HERMENEUTIC AND EMPIRICAL ANALYSES OF GRAPHICALLY INSPIRED METAMATHEMATICS THAT REFLECT CRITICAL CONSCIOUSNESS WITHIN PERSPECTIVES OF PERSONAL AND SOCIAL JUSTICE

Pieter Paul van Jaarsveld

Thesis submitted to Rhodes University, Grahamstown, South Africa
in fulfilment of the requirements for the degree of

Doctor of Philosophy

in the
Department of Education

DECEMBER 2006

DECLARATION

I declare that ***'Hermeneutic and empirical analyses of graphically inspired metamathematics that reflect critical consciousness within perspectives of personal and social justice'*** is my own work, except where indicated, and that it has not been submitted for any degree at any other university.

Pieter Paul van Jaarsveld

December 2006

ACKNOWLEDGEMENTS

I wish to acknowledge with gratitude …

… the support and encouragement of my supervisor, Dr. Marc Schäfer, whose trust has allowed me space to explore and create in the preparation of this thesis,

… the Department of Education, Rhodes University, Grahamstown, and my fellow students for their professional, collegial and caring input during PhD weeks,

… the Carnegie Foundation for bursarial support to complete this work,

… the Scientific and Industrial Leadership Initiative in whose employ it was my privilege to serve disadvantaged education communities,

… my family, friends and colleagues for their love and encouragement,

… all those who have taught me since they have inspired me to teach generously as they did,

… all those whom I have taught because in teaching them they have taught me, and

… a God whose profound infinities have for me become more manifest through the language of mathematics; a language that reflects the order of the universe.

FOR

LAUREN and JENNA

ABSTRACT

My involvement with mathematics education amongst township educators and learners over the past seven years has highlighted the absence of sustained meaning and meaning making of mathematical concepts. It appears though that this instrumental rather than relational understanding of FET mathematics is not unique to township learners but is encountered amongst learners of all socio-economic classes and is representative of many FET mathematics learners. Given that the language of learning and teaching is a major contributory factor in the South African education system, it appears that the language of mathematics itself is a greater exacerbating factor for many learners of mathematics. The exclusive algorithmic approach to classroom mathematics further seems to alienate many learners from the essence of the meaning of mathematical tasks.

This research undertakes to determine whether metateaching and metalearning as forerunners to metacognition facilitates the acquisition of the sustained meaning of mathematical concepts. Metateaching and metalearning refer to the acute and deliberate awareness by educator and learner as to what constitutes concepts. Teaching and learning therefore presupposes the deconstruction of concepts into its subsumed derivative roots. It also assumes an awareness of the tacit degrees of abstraction that characterise tasks and the content of tasks. This in turn has implications for the educator's adopted sequence of topics for instruction. Metacognition implies awareness on the part of the learner (and educator) as to how material is learned and a further awareness as to how that learning can be sustained. Whether we ascribe meaningful learning to radical or social constructivism, or to associationist didactive approaches, or a combination of these, we are making assumptions about how learners acquire and sustain mathematical meaning because mathematics is, by and large a symbolic language often devoid of affective connotation. Furthermore our assessments of learners' tasks amount to clinical corrections of austere formulae wrapped in algorithmic procedures which manifest nothing of a learner's experience of mathematics or the deeper understanding (or misunderstandings) which characterise a learning and / or assessment episode.

To this end the research design of this interpretive case study requires learners to expound in textual accounts their thoughts as they describe the evolution of a mathematical process as they approach a solution and eventually interpret it. The textual account exposes the concept definition for what it really is in a learner's understanding of it and it is the expressiveness of language that indicates whether the understanding of a learner is approaching the concept image. The textual accounts vary in richness in terms of mathematical register and this in turn reflects the conceptual depth. The mechanism which seems to promote the conversion from concept definition to concept image is the graphical representation of the mathematical task or procedure, possibly because of its greater concreteness as opposed to the abstraction of its algebraic form.

TABLE OF CONTENTS

INTRODUCTION 13

MATHEMATICS IN CRISIS 28

AN HISTORICAL OVERVIEW AND CRITIQUE OF LEARNING THEORY WITH REFERENCE TO EPISTEMOLOGICAL FRAMEWORKS IN MATHEMATICS EDUCATION RESEARCH 57

CRITICAL CONSCIOUSNESS AND COMMUNICATIVE ACTION CONSTRUED AS HERMENEUTIC ELEMENTS OF CRITICAL REALISM 84

TOWARD METAMATHEMATICS 109

METACOGNITIVE REFLECTION 150

THE RESEARCH ENVIRONMENT AND PROCESS 176

PROBING AND REFLECTING ON THE DATA 201

Table of Figures

Table	List of Tables	Page

Task sheet	List of task sheets	Page

I

Introduction

'Separateness – the thief of consensus and the mantle of isolation'

1.1 Introduction

This work has been inspired by the experiences of teaching and learning with 'previously disadvantaged'[1] learners, over the past seven years. In particular, my association with learners and teachers from Khayelitsha, a township (squatter settlement) about thirty kilometres outside urban Cape Town has invoked reflection on and confirmed an ontology that underpins this research; an ontology that largely owes its existence to my developmental childhood experiences.

I understand ontology to be developmental and cumulative. As life brings me into new experiential chapters, values akin to those chapters are appropriated and deepened. They have become manifest in pedagogical practices that seek to challenge what in the South African educational context would be accepted as conventional and propositional ways of 'knowing' and 'doing'; pedagogical practices that associate with propositional epistemological hegemonies.

The investigative component of this thesis relates to mathematics in the Further Education and Training phase of the South African education system. My teaching and learning practices find expression in a personalised pedagogy of dialogue and community, both in the practise of caring teaching-learning relationships and the practise of teaching and learning mathematics. The nature and quality of these pedagogical practices are embossments of both the Freirian notion of 'dialogic' (Freire, 1970, 1974, 1987, 1998) and Habermas' 'communicative action' (Habermas, 1990). Dialogical practice is epitomised by caring, conversational classrooms.

[1] This term is used to denote learners or communities that were disadvantaged under the disparate provisioning policies of separate development in apartheid South Africa.

Communicative action is that action that is generated by moral consciousness; it seeks to affirm humanity and by its endeavour of solidarity it labours to emancipate both cognitive and affective domains of endeavour through transformative action. Ethics and social justice are reflected in the practice of communicative action.

There is a recurrence of the theme of social justice throughout this work. Social justice as a value pervades not only the political fabric of our society but also the educational one since the latter is but a subset of the former. Poverty is a political product. I have come to understand that it is no match for knowledge. It is overcome, I believe, with what (Freire, 1974) calls 'critical consciousness'; an indulgence in which I opt to immerse myself as an teacher, and encourage those who form part of my learning communities to appropriate. When sought, found and espoused, it is a liberating habit that unveils dominant thoughts and beliefs about the ownership of knowledge by an established order and its traditional, status quo ways of 'knowing' and 'doing'. It adds a vibrant dimension of 'thinking' and catalyses re-creative meaningful engagement; both self-engagement through radical construction and community-engagement as it emerges in the social construction of knowledge.

Mathematics can prosper too from the political and educational construct of Freirian critical consciousness and the Habermasian moral consciousness and communicative action. I regard both to be radically and socially constructible. Mathematics is also not partial to ethnicity or socio-economic status. It defies separatenesses in this sense, but also if personified would, I think, speak strongly against the separatenesses which are imposed on its discipline that is characterised by its staccato delivery of artificially isolated components. Its discipline is ordered and systematic, its components integrated and its processes synthesise objective and subjective knowing. Its component parts sum to a value more than its whole. Its symmetries are outstandingly beautiful and it strives toward concretising the infinities that the eye cannot behold and that the mind cannot conceive with ease or conceive at all. It describes intricate motions of unfurling petals and fired engines that return tons of space travelling metal to the atmosphere. It is the language of the universe as much as it is a universal language and I imagine if it could, it would clamour to be the heritage of all. Its designated separateness from other disciplines has categorised its elitism bestowing on it a reverence that estranges it with peripheral awe.

And so the commonalities between educational and political justice, for me, begin to emerge from the need for mathematics to be accessible to all and the ideal of enacted social justice. The justice that should ideally characterise democracy should also characterise education and its contexts. I attempt to show that dialogical practice, and communicative action attempt to spark critical consciousness and to value social justice in the ambience of my mathematics classrooms. These are transitive products of epistemological metatheory and ontological values that create accessible ways of knowing. For me it has renewed an energy for mathematics teaching and flamed a passion for its accessibility and renewal in the South African context. For learners it has helped transform an attitude of rote passivity toward mathematics, to a meaningful engagement with and understanding of it, through personal phenomenological discourses and reflection.

I proceed to unravel the values associated with a developing ontology and to demonstrate how these constituent ontological values that have impacted on a self-appropriated epistemology and pedagogy. These ontological values ought to transcend the cognitive rhetoric and translate into a demonstrated, active social justice. I want to return to the real ontological values that emerge in this thesis and seek to provide a perspective from which the work emerges and in which it continues to be immersed.

1.2 An ontological framing of the study

I was born of poor parents, financially and educationally. For them life may have had little consciousness because poverty denied them access to it. Their indigent existence was proletarian and contextualised in an apartheid South Africa. And so, as a child I was constantly in a place of contrasts and contradictions characterised by separatenesses: racial identity and division was entrenched by epidermal pigmentation; residential division was decreed irrespective of socio-economic standing; educational identity enforced by law to accommodate beliefs that ethnicity was strongly correlated with intelligence which in turn led to separate places of learning; separate dignities were associated with forms of employment and linked to a strong belief that knowledge was owned by the educated and / or privileged elite who would give it to those deemed to deserve it; the totalitarianism of the ruling party could never be assailed by the opposition; the separateness of affluence and acute poverty was (is) juxtaposed on the same landscape that was also polarised with the oppressed and the free, to mention a few ugly knots beneath a tightly woven tapestry of abhorrent contradictions.[2]

The 'Then' of another political ideology is, for me, developmentally separate also from the one of 'Now', but ironically they have in common similar separatenesses. Ironically, also, I lived as a child on the boundary of a group area, so called after the Group Areas Act that ensured separate development. It was not unusual therefore that I had Coloured friends as neighbours and with whom my family had contact. It was not till my high school years that I began to sense a tension that had been induced by artificial separateness that characterised our day-to-day living. As a primary school child I was oblivious of these tensions and contravened the law by swimming in a municipal swimming pool designated for persons of colour, brown but not black, on a summer's day in 1965. I undertook that venture with a sure sense of disquiet; a disquiet that I was not able to understand or rationalise. That prohibited, but not prosecuted, experience made me acutely aware of my singularly minor status in an apartheid South Africa. Black South Africans were confined to the homelands and were, by a pass system, permitted migratory status; a system designed to control the influx of black migrants from townships to urban areas. Black South Africans did not constitute the social fabric of my childhood; I never saw fellow Black South Africans. And as the years passed I became aware that separateness isolated Coloured and White, but also that Coloured and Black and Asian had their mapped and imposed identities of separateness. These separatenesses, I realised, began to induce a fearful suspicion of persons of colour other than white and my personal lexical contextualised in my poor upbringing soon became 'them' and 'us'. Propaganda had begun to take its toll on the innocence of a youth that swam in Coloured designated swimming pools; whose presence there was prohibited by red-on-white government sign postings. Yet in the innocent years of social isolation two traditions became part of my psyche and consciousness. I sang my school song many times over and understood and trusted the impact of these words each time they were uttered. They may have subconsciously irrigated roots of my moral consciousness, and in a sense promised an impending epiphany.

---

[2] Relics of these separatenesses still pervade the landscape and ambience of post-Apartheid South Africa and seemingly bedevil our young democracy.

Non Nobis, Domine!
Not unto us, O Lord,
The praise and glory be
Of any deed or word,
For in thy judgement lies
To crown or bring to naught
All knowledge and device
That man has reached or wrought.

In the greater scheme of things all intentions are judged as spurious or honourable and will fall or survive according to its motives. And so the spurious device of apartheid would need to crumble. Crumble it has but the restitution of justice, equity and accessibility through the picking up of the pieces remains an arduous task from a socio-political and a mathematics education perspective for a precariously placed democracy which needs to guard against fashioning its own separatenesses. Further to this, as a member and president of the school's Interact club, a junior rotary service club, I learned always to serve from its motto, '*Service above self*'.

The days drew near, as my schooling ended, where compulsory conscription would have me serve my country; where I was called to defend a racist South Africa; a time of immense pain and confusion where, incarcerated in military protocol, the contradictions of separatenesses in society continued to jostle my consciousness.

When I lost a dear friend in the Angolan border bush war for what I perceived as a useless cause the starkness of those incongruities began to wrestle me more. That tragedy was an epiphany. I became angry that I had not been more alert to the gross injustices that epitomised my childhood social fabric. But with time I have come to understand that where there is an educational vacuum one cannot rationalise the complexities of life, and similarly where there is a vacuum of mathematical content one cannot rationalise or defend its systems, intricacies, rigour, symmetries and power to explain infinities.

For me the social vacuum began with my own poverty and absence of a meaningful dialectic that may have spawned a more acute awareness of injustice. Similarly as much as I loved and valued my years of being taught school mathematics, my own residual questions about mathematics and how its complexities could be more simply represented remained for me dormant and unarticulated. As a first year teacher I was thrust into mathematics teaching and the dormant, contorted residue of questions was disturbed. The questions about how to present abstractions so that learners were receptive to them and acquired a meaningful understanding of them remained. Isolated by the separateness of collegial classrooms I found myself resorting to the presentation of graphical pictures and deictic gestures to concretise what in my own days of learning mathematics were remote algebraic abstractions. It met with success and my love of mathematics increased as I embraced it and as learners began to appreciate mathematics and in many cases lost their fear of it. Mathematics became a collaborative classroom of enacted endeavour. At that time I was engaged in a critical consciousness about the accessibility of mathematics for its learners and an attempt to illustrate its infinite beauty and symmetry even though this was not my conscious purpose twenty-five years ago. The grappling with the literature of theory and practice in mathematics teaching, and the theories and practice of research in the social sciences, and the classroom contexts of second and third language learners, particularly those who have isiXhosa as mother tongue, has sublimated my thoughts around the accessibility of mathematics. The impediment of unfamiliar language presents as another variable in already poorly resourced teaching and learning environments. Having to face the resourcelessness of poverty and the way it renders its subjects mute became another epiphany. Mathematical substance, often enacted substance, needs to begin to fill and replace the cognitive and affective vacuums that in my experience characterise learners' acquaintances with mathematics.

At the completion of my military service I began to work in the private sector continuing to observe the separatenesses of South African life. When I had saved sufficient to get my full-time studies under way I registered as a full-time undergraduate student. Even at that stage I observed the university campus as remarkably ethnically homogeneous. One winter's afternoon while in a tutorial the campus was disturbed by a march. We looked down on a revolutionary black throng of 'toyitoyi–ing' people and a vivid fear emerged in me that sixteenth day of June 1976. The despising of separateness by those who most knew its pained isolation had claimed the lives of black youths that day, and finally my critical consciousness had been pierced. As the disquiet I experienced as a child in a non-white swimming pool welled within me, I began to question the relevance of 'adiabatic and environmental lapse rates', 'tephigrams', 'Freudian dream symbols, and 'De Moivre's theorem and Simpson's Rule' in the context of a divided South Africa.

I began student work amongst black and coloured communities, in orphanages and crèches, and visited Cross Roads in the Nyanga township, as part of university outreach programmes to see first hand that squalor, poverty and ethnicity is also comprised of humanity. I grieved as I learned that people like me had been robbed of their dignity and education, and been relegated to inferior status. It gave perspective to my own poverty; and so it underlined the fact that poverty, and by implication affluence, are relative concepts.

My first teaching post took me to the Eastern Cape where I found myself earning a salary in a 'whites only' designated school but working on weekends amongst black mathematics colleagues and learners in Mdantsane, a black settlement on the outskirts of the city, through the work of the Independent Teacher Enrichment Centre in East London. This consciously entrenched the meaning of Apartheid ideology as the starkness of unequal provisioning was contrasted in poor and affluent educational settings.

I returned as a Deputy Principal to an inner city school in Cape Town in the early nineties, a school that is now demographically representative of the greater population of Cape Town. It was here that I worked and witnessed, after the dismantling of apartheid in 1994, the changing complexion of the school population... a school that has since celebrated the election of its first black head prefect.

My salaried work in township schools since 1999 came about after the rationalisation of teachers in the first years of African National Congress rule. It has at times in the past seven years been depressing working in the heart of squalor and poverty but the rewards of witnessing emerging critical consciousnesses in both the socio-political and educational sense, and the understanding of mathematics, has far outweighed the ache of witnessing acute indigence. Deep poverty has produced aspirant young minds that have excitedly embraced dreams of tertiary education and the beginning of realised personal potential.

Separateness tainted the perceptions of mathematics too; it was (is) for some and not for others. Mathematics did not escape the segregationist attitudes of a separatist ideology. It was the belief of the political and educational hegemonies of the day that some could do it and others could not. Black people were excluded from knowing its richness. It was by statutory law not part of their separate curriculum. And this is our legacy... a country where mathematics and the mathematics community finds itself in crisis and in need of restoring first the stolen dignity of people, and their right to learn and do mathematics. These constitute emotional and cognitive deficits of great magnitude. In our post-apartheid South Africa renovating curricula has been a pre-occupation of education administrators. In many cases these practices of renewal have been enterprising and appropriate, albeit, loosely grounded in the theory of mathematics learning. I cannot but question whether the insertion of Mathematical Literacy as an alternative to Mathematics will not propagate the inequalities that theoretically as a nation we should strive to eliminate. In practice this inserted subject would seem to discriminate against those who are in impoverished communities since it is here that the deficit of a trained teaching corps remains the greatest. The injustice of unequal access will continue since Mathematical Literacy, although rich in content, is not classified sufficient for access to tertiary institutions. It is as if the separateness of mathematical learning communities is designed to continue. Well resourced communities will enrol for mathematics whilst those lacking in infrastructural and human resources may have to elect to do Mathematical Literacy and forfeit the chances of mathematically oriented tertiary and career options.

Living on a Group Area boundary was indicative of our poverty. Ironically, if we were classified differently on the population register we may have been living on the other side of the boundary in a different separateness, one of greater hopelessness.

The close association with poverty and my childhood experiences established deep relational ontological values about people. The social forces of the nineteen-sixties and seventies in South Africa instilled in me empathy toward people, and particularly people of colour. It taught me to appreciate the meaning of solidarity through observance and ultimately abhorrence of separate development. I have learned that the uniqueness of people is to be celebrated. It is uniqueness that contributes to the diversity of thought and heuristic solution. It is the uniqueness that needs to be respected and valued and loved. Love eliminates fear that has its aetiology first in loneliness and then the isolation of separateness. It is the lifeblood that should ideally surge through our veins in an attempt to thread together uniquenesses to create a rich tapestry of experience that has its expression in the concept of 'ubuntu'.[3] Love restores humaneness to the objectification that loneliness imposes. The ontological values of respect for unique identity, nurtured caring relationships, equality of all people and the dignity of individual thought and personal knowing finds expression in my professional educational practice and underpins this research. I have found that I cannot teach people to care about and for each other. I can though, demonstrate care, and how to care, through my action. Through this action I aim to develop trusting human relationships and in turn to trust that the mathematical heuristic has solutions, and the frequency with which they are found diminishes the fearful separateness from it. I cannot only theorise about relationships. I need to live in them and act from a consciousness that has been transformed from a moral theoretical consciousness to a critical consciousness that invokes action and seeks change for good. There should be no hesitation in knowing the reason for active moral engagement with humanity. It is not induced by guilt for having had the privileges of a white education under a separatist ideology. The motive is spontaneous and sincere and embedded in ontological values and beliefs that sustain human endeavour and progress; an engagement that is driven by the hope for liberated thinking, the attainment of self-actualisation and the fulfilment of individual potential and destiny. One thing that is good about separateness is that it instils a tacit knowing that separatenesses invite choices.

Similarly I cannot theorise about my pedagogy without personalising it through practice and in so doing allow the theory to transcend a mere knowing to a level of transitive cognitive consciousness. It must catalyse a transformative action both in my own cognitive machinations about how and why I teach, and for participants in the learning environment as to how and why they learn through engaging with their own metacognition, a metacognition that is accessible through our inner and outer voice and its transcription into language.

In this analysis it would seem therefore that an epistemology that ignores ontology is devoid of humane substance. If I recognise the innate value of individual beings then my epistemological understanding must reflect it. So constructivist epistemologies, both radical and social, characterise this work. From my relational ontology I seek to learn from those I teach as much as those I teach may learn from me as we radically

[3] 'Ubuntu' is a humane South African notion emergent from the African Renaissance and in one word may be equivalent to 'fraternity'. It recognises that 'to be a person I need a person'.

construct our own experiential knowledge through communicative action. Traditional ways of knowing are typically propositional; knowledge that is dispensed to passive recipients by the absolutist elite is not liberating. But knowledge is not a commodity and it is not partial to political ideology. It demands to be accessible to all who aspire to it. Knowledge can be shared. It can be offered for the consideration of others to critically examine for their own acceptance or refutation.

This research seeks to celebrate dialectical knowledge and its emancipatory potential for learners who come to know and value their (mathematical) identity through inner and outer dialogue. Conversation releases the lock on cognitively imprisoned knowing and acclaims and asserts individuality, and contributes to the growth of knowledge, of identity and hope.

My pedagogy cannot be independent of my ontology and epistemology. Teaching and learning are not separate. I am indebted to many learners who have stimulated my thinking through thought provoking questions over twenty-five years of exciting and enriching classroom practice. I have revised teaching strategies in response to understanding how differently we perceive one and the same thing. I have come to learn about cognition and had a peep into how mathematics hones it. I have come to a better understanding of mathematics through being challenged to present it meaningfully. I have been compelled to deconstruct metacognitively complex mathematical concepts that, for my learners and me, have been mystified by abstract algebraic veneers to make it accessible to those whom I have taught. I have been challenged to present mathematical complexities in ways that close the separateness between concept definition and concept image. To reiterate, teaching and learning are not separate. As I teach I learn to teach and I learn about mathematics and how young people think about mathematics. It is my responsibility to teach my learners that to learn is to learn to teach themselves or others to think critically. Their inner dialogue is confirmed and assimilated when they articulate their understanding. Both teaching and learning are metacognitive endeavours. Its rewards are rich when conducted in safe, loving, respectful and nurtured relationships.

Personalised theories of being, knowing and doing translate into actions of social justice. This enacted justice is contrary to the Kantian theoretical idea of justice that would characterise a Utopian society free of the complexities of human interaction or as (Rawls, 1971) posits in abstract, propositional and conceptual ways a seemingly unattainable theoretical justice associated with an ideal society. The unequal accessibility of knowledge and mathematics per se, in a South African educational context, is a rational criterion for engaging in personal enactments of social justice. The identification with social justice as a rational criterion is in solidarity with critical realist theory and hence its selection as a metatheoretical context for this work. Critical consciousness as a reflective practice hopes to transform and emancipate at any level, personal or socio-political. It is as much applicable to educational practices as to political ones. A poignant feature of critical realist theory in terms of its influence on this work is its epistemic relativism that recognises that knowledge is socially generated, transitive and fallible (Danermark et al., 2002). A search for my own meaning of 'learning and teaching mathematics' and my learners' endeavours at metacognitive engagements with similar pedagogic constructs is developmental; an ongoing process of discovery and modification. In my understanding therefore, social justice begins to emerge at two levels; the personal in terms one's entitlement to be taught, to know and understand, and the socio-political entitlement of equal accessibility of knowledge, more specifically in this research the accessibility of the study of mathematics to all who wish to pursue it.

Methodologically the research locates comfortably within critical realist theory that simultaneously accommodates the duality of nomothetic and ideological approaches to the practice of social science research. In order for me as researcher-teacher to draw close to the understanding that learners exhibit when they reflect on their in-situ experiences of describing what they are doing as they engage with mathematical processes, elements of the interpretive phenomenological paradigm come to the fore. As teacher-researcher and learner interact through the spoken and written word, the research artefacts are accumulated and read for understanding. This is characterised by both hermeneutic and empirical phenomenological elements. Learners engage in hermeneutic analyses as they record their personal understanding of mathematical processes. In the analyses of artefacts as teacher-researcher the interpretations may be seen as empirico-phenomenological as I endeavour to make objective sense of individual writing accounts. As teacher-researcher elements of the hermeneutic emerge as I make sense of a personal understanding of my practice. In an attempt to synthesise emerging strands of thinking amongst learners the mathematical language is codified and the empirical evidence is analysed by simple statistical procedures to look for any correlations between the emerging codes. This aspect of the methodology hence draws on a quantitative method. The empirical analysis however, is not independent of learners' accounts of their affective reactions to the task of expressing what they are doing mathematically, and hence the blend of the empiricist and hermeneutic elements, celebrates the fraternity of the methodologies of positivist and interpretive metatheories. Such interdependence to offer the best explanation for research findings, or methodological pluralism, is a feature of critical realist theory associated with Bhaskar (1989). As a metatheory it enfolds Platonic inclusivity as much as it could condone austere Aristotelian exclusivity for educational research.

Mathematics, recognised as a gateway subject, must therefore be accessible to all for the sake of equal opportunity in the social and political domain. So too, it must be accessible in the individualised sense that as human beings our individual innate intelligence clamours for its understanding. It becomes the responsibility of teachers to facilitate the closure of the separateness of innate knowing and the assimilation of mathematical understanding. This is achievable I believe through becoming critically conscious about our own metacognition through action that is served by social justice. Practically this means engaging in personal inner discourses. It is about private conversations with ourselves over critical issues and, continuing that conversation through articulating it through the written or spoken word which leads to dialectical encounters that reorganise and redefine the inner dialogue until we self-actualise at a psycho-social level or assimilate at a mathematically cognitive level, and ultimately both. If learning to interrogate our thinking had a limit the process would tend to rationality and greater concept definition. The investigative research component of this thesis deals with processes that learners are encouraged to engage with in order to articulate their cognitive discourses taking into consideration the constraints and impediments of learning in a second or third language. The concretisation of abstract concepts through graphical depictions is encouraged to be used as schema from which to construct articulations about mathematical concepts, procedures and algorithms.

1.3 Goals of this research

This work aims firstly to engage in a meta-analysis of learners' written accounts of their thought processes as reflected in the completion of a variety of mathematical topics and related tasks, secondly to investigate and understand what, in learners' experiences are essential to them in coming to know and be fluent in mathematics, and more especially in this regard, assess the value of a graphical approach to introducing complex and abstract mathematical concepts, and thirdly to examine and understand the essential role(s) that I as an educator must play in fostering mathematical learning and, more especially, to analyse my metacognitive engagement with the deconstruction of mathematical concepts.

1.4 Significance of the research

South Africa's record of poor mathematics results amongst school leavers as evidenced in Chapter 2, calls for investigation. Besides the poor supply of mathematics teachers (cf. Fig. 2.1) and the apparent lack of content knowledge in large sectors of the current mathematics teaching corps, the language of instruction for the majority of South African school goers is English or Afrikaans; an anomaly for the fact that the majority of South African school goers have an African language as their mother tongue. It is this that one assumes must exacerbate the problem of performance and deepens the stagnancy of progress. This research investigates the value of a concrete pictorial rendition of algebraic abstractions as an intermediary language between English as language of instruction for learners whose mother tongue is isiXhosa. As such the significance of the research is seen to be its contribution to developing the understanding of teachers and learners alike through approaching the problem in terms of a metacognitive strategy that supports a deconstruction of concepts. In so doing I raise for myself the awareness of a pedagogic alternative that may contribute pedagogic theory. And in the final analysis it creates an opportunity for an alternative interpretation of the current National Curriculum Statement for mathematics (DOE, 2003a) that may contribute to the amelioration of the afflictions described briefly in this paragraph and in detail in Chapter 2.

1.5 Organisation of the thesis

The chapter that follows takes an in-depth look at mathematics in the current South African educational landscape. It compares performances between Higher Grade (HG) and Standard Grade (SG) performances nationally, provincially and at township level with specific reference to Khayelitsha, Cape Town. The comparative contrasts of these data are a reflection of the crisis in mathematics education in South Africa and demonstrate perpetuated personal and social injustices in the inaccessibility of mathematics education for the people of this country.

Chapter 3 provides a historical overview of learning theory into which is embedded the epistemological frameworks of mathematics education researchers.

The fourth chapter clarifies firstly the notion of Habermasian communicative action as part of his theory of moral consciousness that aligns with the notions of personal and social injustice, the ontological context of this study. The second part of chapter 4 is devoted to the notion of critical consciousness as defined by Freire. Its seminal value in this study relates to the fact that education is not synonymous with teaching since the former has connotations of developing critiquing individuals whose self-worth is sufficient for them to contribute meaningfully to mathematics discourses, while the latter's notoriety has to do with depositing scholarly content for the purpose of its rote regurgitation.

Chapter 5 probes the concept of metamathematics and views it in terms of Freirian dialogic praxis that embodies theory and reflective practice and Habermasian communicative action that recognises that teaching and learning are complementary processes in open communication systems. Abductive inference, as a thought operation, frames the theoretical analyses that seek to position metacognitive reflections in moving toward metamathematics.

The content of chapter 6 seeks to understand metacognitive mathematics practices by using the counterfactual retroductive thinking, a strategy within critical realist theory. As a mode of thought operation it enables one to search for the basic conditions that make concrete phenomena what they are. As a metacognitive reflective practice retroductive thinking seeks to identify what basic conditions are necessary for mathematics learning to be facilitated, promoted and sustained. I consider also in this chapter the elements of communication and its mediums of transfer.

Chapter 7 describes the research environment, the heterogeneity of the research cohort, the development of a research instrument to quantify the vocabular responses of participants and demonstrates how the instrument is applied to the textual accounts of learners.

The data is probed and reflected on in chapter 8. The analysis of the data seeks to establish whether the applied pedagogy of graphically deconstructed algebraic concepts that takes into account the abstractual order of mathematics constructs, has impacted classroom practice and whether it has value in terms of the accessibility and sustained learning of senior secondary school mathematics in the context of multilingual classrooms.

As the research developed so did my awareness that learning theory and the epistemological frameworks of mathematics education researchers was impacting my practice, and my understanding of how learners conceive mathematical tasks. In tandem with the research practice an associo-constructivist metacycle and metazone model for pedagogic practice developed alongside it. These emerging theoretical notions are elaborated on in chapter 9.

Chapter 10 gives an overview of the thesis and elaborates on the implications of the findings and makes recommendations resulting from this study.

The content of the following chapter seeks to create an awareness of the polarities and inequities in terms of the accessibility of mathematics for learners in South Africa, more particularly, those in poverty. It seeks to show the continued perpetration of personal and social injustice in terms of denied entitlement to quality mathematics education.

II

Mathematics in Crisis

'... our National Strategy identified three key thrusts, namely, to raise participation and performance by historically disadvantaged learners in Senior Certificate Mathematics and Physical Science; to improve on the number and quality of teachers of mathematics, science and technology; and to provide high quality mathematics, science and technology education from grade 1 to grade 12.'

Opening address of the Association of Mathematics Teachers of South Africa (AMESA) conference by the South African National Minister of Education, Naledi Pandor, MP
July 2004

2.1 Introduction

The national, provincial and local enrolment and pass statistics for mathematics are indicators of the inertias in mathematics education in South Africa. This chapter is devoted to the description of these data.

Where 'Higher Grade' or HG appears in this thesis, it is synonymous with the new Further Education and Training (FET) subject nomenclature of 'Mathematics' or 'Mathematics core' implemented in Grade 10 as from 2006. Similarly SG (Standard Grade) is deemed to be commensurate with, but not necessarily synonymous with, 'Mathematical Literacy' in the context of Grade 10, commencing 2006.[4]

The separate tricameral Black, Coloured and Indian ethnically differentiated education departments of the pre-democracy apartheid era (Hartshorne, 1992) contended with problems stemming from disparate national provisioning and personal and social injustices that precluded black South Africans from pursuing scientific fields of study. With the birth of a South African democracy immense hope emerged that the historical mathematics education inequities would be resolved. Yet, after twelve years of democracy, of an approximate 2300 black matriculants in Khayelitsha, Cape Town, 82 matriculants offered higher grade mathematics for their

[4] From the outset it is clarified that the use of Mathematical Literacy as a subject is distinct from 'mathematical literacy' as it refers to the acquisition of skills and comprehension and interpretation as it relates to graphical representations of algebraic concepts in this work.

senior certificate examination, and of these, 39 passed. These 39 learners constitute 47,6% of the enrolled Khayelitsha HG mathematics group and approximately 0,02% of the Khayelitsha senior certificate population of 2005. That 3,3% of the Khayelitsha matriculation population offered HG mathematics for their senior certificate examination reflects a glaring inaccessibility to mathematics for black learners and perpetuated personal and social injustices (cf. Tables 2.1 and 2.2). Khayelitsha is one township of many where the same pattern repeats itself year after year.[5] Setati (2005) writes about multilingualism in mathematics education and the power mathematics taught in English has in offering learners access to social goods, albeit through English rather than mother-tongue instruction. It is the accessibility not only of social goods for black South Africans, but their accessibility to mathematics in the first instance that should be of paramount concern. Without school mathematics irrespective of the medium of instruction there is less chance of accessibility to the social goods to which Setati refers.

Enrolments	Total Senior Certificate Enrolment	National (Mathematics)		Western Cape (Mathematics)		Khayelitsha (Mathematics)	
	(National)	SG	HG	SG	HG	SG	HG
1996	518225	194983	68541	14746	3294		
1997	559233	150046	65015	16335	3703		
1998	552862	89613	79019	18477	3758		
1999	511474	231199	50105	18040	3770	1167	72
2000	489941	245497	38520	18290	3963	1233	27
2001	449371	229075	34870	11020	3464	1289	28
2002	443821	225524	35465	17516	4119	1294	50
2003	440267	222367	35956	17504	4446	1218	51
2004	467985	236155	39939	16933	5093	1343	97
2005	508363	259099	44053	17474	4992	1342	82

Table 2.1

Comparison of National, Western Cape and Khayelitsha mathematics enrolments for the past ten years against national senior certificate enrolments
Source: National Department of Education and Western Cape Education Department

[5] Clynick and Lee (2004: 239) cite the example of the Ekurhuleni townships of Tsakane and KwaThemba, in the heart of the national economy in Gauteng, South Africa, which achieved 1600 senior certificate passes between them but only 12 of these included HG mathematics.

Passes	Total Senior Certificate Enrolment	National (Mathematics)		Western Cape (Mathematics)		Khayelitsha (Mathematics)	
	(National)	SG	HG	SG	HG	SG	HG
1996	518225		25912	9234	2830		
1997	559233	63342	22467	8320	3042		
1998	552862	45826	28094	9748	3094		
1999	511474	72179	19854	9288	3091	206	7
2000	489941	79631	19237	9201	3162	271	5
2001	449371	72301	19504	9555	4321	318	16
2002	443821	96302	20528	11021	3679	483	25
2003	440267	99426	23412	11070	3938	550	26
2004	467985	103721	24143	10751	4268	626	47
2005	508363	106550	26383	10556	4321	522	39

Table 2.2

Comparison of National, Western Cape and Khayelitsha mathematics passes for the past ten years against national senior certificate enrolments
Source: National Department of Education and Western Cape Education Department

The specific example of Khayelitsha mathematics enrolments and passes is a microcosm of the provincial and national statistics. It is therefore not only the injustices perpetrated on the black learning communities in terms of the accessibility of mathematics that is cause for concern. South Africa by international standards is recognised as a poor performer in mathematics as confirmed by the TIMMS-R as reported on by Howie and Vinjevold (2002). Research report 13 of the Centre for Development and Enterprise (2004) entitled 'From Laggard to World Class' comprehensively reports on an ailing mathematics and science education scenario in South Africa (Clynick and Lee, 2004); systems which are shown to have potential but cannot break from its inertia. The report highlights the numerous private, public and foreign sector initiatives that have financed mathematics and science projects in South Africa. Yet despite these benevolences and immense costs over many years South Africa is still not producing close to the desirable number of mathematics (HG) passes that will be an asset to any economy. Cameron Dugmore, MP, Western Cape Minister of Education, in his budget vote refers to the 'establishment of 28 Focus schools in 2005... to provide renewed opportunities for those learners with special interests and aptitudes in disadvantaged communities.' In the same speech he also states 'We have extended our Dinaledi and Maths, Science and Technology schools to a total of altogether 50... as part of our objective of increasing the numbers of learners taking these scarce skill subjects...' (WCED, 2006: 2). It remains to be shown how costly information communication technology installed in schools has made an effective and significant difference to mathematics enrolment and pass rates.

The raw data in Table 2.1 shows the poor national, provincial and local (Khayelitsha) mathematics enrolment and pass rates relative to the national enrolment of all senior certificate candidates for the past ten years. The inertia spoken of earlier is reflected in these descriptive statistics. In 2005 of all senior certificate enrolments 8,7% offered HG mathematics for matriculation. Of this group 59,9% passed the subject. This successful group constitutes only 5,2% of the national senior certificate population of 2005. In no terms can this be regarded as acceptable given that a considerable portion of the national fiscus that is apportioned to education also pays mathematics teachers for consistently producing unacceptably poor mathematics learning returns. The average for the past *ten* years of those learners who nationally acquire HG mathematics successfully is 4,7%. This reflects the inertia in mathematics education development and this despite changes to subject curricula, increased teacher training, institutes, academies, the installation of information communication technology in poor schools, non-governmental organisation contributions, and private, public and foreign investment and the associated extreme expenditure of these inputs, nationally we cannot show improving returns. More disturbing is the fact that the combined HG and SG mathematics passes show that on average over the past ten years 20% of the senior certificate population (See Table 2.3) achieves a pass in mathematics while only 53% of the senior certificate population offers mathematics (HG or SG) as a subject (See Tables 2.1 and 2.2). In a mathematics hungry economy this poses much instability. It would seem conservatively desirable for at least 60% - 70%, from my point of view, for those who offer mathematics to pass even taking constraints in the system into consideration.

National	Enrolments (%)		Passes (%)	
	SG	HG	SG	HG
2005	51	8.7	21	5.2
10 year average 1996 -2005	43	9.8	15.3	4.7

Table 2.3

Comparison of national enrolments and passes of 2005 with the average for the 10 year period, 1996 to 2005

There are several implications that arise from this analysis. One has to do with implications for the mathematics teaching corps, another to do with the policy changes that constitute the pass requirements of the Further Education and Training Certificate that Grade 12 learners will sit for the first time in 2008.

2.2 Mathematics teaching – a fundamental constituent of a national mathematics recovery

The most significant event in education during democratic rule as far as education is concerned was, as I see it, the rationalisation of the profession that took place around 1996 and 1997. It was this process, in my opinion, that robbed the profession of its expertise and experience in mathematics. As much as the reformation of curricula is good and desirable it is almost less than appropriate for those left in the system to replace the stability that characterised the exodus, and then to become acquainted with content matter with which they had been unfamiliar. This together with getting to know the ropes of an 'outcomes based' philosophy of education may be construed as a form of professional injustice. I suspect that these were (are) rough times for teachers, but particularly for mathematics teachers left in the system. The morale in education has been sorely dented and is proving hard to restore. Public perception of the profession is tainted and learners who are in the system themselves opt not to become teachers as demonstrated by enrolment figures (See Table 4) for post graduate education qualifications, especially not mathematics teachers, because for them mathematics would have proved to have been a bad experience, or so the fact that, on average, a combined mathematics HG and SG pass rate of 20% of South African school leavers over the past ten years, seems to suggest. South Africa needs to aspire to making mathematics accessible, attractive and pleasant so as to influence prospective teachers of the subject to consider it as a career. One cannot enjoy mathematics unless one understands it more than one is able to 'do' it. It is maybe this that leads me to think that not enough is being done to stimulate a passion for mathematics amongst the existing mathematics teaching corps that would attract a prospective pool of dedicated mathematics teachers who will sustain sound enrolments and pass rates in the years to come. I have no doubt that incentives may contribute to changed perceptions but these incentives need to be on-going until such time as the crisis is thoroughly managed and addressed, and even beyond that to sustain long term interest in the subject. These incentives further need to recognise teachers, per se, whose mathematics learners, rather than their schools, show desirable progress. There exists currently a perpetuating cycle of poor quality teaching that produces that dampens the spirits of learners and teachers alike, and that ultimately fails to generate prospective professional interest in the subject, quite apart from the attractive employment opportunities that exist for mathematics graduates in the private sector.

I have been part of professional forums that have lobbied for ploughing energy into learners rather than teachers. This desperate crisis management approach serves to assist the current crop of school leavers but does not systemically address the long-term stability of the mathematics education profession. Rewarded, competent teachers in the Further Education and Training band will serve to promote choosing mathematics as a spontaneous option rather than it being selected for reasons that match tertiary faculty entrance requirements. Current state run workshops address the generic implementation of the Further Education and Training Certificate but subject training manuals major on required formats of outcomes based planning, design and assessment procedures and not on teaching methods that may illuminate metacognitive aspects of the mathematics curriculum that may provide new insights, understanding and teaching options for teacher; insights that may also render the subject more stimulating to teach. The most recent National Curriculum Statement training manual for mathematics of the Eastern Cape Department of Education, ECDOE, (2006) delivered in July 2006, devotes three of forty-four (3/44) pages to mathematical content. Some teachers who enter the profession to teach mathematics would in all reality want to teach the subject more than be burdened by constructing time-consuming rubrics for planning, design and assessment. It may be that with metacognitive insights into rich mathematical content, the burdensome task of administration may become more spontaneous. To my thinking I wonder to what extent copious administrative tasks conveniently replace authentic cognitive teaching owing to content incapacity and inexperience.

This leads me to a point made as part of the strategic perspective in Clynick and Lee (2004: 8), I quote: (the report) 'does not propose an alternative theoretical base for maths and physical science education in South Africa. This decision is not meant to imply that all is well theoretically or conceptually with these two subjects, their curricula, or the methods used to teach them. Certainly, in the longer term there is much that should be improved in all these fields.' I believe it is this very issue of a theoretical and conceptual base that holds significant hope for recovery in the ailing subject of mathematics and that it should be a short, and not a long-term aim to address it. I suggest that the nature of the analyses undertaken in this research provides a starting point for inservice and pre-service learning for mathematics teachers. Without ensuring a sound theoretical and conceptual base the subject will be rendered more sterile through mindless pursuits of fitting mathematics contents to plans rather than teachers mastering it as a body of knowledge, of mathematics teachers becoming experts in their classrooms. It is a fundamental requirement to ensure competent teaching if we hope to improve the status of mathematics in our country. It is fundamentally not necessary for teachers to embark on pure mathematics tertiary courses to upgrade their skills and knowledge, because it is the very theoretical nature of these courses that can be impediments to teacher motivation. Teachers require practical courses in what I later develop as generic mathematical knowledge, and metateaching and metalearning insight, that has application across secondary mathematics curricula. Practical courses will equip teachers to be masters of their mathematics classrooms. Advanced theoretical knowledge gained from the study of pure mathematics is valuable but it is not always essential. For most part the National Curriculum Statement (NCS) for Mathematics (Department of Education (DOE), 2003) is rich in content and generic perspective but its interpretation is a function of imaginative and creative teachers. Without a theoretical and conceptual base mathematics teaching in South Africa will remain inert and will fail to supply an economy with the appropriate values, skills and knowledge.

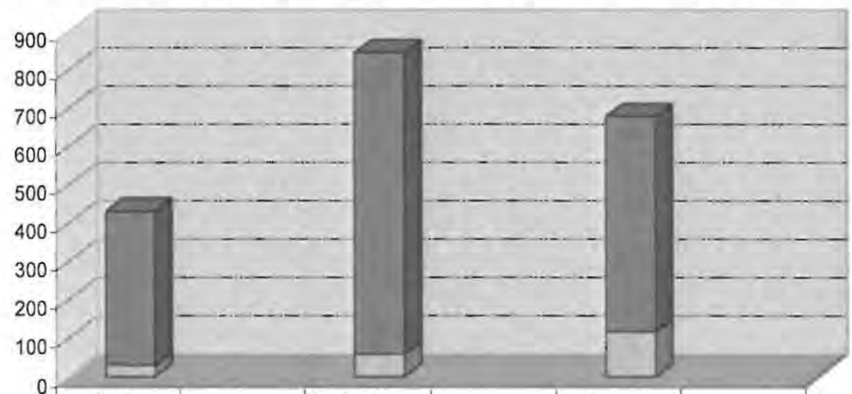
The kind of South African teacher that is envisaged, quoted from the NCS for Mathematics (DOE, 2003a: 5) is indeed a special person: 'All teachers and other teachers are key contributors to the transformation of education in South Africa. The National Curriculum Statement Grades 10 – 12 (General) visualises teachers who are qualified, competent, dedicated and caring. They will be able to fulfil the various roles outlined in the Norms and Standards for Teachers. These include being mediators of learning, interpreters and designers of Learning Programmes and materials, leaders, administrators and managers, scholars, researchers and lifelong learners, community members, citizens and pastors, assessors, and subject specialists.' It is important to recognise that designing mathematics learning material presupposes a thorough grasp of the fundamental content-associated cognitive understanding.

Two observations surface from these expectations. The first is that such a person in the private sector will command a salary that teachers cannot aspire to. His clients and collegial fraternity will also hold such a person in high esteem. This is generally not the case in the teaching profession. With their current earning power, eroded benefits and conditions of service and little professional regard, teachers do not enjoy the virtues of their true professional status. Matched against the expectations of what is statutorily envisaged of them, teachers will always generally be short of the mark.

A second observation is that the extensive list of qualities does not include being a competent teacher. Subject specialists are not necessarily able to teach competently. It is interesting that in such a comprehensive statement on what is expected of teachers that his/her core function, to teach well, is omitted, or taken as implied in such a listing. Teachers of mathematics should not graduate as mathematics teachers until they are thoroughly equipped with the knowledge of the art of teaching mathematics, which in essence is different from being a mathematics specialist. These qualities related to core cognitive understanding will contribute to pride in one's work, be inculcated in one's learners and the resultant classroom satisfaction derived from pleasing outcomes may perpetuate the kind of teacher and mathematics results South Africa so badly wants and needs.

2.3 Curriculum reform amidst mathematics in crisis

Vithal and Volmink (2005: 4) make the following interesting observation about mathematics curriculum reform in South Africa. 'In reviewing research on curriculum reforms and drawing on our own participation and experiences in these processes, we observe that while the South African mathematics curriculum reforms have been shaped and changed by both international and national shifts and developments in mathematics education, theory and practice, very little evidence exists that research has played any significant role in the direction or form taken by the curriculum over time. The question that must be asked is what the driving forces were that shaped the rationale for each new set of curriculum changes.' They continue, '...in reviewing more than a decade of conference proceedings we found virtually no research that directly speaks to mathematics curriculum reforms at a systemic level, with very few exceptions such as the research related to the Third International Mathematics and Science Study (TIMMS).' If *relevance* played an important consideration in reforming the mathematics curriculum, then that is reasonable. However, in the midst of crisis-issues that continue unabated, the increased extent of the renovated curriculum denies real problems with the shorter current Grade 11 and 12 syllabuses. These are often not completed by the time Grade 12 learners sit their final senior certificate mathematics examinations. This injustice unfairly impinges on assessed learning outcomes and their related opportunities for learners. The reasons cited for this is often a shortage of time, so it is difficult to comprehend how a curriculum that desires the searching of mathematical structures more deeply can be accommodated within a teaching year through more time consuming assessment procedures. The added dimension of a lack in teaching expertise that shows little sign of escalation and the poor enrolment and performance data have remained unchanged over the last ten years present as further constraints. Mathematics teachers are also rare commodities, relative to the demand for them, in education faculties of South African tertiary institutions, as reflected in Fig. 2.1.



	Rhodes University	Stellenbosch University	University of Cape Town
Total PGCE enrolment	400	788	563
Mathematics method enrolment	31	58	117

Fig. 2.1

Data available from three South African universities showing total mathematics method enrolments relative to total Post Graduate Certificate in Education (PGCE) enrolments for the years 1996 to 2005

Vithal and Volmink (2005) rightly ask what could have been the defining rationale for curriculum change. If relevance *is* argued as the rationale then we deceive ourselves since the mathematics of the renovated curriculum in most part, is the previous one in new garb. Semantic differential will have no impact on long-term results, and added content will not result in completing content on time. Essentially the changes amount to the interpretation of the curriculum in the light of seven Critical Outcomes, and five Developmental Outcomes, as stated in the National Curriculum Statement for Mathematics (DOE, 2003a: 2). This implies that learners need to take responsibility for their own learning either (in a radical or social constructivist sense) while teachers provide the real life situations for mathematics to be contextualised in the hope of producing citizens equipped with knowledge, skills and values that will help them make sense of their world. These are indeed noble and laudable aspirations but they will not be realised in the absence of a competent and dedicated teaching corps. It will require systemic changes that amount to more than generic administrative approaches to planning, designing, and outcomes based assessment practices. It will require the inert system to acquire meaningful momentum and that momentum can be gained by devoting much time and effort to inservice and pre-service teacher learning programmes that are based on informative practical considerations about how to engage metacognitively with mathematical content for promoting metateaching and metalearning of mathematics; concepts which are developed in Chapter 5.

I reported earlier that a ten year average of 15,3% and 4,7% of the national senior certificate population have passed SG and HG mathematics respectively, meaning 20% of the total senior certificate population passes with some form of mathematics. I raise this again because one wonders whether this statistic prompted the inclusion of Mathematics Literacy as compulsory core component of the Further Education and Training Certificate (FETC) to be issued for the first time to successful school leavers in 2008. In light of the current deficits which are recognised by the National Minister of Education, Naledi Pandor, MP, in her opening speech to the AMESA conference in July 2004, (an excerpt of which is quoted as an introduction to this chapter), one wonders how an impoverished mathematics teaching corps can meet the demands of teaching every school-goer mathematics or mathematical literacy who demonstrate extreme mathematical content deficits at the end of the General Education and Training phase. Fig. 2.2 shows my item analysis of a baseline assessment in February 2006 administered to 204 Grade 10 learners who had already opted for Mathematics as opposed to Mathematical Literacy for the first year of the FETC. The assessment comprised core content required of learners as per GET exit level assessment standards; core content that is fundamental to the commencement of the Grade 10 Mathematics curriculum. Question 3.1 $\frac{6x}{6x^2-12x} \div \frac{x^2-x-6}{x^2-4}$ and Question 3.2 $\frac{x}{x+1} - \frac{x}{x+2}$ of the assessment were omitted owing to it not having been taught in Grade 9.[6] The best item mean is 51% for elementary algebraic manipulations in 1.1 and 1.3 which were the simplification of $\sqrt{100-64}$ and $7a-(-2a)$ respectively. The worst item mean was 0% for item 5.2.3, which required reading off the equation of the drawn oblique linear function $y=-\frac{1}{3}x+1$, while the mean for line graphs was 1%. It is apparent that the linear function, for whatever reasons, was not understood or taught as core content in Grade 9. The group mean for the assessment was 19%. An unabridged test is located for reference purposes as Appendix 1.

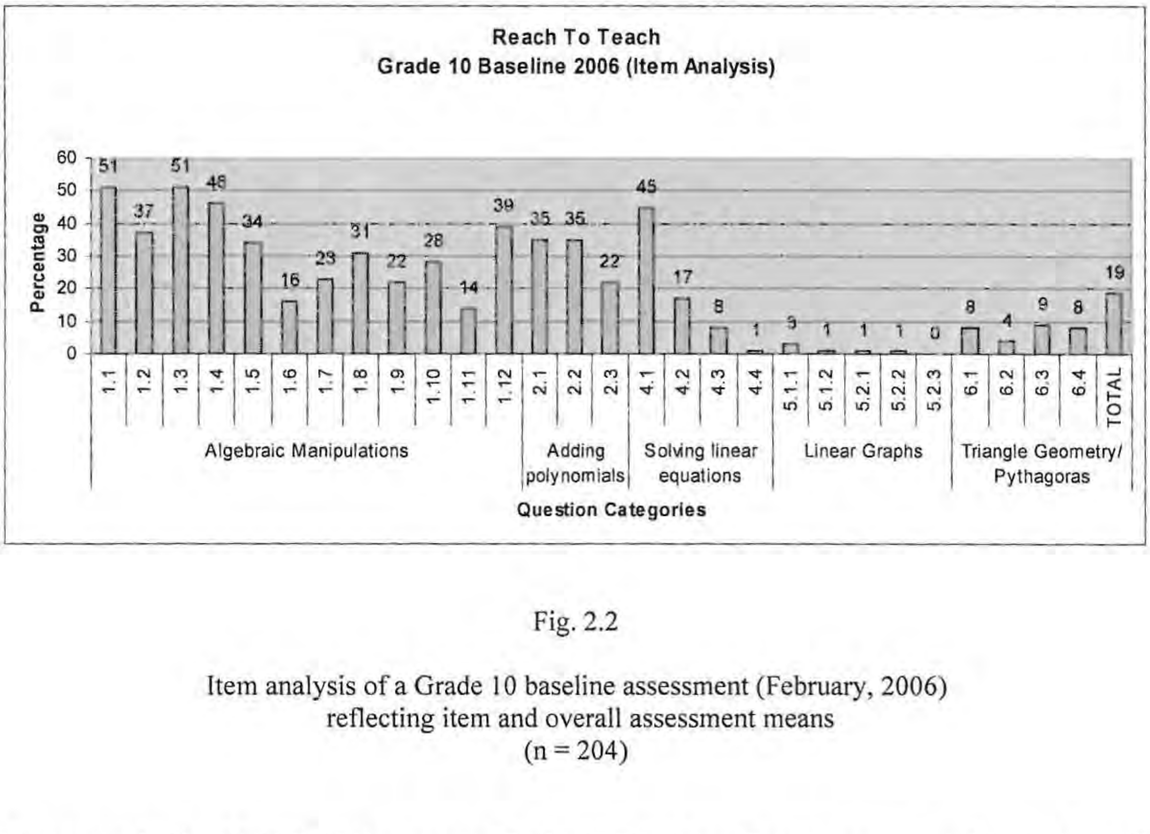


Fig. 2.2

Item analysis of a Grade 10 baseline assessment (February, 2006)
reflecting item and overall assessment means
(n = 204)

Of concern is the fact that the group that sat this baseline assessment were enrolled for Mathematics. If such is the performance of this group, those who did not write the assessment (in the region of 100 learners) would all be Grade 10 Mathematical Literacy learners, whose anticipated performance on this assessment is surmised, would have been weaker. The implications for this group alone are not difficult to

[6] These types of questions comprise the core content of the exit level summative assessment of the GET phase. The assessment was scored out of 87 rather than 100.

contemplate. The logistical implications for all Grade 10 learners in Khayelitsha are worrisome to consider. The impact that poor results will have on the morale of learners and teachers will perpetuate the inertia grounded in deficient teaching, poor results and poor self-esteem. In this regard it is noted that 'The National Curriculum Statement Grades 10 – 12 (General)' seeks to promote human rights, inclusivity, environmental and social justice. All newly developed Subject Statements are infused with the principles and practices of social and environmental justice and human rights as defined in the Constitution of the Republic of South Africa. In particular, the National Curriculum Statement Grades 10 – 12 is sensitive to issues of diversity such as *poverty*, *inequality*, race, gender, *language*, age, disability and other factors.' (Italics mine) (DOE, 2003a: 4) One wonders whether the compulsory nature of Mathematical Literacy contravenes one's constitutional right of choice to enrol for mathematics at all. Little seems to be being achieved in terms of the sensitivity to poverty, inequality and language in mathematics provisioning. Mathematics should be accessible to all those who desire pursuing it, otherwise its compulsory nature is contrary to a principle of social justice upon which the NCS is based. For all those who have no interest in mathematics and who have to participate in it for three final years of schooling compulsorily, will create, one imagines, much frustration, boredom and adverse behaviour that is associated with disinterested learners.

2.4 Declining status of higher grade secondary mathematics as indicated by a time series analysis of enrolments and passes

It is clear from Fig. 2.3 that there is a distinct ascendancy in the preference for SG mathematics amongst final secondary school mathematics students. A survey amongst 72 secondary mathematics teachers, who participated in a teacher inservice course survey presented by me, revealed that this trend may be attributable to a number of factors. Because of the unfavourable learner: teacher ratio of 55:1 as experienced by me in many township classrooms, teachers need to accommodate and teach both HG and SG learners simultaneously. Higher grade learners are therefore disadvantaged in this system and teachers therefore encourage mathematics to be studied on the SG. The dramatic reversal in enrolment trends since 1997 reflected in the comparative symmetry of the enrolment graphs of Fig. 2.3 seems to correspond also with the exodus of skilled mathematics teachers.

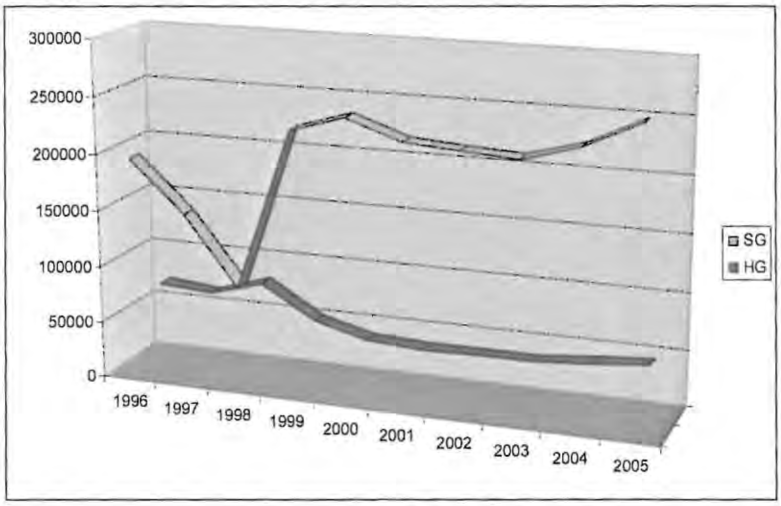


Fig. 2.3

Comparison between HG and SG Grade 12 national mathematics enrolments (1996-2005)
Data source: National Department of Education

This may explain the element of insecurity amongst the residual mathematics teaching corps regarding their own HG mathematical knowledge and consequently lacking teacher confidence in presenting HG lessons and nurturing HG learners. This insecurity prompts recommendations by teachers and (often supported by school management teams in favour of improved overall school performance in matriculation examinations) for learners to pursue SG rather than HG mathematics. This declining HG enrolment is also a feature of mathematics in the Western Cape, a picture of which is found in Fig 2.4.

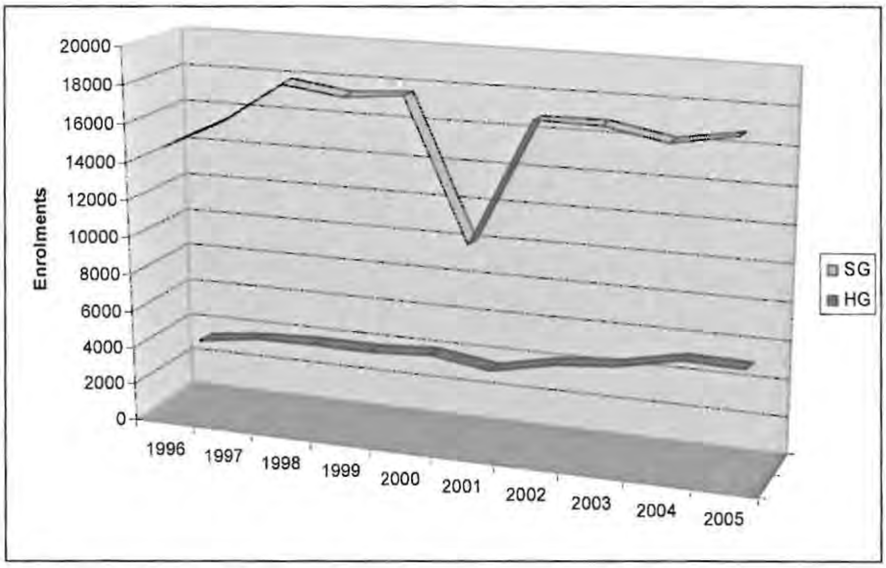


Fig. 2.4

Comparison of HG and SG Grade 12 Western Cape mathematics enrolments (1996 - 2005)
Data source: National Department of Education

The provincial data indicates similar disparity between SG and HG enrolments except for the inexplicable perturbation in the 2001 SG data. Interpolating procedures taken as permissible, the smoothing of the trend around the perturbation of the 2001 data and the resultant near parallel HG and SG trends shows a thick similar difference between the data for the ten year period for which data is available.

The national and provincial trends are repeated in the data of fifteen Khayelitsha high schools shown in Fig 2.5. It would appear though that the similar trends are compounded by greater absolute differences between HG and SG enrolments. This highlights the critical situation in Khayelitsha schools, and one surmises that the phenomenon is as acute in other South African township educational environments. The national data masks the underlying polarity between typically ex 'model C' and township school disparities in HG and SG subject selection and enrolment amongst secondary mathematics learners. The problem of enrolment as it presents at macro-, meso- and micro-scale levels constitutes a serious problem for mathematics education for which a viable solution seems not to be forthcoming at a pace required to match the problem's severity. Increasing the numbers of matriculating secondary learners is not a panacea for the country's economic and technological needs. Many capable black learners are being denied the opportunity of pursuing HG mathematics and are faced with the consequent impoverished credentials when applying for entry into tertiary scientific and mathematical fields of study. There has been and there continues to be an urgent need to address the crisis in township classrooms. The desire to solve the systemic problems of mathematics education is insufficient and short sighted redress. The immediacy of the problem in mathematics education spills over into the tertiary education sector with annual rapidity while the systemic solutions are sought in the General Education and Training (GET) phase. This concurs with the findings of the Centre for Development Enterprise (CDE) whose findings indicate that there should be redress from Grade 12 downwards into the FET grades to complement the systemic interventions, which commence at the Foundation phase (CDE, 2004). Seeking solutions at both General education and Further education training phases (GET and FET) are essential simultaneously to stem the tide of higher grade mathematics impoverishment and denied opportunity, and its adverse impact at tertiary level. Mathematics SG may well fall into this impoverished category too if the quality of content and examining are considered factors.

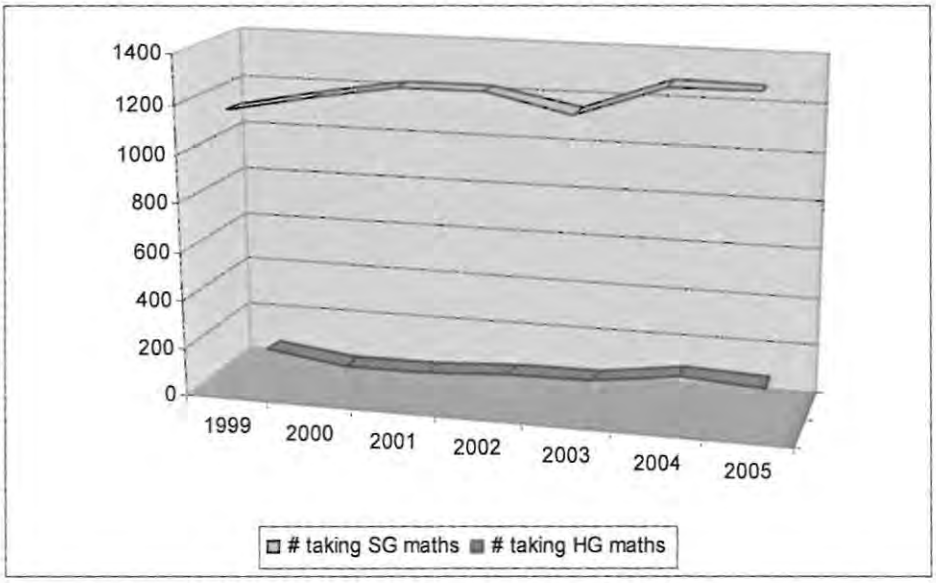


Fig. 2.5

Comparison between HG and SG mathematics enrolments in fifteen Khayelitsha schools from 1999 to 2005
Data source: Western Cape Education Department (WCED)

The raw data of Table 2.5 compares and summarises the enrolments in HG and SG mathematics at national, provincial and local levels. 'Local' refers specifically to the data of fifteen Khayelitsha high schools referred to earlier.

Enrolments	Total Senior Certificate Enrolment	National (Mathematics)		Western Cape (Mathematics)		Khayelitsha (Mathematics)	
	(National)	SG	HG	SG	HG	SG	HG
1996	518225	194983	68541	14746	3294		
1997	559233	150046	65015	16335	3703		
1998	552862	89613	79019	18477	3758		
1999	511474	231199	50105	18040	3770	1167	72
2000	489941	245497	38520	18290	3963	1233	27
2001	449371	229075	34870	11020	3464	1289	28
2002	443821	225524	35465	17516	4119	1294	50
2003	440267	222367	35956	17504	4446	1218	51
2004	467985	236155	39939	16933	5093	1343	97
2005	508363	259099	44053	17474	4992	1342	82

Table 2.4

Raw data reflecting the national, provincial (Western Cape) and local (Khayelitsha) enrolments for HG and SG mathematics in South Africa for the period 1999 to 2005
Data source: National Department of Education and Western Cape Education Department

The starkness of the raw data is informative. The SG enrolments are stable at national level, show a decline in 2001 and an immediate resurgence in 2002 at provincial level, while the local figures are stable, but unacceptably low, over the ten year period.

The dramatic fall in HG numbers from 1999 to 2000, may be explained in terms of the incentives on offer for schools to improve their pass rates (WCED, 2005: 2). There is a noteworthy improvement in provincial HG enrolments in 2002 and 2003 against the previous years and against the national enrolment for the same years. This is reflected in the near doubled enrolments in Khayelitsha schools from 28 in 2001 to 50 and 51 candidates in 2002 and 2003 respectively, notably against a comparatively stable national enrolment. Khayelitsha schools therefore show an 82,1% increase against the 3,1% improvement at national level over the period 2001 to 2003, but this in terms of absolute enrolments is nowhere near satisfactory. The SAILI project of the Cape Higher Education Consortium which is funded by the Gatsby Charitable Foundation, London, on which I served till end 2005, was for six years active in the promotion of HG mathematics in the Khayelitsha area, both through the support and inservice training of teachers and the support and teaching of higher grade classes both on site in Khayelitsha schools and through its Extra Teaching Programme presented at the campus of Stellenbosch University over 18

Saturdays in a year.[7] It is interventions such as these that may contribute to promoting the discipline of HG mathematics and that should aim at sustaining the enrolment in and quality of the subject. The delivery of these interventions will remain to be of critical value until township classrooms and their physical and human resources become the priority of local and national government.

Impromptu attempts at addressing the crisis in mathematics education like incentivising schools' endeavours to improve overall pass rates create indelible damage to both HG and SG mathematics, the effects of which may take many years to reverse. It is a concern that similar developments may occur around the FET subjects of Mathematics and Mathematical Literacy.

Enrolment, per se, is an indicator of 'mathematical aliveness' which is essential for its sustainability. The number and quality of passes is what ultimately adds value to personal opportunity, the country and its economy.

2.5 Parallels between HG and SG Mathematics and Mathematical Literacy of the FET

The enrolment trends of the past ten years at national, provincial and local levels indicate waning or constant characteristics. It is speculated that the trends have been influenced, inter alia, the rationalisation of education profession and the concomitant loss of mathematics teaching expertise; the ensuing absence of the nurturing of HG learners; incentives for schools to improve overall pass rates which has adversely affected the HG intakes because SG enrolments augur better for increased percentage senior certificate pass rates. This has lead to not only a poor supply of suitable candidates for scientific fields of tertiary study, the long-term impact on the national economy, the failure of the private sector to comply meritoriously with Black Economic Empowerment regulations, but also the fact that meagre, inept school exit level qualifications in HG mathematics provides no pool of expertise that potentially can become teachers of mathematics.

---

[7] This programme has over the past two years been restructured owing to funding constraints and does not have the same modus operandi. It is this programme that provided the opportunity for the engaging with learners about their cognitions of senior secondary mathematics.

It was speculated that the introduction of Mathematics Literacy as a compulsory core subject of the FETC may have been born out of 20% of the senior certificate population over the last ten years qualifying with either HG or SG mathematics at the end of their school careers. This not only constitutes personal and constitutional injustice for learners but also for teachers. In the case of learners being compelled to select a subject that has been a millstone for nine years of schooling will undoubtedly increase their frustration and resistance to mathematics. For current under qualified teachers of mathematics the burden increases especially for those who are mathematically inexperienced but trained to fulfil this important task. The burden is compounded by teachers' responsibility to their learners and to the state as their employer. These cautionary observations are all the more important in the light of the Mathematics Literacy curriculum that is not mediocre or mild in its expectations. The National Curriculum Statement Grades 10 – 12 (General), Mathematical Literacy (DOE, 2003b) personally is of an order substantially more demanding than SG mathematics, not only in terms of its mathematical application but also in terms of its demand for a command of language to communicate trends and to posit arguments about mathematical and statistical data.[8] Mathematical Literacy in other words cannot be seen as a soft option. I quote the competence descriptions of Code 3 that indicates adequate competence in the range 40% -49% for Grade 10, from the NCS Mathematical Literacy (DOE, 2003b: 60).

'By the end of Grade 10 the learner with adequate achievement can:

- Interpret and analyse problem statements presented by the teacher and set up a logical sequence to solve multi-step problems by estimating and measuring accurately and calculating with given algorithms and formulae;
- Interpret information in mathematically-based representations such as graphs, tables and diagrams in interpret answers in relation to the problem;
- Communicate clearly using commonly understandable mathematical terminology and retain knowledge of the symbols, definitions, representations, facts and information described in the Learning Outcomes;
- Follow the instructions in formulating questions and organising activities during investigations.'

[8] The language of teaching and learning mathematics is fundamental to the acquisition of acquiring and comprehending deep structure in mathematics as demonstrated in the research aspect of this work. It is an important consideration therefore given that the majority of school going South Africans do not receive instruction in their mother tongue, and that African languages have not had the opportunity of developing a rich mathematical register to accommodate mathematical specificity.

It is observed that each competence description, irrespective of code or level of achievement, as in the assessment standards, as does the one quoted above, requires being able to 'do' mathematics. It may be presumed, but at no point is there a specific requirement of a learner to be able to demonstrate through description or explanation the meaning or demonstrate an *understanding* of mathematical structures. I attempt to show in a later chapter that the metacognitive closure around mathematical concepts is embedded in social and mathematical language that promotes sustained mathematical knowledge.

Similarly, the assessment standards for Mathematical Literacy are framed in a manner of being able to 'do' mathematics. Learning Outcome 1, Number and Operations in Context, envisages 'The learner is able to *use* knowledge of numbers and their relationships to investigate a range of different contexts which include financial aspects of personal, business and national issues.' (Italics mine) (DOE, 2003b: 14).

Given its specialist nature, Mathematical Literacy and the recognised paucity in teaching expertise, the task of successfully complying with the expectations of the NCS Mathematical Literacy (DOE, 2003b) and accomplishing its assessment standards will continue to place pressure on the South African mathematics teaching corps. The subject also assumes widespread mathematical competence emerging from Grade 9 of the GET. The baseline assessment reviewed earlier in this chapter shows that 204 learners in Grade 10 produced an average of 19% on an assessment that requires the core skills for embarking on Mathematics or Mathematical Literacy in the FET band, and more specifically no competence in linear graphicacy that constitutes the very core of Mathematical Literacy. This cannot be construed as just practice. That learners are being failed in their constitutional right to be taught mathematics is reflected in the poor performance in a standard assessment outcome as illustrated in Fig. 2.1 and more generally in South Africa's poor ranking in the TIMMS report. Learners are then compelled to continue with these extreme mathematical content deficits in having to offer Mathematical Literacy as a core requirement of the FETC; a subject that will also not entitle them access to science and mathematics related career options at tertiary institutions.

Alongside this argument one needs to consider the higher cognitive demands of core Mathematics and its effects. The current system allows learners who do not demonstrate sufficient aptitude for HG mathematics to select SG as an option which reduces, but does not eliminate, tertiary study options. The product of attrition amongst Mathematics learners will need to be absorbed by Mathematical Literacy which will further compound the logistical problems of unwieldy class sizes that perpetuates learners' inaccessibility to sound mathematics teaching.

In the final analysis it is my concern that Mathematical Literacy, as a perceived softer option, will compete with Mathematics. This will perpetuate the crisis that is described earlier in the analysis of the enrolment and pass rate data of the past ten years. The cost to the state of the unanticipated consequences cannot be justified in terms of curricular renovation.

I need to make the point that the philosophy that underpins Mathematical Literacy[9] and the quality of its content and its relevance for an emerging South African context is laudable. Its timing in the light of the current crisis around systemic mathematics performance is, however, questionable.

It is important to produce mathematically literate citizens but it seems inappropriate to do so through curricular enforcement of Mathematical Literacy as a core subject of a school leaving certificate especially in the absence of prerequisite mathematical understanding that should emerge as a product of the GET phase and a continuing lack in teaching expertise.

[9] Mathematical Literacy as a subject of the Further Education and Training band is distinctly different from the mathematical literacy that is referred to in this thesis where its meaning concerns the ability to be able to read and interpret graphs that depict real algebraic reality.

2.6 The status of secondary mathematics indicated by trends in performance and pass rates

Fig. 2.6 indicates that, as the trends in enrolments was characterised by dramatic changes around 1997 so the trends in performance data shows the same tendency.

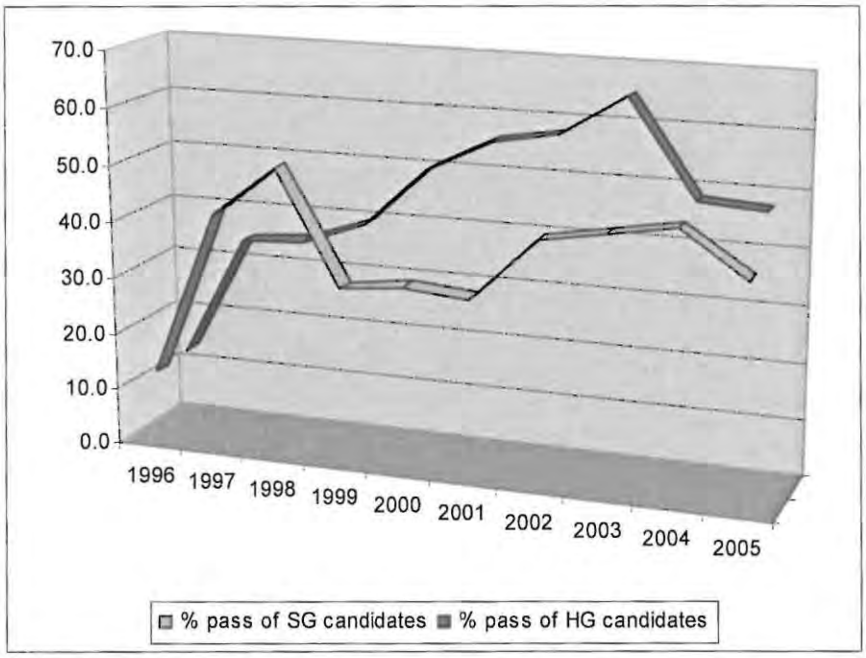
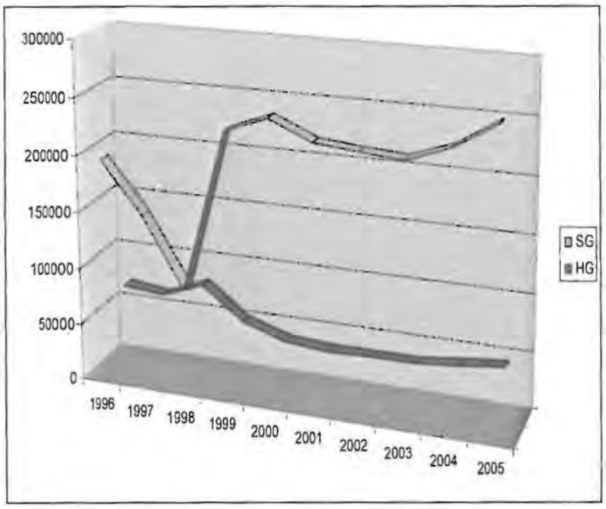


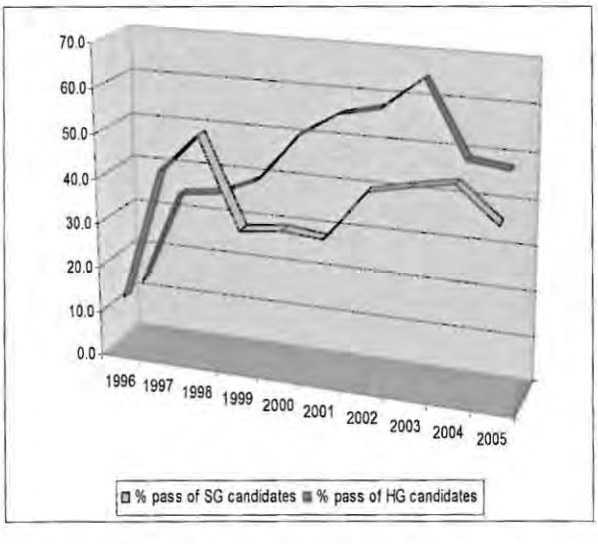
Fig. 2.6

Comparative performances of HG and SG mathematics learners at a national level
Data source: National Department of Education



Replica of Fig. 2.3

Comparison between HG and SG Grade 12 national mathematics enrolments
Data source: National Department of Education



Replica of Fig. 2.6

Comparative performances of HG and SG mathematics learners at a national level
Data source: National Department of Education

When Figures 2.3 and 2.6 are analysed comparatively the discordance in the trends around 1997 is evident.

The sharp fall in SG passes coincides with a sharp rise in SG enrolments. This may be explained by a lobby for mathematics enrolments in recognition of the subject's status and marketability. It would however not be expected that the passes would not decline in the context of more enrolments. The decline in passes may be explained by insufficient and impoverished teaching in the wake of rationalisation. The equally dramatic rise in HG passes is a direct result of and magnified by greatly reduced HG enrolments, which in effect removes from the population of HG learners its potential failures. The incentives provided for schools to demonstrate improved pass rates may further explain these changes since SG mathematics would have offered better chances of matriculation, if not mathematics, passes. The reasons for the trends may further be sought and explained in terms of changes in the standard of questioning in final examination papers.

Fig. 2.7 below, except for the anomaly around 2001 which was discussed earlier, has a stable trend. The graphs do reflect that HG and SG passes for the Western Cape are favourably higher than the national pass rate in the order of 10% on the SG and 30% on the HG this percentage is not comprised of black learners as corroborated by Fig. 2.8 below. It is stressed again that the nationai and provincial data masks the actual performance in township classrooms as reflected in Fig 2.3. Similarly, given the unacceptably poor enrolments in Khayelitsha, these can have no impact on the meso and macro perspective. It similarly confirms that the better meso and macro picture of performances are not contributed to by black learners.

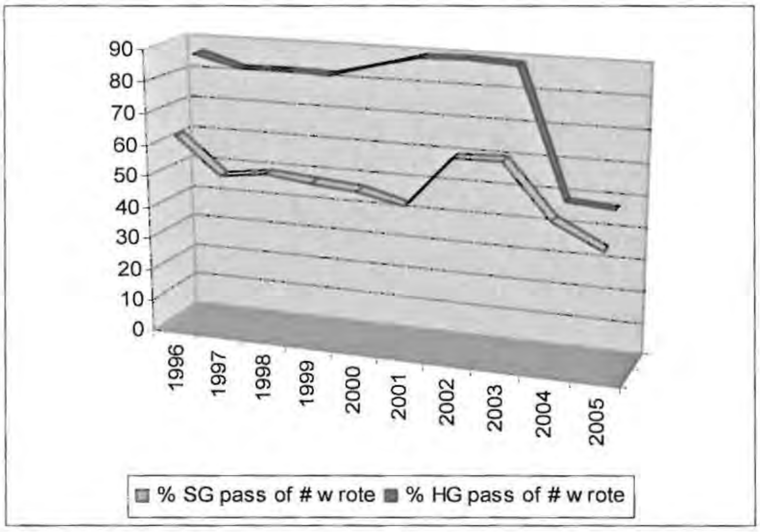


Fig. 2.7

Comparison of HG and SG passes in the Western Cape
Data source: National Department of Education

In Fig. 2.8 below the comparative trends in the HG and SG pass rates of Grade 12 Khayelitsha mathematics learners is illustrated.

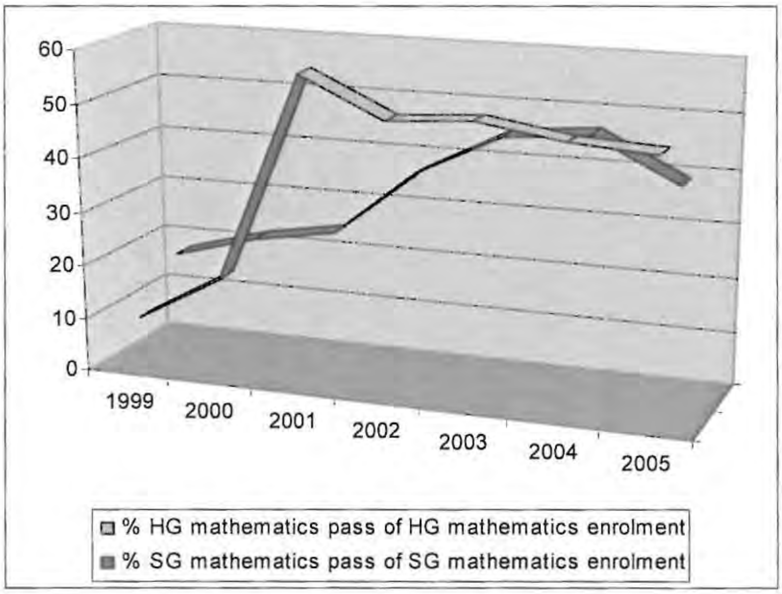
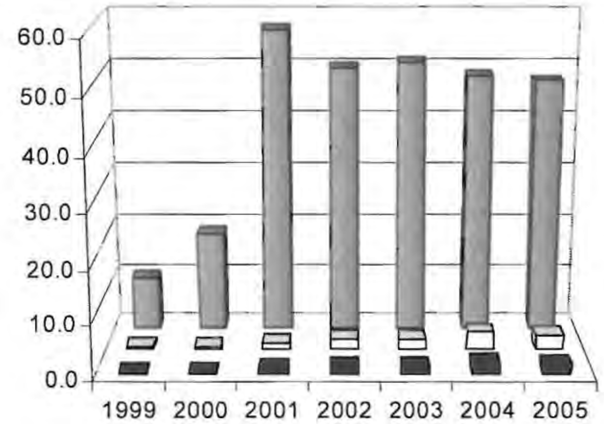


Fig. 2.8

Comparative trends in the HG and SG pass rates of Khayelitsha
Grade 12 mathematics learners
Data source: Western Cape Education Department

The sharp increase in the HG mathematics performance in 2000 relative to the stable to moderate SG mathematics results reflects the decrease in HG enrolments (which affects an exaggerated percentage analysis).

The relative performance of the Khayelitsha cohort of matriculation HG mathematics candidates in the contexts of the Khayelitsha matriculation population, of all mathematics enrolments in Khayelitsha and the HG enrolment in Khayelitsha itself is portrayed in Fig. 2.9.



	1999	2000	2001	2002	2003	2004	2005
■ % pass of whole matric Khayelitsha population	0.3	0.3	0.8	1.2	1.2	2.2	1.8
□ % pass of Khayelitsha HG & SG maths candidates	0.6	0.4	1.2	1.9	2.0	3.3	2.7
▩ % pass of Khayelitsha HG maths candidates	9.7	18.5	57.1	50.0	51.0	48.5	47.6

Fig. 2.9

Data reflecting the performance of Grade 12 Khayelitsha HG mathematics learners relative to three groupings over seven years [10]
Data source: Western Cape Education Department

There is a positive trend of moderately improved performance over the five year period with respect to each of the three groupings of learners. The percentage pass of HG Khayelitsha enrolments is encouraging but is tainted by the fact that in real terms the total cohort for 2005 is 82 learners of whom 39 passed on the HG. For the five years from 1999 to 2003 the SAILI mathematics HG project has worked in teacher inservice training and learner support in four of the fifteen Khayelitsha

[10] Enrolment and performance data for Khayelitsha are not available prior to 1999.

schools where 18 out of 25 and 18 out of 22 learners passed HG mathematics in 2002 and 2003 respectively.[11] This indicates that 18 of the 26 successful Khayelitsha candidates (69,2%) and their teachers had the support of the SAILI project prior to and during the period of improvement. The sharp rise from 18,5% to 57,1% in the pass rate from 2000 to 2001 is likely to be related to the reduction in enrolments in order to magnify overall matriculation performance. The specific HG pass ratio in 2003 of 3 of 6 (School W), 3 of 4 (School X), 6 of 6 (School Y) and 6 of 6 (School Z) is likely to be a combination of the reduction in absolute numbers by way of enrolments and more so, intervention support, when compared with the performances of the previous years. These statistics are summarised in Table 5 below.

With reference to Fig. 2.9 it is of concern that only 2% of all mathematics candidates are successful at HG mathematics. It is common practice however, for this pass rate to be given in terms of the entire matriculation population, which in this case is 1,2%. In essence, though, it is true; 1,2% of all 2003 Khayelitsha matriculants passed HG mathematics. The equivalent statistic at a national level for 2003 is 0,4% calculated from raw data supplied from the Department of Education.

	School W		School X		School Y		School Z	
	No. of HG enrolments	No. of HG passes	No. of HG enrolments	No. of HG passes	No. of HG enrolments	No. of HG passes	No. of HG enrolments	No. of HG passes
1999	1	1	11	2	0	0	10	0
2000	0	0	5	3	0	0	4	0
2001	0	0	7	5	0	0	7	0
2002	5	2	9	8	4	3	7	5
2003	6	3	4	3	6	6	6	6

Table 2.5

Performance of learners at schools where there has been HG mathematics teacher and learner support over five years

[11] This means that these Khayelitsha learners received mathematics instruction according to the methods that constitute a major component of this research.

These observances as detailed in Table 2.5 and descriptive statistics confirm emphatically the situation that HG mathematics finds itself in at national, provincial and local township level with the situation in the last mentioned scenario being increasingly stretched in its capacity to deliver HG mathematics to HG candidates. It is this bleak scenario that begs a solution; a part solution which is sought by way of this doctoral research.

The seven candidates at school Z were part of a HG mathematics intervention which was the initiative of the Western Cape Education Department (WCED) from July 2002 under my tutorial supervision for August and September of 2002 which employed a metacognitive graphical approach to teaching; a strategy which reverses the traditional sequence of generic skills of 'simplification of expressions', the 'solution of (in)equations and the 'sketching of graphs'. It uses graphical mathematical representations as a form of iconic language that is expounded in Chapters 5 and 6.

An intervention to aid Khayelitsha HG mathematics learners who were identified as 'at risk' of failing their final mathematics examination was launched in August 2002. It served as a pilot study for implementation of the contra-typical approach as raised in the previous paragraph. The pilot study showed that learners showed a significant statistical improvement in learners' results from September to December (van Jaarsveld, 2005).

A contra-typical graphical approach to each section characterised each session. The initial purposes of this approach were to provide learners with a graphical visual stimulus upon which to construct abstract algebraic reality, to synthesise common skills and content, to provide a thread of skills and procedures running through the syllabus content, to establish a mental image framework which would facilitate the understanding and contextualising of questions, to inculcate an appreciation for and show the versatility of graphs and its commonality to many aspects of the work at hand and to allow for the completion of the syllabus at a pace compliant with the constraints of time. In essence these are the aims, and advantages, of such a contra-typical graphical approach which is demonstrated in the body of this thesis. From the outset, and from four previous years of township classroom experience, I realised that my group was receiving instruction in their second and sometimes third language and that the iconic value of a graphical approach was greatly aiding and supplementing my language medium of instruction.

2.7 Summary

National, provincial and local senior certificate performances in mathematics attest to the fact that mathematics in South Africa finds itself in a dire position. The democratic change in 1994 from an apartheid government and its disparate educational provision brought with it a freedom for instituting redress in policy and provision both of which came to bear on the teaching and learning of mathematics. The deleterious effects of the rationalisation of teachers have impacted classrooms negatively through the draining away of experience and expertise. The freedom for policy redress has seen the implementation of outcomes based education principles and its concomitant changes reflected in curriculum reform and recorded in the National Curriculum Statements for Mathematics and Mathematical literacy (DOE, 2003a, 2003b), with the inception of the Grade 10 National Curriculum Statement for Mathematics in January 2006. The concurrence of teacher rationalisation with the implementation of policy reform has compounded the crises of classrooms. It is furthermore observed that the polarity of typically former 'model C' and township classrooms within the South African educational system has been exacerbated by the changes since 1994. The effect of these changes has been a deteriorating product of higher grade (HG) mathematics learners exiting the secondary school system and destined for tertiary mathematics study. The deterioration is both in terms of the number and quality of higher grade passes.

It is cautioned that the undesirable way that SG mathematics competed against HG mathematics as a softer option to learn, and an easier option to teach, may repeat itself in the Mathematics and Mathematical Literacy of the FET. The school system will, in this, continue not to supply the mathematics expertise for tertiary study, and mathematics skills and competences that are recognised in the national initiatives, the Accelerated and Shared Growth Initiative for South Africa (ASGISA) and the Joint Initiative for Priority Skills Acquisition (JIPSA)[12], as requiring of redress.

[12] ASGISA and JIPSA are initiatives of the South African Presidency, the latter seen as a building block of the former, was launched by Deputy President, Phumzile Mlambo-Ngcuka, on 27 March 2006.

It would seem that the compulsory nature of Mathematical Literacy reflects injustice towards learners in removing their constitutional right not to choose Mathematical Literacy especially in the light of systemically accumulated content deficits that learners have carried through their compulsory years of schooling. The impact of these parallel developments will serve further to demoralise teachers of mathematics whose rewards in terms of subject performance are already meagre and whose teacher training is incompatible with mathematical fields of study where they are being retrained to teach.

Inservice and pre-service teachers of mathematics need opportunities for inservice and pre-service learning to acquire understanding of mathematical content that allows learners to engage critically with mathematical content, not only in terms of its real life applications but also for the understanding of real mathematics contexts.

The Western Cape Minister of Education, Cameron Dugmore quotes the President in his budget speech on 19 May 2006 in his call to (all) '... to move faster to address challenges of poverty, underdevelopment and marginalisation confronting those caught within the Second Economy, to ensure that the poor in our country share in our growing prosperity...' and further states that 'The challenge of overcoming the legacy of apartheid becomes very apparent when one analyses the matric pass rate of 2005. Whilst the pass rate in this province has been consistently more than 80% in the last five years, the reality is that the results of African and Coloured learners in ex-HOR and ex-DET schools continue to reflect apartheid disparities. (WCED, 2006: 3) This is recognition of the general disparities in education that are magnified in mathematics education roll out.

At a national level ASGISA and JIPSA are encouraging developments but it is to be seen how effective these initiatives prove to be in inducing momentum in an inert mathematics education scenario as it identifies, inter alia, that 'Teacher training for mathematics, science, ICT and language competence in public education' as a working area. Deputy President Mlambo-Ngcuka states 'Our support for skills development includes poorer schools and increased efforts to support maths, science and English language skills in schools (South African Government Information (SAGI), 2006). JIPSA will be focusing specifically on teachers of these subjects. Teachers are being regarded as a scarce and a priority skill.' In terms of granting all South Africans the opportunity to participate in the First Economy, and the growing economy in general, she goes on to say that 'Empowerment through education must be given a big boost in the work of JIPSA.'

Mathematics education in its current delivery needs empowerment in the terms as described by the Deputy President, Phumzile Mlambo-Ngcuka.

Chapter 3 provides an overview of the literature in both learning theory and mathematics education research. The former seeks to establish a basis for later argument that the artificial demarcation between behaviourism and constructivism is a parochialism that does not serve mathematics education well. The epistemological frameworks that provide a means for the understanding of how mathematical knowledge is acquired provides theoretical frameworks that enable the development of the research design and the research instrument.

III

An historical overview and critique of learning theory with reference to epistemological frameworks in mathematics education research

'***The critically transitive consciousness is characterised by*** depth in the interpretation of problems; by the substitution of causal principles for magical explanations; by the testing of one's "findings" and by openness to revision; by the attempt to avoid distortion when perceiving problems and to avoid preconceived notions when analyzing them; by refusing to transfer responsibility; by rejecting passive positions; by soundness of argumentation; by the practice of dialogue rather than polemics; by ***receptivity to the new for reasons beyond mere novelty and by the good sense not to reject the old just because it is old – by accepting what is valid in both old and new.'*** (Bold italics mine)

Freire (1974: 14)

3.1 Introduction

A review of the literature reveals two categories of theory regarding cognition as it relates to learning. On the one hand there is discourse on the notion of theories of learning and on the other a commentary on the theory of teaching or instruction although there is seldom distinct delineation between these two in the discourse itself. It is often that discourse on one implies the other.

3.2 Behaviourism and Constructivism as epistemological opportunities

In the following paragraphs the theories, models and contributions to learning theory are appraised in terms of behaviourist and constructivist thinking. The contributions of key philosophers and authors on the subject of knowledge acquisition and learning are assessed in the context of behaviourism and constructivism as it relates to the process of learning. A précis of each of behaviourism and constructivism prefaces the appraisal of the theories and models. The duality here is not synthetic because deemed to be redundant techniques and contexts are thought to have value for a seeming inertia in South African mathematics classroom research. What appears to emanate from the literature is that, that which is essentially regarded as behaviourist, implies much about teaching as a dominant aspect of the educative process. It contrasts on the other hand with constructivism that makes implicit reference to learning as an aspect of the educative process. The separateness of these schools of thought around learning experience seems to highlight contemporary school learning environments in that teaching is treated as separate from learning when in essence as experiential episodes they are dependent on one another reflected in a reciprocity that needs to be nurtured. How and what I teach must be impacted by what I learn from learners in their response to what I teach, in the same way that their education is dependent on the heard communication that transpires between us. It would seem that elements of both behaviourist and constructivist schools of educational thinking are applicable to contemporary teaching and learning. In mathematics more particularly there seems to be place for deliberate forms of instruction that elicit critical engagements with mathematical concepts; deliberate forms of instruction that impart essential content that may otherwise not be innately accessible to learners, and that may ultimately advertently or inadvertently constitute accumulated learning deficits that tarnish cognitive closure.

Behaviourism as a psychobiological and psychosocial science emerged in the work of Skinner in the 1930's. Learning was regarded as being controlled by affecting the variables of the situation, behaviour and consequences of a learner (Bell, 1978). This gave rise to empirical and quantitative studies of learning, initially amongst rats and extrapolated to human learning.

Educationally the behaviourist philosophy is also referred to, historically in this order, as 'connectionist' or 'associationist', in the literature (Thorndike, 1932). One infers that the term 'connectionist' refers to the link between the learning content and the learning outcome. 'Associationist' relates a learning 'stimulus' and the 'response' that it elicits. The Habermasian notion of communicative action (Habermas, 1990) is developed, in mathematics education terms, in this work and it is seen to have an associationist character in that it seeks to elicit cognitive engagement with the concepts. Behaviourism relates to an empiricist philosophy of science, in that all knowledge originates in physical experience. One is accordingly able to obtain direct and absolute knowledge of any reality since through the senses reality is depicted in one's mind with complete correspondence. The implication of this is that knowledge can be transferred exactly in the teaching process and further presupposes that a passive receiver can acquire knowledge into a vacuous mind state. Current knowledge or an existing schema is irrelevant in the process of learning. A stimulus and response bond characterise behaviourist learning theory. Thorndike (1922) explains that the 'law of exercise' uses repetition as the mechanism for strengthening appropriate stimulus-response bonds that directly affects the longevity of the retention of information. The 'law of effect' states that appropriate stimulus-response bonds are strengthened by success and reward (positive reinforcement) and inappropriate stimulus-response bonds are weakened by failure (negative reinforcement) (idem.) Learning proceeds from the simple to the complex where pockets of knowledge, accumulated as entries, are vigorously rehearsed and stored. It seems that this aligns to a view of compartmentalised learning where 'knowledge' is ultimately equal to the sum of its constituent parts. Von Glasersfeld (1995: 178), states that behaviourism's key notions are still alive and active in the minds of teachers, which depending on how it is viewed, may be of value to mathematics instruction. It is not the operant condition of behaviourism that is attractive, but rather the fundamental component of transferring knowledge that, although didactive, can also be elicitive and inculcative (Jaworski, 1999). It is the latter aspect that would help transform the compartmentalised nature of behaviourist didactics to a gestalt view of the sum being greater than the sum of its parts. The aspersions cast on behaviourist techniques brings to question whether these fundamental forms of knowledge transfer can hold, for mathematics particularly, some value. Is it possible that mathematics teachers embrace social constructivist techniques through a lack of expertise in interrogating their cognition about deeper mathematical structures? This has become a core question, for me, over many years of teaching and of managing mathematics departments. The more learners are asked to take responsibility for their own learning, the more we seem, as mathematics teachers to abdicate our responsibility for delivering, in Skempian terms (Skemp, 1976) vital theory based teaching that underpins the elicitation of critical cognition amongst learners. What is regarded as incompetence amongst learners may be accounted for by loose or shallow teaching.

I understand that constructivism, as a perspective on learning, does not assume that concepts are gained directly from experience. It is rather underpinned by the quality of the existing knowledge through which a learner can grow its knowledge. It implies that a learner is an active participator in the learning process and by implication isolates the teacher from the learning-teaching interaction to a role of facilitator.

Piaget (1970, 1980a) in his work introduced meaning to the unanalysed cognitive function that characterises that domain between 'stimulus and response'. He viewed cognitive epigenesis as the attainment of equilibrium between accommodation and assimilation through reflective abstraction. He viewed the *learner* as being able to 'assimilate' or 'accommodate' additional knowledge in relation to a familiar existing 'schema'. Assimilation enables additional new knowledge to be incorporated into a schema where there are familiar existing cognitive constructs. Accommodation requires a degree of reorganisation of an existing construct to admit a somewhat less familiar cognitive construct. It is this strain of constructivism that is termed 'radical constructivism' of which Piaget was a significant proponent. It is characterised by the 'independent' cognitive functioning (Confrey 1994, 1995a, 1995b), of an active learner. I understand constructivist learning theory to lean toward a phenomenological philosophy of science. It is hermeneutic in that it tries to understand the manner in which knowledge is acquired. It investigates the cognitive domain of human consciousness between stimulus and response, which Lerman (1998) refers to as an unanalysable feature of human behaviour. It may well be unanalysable but by critical realist retroductive and abductive modes of inference one can surmise about these transfactual conditions. It is intuitively evident from this discussion also that *learners* are central elements of the educative process. Without specific reference to the learning required of and intuitively acquired by teachers in the teaching-learning process the implication is that teaching is an isolated aspect of the educative process that bears no learning. It is imperative that teachers also learn from their own hermeneutic radical constructivist endeavours to delve their own cognitive consciousness critically.

The work of Vygotsky (1962, 1978, 1981, 1986, 1987) has found significant support amongst teachers of mathematics and more especially amongst linguists before that. The relevance of his work for second language learning theorists has been critical, but it is the emphasis on language that would make his learning theory equally relevant to mathematics, which is itself a language. Schutz (2002) records Vygotsky as saying that "thought is not merely expressed in words; it comes into existence through them." This gives perspective to the foundations of language in his theory of learning. Vygotsky (1962: 55) is quoted as saying "... a problem must arise that cannot be solved otherwise than through the formation of new concepts." This prompts the question as to how these new concepts come into existence, an issue that I address in a later chapter. There is a suggestion here that the mathematical vocabulary is essential for nascent thought. By implication thinking about mathematical constructs is not possible without its vocabulary. Vygotsky views this cognitive development as taking place in a matrix of the person's social history and being a result of it.

Piaget and Vygotsky's work is representative of radical and social constructivism respectively. Vygotsky theorises that the development of a student's intelligence "results from social interaction in the world and that *speech*, social interaction, and co-operative activity are all important aspects of this social world" (Italics mine) (Sutherland, 1993: 104). Vygotsky defines 'scaffolding' as an adult having conscious control of a concept until a child is able to internalise external knowledge. It is this conscious concept control that I work towards in my teaching in this research.

Behaviourism and constructivism are seen to be different in the following ways. The former concerns itself with the empirical, observable aspects of learning. It considers a learning response being elicited by a stimulus. The effectiveness of the learning is deemed as dependent on mechanical or rote rehearsal, as in 'doing mathematics' to sustain its memory. Success and reward positively reinforce its retention. Constructivism concerns itself with the phenomenological, unobservable cognitive activity which would necessarily occur after the presentation of a stimulus and before its associated response. Constructivism may be radical, in which case it pertains to the independence of the learner in its pursuit of knowledge, or social, which views learning as a function of social interaction; this artificial delineation serves merely to situate this study theoretically. Radical and social constructivism has in common the view that the learner has an active role in the learning process. It is felt strongly that this is not only the domain of the learner but that teachers should embrace the same learning role. More succinctly, teachers should not regard learning as that privilege peculiar to learners. Teachers should indulge in the radical and constructivist learning afforded by their classroom communities. The tenets of these contrasted learning theories are summarised and reflected in Table 3.1 below.

Behaviourist learning theory	Constructivist learning theory	
Concerns empirical, observable stimulus and response	Concerns phenomenological unobservable cognition	
	Radical constructivism	Social constructivism
Learning construed as an independent activity	Learning construed as an independent ***activity***	Learning construed as a social, community based ***activity***
Existing individual knowledge is not significant	Existing individual knowledge is critical	

Fig. 3.1

Comparative view of the critical tenets of Behaviourism and Constructivism as it relates to learning in education

This preamble on an appreciation of behaviourism and constructivism serves as a framework to set in perspective the key aspects of contributions to learning theory by learning theorists as they are summarised in the following paragraphs.

3.3 Perspectives on learning theory with reference to mathematics teaching and learning

Ontogenetically, the Western rudiments of learning theory emanate from the writings of Plato (circa 427 – 347 BC) who considered the nature of knowledge and the way it is acquired. He saw reality as being 'intelligible forms' on the one hand and a 'perceptual world' on the other, the latter being an imperfect copy of the former. This distinction aligns itself with what he calls the transitory nature of true opinion as opposed to the stability of knowledge. True opinions, once tethered by reason, he claimed, become knowledge. In essence, there is a tacit belief that knowledge can be transferred intact from a sender to a receiver. Integral to his beliefs is the idea of 'recollection' which is the recalling of something that has been previously known, a notion, which is incompatible in constructivist rhetoric since it makes no allowance for acquiring knowledge from interactive or collaborative experience. It would seem that behaviourism finds a degree of correspondence with Platonic views in terms of a mechanical transfer of knowledge. There is, in the notion of 'tethered reason' a cognitive element of constructivism. It is the notion of 'true opinion' that also implies much about our authentic personal claims to knowledge that arises from hermeneutic endeavour (Kraut, 1992).

René Descartes (1596 – 1650), regarded as the first great philosopher of the modern era, pursued mathematical and scientific truth. Much of his work was concerned with the provision of a foundation for the advancement of human knowledge through the natural sciences. The progress and certainty of mathematical knowledge, Descartes supposed, provide an emulable model for a similarly productive philosophical method, characterised by four simple rules (Descartes, 1999):

1. Accept as true only what is indubitable.
2. Divide every question into manageable parts.
3. Begin with the simplest issues and ascend to the more complex.
4. Review frequently enough to retain the whole argument at once.

This quasi-mathematical procedure for the achievement of knowledge is typical of a rationalistic approach to epistemology. Behaviourism corresponds to it in its use of the notions of 'manageable parts' and 'frequent review'. Last mentioned has connotations of rehearsal, repetition and rote, as applied in behaviourist learning theory. I attempt to show through the development of this thesis how the rule of beginning with the simplest issues and ascending to the more complex is a neglected rule if these rules are applied to mathematics learning and teaching. The reference to a mathematical gestalt is linked to the rule of 'retaining the whole argument at once.' There is much to learn in terms of mathematics teaching and learning from these Descartian rules that claim to advance the progress and certainty of mathematical knowledge.

Popper (1963, 1977) regarded as one of the greatest philosophers of science of the 20th century, and at one time a teacher of secondary mathematics and physics, viewed the growth of knowledge as proceeding from our attempts to solve our problems. These attempts involve the formulation of theories and necessarily employ 'independent creative imagination' to go beyond existing knowledge. Theories are subjected to deductive testing where conclusions are inferred from a tentative hypothesis. Conclusions and other relevant statements are compared to determine whether they falsify or corroborate the hypothesis. Popper stresses that observation-statements are theory-laden and are as much a function of purely subjective factors (interests, expectations, wishes, etc.) as they are a function of what is objectively real. If it is permissible to position this view on the growth of knowledge within a behaviourist or constructivist paradigm it would be accommodated within 'radical constructivism' from the point of view of its reference to the employment of 'independent creative imagination' in attempting to progress beyond existing knowledge. The point to be made is that without a cognitive catalyst obtained from being deliberately taught or enlightened about mathematics there is little chance of stimulating and fostering the growth of creative independent imagination.

Freudenthal (1973) gives a definition of mathematics which incorporates a notion of how mathematics is learned. In Freudenthal's view, mathematical concepts, structures and ideas serve to organise the phenomena of the real world as well as of the field of (developed) mathematics. The development of mathematics, the learning of mathematics as well as the creating of mathematics, starts from 'common sense'; learning and developing of mathematics are conceived as an ongoing process of mathematisation on different levels and into different directions: basic activities within this process are local and global ordering, schematising, formalising and symbolising, and – most important – deliberately reflecting and communicating these activities. (Keitel, 1993: 162) The value of Freudenthal's perspective to this work is his last mentioned aspect of deliberate reflection and communication of the global ordering and schematising which again requires a starting point that teachers have the privilege of imparting.

From a didactical perspective Freudenthal regards mathematics education as guided reinvention, as connected to reality, as focusing on learning processes with long term orientation. Teaching and learning have to start from rich contexts in which mathematics could be studied and experienced. The applications of mathematics play a major role in this conception. Learning mathematics, in his view, is necessarily based on exchange with others, and consequently social interactions become an indispensable precondition, and the interface between individual and collective experiences must be enforced by intensive communication and experimental work in heterogeneous groups (Keitel, 1993: 162). I interpret rich reality as the real mathematics attached to its deconstruction emanating from its graphical contextuality as demonstrated in Chapter 5.

Freudenthal's perception of mathematics learning is centred on social constructivism. His views on learning are epitomised by cognitive activities whilst, didactically, social interactions are regarded as an 'indispensable precondition' for mathematical learning. This research interprets this social interaction as that which transpires between teacher and learners, and between learners, in desirable discussion or communicative action in pursuit of a mathematical consciousness.

Skemp (1976) is regarded as the first to truly integrate the disciplines of education, psychology and mathematics. Principal ideas in his theory of learning are those of 'relational' and 'instrumental' understanding. The use of the formula $A = \pi r^2$ serves to illustrate the difference between these two ideas. The recall and use of the formula reflects a learner's instrumental understanding of the formula. On the other hand having knowledge of the development of the formula reflects a learner's relational understanding of it. Skemp believes that learners need 'routine' and 'non-routine' methods of solving problems. The former methods are used in the solution of familiar problems. 'Non-routine' problems may need to be aided by activities (the literature seems to imply that 'aided activities' incorporate 'reception learning') which serve to enhance a schema or knowledge structure in which case it will become routine without detracting from the possibility of a future adaptation of the schema. He believes that real world problems are best solved from a *structured knowledge base* and the successful solution of a problem serves to develop a learner's knowledge structure. Skemp distinguishes between two approaches to learning which tacitly make inferences about the act of teaching. A 'direct approach' commences with a real life problem with its required concomitant abstractions, utilises a schema, employs a mathematical model and its routine procedures in solving the problem, and culminates in the re-embodiment in the original problem. There is a tacit understanding here that the learner has sufficiently developed schemas to solve the problem. A 'theory based approach' commences with activities that embody one new concept and proceeds, via a process of abstraction, to the formation of a new concept. These in turn connect with appropriate knowledge structures and the selection of a model and its routines in which the problem is solved. Re-embodiment in the original problem finalises the process. In his 'theory based approach' Skemp finds space for rote learning, a low cognitive element, to assist, or further develop, an existing schema. Schemas are built from encounters with actuality, communication with others, or from within, by our intuitive creativity. These schemas are in turn tested against actual experiments, discussion and self-reflection, respectively. Learners in this work, in engaging with written communicated thoughts to express their understanding of mathematical concepts, give credibility to testing their mathematical schemas. I sense that in contrasting Skemp's direct and theory based approaches to teaching, especially with his recognition of rote as a desirable component, suggests a tacit approval of elements of behaviourist learning theory.

By laying emphasis on a learner's understanding of concepts either 'instrumentally' or 'relationally', Skemp (1976) positions himself as a radical constructivist. His 'direct' and 'theory' approaches to didactics employ 'real world' applications and 'content based' instruction as initial phases in the learning process respectively. These allude to deliberate teaching episodes that precede independent cognitive activity, a scenario that incorporates both behaviourist and constructivist strengths.

Although specific to the learning and teaching of geometry, the van Hiele model has generic traits which are significant for the acquisition of knowledge in general. The van Hiele (1973) model of Geometric thought proposes a structure for reasoning and is characterised specifically by sequential learning and an activity approach. The model asserts that a learner moves through five levels of understanding in strict sequence without omitting a level. The advancement from level to level depends more on content and mode of instruction than on age. Recommended 'sequential phases of learning' accompany the levels to assist learners advance through the levels. It is maybe from this sequential model that my own interrogation of the importance of sequence in this work arises. Sequential learning is associated with orders of abstraction and its associated orders of difficulty and is developed in a later chapter. Sequential learning is a product of deliberate sequential teaching.

Vygotsky (1997) has a view of learning which is regarded as 'social constructivism' in that it views learning as a function of a social context, characterised by the interaction of thoughts and ideas on a topic of learning between a learner and teacher or a learner and more informed peer. He defines his 'zone of proximal development' as that place where a learning individual is drawn into the experiences it may encounter in its future. Vygotsky (1978) maintained the child follows the adult's example and gradually develops the ability to do certain tasks without assistance. He called the difference between what a child can do with help and what it can do without guidance the 'zone of proximal development'. The zone of proximal development is therefore the distance between the actual developmental levels of a child on independent problem solving and the level of potential as determined through problem solving under adult guidance or in collaboration with more capable peers. It is the modelling provided by the teacher that underlines the important role of a teacher who ideally is more advanced in knowledge than his charges, and who therefore has the capacity to teach for a deliberate purpose.

The work of Ausubel (1961) is significant in its claim that there is no dichotomy between rote and meaningful learning and in this regard he emphasises the active nature of reception learning. Ausubel views them as existing on a continuum. His idea of 'subsumption' infers that there are existing forms of knowledge with which new knowledge interacts in the development of more meaningful concepts. Much of his work had its origins in information processing theory and his advocated learning processes held relevance for science based education. Inherent in this understanding of learning, and specifically the ideas of 'active reception learning' and 'subsumption', the latter akin to Piagetian 'accommodation', leans toward a synthesis of behaviourism and constructivism.

The intellectual development theory of Brüner (1957, 1974) advocates that a combination of concrete, pictorial, and then symbolic activities will lead to more effective learning. He is a proponent of 'discovery learning' and as such employs constructivist techniques in the teaching – learning process. Brüner (1966) worked with the perception of infants and young children and stated that a learner's objective is 'to reach a state of symbolic representation through language which is used to describe abstract relations among states and processes'. (Brüner, 1966: 8) This, in my experience, is equally applicable to the learning of mathematics and particularly at the senior secondary school level, if not throughout the mathematical careers of learners since this objective allows learners to proximate in mathematical terms, a level of what Freire (1974) calls critical consciousness. Brüner defines ikonic representation as the child's ability 'to represent the world to himself by an image or spatial schema that is relatively independent of action' (Brüner, 1966: 21). Ikonic representation is characterised by the fact that 'A picture can serve easily as a guide for action or a support for action' (Brüner, 1966: 9) applied in this work as illustrated by Fig. 5.1. With specific reference to this research into what makes a learner approximate deeper understanding of mathematical constructs, I find the following statement particularly useful: '... imagery must be made concrete, and the task of isolating a simple figure within the complex figure must be rendered manipulative.'

(Brüner, 1966: 22). In this regard, I attempt to demonstrate in a later chapter how concreteness of graphical depictions demystifies the concise and abstract forms of algebra. As young children the "ostensive referent of words and their semantic markers the very same words are already embedded in some form of syntactic hierarchy." In other words "the child is learning to use his words and semantic markers for the picturable and ikonic aspects of his world, but the words themselves are enmeshed in a highly abstract and hierarchical system of categories...." (Brüner, 1966: 45). This cognitive element of Brüner's work with young children is equivalent to adolescent mathematical learners using graphical mathematical representations to release the textual forms of mathematical concepts that is deemed to be an integral part of their cognitive mathematical hierarchy. It is however not feasible to accept that these levels of cognitive operation are possible without didactive input, especially the mathematical representations equivalent to Brüner's ikonic representation that learners are exposed to.

Gagne (1983), as an instructional psychologist, focuses on approaches to instruction. He developed a theory of learning hierarchies, often regarded as a 'teaching theory'. Gagne's learning hierarchy comprises increasingly complex skills' levels applicable to five major categories of learning viz., verbal information, intellectual skills, cognitive strategies, motor skills and attitudes. His knowledge hierarchy tacitly assumes instruction from progressively lower-level facts to higher level concepts. This ordered acquisition of knowledge enables learners to reason with higher order concepts if all the lower-level information has been acquired. These thoughts embody the notions of orders of difficulty that I relate to ordered abstractual levels that prescribe a facilitatory *teaching* sequence.

In the following section I make observations about learning theory in terms of the understanding developed in the previous paragraphs as it may relate to mathematics in education.

3.4 Observations arising from an overview of learning theories

1. It would seem that 'cognitivism' viewed as a 'science of learning and understanding' is inclusive of 'constructivism'. Constructivism is, therefore, rather a

school of thought within 'cognitivism' and as such 'cognitivism' is more accurately viewed educationally as a philosophy juxtaposed to 'behaviourism'. Constructivism tacitly refers to the actions of learners and not teachers while behaviourism seems contextualised in its association with teaching.

2. I interpret the literature as being preoccupied with learning, as opposed to teaching, and it seems as such to infer that learning is an unsolicited, voluntary endeavour of those who pursue it. Learning ought rather to be viewed as an element of the knowledge acquisition continuum of which teaching is yet another element. Learning is more than a function of the independent or collaborative accrual of information relative to an existing schema through cognitive engagement. It is also a critical function of sound instruction, and if viewed as such, teaching (as an art and / or science) ought to be more highly regarded as an educational endeavour. What is desired is not only observation or suggestion on the physical and affective configurations of learning environments but on the processes which prompt and induce learning as a natural process.

If teaching is construed as a unidirectional transmission of knowledge then the teacher must be viewed as a unidirectional transmitter and learners are then positioned as a set of passive receivers. The content being delivered impacts its recipients and there may be as many degrees of understanding of the delivered content as there are recipients. The nature of such knowledge transfer is indeed clinical.

If knowledge is transferred so that recipients are engaged, then the knowledge transfer becomes an affective process. A powerful tool of teaching is the art of questioning. If teaching is to encourage bi-directionality of transfer then questioning as a skill needs to pervade any instructional situation where the desire is to develop concepts. It ensures that the recipients of knowledge are active cognitive participators.

In contrast to the clinical unidirectional approach in which knowledge impacts passive recipients of knowledge, an affective bi-directional teaching approach, through questioning techniques extracts engagement from active recipients of knowledge even if such engagements are initially personal and private. These two teaching styles are adequately summed up by the terms 'push' and 'pull' teaching respectively. The quality of questioning in a 'pull' teaching scenario is a function of a teacher's own metacognitive questioning and deliberations about the content prior to a lesson and questioning and deliberations about desired outcomes to be transferred to learners, and the learners' display of understanding, in the teaching process itself. It may be useful for teachers to assess their impact potential on learners as both practising metabehaviourists and metaconstructionists.

3. Material on the learning and conceptualisation of secondary mathematics learners seems scant in comparison with primary mathematics education. Are learners understood to have acquired the skills with which to learn during a specific time of their early psychological and cognitive development after which they are deemed to be in place for all learning that ensues? The irony here is that the product of the school system is assessed at the end of twelve years of schooling but there is a paucity of investigation around the cognitive conceptual mathematics of the further education and training phase.

4. I infer that an implicit assumption of a constructivist paradigm is an orderly and manageable classroom in terms of a favourable teacher to learner ratio which is conducive to effective learning, and meaningful teacher-learner relationships. Established and resourced classrooms have a greater chance of complying with this assumption. Conversely, where the negative Freirian banking element of education as it manifests in behaviourist learning theory dominates poor classrooms it can be ascribed to purposes of control rather than engagement.

5. South African educational thought should derive from its own uniqueness and identity. Our use of international research and experience is essential but with modification or adaptation in respect of its unique skills-impoverished educational landscape. South Africa has, for example, embraced and implemented the constructivist principles of an 'Outcomes Based Education' (DOE, 2003a) and has aligned its implementation with a social constructivist stance in advocating techniques of self-assessment, peer assessment, group work, co-operative learning and collaborative assignments all of which are epitomised in the eight critical outcomes as developed by the South African Qualifications Authority. As much as there is merit in such a system it would seem to be appropriate to the educational

sophistication of mature classrooms and their developed teaching expertise. Its application in a South African context seems to exacerbate the polarity of accessible and inaccessible mathematics in affluent and poor classrooms respectively, defeating the object of seeking advantage for deemed to be overpopulated previously disadvantaged classrooms. Learners and their peers cannot do justice to the cognitive demands and sophistication of mathematics assessment procedures, unless associated with trivial assessment tasks.

6. Discourse on learning theory often embodies and refers to envisaged outcomes for learning and rarely for teaching. An incongruity arises out of this if education is recognised as the reciprocal relationship of teaching and learning.

7. In terms of mathematics education there is scope for the development of a mathematics pedagogy, which I address in chapter 9.

3.5 Constructivist support for mathematics learning that centralises learning as a function of the individual and/or community

I understand contemporary practice, as reflected in the literature, lean toward social constructivist approaches. In the work of Skemp (1976) and Ausubel (1961), however, there is a vestige of the behavioural element of rote learning. It is regarded as an element of low cognition, but nevertheless, features significantly in the respective learning theories. In professional practice there is a poignant aversion to the behaviourist philosophy, possibly because it is seen to dehumanise learning and mathematisation owing to its negatively perceived functions of rote and operant conditioning.

This assertion is substantiated by the following references in the literature which albeit discretely support constructivist theory in mathematics learning environments. Piaget (1970) states that the process of learning has to do with an individual's reorganisation of his own schemata. This is the classical tenet of radical constructivism and embodies independent learning as a function of the quality of an individual's prior accumulated knowledge constructs.

Von Glasersfeld (1991a, 1991b) claims that knowledge cannot simply be transferred ready-made from parent to child or from teacher to student. It needs to be actively built by each learner in his or her own mind.

Gravemeijer (1998) is of the opinion that mathematics can and should be learned on one's own authority and through one's own mental activities. There is a sense that this argues for the independence of the learner in its pursuit of knowledge.

Vygotsky's (1978) view that what a child can imitate today, it will be able to do with assistance tomorrow and alone thereafter, fits the constructivist paradigm.

Cobb (2000) is of the opinion that learning is attributed to learners' participation in particular communities of practice as constituted by a teacher and students in a classroom. Cobb here extends the collaborative notion of Vygotskian social learning into the community of the classroom where teaching as a practice is inferred rather than deemed necessary.

Sfard (1998) sees Vygotsky's 'zone of proximal development' as a margin accounting for all elements of learning which cannot be accounted for by a reference to the learner's own intellectual activity and must therefore be grounded in interpersonal exchange.

Wittman (1998) claims that knowledge is no longer seen as the result of a transmission from the teacher to a passive student but is conceived of as the productive achievement of the student who learns in social interaction with other students and the teacher. Wittman here gives a holistic stance on social constructivism where transmission of knowledge is an element of a one-on-one or a corporate community learning environment where teaching is relegated as a non-integral part of the educative process.

I am however partial to Jaworski (1999: 159) whose view is that 'Where constructivist theory is concerned, the 1990s have seen a shift in focus towards theory-practice rationalization through reporting of classroom research...More is needed, however, in bringing the issues from classrooms to centre stage, to challenge and potentially illuminate theory.' She is also of the opinion that the development of teachers' thinking leads to the development of teachers' teaching. As such a teacher's knowledge ought to be a factor in his students' learning. It is possibly this stance of Jaworski that blends the foregoing discussion with the one that follows.

3.6 Teaching and instructing as an essential component of constructivist learning

One deduces from these accounts that behaviourist elements have been displaced in learning theory discourses from the learning environment. In the ensuing discussion references in the literature which in some way support, albeit not explicitly, elements of behaviourist teaching and learning are highlighted and discussed.

In explaining the concept of Piagetian accommodation, Olivier (1989: 11) states that a new idea may be quite different from existing schemas which are relevant but not adequate to assimilate the new idea. It then becomes necessary to reconstruct and reorganise the schema. Such reconstruction leaves previous knowledge intact as part, or subset or special case of the new modified schema. Olivier (1989) refers to a situation in which Piagetian accommodation requires a reorganisation of a schema to adapt to a new idea with which an individual is not familiar. In the event of there being an adequate existing schema the new idea is assimilated. One questions if there are ever completely novel ideas which may be regarded orphaned as a consequence of not having a parent concept with which to associate. It is this aspect of learning theory which seems to assume that knowledge is evolving from one initial concept since there is the inherent understanding that adequate schemas exist for all new knowledge without due concern for its epistemology. It may be at this point that deliberate instruction may still have a valuable contribution to make to the process of teaching and learning. How is a new concept assimilated if there is no existing schema to accommodate, and ultimately assimilate it? There is therefore reason to debate the contemporary understanding of the nature of schemas; are they Platonic templates of knowledge which are immaculately conceived? Do they evolve from some form of pollination of a dormant knowledge seed? Is an initial construct imparted to a vacuous mind state by instruction and cognitively nurtured by the recipient, by which is inferred that it is conceived through a transfer of knowledge? Are schemas intra- or interdisciplinary? As rational intellectual beings we need to process knowledge irrespective of the origin of existing knowledge; and there would seem to be evidence for knowledge being learned by rote in some cases so that it can form an embryo of a schema. This aligns with Vygotsky (1962: 55) who claims "... a problem must arise that cannot be solved otherwise than through the formation of new concepts." I regard Novak's (1978) claim that reception teaching methods can be effective in developing highly functional concept frameworks amongst learners at all levels as indicative of the fact that the formation of new concepts is commensurate with teaching. He claims further with reference to Kuhn (1962) and Toulmin (1972) that concepts (and not enquiry methods) are at the core of rational human thought.

There is further support for quasi-behaviourist instructional strategies which incorporate a sense of rote learning. Cobb, Yackel and Wood (1992: 28) contend that students must necessarily construct their mathematical ways of knowing in any instructional setting whatsoever, including that of traditional direct instruction, and that the central issue is not whether students are constructing, but the nature and quality of those constructions.

With reference to their understanding of the nature of problem-centred learning Cobb, Wood, Yackel, Nicholls, Wheatley, Trigatti and Perlwitz (1991) view the construction of mathematical knowledge firstly as an individual and secondly a social activity. I concur with this especially where the construction of mathematical knowledge as an individual activity would require some form of active strategy to commit to one's mind the essence of a concept in question and, with that, the terminology and language that binds such a concept. The concept of 'fraction', for example, embodies the notion of sharing; of comparing parts of a whole, but in the teaching of it, the terminology of numerator, denominator and vinculum will facilitate communication about fractions. 'Vinculum' however incorporates notions of the structure of fractions in that, as a line separating numerator and denominator, it implies the grouping of both (by a bracket). Numerator and denominator, to the contrary, are classifiable as terminology. The concept of 'fraction' also assumes knowledge of number systems. By definition, fractions are composed of integers, the definition of which will supplement meaning and understanding of the concept of 'fraction'. It is clear also that the understanding of fractions presupposes an appreciation of the properties of the numbers, one and zero, for a learner to make complete sense of its gestalt conception. This would be sufficient to understand the difficulties which intermediate phase learners encounter in their conception of, and working with, fractions.

It seems therefore that *teaching* strategies which sustain the language and structure of fractions, as for any mathematical concept, need to be a subject of meta-analytical teacher thought before instruction takes place or we run the risk of surface structure teaching that encourages rote memorisation amongst learners. Meta-analytical thought presupposes the deconstruction of concepts so that learners become aware of the derivative notions that constitute concepts.

This stance is succinctly contained in Ernest (1991: 42) who writes that

a. "The basis of mathematical knowledge is linguistic knowledge, conventions and rules, and language is a social construction.
b. Interpersonal processes are required to turn an individual's subjective mathematical knowledge, after publication, into accepted objective mathematical knowledge.
c. Objectivity itself will be understood to be social."

An evaluation of each new idea needs to be analysed to ascertain whether it constitutes a concept for which a schema does not exist, or if it exists in some rudimentary form. Does the concept of 'absolute value', for example, constitute a new concept at the point in time where it is introduced to learners, or is it an extension of an existing concept of 'positivity' as it relates to x^2 ? What is implied here is that teachers need to have undertaken a meta-analysis of the concepts at hand and also have predetermined the concept's occurrence in the anticipated sequence of lessons and modules. This metatask will assist teachers to fathom their own expectations of learners and in turn help learners to develop expectations themselves. Constructivist learning cannot transpire in the absence of such deliberate meta-thought and deliberate teaching of such nuances.

This discussion on the inclusivity of facets of concepts also alludes to the cumulative nature of mathematics. Its most rudimentary levels of instruction have significance for its evolutionary instructional path. It would seem that introspection regarding notions that constitute concepts is desirable not only amongst mathematics teachers. Graves (1977) feels that some geographers recognise the principle of adiabatic cooling but seldom provide detailed explanations of the processes which result in the cooling.

Von Glasersfeld (1991b: xviii, xix) writes "From the constructivist point of view, there can be no doubt that reflective ability is a major source of knowledge on all levels of mathematics ... To verbalise what one is doing ensures that one is examining it. And it is precisely during such examination of mental operating, that insufficiencies, contradictions or irrelevancies are likely to be spotted." This constructivist stance by Von Glasersfeld, inherently suggests that in personal, independent reflection there is a notion of how private learning is manifested in personal cognitive dialogue. This dialogic content needs a starting point and that starting point is taken to be the imparting of knowledge by deliberate instruction, but more importantly it needs to be appropriated by teachers before learners can be expected to operate at these cognitive levels.

Cobb (2000) views an individual's active construction of knowledge as combined with mathematical enculturation and that neither individual students' activities, nor classroom mathematical processes can be accounted for adequately except in relation of one another. Similarly, I believe that the *active construction* of knowledge may be achieved in a number of ways; radical or social constructivism or rehearsal of information to establish a schema, but in mathematics classrooms learner experience cannot substitute for what Skemp (1976) calls theory based instruction; instruction that I regard as a first principles approach from graphical contexts.

It is the intention of this discussion to view elements of behaviourist teaching and learning to mathematics education more favourably, not in lieu of a constructivist paradigm, but rather aside it, and so embrace a more holistic view of the teaching and learning of mathematics. I believe that mathematics education can accommodate an eclecticism of behaviourist elements and constructivist paradigms, in the same way that methodological plurality is defined in critical realist theory. Because of its sophistication of language and symbols and processes, mathematics must be taught by those who have deeply investigated and passionately understood its contents in terms of what matters to its learners. This would be in line with Thom (1973: 202) who sees the real problem which confronts mathematics teaching as the problem of the development of 'meaning' and the 'existence' of mathematical objects and more recently, Ball, Hyman and Hill (2004: 51) who state that "Direct observation of mathematics classes suggests that teachers' knowledge of mathematics and their ability to deploy it in teaching *matter* for the quality of students' opportunities to learn." (Italics mine)

As a subject and discipline its depth, and relative depth for any age cohort, cannot be fully discovered by those who encounter it for the first time by independent or social endeavour alone. I regard even the guided reinvention of mathematics through real applications as proposed by Freudenthal (1973) as relevant only once the new concept as proposed by Skemp (1976) in his 'theory based approach' has been delivered to learners. Mathematics needs to be deliberately and meaningfully taught. One does not become proficient at language without an understanding of its grammar and syntax in my view, and a solid vocabulary, and vocabulary is learned, or acquired by repeated hearing and use in the social environments of mathematics classrooms.

I deem the thinking of Graves (1972), Novak (1978), Von Glasersfeld (1988), Ernest (1994a, 1994b), Olivier (1989), Cobb et al (1992) and Ball et al (2004), as referenced in the paragraphs above as supportive, mostly by suggestion and implication, of this call for some form of instruction that is deliberate about mathematics content and its real mathematics context and I add that it should be underpinned by reflection and meta-analysis. What is envisioned by constructivist thinking seems to be a requirement of learners exclusively. Teachers need to engage with their metacognitive understanding of mathematical content matter through reflection in a mode of radical constructivist thinking as a precursor to engaging with the act of teaching it.

Lesh and Clarke (2000), in contrast to this, state that complex systems are thought of as being nothing more than the sums of their parts; each part is expected to be taught and tested one-at-a-time, in isolation, and out if context; and, the parts are assumed to be defined operationally using naïve checklists of condition action rules. The isolationism and separatenesses inherent in Lesh and Clarke's description of 'complex systems' seems, for me, to be the epitome of the status quo in mathematics classrooms.

The review of the literature reflects the dominance of the contemporary constructivist leanings in comparison with a paucity of quasi-behaviourist support. I have written an account which explicitly advocates an open and plural approach in critical realist terms which itself invites the development of the theory of teaching. This would be in line with Freire's view that I quote at the outset of this chapter. 'The critically transitive consciousness is characterised by ... receptivity to the new for reasons beyond mere novelty and by the good sense not to reject the old just because it is old – by accepting what is valid in both old and new.' (Freire, 1974: 14).

The preceding discussion embraces the historicity of learning theory from behaviourism through radical and social constructivism. It serves to show that learning theory has historically had learners as its central focus. It is argued that teachers need to reverse the looking glass and examine what they learn in both radical and social constructivist terms; radically about their own conceptualisation of mathematical content and socially about what their learners teach them about how to comprehend mathematics.

I move on to the epistemological frameworks espoused by a variety of mathematics education researchers as reflected in the literature where teaching as an integral component of the educative process is more pronounced if not subject to deep analysis.

3.7 Epistemological mathematics frameworks

Epistemological frameworks are seen as ways of viewing mathematics education research episodes, and providing structure from which to understand how learners make meaning of mathematics. In this regard the influence of the notions in the work of Balacheff, Thompson, Skemp, Freudenthal, Kaput, Sfard, Vinner and Dreyfus, come to bear on this research.

Balacheff (1990), in his notion of 'problematique' recognises problem situations as the source of mathematics but incorporates the *role of the teacher* in his framework. Similarly, Thompson (1994a) describes the development of concepts in terms of processes and objects, but his epistemological framework proposes the *development of instruction* which nurtures and extends learners' images in mathematics. Skemp (1993), in his notion of a 'theory based' approach advocates *the necessity to instruct* in cases where learners' heuristic endeavours are counter-productive to understanding new concepts. Freudenthal, in Keitel (1993), epistemologically also sees learning as *'guided reinvention'.* Freudenthal in (Keitel, 1993:162) claims that in learning mathematics, the interface between individual and collective experiences must be enforced by intensive communication and experimental work in heterogeneous groups.

The single feature that distinguishes these epistemological frameworks from those of other authors is the incorporation of the role of teaching. They do not therefore assume that teaching is inferred in the process of learning and suggest that instructional techniques are important considerations, per se, in facilitating concept acquisition.

In terms of this research notion of representation of mathematical ideas is a core consideration. The following researchers incorporate similar notions in their work. Kaput (1985) suggests that learners' understanding of mathematical concepts is reflected in their ability to represent the concept. Learning is deemed to occur when learners' relate a mathematical representation to the interpretation of the same, especially in the context of shared peer meaning. A representation may be in the form of a mental, computer, explanatory, mathematical or symbolic image. It is difficult to infer whether graphs of algebraic relations would fall comfortably into one of these categories. I am of the opinion that its own importance positions graphs as a distinctive category alongside these.

Sfard (1991) sees mental constructions occurring structurally (by object) or operationally (by process) through stages referred to as "interiorization", "condensation" and "reification". Sfard's reference to 'object' is equivalent to graphical representations in this thesis.

Vinner (1992) and Dreyfus (1990) differentiate between 'concept image' and 'concept definition'. The former relates to the object and process of mathematisation, while the latter may describe the schema associated with the concept. Dreyfus (1990) believes that learners' descriptions of a 'concept image' enable teachers to reorganise the 'concept image' into a more coherent structure which is consonant with the 'concept definition'. Concept images are taken as equivalent to the graphical contexts of algebraic structures.

Regarding the interpretation that learners give to teachers' written and spoken questions, Pimm (1987: 8) speaks about what elicits exact responses in interaction with learners? It is learners' responses that embody meanings they attach to mathematic concepts. Pimm's perspective enters the cognitive domain that separates the stimulus and response. Brown (1997: 28) also claims that "Mathematics can only be shared in discourse and the act of realising mathematics in discourse brings to it much beyond the bare symbols." Brown (1997) introduces the hermeneutic in analysing learners' understanding mathematics. Both Pimm and Brown enter the realm of metamathematics in probing cognitive content through descriptions offered by mathematics by learners.

Vinner, Dreyfus, Pimm and Brown expound epistemological frameworks which focus on the cognitive functioning of learners in their acquisition of mathematical concepts.

3.8 Summary

There is reason to believe that personal cognition can be a private endeavour elicited by any number of sensory stimuli, of which deliberately meaningful instructional input may be a vital catalyst in inducing cognitive germination. In seeking to unravel the dilemma of the nature of constructivist teaching (or learning) spoken of by Solomon (1994), there may be validity in viewing radical or constructivist *teaching and learning* environments as supportive of learners in their attempts to consolidate, and hence, confirm cognitive embellishments which they have privately and personally acquired. Lerman (1992) cautions against the replacement of one dogma with another and it is this cautioning which may hint at ushering back the relevance of behaviourist elements or quasi-behaviouristic instruction as a means of knowledge acquisition, albeit in a new garb of metateaching and metalearning.

Towards this end the conception, design and construction of a framework for this study predominantly draws on theoretical notions in the work of Piaget, Vygotsky and Brüner, and on the epistemological frameworks of Vinner, Dreyfus, Pimm and Brown.

Piaget regards cognitive epigenesis as the equilibrium between the mental processes of accommodation and assimilation by way of abstractive reflection. I place emphasis on this theory of learning in that the cohort of learners who are the focus of this case study are required to record written accounts of their reflective thought processes in completing mathematical tasks; the purpose of which is to discern whether the linguistic engagement with the concept at hand facilitates cognitive epigenesis.

Vygotsky (1986:107) states that "real concepts are impossible without words, and thinking in concepts does not exist beyond verbal thinking. That is why the central moment in concept formation, and its generative cause, is a specific use of words as functional tools." Schutz (2002) also records Vygotsky, as saying that thought is not merely expressed in words but that it comes into existence through them. This stance concurs with Taylor *et al., (*1993a) who make the deduction that a learner's new understanding can be formed on the basis of his/her own prior knowledge and experience; which in this case study would constitute existing schemas and linguistic templates for communicating such understanding. I support this deduction with the provision that the meaningfulness of the prior knowledge and experience contributed to the establishment of (or extension of) a cognitive template that facilitates the sublimation of ensuing knowledge constructs. The epigenesis of a learner's individual knowledge can then be the result of the individual's "purposeful and subjective interpretation of his/her experience of the physical and social world." (Taylor et al., 1993a: 4) The equivalent of the 'physical and social world' in this study is the classroom, its instructional time and the subjective interpretations of a variety of mathematical realities. This complies with Gravemeijer (1998) who calls for subject matter that is to be mathematised to be experientially real for students but concedes that it may be mathematics itself that is experientially real.

These assertions form fundamental premises of this study. It prompted me to require of learners to generate worded accounts of their mental processes involved in completing mathematical tasks; a methodology which finds further favour with the sentiments of Doll (1993:156) and Onore (1992) who claim that if curriculum is viewed as a process, then knowledge is achieved through the constant reciprocity of dialogue and reflection. In this study therefore, 'curriculum' would be synonymous with lesson construction via meta-analytical assessments of learners' views of their comprehension of concepts.

The manner in which Goos, Galbraith and Renshaw in Burton (1999: 37) describe mathematics teaching is what this research aims to overcome: 'In the majority of contemporary classrooms, learning mathematics is seen as mastering a predetermined body of knowledge and procedures. The teacher's job involves presenting the subject matter in small easily manageable pieces and demonstrating the correct procedure or algorithm, after which students work individually on practice exercises. However reasonable this approach may appear, numerous research studies (e.g. Schoenfeld, 1988) have shown that such mathematics instruction can leave students with imperfect understanding and flawed beliefs about mathematics. When students' activity is limited to imitating the technique prescribed by the teacher, they can create the appearance of mathematical competence by simply memorizing and reproducing the correct way to manipulate symbols, and may even come to believe that producing the correct form is more important than making sense of what they are doing (Cobb, 1986; Cobb and Bauersfeld, 1995)... As a result of school experiences such as these, students equate mathematics with meaningless practice on routine exercises, and learn that mathematics is not meant to make sense'. This finding corroborates the call for teachers to engage with mathematics content at a level far deeper than surface structure.

In similar vane, Von Glasersfeld (1988) makes the observation that in traditional theories of knowledge the activity of 'knowing' is taken as a matter of course, 'an activity that requires no justification and functions as an initial constituent.' (Von Glasersfeld, 1988: 208) It is this observation that prompts the question as to the manner in which this 'knowing' is, or becomes, part of the cognitive make-up of individuals. It invites debate and paves the way for entry into the domain of metacognition, with metateaching and metalearning as its prerequisite constituents.

At a level of philosophical theory Habermas (1990) provides through his conception of moral consciousness through communicative action an overarching perspective within which the discourses of this work transpires.

The concepts of critical consciousness and a pedagogy for the oppressed provides an understanding of education as opposed to teaching as a process of developing critiquing individuals as well as stimulating the interdependence of theory and practice through his notion of praxis.

Appendices 1, 2 and 3 constructed as summaries, contain, in no particular order, the tenets of teaching and learning theories and epistemological frameworks as referred to in this chapter.

Chapter 4 looks at Habermasian communicative action and Freirian critical consciousness and its impact on the processes that define this study.

IV

Critical Consciousness and Communicative Action construed as hermeneutic elements of Critical Realism

'Authentic education investigates thinking.'

Freire (1970: 90)

'I can't hear her because I can't see what she is saying.'

(Me to a colleague listening to a conference presentation; Windhoek, January 2005)

4.1 Introduction

By way of introduction my retort to a fellow conference goer, bears some explanation. It was uttered spontaneously and only after it had been spoken did its value begin to impact my thinking. Whether this was a residual conscious thought acquired through something I had read or whether it was a transcendental subvocal concern about how learners perceive classroom communication, I cannot say. I do not know either whether it was a vocal translation of the cognitive planning of the methodology of this research. It did generate, however, a further thought on what it was I was actually saying. Because an easel was obscuring the screen, I could not connect the meaning of what I was finding hard to hear with what was supposed to have illustrated the uttered explanation. That thought brought home the idea that when we speak in the mode of Ausubel's 'obliterative subsumption' (Ausubel, 1961) in conveying vital information in classrooms it may be the space separating us from our learners that obscures the meaning of aural algebra! It is the content of this space between teacher and learner that this work endeavours to proximate.

Foucault's (1984) identification with an experimental attitude while confronting the essence of self as one teaches epitomises the hermeneutic analysis that interrogates my own conscience about what makes the learning of mathematics accessible, particularly in the South African context of poverty and resourcelessness.

It therefore concurs with Freire's (1970) 'thinking' that incorporates conscientisation that constitutes authentic education.

The purpose of this chapter is firstly to situate this research in critical realist theory as a research paradigm and secondly to proceed to analyse the implications of Habermasian moral consciousness and communicative action, and Freirian conscientisation, (idem.) for investigating metacognitive aspects of teaching and learning mathematics at senior secondary school level. Critical realist theory embraces hermeneutic analyses, which emerge in both the communicative action and conscientisation as espoused by Habermas and Freire respectively.

4.2 Critical Realist Theory as a metatheoretical context for this research

Both radical and social constructivist epistemological frameworks serve as contexts for this investigation. A relational ontology carries the pedagogical practices. Elements of empirical and hermeneutic phenomenological methods characterise the data analysis; data collected over two years of working with thirty Grade 11 learners who progress to Grade 12 in the second year of the case study. Data constituted by learners' written transfactual artefacts generate qualitative data. As participant-researcher I interpret these data, and situated associated teaching and learning processes, through ongoing reflective practice, and bring modifications realised by these metacognitive practices to bear on the pedagogical processes systemically. The codified data, constituted of the mathematical descriptions and explanations that emerge from the transfactual artefacts, are quantitatively analysed to search for any correlative or explanatory trends between codes. It would seem appropriate at this point to elaborate on the rationale for working in this metatheoretical paradigm.

As a recent development over the past thirty years spearheaded by the work and writing of Bhaskar, this stance on research paradigms proposes to accommodate theoretically opposed traditional philosophies of how knowledge is constituted (Bhaskar, 1989, Archer et al., 1998, Danermark et al., 2002). The tenets of critical realism are briefly identified here accompanied by an attempt to relate it to, and situate this work in relation to it.

1. Critical realist theory combines 'epistemic relativism', the notion that all knowledge is socially produced, or transitive and fallible, with 'judgemental rationalism' that acknowledges that there are rational criteria for preferring one judgement or theory to another. (Archer et al., 1989: xi)

On this matter, I understand that the hermeneutic method that features as an interpretive method for this work, acknowledges that any interpretation, although transcendental in its immediate application, is historical in its development. With changing perceptions the transcendental moments are peculiar to them but contained within historically acquired consciousness and personal epistemologies. The cohort of participants, in these terms, are learners from previously disadvantaged backgrounds, many of them black learners from township schools whose poverty and isiXhosa mother-tongue carry with it socio-political histories that impact on the learning of mathematics. I take into account these personal histories as learners commit to paper their moments of understanding, description or explanation. These moments are in themselves catalytic in transforming knowledge as they radically construct more sophisticated modes of assimilating concepts that in turn contribute to theoretical gestalt constitutions. Socially produced knowledge in the context of this research is therefore a product of a multilingual, multicultural social unit that comprises heterogeneous experiences of a group of thirty, seventeen-year-old learners. My interpretations of transactions in this context are at most momentarily true, a notion that by implication must also include fallibility. But it is this fallibility that constitutes a perturbation that itself constitutes recollection and correction, and hence further transformative practice; a generative mechanism for transformative practice and knowledge growth.

2. Critical realism is plural, open and developing, and dialectical. (Archer et al., 1989: xix)

This tenet is understood as included in the ideas elaborated under point 1 above, but further, critical realism is plural in that it incorporatively embraces empiricist causality and interpretive explanation since both may contribute to explaining the same reality or phenomena in different ways, but equally valid ways. In this research this duality emerges, at one level or stratum (Danermark et al., 2002: 11), in the data analysis where codes are quantified and then subjected to quantitative analyses, while at another level or stratum, qualitative interpretive methods seek understanding and explanations for the tacit understanding in the learners' descriptive and explanatory accounts of mathematical tasks. It is in the second stratum that another duality arises. In the cognitive processing of the meaning I make of these accounts, my own experience of communicative transactions may be described as hermeneutic. My analyses of phenomenological moments constituted by learners' textual mathematical accounts are of an empirical phenomenological nature. It is the meanings associated with these dependent or independent strata that contribute to new enlightenments, referred to in critical realist terms as 'emergent powers' (idem: 11). The transitivity of knowledge accruing from these phenomenological strategies locates the study in an open system, as demonstrated through the intertwining of the quantitative code analysis and the qualitative reflections on the affective responses by learners in chapter 8.

The character of strata is described by Collier (1994: 121) in the following way:
'The 'lower' the strata in the hierarchy of rootedness and emergence, the closer we get to a closed system. For it is possible to isolate, for instance, a chemical process from the interruptions of organic processes, but it is not possible to isolate an organic process from the effects of chemical processes, since it is rooted in them. ... It is often possible to isolate a system from processes generated by 'higher' strata, but never possible to isolate one from those generated by 'lower' strata. Hence the further up the hierarchy we go the more distant our approximations to closure become.'

3. Critical realism explicitly espouses methodological pluralism. (Danermark et al., 2002: 151-152)

Methods are not ends in themselves; they are means of collecting data that strive to seek explanations, even if these explanations do not culminate in generalisations. Where similar work corroborates similar outcomes under different conditions, only then will such corroborations tend to accomplish generalisable theories, that is, if such generalities are contextually meaningful or desirable. According to Bhaskar (1978: 227) 'Scientifically significant generality does not lie on the face of the world, but in the hidden essence of things.' In a single case study the findings may be sufficient in themselves as a situational analysis. To investigate my participant teacher-researcher role I have videotaped sessions that serve as evidence for claims to ontological relationality, engaging in metacognitive practices and for demonstrating what is meant by metacognitive deconstruction, and for evidencing dialogical communicative action as a method of interrogating critical consciousness for those whom I teach and me. I am of the opinion that metacognitive deconstruction is more than a synonym for 'good' teaching.

4. Critical realist theory views 'explanatory power' as that power that serves as an appropriate criterion according to which explanations can be compared (Bhaskar, 1989: 83).

I interpret explanatory power, for the purpose of this research, as being that theoretical framework that most credibly explains a phenomenon or facet of a phenomenon. In this work, I draw on the work of Piagetian radical constructivist learning theory as well as the social constructivist learning, as for example, espoused by Vygotsky and a host of contemporary mathematics education researchers who posit a variety of epistemological frameworks that assist the understanding of how learners make sense of mathematics. This is so because I find no one explanation as singularly satisfactory or exclusive, or generalisable over protracted mathematical realities. A detailed exposition of these theories of learning and epistemological frameworks appears in the literature review. This notion of explanatory power is pertinent to this work in that learning is not seen wholly and exclusively as a product of social constructivism. While social constructivist milieus are seen as collaborative, affirming environments for the sharing of knowledge, I regard knowledge, from a learner perspective, as a necessity, to be radically constructed in a personal sense of knowing before social constructivist engagements are not only possible but also meaningful. From a teacher perspective, on the other hand, learners are participants in a social constructivist communicative action as hearers of information even if this social relationship is one-to-one or one-to-many. The former describes each learner's personal engagement with knowledge inputs, while the latter refers to the ethos that characterises the dynamic reflected in the group's engagement with knowledge inputs. The investigative nature of the case study also incorporates, in terms of my own investigative professional practice, an episode of action research. In every sense therefore there are valid claims to knowledge where such knowledge is insider, personal kinds of knowing that may reflect the personal moments of knowledge that could shed light on what makes the pedagogy of mathematics more meaningful.

5. Critical realist theory is closely aligned with the notion of 'emancipatory science'. Emancipatory science claims to make genuine discoveries that can therefore help to promote human flourishing. (Archer et al., 1989: 392)

As ambitious as this may purport to be, the inequalities in the South African social socio-economic context as it is plagued by poverty and unequal accessibility to (mathematics) education, cries for catharsis; to be liberated from the oppressiveness of poverty and a cruel separateness of inequalities; inequalities that manifest in non-mother tongue pedagogies, resources, an impoverished mathematical register in African languages and the paucity of teaching expertise.

A core notion that arises in elaborating these tenets is that method and theory cannot be treated as separate entities in social science.

The discerning cognitive qualities associated with Descartes' 'Cogito ergo sum' equivalent to 'I think therefore I am' Descartes (1999), make them suitable foci of study, especially in the sense that personal experiential knowledge is held in human consciousness, alongside a socially determined knowledge about reality, both of which are independent of an external real world, that may have been contingent on that real world, or 'personal space' as referred to by Brown (1999). The external real world contains its own mechanisms that explain causality (Danermark et al., 2002) in the same way, I believe, that human consciousness has mechanisms for understanding cause and effect. An important methodological distinction is made by Bhaskar (1989: 93) between the social sciences and the social psychological sciences. Bhaskar claims that if the object of the former is social structure, then that of the latter is social interaction, which reflects the relations between the positions and practices agents reproduce or transform. It is the latter that finds expression in this research in terms of the investigative self-practice that generates mechanisms for understanding, and for facilitating the understanding of classroom mathematics for transformation.

4.3 Freirian 'critical consciousness' and Habermasian 'communicative action' as integrated within a critical realist theoretical perspective

Closely aligned with the tenets of critical realist theory are the notions of communicative action as expounded by Jürgen Habermas' (1990) and Paulo Freire's critical consciousness (1970,1974,1987) respectively, all of which embrace critical engagement through reflective practice.

At this point I seek to situate the conceptualisation of my pedagogic practices as it transpires in the methodology through firstly viewing it in terms of Habermasian communicative action and then against the backdrop of Freirian critical consciousness.

4.4 Habermasian Communicative Action: A strategy for mathematics classrooms

Habermas' seminal work *Moral Consciousness and Communicative Action (1990)* contributes significantly to the understandings reached in this work. Generically it applies to discourse ethics, communication, argumentation and dialogue that promotes voluntary rational agreement for the sake of cooperation. As much as it is generically all of this, it also in my opinion, has far reaching implications for its application in the teaching and learning in mathematics classrooms. Mathematics by virtue of its empiricist rootedness is characterised by impersonal propositional deliveries that often separate it from affective influences that could make it more accessible to those who aspire to learn and teach it. Engagement with communicative action has emancipatory potential for the teaching and learning of mathematics. In Habermas' words, communicative action has this power since 'as empirical research shows him, communicative action from the start of the learning process is in the inescapably social nature of human language.' (Habermas, 1990: 160-165).

I quote McCarthy in the introduction to Habermas (1990: vii) who states that 'Habermas' discourse model, by requiring that perspective-taking be general and reciprocal, builds the moment of empathy *into* the procedure of coming to a reasoned agreement: each must put him or herself into the place of everyone else in discussing whether a proposed norm is fair to all. And this must be done publicly; arguments played out in the individual consciousness or in the theoretician's mind are no substitute for real discourse.'

As teachers and learners we are in the business of communication. It is therefore paramount that the elements of reciprocity, empathy and fairness that constitute this perspective taking be embraced in classroom environments especially where mathematics is concerned. There is much scope here I believe for the application of these solid principles of communication, as alluded to in the quote above, to be practiced in the mathematics classroom. This is done with the knowledge that ethical discourses do not denunciate or disregard mathematical offerings from learners that are not exact in its technical rationality since this would seek to entrench an epistemological hegemony that is oppressive. I offer here an anecdotal account of how this might arise in classroom discourses that reflect these sentiments for the purpose of contextualising this rationale, and for preparing the reader for the analysis that follows later.

In discussing the concept of 'gradient' a learner attempted to describe, in his second language, the difference between two inclines. His attempt used the notion of 'short time' to walk up slope 1 and 'long time' to walk up slope 2, as illustrated in Fig 4.1. In responding I acknowledged that he was correct in his assessment but continued the discussion by suggesting he compare how tired he would be after each ascent. Accepting as valid all other variables that may impact on his physical condition at the end of each walk, the tiredness factor prompted the inclusion of steepness, probably what I would have liked as an initial answer. However the route that his answer steered the discussion in was a direction that allowed deeper meaning for the ultimate understanding of what a gradient is. As a group we soon got onto the notion that the height was constant while the horizontal distance varied, and that this accounted for the differences of the slopes of the lines. The communication culminated in a formal mathematical understanding that gradients depended on the rate of change in the vertical compared to the rate of range in the horizontal ultimately expressed as a ratio. Had the learner's initial suggestion been dismissed as unsuitable the depth of the ensuing discussion would have been lost. It is often these extended dialogues that promote the deconstruction of concepts so that their underlying compositional subconstructs emerge and which in turn facilitate deeper understanding and sustained learning through associated elements of the dialogue. The interaction embraces reciprocity, empathy and fairness and therefore espouses ethical considerations. This discussion prompted help from the learners' classmates and so the interaction was not only one-to-one. It changed from moment to moment, from one-to-many to many-to-many. The pedagogic processes were characterised by dialectical constructivist epistemologies that are dependent on language competences, the importance of which was discussed earlier. The consideration of language, especially in the South African context where the majority of learners receive instruction in their second or third language is an important one. It is important both from the view of teachers and learners. Teachers need to have conceptual depth relating to mathematical structure and process, and the competences to communicate these with a fair degree of exactitude. Learners need to be able to trust that their understanding of the mathematical dialogue contributes to what in Piagetian terms would be regarded as cognitive epigenesis amounting to the attainment of equilibrium between accommodation and assimilation through reflective abstraction. These acquisitions are demonstrated in the ways we express ourselves in both the verbal and written forms.

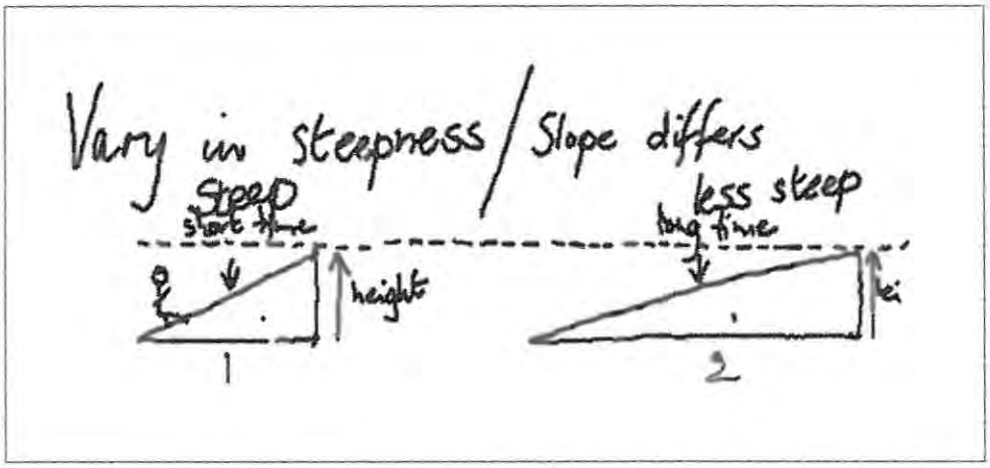


Fig. 4.1

An electronic copy of the lesson development of the concept of 'gradient' as generated by learners and teacher engaged in communicative action

Habermas argues that the model of natural stages is plausible up to the point of what he calls the postconventional break; the point at which the social world loses its quasi-natural validity (Habermas, 1990: ix). From that point on he claims that we are dealing with stages of reflection that have to be assessed and ordered primarily on the basis of moral-philosophical, rather than empirical-psychological, considerations. I have considered that the anecdotal account of unpacking the concept of gradient as described above as being a case of stepping out of a quasi-natural validity into the natural validity of a social identity that is held in our personal consciousness; validity based on our innate knowing, and our personal claims to knowledge. Our personal domains of knowing, and therefore our assigning meaning, cannot be detached from our experience that is made manifest in our dialogue. As novices these dialogues are imperfect and imprecise but they are nonetheless valid.

Concepts are honed through continuing critical and ethical discourses. I cannot express it more succinctly than Habermas (1990: 2) who states that 'Transcendental analysis is a non-empirical reconstruction of the a priori achievements for which there is no alternative: No experience shall be thought possible under *different* conditions.'

Remaining with the episode of communicative action involving the striving toward a deep understanding of gradient, I seek now to analyse the communicative action in terms of the hermeneutic dimension. (Habermas, 1990: 21) reminisces about his call for hermeneutic research not to be abandoned and says that '...any attempt to suppress the problem of interpretation would entail serious distortion.' In speaking about reconstruction and interpretation in the social sciences, Habermas is in agreement, in these terms, with Gadamer (1976) that the task of philosophical hermeneutics is to shed light on ordinary processes of understanding, not on systematic investigations or procedures for collecting and analysing data. Gadamer further conceives method as opposing truth. Truth, according to Gadamer, cited in Habermas (1990) is attained only through the skilled and prudent practice of understanding and interpretation. Method, on the other hand, when associated with empirical causality and generalisation is not inclusive of the cyclical order of understanding, meaning and explanation. I concur with these sentiments. In terms of this work, however, method is construed as an organisational structural notion for the purpose of ordering events within which meaning is sought. This is an important consideration since where quantitative data does not confirm causality interpreted meaning can stand alone as its own truth. Gadamer takes the stance that hermeneutics, 'as an activity is at its best an art but never a method; as far as science is concerned, hermeneutics is a subversive force that undermines all systematic approaches' (Gadamer, 1976). Habermas makes the point that after psychological social science's claim that learning theory would provide a universal explanatory model came to naught and that after it had been hailed as a textbook example of an exact behavioral science, 'These failures opened the door for alternative approaches based on phenomenology...' (Habermas, 1990: 21). The episode of communicative action around the gradient concept emphasises the value of hermeneutic engagement, both of learner and teacher-researcher. From the learner's standpoint there is a move to search for his own understanding by digging deep for ways of expressing himself so that he is understood. This mental exploration serves to stimulate the engagement with one's critical consciousness. From my perspective as a teacher-researcher there is a critically conscious attempt to examine the validity of the learner's response given its initial mathematical inexactness. The reciprocity, empathy and fairness of the communicative action breeds a trust that seeks to find alternative ways of conveying the essential meaning of 'gradient'. The method one resorts to involves the metacognitive deconstruction of abstractions, aided by graphical depictions, to find the derivative subconstructs that ultimately constitute the gestalt of concept definitions.

In his analysis of reconstruction and interpretation in the social sciences, Habermas views language in two modes of use. 'In one mode of language use, *one says what is or is not the case*. In the other, *one says something to someone else in a way that allows him to understand what is being said*.' (Habermas, 1990: 24) (Italics in original). According to Habermas the second mode is tied up with the conditions of communication. I quote Habermas, 'In contrast, when language is used for coming to terms or reaching an understanding with someone else (even if it is ultimately only to agree or disagree), three relations are involved: by giving an expression *of* his belief, the speaker communicates *with* another member of the same speech community *about* something in the world. Epistemology is concerned exclusively with this last relationship, the relationship between language and reality, while hermeneutics deals simultaneously with the three-fold relationship involved in an utterance, which serves as (a) an expression of the speaker's intention, (b) an expression of the establishment of an interpersonal relationship between speaker and hearer, and (c) an expression about something in the world. In addition, any attempt to clarify the meaning of a linguistic utterance reveals a fourth, linguistic relation, namely that between a given utterance and the set of all the utterances possible within the same language.

Hermeneutics watches language at work, so to speak, language as it is used by participants to reach a common *understanding* or shared *view*.' (Idem: 24).

Against this exposition, the 'gradient' teaching-learning episode qualifies as hermeneutic in that both teacher and learner depend on one another for reaching an understanding of the concept of gradient. There is intention to express the desire that I want to facilitate understanding while the learner, by virtue of his attempt at verbalising his thinking, indicates his intention to learn. The interaction between teacher and learner is an embodiment of trust that serves to establish a relationship between us, upon which later communication is founded. It goes without saying that the content involves expressing mathematical meaning around the gradient concept as it occurs in the world and how it is translated into mathematical form. The fourth point in the quote above, of clarifying the meaning of a linguistic utterance, highlights the notion that there is no singular expression regarding the understanding of the gradient concept that is exclusive of others. In another situation, at another time, the conversational dynamic and the dialogue content may have been different owing to more recent personal experiences.

The teaching-learning episode integral to this discussion is assessed now against what Habermas refers to as the interpretation of objective understanding (Habermas, 1990: 26). Firstly, in the hermeneutic, 'interpreters relinquish the superiority that observers have by virtue of their privileged position, in that they themselves are drawn, at least potentially, into negotiations about the meaning and validity of utterances. By taking part in the communicative action, they accept in principle the same status as those whose utterances they are trying to understand. No longer immune to the affirmative or negative positions taken by experimental subjects or laypersons, interpreters give themselves over to a process of reciprocal critique. Within a process of reaching understanding, actual or potential, it is impossible to decide a priori who is to learn from whom.' (Idem: 26).

Secondly, the interpreter also relinquishes a position of authority over his object domain. Consequently he needs to analyse the meaning given to him in terms of its context dependency. The interpreter cannot assume that those with who he is in dialogue are working from the same assumptions and practices. In this regard, Habermas says that, 'The interpreter's own global understanding of the hermeneutic situations can be examined piece by piece but cannot be put into question as a whole.' (Idem: 26).

A fundamental third point is that our everyday language extends to what Habermas refers to as 'nondescriptive utterances and noncognitive claims to validity.' (Idem: 26). In our daily discourses we agree or disagree on the validity of norms, evaluations and the authenticity of our own presentations of our understanding. It is seldom that every-day discourses concern propositional truths. Habermas concludes therefore that 'the knowledge we use when we say something to someone extends beyond strictly propositional truth-related knowledge.'

Essential in the analysis of gradient as described above is therefore that understanding what is said requires participation and not merely observation. As a participant the core analysis of the interpretation of an understanding involves both cognitive and noncognitive claims to validity.

If knowledge is a product of a learning process, that the learning process involves heuristic through participation, and that the learning is guided by a teacher or fellow learner, it is important that the metacognitive aspects of teaching and learning are understood. Habermas refers to this as understanding the learning process from the inside (idem). With this comes the option for a learner to vacillate between consecutive interpretations of the same problem until the concept image aligns with the concept definition through iterative metacognitive personal arguments and dialogue. The emergent descriptions of the development of the concept of gradient through communicative action, serve to establish the validity of this claim.

Habermas (1990: 187) raises the problem of competence related to performance. Theoretical approaches are not easily measured because of the fact that theoretical claims of competence are made manifest in performance. To this end a coding instrument was designed for the purpose of determining the frequency and mathematical aptness of the thoughts and mental dialogues of learners. The rationale and development of the instrument is described at a later point.

4.5 Freirian Critical Consciousness: a route toward metacognitive engagement

Freire's work amongst the Brazilian people was in every sense humane. His writing epitomises his humanity and brandishes emancipatory discourse for both the world of politics and education. It addresses the need of liberating an oppressed proletariat whose emergence from an acritical mutism should offer critique to propositional educational and political hegemonies; their developing ability to critique being that stimulant for establishing their emerging self-actualised identities.

His writing is as pertinent now to the political and educational landscapes in South Africa. The same oppressed mutism characterises the voiceless many whose critique is not forthcoming owing to dominant epistemologies steeped in technical rational delivery. The mute mathematical mass is constituted of teachers and learners alike who are still shackled by the effects of separate development under the apartheid regime. The deficits that accrued under that dispensation still render teachers and learners ineffective in articulating their grievance against continuing disparities in provisioning. The inertia breeds continuing stagnancy. For teachers the cumbersome teacher: learner ratio coupled with rudimentary resources in rural areas (and albeit improving technological resources in urban and peri-urban areas), do not mitigate dire circumstances and serve to entrench that mutism. For learners the opportunities of being introduced to critical reason through mathematics as method and medium are scarce. It is the current school going learners of mathematics that would do well to have their critical faculties challenged if we hope for a mathematical revival in the next generation. What is needed is a relational understanding of mathematics that transcends rote and algorithmic mathematical procedures by teachers and learners alike.

Freire's involvement with adults on a literacy programme induced hope amongst a people who had no voice. It is the breaking into one's critical consciousness that helps one establish one's worth and the value that that personal awareness has for critique. From the perspective of a mathematics classroom I have experienced that learners are also mute insofar as they are not able to offer critique on mathematical issues at hand. In terms of the Vygotskian element of concepts being built on words this foregrounds the need to be conversant with matters mathematical for laying foundations that sustain relational understanding. From a Piagetian learning perspective, that cognitive epigenesis through accommodation and assimilation through reflective abstraction, requires a schema to promote conceptual development, is theoretically sound, but not possible without a dimension of language articulation that refines the concepts at hand. Becoming critically conscious may be facilitated by those who have come to challenge their own critical consciousness, in this case teachers of mathematics, and then communicated in mathematical terms. There are therefore parallels between the political and educational dimensions in Freire's work and the political and educational need of critical consciousness to instigate a metamorphosis around mathematical thinking and doing in the South African mathematics education context.

Freire's equivalent of Habermas' moral consciousness and communicative action is 'conscientisation'. Goulet, in the introduction to Freire (1974: vii) recognises that Freire's conscientisation is 'the symbiosis between reflective action and critical theorising'. This recognises further the concept of 'praxis', which is that reflective action that occurs after critically evaluating, in terms of this work, the mathematical task at hand for 'deconstruction routes' to arrive at its complexity, and for the value that this task has for any mathematics to come. Earlier mathematical teaching and learning must lay the foundation upon which later ideas can be built. This process of conscientising is fundamental to teacher and learner. Learners need to be guided into that place where they can begin to interrogate their own thought processes where for them conscientisation is to know and to understand what they are doing mathematically. At a metacognitive level it is for them to know what it is that makes them know that they know what they are doing. In other words do they know what makes them know mathematics better? To illustrate this I described previously how I have experienced learners fixate on a simple increasing linear function, $y = x + 1$, on the Cartesian plane rendered mute when asked to describe the behaviour of the function before them. I accounted for this mutism as a function of the absence of knowledge that graphs require being read, from left to right, as x increases. This, together with the need to coordinate the behaviour of increasing y values, presents a task that for many learners appears not to be able to taken as assumed known. It seems then that if mathematical graphical representations are regarded as static rather than dynamic phenomena, mathematics curricula that can be facilitated and enriched by graphical literacy will remain inaccessible for learners.

Evidence that learning is meaningful requires that learning be assessed, in whatever form. One of the products of meaningful learning is therefore the ability to articulate that meaning through understandable dialogue, written or spoken, this in addition to being able to demonstrate algorithmic accuracy. I sense that it is this lack of engaging cognitively with the meanings of mathematical processes that distances mathematics from many learners. It is this claim that underpins the purpose for having learners describe and explain their engagement with mathematical tasks. The aim of the coding instrument was to find ways of describing quantitatively learners' articulateness about mathematics. Graphical depictions of algebraic concepts served to provide the visual constructs of the derivative notions, exemplars of which I analysed later. The rationale for this was to assist learners to construct descriptive and explanatory accounts, which was intended to assist learners to articulate their understanding of mathematics. This strategy was seen also as contributing to inducing a level of critical consciousness. Freire's praxis incorporates 'the participant observation of teachers "tuning in" to the vocabular universe of the people' (Freire, 1974: viii). In terms of mathematics teaching we cannot tune-in to learners vocabularies if they do not exist. If they do not exist there is little to substantiate a claim that mathematical concepts are developed, and if concepts are not acquired dialogic teaching is not possible. There cannot therefore be any form of communicative action on which to build agreements or disagreements about mathematical phenomena. So in these terms, dialectical engagement promotes the growth of praxis and generates theory. It can therefore be understood that without dialectical dialogue there is no fodder for praxis, and where there is no fodder for praxis, the substance for generating theory about cognition in learning and teaching remains remote in mathematics classrooms.

The following quote that clearly expresses Freire's views on his envisioned democracy that would transform the Brazilian political landscape is applicable to the democracy of classrooms, in particular mathematics classrooms because the principles of engagement are seen as libertarian. In essence it is dialogical and inclusive.

'The critically transitive consciousness is characterised by depth in the interpretation of problems; by the substitution of causal principles for magical explanations; by the testing of one's "findings" and by openness to revision; by the attempt to avoid distortion when perceiving problems and to avoid preconceived notions when analyzing them; by refusing to transfer responsibility; by rejecting passive positions; by soundness of argumentation; by the practice of dialogue rather than polemics; by receptivity to the new for reasons beyond mere novelty and by the good sense not to reject the old just because it is old – by accepting what is valid in both old and new. Critical transitivity is characteristic of authentically democratic regimes and corresponds to highly permeable, interrogative, restless and dialogical forms of life...' (Freire, 1974: 14).

This serves to contrast the 'power' versus 'powerless' dichotomy. The inability of learners to communicate on matters mathematical, positions teachers in more powerful roles than learners. At whatever level of mathematics learners find themselves, there is a need for them to communicate mathematically for all the reasons already discussed above. It is ironic that the South African teaching context is itself powerless because of historical-political factors and because of language impediments. Both therefore need to strive for emancipatory democracies that promote communicative action. Freire regards this 'power versus powerless' dichotomy as leading to submission. In the mathematics classroom learner submission is characterised by their mathematical muteness, unable to enter into meaningful discourses. For teachers the muteness is manifest in non-contributory attitudes to mathematical policy development and acritical attitudes to propositional imposed change. The dehumanised consciousness of people in Freirian terms is massification. This massification, according to Freire, is associated with a pathological naïveté, better expounded by the excerpt: 'Critical consciousness is integrated with reality; naïve consciousness superimposes itself on reality; and fanatical consciousness, whose pathological naïveté leads to the irrational, adapts to reality.' (Freire, 1974: 39) Teachers and learners need to be confronted with this notion so that genuine theory can be sought in research into mathematics in the South African context. Freire's view according to Goulet (Freire, 1974: ix) is that 'genuine theory can only be derived from some praxis rooted in historical struggles.'

The South African mathematics teaching and learning context provides rich opportunity in this regard as reflected in this discussion.

To activate and engage one's consciousness would appear to require some facilitation or intervention. I have held this view about mathematics learning and teaching for a long time. That phenomenon that invokes dialectical engagement and initial, albeit rudimentary critique, in my experience has been the graphical representation of mathematical concepts used as a catalyst for learners to simply describe what it is they observe before they are asked to explain mathematics. These preliminary encounters would seem to be necessary before learners can be expected to apply the mathematics to some real phenomenon. Freire held the same views about the Brazilian people whose critical consciousness he inspired in terms of their humanity and democratic rights to emerge from poverty. To initiate their critiquing abilities he presented pictorial representations of every-day life for them to describe and in so doing gauged from their responses their affective leanings associated with the pictorial incidents. It was his view that the innate knowing that respondents had was simply dormant behind the unacquired literary ability to express embodied understandings.

Freire writes that 'To acquire literacy is more than to psychologically and mechanically dominate reading and writing techniques. It is to dominate the techniques in terms of consciousness; to understand what one reads and to write what one understands; it is to *communicate graphically.* Acquiring literacy does not involve memorizing sentences, words, syllables – lifeless objects unconnected to an existential universe – but rather an attitude of creation and re-creation, a self-transformation producing a stance of intervention in one's own context.' (Freire, 1974: 49) (Italics mine). The graphical communication in Freirian context purports to use pictorial realities that prompt critical thought, however broken it may be. The Freirian pictorial reality is commensurate with the graphical representation of algebraic conceptual abstractions in this research context.

It is this very same understanding that may hold valuable insights for mathematics teaching and learning. Through describing graphical mathematical representations learners may begin to communicate mathematically sufficiently for teachers to fathom the meaning and understanding learners attach to their remote algebras. The meaning learners demonstrate in terms of rote, mechanical routines of manipulations may be insufficient to ascertain the cognitive authenticity of their relational understanding if it constitutes an exclusive assessment form. The Freirian pictorial stimuli used to 'communicate graphically' is equivalent to the Cartesian-located graphicacy in mathematical terms in this research; a point of departure for accessing what I regard as cognitive indicators of knowing.

Clinically austere education environments that produce mute learners are associated with what Freire calls the 'banking' concept of education (Freire, 1970: 53-67). This form of knowledge deposition is void of conversational learning since the transfer of knowledge is by narrative communiqués of the teacher. Recipients of knowledge adapt to provided knowledge rather than engage with it critically. This has been my experience of mathematics classrooms in my role as an inservice teacher-teacher in twenty-six schools scattered across greater Cape Town and in schools distributed around Worcester in the Overberg in the Western Cape. In contrast to the 'banking' concept Freire describes his dialogic education as being centred on 'problem posing', as opposed to the 'problem solving' that frequents the literature on mathematics learning and teaching. It is characterised by dialectical accounts of negotiating meaning with learners, which requires both learner and teacher to delve cognitively for their conscious understanding of reality. It is this critical attitude that generates an emergence of meaning and which is seen to humanise education. It resolves that education is never a finished task; that new knowledge as it continues to change our thinking also continues to transform our world.

One imagines that other disciplines that are by their very nature discursive could, if necessary, find a way of becoming more inclusive of learners' perceptions and opinions. Mathematics on the other hand finds itself in a position where its content is factually exact. But even stuttered communication about its exactness and application would bring it more in line with being accessible to critical consciousness of teachers and learners.

Dialectical forms of acquiring knowledge produce and enhance personalised forms of knowing. In the personalising of the knowledge there is a sense that the knowledge is sustainable and available for application at another point in time. In these terms, I view dialogical learning as the 'investment' concept in education;

knowledge that has long-term growth and value, particularly for mathematics teaching and learning, particularly since it is our agenda as mathematics teachers to produce learners who ultimately can sustain their mathematical knowledge for application to real life situations.

An investment notion of educative practices is founded on dialogical discourses. Dialogues are sustained by communication, and communication is mediated by language. Language in turn comprises words. In analysing dialogue as a human phenomenon, Freire recognises that words are constituted of the elements of reflection and action (Freire, 1970: 68). These elements he regards as so radically interactive that the absence of one of them contaminates the authenticity of the word itself. If a word or utterance is without reflection it presents as 'activism' in dialogical situations as opposed to action. Similarly, if a word is devoid of action it presents as 'verbalism'. A true word communicated with action and reflection is the epitome of Freirian praxis, where critical theorising has informed practices that has as its agenda to conscientise and transform reality for the emergence of critical thinking. Reflective action is therefore necessary for both teacher and learner for authentic dialogic interaction that could lead to critical conscious learning. Freire states that 'Dialogue is thus an existential necessity.' (Idem: 69). The implications of this analysis for mathematics teaching and learning is that teaching concepts needs a preliminary thorough metacognitive deconstruction with the intention of providing for learners access to a means of thinking critically about the mathematics at hand, and secondly to establish a language, or mathematical register that will facilitate critical mathematical discourses. According to Freire it is 'Only dialogue, which requires critical thinking that can generate critical thinking.' (Freire, 1970: 73). The dialogue however needs to be appropriate and understandable as Freire aptly states, 'Often, teachers and politicians speak and are not understood because their language is not attuned to the concrete situation of the people they address. Accordingly, their talk is just alienated and alienating rhetoric. The language of the teacher..., like the language of the people, cannot exist without thought; and neither language nor thought can exist without a structure to which they refer. In order to communicate effectively, teacher and politician must understand the structural conditions in which the thought and language of the people are dialectically framed.' (Freire, 1970: 77). Often the language in English mother-tongue mathematics classrooms is formal as it supplies concept definitions at initial levels that have circumvented the constituent constructs of the derivative concepts. It is maybe this elevated entry that separates mathematics from other disciplines as elitist. It may seemingly also widen the chasm between those who are inherently privileged by their innate intelligence or by virtue of the fact that their homes provide strong linguistic environments. This is in sharp contrast with English second language speakers in poor communities where mathematics resources are sparse. The polarity emphasises the separateness and entrenches a 'have' and 'have not' dichotomy of knowledge accessibility and ownership.

Freire matches the educational invitation to conscientise with the need to recognise the thought-language assets of learners (in this work, scholars or teachers) as the focus of investigative research. He says, 'Consistent with the liberating purpose of dialogical education, the object of the investigation is not persons (as if they were anatomical fragments), but rather the thought-language with which men and women refer to reality, and their view of the world, in which their generative themes are found.' (Freire, 1970: 77-78).

For individuals, meaning and understanding is intricately interwoven with thought and language. Thought generates meaning through the mediation of words that have assumed conceptual cognitive contexts. Uttered language communicates those thought meanings and serves to convey one's understanding of them. Where such utterance of meaning and understanding are consensually shared we breach the domain of cognition and extend the thought to conceptual levels that become theory and available for wider critique and debate. This is the synthesis of Freirian conscientisation and praxis where critical consciousness sublimates into emancipatory action that seeks to extrapolate meaning and transform realities of dissatisfaction and discontent, and Habermasian communicative action where individuals seek to place themselves in the place of others in discussing whether a proposed norm is fair to all for the sake of ultimate rational agreement. If this could be the changed nature of mathematics classrooms that frequently alienate learners from their own thinking, let alone its content, because of its eloquent abstractions, mathematics may become accessible to more that the relative few who enjoy its powers of elitism and commodity; commodity in the sense described by Setati (2005). The dissatisfaction and discontentment amongst learners in being unable to fulfil their mathematics potential can be dispelled with praxis, by aligning reflective theory with deliberate action. For mathematics teaching and learning, for me, this constitutes careful metacognitive analysis of conceptual content to separate composite concepts into its derivative constructs and depicting these graphically, as Cartesian representations and/or through deictic gesturing, so that they can be described by learners. In describing we acquire the words and language that develops cognition that in turn emancipates us in being able to engage critically in mathematical discourses.

4.6 A hermeneutic perspective on critical reflection and discourse in mathematics classrooms

If our teaching (and research) as mathematics teachers is about the cognitive empowerment of learners then we are engaged with the thinking of our charges. If we wish to educate authentically then this is our business. And if this is our business then we enter the realm of the hermeneutic tradition, and the need to operate within that realm as we teach and learn, and assess the degrees of learning, and the authenticity of our teaching.

Hermeneutic enquiry probes the relationship between understanding and explanation. Modifications that emerge from such cyclical reflective analyses arise in what is referred to as the hermeneutic circle. The continuing influential reciprocity between understanding and explanation and its associated experience are as close as this phenomenological practice gets to be objective. The momentary transcendental experience is ontologically part of a socio-cultural history that has enveloped the sense-maker over time. The subjective amalgam of participants' experiences may lead to some generalisation over time if such subjectivity recurs. In such cases the emerging tendencies may contribute to theory. If not, the transcendental moments still contribute to informing practice and establish for the sense-maker a critical frame of mind upon which to modify current understanding and be able to restate an alternative understanding. This historical influence on transcendental meaning is used by Heidegger (1962) who introduced as a dimension to his teachers' work the insistence that meaning is historical rather than transcendental; a consequence of interpretation through time rather than mere consciousness. Hermeneutics permits interpretations but at no time can it claim any finality to the interpretation. Hermeneutic phenomenology in this way offers an approach to describing how the individual confronts and works with mathematical ideas and for individuals themselves to confront their own understanding in being able represent it textually. These written accounts constitute for the teacher and learner interpretations of mathematical phenomena as they are radically constructed from the learner's cognitive insider perspective, which carries with it past socio-cultural experience all embedded in the social dynamics of classroom interaction.

The anecdotal account of a shared experience of the meaning of 'gradient' demonstrated an increasing awareness of the concept by a learner. Embryonic descriptions associated with the learner's social experience were, within the space of a lesson, modified, as I perceived the learner as approaching a formal understanding of 'gradient'. There is of course the possibility that these awareness's may culminate in a richer gestalt over a more protracted period of time.

This notion of objectivity in hermeneutic enquiry is described by Brown (1997: 36) as follows. 'In a sense the objective world can only be seen through successive overlays; or rather, particular time-dependent partitionings. It is through this route that phenomenology can be seen as offering the opportunity for providing ontological grounding for action.'

The discussion of 'gradient' transpired through operating within communicative action. There is however, also room for hermeneutic interpretation of textual accounts. In this regard, Ricoeur (1981: 197-209) sees text as a "fixation" of a speech event whereby elements of the event are reduced into a set of written words. The written accounts of this hermeneutic study employ learners' momentary experiences through soliciting descriptions of their thought processes as they record their understanding of mathematical processes of solving equations, simplifying expressions and interpreting graphs. In this matter, Brown (1997: 36) states that, 'It is through this sort of fixation that we can employ techniques of interpretation in facilitating both understanding (through signs) and explanation (of "facts").'

Language in hermeneutic enquiry presents as a constraint since it is the vehicle for learners' to sublimate their thoughts. In this study isiXhosa learners who are instructed in English are also required to articulate their understanding. As teacher-researcher I have had to accommodate an absence of fluency by constantly interrogating my own cognition in assessing whether a learner is in the process of aligning concept image and concept definition. It is the immediacy of the graphical representations that have aided articulations. There is a tendency for articulations of even a rudimentary nature to be more successful when such articulations are accompanied by a graphical representation. This is in contrast with Mason (1989: 3) who claims that 'Articulation of a seeing of generality, first in words or pictures, and then increasingly tight and economically succinct expressions, using symbols and perhaps diagrams, is a pinnacle of achievement, often achieved only after a great struggle.' I sense a reversal in natural abstraction levels in this claim. The visual accessibility as described in various passages above indicated by the representative graphs hold the elements of deconstructed concepts that provide the vocabular content for learners to construct the conceptual language. Being aware of inarticulate truth has also been influential in what I regard as ethical considerations in assessing the work of learners whose mother tongue is not English.

Because of the subjectivity of interpretation as teacher-researcher I have been aware that my own perspectives may contaminate the intended meanings of others. I have however, tested my interpretation as far as possible, against my historical experiences and have attempted to avoid introducing extraneous hunches in my interpretations of learners' textual accounts as I seek to make sense of them in the following chapter.

4.7 Summary

Critical realist metatheory accommodates the empiricist scientific analyses alongside the hermeneutic in its stated acceptance of methodological plurality. The core investigative work in the following chapter revolves around hermeneutic elements of cognitive phenomena as they emerge from learners textual accounts of mathematical tasks, and learners' solicited affective responses to these tasks. The textual accounts reflect the mathematical register of learners. These have been codified to ascertain the depth of that register and if there is a related concomitant conceptual depth of mathematical concepts. Hence the qualitative and quantitative research components each exhibit their own explanatory power alongside one another.

Habermasian moral consciousness and communicative action provide the theoretical underpinnings of an ethos of personal and social justice as negotiations occur around mathematical meaning. The invitation to communicative action forges critical consciousness about mathematics tasks that are evoked by iconic representations that are deemed to have concrete visual properties and that manifest the covert derivative subconstructs that are obliteratively subsumed in abstract algebraic forms. The deconstructed concrete graphical depictions facilitate the mathematical register which is seen to develop the cognitive acuity of concepts in promoting conversationally adept mathematics learners in an attempt at emancipatory personal justices that are envisaged as contributing to a fabric of social justice.

Chapter 5 delves into metamathematics as I perceive its cognitive context for both teachers and learners. It begs the question as to what personal discourses initiate public discourses and to what extent the absence of self-interrogatory techniques is indicative of mathematical mutism in classrooms.

V

Toward metamathematics

'For me, one goal of mathematics education should be to ascertain and explore ways of seeing mathematics which provide insight into its learning and teaching. In that sense, mathematics education can be seen as meta-mathematics.'

Pimm (1987: 206)

5.1 Introduction

Abduction and retroduction are modes of thinking with which to make inferences about social scientific phenomena (Danermark, et al, 2002). As such these modes of thinking provide cognitive tools with which to investigate and explore the notions of metacognition and metamathematics and its value for teaching mathematics.

5.2 Critical realist abduction and retroduction

Abduction, as a mode of inference, seeks to interpret phenomena in light of new frames of reference. Since there can be more than one frame of reference possible through the uniqueness of being, there are also therefore any number of frames of reference within which to interpret any given phenomenon. This notion of abduction introduced by Peirce (1962) is spoken of by Habermas (1972: 113) as a thought operation that sees something as it *might be* in contrast with inductive and deductive processes that prove that something *must be* in a certain way (Danermark, 2002: 91) (Italics mine). Recontextualisations have the advantage that they help to see existing and already explained phenomena in a new way. This calls on imaginative resources to redescribe and interpret phenomena through which the potential for extending an understanding of the phenomenon comes into being. In critical realist terms, and in critical scientific analyses, this kind of analysis is seen as integral to the growth of knowledge. Ödman (1979: 78), cited in Danermark, et al., (2002), likens abduction to the hermeneutic circle which advocates the cyclical interpretations of understanding and explanation. Inherent in this cycle is the understanding of parts that relate to the whole and the alternating interpretation, understanding and explanation of part and whole through which mechanism refinements and developments occur. Abduction thus describes how reasoning and interpretation are developed in a process that focuses on the relationship between the specific and the general.

Retroduction although not characterised by formal logic, does enable one to progress from knowledge about one thing to knowledge of something else. Retroductive thinking presupposes a transcendental argumentation that seeks generally to shed light on the basic prerequisites or conditions for social relationships but more pertinently in terms of this work, prerequisite conditions for the acquisition of reasoning ability and knowledge that impacts comprehensible and sustainable mathematics at secondary school level. A feature of this thought operation is its challenge of generally regarded acceptable conditions of normality and provoking others to appear.

With specific reference to this work, abductive processes relate to the deconstruction of algebraic forms of the absolute value, nature of the roots of quadratic equations and the interpretation of simple trigonometric reduction formulae. This deconstruction phase is pertinent to a teacher's preparatory phase of analysing learning material. Abduction sees the standard or typical presentations of these content areas being recontextualised in graphical forms that constitute latent subconstructs at another unobservable cognitive stratum. The commentary in the analysis of these topics that follows is characterised by interchanging teacher and learner perspectives. The first anticipates what it is learners will see more clearly so as to aid their descriptions and explanations. The second perspective reflects, from my personal teaching experiences, what constitutes impediments to conceptual closure. These anticipations are 'redescribed' in the graphical context in order to concretise what may previously have been masked by abstract concept definitions often characteristic of instrumental use of theory as opposed to having a relational understanding of mathematics concepts in Skempian (1976) terms.

In the analysis of learners' artefacts comprising their written communication about mathematical tasks , by retroduction, I use learners' descriptions of the mathematical phenomena to try to ascertain whether expressing themselves mathematically promotes a critical cognitive consciousness about the mathematics in question. This component of the hermeneutic circle indulges thought operations and counterfactual thinking in an attempt to argue towards the transfactual conditions for real cognitive mathematical engagement that transcends the rote application of rules.

Danermark et al., (2002: 113) conclude the following in summary about 'abduction' and 'retroduction'; 'By abduction, individual phenomena are recontextualised with the help of general concepts and categories. By retroduction, accidental circumstances are abstracted in order to arrive at the general and the universal... By abduction and retroduction we can see connections and structures not directly obvious in the empirical reality.'

Critical realist theory makes the distinction between the extrapolations that emerge from the empirical domain and the transfactual conditions pertinent to deep structures of reality, of which I take cognition to be an example. In critical realist terms generality is seen as those 'universal preconditions for an object to be what it is.' (Danermark et al., 2002: 77). Bhaskar's version of this says that 'Scientifically significant generality does not lie on the face of the world, but in the hidden essence of things.' (Bhaskar 1978: 227) These hidden conditions that generate empirically observable events are themselves, in a phenomenological stratum, also connected by their own kinds of generalities. An analysis of these transfactual preconditions are undertaken in critical realist terms through transfactual arguments referred to as abductive and retroductive inference. The analyses of this work attempt to produce knowledge about the constituent properties of learner cognition and aims to establish what facilitates cognitive mathematical consciousness. I am attempting to ascertain whether graphical representation is a facilitatory condition for learners to be able to express and describe their understanding of mathematical structures to promote a mathematical consciousness. Three structures have been selected for purposes of constituting the report of this research. These are,

1. the definition and application of the absolute value function.
2. the nature of the roots of quadratic relationships.
3. the understanding of simple trigonometric reduction formulae.

The reason for selecting three content areas as opposed to a single content area was to observe whether any degree of transferability or replicability emerged to corroborate and contribute to the validity of the knowledge claim about transfactual cognitive conditions that promote mathematical consciousness. Further to this, experience has shown that these particular topics are characterised by Skempian instrumental understanding that translates into algorithmic routines that reflect inert cognitive engagements with the subject matter. In turn mathematical mutism curtails attempts at communicative action which leads one to interrogate the authenticity of the educative purpose. It is these topics also that, from experience, learners' find hard to explain, which I suggest reflects their meagre understanding of what it is they are doing mathematically.

My rationale for pursuing graphical representations as a medium for teaching mathematics is embedded in my claim that the abstractual order of mathematical constructs should play a vital role in the educative processes of mathematics classrooms. If one compares the character and representational capacities associated with sketching graphs, simplifying expressions and solving equations there is an emergent understanding that ensues from the content-related anecdotal account of my experience as an inservice teacher-teacher over seven years of service in the Western Cape, South Africa.

5.3 Metacognitive deconstruction analyses applied to the definition and application of the absolute value

The analyses that follow are constituted of thick descriptions in a mode of critical realist abduction. What is taken as typical presentations or teaching strategies of three specified sections of senior secondary school mathematics, are recontextualised in graphical forms that serve as concrete manifestations of algebraic forms. The analyses that follow are mine as teacher-researcher as I metacognitively deconstruct in lesson preparation so as to overcome what in the first instance is a language impediment for second language English speakers, and which ultimately manifested as facilitatory for all learners in the heterogeneously composed group. In the second instance the iconic language form of graphical representations was intended to have hieroglyphic value as an intermediary form of language between mathematics as a language, and English as the language of teaching and learning. By abductive inference I resort to changing the mathematical object of observation and perception of learners as I recontextualise otherwise factual algebraic mathematics disassociated from its Cartesian context, and reorder tasks on what I perceive as a continuum from concrete to abstract through employing graphical representations.

Working with teachers in the Peninsula and Overberg schools over seven years it became clear that the abstractual order of concepts is not necessarily a consideration in presenting subject matter for learning.

The table 5.1 below summarises this.

Task	Order in which teaching and tasks are usually undertaken	Teachers assessed order of abstraction associated with task.
1. Simplify $\lvert x-3\rvert-\lvert 2-x\rvert,\quad 2<x<3$	1	3
2. Solve for x $\lvert x-3\rvert\geq\lvert 2-x\rvert$	2	2
3. Sketch $y=\lvert x-3\rvert$	3	1

Table 5.1

Comparing mathematical tasks and their perceived abstractual order

For each cohort of teachers that attended the content modules teachers were asked to rank the order in which they taught the tasks, without being aware of the second ranking they would be required to do, so as to eliminate the influence of one ranking on the other. It infers that this exercise is an oral one since the ranking of one process, in written form from experience proved to bias the other. For all cohorts the ranking of the order of undertaking tasks and the teachers' assessed order of abstraction in the task were reversed. It appeared that teachers presented the tasks associated with the absolute value concept in an order which was the reverse order of their assessed level of abstraction associated with the task. In discussion it emerged that the selected order of doing these tasks was largely determined by habit of following textbook presentations, and that teachers did not consider that the tasks differed in abstractual levels. What this seems to indicate is that teachers', for whatever reason, do not necessarily engage with the metacognitive issues that impact their teaching or the learning of their charges.

In the section that follows I analyse what I understand to be the cognitive requirements involved in tasks that occur in a typical sequence of teaching and contrast these with the tasks taking into consideration the metacognitive analyses that lie in the strata of covert algebraic mathematical forms. These analyses further serve to illustrate the notion of abstractual order introduced above. Suggested cognitive arguments and strategies are presented from a learner's perspective and overlaid by my own interpretation of what could make for metacognitive engagements with the deeper structures that contain a host of derivative conceptual constructs.

Task 1

In the task of simplifying the expression, $|x-3|-|2-x|,\ \ 2<x<3$, one needs to have a clear understanding of the definition of the absolute value in addition to having a knowledge about interpreting the restriction on x and how the restriction impacts the meanings attached to the expressions $|x-3|$ and $|2-x|$. If this task, as is usually the case, undertaken from an algebraic point of departure, a learner could argue as follows. If, $2<x<3$, then directly from the continuous interval it is read that $0<-2+x$ [13] or more tidily, $x-2>0$. Similarly, $x-3<0$, and uttered cognitively, one would say to oneself "x minus 2 is positive, but that makes 2 minus x negative, while x minus 3 remains negative." It is surmised that a learner would then depend on the definition that

$$|x| = x,\ if\ x \geq 0,$$
$$or\ |x| = -x,\ if\ x < 0.$$

[13] Reading this inequality in this way "zero is less than 2 minus x" induces a cognitive dissonance in that the order of the inequality makes cognitive sense when it is read that " x minus 2 is positive.", where the variable precedes its assigned value(s). It is at a higher level of cognitive acquirement that the form $0<2-x$ aligns concept image and concept definition on first read.

and hence apply it to the simplification of $|x-3|+|2-x|,\ \ 2<x<3$ in the following way.

$$\begin{aligned}&|x-3|+|2-x|,\quad 2<x<3\\ &=-(x-3)+(-(2-x))\\ &=-x+3-(2-x)\\ &=-x+3-2+x\\ &=1\end{aligned}$$

Task 2

The solution of the inequation $|x-3|\geq|2-x|$ is unlikely to be obtained, by a majority of the group, through inspection, since the solution incorporates an improper fraction, $\frac{5}{2}$. An algebraic point of departure assumes known, not necessarily contextually understood, the definition of the absolute value. Since the structure of the task is constituted of an inequation, dealing with it requires a generic knowledge about inequations. A learner's engagement with the structure may amount to the following kind of personal cognitive discourse, "I need to find value(s) of x for which the left hand side of the inequality will be greater or equal to the right hand side of the inequation." To embellish the meaning of this thought, it is required to know that to cover all cases, $|x-3|$ by definition is $(x-3)$ when $x-3\geq 0$ and that $|x-3|$ is $-(x-3)$ when $x-3<0$. Similar thinking should serve to interpret $|2-x|$. There is therefore a need to apply the definition so that all cases are covered, that is when

1. $(x-3)\geq\ (2-x),\ or$
2. $-(x-3)\geq -(2-x),\ or$
3. $(x-3)\geq -(2-x),\ or$
4. $-(x-3)\geq\ (2-x).$

Having achieved this level of personal discourse it is useful to see further that two cases of the four are sufficient to cover all conditions for a valid solution, in the case of an equation, because of the equivalences in the statements, but not so in the case of the inequality owing to the reversal of the inequality when multiplied or divided through by a negative constant.[14] Statements 1 to 4 therefore produce the following manipulations.

By substituting appropriate values into the original inequality for each case, it becomes evident that the only solution is $x \le \frac{5}{2}$.

$$
\begin{aligned}
(x-3) &\ge (2-x) \\
\therefore 2x &\ge 5 \\
\therefore x &\ge \tfrac{5}{2}
\end{aligned}
$$

Invalid for all $x \ge \frac{5}{2}$.

and

$$
\begin{aligned}
-(x-3) &\ge -(2-x) \\
\therefore (x-3) &\le (2-x) \\
\therefore 2x &\le 5 \\
\therefore x &\le \tfrac{5}{2}
\end{aligned}
$$

Valid for all $x \le \frac{5}{2}$

and

$$
\begin{aligned}
-(x-3) &\ge (2-x) \\
\therefore (x-3) &\le -(2-x) \\
\therefore x-3 &\le x-2 \\
\therefore -3 &\le -2
\end{aligned}
$$

Invalid false statement.

and

$$
\begin{aligned}
(x-3) &\ge -(2-x) \\
\therefore x-3 &\ge -2+x \\
\therefore -3 &\ge -2
\end{aligned}
$$

Invalid false statement.

Frequently the alternative route of squaring both sides amounts to no more than rote manipulation since the final solution is often left untested for its validity. That squaring may introduce extraneous roots is often overlooked. In this case the solution derived by squaring both sides produces the correct solution due to the coefficients of both expressions having the same numerical value (a thought itself requiring cognitive depth about equal gradients of linear functions), so it would not therefore serve as an example that would exact a full cognitive analysis.

[14] This statement regarding the changing direction of an inequality is itself a rule within an algorithm that requires demonstration of how inequality relationships between constants change when the statement is multiplied by a negative constant, for example -1>-3 implies 1<3.

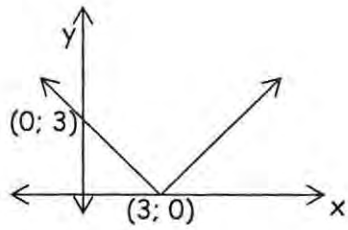
$$|x-3| \geq |2-x|$$
$$\therefore\ (x-3)^2 \geq (2-x)^2$$
$$\therefore\ x^2-6x+9 \geq 4-4x+x^2$$
$$\therefore\ -2x \geq -5$$
$$\therefore\ x \leq \tfrac{5}{2}$$

> The cognitive nuances and deeper mathematical structures that promote cognitive discourses are absent in this example. The routine algorithmic manipulations serve to mask the real mathematics that is contained in the task of solving $|x-3| \geq |2-x|$ for x, or for finding what stretch of the domain, $x \in \mathbb{R}$, for both functions $|x-3|$ and $|2-x|$, where, algebraically, $|x-3|$ has function values greater or equal to $|2-x|$, or graphically for which part of $x \in \mathbb{R}$, $y=|x-3|$ lies on or above $y=|2-x|$.

Task 3

To sketch $y=|x-3|$ requires the same understanding of the definition of the absolute value and the ability to utilise skills for solving equations. There are a variety of ways of approaching this task. From experience it is typically dealt with in a fashion similar to sketching the parabola as it occurs in 'turning point' form. The salient point of the absolute value graph is generally coached as (3; 0). The remaining features are obtained through algebraic manipulation. A typical approach may be as follows. "To find the x intercept, make $y=0$. So, $|x-3|=0$ can be solved by inspection where $x=3$. To obtain the y intercept, make $x=0$. So, $y=|0-3|$, and therefore $y=3$. Since the coefficient of the absolute value bracket is positive 1, the graph has a minimum value at the salient point, so the arms point up." These are factual statements that represent what amounts to 'doing' mathematics. If asked to explain why make $y=0$ for obtaining the x intercepts, learners remain silent or register immense frustration at being at a loss for words.

1. Salient point is (3; 0)
2. x intercept is 3
3. y intercept is 3
4. The arms of the graph point up



The already acquired skills for graph sketching provide an existing schema on which to construct knowledge about the absolute value graph if one assumes that the character and features of the parabola serve as pre-knowledge.

5.4 Reflections on a standard or typical sequence of approaching the absolute value

It becomes evident from this analysis that the abstractual order of the tasks is similar to the order of difficulty associated with the tasks. The standard order therefore of presenting tasks to novice learners of the absolute value appears to be the cognitive reverse in terms of difficulty levels and levels of abstraction. Related to the levels of abstraction is the concreteness with which the tasks can be depicted. The typical order of presenting the simplification of expressions followed by the solution of (in)equations and then the sketching of graphs, as summarised in the table above, is contrary to what may be expected after a metacognitive analysis of the content. Of the three tasks the sketch graph of the relationship appears to hold the greatest levels of concreteness considering the pictorial or iconic elements of the representation. The solution of the (in)equation seems to present as an order of difficulty easier than the analysis of the simplification needed for task 1. Of most significance however, is the fact that a reversal of the typical sequence of instruction provides a graphical iconic template from which to understand the algebraic procedures required for solving (in)equations and for simplifying expressions that may facilitate an understanding of the processes. More than being a sound educational strategy, an iconic representation, as in Fig. 5.1 serves to contextualise tasks in the Cartesian plane. It is from this graphical representation that learners can begin to visualise the latent and covert subconstructs of the concept in question. Staying with the absolute value function the following depiction seeks to illustrate this claim.

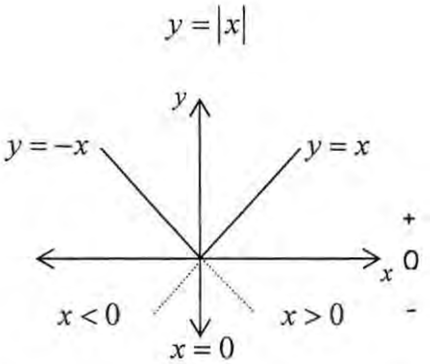
	$\|x\| = x, \quad if\ x \geq 0$ $or\ \|x\| = -x, \quad if\ x < 0$
Deconstructed graphical portrayal of the absolute value definition illustrating its component derivative concepts	Algebraic definition of the absolute value

Fig. 5.1

Comparing the iconic and algebraic language of the absolute value concept

From the graphical depiction of the absolute value the constitutive concepts that are contained in its definition are overt as opposed to the compact and concise definition with which it is contrasted and which masks the covert component derivative concepts. What the graphical depiction serves to illustrate is that the absolute value definition assumes knowledge of the identity function $y = x$ and its reflection in the both the x and y axes. It is observable that where $x > 0$ the absolute value of x is equal to x, and where $x < 0$ the absolute value of x is equal to negative $-x$. At $x = 0$ the absolute value of x is equal to zero. It is often overlooked that the zero condition is incorporated in the positive condition by convention, which for novice learners of the concept, may cloud and reduce comprehension. Further to this, $|x|$ needs to cognitively accommodate as a function value and as such its positivity for all $x \in$ □ can be read off the Cartesian plane since the graphical portrayal of the definition shows the absolute value always resident above the x axis except at zero. At yet another level or stratum, the failure to see x's variability (as opposed to constancy) contaminates comprehension levels. The variability is illustrated when compared, for example with the constant function value of zero of the x axis or any other constant function superimposed on the graphical portrayal of $|x|$. Yet another advantage that emerges from this recontextualisation is the comparative visual solution of the inequalities $|x| < 1$ and $|x| > 1$ illustrated in Fig. 5.2. The algebraic analysis of these inequalities, for the initiated, may incur cognitive reference to its graphical representation, or it may not.[15] The following serves to contrast the algebraic and graphical interpretation of the inequalities.

[15] The extent to which mathematics teachers engage with concepts in terms of Ausubelian obliterative subsumption or with cognitive imagery in referencing mathematical notions provides opportunity for further research into metacognitive mathematical understanding.

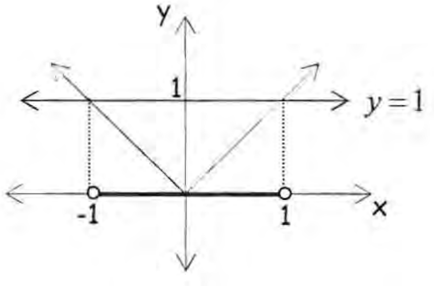
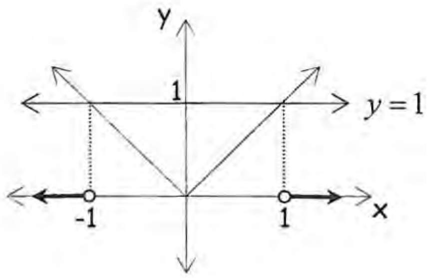
1. $\lvert x \rvert < 1$ $\therefore \quad x < 1, \text{ if } x \geq 0$ *and* $\quad -x < 1, \text{ if } x < 0$ $\therefore \quad x > -1, \text{ if } x < 0$ *and finally,* $-1 < x < 1$	 $\therefore -1 < x < 1, \text{ when } \lvert x \rvert < 1$
2. $\lvert x \rvert > 1$ $\therefore \quad x > 1, \text{ if } x \geq 0$ *and* $\quad -x > 1, \text{ if } x < 0$ $\therefore \quad x < -1, \text{ if } x < 0$ *and finally,* $x < -1, \text{ and } x > 1$	 $\therefore x < -1, \text{ and } x > 1, \text{ when } \lvert x \rvert > 1$

Fig. 5.2

Comparison of the algebraic and graphical version of standard absolute value inequalities

It is reiterated here that it is not advocated that the algebraic context is secondary to a graphical one. They are mutually dependent. The graphicacy holds a visual impact that promotes cognitive imagery for the purpose of developing an immediate reference and develop concept closure.

At yet another level the initial failure to conceive the covert information of the Cartesian plane itself as alluded to earlier seems to contribute to cognitive deficits that accumulate and militate against conceptual closure. These deficits seem to include the failure to read the graph in the direction of increasing x values which essentially renders the graph as a static rather than dynamic relationship. The importance of observing the changing function values in conjunction with increasing x values is a cognitive coordination skill that gives meaning to increasing and decreasing functions. The failure to read graphs in this manner causes learners to fixate on, for example, a simple linear function, $y = x + 1$, as a static phenomenon confined to the boundaries of the page, which further destroys the extrapolatory and predictive potential of functions.

I proceed now to analyse in contrast to the algebraic portrayal of the function how I perceive a learner may perceive the same tasks if the teaching sequence is reversed. In this reversed sequence iconic representation serves as the initial construct that celebrates a first principles approach to teaching. The Cartesian plane is the context of analysis and also serves as a reference system. A fundamental assumption from this point of departure is that the learner is graphically literate, the implication of which is that learners are well grounded in generic graphicacy before commencing with mathematics in the Further Education and Training band.

Task 1 (reversed sequence)

The assumption of graphical literacy as a prerequisite for this teaching strategy also presupposes a first principles approach to teaching where an exposition of the absolute value would have been approached from its graphical depiction.

Sketching $y=|x-3|$ from a perspective of graphical literacy may also be approached in a variety of ways, and it may also be in the variety that understanding is enhanced, which suggests dispelling the notion that more than one method creates confusion rather than understanding.

The first option is to view $y=x$ as the original function that has undergone a sequence of transformations to obtain $y=|x-3|$. A translation of 3 units down obtains $y=x-3$. By applying the affect of the absolute value the negative function values of $y=x-3$ will reflect in the x axis. This is illustrated in Fig. 5.3.

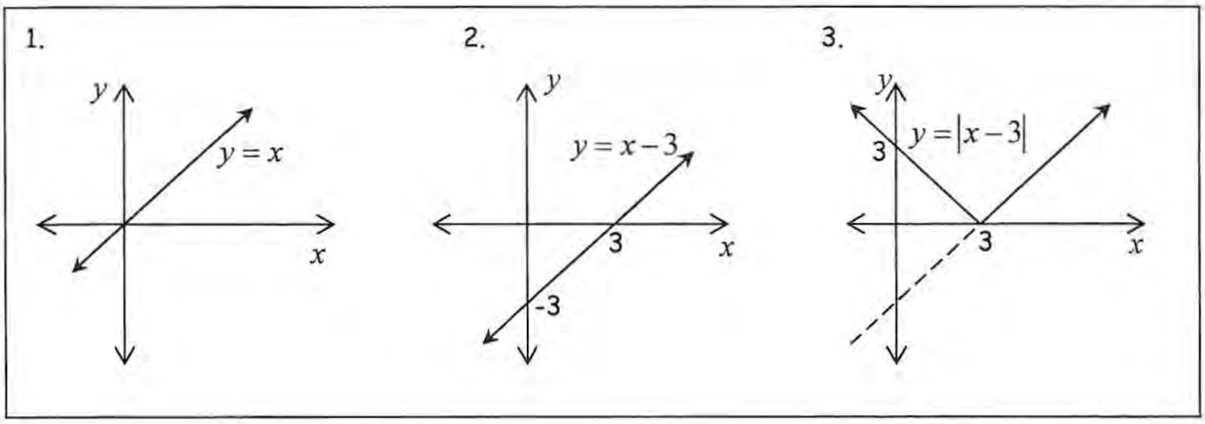


Fig. 5.3

Successive transformations on $y = x$ to obtain $y = |x - 3|$

If $y = |x|$ is taken as the original function then, $y = |x - 3|$ is interpreted as a horizontal translation of 3 units to the right, also with pre-knowledge that *brackets* affect a horizontal translation relative to its variable. In this case the translation is three units in the direction of positive x, or to the right[16], as depicted in Fig. 5.4.

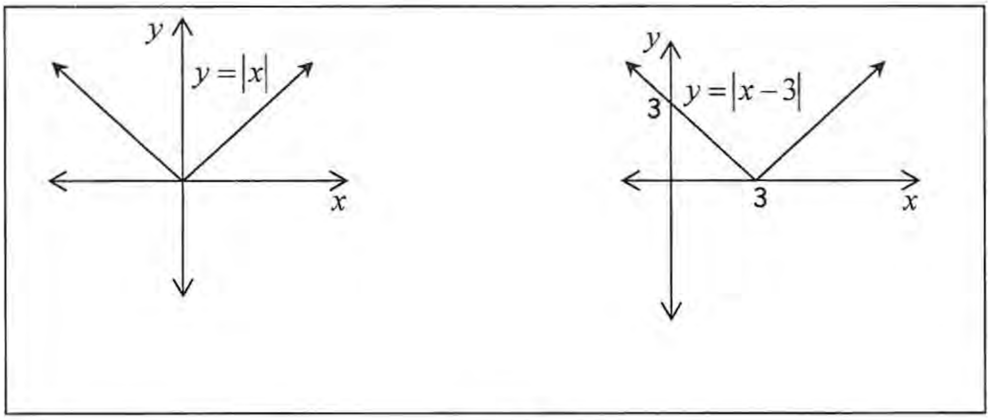


Fig. 5.4

Transforming $y = |x|$ to obtain $y = |x - 3|$ through horizontal translation implied by bracket notation

[16] This is yet another level of comprehension submersed in hierarchy of cognitive strata. It is mathematically important to differentiate between the meaning of y=x+1 and y=(x+1) since the cognitive richness of this understanding contributes to a gestalt about graphical representations that enjoys transferability across components of mathematics curricula.

It would seem to be that this level of interpretation aligns with the expected outcomes of the Further Education and Training National Curriculum Statement for Mathematics in terms of its albeit loosely stated assessment standards. (Department of Education, 2003: 22)

Task 2 (reversed sequence)

In solving $|x-3| \geq |2-x|$ a learner may from a perspective of graphical literacy interpret the solution as being those x values for which the function values of $|x-3|$ lie on or above the function $|2-x|$. To facilitate this interpretation it requires that the learner see the inequality as the simultaneous representation of the functions $y=|x-3|$ and $y=|2-x|$ on the same set of axes as in Fig. 5.5.

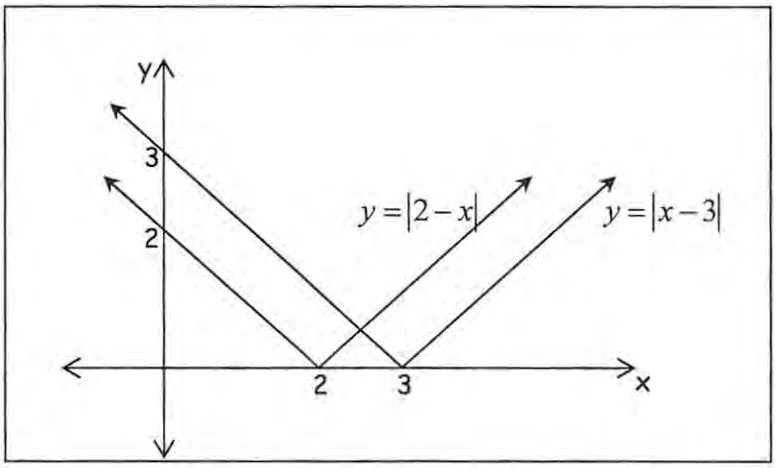


Fig. 5.5
Graphically interpreting $|x-3|=|2-x|$

Analysis of the interpretation of the question indicates that $y=|x-3|$ lies on and above $y=|2-x|$ to the left of the point of intersection of the two functions. From knowing that the component parts of $y=|2-x|$ and $y=|x-3|$, are $y=2-x$ and $y=x-2$, and $y=-x+3$ and $y=x-3$ respectively, it can be seen that the point of intersection is found where the function value of $y=-x+3$ equals the function value of $y=x-2$. From this analysis it should also be clear that there is only one point of intersection owing to the equal gradients of the component graphs, and therefore that they are parallel. This analysis further leads to knowing that solving $3-x \geq x-2$

will produce the correct solution, $x \leq \frac{5}{2}$, without further testing since the validity of the solution is confirmed by the graphical representation as confirmed in Fig. 5.6.

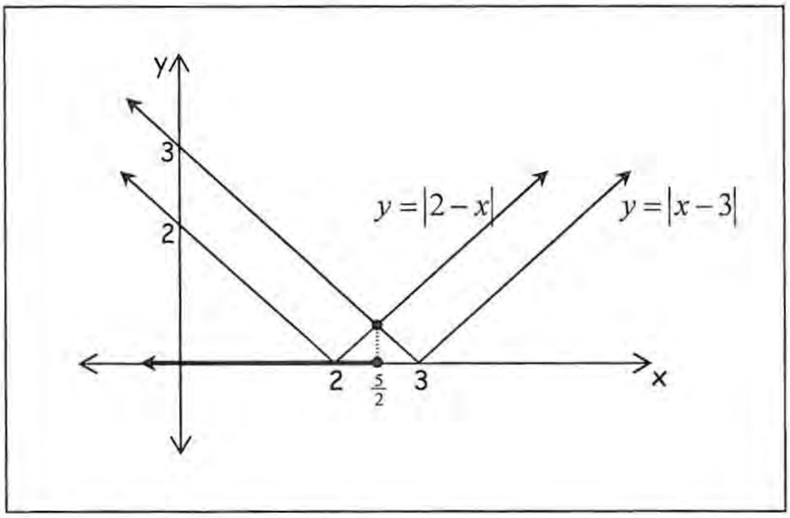


Fig. 5.6

Graphical confirmation of the solution to $|x-3| \geq |2-x|$

Task 3 (reversed sequence)

In attempting to simplify $|x-3|+|2-x|,\ \ 2<x<3$ in a graphical context produces meaning that exceeds the rote memory and application of a definition and subsequent algorithmic algebraic manipulations. Interpreting the task in a graphical context may lead to something of the nature of "I need to find a simplified form of $|x-3|+|2-x|$ on the continuous interval $2<x<3$".

To return to the task of simplifying $|x-3|+|2-x|,\ \ 2<x<3$ within a graphical context, there is sufficient heuristic demand in the task that becomes evident from the graphical analysis that follows, that would satisfy a rigorous mathematical investigation. At face value if the task cannot be seen as obtaining the sum of the function values, for x values between 2 and 3, the purpose of the task would seem superfluous. A cognitive impression of this scenario may correspond to imaging in Fig. 5.7.

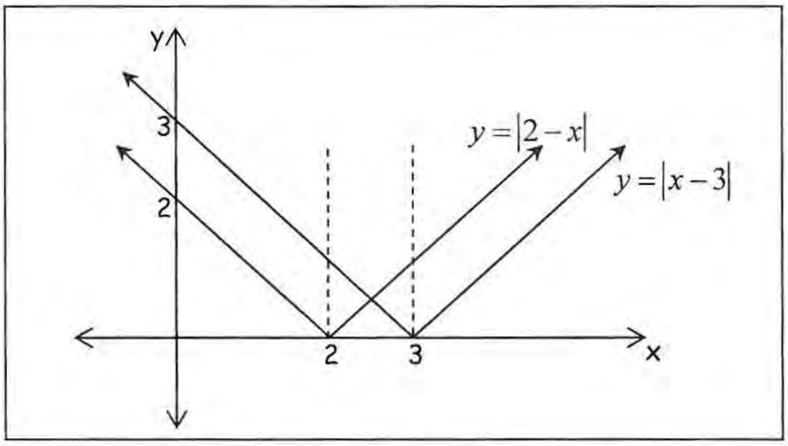


Fig. 5.7

Cognitive imaging of $|x-3|+|2-x|, \quad 2<x<3$ in a graphical context

From an understanding of the definition of the absolute value, algebraic manipulation reduces $|x-3|+|2-x|$, to 1, if $2<x<3$. Fig. 5.8 attempts to show the cognitive imaging of the deconstruction of the restriction.

Interpreting the domain by deconstruction

$$2<x<3$$
$$\Leftrightarrow x>2 \text{ and } x<3$$
$$\Leftrightarrow x-2>0 \text{ and } x-3<0$$
$$\Leftrightarrow 2-x<0 \text{ and } x-3<0$$

Algebraic manipulation

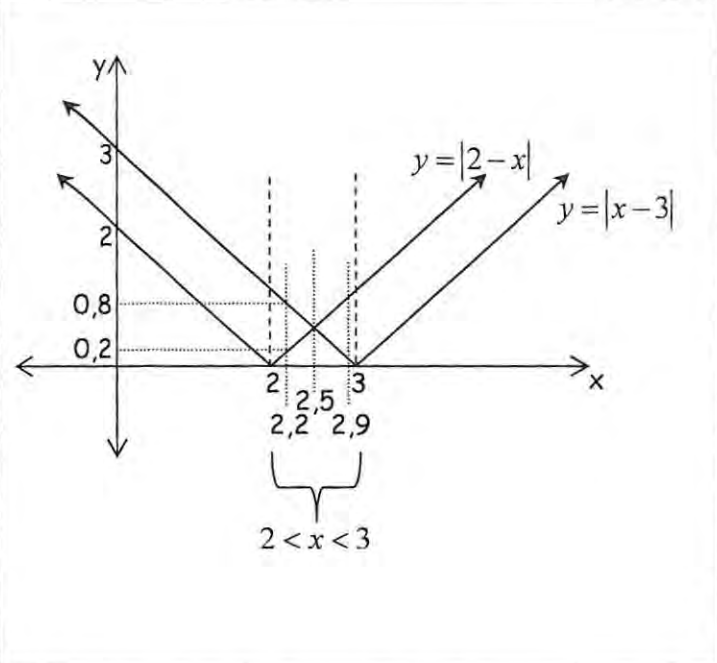
$$|x-3|+|2-x|, \quad 2<x<3$$
$$=-(x-3)+(-(2-x))$$
$$=-x+3-2+x$$
$$=1$$

Fig. 5.8

The deconstruction of the inequality to ascertain the sign of the function values

It is the advantage of the innately competent to be able to infer the deconstructed form of the inequality that specifies the restriction. The added dimension of transforming $x-2>0$ into $2-x<0$ to comply with the form of the second function value, in the original expression, cannot be taken as assumed. It is the power of evidence that convinces. Hence the deconstruction and iconic illustration to emancipate cognitive dormancy or inertia becomes mathematically valuable. It should be reiterated here that teaching these ideas relies on the comprehensive efficacy of the graphical context, accompanied by demonstrative teaching that includes deictic gesturing. Aural algebra is a difficult language to understand.

The task would be incomplete without an interpretation of its simplified form. If $|x-3|+|2-x|, \quad 2<x<3$ is equal to 1, what meaning does this constitute for a learner? With reference to the sketch below, and selected substitutions, it is evident that the sum of the function values for any x value between 2 and 3 is always equal to 1. It would make good mathematical sense to be able to interpret the simplification since this confirms cognitive engagement and understanding of the undertaken task. It is my experience that most mathematics procedures at secondary school level serve often only as meaningless manipulations devoid of mathematical cognition and mathematical context, this being a prime example of that experience. Fig. 5.9 serves to concretise what is surmised to be a cognitive image of the interpretation.



Selected substitutions from the interval $2<x<3$ to show $|x-3|+|2-x|\ for\ 2<x<3\ is\ 1$

1.

$$|2,2-3|+|2-2,2|$$
$$=|-0,8|+|-0,2|$$
$$=0,8+0,2$$
$$=1$$

2.

$$|2,5-3|+|2-2,5|$$
$$=|-0,5|+|-0,5|$$
$$=0,5+0,5$$
$$=1$$

3.

$$|2,9-3|+|2-2,9|$$
$$=|-0,1|+|-0,9|$$
$$=0,1+0,9$$
$$=1$$

From the graphical representation of the task it can be seen that for $x=2,2$ the function values of $|x-3|$ and $|2-x|$ are 0,8 and 0,2 respectively, and that the sum of these function values is 1. Similarly, as a counter example, it is observed that for $x=0$ which falls outside of the specified interval, that the sum of the function values is not 1, but 5.

Fig. 5.9

Making graphical meaning of the simplification of $|x-3|+|2-x|,\quad 2<x<3$

If the order of difficulty is increased by the simplification of the absolute value reducing to a variable expression through altering the domain, so is the need for extended interpretation. The simplification of $|x-5|-|x-1|$, $1<x<5$ reduces to $-2x+6$ which if interpreted by a learner could mean that "the difference between the absolute value functions on the interval $1<x<5$ at any x in the interval is obtained from $f(x)=-2x+6$, so for $1<x<3$, $f(x)=2$ and for $3<x<5$, $f(x)=-2$. The absolute value of the difference between the function values of $|x-5|$ and $|x-1|$ is 2 in this case. The metacognitive value of this analysis is in contrast to arriving at an answer of $-2x+6$ which a learner cannot interpret. Iconically this discussion is represented in Fig. 5.10.

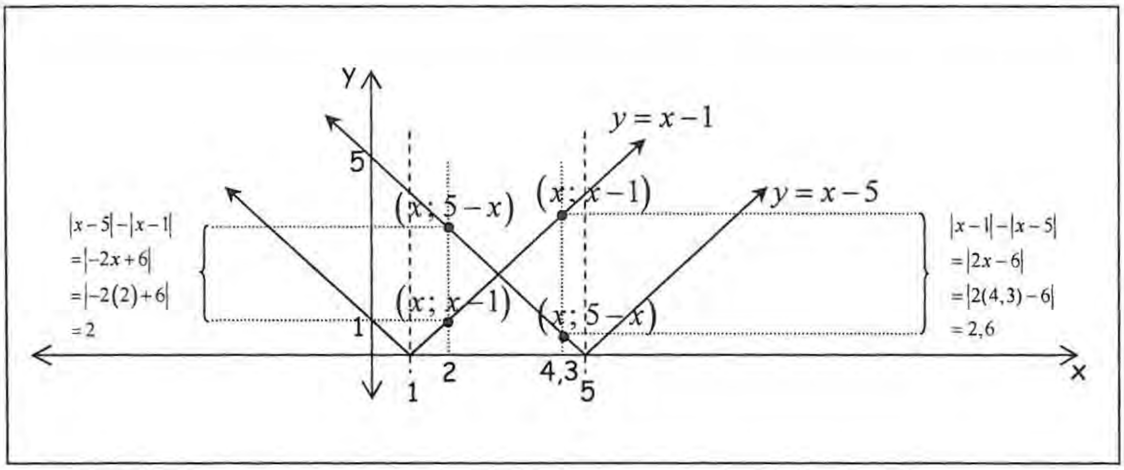


Fig. 5.10

Illustrating an increased order of difficulty related to changing the restriction as being equally graphically interpretable

The reader should not construe the busyness of the graphical representation as an order of difficulty. The ultimate aim of this representation is for teacher and learner to be able to discuss each component of the graphical representation and engage in communicative action that promotes a critical consciousness about the mathematical components. An aspect of this representation, not discussed before on this analysis, is the form $(x; f(x))$ of the points of intersection of the two absolute value functions. This form itself, amongst learners seems not as easily interpreted when given with the coordinates in constant form, but which seems to hold potential for concept closure around how to define, for example, $y = x - 5$, as that relationship constituted of all points of the form $(x; x-5)$.

5.5 Reflections on a reversed sequence of approaching the absolute value

Gravemeijer's (1988) assertion that it may be mathematics that is essentially real is corroborated by these analyses. The reality of the mathematics is situated in its graphical context rather than what is often contrived real life applications in attempts to make mathematics 'real'. The reality of mathematics is construed in this work as the authentic level of the graphical interpretation of its algebra. I attempt to show here that a generic understanding of mathematical structures and interpretations in its graphical context can be regarded as real mathematics. That we try to apply mathematics to artificial and contrived situations, for example, calculating a mother's age if she is now twice as old as her son while three years ago she was three times her son's age, is an indication that the metacognitive analyses that lie between the mathematics and authentic real life applications are seldom addressed, where English as the language of learning and teaching for second-language English speakers is yet another consideration. In seldom addressing the metacognitive components we cannot engage in meta-teaching for facilitating meta-learning. In the contrived situation above the only redeemable merit may be an intention to teach the skill of translating English into algebra; a decided disadvantage for learners whose mother tongue is not English.

These essential and fundamental interpretations of tasks promote a critical consciousness about what we are doing mathematically. The interpretations are linked to cognitive constructs that are embodied in the acquired language or mathematical register that emerge out of the Cartesian context. Without that register and the fundamental thinking we cannot construct the concept. If these constructs are not in place teachers and learners in mathematics learning communities cannot engage in dialogical teaching or communicative action. If we cannot converse mathematically with our charges we cannot delve the depths of their understanding or approximate the meanings they attach to the mathematics they do in rote algorithmic routines. This analysis suggests that mute learners propagate inert learning events. It also suggests that personal justice is denied those who are not innately mathematically adept or satisfied with terse and tacit algebraic forms.

5.6 Metacognitive deconstruction analyses applied to the nature of the roots of quadratic relationships

The case of abstractual order being inherent in tasks of simplification of expressions, solution of (in)equations and the sketching of graphs is as applicable in the nature of roots of quadratic relationships as it is to the concept of absolute value, or another component of the mathematics curriculum. A typical sequence of teaching would be to simplify quadratic expressions, followed by the solution of quadratic (in)equations and then to proceed to the sketching of quadratic functions as is indicated in the National Curriculum Statement for Mathematics (Department of Education, 2003: 22,

26) where quadratic functions of the form $y=ax^2+q$ is stipulated, and a broad reference is made to the solution of quadratic equations in 10.2.5(b). Of greater significance is the absence of an assessment standard that desires the indication of cognitive acquaintance with mathematical concepts. The arguments for applying a reversed sequence of teaching are the same in this case; to facilitate an understanding of the nature of roots or zeros.

The description of the nature of roots of a quadratic equation when viewed solely as an algebraic task implies context is denied. It follows therefore that when equations are taught as solvable by a variety of methods, as a precursor to the understanding of quadratic functions, the one notion remains separate from and unrelated to the other. The separateness precludes a desired tendency toward a critical understanding of the nature of the roots (zeros) and gaining concept closure in militating against acquiring the sense of a mathematical gestalt. It is for this reason that learners know by rote, implying that there is no capacity for explaining what they are doing, that for a point of tangency between two equal functions that reduce to quadratic form, that $\Delta=0$. It therefore presents as a difficult task when learners are required to apply their knowledge of the nature of roots in graphical contexts. It would seem that this incapacity is bedded in the fact that understanding the equivalence of the terms, roots and zeros, is remote.

The simplest expectation related to the nature of roots would be to understand what meaning can be attached to the task. To elicit communication about the equation $x^2-7x+12=0$ from learners produces little more than mutism. The task is deeper than 'find x'. This kind of response is indicative of a tenuous concept development, related to an absence of vocabulary with which to communicate about the equation. It therefore presents as an indicator of rote approaches to the mathematics at hand. In Vygotskian terms, (Vygotsky, 1986) the unavailability of the appropriate language constructs that constitute the concept, prevent the concept itself from being developed. Fig. 5.11 provides a deictic reference to contextualising the meaning of $x^2-7x+12=0$.

It would seem desirable in terms of a gestaltic metacognitive appreciation that $x^2-7x+12=0$, be perceived contextually as finding the values of x for which the quadratic function $y=x^2-7x+12$ has a function value of zero. This cognitive understanding is facilitated by the graphical representation being seen as a mental template on which to make meaning of $x^2-7x+12=0$. Using the equivalence between $y=x^2-7x+12$ and $x^2-7x+12=0$ it is seen that $y=0$. With $y=0$ being seen as the x axis it would be desirable to conclude that we are finding the values of x where the function cuts the x axis. In terms of a later analysis to do with the nature of the roots, it is further of value to comprehend $y=0$ as a member of the family $y=k$, k a constant.

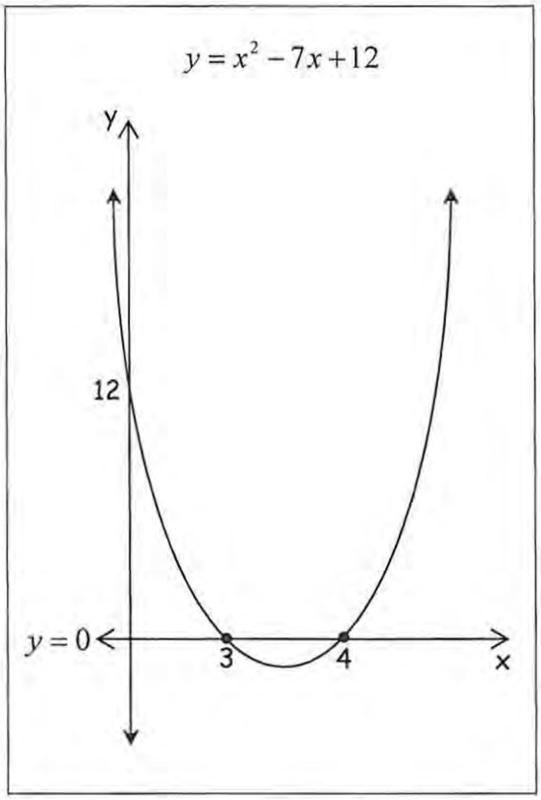


Fig. 5.11

Metacognitively accommodating the metamathematics of $x^2-7x+12=0$

In metacognitive terms the order of teaching that reverses the abstractual order is incommensurate with conceptual development. The graphical contextualisation of the parabola, as in Fig. 5.11, provides reason for solving $x^2-7x+12=0$. It has furthermore become apparent that the very use of the term 'root' essentially requires clarification and definition. I have found that by associating the x intercepts, with specific reference to cuts on the x axis, as being the x values where $x^2-7x+12$ is zero, consolidates that x, in $x^2-7x+12=0$ refers to the roots. The transfer of this information is accompanied by deliberate deictic referencing. The transfer of aural algebraic rules has limited cognitive impact.

Irrespective of the quadratic equation to be solved and the technique required to solve it, the nature of its roots is inherent in its graphical representation. It seems less than sufficient to know that for $ax^2+bx+c=0$, the roots, $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$, are equal if $\Delta=0$, or to merely associate conditions for delta for the equation to have real or non-real roots without cognitively contextualising the information graphically. By associating the equation $x^2 - 7x + 12 = 0$ with its graphical context provides visual confirmation that its roots are real, rational and unequal. This concrete observation may then lead to investigating what component of $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ determines the observed nature of the roots. In this order the concrete, visual depiction of the roots provides a point of departure for analysing the formula for the roots of a quadratic equation. It would have been established from tasks of sketching parabolas that the parameter c is the graph's y intercept. The use of the table below presumes to show the variability of parameter c. In the context of the sketch of $y = x^2 - 7x + 12$ in Fig. 5.11 above, it is inferred that the roots are 3 and 4, hence it can be deduced that in the table below that $c_1 = 12$. The purpose of task sheet 1 below, is to insert possible value(s) of c_1 through c_4, in row 2, that comply with the nature of the roots in the first row of the table, with reference to graph of $y = x^2 - 7x + 12$. In row 3 the selected values of c are inserted in the formula for the roots in row 2 and the roots are generated. Inherent pre-knowledge for this discussion is the symmetry of the parabola around $x = -\frac{b}{2a}$, and that a vertical movement of the graph implies that each unique point is subject to the same movement simultaneously.

Relating the algebraic and graphical interpretations of the nature of the roots for the purpose of establishing the dynamic nature of graphical representational forms				
Nature of the roots	Real, Rational, Unequal	Real, Rational, Equal	Real, irrational, Unequal	Non-real
1. Possible equations: Select a value for c by consulting the graph of $y=x^2-7x+12$ to comply with the specified nature of the roots	$x^2-7x+c_1=0$	$x^2-7x+c_2=0$	$x^2-7x+c_3=0$	$x^2-7x+c_4=0$
2. Selected values of the roots by estimation and consulting the graph of $y=x^2-7x+12$				
3. Formula for the specified roots	$x=\frac{-b\pm\sqrt{b^2-4ac_1}}{2a}$	$x=\frac{-b\pm\sqrt{b^2-4ac_2}}{2a}$	$x=\frac{-b\pm\sqrt{b^2-4ac_3}}{2a}$	$x=\frac{-b\pm\sqrt{b^2-4ac_4}}{2a}$
4. Calculated, exact values of the roots				
5. Discussed conditions for b^2-4ac to comply with the specified nature of the roots				

Task sheet 1[17]

Returning to the graphical representation of the quadratic function the nature of the roots is dependent on the value of the y intercept, since a positive, negative or zero difference resulting from b^2-4ac is dependent on the magnitude of $4ac$, which is in turn dependent on the value of c, the y intercept .

The combination of the graph and task sheet 1 is designed to encourage learners to shift the parabola in their mind's eye to accommodate the required changing positions of c for the desired nature of the roots. With this foundational graphical perspective the otherwise purely algebraic task of "Find c if the roots of $x^2-7x+c=0$ are real", remains therefore, graphically, visually and cognitively uncontextualised. In the data analysis in the following chapter, learners routinely and comfortably apply the conditions on delta for the roots to be real but experience

[17] This task sheet designed to facilitate a metacognitive search for an algebraic and graphical duality of the nature of the roots of a quadratic equation and its parabola.

difficulty expressing what it is they are doing, either by describing the algebra or by interpreting the question graphically.

5.7 Reflections on the value of the interpretation of parabolas for understanding the nature of the roots of quadratic equations

The foregoing analysis aims to show that the visual impact of the graphical representation of the nature of the roots provides a real mathematical schema for facilitating the understanding of what otherwise remains rote, algebraic manipulation. In this way mathematical learning appears to be accommodated and assimilated. In particular, the explicitness of the visual portrayal seems also to impact the positive learning of learners whose mother-tongue is not English. It is surmised that the iconic character of the graphical representation supplements English in the case of second language English speakers through its pictorial value. The graphical portrayal serves also as a catalyst to commence communication through learners being able to refer to components of the representation. It is furthermore the ability to orally and deictically refer to the sketch that prompts and promotes communicative action and dialogic interaction between teacher and learner.

The following graphical analysis that forms Fig. 5.12 serves to show a metacognitive deconstruction of a task as a preparatory phase before engaging with learners.

Determine, by explaining how to read the graph, for which values of p, $x^2+7x+p=0$ will have real roots?

Graphically

The parameter p represents changing y intercepts of the quadratic function. We are therefore looking for values of the constant c that produces real roots. $y=x^2-7x+12$ *will have real roots for all p such that the parabola touches or cuts the x axis. For any c value greater than* $12\frac{1}{4}$ *the parabola will not have roots. So for real roots* $p\le 12\frac{1}{4}$ *because if the turning point of the parabola is at (3; -*$\frac{1}{4}$*) all points lift simultaneously to move the parabola* $\frac{1}{4}$ *unit up. The y intercept will also be raised* $\frac{1}{4}$ *unit to* $12\frac{1}{4}$.

Algebraically

For real roots, $\Delta\ge 0$.

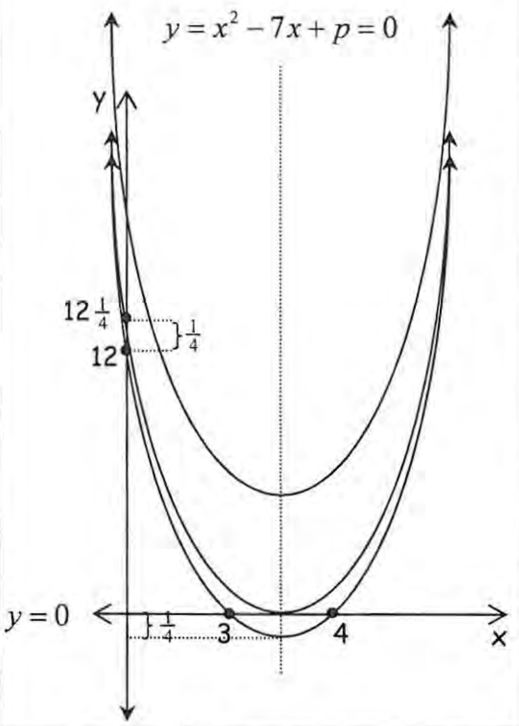
$b^2-4ac\ge 0$

$\therefore (7)^2-4(1)(p)\ge 0$

$\therefore \quad 49-4p\ge 0$

$\therefore \quad -4p\ge -49$

$\therefore \quad p\le 12\frac{1}{4}$



Determine, by explaining how to read the graph, for which values of k, $x^2-7x+12=k$ will have positive, real roots?

Graphically

The parameter k is synonymous with y. Since the question seeks ***values*** *of k, k assumes constant values. y=k, k a constant, is the family of lines parallel to the x axis. It is required to imagine this family of lines superimposed on* $y=x^2-7x-12$. *Those members of the family of lines that intersect the parabola so that they produce two unequal positive roots are the y=k that are being sought. For all positions of k between the minimum value of the function and the y intercept of the function y=k, produces positive, real roots. y=k at the y intercept produces a zero root therefore k=12 is excluded. At* x_2 *the roots are equal and positive and therefore* $k=-\frac{1}{4}$ *is included. Those lines that satisfy the condition are therefore* $-\frac{1}{4}\le k<12$.

Algebraically

If $x^2-7x+12=k$

then, $x^2-7x+12-k=0$

For real roots $\Delta\ge 0$.

$\therefore 49-4(1)(12-k)\ge 0$

$\therefore \quad 4k\ge -1$

$\therefore \quad k\ge -\frac{1}{4}$, *but if* $k\ge 12, x\le 0$

$\therefore \quad -\frac{1}{4}\le k\le 12$

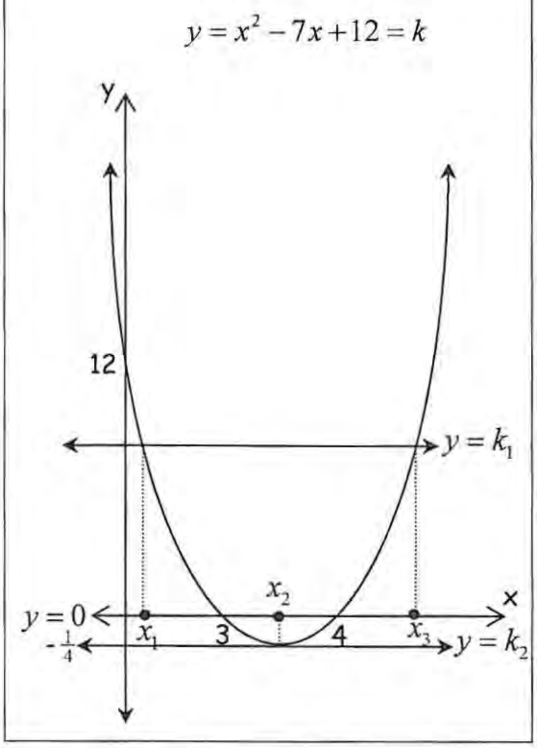


Fig. 5.12
A graphically contextualised search for specific values predetermined by a stated nature of the roots of a quadratic equation related to two parameters p and k.

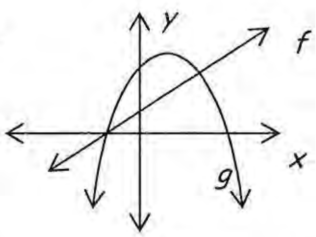
In the analysis of the preparatory phase of the task above it becomes evident that the algebraic routine masks the conceptual depth as it manifests in the textual analysis that accompanies the graphical analysis. What emerges in this comparison is that engagement with the language of mathematics establishes that a learner may not understand the concept although familiar with the algorithmic procedure. The comparison does not serve to show that the two approaches are mutually exclusive. The ability to arrive at a solution succinctly through algebraic economy is a vital part of what constitutes mathematics teaching and learning, but it would seem that to master the ability to communicate about what one is doing is indicative of being critically conscious about the cognitive demands of the task. The two approaches are supplementary. Having accessed one's critical consciousness the confidence of presenting a concise algebraic end to an investigation should be that much more meaningful for learner and teacher. The implication of this observation is that assessing learners' abilities to communicate graphically may be an important consideration in improving the graphical literacy skills of mathematics learners for the purpose of creating communicatively active mathematics classrooms and performances that reflect understanding. To know, metacognitively, how and why one knows mathematics, is itself an empowering and emancipatory notion that embraces a sense of personal justice about accessibility to meaningful mathematics.

5.8 A place for generic mathematics in senior secondary mathematics curricula

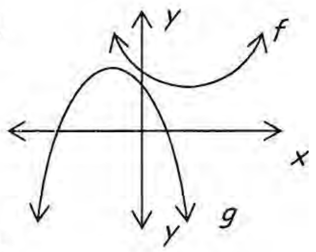
From the above analysis a generic knowledge about mathematical constructs also emerges from the graphically deconstructed approach. There is, if I want to communicate with my learners about an equation, expression or graph, a need for them to be familiar with the essence of these constructs. It is my experience, at senior high school level that the differences between these constructs, and their Cartesian context, cannot be recognised by learners. It appears that what I refer to as a generic knowledge is a stated expectation of the South African National Curriculum Statement for Mathematics (Department of Education, 2003a: 2) which, as critical outcomes, requires learners to be able to 'communicate effectively using visual, symbolic and/or language skills in various modes' and 'to collect, analyse, organise and critically evaluate data'. I regard this work as articulating specifically what these critical outcomes in essence may require but do not explicitly describe.

To extend the notion of generic school mathematics I produce below tasks that may clarify this claim. The hidden generic mathematics can be identified in the following tasks. In each of the four graphical representations two graphs f and g on the same set of axes are given. Learners are requested to state the sign of delta for the equation and describe the nature of the roots of $f(x) = g(x)$. The generic notions regarded as emerging in this analysis are now identified with the aid of task sheet 2 provided below.

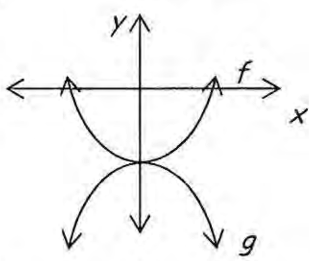
1.



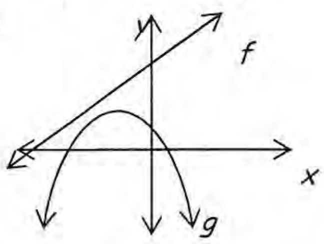
2.



3.



4.



Relating algebraic delta to its graphical context	
Task 1 Δ *for* $f(x) = g(x)$	*Task 2* *Nature of roots of* $f(x) = g(x)$

Task sheet 2

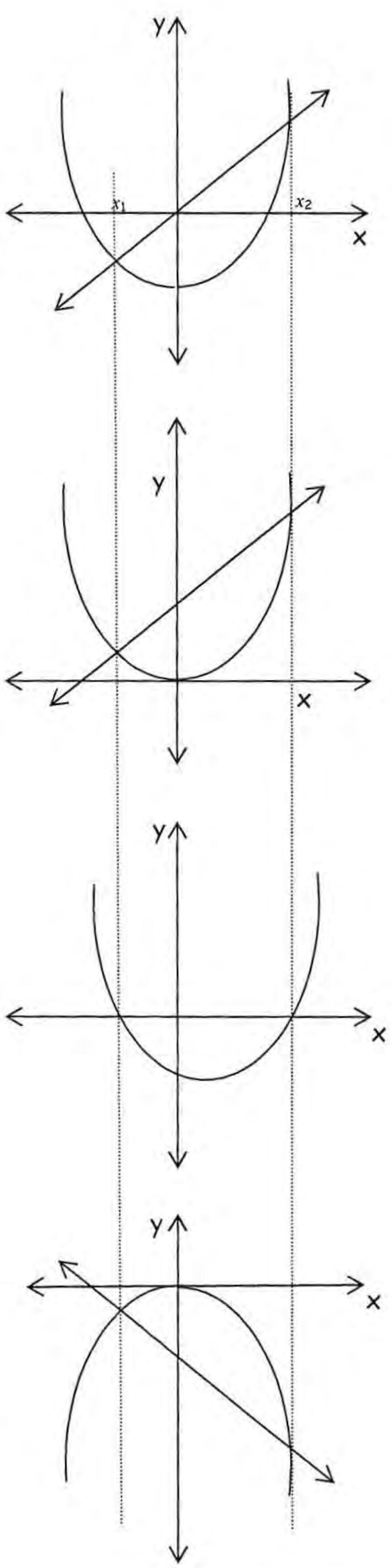
Learners should be able to interpret that each equation $f(x)=g(x)$ reduces to quadratic form from the forms $ax^2+bx+c_1=mx+c_2$ or $a_1x^2+b_1x+c_1=a_2x^2+b_2x+c_2$. The former italicised statement is more obviously a second order equation. The latter will also reduce to a second order since from the graphical depictions it can be deduced that $a_1x^2 \neq a_2x^2$ by comparing the steepness of the arms of the parabolas. It may be further implied that the simultaneous representation of two graphs produce the equation's roots that can be read from their points of intersection. The nature of these roots is also dependent on the information that is contained in the sketch, specifically the proximity of the graphs to one another. The generic notion that an equation, which is constituted of any high school algebra or trigonometry, may be graphically represented by two graphs on the same set of axes is of fundamental importance to interpreting an equation, and accommodating it cognitively.

Task sheet 3 is the sequel to this in which the alternative representations of the equation $x^2-x-1=0$, as $x^2-1=x$ or $x^2=x+1$, or $x^2-x-1=0$, or $-x^2=-x-1$ and other equivalent forms produce the same roots.

By deconstructing concepts and their associated constructs through graphical representations task sheet 4 that appears at the end of this discussion would then be deemed to be appropriate and within learners' comprehension rather than the expectation that learners are adept at these cognitive tasks prior to metacognitive analyses.

Varying graphical representations of the equation $x^2 - x - 1 = 0$

Equations that represent the graphical representations



Solve for x

$$x^2 - x - 1 = 0$$

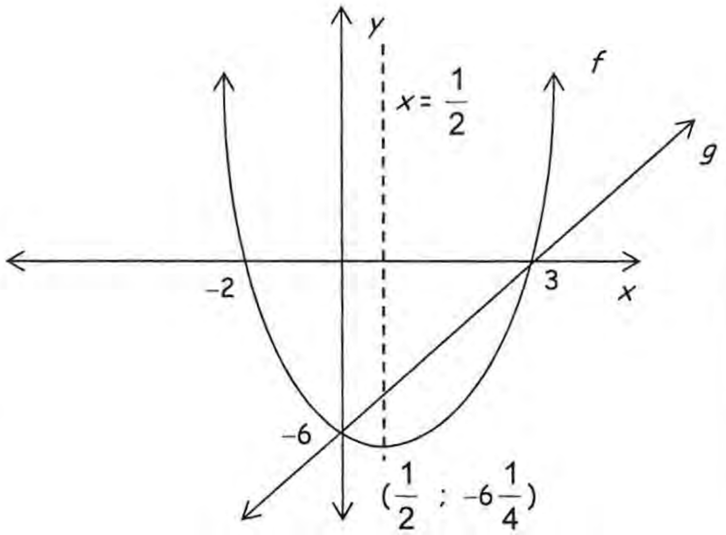
Task sheet 3

NATURE OF THE ROOTS (ZEROS)
INTERPRETIVE EXERCISE

The graphs of

$f(x) = x^2 - x - 6$ and
$g(x) = 2x - 6$ are drawn,

Use the graphs to answer the following questions. No calculations are necessary.



1. For which x value(s) is

1.1 $f(x) = 0$?
1.2 $g(x) = 0$?
1.3 $f(x) > 0$?
1.4 $g(x) > 0$?
1.5 $f(x)$ [0?
1.6 $f(x) = g(x)$?
1.7 $f(x) < g(x)$?
1.8 $f(x) . g(x) > 0$?
1.9 $f(x) . g(x)$ [0?
1.10. $\frac{f(x)}{g(x)} > 0$?
1.11 $2x - 6 = x^2 - x - 6$?
1.12 $2x = x^2 - x$?

2. For which value(s) of k will $x^2 - x + k = 0$ have

2.1 real roots?
2.2 positive roots?
2.3 roots that are opposite in sign?
2.4 two positive, unequal roots?

3. For which value(s) of p will $x^2 - x - 6 = p$ have

3.1 two positive roots?

3.2 roots that are opposite in sign?

4. For which value(s) of t will $x^2 - x - 3 = t$ have equal roots?

5. Confirm your answers to Questions 2.1 and 4 algebraically.

6. Draw on the axes above a line graph that will enable you to read off the solution to

6.1 $x^2 - x - 5 = 0$
6.2 $x^2 - 2x - 6 = 0$

7. Draw the graph of g to show clearly, once it has been moved, where

7.1 it is a tangent of f.
7.2 it produces no roots (non-real or imaginary roots) with that parabola.

Task sheet 4

5.9 Metacognitive deconstruction analyses applied to the reduction of simple trigonometric identities

A standard approach to simplifying $\dfrac{\sin(180°-x)\sec(x-360°)\tan(180+x)\cos(-x)}{\cos(90°+x)\cos ec(90°-x)}$ would be to apply the mnemonic referred to as the CAST diagram that is a familiar diagram in high school textbooks. A typical solution for this typical question, in conjunction with the CAST diagram would be as in Fig 5.13.

$$\frac{\cos(180°-x)\sec(x-360°)\tan(180+x)\cos(-x)}{\cos(90°+x)\cos ec(90°-x)}$$

$$=\frac{-\cos x}{-\sin x}.\frac{\overset{1}{\cancel{\sec x}}.\tan x.\cos x}{\underset{1}{\cancel{\sec x}}}$$

$$=\cot x.\tan x.\cos x$$

$$=\cos x$$

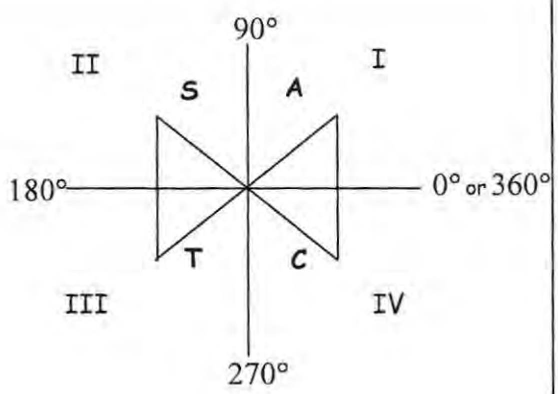


Fig. 5.13

Trigonometric simplification aided by decontextualised mnemonic

From the perspective of a routine simplification an analysis of the solution is seen as a repetition of the independent simplification of each trigonometric ratio that constitutes the complex fraction. For each simplification the CAST diagram is consulted to determine the sign of the ratios. The ratio is also adjusted on whether the $180°$ or $360°$ rule, or the $90°$ rule is applied. Typical thinking for $\cos(180°-x)$ would be to say "$180°-x$ lies in the second quadrant where $\cos$ is negative so $\cos(180°-x)$ is equal to $-\cos x$" as confirmed by the data analysis to follow this chapter.

As demonstrated in the data analyses, learners cannot explain why the identity holds for all values of x, or why it reduces in this fashion. The question therefore arises as to what value such a task holds for learners. In its presented form it serves as a routine manipulation of trigonometric ratios with the aid of an uncontextualised mnemonic. Such manipulation is out of context and devoid of real mathematics that fails to exact levels of critical cognitive mathematical consciousness about the concept.

I seek now to contrast this algorithmic approach with one that is contextualised in the graphical mathematical representation by using one of the component ratios that is part of the compound trigonometric fraction above. I qualify again that the essence of mnemonic use is not in itself a meaningless engagement once the foundational understanding has been established through reference to the authentic mathematical situation from which the real mathematics derives.

If the ratio $\cos(180° + x)$ is considered its graphical context provides visual evidence of its simplification to $-\cos x$ as is confirmed by the graphical representation in Fig. 15 below. It is also vital to the understanding that x by convention is defined as acute for such ratios. The absence of this knowledge otherwise contaminates the cognitive clarity of the reduction process because learners, in trying to locate a possible position of the angle hesitate in doing so, because by their admission, do not recognise the conventional restriction on the magnitude of x. Further to this the use of the CAST diagram as a substitute for the Cartesian context decontextualises the mathematics of the trigonometry as will be evident from the Cartesian reference system illustrated in Fig. 5.14 below that accommodates the mathematics more meaningfully and renders it more real.

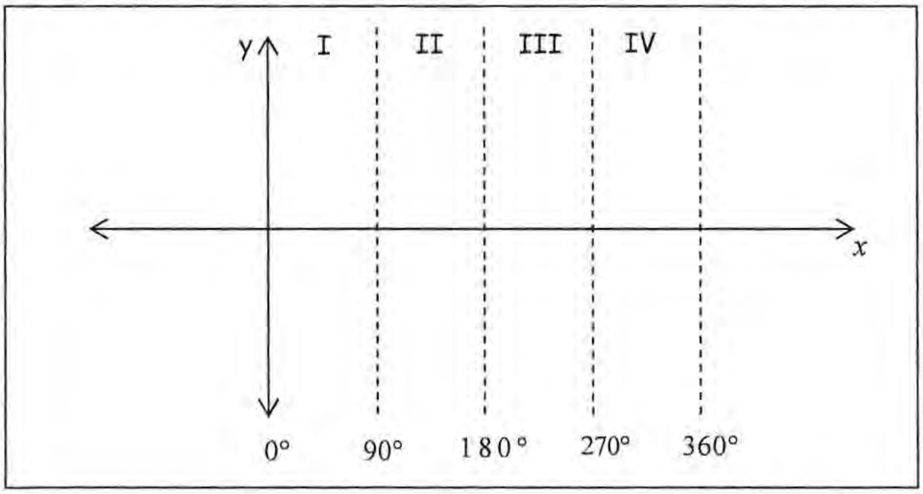


Fig. 5.14

Cartesian plane reflecting quadrant structure as an aid to graphically contextualising trigonometric simplification; a rational alternative to the CAST diagram

When a trigonometric function is superimposed on the Cartesian plane, as in Fig 5.15, the mathematical information it provides is substantially more than provided by the CAST diagram. It can be seen from the following sketch that information regarding the maxima, minima, increasing and decreasing behaviour of the function and the dependent and independent variable values for critical values of the independent variable or function values are immediately evident. The gestalt of this representation also facilitates the complete solution of $\cos x = 0$ rather than the single $90°$ solution that a calculator provides. The same argument applies for the other basic trigonometric functions.

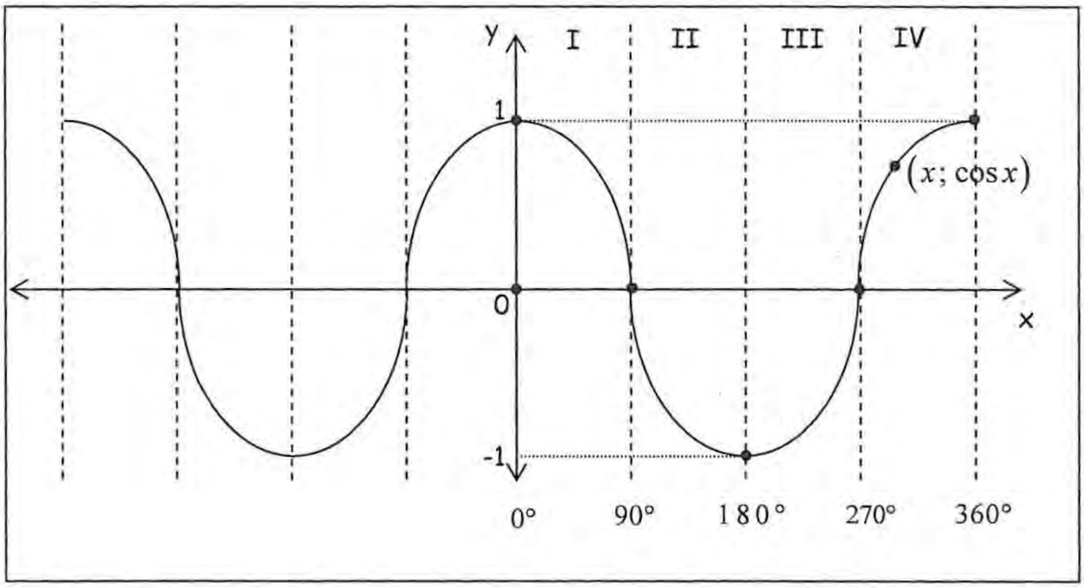


Fig. 5.15

Contextualising generic graphical detail about the function $y = \cos x, \quad x \in [-360°, 360°]$

From the graphical representation of $y = \cos x$ above, the function values of $\cos 0°$, $\cos 90°$, $\cos 180°$, $\cos 270°$, $\cos 360°$ as well as their negative equivalents are evident within the graphical context, as well as the complete solution of the equations $\cos x = -1$, $\cos x = 0$, $\cos x = 1$ for $x \in [-360°, 360°]$, all of which are potentially assimilated knowledge, likened to a cognitive imprint, if the Cartesian representation remains the reference system. The solution of inequalities of the kind $\cos x < 0$ is immediately readable from the graphical form of $y = \cos x$ where the algebraic meaning of this inequality remains cognitively remote and algebraically unattainable. Generically the ability of differentiating between an expression and an equation is synonymous with finding or simplifying ratios of the nature of $\cos 180°$ and solving equations like $\cos x = -1$ and relating these processes to the equivalence of the coordinate pairs $(180°; -1)$, $(x; \cos x)$, $(180°; \cos 180°)$. The directional implications of positive and negative angles can also be contextualised. A quick mental reference to the graph of the basic trigonometric functions provides a cognitive context and aids cognitive acuity in being able to interpret trigonometric structures and statements.

The same context serves to aid the interpretation, meaning and validity of simple trigonometric identities in a generic graphical context. This is dependent, firstly on recognition of the congruency of the cosine arc over each multiple interval of $90°$ commencing at the origin, and the resulting symmetries, and secondly on the one-to-many mappings that takes one incident of x onto multiple mappings of the function values, y. A simple example is selected for purposes of description before going on to multiple function values associated with a given independent variable. Fig. 5.16 below is used to analyse the validity of $\cos(360° - x) = \cos x$.

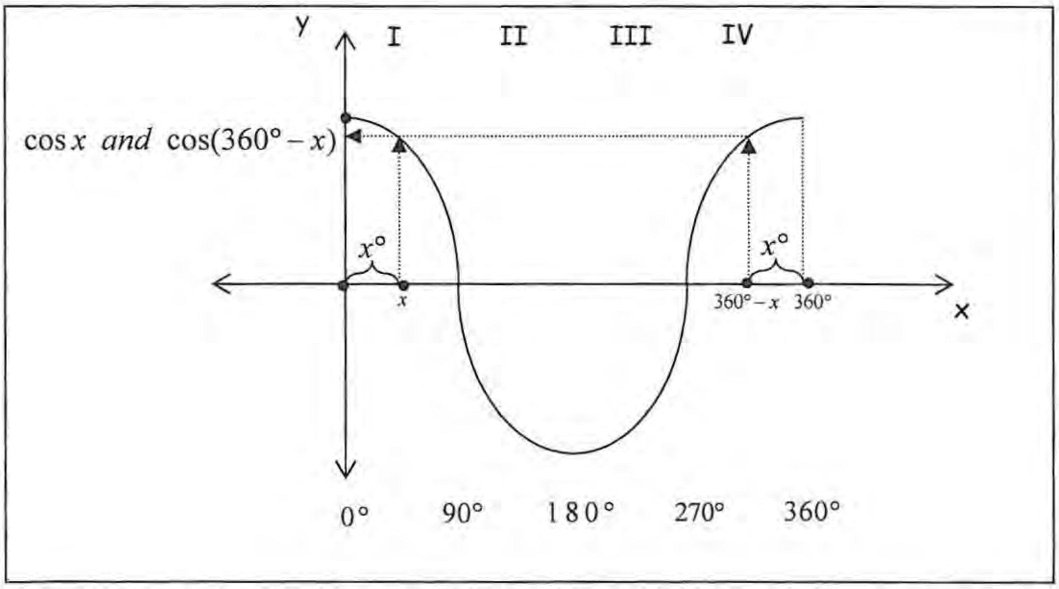


Fig. 5.16

Graphically contextualising the truth that $\cos(360° - x) = \cos x$

Since x is defined as acute, the area beneath the cosine curve $x°$ from the origin is congruent to the area beneath the curve $x°$ backward of $360°$. This implies that the function value at x is $\cos x$ (in quadrant I) and the function value at $360°-x$ (quadrant IV) is $\cos(360°-x)$. But it is the symmetry that ensures that the function values coincide, and hence $\cos(360°-x)=\cos x$. A similar graphical representation establishes the truth of the identity $\sin x=\cos(90°-x)$, for $0°<x<90°$, in Fig. 5.17 by again using congruency and symmetry arguments.

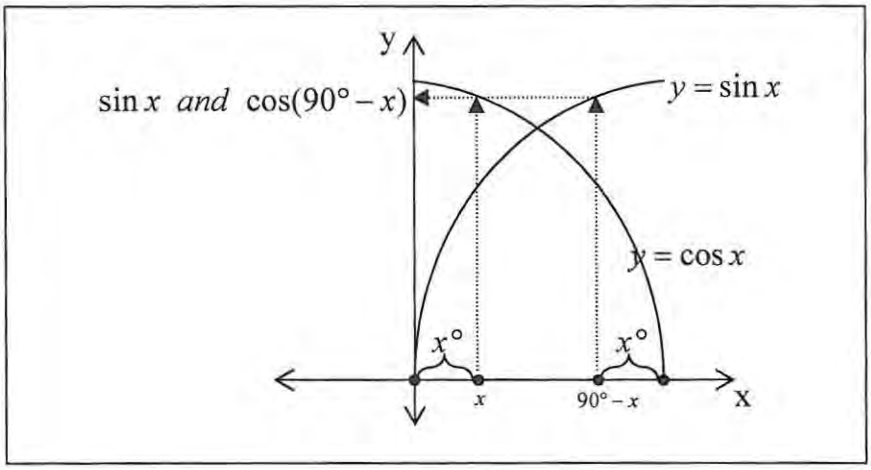


Fig. 5.17

Extending graphical contextualisation to trigonometric co-function reduction formulae

Similarly, by sketching $y=\cos x$ and $y=\sin x$ on the same axes for $x\in[0°, 360°]$ the cognitive awareness of the truth of $\cos(270°+x)=\cos(90°-x)=\sin x$ is able to be established. The visual impact has a sense of being more powerful than calculator substitutions for a chosen x. It is therefore inconsistent to omit the derivation and use of reduction formulae for trigonometric ratios of $270°\pm x$ as recorded in the National Curriculum Statement for Mathematics (Department of Education, 2003a: 57). The question is raised as to the rationale for the omission, as it was in earlier syllabi, since viewed from a perspective of graphical literacy it has a natural place alongside the other reduction formulae. If it is suspected that it is too difficult a concept to grapple with, this line of thinking is contrary to trying to generate perceptions among learners about a mathematical gestalt. Fig. 5.18 provides visual graphical representation confirmation for the following individual cases.

$$\cos(-x) = \cos x$$
$$\cos(180° - x) = -\cos x$$
$$\cos(180° + x) = -\cos x$$
$$\cos(-180° + x) = -\cos x$$
$$\cos(-180° - x) = -\cos x$$

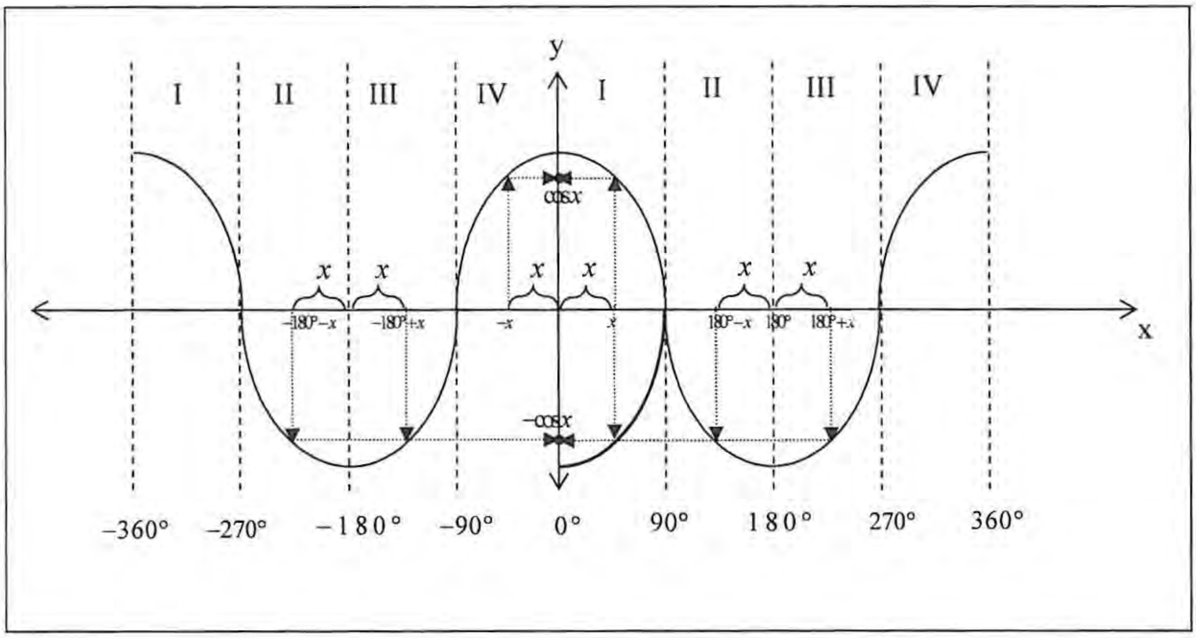


Fig. 5.18

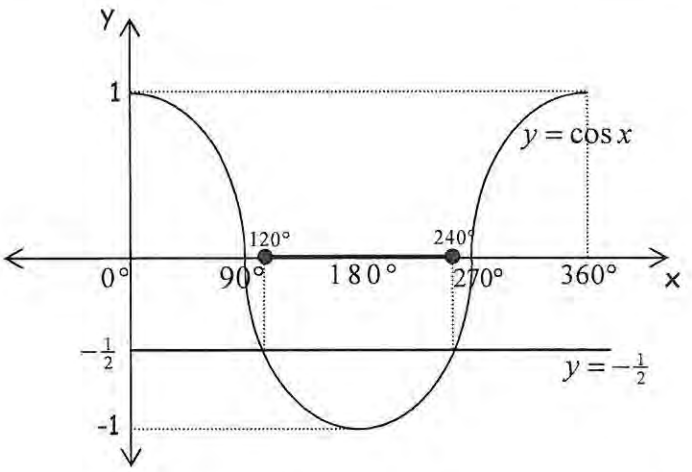
Contextual graphical congruency and symmetry for the establishment of the truth of multiple trigonometric reduction formulae identities of the cosine function

It would seem sounder mathematically for this contextualisation to add more cognitive value than to say that "$\cos(180° + x) = -\cos x$ because $180° + x$ is in the third quadrant and cos is negative there." It is this kind of utterance that is deemed to be rote and instrumental in Skempian terms, and that denies cognitive access to the real contextual mathematics.

5.10 Reflections on graphical procedures for understanding trigonometric compound formulae

In natural progression learners would be required to prove through application of the derived formula for $\cos(A - B) = \cos A \cos B + \sin A \sin B$, that $\cos(180° + x) = -\cos x$. It is in this more advanced stratum that obliterative subsumption may be valid in terms of diminishing opaqueness of lower order constructs, and that the function of proof is free of accumulated cognitive deficits.

In a deeper cognitive stratum, solving for $\cos(180°+x)=0{,}5$ for $x\in[-360°,360°]$ should be facilitated by familiarity with the simplified form of $\cos(180°+x)$ being $-\cos x$, in turn allowing $\cos(180°+x)=0{,}5$ to be transformed to $-\cos x=0{,}5$. Learners are better at ease being able to recognise cognitive templates that accommodate the solution through regular application of graphical interpretations. $-\cos x=0{,}5$ is further interpreted as $\cos x=-0{,}5$ and then interpreted as determining the x values where the functions $y=\cos x$ and $y=-0{,}5$ intersect on the specified interval. It is the graphical recontextualisation that further accommodates an understanding of the general solution of the equation, and further the inequality, $\cos(180°+x)\geq 0{,}5$ being obtainable from $\cos x\leq -0{,}5$ as depicted in Fig. 5.19.



$\cos x\leq -0{,}5\quad x\in[0°,360°]\quad when\quad 120°<x<240°$

Fig. 5.19

Extending the contextuality of the Cartesian plane for the solution of simple trigonometric inequalities

5.11 Summary

Freirian dialogic praxis that embodies theory and reflective practice and Habermasian communicative action that recognises that teaching and learning are complementary processes in open communication systems promote dialectical modes of engagement. Abductive inference, as a thought operation, frames the theoretical analyses that seek to position metacognitive reflections in moving toward metamathematics.

In the preceding sections I have tried to demonstrate the value that graphical representations have for the interpretation of algebraic and trigonometric structures of the senior high school mathematics curriculum that feature particularly as part of Learning Outcome 2, 'Functions and Algebra' of the National Curriculum Statement for Mathematics (Department of Education, 2003) and its assessment standards, and, more particularly how mathematical meaning can be attached to the Critical Outcomes 'to communicate effectively using visual, symbolic and/or language skills in various modes' and 'to collect, analyse, organise and critically evaluate data' where last reference to data need not be confined to statistical data, but would incorporate graphical data that reflects algebraic forms.

The aim of this analysis has been to illustrate, that a reversal of the order of what has been observed as standard or typical sequences of teaching mathematical constructs, is in line with abstractual ordering that further aligns with increasing taxonomic orders of difficulty. The abstractual order of constructs when increasing from opaque to transparent recognises the visual, concrete value of graphical representations that impacts cognitive imagery and that is claimed to facilitate understanding.

In this mode of deconstruction the covert subconstructs of obliteratively subsumed concepts are visually manifested and become available to learners for description, and for recognising the derivative components of higher order concepts. Description of mathematical reality leads to a critical consciousness that promotes communicative action which in turn deepens conceptual acuity and closure. In being able to articulate mathematically, learners are seen to be empowered through the accessibility of mathematics. It is in this empowerment and accessibility that personal and social justices are recognised.

The following chapter is devoted to an in-depth analysis of what constitutes metacognitive reflection in terms of the preparation of teaching incidents. It is deliberate about reversing the abstractual order inherent in mathematical tasks by elevating the graphical representation as primary in terms of delivery of content. The chapter further demonstrates the applicability of this modus operandi to different components of the mathematics curriculum.

VI

Metacognitive reflection

'... a demanding, prudent, "experimental" attitude is necessary; at every moment, step by step, one must confront what one is thinking and saying with what one is doing, with what one is.'

Foucault (1984: 374)

6.1 Introduction

Metacognition is that personal understanding about what it is we know and the assurance we have as to what makes us know how we know. It entails an awareness of our cognitive processes. Colman (2001: 443) defines metacognition as 'knowledge and beliefs about one's own cognition...' In mathematics the need to be cognitively aware and engaged while doing mathematics forms a major part of this work. Without it, I argue that the ability to engage in communicative action cannot take place, and without mathematical discourse we cannot make ourselves understood in mathematical conversation. The conversation in turn feeds one's metacognition about mathematical thinking.

6.2 Metacognition in relation to mathematical tasks

Retroduction is a critical realist mode of inference. Its fundamental thought operation is defined as '... a description and analysis of concrete phenomena to reconstruct the basic conditions for these phenomena to be what they are. By way of thought operations and counterfactual thinking to argue towards counterfactual conditions.' (Danermark, 2002: 80). Counterfactual thinking emerges in this chapter as metacognitive reflection and hermeneutic endeavour that helps me search for the basic conditions that promote mathematical learning. Retroduction in this work features as a preliminary task that aids the preparation of teaching content and strategy that has influenced the development of the research coding instrument. The concrete phenomena are learners' manifested discourses in episodes of communicative action that is constituted by classroom interaction, and the textual artefacts produced by learners in their attempts at describing their engagement in mathematical tasks.

There is much mathematics education research literature that is characterised by research styles that are based on the observation of educational processes (Brüner, 1966; Burton, 1999; Cobb et al, 1999; Dreyfus 1990; Ernest, 1984a; Glasersfeld, 1990; McNeill, 2000; Schoenfeld, 1988; Sfard 2000a; Skemp, 1976; Steffe and Thompson 2000b; Tall, 1999; Vinner, 1992). By this is meant that investigations are made with the research subjects being objects of study; that their performance is described and assessed and conjectures are offered as to what transpires in acts of learning. I intentionally cast myself as participant-researcher because I find that I am as much a learner in the teaching-learning environment as I am a teacher. I regard myself as a research subject as I examine my own cognition as it relates to a rationale for the approaches to teaching and the development of the research instrument. My metacognitive practices constitute not only what I do and say but also what I think and what acts as a catalyst to thinking, both for me and my learners. In these terms this work deviates from traditional approaches in that it locates itself, as far as possible, in the cognitive domain of teacher and learner in order to fathom the conditions that facilitate, promote and sustain mathematical learning. Papert (1980) describes how his affinity with gears provided for him valuable cognitive connections when he first encountered linear equations in two variables. Pimm (1987) makes the point too that teachers' personal images of mathematics are valuable. This metacognitive slant has brought me to the point of wanting to engage in and be engaged with this investigation. What learners say and do as a result of how they think has impacted my deliberations about how I teach. My relational ontology as described in the Introduction embeds itself in my work as teacher and researcher. I value that we all have an innate intelligence that, in my experience, is often dormant or unsolicited amongst learners (of mathematics). It is therefore understood that a mathematical critical consciousness often needs to be awakened so that all aspirant learners can enjoy the benefits of equal opportunity, certainly through equal accessibility to the right to be cognitively challenged and to learn to challenge one's own cognition, or at very least engage in mathematical monologues that are critical to a degree. An attempt at making mathematics an understandable rather than an esoteric experience is a fundamental purpose of this work, desirable for all learners but particularly those whose historical experiences are ones of separateness, isolation and disregard; learners whose rights to mathematical knowledge have been forcibly denied in the past and whose rights now are denied through transformational inertias and the unavailability of teaching expertise.

Pimm (1987) writes convincingly of the 'language of mathematics' and its value as both medium and message (ibid: xvii) in classrooms. Mathematics in British classrooms characterises the context of his work and pertains predominantly therefore to situations where English is the language of learning and teaching. His work therefore invites discourse variants if it is to pertain to multi-lingual classrooms.

Typical modes of communicating are through the spoken and written forms of language. The abilities to communicate incorporate, inter alia, spelling, pronunciation, syntax and structure. Competences in these abilities reflect in one's ability to communicate fluently and clearly in different contexts. One's communicative competence may therefore differ in an informal as opposed to a formal context. Stubbs (1980) cited in Pimm (1987) makes the point that communicative competence in whatever form needs to have a *genuine communicative purpose* (italics mine). From a mathematics perspective, Thom (1973) highlights an absence of a communicative purpose in recognising that the central problem in mathematics education is that the construction of mathematical rigour is primary to the construction of mathematical meaning. My experience as a teacher and inservice teacher of mathematics in secondary school classrooms is that the reality is that mathematics is 'done' after instruction on how to do it. Two decades on from Thom's observation I feel the status quo remains. There is an absence of meaningful communication about the mathematics at hand between learner(s) and teacher, and between learners themselves. I make the point elsewhere that another fundamental reason for an absence of communicative competence is because much mathematics teaching is insufficiently characterised by teachers' metacognitive reflections on firstly the mathematical content, secondly the effective communication of that content and thirdly the purpose of the communication. In short this constitutes a failure to recognise that mathematics is both medium and message.

In my particular context of teaching isiXhosa learners as part of a multilingual group, the added impediment of learning in a second language, compounds the absence of meaning referred to above. This is more so the case if one regards mathematics as a language in its own right.

I want to dwell a while on the competences related to the written and spoken forms of language and how I see these competences differently manifesting levels of cognition and understanding.

In my work with learners and teachers of mathematics I have found that if asked to explicate what it is they are doing the request often meets with an incoherent attempt, if anything at all. I have thought that this incoherence or mutism is attributable to several factors ranging from the apprehension at having to articulate a verbal response, the uncertainty of the answer, and the impediment of insufficient command of the language of teaching and learning to express an answer. The nature of the oral response is that it requires an immediate formulation of the cognitive functions, speaking on one's feet as it were. Once formulated the thoughts need to be expressed. There is insecurity in knowing whether the response is correct, because if incorrect it instantly transfers negative information on how much the speaker knows. The readiness and spontaneity of the response, of course, also has to do with the one's perception of self and one's levels of confidence interlinked with one's content knowledge and language competence.

The written response, on the other hand, permits greater freedom in being able to formulate one's thoughts in response to questions. There is also, in the typical observation, that once something is committed in writing the assessment of it is final, and so once written the associated anxiety is a thing of the past; a sense of 'there's no more I can do about it'. It is this cognitive awareness that may be instrumental in allowing the respondent to produce 'a best attempt' in written attempts at communicating about mathematics. If unprepared, for a question requiring a written response, one can machinate or contemplate, within reason, a meaningful answer. My experience with learners has shown that requesting written descriptive or explanatory responses about what it is they are doing also meets with resistance, for all the same reasons as indicated above. From experience the likelihood of obtaining a meaningful response has seemed more likely when the response has been written.

These written responses immediately become available as research artefacts that provide a point of reference and departure for dialectical discourses, for gaining insight into acquired levels of understanding and for correction.

A further dimension to be noted is that a relationship of trust was established between those I taught and me. Learners have been responsive, albeit, under some level of stress because an understanding was reached that as much as their attendance and agreed participation indicated their desire to achieve a mathematics (HG) pass, it was my hope that I would learn how they related to mathematics and understood it so that I could be more effective in the art and practice of teaching mathematics.

The written response, if prepared for examination conditions, may reflect a rote response in which case it is not indicative of any degree of conceptual depth. In school mathematics, I believe the same applies. The solving of an equation, simplification of an expression and sketching of a graph are skills that may be achieved algorithmically. It does not necessarily reflect any depth of understanding of the structures at hand or that sustainable knowledge has been acquired.

An unprepared written response in my experience has greater value in terms of ascertaining the degree of acquired meaning that learners attach to concepts. It is characterised by an immediacy that is more indicative of the cognitive functions than a prepared response.

Contrasted with this, the nature of oral responses within this research context is such that they produce very little meaningful dialogue simply because a mathematics register has not been developed in learners.[18] This together with the insecurity of not being sure about an appropriate answer, and the second language impediment make interviewing an extremely difficult data collection mode and hence the decision to accumulate written artefacts of learners' accounts of their engagement with mathematical tasks.

[18] I cannot say whether this is a general phenomenon. This investigative case study may be catalytic in stimulating similar investigations on which to build a theoretical claim.

From these considerations I come to conclude, with a degree of intuition and experiential personal knowledge, that effective and meaningful oral responses to required explanations can be good indicators of the speaker's attainment of conceptual depth more so than will be revealed by a premeditated written account. I feel the ability to speak a formulated thought, impromptu, is a dynamic function of all the competences discussed above. The unprepared written account that amounts to learners illustrating, describing or explaining their understanding of the mathematics pertinent to each lesson, on completion of the lesson, is the compromise between the two.

The ideal would therefore be for learners and teachers to reach a level of linguistic competence in both the language of learning and teaching and the language of mathematics. This, I believe, will help to sustain meaningful dialogue and discussion in mathematics classrooms. To obtain linguistic competences in one's mother tongue is a function of one's social milieu. Mathematics language competence acquisition, however, is dependent on the classroom environment and strategies which that environment affords learners.

The learning pathway that best accommodates the favourable development of mathematical understanding, language and register is a feature of this research. It is taken as read that the acquisition of the sophisticated language of mathematics with its tacit abstractions, the sometimes covert information in its notation and its own embodied meanings therefore presents a greater obstacle for learners whose language of learning and teaching is the learner's second language.

The current climate of transformation in the South African education system brings with it challenges that in this regard seem premature, certainly for mathematics if not as intensely so for other subjects. The increasing call for the nature of mathematics classroom activities to be investigative and exploratory aligns with the philosophies of outcomes based educational practices. Such practices call for an increasingly greater facilitatory role of teachers in social constructivist classrooms (as elucidated in Chapter 2 entitled 'Mathematics in Crisis') where debate and discussions are vehicles of knowledge acquisition. It is reasonable to expect learners to participate profitably in these endeavours with the assumption that both the language of learning and teaching, and the language and vocabulary of the curriculum component, satisfactorily permit such collaborative construction of knowledge. A further assumption is that the facilitator is equipped with the necessary skills to promote such learning and that such facilitators have the required depth of knowledge to stimulate and elicit the desired learning outcomes, and further to clarify learners' misconceptions. It may be that the inertia, as I assess it, is related to these two assumptions as invalid premises upon which to base a cognitively higher order, demanding expectation of collaborative mathematical discourses for learning mathematics. I develop this thought further in the following paragraph.

The crisis in mathematics education as described in Chapter 2 forces the question of whether the higher order cognitive skills of analysis, synthesis and evaluation of information associated with the ability to 'discuss' can be expected from learners when learning environments are poorly resourced and lacking in mathematics subject expertise. To analyse, synthesise and evaluate information requires not only the possession of the appropriate language but also the ability to interrogate one's own cognition, and an efficacy to communicate with some confidence the intended messages. Again the point is made that learning in a second language must compound the difficulties associated with radical and social constructivist, and deliberate didactive processes. Irrespective of the form it takes learning requires some form of knowledge transfer. Self-learning for example would require one to read with comprehension in order to learn from the author, who is the transferor of knowledge, before any degree of radically constructed knowledge is gained. I believe increasingly that mathematics is unique in this regard. It has its own sophisticated language through which its own meaning must be conveyed only once the abstract algebraic veneers have been planed away. It needs experts to deliver these meanings and facilitate discussion on it. The national performance data for mathematics in South Africa, detailed in the chapter 'Mathematics in Crisis', confirm the weakness of the subject. My suggestion is that such continuing unacceptably poor national and provincial performances in mathematics are related to the factors raised in this paragraph.

If mathematics learners have a history of deprivation and content deficit in their learning situations the question arises as to how to address these adversities in order to promote the emergence of a meaningful mathematical learning culture. I suggest that mathematics teaching is experiencing incapacitation owing to the cart going before the horse. It seems as if we are losing sight of graded learning. It cannot be disputed that the gradations that characterise the range from simple to complex are essential to the learning of mathematics, together with its cumulative development of prior concepts. From a mathematical perspective there is in my view a reversal of the sequence of ordered complexity that has been expounded in the Chapter 5. From a pedagogical point of view there is a similar pattern in the expectations of having to 'discover mathematics' without the necessary inputs that facilitate these discoveries or inculcate a language or vocabulary in learners that foster the elicitation of these discoveries. Furthermore the cognitive skills required for dialectical discourses are questionably in place for the discoveries to be expressed, discussed and explained by learners.

I examine below, firstly the difference between the mathematical, and, secondly the pedagogical perspectives and its implications and try to show how these have led to and justify the development of the research instrument.

Constructivist learning cannot be effective in the absence of communication, more specifically communication that is constituted by discussion. Discussion in turn requires acquired radical constructivist knowledge (as formulated according to the rationale outlined earlier), which in turn presupposes the appropriate language and register with which to articulate one's, evaluation, analysis, synthesis and explanation of that knowledge.

The art of discourse assumes the incremental development of all the skills and knowledge that constitute it. The skill of description would therefore appear to be an essential cognitive forerunner to episodes of discussion. Practically speaking, in a mathematical context, I regard it more sensible to describe an occurrence before having to discuss, prove or explain it. Through asking learners for thorough or thick descriptions of mathematical phenomena or processes one endeavours to equip learners with the vocabulary that ultimately constitutes the cognitive foundation, infrastructure or schema for the accommodation and assimilation of concepts (Vygotsky, 1986; Piaget, 1980a). These radically constructed schemas serve as inception points for mathematical discourses. These schemas as they manifest in written endeavours, serve also to encourage learners to search cognitively for their assigned meanings to the task at hand. A cognitive schema and an acquired descriptive vocabulary thus provide a platform from which to engage in meaningful discourses.

The design of the teaching strategy therefore hinges on these two fundamental understandings. One, that cognisance is taken firstly, of the inherent gradations on the concrete to abstract continuum as they pertain to mathematical structures, and secondly, to the fact that describing mathematical structures should precede the task of discussing or explaining them, for both teachers and learners.

This work is, indubitably, integrally related to language as a means of learning and teaching and the language of mathematics and the communicative processes and styles that breathe synergic life into them. It probes another dimension though. In its attempts to use the surface structures[19] of mathematical and algebraic language to explore the deeper levels of meaning that learners attribute to mathematical constructs, it seeks to unleash a cognitive self-awareness amongst learners about how they recognise and what they do with mathematical constructs.

The prefix 'meta' is defined in terms of contexts pertinent to its use by the following prepositions; 'beyond', 'above', 'behind', 'between', 'after'. It also has the connotation of 'derivative' (Cassell's dictionary, 1998: 693). This sheds some light on the intended meanings in my use of the ideas of metateaching and metalearning, and in the accepted terms like metamathematics, metalanguage, metalinguistics and metacognition. In each of these notions 'meta' takes on the meanings associated with the prepositions listed above. In the broad sense though, it seems to convey the amalgam of ideas, concepts and knowledge that often resides behind the surface structures of verbally or symbolically manifest knowledge. In a mathematical sense 'beyond', 'above', 'behind', 'between', 'after' in the context of this methodology describes a wealth of knowledge that constitutes all that lies behind the surface structure of a concept. The knowledge that is 'beyond' or 'above' mathematics as a structural language refers to that realm of knowledge that philosophers of

[19] 'Surface structure' is that level at which the observation is made. I regard it as the most superficial level devoid of any underlying subliminal cognitive considerations. Pimm (1987: 165) states that the prevalent language of teachers at this level would include statements like "Cross multiply", "Always do to the top what you do to the bottom" and "Turn it upside down and multiply", without the reasoning for applying such structural rules for purposes of transforming mathematical objects.

mathematics indulge to study and prove theoretical mathematical notions and the logical systems that accommodate them. 'Behind', 'between' and 'after' may reflect the subliminal processes that are assumed in the algorithmic deriving of the solution of an equation or inequality. As an example of this, I resort to identifying possible metacognitive monologues that may exist 'behind, between and after' the steps in arriving at the solution to the following inequality.

$$
\begin{aligned}
& 0,5^x > 5 \\
\therefore \quad & \log 0,5^x > \log 5 \\
\therefore \quad & x < \frac{\log 5}{\log 0,5} \\
\therefore \quad & x < -2,3219... \\
\therefore \quad & x < -\log_2 5
\end{aligned}
$$

There is intentionally no instruction that precedes this task. I concluded after some deliberation that omitting the instruction 'Solve for x' should encourage a monologue like 'What is this and what must I do with it?' These kinds of questions are good in themselves and they indicate an engagement with the mathematical construct. If the mathematical language of the inequality is read correctly the task is overt and subvocally understood, as indicated in the first preferred response below.

An expected ideal positive response, or an equivalent, to this task would be firstly for learners to express what is required of them, something in the line of...
... 'We must find the values of x so that after having multiplied 0,5 by itself that many times the result will (begin to) exceed a value of 5.'

The response to the second line of the inequality may be something of the nature of... 'Because the solution will not be an obvious convenient rational, a strategy would be to take the logarithm of both sides.' A reflective cognitive picture of this may look as illustrated in Fig. 6.1 below.

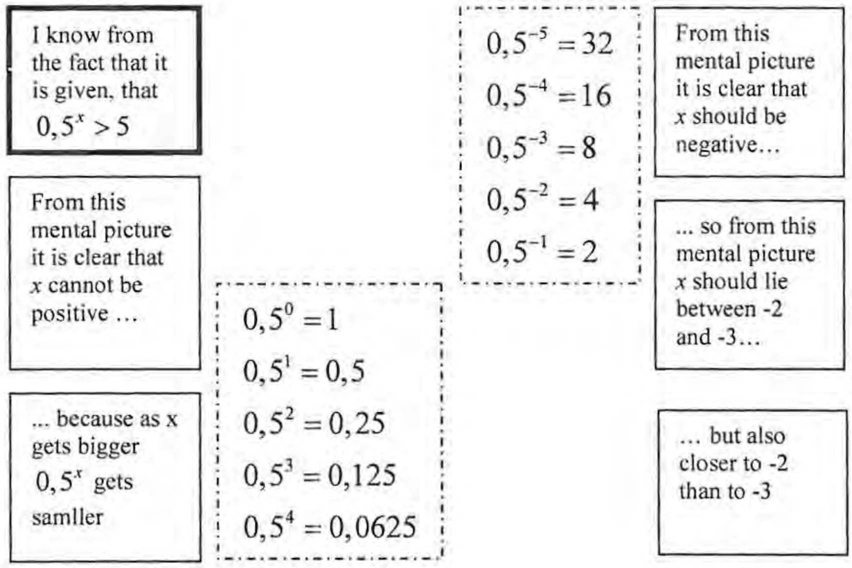


Fig. 6.1

An imagined cognitive reflection about the statement $0,5^x > 5$ prompted by hermeneutic endeavour

The metamorphosis of the syntax in the third line depends on the recognition and understanding that $\log 0,5$ is covertly negative and so a sound metacognitive suggestion may amount to...

... 'Since $\log 0,5$ is negative the inequality would need to reverse direction if both sides were divided by it.'

The 'derivative' connotation of 'meta' most aptly suggests the point I try to make here. For every concept or procedure there is associated any number of derivative constructs or procedures that sum to its composition. This is emphasised by referring again to the desired suggestive metacognitive responses for the procedural steps in the example of the inequality above. Inherent in this exposition are further underlying interpretations and understandings in the analysis.

There is an acceptance of the fact that $0,5^x$ suggests repeated multiplication. There is also the assumption that learners know that $\log_n A^x = x \log_n A$ preferably by proof and not by rote memorisation. It is also an assumption at this level of operation that learners can justify why multiplication throughout an inequality by a negative constant will cause a directional change in the inequality. Recognising the negativity of $\log 0,5$ depends either on the algebraic manipulation and transformation of $\log 0,5$

into $-\log 2$ or for learners to see the graphical representation of $y=\log_n x$ in their mind's eye, as in Fig. 6.2 regard the latter technique as efficient, contextualised and a cognitively sound mental image. Lastly at the algebraic level to test whether the solution is correct will produce further questions dependent on whether the full calculator display or a rounded off version is used to test the validity of the solution.

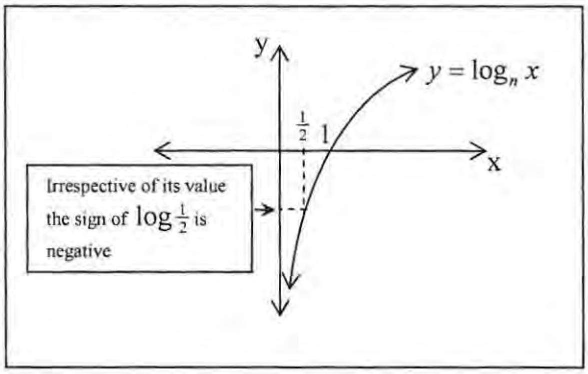


Fig. 6.2
A potential cognitive image that confirms that $\log\frac{1}{2}$ is negative
and which exposes its covert negative sign

I want to suggest that anything less than this level of analysis infers that learners of mathematics learn by rote rather than by understanding. I make a further point. One cannot appreciate the deeper nuances of mathematics unless one is taught to appreciate them. It is the responsibility of teachers to inculcate habits that structural mathematical forms should elicit from learners. This confirms my suggestion that teachers need to engage in the metacognitive deconstruction of concepts and procedures before going about the delivery of what would otherwise simply amount to surface structure exposition (Pimm, 1987). The notions of 'instrumental' and 'relational' understanding as defined in Skemp (1976) relate also to the purpose of teaching. The former associates with teaching that is in the mould of the exposition of surface structures. In this case the surface area of a circle is defined as $A=\pi r^2$ and used as a formula for appropriate calculations. To demonstrate a relational understanding of the formula a learner would be able to explain the development of the formula. It is both learner and teacher who need to be acquainted therefore with these two styles of understanding. Skemp (ibid.) also speaks about a 'direct approach' as opposed to a 'theory based approach' to teaching. The former concerns the use of real situations and the latter would allow, through various strategies, the gradual and systematic development of a new concept. I sense that the 'direct approach' has more to do with the recall and use of the formulae while the 'theory based approach' seeks through a first principles approach to equip learners with the cognitive schemas that promote personal cognitive conversancy, or the radical constructivist learning in Von Glasersfeld (1990). This research rests in aspirations toward relational understanding through 'theory based' instruction that promotes radical constructivist learning; all of which work toward social constructivist classroom milieus. Its aim is for teacher and learner to gain an overview or acquire a gestalt perception of mathematics where the whole is greater than the sum of its parts.

These algebraic procedures are solidly dependent on being able to access the cognitive domain for the language that has allowed the construction of the concept and for accessing the Piagetian schema that accommodates and assimilates concept growth and refinement. In Vygotsky (1986) the point made is that the expression of thought is dependent on the words through which they come into existence. The metacognitive deconstruction and analysis of the exponential inequality serves as proof of this Vygotskian claim. It bears further mention that this level of deconstruction analysis is more characterised by its abstraction than its concreteness. To illustrate this I proceed to what I regard as a more concrete approach to the solution of the exponential inequality, $0,5^x > 5$.

In terms of the senses it may be useful to recognise that as auditory functions are to visual functions so abstractions are to concretisations. If teaching transpires from a perspective of surface structure then receipt of the communicative content is through hearing and associating auditory signals with existing schema. These signals convert the cognitive language into accessible forms established as schemas by learners. It is left for learners to generate visual images associated with the auditory input. These processes are akin to the associative or behaviouristic transfer of knowledge associated with Thorndike (1922) and Skinner (1953). The transfer of knowledge that is entirely dependent on the receipt and conversion of auditory signals reinforces the notion of the 'black box' that holds the mysteries of cognition between stimulus and response. In short teacher-talk is a fundamental component of the dynamic of the learning in mathematics classrooms provided it takes into consideration an attempt to make the context of the underlying cognitive frameworks available to learners through the use of the appropriate mathematical register aided by visual confirmatory images.

In the mathematics classroom the incorporation of a visual component of a concept diminishes the austerity and notoriety of a behaviouristic learning paradigm simply because I believe it promotes cognitive engagement through the provision of a concrete version of abstractions; abstractions that are otherwise amplified by auditory-only inputs. It is my experience that it is the cognitive engagements aided by visual stimuli that lift the rote nature of learning into the cognitive.

I attempt now to translate this theoretical claim into a practicable interpretation continuing to consider the inequality $0,5^x > 5$.

If learners are encouraged to recognise the inequality as the graphical representation of the functions $f(x) = 0,5^x$ and $g(x) = 5$ on the same set of axes the problem becomes more concrete and therefore more easily interpretable, providing an alternative interpretation of real mathematics. The contrast sees real as pertaining to a mathematics context as opposed to real signifying application to real world phenomena. There is no intent here whatsoever to diminish the importance or relevance of real applications, but the observation that applications are facilitated by authentic conceptual closure where concepts are cognitively embedded first in their mathematical contexts. In these terms this alternative interpretation of the expressed problem is constituted and firstly contextualised in the Cartesian plane. The inequality now would need to be interpreted as finding all the x values where the exponential function lies above the constant function. The visual impact of the graphical form serves to concretise and contextualise the problem. This concurs with the sentiments of Gravemeijer (1998) who calls for subject matter that is to be mathematised to be experientially real for learners, but qualifies that experiential reality may exist in the mathematics itself and not only in real life contexts.

The Cartesian context would seem to demystify what otherwise would remain algebraically abstract and cognitively remote. This strategy of contextualising a problem graphically does not mean that there is less need to acquire language with which to articulate the processes that characterise the solution. It is the latter that demands prudent analysis and preparation as part of the teacher's responsibility of teaching after the teacher's metacognitive deconstruction of concepts. I return now to the graphical representation.

The graphical depiction of $0,5^x > 5$ in Fig. 6.3 below illustrates the power of contextualising the problem on the Cartesian plane. It further highlights the power of visualising the solution before it is algorithmically established and serves to illustrate the etymological derivation of associated concepts. It further contextualises the solution and brings into focus the synonymy of the set-builder style solution and the number line representation of the solution. All equations and inequalities pertinent to the South African National Mathematics Curriculum Statement for Further Education and Training (DOE, 2003a) lend themselves to this visual analysis. Another advantage of this approach would be the ability to see whether the solution is reasonable and if in fact it exists, and if so, if it is valid.

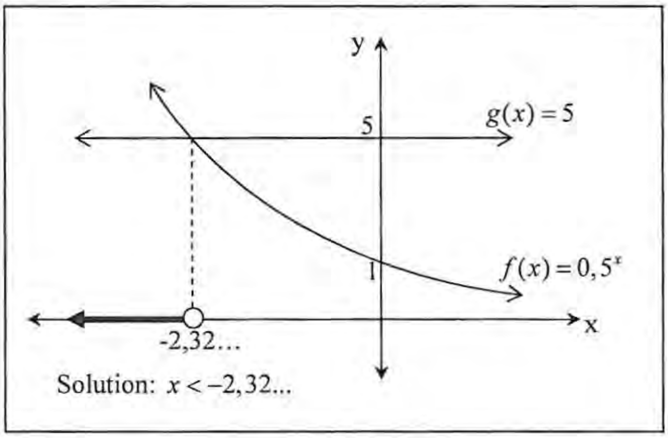


Fig. 6.3

A Cartesian context for $0,5^x > 5$ that confirms all but the arithmetic accuracy of its solution

I return now to consider a first encounter with the inequality $0,5^x > 5$. At the outset the point was made that omitting an instruction for the task served to encourage learners to engage metacognitively by some form of personal, mental dialogue as to what the object communicated to them. In its algebraic form the suggestion is that 'I need to find the number of times I need to multiply $0,5$ by itself until the product first exceeds 5'. This mode of thinking constitutes a viable interpretation in terms of the overt algebraic construct but because of $0,5^x$ is a decreasing function the final solution, $x<-2,32...$ confounds the notion of 'how many times do I multiply $0,5$ by itself...' because in communicating the solution I would need to offer an answer of 'less than $-2,32...$ times.' A reformulation of that thought might be something of the nature of 'To what power(s) must I raise 0,5 so that its value exceeds 5?' since this has more explanatory power in its context, and, in terms of the conventions, rules and definitions of exponents. This reformulation brings the conscious personal dialogue in line with a more rigorous mathematical interpretation. A point more pertinent to the research methodology however, has to do with the interpretation of the task in the more concrete graphical context where covert misconceptions are transcended. In a final analysis the agreeing that in 'doing' $0,5^x > 5$ one is trying to ascertain the values of x for which the decreasing function $f(x) = 0,5^x$ lies above the constant function $g(x) = 5$ for whatever practical application such a task may arise.

Pimm (1987: 96) is quoted as saying '... graphs are symbolic artefacts, and pupils have to learn to read and interpret them as much as other mathematical symbols'. I have come to understand, in my Western Cape mathematics-teaching context, that graphs are regarded as a topic in secondary mathematics teaching separate from rather than integrated with the algebra that it realistically can depict. Experiential evidence that supports this claim is inherent in the following anecdotal account. When learners are asked simply to describe what it is they see when looking at a set of axes in the x-y plane they are hard pressed to produce even simple descriptions of the representations in Fig. 6.4.

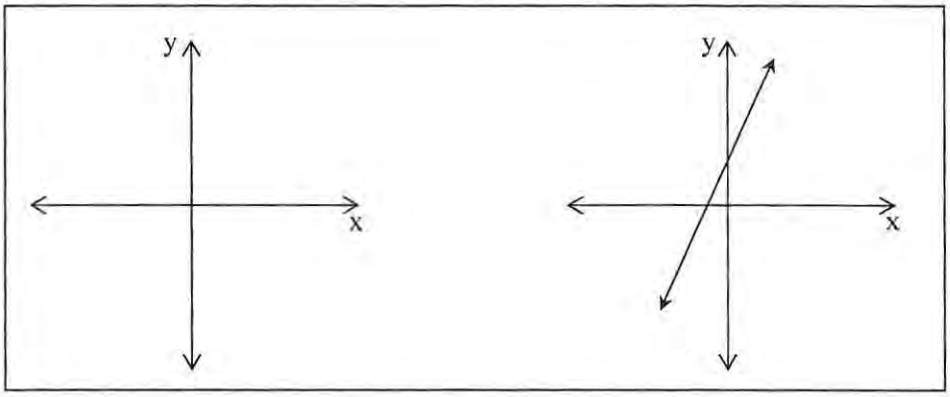


Fig. 6.4

Cartesian contexts perceived as static rather than dynamic constructs

Needless to say the description of a line as depicted on the second set of axes produces less by way of meaningful responses. It is of course the case that, that which is deemed meaningful, is owned by the teacher at the time of asking. I would have formulated a desired response after my own metacognitive deconstruction or I would not have asked the question at all. The mutism that characterises the classroom is disturbing, more so because the sample of learners is heterogeneously constituted. By this heterogeneity is meant that learners are representative of a wide variety of schools in the Cape Peninsula. This being the case one would expect that at least those who have the benefit of functional classrooms be able to produce some degree of meaningful coherence in describing either of the images above. The implications of the absent descriptions are far reaching and demand some reflection in terms of the reasons for non-responsiveness from seventeen-year-old learners who have been learning mathematics for eleven years of their school-going lives. Embedded in metacognitive reflection would be to ask what derivative conceptual constructs constitute the image.

Rudimentary descriptions might be that the images represent a two-dimensional representation of space; that the origin and axes provide a universal concept of referencing; that each point is unique; that the axes divide the two-dimensional plane into four equally sized quadrants, and so on. These would amount to surface structure descriptions.

After continued communicative action to probe the situation further, it became evident to me that there is a fixation on a static notion of the images. By this is meant that learners appeared to look at the axes as a static representation in closed or fixed space limited to the area to which its sketch is confined in the same way that a bar graph depicts absolute data of an occurrence from which nothing can be extrapolated. Inherent in the depictions in Fig. 6.4 though, are dynamic elements that facilitate their interpretation or reading. Learners do not perceive the axes as parts of a greater whole that continue in the direction of both positive and negative infinity in both vertical and horizontal dimensions. Configured in this way it means that the eye needs to traverse the image, left to right, in accordance with the order of the number line as x increases in value. Coordinated with this horizontal movement, the reader needs to recognise the behaviour of the function values relative to a vertical movement, as x increases. This description itself is thick and constitutes accumulative understanding of the contributory concepts. This level of perception, for example, assumes an insight that the axes represent lines and not line segments, which in turn, promotes visualising the extreme infinities in both dimensions. It further promotes the understanding that a graph is read; that it behaves in response to the eye traversing the axes. Alerting learners to the fact that they needed to employ a level of eye movement to read the graph made a crucial difference in their interpretation of graphs. It also means that the predictive and extrapolatory features of functions become cognitively accessible rather than seeing the function in question contained by the limited space of the page on which it appears. This generic value of graphical depictions for all mathematics suggests that we cannot view graphs as an entity isolated and separate from the algebra that it depicts.

Dreyfus (1990), Tall (1989) and Vinner (1992) collaborate on the ideas of 'concept image' and 'concept definition'. These epistemological frameworks theorise how mathematics is learnt. The notions of 'concept image' and 'concept definition' have had a significant bearing on the design of the approach adopted. Concept images as held by learners have been construed as indicators of levels of understanding. Where there is tension between the two I take it to signify shallow understanding. Through the analysis of these tensions teaching can be modified to reduce the discrepancy between the image and definition of the concept. I have aimed at eliciting written verbal accounts to gauge the tensions between concept image and concept definition of learners. I suggest that as the concept deepens so the definition becomes more explicitly coherent. These developments in instruction align with what (Thompson, 1994a) refers to as the nurturing and extending of learners' images in mathematics.

I have alluded earlier that an important dimension of this work is its embeddedness in classrooms where English, although the language of teaching and learning, is not the mother tongue of all learners. IsiXhosa and Afrikaans learners have needed to be accommodated in a heterogeneous multi-lingual classroom culture. The use of graphs and imagistic depictions of algebraic abstraction have been intentionally designed to be part of the teaching strategy. The rationale for this is detailed earlier in the discourse where the advantages of combined visual and auditory images are extolled; this in comparison with static unidirectional auditory-only delivery that characterises surface structure expositions. Graphical images themselves seem to take on the function of symbolic language and provide concrete schema onto which ideas can be attached. Graphical images therefore seem to mediate between the languages of English and Mathematics.

Kaput (1985) regards a learner's ability to employ multiple representations to reflect its mathematical knowledge as an index of concept attainment. The forms that are used in this work are a combination of explanatory and mathematical representations. Kaput continues to say that the learner's notational system is that around which it organises its mathematical experience. Learning, he claims, occurs in the relationship between representing and interpreting mathematics in shared meaning with others. In this work the graphical representations of mathematical objects are in fact shared with me as participant-researcher after reflection on what it is understood to be.

The analytical discourse to this point is indicative of cognitive tussling as to how learners' best learn mathematics and what in everyday teaching episodes constitute impediments to meaningful teaching and learning. I suggest that these interrogatory cognitive monologues are hermeneutic analyses of teaching and learning episodes. These convert to strategies that strive to make school mathematics content accessible; a content which still, for many learners in South Africa, after many costly remedial interventions can still only produce a pass rate in mathematics (HG) of 0,4% of the enrolled school leaving population as reported in the chapter, 'Mathematics in Crisis'.

Elements of the hermeneutic phenomenological method that emerge here are well summed up by Foucault (1984: 374) who states that '... a demanding, prudent, "experimental" attitude is necessary; at every moment, step by step, one must confront what one is thinking and saying with what one is doing, with what one is.' I believe the analysis reflects these bases of participatory research. It involves me in my relationally ontological capacity as I value emancipatory and transformative action through espousing moral and critical consciousnesses that constitute liberatory teaching and learning practices. In it is found the personal and social justice that may be measured through cognitive and socio-economic accessibility to mathematics.

I set out to achieve the following goals at the commencement of this research. Firstly, I wanted to examine and understand the essential role(s) that I as an teacher must play in fostering mathematical learning and, more especially, to analyse my metacognitive engagement with the deconstruction of mathematical concepts. I have tried to illustrate in the analysis above how most high school mathematics is open to this level of interrogation and how amenable it is to being courted in this way, especially through the graphical representations as more concrete objects of abstract algebraic formulations.

My second aim was to investigate and understand what, in learners' experiences are essential to them in coming to know and be fluent in mathematics, and more especially in this regard, assess the value of a graphical approach to introducing complex and abstract mathematical concepts. It began to emerge that the steps toward concretisation began to elicit favourable responses in terms of learners engaging with mathematics at a cognitive level; that there is progress toward Skempian relational understanding. The challenge remained for me to find a means of reducing the tension between 'concept image' and 'concept definition' as described by Dreyfus, Tall and Vinner.

This leads me to my third aim of engaging in a meta-analysis of learners' written accounts of their thought processes as reflected in the completion of a variety of mathematical topics. These written tasks are accompanied by written accounts that relate their responses to questions on the affective elements of what they were required to do. This brings with it a quantitative dimension of analysis as it seeks to codify the mathematical language as used by learners to describe what it is they are doing in a variety of mathematical tasks that employ graphical representations of the concepts at hand. This is elaborated for most part of the data analysis chapter.

The first two aims find a home in the retroductive analyses that form the bulk of this chapter. The third is lodged in the analysis of the data.

By way of drawing this chapter to a close I report on a pilot study that utilises the teaching strategy described in this chapter.

6.3 Findings of an intervention that served as a pilot study

Township learners at risk of failing mathematics at the end of 2002 were identified by the Western Cape Education Department in July of that year for an intervention programme to ameliorate the situation. The improvement of learners' performance over two and half months is largely seen as a result of conscious metacognitive reflections on how to avert the plight of these learners. Given that learners were second or third language English speakers whose mathematics learning deficits were reflected in their June examination performance I needed a pedagogical means of reaching them. I describe a reversed sequence of teaching mathematical structures that takes into account the graded order of difficulty of these structures (van Jaarsveld, 2005). It uses graphical representations of algebraic definitions to concretise abstract concepts as a point of departure. It proceeds to solve equations in these graphical contexts and then contextualises the simplification of expressions in the same way. This sequence recognises an increasing order of abstraction and difficulty inherent in expressions, equations and graphs. The average improvement over ten weeks of 10% in these learners' performances has shown similar improvement amongst similar township cohorts each subsequent year from since 2002 as recorded in Table 6.1 and Table 6.2 below.

Candidates by year 2002	Final Symbol	Final Maths HG %	Baseline symbol	Maths Baseline % [20]	% Deviation
1	F	36	G	21	15
2	C	62	D	50	12
3	FF	30	G	22	8
4	D	54	FF	32	22
5	D	59	F	39	20
6	E	49	F	37	12
7	E	42	G	27	15
2003	Final Symbol	Final Maths HG %	Baseline symbol	Maths Baseline %	% Deviation
1	C	65	B	77	-12
2	B	78	D	50	28
3	E	48	C	67	-19
4	E	49	F	35	14
5	D	55	E	45	10
6	A	82	C	67	15
2004	Final Symbol	Final Maths HG %	Baseline symbol	Maths Baseline %	% Deviation
1	B	70	E	48	22
2	D	55	FF	30	25
3	D	53	D	53	0
4	E	45	G	23	22
5	E	47	G	23	24
6	B	77	D	53	24
7	E	41	F	38	3
2005	Final Symbol	Final Maths HG %	Baseline symbol	Maths Baseline %	% Deviation
1	E	54	B	75	-21
2	D	58	A	83	-25
3	D	58	C	60	-2
4	E	40	FF	30	10
5	A	87	A	85	2
6	C	61	D	53	8
7	C	61	E	48	13
8	D	50	F	38	12
9	D	54	E	45	9

Table 6.1

Performance improvement of isiXhosa learners who were taught according to a reversed sequence of abstraction in a graphical context

Summary	2002	2003	2004	2005
Sum of positive deviations	104	67	120	54
Mean of deviations	15	17	17	9

Table 6.2

A summary of the deviations shown in Table 6.1

[20] The same baseline assessment was administered to each cohort.

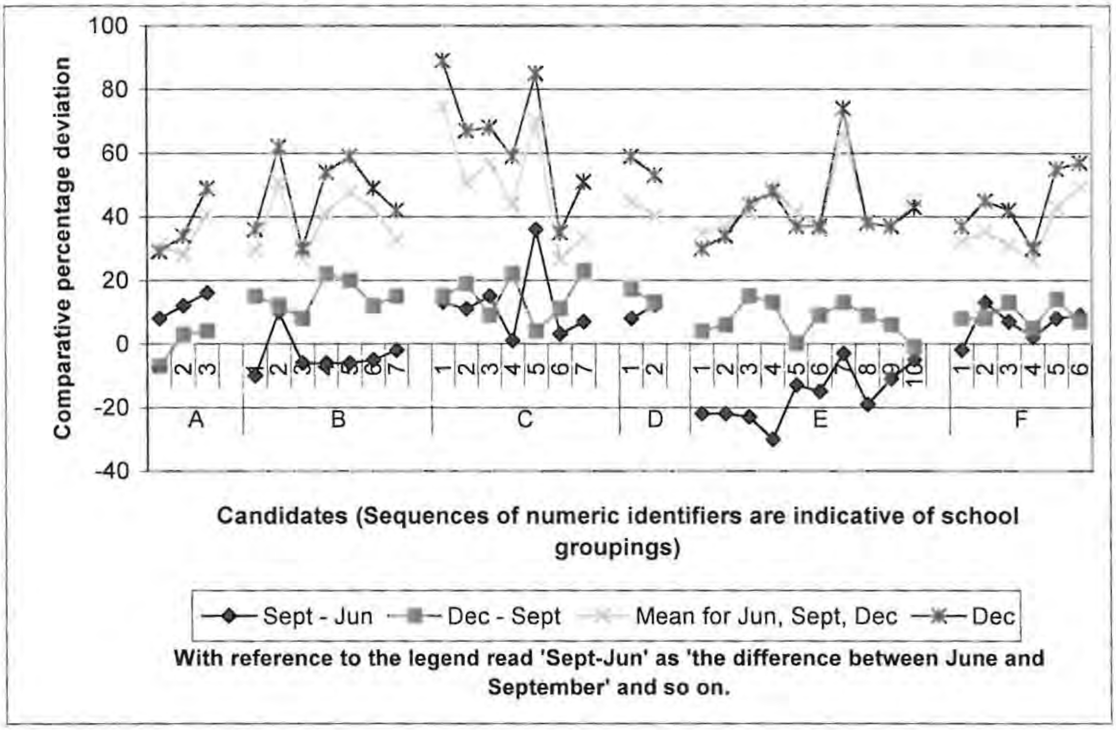


Fig. 6.5

The percentage directional deviation from previous performance in respect of three consecutive examinations for thirty five Khayelitsha candidates in relation to their mean performance for the same examinations and their final performance (All candidates wrote the same assessments[21])

The seven learners' performances of 2002 are contextualised in Fig. 6.5 relative to their 28 Khayelitsha peers. With reference to Fig. 6.5, the sequence of learners (and schools) is in random order. Numeric identifiers remain constant for learners at any point in the analysis, as do school codes A to F. Discontinuities in the graphs serve as school separators. The lighter of the two upper most data lines, learners' mean for June, September and December, is taken as an indicator of learner ability against which performance deviations can be interpreted. The darker line graph of the upper two represents learners' absolute December results. The deviation between these two statistics is taken to read as a measure of achievement. The greater the positive difference between them the greater the final achievement is deemed to be. The 7 learners randomly allocated as my tutorial group that were tracked in this pilot study are located at B. An analysis of the data in Fig. 6.5 follows.

[21] All learners were part of the same intervention programme and selected for participation by the Western Cape Education Department.

6.4 Analysis and interpretation of the quantitative data in Fig. 6.5

1. The graphical display of the data is evidence to infer that the improvement from September to December is appreciable in groups B and E, less so in C and F and anomalous in the case of A. The reason for case A's reversal of expected outcomes is not clear. The positive directional improvements, indicated by the September-December statistics occurring above zero consistently except for two candidates, A1 and E10, may be attributable to learning experiences at a mathematics camp combined with the effects of tutoring sessions. Both groups B and E show negative directional changes from June to September, hence the added significance in their improvement from September to December. The graphical representation of the data suggests significance in the improvements from September to December which prompted a statistical test on the data.

2. These data were subjected to the Wilcoxon Signed Rank Test for non-parametric related samples which assumes a normal distribution with which the data under scrutiny complies. Where sigma is less than 5%, i.e. $p < 0,05$ the improvement in performance is deemed to be significant with a confidence level of 95%. Sigma values are recorded in Table 6.3 below where, for the September to December improvements, groups from schools E, B, C, and F are shown as statistically significant at a 95% confidence level.

Test Statistics[d]

School		September - June	December - September	December - June
A	Z	-1.604 [a]	.000 [b]	-1.604 [a]
	Asymp. Sig. (2-tailed)	.109	1.000	.109
B	Z	-1.279 [c]	-2.375 [a]	-2.366 [a]
	Asymp. Sig. (2-tailed)	.201	.018	.018
C	Z	-2.366 [a]	-2.366 [a]	-2.371 [a]
	Asymp. Sig. (2-tailed)	.018	.018	.018
D	Z	-1.342 [a]	-1.342 [a]	-1.414 [a]
	Asymp. Sig. (2-tailed)	.180	.180	.157
E	Z	-2.805 [c]	-2.554 [a]	-2.245 [c]
	Asymp. Sig. (2-tailed)	.005	.011	.025
F	Z	-1.892 [a]	-2.207 [a]	-2.201 [a]
	Asymp. Sig. (2-tailed)	.058	.027	.028

a. Based on negative ranks.

b. The sum of negative ranks equals the sum of positive ranks.

c. Based on positive ranks.

d. Wilcoxon Signed Ranks Test

Table 6.3

Wilcoxon Signed Rank Test statistics

3. The performance means of the 35 Khayelitsha learners were calculated as 38,3%(June), 38,1% (September) and 48,5% (December). This indicates a notable improvement of 10,4% from September to December. This may infer that tutoring content in an ordered sequence of increasing abstraction commencing with a first principles approach supported by graphical contextualisation had a measure of success.

6.5 Summary

Counterfactual retroductive thinking as a mode of thought operation enables one to search for the basic conditions that make concrete phenomena what they are (Danermark, 2002: 80). As a metacognitive reflective practice such retroductive thinking seeks to identify what basic conditions are necessary for mathematics learning to be facilitated, promoted and sustained.

The content of this chapter considers the elements of communication and its mediums of transfer. In the context of second language English learners the written form of communication is more forthcoming than the spoken form. I come to the understanding that mathematical language or register is a fundamental prerequisite for the successful critical engagement with mathematical descriptions, explanations and discourse. Dialectical communities do not optimally operate without the appropriate language with which to communicate mathematical content. Dialectical communities are further impeded by the constraints of learning in one's second language. The algebraic language of mathematics is rendered more remote in these circumstances. This is compounded by the order of abstraction inherent in a standard approach to the order of mathematical tasks. Through the depiction of algebraic constructs in the Cartesian plane, mathematical tasks are interpreted and visualised. These real, concrete representations provide the catalyst for descriptions that feed the cognitive domain and that assist the establishment of a mathematical register. In describing these concrete mathematical representations a consciousness of mathematical endeavour is irrigated and it is with this consciousness that real mathematical discourse is promoted, and cognitive schemas are established and sustained.

I recognise that metacognitive reflection is dependent upon mentally acquired language with which to think and to communicate. Teachers and learners therefore both benefit from the developed mathematical register that emerges from the graphical depictions that serve also to deconstruct subsumed concepts.

It is recognised that teaching is a fundamental component of the learning process in the sense that social constructivist endeavours do not flourish in the absence of communicated shared thoughts that transpire through the appropriate use of language. An important role and function of the teacher is to acquaint learners with the language of mathematics through using it deliberately in instructional episodes of first principles teaching. It would seem that a pictorial strategy aligns the inherent levels of abstraction associated with increasing orders of difficulty and its associated concepts.

A pilot study hints at the success of a teaching strategy based on the pictorial strength and value of graphical representations as ameliorating the impediments of being taught in one's second language.

The following chapter describes the empirical research environment and the heterogeneous cohort of learners who are the research participants. It also deals with the rationale for the design of the research instrument and demonstrates how this instrument is applied to the work of the research participants.

VII

The Research Environment and Process

'Hermeneutics as an activity is at its best an art but never a method; as far as science is concerned, hermeneutics is a subversive force that undermines all systematic approaches.'

(Gadamer, 1976)

7.1 Introduction

Contrary to the quotation from Gadamer (1976) that introduces this chapter, this interpretive case study's hermeneutic nature is not seen to subvert the essence of scientific enquiry but rather taken as an opportunity to jostle for humane accommodation on the peripheral fringe of scientific method. Its aim is not to make claims about causality. It is rather to make suggestions about what constitutes meaningful mathematics teaching and learning. The case study (Yin, 1994; Lincoln, 1985; Denzin, 1970) serves to probe the outcomes of a method that reverses the abstractual order of mathematical constructs; a reversed order that I regard as more aligned to metacognitive functioning and considerations and that, in terms of existing learning theory, I see as meaningfully promoting sustained mathematics pedagogy. If one commences from the ontological values of personal and social justice then that means that our constitutional entitlement to acquire knowledge is a fundamental purpose that underpins this research. The data in Chapter 2 is evidence of the disequilibrium, in terms of accessibility to mathematics tuition, which exists across ethnic divides. The data accrued by way of written mathematical artefacts and affective data accounts this case study seems to support the notion that the mathematical mutism amongst learners is indicative of tenuous mathematical conceptual depth that militates against conceptual sustainability. The graphical method provides that concrete tool that facilitates and mediates on the continuum that associates didactic input with the cognitive assimilation of new knowledge. It is the graphical method that serves also to permit me as teacher to deconstruct complex concepts whose composition is constituted of lower order conceptual components. The declaration of the lower order components initiates learners into engaging with concept descriptions and the pictorial evidence seems to allow learners to constitute their own cognitive meanings for what would otherwise remain as rote knowledge. The ensuing dialogic communicative action serves to confirm conceptual closure and help establish a critical consciousness about mathematical constructs that aid personal interrogations for teacher and learner about what it is they are doing in mathematical tasks in pedagogic situations.

Because this case study research is a strong blend of both the qualitative and quantitative method within the critical realist tradition it utilises both idiographic and nomothetic techniques. The tools used to probe learner cognition affectively are the artefacts containing textual accounts of learners' mathematical tasks. To supplement these accounts a set of questions designed to gauge learners' feelings in response to having to articulate their experiences of mathematics was obtained at the end of the two-year research period. A coding instrument applied to the textual artefacts, as described later in this chapter, serves as a tool for quantifying the usage of mathematical language used by learners to describe and explain mathematical tasks, and subsequently for discerning trends in the usage of mathematical vocabulary. The numeric data generated in this way serves to support and explain the qualitative data rather than establish causality between the method of combining written and spoken mathematical language and the outcome reflected in learner performance. Fortnightly lessons took place in a lecture theatre that served as the research site for two consecutive years. The composition of the sample, referred to as the cohort in this thesis, is described in section 7.4 of this chapter. In this work I take the roles of teacher, facilitator and participant researcher as I investigate my own practice concurrently with investigating learner cognition.

7.2 Goals of this research

1. To engage in a meta-analysis of learners' written accounts of their thought processes as reflected in the completion of a variety of mathematical topics and related tasks,

2. To investigate and understand what, in learners' experiences are essential to them in coming to know and be fluent in mathematics, and more especially in this regard, assess the value of a graphical approach to introducing and sustaining complex and abstract mathematical concepts,

3. To examine and understand the essential role(s) that I as a teacher must play in fostering mathematical learning and, more especially, to analyse my metacognitive engagement with the deconstruction of mathematical concepts.

A critical realist theoretical perspective (Danermark, et al., 2002) that accommodates this research has been extensively addressed in the preceding chapters. The core research component, to understand how adolescents make meaning of complex mathematical constructs, and how as a teacher I can augment the meaningfulness of these mathematical abstractions, is introduced in this chapter through a description of the research process. Within the critical paradigm, the Freirian notion of critical consciousness and the Habermasian theory of communicative action provide ontological and epistemological perspectives from which the research proceeds and ultimately is embedded.

7.3 Blending of the theoretical and empirical components of the research process

Human consciousness is central to the phenomenological paradigm and as an interpretivist approach it aims primarily to understand rather than explain people. The phenomenological position views people as "continuously constructing, developing, and changing the every-day (common sense) interpretations of their world(s) and it should therefore be taken into account in any conception of social science research." (Babbie & Mouton, 2002: 28). It is understood therefore by these criteria that the explorative and case study investigates the exposition of the learners' and teacher's understanding of the mathematics being exchanged at given times which constitutes moments in the hermeneutic circle. The hermeneutic circle is defined as the circularity between explanation and understanding (Brown, 1997: 37). Put in critical realist terms, the discovery of phenomena reveals deeper levels of reality that require explanations[22] at deeper levels. Bhaskar (1989: 20) states that knowledge of these deeper layers not only explains phenomena at a more superficial level, but may also correct knowledge about those more superficial layers. Scientific knowledge is therefore subject to constant revision. As we, as mathematics learners, progress from our historical realities and its associated aetiologies we tend to converge towards conceptual understanding through our cognitive endeavours, rediscoveries and corrections. As we are exposed to new mathematical realities these may increase the gestaltic envelope that acquires greater meaning than the understanding of the meanings of the mere sum of its constituent mathematical parts. In this way we move toward conceptual refinement, efficiency and fluency on a cognitive journey that may never reach its esoteric destination. Heidegger's view that meaning is historical rather than transcendental has bearing on this investigation; "...meaning is acquired through interpretation through time rather than mere consciousness." (Brown, 1997: 37). For learners of mathematics this constitutes difficulties since 'obliteratively subsumed' (Ausubel, 1961) mathematics characterises the knowledge of teachers who are more travelled on the phenomenological mathematical road, or whose limited experience has not afforded opportunities for reflecting in a critically conscious way about issues of communicative action. The fluent efficiency of mathematical acuity of experienced teachers means that essential component knowledge is lost to them (learners). Teachers and learners are therefore both cognitive travellers; both interpret learning situations and opportunities and new mathematical content and it would seem incumbent on both parties to assume interpretive responsibilities in the learning process. It is therefore important for mathematics teachers to deconstruct metacognitively the mathematical content intended for meaningful transfer; and then to analyse and interpret the meanings attached to learners' texts as they commit to writing the algebraic language and the associated machinations of their cognitive labours. I see these interactive elements of the hermeneutic cycle as instrumental in promoting and enhancing mathematical learning and growing the mathematical body of knowledge of learners and teachers.

Wilhelm Dilthey and Max Weber cited in Babbie and Mouton (2002), as hermeneuticists of the phenomenological paradigm lay emphasis on the 'verstehen'

[22] These explanations are not of people referred to earlier but rather those phenomena that are demonstrated by the critical consciousness of learners as they describe the mathematical processes that reflect their transfactual reality associated with cognitive moments.

or subjective understanding and interpretation of human action, and in this study, human action is constituted of learners' written artefacts. Babbie and Mouton (2002: 30) state "The human sciences ... aim at understanding the internal relations between actions by relating them to ideas, values and purposes which give rise to them." This hermeneutic phenomenological stance contrasts with an empirical phenomenological stance (Schweitzer 2002, van der Mescht 2004) which seeks to comprehend how others make meaning of their experiences. It is my interpretation of learners' meanings attached to their mathematical understanding that situates this aspect of the case study as empirically phenomenological. These two stances of hermeneutic and empirical phenomenology are blended in this case study rather than juxtaposed.

The consciousness that learners, and me as a researcher, display about mathematical tasks, relative to our historical mathematical 'repertoires' in my understanding, akin to the consciousness that is epitomised by the inner mental processes of our subjective experiences, are vestiges of the symbolic interactionism of Mead and Blumer (ibid.: 33). It is perhaps the modification (in strategic mathematical thinking) that the interpretive process hopes to achieve in perpetuating the hermeneutic circle.

7.4 Empirical Research Context

Quoted in Babbie & Mouton (2002: 281), "the case study is an intensive investigation of a single unit." (See also, Handel 1991, Runyan 1982, Yin 1994) and the research qualifies as such. A cohort of 27 Grade 11 learners of 2004 and their progress to Grade 12 in 2005 constitutes the cohort that is the focus group of the case study.

The heterogeneity of the research group is underlined by the fact that individual learners constitute participatory uniqueness in terms of their biography (which incorporates differences in gender, home-language, language of instruction, ethnicity, school, residential geography, attitude to mathematics and historical mathematical exposure and performance) both within the confines of the case study and at their schools. The nature of the composition of the cohort is reflected in Figures 7.1 to 7.3 that follow.

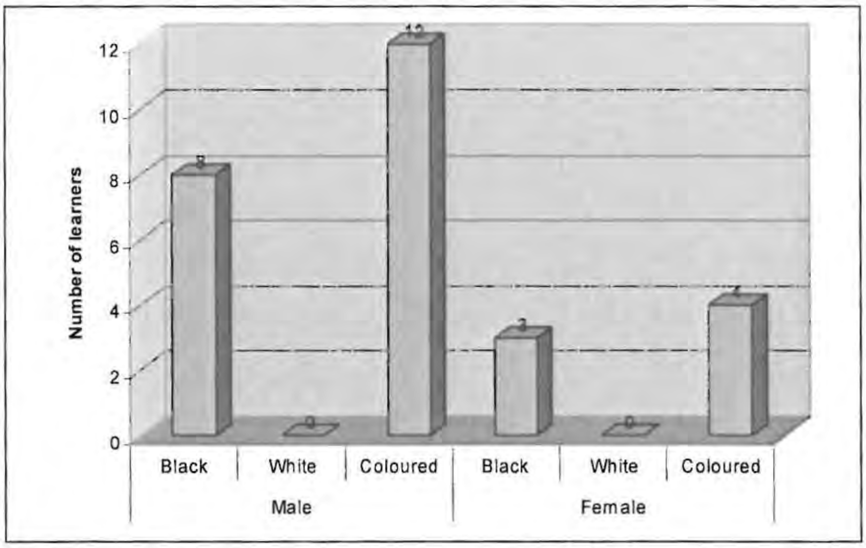


Fig. 7.1

Composition of cohort of learners by ethnic background and gender (n = 27)

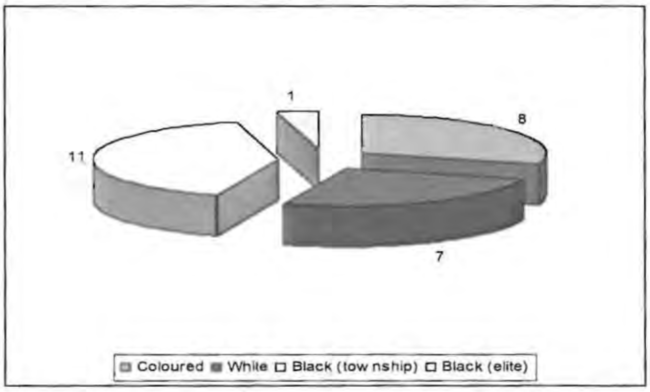


Fig. 7.2

Composition of the cohort of learners by school types [23] represented (n = 27)

[23] These school types correspond with the ethnic divisive categorisation of schools that emerged during the apartheid dispensation and despite regime change have remained. The use of the term 'Black (elite)' designates a school in Khayelitsha that was established in the post-apartheid South Africa to serve promising learners from township schools through recruiting the top performing learners from their primary schools. 'White' and 'Coloured' refers to the ethnically separate schools under the apartheid regime, that have since become multi-racial.

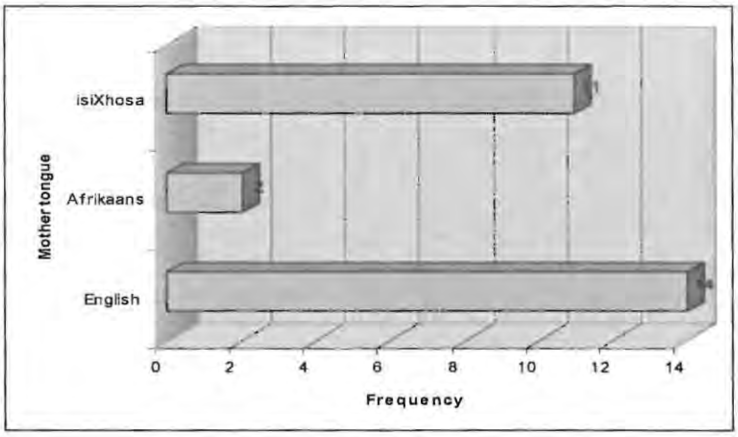


Fig.7.3

Composition of cohort by mother tongue (n = 27)

This case study is located within a HG mathematics and science intervention programme for the benefit of previously disadvantaged learners (who are also currently disadvantaged in terms of the personal and social injustices of inaccessibility to sound and consistent mathematics teaching and perpetuated poverty that stems from it) of the disparate educational dispensation of an apartheid era. It is conducted for three cohorts (Grade 10, 11 and 12) of learners. Learners need to offer HG mathematics and science as matriculation subjects to qualify for selection onto the programme. The selection of some learners was a function of their participation in a previous remedial programme, while most black learners were recruited from schools in Khayelitsha; the sole qualifying criteria again being that they offered HG mathematics and science for matriculation. I teach mathematics to the Grade 11 and 12 cohorts in this programme and have been doing so since the beginning of 2003. The class that is the focus group of this study is inherited in the sense that I had no previous contact with the group prior to the commencement of their Grade 11 year. Because the group is constituted of learners selected from different communities and schools the variance in terms of their exposure to mathematics over the years is emphasised. This gives reason to commence teaching with the assumption that all learners for all intents and purposes need to be nurtured from the viewpoint that nothing is assumed known. All teaching is therefore characterised by a graphical, first principles approach.

Approximately 90 learners comprising these three grades were bussed to Stellenbosch from various points across the Cape Peninsula. These groups of 30 learners each attend 18 Saturday sessions of two hours duration during the academic year. Fifteen of the 18 sessions are for the dedicated teaching of HG topics of the senior secondary mathematics syllabus, while 3 are designated for formal summative assessment of accumulated work. Twenty minutes of each contact session is allocated to the specific formative assessment of the learners' understanding of the lesson content. The remaining time was devoted to pedagogical processes around mathematical content knowledge. The curriculum comprises, in particular those components of the National Curriculum Statement for Mathematics in the Further Education and Training band (DOE, 2003) that from experience are regarded as difficult by teachers, from my experience in teacher training for seven years, and learners alike.

There is substantial emphasis on understanding mathematical objects and processes and the meaning learners attach to it, rather than algorithmic procedures alone, and the time allocated to specific topics therefore varies according to their associated level of difficulty and the time taken for the pre-stated lesson outcomes to be achieved. It is therefore an open system in terms of adjustment to module duration. There is due regard for a wide variety of ability levels within the constraints of having to prepare all learners for the mandate of increasing the number and quality of higher grade mathematics passes amongst previously disadvantaged learners.

7.5 Components of the research process

The process is characterised by its aggregation rather than its linearity. Each of the components as described below has its inception simultaneously and progress alongside each other, rather than each one being characterised by a temporal positioning or a chronological development. The method of teaching, for example, was prompted by the experience of teaching over twenty five years, and second language English speakers in particular over the past seven years, and so becoming aware of the impediment of language to learning; firstly English as a language of learning and instruction, secondly, the condensed, concise, almost terse, language of algebra (from the perspective of those for whom English is a second language), and thirdly the sophisticated mathematical register which describes the language of algebra. These epistemological considerations merge with the ontological values of personal and social justice through seeking how to make mathematics more accessible, and a pedagogy mindful of critical consciousness that is encouraged by communicative action. Inherent in the communication of mathematics, as it is taught, is the associated conceptual depth that we acquire as teachers, and which may obstruct learner understanding and consolidation. Ausubel's (1961) reference to 'obliteratively subsumed' knowledge closely describes this tendency. These experiences prompted thought on how to simplify the languages referred to above and how to concretise the abstractions of mathematical concepts so that my teaching of mathematics might become more accessible especially to those for whom the 'three' languages above were obstacles to understanding mathematics. Out of this arose the development of lessons that are based on the interpretations of graphical representations that illustrate the rudimentary and compositional elements inherent in deconstructed concepts as developed and expounded earlier.

7.6 Teaching as precursor to data collection

Deconstructed concepts are taught by way of graphical representations as motivated and illustrated in chapter 5. Learners are taught with a bi-directional metacognitive approach in mind.[24] Persistent questioning and answering interrogates the learning and teaching situations in an attempt to get learners to articulate their inner mental processes. This is compliant with the bi-directional metateaching and metalearning model, which is being developed alongside this research. Each teaching event is characterised by an increasing order of abstraction correspondent with graphs, equations and expressions, in this order, and where the interpretation of the graph of the concept at hand is intended to develop a cognitive schema.

Learners therefore engaged with their own metacognitive understanding after deliberate teaching paved the way for their recording of textual accounts at the end of each teaching session. The pedagogic process fundamentally incorporates a

[24] A bi-directional metacognitive approach is that stance that recognises that the teaching-learning environment is characterised by the reciprocity of learning and teaching. Learners learn from me, and I from them, about mathematics and mathematics cognition.

graphical didactive element that is in line with the idea of generic graphicacy developed chapter 5, as well as a radical constructivist engagement by learners on the lesson content. On occasion learners were invited to examine content under social constructivist conditions of sharing their understanding with one another. These preliminary processes culminated in the learners recording privately their descriptions of the mathematics tasks that were allocated at the end of each teaching session.[25] These written artefacts constitute the raw data of this case study. I described in detail in an earlier chapter why I relied on textual accounts rather than interviews. That rationale is briefly reiterated here. Firstly, it was dictated by the nature of the task. Learners are accustomed to written mathematical tasks and not oral presentations. Secondly, language proficiency in hermeneutic analyses plays an important role in acquiring meaningful data. The requirement to speak mathematically becomes an impediment in that it stultifies the generation of meaningful data. This is so because learners find it difficult to communicate mathematically in the first instance and further to communicate mathematically in their second language renders their messages more incoherent. The written task serves to overcome these adversities in providing opportunity for learners to engage in a degree of critical cognitive consciousness without the adversarial influence of intimidating recording devices or their linguistic insecurity in making themselves understood. The written accounts serve as permanent references to how learners responded to tasks both in terms of the algorithmic mathematics and their descriptions and explanations of the algorithms. After several attempts at deriving one-on-one oral responses from learners the written accounts proved to reflect more of what transpired cognitively in their describing mathematical tasks. I have made the point also that the level of describing mathematical phenomena is a natural precursor for the development of the appropriate vocabulary with which to communicate and socially construct knowledge, and for learners to become meaningful participants in a communicatively active mathematics teaching and learning environment.

[25] Learners were informed that each lesson culminated in an assessment of the mathematics they had learned in the session and their descriptions of that mathematics. It is assumed that this encouraged learners to be more attentive about the content transfer in the didactive episodes.

7.7 Resources

A teaching venue at Stellenbosch University accommodated lessons each alternate Saturday of the academic year. A heterogeneous cohort of 27 Grade 11 learners who progressed to Grade 12, bussed to the teaching venue from schools across the Peninsula, and which is representative of the demography of the Western Cape. I was positioned as both teacher and researcher and used digital voice recorder and video camera to capture the content of lessons. Learners' specific disclosures in textual accounts demonstrated their deeper understanding of mathematical tasks. Task sheets, as described in Chapter 5 were designed to support the graphical representation approach. I had access to the libraries of the University of Cape Town and Rhodes University.

7.8 Implementation

Learners attended 15 and 8 contact sessions of 2 hours respectively over two consecutive academic years 2004 and 2005; lessons being each alternate Saturday from February to October.

Grade 11:	Year 1	15 X 2 hours = 30 hours
Grade 12:	Year 2	8 X 2 hours = 16 hours
Total contact tuition:		46 hours

7.9 Data

1. Data comprise learners' summary accounts of concept(s) by way of the written and sketched interpretations of these mathematical concepts. These are formative and summative assessments that provide information to the teacher-researcher, and ultimately to the learners themselves. In terms of this study these are the meta-learning and meta-teaching artefacts, the psycho-cognitive analysis of which attempts to provide evidence of learning and understanding and the degree to which conceptual depth has been acquired.

2. Learners' narrative accounts of their experience of mathematics in the teaching-learning context as it occurs in this research context.

3. The quantitative data emerging from the coding of textual accounts.

Table 7.1 that follows serves to summarise the processes of data collection and data analysis whilst Table 7.2 summarises data gathering events.

	Aspect of study	Instruments	Source of data	Purpose of data
1.	Meta-teaching, meta-learning and metacognition in the context of the graphical representation related to the abstractual orders of the FET mathematics classroom	Learners' artefacts: Textual accounts of meaning attached to mathematical concepts	Learners' written responses to instruction based on graphical mathematical stimuli; completed at the end of each contact session	Learners: To engage metacognitively with concepts with the aid of graphical mathematical representations Researcher: To discern learners' depth of knowledge through their use of concept-specific mathematical register
2.	Analysis of learners' textual accounts, which are the artefacts in 1.	Coding Instrument	Learners' textual artefacts	To categorise and quantify the textual accounts in terms of mathematical register (language), mathematical constructs (concept) and mathematical understanding (conceptual depth).
3.	Reflection on my own cognitive episodes by way of action research.	Video and audio tapes of lessons. This is supplemented with learners' unsolicited comments on and about my role in the delivery of content in lessons	My ongoing interaction with learners in lesson preparation and classroom teaching.	A personal assessment of the discourse ethics inherent in engaging with critical consciousness and communicative action in praxis (translating theory into practice).

Table 7.1

Summary of the processes of data collection and analysis

Event date	Event description
15 May 2004	Quadratic theory in relation to its graphical interpretation
26 June 2004	Absolute value graphs leading to equations, inequalities and expressions
24 July 2004	Absolute value graphs leading to equations, inequalities and expressions
21 August 2004	Trigonometric reduction formulae interpreted in its graphical context
18 September 2004	Quadratic theory, Absolute value and Trigonometric reduction formulae
30 October 2004	Quadratic theory, Absolute value and Trigonometric reduction formulae
03 July 2005	Affective responses from learners on their mathematics experiences over the past two years.

Table 7.2

Data gathering events

7.10 Data analysis

Learners are required to engage in reflective abstraction and to this end are required to sketch, write and examine from first principles the concepts in question as they are solicited by instruction and the learners' understanding of them from memory and/or post instruction on the concept.

Learners commit their understanding of selected mathematical topics (quadratic theory, absolute value and simple trigonometric reduction formulae) to text in response to a graphical teaching strategy. Each text, which describes the learner's understanding of a mathematical task, is analysed according to elements of an empirical phenomenological method, where I as teacher-researcher engage with answers to search for the meaning and learners attach to mathematical content.

The mood of this qualitative case study is unfamiliarly complemented by a quantitative analysis that is based on a coding instrument. It is designed to condense learners' textual data. The function of this instrument is firstly to identify categories of linguistic and mathematical content and hence generate thematic trends in understanding. Secondly it aims to identify what textual items learners use to express their acquired meaning. It is language (verbal or written) that is the concretised layer that enables us to reflect our cognitions that are at work at a level of consciousness. There is therefore appreciation for the fact that the language at our disposal is dependent on the amount of time the language (of learning and teaching) has been with us, and the extent to which it is a saturated part of our being. Van der Mescht (2004: 5) cites Giorgi's (1970, 1975, 1985, 1992) articulation of the Duquesne School's phenomenological method and one of the chief characteristics of the approach is the selection of participants on 'the basis of experience of the phenomenon under investigation, as well as their linguistic proficiency in the research language.' Van der Mescht (idem) here refers to the fact that in the phenomenological method that the language proficiency of research participants should be of a standard high enough for them to be able to express themselves meaningfully. In this regard isiXhosa learners fall short of this requirement[26]. In terms of the requirement of experience mathematics has been an integral part of all participants for eleven to twelve years of their lives.

7.11 Development of the Coding Instrument

As much as the phenomenological method subscribes to interpretive analyses to pry into the meaning that learners gain from their mathematical experiences, so the mathematics of this investigation as it relates to mathematical register, seems to demand a means of quantifying its normative nature and recurrence in the textual accounts of learners. I have developed the coding instrument in an attempt to quantify the use and accuracy of the mathematical register, as used by learners, as they recount their interpretation of the mathematics concepts at hand, and interpret annotated graphical representations of the same concept. The meaning and explanation of codes is set out in Table 7.3.

Learners completed six reflective metacognitive exercises between March and October 2004. All these learner responses were first read to derive a sense of what learners could produce by way of metacognitive reflections. A second read-through

[26] This was considered when deciding to use textual accounts of mathematical tasks as primary research artefacts since written language proved to be a more reliable source of meaning than the oral accounts of interviews.

of all learners' work concentrated on the recording of comments that learners used repeatedly in their textual accounts. This listing served as a precursor to establishing categories. The first category that emerged was pertinent to a general mathematical register and the degree of linguistic expressiveness, and the second related to concept-specific vocabulary. This third category synthesises the language and concept categories. In combination the two seemed to indicate the degree of conceptual depth reflected by the learner. This category was designed as a result of a tendency for learners to reflect a degree of understanding evident from their graphical representations of concepts in combination with an incoherent use of language and mathematical register, and particularly amongst second language English speakers. It therefore seemed that where learners were strong at both the articulation of their metacognitive processes and where their use of appropriately correct concept-specific graphical representations was accurate, it indicated a high degree of conceptual depth, which I refer to as 'conceptual closure' in the coding instrument. In cases where learners used (mathematical) language incoherently with weak articulation skills, but showed insights of the concept in their annotated graphical representations, learners were deemed to exhibit what I refer to as 'abridged concept' depth. At the other extreme, conceptual depth was deemed 'tenuous' where learners were neither articulate nor able to produce meaning in terms of their annotated graphical representations.

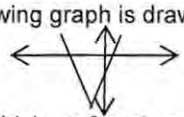
On a third read through the phrases in the learners' textual accounts were coded according to the categories above. The code frequencies per learner per category were captured in an excel spreadsheet. Associated categories were analysed to investigate any statistical correlations between the categories of 'language' and 'concept' and whether these are associated with performance scores and reported on in the next chapter.

Learners responded in July 2005 to a questionnaire soliciting affective reflection on their mathematics experiences on the programme. These supplement the data analysis in the following chapter.

As a set of descriptive statistics in combination with the interpretive analysis, the results will attempt to answer to the goals as set out above.

CODES

LANGUAGE CODES

Code	Meaning	Definition	Examples of learner responses
C+	Coherence (Strong)	The text is logical, and understandable and sequenced	For $\|x-2\| \geq \|2x\|$ the solution can be read off from the graph where $\|x-2\|$ is higher than $\|2x\|$
C+ -	Coherence (Weak)	There is an emerging coherence in the context of the answer	"Where the second line cut the sin graph it will be x . and it will be $= to(180-x)$"
C-	Coherence (None)	The statement or argument is incoherent or unintelligible	"In absolute values you have two answers for x except in the inverse."
E+	Explanation (Full)	A process or procedure is justified	To find the x intercept y=0 because for all the points on the x axis the y co-ordinate is 0.
E-	Explanation (Partial)	A procedure is described but not justified or substantiated	To find the y intercept make x=0
A+	Articulation (Strong)	Command of language	"Calculate the x values of the points of intersection of the two graphs. We do this by setting the graph $y=\|x-2\|$ equal to both possibilities of $y=\|2x\|$ i.e. $y=2x$ and $y=-2x$."
A-	Articulation (Weak)	Characterised by code-switching	"...find out how many number are between the numbers...then half the number from there and you can find the absolute value bracket." "You must find x and he must stand allone." "...the first x how to get it is you must make y=0."
T+	Terminology (Aligned)	Correct contextual use of descriptive terms and constructs	Use of 'zero' and 'root' respectively in terms of graphs and equations
T-	Terminology (Non-Aligned)	Incorrect or inappropriate contextual use of descriptive terms and constructs	Use of 'turning point' to describe a 'salient point'
CA+	Conceptual Accuracy	There is an indication, like the recognition of alternative representation for algebraic structures, that there is an understanding of the concept	"This is an inequality which can be represented by two graphs on the same set of axes."
CA+ -	Emergent Conceptual Accuracy	Rudiments of the construct indicate an initial identification or association with the concept", or sufficient constructs are graphically represented to indicate an emerging conceptual accuracy.	"If you minus x from 90 on the sin graph and put the value you found symmetrically on the cosx graph you will find that they are equal."
CA-	Conceptual Inaccuracy	The graphical representation is not consonant with equivalent algebraic constructs	For $y=-\|2x+4\|-2$ the following graph is drawn  for which a>0, not negative
NA	Not attempted	The description of the process or procedure is omitted	No vocabulary response accompanies algorithm or graph

CONCEPT CODES

Code	Meaning	Definition	Examples of learner responses
I+	Integration appropriate	The current concept is correctly related to its grounding in pre-knowledge	$y = a\|x-p\|+q$ is seen as similar to $y = a(x-p)^2+q$
I-	Integration inappropriate	The current concept is incorrectly related to its grounding in pre-knowledge	The *x* co-ordinate of the salient point of $y = a\|x-p\|+q$ is obtainable by the formula $x = -\frac{b}{2a}$
AA+	Arithmetic accuracy	Mathematical calculations, manipulations and interpretations are correct. These calculations are spontaneous attempts to elucidate an argument or description	$\|2x+4\| = -2$ has no real solution
AA-	Arithmetic inaccuracy	Mathematical calculations, manipulations and interpretations are incorrect	$y = -\|2x+4\|-2$ has a turning point of $(-4;-2)$ instead of $(-2;-2)$
G+	Graph drawn	Graph is drawn as instructed or to aid description	Sketch of graph accompanies text
G-	Graph inappropriate	Attempt at graph is incorrect or inappropriate	Incorrect or inappropriate graph accompanies text

LANGUAGE AND CONCEPT CODES COMBINED AS AN INDEX OF CONCEPT ATTAINMENT AND DEPTH

D+	Depth 'Closure' (Deep)	Conceptual depth is indicated by the use of most of the language and concept codes in conjunction with graphical depictions
D+ -	Depth 'Abridged' (Moderate)	The completion of the concept is tainted by the absence of a critical construct, or arithmetic inaccuracy, or rote
D-	Depth 'Tenuous' (Shallow)	Vestige of concept is evident in minimum use of constituent constructs

Table 7.3

Meaning, Definition and Learner examples of coding categories

The three goals articulated at the outset of this chapter serve also to delineate the levels of data analysis.

Level 1: Meta-analysis of learners' written accounts

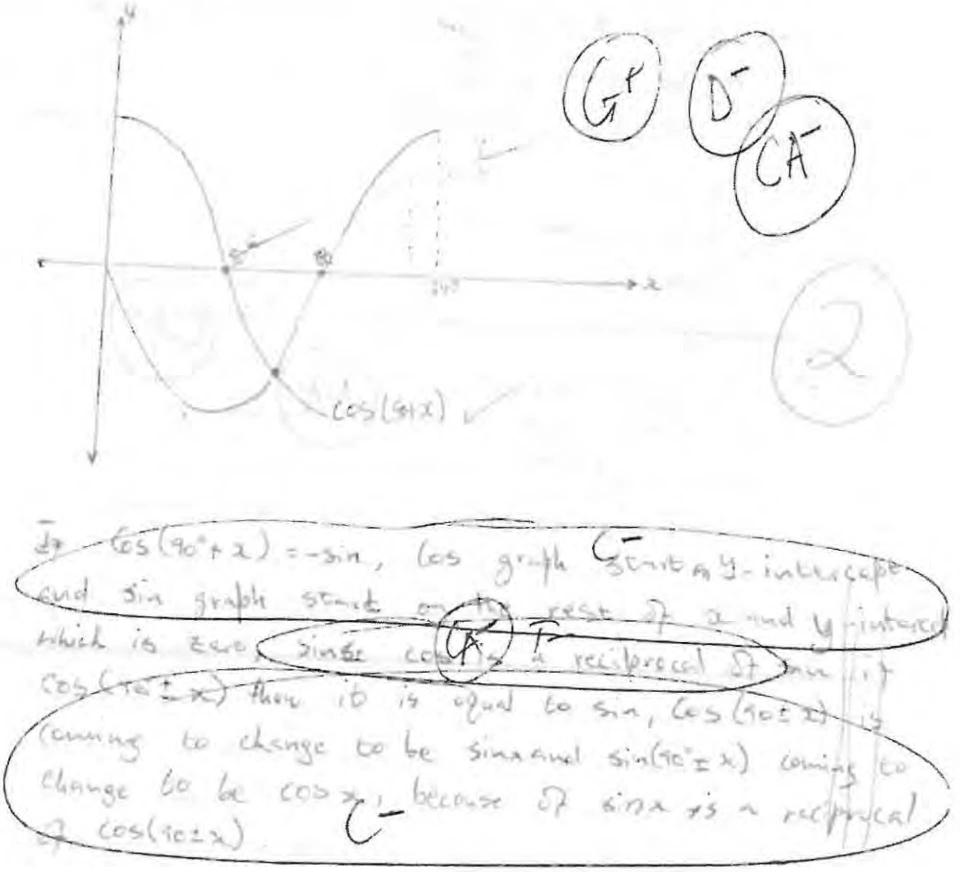
7.12 Application of the coding instrument to learners' textual artefacts

A selection of learners' textual accounts follows to serve to illustrate the application of the coding instrument and the assignment of codes to textual and graphical detail produced by learners. The selected artefacts serve to illustrate the range of articulateness amongst learners in the cohort. The emphasis is on criterion-referenced rather than norm-referenced assessment in terms of the assigned codes and ultimately my intuitive combination of language and concept codes to gauge a degree of acquired conceptual depth.

<u>Artefact 1, participant 9</u>

The explanation for the equivalence statement, $\sin x = \cos(90° - x)$ although incoherent (C -), demonstrates an emergence of the associated conceptual constructs. There is a vestige of the symmetry and congruency properties that associate with the same function value in the making of the statement 'at same dist we sub and add'. E+ is assigned on the grounds that isiXhosa is the mother tongue of this learner, and as such it is incumbent upon me as teacher to fathom the subtleties in the incoherence of the language. T+ indicates the correct and appropriate use of terminology. '$0 + x \equiv 90 - x$' is recognised as the learner having a subliminal appreciation for the congruence in the shapes beneath the sine and cosine curves for $x \in [0°, x]$ and $x \in [90° - x, 90°]$. On the graph itself, G+ indicates that the parts of the sine and cosine graphs are sufficient to be justified by the textual account. CA+ recognises that there is conceptual accuracy and although the language is incoherent there is, for all intents and purposes, an acquired depth in the concept albeit masked by inadequate vocabular usage. This empirical hermeneutic endeavour on my part seeks to make sense of the sense that a learner makes of the mathematical task. There seems to be a closing of the gap between 'concept image' and 'concept definition'. The learner is recognised as demonstrating a critical consciousness about the identity in question through his metacognitive deliberation.

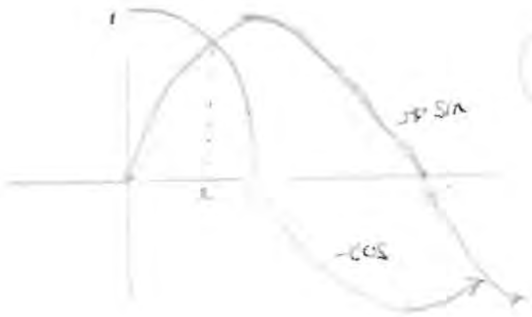
Artefact 2, participant 10



The language used to describe and explain the truth of $\cos(90° + x) = -\sin x$ exhibits incoherence and conceptual inaccuracy. Terminology like ‘reciprocal’ for ‘co-function’ in saying ‘cos is a reciprocal of sin’ indicates that the absence of conceptual depth and a confusion of the terms ‘reciprocal’ and ‘co-function’. Although the form of the graphs is satisfactory, the candidate’s labelling of the graphs confirms the absence of conceptual closure as is evident in the textual account. In combining the language and concept codes the candidate exhibits a tenuous conceptual depth.

Artefact 3, participant 21

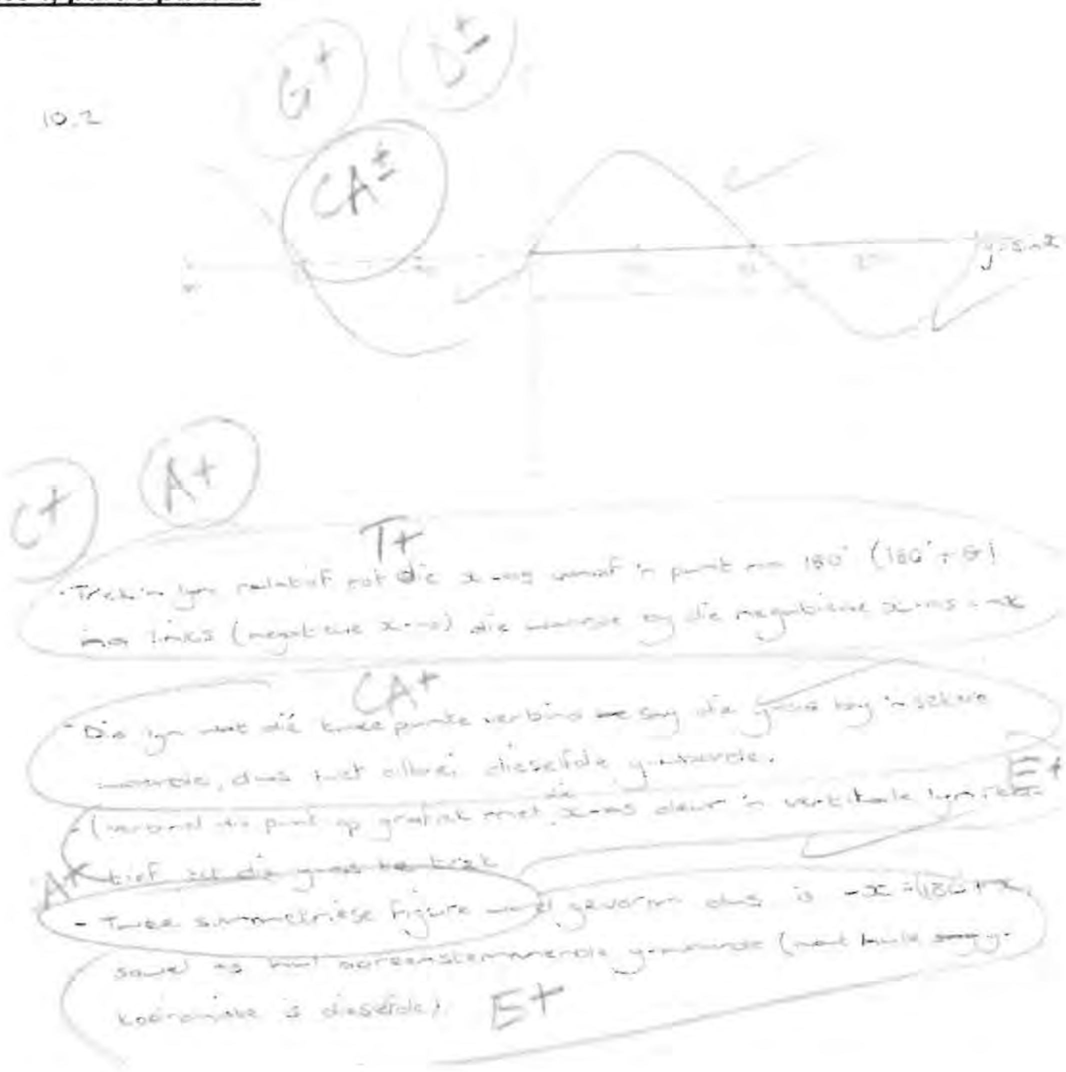
If we produce a cos graph & a sin graph on the same set of axis. T+



- We can see that at the place where the graphs intersect ie. they are equal to each other they meet @ an x value on the x axis. In relation to the sin graph its value is $90° - x$ & then in relation to the cos graph its value is x. We can therefore say that $\sin(90° - x) = \cos x$.

This artefact is characterised by an articulation that is punctuated with inaccuracy as in 'they (the graphs) meet @ an x value on the x axis'. Inasmuch as this is not coherent, it is understood and deemed to be an indicator of attainment given that this constitutes a first attempt. The assignment of a D+ - for abridged conceptual depth relates to the fact that the candidate uses the specific case where the functions are equal, as opposed to the veracity of the general case that would be designated by placing x in a suitable position other than where $x = 45°$. The inherent value of the candidate's choice of the specific case would serve the purpose of verifying the truth of both $\sin x = \cos(90° - x)$ and $\sin(90° - x) = \cos x$, but the candidate's analysis does not proceed this far.

Artefact 4, participant 26



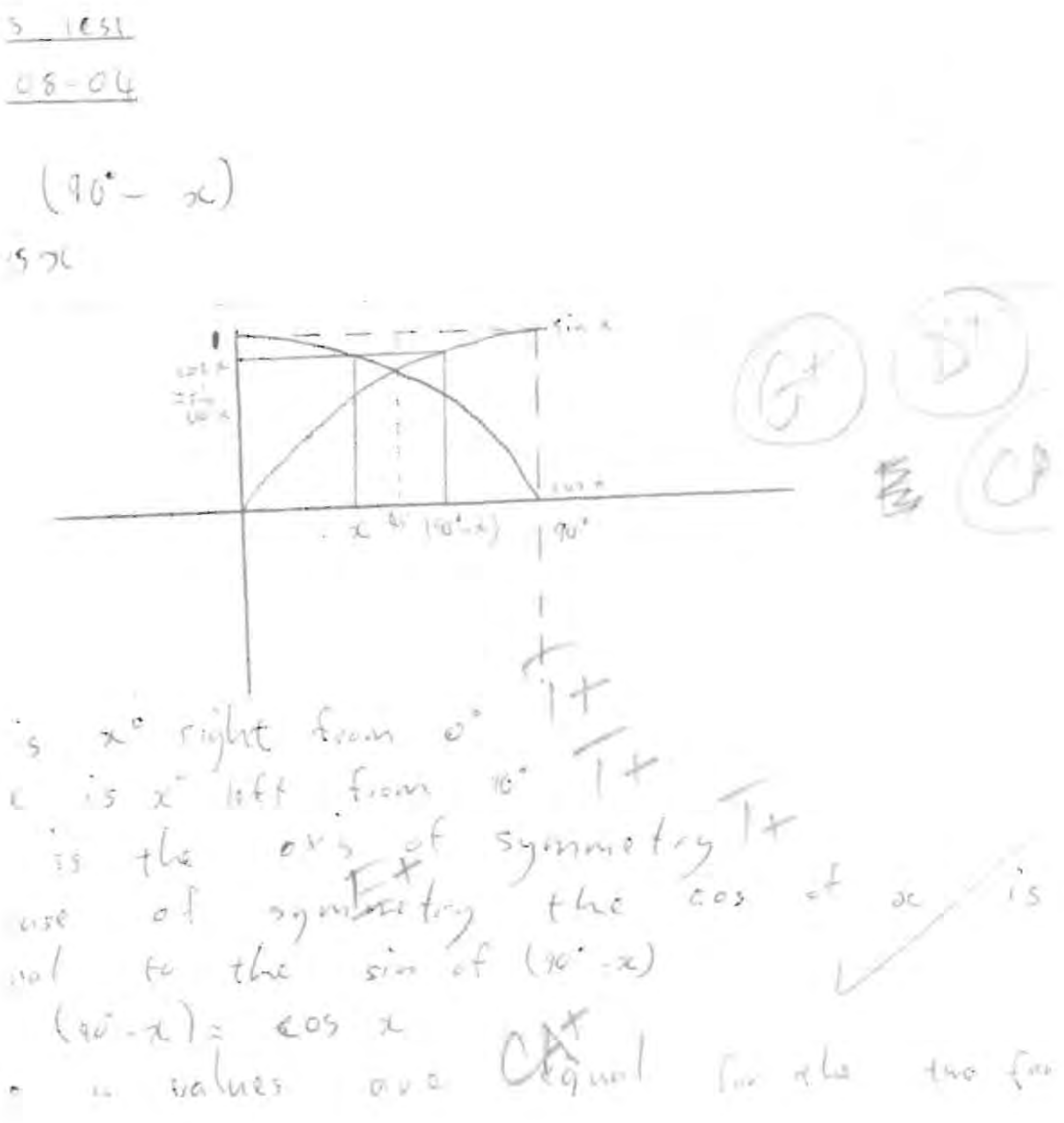
Translation[27]

Draw a line relative to the x axis from a point to 180° (180° + θ)
To the left (negative x axis), the value on the negative x axis = –x.
The line that joins these two points cuts the y-axis at a certain value, therefore both have the same y value.
Join the point on the graph with the x-axis by drawing a vertical line relative to the y-axis.
Two symmetrical figures are formed so (the distance) $-x = (180° + x)$ *as well as their corresponding y-values (because their y co-ordinates are the same)*

In explaining the truth of the identity $\sin(180° + x) = \sin(-x)$ the candidate has the insight to extend the graph to incorporate negative angles. The use of symmetry strengthens the candidate's argument, as does the recognition that the y values or function values are the same. Both the graphical depiction and the use of language, as well as the use of the interval $x < 0°$ are sufficient to accredit the candidate as almost attaining conceptual closure (D+ -) since the argument is not clinched in terms of articulation.

[27] This is my translation. I am a qualified bilingual teacher and in terms of my professional educational qualifications (Higher Education Diploma, 1980) am qualified to teach in English and Afrikaans.

Artefact 5, participant 14



Clarity in the graphical depiction and a conciseness of the description and explanation, that $\cos x = \sin(90° - x)$ demonstrates an acquired conceptual depth inferring conceptual closure in terms of the cognitive consciousness of this participant. The inclusion of the general and specific case positions of x in the graphical depiction seems to support my decision to acknowledge the participants conceptual closure even though the implication of the specific case where the functions are equal is not given special attention by the participant. It is clear that there is no tension between concept image and concept definition.

Level 2: Learners affective experiences

7.13 Learners' affective responses

At the end of the intervention period learners were asked to respond to three questions in writing for the purpose of ascertaining the nature of their experience in terms of the requirements for them to communicate about mathematical tasks.

The questions:

1. Please write about how graphs have affected your ability to explain the mathematics you are doing.
2. Please write about how having to use words to explain the mathematics you are doing has affected your understanding of mathematics.
3. Please write about the feelings you have when you have to explain the mathematics you are doing.

This level of analysis sought to add credibility and validity to the quantification of the coded data. The lived mathematics experience of learners adds a qualitative affective dimension to the data as reflected in the following excerpts. In response to questions 1, 2 and 3 respectively, participant 18 offered the following insights.

'Well, graphs are very useful especially in trigonometry. A lot of the time the easiest way to solve or check a problem / answer is when you draw a graph. The graphs are usually easy to draw and the answer most of the time obvious. For example if you are not to certain of a answer like $\cos 0°$ *you draw a cos graph and you immediately see it is equal to 1. When you don't know where to start when a explain a trigonometric rule, the easiest place to start is by first drawing the graph of the certain ratio. You see straight away the relationship between the values and how to explain it (most of the time).'*

'If you can explain the certain mathematics then it shows that you understand it. The way you explain it also reflects how much you remember since the time you have been taught it. And for me the more I practice it the better I remember it. The better I remember it the better I explain it.'

'I don't enjoy it so much because I don't enjoy it. But I know the way I explain it is a reflection of how I understand it...'

Level 3: My role as teacher in fostering and understanding mathematics through reflective metacognitive deconstruction of concepts ensuing from engaging with learners' textual accounts.

7.14 Cyclical reflection

The cyclical reflection characteristic of the hermeneutic circle progressively focuses learners' understanding of concepts and serves the function of reducing dissonance between concept image and concept definition. The tensions between the two reduce with repeated cognitive forays until I can facilitate their merging for learners in a point of conceptual closure. The meta-analysis is marked by hermeneutic and empirical elements of the phenomenological method, first in terms of the sense making I need to make as a teacher and secondly in terms of the sense I make of the way learners make sense of their written mathematical narrations.

The extent to which these are portrayed result in episodes of communicative action that reflect nurturing and caring in a spirit of recognising personal and social justice. I further play out episodes of communicative action in gestures and gesticulations that aim to provoke a critical consciousness about mathematical reality.

7.15 Summary

This chapter sets out the methodology utilised in this work. The research is theoretically located in the critical realist paradigm and appropriates the methodological plurality of this metatheory in applying both qualitative and quantitative methods in an investigative case study. A heterogeneous cohort of 27 learners comprises the participant group with me serving as teacher, facilitator and participant researcher. The coding instrument and learners' affective responses to questions generate the data, the analysis of which is the focus of Chapter 8.

The work of the following chapter deals with the findings of the research data. It interprets the coding data generated by the coding instrument described in this chapter and undertakes a hermeneutic analysis of the mathematical articulations of participant learners and finally probes for meaning in the affective responses of learners to questions regarding their experiences of having to engage in vocabular action about mathematics tasks.

VIII

Probing and reflecting on the data

$$\sum_{n=1}^{n} 2\left(\tfrac{1}{2}\right)^{n-1} < \sqrt[5]{e^{4.3i}}$$

'Mathematical notation does not simply describe mathematical phenomena, it activates it. Language does not describe action, it is part of it... '

(Brown, 1997:219)

8.1 Introduction

Inasmuch as Piaget's notion of cognitive equilibrium is a fulcrum between the accommodation and assimilation of knowledge so the Vygotskian imperative of language as a social essential for the transfer and acquisition of concepts provides the fluid medium for the flow and articulation of acquired knowledge. Graphicacy akin to the Brünerian construct of ikonic mediation in this work serves as a stimulus to access conceptual knowledge and to facilitate its articulation. Habermasian communicative action provides a theoretical ambience for meaningful mathematical conversation that establishes consensus around meaning. Articulation manifests an acquired degree of Freirian critical consciousness; a consciousness that demonstrates a growing alienation from mathematical mutism that ultimately seeks to recognise learners as dialogically contributing to personal and social mathematical discourses. It is envisioned that in this process emancipated thought restores personal justices through the accessibility of mathematics. In turn the recognition of personal justice may translate into social justices that should pervade our educational landscape and provide opportunities for the citizens of this country and its emerging economy.

This chapter reflects an emerging understanding in terms of the theoretical frameworks described in the previous paragraph. From its hermeneutic endeavours it makes suggestions about what has transpired as a result of reversing the abstractual order of concepts in a pedagogic environment for both teacher and learner and recognises the value of concrete depictions of algebraic constructs through its graphical representations. It seeks to establish whether the method contributes to sustaining mathematical meaning through mapping progressive summative scores as indicators of achievement in addition to analysing learners' mathematical articulateness. Various sub- case studies or cameos emanate from within this case study that seek to investigate authenticity at different levels or strata, since in critical realist terms each strata serves to understand and explain strata adjacent to it.

Throughout the work there is a metacritique of practice as it relates to this intervention. 'Abduction' and 'retroduction' as modes of thinking in critical realist theory seek to interpret phenomena in terms of new frames of reference, and illuminate pre-conditions for social cohesion respectively. The abductive recontextualisations are unique to each learner as each has a personal and social frame of reference. The empirical phenomenological sense making that I engage in tries to correlate learners' assignment of meaning to an established outcome and to learn from their accounts as to what is required to merge concept image and concept definition. The metacritique seeks further to accommodate deviations from expected outcomes as presented by learners in assessing the authenticity of their arguments relative to their articulated mathematical understanding. There is a constant sense of emancipatory and transformative purpose which threads itself through the work.

The application of the coding instrument to the mathematical tasks is described with reference to artefacts 1 to 5 in the previous chapter. The data originating in this way were captured in SPSS (Statistical Package for the Social Sciences) database through recording each occurrence of a code across the data for each learner. Statistical analyses revealed no significant correlations between codes or categories of codes. The data were then subjected to spreadsheet analysis to examine descriptive statistical features over time.

8.2 The occurrence of language and concept indices over time by cohort

Table 8.1 provides an abridged table of codes that are used in the ensuing analysis.

Language codes	Concept codes	Conceptual Depth
Coherence	Conceptual accuracy	Tenuous
Explanation	Arithmetic/Algebraic accuracy	Abridged
Articulateness	Graphing competence	Closed
Terminology	Integration of concepts	

Table 8.1

Abridged table of codes

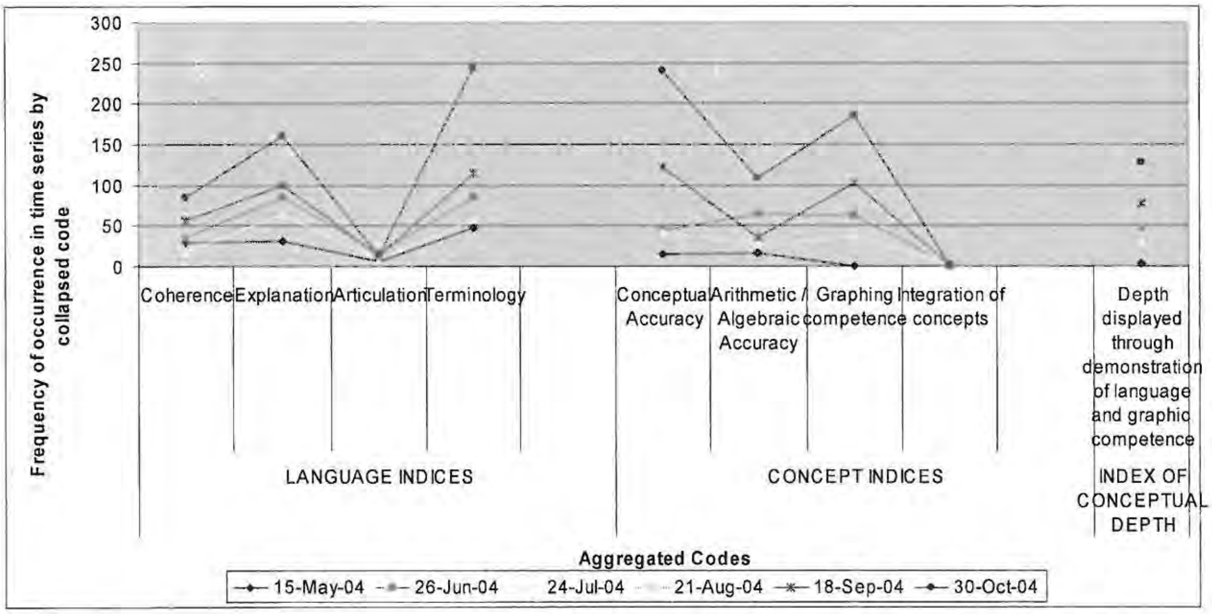


Fig. 8.1

Aggregated[28] language, concept and conceptual depth codes by cohort over time

A feature of the data in Fig. 8.1 is the clear increase in the occurrence of codes generally by learners from May to October which seems to demonstrate that learners were becoming more familiar with the task of explaining or describing mathematical tasks and that there was an increase in mathematical vocabulary that facilitated such linguistic expression. It should be noted that the task size was similar in each event. It is surmised therefore that the emphasis on communicating in mathematical terms

[28] Aggregated codes reflect collapsed levels of competence within codes.

reduced the mutism experienced early in the year as reflected in the low frequency of code occurrence in May 2004, relative to the increase in absolute frequency across codes in October 2004, and a tendency toward greater communicative action. I believe the increased and sustained ability to be attributable to the pictorial value of graphic representations.

Fig. 8.1 shows an increased upward dispersion of frequency markers for all codes except for 'articulation' and 'integration of concepts'. I imagine this may be accounted for by the fact that thirteen of the twenty seven learners do not speak mother tongue English, the medium of instruction. The remaining fourteen learners are from coloured backgrounds and it is surmised that a dialectic influence may stultify 'articulation' to the degree expected by me. The inertia related to 'integration of concepts' is largely, I imagine owing to the higher order cognitive demand of such cognitive capacity, and because it is by nature not spontaneously forthcoming from learners at an early stage of communicative action, or so this research seems to suggest.

It is notable that of the remaining codes the upward (denoting time series) dispersion of markers is pronounced for 'explanation' and 'terminology' under the language codes and for 'conceptual accuracy' and 'graphing competence' under concept codes. Seen as four codes these bridge language / concept divides and are seen to be the effect of the first principles graphical approach to deconstructing dense algebraic concepts. The similarity in the degree of dispersion between the 'conceptual depth' code and the 'explanation' code is observed with interest. In essence therefore it seems that graphicacy promotes a critical consciousness that finds expression in the mathematical discourse of learners.

Fig. 8.2 shows only the positive aggregated codes which reveal a perturbation in the increasing frequency of code occurrence over time. This may be attributed to the fact that the July task took place seven weeks after the commencement of the June mid-year vacation; a long break soon after the commencement of the intervention. The August task involved two components, explanation and description prior to and post-instruction on trigonometric reduction identities for which the rote 'cast' diagram method was vociferously used by learners and which demonstrated tenuous understanding of the concept.

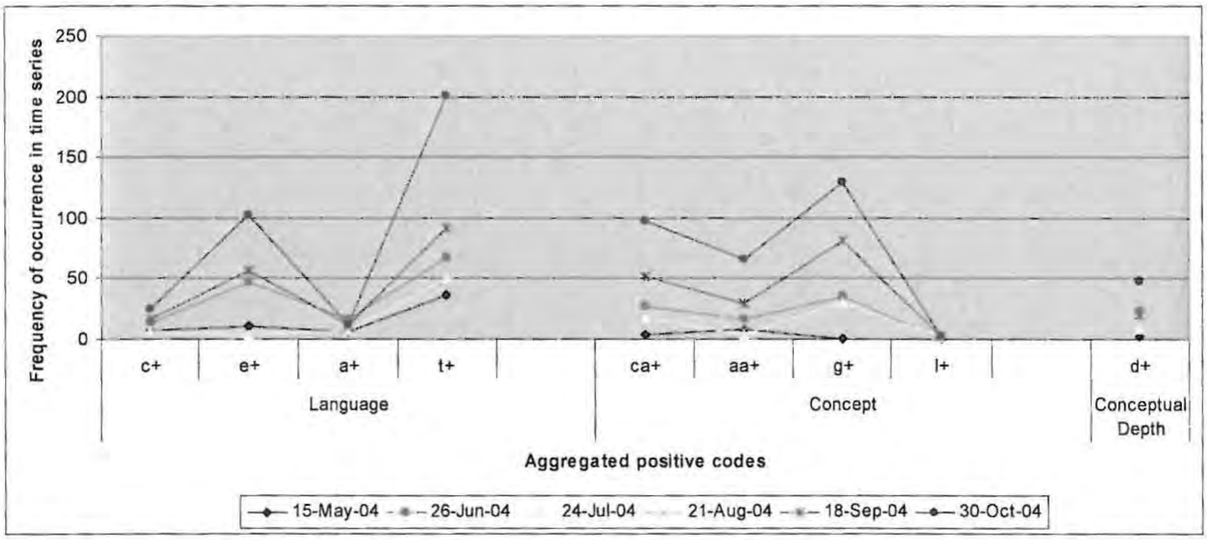


Fig. 8.2

Positive aggregated language, concept and conceptual depth codes by cohort over time

Figures 8.1 and 8.2 reflect similarities in the patterns of code occurrence.

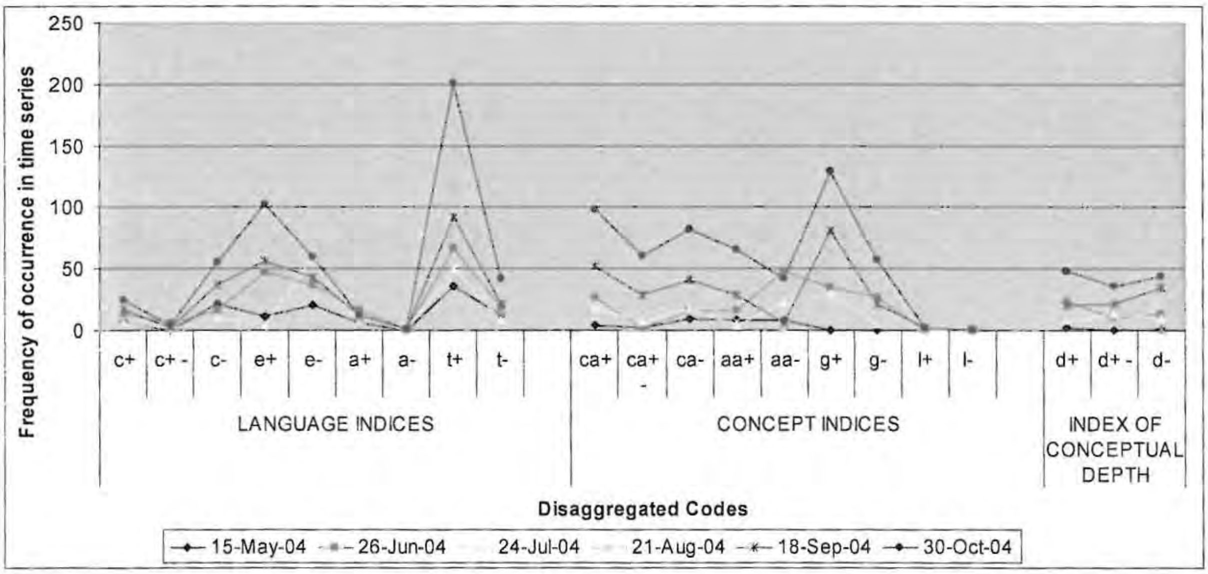


Fig. 8.3

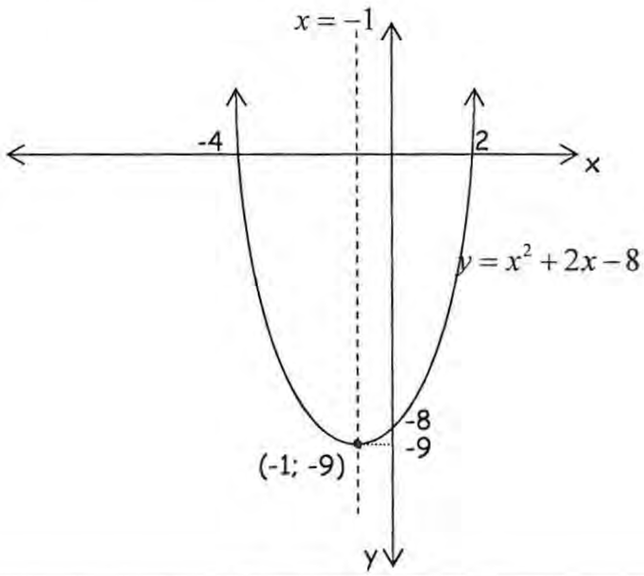
Disaggregated language, concept and conceptual depth codes by cohort over time

Fig. 8.3 showing the disaggregated codes is included for sake of comprehensive reporting and to unmask the fact that 'minus' codes that designate weakness or incompetence in language or concept related codes also show an increase over time and may be a reflection of weaker learners' performances. The compactness of the distribution of the markers for these 'minus' codes in comparison with their positive counterparts indicates possibly that those who are good get better and I venture to suggest that conceptual depth is situated and reflected in the use of mathematical language. The fact that nearly 50% of the cohort has English as a second language may further explain the increase rather than a desired decrease in the frequency of the 'minus' codes.

8.3 Comparing achievement scores by cohort for a quadratic theory task constituted of algebraic and graphical components

Learners responded to questions in a controlled environment independently of one another to quadratic theory questions. Learners were *not* informed that the questions would form part of either assessment. The purpose of this was to ascertain whether the learners sustained understanding and if they would resort to graphical representations to describe their algebraic processes successfully. The explanation code as a dominant indicator was used to assess the vocabular responses that describe the algebra component while the algebra itself was scored normatively. Fig. 4 represents the scores achieved on exactly the same questions fourteen months apart without learners having been instructed to prepare for either task. The tasks are recorded in Table 8.2 below.

The questions below are based on the graph of $y = x^2 + 2x - 8$ which is drawn below.



Task		Question
1	Algebraic/ Algorithmic	Determine algebraically, for which values of k, $x^2 + 2x + k = 0$ will have real roots.
2		Determine algebraically, for which value of t, $x^2 + 2x - 8 = t$ will have equal roots.
3		Show that the equation $x^2 + 2x - 8 = p^2$ has real roots for all $p \in R$.
4	Graphical/ Interpretive	Explain how the answer to question 1 can be obtained by inspecting and moving the graph of $y = x^2 + 2x - 8$.
5		Explain how the answer to question 2 can be obtained by inspecting the graph of $y = x^2 + 2x - 8$.
6		Explain how question 3 can be interpreted graphically?

Table 8.2

Quadratic theory tasks

For purposes of cohesive reporting the questions have been rearranged so that algebraic responses can be compared with responses that offer graphical descriptions. In the original exercise learners were required to interpret the graph immediately after having completed the algebra. The metacognitive construction of the task is bedded in the belief that algebraic procedures tend to be isolated from the graphicacy that depicts it; a separateness that encourages rote manipulation rather than understanding. It is the rote nature of such algebraic tasks that may ruefully establish mutism in terms of mathematical conversancy. The assertion is that the association of the algebra with the graphic representation provides the concrete mechanism that helps learners engage consciously with the reality of the mathematics at hand. For now, the cohort's performances on these tasks are examined and an in-depth analysis of selected learners' attempts is dealt with later in the chapter.

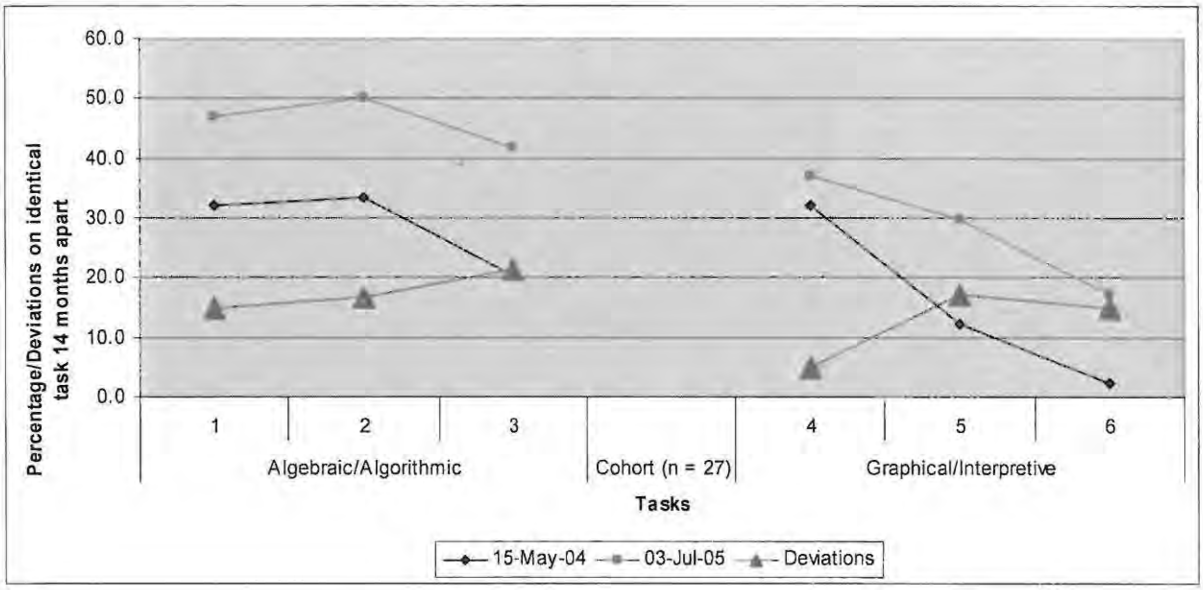


Fig. 8.4

Average percentage per task by cohort on the same unprepared quadratic theory task fourteen months apart, showing also the deviation between the scores for the respective tasks

Fig. 8.4 shows that an increase in algebraic competence in discerning information regarding delta. The relatively poor performance on task 3, however, is associated with the use of p^2 as the constant in the question, and secondly the common weakness of not being able to translate from the English, the need to determine an expression for delta in terms of p and from there proceed to analyse the sign of delta. There is a repetitive tendency to equate delta to zero and find a value for p. This demonstrates the rote behaviour that is associated with finding the values of an arbitrary constant if a condition is imposed on delta; the type of questioning that seems to precede those question types that necessitate analysing delta. Although weaker by performance there is a concomitant increase in achievement on graphical tasks.

The generally weaker performance on interpreting the algebraic solutions in graphical contexts is expected given the linguistic and vocabular deficits of learners. Although weaker there is improvement over time in dealing with the vocabular responses. It is also a question out of the ordinary and the confusion with p^2

continues in that it is regarded as a parabola. The underlying aetiology here is attributed to not seeing p^2 as a constant. For teaching, it would seem essential to ostensibly designate x as the variable as indicated by the labelling of the axis in the sketch that accompanies the question, with the implication that other occurring letters are constants. Further to this, the positivity induced by the square seems not to register amongst learners that a constant squared remains a positive constant. The synonymy of y in $y = x^2 + 2x - 8$ with p^2 in $x^2 + 2x - 8 = p^2$ was rarely observed by learners and it would seem to be required that 'taken for granted' techniques like comparison by equating terms needs to become a deliberate part of a teaching repertoire. Failure to achieve this level of cognition escalates misinterpretation since $y = p^2$ is not seen as a family of parallel lines that lie on or above the x axis and that therefore produce two real roots as each horizontal line intersects the arms of the parabola. Construed in this way the equation will have real roots for all $p \in \mathbb{R}$, since if $p \leq 0$, $p^2 \geq 0$. I believe that it is this level of mathematical discourse that constitutes communicative action, between all classroom participants that promotes consciousness about the mathematics at hand. The graphical interpretation of the solution contextualises the mathematics of the quadratic theory, and as participant 7 suggests relating theory to the appropriate graph is like adding a 'practical' component to mathematics.

The increase in performance in the tasks over time without prior warning of assessment content would seem to suggest that there is sustained mathematical meaning, all the more because of the short period of the intervention, inferring that the systemic use of graphical approaches may have greater power for sustaining mathematical concepts and their meaning over time.

8.4 Comparing the performances of the taught cohort with an untaught cohort on the same tasks

I sensed in me an intuitive restlessness about how learners not represented on the programme would respond to the same questions and so approached an affluent ex model C school where academic standards are high, indicated by a norm of in excess of 40 distinction aggregates per annum of which in the region of 25 per annum are also mathematics and English distinction candidates. My reasoning was that this calibre of learner would respond more favourably to the depth required for analytic descriptions of tasks in graphical contexts. The English prowess of these learners further suggested that the tools for textual graphical analyses would be in place. The taught and untaught cohort is also sharply contrasted in terms of poverty and affluence respectively. A grade 11 HG mathematics group of 89 learners (of three teachers) was given the opportunity to do the same tasks about two weeks after the completion of quadratic theory. Fig. 8.5 compares the performances of the taught and untaught cohorts on the same tasks undertaken at similar times relative to the initial completion of the teaching of the content.

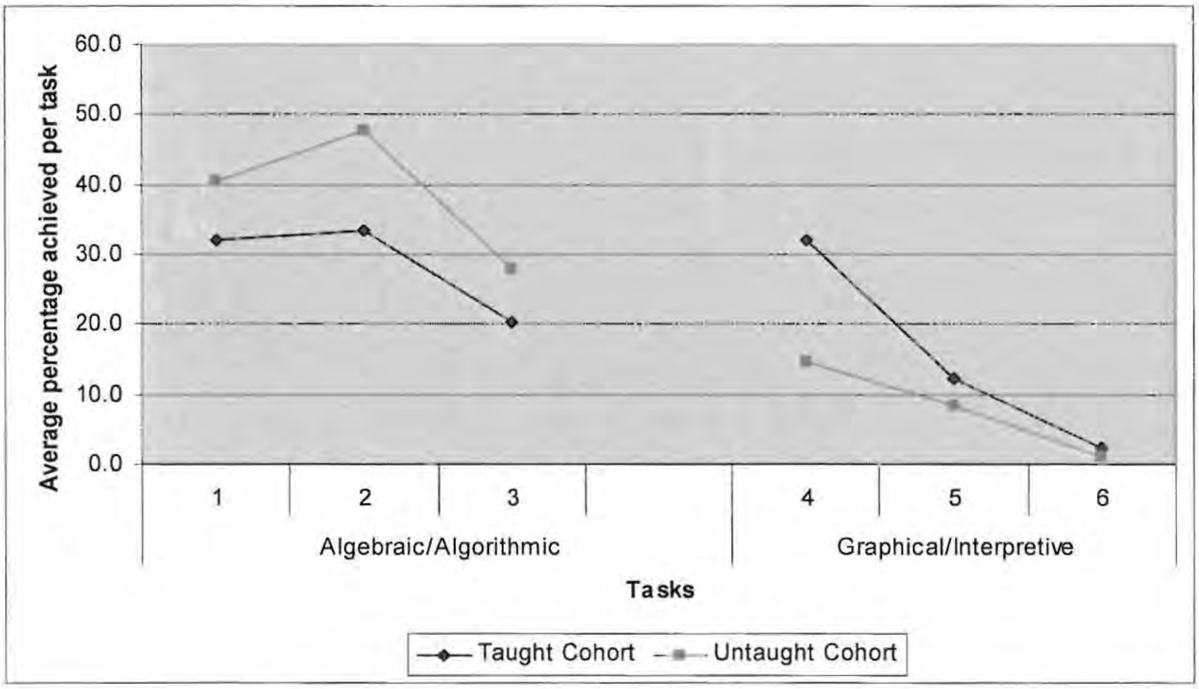


Fig. 8.5

Performance of taught and untaught cohorts on the same quadratic theory tasks

Fig. 8.5 reveals an interesting result. The performance means per item/task for the algebraic components are similar for both cohorts with the same occurrence of the inability to discern from the English the need to derive an expression for delta and proceed to analyse it. Of greater interest however is the performance of the untaught cohort being about 20% less on task 4, about 5% less on task 5 with virtually no difference on task 6. A further observation is that the untaught cohort performs considerably poorer on the simplest interpretive task, task 5, with an average score of about 15% less. These results would seem to suggest that rote performance tarnishes the insights and understanding of the untaught group as well. The failure of a socio-economically and academically elite group on task 6 indicates further that the subtleties of the mathematics relating to quadratic theory in graphical contexts were not addressed. It begs a question about the quality and depth of mathematics teaching and the consciousness that learners exhibit about what it is they are doing in mathematics classrooms in general. It raises questions too about personal justice not only for the disadvantaged but also for the advantaged who seem not to have acquired conceptual depth that may be an indicator of sustained mathematical meaning.

8.5 Comparing entry and exit level means for the taught cohort

One of the objectives of this research, is stated in the goal of investigating and understanding what, in learners' experiences are essential to them in coming to know and be fluent in mathematics, and more especially in this regard, assess the value of a graphical approach to introducing and sustaining complex and abstract mathematical concepts. The overall mathematics performances are therefore used as a measure of this goal. It is acknowledged that no categorical causality is claimed to connect the goal and the final outcome since factors like increasing maturity levels will impact results as will, greater focus toward final examination preparation. The magnitude of the difference between consequent scores may reflect the degree of impact and provide some indication that the method has enjoyed benefit in its application across the mathematics curriculum in its use by learners. This conjecture is corroborated by learners' opinions like that of participant 25 who testifies that, "*By fully understanding graphs, I understand a lot of the algebra which I struggled with previously. The graphs have helped me visualise what most of the equations mean and I can now understand that an equation can be represented by two graphs on the same set of axes. Graphs have explained and helped me explain the logic behind my steps when solving equations. Before, I could never explain why I did the steps, because I never understood the logic, but graphs have made me realise why it is logical... the graphs have helped me understand concepts.*" Participant 14 similarly claims, "*Graphs have definitely made explaining maths easier for me... With explaining maths I think it's all a matter of confidence; confidence in your abilities and confidence in your explanations. With the increasing exposure to explaining my work I feel that I'm slowly but surely developing this confidence to explain what I am* *doing and in this increasing my understanding of mathematics in general, instead of just knowing it.*" These responses seem to suggest that the objectives of sustaining mathematical meaning and inducing critical consciousness about mathematical tasks were in the process of being accomplished by the cohort and for others who may engage with the same experiences of delving to articulate what it is they are doing mathematically.

On entering the programme participants were administered an entry level baseline assessment in order to ascertain the mathematics proficiency of the cohort. This is all the more relevant because of the fact that these learners were at sixteen different high schools of varying academic standard throughout the Cape Peninsula. The entry level assessment further established a baseline against which to maintain some measure of increasing general mathematics proficiency and competence. The entry level assessment was composed of the items stipulated as exit level assessment standards for the participants' previous grade, Grade 10, in the National Curriculum Standard for Mathematics (DOE, 2003a).

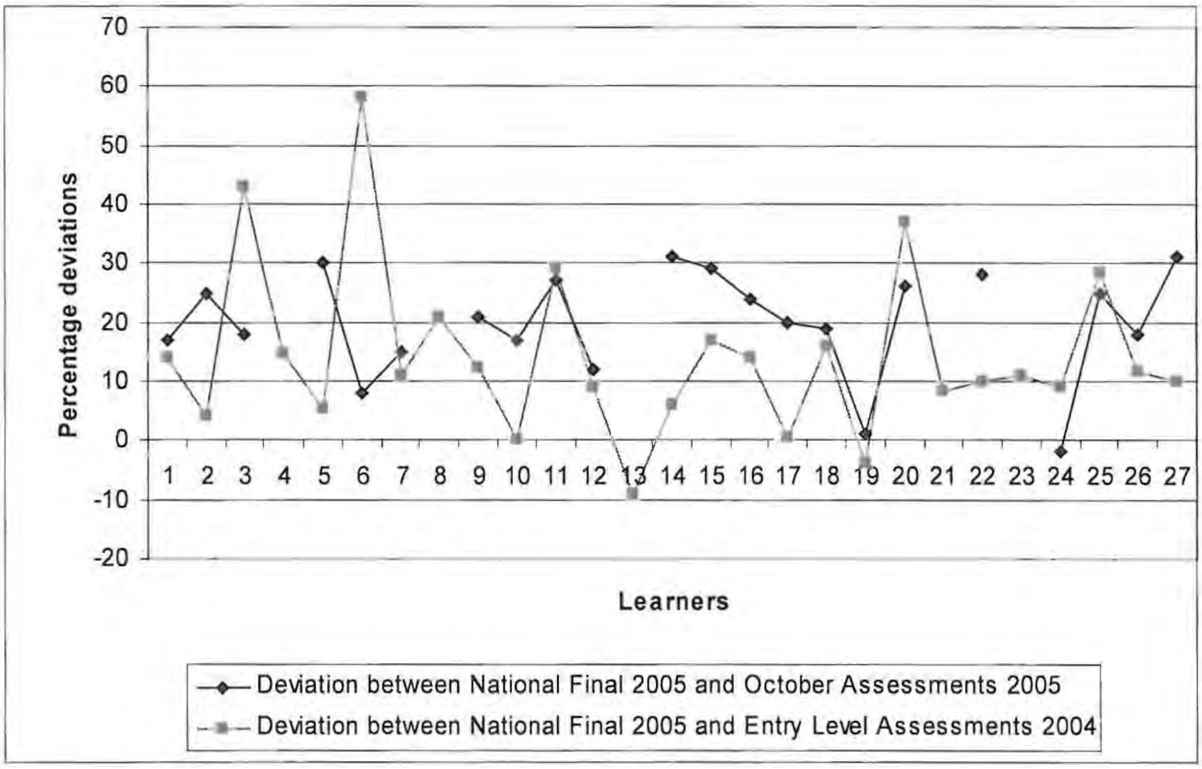


Fig. 8.6

Comparing differences in overall mathematics performance between the final national examination and the internal October assessment

Fig. 8.6 shows the deviation between the final mathematics examination score and the internal October assessment for all participants. Discontinuities reflect learner absence from the October assessment owing to commitments to other final examinations, and closure of the official programme a week earlier. The data reveals a smaller difference between the October and final results when compared with the difference between the entry level assessment and the final result. This is not extraordinary since learners are seen to have grown in mathematical proficiency since the inception of the programme and the lesser deviations between October and the final assessment suggests this. The greater deviations between the entry level and final or exit level scores is a measure of considerable improvement to which some credibility may be added if one considers that the programme's duration was 84 contact hours of tuition relative to an average of 400 contact hours over two standard academic school years. The point being made here is that the final scores for learners may have been considerably higher given an increased number of contact sessions. The validity of such a claim could be disputed were it not for learners' accounts like participants 14 and 25 who attribute their increased understanding and mathematical conversancy to a graphical first principles approach to teaching.

There is a positive deviation between the entry and exit level assessment scores for all but four of the participants. Two of the four, participants 10 and 17 show a zero deviation while participants 13 and 19 show exit level scores less than their entry level scores. The factors that may account for this are not easy to speculate about other than the fact these learners were the weakest in the cohort and that the comprehensiveness nature of exit level assessments may have affected these participants adversely.

Of particular note is the fact that the largest deviation between entry and exit level scores was achieved by two isiXhosa learners. The mean deviations for the isiXhosa, English and Afrikaans language groups respectively are 19%, 11% and 11%. The largest mean deviation achieved by the group of 11 isiXhosa learners is a function of low entry level scores that reflect inaccessibility to mathematics teaching at a level of fundamental analysis. However, in recognition of the fact that these learners receive instruction in their second language, it raises the question of whether the pictorial value of graphical representations serves as a mediating language between algebra and English; a post-doctoral research question worth pursuing. In the final analysis though, it is those learners whose mother tongue is not English whose mathematical proficiency has shown the most profound improvement. In terms of the ontological values that pervade this research it is significant that those whose mathematical deficits are the greatest are also those whose accessibility to mathematics and mathematical meaning has been the greatest. In these terms there is indeed a restoration of personal and social justice.

Assessments	80%+	70-79%	60-69%	50-59%	40-49%	33-39%	30-32%	20-29%	0-19%	Number of participants	Mean %
	A	B	C	D	E	F	FF	G	H	TOTAL	
Entry	2	3	5	5	3	3	1	4	1	27	49,7
Exit	8	3	4	5	6	0	0	1	0	27	64,1

Table 8.3

Improvement of cohort from entry to exit level assessments

Table 8.3 reflects the general improvement from entry level to exit level in terms of a distribution of symbols achieved by participants noting a 14% increase in performance mean from entry to exit level and a migration of symbols into the upper quartile. The value of the data is partly enhanced by the fact that I've observed an acceptable trend for mean performance scores to remain constant or decline over the last two years of schooling. This, one supposes, is largely due to the increase in content volume, increasingly difficult concepts and more sophisticated questioning that searches for deeper understanding and the ability to apply knowledge compositely.

Fig. 8.7 compares the entry and exit level means by mother tongue groupings. The triangular markers reflect the difference between the group means. It is clear from this plot that although the mean for the English and Afrikaans participants was stronger than the isiXhosa group the percentage improvement by isiXhosa participants was double that of the English and Afrikaans groups.

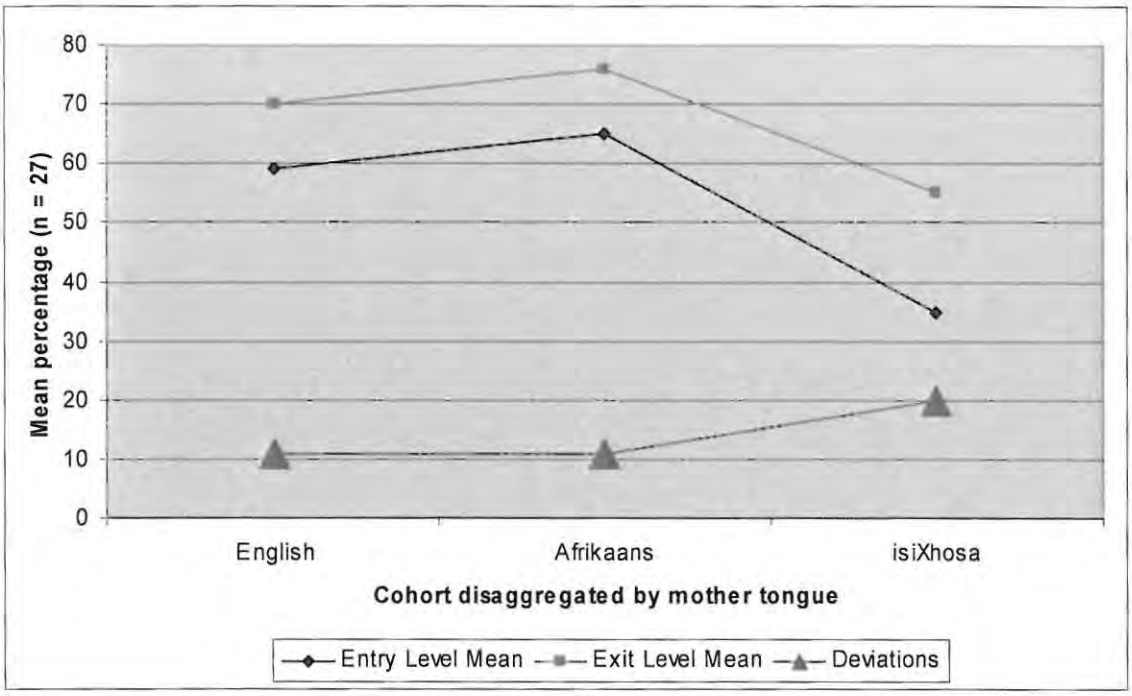


Fig. 8.7

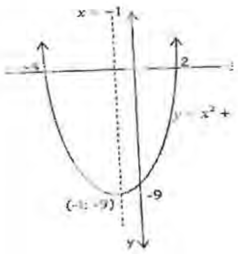
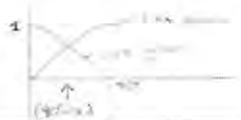
Comparing entry and exit level means by mother tongue

It is tempting to attribute the improvement in performance over the two year period to the use of the reversed abstractual order inherent in a first principles graphical approach to teaching; an improvement - the significance of which is possibly magnified by the 20% difference between entry and exit level means relative to the 10% difference for the English and Afrikaans speaking participants. It is with less confidence that I am prepared to conjecture that the concreteness of the graphicacy through its ability to assist in the deconstruction of dense concepts aided second language English speakers to conscientise the mathematics and to accommodate otherwise covert information in a more tangible context that enhanced the meaning of what may otherwise have remained for them routine algebraic algorithmic manipulations.

8.6 Empirical phenomenological sense making of six selected learners mathematics portfolios

The following section of the analysis aims at understanding written tasks selected from the portfolios of selected learners. Six learners were selected so that the group is constituted of black and coloured learners, from all three language groupings and representing weak to strong ability levels. It aims to show improvement in critical mathematics consciousness over time and also reveal the impact that learners describe about their own changing experiences of mathematics. All participants' responses are typed without erasure or alteration.

Participant 3 Entry Level Mean %: **15** Exit Level Mean %: **58** Mother tongue: **isiXhosa**

Quadratic Theory	Absolute value	Trigonometry	Learners affective written responses to questions
Explain how to read the graph to obtain the values of t for which $x^2+2x-8=t$ has two negative roots.  Not attempted	*Sketch* $y=-\|2x+4\|-2$ This is a graph of an absolute value. Just by looking at the equation $y=-\|2x+4\|-2$ you can tell the value of the salient point because the equation is represented as $y=\|a+p\|-q$ you can see that the SP is $(p;q)\therefore p$ is the x-co-ordinate of the SP and q is the y-co-ordinate. In this case $y=-\|2x+4\|-2 \therefore if$ $2x+4<0$ $-2x<4$ $x<-2$ $and\ if\ 2x+4>0$ $\therefore 2x>-4$ $x>-2$ 	*Explaining* $\sin(180°-x)=\sin x$ Step1: which quadrant is it located $(180°-x)$ =2nd quadrant 2: is the ratio positive or negative = positive (because it is in the first quad) 3: $(180°\pm)$ Rule use same ratio *Explaining* $\sin(90°-x)=\cos x$ If symmetrical line were to be drawn at the point $(90°-x)$ where these two graphs intersect one half would be a mirror image of the other $\therefore \sin=\cos^{-1}$ (e.g. sin is equal to the inverse of cos, that is why $\sin(90°-x)=\cos x$ 	Responding to how graphs has affected his ability to explain mathematics tasks, he says, 'Graphs are where equations are derived from, so by knowing how graphs work, I can be able to understand the principles of mathematics, so that I can be able to say wether a statement is true or False. And explaining mathematics using graphs gives learners a much broader understanding.' In responding to how using words to explain mathematics has impacted him, he responds, '… (it) helps me very much, because by using words to explain the work that I am doing, I am able to recognize my mistakes, and I learn to know exactly what I am doing and more importantly, why? Because it does not help knowing how to solve an equation if you don't no why you are solving it. I think using words and graphs to explain mathematics goes hand in hand.' About his feelings about having to explain or describe the mathematics he is doing he responds, 'At first, I did not like writing/explaining the maths in words. The reason was the fact that I did not know exactly what I was doing. I was just solving equations and doing mathematics, not knowing the reason why… then I learnt that knowing the principles of mathematics helps when solving any equation. This is why I think explaining the work… helps me understand, it even better every time I explain it… and explaining mathematics in my own way of seeing it means that I learn in the process.'

8.6.1 Reflections on the work of Participant 3

For a second language English speaker to produce this level of coherence in expressing a perceived understanding of mathematical constructs is an indication of deliberate endeavour.

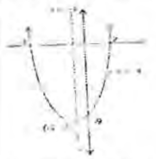
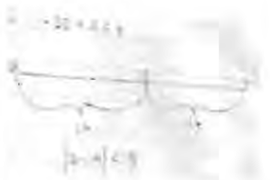
The unattempted quadratic theory task presented difficulties not only for this participant. It would seem that the task is made complex by the need to visually co-ordinate the possible positions of t as a family of constant functions relative to the fixed position of the quadratic function.

Elaborating on the procedure for sketching the absolute graph, the participant uses correct terminology like 'salient point' but seems to resort to rote understanding of the definition of $|2x+4|$ with some convolution without realising that his endeavour has been to find that the function is decreasing and increasing on the left and right of -2 respectively. Although the graph reflects this (albeit with an omission of sign in incorrectly designating 2 as the salient point) his misinterpretation relates also to the use of $(p;q)$ as the salient point for $y=a|x+p|+q$. His error in failing to recognise the negative coefficient of the absolute value bracket in, $y=-|2x+4|-2$ produces a sketch far from correct. However, the exercise of self dialogue about what he is doing indicates a conscious engagement with the concept.

In the first trigonometry attempt, with no instruction on how to use the graph, the cognitive product is one of the rote use of the CAST diagram. Other than the reference to $(180°-x)$ being in the first quadrant, the argument is correct, but lacking in terms of a critical mathematical consciousness because the attempt is far removed from its authentic context. The CAST diagram is an artificial reference system of value in instances only once the contextual logic is established. In his second attempt the domain that is generally represented by quadrant I, situates an authentic context for the explanation of the truth of the identity. His graphical representation is in terms of a rough sketch sufficiently accurate given his reference to symmetry in his exposition. The fact that he opts for the special case where $\sin 45° = \cos 45°$ also reduces the generality of the identity, but provides for the teacher an excellent interlude to expound the value of generalising. The use of $\sin^{-1}$ to designate an inverse is probably linked erroneously to exponential theory. The use of 'inverse' I believe, at this early stage of dialogic endeavour as synonymous with 'co-function' is not uncommon. It does provide for me purpose for remediation and for teaching the necessity for rigour in mathematical register, and opportunity to introduce and differentiate between the terms, 'inverse', 'reciprocal' and 'co-function'. And ultimately, had this attempt at personal dialogic endeavour not have taken place the opportunity of knowing the cognitive discordances of this participant would have been lost.

What is demonstrated in his affective responses seems to be a genuine awareness of the generic roots of mathematical constructs and an appreciation for the fact that having to explain his processes has enhanced his understanding, seemingly confirmed by his 43% improvement from entry to exit level mean score.

Participant 6 Entry Level Mean %: **29** Exit Level Mean %: **87** Mother tongue: **isiXhosa**

Quadratic Theory	Absolute value	Trigonometry	Learners affective written responses to questions
1. Explain how to read the graph to obtain the values of t for which $x^2+2x-8=t$ *has two negative roots.*  You use the formula (delta) that you use when you are checking the nature of the roots, but in this we are told that t.... 2. *Explain how to read the values of* k *for which* $x^2+2x+k=0$ *will have real roots.* k is taken to be the y-int, if we want the roots to be equal, the parabola must be shifted upward such that the x-axis is tangent to the curve. By doing that the parabola will cut the y-axis at 1, $\therefore$ if k is equal to 1,... the parabola will have equal roots, (so) for any value of k <u>greater</u> than 1, the roots will be real.	*Write* $-32<x<4$ *in the form* $\|x+a\|<b$ You must first add the two number and find the half of these two numbers. After that draw a number line and write 18 down; 32 and find the no. 4 of the same distance. After that put an absolute value that is my understanding 	*An explanation for the truth of* $\sin(90°-x)=\cos x$ When you have an expression of $\sin(90°-x)$, that is $90°$ rule, and it have to change then you will have $\cos x$, it is no longer $\sin(90°-x)$, even if we had $\cos(90°-x)$ was going to change to be $\sin x$ ...my teacher taught... you must just take off the thing that was putted and then you put another one if it has s you then put c... 	Responding to how graphs have affected his ability to explain mathematics tasks, he says 'Graphs have increased my maths understanding.' In responding to how using words to explain mathematics, he responds, 'I think it has helped me to achieve better result and to understand what mathematics is all about, by explaining something makes you to understand more and more, but sometimes it may seem as a frustrating thing, but at the end it really helps. I think it is good for one to be able to explain maths because you will also explain it to other people then you also gain knowledge and experience.' In responding to his feelings when he has to explain mathematics, he says, 'Sometimes I feel frustrated, because mathematics is not an easy thing, it needs someone who is committed to his work/books, in order for one to be good for explaining maths, you must be a workerholic, and then it becomes difficult for one who is taught, by a teacher who do not explain.'

8.6.2 Reflections on the work of Participant 6

Participant 6's first task on quadratic theory demonstrates the reliance on tools like 'delta' to obtain a solution despite being given the sketch from which to interpret. His complete argument is curtailed since it does not answer the question. In the second recorded attempt the progress in terms of graphical contextualisation is evident. The only omissions that may tarnish a handsome display of conceptual depth is the failure to note the difference between the function values of the turning point and the y-intercept as being 1 and the congruency that results in the parabola's translated position with the x-axis tangent to it; as well as the inaccuracy of not including 1 as part of the solution.

In writing $-32 < x < 4$ in the form $|x+a| < b$ he shows his argument graphically. The description shows instrumental rather relational understanding but there is a vestige of the argument that recognises the notion of centrality akin to the absolute value function. Despite identifying the central value as 18 and the left and right displacement as 14, his final notation is contaminated by a reversal of their correct positions. One cannot say with certainty the cause of the error, but given his discourse it would take little to rectify the error. What is sure is that had his answer been recorded without commentary it would have been deemed to have been incorrect without further positive consequence.

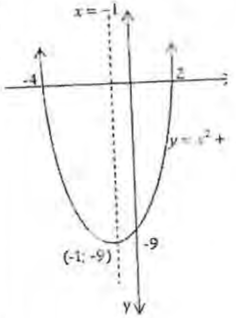
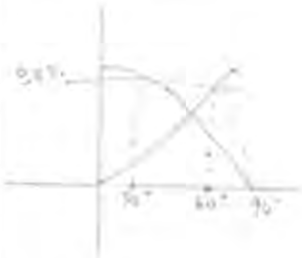
The trigonometric argument relates a strong sense of rote and a sense of frustration in having to resort to rote. He says, '... my teacher taught you must just take off the thing that was putted and then you put another one if it has s you then put c...' This statement is evidence for not only instrumental understanding but also instrumental teaching devoid of any relational understanding.

The second quadratic theory account when compared to the trigonometric exposition clearly demonstrates a recognisably dramatic improvement in the command of English but also in the ability to construct coherent mathematical statements.

Participant 6 acknowledges that graphs have increased his understanding of mathematics. The excerpts from his portfolio of work seem to corroborate this. His frustration with the teacher at school 'who do not explain' is an indication of personal injustices that are continually perpetrated in unequal systems, especially where learners like participant 6 is denied access to mathematics teaching and its concomitant opportunity.

This participant commenced the programme as a SG (standard grade) mathematics candidate, with an entry level assessment mean of 29%. At exit level he obtained 87% on the HG (higher grade). He achieved a B aggregate on leaving school and was admitted as a B.Sc undergraduate in Chemical Engineering at the University of Cape Town, and has had an extremely successful first year at university. I venture to suggest that this cameo case of the accessibility to mathematics through tuition and personal conscious accessibility is possible and beneficial despite learning deficits for many others who are mute subjugates of a failing system.

Participant 20 Entry Level %: **26** Exit Level %: **63** Mother tongue: **English**

Quadratic Theory	Absolute value	Trigonometry	Learners affective written responses to questions
Explain how to read the graph to obtain the values of t for which $x^2+2x-8=t$ has two negative roots. Looking at the graph the roots are real, unequal, rational. Moving the graph upwards until it reaches zero as the turning point, and cuts the graph at one point only, roots will be real, equal and rational. 	*Sketch* $y=-\|2x+4\|-2$ Firstly find the x-int.- make $y=0$. Second step is to find the y-int. by making $x=0$. Determining the x int you make the product of ($2x+4$) positive and solve, thereafter negative. Third step is to find the x turning point. The x turning point is the x value inside the absolute value bracket which makes the bracket equal zero. The y turning point is the – or + number that follows the abs. value bracket, -2. a<0 so arms point downwards. 	*Explaining the truth of* $\sin(90°-x)=\cos x$ The angle x will lie between $0°$ & $90°$ The angle from the origin will be the same as the angle of $(90°-x)$. So if you find where the two graphs cut the y-axis at that angle x the y value will be the same e.g. say for instance x is $30°$, $\cos 30°=0{,}87$ and $\sin(90°-30°)=\sin 60°=0{,}87$ which clearly shows that the y values are equal. Therefore we can deduce that $\sin(90°-x)=\cos x$ 	Responding to how graphs has affected his ability to explain mathematics tasks, he says, 'Having the ability to interpret graphs enables a person not to use calculations has it would cause complications. You could see were graphs are > or < and its often given in tests. This also often saves time has limited times are given…' Responding to how using words has affected his understanding of mathematics, he says, 'This enabled me to see how I can explain mathematics and from that you learn as well. You see your week points and thereafter can practise on it.' Responding to his feelings when having to explain mathematics, he says, 'I don't like explaining mathematics has I'm not quite good at it. I however feel that after doing this exercise I gained knowledge about maths. IT is vitally important that one doing maths should be able to explain it as you could see how it is done.'

8.6.3 Reflections on the work of Participant 20

In the quadratic theory exercise participant 20 demonstrates that by translating the graph upward to a position of tangency with the x-axis he can determine the value of t that will produce real, equal and rational roots for the equation $x^2+2x-8=t$. He however does not clinch the argument. There is nonetheless coherence and an ability to engage meaningfully in terms of his using apt mathematical register.

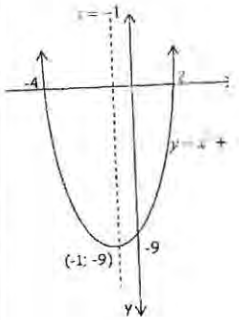
In recording the process with which to establish parameter values of $y=-|2x+4|-2$ he displays instrumental understanding. Expressing understanding is found in the clarity of explanation. This account is largely a set of instructions that does not show any reasoning. This understanding referred to here is demonstrated by other participants when they say that to find the x-intercept, y should be zero because on the x-axis the y value of every point is constantly zero while its x-co-ordinate varies. He also fails to calculate values despite being able to say how to obtain them. There is consequently no sketch to accompany his deliberations.

The trigonometric exercise shows conceptual depth in the correct use of mathematical register, applied meaningfully to the graphic representation. His use of a specific case diminishes his account in that it fails to recognise the general case. In essence though, his recognition of the fact that the function value for $\sin 60°$ and $\cos 30°$ is equal, is sufficient at this stage of communicative action to demonstrate conceptual depth.

His acknowledgement of the value of graphs is clear in his affective responses although not as articulate as some of his peers.

His 37% improvement from entry to exit level is indicative of marked improvement in his understanding of mathematics.

Participant 7 Entry Level %: **50** Exit Level %: **61** Mother tongue: **isiXhosa**

Quadratic Theory	Absolute value	Trigonometry	Learners affective written responses to questions
Explain how to read the graph to obtain the values of t for which $x^2+2x-8=t$ *has two negative roots.* The value of t is the value of y of the turning point. 	*Write* $-32<x<4$ *in the form* $\|x+a\|<b$ This is an inequality to put it in absolute brackets you have to find the difference between them and half it and take the sign of a bigger number. Look for how much space between them. In side brackets the sign will be the opposite to your half. $-32<x<4$ $\|x+14\|<18$	*Explaining the truth of* $\sin(180-x)=\cos x$ At $180°$ the graphs are equal $\therefore$ $180-x$ is backward so the $180-x$ of both graphs will be at the same point at $30°$ they have equal value which is = to $180-x$ $\therefore \sin(180-x)=\cos x$ 	Responding to how graphs have affected her ability to explain mathematics, she says, 'Graphs to me are like practical work, they are another system of expressing information or given statement and thus graphs give me a clear picture or better explanation of mathematics. Through graphs I can explain maths more easy but I am still struggling with the mathematical language. Graphs to me serve as proof of what is being said.' Responding to how the use of words to explain mathematics has affected her understanding of mathematics she says, 'Firstly I think language is very important in maths and mathematics is not only about calculating as I used to think so being able to communicate with mathematics has helped me to solve or calculate with reasons. Now I know what I am doing and why I found the use of words a very effective way of understanding mathematics which I have never been taught at school. It is not very easy at the moment but step by step I am getting there and I can see the difference this makes. Before I learnt this use of words method there are mathematical rules I used to memorize but now I know and can justify those rules & don't memorize anymore.' Responding to her feelings when having to explain mathematics, she says, 'I am feeling good because I can be able to determine whether I understand or do not understand what am I doing. For the moment the method is not easy but it has impacted my mathematics hugely. Sometimes I struggle to explain but I am happy that when I struggle, I simple realise that I need more practise in that particular topic. What makes me more willing to learn the mathematics language is that the more I try and express in writing the more what I am doing is anchored in my head.'

8.6.4 Reflections on the work of Participant 7

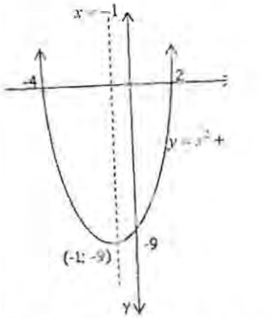
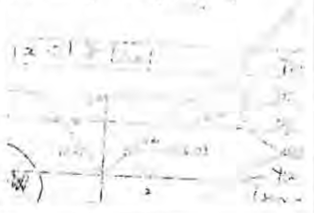
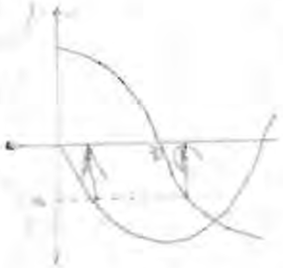
Participant 7's attempt at the quadratic theory exercise indicates no meaningful point of commencing and she makes an incorrect statement about the meaning of t.

In trying to write $-32 < x < 4$ using absolute value form, she is aware that the distance between the extremes needs to be found and by halving that distance she establishes the midpoint on the interval. Initially it was not easy for me to fathom the meaning of 'take the sign of the bigger number'. I imagine what this second language English learner is saying is that the midpoint of the interval, lies closer to -32 so the sign of 14 on the interval is negative because 32 is bigger than 4 and 32 is negative. 'In side your brackets the sign will be the opposite of your half' although correct is without explanation or reason. It might have been said that at the midpoint, -14, the displacement is zero and the additive inverse added to x is necessary for that condition to be satisfied. The absence of a number line sketch as in participant 6's work has not prevented her from obtaining a correct solution. It was not established if this learner had cognitively visualised the number line in preparing her solution or whether cognitive functioning revolved around arithmetic practises only.

The learner incorrectly transcribed the question for the trigonometry exercise. The argument is therefore invalid from the outset.

Participant 7's equating graphs to a practical side of the subject is interesting. Her admission is that she still struggles to find mathematical language to describe what she is doing. She articulates well in saying 'Graphs... serve as proof of what is being said'. She eloquently states that 'Before I learnt this method (graphical representation) there are mathematical rules I used to memorize but now I know and can justify those rules & don't memorize anymore.' Her confidence has improved. She says, 'I am feeling good because I can be able to determine whether I understand or do not understand what I am doing... the more I try and express in writing the more what I am doing is anchored in my head.' These assertions seem to demonstrate that a first principles graphic approach has the effect of diminishing the need for rote learning.

Participant 15 Entry Level %: **79** Exit Level %: **96** Mother tongue: **English**

Quadratic Theory	Absolute value	Trigonometry	Learners affective written responses to questions
Explain how to read the graph to obtain the values of t for which $x^2+2x-8=t$ *has two negative roots.*  By shifting the graph up so that both x values are negative, you will find the value of t. At the point where TP is on the x axis both roots are (negative), t will then be 1. t cannot equal less than zero because then it will only have one (negative) root! It also cannot be more than 1, because then there will be no roots.	*Explain how to solve* $\|x-2\|\geq\|2x\|$ $\|x-2\|\geq\|2x\|$ can be represented by 2 graphs on the same set of axes. $y=\|x-2\|$ and $y=\|2x\|$. To get x int, make $y=0$ & solve. To get y int, make $x=0$ & solve.To get x_{TP}, substit a value which will make absolute value =0. This is your x_{TP}. y_{TP} is equal to the number after the absolute value.The shape is represented by the sign i.e. a>0⇒V, a<0⇒ ∧. To find where $\|x-2\|\geq\|2x\|$ you have to find the intersection points. At the places above the intersection points $\|x-2\|\geq\|2x\|$ 	*Explaining the truth of* $\cos(90°+x)=-\sin x$  The point where the cos graph cuts the x axis is $90°$. When moving x values (units) to the right on the cos graph to get $\cos(90°+x)$, let the y value = a. At that same y value (a) on the sin-graph the x value is also equal to x i.e. it is x values from the start point $\therefore\ \cos(90°+x)=-\sin x$ because the y value on $\cos(90°+x)$ is equal to the y value on $-\sin x$.	Responding to how graphs has affected her ability to explain mathematics, she says, 'Graphs has always confused me as I could never make the connection between the algebra & the graphical representation…I have learnt to make this connection. In the Trigonometry section, it has demonstrated to me how to attach pieces of information onto a graph. From here, the necessary information/answers can be read off. The graphical representation has given me a deeper understanding of the reasons for the statements that are made. It really helps me when I am stuck on a problem, & I don't understand what is being asked. A simple graph helps me to sort out all the troubles.' Responding to having to use words to explain mathematics she says, 'Using words to explain the maths has not really helped me that much. I know that I understand the work, I can go about step by step, doing each problem, & know what I am doing. The only reason I do not enjoy explaining the work is that I find it quite difficult to formulate my sentences & actually explain what I am doing… Using words to explain maths can be a very powerful tool to show that you have an understanding of it…' Responding to how she feels when she has to explain mathematics she says, 'When I have to explain what I do while I work, I do get a bit irritated. I find that I am a thinker & I work best when I think things through and write out the mathematical explanations/answers thereof. When I have to formulate actual sentences about what I do, it usually takes me quite a while as I have to sit back & think about something I am not so good at: WORDS. When a friend asks me to explain something to them, I always try to do it with as much grace as I can, as I would like to share my understanding with them, so that they can also gain the pleasure I do get out of my maths work. I do however get frustrated when I'm not able to put into words what I so clearly know…'

8.6.5 Reflections on the work of Participant 15

In the quadratic theory discussion participant 15 misinterprets t as a y-intercept and therefore does not establish the values of t that for this equation produce two negative roots. Her attempt is clear, concise, coherent and articulate but all these qualities serve to establish her erroneous unstated assertion.

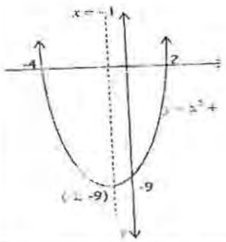
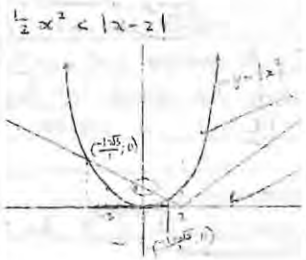
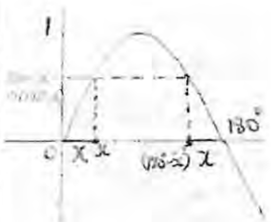
The absolute value exercise is characterised by a command of English as well as a good mathematical register and a grasp of the graphical depiction of the problem. If it can be faulted it is again, as in previous accounts where reasons are not offered for statements. In the last sentence there is an element of vagueness in her saying 'At the places above the intersection points $|x-2| \geq |2x|$'. One gathers that her intention was to say that for all x values for where $|x-2|$ is on or above $|2x|$ constitutes the solution of the inequality.

The explanation that $\cos(90° + x) = -\sin x$ is handled well except for the inaccuracy in her saying 'When moving x units to the right on the cos graph to get $\cos(90° + x)$ she in fact intends the movement to be x units right of $90°$ on the x-axis to get $(90° + x)$ a statement that indicates no conceptual closure on the meaning of independent and dependent variables which in turn becomes a point for communicative action.

Participant 15 has an excellent command of English and is thus able to communicate her consciousness about mathematical tasks. Her work does however demonstrate that there is a need for rigour in language so as to demonstrate unequivocal meaning. She was animatedly resistant to having to express what she was doing mathematically. In discussion I sensed in her some opposition because she believed it wasn't necessary in mathematics at school and neither was it necessary for the purpose of scoring marks. In her view the logic of describing her mathematics with the aid of graphic representations was seen as a burden. This is substantiated by her statement in the final affective assessment when she says, 'When I have to explain what I do while I work, I do get a bit irritated.' On a more positive note she says, 'The graphical representation has given me a deeper understanding of the reasons for the statements that are made. It really helps when I am stuck on a problem & I don't understand what is being asked. A simple graph helps me to sort out all the troubles.'

A difference of 17% between entry and exit level scores seems to confirm her claim that graphical representations have given her a deeper understanding.

Participant 27 Entry Level %: **60** Exit Level %: **70** Mother tongue: **Afrikaans**

Quadratic Theory	Absolute value	Trigonometry	Learners affective written responses to questions
Explain how to read the graph to obtain the values of t for which $x^2+2x-8=t$ *has two negative roots.* t is in place of y, therefore t represents a function value. As shown on the graph, the parabola cuts the y axis at -8, Therefore if we move the parabola up to cut the y axis at at zero we will have a negative root and a root of zero, but if the parabola cuts the graph at 1, the parabola will have two negative and equal roots. To make this possible c has to become 1 therefore $t=-9$. 	*Solve and illustrate* $\frac{1}{2}x^2<\|x-2\|$  The first thing I do before solving this… is to find for which value(s) of x it is valid on the Cartesian plane. This is finding the points of intersection between the two graphs, Now because the negative part of the absolute value graph cuts the parabola the region … will be valid for the negative region of x.	*Explaining the truth of* $\sin x=\sin(180°-x)$ As shown in the graph… if measured x units from the origin to the right (provided x is acute) to the point x and measure x from $180°$ to the point $(180°-x)$ then it is clearly shown that the points x and $(180°-x)$ have the same 'y' function value. Therefore $\sin x=\sin(180°-x)$ 	Responding to how graphs has affected his ability to explain mathematics he says. 'Graphs have really improved my ability to explain maths as well as improving my understanding of maths very well. It makes explaining maths more easier because I can actually see the graph infront of me and point out co-ordinates or determine functions or restrictions as was the domain/range by simply consulting the graph. It helps me to explain to myself how different factors influence the graphs by making changes to the equation algebraically and it allows me to see how the graph changes.' Responding to how having to use words to explain mathematics has affected his understanding he says, ' It affects me in a way that I can see how much I really know or understand mathematics. It also helps me to really thinks about mathematics and really try figure and solve mathematical problems by myself, by making and changing the question to a frame of mind which I can understand.' Responding to how he feels when having to explain the mathematics he is doing, he says, ' I feel irritated and bothered because it's something new and most of the time I don't know how to explain the mathematical problems or why I should explain it. It try sometimes to explain mathematical problems, but I feel very unsure of my elaboration or answer. I doubt myself and usually make a mess out of the whole entire elaboration. When I can't explain one question then it affects the way I try to explain the next question and then I really start to work backward and write anything that comes to mind. Although it is irrelevant and doesn't make any sense at all. I feel also disappointed, but sometimes I would be totally motivated to try harder and do better…'

8.6.6 Reflections on the work of Participant 27

Participant 27's reasoning around arriving at the values of t for which the equation $x^2 + 2x - 8 = t$ has two negative roots is like his peers fairly circuitous but the acquisition of mathematical register is a favourable by-product of his endeavours. He recognises the synonymy of y and t as function values by saying 't is in the place of y'. He makes the statement that for equal roots the parabola cuts the 'graph' at 1, where he means to say 'cuts the y-axis at 1'. His articulation and coherence is good for a second language English speaker. In the final analysis his account is marginally short of a complete exposition in coming up with a single value of t that produces two negative roots instead of the valid range of values for the interval $-9 < t < -8$.

His absolute value attempt shows a graph of the two functions. His aim 'to find for which value(s) of x it is valid on the Cartesian plane' is to show graphically where the solution can be read off on the x-axis. His reference to 'the negative part of the absolute value graph' refers to the decreasing component of the function where $x < 2$. His attempt is admirable albeit with an absence of rigour in the last quote. It is incumbent on the teacher in such situations to make sense of the learner's situated cognition in coming to understand whether there is an approach towards a mathematical conscientisation.

This participant's attempt at explaining the equivalence statement $\sin x = \sin(180° - x)$ shows a clear appreciation for the distinction between independent and dependent variables and deals with the general case of x rather than a special case.

This is another learner whose resistance to having to speak mathematically was evident. His perseverance though was admirable and at the end he says, 'Graphs have really improved my ability to explain mathematics and my understanding of maths very well.'

I proceed to conclude reflecting on the data by presenting the quadratic theory tasks of participant 14 that highlights vocabular variety and alternative viewpoints on the same task, and establishes what I think is a meaning of conceptual depth. Inherent in it is for me a synthesis of the work of the previous six participants on whose work I have reflected.

Participant 14 Enty Level %: **80** Exit Level %: **86** Mother tongue: **English**

	Explain how to find using the graph of $y = x^2 + 2x - 8$, for which values of k, $x^2 + 2x + k = 0$ will have real roots.	Explain by using the graph of $y = x^2 + 2x - 8$ how to find for which value of t, $x^2 + 2x - 8 = t$ will have equal roots.	Explain how the equation $x^2 + 2x - 8 = p^2$ can be interpreted graphically as having has real roots for all $p \in R$.
May 2004	By moving $x^2 + 2x - 8$ up and keeping it's TP on the axis of symmetry. If $x^2 + 2x - 8$ is moved up so that it cuts the x-axis only once, k represents the y-intercept so that $k = 1$ is the biggest value of k for real roots, else if the graph is moved up further it will have non-real roots	As alongside the graph should be move up to where the TP is (-1;0) thus it will have equal roots. $-8-t$ represent the y intercept for the graph at this point. The y-intercept is 1 thus $-8-t=1$, $-8-1=t$, $t=-9$	$x^2 + 2x - 8$ the y-intercept is -8 which is negative in $x^2 + 2x - 8 = p^2$, $-8-p^2$ is the y-intercept. p^2 is always negative or 0 thus $-8-p^2$ will always be negative for real root the y intercept has to be less than or equal to 1 and if the y- intercept is negative which it is, it is less than 1 and thus the roots are real
Sept 2004	If the graph is move up to where the equation has two equal root it can be read off. The graph has to be moved up by 9 units. k represents the y-intercept thus where the roots are equal $k = 1$. Any k value greater than 1 will result in non-real roots (ie not touching x-axis) so for real roots $k \leq 1$	$-8-t$ represents the y-intercept so the graph has been moved up 9 units to where it has equal roots i.e. cuts the x-axis twice but at the same place or once, in other words. 9 units up the y-intercept is 1 so just set $-8-t=1$ and t is thus equal to -9	p^2 is always positive, the y-intercept represented by $-8-p^2$ $\therefore$ the y-intercept will always be a number less than or equal to -8. Thus the roots will always be real for $p \in$ □ as the y-intercept will always be less than 1
July 2005	If the graph is moved up 9 units the TP: (-1; 0) and the y int:1 this is where the graph has 2 equal roots any y int (k) above this will result in non-real roots any y below or equal to this will result in real roots	If the graph is moved up to where it has equal root you can use $-t-8$ as the y-int thus $-t-8=1$	draw the line $y = p^2$ p^2 is a positive or zero constant thus it will always cut the graph of $x^2 + 2x - 8$ twice producing two real, unequal roots, thus it can be determined that the roots are always real.

8.7 Reflections on the work of Participant 14

The work of participant 14 who has a good command of English demonstrates the performance on the quadratic theory task over three successive attempts from May to September 2004 and again in October 2005.

It is noticeable in the first task that his third attempt falls short in not clinching the exposition through stating the desired values of k. Despite this his description is accurate and coherent. In the second task a similar pattern occurs. There is a sense though that the diminishing vocabular intensity over time may indicate a move toward conceptual closure aligned to the notion of obliterative subsumption which recognises a cognitive tendency toward a more economic use of language to describe dense concepts. I find it interesting that participant 14, as does many of his peers, moves the parabola into a position where the x axis is tangent to it, and changes the form of the equation to accommodate and interpret $-8-t$ as being the required y-intercept equal to 1, which after manipulation reduces to $t=-9$. The expected engagement might have been to see $y=t$ as a family of horizontal lines and selecting the one which is tangent to the parabola, i.e. $t=-9$. I reasoned that the synonymy of y and t when using the 'family of horizontal lines' argument is still rooted in algebraic equivalence and may be cognitively more demanding to argue as follows even if the graphic representation is available for consultation. He would firstly need to recognise the synonymy of y and t as function values, then reconcile equal roots with the concept of tangency to obtain $y=-9$ as the required tangent line and finally syllogistically argue that

$y = t$ *(by inspection)*

$y = -9$ *(the tangent that produces equal roots)*

$\therefore\ t = -9$

In the third task the difficulty is evident from the fact this participant renders confused arguments for his first and second attempts, where he resorts to moving the graph in order to graphically interpret the algebra. In his third attempt there is a coherent and articulate argument that recognises the constancy of p^2 as a family of lines and infers rather than states that because these lines occupy positions on or above the x-axis they will cut the parabola twice producing real roots for all $p \in \mathbb{R}$.

In the final analysis his arguments show a cognitive consciousness about the mathematics he is doing that permits communicative action in what can contribute to meaningful radical and social constructivist classroom milieus. There is a real sense that without the graphical interpretation the algebra will remain algorithmic and the vocabulary may not indicate an acquisition of conceptual depth. The observation that is most probably the most enlightening from a research perspective and in line with the goals of this work is participant 14's different interpretations of task 3 over time and his demonstration of varied vocabularies to achieve the outcome of graphical interpretation. The ability to utilise vocabular and algebraic methods and recognise solutions from different perspectives seems to be a sound indicator of conceptual depth. His normative entry and exit level scores although indicative of ability is not a categorical indicator of conceptual depth.

8.8 Summary

I have reflected in the preceding pages on the quantitative data associated with the codification of mathematical register and concept attainment as demonstrated in textual accounts of participants. The descriptions and explanations that are offered by participants of mathematical tasks are also subjected to qualitative and interpretive analyses to aim to enlighten me about the possible cognitive endeavours of learners. Further to this I have undertaken an analysis of learners' affective responses offered on questions pertaining to their feelings about how graphically inspired experiences of mathematics have impacted them. The analysis depicts an intertwining of quantitative and qualitative data that are mutually supportive and that further confirms the applicability of the methodological plurality of critical realist theory.

From a quantitative perspective, an analysis at different levels, like aggregated and disaggregated codes, there is evidence for an increase in the frequency of language and concept codes over time. Within the language index, explanation and terminology show marked increase over time while coherence and articulation generally show little upward mobility, speculated to be a function of the cohort being of nearly 50% isiXhosa mother-tongue. Within the concept index, conceptualisation and graphicacy codes show increased frequency of occurrence. The conceptual depth code, as a composite of language and concept indices, shows improvement over time.

An examination of participants' performance on a quadratic theory task revealed that algebraic performance exceeds graphic interpretation but that both aspects of the task show improvement over time; probably indicative of the general difficulty learners have with graphical interpretation. When a taught disadvantaged group and untaught advantaged cohort's performances on the same quadratic theory task are evaluated it is evident that graphical contexts of algebraic tasks are not pertinent considerations in advantaged communities either. In the light of the taught cohorts' improvement this is seen as inaccessibility of contextualised mathematics for both currently disadvantaged and advantaged learners.

A comparison of entry and exit level scores for all participants shows an improvement from 50% to 64% over two years. This is regarded as meaningful given that group means generally deteriorate or remain similar from Grade 11 to Grade 12. The fact that the contact time for the use of graphically inspired strategy constituted 21% of the contact time for teaching mathematics in a standard academic year, more systemic time devoted to graphically inspired methods may have improved results by a greater margin.

When viewed in terms of language groupings, isiXhosa participants showed the greatest increase in improvement based on entry and exit level means for mathematics. This is an interesting occurrence from the point of view that since the language of instruction is English there was an expectation that black learners may fare poorer. The outcome to the contrary may be attributable, inter alia, to two things that come to mind. The first is that the contrast in consistency between township and programme teaching may amplify the personal and social injustice of the inaccessibility of mathematics and emphasise that sound content teaching contributes to dramatic improvements in mathematics performances. The other is that the pictorial value of a graphically inspired first principles approach has added cognitive value more for currently disadvantaged learners in its concretisation of abstract algebraic constructs.

The relative poor performance of an elite, affluent group of learners may indicate that mathematics is not only inaccessible to disadvantaged communities but also to advantaged communities. The nature of this kind of inference resonates with the crisis in mathematics in South Africa for all its learners and communities.

At a qualitative level of analysis the textual accounts of the cognitive endeavours of participants provides insights that inform pedagogic practises. Mathematical mutism prevents both radical and constructivist modes of knowledge acquisition and reflects a lack of critical consciousness without which meaningful educational practises are seem stultified. Participants' accounts of mathematical tasks provide an abundance of information both in terms of insight into cognitive functioning of learners and its revelation of misperceptions that provide critical teaching content and opportunities. Graphically inspired communicative action develops not only dialogic possibilities but also a concrete pictorial representation of algebra which respects a natural cognitive hierarchy from concrete to abstract representational forms. Using this as a starting point conceptual depth may well be a function of mathematical conversancy.

Participants attest to their frustration at having to delve deep to find language with which to communicate mathematical ideas and processes but ultimately also attest to the value that graphically inspired methods have had for them in coming to a fuller understanding of mathematics. There seems to be fair claim therefore for a generic mathematics that situates algebra in the Cartesian plane and that enables metacognition for both teacher and learner alike.

The following chapter draws together the practicalities of the research process and reflections on the data and places it in theoretical perspectives that seek to move toward a theory of pedagogy.

Having probed the data for meaning chapter 9 seeks to position the research experience in such as a way as to derive from it insights as to how theory may be a generated to incorporate the teacher in a bi-directional research environment. It aims at contributing to pedagogic theory with specific reference to mathematics.

ix

Towards emerging theory for mathematics pedagogy

'Where constructivist theory is concerned, the 1990s have seen a shift in focus towards theory-practice rationalization through reporting of classroom research... More is needed, however, in bringing the issues from classrooms to centre stage, to challenge and potentially illuminate theory.'

Jaworski (1999: 159)

9.1 Introduction

My engagement with a metacritique of personal practice in relation to episodes of Habermasian communicative action that seeks to probe and understand what constitutes effective and sustained mathematics teaching and learning has deepened my own awareness of the value of a Freirian-type critical consciousness. This consciousness theoretically has the potential to emancipate and transform thinking and self-esteem and transpose personal and social inertias contained in injustice, into opportunities that transform mute personal states into critical voices; mathematical voices whose echoes return with social opportunity as they rise from the inertias of poverty and injustice. It is in this awareness that I have found rewards in mathematics education as opposed to mathematics teaching. Education, for me, has constituted dialogic encounters that have probed my own cognitive understanding and the meaning learning participants make of mathematical tasks in reflecting on classroom based conversational pieces and textual accounts of graphically inspired discourses. Phenomenologically, both hermeneutic (reflections on my understanding of dialogic moments) and empirical (probing learners' accounts of mathematical tasks and experiences for meaning) engagements has fore-grounded the metacritique of practice as a shared responsibility of learner and teacher. There is for me a sense that the art of teaching emanates from a desire to educate. And the desire to educate summons thought as to how learners think, and what makes me know that they are thinking mathematically. One of those indicators, I have come to understand, is the degree of conversancy about matters mathematical that learners have and that in Vygotskian terms is the manifestation of thought, and in this study, metacognitive thought about matters mathematical. One cannot claim mathematics, as indeed any other subject, to be integral to one's psycho-socio cognitive make up if one cannot engage in its discourses. The most fundamental impact that this research has had on my own knowledge about mathematics education is that mathematical mutism is characteristic of learners across all socio-economic boundaries and that this mutism goes hand in hand with rote algorithmic routines that teachers too often erroneously take as indicative of mathematical competence. And, if language is construed as a social construct that envelopes interaction it is only through interaction and exchange of a language register that we are initiated into the fabric of language and integrated into its societal ambiences. It is no different in mathematics. We are shy and short of its richness if we cannot speak mathematics. I say with a degree of confidence that mathematics, like Latin, largely remains unspoken. In what I regard as emerging theory for mathematics pedagogy, the dialogic teacher-learner reciprocity in oral and/or written form constitutes a fundamental part of the teaching and learning of mathematics. If we are socialised by language into cultures then it is by mathematical language that we will be conscientised about our cognition of it. If I need to argue economically, for example, I need a command of an economic register to make sense by way of communicative action. If I cannot argue with a mathematical register I cannot make mathematical sense in terms of communicative action. If I cannot through discourse make rational contributions for the purpose of arriving at understanding and consensus then I remain cognitively ignorant. It seems to imply that if mathematics is going to be socially constructed it requires of learners to have a competent mathematical register to be able to achieve it. To this end the single most important strategy in this work has been the graphical representation of algebraic constructs, and its associated abstractual order, that has been catalytic in helping me and learners probe cognitively for meaning and articulation, and for the deconstruction of concepts that are composed of orders of constituent derivative concepts.[29]

As in all languages the level of articulation is an indicator of competence in that language. The written language as a fundamental level of proficiency is still able to

[29] I venture to suggest that the deepening crisis in mathematics education in South Africa, as described in Chapter 2, may be attributed to both the mutism of teachers and learners in mathematics classrooms.

mask incompetences since rote neutralises cognition. The argument is the same, I believe in mathematics, as has been demonstrated in the reflection on the often incongruity between algorithmic and textual accounts of learners in the previous chapter. An ultimate test of mathematical acuity is the ability to convey unequivocally the meaning of mathematical constructs. The emphasis I try to make here is that the communicative role of the mathematics teacher is vital to opening cognitive pathways for learners through deep and thorough descriptions and explanations, rich in vocabular exchanges that impart correct terminologies. This seems needed at least until learners themselves are enveloped in the social and linguistic cultures of mathematics classrooms and are able through their radical constructivist learning contribute to a functional social constructivist learning environment. Investigative and exploratory mathematical tasks are often jeopardised by deficient mathematical register with which learners can express their mathematical discoveries.

What I have described is what I term vocabular action around graphic representation. Ideally, it occurs in two directions, between teacher and learner, but as in any cycle, the circularity is dependent on inception. In terms of mathematics classrooms the competent, cognitive and articulate teacher is the beginning of vocabular reciprocity. I suggest that instruction is a fundamental requirement for the initiation into mathematical conversancy. What emerges from this is that the transparent teacher-facilitator of constructivist explorations can only be a position held once the first principles of its mathematics has been conveyed so that it can become a concept image that will ultimately develop into concept definition.[30] And this observation seems contrary to current practice where excellent teaching is not overtly designated as a prime desirable quality of teachers (DOE, 2003a: 5). Competent mathematics teaching is not necessarily implicit in the professional designation and occupation as a teacher.

I have in the preceding exposition tried to lay the foundation for a pedagogic model that incorporates the depositional elements of behaviourist learning theory. Thorough instruction implies the transfer of knowledge from one to another akin to the Freirian notion of banking or depositing information into receptive learners. The malignancy of such behaviourist elements is to be found only in its exclusivity. Its recognition and

[30] The Brünerian ikonicism finds expression in this study as algebra being portrayed as concrete graphical and representations that have pictorial value.

incorporation into a pedagogic practice has the benefits of catalysing the hermeneutic in both teacher and learner and propagating cyclical enquiry. It constitutes the stimulus that derives a response in a bi-directional fashion. The instructional first principles teaching content initially acts as a stimulus in its most positive sense as it evokes a response from learners. The response of learners in turn constitutes a stimulus for the teacher whose further response constitutes an episode of communicative action that repeats through mathematical conversancy. The exchange of mathematical register provides the cognitive underpinnings of understanding that are manifested in learners' algorithms and textual accounts. The increasing cognitive acuity of the teacher is reflected in refining interrogative cognitive questioning and the recording of continuing practice based enquiry as research findings. The learning afforded by episodes of communicative action is not uni-directional. Within this cyclical exchange of learning between providers (traditionally teachers) and receivers (traditionally learners) of knowledge the critical consciousness that finds itself between stimulus and response, I believe, becomes more accessible; a consciousness that may become more analysable as it manifests in associo-constructivist[31] learning conversations. Radical constructivism is not the exclusive domain of learners. In concentrating on the cognition of the individual it could recognise both teacher and learner as individual constructors of knowledge. More pointedly, mathematics content knowledge is only a precursor to cognitive appreciation and understanding of mathematics. Singularly constructivist South African classrooms, radical and / or social, and more especially poverty situated classrooms, therefore may, not be the most effective environments for learners to acquire mathematical language and its concomitant cognitive concepts.

Experience has shown that teachers assume learners to know the fundamental structure of mathematics. I make the point elsewhere that teachers make use of the most economical form of a concept at their disposal and thereby overlook the embeddedness of a concept's constituent derivatives. This I believe not to be a conscious act; had cognitive consciousness been present the most economical form would have been transformed into deconstructed concepts. From this viewpoint, I reiterate that teaching, per se, is integrally important to the learning process.

[31] I coin the term 'associo-constructive' to include and draw attention to what I deem to be an essential instructional component of the didactic process in a first principles graphical approach to mathematics teaching.

In search of a didactic alternative for the pedagogy of mathematics one needs to consider that various modes of cognitive concept acquisition are psychological activities which present as strong aetiological factors. I suspect that it is at the micro level of methods of transfer of mathematical knowledge where the solution to improved mathematics teaching lies, elaborated in the discussion of the preceding paragraphs.

The bi-directionality of the teaching-learning process emphasises its interactive mutuality. Learners learn from teachers and teachers learn from learners. This conception of mutual learning has helped me to design an associo-constructivist[32] metacycle which is schematised in Fig. 9.1.

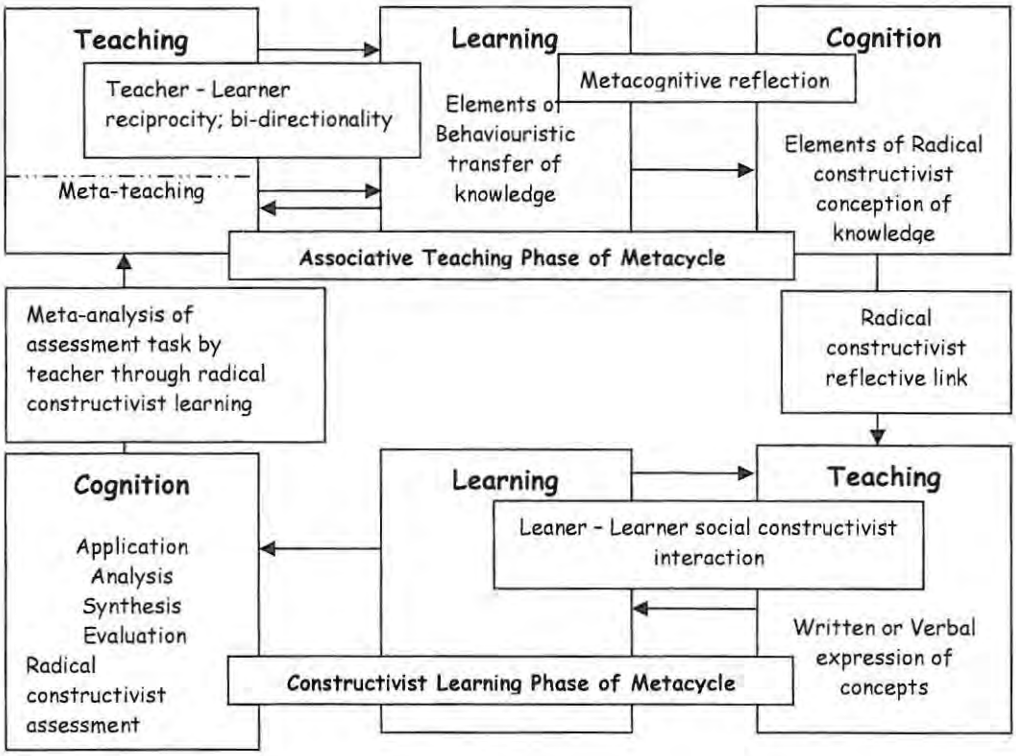


Fig. 9.1

Associo – Constructive Learning Metacycle

[32] The prefix 'associative' derives from early behaviourist learning theory terminology where the *association* between stimulus and response was emphasised. Its application here is within similar connotations but is not exclusive of the cognitive transactions that occur between stimulus and response in acts of comprehending mathematical constructs.

9.2.1 Associative Teaching Phase of Metacycle

The initial inception of this cycle presumes that a teaching incident occurs when framed by a meta-analytical approach. Learning comprises elements of behaviouristic teaching of deliberate content and is reinforced by systematic and strategic questioning prompted by in situ metacognitive reflection of the teacher regarding learners' verbal or written responses to the lesson content. The reflective process continues in learners through involvement in independent tasks. This stage of cognitive enquiry is characterised by radical constructivist learning by the teacher regarding learners' facilitated understanding or their misconceptions. Learners themselves are involved in radical constructivist learning in assimilating new content. Subsumptive processes allow the integration of new knowledge. In terms of this research this phase is characterised by my analysis of a concept and translating it from algebraic to graphic form so as to expose its derivative constructs. I utilise the graphical representation to deliver a first principles approach teaching incident that becomes the inception point of the cycle. These incidents provided opportunity for dialogic reciprocity and the basis for learners to reflectively respond through textual descriptions of their understanding.

9.2.2 Constructivist Learning Phase of Metacycle

Continued reflection by learners permits their verbal or written query about the conceptual content. This phase of the process is characterised by social constructivist learning in pairs, groups or with the teacher. In answering queries learners fill the role of the teacher and confirm their understanding in instructing peers. Increased fluency in communicating around the topic serves to reassure the learner. Affirmation by peers and teacher instil learner confidence and concepts coalesce to form a meaningful gestalt. Cognitive tasks of application, analysis, synthesis and evaluation are assessed. Such assessments are opportunities for learners to display their understanding but are also opportunities for further refining of concepts through engaging with unseen cognitive tasks of synthesising knowledge and integrating concepts. In assessing such assignments teachers engage in a meta-analysis and accordingly refine content and didactical aspects so as to commence another learning metacycle.

In the continuum of knowledge transfer from transmitter to receiver, knowledge is processed before becoming a psychological construct by the receiver. It is, I believe, the efficiency of the processing which promotes and enhances sustained acquisition of conceptual depth. The singular receipt of auditory signals (incorporating language) in the absence of other simultaneous sensory stimuli impedes the efficiency of the processing of information, more so when the nature of the input is abstract as in the algebraic definition of an absolute value, for example. Simultaneous, multiple modes of sensory input will facilitate efficient and sustained acquisition of conceptual depth, as in the graphical portrayal of algebraic concepts being the point of departure for teaching incidents. A proposed sequence for the teaching of a mathematical topic would be from intuitive graphing and interpreting the resultant characteristics and features, (providing a visual stimulus), to the solution of equations in this graphical context, followed by simplification of expressions, (where applicable also in this context). The graphical portrayal of intuitively abstract concepts like the absolute value serves as the most concrete level from which to initiate instruction and provides a learner with a conceptual adhesion on which to begin to structure and concretise and abstract a new concept. The instructional ordering of the features of graphs, followed by the solution of equations and then the simplification of expressions is furthermore consonant with increasing levels of abstraction. It is this very sequence which provides the cognitive infrastructure that permits and facilitates the progressive continuity in concept construction which is more inclined to sustain conceptual depth. It may also limit a tendency to commit unnecessary detail to memory which may ameliorate a learner becoming cognitively saturated. **Cognitive saturation** or the over-storage of rote detail, I see as impeding the spontaneous recovery of conceptual depth that may relate to the mutism in mathematics that this work exposes.

This sequence of instruction is contrary to current practice where the reverse of the sequence is adopted by teachers, often dictated by the sequencing of topics in textbooks and entrenched because of a paucity of metateaching engaged in by teachers. It is this reversed sequencing of constructs, I believe, that changes the tendency to commit mathematical knowledge to rote memory, to facilitating a meaningful, sustained acquisition of knowledge. In the latter case knowledge is readily recalled, applied and transferred across typical topics of the National Curriculum Statement for mathematics, serving also to destruct the compartmentalisation of mathematical knowledge where, for example, the similarities of the parabola and the absolute value are viewed, per se, as topics and not as separate entities. The approach allows mathematics as a learning area and subject to be viewed as a gestalt, where mathematics as a whole, is greater than the sum of its parts.

Further to the mutuality of the meta-teaching and meta-learning episodes I posit a construal of how meta-teaching and meta-learning are confluent with metacognition. The model integrates the components of teaching, cognition and learning. The description that follows details the operation of the model and is best read in conjunction with its schematic representation as found in Fig. 9.2. The conception, design and construction of the theory and the epistemological framework which I espouse draws on the theoretical work of Piaget, Skemp, Ausubel and Vygotsky. The model is essentially described here as it relates to a micro-teaching situation. Its wider applications are referred to in the text.

The bi-directional associo-constructive pedagogic model, illustrated in Fig. 9.2 that I proceed to describe serves to confirm the opinion that I offer in the preceding paragraphs. It seeks to show that no one pedagogic model is exclusive of the other and that one pedagogy comfortably replaces another as circumstances change. In this sense it is the teacher that is adapting pedagogy rather than adopting perennial pedagogy for practice.

The schematic representation, Fig. 9.2, is such that time is configured as consonant with teaching and **learning incidents** along its horizontal axis, while **learning processes** are configured in the vertical.

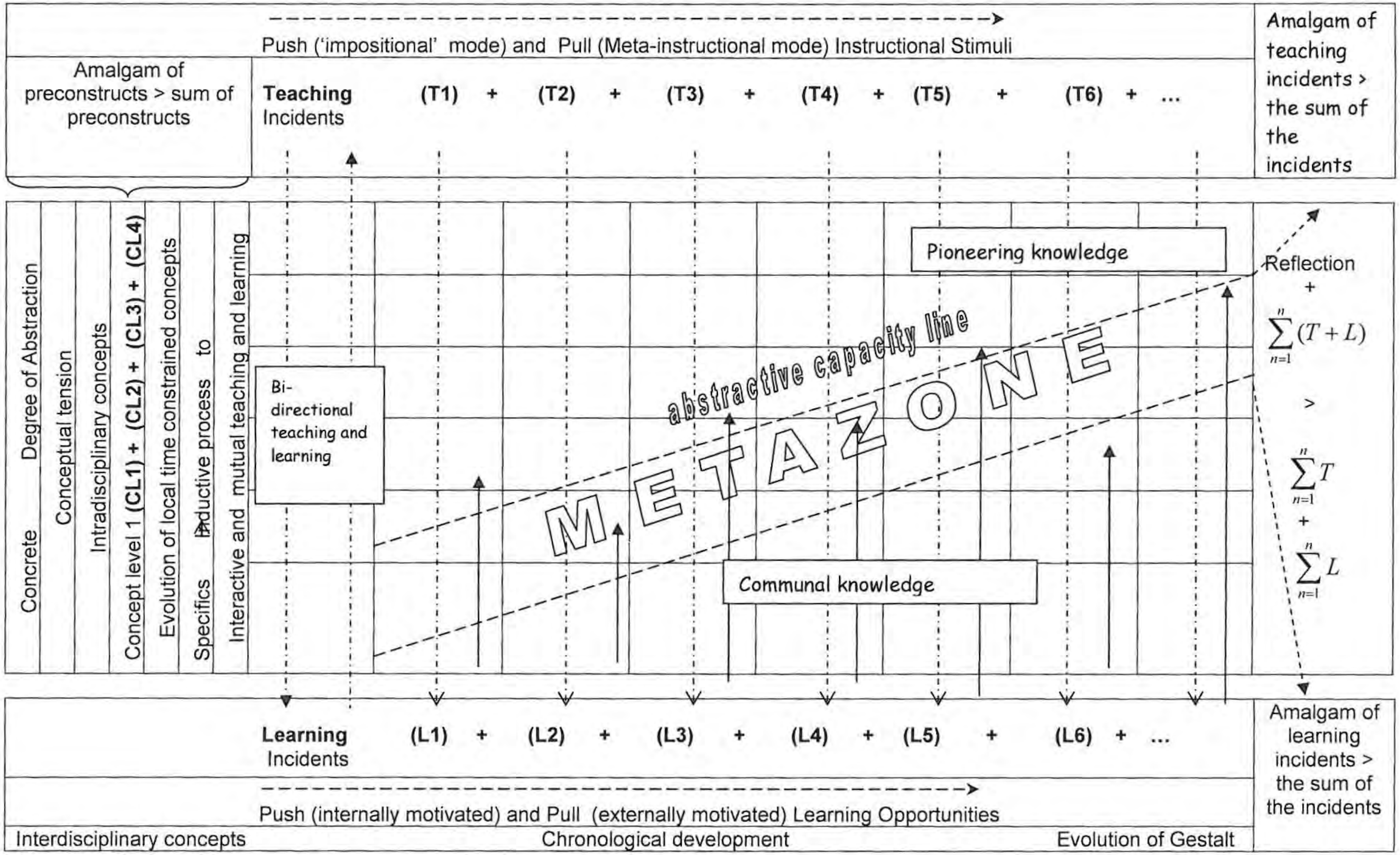


Fig. 9.2 A bi-directional Associo-Constructive Metazone model of teaching, and learning theory

9.3.1 A bi-directional Associo-Constructive Metazone model of teaching, and learning theory

The horizontal components comprise the progressive learning incidents and corresponds to (but not a function of) the chronological development of a learner. The model is dynamic in that it is pertinent to micro-, meso- and macro-scale time frames. A single lesson would be considered a micro-scale application, in which case, teaching incidents T(1), T(2), ... and their correspondence with learning incidents L(1), L(2) ... and so on, represent the presentation and the assimilation of the concept derivatives at hand, respectively. In a meso-scale time frame the same incidents would be modules pertaining to a topic, for example, trigonometry, and at a macroscale level a global representation of a discipline, for example, mathematics which is inclusive of all the learning which developed into this amalgam of mathematical knowledge. Conceived in this way the model is rendered dynamic in terms of time in that it represents learning as it occurs at any point in an individual's life-time; it also emphasises that the degree of learning at any given learning incident varies from concrete to abstract and that the degree of abstraction at the end of any one learning incident varies also from individual as a function of a matrix of social factors. As such, the schematic model may therefore serve as a planning framework for a lesson, for modules, topics, subjects or discipline. The horizontal component accommodates and reflects the interdisciplinary nature of knowledge and allows for the merging of concepts from separate disciplines. As a tool, the schematic model may therefore aid the process of structuring lessons and ensures structural consistency across departments and teachers. It may also serve to prompt teachers to engage in meta-mathematical thought in the preparation of a lesson. It is also on this horizontal continuum that a knowledge gestalt evolves.

Learning opportunities may be considered to be either 'push' or 'pull' experiences. The former refers to an individual's personal motivation and expectations of a lesson and is reflected in an attitude to learning. These are largely affective factors, such as the initial resistance of learners in this work to textualising their understanding of procedures, and considerations which are only controlled by an teacher insofar as the product of teaching is a worthwhile experience for a learner. 'Pull' factors in a learning opportunity amount to the degree of meaningful response elicited by a teaching stimulus, as in the case of a graphical representation and elicitation of dialogue, and manifested in a learner's participation in, and understanding of, a lesson. Furthermore, 'understanding' itself, is considered to be an internal motivating stimulus since it inevitably promotes further interest, enquiry and learning. The accumulation of experience, understanding, reflection and knowledge acquisition associated with progressive learning incidents contribute to an amalgam of knowledge which forms a gestaltic construct.

Teaching incidents are correspondingly also classified 'push' or 'pull'. The former is characterised by 'impositional' teaching, devoid of understanding and meta-analysis of concepts. This style of teaching is essentially behaviourist and aligns with the stimulus-response bond theory of the 1920's and is described by Freire (1970) as a banking notion of education, characterised by the deposition of knowledge. 'Pull' teaching incidents are incorporative of learners' current understanding, and demonstrates unequivocally an understanding of a concept and is aware of its location within a gestaltic framework and comply with the principles of social constructivism of the 1970's (Von Glasersfeld, 1988, 1990). It shows evidence of meta-analysis and awareness of a concept's lower order derivatives. It is essentially constructivist in its attempt to engage learners collaboratively either through, teacher and learner, or learner(s) and learner(s) interaction and their exchange or interchange of ideas. It is not advocated that 'pull' styles are preferable to 'push' styles. Both have their rightful place in a teaching incident. It may further be required that the content of a 'push' teaching incident be consolidated by a learner through a process of repetition and rehearsal, for purposes of establishing a schema and becoming confidently conversant with the concept at hand. For the teacher the summation of teaching incidents accumulate as experience and form an amalgam of knowledge which is itself greater in meaning than the sum of its parts. It also comprises elements of experience and learning gained from the instructional process which in turn aids the modification of instructional content. In short, teachers, while teaching, learn about themselves, their learners, teaching and mathematics. Meta-instructional techniques will present as externally motivating or 'pull' factors for a learner.

I use the term, metazone to designate those cognitive activities which transpire between a teaching and learning incident. It is essentially the vertical component of the schematic representation in Fig 9.2. This component comprises ordered or graded concept levels progressing from a point of reception learning to increased levels of concepts. This directional nature of increasing concept levels aligns with correspondently concrete to abstract conceptions. The higher an individual's abstractive capacity the higher the order of concept level attainment. The incompleteness of a concept, owing to misconception, premature or elevated content delivery, induces a conceptual tension which may itself manifest as a 'pull' learning factor.

Its directional ordering aligns with an inductive process at the end of which specific content of a learning incident may become generalised. The accumulative vertical effect culminates in an amalgam of preconstructs which is also more in meaning than a sum of its preconstructs.

Reverting to a discussion of metazone activity, a learner's abstractive capacity is seen to increase as a function of time and as a function of the accumulative effect of teaching and learning incidents and especially so given the cumulative nature of mathematics knowledge. This is the significance of the positive gradient of the 'abstractive capacity line'. As the number of learning incidents increases over time, metazone activity is characterised by an accumulative gestalt of teaching experiences, learning experiences and a learner's private reflections on the analysis and synthesis of knowledge at hand. In this work the learners' final textual accounts seem to reveal a richness of gestalt that is portrayed in increasing levels of coherence of descriptions and explanations. It resembles a macro-conceptual construct. It is seen to have further significance in that it symbolises a break-point between communal knowledge and pioneering knowledge, the latter resembling the work of Descartes, Newton, Leibnitz and similar inventive contributions to mathematics and science. It is represented in the table by $\sum_{n=1}^{n}(T+L)+reflection>\sum_{n=1}^{n}T+\sum_{n=1}^{n}L$. The columnar representation in the table of this inequality reflects a learner's potential to use all orders of acquired constructs and concepts to engage in pioneering thought.

The abstractive capacity line is represented linearly in this schematisation. If a learner exhibits rapid understanding because of affective influences or natural intellect, an exponential curve may be more representative of the situation. The transfer of conceptual conglomerates, as in the integration of ideas across mathematics content knowledge, through the abstractive capacity line renders the line permeable in nature. As detailed earlier the accumulation of conceptual knowledge is variable at each teaching and learning incident, inferring that understanding and conceptualisation may not be complete. The completeness of the concept seems to be a function of a combination of sound teaching, natural affinity, natural ability and pre-construct knowledge.

9.3.2 Features of a bi-directional metazone model of teaching and learning

The model draws on **cognitivist** and **behaviourist** learning theory to link teaching and learning. It proposes that teaching is a fundamental component of any learning incident and thereby infers that **metateaching** is a vital component of any pre-service or in-service training of teachers. It allows for the incorporation of behaviourist rehearsal techniques where these benefit **mathematical vocabulary** and provide a **cognitive infrastructure** for the sound development of constructs. There is a place for algorithmic mathematics provided it is substantiated by **meta-analysis** of the associated content and processes. It proposes that the optimal manner, economically both in terms of time and meaning, of acquiring conceptual knowledge is along a concrete to abstract gradation of knowledge and therefore by meta-analysis, seeks ways of teaching that align with this continuum. It views social constructivist activities as a means of refining private thought through collaborative engagements and it regards written and verbal accounts of concepts so refined as being **expressive indices** of sound understanding contributing to the **confidence levels** of learners. It recognises that **cognitive individualities** are brought to a learning incident by virtue of innate intelligence, and social and environmental influences. The model is essentially **developmental** in that its developmental attributes may be dynamically placed along a chronological developmental path. The model recognises each learning incident as a **cognitive growth point** which is contextualised as a micro-scale cognitive event having all the features of the model, in the same way that a meso-scale cognitive event and macro-scale cognitive event can be contextualised by this model. It views the accumulation of knowledge as a **gestalt** where the whole is always greater than the sum of the parts. Ultimately knowledge is a mega metaconstruct of amalgams of teaching, cognitive and learning experiences.

9.3.3 Metazone

I define the **metazone** as that area of cognitive functioning that permits one to express, in one's own, but relevant terms, by written or verbal means, the meaning of a concept so that it is unequivocally understood by a receiver. It is necessarily a feature of every individual's cognitive fabric. Acquisition of this stage is an index of the attainment of **conceptual depth** as opposed to **conceptual recall** which is the reproduction of rote acquired knowledge. I regard the original written accounts on reflection of a concept as an order of understanding one level less than the accurate verbal account of the same. The latter may be a function of personality but, if indulged, indisputably renders the transmitter more confident and assured of its understanding of the concept in point. The ability to give alternative accounts of the same construct so that its original and fullest meaning is maintained may be considered another level removed from the previous. This level is necessarily a function of one's level of **language proficiency**.[33] There is no distinction in the discussed expressive indices in terms of one being more desirable than another. The implication is however, that the ability to engage in multiple expressions of the same concept enhances the abstractive capacity of the learner (and teacher). This is a necessary characteristic of an teacher of mathematics where, for example, a problem may be conceived of in different ways by different learners and hence produce a variety of equally correct and equivalent solutions that need to be assessed as demonstrated by the experience in a Grade 10 classroom, for example, where a learner's simplification of an algebraic fraction resolved to $-\frac{1}{b-a}$ and was deemed incorrect in comparison with $\frac{1}{a-b}$.

It is the attainment of this stage that permits communicable interchange and exchange between parties, as episodes of communicative action and dialogical discourses. Continued exchange and interchange of conceptual knowledge serves to

[33] There is concurrence here with the claims of Ball, Hyman and Hill (2004) who, in their analysis of what enhances the quality of students learning opportunities, assert that each step of the teaching process entails a deeper and more explicit knowledge (of multiplication) than simply performing the calculation. They further make the point that teaching involves more than recognising that students' answers are wrong. It requires an analysis of the site and source of the error. (Although 'multiplication' is the subject reference in this case, it could be substituted by any other mathematical topic, structure or process.)

promote communal knowledge and personal knowledge by the assimilation of constructs.

A written means of communication reflects a subliminal level of conceptual development, whilst being able to express concepts verbally is a reflection of an actualised level of understanding characterised by successfully abstracted **pure concepts.**[34] A learner's conceptual recall and conceptual depth are indistinguishable from written algorithmic accounts, but unaided or unprompted verbal expression of the same concept seems to reflect whether a concept has attained depth.

Teachers employ the purest or most economical form of a concept with progress along the **conceptual gradations** of the **continuum**. One, for example, would expect adolescents to make little sense of, which for them would be an esoteric debate, between Popper and Kuhn on the nature of science and the growth of knowledge. For a debate of this nature to be of value to its participants, would require a facilitator to unravel a myriad of concepts embedded in concept specific terminology and language and, in the process extend the physical length of the debate considerably for its intended audience. In short, **conceptual sophistication** amounts to the ability to use the most concise, economical means of expression in meaningful communicable interchange and exchange.

The implications of this for teachers, at any level, are critical. A teacher, by virtue, not of age, but conceptual development, will use the purest acquired form of concept to instruct learners on a certain topic. Teachers therefore need to be exposed to the metacognitive deconstruction of concepts and their meta-analysis before they venture to teach them; the advantage being that in so doing the spectrum of attainment levels in a class may be covered. This may indirectly remediate misconceptions and enable the reconnection of discontinuities on the concept continuum. It is common for the **constituent derivatives** of which a concept is composed to be assumed known in the teaching process that I illustrate as follows. An teacher of 13 year olds will use a term 'power' in the context of introductory algebra and may not elucidate that a 'base' and an 'exponent' are component parts of the term 'power'. This, from experience, leads learners to use the terms 'exponent'

[34] Von Glasersfeld (1991, p. xviii, xix) states that "... leading students to discuss their view of a problem and their own tentative approaches, raises their self-confidence and provides opportunities for them to reflect and to devise new and perhaps more viable conceptual strategies."

and 'power' synonymously. (It is, of course, not made easier by the fact that we verbalise '2^3' as '2 to the (power of) three'.) This in itself is a reflection of an teacher's advancement on the concept continuum since it is tacitly understood that the result of the operation is a composite notion (not in terms of algebraic function composition, but in the **subconstruct** composition of a concept!) amounting to 8 which is conceived of as the 'power'. It is understood that if '$2^3 = 8$' and '2^3' is a power, then 8 is a power. This is a valid form of deduction by way of syllogistic reasoning, but induces a degree of cognitive dissonance since if 8 is a number how is it also a power? The **cognitive tension** brought about by the inconsistency in understanding may be a negative factor, but more importantly may serve to motivate further cognitive enquiry on the part of the learner in attempting to ascertain clarity regarding the concept in question. It may, in terms of learning theory, serve as a useful perturbation. This example illustrates the **'embeddedness'** of concepts where ultimate concepts are wholly inclusive of its derivatives and where the **ultimate concept** has gestaltically greater meaning than the sum of its derivatives. This illustration serves an extension of the concept well. To accommodate a solution to the equation $2^x = 6$ a learner, in a mode of relational understanding as expounded by Skemp (1996), may argue that if $2^2 = 4$ and $2^3 = 8$ then x will assume a valid value somewhere between 2 and 3, and may record such an observation as $2 < x < 3$. Compare this with a learner who has received instruction in the mode of Skempian instrumental understanding. A learner may offer a solution of 3 as a result of incorrectly dividing or multiplying throughout by 2. An analysis of this error reveals a faulty understanding of the structure of the symbol 2^x and its implied repeated operation. It is interpreted as '2 multiplied by x ', and not as ' 2 to the x'. Such a learner would be further hard pressed to arrive at a solution to the equation albeit by intuitive iteration. Where the symbolic structure and implied operation of 2^x is correctly conceptualised, a learner may offer a solution of 'somewhere between 2 and 3'. To further estimate a root closer to one of the two rational extremes, induces a degree of **ambivalence** in both instrumentally and relationally instructed learners owing to the fact that inspection of the feasible interval shows it to be evenly distributed between its extremes relative to the anticipated position of the valid root, prompting an intuitive solution of 2,5 , the midpoint of the interval. In doing so a learner reveals its **current conceptual limit** in inadvertently presuming that 2^x is linear instead of exponential. **Discontinuities** of this nature in the **concept continuum** present further **cognitive dissonance** and impede the **coalescence** of

extension constructs in the cognitive development of an individual. In order to obtain an exact, real solution for the equation $2^x = 6$, albeit an irrational root, the introduction of the concept of a 'logarithm' becomes meaningful. From this discourse it is evident that the **first order derivative constructs** of the concept of 'logarithm' are 'base', 'exponent' ('index'), and 'power'. One level removed from this, or **second order constructs**, are the constructs of 'root', 'rational' and 'irrational'. The absence of, or poorly developed first or second order derivative constructs, may create discontinuities in the cognitive continuum, resulting in impediments to the acquisition of a pure concept of 'logarithm'. Mathematical concepts at any stage of the curriculum are characterised by their respective derivative constructs. Progress towards abstracting at the advanced end of the concept continuum is thus thwarted.

It does imply that teachers need to have an understanding of mathematics that will serve the needs of a learner in being able to unravel and deconstruct concepts so that learners attain mature levels of understanding. In so doing teachers are further able to facilitate the restoration of concept discontinuities in development that impede the attainment of purest known level of concepts. The qualification 'known level' permits the continued evolution of the concept. I refer to this approach to teaching and learning as **metateaching** which would be the natural precursor of **metalearning** and metacognition.

The **metazone** is characterised by intensive self reflection, ordering and refining of subconstructs that coalesce to distil pure concepts that are uncontaminated by confusion or uncertainty. Pure concepts are composite and are locally complete, subject to further extension where such conceptual evolution by coalescence, becomes necessary for further meaningful learning, for example knowing that $2^x = 6$ has a solution $2 < x < 3$ prior to the introduction to logarithms is sufficient to be deemed locally complete. It in fact may serve as a perturbation for the introduction of a logarithm.

The attainment of this level of metacognitive functioning presupposes the systematic and accumulative development of **preconstructs** which progressively coalesce to form pure composite concepts. The developments are essentially continuous. It is the discontinuity in development that disrupts concept formation and impedes the coalescence of **subconstructs** and the formation of pure concepts. Frequent, repeated assessment of the appropriate form, as in textual descriptions, with affirmative and/or corrective feedback, will nurture and refine a learner's understanding of a concept.

The attainment of this level of cognitive functioning facilitates the higher order cognitive skill of evaluating information by analysing and synthesising knowledge to ascertain similarities and differences in bodies of information. Information only acquires a status of knowledge once it is internalised or assimilated and located within a gestalt, when meaning and understanding amount to more than the sum of the strands of meaning and understanding of individual preconstructs.

The models of learning that frame this study seem not to address the range of cognitive abilities. A denominator common to them seems to be the assumption that all learners of the same age operate at the same cognitive level, are at the same stage of cognitive functioning and are static in terms of their individual cognitive potential. This assertion is borne out by the fact that there is, generally in all forms of teaching, a standard presentation of content to a class of learners in the teaching process. Teachers, generally in my experience, do not differentiate in their delivery of new content knowledge in terms of cognitive levels, in accordance with the varying cognitive levels of their learners. The **cognitive uniqueness** or **cognitive identity** of the individual, in every sense, is mostly ignored. The impracticality of the implied suggestion here is stark, but it should trigger amongst teachers an attempt to consider a personal philosophy of teaching and a didactical approach empathetic to the varied cognitive needs of their learning audiences; and it may be that some learners optimally acquire knowledge by radically constructivistic means rather than by socially constructivistic means as suggested by the cohort of learners in this study. The widely accepted contemporary notion that knowledge is socially constructed may incorrectly make assumptions about the psychological social nature and preferences of all learners.

There is a tendency for developmental theories of learning that frame this study to assume that an individual's achievable degree of sophistication of a concept is a function of age. In Piaget's learning theory for example, the egocentricity of an individual as an entity dependent on its environment, characterises the 0 – 2 age group. There is no apparent cognitive thought at this end of the developmental spectrum, although there is an interaction with objects. The other extreme is characterised by the ability to engage in advanced thought by way of reasoning and abstraction. The gradation between these extremes amounts to progress by age where the upper extreme may be attained at age 12 and older. The Piagetian model also seems to imply that cognitive maturing occurs over the first 12 -14 years of life, after which the mental functions are assumed developed for life. I suggest that as human beings we are innately life long learners, even if learning is attained at a variety of levels or forms. As cognitive beings all individuals are capable of thought of varying degrees of intensity either consciously or subconsciously. The expression of the thought is facilitated by an increase in one's written communicative language skills, and more so, by increased verbal skills. I am sure that there exists at each **point of growing** a **gradation** of **cognitive capacity**. The ability to abstract or achieve extreme levels of cognition seems to be peculiar to each growing point and independent of chronological age. It is dependent, however, as in other theories of learning, on perturbations for its initiation. The perturbations may be anything from a need to solve a problem, to a rich and exciting provocation of thought coupled with an individual's desire for learning and affinity with specific subject matter and / or a combination of these. The extent to which the degree of abstraction is achieved at each growing point is therefore dependent on the stimulus that elicits cognitive responses. The subject and content of the thought process is furthermore peculiar to, and appropriate to, one's experiences. The concrete to abstract gradations in the thought processes between any individuals of different age seem similar but not congruent. A five year old eating a sugar stick may be tempted to begin to contemplate what it is about the sugar stick that gives satisfaction, what it is actually that it is eating, where it comes from, how it was made and possibly how it could make something similar. All these projective thoughts, given here subjectively, in a possible order of gradation, require levels of abstractive thought. They may have been elicited by any number of stimuli, one of which may have been a question designed to provoke its imagination or have been self initiated by its own intuitive curiosity, and in which case, was prompted by the sugar stick itself. *It would seem also that this five year old is not just indulging in an eating experience to satisfy the lowest level of the Maslowian needs hierarchy,* (Maslow, 1970) but is at the lower level of the hierarchy that young people operate at as opposed to the levels of self actualisation that older people operate. Self actualisation requires the experiences protracted over time. Even at these affective levels therefore there seems to be gradations of experience. It is the last mentioned aspect which I refer to as **intuitive curiosity**, which Popper emphasises has an important role as 'independent creative imagination' for the rudiments of theory and therefore ultimately, for the growth of knowledge cited in Thornton (2002).

An individual's pursuit of knowledge can therefore be sustained from an early age and the degree to which abstractive thought is engaged in at an early age, I suspect, is correspondent with the degree and extent to which it is engaged as we grow older. A splinter of this idea is reminiscent of the behaviourist tendency toward repetition by rote learning in the process of acquiring knowledge, except here rote is not committing content to memory but, almost unknowingly, the repetition of process which entrenches or conditions a skill.

In the same way that human knowledge grows by deductive procedure, according to Popper (cf. Thornton,2002), so it seems that the growth of individual knowledge parallels it albeit at a lower order of scale.

The **cognitive intensity** of the abstraction in the metazone referred to above will **sublimate** the development of new knowledge, the latter being referred to as **pioneering knowledge** and the former as **communal knowledge** in the schematisation of the model.

Darwin's theory of evolution of the species serves as a metaphor to elucidate a further facet of these ideas. Metaphorically it is not the individual that is the species; it is knowledge itself. And so if knowledge is to survive it needs to be fit, or in Karl Popper's terms (cf. Thornton, 2002), it will be falsified if it cannot be corroborated by valid conclusions and evidence statements. Inter species evolution suggests that one species develops from another and environmental change and cataclysm prompts adaptation. Similarly, learning theories like those of Piaget are developmental. Embryonic concepts progressively coalesce with cumulative constructs which ultimately obtain levels of abstraction. As in the theory of evolution of the species development is dependent on time so the attainment of abstractive levels of thought is dependent on a meso-level notion of time. An intra species view of evolution accounts for the development of a species within itself. To extend the metaphor therefore knowledge domains evolve within themselves. There may be similarities in the manner in which growth and development occurs between knowledge domains. Where similarities do occur between domains the development of hybrid systems result from their merging and these may serve to strengthen concepts.

Piaget's theory of learning is dynamic only in that it posits changing modes of cognitive functioning in accordance with chronological age, but ignores different cognitive capacities of individuals whether these different capacities are attributable to genetic inheritance or other variables in the education process be it, the teacher, the curriculum, social factors, emotional factors, affective factors, motivation, attitude or interactions between these factors, or a child's current level of concept attainment.

Individuals irrespective of chronological age display different cognitive reasoning capacities and singular or multiple capacities by way of linguistic or numeric strength as seen in the verbal and non-verbal component of intelligence quotient scores. In as much as assigning a static score to define an individual's cognitive capacity is parochial, current learning strategies (techniques) like heterogeneous group work, co-operative learning, peer assessment etc., inherently recognises innate differences in cognitive capacity in that learners who have acquired concepts become the expected transferors of knowledge to peers.

Table 9.2 provides a glossary of terms integral to an associo-constructive metazone pedagogic model that are coined in this exposition.

Key stages and features and defining terms of an Associo-Constructive Metazone model of teaching and learning theory
1. Bi-directional Metazone model of teaching and theory of learning
2. Learning is a summation of Concept levels, Teaching incidents and Learning incidents and the amalgam is greater than the sum of its components;
3. Cognitive uniqueness / identity, cognitive capacity, intuitive curiosity, cognitive tension, cognitive dissonance, conceptual ambivalence, current conceptual limit, conceptual gradation: concrete recall to abstract depth (sophistication) composed of constituent derivatives
4. Metateaching + metalearning = metacognition
(Discontinuities in conceptual gradation are impediments to acquisition of pure concepts)

Table 9.1

Glossary of terms -
Key stages, features and defining terms of an associo-constructive metazone model of teaching and learning

9.4 A theoretical positioning of myself as teacher in terms of pedagogy

Bourne (2006) presents dynamic continua that incorporate both learning and teaching styles in pedagogic models that derive from Bernstein (1990) as depicted in Fig. 9.3. Rather than viewing the research classroom that is central to this case study as one or other of these modalities (Blair and Bourne, 1998; Kress, Jewitt, Bourne et al, 2004), I perceive that each modality arises circumstantially according to the prevailing needs.

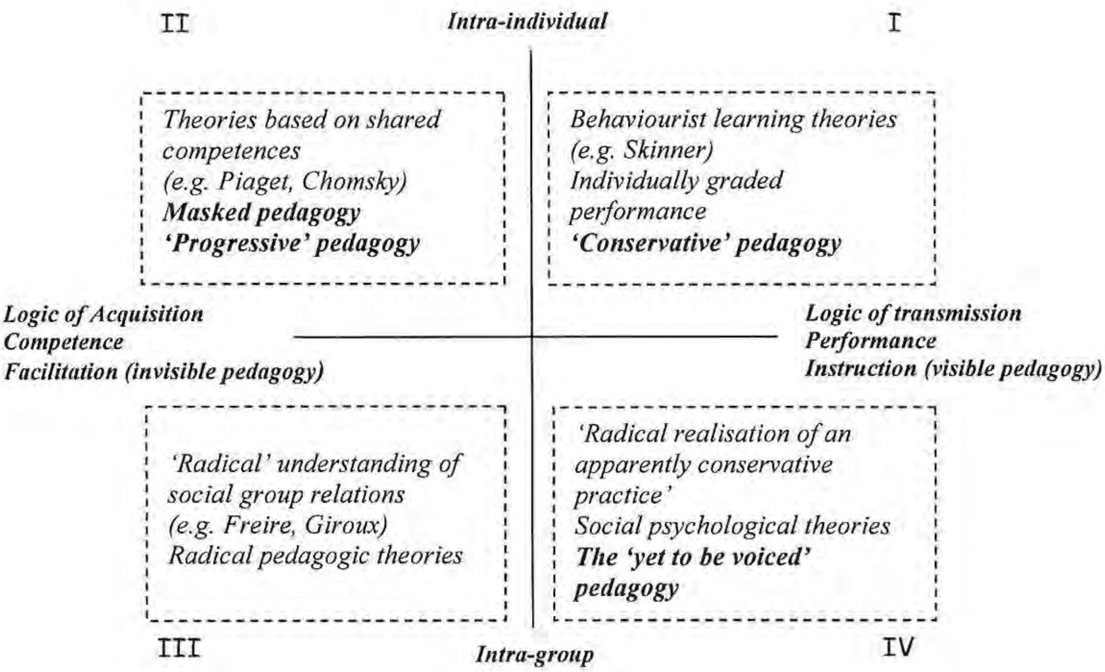


Fig. 9.3

Pedagogic modalities along two continua modified from Bourne (2006)

With reference to Fig. 9.1 that depicts what, in my view, is a metacycle of teaching, the conservative pedagogy of the Bernstein / Bourne categorisation is synonymous with the associative aspect of a first principles approach that is advocated in this work. In terms of the innate, nurtured talents of learners a prevailing pedagogy tends toward the progressive pedagogy of quadrant II. The Freirian notion of pedagogy for oppressed societies the work is located in quadrant III in terms of the disadvantaged communities in the South African context. The communicative action that characterises this case study and the emerging critical consciousness of learners in terms of vocabular reciprocity, theoretically falls within the ambiences of Bernstein's pedagogic model of quadrant IV, that positions the 'yet to be voiced' model as a radical transmission based pedagogy. Bernstein asserts this pedagogic modality as one that 'shows a radical realisation of an apparently conservative practice' (Bernstein, 1990: 73) cited in Bourne (2006).

The dynamism of the continua is representative of the competence based pedagogy transforming to performance based pedagogy along the horizontal continuum, while the vertical continuum traverses the intra-group to intra-individual learning styles. It is the dynamism of the horizontal and vertical continua that permits the mobility and hence transforming pedagogies that I see as operating in this study.

9.5 Summary

I use a reflective piece to describe my research experience as an introduction to this chapter. It served to draw together theory and practice that are threaded through this case study with the purpose of using classroom based communicative action to awaken a critical mathematics consciousness in learners to illuminate theoretical notions about mathematics pedagogy.

An associo-constructivist metacycle is used to describe the vocabular reciprocity between teacher and learner that emphasises the bi-directional communication streams that serve to establish social and cognitive milieus that promote mathematical consciousness and that ultimately affords teacher and learner the dialectical opportunity to construct increasingly refined concepts.

I proceed to show how mathematics learning because of its accumulative nature constitutes co-operative amalgams of teaching and learning events that developmentally contribute to the integration of mathematics concepts for both learner and teacher. A fundamental product of the metateaching and metalearning as they are defined in this exposition is metacognition which I deem to be situated within a mental construct called the metazone, a capacity to cognitively construe mathematics ideas in relation to past and current experiences that help sustain understanding. These ideas are contained in a schematic representation of a bi-directional associo-constructive metazone model of teaching and learning.

The final chapter looks for the implications of the findings, opportunities for the application of the findings and recommendations in terms of mathematics of the Further Education and Training band of the South African mathematics education system.

X

Conclusions and Recommendations

10.1 Introduction

In this final chapter I set about an overview of this work by referring to the introductory chapter quotes. I proceed to draw observations from the data examined in Chapter 8 and make recommendations that stem from this research experience. Finally I suggest ways in which this work may serve as a platform for continuing mathematics education research, and reflect on the impact of this research on my own practice.

10.2 Overview

1. 'Separateness – the thief of consensus and the mantle of isolation'

The polarities that still characterise our young South African democracy perpetuate personal and social injustices in failing to recognise that mathematics as a gateway subject is inaccessible to the vast majority of South Africa's poverty stricken township youth. It has emerged too from this work that mutism in, and about, mathematics is not only confined to circumstances of poverty. Middle-class and affluent mathematics classrooms show similar trends in demonstrating a paucity of mathematics vocabular acuity. I recognise this muteness as an indicator of personal injustice across socio-economic class distinctions since an absence of appropriate language with which to reach consensus in communicative action about mathematics concepts suggests that acritical cognition about mathematics constructs does no justice to mathematics or its learners. The personal inaccessibility of mathematics both in terms of an absent quality teacher corps, and the cognitive inaccessibility of mathematics understanding, translates into social injustice through preventing learners from pursuing careers commensurate with their ability and in so doing depriving an economy of the force that should help to drive it, and that would ultimately alleviate poverty. The separatenesses thus remain in terms of socio-economic extremes and the inertias that characterise a stagnating and deteriorating crisis in mathematics education in South Africa this despite the intentions of government as far back as July 2004 when national Minister of Education, Naledi Pandor, identified learner participation and performance, and quality teaching as national strategy in her opening address to the Association of Mathematics Teachers of South Africa.

'… our National Strategy identified three key thrusts, namely, to raise participation and performance by historically disadvantaged learners in Senior Certificate Mathematics and Physical Science; to improve on the number and quality of teachers of mathematics, science and technology; and to provide high quality mathematics, science and technology education from grade 1 to grade 12.'

Strategy without measurable action remains idle rhetoric. If we are to take the crisis in mathematics seriously it will demand an unparalleled effort to engage in measurable action that makes a difference to the recurring statistics that tell the story of meagre participation and dismal performance in mathematics over the past 12 years as detailed in chapter 2.

2. '***The critically transitive consciousness is characterised by*** depth in the interpretation of problems; by the substitution of causal principles for magical explanations; by the testing of one's "findings" and by openness to revision; by the attempt to avoid distortion when perceiving problems and to avoid preconceived notions when analyzing them; by refusing to transfer responsibility; by rejecting passive positions; by soundness of argumentation; by the practice of dialogue rather than polemics; by ***receptivity to the new for reasons beyond mere novelty and by the good sense not to reject the old just because it is old – by accepting what is valid in both old and new.'*** (Bold italics mine)

Freire (1974: 14)

'Authentic education investigates thinking.'

Freire (1970: 90)

Paulo Freire's pedagogy of the Brazilian oppressed has profound parallels for contemporary South Africa, both in terms of education in general and particularly for mathematics education. The critical consciousness that should be the product of the educative process should similarly be the product of mathematics learning. I have attempted to show how mathematics register acquired through concrete graphical representations of more abstract mathematics concepts has inspired a consciousness about mathematics that has lead to improved performance, particularly among isiXhosa speaking learners. Pedagogic modalities vary according to changing classroom circumstances and the conservative pedagogy of behaviouristic instruction is shown to be a crucial element of mathematics pedagogy that is the foundation of relational understanding.

3. 'For me, one goal of mathematics education should be to ascertain and explore ways of seeing mathematics which provide insight into its learning and teaching. In that sense, mathematics education can be seen as meta-mathematics.'

Pimm (1987: 206)

Pimm's challenge to explore ways that may provide insight into mathematics learning and teaching is an integral focus of this research. Its outcome has been the depth to which learners have acknowledged how their mathematics has improved. Participants attribute their improvements to being able to interpret mathematics constructs and its graphical counterpart in terms of the vocabulary that such graphicacy solicits and enables. Learner performance attests to the veracity of such claims.

4. '… a demanding, prudent, "experimental" attitude is necessary; at every moment, step by step, one must confront what one is thinking and saying with what one is doing, with what one is.'

Foucault (1984: 374)

For teachers the meta-analysis of tasks provides cognitive challenges that can inspire critical consciousness and communicative action both for themselves and for classroom participants. Foucault's 'awareness of self' demonstrates the critical consciousness that should epitomise teaching for the sake of education. It is my firm belief that mathematics teaching clamours for advancing engagement with participant learners for effective and sustained mathematics learning.

It is this self awareness that also suggests that as mathematics teachers we should be constantly engaged in personal enquiry based practices that continually advance our own cognition about pedagogy and refined strategies for teaching. Cognitive dialectical discourses are personal mental conversations about what we are doing and thinking.

5. 'Hermeneutics as an activity is at its best an art but never a method; as far as science is concerned, hermeneutics is a subversive force that undermines all systematic approaches.'

Gadamer (1976)

To this end the engagement with participant learners constitutes dialectical and dialogic encounters that are realised by written or spoken vocabular responses which are indicative of cognitive depth, quite apart from algorithmic algebraic accuracy that can effectively mask comprehension. The qualitativeness of the hermeneutic delves for participant cognition and acquired conceptual depth. In terms of the quote from Gadamer, above this may not be so subversive since the quantitative analysis of the coding data systematically corroborates such findings. The complementary nature of the qualitative and quantitative methods in this work affirms the critical realist tenet of methodological plurality and the interwoven dependency of interpretive and normative approaches to data.

6. 'Mathematical notation does not simply describe mathematical phenomena, it activates it. Language does not describe action, it is part of it... '

Brown (1997:219)

Where mathematical graphicacy is construed as mathematical notation, Brown's assertion above is all the more relevant to this study. Graphical representations of mathematics constructs are not merely pictures of mathematical reality; they activate cognitive engagement with such mathematics. When participants do not possess a mathematical register, there can be no discourse about cognitive reality, and without discourse there is no rational contribution towards consensus about mathematical meaning. Improved meaning is demonstrated by the increasing levels of coherent articulation amongst participants over two academic years, and is corroborated by affective accounts of such acquired meaning. Cognitive dialectical discourses are proof that learners are critically aware about what they are doing and thinking about mathematical tasks.

7. ***'Where constructivist theory is concerned, the 1990s have seen a shift in focus towards theory-practice rationalization through reporting of classroom research... More is needed, however, in bringing the issues from classrooms to centre stage, to challenge and potentially illuminate theory.'***

Jaworski (1999: 159)

This work has attempted to draw together theory and practice and as such has intended to illuminate my theoretical understanding of what constitutes meaningful mathematical learning and teaching. Reflective moments have merged in a pedagogic mathematics model where the didactive or conservative pedagogy of instruction is seen as a core component of mathematics classrooms without behaviourist malignancy. A first principles approach to mathematics teaching embodies the very cognitive underpinnings of thought that transcends rote instrumental understanding and takes it into the realm of relational cognition. A first principles approach based on the concrete graphical representation of algebraic abstraction facilitates the reversal of the abstractual order of mathematical constructs. Progressing from graphs to equations to expressions moves from concretised real mathematics to more abstracted real mathematics.

The theoretical first principles approach to mathematics teaching is a fundamental strategy for the inculcative transfer of mathematical concepts and for acquainting learners with the essential mathematical register. It is the cognitive acquisition of this register that provides learners with the language with which to think and communicate about mathematical constructs. The communication about mathematical concepts in verbal or written form establishes the degree to which the concept is saturated. This research has shown that the cognitive engagement with describing processes, promotes both awareness and alertness about mathematical procedures.

The goals of this research were

1. *to engage in a meta-analysis of learners' written accounts of their thought processes as reflected in the completion of a variety of mathematical topics and related tasks,*

2. *to investigate and understand what, in learners' experiences are essential to them in coming to know and be fluent in mathematics, and more especially in*

this regard, assess the value of a graphical approach to introducing complex and abstract mathematical concepts,

3. *to examine and understand the essential role(s) that I as a teacher must play in fostering mathematical learning and, more especially, to analyse my metacognitive engagement with the deconstruction of mathematical concepts.*

In terms of these goals the overview in the preceding paragraphs would suggest that the cohort of learners and I have progressed in the following ways.

- *Graphical representation of algebra concepts provides a concrete visual stimulus that facilitates the description of mathematical constructs.*
- *Graphicacy aids the deconstruction of concepts so as to expose its evolutionary derivative concepts that amalgamate in closed concepts and that promote a critical consciousness about mathematics.*
- *Attempts at describing concepts establish mathematical registers that promote vocabular exchanges that in turn diminish mutism and that reflect radical constructivist learning.*
- *Diminished mutism promotes cognitive dialectical discourses that contribute to communicative action.*
- *Communicative action contributes to social constructivist environments.*
- *Social constructivist environments promote co-operative learning.*
- *Forums of learning encourage cognitive consensus about concepts through coherent rational argumentation through mathematical language.*
- *Rational mathematical argumentation promotes good performance.*
- *Good performance breeds success reflected in continuing good results.*

The recommendations that arise from this study are detailed below.

10.3 Recommendations

1. *The accessibility of mathematics*

   Mathematics by virtue of its elitist and once iconoclastic apartheid status was not accessible to the majority of South Africans. This denied entitlement was statutory. The current inaccessibility is denied entitlement through inopportunity arising out of absent expertise and pedagogy and poverty.

The short term solution to such inaccessibility would seem to be in the form of projects that deliver expert tuition into classrooms for the purpose of daily curriculum delivery for the purpose of reaching disadvantaged learners and to provide learning opportunities for inservice teachers in an attempt to alleviate personal and social injustice. Such projects need to be recognised by national and provincial government to be priority more than the provisioning of schools with costly information communication technology requiring unavailable expertise for on-site sustainability and which cannot be measured through an improved quantity and quality of mathematics performances.

2. *Professional development*

Both the inservice and pre-service equipping of mathematics teachers begs attention.

Inservice learning, as already suggested, can be effected through videoconferencing mathematics, the likes of which I am currently pioneering. Daily curriculum delivery using the strategies and pedagogy expounded in this thesis serve to reach teachers present in their remote classrooms connected to a videoconferencing studio. The merits of this approach are that teachers and learners remain in their classrooms and many classrooms can be accessed simultaneously. Inservice teachers need to engage in a metacritique of their own practice. Such would constitute meaningful professional development and could well replace the arduous and meaningless administrative tasks that seem to preoccupy what would otherwise be valuable academic reflection.

Pre-service professional development needs to recognise that mathematics content knowledge is insufficient for novice teachers to be competent teachers of mathematics. The chapters of this thesis aim to demonstrate that tertiary mathematics method courses could invoke the cognitive dialectical voice in prospective teachers. Metacognitive deconstruction of concepts facilitates metateaching and metalearning. A module of this nature may have the value of inducing a pedagogic critical consciousness in prospective mathematics teachers.

3. *Curriculum development*

In as much as the current curriculum provides creative opportunity for teachers to deal expansively and innovatively with mathematics the expertise is lacking. The chapters of this thesis aim to demonstrate that there is space for what I have termed 'generic mathematics'; the structure of mathematics and the development of the mathematical literacy that confines itself to the teaching of the construct of graphs, equations and expressions and the embeddedness of these in the Cartesian plane. The interpretation and the assignment of meaning to mathematics phenomena of the likes of $y=-(x+2)^2-1$, $0,5^x>5$, $|x-1|-|3-x| \ for \ 2<x<3$, $\frac{2-x}{x-2}$ without instruction is indicative of a critical consciousness about the mathematics that these phenomena elicit. Each, in its own right, can be contextualised in the Cartesian plane and from that contextualisation emanates its meaning. Generic mathematics recognises the transferability of process across topics. As such it recognises for example the difference and similarity between $y=x-1$ and $y=(x-1)$, and the implication of this for $y=(x-1)^2$ and $y=\sqrt{x-1}$. Generic mathematics deals with the recognition of structure and the assignment of meaning to such structure; it recognises the transferability of notational forms across topics. It therefore deals with mathematics holistically. Generic mathematics provides an ambient gestalt that sees the whole as greater than the sum of its parts. It permits accessing strategy, and encourages sustainable meaning.

3. *Assessment*

It is acknowledged that outcomes based assessments encompass a wide variety of assessment modes, inter alia, group assessment, peer assessment, formative and summative assessment, of a variety of tasks like homework assignments, class assessments, investigations, explorations, projects and the like. Each assessment however is either criterion- or norm-referenced. I suggest that in mathematics criterion- and norm-referenced assessment still falls short of ascertaining whether learners have achieved concept closure through acquiring conceptual depth. The point has been made that algorithmic algebraic accuracy masks understanding since rote procedures produce valid outcomes. I have demonstrated in the course of this work that vocabular responses that describe the mathematics of algebra or illustrates its Cartesian context, reveals the depth that characterises meaningful mathematics, and at the same time obviates superfluous algebra. For example, $|x-1|=-x$, is immediately solvable where $x-1=-x$, so $x=\frac{1}{2}$. Without a generic understanding the algebra of considering both cases, although important for algebraic argumentation, is superfluous. The mind's eye visual graphic of $|x-1|=-x$ confirms the single valid solution and adds strength and credibility to the algebraic argument.

In the light of this my recommendation is that the final exit level examination of the Further Education and Training band comprise questions that allow learners to demonstrate their understanding of mathematics through their use of acquired mathematical register to describe, explain or account for mathematical constructs or phenomena. Such an assessment, preferably as a short third examination paper, has value in revealing learners who are competent, coherent and articulate and who are prospective teachers of mathematics and in turn whose recruitment to the profession will serve the needs of mathematics education well. Its other value is that it will require of inservice teachers of mathematics to acquire explanatory skills that transcend instruction at the level of surface structure; teaching that manifests as rote instruction and learning. It has been shown that the pictorial value of graphic representations of mathematics constructs has further facilitated vocabular responses from isiXhosa learners showing that the anticipated impediment of English as language of instruction is surmountable.

4. *Pedagogy*

There appears much to be gained from incorporating the written and spoken forms of mathematics through incorporating communicative action in mathematics classrooms. An articulated mathematics vocabulary born out of graphical representation seems to indicate the degree or depth of concept development and such an approach is therefore advocated.

5. *Material development*

If the pedagogic strategies described in this work are recognised as contributing to meaningful and sustained mathematics learning then there exists opportunity and scope for recontextualising mathematics learning materials in terms of graphical representations.

10.4 Opportunities for continuing research

Metacritique, that encompasses bi-directional metateaching and metalearning, that facilitates and promotes metacognition is a valuable method for assisting teachers to engage in reflective enquiry based classroom research practice suitable for post-graduate qualification, and for enhancing mathematics pedagogy.

Further to this each of the areas to which the recommendations are pertinent may also become areas of research opportunity.

It is my hope also that this work will contribute to a debate on the theory of mathematics pedagogy.

Much is required in terms of ensuring the accessibility of mathematics for South Africa's poverty stricken communities and the application of the teaching strategy that pervades the chapters of this thesis has the potential of being piloted in other poverty stricken communities, more especially in the ascertaining of the intermediary value of graphical representations for those whose mother tongue is not English.

There is also a case for investigating mathematical conceptual depth in terms of mathematical conversancy amongst articulate learners.

This research I believe asks questions of mathematics teacher development at inservice and pre-service levels. Module and course development could ensue from these findings.

Continued work on teacher and learner metacognition through the use of graphical representation as a means of eliciting vocabular responses, and for comparison across socio-economic groupings can be used to ascertain whether accessibility in terms of personal justice is confined only to disadvantaged groupings.

Much remains to be developed in terms of generic mathematics as an introduction in Grade 10 as a compulsory foundation module for mathematics. Curriculum development presents as an opportunity for research.

In terms of the need of competent mathematics teachers for South Africa the implementation of a short third exit level examination paper may provide valuable insights as to whom to incentivise to enter the mathematics teaching profession.

The cognitive assessment of learners in terms of their acquisition of conceptual depth through the evaluation of descriptions and explanations may serve to suggest that the understanding of mathematics is more valid as an assessment than criterion- or norm-referenced assessments, that coherence and articulation in conjunction with graphical interpretation are fundamental indicators of conceptual depth.

And finally, the differentiation between Higher Grade (HG) and Standard Grade (SG) mathematics that characterises this work may hold meaning for the delineation of Mathematics and Mathematical Literacy as subjects of the Further Education and Training band. The issues of interpretation and pedagogy pertain as much to mathematics in its new nomenclature as it did in its old.

10.5 Self reflection and factors seen to be limiting on the research

My ontological beliefs and values regarding the accessibility of mathematics especially for those afflicted by poverty has played a major role in the development of this research, as has a concern about the inferior quality and quantity of mathematics passes that the education system in this country produces. It seems unreasonable that inertias remain despite both government and non-government continual interventions. Infrastructural development of schools and the provisioning of state-of-the-art technology seem not to be able to substitute for competent mathematics teaching. It is to this end that I was prompted to investigate what pedagogic practices promote sustained mathematics learning. The results seem to indicate that the inaccessibility of mathematics is not peculiar to poverty or poorly resourced classrooms only; a factor that emphasises the critical need to address the lack experienced by inservice and pre-service mathematics teachers.

The research experience has confirmed my belief that successful mathematics learning need not be exclusively that of the affluent. Despite deficits in the language of instruction and prior mathematics content knowledge, black and coloured South African learners have shown that a pedagogy characterised by communicative action around graphical representation in combination with nurturing, helps develop a critical consciousness about mathematics.

The hunches and suspicions about the pedagogic value of the pictorial impact of graphical representations have been confirmed through this work. The research process has not only confirmed my hunches but has also initiated an acute awareness of how learners view mathematical constructs. It has also revealed that mute mathematics learners, and teachers, seem to lack conceptual depth about mathematics constructs and that muteness reflects meagre understanding.

The research process and pedagogic strategy has developed and refined my ability to deconstruct and recontextualise mathematics concepts; a practice that has improved the accessibility of mathematics for those I teach, both learners and teachers. It has further made me critically aware of the importance of speaking mathematics accurately.

Limitations

A limitation of the research technique is the fact that sound hermeneutic analyses are dependent on a good command of the language of interaction. Eleven of the twenty- seven learner participants have isiXhosa as a home language and this is seen to have affected the articulateness of the descriptions of the black participants. But it appears that where language has been sparse the graphical representation has sufficiently compensated for a lack of words. However, an unexpected outcome has been the improved English language proficiency of participants like participant 6 whose progress also from 27% for SG mathematics to 87% for HG mathematics over 18 months is a highlight of this research.

The mathematical muteness amongst teachers and learners is a focus of this research. A case study approach was regarded as the best tool for the research given that the muteness referred to above would be a limitation in terms of learner and educator peers that may have served as a qualitative control, through triangulation, peer debriefing and member checks. A more communicatively active participant group would have served well as qualitative controls. This limitation, within the qualitative aspect of this work, forces me as teacher-researcher to seek hermeneutic phenomenological meaning, since the would-have-been empirical hermeneutic expectation is flawed by the mathematical muteness of the research participants.

I have gained immensely as a teacher, learner, facilitator, researcher and materials developer from this enriching research experience.

REFERENCES

AANSTOOS, C. M. (1983) *A Phenomenological study of thinking*, Duquesne University Press.

ADLER, J. (2001) *Teaching Mathematics in Multilingual Classrooms*, Kluwer Academic Publishers.

APPLE, M. (1995) Taking power seriously: New directions in equity in mathematics education and beyond. IN SECADA, W. G., FENNEMA, E., ADAJIAN, L. B. (Ed.) *New Directions for Equity in Mathematics Education.* Cambridge, Cambridge University Press.

ARCHER, M., BHASKAR, R., COLLIER, A., LAWSON, T. AND NORRIE, A. (Ed.) (1998) *Critical Realism - Essential Readings,* London and New York, Routledge.

ARTIGUE, M. (1992) Functions from an algebraic and graphic point of view: Cognitive difficulties and teaching practices. IN HAREL, G., DUBINSKY, E. (Ed.) *The concept of function: Aspects of epistemology and pedagogy, MAA notes 25.* Washington, DC, Mathematical Association of America.

AUSUBEL, D. P. (1961) Reception versus Discovery Learning in Classroom Instruction. *Educational Theory,* 2, 21 - 24.

BABBIE, E., MOUTON, J. (Ed.) (2002) *The Practice of Social Research,* Cape Town, Oxford University Press Southern Africa.

BALACHEFF, N. (1990) Towards a problematique for research on mathematics teaching. *Journal for Research in Mathematics Education,* 21, 258-272.

BALL, D. L. (1993) Halves, pieces and twoths: Constructing representational contexts in teaching fractions. IN CARPENTER, T., FENNEMA, E., ROMBERG, T. (Ed.) *Rational Numbers: An Integration of Research.* Hillsdale, NJ, Lawrence Erlbaum.

BALL, D. L., BASS, H., HILL, H. C. (2004) Knowing and Using Mathematical Knowledge in Teaching: Learning what matters. IN BUFFLER, A., LAUGKSCH, R. (Ed.) *Proceedings of the 12th Annual Conference of the South African Association for Research in Mathematics, Science and Technology Education.* Durban, SAARMSTE.

BATTISTA, M. T. (1994) On Greeno's environmental/model view of conceptual domains: A spatial/geometric perspective. *Journal for Research in Mathematics Education,* 25, 86-94.

BAUERSFELD, H., KRUMMHEUER, G., VOIGT, J. (1988) Interaction theory of learning and teaching mathematics and related microethnographical studies. IN STEINER, H. G., VERMANDEL, A. (Ed.) *Foundations and Methodology of the Discipline of Mathemtics Education.* Antwerp, Proceedings of the Theory of Mathematics Education Conference

BELL, F. (1978) *Teaching and learning mathematics (in secondary schools),* Dubuque, lA, Wm. C. Brown.

BERNSTEIN, B. (1996) *Pedagogy, Symbolic Control and Identity: Theory, Research, Critique,* London, Taylor and Francis.

BERNSTEIN, B. (1990) *Class, Codes and Control Vol IV. The Structuring of Pedagogic Discourse.* London: Routledge

BETZ, W. (1936) *The reorganisation of secondary education. In: The place of mathematics in modern education. 11th Yearbook of NCTM,* New York: Bureau of Publications, Teachers College, Columbia University.

BHASKAR, R. (1978) *A Realist Theory of Science,* Hassocks, Harvester Press.

BHASKAR, R. (1989) *Reclaiming Reality: A Critical Introduction to Contemporary Philosophy,* London and New York, Verso.

BLAIR, M., BOURNE, J. (1998) *Making the Difference: Teaching and learning strategies in multi-ethnic schools. (Research Report 59).* GB Dept. of Education and Employment, London.

BOURNE, J. (2006) Lecture notes: PhD week - Rhodes University, Grahamstown.
BROWN, T., COLLINS, A., DUGUID, P. (1989) Situated Cognition and the culture of learning. *Educational Researcher,* 18, 32-42.
BROWN, T. (1991) Hermeneutics and mathematical activity. *Educational Studies in Mathematics,* 22, 475-480.
BROWN, T. (1994a) Towards a hermeneutical understanding of mathematics and mathematical learning. IN ERNEST, P. (Ed.) *Constructing Mathematical Knowledge: Epistemology and Mathematics Education.* London, Falmer Press.
BROWN, T. (1994b) Creating and knowing mathematics through language and experience. *Educational Studies in Mathematics,* 27, 79-100.
BROWN, T. (1994c) Creating evidence towards making statements about classroom activity. *Research in Education,* 51, 33-40.
BROWN, T. (1994d) Describing the mathematics you are part of: a post-structuralist account of mathematical learning. IN ERNEST, P. (Ed.) *Mathematics, Education and Philosophy: an International Perspective.* London, Falmer Press.
BROWN, T. (1996a) Creating data in practitioner research. *Teaching and Teacher Education,* 12, 261-270.
BROWN, T. (1996b) The phenomenology of the mathematics classroom. *Educational Studies in Mathematics,* 31, 115-150.
BROWN, T. (1997) *Mathematics Education and Language - Interpreting Hermeneutics and Post-Structuralism,* Dordrecht, London, Boston, Kluwer Academic Publishers.
BROWNELL, W. A. (1935) Psychological considerations in the learning and teaching of arithmetic. *The teaching of arithmetic. 10th Yearbook of NCTM.* New York, Bureau of Publications, Teachers College, Columbia University.
BROWNELL, W. A. (1945) When is arithmetic meaningful? *Journal of Educational Research,* 38, 481-498.
BRÜNER, J. (1957) *Contemporary Approaches to Cognition,* Cambridge, Harvard University Press.
BRÜNER, J. (1960) *The Process of Education,* Harvard University Press.
BRÜNER, J. (1974) *Beyond the Information Given. Studies in the Psychology of Knowing,* London, George Allen & Unwin Ltd.
BRÜNER, J. & (1985) Vygotsky: A historical and conceptual perspective. IN WERTSCH, J. V. (Ed.) *Culture, communication, and cognition: Vygotskian perspectives* Cambridge, Cambridge University Press.
BRÜNER, J. S., OLVER, R. R. AND GREENFIELD, P. M. (Ed.) (1966) *Studies in Cognitive Growth,* New York, London, Sydney, John Wiley & Sons, Inc.
BUERK, D. (1985) The voices of women making meaning in mathematics. *Journal of Education,* 167, 393-416.
BURTON, L. (1996) Mathematics, and its meaning as narrative - a literacy for the twenty-first century. IN BAKER, D., CLAY, J., FOX, C. (Ed.) *Challenging Ways of Knowing: In English, Mathematics and Science.* London, Falmer Press.
BURTON, L. (Ed.) (1999) *Learning Mathematics - From Hierarchies to Networks,* Falmer Press.
BUXTON, L. (1981) *Do You Panic about Maths?,* London, Heinemann.
CARR, W., KEMMIS, S. (1986) *Becoming Critical: Education, Knowledge and Action Research,* Lewes, Falmer Press.
CLYNICK, T., LEE, R. (2004) From Laggard to World Class - Reforming maths and science education in South Africa's schools. *Centre for Development and Enterprise (CDE), Research report no.13.*
COBB, P., STEFFE, L. P. (1983) The constructivist researcher as teacher and model builder. *Journal for Research in Mathematics Education,* 14, 83-94.
COBB, P. (1986) Contexts, goals, beliefs and learning mathematics. *For the learning of mathematics,* 6, 2-9.

COBB, P., YACKEL, E., WOOD, T. (1992) A Constructivist alternative to the representational view of mind in mathematics education. *Journal for Research in Mathematics Education* 23 2-33.

COBB, P. (1994) Where is the mind? Constructivist and socio-cultural perspectives on mathematical development. *Educational Researcher,* 23, 13-19.

COBB, P. (1995) Cultural tools and mathematical learning: A case study. *Journal for Research in Mathematics Education,* 26, 362-385.

COBB, P., BAUERSFELD, H. (1995) Introduction: The coordination of psychological and sociological perspectives in mathematics education. IN COBB, P., BAUERSFELD, H. (Ed.) *The Emergence of Mathematical Meaning: Interaction in Classroom Cultures* Hillsdale, Erlbaum.

COBB, P., BOUFI, A., MCCLAIN, K., WHITENACK, J. (1997) Reflective discourse and collective reflection. *Journal for Research in Mathematics Education,* 14, 83-94.

COBB, P. (2000) Conducting Teaching Experiments in Collaboration With Teachers. IN KELLY, A. E., LESH, R. A. (Ed.) *Handbook of Research Design in Mathematics and Science Education.* Lawrence Erlbaum Associates.

COBB, P., WOOD, T., YACKEL, E., NICHOLLS, J., WHEATLEY, G., TRIGATTI, B., PERLWITZ, M. & (1991) Assessment of a problem-centered second-grade mathematics project. *Journal for Research in Mathematics Education,* 22, 3-29.

COBERN, W. W. (1991) *World view theory and science education research. Monograph No.3,* Manhattan, KS, National Association for Research in Science Teaching.

COHEN, L., MANION, L. (1980) *Research Methods in Education,* Croom Helm.

COLLINS, P. H. (1991) *Black Feminist Thought: Knowledge, Consciousness, and the Politics of Empowerment,* London, Routledge.

COLMAN, A. M. (2001) *Dictionary of Psychology.* Oxford University Press

CONFREY, J. (1994) A theory of intellectual development: Part 1. *For the Learning of Mathematics,* 14, 2-8.

CONFREY, J. (1995) How compatible are radical constructivism, sociocultural approaches, and social constructivism? IN STEFFE, L. P., GALE, J. (Ed.) *Constructivism in Education.* Hove, UK, Lawrence Erlbaum.

CONFREY, J. (1995a) A theory of intellectual development. Part 2. *For the Learning of Mathematics,* 15, 38-48.

CONFREY, J. (1995b) A theory of intellectual development: Part 3. *For the Learning of Mathematics,* 15, 36-45.

CONFREY, J. (2000) Improving Research and Systemic Reform Toward Equity and Quality. IN KELLY, A. E., LESH, R.A. (Ed.) *Handbook of Research Design in Mathematics and Science Education.* Lawrence Erlbaum Associates.

CONFREY, J. L., A. (2000,) Transformative Teaching Experiments Through Conjecture-Driven Research Design. IN KELLY, A. E., LESH, R.A. (Ed.) *Handbook of Research Design in Mathematics and Science Education.* Lawrence Erlbaum Associates.

COTTRILL, J. (2003) An Overview of Theories of Learning in Mathematics Education Research.

DANERMARK, B., EKSTRÖM, M., JAKOBSEN, L. AND KARLSSON, J. CH. (Ed.) (2002) *Explaining Society - Critical Realism in the Social Sciences,* London and New York, Routledge.

DAVIS, B. (1997) Listening for Differences. *Journal for Research in Mathematics Education,* 28, 355 - 376

DAVIS, R. B. (1984) *Learning Mathematics. The Cognitive Science Approach to Mathematics Education,* London, Croom Helm.

DAVIS, R. B. (1992) Understanding "understanding". *Journal of Mathematical Behavior,* 11, 225-241.

DESCARTES, R. (1999) *Discourse on Method and Meditations on First Philosophy.* Translated by CRESS, D. A. Hackett.

DENZIN, N. K. (1970) *The Research Act in Sociology: A Theoretical Introduction to Sociological Methods,* London, The Butterworth Group.
DENZIN, N. K., LINCOLN, Y. S. (Ed.) (2000) *Handbook of Qualitative Research,* Sage Publications.
DOE (1997) Senior Phase (Grades 7 to 9) Policy Document. Pretoria: Government Publication.
DOE (2001) National Plan for Higher Education in South Africa. Pretoria, Government Printer.
DOE (2001) National Strategy for Mathematics, Science and Technology Education in General and Further Education and Training. Pretoria, Government Printer.
DOE (2003a) National Curriculum Statement: Grades 10 -12 (General). Mathematics. Seriti Publishing.
DOE (2003b) National Curriculum Statement: Grades 10 -12 (General). Mathematical Literacy. Seriti Printing.
DOLL, W. E. (1993) *The post-modern perspective on curriculum,* New York, Teachers College Press.
DREYFUS, T. (1990) Advanced mathematical thinking. IN NESHER, P., KILPATRICK, J. (Ed.) *Mathematics and cognition: A research synthesis by the International Group for the Psychology of Mathematics Education.* Cambridge, Cambridge University Press.
DRIVER, R., OLDHAM, V. (1986) A constructivist approach to curriculum development in science. *Studies in Science Education,* 13, 105 - 122.
DUBINSKY, E. (1991) Reflective abstraction in advanced mathematical thinking. IN TALL, D. (Ed.) *Advanced mathematical thinking.* Dordrecht, Kluwer Academic Publishers.
ECDOE (2006) National Curriculum Statement for Mathematics - Grades 10-12 - Training Manual.
EDWARDS, L. D. (2005) The Role of Gestures in Mathematical Discourse: Remembering and Problem Solving. IN CHICK, H. L., VINCENT J. L. (Ed.) *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education.* Melbourne, PME.
ELLIOTT, J. (1993) The relationship between 'understanding' and 'developing' teachers' thinking. IN ELLIOTT, J. (Ed.) *Reconstructing Teacher Education.* London, Falmer Press.
ERLWANGER, S. (1973) Benny's conception of rules and answers in IPI mathematics. *Journal of Children's Mathematical Behaviour,* 1, 7-26.
ERNEST, P. (1991) *The Philosophy of Mathematics Education,* London, Falmer Press.
ERNEST, P. (Ed.) (1994a) *Constructing Mathematical Knowledge: Epistemology and Mathematical Education*
London, Falmer Press.
ERNEST, P. (1994b) What is constructivism in the psychology of mathematics education? *Proceedings of the Eighteenth Conference of the Group on the Psychology of Mathematics Education.* Lisbon.
ERNEST, P. (1998) A Postmodern Perspective on Research in Mathematics Education. IN SIERPINSKA, A., KILPATRICK, J. (Ed.) *Mathematics as a Research Domain: A Search for Identity.* Books I and II. Kluwer Academic Publishers.
FERRARA, F., NEMIROVSKY, R. (2005) Connecting Talk, Gesture and Eye Motion for the Microanalysis of Mathematics Learning. IN CHICK, H. L., VINCENT J. L. (Ed.) *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education.* Melbourne, PME.
FOUCAULT, M. (1984) Politics and ethics: an interview. IN RABINOW, P. (Ed.) *The Foucault Reader.* Penguin, Harmondsworth.
FREIRE, P. (1970) *Pedagogy of the Oppressed,* London, Penguin Books.
FREIRE, P. (1974) *Education for Critical Consciousness,* London, Continuum

FREIRE, P. (1987) *A Pedagogy for Liberation*, Macmillan.
FREIRE, P. (1998) *Teachers as Cultural Workers*, Westview Press.
FREUDENTHAL, H. (1973) *Mathematics as an Educational Task,* Dordrecht, D. Reidel.
FREUDENTHAL, H. (1991) *Revisiting Mathematics Education: China Lectures,* Dordrecht, Kluwer.
GADAMER, H. (1976) Philosophical Hermeneutics. IN GADAMER, H. (Ed.) *On the Scope and Function of Hermeneutical Reflection* Berkeley.
GAGNE, R. M. (1983) Some issues in the Psychology of Mathematics Instruction. *Journal for Research in Mathematics Education,* 14, 7-18.
GATTEGNO, C. (1971) *What we owe Children,* London, Routledge and Kegan Paul.
GERGEN, K. J. (1995) Social construction and the educational process. IN STEFFE, L. P., GALE, J. (Ed.) *Constructivism in Education.* Hove, UK, Lawrence Erlbaum.
GIORGI, A. (1970) Toward phenomenologically based research in psychology. *Journal of Phenomenological Psychology,* 1, 75-78.
GIORGI, A. (1975) An application of phenomenological method in psychology. IN GIORGI, A., FISCHER, C., MURRAY, E. (Ed.) *Duquesne studies in phenomenological psychology* Pittsburgh, Duquesne University Press.
GIORGI, A. (1985) *Phenomenology and Psychological Research,* Pittsburgh, Duquesne University Press.
GIORGI, A. (1992) An exploratory phenomenological psychological approach to the experience of the moral sense. *Journal of Phenomenological Psychology,* 23, 50-86.
GOODMAN, Y. M., GOODMAN, K. S. (1990) Vygotsky in a whole language perspective. IN MOLL, L. C. (Ed.) *Vygotsky and education: Instructional implications and applications of sociohistorical psychology.* New York, Cambridge University Press.
GOOS, M., GALBRAITH, P., RENSHAW, P. (1994) Collaboratioin, dialogue and metacognition: The mathematics classroom as a "community of practice". IN BELL, G., WRIGHT, R., LEESON, N., GEAKE, J. (Ed.) *Proceedings of the Seventeenth Annual Conference of the Mathematics Education Research Group of Australasia.* Lismore, Australia, Mathematics Education Research Group of Australasia
GOOS, M., GALBRAITH, P., RENSHAW, P. (1999) Establishing a Community of Practice in a Secondary Mathematics Classroom. IN BURTON, L. (Ed.) *Studies in Mathematics Education Series: Learning Mathematics - From Hierarchies to Networks.* London, Falmer Press.
GRAVEMEIJER, K. (1998) Developmental Research as a Research Method. IN SIERPINSKA, A., KILPATRICK, J. (Ed.) *Mathematics as a Research Domain: A Search for Identity.* Books I and II. Kluwer Academic Publishers.
GRAVES, N. J. (1972) *Geography in Education,* Heinemann Educational Books
HABERMAS, J. (1972) *Knowledge and Human Interests,* Boston MA, Beacon Press.
HABERMAS, J. (1984) *The Theory of Communicative Action,* Cambridge, Polity Press.
HABERMAS, J. (1990) *Moral Consciousness and Communicative Action,* Cambridge, Polity Press.
HALLIDAY, M. A. K. (1975) Some aspects of socio-linguistics. *Interactions between linguistics and mathematical education*, 64-73.
HANDEL, G. (1991) Case study in family research. IN FEAGIN J. R., O., A. M., SJOBERG, G. (Ed.) *A case for the case study.* Chapel Hill, University of North Carolina Press.
HARTSHORNE, K. (1992) *Crisis and Challenge: Black Education 1910 -1990,* Cape Town, Oxford University Press.
HEIDEGGER, M. (1962) *Being and Time,* Oxford, Blackwell.
HIEBERT, J., CARPENTER, T., FENNEMA, F., FUSON, K., HUMAN, P.,MURRAY, H., OLIVIER, A., WEARNE, D. (1996) Problem Solving as a Basis for Reform in Curriculum and Instruction: The Case for Mathematics. *Educational Researcher*, 12-21.

HIEBERT, J., CARPENTER, T., FENNEMA, E., FUSON, K., WEARNE, D., MURRAY, H., OLIVIER, A., HUMAN, P. (1997) *Making Sense: teaching and learning mathematics with understanding,* Portsmouth, NH, Heinemann.

HILL, J. E., KERBER, A. (1999) *Models, Methods, and Analytical Procedures in Educational Research,* Detroit, Wayne State University Press.

HOWIE, S., VINJEVOLD, P. (2002) Report on 11 countries that participated in the TIMMS-R. *Centre for Development and Enterprise (CDE) background research report*

HUGHES, M. (1986) *Children and Number,* Oxford, Basil Blackwell.

ISAACSON, Z. (1990) "They look at you in absolute horror": Women talking and writing about mathematics. IN BURTON, L. (Ed.) *Gender and Mathematics: An International Perspective.* London, Cassell.

JAWORSKI, B. (1999) Tensions in Teachers' Conceptualizations of Mathematics and of Teaching. IN BURTON, L. (Ed.) *Learning Mathematics: From Hierarchies to Networks.* London, Falmer Press.

JENKINS, J. G., DALLENBACH, K. M. (1924) Oblivescence during sleep and waking. *American Journal of Psychology,* 35.

JORDAAN, W. J., JORDAAN, J. J., NIEUWOUDT, J. M. (Ed.) (1975) *General Psychology: A Psychological Approach*, McGraw-Hill Book Company.

KAPUT, J. J. (1985) Representation and problem solving: Methodological issues related to modeling. IN SILVER, E. A. (Ed.) *Teaching and learning mathematical problem solving: Multiple research perspectives.* Hillsdale, NJ: Erlbaum.

KEITEL, C. (1993) Hans Freudenthal: Revisiting Mathematics Education - China Lectures. *Educational Studies in Mathematics,* 25, 161 -164.

KRAUT, R. (1992) (Ed.) *The Cambridge Companion to Plato*. Cambridge.

KRESS,G., JEWITT,C., BOURNE, J. et al (2004) *English in Urban Classrooms: a multimodal perspective on teaching and learning.* London: RoutledgeFalmer

KUHN, T. S. (1962) *The Structure of Scientific Revolutions*, University of Chicago Press.

LAKOFF, G., NUNẼŹ, R. E. (2000) *Where mathematics comes from: How the embodied mind brings mathematics into being,* New York, Basic Books.

LAROCHELLE, M. (2000) Radical Construction: Notes on Viability, Ethics and Other Educational Issues. IN STEFF, L. P., THOMPSON, P. W. (Ed.) *Radical Constructivism in Action - Building on tne work of Ernst von Glasersfeld.* London, Routledge, Falmer.

LAVE, J. (1988) *Cognition in Practice,* Cambridge Cambrdige University Press.

LAVE, J., WENGER, E. (1991) *Situated Learning: Ligitimate Peripheral Participation,* Cambridge, Cambridge University Press.

LERMAN, S. (1992) The position of the individual in radical constructivism: In search of the subject. *Proceedings of the 7th International Congress on Mathematics Education.* Quebec, Canada.

LERMAN, S. (1996) Intersubjectivity in mathematics learning: a challenge to the radical constructivist paradigm. *Journal for Research in Mathematics Education,* 27.

LERMAN, S. (1998) Research on Socio-Cultural Perspectives of Mathematics Teaching and Learning. IN SIERPINSKA, A., KILPATRICK, J. (Ed.) *Mathematics as a Research Domain: A Search for Identity.* Books I and II. Kluwer Academic Publishers.

LESH, R., CLARKE, D. (2000) Formulating Operational Definitions of Desired Outcomes of Instruction in Mathematics and Science Education. IN KELLY, A. E., LESH, R.A. (Ed.) *Handbook of Research Design in Mathematics and Science Education.* Lawrence Erlbaum Associates.

LESH, R., HOOVER, M., HOLE, B., KELLY, A., POST, T. (2000) Principles for Developing Thought-Revealing Activities for Students and Teachers. IN KELLY, A. E., LESH, R.A. (Ed.) *Handbook of Research Design in Mathematics and Science Education.* Lawrence Erlbaum Associates.

LINCOLN, Y. S., GUBA, E.G. (1985) *Naturalistic Inquiry,* London, SAGE.

MASLOW, A. (1970) *Motivation and Personality,* London, Harper and Row.
MASON, P. (1989) Mathematical abstraction as the result of a delicate shift of attention. *For the learning of mathematics,* 9, 2-8.
MASON, P. (1994) Researching From the Inside in Mathematics Education: Locating an I-You Relationship. *Proceedings of the Eighteenth Conference of the Group on the Psychology of Mathematics Education.* Lisbon, Extended version:Centre for Mathematics Education, Open University, Milton Keynes.
MCNEILL, D. (1992) *Hand and Mind: What gestures reveal about thought,* Chicago, Chicago University Press.
MCNEILL, D. (2000) *Language and Gesture,* Cambridge, UK, Cambridge University Press.
MINIUM, E. W. (1970) *Statistical Reasoning in Psychology and Education,* John Wiley & Sons, Inc.
MOUTON, J. (2003) *How to succeed in your Master's and Doctoral Studies. A South African guide and resource book,* Pretoria, Van Schaik.
NEWMAN, D., GRIFFIN, P., COLE, M. (1989) *The construction zone: Working for cognitive change in school,* New York, Cambridge University Press.
NOLDER, R. (1991) Mixing Metaphor and Mathematics in the Secondary Classroom. IN DURKIN, K., SHIRE, B. (Ed.) *Language in Mathematical Education Research and Practice.* Open University Press.
NOVAK, J. D. (1978) An alternative to Piagetian Psychology for Science and Mathematics education. *Studies in Science Education,* 5, 1-30.
ÖDMAN, P.-J. (1979) *Tolkning, förståelse, vetande,* Stockholm, AWE/Gebers.
OLIVIER, A. (1989) Handling Pupils' Misconceptions. *Presidential address delivered at the Thirteenth National Convention on Mathematics, Physical Science and Biology Education.* Pretoria.
ONORE, C. S. (1992) Why we learn is what and how we learn: Curriculum as possibility. IN BOOMER. G., L., N., ONORE, C., COOK, J. (Ed.) *Negotiating the curriculum: Education for the 21st century.* London, Falmer Press.
OSGOOD, C. E. (1957) A behavioristic analysis fo perception and language as cognitive phenomena. *Cognition - The Colorado Symposium.* Cambridge, Harvard University Press.
PA, N. A. N. (1986) Meaning in Arithmetic from Four Different Perpectives. *For the Learning of Mathematics,* 6, 11-16.
PAPERT, S. (1980) *Mindstorms: Children, Computers and Powerful Ideas,* Brighton, Harvester.
PEIRCE, C. (1932) Collected Papers of Charles Sanders Peirce. IN HARTSHORNE, C., WEISS, P. (Ed.) Cambridge MA, Belknap Press.
PERKINS, D. N., SOLOMON, G. (1988) Teaching for transfer. *Educational Leadership,* 46, 22-32.
PERKINS, D. N., SOLOMON, G. (1989) Are cognitive skills context-bound? *Educational Researcher,* 18, 16-25.
PIAGET, J. (1970) *Genetic epistemology,* New York, Columbia University Press.
PIAGET, J. (1980a) The psychogenesis of knowledge and its epistemological significance. IN PIATELLI-PALMARINI, M. (Ed.) *Language and learning: The debate between Jean Piaget and Noam Chomsky.* Cambridge, MA:, Harvard University Press.
PIAGET, J. (1980b) *Adaptation and intelligence: Organic selection and phenocopy,* Chicago, University of Chicago Press.
PIMM, D. (1982) Why the history and philosophy of mathematics should not be rated X. *For the learning of mathematics,* 3, 12-15.
PIMM, D. (1987) *Speaking Mathematically - Communication in Mathematics Classrooms,* London and New York, Routledge.
PIMM, D. (1992) *Mathematics: Symbols and Meanings,* The Open University.

PIMM, D. (1995) *Symbols and Meanings in School Mathematics,* London and New York, Routledge.

POLANYI, M. (1978) *Personal Knowledge,* London, Routledge.

POLYA, G. (1948) *How to solve it,* New York, Dover.

POPPER, K. (1963) *Conjecture and Refutation: The Structure of Scientific Knowledge.* London, Routledge.

POPPER, K. (1977) *The logic of Scientific Discovery.* 14th printing. London, Routledge.

POTTER, C. (1999) Programme evaluation. IN TERRE BLANCHE, M., DURHEIM, K. (Ed.) *Research in practice: Applied methods for social sciences.* Cape Town, University of Cape Town Press.

PRESMEG, N. C. (1998) Balancing Common Worlds: Mathematics Education as an Emergent Discipline in its Own Right. IN SIERPINSKA, A., KILPATRICK, J. (Ed.) *Mathematics as a Research Domain: A Search for Identity.* Books I and II. Kluwer Academic Publishers.

PULGRAM, E. (1976) The typologies of writing systems. IN HAAS, W. (Ed.) *Writing Without Letters.* Manchester, Manchester University Press.

RADFORD, L. (2003) Gestures, speech and the sprouting of signs. *Mathematical Thinking and Learning,* 5, 37-70.

RICHARDS, I. & (1991) Mathematical Discussions. IN VON GLASERSFELD, E. (Ed.) *Radical Constructivism in Mathematics Education.* Dordrecht, Kluwer Academic Publishers.

RICHARDS, J. (1991) Mathematical discussions. IN GLASERSFELD, E. V. (Ed.).

RICOEUR, P. (1981) *Hermeneutics and the Human Sciences,* Cambridge, Cambridge University Press.

RUNYAN, W. M. (1982) *Life histories and psychobiography: Explorations in theory and method,* New York, Oxford University Press.

SAGI (2006) Address delivered by the Deputy President, Phumzile Mlambo-Ngcuka, at the launch of the Joint Initiative for Priority Skills Acquisition (JIPSA) at http://www.info.gov.za/speeches/2006/06032810451001.htm Retrieved, July 2006

SCHOENFELD, A. H. (1985) *Mathematical problem solving.,* Orlando, FL: Academic Press.

SCHOENFELD, A. H. (1988) When good teaching leads to bad results: The disasters of "well taught" mathematics courses'. *Educational Psychologist,* 23.

SCHOENFELD, A. H. (1992) Learning to think mathematically: Problem solving, metacognition and sense making in mathematics'. IN GROUWS, D. A. (Ed.) *Handbook of Research on Mathematics Teaching and Learning.* New York, Macmillan.

SCHÜTZ, R. (2002) Vygotsky & Language Acquisition.

SCHWEITZER, R. (2002) Editorial. *Indo-Pacific Journal of Phenomenology,* 2, 1-2.

SCOTT, D., USHER, R. (1999) *Researching education: Data, methods and theory in educational enquiry,* London, Cassell.

SETATI, M. (2005) Power and Access in Multilingual Mathematics Classrooms. IN GOOS, M., KANES, C., BROWN, R. (Ed.) *Proceedings of the 4th International Mathematics Education and Society Conference.* Centre for Learning Research, Griffith University.

SFARD, A. (1991) On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the some coin. *Educational Studies in Mathematics,* 22, 1-36.

SFARD, A. (1998) The Many Faces of Mathematics: Do Mathematicians and Researchers in Mathematics Education Speak about the Same Thing? IN SIERPINSKA, A., KILPATRICK, J. (Ed.) *Mathematics as a Research Domain: A Search for Identity.* Books I and II. Kluwer Academic Publishers.

SIERPINSKA, A. (1995) Mathematics: "in context", "pure", or "with applications"? *For the Learning of Mathematics,* 15, 2-15.

SIMON, M. A. (2000) Constructivism, Mathematics Teacher Education, and Research in Mathematics Teacher Development. IN STEFF, L. P., THOMPSON, P. W. (Ed.) *Radical Constructivism in Action -Building on the work of Ernst von Glasersfeld.* London, Routledge, Falmer.
SKEMP, R. R. (1976) Relational and Instrumental Understanding in the Learning of Mathematics,. *Mathematics Teaching,* 77.
SKEMP, R. R. (1986) *The Psychology of Learning Mathematics. 2nd edition,* Harmondsworth, Penguin Books.
SKEMP, R. R. (1993) Theoretical Foundations of Problem Solving - A Position Paper.
SKINNER, B. F. (1953) *Science and Human Behaviour,* New York, Macmillan.
SKINNER, B. F. (1969) *Contingencies of Reinforcement: A Theoretical Analysis,* New York, Appleton-Century-Crofts.
SKOVSMOSE, O. (1994) *Towards a Philosophy of Critical Mathematics Education,* Dordrecht, Kluwer.
SOLOMON, J. (1994) The rise and fall of constructivism. *Studies in Science Education,* 23, 1-19.
STEFFE, L. P. (1991) The constructivist teaching experiment. IN GLASERSFELD, E. V. (Ed.).
STEFFE, L. P., NESHER, P. (Ed.) (1996) *Theories of mathematical learning,* Mahwah, N Lawrence Erlbaum Associates.
STEFFE, L. P. (2000a) Perspectives on Practice in Mathematics Education. IN STEFFE, L. P., THOMPSON, P. W. (Ed.) *Radical Constructivism in Action - Building on the Pioneering work of Ernest von Glasersfeld.* London, Routledge, Falmer.
STEFFE, L. P., THOMPSON, P. W. (Ed.) (2000b) *Radical Constructivism in Action,* London, Routledge Falmer.
STREEFLAND, L. (1990) Realistic Mathematics Education (RME) What does it mean? IN GRAVEMEIJER, K., VAN DEN HEUVEL, M., STREEFLAND, L. (Ed.) *Contexts Free Productions Tests and Geometry in Realistic Mathematics Education.* Utrecht, Research Group for Mathematics Education and Educational Computer Centre State University.
STUBBS, M. (1980) *Language and Literacy,* London, Routledge & Kegan Paul.
SUTHERLAND, R. (1993) Connecting theory and practice: Results from the teaching of Logo. *Educational Studies in Mathematics,* 24, 95-113.
TALL, D. (1989) Concept images, generic organizers, computers, and curriculum change. *For the Learning of Mathematics,* 9, 37-42.
TAYLOR, P., FRASER, B., FISHER, D. (1993a) Monitoring the Development of Constructivist Learning Environments. *Annual Convention of the National Science Teachers Association (NSTA).* Kansas City, MO.
TAYLOR, P., CAMPBELL-WILLIAMS, M. (1993b) Discourse towards balanced rationality in the high school mathematics classroom: Ideas from Habermas's critical theory. IN TAYLOR, P. C. S., MALONE, A. J. (Ed.) *Constructivist Interpretations of Teaching and Learning Mathematics.* Perth, Australia, Curtin University of Technology.
TEPPO, A. (1991) Van Hiele Levels of Geometric thought Revisited. *Mathematics Teacher.*
THARP, R. G., GALLIMORE, R. (1988) *Rousing minds to life: Teaching, learning, and schooling in social context,* New York, Cambridge University Press
THOM, R. (1973) Modern Mathematics: Does it exist? IN HOWSON, A. G. (Ed.) *Developments in Mathematics Education. Proceedings of the Second International Congress on Mathematical Education.* Cambridge, Cambridge University Press.
THOMPSON, P. W. (1994a) Images of rate and operational understanding of the fundamental theorem of calculus. *Educational Studies in Mathematics,* 26, 229-274.

THOMPSON, P. W. (1994b) Students, functions, and the undergraduate curriculum. IN DUBINSKY, E., SCHOENFELD, A. H., KAPUT, J. (Ed.) *Research in collegiate mathematics education. Issues in Mathematics Education.* Providence, RI, American Mathematical Society.

THORNDIKE, E. L. (1922) *The psychology of arithmetic,* New York, Macmillan Co.

THORNDIKE, E. L. (1932) *The Fundamentals of Learning,* New York, Teachers College Bureau of Publications.

THORNTON, S. (2002) Karl Popper. IN ZALTA, E. N. (Ed.) *The Stanford Encyclopedia of Philosophy.*

TOULMIN, S. (1972) *Human Understanding -The Collective Use and Evolution of Concepts,* Princeton University Press.

TREFFERS, A. (1987) *Three dimensions,* Dordrecht, D. Reidel.

TZUR, M. (1995) Interaction and Children's Fraction Learning. Unpublished PhD dissertation. Athens, University of Georgia.

VAN DER MESCHT, H. (2004) Phenomenology in Education: A Case Study in Educational Leadership. *Indo-Pacific Journal of Phenomenology,* 4, 1-16.

VAN ENGEN, H. (1949) An analysis of meaning in arithmetic. *Elementary School Journal,* 49, 321-329, 395-400.

VAN HIELE, P. M. (1973) *Begrip en Inzicht*, Muusses Purmerend.

VAN JAARSVELD, P. P. (2005) The tracking of the pre- and post-intervention performance of seven Khayelitsha learners in higher grade mathematics in response to a teaching and learning intervention from August to October 2002. *Confrerence proceedings of the South African Association for Research in Mathematics, Science and Technology Education.* Windhoek, Namibia, UNESCO.

VINNER, S. (1992) The function concept as a prototype for problems in mathematics learning. IN HAREL, G., DUBINSKY, E. (Ed.) *The concept of function: Aspects of epistemology and pedagogy, MAA notes 25.* Washington, DC, Mathematical Association of America.

VITHAL, R., VOLMINK, J. (2005) Mathematics curriculum research: roots, reforms, reconciliation and relevance. IN VITHAL, R., ADLER, J., KEITEL, C. (Ed.) *Researching Mathematics Educationi in South Africa: Perspectives, Practices and Possibilities.* Cape Town, HSRC Press.

VITHAL, R., ADLER, J., KEITEL, C. (Ed.) (2005) *Researching Mathematics Education in South Africa: Perspectives, Practices and Possibilities,* Cape Town, HSRC Press.

VON GLASERSFELD, E. (1982) An interpretation of Piaget's constructivism. *Revue Internationale de Philosophie*, 142-143, 612-635.

VON GLASERSFELD, E. (1988) *The Construction of Knowledge: A Contribution to Conceptual Semantics.,* New York, Intersystems Publication.

VON GLASERSFELD, E. (1990) An exposition of constructivism: Why some like it radical. IN DAVIS, R. B., MAHER, C. A., NODDINGS, N. (Ed.) *Constructivists views on the Learning and Teaching of Mathematics.* Reston, Virginia, National Council of Teachers of Mathematics.

VON GLASERSFELD, E. (1991a) Introduction. IN VON GLASERSFELD, E. (Ed.) *Radical Constructivism in Mathematics Education.* Dordrecht, Kluwer Academic Publishers.

VON GLASERSFELD, E. (1991b) Abstraction, re-presentation and reflection: An interpretation of experience and Piaget's approach. IN STEFFE, L. P. (Ed.) *Epistemological Foundations of Mathematical Experience.* New York, Springer Verlag.

VON GLASERSFELD, E. 1995) *Radical Constructivism: A Way of Knowing and Learning,* London, Falmer Press.

VYGOTSKY, L. (Ed.) (1986) *Thought and Language, (Revised Edition),* Cambridge, MIT Press.

VYGOTSKY, L. S. (1962) *Thought and Language,* Cambridge MA, MIT Press.

VYGOTSKY, L. S. (1978) *Mind in Society: The development of higher psychological processes,* Cambridge, MA, Harvard University Press.

VYGOTSKY, L. S. (1981) The genesis of higher mental functions. IN WERTSCH, J. V. (Ed.) *The concept of activity in Soviet Psychology.* Armonk, Sharpe.

VYGOTSKY, L. S. (1997) Collected Works, 4. IN BIEBER, R. (Ed.) New York, Plenum.

WALKERDINE, V. (1988) *The Mastery of Reason,* London, Routledge.

WANT, C., KLIMOWSKI, A. (1996) *Introducing Kant,* Cambridge.

WCED (1995) Syllabus for Mathematics Higher Grade and Standard Grade; Standards 8 - 10. Government Printer.

WCED (2005) Senior Certificate Awards 2005. *Media Release at http://wced.wcape.gov.za/home/service/87_awards.html.*

WCED (2006) Education Budget Vote at *http://wced.wcape.gov.za/ministry/speeches06/edbudget2006.html*

WEAVER, J. F., SUYDAM. M.N. (1972) *Meaningful instruction in mathematics education,* ERIC/SMEAC.

WERTHEIMER, M. (1959) *Productive Thinking (revised edition),* New York, Harper and Row.

WILLIAMS, J. (2005) Gestures, Signs and Mathematisation. IN CHICK, H. L., VINCENT J. L. (Ed.) *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education.* Melbourne, PME.

WITTGENSTEIN, L. (1958) *Philosophical Investigations,* London, Basil Blackwell.

WITTMAN, E. C. H. (1998) Mathematics Education as a 'Design Science'. IN SIERPINSKA, A., KILPATRICK, J. (Ed.) *Mathematics as a Research Domain: A search for identity.* Kluwer Academic Publishers.

WOODROW, D. (1995) The quest for multiple beliefs in learning theories - and its frustration by single faiths. *Chreods,* 9, 10-14.

YIN, R. K. (1994) *Case study research: design and methods,* London, Sage.

APPENDICES

APPENDIX 1

REACH TO TEACH

ENTRY LEVEL BASELINE ASSESSMENT

Grade 10

Calculators may not be used

Examiner: P P van Jaarsveld **Time: 80 minutes**

QUESTION 1 :

Simplify

1.1	$\sqrt{100-64}$	(1)	1.7	$(x+x)(x+x)$	(3)
1.2	$2^3-(-1)^5$	(3)	1.8	$a+a+a-a.a.a$	(3)
1.3	$7a-(-2a)$	(2)	1.9	$\sqrt{49x^{14}}$	(2)
1.4	$(-3a^3b)(-5ab^2)$	(3)	1.10	$(-2x^4)^3$	(3)
1.5	$-x-2x-3x$	(2)	1.11	$\frac{-9x^3+6x^2-3x}{3x^2}$	(3)
1.6	$6a-a(a-1)-a^2$	(4)	1.12	$(x+2)(x+3)$	(3)

[32]

QUESTION 2 :

2.1 Add : $5a-2b+c$; $a+b-c$; $-7a-c$ (3)

2.2 Subtract $x-y-3z$ from $2x+3y-4z$ (3)

2.3 Find the value of $(2x+y^2)^x$ if $x = 2$ and $y = -1$. (3)

[9]

QUESTION 3 :

Simplify

3.1.1 $$\frac{6x}{6x^2-12x} \div \frac{x^2-x-6}{x^2-4}$$ (8)

3.1.2 $$\frac{x}{x+1}-\frac{x}{x+2}$$ (5)

[13]

QUESTION 4 :

Solve for x

4.1 $2x - 3 = 5x + 9$ (4)

4.2 $1 - (x + 2) = 3(x - 7)$ (4)

4.3 $\frac{1}{2}x + 8 - \frac{1}{5}x - 2 = 0$ (4)

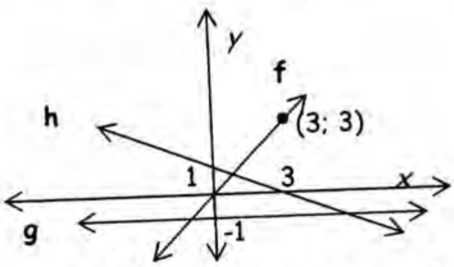
4.4 $\frac{7}{x+2} = \frac{3}{x}$ (4)

[16]

QUESTION 5 :

5.1.1 Sketch the graphs of $y = -x + 3$ and $y = \frac{1}{2}x$ on the same set of axes. (4)

5.1.2 Calculate the point of intersection of the graphs in 5.1.1. (4)



5.2.1 What is the equation of the line labelled **f** in the sketch above? (2)

5.2.2 What is the equation of the line labelled **g** in the sketch above? (2)

5.2.3 What is the equation of the line labelled **h** in the sketch above? (2)

[14]

QUESTION 6 :

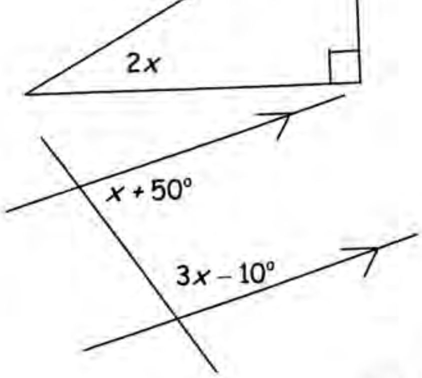
Calculate the value of x in each of the figures below by first making an equation.
Give reasons in brackets.

6.1

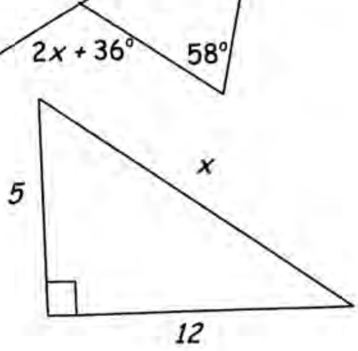
6.2

(8)

6.3



6.4



(8)

[16]

TOTAL 100 marks

APPENDIX 2

Author	Critical tenets of learning or teaching theory or model						Classification
Piaget Theory of Intellectual Development	**Sensorimotor** 0 -2 years		**Pre-operational** 2 – 7 years	**Concrete Operational** 7 -12 years	**Formal** 12 + years		Cognitivist
	Egocentricity Entity of environment		**Pre-conceptual (2-4)** Language impeding Symbolical representations **Intuitive (4-7)** Simple concepts				Radical constructivist
			Views space topologically	Views space projectively	Views space in Euclidean terms		Developmental
Van Hiele Model of Geometric Thought	**Inquiry / Information** Objects of discussion from observation Vocabulary	**Visualisation** Figures are viewed as total entities	**Analytic thought** Component parts of figures analysed	**Informal deduction** Properties within entities	**Deduction** Within axiom systems More than one proof seen as feasible	**Rigour** Compares different axiom systems Abstraction	Cognitivist Social constructivist Non - Developmental
	Directed orientation Exploration of topic through sequenced material Structures						
	Explication Learners debate their views on structures System of relations	Phases of learning applicable to each level of the van Hiele model					
	Free Orientation Resolve problems independently in many ways Open ended multiple solutions Teacher a facilitator						
	Integration Internalise Unify relations						

Appendix 2.1 Comparative tabulation of the critical tenets of the learning theories of Piaget and Van Hiele
Broken lines signify overlapping stage / phase boundaries

<table>
<tr><th>Author</th><th colspan="5">Critical tenets of learning or teaching theory or model</th><th>Classification</th></tr>
<tr><td rowspan="2">**Ausubel**</td><td>**Subsumption**
New knowledge interacts with existing, meaningful, relevant schemas to coalesce into a more composite concept
Obliterative subsumption
Meaningful forgetting of residual detail but retention of the original concept</td><td colspan="2">**Progressive differentiation**
Existing schemas are modified in accordance with new knowledge</td><td>**Superordinate learning**
Strengthens progressive differentiation since subordinate concepts acquire new integrative meaning (Classificatory type ordering)</td><td>**Integrative Reconciliation**
Previously construed independent concepts are integrated into higher concept meanings (or it is recognised that several terms represent the same concept)</td><td rowspan="2">Cognitivist
Radical constructivist
Behaviourist element of rote learning
Non-Developmental</td></tr>
<tr><td colspan="5">**Emphasised the active nature of reception learning; 'Missing words, underlining' to 'worksheet'**
No dichotomy between rote and meaningful learning…they exist on a continuum
Interaction with new knowledge enhances meaning and develops concepts
Advance organiser, which considers the impact of prior learning, activates existing relevant schema to facilitate the subsumption of new ideas
To operate with mental abstractions is not a function of age, but rather a function of the degree of sophistication of a child's cognitive repertoire</td></tr>
<tr><td rowspan="3">**Bruner**</td><td>**Enactive**</td><td>**Iconic**</td><td></td><td></td><td>**Symbolic**</td><td></td></tr>
<tr><td>Actions on objects</td><td>Pictorial representations and models</td><td></td><td></td><td>Abstract thought</td><td rowspan="2">Cognitivist
Radical construcitivist
Non- evelopmental</td></tr>
<tr><td colspan="5">**Structure of subject should be taught through teaching the 'real' process e.g. modelling of real problems**
Experiential and Discovery learning</td></tr>
<tr><td>**Gagne**</td><td colspan="5">**Investigated a skills hierarchy for skills and knowledge to impact improved performance.**
Defined 'Verbal information', 'Intellectual skills', 'Cognitive strategies', 'Motor skills', 'Attitudes' as categories of learning
Work assumes lower level facts are taught before higher level concepts
Learners are able to reason with higher level concepts if lower-level information is in place</td><td>Cognitivist
Radical constructivist
Non-developmental</td></tr>
</table>

Appendix 2.2 Comparative tabulation of the critical tenets of the learning theories of Ausubel, Bruner and Gagne
Broken lines signify overlapping stage / phase boundaries

Author	Critical tenets of learning or teaching theory or model	Classification
Freudenthal	Mathematical concepts are seen to organise real world phenomena and developed mathematics Learning, creating and developing mathematics viewed as commencing with 'common sense' Learning and developing mathematics is an ongoing process of mathematisation Mathematisation includes activities of; local and global ordering, schematising, formalising, symbolising and reflecting and communicating about these activities	Cognitivist Constructivist Non-developmental
Skemp	Relational and Instrumental Understanding Non – routine and Routine methods of solving problems Real world problems are best solved from a structured knowledge base Rote (habit learning) with low cognitive elements Problem solving involves developing a knowledge structure Direct approach: from real-life problem to abstraction to schema to mathematical model using necessary routines to re-embodiment in the original problem Theory based approach: activities that embody one new concept to abstraction to formation of new concept to connections with appropriate knowledge structure to appropriate mathematical model to solution of problem in mathematical model to re-embodiment in original problem Schema building From our encounters with actuality (by experiment) tested against actual experiments From schemas of others (by communication) tested against discussion From within (by intuitive creativity) tested against self reflection	Cognitivist Social constructivist Behaviourist element of rote learning Non-developmental
Vygotsky	Zone of Proximal Development The difference between a learner's actual performance on an independent problem solving task and its potential performance with guidance by a teacher or collaboration with a more informed peer.	Cognitivist Social constructivist Non - Developmental
Cobb	Combines the language stance of Vygotsky and the work of Steffe and Von Glasersfeld who use teaching experiments and co-operative learning to explore the ways in which children learn.	
Confrey	Combines the work of Piaget and Vygotsky	

Appendix 2.3 Comparative tabulation of the critical tenets of the learning theories of Freudenthal, Skemp and Vygotsky, Cobb and Confrey