(\tfrac{2a_0}{\lambda_i}t - \sin(\tfrac{2a_0}{\lambda_i}t)\right) \\ x_i(t) &= \tfrac{a_i}{a_0}\sin(\tfrac{a_0}{\lambda_i}t) \\ y_i(t) &= \tfrac{a_i}{a_0}(1 - \cos(\tfrac{a_0}{\lambda_i}t)) \end{aligned}$$

if $a_0 > 0$ and $g(t) = m(0, x(t), 0)$, $x_i(t) = \frac{a_i}{\lambda_i}t$ if $a_0 = 0$.

The length of the curve $g(\cdot) = m(z(t), x(t), y(t))$ from $\mathbf{1}$ to $g(T)$ is

$$\ell(g(\cdot)) = \int_0^T \sqrt{\textstyle\sum_{i=1}^{n}\lambda_i(\dot{x}_i^2 + \dot{y}_i^2)}\, dt = T\sqrt{\textstyle\sum_{i=1}^{n}\frac{a_i^2}{\lambda_i}}.$$

Hence, replacing t by $\frac{1}{\sqrt{\sum_{i=1}^{n} \frac{a_i^2}{\lambda_i}}} t$ yields the required unit speed curve. Let $c_0 = \frac{a_0}{\sqrt{\sum_{i=1}^{n} \frac{a_i^2}{\lambda_i}}}$ and $c_i = \frac{a_i}{\sqrt{\sum_{i=1}^{n} \frac{a_i^2}{\lambda_i}}}$. (Note that $\frac{a_i}{a_0} = \frac{c_i}{c_0}$.) Then $\sum_{i=1}^{n} \frac{c_i^2}{\lambda_i} = 1$ and we get the given expression. □

We say that a submanifold N of G is a *totally geodesic submanifold* of the sub-Riemannian (or Riemannian) structure $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ if it satisfies the following property: whenever a geodesic $g(\cdot)$ is tangent to N at some point $g \in \mathsf{N}$, then the entire trace of $g(\cdot)$ is contained in N.

Corollary 4.13. *The subgroups with Lie algebra spanned by*

$$(Z, X_{i_1}, Y_{i_1}, \ldots, X_{i_k}, Y_{i_k}), \qquad 1 \leq i_1 < i_2 < \ldots < i_k \leq n, \quad 1 \leq k \leq n$$

are totally geodesic submanifolds of (4.2).

Remark 4.14. In the case of minimal symmetry, i.e., $\lambda_1 < \lambda_2 < \cdots < \lambda_n$, no further normalization of the geodesics (as stated in Theorem 4.12) is possible. However, in the other cases, more normalization is possible. We give details here for the case of maximal symmetry (i.e., $\lambda_1 = \lambda_2 = \cdots = \lambda_n$). We have

$$\psi = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & \cos\theta & 0 & \sin\theta \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & -\sin\theta & 0 & \cos\theta \end{bmatrix} \in \mathsf{U}(2) = \mathsf{Sp}(2, \mathbb{R}) \cap \mathsf{O}(4).$$

Accordingly, there exist $\tilde{\psi}_j \in d\,\mathsf{Iso}_{\mathbf{1}}(\mathsf{H}_{2n+1}, \mathcal{D}, \mathbf{g}^\lambda)$, $j = 1, \ldots, n-1$ such that

$$\tilde{\psi}\Big|_{\mathrm{span}(X_j, Y_j, X_{j+1}, Y_{j+1})} = \psi$$

and $\tilde{\psi}(Z) = Z$, $\tilde{\psi}(X_i) = X_i$, $\tilde{\psi}(Y_i) = Y_i$, $i = 1, \ldots, j-1, j+2, \ldots, n$. Notice that $\psi \cdot (r\cos\theta, 0, r\sin\theta, 0) = (r, 0, 0, 0)$. Accordingly, the unit speed geodesics are, up to composition with an isometry, given by

(i) $g(t) = m(z(t), x(t), y(t))$, where

$$z(t) = \tfrac{1}{2c_0} t - \tfrac{\lambda_1}{4c_0^2} \sin(\tfrac{2c_0}{\lambda_1} t)$$
$$x_1(t) = \tfrac{\sqrt{\lambda_1}}{c_0} \sin(\tfrac{c_0}{\lambda_1} t) \qquad y_1(t) = \tfrac{\sqrt{\lambda_1}}{c_0}(1 - \cos(\tfrac{c_0}{\lambda_1} t))$$
$$x_j(t) = y_j(t) = 0,\ j = 2, \ldots, n, \qquad c_0 > 0.$$

(ii) $g(t) = m(0, x(t), 0)$, where $x_1(t) = \frac{1}{\sqrt{\lambda_1}} t$ and $x_j(t) = 0$, $j = 2, \ldots, n$.

Hence any geodesic is the image, under an isometry, of a geodesic of the totally geodesic subgroup with Lie algebra generated by Z, X_1, Y_1. Accordingly, in the case of maximal symmetry the problem of determining minimal geodesics on H_{2n+1} reduces to determining minimal geodesics on H_3 (indeed, $g(\cdot)$ is a minimising geodesic if and only if its image $\phi \circ g(\cdot)$, evolving on the totally geodesic subgroup H_3, is a minimising geodesic).

We now proceed to the Riemannian case. The proof of the following theorem is very similar to that of Theorem 4.12 and hence omitted. The main difference is that in the equations for the geodesics $\dot{z} = \sum_{i=1}^{n} \frac{1}{\lambda_i} x_i p_{Y_i}$ is replaced by $\dot{z} = p_Z + \sum_{i=1}^{n} \frac{1}{\lambda_i} x_i p_{Y_i}$. Remarkably, the only difference in the expressions for the Riemannian geodesics is the introduction of the $c_0 t$ term for $z(t)$; this similarity can be explained by the fact that the Riemannian structure is a central expansion of the sub-Riemannian structure (see the remarks that follow).

Theorem 4.15. *The unit speed geodesic $g(\cdot)$ through $g(0) = \mathbf{1}$ with $\dot{g}(0) = a_0 Z + \sum_{i=1}^{n} \frac{a_i}{\lambda_i} X_i + \frac{b_i}{\lambda_i} Y_i$ for the Riemannian structure* (4.3) *is, up to a composition with an isometry, given by*

(i) $g(t) = m(z(t), x(t), y(t))$, *where*

$$\begin{aligned} z(t) &= c_0 t + \tfrac{1}{4} \sum_{i=1}^{n} \tfrac{c_i^2}{c_0^2} \left(\tfrac{2c_0}{\lambda_i} t - \sin(\tfrac{2c_0}{\lambda_i} t) \right) \\ x_i(t) &= \tfrac{c_i}{c_0} \sin(\tfrac{c_0}{\lambda_i} t) \\ y_i(t) &= \tfrac{c_i}{c_0} (1 - \cos(\tfrac{c_0}{\lambda_i} t)) \end{aligned}$$

if $a_0 \neq 0$*;*

(ii) $g(t) = m(0, x(t), 0)$, *where* $x_i(t) = \frac{c_i}{\lambda_i} t$ *if* $a_0 = 0$.

Here $c_i = \sqrt{a_i^2 + b_i^2}$ *and* $c_0 = |a_0|$.

Corollary 4.16. *The subgroups with Lie algebra spanned by*

$$(Z, X_{i_1}, Y_{i_1}, \ldots, X_{i_k}, Y_{i_k}) \qquad 1 \leq i_1 < i_2 < \ldots < i_k \leq n, \quad 1 \leq k \leq n$$

are totally geodesic submanifolds of (4.3).

Remark 4.17. In the case of maximal symmetry (i.e., $\lambda_1 = \lambda_2 = \cdots = \lambda_n$) any geodesic is the image, under an isometry, of a geodesic of the totally geodesic subgroup with Lie algebra spanned by Z, X_1, Y_1. This follows in the same way as for the sub-Riemannian case (see Remark 4.14).

4.5 Remarks

The Riemannian structures (4.3) are central expansions of the sub-Riemannian structures (4.2) with respect to $\mathsf{Z}(\mathsf{H}_{2n+1})$; this is most easily seen by inspection of the given orthonormal frames (see Lemma 1.49). Accordingly, the images of the respective sets of geodesics under the quotient map

$$q : \mathsf{H}_{2n+1} \to \mathbb{R}^{2n} \cong \mathsf{H}_{2n+1} / \mathsf{Z}(\mathsf{H}_{2n+1}), \qquad m(z, x, y) \mapsto (x, y)$$

are identical. In terms of the parametrisations of the geodesics given in Theorems 4.12 and 4.15, this means that the expressions for the $(x_1, y_1, \ldots, x_n, y_n)$ coordinates should match (see Theorem 1.50 and its corollary), which is indeed the case.

On the other hand, we note that the subclass $g(t) = m(0, x(t), 0)$, $x_i(t) = \frac{c_i}{\lambda_i} t$ of geodesics of (4.2) are the lifts of geodesics (being straight lines) of an invariant Riemannian structure on $\mathbb{R}^{2n}$ (compare with Proposition 1.59; see also Corollary 1.36). Indeed, we have that

$$q : (\mathsf{H}_{2n}, \mathcal{D}, \mathbf{g}^{\lambda}) \to (\mathbb{R}^{2n}, \bar{\mathbf{g}}^{\lambda})$$

is a **LiSR**-epimorphism; here $\bar{\mathbf{g}}^{\lambda}_{\mathbf{1}}$ is the inner product on $\mathbb{R}^2$ with respect to which $(T_{\mathbf{1}}q \cdot \frac{1}{\sqrt{\lambda_1}} X_1, T_{\mathbf{1}}q \cdot \frac{1}{\sqrt{\lambda_1}} Y_1, \ldots T_{\mathbf{1}}q \cdot \frac{1}{\sqrt{\lambda_n}} X_n, T_{\mathbf{1}}q \cdot \frac{1}{\sqrt{\lambda_n}} Y_n)$ is orthonormal.

Conclusion

This thesis investigated equivalences and interrelations between cost-extended systems (associated to invariant optimal control problems), invariant sub-Riemannian structures, and quadratic Hamilton-Poisson systems, both within the context of each of these classes as well as between these classes. In Chapter 1 we developed a (simple and elegant) framework for these purposes. By formulating each class as a category, equivalence relations for cost-extended systems, sub-Riemannian structures, and Hamilton-Poisson systems were introduced. The category of sub-Riemannian structures was shown to be functorially equivalent to a subcategory of the category cost-extended systems. Accordingly, sub-Riemannian structures are equivalent (i.e., $\mathfrak{L}$-isometric up to rescaling) if and only if the corresponding cost-extended systems are equivalent. Furthermore, equivalence of cost-extended systems (resp. sub-Riemannian structures) implies equivalence of the associated Hamilton-Poisson systems. It was also shown that the geodesics of central expansion of an invariant sub-Riemannian structure are closely related to the geodesics of that structure. The primary aim in our formalism is to facilitate the systematic investigation of these various systems and structures.

A few examples illustrating some of the main results of Chapter 1 were discussed (at the end of Chapter 1); these examples pertained to the properties of structures on the three-dimensional Heisenberg group. In Section 4.5 we noticed that some properties generalized to the $(2n+1)$-dimensional Heisenberg groups. With regards to quotient objects (in the respective categories) and central expansions (of sub-Riemannian structures), we would like to briefly point out two other significant cases. Any rank-two invariant sub-Riemannian structure (i.e., $\dim \mathcal{D}(\mathbf{1}) = 2$) on the four-dimensional Engel group (see, e.g., [3, 17]) admits both an Abelian Riemannian quotient structure (by taking the quotient of the group by its commutator subgroup) and a sub-Riemannian quotient structure on the three-dimensional Heisenberg group (by taking the quotient of the group by its center). Accordingly, some subclasses of geodesics on the Engel group will be intimately related to the geodesics on these quotient structures. Moreover, the rank-two invariant sub-Riemannian structures on the Engel group admit rank-three central expansions. For a non-nilpotent example, we have the rank-two sub-Riemannian structures on the four-dimensional oscillator group ([39]). These structures admit sub-Riemannian quotient structures on the three-dimensional Euclidean group $\mathsf{SE}(2)$ (by taking the quotient of the group by its center). Hence, a subclass of geodesics for these structures will be intimately related to the sub-Riemannian geodesics on $\mathsf{SE}(2)$ (which were studied in [92, 111]). The rank-two sub-Riemannian structures on the oscillator group likewise admit rank-three central expansions. Hence, we believe that quotients and central expansions will be useful new tools for the investigation of invariant sub-Riemannian structures.

In Chapter 2 we classified the positive semidefinite quadratic Hamilton-Poisson systems on three-dimensional Lie-Poisson spaces. An exhaustive and nonredundent list of normal forms was obtained. Some simple invariants for these systems allow for a taxonomy of systems on each Lie-Poisson space, thus giving us the means (in most cases) to easily identify the normal form of an arbitrary system. The primary aim of this chapter was to unite and systematise the treatment of these systems, especially as several systems may be equivalently realized on a number of non-isomorphic Lie-Poisson spaces (see Tables 2.1 and 2.2). By studying normal forms for systems, one can accomplish more elegant and clear results. A systematic treatment of the stability of equilibria and computation of integral curves for this class of systems is currently in preparation. Avenues of further investigation include considering the class of all quadratic Hamilton-Poisson systems in three dimensions (i.e., not only the positive semidefinite ones), a systematic treatment of the inhomogeneous systems (for $\mathfrak{se}(2)^*_-$ some partial results were obtained in [5, 9]), and a classification of homogeneous systems in four dimensions.

In Chapter 3 we classified the invariant sub-Riemannian structures in three dimensions and determined the subgroup of isometries fixing the identity for each normal form. By comparing our results to that of Agrachev and Barilari [10], we were able to show that, quite remarkably, most isometries are the composition of a left-translation and a group automorphism. This begs the question as to whether this property generally holds true for some class of structures beyond nilpotent Riemannian structures and sub-Riemannian Carnot groups (cf. Theorems 1.43 and 1.45). Towards this end, it would be of interest to carry out a similar study (including classifications) of the invariant Riemannian structures in three dimensions, as well as the sub-Riemannian structures in four dimensions. Such studies would of course also be of interest in their own right. So far, the invariant three-dimensional Riemannian structures have been classified up to $\mathfrak{L}$-isometry in [62] and the larger class of homogeneous Riemannian structures has been classified up to isometry in [99]; some partial classifications of sub-Riemannian structures in four dimensions have also been obtained (see, e.g., [15, 56]). We are currently preparing a classification of the invariant distributions on four-dimensional Lie groups (up to group automorphism).

The classification of invariant sub-Riemannian structures in Chapter 3 can be reinterpreted as a classification of the two-input cost-extended systems (where the system is linear in the control and the cost is homogeneous). It would be of interest to expand this classification to cover all cost extended systems in three-dimensions (cf. Proposition 1.56). Another avenue of investigation would be to formalise the inhomogeneous cost-extended systems as affine distributions together with quadratic forms and consider diffeomorphisms relating such structures; such an approach has been followed to study the geometry of more general (point-)affine distributions on manifolds in [49, 50].

In Chapter 4 we classified the Riemannian and sub-Riemannian structures on the Heisenberg groups, determined the associated isometry groups, and briefly investigated the geodesics. By making use of the isometries (and normal forms for geodesics) we suspect that we may be able to find a simpler derivation and description of the minimising geodesics in the general case (cf. [27]). Furthermore, investigation of the similarities between the Riemannian and sub-Riemannian structures may elucidate some more general properties for central expansions. As prototypical cases, it would also be of interest to classify and study more generally the cost-extended systems (or affine distributions) on the Heisenberg groups.

Appendix A

Three-dimensional Lie algebras and groups

A.1 Classification

There are eleven types of three-dimensional real Lie algebras; in fact, nine algebras and two parametrized infinite families of algebras (see, e.g., [76, 85, 95]). In terms of an (appropriate) ordered basis (E_1, E_2, E_3), the commutation operation is given by

$$\begin{aligned}[E_2, E_3] &= n_1 E_1 - \alpha E_2 \\ [E_3, E_1] &= \alpha E_1 + n_2 E_2 \\ [E_1, E_2] &= n_3 E_3.\end{aligned}$$

The structure parameters α, n_1, n_2, n_3 for each type are given in Table A.1.

Note. Throughout this thesis we shall use a basis for $\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1$ different from the one listed in Table A.1. More precisely, we use the basis $E_1' = \frac{1}{2}(E_1 - E_2)$, $E_2' = -\frac{1}{2}E_3$, $E_3' = \frac{1}{2}(E_1 + E_2)$; the only nonzero commutator is $[E_1', E_2'] = E_1'$.

A classification of the three-dimensional (real connected) Lie groups can be found in [59]. Let G be a three-dimensional (real connected) Lie group with Lie algebra $\mathfrak{g}$.

1. If $\mathfrak{g}$ is Abelian, i.e., $\mathfrak{g} \cong 3\mathfrak{g}_1$, then G is isomorphic to $\mathbb{R}^3$, $\mathbb{R}^2 \times \mathbb{T}$, $\mathbb{R} \times \mathbb{T}$, or $\mathbb{T}^3$.
2. If $\mathfrak{g} \cong \mathfrak{g}_{2.1} \oplus \mathfrak{g}_1$, then G is isomorphic to $\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}$ or $\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{T}$.
3. If $\mathfrak{g} \cong \mathfrak{g}_{3.1}$, then G is isomorphic to the Heisenberg group H_3 or the Lie group $\mathsf{H}_3^* = \mathsf{H}_3 / Z(\mathsf{H}_3(\mathbb{Z}))$, where $Z(\mathsf{H}_3(\mathbb{Z}))$ is the group of integer points in the centre $Z(\mathsf{H}_3) \cong \mathbb{R}$ of H_3.
4. If $\mathfrak{g} \cong \mathfrak{g}_{3.2}$, $\mathfrak{g}_{3.3}$, $\mathfrak{g}_{3.4}^0$, $\mathfrak{g}_{3.4}^\alpha$, or $\mathfrak{g}_{3.5}^\alpha$, then G is isomorphic to the simply connected Lie group $\mathsf{G}_{3.2}$, $\mathsf{G}_{3.3}$, $\mathsf{G}_{3.4}^0 = \mathsf{SE}(1,1)$, $\mathsf{G}_{3.4}^\alpha$, or $\mathsf{G}_{3.5}^\alpha$, respectively (the centres of these groups are trivial.)
5. If $\mathfrak{g} \cong \mathfrak{g}_{3.5}^0$, then G is isomorphic to the Euclidean group $\mathsf{SE}(2)$, the n-fold covering $\mathsf{SE}_n(2)$ of $\mathsf{SE}_1(2) = \mathsf{SE}(2)$, or the universal covering group $\widetilde{\mathsf{SE}}(2)$.
6. If $\mathfrak{g} \cong \mathfrak{g}_{3.6}$, then G is isomorphic to the pseudo-orthogonal group $\mathsf{SO}(2,1)_0$, the n-fold covering A_n of $\mathsf{SO}(2,1)_0$, or the universal covering group $\widetilde{\mathsf{A}}$. Here $\mathsf{A}_2 \cong \mathsf{SL}(2,\mathbb{R})$.

Table A.1: Three-dimensional Lie algebras

	α	n_1	n_2	n_3	Unimodular	Nilpotent	Compl. Solv.	Exponential	Solvable	Simple	Connected Groups
$3\mathfrak{g}_1$	0	0	0	0	•	•	•	•	•		$\mathbb{R}^3,\ \mathbb{R}^2\times\mathbb{T},\ \mathbb{R}\times\mathbb{T}^2,\ \mathbb{T}^3$
$\mathfrak{g}_{2.1}\oplus\mathfrak{g}_1$	1	1	-1	0			•	•	•		$\mathsf{Aff}\,(\mathbb{R})_0\times\mathbb{R},\ \mathsf{Aff}\,(\mathbb{R})_0\times\mathbb{T}$
$\mathfrak{g}_{3.1}$	0	1	0	0	•	•	•	•	•		$\mathsf{H}_3,\ \mathsf{H}_3^*$
$\mathfrak{g}_{3.2}$	1	1	0	0			•	•	•		$\mathsf{G}_{3.2}$
$\mathfrak{g}_{3.3}$	1	0	0	0			•	•	•		$\mathsf{G}_{3.3}$
$\mathfrak{g}_{3.4}^0$	0	1	-1	0	•		•	•	•		$\mathsf{SE}\,(1,1)$
$\mathfrak{g}_{3.4}^\alpha$	$\alpha>0$, $\alpha\neq1$	1	-1	0			•	•	•		$\mathsf{G}_{3.4}^\alpha$
$\mathfrak{g}_{3.5}^0$	0	1	1	0	•				•		$\widetilde{\mathsf{SE}}\,(2),\ \mathsf{SE}_n(2),\ \mathsf{SE}\,(2)$
$\mathfrak{g}_{3.5}^\alpha$	$\alpha>0$	1	1	0				•	•		$\mathsf{G}_{3.5}^\alpha$
$\mathfrak{g}_{3.6}$	0	1	1	-1	•					•	$\widetilde{\mathsf{A}},\ \mathsf{A}_n,\ \mathsf{SL}\,(2,\mathbb{R}),\ \mathsf{SO}\,(2,1)_0$
$\mathfrak{g}_{3.7}$	0	1	1	1	•					•	$\mathsf{SU}\,(2),\ \mathsf{SO}\,(3)$

7. If $\mathfrak{g}\cong\mathfrak{g}_{3.7}$, then G is isomorphic to either the unitary group $\mathsf{SU}\,(2)$ or the orthogonal group $\mathsf{SO}\,(3)$.

Among these Lie groups, only H_3^*, A_n, $n\geq3$, and $\widetilde{\mathsf{A}}$ are not matrix Lie groups.

A.2 Matrix Lie groups

We have the following parametrizations of the solvable three-dimensional matrix Lie groups and their Lie algebras (cf. [37, 58]). We omit the Abelian groups.

$$\mathsf{Aff}\,(\mathbb{R})_0\times\mathbb{R} : \begin{bmatrix}1 & 0 & 0\\ x & e^{-y} & 0\\ 0 & 0 & e^{z}\end{bmatrix} \qquad \mathfrak{g}_{2.1}\oplus\mathfrak{g}_1 : \begin{bmatrix}0 & 0 & 0\\ x & -y & 0\\ 0 & 0 & z\end{bmatrix}$$

$$\mathsf{Aff}\,(\mathbb{R})_0\times\mathbb{T} : \begin{bmatrix}1 & 0 & 0\\ x & e^{-y} & 0\\ 0 & 0 & e^{iz}\end{bmatrix} \qquad \mathfrak{g}_{2.1}\oplus\mathfrak{g}_1 : \begin{bmatrix}0 & 0 & 0\\ x & -y & 0\\ 0 & 0 & iz\end{bmatrix}$$

$$\mathsf{H}_3 : \begin{bmatrix}1 & y & x\\ 0 & 1 & z\\ 0 & 0 & 1\end{bmatrix} \qquad \mathfrak{h}_3 : \begin{bmatrix}0 & y & x\\ 0 & 0 & z\\ 0 & 0 & 0\end{bmatrix}$$

$$\mathsf{G}_{3.2}: \begin{bmatrix} 1 & 0 & 0 \\ y & e^z & 0 \\ x & -ze^z & e^z \end{bmatrix} \qquad \mathfrak{g}_{3.2}: \begin{bmatrix} 0 & 0 & 0 \\ y & z & 0 \\ x & -z & z \end{bmatrix}$$

$$\mathsf{G}_{3.3}: \begin{bmatrix} 1 & 0 & 0 \\ y & e^z & 0 \\ x & 0 & e^z \end{bmatrix} \qquad \mathfrak{g}_{3.3}: \begin{bmatrix} 0 & 0 & 0 \\ y & z & 0 \\ x & 0 & z \end{bmatrix}$$

$$\mathsf{SE}(1,1): \begin{bmatrix} 1 & 0 & 0 \\ x & \cosh z & -\sinh z \\ y & -\sinh z & \cosh z \end{bmatrix} \qquad \mathfrak{se}(1,1): \begin{bmatrix} 0 & 0 & 0 \\ x & 0 & -z \\ y & -z & 0 \end{bmatrix}$$

$$\underset{\substack{a>0 \\ a\neq 1}}{\mathsf{G}^{\alpha}_{3.4}}: \begin{bmatrix} 1 & 0 & 0 \\ x & e^{\alpha z}\cosh z & -e^{\alpha z}\sinh z \\ y & -e^{\alpha z}\sinh z & e^{\alpha z}\cosh z \end{bmatrix} \qquad \underset{\substack{a>0 \\ a\neq 1}}{\mathfrak{g}^{\alpha}_{3.4}}: \begin{bmatrix} 0 & 0 & 0 \\ x & \alpha z & -z \\ y & -z & \alpha z \end{bmatrix}$$

$$\widetilde{\mathsf{SE}}(2): \begin{bmatrix} 1 & 0 & 0 & 0 \\ x & \cos z & -\sin z & 0 \\ y & \sin z & \cos z & 0 \\ 0 & 0 & 0 & e^z \end{bmatrix} \qquad \widetilde{\mathfrak{se}}(2): \begin{bmatrix} 0 & 0 & 0 & 0 \\ x & 0 & -z & 0 \\ y & z & 0 & 0 \\ 0 & 0 & 0 & z \end{bmatrix}$$

$$\mathsf{SE}_n(2): \begin{bmatrix} 1 & 0 & 0 & 0 \\ x & \cos z & -\sin z & 0 \\ y & \sin z & \cos z & 0 \\ 0 & 0 & 0 & e^{\frac{iz}{n}} \end{bmatrix} \qquad \mathfrak{se}_n(2): \begin{bmatrix} 0 & 0 & 0 & 0 \\ x & 0 & -z & 0 \\ y & z & 0 & 0 \\ 0 & 0 & 0 & \frac{iz}{n} \end{bmatrix}$$

$$\mathsf{SE}(2): \begin{bmatrix} 1 & 0 & 0 \\ x & \cos z & -\sin z \\ y & \sin z & \cos z \end{bmatrix} \qquad \mathfrak{se}(2): \begin{bmatrix} 0 & 0 & 0 \\ x & 0 & -z \\ y & z & 0 \end{bmatrix}$$

$$\mathsf{G}^{\alpha}_{3.5}: \begin{bmatrix} 1 & 0 & 0 \\ x & e^{\alpha z}\cos z & -e^{\alpha z}\sin z \\ y & e^{\alpha z}\sin z & e^{\alpha z}\cos z \end{bmatrix} \qquad \mathfrak{g}^{\alpha}_{3.5}: \begin{bmatrix} 0 & 0 & 0 \\ x & \alpha z & -z \\ y & z & \alpha z \end{bmatrix}$$

An appropriate ordered basis for the Lie algebra in each case is given by setting $(x, y, z) = (1, 0, 0)$ for E_1, $(x, y, z) = (0, 1, 0)$ for E_2, and $(x, y, z) = (0, 0, 1)$ for E_3.

Matrix Lie groups with algebra $\mathfrak{g}_{3.6}$

There are only two connected matrix Lie groups with Lie algebra $\mathfrak{g}_{3.6}$, namely the pseudo-orthogonal group $\mathsf{SO}(2,1)_0$ and the special linear group $\mathsf{SL}(2,\mathbb{R})$; $\mathsf{SL}(2,\mathbb{R})$ is a double cover of $\mathsf{SO}(2,1)_0$.

The pseudo-orthogonal group

$$\mathsf{SO}(2,1) = \{g \in \mathbb{R}^{3\times 3} \,:\, g^\top J g = J, \det g = 1\}$$

has two connected components. Here $J = \mathrm{diag}(1, 1, -1)$. The identity component of $\mathsf{SO}(2,1)$ is $\mathsf{SO}(2,1)_0 = \{g \in \mathsf{SO}(2,1) \,:\, g_{33} > 0\}$ where $g = \begin{bmatrix} g_{ij} \end{bmatrix}$ (for $g \in \mathsf{SO}(2,1)$). Its Lie algebra is given by

$$\begin{aligned}\mathfrak{so}(2,1) &= \{A \in \mathbb{R}^{3\times 3} \,:\, A^\top J + JA = 0\} \\ &= \left\{ \begin{bmatrix} 0 & z & y \\ -z & 0 & x \\ y & x & 0 \end{bmatrix} \,:\, x,y,z \in \mathbb{R} \right\}.\end{aligned}$$

On the other hand, the special linear group is given by

$$\mathsf{SL}(2,\mathbb{R}) = \{g \in \mathbb{R}^{2\times 2} \,:\, \det g = 1\}.$$

Its Lie algebra is given by

$$\mathfrak{sl}(2,\mathbb{R}) = \left\{ \begin{bmatrix} \frac{x}{2} & \frac{y-z}{2} \\ \frac{y+z}{2} & -\frac{x}{2} \end{bmatrix} \,:\, x,y,z \in \mathbb{R} \right\}.$$

Matrix Lie groups with algebra $\mathfrak{g}_{3.7}$

There are exactly two connected Lie groups with Lie algebra $\mathfrak{g}_{3.7}$; both are matrix Lie groups. The special unitary group and its Lie algebra are given by

$$\begin{aligned}\mathsf{SU}(2) &= \left\{ g \in \mathbb{C}^{2\times 2} \,:\, gg^\dagger = \mathbf{1},\, \det g = 1 \right\} \\ \mathfrak{su}(2) &= \left\{ \begin{bmatrix} \frac{i}{2}x & \frac{1}{2}(iz+y) \\ \frac{1}{2}(iz-y) & -\frac{i}{2}x \end{bmatrix} \,:\, x,y,z \in \mathbb{R} \right\}\end{aligned}$$

$\mathsf{SU}(2)$ is a double cover of the orthogonal group $\mathsf{SO}(3)$. The orthogonal group $\mathsf{SO}(3)$ and its Lie algebra are given by

$$\begin{aligned}\mathsf{SO}(3) &= \left\{ g \in \mathbb{R}^{3\times 3} \,:\, gg^\top = \mathbf{1},\, \det g = 1 \right\} \\ \mathfrak{so}(3) &= \left\{ \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} \,:\, x,y,z \in \mathbb{R} \right\}.\end{aligned}$$

Note. Again, for both $\mathfrak{g}_{3.6}$ and $\mathfrak{g}_{3.7}$, an appropriate ordered basis for the Lie algebra in each case is given by setting $(x,y,z) = (1,0,0)$ for E_1, $(x,y,z) = (0,1,0)$ for E_2, and $(x,y,z) = (0,0,1)$ for E_3.

A.3 Automorphism groups

A standard computation yields the automorphism group for each three-dimensional Lie algebra (see, e.g., [64]). With respect to the given ordered basis (E_1, E_2, E_3), the automorphism group of each solvable Lie algebra has parametrization:

$$\mathsf{Aut}(\mathfrak{g}_{3.1}) \ : \ \begin{bmatrix} yw - vz & x & u \\ 0 & y & v \\ 0 & z & w \end{bmatrix} \qquad \mathsf{Aut}(\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1) \ : \ \begin{bmatrix} x & y & 0 \\ 0 & 1 & 0 \\ 0 & u & v \end{bmatrix}$$

$$\mathsf{Aut}(\mathfrak{g}_{3.2}) \ : \ \begin{bmatrix} u & x & y \\ 0 & u & z \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathsf{Aut}(\mathfrak{g}_{3.3}) \ : \ \begin{bmatrix} x & y & z \\ u & v & w \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathsf{Aut}(\mathfrak{g}^0_{3.4}) \ : \ \begin{bmatrix} x & y & u \\ y & x & v \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} x & y & u \\ -y & -x & v \\ 0 & 0 & -1 \end{bmatrix} \qquad \mathsf{Aut}(\mathfrak{g}^a_{3.4}) \ : \ \begin{bmatrix} x & y & u \\ y & x & v \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathsf{Aut}(\mathfrak{g}^0_{3.5})) \ : \ \begin{bmatrix} x & y & u \\ -y & x & v \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} x & y & u \\ y & -x & v \\ 0 & 0 & -1 \end{bmatrix} \qquad \mathsf{Aut}(\mathfrak{g}^a_{3.5}) \ : \ \begin{bmatrix} x & y & u \\ -y & x & v \\ 0 & 0 & 1 \end{bmatrix}$$

For the semisimple Lie algebras, we have

$$\mathsf{Aut}(\mathfrak{g}_{3.6}) = \mathsf{SO}\,(2,1) \qquad \text{and} \qquad \mathsf{Aut}(\mathfrak{g}_{3.7}) = \mathsf{SO}\,(3).$$

Appendix B

Tables

B.1 Quadratic Hamilton-Poisson systems in three dimensions

Tabulation of systems and equilibria

Table B.1: Ruled systems (Equations of motion and equilibria)

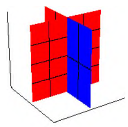
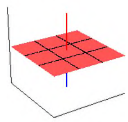
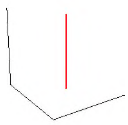
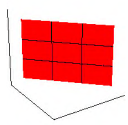
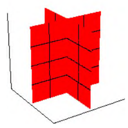
System	Eqn. of motion	Equilibria
$\mathsf{R}(1)\ :\ ((\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1)^*_-, p_2^2)$	$\begin{cases} \dot{p}_1 = -2p_1p_2 \\ \dot{p}_2 = 0 \\ \dot{p}_3 = 0 \end{cases}$	
$\mathsf{R}(2)\ :\ ((\mathfrak{g}_{3.3})^*_-, p_1^2 + p_2^2)$	$\begin{cases} \dot{p}_1 = 2p_1p_3 \\ \dot{p}_2 = 2p_2p_3 \\ \dot{p}_3 = 0 \end{cases}$	
$\mathsf{R}(3)\ :\ ((\mathfrak{g}_{3.3}, p_3^2)$	$\begin{cases} \dot{p}_1 = 0 \\ \dot{p}_2 = 0 \\ \dot{p}_3 = -2(p_1^2 + p_2^2) \end{cases}$	
$\mathsf{R}(4)\ :\ ((\mathfrak{g}^0_{3.4})^*_-, (p_1 + p_2)^2)$	$\begin{cases} \dot{p}_1 = 0 \\ \dot{p}_2 = 0 \\ \dot{p}_3 = 2(p_1 + p_2)^2 \end{cases}$	
$\mathsf{R}(5)\ :\ ((\mathfrak{g}^0_{3.5})^*_-, p_2^2)$	$\begin{cases} \dot{p}_1 = 0 \\ \dot{p}_2 = 0 \\ \dot{p}_3 = 2p_1p_2 \end{cases}$	

Table B.2: Planar systems (Equations of motion and equilibria)

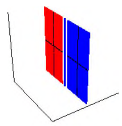
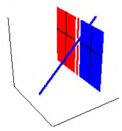
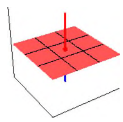
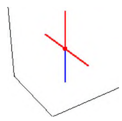
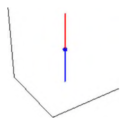
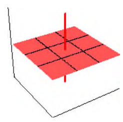
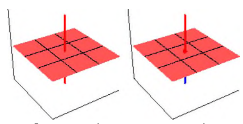
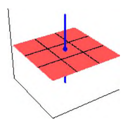
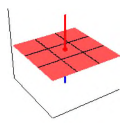
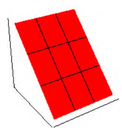
System	Eqn. of motion	Equilibria
$\mathsf{P}(1) \; : \; ((\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1)^*_-, p_1^2 + p_2^2)$	$\begin{cases} \dot{p}_1 = -2p_1p_2 \\ \dot{p}_2 = 2p_1^2 \\ \dot{p}_3 = 0 \end{cases}$	
$\mathsf{P}(2) \; : \; ((\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1)^*_-, p_2^2 + (p_1 + p_3)^2)$	$\begin{cases} \dot{p}_1 = -2p_1p_2 \\ \dot{p}_2 = 2p_1(p_1 + p_3) \\ \dot{p}_3 = 0 \end{cases}$	
$\mathsf{P}(3) \; : \; ((\mathfrak{g}_{3.2})^*_-, p_3^2)$	$\begin{cases} \dot{p}_1 = 2p_1p_3 \\ \dot{p}_2 = 2(-p_1 + p_2)p_3 \\ \dot{p}_3 = 0 \end{cases}$	
$\mathsf{P}(4) \; : \; ((\mathfrak{g}_{3.3})^*_-, p_1^2 + p_3^2)$	$\begin{cases} \dot{p}_1 = 2p_1p_3 \\ \dot{p}_2 = 2p_2p_3 \\ \dot{p}_3 = -2p_1^2 \end{cases}$	
$\mathsf{P}(5) \; : \; ((\mathfrak{g}_{3.3})^*_-, p_1^2 + p_2^2 + p_3^2)$	$\begin{cases} \dot{p}_1 = 2p_1p_3 \\ \dot{p}_2 = 2p_2p_3 \\ \dot{p}_3 = -2(p_1^2 + p_2^2) \end{cases}$	
$\mathsf{P}(6) \; : \; ((\mathfrak{g}_{3.4}^0)^*_-, p_3^2)$	$\begin{cases} \dot{p}_1 = -2p_2p_3 \\ \dot{p}_2 = -2p_1p_3 \\ \dot{p}_3 = 0 \end{cases}$	
$\mathsf{P}(7) \; : \; ((\mathfrak{g}_{3.4}^\alpha)^*_-, p_3^2)$	$\begin{cases} \dot{p}_1 = 2(\alpha\, p_1 - p_2)p_3 \\ \dot{p}_2 = -2(p_1 - \alpha\, p_2)p_3 \\ \dot{p}_3 = 0 \end{cases}$	 $0<\alpha<1$ $\quad$ $\alpha>1$
$\mathsf{P}(8) \; : \; ((\mathfrak{g}_{3.5}^0)^*_-, p_3^2)$	$\begin{cases} \dot{p}_1 = 2p_2p_3 \\ \dot{p}_2 = -2p_1p_3 \\ \dot{p}_3 = 0 \end{cases}$	
$\mathsf{P}(9) \; : \; ((\mathfrak{g}_{3.5}^\alpha)^*_-, p_3^2)$	$\begin{cases} \dot{p}_1 = 2(\alpha\, p_1 + p_2)p_3 \\ \dot{p}_2 = -2(p_1 - \alpha\, p_2)p_3 \\ \dot{p}_3 = 0 \end{cases}$	
$\mathsf{P}(10) \; : \; ((\mathfrak{g}_{3.6})^*_-, (p_2 + p_3)^2)$	$\begin{cases} \dot{p}_1 = 2(p_2 + p_3)^2 \\ \dot{p}_2 = -2p_1(p_2 + p_3) \\ \dot{p}_3 = 2p_1(p_2 + p_3) \end{cases}$	

Table B.3: Non-planar systems (Equations of motion and equilibria)

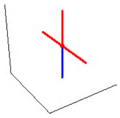
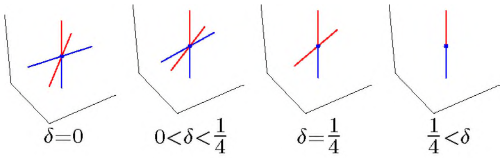
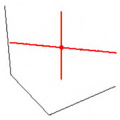
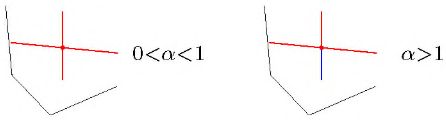
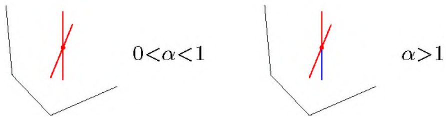
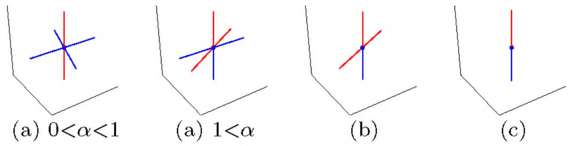
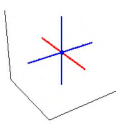
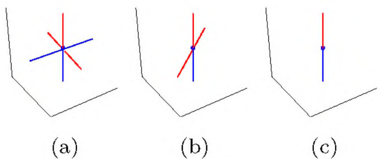
System	Equilibria
$\mathsf{Np}(1)\ :\ ((\mathfrak{g}_{3.2})^*_-, p_1^2+p_3^2)$ $\begin{cases}\dot p_1 = 2p_1p_3\\ \dot p_2 = 2(-p_1+p_2)p_3\\ \dot p_3 = -2p_1^2\end{cases}$	
$\mathsf{Np}(2)\ :\ ((\mathfrak{g}_{3.2})^*_-, \delta\, p_1^2+p_2^2+p_3^2)$ $\begin{cases}\dot p_1 = 2p_1p_3\\ \dot p_2 = 2p_1(-p_1+p_2)p_3\\ \dot p_3 = -2(\delta\, p_1^2-p_1p_2+p_2^2)\end{cases}$	
$\mathsf{Np}(3)\ :\ ((\mathfrak{g}^0_{3.4})^*_-, (p_1+p_2)^2+p_3^2)$ $\begin{cases}\dot p_1 = -2p_2p_3\\ \dot p_2 = -2p_1p_3\\ \dot p_3 = 2(p_1+p_2)^2\end{cases}$	
$\mathsf{Np}(4)\ :\ ((\mathfrak{g}^\alpha_{3.4})^*_-, (p_1+p_2)^2+p_3^2)$ $\begin{cases}\dot p_1 = 2(\alpha\, p_1-p_2)p_3\\ \dot p_2 = -2(p_1-\alpha\, p_2)p_3\\ \dot p_3 = -2(\alpha-1)(p_1+p_2)^2\end{cases}$	
$\mathsf{Np}(5)\ :\ ((\mathfrak{g}^\alpha_{3.4})^*_-, (p_1-p_2)^2+p_3^2)$ $\begin{cases}\dot p_1 = 2(\alpha\, p_1-p_2)p_3\\ \dot p_2 = -2(p_1-\alpha\, p_2)p_3\\ \dot p_3 = -2(1+\alpha)(p_1-p_2)^2\end{cases}$	
$\mathsf{Np}(6)\ :\ ((\mathfrak{g}^\alpha_{3.4})^*_-, \beta\, p_1^2+p_2^2+p_3^2)$ $\begin{cases}\dot p_1=2(\alpha\, p_1-p_2)p_3\\ \dot p_2=-2(p_1-\alpha\, p_2)p_3\\ \dot p_3=-2(\alpha\beta\, p_1^2-(\beta+1)p_1p_2+\alpha\, p_2^2)\end{cases}$	
$\mathsf{Np}(7)\ :\ ((\mathfrak{g}^0_{3.5})^*_-, p_2^2+p_3^2)$ $\begin{cases}\dot p_1 = 2p_2p_3\\ \dot p_2 = -2p_1p_3\\ \dot p_3 = 2p_1p_2\end{cases}$	
$\mathsf{Np}(8)\ :\ ((\mathfrak{g}^\alpha_{3.5})^*_-, \beta\, p_1^2+p_2^2+p_3^2)$ $\begin{cases}\dot p_1=2(\alpha\, p_1+p_2)p_3\\ \dot p_2=-2(p_1-\alpha\, p_2)p_3\\ \dot p_3=-2(\alpha\beta\, p_1^2+(\beta-1)p_1p_2+\alpha\, p_2^2)\end{cases}$	

Table B.4: Equilibria of systems

Normal form	Set of equilibria ($\mu, \nu \in \mathbb{R}$)	Eq. Index
$\mathsf{R}(1)$	$(0,\nu,\mu),\ (\nu,0,\mu)$	$(0,2)$
$\mathsf{R}(2)$	$(\nu,\mu,0),\ (0,0,\mu)$	$(1,1)$
$\mathsf{R}(3)$	$(0,0,\mu)$	$(1,0)$
$\mathsf{R}(4)$	$(\nu,-\nu,\mu)$	$(0,1)$
$\mathsf{R}(5)$	$(0,\nu,\mu),\ (\nu,0,\mu)$	$(0,2)$
$\mathsf{P}(1)$	$(0,\nu,\mu)$	$(0,1)$
$\mathsf{P}(2)$	$(0,\nu,\mu),\ (\mu,0,-\mu)$	$(1,1)$
$\mathsf{P}(3)$	$(\nu,\mu,0),\ (0,0,\mu)$	$(1,1)$
$\mathsf{P}(4)$	$(0,\mu,0),\ (0,0,\mu)$	$(2,0)$
$\mathsf{P}(5)$	$(0,0,\mu)$	$(1,0)$
$\mathsf{P}(6)$	$(\nu,\mu,0),\ (0,0,\mu)$	$(1,1)$
$\mathsf{P}(7)$	$(0,0,\mu),\ (\nu,\mu,0)$	$(1,1)$
$\mathsf{P}(8)$	$(\nu,\mu,0),\ (0,0,\mu)$	$(1,1)$
$\mathsf{P}(9)$	$(\nu,\mu,0),\ (0,0,\mu)$	$(1,1)$
$\mathsf{P}(10)$	$(\nu,\mu,-\mu)$	$(0,1)$
$\mathsf{Np}(1)$	$(0,\mu,0),\ (0,0,\mu)$	$(2,0)$
$\mathsf{Np}(2)$ $0\leq\delta<\frac{1}{4}$	$(0,0,\mu),\ (\mu,\frac{1}{2}(1\pm\sqrt{1-4\delta})\mu,0)$	$(3,0)$
$\mathsf{Np}(2)$ $\delta=\frac{1}{4}$	$(0,0,\mu),\ (\mu,\frac{1}{2}\mu,0)$	$(2,0)$
$\mathsf{Np}(2)$ $\delta>\frac{1}{4}$	$(0,0,\mu)$	$(1,0)$
$\mathsf{Np}(3)$	$(\mu,-\mu,0),\ (0,0,\mu)$	$(2,0)$
$\mathsf{Np}(4)$	$(\mu,-\mu,0),\ (0,0,\mu)$	$(2,0)$
$\mathsf{Np}(5)$	$(\mu,\mu,0),\ (0,0,\mu)$	$(2,0)$
$\mathsf{Np}(6)$ $(0<\alpha<1$ and $0\leq\beta\leq1)$ or $(\alpha>1$ and $0\leq\beta<-1+2\alpha^2-2\alpha\sqrt{\alpha^2-1})$		
	$(0,0,\mu),\ (\mu,\frac{(1+\beta)\pm\sqrt{1+\beta(2-4\alpha^2+\beta)}}{2\alpha}\mu,0)$	$(3,0)$
$\mathsf{Np}(6)$ $\alpha>1$ and $\beta=-1+2\alpha^2-2\alpha\sqrt{\alpha^2-1}$	$(0,0,\mu),\ (\mu,(\alpha-\sqrt{\alpha^2-1})\mu,0)$	$(2,0)$
$\mathsf{Np}(6)$ $\alpha>1$ and $-1+2\alpha^2-2\alpha\sqrt{\alpha^2-1}<\beta\leq1$	$(0,0,\mu)$	$(1,0)$
$\mathsf{Np}(7)$	$(\mu,0,0),\ (0,\mu,0),\ (0,0,\mu)$	$(3,0)$
$\mathsf{Np}(8)$ $0\leq\beta<1+2\alpha^2-2\alpha\sqrt{\alpha^2+1}$	$(0,0,\mu),\ (\mu,\frac{1-\beta\pm\sqrt{1-2\beta-4\alpha^2\beta+\beta^2}}{2\alpha}\mu,0)$	$(3,0)$
$\mathsf{Np}(8)$ $\beta=1+2\alpha^2-2\alpha\sqrt{\alpha^2+1}$	$(0,0,\mu),\ (\mu,(-\alpha+\sqrt{1+\alpha^2})\mu,0)$	$(2,0)$
$\mathsf{Np}(8)$ $1+2\alpha^2-2\alpha\sqrt{\alpha^2+1}<\beta\leq1$	$(0,0,\mu)$	$(1,0)$

Taxonomy of systems

Table B.5: Taxonomy of 3D systems, semisimple algebras

Algebra	Class	Eq. Index	Normal Forms
$\mathfrak{g}_{3.6}$	planar	$(1,1)$	$\mathsf{P}(6)$; $\mathsf{P}(8)$
		$(0,1)$	$\mathsf{P}(10)$
	non-planar	$(2,0)$	$\mathsf{Np}(3)$
		$(3,0)$	$\mathsf{Np}(7)$
$\mathfrak{g}_{3.7}$	planar	$(1,1)$	$\mathsf{P}(8)$
	non-planar	$(3,0)$	$\mathsf{Np}(7)$

Table B.6: Taxonomy of 3D systems, solvable algebras I

Algebra	Class	Eq. Index	Normal Forms
$\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1$	ruled	$(0,1)$	$\mathsf{R}(4)$
		$(0,2)$	$\mathsf{R}(1)$; $\mathsf{R}(5)$
	planar	$(1,1)$	$\mathsf{P}(2)$
		$(0,1)$	$\mathsf{P}(1)$
$\mathfrak{g}_{3.1}$	ruled	$(0,2)$	$\mathsf{R}(5)$
	planar	$(1,1)$	$\mathsf{P}(8)$
$\mathfrak{g}_{3.2}$	linear	$(1,0)$	$\mathsf{R}(3)$
		$(0,1)$	$\mathsf{R}(4)$
		$(0,2)$	$\mathsf{R}(5)$
	planar	$(1,1)$	$\mathsf{P}(3)$
	non-planar	$(1,0)$	$\mathsf{Np}(2)$, $\delta > \frac{1}{4}$
		$(2,0)$	$\mathsf{Np}(1)$; $\mathsf{Np}(2)$, $\delta = \frac{1}{4}$
		$(3,0)$	$\mathsf{Np}(2)$, $0 \leq \delta < \frac{1}{4}$
$\mathfrak{g}_{3.3}$	ruled	$(1,0)$	$\mathsf{R}(3)$
		$(1,1)$	$\mathsf{R}(2)$
		$(0,1)$	$\mathsf{R}(4)$
	planar	$(1,0)$	$\mathsf{P}(5)$
		$(2,0)$	$\mathsf{P}(4)$

Table B.7: Taxonomy of 3D systems, solvable algebras II

Algebra	Class	Eq. Index	Normal Forms
$\mathfrak{g}_{3.4}^0$	ruled	$(0,1)$	$\mathsf{R}(4)$
		$(0,2)$	$\mathsf{R}(5)$
	planar	$(1,1)$	$\mathsf{P}(6)$
	non-planar	$(2,0)$	$\mathsf{Np}(3)$
		$(3,0)$	$\mathsf{Np}(7)$
$\mathfrak{g}_{3.4}^\alpha$	ruled	$(1,0)$	$\mathsf{R}(3)$
		$(0,1)$	$\mathsf{R}(4)$
		$(0,2)$	$\mathsf{R}(5)$
	planar	$(1,1)$	$\mathsf{P}(7)$
	non-planar	$(1,0)$	$\mathsf{Np}(6)$, case d
		$(2,0)$	$\mathsf{Np}(4)$; $\mathsf{Np}(5)$;
			$\mathsf{Np}(6)$, case c
		$(3,0)$	$\mathsf{Np}(6)$, cases a&b
$\mathfrak{g}_{3.5}^0$	ruled	$(0,2)$	$\mathsf{R}(5)$
	planar	$(1,1)$	$\mathsf{P}(8)$
	non-planar	$(3,0)$	$\mathsf{Np}(7)$
$\mathfrak{g}_{3.5}^\alpha$	ruled	$(1,0)$	$\mathsf{R}(3)$
		$(0,1)$	$\mathsf{R}(4)$
		$(0,2)$	$\mathsf{R}(5)$
	planar	$(1,1)$	$\mathsf{P}(9)$
	non-planar	$(1,0)$	$\mathsf{Np}(8)$, $\kappa_\alpha^- < \beta \leq 1$
		$(2,0)$	$\mathsf{Np}(8)$, $\beta = \kappa_\alpha^-$
		$(3,0)$	$\mathsf{Np}(8)$, $0 \leq \beta < \kappa_\alpha^-$

B.2 Three-dimensional sub-Riemannian structures

Table B.8: Invariant structures on simple groups

Group & algebra	Orthonormal frame	$d\,\mathsf{Iso}_1(\mathsf{G}, \mathcal{D}, \mathbf{g})$	$\mathfrak{H}(\mathsf{G}, \mathcal{D}, \mathbf{g})$
$\tilde{\mathsf{A}}$ $[E_2, E_3] = E_1$ $[E_3, E_1] = E_2$ $[E_1, E_2] = -E_3$	$(\frac{1}{\sqrt{\alpha}} E_1^L, E_2^L)$ $0<\alpha<1$	$\left\{ \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_1\sigma_2 \end{bmatrix} : \sigma_1, \sigma_2 = \pm 1 \right\}$	$\mathsf{Np}(7)$
	(E_1^L, E_2^L)	$\left\{ \begin{bmatrix} g & & 0 \\ & & 0 \\ 0 & 0 & \det g \end{bmatrix} : g \in \mathsf{O}(2) \right\}$	$\mathsf{P}(8)$
	$(\frac{1}{\sqrt{\alpha}} E_2^L, E_3^L)$ $0<\alpha$	$\left\{ \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_1\sigma_2 \end{bmatrix} : \sigma_1, \sigma_2 = \pm 1 \right\}$	$\mathsf{Np}(7)$
$\mathsf{SU}(2)$ $[E_2, E_3] = E_1$ $[E_3, E_1] = E_2$ $[E_1, E_2] = E_3$	$(\frac{1}{\sqrt{\alpha}} E_2^L, E_3^L)$ $0<\alpha<1$	$\left\{ \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_1\sigma_2 \end{bmatrix} : \sigma_1, \sigma_2 = \pm 1 \right\}$	$\mathsf{Np}(7)$
	(E_2^L, E_3^L)	$\left\{ \begin{bmatrix} \det g & 0 & 0 \\ 0 & & \\ 0 & & g \end{bmatrix} : g \in \mathsf{O}(2) \right\}$	$\mathsf{P}(8)$

Table B.9: Invariant structures on solvable groups

Group & algebra	Orthonormal frame	$d\,\mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathcal{D}, \mathbf{g})$	$\mathfrak{H}(\mathsf{G}, \mathcal{D}, \mathbf{g})$
$\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}$ $[E_2, E_3] = 0$ $[E_3, E_1] = 0$ $[E_1, E_2] = E_1$	$(E_1^L + E_3^L, E_2^L)$	$\left\{ \begin{bmatrix} \sigma\cos\theta & -\sigma\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ -\sigma + \sigma\cos\theta & -\sigma\sin\theta & \sigma \end{bmatrix} \,:\, \theta \in \mathbb{R},\, \sigma = \pm 1 \right\} \cong \mathsf{O}(2)$	$\mathsf{P}(2)$
H_3 $[E_2, E_3] = E_1$ $[E_3, E_1] = 0$ $[E_1, E_2] = 0$	(E_2^L, E_3^L)	$\left\{ \begin{bmatrix} \det g & 0 \quad 0 \\ 0 & \multirow{2}{*}{g} \\ 0 & \end{bmatrix} \,:\, g \in \mathsf{O}(2) \right\}$	$\mathsf{P}(8)$
$\mathsf{G}_{3.2}$ $[E_2, E_3] = E_1 - E_2$ $[E_3, E_1] = E_1$ $[E_1, E_2] = 0$	(E_2^L, E_3^L)	$\left\{ \begin{bmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & 1 \end{bmatrix} \,:\, \sigma = \pm 1 \right\}$	$\mathsf{Np}(2)_{\delta=0}$
$\mathsf{SE}(1,1)$ $[E_2, E_3] = E_1$ $[E_3, E_1] = -E_2$ $[E_1, E_2] = 0$	(E_2^L, E_3^L)	$\left\{ \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_1\sigma_2 & 0 \\ 0 & 0 & \sigma_2 \end{bmatrix} \,:\, \sigma_1, \sigma_2 = \pm 1 \right\}$	$\mathsf{Np}(7)$
$\mathsf{G}_{3.4}^{\alpha}$ $[E_2, E_3] = E_1 - \alpha\, E_2$ $[E_3, E_1] = \alpha\, E_1 - E_2$ $[E_1, E_2] = 0$	(E_2^L, E_3^L)	$\left\{ \begin{bmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & 1 \end{bmatrix} \,:\, \sigma = \pm 1 \right\}$	$\mathsf{Np}(6)_{\beta=0}$
$\widetilde{\mathsf{SE}}(2)$ $[E_2, E_3] = E_1$ $[E_3, E_1] = E_2$ $[E_1, E_2] = 0$	(E_2^L, E_3^L)	$\left\{ \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_1\sigma_2 & 0 \\ 0 & 0 & \sigma_2 \end{bmatrix} \,:\, \sigma_1, \sigma_2 = \pm 1 \right\}$	$\mathsf{Np}(7)$
$\mathsf{G}_{3.5}^{\alpha}$ $[E_2, E_3] = E_1 - \alpha\, E_2$ $[E_3, E_1] = \alpha\, E_1 + E_2$ $[E_1, E_2] = 0$	(E_2^L, E_3^L)	$\left\{ \begin{bmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & 1 \end{bmatrix} \,:\, \sigma = \pm 1 \right\}$	$\mathsf{Np}(8)_{\beta=0}$

Appendix C

Mathematica notebooks

Wolfram Mathematica 8 was used for various computations; we include here some simplified notebooks as typical examples. Input is presented in **bold**, and output not bold.

C.1 Hamilton-Poisson systems on $(\mathfrak{g}_{3.2})^*_-$

Algebra

Mg32[{x_, y_, z_}]:= $\begin{pmatrix} 0 & 0 & 0 \\ y & z & 0 \\ x & -z & z \end{pmatrix}$**;**

Common functions

cc[A_, B_]:= $A.B - B.A$**;**

pp = { p_1, p_2, p_3 **};**

Minv[MP_, A_]:=

Module[{ss, z1, z2, z3},

ss =

Solve[

A **== z1MP[{1, 0, 0}]+**

z2MP[{0, 1, 0}]+

z3MP[{0, 0, 1}],

{z1, z2, z3}];

{z1, z2, z3}/.ss[[1]]

];

PB[MP_, F_, G_]:=

−pp.Minv[MP, cc[MP@ $D[F$ **, {pp}],**

MP@ $D[G$ **, {pp}]]];**

Hvec[MP_, H_, pv_]:=

Table [PB [MP, p_i, H **] , {** i **, 3}] /.**

{ p_1 **→ pv[[1]],** p_2 **→ pv[[2]],**

p_3 **→ pv[[3]]} ;**

Hvec[MP_, Hmax_]:=Hvec[MP, Hmax, pp]

Family $\beta {p_1}^2 + {p_2}^2 + {p_3}^2$

ψ **=** $\begin{pmatrix} \psi 11 & \psi 12 & \psi 13 \\ \psi 21 & \psi 22 & \psi 23 \\ \psi 31 & \psi 32 & \psi 33 \end{pmatrix}$**;**

H1 = $\beta {p_1}^2 + {p_2}^2 + {p_3}^2$**;**

H2 = $\gamma {p_1}^2 + {p_2}^2 + {p_3}^2$**;**

Alg = {Mg32, Mg32};

ψ**.Hvec[Alg[[1]], H1]//**

FullSimplify

Hvec[Alg[[2]], H2, ψ**.pp]//**

FullSimplify

Print["— Conditions —"];

Thread[%% == %%%]

Print[

"— Coefficients equated —"];

Flatten[D[D[%%, {pp}], {pp}]]

Print["— Reduced equations —"];

Reduce[Flatten[{%%, Det[ψ] ≠ 0}]]

$\{-2\psi 13\left(\beta p_1^2 - p_1p_2 + p_2^2\right) +$
$2\left((\psi 11 - \psi 12)p_1 + \psi 12 p_2\right)p_3,$
$-2\psi 23\left(\beta p_1^2 - p_1p_2 + p_2^2\right) +$
$2\left((\psi 21 - \psi 22)p_1 + \psi 22 p_2\right)p_3,$
$-2\psi 33\left(\beta p_1^2 - p_1p_2 + p_2^2\right) +$
$2\left((\psi 31 - \psi 32)p_1 + \psi 32 p_2\right)p_3\}$

$\{2\left(\psi 11 p_1 + \psi 12 p_2 + \psi 13 p_3\right)$
$\left(\psi 31 p_1 + \psi 32 p_2 + \psi 33 p_3\right),$
$2\left((-\psi 11 + \psi 21)p_1 + (-\psi 12 + \psi 22)p_2 +\right.$
$\left.(-\psi 13 + \psi 23)p_3\right)$
$\left(\psi 31 p_1 + \psi 32 p_2 + \psi 33 p_3\right),$
$-2\left(\psi 11 p_1 + \psi 12 p_2 + \psi 13 p_3\right)$
$\left((\gamma\psi 11 - \psi 21)p_1 + (\gamma\psi 12 - \psi 22)p_2 +\right.$
$\left.(\gamma\psi 13 - \psi 23)p_3\right) -$
$2\left(\psi 21 p_1 + \psi 22 p_2 + \psi 23 p_3\right)^2\}$

— Conditions —

$\{2\left(\psi 11 p_1 + \psi 12 p_2 + \psi 13 p_3\right)$
$\left(\psi 31 p_1 + \psi 32 p_2 + \psi 33 p_3\right) ==$
$-2\psi 13\left(\beta p_1^2 - p_1p_2 + p_2^2\right) +$
$2\left((\psi 11 - \psi 12)p_1 + \psi 12 p_2\right)p_3,$
$2\left((-\psi 11 + \psi 21)p_1 + (-\psi 12 + \psi 22)p_2 +\right.$
$\left.(-\psi 13 + \psi 23)p_3\right)$
$\left(\psi 31 p_1 + \psi 32 p_2 + \psi 33 p_3\right) ==$
$-2\psi 23\left(\beta p_1^2 - p_1p_2 + p_2^2\right) +$
$2\left((\psi 21 - \psi 22)p_1 + \psi 22 p_2\right)p_3,$
$-2\left(\psi 11 p_1 + \psi 12 p_2 + \psi 13 p_3\right)$
$\left((\gamma\psi 11 - \psi 21)p_1 + (\gamma\psi 12 - \psi 22)p_2 +\right.$
$\left.(\gamma\psi 13 - \psi 23)p_3\right) -$
$2\left(\psi 21 p_1 + \psi 22 p_2 + \psi 23 p_3\right)^2 ==$
$-2\psi 33\left(\beta p_1^2 - p_1p_2 + p_2^2\right) +$
$2\left((\psi 31 - \psi 32)p_1 + \psi 32 p_2\right)p_3\}$

— Coefficients equated —

$\{4\psi 11\psi 31 == -4\beta\psi 13,$
$2\psi 12\psi 31 + 2\psi 11\psi 32 == 2\psi 13,$
$2\psi 13\psi 31 + 2\psi 11\psi 33 == 2(\psi 11 - \psi 12),$
$2\psi 12\psi 31 + 2\psi 11\psi 32 == 2\psi 13,$
$4\psi 12\psi 32 == -4\psi 13,$
$2\psi 13\psi 32 + 2\psi 12\psi 33 == 2\psi 12,$
$2\psi 13\psi 31 + 2\psi 11\psi 33 == 2(\psi 11 - \psi 12),$
$2\psi 13\psi 32 + 2\psi 12\psi 33 == 2\psi 12,$
$4\psi 13\psi 33 == 0,$
$4(-\psi 11 + \psi 21)\psi 31 == -4\beta\psi 23,$
$2(-\psi 12 + \psi 22)\psi 31 + 2(-\psi 11 + \psi 21)\psi 32 ==$
$2\psi 23, 2(-\psi 13 + \psi 23)\psi 31 +$
$2(-\psi 11 + \psi 21)\psi 33 == 2(\psi 21 - \psi 22),$
$2(-\psi 12 + \psi 22)\psi 31 + 2(-\psi 11 + \psi 21)\psi 32 ==$
$2\psi 23, 4(-\psi 12 + \psi 22)\psi 32 == -4\psi 23,$
$2(-\psi 13 + \psi 23)\psi 32 + 2(-\psi 12 + \psi 22)\psi 33 ==$
$2\psi 22, 2(-\psi 13 + \psi 23)\psi 31 +$
$2(-\psi 11 + \psi 21)\psi 33 == 2(\psi 21 - \psi 22),$
$2(-\psi 13 + \psi 23)\psi 32 + 2(-\psi 12 + \psi 22)\psi 33 ==$
$2\psi 22, 4(-\psi 13 + \psi 23)\psi 33 == 0,$
$-4\psi 11(\gamma\psi 11 - \psi 21) - 4\psi 21^2 == -4\beta\psi 33,$
$-2\psi 12(\gamma\psi 11 - \psi 21) -$

$2\psi 11(\gamma\psi 12 - \psi 22) - 4\psi 21\psi 22 == 2\psi 33,$

$-2\psi 13(\gamma\psi 11 - \psi 21) -$

$2\psi 11(\gamma\psi 13 - \psi 23) - 4\psi 21\psi 23 ==$

$2(\psi 31 - \psi 32), -2\psi 12(\gamma\psi 11 - \psi 21) -$

$2\psi 11(\gamma\psi 12 - \psi 22) - 4\psi 21\psi 22 == 2\psi 33,$

$-4\psi 12(\gamma\psi 12 - \psi 22) - 4\psi 22^2 == -4\psi 33,$

$-2\psi 13(\gamma\psi 12 - \psi 22) -$

$2\psi 12(\gamma\psi 13 - \psi 23) - 4\psi 22\psi 23 == 2\psi 32,$

$-2\psi 13(\gamma\psi 11 - \psi 21) -$

$2\psi 11(\gamma\psi 13 - \psi 23) - 4\psi 21\psi 23 ==$

$2(\psi 31 - \psi 32), -2\psi 13(\gamma\psi 12 - \psi 22) -$

$2\psi 12(\gamma\psi 13 - \psi 23) - 4\psi 22\psi 23 == 2\psi 32,$

$-4\psi 13(\gamma\psi 13 - \psi 23) - 4\psi 23^2 == 0\}$

— Reduced equations —

$\psi 33 == 1\&\&\psi 32 == 0\&\&\psi 31 == 0\&\&\psi 23 ==$

$0\&\&$

$(\psi 22 == -1 \| \psi 22 == 1)\&\&\psi 21 == 0\&\&$

$\psi 13 == 0\&\&\psi 12 == 0\&\&\psi 11 == \psi 22\&\&\beta == \gamma$

C.2 Contact structure for $(\widetilde{\mathsf{SE}}(2), \mathcal{D}, \mathbf{g})$

Note. This notebook relies on the *Differential Forms* Mathematica package (Version 3.1, February 2007) by Frank Zizza (see Section 3.1.2).

Setup

```
crd = {x1, x2, x3};
XB = {X[x1], X[x2], X[x3]};
crdv = IdentityMatrix[Length[crd]];
ip[a_, b_]:=InteriorProduct[a, b];
LB[X_, Y_]:=
Sum[
Sum[
X[[i]]D[Y[[j]], crd[[i]]]
crdv[[j]]−
Y[[j]]D[X[[i]], crd[[j]]]
crdv[[i]], {i, 1, Length[crd]}],
{j, 1, Length[crd]}];
SetAttributes[{λ, a1, a2, a3, α},
Constant]
```

Group and algebra

$m[\{x\_, y\_, z\_\}] := \begin{pmatrix} 1 & 0 & 0 & 0 \\ x & \mathrm{Cos}[z] & -\mathrm{Sin}[z] & 0 \\ y & \mathrm{Sin}[z] & \mathrm{Cos}[z] & 0 \\ 0 & 0 & 0 & e^z \end{pmatrix};$

$M[\{x\_, y\_, z\_\}] := \begin{pmatrix} 0 & 0 & 0 & 0 \\ x & 0 & -z & 0 \\ y & z & 0 & 0 \\ 0 & 0 & 0 & z \end{pmatrix};$

Maurer-Cartan frame

```
Tm[{x1_, x2_, x3_}, {v1_, v2_, v3_}]:=
(D[m[{x1 + tv1, x2 + tv2, x3 + tv3}],
t]/.t → 0);
Tminv[{x1_, x2_, x3_}, mxA_]:=
({v1, v2, v3}/.
(Solve[Thread[
Flatten[Tm[{x1, x2, x3},
```

```
{v1, v2, v3}]] ==
Flatten[mxA]],
{v1, v2, v3}])[[1]]);
Tminv[{x1, x2, x3},
m[{x1, x2, x3}].M[{u1, u2, u3}]];
ttt = %//FullSimplify;
X1 = ttt/.{u1 → 1, u2 → 0, u3 → 0}
X2 = ttt/.{u1 → 0, u2 → 1, u3 → 0}
X3 = ttt/.{u1 → 0, u2 → 0, u3 → 1}
Print["—"]
LB[X2, X3]//Simplify
LB[X3, X1]
LB[X1, X2]
```

{Cos[x3], Sin[x3], 0}

{−Sin[x3], Cos[x3], 0}

{0, 0, 1}

—

{Cos[x3], Sin[x3], 0}

{−Sin[x3], Cos[x3], 0}

{0, 0, 0}

```
{X1.XB, X2.XB, X3.XB}//MatrixForm
```

$$\begin{pmatrix} (\mathrm{Cos}[x3])\ X[x1] + (\mathrm{Sin}[x3])\ X[x2] \\ (\mathrm{Cos}[x3])\ X[x2] + (-\mathrm{Sin}[x3])\ X[x1] \\ (1)\ X[x3] \end{pmatrix}$$

Coframe

```
νt = a1 d[x1] + a2d[x2] + a3d[x3];
ν1 =
(νt/.
(Solve[Thread[
{ip[X1.XB, νt],
ip[X2.XB, νt],
ip[X3.XB, νt]} == {1, 0, 0}],
{a1, a2, a3}])[[1]]//
FullSimplify);
ν2 =
(νt/.
(Solve[Thread[
{ip[X1.XB, νt],
ip[X2.XB, νt],
ip[X3.XB, νt]} == {0, 1, 0}],
{a1, a2, a3}])[[1]]//
FullSimplify);
ν3 =
(νt/.
(Solve[Thread[
{ip[X1.XB, νt],
ip[X2.XB, νt],
ip[X3.XB, νt]} == {0, 0, 1}],
{a1, a2, a3}])[[1]]//
FullSimplify);
{ν1, ν2, ν3}//MatrixForm
```

$$\begin{pmatrix} (\mathrm{Cos}[x3])\ dx1 + (\mathrm{Sin}[x3])\ dx2 \\ (\mathrm{Cos}[x3])\ dx2 + (-\mathrm{Sin}[x3])\ dx1 \\ (1)\ dx3 \end{pmatrix}$$

{{ip[X1.XB, ν1], ip[X2.XB, ν1],
ip[X3.XB, ν1]},
{ip[X1.XB, ν2], ip[X2.XB, ν2],
ip[X3.XB, ν2]},
{ip[X1.XB, ν3], ip[X2.XB, ν3],
ip[X3.XB, ν3]}}//FullSimplify//
MatrixForm
$\nu 1 \wedge \nu 2 \wedge \nu 3$//Simplify

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$(1)\ dx1 \wedge dx2 \wedge dx3$

Contact one form

$\mathrm{Y1} = \lambda(\mathrm{X2})$
$\mathrm{Y2} = \lambda(\mathrm{X3})$

$\{-\lambda \mathrm{Sin}[x3], \lambda \mathrm{Cos}[x3], 0\}$

$\{0, 0, \lambda\}$

$\omega = \mathrm{a1}\nu 1 + \mathrm{a2}\nu 2 + \mathrm{a3}\nu 3$;
ip[Y1.XB, ω]//FullSimplify
ip[Y2.XB, ω]

$\mathrm{a2}\lambda$

$\mathrm{a3}\lambda$

$\omega = \mathrm{a1}\nu 1$;
ip[Y1.XB, ω]
ip[Y2.XB, ω]
$d[\omega]$//FullSimplify
ip[Y2.XB, ip[Y1.XB, $d[\omega]$]]//
FullSimplify

0

0

$(-\mathrm{a1Cos}[x3])\ dx2 \wedge dx3 +$
$(\mathrm{a1Sin}[x3])\ dx1 \wedge dx3$

$-\mathrm{a1}\lambda^2$

$\omega = \frac{-1}{\lambda^2}\nu 1$
ip[Y1.XB, ω]
ip[Y2.XB, ω]
ip[Y2.XB, ip[Y1.XB, $d[\omega]$]]//
Simplify

$-\frac{(\mathrm{Cos}[x3])\ dx1 + (\mathrm{Sin}[x3])\ dx2}{\lambda^2}$

0

0

1

$d[\omega]$//Simplify
$d[\omega]-$
$\left(0\nu 1 \wedge \nu 2 + \frac{1}{\lambda^2}\nu 2 \wedge \nu 3 + 0\nu 3 \wedge \nu 1\right)$//
FullSimplify

$\left(\frac{\mathrm{Cos}[x3]}{\lambda^2}\right) dx2 \wedge dx3 +$
$\left(-\frac{\mathrm{Sin}[x3]}{\lambda^2}\right) dx1 \wedge dx3$

0

Reeb vector field

Y0 = a1X1 + a2X2 + a3X3;
ip[Y0.XB, d[ω]]//Simplify
ip[Y0.XB, ω]//Simplify

$\left(\frac{\mathrm{a2}}{\lambda^2}\right) dx3 + \left(-\frac{\mathrm{a3Cos[x3]}}{\lambda^2}\right) dx2 + \left(\frac{\mathrm{a3Sin[x3]}}{\lambda^2}\right) dx1$

$-\frac{\mathrm{a1}}{\lambda^2}$

Y0 = $-\lambda^2$X1 + 0X2 + 0X3//FullSimplify
ip[Y0.XB, d[ω]]//Simplify
ip[Y0.XB, ω]//Simplify

$\{-\lambda^2\mathrm{Cos[x3]}, -\lambda^2\mathrm{Sin[x3]}, 0\}$

0

1

Evaluation of invariants

csln =
Solve[LB[Y1, Y0] == c101Y1 + c201Y2,
{c101, c201}];
csln =
(Append[csln,
Solve[
Thread[LB[Y2, Y0] ==
c102Y1 + c202Y2]//
FullSimplify, {c102, c202}]]);
csln =
(Flatten[Append[csln,
Solve[
Thread[LB[Y2, Y1] − Y0 ==
c112Y1 + c212Y2]//
FullSimplify,
{c112, c212}]]]]);

$$\begin{pmatrix} \mathrm{c101} & \mathrm{c201} \\ \mathrm{c102} & \mathrm{c202} \\ \mathrm{c112} & \mathrm{c212} \end{pmatrix} /.\mathrm{csln}//\mathrm{MatrixForm}$$

hh0 =

$$\begin{pmatrix} \mathrm{c101} & \frac{1}{2}(\mathrm{c201}+\mathrm{c102}) \\ \frac{1}{2}(\mathrm{c201}+\mathrm{c102}) & \mathrm{c202} \end{pmatrix} /.$$

csln;
Print["—χ—"];
$\chi = \sqrt{-\mathrm{Det[hh0]}}$//
FullSimplify[#, $\lambda > 0$&&$\alpha > 0$]&
Print["—κ—"];
κ =
$\left(\left(-(\mathrm{c112})^2 - (\mathrm{c212})^2 + \frac{\mathrm{c201}-\mathrm{c102}}{2}\right)\right.$ /.
csln)//
FullSimplify[#, $\lambda > 0$&&$\alpha > 0$]&
Print["– {χ,κ} normalized –"];
λv =
(λ/. (Solve [$\chi^2 + \kappa^2 == 1$&&$\lambda > 0, \lambda$]) [[
1]]//FullSimplify)
{χ, κ}/.$\lambda \to \lambda$v

$$\begin{pmatrix} 0 & 0 \\ -\lambda^2 & 0 \\ 0 & 0 \end{pmatrix}$$

—χ—

$\frac{\lambda^2}{2}$

—κ—

$\frac{\lambda^2}{2}$

– {χ,κ} normalized –

$2^{1/4}$

$\left\{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\}$

C.3 Isometries of $(\mathsf{G}_{3.2}, \mathcal{D}, \mathrm{g})$

Setup algebra

$M[\{x\_, y\_, z\_\}]{:=}\begin{pmatrix} 0 & 0 & 0 \\ y & z & 0 \\ x & -z & z \end{pmatrix};$

```
Minv[MM_]:=Module[{ss, zv},
zv = Table [z_i, {i, 1, 3}] ;
ss =
Solve[
MM ==
zv.{M[{1,0,0}], M[{0,1,0}],
M[{0,0,1}]}, zv];
zv/.ss[[1]]
];
cc[A_, B_]:=A.B − B.A;
```

Riemannian expansion

```
rE1 = ({0, 1, 0});
rE2 = ({0, 0, 1});
rE3 = ({1, 1, 0});
gm = Inverse[{rE1, rE2, rE3}].
Inverse[{rE1, rE2, rE3}^T];
gm//MatrixForm
ONB = {rE1, rE2, rE3};
Table[ONB[[i]].gm.ONB[[j]],
{i, 1, 3}, {j, 1, 3}]//MatrixForm
```

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Connection and curvature

```
Conn[x_, y_]:=Module[{Connxyz},
Connxyz[xs_, ys_, zs_]:=
1/2(Minv[cc[M[xs], M[ys]]].gm.zs−
Minv[cc[M[ys], M[zs]]].gm.xs+
Minv[cc[M[zs], M[xs]]].
gm.ys);
Connxyz[x, y, ONB[[1]]]ONB[[1]]+
Connxyz[x, y, ONB[[2]]]ONB[[2]]+
Connxyz[x, y, ONB[[3]]]ONB[[3]]
];
R[x_, y_, z_]:=
Conn[Minv[cc[M[x], M[y]]], z]−
Conn[x, Conn[y, z]]+
Conn[y, Conn[x, z]];
codR[Y_, Z1_, Z2_, Z3_]:=
Conn[Y, R[Z1, Z2, Z3]]−
R[Conn[Y, Z1], Z2, Z3]−
R[Z1, Conn[Y, Z2], Z3]−
R[Z1, Z2, Conn[Y, Z3]];
```

Linear maps preserving $\mathcal{D}(1)$, R, and ∇R

$$\psi = \mathrm{ONB}^{\mathsf{T}}.\begin{pmatrix} \sigma1\mathrm{Cos}[\theta] & -\sigma1\mathrm{Sin}[\theta] & 0 \\ \mathrm{Sin}[\theta] & \mathrm{Cos}[\theta] & 0 \\ 0 & 0 & \sigma2 \end{pmatrix};$$

$\mathrm{Inverse}[\mathrm{ONB}^{\mathsf{T}}];$

%//MatrixForm

$$\begin{pmatrix} \sigma2 & 0 & 0 \\ \sigma2 - \sigma1\mathrm{Cos}[\theta] & \sigma1\mathrm{Cos}[\theta] & -\sigma1\mathrm{Sin}[\theta] \\ -\mathrm{Sin}[\theta] & \mathrm{Sin}[\theta] & \mathrm{Cos}[\theta] \end{pmatrix}$$

A1 = {1, 0, 0};

A2 = {0, 1, 0};

A3 = {0, 0, 1};

$\psi.R$[A1, A2, A3];

$R[\psi.\mathrm{A1}, \psi.\mathrm{A2}, \psi.\mathrm{A3}];$

%%//Simplify

%//Simplify

$\{0, 0, 0\}$

$\{0, \sigma2\mathrm{Sin}[2\theta], 2\sigma1\sigma2\mathrm{Sin}[\theta]^2\}$

A1 = {0, 0, 1};

A2 = {0, 1, 0};

A3 = {0, 0, 1};

$\psi.R$[A1, A2, A3];

$R[\psi.\mathrm{A1}, \psi.\mathrm{A2}, \psi.\mathrm{A3}];$

%%//Simplify

%//Simplify

$\{-2\sigma2, -2(\sigma2 + 2\sigma1\mathrm{Cos}[\theta]), -4\mathrm{Sin}[\theta]\}$

$\{-2\sigma1\mathrm{Cos}[\theta],$
$-6\sigma1\mathrm{Cos}[\theta], -4\sigma1^2\mathrm{Sin}[\theta]\}$

ψ/.Sin[θ] → 0/.Cos[θ] → $\sigma1\sigma2$;

$\psi = \%/.\sigma1^2 \to 1;$

%//MatrixForm

$$\begin{pmatrix} \sigma2 & 0 & 0 \\ 0 & \sigma2 & 0 \\ 0 & 0 & \sigma1\sigma2 \end{pmatrix}$$

A1 = {1, 0, 0};

A2 = {0, 1, 0};

A3 = {0, 0, 1};

A4 = {1, 0, 0};

ψ.codR[A1, A2, A3, A4];

codR[ψ.A1, ψ.A2, ψ.A3, ψ.A4];

%%//Simplify

%//Simplify

$\{-4\sigma2, -8\sigma2, 0\}$

$\{-4\sigma1\sigma2^4, -8\sigma1\sigma2^4, 0\}$

Bibliography

[1] Abraham R., Marsden J.E., Foundations of Mechanics, 2nd ed., Benjamin/Cummings, Reading, Mass., 1978.

[2] Adámek J., Herrlich H., Strecker G.E., Abstract and Concrete Categories, John Wiley & Sons, Inc., New York, 1990, http://katmat.math.uni-bremen.de/acc/acc.pdf.

[3] Adams M.R., Tie J., On sub-Riemannian geodesics on the Engel groups: Hamilton's equations, *Math. Nachr.* **286** (2013), 1381–1406.

[4] Adams R.M., Biggs R., Remsing C.C., Equivalence of control systems on the Euclidean group $\mathsf{SE}(2)$, *Control Cybernet.* **41** (2012), 513–524.

[5] Adams R.M., Biggs R., Remsing C.C., Single-input control systems on the Euclidean group $\mathsf{SE}(2)$, *Eur. J. Pure Appl. Math.* **5** (2012), 1–15.

[6] Adams R.M., Biggs R., Remsing C.C., Control systems on the orthogonal group $\mathsf{SO}(4)$, *Commun. Math.* **21** (2013), 107–128.

[7] Adams R.M., Biggs R., Remsing C.C., On some quadratic Hamilton-Poisson systems, *Appl. Sci.* **15** (2013), 1–12.

[8] Adams R.M., Biggs R., Remsing C.C., Quadratic Hamilton-Poisson systems on $\mathfrak{so}^*_-(3)$: classifications and integration, in Fifteenth International Conference on Geometry, Integrability and Quantization, Editors I.M. Mladenov, A. Ludu, A. Yoshioka, Bulgarian Academy of Sciences, Varna, Bulgaria, 2013, 55–66.

[9] Adams R.M., Biggs R., Remsing C.C., Two-input control systems on the Euclidean group $\mathsf{SE}(2)$, *ESAIM Control Optim. Calc. Var.* **19** (2013), 947–975, http://dx.doi.org/10.1051/cocv/2012040.

[10] Agrachev A., Barilari D., Sub-Riemannian structures on 3D Lie groups, *J. Dyn. Control Syst.* **18** (2012), 21–44, http://dx.doi.org/10.1007/s10883-012-9133-8.

[11] Agrachev A.A., Exponential mappings for contact sub-Riemannian structures, *J. Dyn. Control Syst.* **2** (1996), 321–358, http://dx.doi.org/10.1007/BF02269423.

[12] Agrachev A.A., Feedback-invariant optimal control theory and differential geometry, II. Jacobi curves for singular extremals, *J. Dyn. Control Syst.* **4** (1998), 583–604, http://dx.doi.org/10.1023/A:1021871218615.

[13] Agrachev A.A., Gamkrelidze R.V., Feedback-invariant optimal control theory and differential geometry, I. Regular extremals, *J. Dyn. Control Syst.* **3** (1997), 343–389, http://dx.doi.org/10.1007/BF02463256.

[14] Agrachev A.A., Sachkov Y.L., Control Theory from the Geometric Viewpoint, Springer-Verlag, Berlin, 2004, http://dx.doi.org/10.1007/978-3-662-06404-7.

[15] Almeida D.M., Sub-Riemannian homogeneous spaces of Engel type, *J. Dyn. Control Syst.* **20** (2014), 149–166, http://dx.doi.org/10.1007/s10883-013-9194-3.

[16] Ambrosio L., Rigot S., Optimal mass transportation in the Heisenberg group, *J. Funct. Anal.* **208** (2004), 261–301, http://dx.doi.org/10.1016/S0022-1236(03)00019-3.

[17] Ardentov A.A., Sachkov Y.L., Extremal trajectories in the nilpotent sub-Riemannian problem on the Engel group, *Mat. Sb.* **202** (2011), 31–54, http://dx.doi.org/10.1070/SM2011v202n11ABEH004200.

[18] Ardentov A.A., Sachkov Y.L., Conjugate points in nilpotent sub-Riemannian problem on the Engel group, *J. Math. Sci. (N. Y.)* **195** (2013), 369–390, translation of Sovrem. Mat. Prilozh. No. 82 (2012), http://dx.doi.org/10.1007/s10958-013-1584-2.

[19] Arnol'd V.I., Mathematical Methods of Classical Mechanics, 2nd ed., Springer-Verlag, New York, 1989, http://dx.doi.org/10.1007/978-1-4757-2063-1.

[20] Aron A., Craioveanu M., Pop C., Puta M., Quadratic and homogeneous Hamilton-Poisson systems on $A^*_{3,6,-1}$, *Balkan J. Geom. Appl.* **15** (2010), 1–7.

[21] Aron A., Dăniasă C., Puta M., Quadratic and homogeneous Hamilton-Poisson system on (so(3)), *Int. J. Geom. Methods Mod. Phys.* **4** (2007), 1173–1186, http://dx.doi.org/10.1142/S0219887807002491.

[22] Aron A., Pop C., Quadratic and homogeneous Hamilton-Poisson systems on the 13th Lie algebra from Bianchi's classification, in International Conference of Differential Geometry and Dynamical Systems (DGDS-2009), *BSG Proc.*, Vol. 17, Geom. Balkan Press, Bucharest, 2010, 12–20.

[23] Aron A., Pop C., Puta M., Some remarks on (sl(2, $\mathbb{R}$)) and Kahan's integrator, *An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.)* **53** (2007), 49–60.

[24] Barrett D.I., Biggs R., Remsing C.C., Affine subspaces of the Lie algebra $\mathfrak{se}(1,1)$, *Eur. J. Pure Appl. Math.* **7** (2014), 140–155.

[25] Barrett D.I., Biggs R., Remsing C.C., Optimal control of drift-free invariant control systems on the group of motions of the Minkowski plane, in Proc. European Control Conf., Strasbourg, France, 2014, 2466–2471.

[26] Barrett D.I., Biggs R., Remsing C.C., Quadratic Hamilton-Poisson systems on $\mathfrak{se}(1,1)^*$: the homogeneous case, *Int. J. Geom. Methods Mod. Phys.* (2015), in press, http://dx.doi.org/10.1142/S0219887815500115.

[27] Beals R., Gaveau B., Greiner P.C., Hamilton-Jacobi theory and the heat kernel on Heisenberg groups, *J. Math. Pures Appl. (9)* **79** (2000), 633–689, http://dx.doi.org/10.1016/S0021-7824(00)00169-0.

[28] Biggs J., Holderbaum W., Planning rigid body motions using elastic curves, *Math. Control Signals Systems* **20** (2008), 351–367, http://dx.doi.org/10.1007/s00498-008-0033-0.

[29] Biggs J., Holderbaum W., Integrable quadratic Hamiltonians on the Euclidean group of motions, *J. Dyn. Control Syst.* **16** (2010), 301–317, http://dx.doi.org/10.1007/s10883-010-9094-8.

[30] Biggs R., Nagy P.T., A classification of sub-Riemannian structures on the Heisenberg groups, *Acta Polytech. Hungar.* **10** (2013), 41–52, http://dx.doi.org/10.12700/APH.10.07.2013.7.4.

[31] Biggs R., Remsing C.C., A category of control systems, *An. Ştiinţ. Univ. "Ovidius" Constanţa Ser. Mat.* **20** (2012), 355–367, http://dx.doi.org/10.2478/v10309-012-0024-5.

[32] Biggs R., Remsing C.C., On the equivalence of cost-extended control systems on Lie groups, in Recent Researches in Automatic Control, Systems Science and Communications, Editor H.R. Karimi, WSEAS Press, Porto, Portugal, 2012, 60–65.

[33] Biggs R., Remsing C.C., A classification of quadratic Hamilton-Poisson systems in three dimensions, in Fifteenth International Conference on Geometry, Integrability and Quantization, Editors I.M. Mladenov, A. Ludu, A. Yoshioka, Bulgarian Academy of Sciences, Varna, Bulgaria, 2013, 67–78.

[34] Biggs R., Remsing C.C., Control affine systems on semisimple three-dimensional Lie groups, *An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.)* **59** (2013), 399–414, http://dx.doi.org/10.2478/v10157-012-0038-3.

[35] Biggs R., Remsing C.C., Control affine systems on solvable three-dimensional Lie groups, I, *Arch. Math. (Brno)* **49** (2013), 187–197, http://dx.doi.org/10.5817/AM2013-3-187.

[36] Biggs R., Remsing C.C., Control affine systems on solvable three-dimensional Lie groups, II, *Note Mat.* **33** (2013), 19–31.

[37] Biggs R., Remsing C.C., Control systems on three-dimensional Lie groups: equivalence and controllability, *J. Dyn. Control Syst.* **20** (2014), 307–339, http://link.springer.com/article/10.1007/s10883-014-9212-0.

[38] Biggs R., Remsing C.C., Cost-extended control systems on Lie groups, *Mediterr. J. Math.* **11** (2014), 193–215, http://dx.doi.org/10.1007/s00009-013-0355-0.

[39] Biggs R., Remsing C.C., Some remarks on the oscillator group, *Differential Geom. Appl.* **35** (2014), 199–209, http://dx.doi.org/10.1016/j.difgeo.2014.03.003.

[40] Biggs R., Remsing C.C., Equivalence of control systems on the pseudo-orthogonal group $\mathsf{SO}(2,1)$, submitted.

[41] Biggs R., Remsing C.C., On the equivalence of control systems on Lie groups, submitted.

[42] Bloch A.M., Nonholonomic Mechanics and Control, Springer-Verlag, New York, 2003, http://dx.doi.org/10.1007/b97376.

[43] Bloch A.M., Crouch P.E., Nordkvist N., Sanyal A.K., Embedded geodesic problems and optimal control for matrix Lie groups, *J. Geom. Mech.* **3** (2011), 197–223, http://dx.doi.org/10.3934/jgm.2011.3.197.

[44] Boscain U., Rossi F., Invariant Carnot-Caratheodory metrics on S^3, SO(3), SL(2), and lens spaces, *SIAM J. Control Optim.* **47** (2008), 1851–1878, http://dx.doi.org/10.1137/070703727.

[45] Brockett R.W., System theory on group manifolds and coset spaces, *SIAM J. Control* **10** (1972), 265–284.

[46] Butt Y.A., Sachkov Y.L., Bhatti A.I., Extremal trajectories and Maxwell strata in sub-Riemannian problem on group of motions of pseudo-Euclidean plane, *J. Dyn. Control Syst.* **20** (2014), 341–364, http://dx.doi.org/10.1007/s10883-014-9239-2.

[47] Calin O., Chang D.C., Sub-Riemannian Geometry, Cambridge University Press, Cambridge, 2009, http://dx.doi.org/10.1017/CBO9781139195966.

[48] Capogna L., Le Donne E., Smoothness of subRiemannian isometries, *arXiv* (2014), 1305.5286, http://arxiv.org/abs/1305.5286.

[49] Clelland J.N., Moseley C.G., Wilkens G.R., Geometry of control-affine systems, *SIGMA Symmetry Integrability Geom. Methods Appl.* **5** (2009), Paper 095, 28, http://dx.doi.org/10.3842/SIGMA.2009.095.

[50] Clelland J.N., Moseley C.G., Wilkens G.R., Geometry of optimal control for control-affine systems, *SIGMA Symmetry Integrability Geom. Methods Appl.* **9** (2013), Paper 034, 31, http://dx.doi.org/10.3842/SIGMA.2013.034.

[51] Craioveanu M., Pop C., Aron A., Petrişor C., An optimal control problem on the special Euclidean group $\mathsf{SE}(3,\mathbb{R})$, in International Conference of Differential Geometry and Dynamical Systems (DGDS-2009), *BSG Proc.*, Vol. 17, Geom. Balkan Press, Bucharest, 2010, 68–78.

[52] D'Alessandro D., Dahleh M., Optimal control of two-level quantum systems, *IEEE Trans. Automat. Control* **46** (2001), 866–876, http://dx.doi.org/10.1109/9.928587.

[53] Dăniasă C., Gîrban A., Tudoran R.M., New aspects on the geometry and dynamics of quadratic Hamiltonian systems on $(\mathfrak{so}(3))^*$, *Int. J. Geom. Methods Mod. Phys.* **8** (2011), 1695–1721, http://dx.doi.org/10.1142/S0219887811005889.

[54] de Gosson M., Symplectic Geometry and Quantum Mechanics, Birkhäuser Verlag, Basel, 2006.

[55] Eberlein P., Geometry of 2-step nilpotent groups with a left invariant metric, *Ann. Sci. École Norm. Sup. (4)* **27** (1994), 611–660, http://www.numdam.org/item?id=ASENS_1994_4_27_5_611_0.

[56] Falbel E., Gorodski C., Sub-Riemannian homogeneous spaces in dimensions 3 and 4, *Geom. Dedicata* **62** (1996), 227–252, http://dx.doi.org/10.1007/BF00181566.

[57] Gamkrelidze R.V., Discovery of the maximum principle, *J. Dyn. Control Syst.* **5** (1999), 437–451, http://dx.doi.org/10.1023/A:1021783020548.

[58] Ghanam R., Strugar I., Thompson G., Matrix representations for low dimensional Lie algebras, *Extracta Math.* **20** (2005), 151–184.

[59] Gorbatsevich V.V., Onishchik A.L., Vinberg E.B., Lie Groups and Lie Algebras, III, Springer-Verlag, Berlin, 1994, http://dx.doi.org/10.1007/978-3-662-03066-0.

[60] Gorbatsevich V.V., Onishchik A.L., Vinberg E.B., Foundations of Lie Theory and Lie Transformation Groups, Springer, Berlin, 1997.

[61] Goyal S., Perkins N.C., Lee C.L., Nonlinear dynamics and loop formation in Kirchhoff rods with implications to the mechanics of DNA and cables, *J. Comput. Phys.* **209** (2005), 371–389, http://dx.doi.org/10.1016/j.jcp.2005.03.027.

[62] Ha K.Y., Lee J.B., Left invariant metrics and curvatures on simply connected three-dimensional Lie groups, *Math. Nachr.* **282** (2009), 868–898, http://dx.doi.org/10.1002/mana.200610777.

[63] Hamenstädt U., Some regularity theorems for Carnot-Carathéodory metrics, *J. Differential Geom.* **32** (1990), 819–850.

[64] Harvey A., Automorphisms of the Bianchi model Lie groups, *J. Math. Phys.* **20** (1979), 251–253, http://dx.doi.org/10.1063/1.524073.

[65] Holm D.D., Geometric Mechanics. Part I: Dynamics and Symmetry, Imperial College Press, London, 2008.

[66] Holm D.D., Geometric Mechanics. Part II: Rotating, Translating and Rolling, 2nd ed., Imperial College Press, London, 2011, http://dx.doi.org/10.1142/p802.

[67] Iwai T., Tarama D., Classical and quantum dynamics for an extended free rigid body, *Differential Geom. Appl.* **28** (2010), 501–517, http://dx.doi.org/10.1016/j.difgeo.2010.05.002.

[68] Jakubczyk B., Equivalence and invariants of nonlinear control systems, in Nonlinear Controllability and Optimal Control, Editor H.J. Sussmann, M. Dekker, New York, 1990, 177–218.

[69] Jurdjevic V., The geometry of the plate-ball problem, *Arch. Rational Mech. Anal.* **124** (1993), 305–328, http://dx.doi.org/10.1007/BF00375605.

[70] Jurdjevic V., Geometric Control Theory, Cambridge University Press, Cambridge, 1997.

[71] Jurdjevic V., Optimal control on Lie groups and integrable Hamiltonian systems, *Regul. Chaotic Dyn.* **16** (2011), 514–535, http://dx.doi.org/10.1134/S156035471105008X.

[72] Jurdjevic V., Monroy-Pérez F., Variational problems on Lie groups and their homogeneous spaces: elastic curves, tops, and constrained geodesic problems, in Contemporary Trends in Nonlinear Geometric Control Theory and its Applications (México City, 2000), World Sci. Publ., River Edge, NJ, 2002, 3–51, http://dx.doi.org/10.1142/9789812778079_0001.

[73] Jurdjevic V., Sussmann H.J., Control systems on Lie groups, *J. Differential Equations* **12** (1972), 313–329.

[74] Jurdjevic V., Zimmerman J., Rolling sphere problems on spaces of constant curvature, *Math. Proc. Cambridge Philos. Soc.* **144** (2008), 729–747, http://dx.doi.org/10.1017/S0305004108001084.

[75] Kishimoto I., Geodesics and isometries of Carnot groups, *J. Math. Kyoto Univ.* **43** (2003), 509–522.

[76] Krasiński A., Behr C.G., Schücking E., Estabrook F.B., Wahlquist H.D., Ellis G.F.R., Jantzen R., Kundt W., The Bianchi classification in the Schücking-Behr approach, *Gen. Relativity Gravitation* **35** (2003), 475–489.

[77] Krener A.J., On the equivalence of control systems and linearization of nonlinear systems, *SIAM J. Control* **11** (1973), 670–676.

[78] Krishnaprasad P.S., Optimal control and Poisson reduction, Tech. Rep. T.R. 93-87, Inst. Systems Research, University of Maryland, 1993.

[79] Laurent-Gengoux C., Pichereau A., Vanhaecke P., Poisson Structures, Springer, Heidelberg, 2013, http://dx.doi.org/10.1007/978-3-642-31090-4.

[80] Lauret J., Homogeneous nilmanifolds attached to representations of compact Lie groups, *Manuscripta Math.* **99** (1999), 287–309, http://dx.doi.org/10.1007/s002290050174.

[81] Lauret J., Modified H-type groups and symmetric-like Riemannian spaces, *Differential Geom. Appl.* **10** (1999), 121–143, http://dx.doi.org/10.1016/S0926-2245(99)00002-9.

[82] Le Donne E., Ottazzi A., Isometries of Carnot groups and subFinsler homogeneous manifolds, *arXiv* (2012), 1210.5097, http://arxiv.org/abs/1210.5097.

[83] Leonard N.E., Krishnaprasad P.S., Motion control of drift-free, left-invariant systems on Lie groups, *IEEE Trans. Automat. Control* **40** (1995), 1539–1554.

[84] Liu W., Sussman H.J., Shortest paths for sub-Riemannian metrics on rank-two distributions, *Mem. Amer. Math. Soc.* **118** (1995), x+104.

[85] MacCallum M.A.H., On the classification of the real four-dimensional Lie algebras, in On Einstein's Path, Springer, New York, 1999, 299–317, http://dx.doi.org/10.1007/978-1-4612-1422-9_21.

[86] MacLane S., Categories for the Working Mathematician, Springer-Verlag, New York, 1971.

[87] Marenich V., Geodesics in Heisenberg groups, *Geom. Dedicata* **66** (1997), 175–185, http://dx.doi.org/10.1023/A:1004916117293.

[88] Marsden J.E., Ratiu T.S., Introduction to Mechanics and Symmetry, 2nd ed., Springer-Verlag, New York, 1999, http://dx.doi.org/10.1007/978-0-387-21792-5.

[89] Mazhitova A.D., Sub-Riemannian geodesics on the three-dimensional solvable non-nilpotent Lie group SOLV^-, *J. Dyn. Control Syst.* **18** (2012), 309–322, http://dx.doi.org/10.1007/s10883-012-9145-4.

[90] Meyer C., Matrix Analysis and Applied Linear Algebra, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2000, http://dx.doi.org/10.1137/1.9780898719512.

[91] Milnor J., Curvatures of left invariant metrics on Lie groups, *Advances in Math.* **21** (1976), 293–329.

[92] Moiseev I., Sachkov Y.L., Maxwell strata in sub-Riemannian problem on the group of motions of a plane, *ESAIM Control Optim. Calc. Var.* **16** (2010), 380–399, http://dx.doi.org/10.1051/cocv/2009004.

[93] Monroy-Pérez F., Anzaldo-Meneses A., Optimal control on the Heisenberg group, *J. Dyn. Control Syst.* **5** (1999), 473–499, http://dx.doi.org/10.1023/A:1021787121457.

[94] Montgomery R., A Tour of Subriemannian Geometries, Their Geodesics and Applications, American Mathematical Society, Providence, RI, 2002.

[95] Mubarakzyanov G.M., On solvable Lie algebras, *Izv. Vysš. Učehn. Zaved. Matematika* **1963** (1963), 114–123.

[96] Nagy P.T., Puta M., Drift-free left invariant control systems on $\mathrm{SL}(2,\mathbb{R})$ with fewer controls than state variables, *Bull. Math. Soc. Sci. Math. Roumanie (N.S.)* **44(92)** (2001), 3–11.

[97] Pappas G.J., Lafferriere G., Sastry S., Hierarchically consistent control systems, *IEEE Trans. Automat. Control* **45** (2000), 1144–1160, http://dx.doi.org/10.1109/9.863598.

[98] Patera J., Sharp R.T., Winternitz P., Zassenhaus H., Invariants of real low dimension Lie algebras, *J. Mathematical Phys.* **17** (1976), 986–994.

[99] Patrangenaru V., Classifying 3- and 4-dimensional homogeneous Riemannian manifolds by Cartan triples, *Pacific J. Math.* **173** (1996), 511–532, http://projecteuclid.org/euclid.pjm/1102365636.

[100] Petersen P., Riemannian Geometry, 2nd ed., Springer, New York, 2006.

[101] Popovych R.O., Boyko V.M., Nesterenko M.O., Lutfullin M.W., Realizations of real low-dimensional Lie algebras, *J. Phys. A* **36** (2003), 7337–7360.

[102] Puta M., Stability and control in spacecraft dynamics, *J. Lie Theory* **7** (1997), 269–278.

[103] Puta M., Optimal control problems on matrix Lie groups, in New Developments in Differential Geometry, Editor J. Szenthe, Kluwer, Dordrecht, 1999, 361–373.

[104] Remsing C.C., Optimal control and integrability on Lie groups, *An. Univ. Vest Timiş. Ser. Mat.-Inform.* **49** (2011), 101–118.

[105] Respondek W., Tall I.A., Feedback equivalence of nonlinear control systems: a survey on formal approach, in Chaos in Automatic Control (Control Eng.), CRC Press (Taylor & Francis), Boca Raton, FL, 2006, 137–262.

[106] Saal L., The automorphism group of a Lie algebra of Heisenberg type, *Rend. Sem. Mat. Univ. Politec. Torino* **54** (1996), 101–113.

[107] Sachkov Y.L., Optimality of Euler elasticae, *Dokl. Akad. Nauk* **417** (2007), 23–25, http://dx.doi.org/10.1134/S106456240706004X.

[108] Sachkov Y.L., Conjugate points in the Euler elastic problem, *J. Dyn. Control Syst.* **14** (2008), 409–439, http://dx.doi.org/10.1007/s10883-008-9044-x.

[109] Sachkov Y.L., Maxwell strata in the Euler elastic problem, *J. Dyn. Control Syst.* **14** (2008), 169–234, http://dx.doi.org/10.1007/s10883-008-9039-7.

[110] Sachkov Y.L., Control theory on Lie groups, *J. Math. Sci.* **156** (2009), 381–439.

[111] Sachkov Y.L., Conjugate and cut time in the sub-Riemannian problem on the group of motions of a plane, *ESAIM Control Optim. Calc. Var.* **16** (2010), 1018–1039, http://dx.doi.org/10.1051/cocv/2009031.

[112] Strichartz R.S., Sub-Riemannian geometry, *J. Differential Geom.* **24** (1986), 221–263, http://projecteuclid.org/euclid.jdg/1214440436.

[113] Sussmann H.J., An extension of a theorem of Nagano on transitive Lie algebras, *Proc. Amer. Math. Soc.* **45** (1974), 349–356.

[114] Sussmann H.J., Lie Brackets, Real Analyticity and Geometric Control, in Differential Geometric Control Theory, Editors R.W. Brockett, R.S. Millman, H.J. Sussmann, Birkhäuser, Boston, MA, 1983, 1–116.

[115] Tabuada P., Pappas G.J., Bisimilar control affine systems, *Systems Control Lett.* **52** (2004), 49–58, http://dx.doi.org/10.1016/j.sysconle.2003.09.013.

[116] Tabuada P., Pappas G.J., Quotients of fully nonlinear control systems, *SIAM J. Control Optim.* **43** (2005), 1844–1866 (electronic), http://dx.doi.org/10.1137/S0363012901399027.

[117] Tan K.H., Yang X.P., Characterisation of the sub-Riemannian isometry groups of H-type groups, *Bull. Austral. Math. Soc.* **70** (2004), 87–100, http://dx.doi.org/10.1017/S000497270003584X.

[118] Testa F.J., Congruence diagonalization and Lie groups, *SIAM J. Appl. Math.* **28** (1975), 54–59.

[119] Tudoran R.M., On a class of three-dimensional quadratic Hamiltonian systems, *Appl. Math. Lett.* **25** (2012), 1214–1216, http://dx.doi.org/10.1016/j.aml.2012.02.048.

[120] Tudoran R.M., The free rigid body dynamics: generalized versus classic, *J. Math. Phys.* **54** (2013), 072704, 10, http://dx.doi.org/10.1063/1.4816550.

[121] Tudoran R.M., Tudoran R.A., On a large class of three-dimensional Hamiltonian systems, *J. Math. Phys.* **50** (2009), 012703, 9, http://dx.doi.org/10.1063/1.3068405.

[122] Vershik A.M., Gershkovich V.Y., Nonholonomic problems and the theory of distributions, *Acta Appl. Math.* **12** (1988), 181–209, http://dx.doi.org/10.1007/BF00047498.

[123] Vershik A.M., Gershkovich V.Y., Nonholonomic dynamical systems, geometry of distributions and variational problems, in Dynamical Systems VII, Editors V.I. Arnol'd, S.P. Noviko, Springer-Verlag, Berlin, 1994, 1–81, http://dx.doi.org/10.1007/978-3-662-06796-3.

[124] Walsh G.C., Montgomery R., Sastry S.S., Optimal path planning on matrix Lie groups, in Proc. 33rd Conf. Dec. & Control, Lake Buena Vista, 1994, 1258–1263.

[125] Wilson E.N., Isometry groups on homogeneous nilmanifolds, *Geom. Dedicata* **12** (1982), 337–346, http://dx.doi.org/10.1007/BF00147318.