CASE STUDIES OF EQUIVALENT FUZZY SUBGROUPS OF FINITE ABELIAN GROUPS

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Abstract

The broad goal is to classify all fuzzy subgroups of a given type of finite group. P.S Das introduced the notion of level subgroups to characterize fuzzy subgroups of finite groups. The notion of equivalence of fuzzy subgroups which is used in this thesis was first introduced by Murali and Makamba. We use this equivalence to characterize fuzzy subgroups of finite Abelian groups (p-groups in particular) for a specified prime p. We characterize some crisp subgroups of p-groups and investigate some cases on equivalent fuzzy subgroups.

KEYWORDS: *Fuzzy subgroups, Equivalence, Maximal chains, Key chains, p-groups, Isomorphic subgroups, Equivalent fuzzy subgroups, Cyclic subgroups.*

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0.2 Preface

The broad goal of the theory of fuzzy subgroups of a finite group is to arrive at a classification of fuzzy subgroups of a given type of finite group. In this thesis, we achieve a modest step in this problem. We classify all fuzzy subgroups of finite Abelian groups of a given type, namely, p- groups for a specified prime p. We use the notion of equivalence of fuzzy subgroups first introduced by Murali and Makamba [16]. Earlier Das [4] introduced the notion of level subgroups to characterize fuzzy subgroups of finite groups. The reference [23] contains an extensive literature in fuzzy algebra (in general), but fuzzy groups in particular. The equivalence of fuzzy subgroups preserves level subgroups but fuzzy isomorphism is a more general concept.

In [16] Murali and Makamba study equivalence classes of fuzzy subgroups of a given group under a suitable equivalence relation. They use the relation to characterize a number of fuzzy subgroups of finite Abelian groups (p- groups in particular). In [17] they determine the number of distinct equivalence classes of fuzzy subgroups of $\mathbb{Z}_{p_1} + \cdots + \mathbb{Z}_{p_n}$, where $p_1, \ldots, p_n$ are distinct primes. They go on to introduce the notion of a key chain of length $n+1$ to be used a lot in this thesis.

In chapter 1 we provide the background material for the theory of fuzzy sets. We state the main theorem of finite Abelian groups. We present the results of the application of the theorem in the construction of Abelian groups. In section 1.3 we present the notion of fuzzy subgroups and some related concepts such as level subgroups. Further, we illustrate the characterization of fuzzy subgroups by means of level subgroups and normality as appearing in [4].

In chapter 2 we introduce the main types of groups we shall work with, namely, the p-groups. We collect and present some known results from group theory. It is in this chapter where we bring along the building blocks of finite Abelian groups as well as maximal chains. We provide some results on maximal chains in the form of lemmas. We end the chapter by classifying the crisp subgroups of finite Abelian groups of the form $\mathbb{Z}_{p^n} \times \mathbb{Z}_p$ for any prime p and any integer n.

In chapter 3 we introduce the notion of equivalence relation. We do this in style by giving the general case first as in group theory. In the next section we give the theory of fuzzy relations as studied by the author [15]. This we do with the view to state the notion of fuzzy equivalence which can be viewed as a special case of fuzzy isomorphism. The result is consistent with the classical one that *isomorphism is an equivalence relation.* Also, fuzzy equivalence implies fuzzy isomorphism and not the other way round.

The determination of the number of fuzzy subgroups of a finite Abelian group $\mathbb{Z}_p$ will begin with one coordinate p^n, n a positive integer, and hopefully spread to a multiple coordinate of primes. It is in this chapter(4) that we have decided to incorporate the results of counts for $\mathbb{Z}_{p^n}$ as studied in [16] for the sake of consistency.

In chapter 5 we are trying to systematize the method of determination of fuzzy subgroups. The maximal chains and key chains have been useful throughout this determination. We classify the fuzzy subgroups of finite Abelian groups of the form $\mathbb{Z}_{p^n} \times \mathbb{Z}_p$ for any prime p and $n = 1, 2, 3$. Much of the results appearing in tables and pictures have been gathered around this chapter. For the group $\mathbb{Z}_{p^n} \times \mathbb{Z}_{p^m}$, and for the case when all the nontrivial subgroups are cyclic, we have developed the general formulae for the number of maximal chains, the number of crisp subgroups or generators, and, for the case $n = m$, the number of fuzzy subgroups. We have also included some special results for n and m. These results appear in the section called **appendix**.

The figures and tables given at the back form part of the text. They have been arranged together for ease of reference.

Note 1: In figures, the nodes that appear in one row fall in the same level. Such nodes represent those subgroups of G with the same order. Each subgroup is contained in the subgroup above it, as indicated by the line.

Note 2: Each node ab represents a subgroup generated by (a, b), written as $< (a, b) >$.

0.3 Introduction

The theory of fuzzy subsets which was developed by L.Zadeh finds wide applications in various fields.
For instance it was used by A.Rosenfeld in 1971 to develop the theory of fuzzy groups in which many of the basic properties found in group theory have been carried over to fuzzy groups [22].

Researchers became heavily involved in the classification of fuzzy subgroups of various groups some years later. P.S Das characterized fuzzy subgroups of cyclic groups and other finite groups by means of level subgroups of a fuzzy subgroup [4]. Then later, Alkhamees followed up with the study of fuzzy cyclic subgroups and, in particular, fuzzy cyclic p-subgroups [1]

In this thesis, among other things, we characterize fuzzy subgroups of finite Abelian groups under the natural equivalence studied in [16].
But we first give the results of characterization by means of level and normal subgroups [4].

Without any equivalence on fuzzy subsets of a set, the number of fuzzy subgroups of every group was found to be infinite no matter what the group is [16].
It is with the use of this equivalence that we will attempt to determine the number of all the distinct equivalence classes of fuzzy subgroups of a p-group.

Contents

Chapter 1

Background

1.1 Fuzzy set theory

For completeness sake, we collect together the following concepts and some of the known results in the fuzzy set theory literature.

An element x from a universal set $\cup$ belongs or does not belong to a subset A of $\cup$. This is written as

$$x \in A \ \ or \ \ x \notin A.$$

The above can be expressed as a function χ_A from $\cup$ onto two-element set $\{0, 1\}$

$$\chi_A(x) = \begin{cases} 1 \ \text{ if } \ x \in A \\ \\ 0 \ \text{ if } \ x \notin A \\ \end{cases}$$

For example, if $\cup = \mathbf{N}$, a subset A described by all numbers greater than 15 can be written as

$$\chi_A(x) = \begin{cases} 1 \ \text{ if } \ x > 15 \\ \\ 0 \ \text{ if } \ 1 \leq x \leq 15 \\ \end{cases}$$

The above translates to fuzzy sets in the following way:
A fuzzy subset μ_A, is a function

$$\mu_A : \cup \to I,$$

with the number $\mu_A(x)$ in $[0,1]$ interpreted as the degree to which x belongs to A, where $\mu_A(x) = 1$ means that x belongs to A absolutely, and $\mu_A(x) = 0$ means x does not belong to A absolutely.
$\mu_A(x)$ close to 1 is interpreted as strong belonging, while $\mu_A(x)$ close to 0 means x belongs to A to a less extent.
Throughout our discussion, fuzzy sets will simply be denoted by lowercase Greek letters μ, ν etc.

Thus if we let X be the universal set, we have the following definition:
A mapping $\mu : X \to [0,1]$ is called a fuzzy subset of X. Thus $\mu \in [0,1]^X$

The support of μ, denoted as $supp(\mu)$, is defined to be the crisp subset of X given by

$$supp(\mu) = \{x \in X : \mu(x) > 0\}.$$

1.1.1 Definition

Let μ be a fuzzy subset of a set X. If $t \in [0,1]$, the set

$$\mu_t = \{x \in X : \mu(x) \geq t\}$$

is called a level subset of μ, and this μ_t is the subset of X in the ordinary sense.

Notation: If μ is a mapping, then $Im(\mu)$ shall denote the image set of μ.
$Im(\mu) = \{i \in I \mid \exists \ x \in X : \mu(x) = i\}$.

Union, intersection, and complementation of fuzzy sets are defined by taking max, min and $\prime$ pointwise for the degree of membership. That is, if A, B are two fuzzy subsets of $\cup$, then $A \cap B, A \cup B, A'$ are given by

$$\mu_{A \cap B}(x) = \mu_A(x) \wedge \mu_B(x)$$

$$\mu_{A \cup B}(x) = \mu_A(x) \vee \mu_B(x)$$

$$\mu_{A'}(x) = 1 - \mu_A(x)$$

Thus in our notation, if μ, ν are fuzzy sets of X then

$$\mu \cap \nu = \mu \wedge \nu$$

$$\mu \cup \nu = \mu \vee \nu$$

$$\mu' = 1 - \mu$$

For two fuzzy sets μ and ν of X,

$$\mu = \nu \Longleftrightarrow \mu(x) = \nu(x) \ \ \forall x \in X$$

$$\mu \subseteq \nu \ \text{ or } \ \mu \leq \nu \Longleftrightarrow \mu(x) \leq \nu(x) \ \ \forall x \in X$$

$$\mu < \nu \Longleftrightarrow \mu(x) \leq \nu(x) \ \ \forall x \in X \ \text{ and for at least one } x \in X, \ \ \mu(x) < \nu(x).$$

1.1.2 Definition

Suppose μ is a fuzzy subset of X. Then by $f(\mu)$,the image of μ under the function $f : X \to Y$, we mean a fuzzy subset of Y, $f(\mu) : Y \to I$ defined by

$$f(\mu)(y) = \begin{cases} \sup\{\mu(x) \mid x \in X, \ \ f(x) = y\} \\ \\ 0 \text{ if there are no such } x \in X \text{ for which } f(x) = y \end{cases}$$

This definition is called Zadeh's extension principle.

Note : Suppose $f : X \to Y$ is a function. The pre image $f^{-1}(\nu)$ of a fuzzy set ν of Y is defined as the fuzzy subset of X given by

$$(f^{-1}(\nu))(x) = \nu(f(x)) \;\; \forall \;\; x \in X$$

Results

1. $(g \circ f)(\mu) = g(f(\mu))$, where $f : X \to Y, \;\; g : Y \to Z$.
2. If $i : X \to X$ is the identity mapping, then $i(\mu) = \mu$.
3. $f^{-1}(f(\mu)) \geq \mu$ for all $\mu \in I^X$.
4. $f(f^{-1}(\nu)) \leq \nu$ for all $\nu \in I^Y$.
5. if $\mu_1 \leq \mu_2$ then $f(\mu_1) \leq f(\mu_2)$

1.2 Finite Abelian Groups

In this section we shall build on the theory on Abelian groups that has been gathered so far. We shall also present, without proof, the fundamental theorem of finite Abelian groups and some of its consequences.
In his book [6], Joseph Gallian remarks that 'the fundamental theorem is extremely powerful and can be used as an algorithm for constructing all Abelian groups of any order up to isomorphism'. Although we are primarily concerned with Abelian groups, we shall continue using multiplicative notation whenever it is convenient.
As we remarked earlier, we will see the role played by cyclic groups as building blocks for all finite Abelian groups.

1.2.1 Theorem

Every finite Abelian group is a direct product of cyclic groups of prime-power order. Moreover, the factorization is unique except for rearrangement of factors.

Theorem 1.2.1 settles the problem of determining all isomorphism classes of finite Abelian groups. In essence it shows that every finite Abelian group is isomorphic to a group of the form $\mathbb{Z}_{p_1^{n_1}} \times \mathbb{Z}_{p_2^{n_2}} \times \cdots \times \mathbb{Z}_{p_k^{n_k}}$, where the p_i need not be distinct.

To start off we begin with the Abelian groups of prime-power order, and towards the end we consider the prime-power decomposition case.

Note:

1. Any cyclic group of order n is isomorphic to $\mathbb{Z}_n$.
2. Each group G of prime order is cyclic, generated by na of its non identity elements.
3. There is only one group of order n up to isomorphism whenever n is a prime or a product of distinct primes $p_1, p_2, \ldots, p_k$ such that p_j does not divide $(p_i - 1)$ for $1 \leq i, j \leq k$.

We mention the following consequences of n in the form of examples:

1.2.2 Example

Let $n = p$ and G be a group of order p, where p is a prime.

Then G is isomorphic to $\mathbb{Z}_p$.

1.2.3 Example

For $n = p^2$, there are two isomorphism classes of groups of order p^2. They are $\mathbb{Z}_{p^2}$

and $\mathbb{Z}_p \times \mathbb{Z}_p$. These groups of order p^2 are all Abelian.

Examples 1.2.2 and 1.2.3 are special cases to the problem of looking at groups whose orders have the form p^k ,where p is a prime and k is a positive integer.

If $n = 4$, then there are five isomorphism classes of G, namely

$$\mathbb{Z}_{p^4}$$
$$\mathbb{Z}_{p^3} \times \mathbb{Z}_p$$
$$\mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2}$$
$$\mathbb{Z}_{p^2} \times \mathbb{Z}_p \times \mathbb{Z}_p$$
$$\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$$

corresponding to each partition of k. Thus there is one group of order p^k for each sum of positive integers that give k.

In general, if k can be written as $k = n_1 + n_2 + \cdots + n_s$, where $n_i \in Z^+$, then $\mathbb{Z}_{p^{n_1}} \times \mathbb{Z}_{p^{n_2}} \times \cdots \times \mathbb{Z}_{p^{n_s}}$ is an Abelian group of order p^k.

The uniqueness part of the theorem ensures that the distinct partitions of k give rise to distinct isomorphism classes.

This means that, for instance, $\mathbb{Z}_4 \times \mathbb{Z}_2$ is not isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Now we know how to construct all the Abelian groups of prime-power order. We proceed to part two of the problem, that is, the case when n has two or more distinct prime divisors.

Suppose we are asked to solve the following problem:

1.2.4 Example

List all different Abelian groups of order 180 up to isomorphism.

Now $180 = 5.2^2.3^2$, so that $G = \mathbb{Z}_{5.2^2.3^2}$. Then we have the following Abelian groups, all of order 180:

$$(i)\ \mathbb{Z}_5 \times \mathbb{Z}_4 \times \mathbb{Z}_9$$
$$(ii)\ \mathbb{Z}_5 \times \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_3$$
$$(iii)\ \mathbb{Z}_5 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_9$$
$$(iv)\ \mathbb{Z}_5 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$$

The above groups can be determined using the following procedure:
(i) write n in the form $n = {p_1}^{n_1}{p_2}^{n_2} \dots {p_k}^{n_k}$
(ii) form all Abelian groups of order ${p_1}^{n_1}, {p_2}^{n_2}$ etc.
(iii) form all possible direct products of these groups.

The question we need to answer, which of the groups is the big group G isomorphic to?

Note:

$$\begin{aligned} \mathbb{Z}_5 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 &\approx \mathbb{Z}_{10} \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \\ &\approx \mathbb{Z}_{10} \times \mathbb{Z}_6 \times \mathbb{Z}_3 \\ &\approx \mathbb{Z}_{15} \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \\ &\approx \mathbb{Z}_{15} \times \mathbb{Z}_6 \times \mathbb{Z}_2 \end{aligned} \tag{1.1}$$

Since it can be shown that *two Abelian groups are isomorphic if and only if they have the same number of elements of each order*, we can answer the question by comparing the orders of elements of the big group with the orders of elements in the 4 direct products in the above example.

The question we can ask, for instance, is whether the big group G has any elements of order 9. For this would mean that (i) and (iv) are candidates for isomorphism. Further, we may need to determine if G has any elements of order 4. Hence it is isomorphic to group (i).

Given a positive integer n, how can we tell if an Abelian group has a subgroup of order n? Proposition 1.2.5 gives the criteria for testing for such existence:

1.2.5 Proposition

If m divides the order of a finite Abelian group G, then G has a subgroup of order m.

Proof

Let $|G| = n$ and let $n = {p_1}^{r_1}{p_2}^{r_2} \dots {p_k}^{r_k}$, where p_i are prime. Now G can be expressed as $G = \mathbb{Z}_{{p_1}^{r_1}} \times \mathbb{Z}_{{p_2}^{r_2}} \times \dots \mathbb{Z}_{{p_k}^{r_k}}$, by the fundamental theorem of finite Abelian groups.

Now if m divides n, then m must be of the form ${p_1}^{s_1}{p_2}^{s_2} \dots {p_k}^{s_k}$ where $0 \leq s_i \leq r_i$. Since ${p_i}^{s_i}$ divides ${p_i}^{r_i}$, we have that $\mathbb{Z}_{{p_i}^{r_i}}$ has exactly one subgroup of order ${p_i}^{s_i}$ (fundamental theorem of cyclic groups). Hence $\mathbb{Z}_{{p_1}^{s_1}} \times \mathbb{Z}_{{p_2}^{s_2}} \times \dots \mathbb{Z}_{{p_k}^{s_k}}$ is a subgroup of G of order m.

Finally, we show that every finite Abelian group can be expressed as the direct product of cyclic groups of orders $n_1, n_2, \dots, n_t$ where n_{i+1} divides n_i for $i = 1, 2, \dots, t-1$. Every finite Abelian group G is isomorphic to some direct product of cyclic groups of prime-power order. We proceed in the following manner:

For each distinct prime, choose the largest factor of that prime power, and form one factor from all of these. Let the order be n_1, say. Repeat the process with the remaining original factors to obtain a factor of order n_2, say.

Now since each prime divisor of n_2 also divides n_1, we have that n_2 divides n_1. Continue the process until all the factors have been considered. This gives the result.

To illustrate this process, we consider the following example:

1.2.6 Example

Suppose

$$G \approx \mathbb{Z}_{27} \times \mathbb{Z}_3 \times \mathbb{Z}_{125} \times \mathbb{Z}_{25} \times \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$$
$$= \mathbb{Z}_{3^3} \times \mathbb{Z}_3 \times \mathbb{Z}_{5^3} \times \mathbb{Z}_{5^2} \times \mathbb{Z}_{2^2} \times \mathbb{Z}_2 \times \mathbb{Z}_2$$

Then

$$G \approx \mathbb{Z}_{3^3.5^3.2^2} \times \mathbb{Z}_{3.5^2.2} \times \mathbb{Z}_2$$

1.3 Fuzzy Subgroups

In this section we shall give a brief review of fuzzy subgroups and some of the group-theoretic analogues. We imagine that the set X considered in 1.1 has a group structure, and call the group G. We then proceed to characterize fuzzy subgroups by their level subgroups with special emphasis on normal subgroups.

Let G be any group. A mapping $\mu : G \to [0, 1]$ is called a fuzzy subgroup of G if the following conditions hold:
(i) $\mu(x + y) \geq \min(\mu(x), \mu(y))$ for all $x, y \in G$
(ii) $\mu(-x) = \mu(x)$ for all $x \in G$

If G is a group and μ is a fuzzy subgroup of G, then for any x in G, we have

$$\mu(x - x) = \mu(e) \geq \min(\mu(x), \mu(-x)) = \mu(x),$$

where e denotes the identity of G. Hence

$$\mu(x) \leq \mu(e) \;\; \forall x \in G.$$

Remark: If G is a group and μ is a fuzzy subgroup of G, then the level subset μ_t, $t \in [0, 1]$, $t \leq \mu(e)$, is an ordinary subgroup of G and conversely [Das].

Let μ be a fuzzy subgroup of a group G. The subgroups

$$\mu_t, \ \ t \in [0,1], \ \ t \leq \mu(e)$$

are called level subgroups of G.

1.3.1 Proposition

Suppose μ is a fuzzy subgroup of G. If $Im(\mu) = \{t_1, \ldots, t_n\}$, then the family of subgroups $\mu_{t_i}, \ \ 1 \leq i \leq n$, constitutes all the level subgroups of μ. Furthermore, the family of level subgroups of a fuzzy subgroup μ form a chain

$$\mu_{t_0} < \mu_{t_1} < \cdots < \mu_{t_n},$$

where

$$t_0 > t_1 > \cdots > t_n$$

in $Im(\mu)$, and $\mu(e) = t_0$.

Remark: If G is a finite group, then the number of subgroups of G is finite. But the number of level subgroups of a fuzzy subgroup A appears to be infinite. But then the fact is, every level subgroup is a subgroup of G. Hence, not all of these level subgroups are different.
How can we tell which level subgroups are equal?
The characterization as stated in proposition 1.3.2 gives a clue:

1.3.2 Proposition

Let G be a finite group and let μ be a fuzzy subgroup of G. If $t_i, \ \ t_j \ \in Im(\mu)$ such that $\mu_{t_i} = \mu_{t_j}$, then $t_i = t_j$.

Proof
Let $x_i \in \mu_{t_i}$ and $x_j \in \mu_{t_j}$. Since $\mu_{t_i} = \mu_{t_j}$ we have that $x_i \in \mu_{t_j}$ so that $\mu(x_i) = t_i \geq t_j$, that is $t_i \geq t_j$. Similarly, it can be shown that $t_j \geq t_i$ which gives the result.

1.3.3 Definition

Let μ be a fuzzy subgroup of a group G, and let $\theta : G \to G$ be a map from G into itself. Then we define a map

$$\mu \circ \theta : G \to [0,1] \text{ by } \mu \circ \theta(g) = \mu(\theta(g)) \ \forall g \in G.$$

Note:(in group theory) A subgroup H of a group G is called a characteristic subgroup if $\theta(K) = K$ for every automorphism ϕ of G.

Now for a fuzzy subgroup we have the following analogue:

1.3.4 Definition

A fuzzy subgroup μ on a group G is called a characteristic fuzzy subgroup of G if $\mu \circ \theta = \mu$.

Utilizing the information given in definitions 1.3.3 and 1.3.4, we make the following proposition:

1.3.5 Proposition

Let μ be a fuzzy subgroup of a group G and let θ be a homomorphism on G. Then $\mu \circ \theta$ is a fuzzy subgroup of G.

Proof

For any $a, b \in G$ we need to show that

$$\mu \circ \theta(a+b) \geq \min(\mu \circ \theta(a), \mu \circ \theta(b))$$

and

$$\mu \circ \theta(-a) = \mu \circ \theta(a).$$

Let $a, b \in G$. Now

$$\begin{aligned} \mu \circ \theta(a+b) &= \mu(\theta(a+b)) \\ &= \mu(\theta(a)\theta(b)) \\ &\geq \min(\mu(\theta(a)), \mu(\theta(b))) \\ &= min(\mu \circ \theta(a), \mu \circ \theta(b)) \end{aligned} \tag{1.2}$$

Also

$$\begin{aligned} \mu \circ \theta(-a) &= \mu(\theta(-a)) \\ &= \mu(-\theta(a)) \\ &= \mu(\theta(a)) \\ &= \mu \circ \theta(a) \end{aligned} \tag{1.3}$$

Thus $\mu \circ \theta$ is a fuzzy subgroup.

If μ is a fuzzy characteristic subgroup of a group G, it can be shown that μ is fuzzy normal, since this is the case in ordinary groups.

Next we extend the concepts of conjugacy and normalizers to fuzzy groups.

1.3.6 Definition

If μ and ν are fuzzy subgroups of a group G, then we say that μ is conjugate to ν if there exists a $x \in G$ such that

$$\mu(g) = \nu(x^{-1}(g)x) \ \ \forall g \in G.$$

Note: It is shown in 3.1.5 that conjugacy is an equivalence relation in the family of fuzzy subgroups of a group G.

Notation: Let μ be a fuzzy subgroup of a group G and let $g \in G$. Then μ^g denotes the mapping

$$\mu^g(a) = \mu(g^{-1}ag) \ \forall a \in G$$

of G into G. Now μ^g is a fuzzy subgroup of G.
For, for any $a, b \in G$,

$$\begin{aligned} \mu^g(ab) &= \mu(g^{-1}(ab)g) \\ &= \mu(g^{-1}abg) \\ &= \mu((g^{-1}ag)(g^{-1}bg)) \\ &\geq \min(\mu(g^{-1}ag), \mu(g^{-1}bg)) \\ &= \min(\mu^g(a), \mu^g(b)) \end{aligned} \quad (1.4)$$

Also

$$\begin{aligned} \mu^g(a^{-1}) &= \mu(g^{-1}a^{-1}g) \\ &= \mu(g^{-1}ag) = \mu^g(a). \end{aligned} \quad (1.5)$$

Another important concept in group theory is that of a *normalizer*[1]. We shall see how it translates into fuzzy subgroups.

1.3.7 Definition

Let μ be a fuzzy subgroup of a group G. Then the set given by $N(\mu) = \{g \in G : \mu^g = \mu\}$ is called the normalizer of μ.

[1]Let G be a group and let H be a subgroup of G. Then $N(H) = \{x \in G : xHx^{-1} = H\}$ is called the normalizer of H.

1.3.8 Proposition

If μ is a fuzzy subgroup of a group G, then $N(\mu)$ is a subgroup of G.

Proof

For any $x, y \in N(\mu)$ we want to show that $xy \in N(\mu)$ and $x^{-1} \in N(\mu)$. Let $x, y \in N(\mu)$, then

$$\mu^{xy} = \mu^{yx} = \mu^{y} = \mu.$$

Hence $x, y \in N(\mu)$ implies $xy \in N(\mu)$.

Next, let $y = x^{-1}$, then

$$\begin{aligned} \mu^{y}(g) &= \mu(y^{-1}gy) \\ &= \mu(xgx^{-1}) \\ &= \mu((x^{-1}g^{-1}x)^{-1}) \\ &= \mu(x^{-1}g^{-1}x) \\ &= \mu^{x}(g^{-1}) \\ &= \mu(g^{-1}) = \mu(g) \end{aligned} \tag{1.6}$$

so that $x^{-1} \in N(\mu)$, that is $\mu^{x^{-1}} = \mu$. Thus $N(\mu)$ is a subgroup of G.

1.3.9 Proposition

Let μ be a fuzzy subgroup of a finite group G. Let $H = \{a \in G : \mu(a) = \mu(e)\}$, where e denotes the identity of G. Then H is a subgroup.

Finally, we define what is meant by the intersection of two fuzzy subgroups and prove a proposition based on the intersection.

1.3.10 Definition

Let μ and ν be two fuzzy subgroups of a group G. Then by their intersection $\mu \cap \nu$

is meant:

$$(\mu \cap \nu)(x) = \min(\mu(x), \nu(x)) \;\; \forall x \in G.$$

We now make the following proposition involving intersection:

1.3.11 Proposition

If μ and ν are fuzzy subgroups of a group G, then their intersection $(\mu \cap \nu)$ is a fuzzy subgroup.

Proof

Suppose μ and ν are fuzzy subgroups of a group G. Let $x, y \in G$. We want to show that

$$(\mu \cap \nu)(xy) \geq \min((\mu \cap \nu)(x), (\mu \cap \nu)(y)$$

and

$$(\mu \cap \nu)(x^{-1}) = (\mu \cap \nu)(x)$$

Now

$$\begin{aligned}(\mu \cap \nu)(xy) &= \min(\mu(xy), \nu(xy)) \\ &\geq \min(\min(\mu(x), \mu(y)), \min(\nu(x), \nu(y)) \qquad (1.7) \\ &\geq \min(\min(\mu(x), \nu(x)), \min(\mu(y), \nu(y)) \\ &= \min((\mu \cap \nu)(x), (\mu \cap \nu)(y))\end{aligned}$$

(1.8)

Also

$$\begin{aligned}(\mu \cap \nu)(x^{-1}) &= \min(\mu(x^{-1}), \nu(x^{-1})) \\ &= \min(\mu(x), \nu(x)) \\ &= (\mu \cap \nu)(x).\end{aligned}$$

(1.9)

1.3.1 Level Subgroups

If G is a group and μ is a fuzzy subgroup of G, then for any x in G, we have

$$\mu(x-x) = \mu(e) \geq min(\mu(x), \mu(-x)) = \mu(x),$$

where e denotes the identity of G. Hence

$$\mu(x) \leq \mu(e) \ \ \forall x \in G.$$

1.3.2 Classification of fuzzy subgroups by their level subgroups

We first show that if two fuzzy subgroups of a group have identical family of level subgroups, it does not necessarily mean that the fuzzy subgroups are equal. [3] was useful in the preparation of this section.

Consider $G = \{a, b : a^2 = b^2 = (ab)^2 = e\}$, that is $G = \{e, \ a, \ b, ab\}$. Let $t_i \in [0,1], \ \ 0 \leq i \leq 2$, with $t_0 > t_1 > t_2$. Define $\mu : G \to [0,1]$ by $\mu(e) = t_0, \ \ \mu(a) = t_1, \ \ \mu(b) = t_2, \ \ \mu(ab) = t_2$. It is clear that μ is a fuzzy subgroup of G.
Thus if we define

$$\mu(x) = \begin{cases} t_0 & \text{if } \ x = e \\ t_1 & \text{if } \ x = a \\ t_2 & \text{otherwise.} \end{cases}$$

then

$$Im(\mu) = \{t_0, \ \ t_1, \ \ t_2\}.$$

Also suppose $s_0, \ \ s_1, \ \ s_2$ are three numbers in $[0,1]$ with $s_0 > s_1 > s_2$ such that

$\{s_i\} \cap \{t_i\} = \phi$. Now define $\nu : G \to [0,1]$ as follows:

$$\nu(x) = \begin{cases} s_0 & \text{if } x = e \\ s_1 & \text{if } x = a \\ s_2 & \text{else.} \end{cases}$$

then $Im(\nu) = \{s_0, \ s_1, \ s_2\}$. Note that the level subgroups of μ and ν are the same, but μ and ν are not equal.

1.3.12 Lemma

Let G be a finite group and let μ be a fuzzy subgroup of G. If $t_i, \ t_j \ \in Im(\mu)$ such that $\mu_{t_i} = \mu_{t_j}$, then $t_i = t_j$.

Proof [See 1.3.2]

1.3.13 Proposition

Let G be a finite group. A fuzzy subset μ of G is a fuzzy subgroup of G if there exists a maximal chain of subgroups

$$H_0 \subset H_1 \subset \cdots \subset H_m = G$$

such that for the numbers $t_0, \ldots, t_m$ in $Im(\mu)$ with $t_0 > t_1 > \cdots > t_m$ we have

$$\mu(H_0) = t_0, \ \ \mu(\hat{H}_1) = t_1, \ldots, \ \ \mu(\hat{H}_m) = t_m$$

where $\hat{H}_i = H_i \setminus H_{i-1}, \ \ 1 \leq i \leq m$.

Proof

Suppose $H_0 \subset H_1 \subset \cdots \subset H_m$ is a maximal chain of subgroups of G with $\mu : G \to [0,1]$ satisfying

$$\mu(H_0) = t_0, \ \ \mu(\hat{H}_1) = t_1, \ldots, \ \ \mu(\hat{H}_m) = t_m.$$

We need to show that μ is a fuzzy subgroup of G. To do this let $x, y \in G$ and consider the cases $x, y \in H_i \setminus H_{i-1}$ and $x \in \hat{H}_i,\ \ y \in \hat{H}_j,\ \ i > j$.

<u>case 1</u>: Let $x, y \in H_i \setminus H_{i-1}$. Then $\mu(x) = t_i$ and $\mu(y) = t_i$ and $xy \in H_i$. So

$$\mu(xy) \geq t_i = \min(\mu(x), \mu(y)), \text{ and } \mu(x^{-1}) = \mu(x) = t_i \text{ since } H_i \text{ is a group.}$$

<u>case 2</u>: Let $x \in \hat{H}_i,\ \ y \in \hat{H}_j$ with $i > j$. So we have $\mu(x) = t_i$ and $\mu(y) = t_j$ and $xy \in H_i$. As in case 1,

$$\mu(xy) \geq t_i = \min(\mu(x), \mu(y)), \text{ and } \mu(x^{-1}) = \mu(x) = t_i.$$

This completes the proof.

Note: The converse to the above proposition also holds.

We next state the following theorem:

1.3.14 Theorem

Suppose μ and ν are two fuzzy subgroups of a finite group G whose family of level subgroups are identical. If $Im(\mu) = \{t_0, \ldots, t_r\}$ and $Im(\nu) = \{s_0, \ldots, s_k\}$ with $t_0 > \cdots > t_r$ and $s_0 > \cdots > s_k$, then the following are true:

(i) $r = k$

(ii) $\mu_{ti} = \nu_{si},\ \ 0 \leq i \leq r$

(iii) if $x \in G$ such that $\mu(x) = t_i$, then $\nu(x) = s_i,\ \ 0 \leq i \leq r$.

Proof: See [4]

Before we engage in the investigation into the level subgroups of fuzzy normal subgroups, we shall first characterize the fuzzy normal subgroups. This will be done in two ways:

(i) use of conjugacy classes of a group G.

(ii) the idea of commutators of G.

1.3.15 Definition

A fuzzy subgroup μ on a group G is called a fuzzy normal subgroup if

$$\mu(xy) = \mu(yx) \ \forall x, \ \ y \in G.$$

1.3.16 Definition

Let G be a group. For each $a \in G$, the set $\{xax^{-1} : x \in G\}$ is called the conjugacy class of a.

1.3.17 Proposition

A fuzzy subgroup μ of a group G is a fuzzy normal subgroup if and only if μ is constant on the conjugate classes of G.

Proof

($\Longrightarrow$) Suppose μ is a fuzzy normal subgroup of G. Now for all $x, y \ \ \in G$,

$$\mu(y^{-1}xy) = \mu(xyy^{-1}) = \mu(xe) = \mu(x),$$

that is

$$\mu(y^{-1}xy) = \mu(x) \ \forall x, y \ \ \in G.$$

($\Longleftarrow$) Assume that μ is constant on each conjugate class of G. We must show that $\mu(xy) = \mu(yx)$ for all $x, y \ \ \in G$. Now

$$\mu(yx) = \mu(eyx) = \mu(x^{-1}xyx) = \mu(x^{-1}(xy)x).$$

By hypothesis,

$$\mu(x^{-1}(xy)x) = \mu(xy) \ \forall x, y \ \ \in G.$$

Thus μ is fuzzy normal in G.

1.3.18 Theorem

Let μ be a fuzzy normal subgroup of a group G. Let $t \in [0, 1]$ such that $t \leq \mu(e)$, where e denotes the identity of G. Then the set $\mu_t = \{x \in G : \mu(x) \geq t\}$ is a normal subgroup of G.

Proof

We already know that μ_t is a subgroup of G. We need only show that μ_t is a normal subgroup under the given conditions. To do this, let $x \in \mu_t$ and $y \in G$. Now μ is a fuzzy normal subgroup of G, hence μ is constant on the conjugate classes of G. So we have that $\mu(y^{-1}xy) = \mu(x) \geq t$, that is, $\mu(y^{-1}xy) \geq t$. Hence $(y^{-1}xy) \in \mu_t$. Thus $(y^{-1}xy) \in \mu_t \; \forall \; x \in \mu_t$ and $y \in G$. Thus μ_t is normal in G, written as $\mu_t \lhd G$.

(ii) Characterization using the idea of commutator subgroups.

1.3.19 Definition

Let G be a group, and let $x, y \in G$. The commutator of x, y is the element $[x, y]$ denoting the element $x^{-1}y^{-1}xy$.

Note: If x and y commute with each other, then $[x, y] = e$, so the idea of the terminology is justified.

Thus the commutator subgroup of a group G is the group generated by the set $\{x^{-1}y^{-1}xy : x, y \in G\}$.

1.3.20 Proposition

Let μ be a fuzzy subgroup of a group G. Then μ is a fuzzy normal subgroup if and only if $\mu([x, y]) \geq \mu(x) \; \forall \; x, y \in G$.

Proof

($\Longrightarrow$) Suppose μ is a fuzzy normal subgroup of G. Let $x, y \in G$ be given. Then

$$\begin{aligned} \mu(x^{-1}y^{-1}xy) &\geq min(\mu(x^{-1}, \mu(y^{-1}xy)) \text{ (since } \mu \text{ is a fuzzy subgroup)} \\ &= \min(\mu(x^{-1}, \ \mu(x)) \quad (\mu \text{ is fuzzy normal}) \\ &= \mu(x) \end{aligned} \tag{1.10}$$

($\Longleftarrow$) follows immediately.

We will first give a general discussion on some results on the equivalence of fuzzy subsets of a given set. We will do this by defining and studying a natural equivalence on the set of all fuzzy subsets of a given set. We will then use this equivalence to determine the number of distinct equivalence classes of fuzzy subgroups of a p-group. Right through our discussion we assume that $\mu(0) = 1$, and notice that $\mu(0) = 1$ is the only admissible fuzzy subgroup of a trivial group.

Chapter 2

On maximal chains and subgroup generators of p-groups

2.1 p- Group

For a prime p, a group G is a p-group if every element of G except the identity has order a power of the prime p.

Alternatively,

2.1.1 Proposition

A finite group G is a p-group if and only if the order of G is a power of p.

Proof

($\Longleftarrow$)Assume that G has order a power of p. Since the order of an element divides the order of the group, then every element of G must have the order a power of p. Hence G is a p-group.

($\Longrightarrow$)Assume G is a p-group. Let $\mid G \mid= p^r m, \ \ p \mid m$. Suppose q is a prime such that m is divisible by q. By Sylow first theorem, G has a subgroup H of prime order q. Moreover H is cyclic ($H =< a >$, with $\mid a \mid \ = \ \ q \neq p$). This contradicts the fact that G is a p-group. Thus $m = 1$ and $\mid G \mid= p^r$.

We discuss the most important concept of finite Abelian groups, the cyclic group. We also cite some results from group theory.

2.1.2 Definition

A group G is said to be cyclic if there is an element a in G such that $G = \{na : n \in Z\}$. Such an element is called a generator of G, and we write $G =< a >$ to indicate that G is a cyclic group generated by a.

2.1.3 Example

$\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}, \quad n \geq 1$, is a cyclic group under addition modulo n. Its generators are 1 and $-1 = n - 1$.

2.1.1 Results from group theory

(1) Every cyclic group is Abelian.

Proof

Let $G = \{na : n \in Z\}$ and let $x, y \in G$. Then $x = ra, y = sa$ for some integers r and s. Thus

$$\begin{aligned} x + y &= ra + sa \\ &= (r+s)a \\ &= (s+r)a \\ &= sa + ra \\ &= y + x \end{aligned} \tag{2.1}$$

2.1.4 Proposition

Let G be a group and let H be a subgroup of G. For any $x \in G$, define

$$xHx^{-1} = \{xhx^{-1} : h \in H\}.$$

Then the following are true:

(i) xHx^{-1} is a subgroup of G,

(ii) if H is Abelian, then xHx^{-1} is Abelian.

Proof

(i) Let xh_1x^{-1} and xh_2x^{-1} be in xHx^{-1}. Now

$$(xh_1x^{-1})(xh_2x^{-1}) = xh_1h_2x^{-1} \quad \in \quad xHx^{-1}.$$

Also,

$$(xhx^{-1})^{-1} = xh^{-1}x^{-1} \quad \in \quad xHx^{-1}.$$

(ii) Assume H is Abelian. Let xh_1x^{-1} and xh_2x^{-1} be in xHx^{-1}. Then

$$\begin{aligned}(xh_1x^{-1})(xh_2x^{-1}) &= xh_1h_2x^{-1} \\ &= xh_2h_1x^{-1} \\ &= (xh_2x^{-1})(xh_1x^{-1}).\end{aligned} \tag{2.2}$$

2.2 Cyclic p-groups

A cyclic p-group is of the form $\mathbb{Z}_{p^n}$ for some positive integer n.

Consider $\mathbb{Z}_p$ for any prime p. $(\mathbf{Z}_p, +)$ is a finite Abelian group. The only subgroups of $\mathbb{Z}_p$ are $\{0\}$ and $\mathbb{Z}_p$ itself. The subgroups of a given group form a chain. Thus for $\mathbb{Z}_p$ we have $\{0\} \subset \mathbb{Z}_p$ as the chain with only two components.

A chain is said to be **maximal** if it cannot be refined[1] any more.

[1] no more subgroups can be inserted in the chain.

2.2.1 Example

$\{0\} \subset \mathbb{Z}_2 \subset \mathbb{Z}_4 \subset \mathbb{Z}_{12} \subset \mathbb{Z}_{36}$, where $\mathbb{Z}_2 =< 18 >, \mathbb{Z}_4 =< 9 >, \mathbb{Z}_{12} =< 3 >$, is a maximal chain of subgroups of $\mathbb{Z}_{36}$.

For the cyclic group $\mathbb{Z}_{p^n}$ we will use the notation $\mathbb{Z}_{p^k} \subset \mathbb{Z}_{p^n}$, for $k \leq n$, to mean that $\mathbb{Z}_{p^n}$ contains the cyclic subgroup of order p^k generated by p^{n-k}.
For example, we will write $\mathbb{Z}_{3^2} \subset \mathbb{Z}_{3^4}$ to mean that $< 3^2 > \subset \mathbb{Z}_{3^4}$.

2.3 Classification of Crisp Subgroups

In this section we classify the subgroups of some finite Abelian groups of various types by means of maximal chains as in the following discussion. We will start by giving a number of lemmas before proving a general result(where possible). The proofs of lemmas will be provided in chapter 5. The figures at the back may be consulted for clarity.

2.3.1 Maximal chains of $\mathbb{Z}_{p^n} \times \mathbb{Z}_p$

For any natural number n we have the following specific number of maximal chains:

2.3.1 Lemma

$G = \mathbb{Z}_p \times \mathbb{Z}_p$ has $p + 1$ maximal chains.

When $n = 2$ we have the following lemma:

2.3.2 Lemma

$G = \mathbb{Z}_{p^2} \times \mathbb{Z}_p$ has 2p+1 maximal chains.

When $n = 3$ we have the following lemma:

2.3.3 Lemma

$G = \mathbb{Z}_{p^3} \times \mathbb{Z}_p$ has 3p+1 maximal chains.

In general, for $n \in \mathbb{N}$ and any prime p, we have the following:

2.3.4 Proposition

$G = \mathbb{Z}_{p^n} \times \mathbb{Z}_p$ has $(n-1)(p-1) + (p+1) + (n-1)$ maximal chains.

In view of the discussion on maximal chains, we are able to count the number of subgroup-generators and give the results in the form of the following propositions. The figures at the back facilitated the process.

2.3.2 Crisp subgroups of $\mathbb{Z}_{p^n} \times \mathbb{Z}_p$

2.3.5 Proposition

$G = \mathbb{Z}_p \times \mathbb{Z}_p$ has $p + 3$ crisp subgroups

Proof

There are obviously two trivial subgroups, namely, $\{0\}$ and the big group. According to 5.2.1, $\{0\}$ is contained in $p + 1$ further subgroups, each of which is contained in G.

2.3.6 Proposition

$G = \mathbb{Z}_{p^2} \times \mathbb{Z}_p$ has $2p + 4$ crisp subgroups

Proof

As in 2.3.5, there are two trivial subgroups, with $\{0\}$ contained in $p+1$ other subgroups which are themselves the subgroups of a non-cyclic subgroup of G of order p^2, viz $<(01, p0)>$. The remaining subgroups of G which have not been counted are the cyclic subgroups of order p^2 which contain $(p, 0)$, and there are p of them.

Example: $\mathbb{Z}_9 \times \mathbb{Z}_3$,

In general,

2.3.7 Proposition

$G = \mathbb{Z}_{p^n} \times \mathbb{Z}_p$ has $n(p+1)+2$ crisp subgroups

Chapter 3

Equivalence relation

3.1 An equivalence relation in general

Things that are considered different in one context may be viewed as equivalent in another context. In this section we discuss an appropriate generalization or mechanism for specifying whether or not two quantities are the same in a given situation. Such a mechanism is an *equivalence relation.*

In his book, J. Gallian gives the following definition of an equivalence relation:

3.1.1 Definition

An equivalence relation on a set S is a set R of ordered pairs of elements of S such that

1. $(a, a) \in R$ for all $a \in S$ (reflexive property).
2. $(a, b) \in R$ implies $(b, a) \in R$ (symmetric property).
3. $(a, b) \in R$ and $(b, c) \in R$ imply $(a, c) \in R$ (transitive property).

It is customary to write $a\Re b$ instead of $(a, b) \in R$.

3.1.2 Definition

An isomorphism ϕ from a group G to a group $\overline{G}$ is a one-to-one correspondence

$\phi : G \to \overline{G}$ that preserves the group operation. We then say that G and $\overline{G}$ are isomorphic and write $G \approx \overline{G}$.

3.1.3 Example

Let $G = (\mathbf{R}, +)$ and $\overline{G} = (\mathbf{R}^{+}, \mathbf{x})$. Define a function $\phi : G \to \overline{G}$ by $\phi(x) = 3^x,\ x \in \mathbf{R}$. Then ϕ is an isomorphism.

But if $\phi : R \to R$ under addition operation, then $\phi(x) = x^3$ is not an isomorphism because ϕ is not operation-preserving.

3.1.4 Theorem

Isomorphism is an equivalence relation on the class of all groups.

Proof

Suppose G, H and K are groups. We must show that:

(i) $\approx$ is reflexive, that is , $G \approx G$.

(ii) $\approx$ is symmetric, that is, $G \approx H \Longrightarrow H \approx G$.

(iii) $\approx$ is transitive, that is, $G \approx H$ and $H \approx K \Longrightarrow G \approx K$.

Now, the identity mapping $i : G \to G$ is an isomorphism. So $G \approx G$ and thus (i) is true.

Assume $G \approx H$ and define $\phi : G \to H$ such that ϕ is an isomorphism. We must show that $H \approx G$. Now ϕ is one-to-one and onto, so the inverse $\phi^{-1} : H \to G$ exists. We need to show that ϕ^{-1} is also an isomorphism. Clearly, ϕ^{-1} is one-to-one and onto. We need only show that ϕ^{-1} is operation preserving, that is, we must show that for all $a, b \in H$, $\phi^{-1}(ab) = \phi^{-1}(a)\phi^{-1}(b)$. Now let $a, b \in H$ and let $\phi^{-1}(a) = x$ and $\phi^{-1}(b) = y$. Then $a = \phi(x)$ and $b = \phi(y)$ and so $ab = \phi(x)\phi(y) = \phi(xy)$ since ϕ is operation preserving. Hence $xy = \phi^{-1}(ab)$, that is, $\phi^{-1}(ab) = \phi^{-1}(a)\phi^{-1}(b)$ as required. So we have that $H \approx G$, thus (ii) is true.

Finally, let G, H and K be groups with isomorphisms $\theta : G \to H$ and $\phi : H \to K$. We must show that $\psi = \phi \circ \theta : G \to K$ is also an isomorphism.

Now $\phi \circ \theta$ is a one-to-one correspondence. To show that ψ is operation preserving, let $x, y \in G$.

Then

$$\begin{aligned}\psi(xy) = \phi \circ \theta(xy) &= \phi(\theta(xy)) \\ &= \phi(\theta(x)\theta(y)) \\ &= \phi(\theta(x))\phi(\theta(y)) \\ &= \phi \circ \theta(x)\phi \circ \theta(y) \\ &= \psi(x)\psi(y)\end{aligned}$$

so that ψ is operation preserving. Thus $\psi = \phi \circ \theta$ is an isomorphism, hence (iii) is true.

3.1.5 Example

Conjugacy is an equivalence relation

Let $a\Re b$ mean that a is conjugate to b . Thus $a\Re b$ if and only if $\exists \quad x \in G$ s.t $xax^{-1} = b$.

Now $a = eae^{-1}$ so that $a\Re a$. Therefore, $\Re$ is reflexive.

Next assume that $a\Re b$. Then there exist x, y such that $xax^{-1} = b$. So

$$\begin{aligned} x^{-1}xax^{-1}x &= x^{-1}bx \\ &\Rightarrow eae = x^{-1}bx \\ &\Rightarrow a = (x^{-1})b(x^{-1})^{-1} \end{aligned} \tag{3.1}$$

so that $b\Re a$. therefore, $\Re$ is symmetric.

Finally, assume that $a\Re b$ and $b\Re c$. Then $xax^{-1} = b$ and $yby^{-1} = c$ for some $x, y \in G$. Now

$$(yx)a(x^{-1}y^{-1}) = c$$

$\Rightarrow$

$$(yx)a(yx)^{-1} = c$$

so that $a\Re c$. Therefore $\Re$ is transitive.

3.2 Fuzzy Relations

We shall see how the idea of equivalence relation translates into fuzzy sense.

3.2.1 Definition

A fuzzy relation μ on X is a fuzzy binary relation given by

$$\mu : X \times X \to [0, 1],$$

where X is a nonempty set.

3.2.2 Definition

A fuzzy relation μ on a group G is said to be a fuzzy equivalence relation on G if:

1. $\mu(x, x) = 1 \ \forall \ x \in G$
2. $\mu(x, y) = \mu(y, x) \ \forall \ x, y \in G$
3. $\mu \circ \mu \leq \mu$

Note 1: If μ_1 and μ_2 are two fuzzy relations on X, then their composition, denoted by $\mu_1 \circ \mu_2$, is defined as

$$(\mu_1 \circ \mu_2)(x, y) = sup_z(\mu(x, z) \wedge \mu(z, y)).$$

Note 2: It can be shown that the composition as defined above is associative, that is

$$(\mu_1 \circ \mu_2) \circ \mu_3 = \mu_1 \circ (\mu_2 \circ \mu_3).$$

Note 3: A fuzzy relation μ is said to be idempotent if $\mu \circ \mu = \mu$.
A partial ordering $\leq$ in the set of all fuzzy relations on X is given by

$$\mu_1 \leq \mu_2 \text{ if and only if } \mu_1(x, y) \leq \mu_2(x, y) \ \forall x, y \in X.$$

3.2.3 Proposition

If μ is a fuzzy equivalence relation on X, then μ is idempotent.

Proof

Assume μ is a fuzzy equivalence relation, and let $x, y \in X$. We must show that

$$(\mu \circ \mu)(x, y) \geq \mu(x, y) \text{ and } (\mu \circ \mu)(x, y) \leq \mu(x, y).$$

By definition 3.2.2(3), we need only show that

$$(\mu \circ \mu)(x, y) \geq \mu(x, y) \text{ for any } x, y \in X.$$

Now

$$\begin{aligned} \mu \circ \mu(x, y) &= sup_z(\mu(x, z) \wedge \mu(z, y)) \\ &\geq \mu(x, x) \wedge \mu(x, y) \\ &= \mu(x, y) \quad (\text{since } \mu(x, x) = 1) \end{aligned} \tag{3.2}$$

3.3 Fuzzy isomorphism

In this section we are going to give two notions of fuzzy isomorphism. The first will be the one that uses the concept of a norm [12], and the second will be the one on which this work on equivalence of fuzzy subgroups has been built. In the end we are going to notice that the fuzzy equivalence to be discussed below is finer than the notion of fuzzy isomorphism in the sense that it will always guarantee fuzzy isomorphism, whereas fuzzy isomorphism does not imply fuzzy equivalence.

Notion 1

The discussion we are about to give on fuzzy isomorphism will require the knowledge of a number of concepts. These will include among others the concepts of t-norms, Ω-admissibility, T-fuzzy subgroups, to name but a few. We shall then give a preliminary information, prove some results that come along, and towards the end give an extended version of Ray's definition of fuzzy isomorphism by the authors in [12]. We shall also encompass the discussion of fuzzy equivalence through the study of fuzzy relations by the author [15].

3.3.1 Definition

A t-norm, denoted by T, is a binary operation

$$T : [0,1] \times [0,1] \to [0,1]$$

which satisfies the following properties:

1. $T(x, 1) = x$
2. $T(x, y) = T(y, x)$
3. if $x \leq x',\ T(x, y) \leq T(x', y)$
4. $T(x, T(y, z)) = T(T(x, y), z)$

Remark: If (1) is replaced by $T(x, 0) = x$, then T becomes a t-conorm.

Another important concept is that of T-fuzzy subgroup.

3.3.2 Definition

A T-fuzzy subgroup μ of a group G is an element μ of $F(G)$, the set of all fuzzy subgroups of G, which satisfies $\forall\ x, y \in G$:

1. $\mu(x, y) \geq T(\mu(x), \mu(y))$
2. $\mu(e) = 1$

Note: μ said to be an anti T-fuzzy subgroup if (1) and (3) in the above definition are replaced by $\mu(x, y) \leq T(\mu(x), \mu(y))$ and $\mu(e) = 0$, respectively.
A T-fuzzy subgroup of G will be denoted as $\mu <_T G$. Similarly $\mu <_{AT} G$ will denote

the anti T-fuzzy subgroup μ of G.

We now introduce the concept of a complement.

3.3.3 Definition

By a complement C on [0,1], we mean an operation

$$C : [0,1] \to [0,1] \text{ satisfying } \forall\ x, y \in [0,1]:$$

1. $x < y \Longrightarrow C(y) < C(x)$
2. $C(C(x)) = x$
3. $C(0) = 1$

Notation: If $\mu \in F(X)$, by a complement $\mu^c \in F(X)$ we will mean that

$$\mu^c(x) = C(\mu(x)),\ \ \forall\ x \in X.$$

Result 1: Let C be a complement on [0,1], and let T be a t-norm. Then T^c, defined by

$$T^c(x, y) = C(T(C(x), C(y))),$$

is a t-conorm.

Proof: Let C be a complement on [0,1] and T be a t-norm.
Now

$$\begin{aligned} T^c(x, y) &= C(T(C(x), C(y))) \\ &= C(T(C(y), C(x))) \\ &= T^c(y, x) \end{aligned} \tag{3.3}$$

Also

$$\begin{aligned} T^c(x,0) &= C(T(C(x),C(0))) \\ &= C(T(C(x),1)) \\ &= C(C(x)) \text{ by } 3.3.1(1) \\ &= x \end{aligned} \tag{3.4}$$

Finally, assume $x \leq x'$, then

$$\begin{aligned} T^c(x,y) &= C(T(C(x),C(y))) \\ &\leq C(T(C(x'),C(y))) \cdots * \\ &= T^c(x',y) \end{aligned} \tag{3.5}$$

*[For, if $x \leq x'$ then $C(x) \geq C(x')$. So $T(C(x),C(y)) \geq T(C(x'),C(y))$ and thus $C(T(C(x),C(y))) \leq C(T(C(x'),C(y)))$.]

Result 2: $\mu <_T G$ if and only if $\mu^c <_{AT^c} G$

Proof

$\Longrightarrow$) Assume $\mu <_T G$. Let $x,y \in G$. Then

$$\mu^c(xy) = C(\mu(xy)) \leq C(T(C(\mu(x)),C(\mu(y))))$$

$$= T^c(\mu(x),\mu(y)), \ \forall x,y \in G, \text{ which proves property 3.3.2(1)}$$

Also

$$\mu^c(x) = C(\mu(x)) = C(\mu(x^{-1}))$$

$$= \mu^c(x^{-1}), \ \forall x \in G, \text{ hence 3.3.2(2)}$$

Finally

$$\mu^c(e) = C(\mu(e)) = C(1) = 0, \text{ property 3.3.2(3).}$$

$\Longleftarrow$) Assume $\mu^c <_{AT^c} G$. Let $x, y \in G$.

Now

$$\begin{aligned}\mu(xy) &= C(C(\mu(xy))) \\ &= C(\mu^c(xy)) \leq C(C(T(\mu(x), \mu(y)))) \\ &= T(\mu(x), \mu(y))\end{aligned} \tag{3.6}$$

Also

$$\begin{aligned}\mu(x) &= C(C(\mu(x))) \\ &= C(\mu^c(x)) \\ &= C(\mu^c(x^{-1})) \\ &= C(C(\mu(x^{-1}))) \\ &= \mu(x^{-1})\end{aligned} \tag{3.7}$$

And

$$\begin{aligned}\mu(e) &= C(C(\mu(e))) \\ &= C(\mu^c(e)) \\ &= C(0) \\ &= 1\end{aligned} \tag{3.8}$$

We proceed to define Ω-admissibility as discussed in [12]. We first define what is meant by an Ω-group, where Ω is an *operator domain*[1] on G. When G is equipped with an operator domain Ω, we say G is an Ω-group.

[1]Let Ω be a nonempty set. Assume to each $x \in G$ and $\omega \in \Omega$ there corresponds a unique element $\omega(x) \in G$, and $\omega(xy) = \omega(x)\omega(y)$ $\forall x, y \in G$ and $\omega \in \Omega$. Then every $\omega \in \Omega$ is called an operator and Ω is an operator domain on G.[23]

3.3.4 Definition

By an Ω-group or operator group, we mean an ordered pair $(G, *)$ in which

$$* : G \times \Omega \to G, \text{ with } (a, \omega) \mapsto a * \omega,$$

satisfies the property:

$$(a, b) * \omega = (a * \omega)(b * \omega) \text{ for all } a, b \in G, \text{ and } \omega \in \Omega.$$

Now for a subgroup $H < G$ to be called an Ω-subgroup of an Ω-group G, H has to be Ω-admissible, that is

$$h * \omega \in H, \; \forall \; h \in H, \; \forall \; \omega \in \Omega.$$

Extending this admissibility idea to the case of fuzzy sets, we have the following definition:

3.3.5 Definition

Let μ be a fuzzy subset of the Ω-group G. Then μ is called Ω-admissible on G if

$$\mu(x) \leq \mu(x * \omega), \; \forall \; x \in G, \; \forall \; \omega \in \Omega.$$

However, if

$$\mu(x) \geq \mu(x * \omega), \; \forall \; x \in G, \; \forall \; \omega \in \Omega,$$

then μ is said to be anti Ω-admissible.

Result: μ is Ω-admissible if and only if μ^c is anti Ω-admissible.

Proof

$\Longrightarrow$)Assume μ is Ω-admissible. Then $\forall x \in G, \; \forall \omega \in \Omega$, we have $\mu(x) \leq \mu(x * \omega)$ so that

$$C(\mu(x)) \geq C(\mu(x * \omega)),$$

i.e

$$\mu^c(x) \geq \mu^c(x * \omega).$$

$\Longleftarrow$)Follows immediately by reverse of the above argument.

In preparation for the next proposition we give the following definition:

3.3.6 Definition

Suppose $G(G')$ is an $\Omega(\Omega')$-group, where $G(G')$ represents a group with identity $e(e')$, and let $f : G \to G'$ be a homomorphism. Then:

1. Ω' is said to be f-equivalent to Ω if for any $\omega' \in \Omega'$ there exists an $\omega \in \Omega$ such that
$$f(x * \omega) = f(x) * \omega', \ \ \forall \ x \in G.$$

2. Ω is said to be f-equivalent to Ω' if for any $\omega \in \Omega$ there exists an $\omega' \in \Omega'$ such that
$$f(x * \omega) = f(x) * \omega', \ \ \forall \ x \in G.$$

3. Ω and Ω' are said to be f-equivalent if Ω' is f-equivalent to Ω and Ω is f-equivalent to Ω'.

3.3.7 Proposition

Let G, G', G'' be $\Omega, \Omega', \Omega''$-groups respectively, and let $f : G \to G'$, $\ g : G' \to G''$ be homomorphisms. If Ω' is f-equivalent to Ω and Ω'' is g- equivalent to Ω', then Ω'' is $g \circ f$-equivalent to Ω.

Proof

Let $f : G \to G'$, $\ g : G' \to G''$ be the given homomorphisms of groups G, G', G''.
Now Ω' is f-equivalent to Ω implies (1) of definition 3.3.6
Also Ω'' is g- equivalent to Ω' implies that for any $\omega' \in \Omega' \ \exists \ \omega'' \in \Omega''$ such that

$$g(x * \omega') = g(x) * \omega'', \ \ \forall \ x \in G.$$

Now

$$\begin{aligned}(g \circ f)(x * \omega) &= g(f(x * \omega)) \\ &= g(f(x) * \omega') \text{ by } (1) \\ &= g(f(x)) * \omega'' \\ &= g \circ f(x) * \omega''\end{aligned} \tag{3.9}$$

so that Ω'' is $g \circ f$-equivalent to Ω.

We now give an extended version of Ray's definition of a fuzzy homomorphism(isomorphism) by the authors in [12] which includes an $\Omega(\Omega')$-group $G(G')$.

3.3.8 Definition

Let $G(G')$ be an $\Omega(\Omega')$-group. Then $\lambda \in F(G')$ is homomorphic(isomorphic) to $\mu \in F(G)$ if there exist a homomorphism(resp. isom) f from G onto G' and a function $h_f : Im^+(\mu) \to [0,1]$ such that:

1. h_f is an operation preserving (O.P) function satisfying $h_f(0) = 0$ and $h_f(1) = 1$
2. Ω' is f-equivalent to Ω (resp. Ω and Ω' are f-equivalent).
3. $\lambda(y) = sup_{x \in f^{-1}(y)} h_f(\mu(x))$ (resp. $\lambda(f(x)) = h(\mu(x))$), $\forall\ y \in G'$.

3.3.9 Proposition

Let μ be a fuzzy subgroup of G. Then μ is said to be fuzzy normal if and only if $x\mu = \mu x\ \forall\ x \in G$.

3.3.10 Theorem

Let f be a homomorphism of the group G onto the group G'. If μ is a normal fuzzy subgroup of G, then $f(\mu)$ is a normal fuzzy subgroup of G'.

Proof

Assume μ is a normal fuzzy subgroup of G. Let $f : G \rightarrow G'$ be a homomorphism. Firstly, we must show that $f(\mu)$ is a fuzzy subgroup of G'. To do this we must show that $f(\mu)(f(x)f(y)) \geq \min(f(\mu)(f(x)), f(\mu)(f(y)))$ and $f(\mu)(f(x)) = f(\mu)(f(x)^{-1}))$. By definition, $f(\mu)(f(x)) = sup\ \mu(a)_{f(a)=f(x)}$ and $f(\mu)(f(y)) = sup\ \mu(a)_{f(a)=f(y)}$.

To show that $f(\mu)(f(x)f(y)) \geq f(\mu)(f(x)) \wedge f(\mu)(f(y))$ we choose any $\epsilon > 0$ such that $\epsilon > \min(f(\mu)(f(x)), f(\mu)(f(y)), f(\mu)(f(x)f(y)))$. Then there exist some a_1, a_2 with $f(a_1) = f(x)$ and $f(a_2) = f(y)$ such that $(f(\mu)(f(x)) < \mu(a_1) + \epsilon$ and $(f(\mu)(f(y)) < \mu(a_2) + \epsilon$.

Hence we have

$$\begin{aligned}(f(\mu)(f(x)) - \epsilon) \wedge (f(\mu)(f(y)) - \epsilon) &< \mu(a_1) \wedge \mu(a_2) \\ &\leq \mu(a_1 a_2) \\ &\leq \sup \mu(a)_{f(a)=f(x)f(y)} \\ &= f(\mu)(f(x)f(y))\end{aligned}$$

Thus $f(\mu)(f(x)f(y)) \geq f(\mu)(f(x)) \wedge f(\mu)(f(y))$. Clearly $f(\mu)(f(x)) = f(\mu)(f(x)^{-1}))$ so that $f(\mu)$ is a fuzzy subgroup of G'.

Note: If $f(\mu)(f(x)f(y)) = 0$ then either $(f(\mu)(f(x)) = 0$ or $(f(\mu)(f(y)) = 0$.

We proceed to show that $f(\mu)$ is normal in G'. For this purpose, let K be the kernel of the homomorphism f. Hence $f(x) = y$ implies that $f^{-1}(y) = xK$, where $x \in G,\ y \in G'$.

Thus

$$\begin{aligned}f(\mu)(f(x)) &= \sup_{z \in f^{-1}(f(x))} \mu(z) \\ &= \sup_{z \in xK} \mu(z) \\ &= \sup \mu(xk) \quad \cdots *\end{aligned}$$

for $k \in K$, and for any $x \in G$.

Now

$$\begin{aligned} f(\mu)(f(a)f(b)) &= f(\mu)(f(ab)) \\ &= \sup_{k\in K}\mu(abk) \quad by(*) \\ &= \sup_{k\in K}\mu(bka) \\ &= \sup_{k\in K}\mu(baa^{-1}ka) \end{aligned}$$

Now since $a^{-1}Ka = K$, we have that

$$\begin{aligned} f(\mu)(f(a)f(b)) &= \sup_{k\in K}\mu(ba(a^{-1}ka)) \\ &= \sup_{k\in K}\mu(bak) \\ &= f(\mu)(f(ba)) \\ &= f(\mu)(f(b)f(a)). \end{aligned}$$

This proves the result.

Finally, we state the following proposition based on the previous results:

3.3.11 Proposition

Let $f : G \to K$ be an isomorphism from a group G onto a group K. Then the mapping $\mu \to f(\mu)$ sets up a one-to-one correspondence between the set of all fuzzy subgroups of K such that the corresponding groups μ and $f(\mu)$ are isomorphic for all μ. Moreover, under this correspondence, the set of all normal fuzzy subgroups of G are mapped onto the set of all normal fuzzy subgroups of K.

Proof: See [21].

Notion 2: [16]

3.3.12 Definition

Let G be a group and μ and ν be fuzzy subgroups of G. Then μ is said to be fuzzy isomorphic to ν if there exists an isomorphism $f : supp(\mu) \rightarrow supp(\nu)$ such that

$$\mu(a) > \mu(b) \Longleftrightarrow \nu(f(a)) > \nu(f(b))$$

for any $a, b \in supp(\mu)$. We then write $\mu \approx \nu$.

Whilst on fuzzy homomorphism, we give the following definition:

3.3.13 Definition

Let G and G' be groups, and let $f : G \rightarrow G'$ be a homomorphism. Let μ be a fuzzy subgroup of G. By the image of μ under f, $f(\mu)$, we mean a fuzzy subset of $f(G)$ defined by

$$f(\mu)(f(x)) = sup\{\mu(y) : f(y) = f(x)\}.$$

Define $f(\mu)(y) = 0$ if $y \notin f(G)$. Then, claim: $f(\mu)$ **becomes a fuzzy subgroup of** G'.

For proof of the claim, see [11].

3.4 Fuzzy Equivalence Relation

3.4.1 Definition

Let μ and ν be any fuzzy sets on I^X. An equivalence relation on I^X is defined as follows:

$\mu \sim \nu$ if and only if for all $x, y \in X, \mu(x) > \mu(y)$ if and only if $\nu(x) > \nu(y)$ and $\mu(x) = 0$ if and only if $\nu(x) = 0$. The second condition , namely, $\mu(x) = 0$ if and only if $\nu(x) = 0$ implies that the supports of μ and ν are equal.

Note: The strict inequality in the above conditions can be relaxed to the $\geq$ inequality without the latter affecting the results. In fact the two are equivalent as is shown below:

3.4.2 Proposition

$\mu \sim \nu$ if and only if

$$I. \;\; \mu(x) > \mu(y) \text{ iff } \nu(x) > \nu(y) \text{ and } \mu(x) = 0 \text{ iff } \nu(x) = 0$$

$$\Updownarrow$$

$$II. \ \mu(x) \geq \mu(y) \text{ iff } \nu(x) \geq \nu(y) \text{ and } \mu(x) = 0 \text{ iff } \nu(x) = 0.$$

Proof

($\Longrightarrow$) Assume $\mu \sim \nu$ as in I. Suppose $\mu(x) \geq \mu(y)$, then either $\mu(x) > \mu(y)$ or $\mu(x) = \mu(y)$. If $\mu(x) > \mu(y)$, then $\nu(x) > \nu(y)$.
Suppose $\mu(x) = \mu(y)$ and $\nu(x) \neq \nu(y)$. Assume $\nu(x) > \nu(y)$, then $\mu(x) > \mu(y)$ by I (contradiction).
Similarly, if $\nu(y) > \nu(x)$, then $\mu(y) > \mu(x)$ by I (contradiction). Therefore, $\nu(x) = \nu(y)$. Thus $\mu(x) \geq \mu(y)$ implies $\nu(x) \geq \nu(y)$. Similarly $\nu(x) \geq \nu(y)$ implies $\mu(x) \geq \mu(y)$.

($\Longleftarrow$) Assume $\mu \sim \nu$ as in II. That is $\mu(x) \geq \mu(y) \Longleftrightarrow \nu(x) \geq \nu(y)$. Suppose $\mu(x) > \mu(y)$. We claim that $\nu(x) > \nu(y)$. If not then $\nu(x) \leq \nu(y)$. This means that $\mu(x) \leq \mu(y)$ by II, contradiction. Therefore, $\nu(x) > \nu(y)$.

The condition $\mu(x) = 0$ iff $\nu(x) = 0$ is an essential part of the equivalence relation as the following example illustrates.
Consider $D_4 = \{e, f, f^2, f^3, g, fg, f^2g, f^3g\}$ and define the fuzzy sets μ and ν on D_4 as follows:

$$\mu(x) = \begin{cases} 1 & \text{if } x = e \\ \frac{1}{2} & \text{if } x = f, f^2, f^3 \\ \frac{1}{3} & \text{else.} \end{cases}$$

$$\nu(x) = \begin{cases} 1 & \text{if } x = e \\ \frac{1}{2} & \text{if} x = f, f^2, f^3 \\ 0 & \text{else.} \end{cases}$$

It can be seen that $supp(\mu) \neq supp(\nu)$ even though $\mu(x) > \mu(y)$ iff $\nu(x) > \nu(y), \ \forall x, y \in D_4$.

At this stage it is natural to expect that the equivalence of μ and ν implies that their images have the same cardinality. Indeed this is the case as the following proposition shows, but unfortunately the converse need not be true.

3.4.3 Proposition

If $\mu \sim \nu$ then $\mid Im(\mu) \mid = \mid Im(\nu) \mid$.

Proof: Let $x \in X$.
Define $f : Im(\mu) \rightarrow Im(\nu)$ by $f(\mu(x)) = \nu(x)$. Assume $\mu(x) > \mu(y)$, then $f(\mu(x)) > f(\mu(y))$. And if $\mu(x) = 0$ we have $\nu(x) = f(\mu(x)) = 0$ so that f is well-defined. To show that f is one-one, we need to show that $\mu(x_1) \neq \mu(x_2) \Longrightarrow f(\mu(x_1)) \neq f(\mu(x_2)), \ x_1, x_2 \in X$. Assume $\mu(x_1) \neq \mu(x_2)$, then $\mu(x_1) > \mu(x_2) \Longrightarrow \nu(x_1) > \nu(x_2)$, that is, $f(\mu(x_1)) \neq f(\mu(x_2))$.

However, if $\mid Im(\mu) \mid = \mid Im(\nu) \mid$, it is not necessarily true that $\mu \sim \nu$, as the following example illustrates:

3.4.4 Example

Let $S_3 = \{e, f, f^2, g, fg, f^2g\}$ be the symmetric group on 3 symbols. Define fuzzy subgroups μ and ν on S as follows:

$$\mu(x) = \begin{cases} 1 & \text{if } x = e \\ \frac{1}{4} & \text{if } x = fg \\ \frac{1}{5} & \text{else.} \end{cases}$$

$$\nu(x) = \begin{cases} 1 & \text{if } x = e \\ \frac{1}{4} & \text{if } x = g \\ \frac{1}{5} & \text{else.} \end{cases}$$

It is easily seen that $Im(\mu) = Im(\nu)$ and $Supp(\mu) = Supp(\nu)$. However, μ is not equivalent to ν because $\mu(fg) > \mu(g)$ whereas $\nu(fg) \not> \nu(g)$.

We next state the following propositions, with 3.4.6 being a partial converse to 3.4.5. For their proofs, see [16]:

3.4.5 Proposition

Let μ and ν be two fuzzy subsets of X. Suppose for each $t > 0$ there exists an $s > 0$ such that $\mu^t = \nu^s$. Then $\mu \sim \nu$.

3.4.6 Proposition

Suppose μ and ν are two fuzzy subsets of X such that μ is equivalent to ν. Then for each $t \in [0, 1]$ there is an $s \in [0, 1]$ such that $\mu^t = \nu^s$.

Chapter 4

Equivalent Fuzzy Subgroups

[16] and [17] were useful in the preparation of this material.

4.1 Chains and key-chains

The following example serves to illustrate how we classify the fuzzy subgroups of a given group.

4.1.1 Example

Let $G = \mathbb{Z}_{16}$. Define $\mu : G \to I$ as follows:

$$\mu(x) = \begin{cases} 1 & \text{if } x = 0 \\ \frac{1}{2} & \text{if } x \in \mathbb{Z}_2 \setminus \{0\} \\ \frac{1}{4} & \text{if } x \in \mathbb{Z}_4 \setminus \mathbb{Z}_2 \\ \frac{1}{8} & \text{if } x \in \mathbb{Z}_8 \setminus \mathbb{Z}_4 \\ \frac{1}{16} & \text{if } x \in \mathbb{Z}_{16} \setminus \mathbb{Z}_8 \end{cases}$$

μ can be represented by

$$\{0\}^1 \subset \mathbb{Z}_2^{\frac{1}{2}} \subset \mathbb{Z}_4^{\frac{1}{4}} \subset \mathbb{Z}_8^{\frac{1}{8}} \subset \mathbb{Z}_{16}^{\frac{1}{16}},$$

where $1, \frac{1}{2}$, etc are the values or weights in each subgroup.
All fuzzy subgroups ν that are equivalent to μ can be represented by

$$\nu : \{0\}^1 \subset \mathbb{Z}_2^{\lambda} \subset \mathbb{Z}_4^{\beta} \subset \mathbb{Z}_8^{\gamma} \subset \mathbb{Z}_{16}^{\delta}$$

such that

$$1 \geq \lambda \geq \beta \geq \gamma \geq \delta \geq 0.$$

The notation $111\lambda\beta$ denotes the fuzzy subgroup represented by the flagged chain

$$\mathbb{Z}_4^1 \subset \mathbb{Z}_8^{\lambda} \subset \mathbb{Z}_{16}^{\beta}.$$

In general, a maximal chain

$$\{0\}^1 \subset \mathbb{Z}_p^{\lambda_1} \subset \cdots \subset \mathbb{Z}_{p^{n-1}}^{\lambda_{n-1}} \subset \mathbb{Z}_{p^n}^{\lambda_n}$$

defines a fuzzy subgroup μ in the following way:
μ assumes

$$1 \text{ on } 0, \ \lambda_1 \text{ on } \mathbb{Z}_p, \ldots, \lambda_{n-1} \text{ on } \mathbb{Z}_{p^{n-1}} \text{ and } \lambda_n \text{ on } \mathbb{Z}_{p^n},$$

where,

$$1 \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq \lambda_n \geq 0.$$

We then denote the fuzzy subgroup defined above simply by $1\lambda_1\lambda_2 \ldots \lambda_{n-1}\lambda_n$ in the descending order, where the last entry may or may not be zero.

The numbers $1\lambda_1\lambda_2 \ldots \lambda_{n-1}\lambda_n$ are called **pins**. Notice that 1 occupies the first position, and λ_i occupies the $(i+1)$-th position, for $i = 1, 2, \ldots, n$. Hence the length of an n-chain is $n+1$, and so the n-chain has $n+1$ available positions. It is these positions that will play a crucial role throughout our discussion.

Note: (i) A finite n-chain is a collection of numbers on [0,1] of the form $1 > \lambda_1 > \lambda_2 > \cdots > \lambda_{n-1} > \lambda_n$. An n-chain is called a key chain if $1 \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq \lambda_n$

(ii) Interlocked pins are called components, with the exception of a 1 standing alone in the first position and **not** interlocked with any of the λ's. (A consecutive occurrence of equality signs is said to be in an interlocking position of pins).

(iii) A k-pad, for $1 \leq k \leq n$, is a key chain containing k distinct components.

(iv) The number of pins found in the component formed by interlocked positions is called the padidity of the component. So in example 4.1.2, the padidities are respectively 1, 3, 1 for a 4-pad key chain, while they are 3, 2 for a 2-pad key chain.

4.1.2 Example

Consider the chain $1 > \lambda_1 > \lambda_2 = \lambda_3 = \lambda_4 > \lambda_5$. This is a 4-pad key chain of a 6-chain.

$1 = \lambda_1 = \lambda_2 > \lambda_3 = \lambda_4$ is a 2-pad key chain of a 5-chain.

4.2 Cases of n in $\mathbb{Z}_{p^n}$

If $n = 1$, then the group is $\mathbb{Z}_p$ and the maximal chain is $\{0\}^1 \subset \mathbb{Z}_p{}^\lambda$ with $1 \geq \lambda \geq 0$. A typical subgroup is

$$\mu(x) = \begin{cases} 1 & \text{if } x = 0 \\ \\ \lambda & \text{if } x \in \mathbb{Z}_p \setminus \{0\} \end{cases}$$

The possibilities given by the inequalities in $1 \geq \lambda \geq 0$ give rise to the following three combinations $11, 1\lambda, 10$. Any fuzzy subgroup of $\mathbb{Z}_p$ is equivalent to one of these three symbols. One can see that these are the only distinct equivalence classes of fuzzy subgroups on $\mathbb{Z}_p$ with the following interpretations: 11 represents the crisp trivial subgroup $\mathbb{Z}_p$ given by $\mu(x) = 1 \ \forall x \in \mathbb{Z}_p$, 1λ represents the fuzzy subgroup

$$\mu(x) = \begin{cases} 1 & \text{if } x = 0 \\ \\ \lambda & \text{if } x \neq 0 \end{cases}$$

and 10 represents the trivial subgroup $\{0\}$ given by

$$\mu(x) = \begin{cases} 1 & \text{if } x = 0 \\ \\ 0 & \text{otherwise} \end{cases}$$

Here we make an assumption that $1 > \lambda \geq 0$.

Proceeding to the other cases of n, we make the following example:

4.2.1 Example

Consider the case when $n = 2$. Then we have the maximal chain $\{0\} \subset \mathbb{Z}_p \subset \mathbb{Z}_{p^2}$ defining a fuzzy subgroup μ as follows:

μ assumes 1 on $\{0\}$, λ on $\mathbb{Z}_p$ and β on $\mathbb{Z}_{p^2}$.

Corresponding to the maximal chain, there are seven distinct equivalence classes of fuzzy subgroups. These subgroups are given by the symbols 111, 11λ, 110, $1\lambda\lambda$, $1\lambda\beta$, $1\lambda 0$, 100 where for instance by $1\lambda\beta$ we mean the fuzzy subgroup:

$$\mu(x) = \begin{cases} 1 & \text{if } x = 0 \\ \\ \lambda & \text{if } x \in \mathbb{Z}_p \setminus \{0\} \\ \\ \beta & \text{otherwise} \end{cases}$$

Here again we make the assumption that $1 > \lambda > \beta \geq 0$.

It can be observed that the number of fuzzy subgroups whose support is $\mathbb{Z}_{p^2}$ is one more than the number of fuzzy subgroups whose supports are properly contained in $\mathbb{Z}_{p^2}$.

Again it is observed that the number 7 has the following partition:

$7 = 4 + 2 + 1$, where the components in the partition have the following meaning:

- 4 is the number of fuzzy subgroups whose support is $\mathbb{Z}_{p^2}$
- 2 is the number of fuzzy subgroups whose support is $\mathbb{Z}_p$
- 1 is the number of fuzzy subgroups whose support is $\{0\}$.

Hence we have the following:

$$7 = \sum_{k=0}^{2} 2^k = 2^3 - 1.$$

Another example to illustrate the pattern is the following:

4.2.2 Example

When $n = 3$ the group is $\mathbb{Z}_{p^3}$.

The corresponding maximal chain is $\{0\} \subset \mathbb{Z}_p \subset \mathbb{Z}_{p^2} \subset \mathbb{Z}_{p^3}$. It will be evident how the fuzzy subgroups of the previous group give rise to the fuzzy subgroups of the next group of the chain as we proceed.

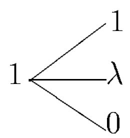
If we attach the symbols in the order $1\lambda\beta\gamma$ to the groups in the chain, then ,by counting, there are 15 distinct fuzzy subgroups of $\mathbb{Z}_{p^3}$. If we write $15 = 8+4+2+1$, then we have

$$15 = \sum_{k=0}^{3} 2^k = 2^{3+1} - 1,$$

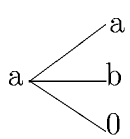
where each 2^k is the number of fuzzy subgroups of $\mathbb{Z}_{p^3}$ whose support is $\mathbb{Z}_{p^k}$, for $k = 0, 1, 2, 3$.

Marrying the previous examples, we can find all fuzzy subgroups of $\mathbb{Z}_{p^3}$ from those of $\mathbb{Z}_{p^2}$ as follows:
Each fuzzy subgroup of $\mathbb{Z}_{p^2}$ whose symbols are all different from zero gives rise to three fuzzy subgroups of $\mathbb{Z}_{p^3}$.The rest each gives rise to one fuzzy subgroup of $\mathbb{Z}_{p^3}$ by simply attaching 0.
For instance, 111 of $\mathbb{Z}_{p^2}$ gives rise to $1111, 111\lambda, 1110$ whereas 110 will give rise to 1100.

We can represent the fuzzy subgroups by means of the following tree diagram:



To construct a tree diagram for bigger fuzzy subgroups, we attach to each node of the previous tree having a nonzero pin a branch of the form:



where $a = 1, \lambda$, or β; $b = \lambda, \beta$, or $\delta \neq 0$ and 0 can only be attached to 0.

4.3 Main result

Now for any group $\mathbb{Z}_{p^n}$ it appears that there is a one-to-one correspondence between the distinct fuzzy subgroups whose support is $\mathbb{Z}_{p^n}$ and the distinct fuzzy subgroups whose support is properly contained in $\mathbb{Z}_{p^n}$. This can be verified by writing the subgroups fully. However, the subgroup 111...1 is not included in this correspondence.

On the basis of the discussion above we can make the following proposition:

4.3.1 Proposition

Given any natural number n, there are $2^{n+1} - 1$ distinct equivalence classes of fuzzy subgroups of $\mathbb{Z}_{p^n}$.

Proof [1]

We shall prove that the number of non-equivalent fuzzy subgroups of $\mathbb{Z}_{p^n}$ is given by $\sum_{k=0}^{n} 2^k = 2^{n+1} - 1$, where $n \in N$. We shall do this by induction on n.

From the above examples, the statement is true for $n = 1, 2, 3$. We next assume that the statement is true for $n = k$, that is, the number of distinct fuzzy subgroups of $\mathbb{Z}_{p^k}$ is $2^{k+1} - 1$. We must show that the statement is true for $n = k + 1$, that is the number of non-equivalent fuzzy subgroups of $\mathbb{Z}_{p^{k+1}}$ is $2^{k+2} - 1$.

Now $2^{k+1} - 1 = \frac{2^{k+1}}{2} + \frac{2^{k+1}}{2} - 1$. By one-to-one correspondence, there are $\frac{2^{k+1}}{2}$ distinct fuzzy subgroups of $\mathbb{Z}_{p^k}$ whose support is $\mathbb{Z}_{p^k}$. Each gives rise to two fuzzy subgroups of $\mathbb{Z}_{p^{k+1}}$ whose support is $\mathbb{Z}_{p^{k+1}}$ and one fuzzy subgroup whose support is $\mathbb{Z}_{p^k}$.

In other words, $\frac{2^{k+1}}{2}$ fuzzy subgroups of $\mathbb{Z}_{p^k}$ give rise to $2(\frac{2^{k+1}}{2}) + \frac{2^{k+1}}{2}$ fuzzy subgroups of $\mathbb{Z}_{p^{k+1}}$.

The remaining $\frac{2^{k+1}}{2} - 1$ fuzzy subgroups of $\mathbb{Z}_{p^k}$ have supports that are properly

[1] This proof has been reproduced from the reference 'On an equivalence of fuzzy subgroups 1' for consistency and convenience.

contained in $\mathbb{Z}_{p^k}$ which give rise to $\frac{2^{k+1}}{2} - 1$ fuzzy subgroups of $\mathbb{Z}_{p^{k+1}}$ by simply attaching zero to each. This means that the number of distinct fuzzy subgroups of $\mathbb{Z}_{p^{k+1}}$ is $2(\frac{2^{k+1}}{2}) + \frac{2^{k+1}}{2} + \frac{2^{k+1}}{2} - 1 = 2^{k+1} + 2^{k+1} - 1$

$$= 2(2^{k+1}) - 1 = 2^{k+2} - 1.$$

Hence the statement is true for $n = k + 1$. Thus, by the principle of induction, the statement is true for all $n \in N$.

Chapter 5

Classification of Abelian groups of the form $\mathbb{Z}_{p^n} \times \mathbb{Z}_p$

5.1 The Counting Technique of Subgroups

In preparation for the smooth discussion in subsequent chapters, we shall first build the background from the known results from group theory. For this purpose [5], and [6] were useful.

5.1.1 Theorem

The order of an element of a direct product of a finite number of finite groups is the least common multiple of the orders of the components of the element. That is $\mid (a_1, a_2, \ldots, a_n) \mid = \text{lcm } (\mid a_1 \mid, \mid a_2 \mid, \ldots, \mid a_n \mid)$.

5.1.2 Example

Determine the number of elements of order 9 in $\mathbb{Z}_{81} \times \mathbb{Z}_3$.

By 5.1.1, we can count the number of elements by using the fact that $\mid (a, b) \mid = 9 = \text{lcm}(\mid a, \mid b)$. Obviously we must have $\mid a \mid = 9$ and $\mid b \mid = 1$ or 3.

If $\mid a \mid = 9$ and $\mid b \mid = 1$, then there are 6 choices for a and one for b, which yields 6 elements of order 9.

If $\mid a \mid = 9$ and $\mid b \mid = 3$, there are 6 choices for a and 2 for b, giving 12 elements of order 9.

Thus $\mathbb{Z}_{81} \times \mathbb{Z}_3$ has 18 elements of order 9.

Similarly $\mathbb{Z}_{25} \times \mathbb{Z}_5$ has 24 elements of order 5.

5.1.3 Theorem

If d is a divisor of n, then the number of elements of order d in a cyclic group of order n is $\phi(d) = \mid U(d) \mid$.

Notation: Let $(a, b) \in \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^m}$. By $h(a, b)$ we shall mean $(ha \mod p^n, \ hb \mod p^m)$ ... **

5.1.4 Theorem

If a is a generator of a finite cyclic group G of order n, then the other generators of G are elements of the form ra, where $\gcd(r, n) = 1$.

5.1.5 Example

Suppose that $< a >, < b >$ and $< c >$ are cyclic groups of orders 6, 8 and 20 respectively. Find all generators of $< a >, < b >$ and $< c >$.

We know that $U(6) = \{1, 5\}, U(8) = \{1, 3, 5, 7\}$ and $U(20) = \{1, 3, 7, 9, 11, 13, 17, 19\}$. $[U(n) = \{x \in \mathbb{N} : \gcd(x, n) = 1\}]$

Thus by theorem 5.1.4,

$$\begin{aligned} < a > &= < 5a > \\ < b > = < 3b > = < 5b > &= < 7b > \\ < c > = < 3c > = \cdots &= < 19c > \end{aligned} \tag{5.1}$$

5.1.6 Example

Determine the number of cyclic subgroups of order 5 in $\mathbb{Z}_{25} \times \mathbb{Z}_5$.

To start off, we need to count the number of elements (a, b) of order 5 in $\mathbb{Z}_{25} \times \mathbb{Z}_5$. By example 5.1.2, there are 24 such elements. But each cyclic subgroup of order 5 has four elements of order 5 and no two of them can share an element of order 5. This gives 6 cyclic subgroups of order 5 in $\mathbb{Z}_{25} \times \mathbb{Z}_5$.

5.1.7 Example

The number of cyclic subgroups of order p in $\mathbb{Z}_{p^2} \times \mathbb{Z}_p$ can be found by counting the number of elements using the fact that $\mid (a, b) \mid = p = \operatorname{lcm}(\mid a \mid, \mid b \mid)$. This requires that both $| a |$ and $| b |$ be p or $| a |= p$ and $| b |= 1, p$ and vice versa. The first case yields $(p-1)^2$ elements, while the second case yields $2(p-1)$ elements of order p. But each cyclic subgroup of order p has $p-1$ elements of order p no two of which have an element of order p in common. Thus $\mathbb{Z}_{p^2} \times \mathbb{Z}_p$ has $p+1$ cyclic subgroups of order p.

By similar argument, $\mathbb{Z}_{p^2} \times \mathbb{Z}_p$ has p cyclic subgroups of order p^2.

5.1.8 Theorem

The number of subgroups of order p^β in a group of order $p^\alpha \equiv 1 (\text{mod } p)$.

5.1.9 Example

The number of subgroups of order 9 in a group of order $27 \equiv 1 (\text{mod } 3)$.

Let the group be $\mathbb{Z}_{3^2} \times \mathbb{Z}_3$, and let x be the number of subgroups. Now $x = 1 (\text{mod } 3)$ implies that $x - 1 = 3k$ for some k.

Hence $x \in \{0, \pm 3, \pm 6, \pm 9, \dots\}$. Thus $x = 4$.

5.1.10 Theorem

Every subgroup of a cyclic group is cyclic. Moreover, if $|< a >|= n$, then the order of any subgroup of $< a >$ is a divisor of n; and for each divisor k of n , the group $< a >$ has exactly one subgroup of order k which is $< a^{n/k} >$.

Proof (Latter part)

Let k be any divisor of n. Now $k(\frac{n}{k}a) = na = e$ and $t(\frac{n}{k}a) \neq e$.

claim: $< (\frac{n}{k}a) >$ is a unique subgroup of order k.

To verify our claim, we let H be any subgroup of order k. It can be shown that $H =< ma >$ for any least positive integer m such that $ma \in H$. By the division algorithm, there exist integers q and r such that $n = mq + r$ where $0 \leq r < m$. Then we have $na = (mq + r)a$ so that $ra = -q(ma) \in H$. This implies that $r = 0$ and so $n = mq$. So $k =| H |=|< ma >|= n/m$. It follows that $m = n/k$ and so $H =< ma >=< \frac{n}{k}a >$.

5.2 Maximal Chains of $\mathbb{Z}_{p^n} \times \mathbb{Z}_p$

In this section we determine the number of maximal chains of $G = \mathbb{Z}_{p^n} \times \mathbb{Z}_p$ for the cases $n = 1, 2, 3$. We shall use the acquired results to determine the number of fuzzy subgroups of the same groups in the next section. For illustration, see figures at the back.

Note: All maximal chains between a and b have the same length.(Jordan-Dedekind maximal chain condition)

5.2.1 Lemma

$G = \mathbb{Z}_p \times \mathbb{Z}_p$ has $p + 1$ maximal chains.

Proof:

$\mathbb{Z}_p \times \mathbb{Z}_p$ is a group of order p^2. So every nontrivial subgroup of $\mathbb{Z}_p \times \mathbb{Z}_p$ is cyclic and has order p. Hence all maximal chains of subgroups of G are of the form $\{0\} \subset \; < a > \; \subset G$ in which a has the form (α, β), where $1 \leq \alpha < p$ and $1 \leq \beta < p$. Now a generates all the distinct subgroups of the form: (0,1), (1,0), (1,1), (1,2), (1,3),..., (1,p-2), $(\alpha, p - \alpha)$ for $1 < \alpha < p - 1$. For the generator (α, β) we consider the two cases, namely, $\alpha \neq \beta$ and $\alpha + \beta \neq p$ as well as $\alpha = \beta$ and $\alpha + \beta = p$. Now the subgroup generated as in the first case contains one of (1,2), (1,3),..., (1,p-2). For the latter case we have (1,1) and $(\alpha, p - \alpha)$. The remaining two other subgroups outside these cases are (1,p) and (p,1) which have been listed already.
(see fig.1)

Example: $\mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_5 \times \mathbb{Z}_5$.

5.2.2 Lemma

$G = \mathbb{Z}_{2^2} \times \mathbb{Z}_2$ has five maximal chains.

Proof

The order of group G is eight, hence G has subgroups of orders 1,2,4 and 8. According to examples 5.1.2 and 5.1.6 , G has three cyclic subgroups of order two, and two of order four. Clearly the cyclic subgroups of order two are all generated by the elements (2,0), (2,1) and (0,1).
Similarly, the generators h(1,0) and h(1,1), where $(h, 4) = 1$, generate the list of all cyclic subgroups of order four in G. In addition, G has a unique non-cyclic subgroup of order four which does not contain any element of $< h(1, 0) >$ or $< h(1, 1) >$. Such a subgroup is generated by any of the pairs $< (0, 1), (2, 0) >$ or $< (0, 1), (2, 1) >$.
The subgroups of G form maximal chains of the form $\{0\} \subset < a > \subset < b > \subset G$ of four levels each. By simple counting, G has $3 + 2 = 5$ maximal chains.

Similarly $\mathbb{Z}_{3^2} \times \mathbb{Z}_3$ has seven maximal chains; $\mathbb{Z}_{5^2} \times \mathbb{Z}_5$ has eleven maximal chains; etc.

In general, for any prime p:

5.2.3 Lemma

$G = \mathbb{Z}_{p^2} \times \mathbb{Z}_p$ has 2p+1 maximal chains.

Proof

$\mathbb{Z}_{p^2} \times \mathbb{Z}_p$ is a group of order p^3 and has nontrivial subgroups of orders p and p^2. Each nontrivial subgroup of order p is cyclic, and therefore has the form $< a >$ where a is a generator of the form (x, y), $0 \leq y < p$ and $x = p$ or $x = 0$ when $x + y = 1$. Now a generates $p + 1$ distinct cyclic subgroups according to example 5.1.7. Such distinct subgroups are given by the following ordered pairs:

$$(0, 1), (p, 1), (p, 2), \ldots, (p, p - 1), (p, 0) \tag{5.2}$$

As can be checked out, each of these subgroups is contained in the only non-cyclic subgroup of order p^2 which is generated by any pair of subgroups in (5.2), say $< (0, 1), (p, 0) >$. Hence these subgroups form maximal chains of the form $\{0\} \subset <$

$a > \subset < (0,1), (p,0) > \subset G$.

By example 5.1.7, G has p cyclic subgroups of order p^2 that are given by the ordered pairs

$$(1,0), (1,1), (1,2), \ldots, (1, p-1) \tag{5.3}$$

As per the notation ** above, it can be seen that these subgroups together with $h < b >$ contain in common the subgroup $(p, 0)$, where $< b >$ is any ordered pair in (5.3) with $\gcd(h, p^2) = 1$. Hence all the resulting maximal chains in this case are of the form $\{0\} \subset < (p,0) > \subset < b > \subset G$. The chain $\{0\} \subset < (p,0) > \subset < (0,1), (p,0) > \subset G$ has already been listed.

5.2.4 Lemma

$G = \mathbb{Z}_{p^3} \times \mathbb{Z}_p$ has 3p+1 maximal chains.

5.3 Fuzzy Subgroups of $\mathbb{Z}_{p^n} \times \mathbb{Z}_p$ for $n = 1, 2, 3$

5.3.1 Proposition

$G = \mathbb{Z}_p \times \mathbb{Z}_p$ has $4p + 7$ distinct fuzzy subgroups.

Proof

Each of the maximal chains of $\mathbb{Z}_p \times \mathbb{Z}_p$ has only three levels. As in example 4.2.1, the three symbols will give rise to 7 key chains. Of the 7, three will yield identical fuzzy subgroups. The remaining four will each give rise to different fuzzy subgroups. Thus according to lemma 5.2.1, there will be $4(p+1)+3$ fuzzy subgroups as claimed in the proposition.

5.3.2 Proposition

$G = \mathbb{Z}_{p^2} \times \mathbb{Z}_p$ has $16p + 15$ distinct fuzzy subgroups.

Proof

Each of the maximal chains of $\mathbb{Z}_{p^2} \times \mathbb{Z}_p$ has only four levels. Hence the four symbols will give rise to 15 key chains. Of the 15 key chains, three will yield identical fuzzy subgroups. These key chains are 1111, $1\lambda\lambda\lambda$, 1000. Of the remaining 8 will give rise to $p+1$ distinct fuzzy subgroups. Such key chains have a duplicated symbol in their last three positions . e.g 111λ, $1\lambda00$, $1\lambda\lambda\beta$ etc. The remaining four key chains will each give rise to $2p+1$ distinct fuzzy subgroups.

5.3.3 Proposition

$G = \mathbb{Z}_{p^3} \times \mathbb{Z}_p$ has $48p + 31$ distinct fuzzy subgroups.

Proof

Each maximal chain of $\mathbb{Z}_{p^3} \times \mathbb{Z}_p$ has only five levels, corresponding to the key chain $1 \geq \lambda \geq \beta \geq \delta \geq \gamma \geq 0$. The five symbols will yield 31 key chains. All the resulting key chains have a symbol 1 in common at the first position while the last four symbols may differ. It is these last four symbols that will be vital to the proof given here. Of the 31 key chains, three will give rise to identical fuzzy subgroups. Such key chains have identical symbols in the last four positions. e.g $11111, 1\lambda\lambda\lambda\lambda, 10000$. Of the remaining, those key chains with exactly one double in the last four positions will each yield $2p+1$ distinct fuzzy subgroups, and there are twelve of these. The key chains with distinct last four symbols will each yield $3p+1$ distinct fuzzy subgroups according to lemma 5.2.4, and there are four such chains. The rest will each yield $p+1$ distinct fuzzy subgroups. Such chains have either two doubles or a triple in their last four symbols, and by counting there are twelve such key chains.

11111	$1\lambda\lambda\lambda\lambda$	1111λ	$1\lambda\lambda\lambda\beta$
11110	$1\lambda\lambda\lambda0$	$111\lambda\lambda$	$1\lambda\lambda\beta\beta$
$111\lambda\beta$	$1\lambda\lambda\beta\delta$	$111\lambda0$	$1\lambda\lambda\beta0$
11100	$1\lambda\lambda00$	$11\lambda\lambda\lambda$	$1\lambda\beta\beta\beta$
$11\lambda\lambda\beta$	$1\lambda\beta\beta\delta$	$11\lambda\lambda0$	$1\lambda\beta\beta0$
$11\lambda\beta\beta$	$1\lambda\beta\delta\delta$	$11\lambda\beta\delta$	$1\lambda\beta\delta\gamma$
$11\lambda\beta0$	$1\lambda\beta\delta0$	$11\lambda00$	$1\lambda\beta00$
11000	$1\lambda000$	10000	

Chapter 6

Mixed Primes

We next turn our attention to the case of mixed primes. The research group members, namely, Murali and Makamba, have worked on the distinct primes p and q to determine the number of distinct equivalence classes of fuzzy subgroups of $G = \mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m}$, where n and m are any two natural numbers. We shall state the results on this work, and for detailed proofs one is referred to [18], [19].

6.1 The case when $m = 1$

We want to characterize all equivalent fuzzy subgroups of $\mathbb{Z}_{p^n} \times \mathbb{Z}_q$.
As we have noticed in the previous work, we will see the crucial role played by the maximal chains in facilitating this characterization.
We begin with the case $n = 1 : \mathbb{Z}_p \times \mathbb{Z}_q$. As can be noticed, $\mathbb{Z}_p \times \mathbb{Z}_q$ has the maximal chains $0 \subset \mathbb{Z}_p \times 0 \subset \mathbb{Z}_p \times \mathbb{Z}_q$ and $0 \subset 0 \times \mathbb{Z}_q \subset \mathbb{Z}_p \times \mathbb{Z}_q$ each of which can be identified with the chain $0 \subset \mathbb{Z}_p \subset \mathbb{Z}_{p^2}$. Now according to 4.2.1, each of these chains will yield 7 distinct fuzzy subgroups. It can be checked out by writing it out fully that $\mathbb{Z}_p \times \mathbb{Z}_q$ turns out to have 11 non-equivalent fuzzy subgroups according to this classification: Of these 7 distinct fuzzy subgroups, three will give rise to identical fuzzy subgroups, and these are $111, 1\lambda\lambda, 100$. The remaining 4, namely, $11\lambda, 110, 1\lambda\beta, 1\lambda 0$ will give rise to distinct fuzzy subgroups,i.e $3 + 4 \times 2 = 11$.

Examples: $\mathbb{Z}_2 \times \mathbb{Z}_3, \mathbb{Z}_2 \times \mathbb{Z}_5, \mathbb{Z}_3 \times \mathbb{Z}_7$.

Case $n = 2 : \mathbb{Z}_{p^2} \times \mathbb{Z}_q$

Again as in the above case, $\mathbb{Z}_{p^2} \times \mathbb{Z}_q$ has 3 maximal chains each of which can be identified with the chain $0 \subset \mathbb{Z}_p \subset \mathbb{Z}_{p^2} \subset \mathbb{Z}_{p^3}$. Each of these chains will yield 15 distinct fuzzy subgroups. Again by writing it out fully, it can be seen that $\mathbb{Z}_{p^2} \times \mathbb{Z}_q$ has 31 non-equivalent fuzzy subgroups. For, of the 15 distinct fuzzy subgroups, three will yield identical fuzzy subgroups, 8 will yield two and 4 will yield three distinct fuzzy subgroups, i.e $3 + 8 \times 2 + 4 \times 3 = 31$

Examples: $\mathbb{Z}_4 \times \mathbb{Z}_3, \mathbb{Z}_4 \times \mathbb{Z}_7, \mathbb{Z}_{5^2} \times \mathbb{Z}_2$

Continuing in this fashion, it appears that $\mathbb{Z}_{p^3} \times \mathbb{Z}_q$ will give rise to 79 distinct fuzzy subgroups, as the following generalization suggests for any n:

6.1.1 Proposition

$\mathbb{Z}_{p^n} \times \mathbb{Z}_q$ has $2^{n+1}(n + 2) - 1$ distinct fuzzy subgroups.

Proof: (see Theorem 3.4 [16]).

6.2 Maximal chains of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m}$

In this section we determine the number of maximal chains of $G = \mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m}$, and state some results without proofs.

6.2.1 Proposition

The number of maximal chains of $G = \mathbb{Z}_{p^n} \times \mathbb{Z}_q$, where p and q are distinct primes, is $n+1$.

Note. By symmetry, we can conclude that $\mathbb{Z}_p + \mathbb{Z}_{q^m}$ has $m+1$ maximal chains.

The next proposition extends to $G = \mathbb{Z}_{p^2} + \mathbb{Z}_{q^m}$.

6.2.2 Proposition

Let $G = \mathbb{Z}_{p^2} + \mathbb{Z}_{q^m}$ where p and q are distinct primes. Then the number of maximal chains of G is $\sum_{i=-1}^{m-1} r_i(m-i)$, where $r_i = \frac{(2+i-1)!}{(2-2)!(1+i)!}$.

Again we observe by symmetry that $\mathbb{Z}_{p^n} + \mathbb{Z}_{q^2}$ has $\sum_{i=-1}^{n-1} r_i(n-i)$ maximal chains. For the inductive process of finding the maximal chains of $G = \mathbb{Z}_{p^n} + \mathbb{Z}_{q^m}$ in general , to become transparent for higher indices n, it may be clarifying to consider one more case as follows :

6.2.3 Proposition

Let $G = \mathbb{Z}_{p^3} + \mathbb{Z}_{q^m}$ where p and q are distinct primes. Then the number of maximal chains of G is $\sum_{i=-1}^{m-1} r_i(m-i)$, where $r_i = \frac{(3+i-1)!}{(3-2)!(1+i)!}$, $\quad m \geq 2$.

In general, we have the following theorem:

6.2.4 Theorem

Let $G = \mathbb{Z}_{p^n} + \mathbb{Z}_{q^m}$ where p and q are distinct primes. Then the number of maximal chains of G is

$$\sum_{i=-1}^{m-1} r_i(m-i), \quad \text{where} \quad r_i = \frac{(n+i-1)!}{(n-2)!(1+i)!}, \ n \geq 2.$$

6.3 Fuzzy subgroups of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m}$

In this section we wish to find a formula for the number of equivalence classes of fuzzy subgroups of G given any positive integers m and n.

It may be noted that for $m = 1$ and any positive integer n, the number of distinct equivalence classes was found to be $2^{n+1}(n+2)-1$ (see 6.1.1)

We advance the value of m with the hope of ultimately classifying for all positive integers m and n.

For any n and $m = 2, 3$ we have the following propositions:

6.3.1 Proposition

$G = \mathbb{Z}_{p^n} \times \mathbb{Z}_{q^2}$, where $n \geq 2$, has

$$2^{n+2+1} \sum_{r=0}^{2} 2^{-r} \binom{n}{n-r} \binom{2}{r} - 1$$

distinct fuzzy subgroups

6.3.2 Proposition

$G = \mathbb{Z}_{p^n} \times \mathbb{Z}_{q^3}$, where $n \geq 2$, has

$$2^{n+2+1} \sum_{r=0}^{3} 2^{-r} \binom{n}{n-r} \binom{3}{r} - 1$$

distinct fuzzy subgroups

In general, with the observation in 6.3.1 and 6.3.2, we have the following formula for the number of distinct fuzzy subgroups of $G = \mathbb{Z}_{p^n} \times \mathbb{Z}_{q^m}$:

6.3.3 Theorem

$G = \mathbb{Z}_{p^n} + \mathbb{Z}_{q^m}$, where $n \geq m$, has

$$2^{n+m+1} \sum_{r=0}^{m} 2^{-r} \binom{n}{n-r} \binom{m}{r} - 1$$

distinct fuzzy subgroups.

This process of characterization is ongoing with more results and developments expected in the next work of the same nature. The goal is not achieved until all possible cases have been considered.

Chapter 7

List of Tables

With the exception of Table 1, the rest of the tables are for the case when all the nontrivial subgroups of G are cyclic.

Table 1

Group	Maximal Chains	Crisp Subgroups	Fuzzy Subgroups
$\mathbb{Z}_p \times \mathbb{Z}_p$	$p + 1$	$p + 3$	$4p + 7$
$\mathbb{Z}_{p^2} \times \mathbb{Z}_p$	$2p + 1$	$2p + 4$	$16p + 15$
$\mathbb{Z}_{p^3} \times \mathbb{Z}_p$	$3p + 1$	$3p + 5$	$48p + 31$

Table 2

Group	Maximal Chains	Fuzzy Subgroups
$\mathbb{Z}_p \times \mathbb{Z}_p$	$p+1$	$4p+7$
$\mathbb{Z}_{p^2} \times \mathbb{Z}_p$	$2p$	$12p+7$
$\mathbb{Z}_{p^3} \times \mathbb{Z}_p$	$3p-1$	$28p+7$
$\mathbb{Z}_{p^4} \times \mathbb{Z}_p$	$4p-2$	$60p+7$
$\mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2}$	p^2+p	$8p^2+12p+7$
$\mathbb{Z}_{p^3} \times \mathbb{Z}_{p^2}$	$2p^2$	$24p^2+12p+7$
$\mathbb{Z}_{p^4} \times \mathbb{Z}_{p^2}$	$3p^2-p$	$56p^2+12p+7$
$\mathbb{Z}_{p^5} \times \mathbb{Z}_{p^2}$	$4p^2-2p$	$120p^2+12p+7$
$\mathbb{Z}_{p^3} \times \mathbb{Z}_{p^3}$	p^3+p^2	$16p^3+24p^2+12p+7$
$\mathbb{Z}_{p^4} \times \mathbb{Z}_{p^3}$	$2p^3$	$48p^3+24p^2+12p+7$
$\mathbb{Z}_{p^5} \times \mathbb{Z}_{p^3}$	$3p^3-p^2$	$112p^3+24p^2+12p+7$
$\mathbb{Z}_{p^6} \times \mathbb{Z}_{p^3}$	$4p^3-2p^2$	$240p^3+24p^2+12p+7$
$\mathbb{Z}_{p^4} \times \mathbb{Z}_{p^4}$	p^4+p^3	$32p^4+48p^3+24p^2+12p+7$
$\mathbb{Z}_{p^5} \times \mathbb{Z}_{p^4}$	$2p^4$	$96p^4+48p^3+24p^2+12p+7$
$\mathbb{Z}_{p^6} \times \mathbb{Z}_{p^4}$	$3p^4-p^3$	$224p^4+48p^3+24p^2+12p+7$
$\mathbb{Z}_{p^7} \times \mathbb{Z}_{p^4}$	$4p^4-2p^3$	$480p^4+48p^3+24p^2+12p+7$
$\vdots$		
	$\vdots$	$\vdots$
$\mathbb{Z}_{p^n} \times \mathbb{Z}_p$	$(n-1)(p-1)+(p+1)$	$\sum_{k=1}^{n} 2^{k+1}p+7$
$\mathbb{Z}_{p^n} \times \mathbb{Z}_{p^2}$	$[(n-2)(p-1)+(p+1)]p$	$2^3\sum_{k=0}^{n}(2^k-1)p^2+(12p+7)$
$\mathbb{Z}_{p^n} \times \mathbb{Z}_{p^3}$	$[(n-3)(p-1)+(p+1)]p^2$	-
$\vdots$	$\vdots$	$\vdots$
$\mathbb{Z}_{p^n} \times \mathbb{Z}_{p^m}$	$[(n-m)(p-1)+(p+1)]p^{m-1}$	-

Table 3

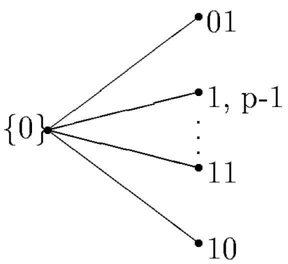
Group	Maximal Chains	Subgroups
$\mathbb{Z}_p \times \mathbb{Z}_p$	$p+1$	$p+3$
$\mathbb{Z}_{p^2} \times \mathbb{Z}_p$	$2p$	$2p+3$
$\mathbb{Z}_{p^3} \times \mathbb{Z}_p$	$3p-1$	$3p+3$
$\mathbb{Z}_{p^4} \times \mathbb{Z}_p$	$4p-2$	$4p+3$
$\mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2}$	p^2+p	p^2+2p+3
$\mathbb{Z}_{p^3} \times \mathbb{Z}_{p^2}$	$2p^2$	$2p^2+2p+3$
$\mathbb{Z}_{p^4} \times \mathbb{Z}_{p^2}$	$3p^2-p$	$3p^2+2p+3$
$\mathbb{Z}_{p^5} \times \mathbb{Z}_{p^2}$	$4p^2-2p$	$4p^2+2p+3$
$\mathbb{Z}_{p^3} \times \mathbb{Z}_{p^3}$	p^3+p^2	p^3+2p^2+2p+3
$\mathbb{Z}_{p^4} \times \mathbb{Z}_{p^3}$	$2p^3$	$2p^3+2p^2+2p+3$
$\mathbb{Z}_{p^5} \times \mathbb{Z}_{p^3}$	$3p^3-p^2$	$3p+3$
$\mathbb{Z}_{p^6} \times \mathbb{Z}_{p^3}$	$4p^3-2p^2$	$2p^2+2p+3$
$\mathbb{Z}_{p^4} \times \mathbb{Z}_{p^4}$	p^4+p^3	-
$\mathbb{Z}_{p^5} \times \mathbb{Z}_{p^4}$	$2p^4$	-
$\mathbb{Z}_{p^6} \times \mathbb{Z}_{p^4}$	$3p^4-p^3$	-
$\mathbb{Z}_{p^7} \times \mathbb{Z}_{p^4}$	$4p^4-2p^3$	-
$\vdots$		
	$\vdots$	$\vdots$
$\mathbb{Z}_{p^n} \times \mathbb{Z}_p$	$(n-1)(p-1)+(p+1)$	$np+3$
$\mathbb{Z}_{p^n} \times \mathbb{Z}_{p^2}$	$[(n-2)(p-1)+(p+1)]p$	$(n-1)p^2+2(p+1)+1$
$\mathbb{Z}_{p^n} \times \mathbb{Z}_{p^3}$	$[(n-3)(p-1)+(p+1)]p^2$	$(n-2)p^3+2(p^2+p+1)+1$
$\vdots$	$\vdots$	-
$\mathbb{Z}_{p^n} \times \mathbb{Z}_{p^m}$	$[(n-m)(p-1)+(p+1)]p^{m-1}$	$(n-m+1)p^m+2(\frac{p^m-1}{p-1})+1$

Table 4

Group	Maximal Chains	Fuzzy Subgroups
$\mathbb{Z}_{p^n} \times \mathbb{Z}_{p^n}$	$(p+1)p^{n-1}$	$\sum_{k=1}^{n} 2^{k+1}(p^k + p^{k-1}) + 3$
$\mathbb{Z}_{p^n} \times \mathbb{Z}_{p^{n-1}}$	$2p^{n-1}$	$3\sum_{k=1}^{n-1} 2^{k+1}p^k + 7$
$\mathbb{Z}_{p^n} \times \mathbb{Z}_{p^{n-2}}$	$(3p-1)p^{n-3}$	$\sum_{k=0}^{n-1}(2^k - 1)p^2 + (2p+3)$
$\mathbb{Z}_{p^n} \times \mathbb{Z}_{p^m}$	$[(n-m)(p-1) + (p+1)]p^{m-1}$	-

Chapter 8

List of Figures



$G = \mathbb{Z}_p \times \mathbb{Z}_p$

fig.1

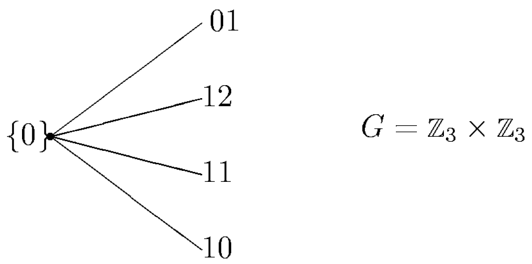

01
12
{0}
11
10
$G = \mathbb{Z}_3 \times \mathbb{Z}_3$

fig.2

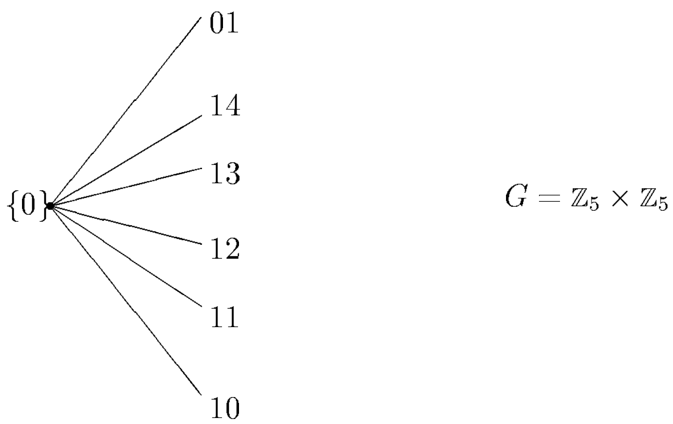

01
14
13
{0}
12
11
10
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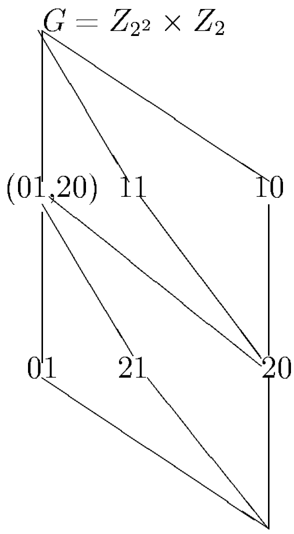
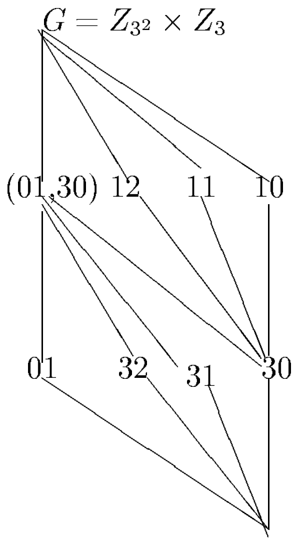
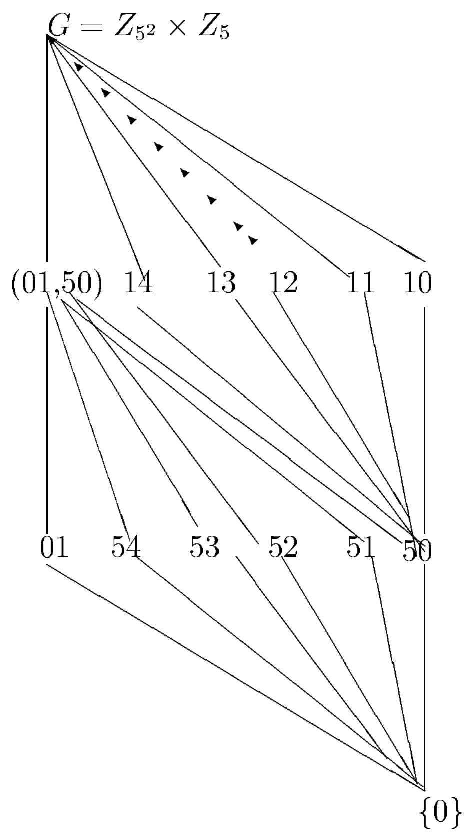
fig.3


$G = Z_{2^2} \times Z_2$
(01,20)
11
10
01
21
20


$G = Z_{3^2} \times Z_3$
(01,30)
12
11
10
01
32
31
30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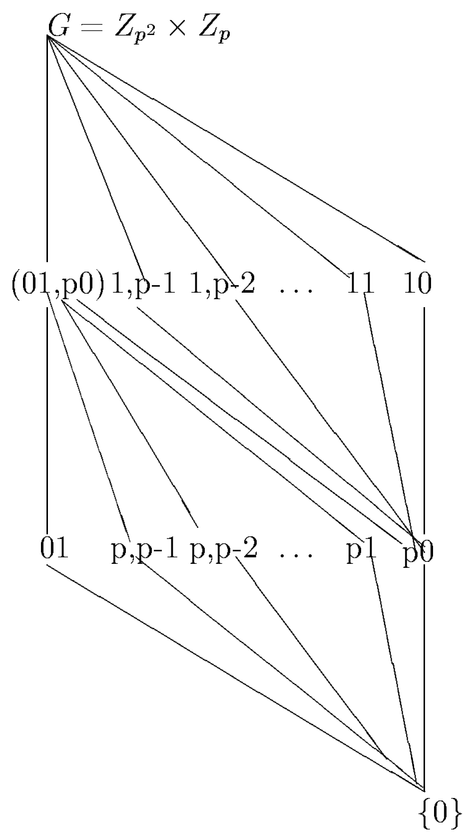
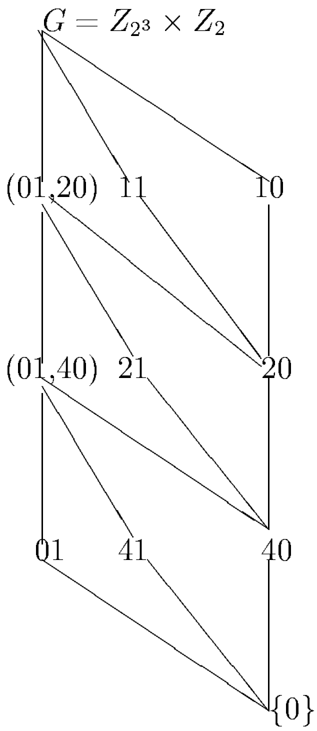
(01,50)
14
13
12
11
10
01
54
53
52
51
50
{0}


$G = Z_{p^2} \times Z_p$
(01,p0) 1,p-1 1,p-2 ... 11 10
01 p,p-1 p,p-2 ... p1 p0
{0}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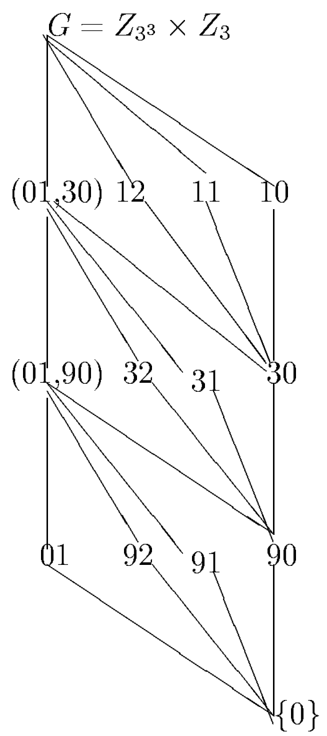
(01,20)
11
10
(01,40)
21
20
01
41
40
{0}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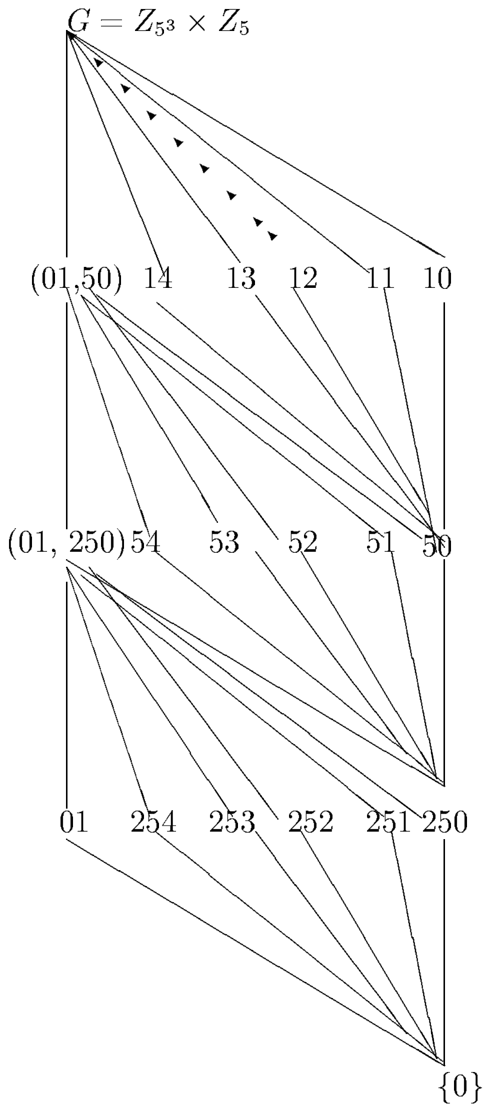
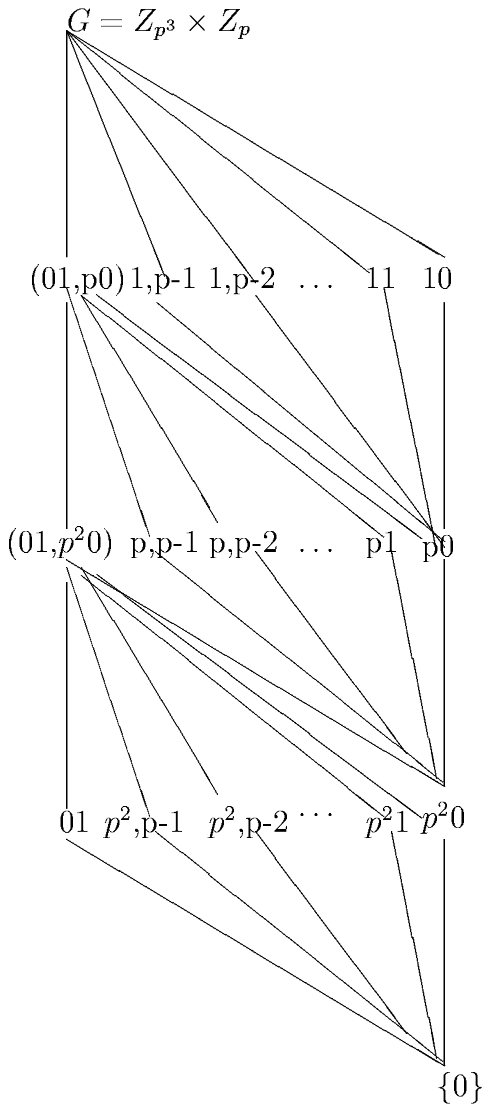
(01,30)
12
11
10
(01,90)
32
31
30
01
92
91
90
{0}


$G = Z_{5^3} \times Z_5$
(01,50)
14
13
12
11
10
(01, 250)
54
53
52
51
50
01
254
253
252
251
250
{0}


$G = Z_{p^3} \times Z_p$
(01,p0) 1,p-1 1,p-2 … 11 10
(01,$p^2$0) p,p-1 p,p-2 … p1 p0
01 p^2,p-1 p^2,p-2 ⋯ $p^2$1 $p^2$0
{0}

Chapter 9

Appendix: Classification of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{p^m}$ form

All the results in this chapter pertain to the case when all the nontrivial subgroups are cyclic.
The algorithm at the end is applicable for cyclic - subgroup case.

Note: In figures, the nodes that appear in one column fall in the same level. Such nodes represent those subgroups of G with the same order. Each subgroup is contained in the subgroup on its right, as indicated by the line.

9.1 Maximal chains of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{p^m}$

In this section we determine the number of maximal chains of $G = \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^m}$. We develop a method to represent the maximal chains in the form of diagrams and explain it using an algorithm. Each case of n and m is important in its own right and deserves its rightful treatment. We will, however, give a few cases and collect the rest of the results in tabular form.

9.1.1 Lemma

$G = \mathbb{Z}_p \times \mathbb{Z}_p$ has $p + 1$ maximal chains.

9.1.2 Lemma

$G = \mathbb{Z}_{p^2} \times \mathbb{Z}_p$ has 2p maximal chains.

9.1.3 Lemma

$G = \mathbb{Z}_{p^3} \times \mathbb{Z}_p$ has $3p - 1$ maximal chains.

Proof

The subgroups of G form maximal chains of the form

$$\{0\} \subset \; < a > \; \subset G; \{0\} \subset \; < a > \; \subset \; < b > \; \subset G$$

and

$$\{0\} \subset \; < a > \; \subset \; < b > \; \subset \; < c > \; \subset G,$$

where a, b, c are of the form (i, j) for $p \leq i \leq p^2$ and $1 \leq j < p$. The distinct nontrivial subgroups of G generated by a are all given by

$$(p, 1), (p^2, 1), (p^2, 2), \ldots, (p^2, p - 1).$$

The generator (p^2, p) is contained in $(p, 1), (p, 2), \ldots, (p, p-1)$ generated by b which together form the maximal chain

$$\{0\} \subset \; < a > \; \subset \; < b > \; \subset G.$$

The only subgroup not yet considered is (p, p) which contains (p^2, p) and is contained in the subgroups of the form

$$(1, p), (1, 1), (1, 2), \ldots, (1, p - 1)$$

which together constitute

$$\{0\} \subset \; < a > \; \subset \; < b > \; \subset \; < c > \; \subset G.$$

Example:$\mathbb{Z}_8 \times \mathbb{Z}_2, \ \ \mathbb{Z}_{27} \times \mathbb{Z}_3$

For any natural number n, we have the following general case:

9.1.4 Proposition

$G = \mathbb{Z}_{p^n} \times \mathbb{Z}_p$ has $(n-1)(p-1) + (p+1)$ maximal chains.

Proof

We shall prove by induction on n. For $n = 1, 2, 3$ the statement is true as seen in 9.1.1, 9.1.2, 9.1.3.

Assume that the statement is true for $n = k$, that is, assume that $G = \mathbb{Z}_{p^k} \times \mathbb{Z}_p$ has $(k-1)(p-1) + (p+1)$ maximal chains. We must prove that $G = \mathbb{Z}_{p^{k+1}} \times \mathbb{Z}_p$ has $k(p-1) + (p+1)$ maximal chains.

Now for each k, each $\mathbb{Z}_{p^k} \times \mathbb{Z}_p$ has one more $(p-1)$ set of maximal chains than $\mathbb{Z}_{p^{k-1}} \times \mathbb{Z}_p$ with the other 1 already counted under $p+1$. Hence $\mathbb{Z}_{p^{k+1}} \times \mathbb{Z}_p$ must have $(k-1)(p-1) + (p+1) + (p-1)$ maximal chains. This means that $\mathbb{Z}_{p^{k+1}} \times \mathbb{Z}_p$ has $(k-1+1)(p-1) + (p+1) = k(p-1) + (p+1)$ maximal chains, hence the statement is true for $n = k+1$.

Thus, by the principle of mathematical induction, the statement is true for all n.

The next lemma advances to the case when $m = 2$ and $n = 3$.

9.1.5 Lemma

$G = \mathbb{Z}_{p^3} \times \mathbb{Z}_{p^2}$ has $2p^2$ maximal chains.

Proof

The maximal chains formed by the subgroup generators of G are of the form

$$\{0\} \subset \ < a > \ \subset \ < b > \ \subset G;$$

and

$$\{0\} \subset \ < a > \ \subset \ < b > \ \subset \ < c > \ \subset G,$$

where a, b, c are of the form (i, j) for $p \le i \le p^2$ and $1 \le j < p$.
a generates $p+1$ distinct subgroups of G, p of which have the form (p^2, np) for $n = 1, 2, \ldots, p$ and 1 of the form $(0, p)$. Now each (p^2, np), for a particular n, is contained in p other generators of the form

$$(p, n), (p, n+p), (p, n+2p), (p, n+3p), \ldots \tag{9.1}$$

The two cases being treated here are $(n, p) = 1$ as well as $n = kp$ for some natural k. The latter suggests that each $(p, kp + p)$ is further contained in the nontrivial subgroups

$$(1, k+1), (1, (k+1)+p), (1, (k+1)+2p), \ldots \tag{9.2}$$

The generator that has not been accounted for yet is $(0, p)$ and is contained in

$$(0, 1), (p^2, 1), (p^2, 2), \ldots, (p^2, p-1). \tag{9.3}$$

Thus the subgroups $\{0\}, < (0, p) > \ and \ < (p^2, np) >, n \neq p$, together form the 4-component chains as explained in (9.1) and (9.3), and there are p^2 of these chains. The 5-component chains are formed by $\{0\}, < (p^2, np) >, n = p$, as suggested by (9.1) and (9.2), again p^2 of these.

Example: $\mathbb{Z}_8 \times \mathbb{Z}_4$.

9.1.6 Lemma

$G = \mathbb{Z}_{p^4} \times \mathbb{Z}_{p^3}$ has $2p^3$ maximal chains.

Example: $\mathbb{Z}_{16} \times \mathbb{Z}_8$.

9.2 The case when $n = m$

We need to classify all possible distinct fuzzy subgroups of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{p^m}$, where $m = n$. For clarity in induction, we prove at least three cases of n.

We first look at $\mathbb{Z}_p \times \mathbb{Z}_p$, where p is a prime. To do this we need the number of maximal chains of $\mathbb{Z}_p \times \mathbb{Z}_p$. For this, refer to lemma 5.2.1.

We make the following propositions:

9.2.1 Proposition

The number of distinct fuzzy subgroups of $\mathbb{Z}_p \times \mathbb{Z}_p$ is $4p + 7$.

9.2.2 Proposition

$G = \mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2}$ has $8p^2 + 12p + 7$ distinct fuzzy subgroups.

Proof

Since $\mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2}$ has maximal chains of 4 levels, as in example 4.2.2, the 4 symbols will give rise to 15 possible combinations. Of these 15, 3 will give rise to one fuzzy subgroup, and these are $1111, 1\lambda\lambda\lambda, 1000$. Of the remaining, 8 will give rise to distinct fuzzy subgroups. These are those that contain distinct last two symbols, e.g $111\lambda, 1\lambda\beta 0, etc.$ By the previous lemma, this means that there will be $8(p^2 + p)$ distinct fuzzy subgroups. The rest will give rise to non-distinct fuzzy subgroups, and there will be $p + 1$ of these. This gives the result.

9.2.3 Proposition

$G = \mathbb{Z}_{p^3} \times \mathbb{Z}_{p^3}$ has $16p^3 + 24p^2 + 12p + 7$ distinct fuzzy subgroups.

Proof

in every maximal chain of $\mathbb{Z}_{p^3} \times \mathbb{Z}_{p^3}$ there are 5 levels. As in the previous work, there will be 31 key chains, namely

$11111 \quad 1\lambda\lambda\lambda\lambda \quad 1111\lambda \quad 1\lambda\lambda\lambda\beta$

$11110 \quad 1\lambda\lambda\lambda 0 \quad 111\lambda\lambda \quad 1\lambda\lambda\beta\beta$

$111\lambda\beta \quad 1\lambda\lambda\beta\delta \quad 111\lambda 0 \quad 1\lambda\lambda\beta 0$

$11100 \quad 1\lambda\lambda 00 \quad 11\lambda\lambda\lambda \quad 1\lambda\beta\beta\beta$

$11\lambda\lambda\beta \quad 1\lambda\beta\beta\delta \quad 11\lambda\lambda 0 \quad 1\lambda\beta\beta 0$

$11\lambda\beta\beta \quad 1\lambda\beta\delta\delta \quad 11\lambda\beta\delta \quad 1\lambda\beta\delta\gamma$

$11\lambda\beta 0 \quad 1\lambda\beta\delta 0 \quad 11\lambda 00 \quad 1\lambda\beta 00$

$11000 \quad 1\lambda 000 \quad 10000$

Of these, 3 will give rise to one fuzzy subgroup. Of the remaining, 16 will give rise to distinct fuzzy subgroups, and these are those with no duplication of symbols in the last two positions e.g $11\lambda\lambda\beta$. Thus there will be $16(p^3 + p^2)$ distinct fuzzy subgroups. The 4 combinations with a symbol repeated three times at the end will give rise $p+1$ fuzzy subgroups. e.g $1\lambda\boxed{\beta\beta\beta}, 11\boxed{000}$. The rest will give rise to p^2+p fuzzy subgroups, and these have a duplicated symbol at the end, e.g $11\lambda 00, 111\boxed{\lambda\lambda}$. Thus the proposition is proved.

In general,

9.2.4 Proposition

$G = \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^n}$ has $\sum_{k=0}^{n} 2^{k+2}(p^{k+1} + p^k) + 3 = 4(p+1)[(2p)^n - 1]/(2p-1)$ distinct fuzzy subgroups.

Proof

In every maximal chain of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{p^n}$ there are $n+2$ levels. These $n+2$ levels yield $2^{n+2} - 1 = 2^{n+2} - 4 + 3$ key chains as noticed in the previous examples. Of these key chains, three will yield identical fuzzy subgroups, and these are $111\ldots 1, 1\lambda\lambda\ldots\lambda, 100\ldots 0$. 2^{n+1} will yield $p^n + p^{n-1}$ distinct fuzzy subgroups, and such key chains have no duplicated symbol at the last two positions. e.g $1\lambda\lambda\beta\ldots\beta\delta, 111\lambda\lambda\ldots\lambda\beta 0$.

Of the remaining, 2^n key chains with a pair of duplicated symbols at the end will yield $p^{n-1} + p^{n-2}$ distinct fuzzy subgroups. e.g $11\ldots 1\lambda\beta\delta\delta$. The process goes on with key chains of the form $11\lambda\lambda\lambda\beta\ldots\beta\gamma\delta\delta\delta; 1\lambda\lambda\lambda\beta\gamma\gamma\gamma\gamma$ etc. The last 2^2 key chains will give rise to $p+1$ distinct fuzzy subgroups, and these have a repetition of n symbols at the end. The key chains with $n+1$ repetition of symbols at the end have been considered.

Thus we have

$$2^{n+1}(p^n + p^{n-1} + 2^n(p^{n-1} + p^{n-2}) + 2^{n-1}(p^{n-2} + p^{n-3}) + \ldots 2^2(p+1) + 3$$

which can be seen to give $4(p+1)[\frac{(2p)^n-1}{2p-1}] + 3$ distinct fuzzy subgroups.

Now

$$\sum_{k=0}^{n} 2^{k+2}(p^{k+1} + p^k) + 3 = 4(p+1)[(2p)^n - 1]/(2p-1)$$

as can be checked out by the principle of mathematical induction.

{Assume that the statement is true for n. Show that the statement is true for $n+1$. This is achieved by adding $2^{n+2}(p^{n+1} + p^n)$ on both sides of the equality. Hence

$$4(p+1)[(2p)^n - 1]/(2p-1) + 2^{n+2}(p^{n+1} + p^n)$$

$$= 4(p+1)\{(2p)^n - 1 + (2p)^n(2p-1)\}/(2p-1)$$

$$= 4(p+1)[(2p)^{n+1} - 1]/(2p-1)$$

which gives the result}.

9.3 The case when $n \neq m$

As in section 5.1 we consider a number of cases of n and m.

We consider a specific case when $n = 3$ and $m = 1$.

9.3.1 Proposition

$G = \mathbb{Z}_{p^3} \times \mathbb{Z}_p$ has $28p + 7$ distinct fuzzy subgroups.

9.3.2 Proposition

$G = \mathbb{Z}_{p^3} \times \mathbb{Z}_{p^2}$ has $24p^2 + 12p + 7$ distinct fuzzy subgroups.

Proof

As we have remarked in 9.1.5, the group $\mathbb{Z}_{p^3} \times \mathbb{Z}_{p^2}$ has p^2 4-level and p^2 5-level maximal chains which together make up $2p^2$.

Now for the 5-level chains, the 5 symbols will yield 31 different combinations. 3 of these will give rise to one fuzzy subgroup, and these are $11111, 1\lambda\lambda\lambda\lambda$ and 10000. Four other combinations with triples at the end will each yield one fuzzy subgroup different from the above, and these are of the form $11\lambda\lambda\lambda, 1\lambda000$,etc. The combinations with doubles at the end will each give rise to p distinct fuzzy subgroups, and there are 8 such combinations hence $8p$ distinct fuzzy subgroups. The rest will yield p^2 distinct fuzzy subgroups. Thus so far we have $16p^2 + 8p + 7$ distinct fuzzy subgroups of G.

Turning to the 4-level maximal chains, we shall have 15 different combinations of symbols as expected, three of which have been accounted for already.

Of the remaining twelve, 4 will yield p distinct fuzzy subgroups, and these have doubles at the end. The rest will give rise to p^2 distinct fuzzy subgroups.

By collection, we have $(16 + 8)p^2 + (8 + 4)p + 4 + 3 = 24p^2 + 12p + 7$ distinct fuzzy subgroups as claimed in the proposition.

9.3.3 Proposition

The number of distinct fuzzy subgroups of $G = \mathbb{Z}_{p^4} \times \mathbb{Z}_{p^3}$, where p is a prime, is $48p^3 + 24p^2 + 12p + 7$.

Proof

The group $\mathbb{Z}_{p^4} \times \mathbb{Z}_{p^3}$ has two sets of maximal chains, the 6-level as well as the 5-level chains.

With the 6-level maximal chains we are in the situation of 63 different combinations of symbols,

$$1 \geq \lambda \geq \beta \geq \gamma \geq \delta \geq \eta \geq 0.$$

Of the 63, three will give rise to one fuzzy subgroup, and these are $111111, 1\lambda\lambda\lambda\lambda\lambda, 100000$.

The arrangements with quads[1] at the end will each yield one fuzzy subgroup different from the above, and there are four such arrangements. Those arrangements with triples at the end will each give rise to p distinct fuzzy subgroups, so we have $8p$ of those. Of the remaining arrangements, those with doubles at the end will each yield p^2 distinct fuzzy subgroups, and these are of the form

$$1111\lambda\lambda, 11\lambda\beta\gamma\gamma, 1\lambda\beta\gamma\delta\delta, \text{etc.}, 16 \text{ of these.}$$

The rest of the arrangements will each give rise to p^3 distinct fuzzy subgroups. We have, thus far, $32p^3 + 16p^2 + 8p + 4 + 3$.
Now for the 5-level chains, the symbols give rise to 31 different arrangements, three have been accounted for already, and these are $11111, 1\lambda\lambda\lambda\lambda, 10000$. The rest of the argument is similar to the case of the 6-level chains. That is, arrangements with triples yield p, those with doubles p^2 and the rest p^3 distinct fuzzy subgroups. There will be four, eight and sixteen, successively, of these distinct fuzzy subgroups.
Thus, collectively, there are

$$(32 + 16)p^3 + (16 + 8)p^2 + (8 + 4)p + 4 + 3 = 48p^3 + 24p^2 + 12p + 7$$

distinct fuzzy subgroups of $\mathbb{Z}_{p^4} \times \mathbb{Z}_{p^3}$ as claimed in the proposition.

9.4 Special Cases

Other interesting cases of n are given in the form of the following lemmas:

9.4.1 Lemma

$G = \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^{n-1}}$ has $2p^{n-1}$ maximal chains.

[1] a symbol repeated four times

9.4.2 Lemma

$G = \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^{n-2}}$ has $(3p-1)p^{n-3}$ maximal chains.

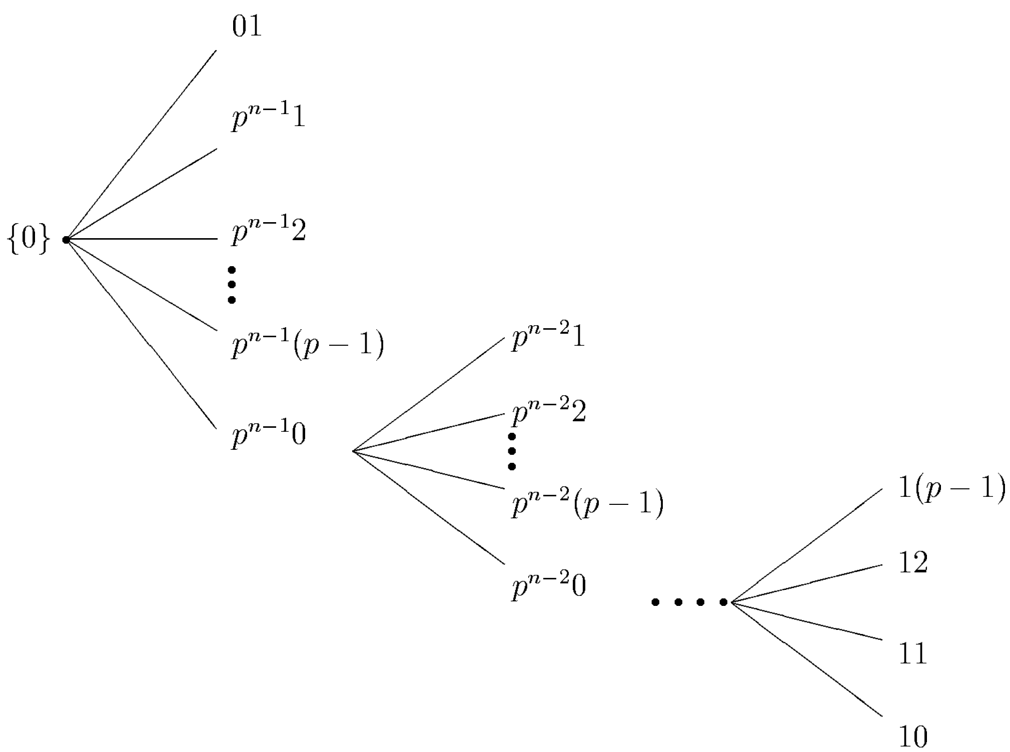
9.4.3 Proposition

$G = \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^{n-1}}$ has $3\sum_{k=1}^{n-1} 2^{k+1}p^k + 7 = 12p[(2p)^{n-1} - 1]/(2p-1) + 7$ distinct fuzzy subgroups.

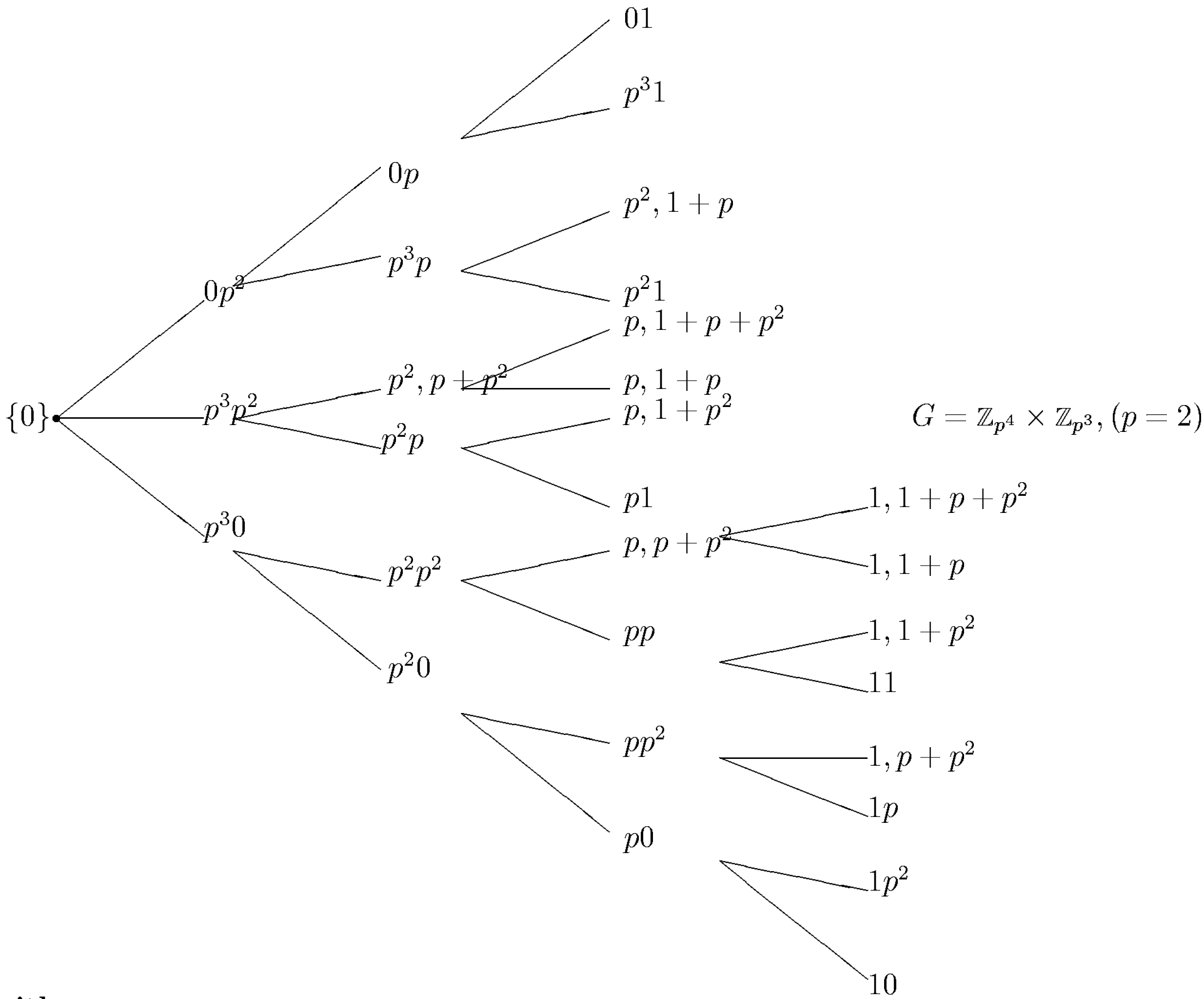
Examples: $\mathbb{Z}_{p^2} \times \mathbb{Z}_p$, $\mathbb{Z}_{p^4} \times \mathbb{Z}_{p^3}$

9.4.4 Proposition

$G = \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^{n-2}}$ has $\sum_{k=0}^{n-1}(2^k - 1)p^2 + (2p+3)$ distinct fuzzy subgroups.



$G = \mathbb{Z}_{p^n} \times \mathbb{Z}_p$



Algorithm

It follows then that to count the number of maximal chains of the group $\mathbb{Z}_{p^n} \times \mathbb{Z}_{p^m}$, for $n, m \in Z^+$, $n \geq m$, one can use the sketch of a tree diagram produced by use of the following algorithm:

1. Sketch $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^m}$ in the following manner:

(i) $\{0\}$ splits into $p+1$ branches, here referred to as main branches

(ii) Each branch splits into p branches

(iii) After $m-1$ times the process of splitting stops

2. (i) Extend the main branch or p^{m-1} of its branches to p branches

(ii) On the same wing, repeat (i), and so on

(iii) After $n - m$ times Stop.

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