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THE PRINCIPLE OF INCLUSION-EXCLUSION AND MÖBIUS FUNCTION AS COUNTING TECHNIQUES IN FINITE FUZZY SUBSETS.

by

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Abstract

The broad goal in this thesis is to enumerate elements and fuzzy subsets of a finite set enjoying some useful properties through the well-known counting technique of the principle of inclusion-exclusion. We consider the set of membership values to be finite and uniformly spaced in the real unit interval. Further we define an equivalence relation with regards to the cardinalities of fuzzy subsets providing the Möbius function and Möbius inversion in that context.

KEYWORDS:
Fuzzy sets, Posets, Chains, Lattices, Cardinality, Equivalence, Möbius Function, Möbius Inversion, Principle of inclusion and exclusion .

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Contents

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PREFACE

Since the seminal work of L. Zadeh on the fuzzy logic, there have been various attempts to widen the notion, extending it to other fields of mathematics and computer science. Many crisp concepts are being "fuzzified". One example of this is the concept of cardinality. The purpose of our study is to enumerate different entities of fuzzy subsets of a finite set. From that enumeration,we provide the Möbius Function and Möbius Inversion formula for the lattice of fuzzy subsets of a finite set.

In Chapter 1, we recall the well known Principle of Inclusion and Exclusion (PIE) in crisp sets theory. This method expresses the function of a union of finitely many sets as an alternating sum of function of their intersections. Later in Chapter 1, we propose and prove a dual formula of the usual PIE. This dual of PIE was mentioned no where in the literature we consulted.

In chapter 2 we provide the background material for the theory of partially ordered sets and lattices. We briefly discuss concepts such as length, height and width of a chain in posets and lattices.

It is in chapter 3 that we review some definitions and properties pertaining to fuzzy subsets, brushing a little the notions such as cardinality of a fuzzy subset and distance (Hamming) between two fuzzy subsets, and that of the number of fuzzy subsets at a distance d from a given one.

In chapter 4 we extend the usual PIE for crisp sets to the PIE in the poset

$\mathcal{F}(\mathcal{X})$ of fuzzy subsets of a finite set X. Here we enumerate elements of a finite set X using fuzzy subsets of X. Later we propose some enumerating functions of fuzzy subsets either using $\alpha - cuts$ or cardinalities. When considering fuzzy subsets as mere functions from X to $M \subset I$, we state and prove the theorem determining the number $(N_\mu \leq \alpha)$ of fuzzy subsets such that no membership value to the fuzzy subsets is above a real number α. In the course of this chapter we enumerate also the fuzzy subsets of a set with cardinality a real number p. This was first studied by B.Bouchon and G. Cohen in [2] . We use their idea and rewrite the result as a proposition using a new set of notations. We extend from this proposition to the enumeration of the fuzzy subsets of a set with cardinality greater than or equal to a real number p. We use the notion of cardinality and establish an equivalence relation in $\mathcal{F}(\mathcal{X})$ and provide a way of enumerating the equivalence classes for the said relation. At the end of this chapter we have a discussion on other applications of PIE in the lattice $M \subseteq [0,1]$ of membership values of fuzzy subsets of a set X where M is a finite set.

In Chapter 5 we review some definitions related to Möbius Function and Möbius Inversion and compare PIE to Möbius Inversion. Later in the chapter we define the Möbius function in the lattice of fuzzy subsets $\mathcal{F}(\mathcal{X})$ and using this function we do the Möbius inversion in the lattice $\mathcal{F}(\mathcal{X})$. For illustrative purpose, we include at the end of the thesis the diagram of the lattice of a 3-element set, the diagram of the lattice $\mathcal{F}(\mathcal{X})$ as well as the Zeta matrix and the Möbius matrix of the same set. With this tool we wish to illustrate the way the Möbius inversion operates in a $\mathcal{F}(\mathcal{X})$.

We have acknowledged accordingly sources of materials that we used in this thesis. Any other topic mentioned here is fruit of our humble contribution. For instance Theorem 1.2.1, Theorem 4.1.2, Theorem 4.2.3, Theorem 4.4.1, Proposition 4.4.5, Proposition 4.5.1, Proposition 4.6.3, Theorem 4.8.1, Theorem 5.4.4.

Chapter 1

The Principle of inclusion-exclusion

The Principle of inclusion-exclusion hereafter called (PIE) is a topic well-studied in Mathematics nearly for the last 150 years, see [16]. The idea occured as early as 1854 in a paper by Daniel da Silva (this was picked up by a "Google" search); later developed by Sylvester in 1883 in his paper. This principle is also known as " sieve formula". It is a counting technique used to count the elements of a specified set X satisfying a finite collection of dichotomous properties which are not necessarily mutually exclusive. Complimentarily, sometimes it is desirable to count the number of elements of X that do not have any of the properties. In this chapter we state the principle in its primary form, discuss its basic properties and write down a generalization of the principle. We also state the principle in a dual form using the duality of set union and intersection. We end the chapter illustrating how PIE is applied in specific counting problems.

1.1 The idea of PIE as a counting tool

Let S be a finite set and A be a non-empty subset of S. We write $\mid A \mid$ for the number of elements of A and $S \setminus A$ for those elements of S that do not belong to A. Since every element of S either belongs to A or does not belong to A exclusively, it is clear that $\mid S \setminus A \mid = \mid S \mid - \mid A \mid$.

Suppose A and B are non-empty subsets of a finite set S which are not necessarily disjoint from each other and we wish to enumerate the elements of S not in $A \cup B$. That is we have to compute $\mid S \setminus (A \cup B) \mid = \mid S \mid - \mid A \cup B \mid$ which is not $\mid S \mid - \mid A \mid - \mid B \mid$ because the number of elements in $A \cap B$ would have been subtracted twice; as part of A and as part of B. Therefore

$$\mid S \mid - \mid A \mid - \mid B \mid + \mid A \cap B \mid \tag{1.1.1}$$

is the correct number of elements of $S \setminus A \cup B$. This follows from the

Lemma 1.1.1 For the subsets A and B of S, $\mid A \cup B \mid = \mid A \mid + \mid B \mid - \mid A \cap B \mid$

Proof : Firstly it is clear that $\mid A \cup B \mid = \mid A \mid + \mid B \setminus A \mid$ and secondly $\mid A \cap B \mid + \mid B \setminus A \mid = \mid B \mid$. These two equations yield $\mid A \cup B \mid = \mid A \mid + \mid B \mid - \mid A \cap B \mid$. Hence the result.

We note that the dual identity $\mid A \cap B \mid = \mid A \mid + \mid B \mid - \mid A \cup B \mid$ to the above lemma is also true. The following theorem is the well known formula for PIE when n subsets of a set S are considered and is a generalization of Lemma 1.1.1. Its proof is based on mathematical induction on n.

Theorem 1.1.2 Let $A_1, A_2, \cdots A_n$ be subsets of a finite set S, Then

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} \mid A_i \mid - \sum_{1\leq i<j} \mid A_i \cap A_j \mid + \sum_{1\leq i<j<k} \mid A_i \cap A_j \cap A_k \mid$$
$$+ \cdots + (-1)^{n-1} \mid A_1 \cap A_2 \cap \cdots \cap A_n \mid \quad (1.1.2)$$

Alternatively taking complements in S we have

$$\left|S \setminus \bigcup_{i=1}^{n} A_i\right| = \mid S \mid - \Sigma_{i=1}^{n} \mid A_i \mid + \Sigma \mid A_i \cap A_j \mid + \cdots + (-1)^n \mid A_1 \cap A_2 \cap \cdots A_n \mid \quad (1.1.3)$$

That is to say that the cardinality of union $A_1 \cup A_2 \cup A_3 \cdots \cup A_n$ can be calculated by including (adding) the size of all the sets together, then excluding (subtracting) the size of the intersection of all pairs of sets, then including the size of the intersection of all triples, excluding the size of the intersection of all quadruples, and so on until finally the size of the intersections of all the sets has been included or excluded, as appropriate. Thus if n is odd it is included, and if n is even it is excluded.

The PIE can be stated in an other useful form with a different notation. Let $a_1, a_2, \cdots a_n$ be dichotomous properties enjoyed by elements of S and $A_1, A_2, \cdots A_n$ be n subsets of S. Then the subsets A_i's are described by $x \in A_i$ (or $x \notin A_i$) if and only if x has (or does not have) the property a_i for $i = 1, 2, \cdots n$. Let $N(a_i)$ and $N(a_i')$ denote the number of elements in A_i and the number of those elements not in A_i respectively. Now let $N(a_i a_j)$ represent the number of elements of S with two of the properties a_i and a_j while $N(a_i' a_j')$ will be used for the number of objects of S that have neither the property a_i nor the property a_j. Then the expression $\mid S \setminus A \cup B \mid = \mid S \mid - \mid A \mid - \mid B \mid + \mid A \cap B \mid$ can be captured in this notation as

$$N(a_1' a_2') = N - N(a_1) - N(a_2) + N(a_1 a_2)$$

with N being the number of elements in S.

In general the formula

$$\left| S \setminus \bigcup_{i=1}^{n} A_i \right| = \mid S \mid - \Sigma_{i=1}^{n} \mid A_i \mid + \Sigma \mid A_i \cap A_j \mid + \cdots + (-1)^n \mid A_1 \cap A_2 \cap \cdots A_n \mid \tag{1.1.4}$$

can be written as

$$N(a_1' a_2' ... a_n') = N - [N(a_1) + N(a_2) + ... + N(a_n)] + [N(a_1 a_2) + ... + N(a_{n-1} a_n]$$
$$+ ... + (-1)^n N(a_1 a_2 ... a_n) \tag{1.1.5}$$

which means taking the number of all elements of S, subtract the number of those with one property at a time; add the number of those with two properties and so on until the number of those with n properties are added or subtracted appropriately.

1.2 Dual of the usual PIE formula

In most text books or papers, PIE is expressed in the way theorem 1.1.2 is stated. But sometimes we find it useful to write in its dual form interchanging the roles of union and intersection of sets. The following is an expression of that dual form of the PIE formula.

Theorem 1.2.1 If $A_1, A_2, \cdots A_n$ are subsets of a finite set S, Then

$$\left|\bigcap_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} \mid A_i \mid \; - \sum_{1\le i<j} \mid A_i \cup A_j \mid + \sum_{1\le i<j<k} \mid A_i \cup A_j \cup A_k \mid$$

$$+ \cdots + (-1)^{n-1} \mid A_1 \cup A_2 \cup \cdots \cup A_n \mid \qquad (1.2.1)$$

Proof : We prove the theorem by induction on n. Firstly it is clear that from the usual PIE $\mid A \cup B \mid = \mid A \mid + \mid B \mid \; - \mid A \cap B \mid$, we have the following equation

$$\mid A \cap B \mid = \mid A \mid \; + \mid B \mid - \mid A \cup B \mid \qquad (1.2.2)$$

to be true. Thus the formula is valid for two subsets A and B, that is for $n = 2$. Now suppose the formula is true for the intersection of n subsets $A_1, A_2, \cdots A_n$ of S. That is :

$$\mid A_1 \cap A_2 \cdots \cap A_n \mid = \sum_{i=1}^{n} \mid A_i \mid - \sum_{1\le i<j} \mid A_i \cup A_j \mid + \sum_{1\le i<j<k} \mid A_i \cup A_j \cup A_k \mid +$$

$$\cdots + (-1)^{n-1} \mid A_1 \cup A_2 \cup \cdots \cup A_n \mid . \qquad (1.2.3)$$

Then we have to prove that the theorem is true for $n+1$. That is to show that

$$\mid A_1 \cap A_2 \cap \cdots \cap A_n \cap A_{n+1} \mid = \sum_{i=1}^{n+1} \mid A_i \mid - \sum_{1\le i\le j\le n+1} \mid A_i \cup A_j \mid + \cdots$$

$$+ \cdots + (-1)^{n} \mid A_1 \cup A_2 \cup \cdots \cup A_{n+1} \mid \qquad (1.2.4)$$

is true. Let $A = A_1 \cap A_2 \cap \cdots \cap A_n$, then the left side of the above expression becomes $\mid A \cap A_{n+1} \mid$ which gives rise to the following equality

$\mid A \cap A_{n+1} \mid = \mid A \mid + \mid A_{n+1} \mid - \mid A \cup A_{n+1} \mid$ as seen above in the case of two subsets. Firstly

$$\mid A \mid + \mid A_{n+1} \mid = \sum_{i=1}^{n+1} \mid A_i \mid - \sum_{1 \leq i < j \leq n} \mid A_i \cup A_j \mid + \sum_{1 \leq i < j < k \leq n} \mid A_i \cup A_j \cup A_k \mid + \cdots + (-1)^{n-1} \mid A_1 \cup A_2 \cup \cdots \cup A_n \mid \tag{1.2.5}$$

Now by using distributivity we get

$\mid A \cup A_{n+1} \mid = \mid (A_1 \cup A_{n+1}) \cap \cdots \cap (A_n \cup A_{n+1}) \mid$. Setting $A \cup A_{n+1} = A'$ and $A_i \cup A_{n+1} = A'_i$ for each i, we further obtain, again by inductive hypothesis,

$$\mid A' \mid = \mid A'_1 \cap \cdots \cap A'_n \mid = \sum_{i=1}^{n} \mid A'_i \mid - \sum_{1 \leq i < j \leq n} \mid A'_i \cup A'_j \mid + \sum_{1 \leq i < j < k \leq n} \mid A'_i \cup A'_j \cup A'_k \mid + \cdots + (-1)^{n-1} \mid A'_1 \cup A'_2 \cup \cdots \cup A'_n \mid \tag{1.2.6}$$

The rule of set union allows us to write $A'_{i_1} \cup A'_{i_2} \cup \cdots \cup A'_{i_k}$ as $A_{i_1} \cup A_{i_2} \cup \cdots \cup A_{i_k} \cup A_{n+1}$ for any subset of indices $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ where $1 \leq k \leq n$. Therefore

$$\mid A \cup A_{n+1} \mid = \sum_{i=1}^{n} \mid A_i \cup A_{n+1} \mid - \sum_{1 \leq i < j \leq n} \mid A_i \cup A_j \cup A_{n+1} \mid + \sum_{1 \leq i < j < k \leq n} \mid A_i \cup A_j \cup A_k \cup A_{n+1} \mid + \cdots + (-1)^{n-1} \mid A_1 \cup A_2 \cup \cdots \cup A_n \cup A_{n+1} \mid \tag{1.2.7}$$

Now subtracting each side of the equation 1.2.7. from the corresponding side of the equation 1.2.5., we see that the inductive process on n is valid. This completes the proof. □

1.3 Applications of the PIE

Among the many applications of PIE as a counting technique, we will focus on two, since they have some applications in our further work later on in the thesis, for instance, counting fuzzy subsets of finite sets satisfying some stringent conditions:

1.Enumeration of surjections of a n-element set M to a m-element set K;

2.Enumeration of derangements, which are permutations π of an n-element X such that $\pi(i) \neq i \ \forall i$

1°. The enumeration of surjections.

Let M and K be an m-element set and a k-element set respectively. A function $f : M \to K$ is a surjection if each element of K is image of some element x of M. The number of functions from M to K is k^m. This number is easily used to compute the number of surjections from M to K when K has only few elements. For example let us consider K to be $\emptyset, \{a\}, \{a_1, a_2\}$ respectively. If $K = \emptyset$, There is only one function, namely the empty function from M to K. If $K = \{a\}$, again there is only one function from M to K, but this function is a surjection. If $K = \{a_1, a_2\}$: There are 2^m functions from M to K. Of these one only skips a_1 and one only skips a_2. This means two of the 2^m functions are not surjections. Therefore $2^m - 2$ are surjections.

Generally consider that $K = \{a_1, a_2, \cdots a_k\}$. Let $p_1, p_2, \cdots, p_k$ be the properties that $a_1, a_2, \cdots a_k$ are not in the range of the function respectively. Also let $p_1^{'}, p_2^{'}, \cdots, p_k^{'}$ be the properties that $a_1, a_2, \cdots a_k$ are in the range of the function respectively.

We denote by $N(p_i)$ and $N(p_i^{'})$ the number of functions that do not have a_i in their range and the number of functions that do have a_i in their range respectively for $i = 1, 2, .$ Also we denote by $N(p_ip_j)$ and $N(p_i^{'}p_j^{'})$ the number of functions that do not have both a_i and a_j in their range and the number of functions that do have both a_i and a_j in their range respectively.

Using the PIE to enumerate functions that do have every element of K in their range we can write :$N(p_1^{'}p_2^{'}\cdots p_m^{'}) = N - [N(p_1) + N(p_2) + \cdots + N(p_k)] + [N(p_1p_2)+\cdots+N(p_ip_j+\cdots+N(p_{k-1}p_k)]-[N(p_1p_2p_3)+\cdots+N(p_ip_jp_k)+\cdots+ [N(p_1p_2\cdots p_k)]$. Where N is k^n. For each a_i not in the range there are $k-1$ choices for the value of the function at each element of the domain. Therefore there are $(k-1)^m$ functions. That is to mean that there are $\begin{pmatrix} k \\ 1 \end{pmatrix}(k-1)^m$ functions skipping one element of K. If two elements a_i and a_j are not in the range, then there are $k-2$ choices for the value of the function at each element of the domain. That means there are $(k-2)^m$ functions that skip any two elements of K. There are $\begin{pmatrix} k \\ 2 \end{pmatrix}(k-2)^m$ functions skipping any two elements of K. Since any function skipping two elements of K, skips at least one element of K. These functions are counted among the $N(p_1^{'}p_2^{'}\cdots p_m^{'})$. Continuing this way until none of the $a_i\forall i$ is in the range of the functions.That is to mean that $N(p_1^{'}p_2^{'}\cdots p_m^{'}) = 0$. In which case there are no such functions. Thus the number of surjections for an m-elements set M to a k-elements-set is $k^m - \begin{pmatrix} k \\ 1 \end{pmatrix}(k-1)^m + \begin{pmatrix} k \\ 2 \end{pmatrix}(k-2)^m + \begin{pmatrix} k \\ 3 \end{pmatrix}(k-3)^m + \cdots \pm \begin{pmatrix} k \\ k-1 \end{pmatrix}1^m$

When the above number is multiplied by $\frac{1}{k!}$, that is $\frac{1}{k!}(k^m - \begin{pmatrix} k \\ 1 \end{pmatrix}(k-1)^m +$

$\cdots \pm \begin{pmatrix} k \\ k-1 \end{pmatrix} 1^m)$, the resulting number is called Stirling Number of the second kind and is denoted $S(m, k)$.

2°. In the above example the range set K was not ordered in any fashion. But it is useful in our further work to have an ordering and more particularly a total ordering on K. So we impose that K is a totally ordered set with $a_1 \leq a_2 \leq \cdots \leq a_k$. We could count using PIE the number f of functions from M to K, each satisfying a property that all the elements of the range of the function exceed a certain specified element of K. With that specification, we associate a number p which is the number of elements exceeding the specified element. Without going through the details here (a proof will be supplied later), we state a formula based on PIE for such counting

$$f = k^n - \sum_{i=0}^{n} C(n, i)(-1)^{n-i} k^i p^{n-i}$$

Chapter 2

Partially ordered set and Lattices

In this Chapter we review some basic concepts of partially ordered sets and lattices in a way that will be suitable for use later in the thesis. We briefly discuss such concepts as length, height and width of a chain in posets and lattices.

2.1 Partially ordered sets.

A partially ordered set (often called poset) $(X; \leq)$ is a pair consisting of a set X and a binary relation $\leq$ on X, satisfying the following properties:
(a) reflexivity: $x \leq x \ \forall \ x \in X$,
(b) transitivity: If $x \leq y$ and $y \leq z$ then $x \leq z$ for $x, y, z \in X$, and
(c) anti-symmetry: If $x \leq y$ and $y \leq x$ then $x = y$ for $x, y \in X$.

The examples of posets are myriad and come in a variety of ways including integers under the usual ordering $\leq$, a non-empty set under inclusion $\subseteq$, The integers between 1 and a positive integer n under division $|$.

In a poset $(X;\leq)$, two elements x and y are *comparable* if either $x \leq y$ or $y \leq x$. Conversely elements x and y are said to be *incomparable*, if $x \leq y$ is false and $y \leq x$ is also false.

In the poset $(\mathcal{P}(\mathcal{X});\subseteq)$ with $X = \{x_1, x_2, x_3,\}$ under the partial order $\subseteq$, elements $\{x_1\}$ and $\{x_2\}$ are incomparable while $\{x_1\}$ and $\{x_1; x_2\}$ are comparable. Similarly 2 and 3 are comparable under usual $\leq$ while they are not comparable under division $|$. A *segment*, $[x, y]$, for x and y in a partially ordered set X, is the set of all elements z between x and y, that is $[x, y] = \{z \in X : x \leq z \leq y\}$.

The notion of *duality* is quite simple and useful in the discussion of posets. It is defined as that ordering denoted by $\geq$ which is related to $\leq$ by

$$x \leq y \text{ if and only if } y \geq x \text{ for all } x, y \in X.$$

If for every two elements x and y, either $x \leq y$ or $y \leq x$ is true, then $(X, \leq)$ is called linearly ordered set or a chain. Any finite chain has unique minimal and maximal elements, in the sense that there are elements 0 and 1 in a finite chain X with the property $0 \leq x \leq 1$ for all $x \in X$. For example in the poset $(\mathcal{P}(\mathcal{X});\leq)$, $\{\emptyset, \{x_1\}, \{x_1, x_2\}, \{x_1, x_2, x_3\}\}$ is a chain with minimal element Φ and maximal element $\{x_1, x_2, x_3\}$.

The set of non-empty chains of the poset X, denoted $Ch(X)$, ordered by inclusion pointwise, is itself a poset.

The *length* of chain C, denoted by $\mathcal{L}(\mathcal{C})$ is one less than the number of elements in C [4].

A set of elements in a poset, no two of which are comparable is called an *antichain*. Suppose A is an antichain in X such that any subset B containing A fails to be an antichain. In this case we call A a maximal antichain. The cardinality of a maximal antichain is called the *width* of $(X, \leq)$.

The *height* of an element $x \in X$ is the cardinality of the largest chain such that each element of the chain is strictly below x. The height of the poset $(X, \leq)$ is the largest height of any element of X and thus is 1 less the size of the largest chain. For example , the width of the poset $\mathcal{P}(\mathcal{X})$ of subsets of $X = \{x_1, x_2, x_3\}$ ordered by inclusion is 3. Its height is also 3, whereas the length of a chain may be 1,2,3 or 4.

In any partially ordered set, an element u is an upper bound for x and y if u is greater than or equal to x and y. This upper bound u is the least upper bound of x and y if for any v with $x \leq v$ and $y \leq v$, the element u is less than or equal to v. We write $u = x \vee y$ and also call u the "join" of x and y if u is the least upper bound of x and y.

Dually w is the greatest lower bound of x and y and we write $w = x \wedge y$ if $w \leq x$ and $w \leq y$, and whenever $v \leq x$ and $v \leq y$, then $v \leq w$. We call $x \wedge y$ the " meet" of x and y.

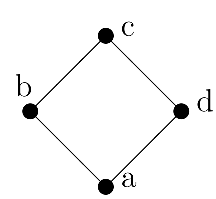
Let $v = (k_1, k_2, \cdots, k_n)$ and $v' = (k_1^{'}, k_2^{'}, \cdots, k_n^{'})$ be two ordered n-tuples in which the k_i and the $k_i^{'}$ belong to the same partially or totally ordered set $(K, \leq)$. If $\forall i = 1, 2 \cdots, n$ we have that $k_i^{'} \geq k_i$, we then say that v' dominates v and we write $v' \geq v$. The set containing the v's is therefore ordered by *dominance*.

A partially ordered set is locally finite if every interval has a finite number of elements. This means that for any element m in X there are finitely many elements n such that $n \leq m$. We shall only deal with locally finite partially ordered sets in this thesis.

2.2 Hasse diagram.

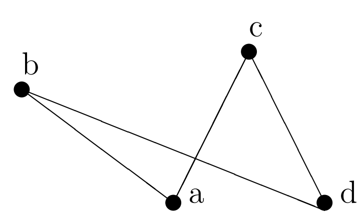
There are many ways of portraying posets. One of them is called the Hasse diagram. In this diagram, a point (also called vertex) is plotted for each object a; b is located higher on the paper whenever $a \leq b$. Secondly a and b are connected by a straight line segment (also called edge) if b covers a ($a \leq b$ and there is no z such that $a \leq z \leq b$). Because of the property of transitivity we do not include edge such as (a, c) since (a, b) and (b, c) guarantees its existence.

Figure 2.2.1 .



This is the Hasse Diagram of the power set $\mathcal{P}(X)$ of two element set $X = \{x_1, x_2\}$, where a is the empty set $\emptyset$, b is $\{x_1\}$ and d is $\{x_2\}$ while c is X.

Figure 2.2.2 .



This is a poset diagram but not a lattice diagram (See below) as there is no least upper bound and greatest lower bound for a and d.

In chapter 5 we have provided an example of the Hasse diagram of the poset of fuzzy subsets of set $\{x_1, x_2, x_3\}$ with membership values in the set $\{0, \frac{1}{2}, 1\}$.

2.3 Lattices

A *lattice* L is a partially ordered set in which any two elements have a "join" and a "meet". As before we write $\vee$ and $\wedge$ for join and meet respectively. Then a poset $(L, \leq)$ is a lattice if and only if $x \vee y$ and $x \wedge y$ both exist in L for all $x, y \in L$.
In a lattice, the *bottom* and the *top* elements are respectively 0 and 1 such that $0 \leq x \leq 1$ for all x in L. In an alternative notation we denote a lattice by $(L, \vee, \wedge)$.
The familiar examples of lattices include the number systems such as $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ and $\mathbb{R}$ with usual ordering less than or equal to, $\leq$. On the other hand a more interesting example which is useful in combinatorics is the following

Example 2.3.1 .
The set $\mathbb{N}$ under the divisibility ordering, namely, $x \leq y$ if and only if x divides y for $x, y \in \mathbb{N}$ is clearly a poset. In particular, the set $\mathcal{D}_n$ of all positive divisors of a fixed positive integer n under the same ordering of divisibility is a lattice. For instance in that ordering $x \wedge y = g.c.d(x, y)$ and $x \vee y = l.c.m(x, y)$ where $g.c.d(x, y)$ and $l.c.m(x, y)$ are the *greatest common divisor* and *least common multiple* of x and y respectively.

Lattices are posets but not all posets need be lattices. That is there are posets which are not lattices. This is due to the failure of the existence of either the meet or the join of any two elements in the poset. We have given in the previous section an example of a poset which is not a lattice.

Example 2.3.2 .

Another classic example of a lattice is the power set $\mathcal{P}(X)$ of a non-empt set X under the usual inclusion of subsets as partial order. In this lattice the join of two subsets S and T of X is the union $S \vee T = S \cup T$ and the meet is the intersection $S \wedge T = S \cap T$.

A lattice L is called a *complete lattice*, if $\sup U$ and $\inf U$ exist for any subset U of L where $\sup U = \bigvee_{u \in U} u = \sup\{u : u \in U \subseteq L\}$. The idea of $\inf U$ is defined similarly. When U is empty $\sup U$ and $\inf U$ are conventionally taken to be the bottom and top elements respectively.

Example 2.3.3 .

1. $\mathbb{Q}$ is not complete with the usual ordering since there is no top element.
2.$\mathbb{R} \cup \{\infty\} \cup \{-\infty\}$ is complete with the usual ordering.
3.$\mathcal{P}(\mathcal{X})$ is complete lattice under the usual inclusion with the top element X and the bottom element $\emptyset$.
4. Generally any finite lattice is complete.

The operations $x \vee y$ and $x \wedge y$, defined in a lattice L, are similar to the algebraic operations of $x + y$ and $x.y$. They satisfy some equations such as associativity, commutativity and idempotency. In this respect they can be considered as equational algebras (Varieties). Moreover the two operations individually and equivalently induce a partial order. Conversely, as we have already noted, the partial order of the lattice induces these two operations. This well known fact is summarized in the following theorem.

Theorem 2.3.4 Let $(L, \vee, \wedge)$ be a lattice. Then
(1) $(x \vee y) \vee z = x \vee (y \vee z)$ and $(x \wedge y) \wedge z = x \wedge (y \wedge z)$ $\forall x, y, z \in L$,
(2) $x \vee x = x = x \wedge x = x$, $\forall x \in L$,
(3) $x \vee y = y \vee x$ and $x \wedge y = y \wedge x$, $\forall x, y \in L$,
(4) $x \vee (x \wedge y) = x$ and $x \wedge (x \vee y) = x$, $\forall x, y \in L$.
Conversely any mathematical system L with two operations $\wedge$ and $\vee$ defined for all pairs of elements of L and satisfying the equations 1 through to 4 is a lattice with the partial ordering defined by $x \leq y$ if and only if $x = x \wedge y$ or equivalently $x \leq y$ if and only if $y = x \vee y$.

The distributivity, namely, $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ or equivalently $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ for all $x, y, z \in L$ is not generally satisfied in a lattice. If L has distributivity, then it is called a *distributive lattice*. It is an important class of lattices with many pleasant properties.

2.4 Complementation

Let a be an element of a lattice L with bottom element 0 and top element 1. An element $a^{'}$ is said to be a *complement* of a in L if $a \vee a^{'} = 1$ and $a \wedge a^{'} = 0$. A lattice L is said to be complemented when each $x \in L$ possesses a complement in L. A distributive lattice which is not complemented is called a $Vectorial\, lattice$. On the other hand a complemented distributive lattice is called a $Boolean\, Algebra$. For example , it is well known that the power set $(\mathcal{P}(\mathcal{X}), \leq, \vee, \wedge)$ of a non- empty set X is a complemented distributive lattice, thus a Boolean Algebra with top element X and bottom element ϕ with the usual set complementation. It is well known that the complement of an element in a lattice, when it exists, need not be unique. [20]. Many text books deal with complements and their properties in a nice fashion, see for instance, [20]. We reproduce two of the important properties below.

Lemma 2.4.1 The complement $a^{'}$ of a is the greatest element u in L such that $a \wedge u = 0$.

In fact if, if $a \wedge u = 0$, then $u = u \wedge 1 = u \wedge (a \vee a^{'}) = (u \wedge a) \vee (u \wedge a^{'}) = 0 \vee u \wedge a^{'} = u \wedge a^{'}$. That is $u = u \wedge a^{'}$ or that $u \leq a^{'}$.

Lemma 2.4.2 In a distributive lattice, complements are unique.

$Proof$: Suppose that $a^{'}$ and b were both complements of a in a distributive lattice L. Then $a \wedge (b \vee a^{'}) = (a \wedge b) \vee (a \wedge a^{'}) = 0 \vee 0 = 0$. This is to mean $a \wedge (b \vee a^{'}) = 0$. By the above lemma, $b \vee a^{'} \leq a^{'}$. This is not possible since $a^{'} \leq b \vee a^{'}$. Hence $b \leq a^{'}$. Reversing the roles of $a^{'}$ and b we get, $a^{'} \leq b$. Therefore there is only one complement $b = a^{'}$ for a.□

Chapter 3

Fuzzy Subsets

This chapter introduces the basic definitions, notations and operations for fuzzy subsets of a non-empty set relevant for our study of the principle of Inclusion-Exclusion and the associated concept of Mobius inversion. Research on fuzzy subsets has been underway for over 40 years now. It is therefore impossible to cover all aspects in this field nor it is necessary to look into all aspects since we are only interested in the applications of PIE to a collection of fuzzy subsets of a finite set. We merely aim to provide a summary of the basic concepts central to the study of fuzzy subsets and refer to various excellent text books available in the literature, [6].

3.1 Definition of a fuzzy subset

Let X be a nonempty set. A $fuzzy\, subset$, A of X is characterized by a membership function: $\mu_A : X \longrightarrow [\,0, 1\,] = I$ such that the number $\mu_A(x)$ in the unit interval I is interpreted as the degree of membership of element x to the fuzzy subset A for each $x \in X$. The set X is referred to as the universe of discourse.

Every $\{0, 1\}$ -valued fuzzy subset with membership function taking only either 0 or 1 is called a crisp subset, that is just a subset of X in the usual sense of the term. This means each object x of X either belongs to A when the degree of membership is 1 or does not belong to the subset A (membership-0) of X. Therefore we can identify a subset A with its characteristic function $\chi_A : X \to I$ such that

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{otherwise, that is, } x \notin A \end{cases}$$

for all $x \in X$.

Unlike the crisp subset, a fuzzy subset expresses the degree to which an element belongs to the fuzzy subset. Hence the extended characteristic function of A, $\chi_A : X \to I$ of a genuine fuzzy subset A that is not crisp is allowed to have values strictly between 0 and 1, which denotes the degree of membership, that is partial membership of an element in a given set.

The fuzzy subset A is completely determined by the set of tuples: $A = \{(x, \mu(x))\}, x \in X\}$ provided either X is finite or countably infinite. For example,

Example 3.1.1 .

Let $X = \{x_1, x_2, x_3, x_4, \}$ such that $\mu(x_1) = 0.2$, $\mu(x_2) = 0$, $\mu(x_3) = 0.3$, $\mu(x_4) = 0.8$. Therefore $\mu_A = \{(x_1; 0.2), (x_2; 0), (x_3; 0.3), (x_4; 0.8)\}$ is a fuzzy subset of X with a four-tuple representation.

The set such as $M = \{0.2; 0; 0.3; 0.8\}$ in the above case is called the membership set of the fuzzy subset μ_A.

3.2 Family of fuzzy subsets of a set.

Throughout the remainder of this thesis $X = \{x_1, x_2, \cdots, x_n\}$ is a finite set with $1 \leq n$ elements and all fuzzy subsets μ of X take n membership values not all necessarily distinct and hence take m values with $1 \leq m \leq n$. The membership values in the interval $I = [0, 1]$ are taken to be uniformly spaced, with the usual ordering, given by $M_m = \{0, \frac{1}{m-1}, \frac{2}{m-1}, \cdots, \frac{m-1}{m-1} = 1\}$. This uniform choice of values in M_m does not affect the counting of fuzzy subsets with special property and also is in line with preferential equality discussed elsewhere, [10].

The family of all fuzzy subsets in X is denoted by $\mathcal{F}(\mathcal{X})$ or I^X

Here it is useful to have the notation $\mid X \mid$ to stand for the cardinality of a set X. In general if $\mid X \mid= n$ and $\mid M \mid= m$, then there are m^n possible fuzzy subsets in total which is $\mid \mathcal{F}(\mathcal{X}) \mid= m^n$.

$\mathcal{F}(\mathcal{X})$ is finite if both X and M are finite.

Note that the set $\mathcal{P}(\mathcal{X})$ of crisp subsets of X has 2^n elements.

Example 3.2.1 .

$X = \{x_1, x_2, x_3\}$ and $M = \{0, \frac{1}{2}, 1\}$. There are $27 = 3^3$ distinct fuzzy subsets as members in $\mathcal{F}(\mathcal{X})$ among which $2^3 = 8$ are crisp subsets of X.

3.3 Operations on Fuzzy Subsets.

In this section we extend some of the operations of crisp set theory such as inclusion, intersection, union and complementation. These extensions are done in such as way that the extended operations restricted to two-valued subsets, namely, crisp subsets coincide with the usual operations.

1°. Inclusion.

Let μ , $\lambda \in \mathcal{F}(\mathcal{X})$ be two fuzzy subsets of X. μ is said to be *included* in λ (or, equivalently, μ is *contained* in λ, or μ is smaller than or equal to λ) if and only if $\forall x \in X, \quad \mu(x) \leq \lambda(x)$. We express the containment or inclusion also as *domination* or *dominance* in the sense that λ dominates μ if and only if $\forall x \in X, \quad \lambda(x) \geq \mu(x)$. Thus, clearly the set $(\mathcal{F}(\mathcal{X}), \leq)$ is a partially ordered set. This means that the relation $\leq$ defined on $\mathcal{F}(\mathcal{X})$ is reflexive, anti-symmetric and transitive since the ordering is the point-wise usual ordering of real numbers. Thus we may view the partial orders containment and dominance as dual to each other on the set of all fuzzy subsets.

Let μ and λ be two fuzzy subsets of a set X. Then μ is said to be *equal* to λ ($\mu = \lambda$) if and only if $\mu \leq \lambda$ and $\lambda \leq \mu$.

We say that λ is strictly contained in μ written as $\lambda < \mu$ in the sense that there is at least one $x \in X$ for which $\lambda(x) < \mu(x)$.

2°. Intersection.

Let μ and λ be two fuzzy subsets of a set X. The *intersection* of μ and λ is a fuzzy subset γ defined as:

$\gamma(x) = (\mu \cap \lambda)(x) = \min(\mu(x), \lambda(x)) = \mu(x) \wedge \lambda(x)\ \forall\ x$ in X. The fuzzy subset γ is sometimes denoted by $(\mu \wedge \lambda)$. Two fuzzy subsets μ and λ are disjoint if and only if $(\mu \wedge \lambda)(x) = 0$ for all $x \in X$. That means wherever $\mu(x) = 0$, then $\lambda(x) \neq 0$ and vice versa. The intersection of μ and λ is the " largest" fuzzy subset which is contained in both μ and λ.

3°. Union.

The *union* of μ and λ is a fuzzy subset γ defined as: $\gamma(x) = (\mu \cup \lambda)(x) = \max$ $(\mu(t), \lambda(x)) = \mu(x) \vee \lambda(x)\ \forall\ x$ in X. The fuzzy subset union γ can be denoted as $(\mu \vee \lambda)$. The union of μ and λ is the "smallest" fuzzy subset that contains both μ and λ.

4°. Complementation.

Let μ and λ be two fuzzy subsets of a set X. The two fuzzy subsets are said to be complementary if $\forall x \in X; \lambda(x) = 1 - \mu(x)$ or $\mu(x) = 1 - \lambda(x)$. In case λ is complement of μ, we write $\lambda = \overline{\mu}$.

Remark 3.3.1

1. A fuzzy subset μ has a unique complement $\overline{\mu}$ by definition of the complement of a fuzzy subset. In fact if we assume that λ and η were two complements of μ. Then by definition of complement, $\lambda(x) = 1 - \mu(x)$ and $\eta(x) = 1 - \mu(x)$. This says $\lambda(x) = \eta(x)$ and means that μ has only one complement.

2. In general the intersection of a fuzzy subset and its complement is not the empty fuzzy subset. Suppose μ be a fuzzy subset which is not a crisp subset. Let $\overline{\mu}$ be its complement. Now assume $\forall\ x\ \ \in X$,
$min(\mu(x), \overline{\mu}(x)) = min(\mu(x), (1 - \mu(x)) = 0$. Then for every x either $\mu(x) = 0$ or $1 - \mu(x) = 0$. But since μ is not a crisp subset, there is at least one $x \in X$

such that $0 < \mu(x) < 1$. This implies $1 - \mu(x)$ is also strictly between 1 and 0. This is a contradiction. So we can conclude the intersection of a fuzzy subset and its complement is the empty fuzzy set if and only if the fuzzy subset is a crisp subset. See [14]

The following is an example of a fuzzy set whose intersection with its complement is not empty fuzzy set.

Example 3.3.2 .

Consider a fuzzy subset μ such that:
$\mu = \{(x_1/0.13); (x_2/0.61); (x_3/0); (x_4/0); (x_5/1); (x_6/0.03)\}$of a finite set $X = \{x_1; x_2, x_3, x_4, x_5, x_6\}$ and $\overline{\mu} = \{x_1/0.87); (x_2/0, 39); (x_3/1); (x_4/1); (x_5/0); (x_6/0.97)\}$
$\mu \cap \overline{\mu} = \{(x_1/0.13); (x_2/0.39); (x_3/0); (x_4/0); (x_5/0); (x_6/0.03)\}$ which is not empty.

Note 3.3.3

1. The complementation used for fuzzy subsets is not the same as the complementation of Boolean lattice. The only time these two coincide is when the membership set $M = \{0, 1\}$.
2.The lattice $(\mathcal{F}(\mathcal{X}), \leq)$ is distributive but not complementary,that is, a vectorial lattice and not a Boolean lattice.
3. We observe that the union of a fuzzy subset and its complement is not the universal set X.

Example 3.3.4 .

Consider again the $\mu = \{(x_1/0.13); (x_2/0.61); (x_3/0); (x_4/0); (x_5/1); (x_6/0.03)\}$

and $\overline{\mu} = \{(x_1/0.87); (x_2/0,39); (x_3/1); (x_4/1); (x_5/0); (x_6/0.97)\}$ as in the example above.
$\mu \cup \overline{\mu} = \{(x_1/.87); (x_2/0.61); (x_3/1); (x_4/1); (x_5/1); (x_6/0.97)\}$ which does not represent X. In addition $\forall \mu, \lambda \in (F(X), \leq)$, $\mu \wedge \lambda$; $\mu \vee \lambda$ exist in $(F(X), \leq)$ as we have defined above. Therefore $(\mathcal{F}(\mathcal{X}), \leq)$ is a lattice.
Furthermore this lattice is bounded because $\forall \mu \in (\mathcal{F}(\mathcal{X}), \leq)$, $\exists \lambda$ such that $\lambda \leq \mu$. This λ is defined as the empty fuzzy subset of X such that $\chi_{\emptyset}(x) = 0$ $\forall x \in X$ and $\forall \mu \in (\mathcal{F}(\mathcal{X}), \leq)$, $\exists \lambda'$ such that $\mu \leq \lambda'$. This λ' is the fuzzy subset χ_X.
5°. Difference.
Let μ and λ be two fuzzy subsets of a set X, with $\overline{\mu}$ the complement of μ. The fuzzy subset difference of λ and μ, noted $(\lambda - \mu)$ is defined as $\lambda \cap \overline{\mu}$.

Note 3.3.5

Throughout the remaining part of this thesis we will represent a fuzzy subset $\mu = \{(x_1, \mu(x_1), (x_2, \mu(x_2)), \cdots (x_n, \mu(x_n)\}$ of X simply by writing the respective membership values with a particular ordering of the elements of the set X as $\mu(x_1)\mu(x_2)\cdots\mu(x_n)$. This is done in order to simplify the identification of fuzzy subsets. For instance $\frac{1}{2}01$ means for the fuzzy subset that

$$\mu(x_i) = \begin{cases} \frac{1}{2}, & \text{if i = 1} \\ 0, & \text{if i = 2} \\ 1, & \text{1 if i = 3} \end{cases}$$

for all $x_i \in X$. [6]
The above is a simplification of Kaufmann's notation $\{(x_1|\frac{1}{2}); (x_2|0); (x_3|1)\}$.

3.4 The Hamming distance

The notion of distance is quite an interesting one. We wish to define it with regards to the notion of fuzzy subsets. Later on we will attempt to explore the idea of the counting of fuzzy subsets of a finite set X, at a distance from a fixed one.

The distance "d" between two fuzzy subsets μ and λ of a set X can be expressed in two different ways: A linear distance, also called Generalized Hamming distance and a quadratic distance. See [6]

$d_1(\mu, \lambda) = \sum_{i=1}^{n} \mid \mu(x_i) - \lambda(x_i) \mid$ is the expression for linear distance (generalized Hamming distance),

and the quantity $d_2(\mu, \lambda) = \sqrt{\sum_{i=1}^{n} (\mu(x_i) - \lambda(x_i))^2}$ is the expression of the quadratic distance or Euclidian distance while $\sum_{i=1}^{n} (\mu(x_i) - \lambda(x_i))^2$ is called the Euclidean norm.

One can verify that these two definitions of distance satisfy the necessary conditions of distance. That is to say that for any three fuzzy subsets μ, λ and α of X:

1. $d(\mu, \lambda) \geq 0$
2. $d(\mu, \lambda) = d(\lambda, \mu)$
3. $d(\mu, \alpha) \leq d(\mu, \lambda) + d(\lambda, \alpha)$

In addition to these three conditions $0 \leq d_1(\mu, \lambda) \leq n$ for any two fuzzy subsets of a n-element set X.

In fact $0 \leq d(\mu, \lambda)$ by definition of distance. The second inequality is justified by definition of $Card_\mu$ dealt with in Chap 4.

Example 3.4.1 .

Consider $\mu = \{(x_1/0.87);(x_2/0.39);(x_3/1);(x_4/1);(x_5/0);(x_6/0.97)\}$ and $\lambda = \{(x_1/0.2);(x_2/0);(x_3/0);(x_4/0.6);(x_5/0.8);(x_6/1)\}$. Then $d(\mu, \lambda) = \mid 0.87 - 0.2 \mid + \mid 0.39 - 0 \mid + \mid 1 - 0 \mid + \mid 1 - 0.6 \mid + \mid 0 - 0.8 \mid + \mid 0.97 - 1 \mid = 0.67 + 0.39 + 1 + 0.4 + 0.8 + 0.03 = 3.29$

To end this section we wish to ask a question, namely, how many fuzzy subsets of a finite set are there at a given distance from a specified fuzzy subset? A search in google revealed that very few people are interested in such a question, but we feel that it is an important one.

Let μ be a fuzzy subset a finite set X. How many fuzzy subsets are at a distance d from a fixed fuzzy subset μ ?

We attempt answering the question by first considering the case of two fuzzy subsets λ and γ, both at the distance d from μ. How are λ and γ related ?

Using linear distance formula we can write that:

$d_1(\mu, \lambda) = \mathrm{d} = d_1(\mu, \gamma)$ and that $d_1(\lambda, \gamma) \leq d_1(\lambda, \mu) + d_1(\mu, \gamma)$ or $d_1(\lambda, \gamma) \leq 2d_1(\mu, \gamma)$. This means that γ is such that $d_1(\lambda, \gamma) \leq 2d_1(\mu, \lambda)$.

3.5 The α- cut

The weak α - cut or also called α- level set of a fuzzy subset μ of a set X is a crisp subset of X denoted by μ^α where $\mu^\alpha = \{x \in X/\ \mu(x) \geq \alpha\}\, \forall \alpha \in [0, 1]$. μ^α can also be defined as $\mu^{-1}([\alpha, 1])$ [22]. The complement of μ^α is the set

$\{x \in X /\ \mu(x) < \alpha\}\, \forall \alpha\ \in\ [0,1]$.

Theorem 3.5.1 If μ_1 and μ_2 are two fuzzy subsets of a set X, then
(1): $(\mu_1 \wedge \mu_2)^\alpha = \mu_1^\alpha \cap \mu_2^\alpha$ and
(2): $(\mu_1 \vee \mu_2)^\alpha = \mu_1^\alpha \cup \mu_2^\alpha$

$Proof$: Let $A = (\mu_1 \wedge \mu_2)^\alpha$ and $B = \mu_1^\alpha \cap \mu_2^\alpha$ be two subsets of X. We show that $A = B$ by showing $A \subseteq B$ and $B \subseteq A$.
Let $a \in A$, that is $a \in (\mu_1 \wedge \mu_2)^\alpha$, then $(\mu_1 \wedge \mu_2)(a) \geq \alpha$. That means $\mu_1(a) \wedge \mu_2(a) \geq \alpha$. It follows that both $\mu_1(a) \geq \alpha$ and $\mu_2(a) \geq \alpha$. In other words it means $a \in \mu_1^\alpha$ and $a \in \mu_2^\alpha$. That is to say $a \in \mu_1^\alpha \cap \mu_2^\alpha = B$. Which concludes that $a \in B$ and therefore $A \subseteq B$.
Conversely let $x \in B = \mu_1^\alpha \cap \mu_2^\alpha$; then $x \in \mu_1^\alpha$ and $x \in \mu_2^\alpha$. That is to say that $\mu_1^\alpha(x) \geq \alpha$ and $\mu_2^\alpha(x) \geq \alpha$ or that $\mu_1(x) \wedge \mu_2(x) \geq \alpha$ which also means $(\mu_1 \wedge \mu_2)(x) \geq \alpha$. This expresses the fact that $x \in (\mu_1 \wedge \mu_2)^\alpha = A$ and that $B \subseteq A$. We conclude therefore that $A = B$. That is $(\mu_1 \wedge \mu_2)^\alpha = \mu_1^\alpha \cap \mu_2^\alpha$
Similarly it is clear to prove that $(\mu_1 \vee \mu_2)^\alpha = \mu_1^\alpha \cup \mu_2^\alpha.\square$.

Theorem 3.5.2
1.A fuzzy subset μ_1 of X is greater than another fuzzy subset μ_2 of X if *all* of the α-cut of μ_1 contains the α-cut of μ_2.
2.A fuzzy subset μ_1 of X is equal to another fuzzy subset μ_2 of X if *all* of the α-cut of μ_1 is equal to the α-cut of μ_2.
3.A fuzzy subset μ_1 of X is smaller than another fuzzy subset μ_2 of X if the α-cut of μ_1 is contained by the α-cut of μ_2. [23]

Proof: Let $\mu_1 \geq \mu_2$. Then $\forall x \in X$, $\mu_1(x) \geq \mu_2(x)$. Show that any α-cut of μ_1 contains the α-cut of μ_2 that is $\mu_2^\alpha \subseteq \mu_1^\alpha$.

Now consider $\alpha \in I = [0, 1]$, such that $x \in \mu_2^\alpha$, that is $\mu_2(x) \geq \alpha$ which also means by transitivity of $\leq$, for the same α, $\alpha \leq \mu_2(x) \leq \mu_1(x)$, as per our hypothesis $\mu_1(x) \geq \mu_2(x)$. Therefore $\mu_1(x) \geq \alpha$ or that $x \in \mu_1^\alpha$. Which conclude that $\mu_2^\alpha \subseteq \mu_1^\alpha$.

Again, let $\mu_1 = \mu_2$. That is to say $\forall x \in X$, $\mu_1(x) = \mu_2(x)$. We show that any α-cut of μ_1 is an α-cut of μ_2. Consider $x \in \mu_1^\alpha$. That is $\mu_1(x) \geq \alpha$. By the equality $\mu_1(x) = \mu_2(x)$ we have that $\mu_2(x) \geq \alpha$ or $x \in \mu_2^\alpha$ and conclude that $\mu_1^\alpha \subset \mu_2^\alpha$. In the same way we show that $\mu_2^\alpha \subset \mu_1^\alpha$ and prove that $\mu_2^\alpha = \mu_1^\alpha$.

Finally Let $\mu_1 < \mu_2$. Then $\forall x \in X$, $\mu_1(x) < \mu_2(x)$. Now if $x \in \mu_1^\alpha$, then $\alpha \leq \mu_1(x)$. By transitivity $\alpha \leq \mu_2(x)$ and therefore $x \in \mu_2^\alpha$. The α-cut of μ_1 is contained in the α-cut of μ_2.

The characteristic function of the alpha cut of μ is denoted by χ_{μ^α}.

Proposition 3.5.3 .

Any fuzzy subset μ of X can be represented by its α-cuts as follows:

$$\mu = \bigvee\{\alpha\chi_{\mu^\alpha} : 0 < \alpha < 1\}.$$

Proof:

For any fixed $x \in X$, we need to prove that

$\mu(x) = \bigvee_\alpha(\alpha\chi_{\mu^\alpha})(x) \qquad *$

The R.H.S. of * is equal to $(\bigvee \alpha\chi_{\mu^\alpha})(x) = \bigvee(\alpha\chi_{\mu^\alpha})(x)$.

If $x \in \mu^\alpha$ then $\alpha\chi_{\mu^\alpha}(x) = \alpha$, otherwise it is 0.

Suppose $x \in \mu^\alpha$ then $\mu(x) \geq \alpha$. Also if $\mu(x) \geq \alpha$ then $x \in \mu^\alpha$.

The above imply that $x \in \mu^\alpha$ if and only if $\mu(x) \geq \alpha$.
Also if $\mu(x) > \alpha$ implies that $\alpha\chi_{\mu^\alpha}(x) = \alpha$.
Therefore $\mu(x) \geq \alpha\chi_{\mu^\alpha}(x)$ for every α such that $\mu(x) > \alpha$.
Thus $\mu(x) \geq \bigvee(\alpha\chi_{\mu^\alpha})(x)$. The L.H.S of * > R.H.S of *.

Conversely for a fixed $x \in X$, $\mu(x) = \beta$. If $\mu(x) = \beta < \alpha < 1$ then $\alpha < 1$, $\alpha\chi_{\mu^\alpha}(x) = 0$. Also if $0 < \alpha \leq \beta = \mu(x)$ then $\alpha\chi_{\mu^\alpha}(x) = \alpha$.
Therefore $\bigvee(\alpha\chi_{\mu^\alpha})(x) = \bigvee \alpha \leq \beta = \mu(x)$. We claim the equality holds.
Suppose not. Then $\bigvee \alpha\chi_{\mu^\alpha}(x) < \beta = \mu(x)$.
If we choose φ such that $\bigvee(\alpha\chi_{\mu^\alpha})(x) < \varphi < \beta = \mu(x)$ then $\varphi\chi_{\mu^\varphi}(x) = \varphi$
Thus $\bigvee(\alpha\chi_{\mu^\alpha})(x) < \varphi$ is a contradiction. Therefore $\bigvee \alpha\chi_{\mu^\alpha} = \mu$.

3.6 Cardinality of a fuzzy subset

Let X be a set. The cardinality of a subset $A \subseteq X$ is the number of elements of X contained in A. The cardinality of a crisp set is a natural number except when the set is empty in which case we assign zero. That is the cardinality of the empty set is zero.
Let X be an n-element set and let M be a subset of $[0, 1]$ to be defined suitably in the context of discussions, for instance see chapter 4 or 5 or in this chapter section 2. We define below the cardinality of a fuzzy set. It was first defined by Kaufmann in [6].

Definition 3.6.1 Suppose μ is a fuzzy subset of X. Then the *cardinality* of the fuzzy subset μ of X, denoted $\mid \mu \mid$ is defined as $\sum_{1}^{n} \mu(x_i) \; \forall \; x_i \in X$. We normally restrict the membership values of μ to the subset M of I.

This number is not necessarily a natural number. It is a sum of some real numbers in the interval $[0, 1]$. In practice we only allow membership values to be rational numbers or more useful numbers such as $\frac{1}{n}$ or $\frac{m}{n}$ for $1 \leq m \leq n$. Therefore the cardinality for all practical purposes is a manageble rational number being the sum of a finite number of rational numbers.
The cardinality of a finite fuzzy subset is finite. Generally the cardinality of a fuzzy set of an infinite set is infinite according to our definition above. But by assigning suitable membership values to certain elements of an infinite fuzzy subset, we can make the cardinality of an infinite fuzzy subset to be finite. The cardinality of a crisp subset A of X coincides with the concept of the cardinality of a fuzzy subset when we assign to each element of A the membership value 1 and assign 0 to other elements of X not in A.

Example 3.6.2 .
$\mu : \{(x_1; 0.8), (x_2; 0.7), (x_3; 0.5), (x_4; 1)\}$

$$\mid \mu \mid = \sum_{1}^{4} \mu(x_i) = 0.8 + 0.7 + 0.5 + 1 = 3$$

Example 3.6.3 .
Consider the set $\mathbb{N} = \{1, 2, 3, \cdots\}$ of natural numbers. If we assign to the first 2000 members of the set the membership $\frac{1}{2}$ and assign 0 as membership value to the remaining elements of the set, then the fuzzy subset obtained in this fashion has cardinality of 1000, which is finite.

On the other hand the cardinality of the fuzzy set $\mu : \mathbb{N} \longrightarrow [0,1]$ with the assignment $\mu(n) = \frac{1}{n}$ for all $n \in \mathbb{N}$ is infinite since it is equal to $\sum_{1}^{\infty} \frac{1}{n}$.

Proposition 3.6.4 Let μ_1 and μ_2 be two fuzzy subsets of a set $X = \{x_1, x_2, \cdots, x_n\}$ with cardinalities $|\mu_1|$ and $|\mu_2|$ respectively. The cardinality of the fuzzy subset union of μ_1 and μ_2 is obtained as follows:

$$|\mu_1 \vee \mu_2| \;=\; |\mu_1| + |\mu_2| \;-\; |\mu_1 \wedge \mu_2|$$

Proof: It is clear that for any two real numbers a and b such that $0 \leq a \leq 1$ and $0 \leq b \leq 1$ we have $a + b = \min(a, b) + \max(a, b)$. Based on this equality for each $i = 1, 2, \cdots, n$, we have

$$|\mu_1(x_i)| + |\mu_2(x_i)| = \min(|\mu_1(x_i)|, |\mu_2(x_i)|) + \max(|\mu_1(x_i)|, |\mu_2(x_i)|)$$

Summing up the value of each term on either side of the above equality from $i = 1$ through to $i = n$, we get

$$|\mu_1| + |\mu_2| \;=\; \min(|\mu_1|, |\mu_2|) \;+\; \max(|\mu_1|, |\mu_2|)$$

from which we get the required equality of the proposition. $\square$.

Chapter 4

Principle of Inclusion-Exclusion in $\mathcal{F}(\mathcal{X})$

In general the Principle of Inclusion-Exclusion (PIE) is applied to subsets where the membership of elements to the subsets are either 0 or 1. In our opinion this is a limitation. In that case an element either strictly enjoys or does not enjoy a given property in the set. There is no room for an element to enjoy a property partially, such as in properties that are vaguely expressed. We intend to apply the principle of inclusion-exclusion in a more general setting of the lattice $(\mathcal{F}(\mathcal{X}), \leq)$ of fuzzy subsets of a finite set. We will first enumerate elements of a finite set X using fuzzy subsets of X and their α-cuts. In section 4.1 we count elements of a set X belonging to either in the intersection or in the union of fuzzy subsets; Propositions 4.1.1, 4.1.2 and 4.1.3 support our discussion in this section. Next we will engage in the counting of elements of X of minimum and of maximum degree of membership to a number of fuzzy subsets of X. This will lead us to the enumeration of elements with *no worth* and elements with *absolute desirability* as well as the enumeration of elements of minimum degree of membership. We end the section 4.2 with a mention on the usefulness of the counting of fuzzy subsets in the market place. We propose some functions to enumerate fuzzy subsets of X having some given conditions. To that effect we present some propositions and proving some of them. In the course of this chapter we will also discuss an equivalence relation $\mathcal{R}$ in the set $\mathcal{F}(\mathcal{X})$ with respect to cardinality. The number of fuzzy subsets of X of cardinality p, which is exactly the number of elements in an equivalence class of cardinality p. Here we acknowledge and have taken note of the work done by B. Bouchon and G.Cohen [Bouchon and Cohen, page 102, Advances in fuzzy sets. Possibility theory and applications] on counting fuzzy subsets of cardinality equal to an $\alpha \in [0, 1]$ and have extended it by including the counting of fuzzy subsets of cardinality greater than or equal to α. In this case α is not restricted to be in $[0, 1]$. In a later section we will discuss some other applications of the PIE in the sets I of membership values $\alpha_1\alpha_2, \cdots, \alpha_n$ of elements of X and their pull back $\mu^{-1}(\alpha_i)$ in X. Two such applications will be discussed here and another one will be just mentioned without discussion.

4.1 Elements of minimum membership degree

In this section and subsequent sections we have used a natural number n for the number of elements of the set X. In other instances we have used n when counting some fuzzy subsets of X. These two notations do not give rise to confusion and are used in separate instances but clear from the context.

Let X be an n-element set for a natural number n. As before let M be a finite subset of the unit interval consisting of m-elements. Consider the set $\mathcal{F}(\mathcal{X})$ of all possible fuzzy subsets of X with membership values in M. We noted in section 3.2. that there are m^n possible fuzzy subsets of X taking membership values in M. Suppose a value α is given in the unit interval not necessarily in M. Consider any two fuzzy subsets μ_1 and μ_2 in $\mathcal{F}(\mathcal{X})$ and their respective α-cuts μ_1^α and μ_2^α such that $\mu_1^\alpha = \{x \in X \ \ \mu_1(x) \geq \alpha\}$ and $\mu_2^\alpha = \{x \in X \ \ \mu_2(x) \geq \alpha\} \ \ \alpha \in [0;1]$.

In the following subsections we will look at different cases involving either many fuzzy subsets and one specified membership value, or many fuzzy subsets and an equal number of membership values.

1°. Two fuzzy subsets and one specified membership value.

We enumerate elements of X that belong to two fuzzy subsets of X to a minimum specified degree of membership. Now the set of elements of X that belong to both μ_1 and μ_2 to a minimum degree of membership value α is therefore the set intersection $\mu_1^\alpha \cap \mu_2^\alpha$. Similarly $\mu_1^\alpha \cup \mu_2^\alpha$ would be the set of elements of X that belong to either μ_1 or μ_2 to a minimum degree of membership value α. The following proposition sets the PIE in this simple context.

Proposition 4.1.1 Suppose μ_1 and μ_2 are fuzzy subsets of X and α is a value in $[0, 1]$. Then

1. $\mid (\mu_1 \wedge \mu_2)^\alpha \mid = \mid \mu_1^\alpha \cap \mu_2^\alpha \mid = \mid \mu_1^\alpha \mid + \mid \mu_2^\alpha \mid - \mid \mu_1^\alpha \cup \mu_2^\alpha \mid$,

and dually

2. $|(\mu_1 \vee \mu_2)^\alpha| = |\mu_1^\alpha \cup \mu_2^\alpha| = |\mu_1^\alpha| + |\mu_2^\alpha| - |\mu_1^\alpha \cap \mu_2^\alpha|$.

Proof: Recall that μ_1^α and μ_2^α are crisp subsets of X. Therefore applying PIE as in Lemma 1.1.1 we can write: $|\mu_1^\alpha \cap \mu_2^\alpha| = |\mu_1^\alpha| + |\mu_2^\alpha| - |\mu_1^\alpha \cup \mu_2^\alpha|$ and $|\mu_1^\alpha \cup \mu_2^\alpha| = |\mu_1^\alpha| + |\mu_2^\alpha| - |\mu_1^\alpha \cap \mu_2^\alpha|$. This proves the proposition.□.

If we consider $\mu_1 = \mu_2$, then $|\mu_1^\alpha \cap \mu_1^\beta| = |\mu_1^\alpha| + |\mu_1^\beta| - |\mu_1^\alpha \cup \mu_1^\beta|$. Now if $\alpha \leq \beta$, then $\mu_1^\beta \subseteq \mu_1^\alpha$ and $|\mu_1^\alpha \cup \mu_1^\beta| = |\mu_1^\alpha|$. Therefore $|\mu_1^\alpha \cap \mu_1^\beta| = |\mu_1^\alpha| + |\mu_1^\beta| - |\mu_1^\alpha| = |\mu_1^\beta|$, which means $|\mu_1^\alpha \cap \mu_1^\beta| = |\mu_1^\beta|$ is the set of elements of X with membership at least β as expected.□.

2°. n fuzzy subsets and one specified membership value in $[0,1]$.

We enumerate elements of X using n fuzzy subsets and a single membership value in $[0,1]$.

Now if we consider n fuzzy subsets of $\mu_1, \mu_2, \cdots, \mu_n$ of X. We can write that the number of elements of X belonging in each of the n fuzzy subsets of X at a degree at least α with $\alpha \in [0,1]$ is given by :

Theorem 4.1.2 [12],[13] Suppose μ_i for i equals to $1, 2, \cdots, n$ are fuzzy subsets of X and α is a value in $[0,1]$. Then

1. $|(\mu_1 \wedge \mu_2 \wedge \cdots \wedge \mu_n)^\alpha| = |\mu_1^\alpha \cap \mu_2^\alpha \cap \cdots \cap \mu_n^\alpha| = \sum_{i=1}^{n} |\mu_i^\alpha| - \sum_{1 \leq i \leq j \leq n} |\mu_i^\alpha \cup \mu_j^\alpha| + \sum_{1 \leq i \leq j \leq k \leq n} |\mu_i^\alpha \cup \mu_j^\alpha \cup \mu_k^\alpha| + \cdots + (-1)^{n-1} |\mu_1^\alpha \cup \mu_2^\alpha \cup \cdots \cup \mu_n^\alpha|$

and dually

2. The number of elements of X belonging to at least one of the n fuzzy subsets $\mu_1\, \mu_2\, \cdots, \mu_n$ of X at a degree at least α is

$$| (\mu_1 \vee \mu_2 \vee \cdots \vee \mu_n)^\alpha | = | \mu_1^\alpha \cup \mu_2^\alpha \cup \cdots \cup \mu_n^\alpha |$$
$$= \sum_{i=1}^{n} | \mu_i^\alpha | - \sum_{1 \le i \le j \le n} | \mu_i^\alpha \cap \mu_j^\alpha | + \cdots + (-1)^{n-1} | \mu_i^\alpha \cap \mu_2^\alpha \cap \cdots \cap \mu_n^\alpha |.$$

Proof: Since the α-cut of the fuzzy intersection is the crisp intersection of their α-cuts, and that the α-cut of the fuzzy union is the crisp union of their α-cuts therefore $| (\mu_1 \wedge \mu_2 \wedge \cdots \wedge \mu_n)^\alpha | = | \mu_1^\alpha \cap \mu_2^\alpha \cap \cdots \cap \mu_n^\alpha |$. Also $| (\mu_1 \vee \mu_2 \vee \cdots \vee \mu_n)^\alpha | = | \mu_1^\alpha \cup \mu_2^\alpha \cup \cdots \cup \mu_n^\alpha |$. Now applying the PIE to n crisp subsets $\mu_i^\alpha, 1 \le i \le n$ of the set X, we can conclude the validity of the above theorem as per Theorem 1.1.2 and Theorem 1.2.1.

We note that the above formula can be rewritten in terms of α-cuts of union of fuzzy subsets as

$$| (\mu_1 \wedge \mu_2 \wedge \cdots \wedge \mu_n)^\alpha | = | \mu_1^\alpha \cap \mu_2^\alpha \cap \cdots \cap \mu_n^\alpha | = \sum_{i=1}^{n} | \mu_i^\alpha | - \sum_{1 \le i \le j \le n} | (\mu_i \vee \mu_j)^\alpha | + \sum_{1 \le i \le j \le k \le n} | (\mu_i \vee \mu_j \vee \mu_k)^\alpha | + \cdots + (-1)^{n-1} | (\mu_1 \vee \mu_2 \vee \cdots \vee \mu_n)^\alpha | . \quad (4.1.1)$$

and

$$| (\mu_1 \vee \mu_2 \vee \cdots \vee \mu_n)^\alpha | = | \mu_1^\alpha \cup \mu_2^\alpha \cup \cdots \cup \mu_n^\alpha | = \sum_{i=1}^{n} | \mu_i^\alpha | - \sum_{1 \le i \le j \le n} | (\mu_i \wedge \mu_j)^\alpha | + \sum_{1 \le i \le j \le k \le n} | (\mu_i \wedge \mu_j \wedge \mu_k)^\alpha | + \cdots + (-1)^{n-1} | (\mu_1 \wedge \mu_2 \wedge \cdots \wedge \mu_n)^\alpha | . \quad (4.1.2)$$

Illustration

Consider $X = \{x_1, x_2, x_3\}$, $\alpha = \frac{1}{2}$, $\mu_1 = 1\frac{1}{2}\frac{1}{2}$, $\mu_2 = \frac{1}{2}01$, $\mu_3 = 0\frac{1}{2}\frac{1}{2}$. To find the number of elements of X belonging to each of the μ_1, μ_2 and μ_3 to a minimum degree of $\frac{1}{2}$ we apply the above proposition,

$\mid \mu_1^\alpha \cap \mu_2^\alpha \cap \mu_3^\alpha \mid = \{\mid \mu_1^\alpha \mid + \mid \mu_2^\alpha \mid + \mid \mu_3^\alpha \mid\} - \{\mid \mu_1^\alpha \cup \mu_2^\alpha \mid + \mid \mu_1^\alpha \cup \mu_3^\alpha \mid + \mid \mu_2^\alpha \cup \mu_3^\alpha \mid\} + \mid \mu_1^\alpha \cup \mu_2^\alpha \cup \mu_3^\alpha \mid$.

(3 + 2 + 2) - (3 + 3 + 3) + 3 = 1.

By inspection we can see immediately that only one element, namely x_3 that belongs to all the three fuzzy subsets to a minimum degree 1/2. In general we need the method of PIE when large number of fuzzy subsets are involved in a large set of elements.

3°. n fuzzy subsets and n real numbers in $[0, 1]$.

Here we enumerate elements of X using n fuzzy subsets and n membership values in $[0, 1]$. Consider n fuzzy subsets $\mu_1\, \mu_2, \cdots \mu_n$ of X and $\alpha_1,\ \alpha_2,\ \cdots \alpha_n \in I$. We intend to count the elements of X which are simultaneously such that $\mu_1(x) \geq \alpha_1,\ \ \mu_2(x) \geq \alpha_2, \cdots, \mu_n(x) \geq \alpha_n$. Then the subset of such elements of X is given by $\mu_1^{\alpha_1} \cap \mu_2^{\alpha_2} \cap \cdots \cap \mu_n^{\alpha_n}$. Their number is given by $\mid \cap_{i=1}^n \mu_i^{\alpha_i} \mid$ and can be expressed using the PIE by:

Theorem 4.1.3 $\mid \cap_{i=1}^n \mu_i^{\alpha_i} \mid = \sum_{i=1}^n \mid \mu_i^{\alpha_i} \mid - \sum_{1 \leq i \leq j \leq n} \mid \mu_i^{\alpha_i} \cup \mu_j^{\alpha_j} \mid + \cdots + (-1)^n \mid \mu_1^{\alpha_1} \cup \cdots \cup \mu_n^{\alpha_n} \mid$.

Proof: Follows the same steps as in the previous theorem.□.

Note 4.1.4

The concept of elements of X to a maximum degree can be obtained by taking the complement in X of the subset of elements with a minimum prescribed membership degree.

4.2 Elements of minimum & maximum degree

We consider in the following subsections two types of enumerations. In the first case we take n fuzzy subsets and enumerate elements of X with a minimum degree and a maximum degree of membership. In the second case we are given n fuzzy subsets and two special fixed numbers, namely 0 and 1, are considered. In this case we count elements of the set X with membership value equal to 1 or those elements of X with membership values greater than 0.

1°. n fuzzy subsets and Two specified membership values.

In this section we enumerate elements of X that belong to fuzzy subsets of X with minimum degree of membership α and a maximum degree membership β. Obviously α has to be smaller than or equal to β. Suppose $\mu \in \mathcal{F}(\mathcal{X})$ is a given fuzzy subset of X. We want to enumerate the elements of X which satisfy $\mu(x) \geq \alpha$ and $\mu(x) \leq \beta$ simultaneously. That means the elements $x \in X$ must satisfy $\alpha \leq \mu(x) \leq \beta$. We introduce the following simple and natural notation which is very useful later.

$$\mu^{\alpha\beta} = \{x \in X : \alpha \leq \mu(x) \leq \beta\}.$$

Therefore there are two properties that these elements of X must have.

The two properties are:

-1. Having $\mu(x) \geq \alpha$,

-2. Having $\mu(x) \leq \beta$.

Let α and β be two numbers in the unit interval such that $\alpha \leq \beta$. Consider a fuzzy subset μ of X. The number of elements of X belonging to μ at a minimum degree α and at a maximum degree β is $\mid \mu^{\alpha} \setminus \mu^{\beta} \mid$. It is clear that these elements are in μ^{α} since $\mu(x) \geq \alpha$. They do not belong to μ^{β} since $\mu(x) \leq \beta$. In brief, they are in $\mu^{\alpha} \setminus \mu^{\beta}$. Therefore their number is $\mid \mu^{\alpha\beta} \mid = \mid \mu^{\alpha} \setminus \mu^{\beta} \mid$.

It is necessary to mention that α is not equal to β. Otherwise if $\alpha = \beta$, then the number of elements sought would simply be $\mid \mu^{-1}(\alpha) \mid$.

Let us introduce useful notations $\mid \mu^{\alpha} \mid$ and $\mid \mu^{\alpha'} \mid$ as the number of elements in μ^{α} and the number of elements not in μ^{α}, respectively. $\mid \mu^{\alpha'} \cap \mu^{\beta'} \mid$ is the number of elements neither in μ^{α} nor in μ^{β}.

We recall that μ^{α} and $\mu^{\alpha'}$ are crisp complementary subsets of X. We can use the PIE on the set X to enumerate these elements.

Lemma 4.2.1 . Let μ be a fuzzy subset of X while , $\mid \mu^{\alpha} \mid$ and $\mid \mu^{\alpha'} \mid$ are defined as above, the number of elements of X such that $\alpha \leq \mu(x) \leq \beta$ can be obtained by solving

$$|\mu^{\alpha\beta}| = \mid X \mid - (\mid \mu^{\alpha'} \mid + \mid \mu^{\beta} \mid) + \mid \mu^{\alpha'} \cap \mu^{\beta} \mid \tag{4.2.3}$$

where $\mu^{\alpha\beta}$ denotes the set of elements such that $\alpha \leq \mu(x) \leq \beta$.

Proof : By using PIE in X to enumerate elements enjoying two properties, we subtract from $\mid X \mid$ the number of those elements x without one property at the time, then add those without both properties simultaneously. We have this result:

$$|\mu^{\alpha\beta}| = \mid X \mid - (\mid \mu^{\alpha'} \mid + \mid \mu^{\beta} \mid) + \mid \mu^{\alpha'} \cap \mu^{\beta} \mid$$

Now since $\alpha \leq \beta$, $\mu^{\beta} \subseteq \mu^{\alpha}$ and $\mu^{\alpha} \cap \mu^{\alpha'} = \emptyset$ are all true, we have $\mu^{\beta} \cap \mu^{\alpha'} = \emptyset$. Therefore $\mid \mu^{\alpha'} \cap \mu^{\beta} \mid = 0$. This says that the number $\mid X \mid - (\mid \mu^{\alpha'} \mid + \mid \mu^{\beta} \mid)$

is equal to $| \mu^{\alpha} \setminus \mu^{\beta} | = |\mu^{\alpha\beta}|$. This completes the proof.□.

Consider a fuzzy subset μ of X as above. The set $\mu^{-1}([\alpha, \beta])$ which consists of elements of X such that $\mu(x) \geq \alpha$ and $\mu(x) \leq \beta$ describes $\mu^{\alpha\beta}$. By using the PIE we are also able to determine the elements of X in the complement of $\mu^{\alpha\beta}$.

Example 4.2.2 . We refer to diagram 1 [6], attached at the end of this thesis. For $X = \{x_1, x_2, x_3\}$; $n = 3$ and $M = \{0, \frac{1}{2}, 1\}$, $\mu_i = 1\frac{1}{2}\frac{1}{2}$ Consider $\alpha = \frac{1}{2}$ and $\beta = 1$.

We have $| \mu_i^{\alpha} | = 3$; therefore $| \mu_i^{\alpha'} | = 0$; while $| \mu^{\beta} | = 1$

Therefore $3 - [0 + 1] + 0 = 2$.

Now assume that $\mu_1 , \mu_2 , \cdots , \mu_n$ are n given fuzzy subsets of X such that $\forall i$, $\alpha \leq \mu_i(x) \leq \beta$. We are now interested in enumerating elements of X which are such that $\alpha \leq \cap_{i=1}^{n} \mu_i(x) \leq \beta$. Similarly it is interesting to enumerate the elements of X such that $\alpha \leq \cup_{i=1}^{n} \mu_i(x) \leq \beta$. First we take up the case of intersection. Before we state the proposition, it is useful to observe that $\{x \in X : \alpha \leq \cap_{i=1}^{n} \mu_i(x) \leq \beta\} = (\wedge_{i=1}^{n} \mu_i)^{\alpha\beta}$.

Theorem 4.2.3 Let $\mu_1 , \mu_2 , \cdots , \mu_n$ be fuzzy subsets of X and $\forall i$, $\alpha \leq \mu_i(x) \leq \beta$. Then the number of elements of X such that $\alpha \leq \cap_{i=1}^{n} \mu_i(x) \leq \beta$ is

$$(\wedge_{i=1}^{n} \mu_i)^{\alpha\beta} = | X | - (| \cap_{i=1}^{n} \mu^{\alpha'} | + | \cap_{i=1}^{n} \mu^{\beta} |), \tag{4.2.4}$$

which can be expressed using PIE as

$$|(\wedge_{i=1}^{n}\mu_i)^{\alpha\beta}| = |X| - \sum_{i=1}^{n}(|\mu_i^{\alpha'}| + |\mu_i^{\beta}|) + \sum_{1\le i\le j\le n}(|(\mu_i^{\alpha'} \cup \mu_j^{\alpha'}| + |\mu_i^{\beta} \cup \mu_j^{\beta}|) + \cdots$$

Proof: Set $\gamma = \cap_{i=1}^{n}\mu_i$. Then this takes us back to the case of a given $\mu = \gamma$ dealt with earlier in Lemma 4.2.1. Generally we can use Theorem 4.1.2 to expand $| \cap_{i=1}^{n}\mu^{\alpha'} |$ and $| \cap_{i=1}^{n}\mu^{\beta} |$ as follows:

$$| \cap_{i=1}^{n}\mu^{\alpha'} |= \sum_{i=1}^{n} | \mu_i^{\alpha'} | - \sum_{1\le i\le j\le n} | \mu_i^{\alpha'} \cup \mu_j^{\alpha'} | + \sum_{1\le i\le j\le k\le n} | \mu_i^{\alpha'} \cup \mu_j^{\alpha'} \cup \mu_k^{\alpha'} | + \cdots + (-1)^{n-1} | \mu_1^{\alpha'} \cup \mu_2^{\alpha'} \cup \cdots \cup \mu_n^{\alpha'} |$$

and $$| \cap_{i=1}^{n}\mu^{\beta} |= \sum_{i=1}^{n} | \mu_i^{\beta} | - \sum_{1\le i\le j\le n} | \mu_i^{\beta} \cup \mu_j^{\beta} | + \sum_{1\le i\le j\le k\le n} | \mu_i^{\beta} \cup \mu_j^{\beta} \cup \mu_k^{\beta} | + \cdots + (-1)^{n-1} | \mu_1^{\alpha} \cup \mu_2^{\alpha} \cup \cdots \cup \mu_n^{\beta} |$$

Now collecting and regrouping the terms carefully of the α-cuts and β-cuts of each of n fuzzy subsets corresponding to indices, we can rewrite the sum in 4.2.4 as:

$$| X | -(| \cap_{i=1}^{n}\mu^{\alpha'} | + | \cap_{i=1}^{n}\mu^{\beta}) | =$$

$$|X| - \sum_{i=1}^{n}(| \mu_i^{\alpha'} | + | \mu_i^{\beta} |) + \sum_{1\le i\le j\le n}(| \mu_i^{\alpha'} \cup \mu_j^{\alpha'} | + | \mu_i^{\beta} \cup \mu_j^{\beta} |) \cdots$$ This completes the proof

Similarly we can express $\eta = \cup_{i=1}^{n}\mu_i$ in terms of α-cuts of individual fuzzy subsets and their intersections using PIE.

If we consider the set of all m^n fuzzy subsets of X then $\gamma = \cap_{i=1}^{n}\mu_i$ would be the fuzzy subset $\emptyset$ which takes the membership value zero for all $x \in X$. Hence if $\alpha = 0$ and $\beta > 0$ then every $x \in X$ satisfies the property $x \in (\gamma)^{\alpha\beta}$. Hence $| (\gamma)^{\alpha\beta} |=| X |$. With the same consideration $\eta = \cup_{i=1}^{n}\mu_i$ would be the fuzzy subset X and $| X |$ would be the number of elements sought provided $\beta = 1$ and $\alpha < 1$.

Suppose we have k subsets $[\alpha_1, \beta_1], [\alpha_2, \beta_2], \cdots [\alpha_k, \beta_k]$ of $[0, 1]$. Suppose also that these intervals do not intersect. We can generalize 4.2.3 to enumerate elements of X that are such that $\forall j,\ \alpha_j \leq \cap_{i=1}^{n} \mu_i(x) \leq \beta_j$ as follows:

$$|X| - \sum_{i=1}^{n}(\sum_{j=1}^{k}(| \mu_i^{\alpha_j^{'}} | + | \mu_i^{\beta_j} |)) + \cdots$$

Illustration

We refer to the diagram 1 attached at the end of this thesis. Consider as previously $X = \{x_1, x_2, x_3\}$ $\alpha = \frac{1}{2}$, $\mu_1 = 1\frac{1}{2}\frac{1}{2}$, $\mu_2 = \frac{1}{2}01$, $\mu_3 = 0\frac{1}{2}\frac{1}{2}$

$| \mu_1^\alpha \cup \mu_2^\alpha \cup \mu_3^\alpha | = \{| \mu_1^\alpha | + | \mu_2^\alpha | + | \mu_3^\alpha |\} - \{| \mu_1^\alpha \cap \mu_2^\alpha | + | \mu_1^\alpha \cap \mu_3^\alpha | + | \mu_2^\alpha \cap \mu_3^\alpha |\} + | \mu_1^\alpha \cap \mu_2^\alpha \cap \mu_3^\alpha |$.

$= \{3 + 2 + 2\} - \{2 + 2 + 1\} + 1$

$7 - 5 + 1 = 3$

The three elements of X are either in μ_1^α , in μ_2^α or in μ_3^α.

2°. n fuzzy subsets and two special numbers in $[0, 1]$.

In this subsection we will recall the definitions of the *support*, as well as that of the *core*, of a fuzzy subset written as *supp*μ and *core*μ respectively.

supp$\mu = \{x \in X : \mu(x) > 0\}$ and *core*$\mu = \{x \in X : \mu(x) = 1\}$. Using these definitions, we will use $|\mu_i|_s$ to represent the cardinality of the *support* of μ_i and $|\mu_i|_c$ to represent the cardinality of the *core* of μ_i. Imagine a set X as before and n fuzzy subsets of X. Each element of X in *supp*μ has a certain degree of desirability or membership. Note here that any element of X not in *supp*μ has membership value *zero*. Now the number of elements of X with no *worth*, that is with membership value *zero* to a fuzzy subset μ_i, denoted here by $|\overline{\mu}_i|_s$ is $|X| - |\mu_i|_s$. Each element of X in $core\mu$ has absolute desirability. If we consider the n fuzzy subsets, then the number of elements with *no worth* at all to any of the n fuzzy subsets is obtained in this fashion:

Proposition 4.2.4 $| \cap_{i=1}^{n} \overline{\mu}_i|_s = |X| - \sum_{i=1}^{n} |\mu_i|_s + \sum_{1 \leq i \leq j \leq n} |\mu_i \cup \mu_j|_s + \cdots + (-1)^n |\mu_1 \cup \mu_2 \cup \cdots \cup |\mu_n|_s$

The usefulness of counting the number of elements with *no worth* is that if a great number of elements of X are *worthless*, then we need to set up the fuzzy subsets differently.

As for the $core\mu$, we may denote by $|\overline{\mu}_i|_c$ the set of elements of X with *no absolute desirability* to the fuzzy subset μ. Therefore the number of elements with *no absolute desirability* to the n fuzzy subsets is:

Proposition 4.2.5 $| \cap_{i=1}^{n} \overline{\mu}_i|_c = |X| - \sum_{i=1}^{n} |\mu_i|_c + \sum_{1 \leq i \leq j \leq n} |\mu_i \cup \mu_j|_c + \cdots + (-1)^n |\mu_1 \cup \mu_2 \cup \cdots \cup |\mu_n|_c$

In this case knowing the number of elements with *absolute desirability* to the n fuzzy subsets is necessary because if their number is $|X|$, then no good selection was done and thus new setting up of fuzzy conditions is required .

Note 4.2.6

1. In the market place, properties such as length, mass, composition of ingredients are considered when pricing an item. But these are all crisp properties. That is all the elements under consideration will show these properties no

matter who checks them. On the other hand our interest is to involve some fuzzy properties like appearance, colour, texture, size etc... to commodities such as fruit, vegetables, meat so that it would enable us to fix a realistic and affordable price according to quality. This will entail the enumeration of such commodities in numbers that possess such vague properties at a certain level of satisfaction for fixing the price. That is such commodities should make their way to the shelves and be priced accordingly. Just imagine the case where a washing machine would select the amount of soap not in relation with the amount/mass of clothes, but according to the cloth's degree of dirtiness. Or think of a toilet that would flush a volume of water according to the degree of *souillure* a user would have left as compared to the usual, not selective and wasteful way we are used to so far. There are numerous other instances where the counting of fuzzy subsets is relevant to our daily life.

4.3 PIE in crisp sets vs PIE in fuzzy subsets

The PIE for crisp sets as seen in Theorem 1.1.2 and Theorem 1.2.1 has many shortfalls compared to the PIE used for genuine fuzzy subsets– those that are not crisp subsets. There are many reasons why PIE for genuine fuzzy subsets is more appropriate. In the following we illustrate with two instances.

1. For crisp sets, for any $\alpha\ \beta \in [0,1]$, $\mu^\alpha = \mu^\beta$. Therefore the expression in Theorem 4.1.3 which reads as follows $\mid \cap_{i=1}^n \mu_i^{\alpha_i} \mid = \sum \mid \mu_i^{\alpha_i} \mid - \sum \mid \mu_i^{\alpha_i} \cup \mu_j^{\alpha_j} \mid + \cdots + (-1)^n \mid \mu_1^{\alpha_1} \cup \cdots \cup \mu_n^{\alpha_n} \mid$ would just turn out to be equal to $\mid \cap_{i=1}^n \mu_i^{\alpha_i} \mid = \sum_{i=1}^n \mid \mu_i^{\alpha_i} \mid - \sum_{1 \leq i \leq j \leq n} \mid \mu_i^{\alpha_i} \mid + \cdots + (-1)^n \mid \mu_1^{\alpha_1} \mid$. That is PIE in

crisp sets would fail to capture elements with some interesting properties. In fact the absolute desirability and desirability to a certain extent coincide.

2. The notion of elements with either a minimum membership value or a maximum membership value and even the idea of elements with membership value between two given numbers would become irrelevant. Thus we would lack some interesting ways of selecting elements of a given set.

4.4 Enumeration of fuzzy subsets of a finite set X.

In the previous sections we derived several formulae arising from the idea of PIE in fuzzy subsets when enumerating elements of the set X. Now we will set some conditions or properties for elements in the set $\mathcal{F}(\mathcal{X})$ and count fuzzy subsets of X satisfying these properties.

1°. Fuzzy subsets enjoying n properties.

Consider the set $\mathcal{F}(\mathcal{X})$ of all fuzzy subsets of a finite set $\{x_1, x_2, \cdots, x_n\}$ taking values in an m-elements set M. We consider for each $i = 1, 2, \cdots, n$, a property p_i on $\mathcal{F}(\mathcal{X})$ for a given α in the unit interval as,

$$p_i : \mu(x_i) \leq \alpha \text{ for some } \mu \text{ in } \mathcal{F}(\mathcal{X}).$$

Let $P = \{p_1, p_2, \cdots, p_n\}$ be the set of n properties on $\mathcal{F}(\mathcal{X})$. We say that a fuzzy subset $\mu = \mu(x_1)\mu(x_2)\cdots\mu(x_n)$ satisfies the property p_i if
$\mu(x_i) \leq \alpha$, $\alpha \in [0, 1]$. We denote by $(N_\mu^{p_i} \leq \alpha)$ and $(N_\mu^{p_i} \geq \alpha)$ respectively, the number of fuzzy subsets μ of X in which element x_i has a membership value

to μ in a way such that $\mu(x_i) \leq \alpha$ and the number of those fuzzy subsets μ of X in which element x_i has a membership value $\mu(x_i)$ such that $\alpha \leq \mu(x_i) \leq 1$. In other words $(N_\mu^{p_i} \leq \alpha)$ is the number of fuzzy subsets that satisfy p_i to a value less than or equal to α and $(N_\mu^{p_i} \geq \alpha)$ is the number of fuzzy subsets that satisfy p_i to a value greater than or equal to α. We will denote by $(N_\mu^{p_i} < \alpha)$ and $(N_\mu^{p_i} > \alpha)$ respectively the number of fuzzy subsets that satisfy p_i to a value strictly less than α and the number of fuzzy subsets that satisfy p_i to a value strictly greater than α. Further we wish to extend our notation by using $(N_\mu^{p_i} \geq \alpha) \cap^* (N_\mu^{p_j} \geq \beta), \quad \forall i, j$ where $i \neq j$ and they are between 1 and n, for the number of fuzzy subsets satisfying both property p_i and property p_j to values greater than or equal to α and β respectively. Finally we will denote by $(N_\mu^{p} \leq \alpha)$ the number of fuzzy subsets satisfying at least one property. We recall that there are m^n possible fuzzy subsets in $\mathcal{F}(\mathcal{X})$ enjoying some of the properties p_i, $1 \leq i \leq n$ defined in $\mathcal{F}(\mathcal{X})$.

We can enumerate those fuzzy subsets in the lattice $(\mathcal{F}(\mathcal{X}), \mathcal{M}, \leq)$ without any of the n properties. Their number which is denoted here by $(N_\mu^{\overline{p}})$ is $m^n - (N_\mu^{p} \leq \alpha)$ and is obtained by subtracting from m^n all fuzzy subsets with *one* property at a time; then add the number of those with two properties simultaneously; subtract the *ones* with three properties $\cdots$ etc; each time add or subtract the extra as per the proposition below.

Theorem 4.4.1 Let X be an n-element set, $(\mathcal{F}(\mathcal{X}), \mathcal{M}, \leq)$ be the lattice of fuzzy subsets μ_i of X with memberships values in an m-element set $M \subset [0, 1]$ and α be a given value in $[0, 1]$. Then the number of fuzzy subsets of X in $(\mathcal{F}(\mathcal{X}), \leq)$ without any of the n properties in $P = \{p_1, p_2, \cdots, p_n\}$ is given by:

$(N_{\mu}^{\overline{p}}) = m^n - \{(N_{\mu}^{p_1} \leq \alpha) + (N_{\mu}^{p_2} \leq \alpha) + (N_{\mu}^{p_3} \leq \alpha) + \cdots + (N_{\mu}^{p_n} \leq \alpha)\} + \{((N_{\mu}^{p_1} \leq \alpha) \cap^* (N_{\mu}^{p_2} \leq \alpha)) + ((N_{\mu}^{p_1} \leq \alpha) \cap^* (N_{\mu}^{p_3} \leq \alpha)) + \cdots + ((N_{\mu}^{p_i} \leq \alpha) \cap^* (N_{\mu}^{p_j} \leq \alpha)) \cdots\} - \{((N_{\mu}^{p_1} \leq \alpha) \cap^* (N_{\mu}^{p_2} \leq \alpha) \cap^* (N_{\mu}^{p_3} \leq \alpha)) + \cdots + (-1)^n((N_{\mu}^{p_1} \leq \alpha) \cap^* (N_{\mu}^{p_2} \leq \alpha) \cdots \cap^* (N_{\mu}^{p_n} \leq \alpha))\}$,

and

$m^n - (N_{\mu}^{p} \geq \alpha) = m^n - \{(N_{\mu}^{p_1} \geq \alpha) + (N_{\mu}^{p_2} \geq \alpha) + (N_{\mu}^{p_3} \geq \alpha) + \cdots + (N_{\mu}^{p_n} \geq \alpha)\} + \{((N_{\mu}^{p_1} \geq \alpha) \cap^* (N_{\mu}^{p_2} \geq \alpha)) + ((N_{\mu}^{p_1} \geq \alpha) \cap^* (N_{\mu}^{p_3} \geq \alpha)) + \cdots + ((N_{\mu}^{p_i} \geq \alpha) \cap^* (N_{\mu}^{p_j} \geq \alpha)) \cdots\} - \{((N_{\mu}^{p_1} \geq \alpha) \cap^* (N_{\mu}^{p_2} \geq \alpha) \cap^* (N_{\mu}^{p_3} \geq \alpha)) + \cdots + (-1)^n((N_{\mu}^{p_1} \geq \alpha) \cap^* (N_{\mu}^{p_2} \geq \alpha) \cdots \cap^* (N_{\mu}^{p_n} \geq \alpha))\}$.

Proof: Here we use the PIE on the set $\mathcal{F}(\mathcal{X})$ as seen in Lemma 1.1.3. In the first instance, we enumerate fuzzy subsets without any of the property p_i and secondly we enumerate fuzzy subsets which satisfy p_i to values less than or equal to α. $\square$.

Note 4.4.2 From the above equation we can say that the number $(N_{\mu}^{p} \leq \alpha)$ of fuzzy subsets satisfying at least one property is $\{(N_{\mu}^{p_1} \leq \alpha) + (N_{\mu}^{p_2} \leq \alpha) + (N_{\mu}^{p_3} \leq \alpha) + \cdots + (N_{\mu}^{p_n} \leq \alpha)\} - \{((N_{\mu}^{p_1} \leq \alpha) \cap^* (N_{\mu}^{p_2} \leq \alpha)) + ((N_{\mu}^{p_1} \leq \alpha) \cap^* (N_{\mu}^{p_3} \leq \alpha)) + \cdots + ((N_{\mu}^{p_i} \leq \alpha) \cap^* (N_{\mu}^{p_j} \leq \alpha)) \cdots\} + \{((N_{\mu}^{p_1} \leq \alpha) \cap^* (N_{\mu}^{p_2} \leq \alpha) \cap^* (N_{\mu}^{p_3} \leq \alpha)) + \cdots + (-1)^n((N_{\mu}^{p_1} \leq \alpha) \cap^* (N_{\mu}^{p_2} \leq \alpha) \cdots \cap^* (N_{\mu}^{p_n} \leq \alpha))\}$.

Illustration I

For illustration purpose we will use the diagram 1 which is attached at the end of this thesis. This diagram will be referred to for the following two examples and for the subsequent illustrations in the remaining part of the thesis.

Consider $X = \{x_1, x_2, x_3\}$, $M = \{0, \frac{1}{2}, 1\}$, $\mid X \mid = n = 3$. Set p_1: $\mu(x_1) \geq \frac{1}{2}$,

p_2: $\mu(x_2) \geq 1$, p_3: $\mu(x_3) \geq \frac{1}{2}$.

Enumerate the of fuzzy subsets that do not satisfy the three above properties. That is $\mu(x_i) < \frac{1}{2}$, $\mu(x_2) < 1$ or $\mu(x_3) < \frac{1}{2}$. By using the PIE we obtain: $3^3 - (18 + 9 + 18) + (6 + 6 + 12) - 4 = 2$. We have obtained the numbers 18, 9, 6,$\cdots$ using the Diagram of the lattice of set $X = \{x_1, x_2, x_3\}$ See diagram 1 attached at the end of this thesis.

The two fuzzy subsets obtained above are in fact $\mu_1 = 000$ and $\mu_2 = 0\frac{1}{2}0$. Clearly one can see that $\mu_1(x_1) = 0 < \frac{1}{2}$, $\mu_1(x_2) = 0 < 1$ and $\mu_1(x_3) = 0 < \frac{1}{2}$. The same applies to μ_2.

Illustration II

Consider X and M as above. Suppose p_1: $\mu(x_1) \geq \frac{1}{2}$ and p_2: $\mu(x_2) \geq 0$, p_3: $\mu(x_3) \geq 1$.

We set our three properties as:

property p_1: membership $\mu(x_1)$ greater than $\frac{1}{2}$,

property p_2: membership $\mu(x_2)$ equal to 0,

property p_3: membership $\mu(x_3)$ equal to 1

then

$3^3 - \{(N_\mu^1 \leq \frac{1}{2}) + (N_\mu^2 = 0) + (N_\mu^3 = 1)\} + \{((N_\mu^1 \leq \frac{1}{2}) \cap (N_\mu^2 = 0)) + ((N_\mu^1 \leq \frac{1}{2} \cap (N_\mu^3 = 1)) + ((N_\mu^2 = 0) \cap (N_\mu^3 = 1))\} - \{(N_\mu^1 \leq \frac{1}{2}) \cap (N_\mu^2 = 0) \cap (N_\mu^3 = 1)\}$
$= 27 - (18 + 9 + 9) + (6 + 6 + 3) - 2$ $27 - 36 + 15 - 2 = 4$.

These 4 fuzzy subsets are $\mu_1 = 0\frac{1}{2}0$, $\mu_1 = 0\frac{1}{2}\frac{1}{2}$, $\mu_3 = 010$ and $\mu_4 = 01\frac{1}{2}$ as can easily be seen from the diagram.

2°. Fuzzy subsets such that $\alpha \leq \mu_i(x) \leq \beta$

Now let us consider the sets X and M as defined previously. Taking α and β in the unit interval with $\alpha \leq \beta$, we count the fuzzy subsets μ_i, among those in $\mathcal{F}(\mathcal{X})$, which are such that for a given fixed element $x \in X$, $\mu_i(x)$, the membership value of x is comprised of numbers between α and β. These fuzzy subsets μ_i of X are such that for a given fixed $x \in X$, $\alpha \leq \mu_i(x) \leq \beta$.

Here the members of $\mathcal{F}(\mathcal{X})$ may or may not satisfy either one or two or both of the following two properties. We state them as:

$$p_1 : \mu(x_i) \geq \alpha \text{ for some } \mu \text{ in } \mathcal{F}(\mathcal{X})$$

$$p_2 : \mu(x_i) \leq \beta \text{ for some } \mu \text{ in } \mathcal{F}(\mathcal{X}).$$

In a way different from the previous case, we wish to use this time the notations $(N_\mu^1 \leq \alpha)$ and $N_\mu^2 \geq \beta)$ for the number of fuzzy subsets satisfying p_1 and p_2 respectively. Further we use the notation $(N_\mu^1 \leq \alpha) \cap^* (N_\mu^2 \geq \beta)]$ for the number of subsets μ_i satisfying both properties simultaneously. The number of such fuzzy subsets of X having both properties is obtained by using the PIE as follows:

Subtract from $\mid \mathcal{F}(\mathcal{X}) \mid$ the number of those fuzzy subsets without the property 1 and without the property 2 but not those without both properties. Then add those without both properties. This result is recorded in the following lemma the proof of which is similar to that of Theorem 4.4.1.

Lemma 4.4.3 The number of fuzzy subsets of X satisfying the two properties stated above is given by: $m^n - [(N_\mu^1 \leq \alpha) + (N_\mu^2 \geq \beta)] + [(N_\mu^1 \leq \alpha) \cap^* (N_\mu^2 \geq \beta)]$.

In the following example, we refer to diagram 1 attached at the end of this thesis.

Example 4.4.4 . Consider $X = \{x_1, x_2, x_3\}$, and $M = \{0, \frac{1}{2}, 1\}$ so that $\mid \mathcal{F}(\mathcal{X}) \mid = 27$. Suppose $\alpha = \frac{1}{4}$ and $\beta = \frac{3}{4}$ and $x = x_1$
$(N_\mu^1 \leq \frac{1}{4}) = 9$; $(N_\mu^2 \leq \frac{3}{4}) = 9$ Therefore:
$m^n - [(N_\mu^1 \leq \frac{1}{4}) + (N_\mu^2 \geq \frac{3}{4})] + [(N_\mu^1 \leq \frac{1}{4}) \cap (N_\mu^2 \geq \frac{3}{4})] =$
$27 - [9 + 9] + 0 = 9.$
These 9 fuzzy subsets are $\mu_1 = \frac{1}{2}00$, $\mu_2 = \frac{1}{2}\frac{1}{2}0$, $\mu_3 = \frac{1}{2}0\frac{1}{2}$, $\mu_4 = \frac{1}{2}10$, $\mu_5 = \frac{1}{2}\frac{1}{2}\frac{1}{2}$, $\mu_6 = \frac{1}{2}01$, $\mu_7 = \frac{1}{2}11$, $\mu_8 = \frac{1}{2}1\frac{1}{2}$, $\mu_9 = \frac{1}{2}\frac{1}{2}1$.

3°. Fuzzy subsets with at least one membership value greater than α.
In this section we count those fuzzy subsets μ of X which are such that the membership value of at least one element of X to μ is greater than or equal to α. We use the notation $(N_\mu \geq \alpha)$ to represent the number of fuzzy subsets μ of X for which the membership value of at least one element of X to the fuzzy subset μ is greater than or equal to α. The notation $(N_\mu = \alpha)$ will be used for the number of fuzzy subsets μ of X for which the membership value of at least one element $x \in X$ equal α. We extend the notation by writing $(N_{\mu(x_i)} = \alpha)$ for the number of fuzzy subsets μ of X for which the membership value for a specific $x_i \in X$ equals α. Again we will write $(N_{\mu_{(x_i)}} \geq \alpha)$ and $(N_{\mu(x_i x_j)} = \alpha)$ respectively for the number of fuzzy subsets of X for which the membership value of a specific $x_i \in X$ to the fuzzy subset μ is greater than or equal to α and the number of fuzzy subsets for which the membership values of elements x_i and x_j of X equal to α. It follows that $(N_{\mu(x_{i_1} x_{i_2} x_{i_3} \cdots x_{i_k})} \geq \alpha)$ will stand

for the number of fuzzy subsets for which the membership values of elements $x_{i_1}, x_{i_2}, \cdots x_{i_k}$ of X for $1 \leq k \leq n$, is greater than or equal to α and so on.

The following proposition determines the way of enumerating $(N_\mu \geq \alpha)$, the number of fuzzy subsets μ of X for which the membership value of at least one element $x \in X$ is greater or equal to α, considering all the above notations.

Proposition 4.4.5 $(N_\mu \geq \alpha) = \sum_{i=1}^{n}(N_{\mu(x_i)} \geq \alpha) - \sum_{1\leq i<j\leq n}(N_{\mu(x_i x_j)} \geq \alpha) + \sum_{1\leq i<j<k\leq n}(N_{\mu(x_i x_j x_k)} \geq \alpha) - \cdots + (-1)^{n-1}(N_{\mu(x_1 x_2 \cdots x_n)} \geq \alpha)$.

Proof: The proof will apply the PIE on the set $\mathcal{F}(\mathcal{X})$ of fuzzy subsets of X in which there are a certain number of properties. These properties are:

- Having a membership value of at least one element x_i of X exceeding α,
- Having membership values of at least two elements x_i and x_j exceeding α
- and so on.

The method used in counting the subsets with at least one property is to consider the sum of all fuzzy subsets where at least one element has a membership value to μ equal to or exceeding α. To this number we exclude the number of those with at least two elements having membership values equal to or exceeding α simultaneously, since they would have been counted twice among the ones with the first property and among those with the second property. To the number obtained at this stage, we include the number of the subsets where any three elements have membership values equal to or exceeding α. We carry on the process –add or subtract– until we get to the situation of the number of fuzzy subsets where all elements of X have a membership value equal to or exceeding α.$\square$.

Illustration III

In the following two examples, we refer to diagram 1 attached at the end of the thesis.

Take as in our previous example $\alpha = \frac{1}{2}$; $X = \{x_1, x_2, x_3\}$ and $M = \{0; \frac{1}{2}; 1\}$;

$(N_\mu \geq \frac{1}{2}) = \sum_{i=1}^{3}(N_{\mu(x_i)} \geq \frac{1}{2}) - \sum_{1 \leq i < j \leq 3}(N_{\mu(x_i x_j)} \geq \frac{1}{2}) + \sum (N_{\mu(x_i x_j x_k)} \geq \frac{1}{2})$

= (18 + 18 + 18) - (12 + 12 + 12) + (8)

= 26. The only fuzzy subset of X satisfying neither of the properties in this case is $\mu = 000$.

Illustration IV

Take $\alpha = 1$, $X = \{x_1; x_2; x_3\}$ and $M = \{0; \frac{1}{2}; 1\}$ then

$(N_\mu \geq 1) = [\,(N_{\mu(x_1)} \geq 1) + (N_{\mu(x_2)} \geq 1) + (N_{\mu(x_3)} \geq 1)\,] - [\,(N_{\mu(x_1 x_2)} \geq 1) + (N_{\mu(x_1 x_3)} \geq 1) + (N_{\mu(x_2 x_3)} \geq 1)\,] + (N_{\mu(x_1 x_2 x_3)} \geq 1)$

$= [9 + 9 + 9] - [3 + 3 + 3] + 1$

$=27 - 9 + 1$

$= 19.$

4.5 Fuzzy subsets as ordinary functions.

In this section, we consider fuzzy subsets of a finite set X with membership values in a finite set $M \subset [0, 1]$. Assume $|X| = n$ and $|M| = k$. As seen earlier there are k^n possible fuzzy subsets of X in $\mathcal{F}(\mathcal{X})$. Consider $\alpha \in M$. In this section we count among the k^n fuzzy subsets those which are such that no element of X has a membership value that exceeds α. Let us use the notation ($N_\mu \geq \alpha$) for the number of fuzzy subsets which are such that at least one element of X has a membership value that is equal to or exceeds α. We choose an approach different from our discussions as per section 4.4 above. This time we consider fuzzy subsets of X as functions from X to M, membership values are therefore images under this consideration. Now the number of (fuzzy subsets) functions sought is given by the expression in the following lemma.

Proposition 4.5.1 $(N_\mu \geq \alpha) = k^n - \sum_{i=0}^{n} C(n,i)(-1)^{n-i}k^i p^{n-i}$

Proof: Let $\alpha \in M$ and $\mid M \mid = k$, while $\mid X \mid = n$.

Since M is finite and ordered with the usual ordering in $\mathbb{Z}$, and M has the minimum element denoted by l; and the maximum elements denoted here as h; We observe that $l \leq \alpha \leq h$.

Let us now find the number of fuzzy subsets such that no element of X has a membership value (image)equal to or exceeding α.

This problem is the same as that of finding the number of functions (fuzzy subsets) from X with (membership) values in M skipping $\mid [\alpha, h] \mid$ values in M.

Call $\mid [\alpha, h] \mid = p$. Since $\mid M \mid = k$, there are $k - p$ values in M not exceeding α. Therefore there are $(k-p)^n$ functions (fuzzy subsets)from X with (membership) values in M with no image (membership) exceeding α out of a total k^n functions.

This means there are $k^n - (k-p)^n$ functions (fuzzy subsets) with an image (membership value) greater or equal to α.

This also means $(N_\mu \geq \alpha) = k^n - (k-p)^n \ = \ p^n$.

Expanding $k^n - (k-p)^n$ we get:

$k^n - (k-p)^n = k^n - [\sum_{i=0}^{n} C(n,i).k^i.(-p)^{n-i}\,]$

And $(N_\mu \geq \alpha) = k^n - \sum_{i=0}^{n} C(n,i)(-1)^{n-i}k^i p^{n-i}.\square$.

In the following two examples, we refer to diagram 1 attached at the end of this thesis.

Example 4.5.2 . Consider as previously $M = \{0, \frac{1}{2}, 1\}$, $X = \{x_1, x_2, x_3\}$, $\alpha = \frac{1}{2}$.

$h = 1$, $\mid M \mid = k = 3$, $\mid [\alpha, h] \mid = p = 2$.

There are $(k-p)^3 = (3-2)^3 = 1^3 = 1$ fuzzy subset with no membership value greater or equal to $\alpha = \frac{1}{2}$.

It is the fuzzy subset $\mu = 000$.

Therefore there are $k^3 - (k-p)^3 = 3^3 - (3-2)^3 = 26$ fuzzy subsets with membership value greater or equal to $\alpha = \frac{1}{2}$ as seen earlier in subsection **4°** above.

Example 4.5.3 . We refer to the diagram 1 attached at the end of this thesis.

If we take $\alpha = 1$, $|X| = 3$, $\;k = 3$. Then

$\mid [\alpha, h] \mid = \mid \ [1,1] \ \mid = p = 1$

$(k-p)^3 = (3-1)^3 = 2^3 = 8$.

There are $3^3 - 8 = 19$ fuzzy subsets with membership value greater or equal to $\alpha = 1$.

4.6 Enumeration of Fuzzy subsets of a given cardinality.

In this section we wish to enumerate the fuzzy subsets of an n-element set X with membership in an m-element set M. We group the fuzzy subsets in terms of their cardinalities as defined in 3.6.1. For $\alpha \in \mathbb{R}$, we count first in subsection 1, the fuzzy subsets with cardinality greater than or equal to α. Clearly $\alpha > 0$. For otherwise we would have counted every fuzzy subset since the cardinality of any fuzzy subset is non-negative. Later in subsection 2, we will count fuzzy subsets with cardinality equal to α.

1°. Number of fuzzy subsets of X with cardinality greater than or equal to α.

Suppose α is a positive real number, not necessarily confined to be in $[0, 1]$, $\mid X \mid = n$ and $\mid M \mid = m$. Let us denote by $(N_\mu(\mid . \mid \geq \alpha))$ the number of fuzzy subsets of X of cardinality greater than or equal to α and by $(N_\mu(\mid . \mid = \alpha))$ for the number of fuzzy subsets of X with cardinality equal to α. We recall from chapter 3 that $M_m = \{0, \frac{1}{m-1}, \frac{2}{m-1}, \cdots, \frac{m-1}{m-1} = 1\}$ as a possible set of membership values. If we denote by $Card_\mu$ the set of all possible distinct cardinalities of fuzzy subsets of X with membership values in M, then $Card_\mu$

is in fact made up of sums of elements in M_m.

$$Card_\mu = \{0, \frac{1}{m-1}, \frac{2}{m-1}, \cdots, \frac{m-1}{m-1}, \frac{m}{m-1}, \cdots, 2, \cdots, 3, \cdots, n\}.$$

The top element of $Card_\mu$ is n while the bottom element is 0. The set $(Card_\mu, \leq)$ is totally ordered with the usual order in $\mathbb{R}$.

We observe that for a fixed M, the cardinalities of fuzzy subsets depend on the size of the set X. This is also true for $Card_\mu$. Let us denote by $Card_{\mu_{X_n}}$ the set of all possible distinct cardinalities of fuzzy subsets of $X_n = \{x_1, x_2, \cdots, x_n\}$ with membership values in M. Subsequent to the above notation we will denote by $Card_{\mu_{X_{n+1}}}$ the set of all possible distinct cardinalities of fuzzy subsets of $X_{n+1} = \{x_1, x_2, \cdots, x_n, x_{n+1}\}$ with membership values in M. It follows that $Card_{\mu_\emptyset}$ will denote the set of cardinalities of fuzzy subset of the empty set.

Lemma 4.6.1 Let $|X| = n$ and $M = \{0, \frac{1}{2}, 1\}$.

$$|Card_{\mu_{X_{n+1}}}| = |Card_{\mu_{X_n}}| + |Card_{\mu_{\{x_{n+1}\}}}| - |Card_{\mu_\emptyset}|$$

Proof: We first note that $\mid Card_{\mu_\emptyset} \mid = 1$. [10]. Secondly we observe that $|Card_{\mu_{\{x_{n+1}\}}}| = 3$ and recall that the top element of $Card_{\mu_{X_n}}$ is n. Now $Card_{\mu_{X_{n+1}}}$ is obtained by summing members of $Card_{\mu_{X_n}}$ to any possible membership value the extra element x_{n+1} might have in M. Therefore this results in including into $Card_{\mu_{X_n}}$ the two numbers $n+\frac{1}{2}$ and $n+1$. Briefly we may say that $|Card_{\mu_{X_{n+1}}}| = |Card_{\mu_{X_n}}| + 3 - 1$. That is $|Card_{\mu_{X_{n+1}}}| = |Card_{\mu_{X_n}}| + 2$. Therefore it is clear that

$$|Card_{\mu_{X_{n+1}}}| = |Card_{\mu_{X_n}}| + |Card_{\mu_{\{x_{n+1}\}}}| - |Card_{\mu_\emptyset}|$$

.□. How many elements are there in $Card_\mu$ for a fixed $m = 3$? The following proposition gives an expression for the number of elements in the set of cardinalities of fuzzy subsets of a finite set X in term of the cardinality n of X.

Proposition 4.6.2 Let X be a set such that $\mid X \mid= n$ and $\mid M \mid= 3$, then $\mid Card_\mu \mid = \ (2n+1)$.

Proof: Suppose $X = \emptyset$ and $M = \{1, \frac{1}{2}, 1\}$. $\mid X \mid= n = 0$, there is only one fuzzy subset which takes membership value 1. This means $\mid Card_\mu \mid= 1$. [10]. Let $X = \{x_1\}$ and M is defined as above. Then $\mid X \mid = n = 1$. It follows that $Card_\mu = \{0, \frac{1}{2}, 1\}$. This means $\mid Card_\mu \mid= 3$.
Now assume that for $n = k$, that is $X_k = \{x_1, x_2, x_3, \cdots, x_k\}$, $\mid Card_\mu \mid= 2k + 1$.
Let us show that for $n = k+1$, we have $\mid Card_\mu \mid= 2(k+1)+1 = 2k+3$.
We observe that $X_{k+1} = X_k \cup \{x_{k+1}\}$ which also means:
$\mid X_{k+1} \mid=\mid X_k \cup \{x_{k+1}\} \mid = \mid X_k \mid + \mid \{x_{k+1}\} \mid - \mid X_k \cap \{x_{k+1}\} \mid$, using the PIE principle when two crisp sets are considered.
Now applying the Lemma 4.6.1 for the sets X_k and $\{x_{k+1}\}$ we see that $\mid Card_\mu \mid$ associated with X_{k+1} equals $\mid Card_\mu \mid$ associated with X_k + $\mid Card_\mu \mid$ associated with singleton $\{x_{k+1}\}$ - $\mid Card_\mu \mid$ associated with the intersection $X_k \cap \{x_{k+1}\}$. But $X_k \cap \{x_{k+1}\} = \emptyset$. Therefore:
$\mid Card_\mu \mid$ associated with $X_{k+1} = (2k+1)+(3)-1 = 2k+3 = 2(k+1)+1 = 2k+3$. The proof is completed.□.
The above lemma and proposition are valid for any value of n and for value

of m restricted to the value 3. Now in the next proposition we establish in general the number of elements in $Card_\mu$ in terms of n and m.

Proposition 4.6.3 Let X be a set such that $\mid X \mid= n$ and $\mid M \mid= m$ then $\mid Card_\mu \mid = (m-1)n+1$.

Proof: Let m be any number. This implies as per our definition of M that $M = \{0, \frac{1}{m-1}, \frac{2}{m-1}, \cdots, \frac{m-1}{m-1} = 1\}$. Now if $X = \emptyset$, $n = 0$ then $Card_\mu = 1$ as seen before. Suppose $n = 1$. Then $Card_\mu = m = (m-1)1+1$. This is evident since the only one element of X would have m possible distinct membership values in M. Suppose for $n = k$, $Card_\mu = (m-1)k+1$. We observe that for $n = k+1$, $Card_{\mu_{X_{n+1}}}$ is obtained by including into $Card_{\mu_{X_n}} = \{0, \frac{1}{m-1}, \frac{2}{m-1}, \cdots, \frac{m-1}{m-1}, \cdots n\}$, $(m-1)$ additional members. These numbers are $n+\frac{1}{m-1}, n+\frac{2}{m-1}, \cdots, n+\frac{m-1}{m-1} = n+1$. Therefore $|Card_{\mu_{X_{n+1}}}| = |Card_{\mu_{X_n}}|+(m-1)$. That is $(m-1)n+1+(m-1) = m(n+1)-(n+1)+1$. This confirms that for $n = k+1$, $|Card_{\mu_{X_{n+1}}}| = m(n+1)-(n+1)+1$. $\Box$.

Now we want to concentrate on the counting of fuzzy subsets having some properties. The properties we are referring to here are related to the concept of cardinality of fuzzy subsets. First we have a summation of the numbers of fuzzy subsets of cardinality each $p \in Card_\mu$. Later on, by varying either p, or m or n we enumerate the fuzzy subsets of X that have cardinalities according to the new set up.

Proposition 4.6.4 Let X, M and $Card_\mu$ be defined as previously.
For each $p \in Card_\mu$, we have:

$$\sum_{p \in Card_\mu} N_\mu(|\ .\ |= p) = m^n.$$

Proof: Let $p \in Card_\mu$; The left hand-side of the above proposition corresponds to enumerating the number of ways of associating every fuzzy subset of X to its cardinality. This is precisely the number of fuzzy subsets of X in M, which is equal to m^n.□.

Consider a number p in the set $Card_\mu$ of cardinalities of fuzzy subsets of an n-element set X. The following proposition is concerned with finding how many fuzzy subsets of cardinality p are there if we assume that $p = 0$ or that $p = n$?

Proposition 4.6.5 Let $Card_\mu$ be the set of cardinalities of fuzzy subsets of an n-element set X and $p \in Card_\mu$.

1. If $p = 0$, then $\sum_p^n N_\mu(|\ .\ |\geq p) = m^n$.
2. If $p = n$, then $\sum_p^n N_\mu(|\ .\ |\geq p) = N_\mu(|\ .\ |= n) = 1$.

Proof: Let $p = 0$ and $Card_\mu = \{0, \frac{1}{m-1}, \frac{2}{m-1}, \cdots, \frac{m-1}{m-1}, \frac{m}{m-1}, \cdots, n\}$. We recall that $(Card_\mu, \leq)$ is totally ordered with top element n and bottom element 0. The fuzzy subset of cardinality n is such that each membership value of the n elements of X is 1. It is therefore clear that $\sum_p^m N_\mu(|\ .\ |\geq 0) = \sum_{i \in Card_\mu} N_\mu(|\ .\ |= i) = m^n$ as per Proposition 4.6.4.

If $p = n$, then $\sum_p^n N_\mu(|\ .\ |\geq p) = \sum_n^n N_\mu(|\ .\ |\geq n) = 1$ This is the only fuzzy subset of X with every membership value of each of the n elements of X equals to 1.□.

Proposition 4.6.6 Let $Card_\mu$ be the set of cardinalities of fuzzy subsets of a set X with membership values in M. Consider $p \in M$. If $\mid X \mid= 1$ then $(N_\mu(\mid . \mid= p)) = 1$.

Proof: Let $\mid X \mid= 1$ and p a fixed element in M. The only one element of X may have m distinct membership values in M. But only one of these membership values in M would be equal to p. Therefore there is only one fuzzy subset of X of cardinality p.$\Box$.

Proposition 4.6.7 Let X be a singleton and $p \notin M$ then the number of fuzzy subsets of X of cardinality p denoted by $(N_\mu(\mid . \mid= p))$ is 0.

From the above proposition, the only fuzzy subset of X associated with the unique element of X should have its membership value, which in this case is its cardinality, in M. If this number is not in M, then there is no such fuzzy subset of X and therefore $(N_\mu(\mid . \mid= p)) = 0$.$\Box$.

Proposition 4.6.8 [2] If $\mid X \mid \geq 1$ then $(N_\mu(\mid . \mid= 0)) = 1$

This is the only fuzzy subset $\mu = 00 \cdots 0$ of X with every membership value equal to 0.

In the special case where $m = 2$, $M = \{0, 1\}$, and $Card_\mu = \{0, 1\}$. In this case we consider only crisp subsets of X. Then the number of crisp subsets of X with cardinality a given $p \in Card_\mu$ is given by the following proposition.

Proposition 4.6.9 The number of crisp subsets of an n-element set X with cardinality p denoted by $N_\mu(|\ .\ |= p)$ is equal to $N_\mu(|\ .\ |= p) = \begin{pmatrix} n \\ p \end{pmatrix} = \frac{n!}{p!(n-p)!}$.

Proof: Let $p \in Card_\mu$. With $m = 2$, that is to say that $M = \{0, 1\}$ and $Card_\mu = \{0, 1, 2, \cdots, n\}$, then either $p = 0, p = 1, p = 2, \cdots$ or $p = n$.
If $p = 0$, there is only $\begin{pmatrix} n \\ 0 \end{pmatrix} = \frac{n!}{0!(n-0)!} = 1$ way in which the n membership values of the n elements of X would sum up to $p = 0$. Here each membership value is therefore 0.
If $p = 1$, there are $\begin{pmatrix} n \\ p \end{pmatrix} = \frac{n!}{p!(n-p)!} = \begin{pmatrix} n \\ 1 \end{pmatrix} = \frac{n!}{1!(n-1)!} = n$ ways the n membership values of the n elements of X sum up to 1. This happens when $n - 1$ membership values are 0 except one which is 1.
If $p = 2$, there are n ways the memberships values of the n elements of X sum up to 2.
If $p = n$, there is only one way in which the n membership values of elements of X sum up to n. In this case each one of the n membership values is 1. □.

Note: For the following two examples we refer to the diagram 1 attached at the end of the thesis.

Example 4.6.10 . If $M = \{0, 1\}$ and $X = \{x_1, x_2, x_3\}$; then
$N_\mu(|\ .\ |= 0) = \begin{pmatrix} 3 \\ 0 \end{pmatrix} = \frac{3!}{0!.3!} = 1$, this is the fuzzy subset 000.
$N_\mu(|\ .\ |= 1) = \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \frac{3!}{1!.2!} = 3$, the 3 fuzzy subsets are: 100, 010 001.
$N_\mu(|\ .\ |= 2) = \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \frac{3!}{2!.1!} = 3$, the fuzzy subsets are 110, 101, 011.
$N_\mu(|\ .\ |= 3) = \begin{pmatrix} 3 \\ 3 \end{pmatrix} = 1$, the fuzzy subset in question is 111.

We note that $\begin{pmatrix} n \\ p \end{pmatrix} = \begin{pmatrix} n-1 \\ p \end{pmatrix} + \begin{pmatrix} n-1 \\ p-1 \end{pmatrix}$.

We interpret this result in the following proposition assuming that $M = \{0, 1\}$, $Card_\mu = \{0, 1, 2, \cdots, n\}$, and either $p = 0, p = 1, p = 2 \cdots$ or $p = n$.

Proposition 4.6.11 The number of fuzzy subsets of X with cardinality $p \in Card_\mu$ is equal to the sum of the number of fuzzy subsets of $X - \{x_i\}\ \forall\, x_i \in X$ with cardinalities p and the number of fuzzy subsets of $X - \{x_i\}\ \forall\, x_i \in X$ with cardinalities $(p-1)$. In symbols we write
$N_\mu(|\ .\ |= p)_{inX} = N_\mu(|\ .\ |= p)_{inX-\{x_i\}} + N_\mu\ |\ .\ |= p-1)_{inX-\{x_i\}}$.

Now if $m = 2$ and $p = 0$, $Card_\mu = \{0, 1, 2, 3, \cdots n\}$ and with $N_\mu(|\ .\ |= p) = \begin{pmatrix} n \\ p \end{pmatrix}$,
then $N_\mu(|\ .\ |\geq p) = \begin{pmatrix} n \\ 0 \end{pmatrix} + \begin{pmatrix} n \\ 1 \end{pmatrix} + \begin{pmatrix} n \\ 2 \end{pmatrix} + \begin{pmatrix} n \\ 3 \end{pmatrix} + \cdots + \begin{pmatrix} n \\ n \end{pmatrix}$

Example 4.6.12 . Take $m = 2$, $n = 3$.
$card_\mu = \{0, 1, 2, 3\}$ and
$N_\mu(|\ .\ |\geq p) = \begin{pmatrix} n \\ 0 \end{pmatrix} + \begin{pmatrix} n \\ 1 \end{pmatrix} + \begin{pmatrix} n \\ 2 \end{pmatrix} + \begin{pmatrix} n \\ 3 \end{pmatrix}$
$= \begin{pmatrix} n \\ 0 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ 3 \end{pmatrix}$

$= 1 + 3 + 3 + 1$

$= 8$

These are the 8 crisp subsets of $X = \{x_1, x_2, x_3\}$ with membership values in $M = \{0, 1\}$. See the diagram 1 attached at the end of this thesis.

In the following subsection, we will have a discussion on the number of fuzzy subsets with cardinality equal to an $\alpha \in \mathbb{R}$. Here we extend Proposition 4.6.11 to cases of genuine fuzzy subsets in general. In this case the cardinality of a fuzzy subset is not restricted to a value that is a positive integer. This topic was studied by Bouchon and Cohen in [2]. We have used their result and have written the following proposition in terms of our new notations. Here are two useful notations. $(N_\mu(|\ .\ |= \alpha))_n$ will stand for the number of fuzzy subsets of an n-element set with cardinality α while $(N_\mu(|\ .\ |= (\alpha - i)))_{n-1}$ stands for the number of fuzzy subsets of an $(n-1)$-element set with cardinality $(\alpha - i)\, i \in Card_\mu$.

2°. Number of fuzzy subsets of X with cardinality equal to α.

Proposition 4.6.13 Let the set X be of cardinality n, M be a m-elements set and α a number in the unit interval I. Then the number of fuzzy subsets of X of cardinality equal to α, denoted here as $(N_\mu(|\ .\ |= \alpha))_n$, is given by the expression: $(N_\mu(|\ .\ |= \alpha))_n = \sum_{i \in M} (N_\mu(|\ .\ |= \alpha - i))_{n-1}$.

Proof: On the left of the above equation we are counting the number of fuzzy subsets of set $X = \{x_1, x_2, x_3, \cdots, x_n\}$, while on the right of the equation we count the fuzzy subsets of the set $X' = \{x_1, x_2, x_3, \cdots, x_{n-1}\} = X \setminus \{x_n\}$.

Any fuzzy subset μ of set X is obtained from a fuzzy subset μ' of set X' by associating with x_n the difference between $\mid \mu \mid$ and $\mid \mu' \mid$, if this difference belongs to M. This proves the proposition. □.

For illustration we will refer to diagram 1 attached at the end of this thesis.

Example 4.6.14 . Consider $X = \{x_1, x_2, x_3\}$, $X' = \{x_1, x_2\}$ and $M = \{0, \frac{1}{2}, 1\}$.

1. $(N_\mu(\mid . \mid = 3))_3 = \sum_{i \in M} (N_\mu(\mid . \mid = 3 - i))_2 =$
$(N_\mu(\mid . \mid = 3 - 0))_2 + (N_\mu(\mid . \mid = 3 - \frac{1}{2}))_2 + (N_\mu(\mid . \mid = 3 - 1)) =$
$0 + 0 + 1 = 1$
2. $(N_\mu(\mid . \mid = 2))_3 = \sum_{i \in M} (N_\mu(\mid . \mid = 3 - i))_2 =$
$(N_\mu(\mid . \mid = 2 - 0))_2 + (N_\mu(\mid . \mid = 2 - \frac{1}{2}))_2 + (N_\mu(\mid . \mid = 2 - 1)) =$
$(N_\mu(\mid . \mid = 2))_2 + (N_\mu(\mid . \mid = 1\frac{1}{2}))_2 + (N_\mu(\mid . \mid = 1)) =$
$1 + 2 + 3 = 6$

The proposition 4.6.13 can be illustrated by the following table of $N_\mu(\mid . \mid = p)$. We call this table *Pascal Rectangle* since it is rectangular and the entries are obtained in a fashion similar to the process in the Pascal triangle.

n/p	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4
1	1	1	1	0	0	0	0	0	0
2	1	2	3	2	1	0	0	0	0
3	1	3	6	7	6	3	1	0	0
4	1	4	10	16	19	16	10	4	1

The entries in each line n represent the number of fuzzy subsets of a set with

n elements and with cardinality the number on the column p. In short, the entry on line n and column p is the sum of the three entries on line $n-1$ and columns $p-\frac{0}{2}$, $p-\frac{1}{2}$, $p-\frac{2}{2}$.

Check that when $n = 3$ and $M = \{0, \frac{1}{2}, 1\}$, the number of fuzzy subsets of cardinality , say $p = 1\frac{1}{2}$ is 7.

This number is the sum of the number of fuzzy subsets when $n = 2$ and with cardinalities $p = 1\frac{1}{2} - 0 = 1\frac{1}{2}$; $\quad p = 1\frac{1}{2} - \frac{1}{2} = 1$; $\quad p = 1\frac{1}{2} - 1 = \frac{1}{2}$ as stated in the above proposition.

We write: $7 = 2 + 3 + 2$. See the diagram 1 attached at the end of this thesis.

4.7 Equivalence in $\mathcal{F}(\mathcal{X})$.

We define in $\mathcal{F}(\mathcal{X})$ the relation $\mathcal{R}$ such that two fuzzy subsets μ_1 and μ_2 are in relation $\mathcal{R}$ and we write $\quad \mu_1 \mathcal{R} \mu_2$ if and only if $\quad | \mu_1 |=| \mu_2 |$.

Properties of the relation $\mathcal{R}$.

1. Reflexivity: $\forall \mu \in \mathcal{F}(\mathcal{X})$, $\mu_1 \ \mathcal{R} \ \mu_1$ since $| \mu_1 |=| \mu_1 |$.

2. Symmetry: $\forall \mu_1, \mu_2 \in \mathcal{F}(\mathcal{X})$, If $\mu_1 \ \mathcal{R} \ \mu_2$, then $\mu_2 \ \mathcal{R} \ \mu_1$. In fact if $| \mu_1 |=| \mu_2 |$ then $| \mu_2 |=| \mu_1 |$, i.e. $\mu_2 \ \mathcal{R} \ \mu_1$.

3. Transitivity: $\forall \mu_1, \mu_2, \mu_3 \in \mathcal{F}(\mathcal{X})$, If $\mu_1 \ \mathcal{R} \ \mu_2$ and $\mu_2 \ \mathcal{R} \ \mu_3$, then $\mu_1 \ \mathcal{R} \ \mu_3$. In fact if $| \mu_1 |=| \mu_2 |$ and $| \mu_2 |=| \mu_3 |$, then $| \mu_1 |=| \mu_3 |$ which means $\mu_1 \ \mathcal{R} \ \mu_3$.

From the above, we can say that $\mathcal{R}$ is an equivalence relation in the set $\mathcal{F}(\mathcal{X})$.

Equivalence classes are disjoint subsets of a set whose union is the set.

How many equivalence classes are there for the relation $\mathcal{R}$ defined on an n-

element set X?
If the equivalence classes had all the same size, then it would be easy to apply the multiplication principle which says that the union of m disjoint sets each of size s has $m.s$ elements. This means that the relation $\mathcal{R}$ would have had $\frac{n}{s}$ equivalence classes. Each equivalence class is represented by the actual unique cardinality of each member of the class. Now knowing that $Card_\mu$ is the set of cardinalities of fuzzy subsets of X, we therefore state in the following proposition that:

Proposition 4.7.1 If X is an n-element set , M is the set of membership values of fuzzy subsets of X and $\mathcal{R}$ is the equivalence relation defined in $\mathcal{F}(\mathcal{X})$ as defined above. Then the number of equivalence classes for $\mathcal{R}$ as defined in $\mathcal{F}(\mathcal{X})$ is the number of elements in $Card_\mu$.

Proof: For a fixed m, there are $(m-1)n+1$ members in $Card_\mu$ as stated above in proposition 4.6.3. Each of these $(m-1)n+1$ members is a cardinality of a fuzzy subset, thus representing a class. Therefore there are $(m-1)n+1$ equivalence classes for the relation $\mathcal{R}$ as defined in $\mathcal{F}(\mathcal{X})$.$\square$.
The members of a class are different from those of another class if the difference of their cardinalities is a positive multiple of $\frac{1}{m-1}$.
How many elements of $\mathcal{F}(\mathcal{X})$ have the same cardinality. In other words, How many elements are there in a class? In the following proposition we use the notation $(N_\mu(|\ .\ |= \alpha))_n$ for the number of fuzzy subsets of an n-element set with cardinality α. Extending our notation we say that $(N_\mu(|\ .\ |= (\alpha - i)))_{n-1}$ is the number of fuzzy subsets of an $(n-1)$-element set with cardinality $(\alpha -$

i) $i \ \in Card_{\mu}$.

Proposition 4.7.2 Let X be an n-element set and M be the set of membership values of fuzzy subsets of X. Let $\mathcal{R}$ be the equivalence relation defined on $\mathcal{F}(\mathcal{X})$. Consider $\alpha \in \mathbb{R}$. Then the number of elements in each equivalence class is the number $(N_{\mu}(| \ . \ |= \alpha))_n$ such that: $(N_{\mu}(| \ . \ |= \alpha))_n = \sum_{i \in M}(N_{\mu}(| \ . \ |= \alpha - i))_{n-1}$

Proof: We know that an equivalence class is made of elements of $\mathcal{F}(\mathcal{X})$ with the same cardinality, by definition of $\mathcal{R}$. Now proposition 4.6.13 gives us the number of fuzzy subsets of X of cardinality α. This number is exactly the number of elements in a class of cardinality α.$\square$.

Since all members of an equivalence class have the same cardinality by definition of $\mathcal{R}$, we can determine for special classes, for instance equivalence classes containing crisp subsets, the number of elements in that equivalence class.

Illustration III

We refer to diagram 1 attached at the end of the thesis.

Here $|X| = n = 3, \quad |M| = 3$. There are 7 = (3-1)3 + 1 equivalence classes.

If we denote by $\overline{p} \forall p$ the equivalence class of cardinality p, then there is one element in the class $\overline{0}$ of cardinality 0.

There is also one element in the class $\overline{n} = \overline{3}$ of cardinality $n = 3$.

There are $n = 3$ elements in the class $\overline{\frac{1}{n-1}} = \overline{\frac{1}{3-1}} = \overline{\frac{1}{2}}$ and so on.

4.8 Some Other Applications of PIE.

In this section we want to look at some other ways of applying the PIE in the set of fuzzy subsets. We will include a discussion on the PIE in the set I of membership values $\alpha_1, \alpha_2, \cdots, \alpha_n$ of elements of X and their pull back $\mu^{-1}(\alpha_i)\,\forall\, i = 1, 2, \cdots, n$ in X. Two of such applications only will be mentioned here without details.

1°. Consider a fuzzy subset μ of set X with distinct membership values $\alpha_1, \alpha_2, \cdots, \alpha_n$ in $I \subset [0, 1]$. It is clear that $\{\mu^{-1}(\alpha_i)\}_{i=1}^{n}$ is a partition of X and that $\sum_{i=1}^{n} \mid \mu^{-1}(\alpha_i) \mid = \mid X \mid$ while $\mid \bigcap_{i=1}^{n} \mu^{-1}(\alpha_i) \mid = |\emptyset|$.

We define in X a relation $\mathcal{R}$ such that two elements x_i and x_j are in relation and we write $x_i \mathcal{R}\, x_j$ if and only if $\mu(x_i) = \mu(x_j)$. In other words we may say that $\forall\, \alpha \in [0, 1]$ $x_i \mathcal{R}\, x_j$ if and only if $x_i \in \mu^{-1}(\alpha)$ and $x_j \in \mu^{-1}(\alpha)$
This relation $\mathcal{R}$ is an equivalence on X. The equivalence classes are the $\mu^{-1}(\alpha_i)\ \forall i$. The cardinality of $I \subset [0, 1]$ is the number of equivalence classes in X.
We can express $\mid \bigcup_{i=1}^{n} \mu^{-1}(\alpha_i) \mid$ as in the theorem below:

Theorem 4.8.1 Let μ be a fuzzy subset of set X with membership values $\alpha_1, \alpha_2, \cdots, \alpha_n$ in $I \subset [0, 1]$, then:

$$\mid \bigcup_{i=1}^{n} \mu^{-1}(\alpha_i) \mid \;=\; (\sum_{i=1}^{n} \mid \mu^{-1}(\alpha_i) \mid) - \sum_{1 \le i < j} \mid \mu^{-1}(\alpha_i) \cap \mu^{-1}(\alpha_j \mid + \sum_{1 \le i < j < k} \mid \mu^{-1}(\alpha_i) \cap \mu^{-1}(\alpha_j) \cap \mu^{-1}(\alpha_k) \mid + \cdots + (-1)^{n-1} \mid \mu^{-1}(\alpha_1) \cap \mu^{-1}(\alpha_2) \cap \cdots \cap \mu^{-1}(\alpha_n) \mid$$

and dually

$\mid \bigcap_{i=1}^{n} \mu^{-1}(\alpha_i)) \mid = (\sum_{i=1}^{n} \mid \mu^{-1}(\alpha_i) \mid) - \sum_{1 \leq i < j} \mid \mu^{-1}(\alpha_i) \cup \mu^{-1}(\alpha_j) \mid + \sum_{1 \leq i < j < k} \mid \mu^{-1}(\alpha_i) \cup \mu^{-1}(\alpha_j) \cup \mu^{-1}(\alpha_k) \mid + \cdots + (-1)^{n-1} \mid \mu^{-1}(\alpha_1) \cup \mu^{-1}(\alpha_2) \cup \cdots \cup \mu^{-1}(\alpha_n) \mid.$

Proof: For the fact that $\{\mu^{-1}(\alpha_i)\}_{i=1}^{n}$ is a partition of X; each intersection $\mu^{-1}(\alpha_i) \cap \mu^{-1}(\alpha_j) = \emptyset$ for any two i, j, and furthermore it is clear that $\mid \bigcup_{i=1}^{n} \mu^{-1}(\alpha_i) \mid = (\sum_{i=1}^{n} \mid \mu^{-1}(\alpha_i) \mid) = \mid X \mid$.

Similarly if we express each union of the kind $\mid \mu^{-1}(\alpha_i) \cup \mu^{-1}(\alpha_j) \mid$ or even of the type $\mid \mu^{-1}(\alpha_i) \cup \mu^{-1}(\alpha_j) \cup \mu^{-1}(\alpha_k) \mid$ found in $\mid (\bigcap_{i=1}^{n} \mu^{-1}(\alpha_i)) \mid$ as $\mid \mu^{-1}(\alpha_i) \mid + \mid \mu^{-1}(\alpha_j) \mid - \mid \mu^{-1}(\alpha_i) \cap \mu^{-1}(\alpha_j) \mid$, then each term on the right side will cancel each other. As a result we will be able to write that $\mid \bigcap_{i=1}^{n} \mu^{-1}(\alpha_i) \mid = 0$. This confirms the fact that $\bigcap_{i=1}^{n} \mu^{-1}(\alpha_i) = \emptyset$ since the $\mu^{-1}(\alpha_i)$ $\forall i$ make a partition of X.□.

2°. In our previous discussion we considered finitely many fuzzy subsets of a set X and one specific membership value α. Now we consider one fuzzy subset μ of set X and finitely many membership values. We take them to be $\alpha_1, \alpha_2, \cdots \alpha_n$ such that $\alpha_1 \leq \alpha_2 \leq \cdots, \leq \alpha_n$. We know that $\mu^{\alpha_n} \subseteq \mu^{\alpha_{n-1}} \subseteq \cdots \subseteq \mu^{\alpha_2} \subseteq \mu^{\alpha_1}$. These μ^{α_i} $\forall$ $i, 1 \leq i \leq n$ are subsets of X. We can therefore apply the PIE on these finite subsets in this manner:

$$\mid \bigcap_{i=1}^{n} \mu^{\alpha_i} \mid = \sum_{i=1}^{n} \mid \mu^{\alpha_i} \mid - \sum_{1 \leq i < j} \mid \mu^{\alpha_i} \cup \mu^{\alpha_j} \mid + \sum_{i < j < k} \mid \mu^{\alpha_i} \cup \mu^{\alpha_j} \cup \mu^{\alpha_k} \mid$$
$$\cdots (-1)^n \mid \mu^{\alpha_1} \cup \mu^{\alpha_2} \cup \cdots \cup \mu^{\alpha_n} \mid \tag{4.8.5}$$

Now, if $\alpha_1 \leq \alpha_j \leq \alpha_k$ then $\mu^{\alpha_1} \cup \mu^{\alpha_j} = \mu^{\alpha_1}$ and $\mu^{\alpha_1} \cup \mu^{\alpha_j} \cup \mu^{\alpha_k} = \mu^{\alpha_1}$. Therefore

μ^{α_1} appears $(n-1) = \begin{pmatrix} n-1 \\ 1 \end{pmatrix}$ times in expressions of the type $\mu^{\alpha_1} \cup \mu^{\alpha_j}$, $\begin{pmatrix} n-1 \\ 2 \end{pmatrix}$ in expressions of the form $\mu^{\alpha_1} \cup \mu^{\alpha_j} \cup \mu^{\alpha_k}$ and so on. Finally because of the alternating signs, the sum

$$\begin{pmatrix} n-1 \\ 0 \end{pmatrix} - \begin{pmatrix} n-1 \\ 1 \end{pmatrix} + \begin{pmatrix} n-1 \\ 2 \end{pmatrix} \cdots (-1)^{n-1} \begin{pmatrix} n-1 \\ n-1 \end{pmatrix} = 0.$$

See [7]. Therefore the term $\mid \mu^{\alpha_1} \mid$ will actually vanish in the right hand side of 4.8.5.

Similiarly μ^{α_2} appears $\begin{pmatrix} n-2 \\ 1 \end{pmatrix} = (n-2)$ times in expression of the form $\mu^{\alpha_2} \cup \mu^{\alpha_j}$; it appears $\begin{pmatrix} n-2 \\ 2 \end{pmatrix}$ in expressions of the type $\mu^{\alpha_2} \cup \mu^{\alpha_j} \cup \mu^{\alpha_k}$. As before we conclude that the term $|\mu^{\alpha_2}|$ vanishes in the right hand side of 4.8.5. We use this reasonning for each term of the type μ^{α_i} for $1 \leq i \leq n$ repeatedly until the term μ^{α_n} which is contained in each union $\mu^{\alpha_i} \cup \mu^{\alpha_n}$ $\forall i$ as per our assumption $\alpha_1 \leq \alpha_2 \leq \cdots, \leq \alpha_n$ remains. Therefore it appears only once in the sum $\mid \bigcap_{i=1}^{n} \mu^{\alpha_i} \mid = \sum_{i=1}^{n} \mid \mu^{\alpha_i} \mid - \sum_{1 \leq i \leq j} \mid \mu^{\alpha_i} \cup \mu^{\alpha_j} \mid + \sum_{i \leq j \leq k} \mid \mu^{\alpha_i} \cup \mu^{\alpha_j} \cup \mu^{\alpha_k} \mid \cdots (-1)^n \mid \mu^{\alpha_1} \cup \mu^{\alpha_2} \cup \cdots \cup \mu^{\alpha_n} \mid$. In conclusion the right hand side of the expression 4.8.5 which solves $\mid \bigcap_{i=1}^{n} \mu^{\alpha_i} \mid$ is made up of only one term $\mid \mu^{\alpha_n} \mid$. This confirms the already known fact that $\mid \bigcap_{i=1}^{n} \mu^{\alpha_i} \mid = \mid \mu^{\alpha_n} \mid$.

3°. Let μ be a fuzzy subset of X with membership values in I such that $\mu(X) = \{\alpha_1, \alpha_2, \cdots, \alpha_n\} \subset I$. Now we take finitely many subsets $J_1, J_2, \cdots, J_k$ of $\mu(X)$ with $k \leq n$ which are not necessary disjointed subsets of $\mu(X)$. We denote by J_i^C the complement of J_i in $\mu(X)$. We consider the set $(J_1 \cap J_2 \cap$

$\cdots \cap J_k)^C$ which is the same as the set $(J_1^C \cup J_2^C \cup \cdots \cup J_k^C)$. Applying the PIE on this set to find the number of elements in the union we get

$$| (J_1^C \cup J_2^C \cup \cdots \cup J_k^C) | = | \mu(X) | - \sum_{i=1}^{k} | J_i | + \sum_{1 \leq i \leq j \leq k} | J_i \cap J_j | + \ldots$$

We are interested to study in the set X the number $| (\bigcup_{\alpha \in J_i^C}^{k} \mu^{-1}(\alpha)) |$. This number is obtained as follows

$$| (\bigcup_{\alpha \in J_i^C}^{k} \mu^{-1}(\alpha)) | = | X | - \sum_{\alpha \in \cup_{i=1}^{k} J_i} | \mu^{-1}(\alpha) |$$

We have realized that PIE as a tool can be used for counting elements of crisp sets as well as elements of fuzzy subsets with fractional non-zero membership values or counting the number of fuzzy subsets themselves with some restricted properties. In this chapter we have dealt with a variety of such problems but by no means have we exhausted answering all different related questions in this topic.

Chapter 5

Möbius function and Möbius inversion formula

5.1 Introduction

In this chapter we study the Möbius function and Möbius inversion formula on the lattice $(\mathcal{F}(\mathcal{X}), \mathcal{M}, \leq)$ of fuzzy subsets μ_i of X with membership values in an m-element set $M \subset [0,1]$ where $m \geq 2$. As in Section 3.2 we assume that $M_m = \{0, \frac{1}{m-1}, \frac{2}{m-1}, \cdots, \frac{m-1}{m-1} = 1\}$.

An important topic in Combinatorics is the study of Möbius functions and their application to inversion formulae for counting functions including the context of PIE. The earliest form of Möbius function dealt with number theoretic considerations. Möbius inversion is an overcounting-undercounting, or sieve procedure. We keep track of the over and undercount by indexing with the elements of a partially ordered set which classically was taken to be the subsets of a finite set. The Möbius inversion formula of number theory as given in [5] uses functions with the set of positive integers under the divisibility order as indices. The classical PIE is a special case [3] of inversion problem studied under Möbius inversion techniques.

The statement of Möbius inversion formula in a lattice was first given independently by Weisner [21] and Philip Hall [15]. In a fundamental paper on Möbius functions, Rota [11] showed the importance of the theory in Combinatorics. He noted the relation between Möbius inversion and the principle of inclusion-exclusion.

5.2 The Incidence and Möbius functions

1°. The Incidence function, Incidence algebra.

A partially ordered set is locally finite if every interval has a finite number of elements. Let X be a locally finite partially ordered set. A function θ: $X \times X \to \mathbb{Z}$ is an *incidence function* provided that for $x, y \in X$ if $\theta(x,y) \neq 0$ then $x \leq y \in X$. In other words $\theta(x,y) = 0$ if $x \not\leq y$ in X. [15] [11]

The set of all such functions is denoted by $\mathcal{I}(\mathcal{X})$

Scalar multiples and sums of incidence functions are also incidence functions.

The product of two incidence functions $\theta, \epsilon \in \mathcal{I}(\mathcal{X})$ is defined by

$$(\theta\epsilon)(x,y) = \sum_{z \in X} \theta(x,z)\epsilon(z,y). \tag{5.2.1}$$

By our assumption of X being locally finite, the above sum has finitely many non-zero terms. and hence it is a finite sum.

This gives $\mathcal{I}(\mathcal{X})$ the structure of an associative algebra over $\mathbf{Z}$, called the *incidence algebra of* X. The identity element in $\mathcal{I}(\mathcal{X})$ is the Kronecker delta function:

$$\delta(x,y) = \begin{cases} 1, & \text{if } x = y \\ 0, & \text{otherwise} \end{cases}$$

2°. The Zeta function and the Zeta matrix.

The Zeta function of the poset X, which is denoted by ζ is an incidence function on $X \times X$ defined by:

$$\zeta(x,y) = \begin{cases} 1, & \text{if } x \leq y \\ 0, & \text{otherwise} \end{cases}$$

The matrix associated with the Zeta function of X is called the $Zeta-Matrix$. It is denoted by $Z(i,j)$. It is a square matrix whose row and whose columns are labeled by the members of the poset.

The entries of the Zeta matrix are either 0 or 1 as follows:

$$\zeta(x,y) = \begin{cases} 1, & \text{if } x \leq y \\ 0, & \text{otherwise.} \end{cases}$$

When we read across row (i) of a Zeta matrix, the presence of a 1 means that *the column label is greater or equal to* i. Similarly, reading down column j, each occurrence of a 1 means that *the row label is less than or equal to* j.

The properties of reflexivity is satisfied since the Zeta matrix has 1 along its diagonal. The antisymmetry is satisfied since $Z(a,b)$ and $Z(b,a)$ cannot be both 1. The transitivity is satisfied because $Z(a,b) = 1$ whenever $\exists x$ such that $Z(a,x) = 1$ and $Z(x,b) = 1$.

The elements of the poset are arranged in a way consistent with the poset "ordering" so that the Zeta matrix will be an upper triangular matrix.

From this we conclude that the Zeta matrix is invertible since its determinant is 1 and diagonal elements are all 1.

Its inverse is called *the Möbius matrix* M_{ij} and is definitely an upper triangular matrix.

The inverse of the Zeta function is the Möbius function μ of the poset. In other words, μ satisfies

$$\sum_{x \leq z \leq y} \mu(x,y) = \begin{cases} 1, & \text{if } x = y \\ 0, & \text{otherwise.} \end{cases}$$

In particular $\mu(x,x) = 1$ for all x. Moreover, if we know $\mu(x,z)$ for $x \leq z \leq y$, then we can calculate

$$\mu(x,y) = -\sum_{x \leq z \leq y} \mu(x,z). \qquad (5.2.2)$$

In particular the values of the Möbius function are all integers.

Example 5.2.1 . We refer to tables 1 and 2 attached at the end of the thesis for the Zeta matrix and Möbius matrix of the poset $(\mathcal{P}(\mathcal{X}), \subseteq)$ when $X = \{1,2,3\}$ respectively.

3°. Definition of the Möbius function.

Let $(X, \leq)$ be a locally finite poset. Then there exists a unique function μ $X \times X \to \mathbf{Z}$, called Möbius function such that $\mu(x,y) \neq 0$ if $x \leq y$ and such that whenever $f, g \in \mathcal{I}(\mathcal{X})$ the following conditions are equivalent:

$$(a)\ g(x,y) = \sum_{x \le z \le y} f(x,z);$$

$$(b)\ f(x,y) = \sum_{x \le z \le y} g(x,z)\mu(z,y)$$

Example 5.2.2 . Consider the power-set of X ordered by *inclusion*. [19] Let$f$ and g be two functions from $\mathcal{P}(\mathcal{X})$ to the set of real(or complex) numbers, and

$f(X) = \sum_{A \subseteq X} g(A)$. We can express g in terms of f in this fashion

$$g(X) = \sum_{A \subseteq X} (-1)^{|A|} f(A)$$

This means $\mu(A,B) = \begin{cases} (-1)^{|B|-|A|} \ if A \subseteq B \\ 0 \ otherwise \end{cases}$.

5.3 Möbius inversion.

Let $(X; \le)$ be a locally finite partially ordered set. Let f be a given function f on X taking values in an additive group. We define a summation function g in X such that $g(m) = \sum_{n \le m} f(n)$. This summation function is with respect to the given *ordering* $\le$ in X and is therefore over all elements n of X such that $n \le m$.

We solve for the given function f in terms of the summation function g. Therefore we are to invert a system of linear equations. In other words solving f in terms of g is an inversion problem in the poset $(X, \le)$. The inversion is done as in the following well known proposition

Proposition 5.3.1 [19] Let f and g be two functions defined with respect to the order in a poset $(X, \leq)$ such that $g(m) = \sum_{n \leq m} f(n)$, then by inversion we have: $f(m) = \sum_{n:\, n \leq m} \mu(n, m) g(n)$

where μ is the Möbius function of the poset.

Example 5.3.2 . Consider that $f(n)$ and $g(n)$ are functions from the set of positive integers to the set of real (complex) numbers. Suppose for every $n \geq 0$ $f(n) = \sum_{i|n} g(i)$. Suppose that we only know a formula for the function $f(n)$, not for $g(n)$. We wish to deduce somehow a formula for $g(n)$ in terms of values of $f(i)$ for $i|n$. This is also an inversion exercise. It is known classically and attributed to Möbius that the function named after him is the right tool to help us carry through the process successfully. In this example the Möbius function μ (for the lattice of all divisors of an integer n) is expressed by:

$$\mu(x, y) = \begin{cases} (-1)^s, & \text{if y/x is the product of s distinct primes} \\ 0, & \text{if x does not divide y or if y/x is not squarefree.} \end{cases}$$

See for instance [19].

Example 5.3.3 . Let f be a function on the positive integers to the set of real (or complex) numbers. Form the series $f(1) + f(2) + f(3) + \cdots$ and let $g(n) = \sum_{m<n} f(m)$. By inverting the sum, that is expressing f in terms of g we have $f(n) = g(n) - g(n-1)$, see reference [1].

We note that the Möbius inversion and the PIE have something in common, see [18][17]. In each case we have a partially ordered set. The ordering has the property that given any two elements in the set there are finitely many other elements between them(locally finite poset). Each of the posets contains the smallest element. But in Möbius inversion, there is no mention of properties that some of the elements may or may not satisfy; but instead the inversion is only based on the way the two functions f and g, defined in the poset, are related. For this reason the Möbius inversion is a better tool compared to PIE since it can be generalized to any poset.

5.4 Möbius function, Möbius inversion in the lattice $(\mathcal{F}(\mathcal{X}), \leq)$

In the rest of this thesis we will denote the Möbius function of the poset exclusively by the letter μ and will use letters such as λ, γ for fuzzy subsets. We make this arrangement to avoid confusion with the usual notation of μ for fuzzy subsets.

The Möbius function in the poset of positive integers ordered by divisibility and the Möbius function in the poset of subspaces of a finite vector space are a few cases where this function has been extensively studied and have been dealt with. We wish to study for an n-element set X the Möbius function in the poset $(\mathcal{F}(\mathcal{X}), \leq)$ of fuzzy subsets of X, naturally ordered by: $\lambda_1 \leq \lambda_2 \Leftrightarrow \lambda_1(x) \leq \lambda_2(x)$. for all x in X. The pointwise ordering in $\mathcal{F}(\mathcal{X})$ is also called dominance order.

We will denote fuzzy subsets as elements of $\mathcal{F}(\mathcal{X})$ by λ_1, λ_2 etc. using the subscripted λ's. We also recall that $(\mathcal{F}(\mathcal{X}), \leq)$ is a distributive but not complemented lattice, thus a vectorial lattice. We note that $\forall \lambda \in \mathcal{F}(\mathcal{X})$ the set $\{\lambda_i \in \mathcal{F}(\mathcal{X}),\ \lambda_i \leq \lambda\}$ is finite. Therefore $\mathcal{F}(\mathcal{X})$ is locally finite.

In the next lemma we show that the dominance order on fuzzy subsets preserves the usual ordering of cardinalities as numbers but not conversely.

Lemma 5.4.1 Let X be an n-element set and $\lambda_i = \lambda_i(x_1)\lambda_i(x_2)\cdots\lambda_i(x_n)$ and $\lambda_j = \lambda_j(x_1)\lambda_j(x_2)\cdots\lambda_j(x_n)$ be two fuzzy subsets of X, then:

1. $\lambda_i \ \leq\ \lambda_j$ implies $\mid \lambda_i \mid \leq \mid \lambda_j \mid$,
2. $\mid \lambda_i \mid \leq \mid \lambda_j \mid$ does not necessary mean that $\lambda_i \ \leq\ \lambda_j$.

Proof: 1. Let $\lambda_i = \lambda_i(x_1)\lambda_i(x_2)\cdots\lambda_i(x_n)$ and $\lambda_j = \lambda_j(x_1)\lambda_j(x_2)\cdots\lambda_j(x_n)$ be two fuzzy subsets such that $\lambda_i \leq \lambda_j$. Then for each $x_k, 1 \leq k \leq n$, $\lambda_i(x_k) \leq \lambda_j(x_k)$. Therefore $\mid \lambda_i \mid = \sum \lambda_i(x_k) \leq \sum \lambda_j(x_k) = \mid \lambda_j \mid$.

2. It is clear that the sum of membership values of two fuzzy subsets λ_i and λ_j being equal does not imply that $\lambda_i(x_k) \leq \lambda_j(x_k)$ for each x_k. $\square$.

1°. The Möbius function in $\mathcal{F}(\mathcal{X})$.

Let $\mathcal{F}(\mathcal{X})$ be the set of fuzzy subsets of an n-element set X. Consider an α in the unit interval I and two fuzzy subsets $\lambda_1,\ \lambda_2$ of X.

Lemma 5.4.2 For the lattice $(\mathcal{P}(\mathcal{X}), \subseteq)$ of crisp subsets of X,

$$\mu(\lambda_1^\alpha, \lambda_2^\alpha) = \begin{cases} (-1)^{|\lambda_1^\alpha| - |\lambda_2^\alpha|},\ if \lambda_1 \leq \lambda_2 \\ 0,\ otherwise. \end{cases}$$

Proof: For any two fuzzy subsets λ_1, λ_2 of X, $\lambda_1^\alpha \subseteq X$ and $\lambda_1^\alpha \subseteq X$. Therefore using the result in Example 5.2 we conclude the proof.$\square$.

Now we want to consider two real numbers α_1, $\alpha_2 \in [0,1]$ and two distinct fuzzy subsets λ_1, λ_2 of X. As for the above case we can write the following lemma whose proof is similar to that of lemma 5.4.2.

Lemma 5.4.3 For the lattice $(\mathcal{P}(\mathcal{X}), \subseteq)$ of crisp subsets of X,

$$\mu(\lambda_1^{\alpha_1}, \lambda_2^{\alpha_2}) = \begin{cases} (-1)^{|\lambda_1^{\alpha_1}|-|\lambda_2^{\alpha_2}|}, \ if \lambda_1 \leq \lambda_2 \\ 0, \ otherwise. \end{cases}$$

Let $(\mathcal{F}(\mathcal{X}), \leq))$ be the lattice of fuzzy subsets of an n-element set X. We recall that $M = \{0, \frac{1}{m-1}, \frac{2}{m-1}, \cdots, \frac{m-1}{m-1} = 1\}$ and
$Card_\mu = \{0, \frac{1}{m-1}, \ \frac{2}{m-1}, \cdots, \frac{m-1}{m-1}, \ \frac{m}{m-1}, \cdots, n\}$
Let $\lambda_i = \lambda_i(x_1)\lambda_i(x_2)\cdots\lambda_i(x_n)$ and $\lambda_j = \lambda_j(x_1)\lambda_j(x_2)\cdots\lambda_j(x_n)$ be two fuzzy subsets of X such that $\lambda_i \leq \lambda_j$. Now if $\lambda_i \leq \lambda_j$, then $\lambda_i(x_1) \leq \lambda_j(x_1)$, $\lambda_i(x_2) \leq \lambda_j(x_2)$, $\cdots$, $\lambda_i(x_n) \leq \lambda_j(x_n)$.

Theorem 5.4.4 Let $(\mathcal{F}(\mathcal{X}), \leq))$, X, M, and $Card_\mu$ be defined as above. The Möbius function μ in $\mathcal{F}(\mathcal{X})$ is defined as follows:

$$\mu(\lambda_1, \lambda_2) = \begin{cases} (-1)^t, \ if \ \lambda_1 \leq \lambda_2 \\ 0, \ otherwise, \end{cases}$$

where t is a natural number $0 \leq t \leq |X|(m-1)$ and such that
$\frac{t}{m-1} = \mid \lambda_2 \mid - \mid \lambda_1 \mid$.

Proof. Because of the nature of members of the set $Card_\mu$, if $\lambda_i \leq \lambda_j$, then there exists t with $0 \leq t \leq \mid X \mid (m-1)$ such that $\mid \lambda_j \mid = (\mid \lambda_i \mid + \frac{t}{m-1})$. That is to say $\frac{t}{m-1} = \mid \lambda_j \mid - \mid \lambda_i \mid$,

If $t = 0$, then $\frac{t}{m-1} = 0$ and $\lambda_j = \lambda_i$.
If $t = 1$, then $|\lambda_j| = |\lambda_i| + \frac{1}{m-1}$. Then $\lambda_i < \lambda_j$ for a given λ_j implies in this context that the membership value of one element at a time, x_1 through x_n, to λ_i is $\frac{1}{m-1}$ less than the membership value of the corresponding element to λ_j.
When $t = 2$, it means that the membership values of two elements of X at a time, x_1 through x_n, to λ_i are $\frac{1}{m-1}$ less than their corresponding membership values to λ_j. We can vary t until we reach the value of $t = n(m-1)$ which makes each $\lambda_i(x_k)$ to be $\frac{1}{m-1}$ less than $\lambda_j(x_k)$. That is to say that all x_k in X are such that their membership values to λ_i are $\frac{1}{m-1}$ less than their corresponding values to λ_j. If $\lambda_i \not\leq \lambda_j$ then nothing can be deduced for the value of t.
Thus the stated function μ is indeed the correct Möbius function, as can be seen by recursion.
The above proposition is in line with the famous Hall Lemma [4] which says :
If a and b are elements of the poset X, then

$$\mu_X(a, b) = \sum (-1)^{\mathcal{L}(\mathcal{C})}, \tag{5.4.3}$$

where the sum is over all chains C in X with minimal element a and maximal element b. The above can also be expressed as in [15]. Let X is a finite bounded poset. For each $j \geq 1$ let c_j denote the number of j-element chains $C \subseteq X$ such that $C^{min} = \{0\}$ and $C^{max} = \{1\}$. Then

$$\mu(X) = c_1 - c_2 + c_3 - c_4 + \cdots \tag{5.4.4}$$

2°. The Möbius inversion in $\mathcal{F}(\mathcal{X})$.

Let f and g be two functions in $\mathcal{F}(\mathcal{X})$ such that $f(\lambda_j) = \sum_{\lambda_i \leq \lambda_j} g(\lambda_j)$.

Let λ_j be an element of $\mathcal{F}(\mathcal{X})$. We can solve g in terms of f by writing $g(\lambda_j) = \sum_{\lambda_i \leq \lambda_j} f(\lambda_i)\mu(\lambda_i, \lambda_j)$ where $\mu(\lambda_i, \lambda_j)$ is the Möbius function . That is $g(\lambda_j)$ is defined in terms of $f(\lambda_i)$ where $\lambda_i \leq \lambda_j$ and $\mid \lambda_i \mid \leq \mid \lambda_j \mid$.

In other words we express $g(\lambda_j)$ as a sum of the $f(\lambda_i)$ which are such that there exists t with $\mid \lambda_j \mid = (\mid \lambda_i \mid + \frac{t}{m-1})$.

Theorem 5.4.5 Let X, M, $\mathcal{F}(\mathcal{X})$ and $Card_\mu$ be defined as before. For a natural number t; $0 \leq t \leq \mid X \mid$ such that $\mid \lambda_j \mid = (\mid \lambda_i \mid + \frac{t}{m-1})$ and a given λ_j in $(\mathcal{F}(\mathcal{X}), \leq)$, $g(\lambda_j) = \sum_{\lambda_i \leq \lambda_j \ 0 \leq t \leq |X|} (-1)^t f(\lambda_i)$

Proof. Let $\lambda_j = \lambda_j(x_1)\lambda_j(x_2)\cdots\lambda_j(x_n)$ be written as $\lambda_j = a_1a_2a_3\cdots a_n$ then using the definition of f and solving for g we write:

$g(\lambda_j) = f(a_1a_2a_3\cdots a_n)$

$-\,(\, f(a_1 - \frac{1}{m-1}; a_2, \cdots, a_n) + \cdots + f(a_1; a_2; \cdot; a_n - \frac{1}{m-1})\,)$

$+(\, f(a_1 - \frac{1}{m-1}; a_2 - \frac{1}{m-1}; a_3; \cdots; a_n) + \cdots, + f(a_1; a_2, \cdots, a_{n-1} - \frac{1}{m-1}; a_n - \frac{1}{m-1}))$

$- \cdots$

$\vdots$

$+ (-1)^n f(a_1 - \frac{1}{m-1}; a_2 - \frac{1}{m-1}, a_3 - \frac{1}{m-1}, \cdots, a_n - \frac{1}{m-1})$.

The above can be written using the value of t; $0 \leq t \leq n(m-1) = \mid X \mid$ as :

$g(\lambda_j) = (-1)^0 f(a_1; a_2, a_3, \cdots, a_n) + (-1)^1(\, f(a_1 - \frac{1}{m-1}; a_2, \cdots, a_n) + f(a_1; a_2 - \frac{1}{m-1}; a_3; \cdots, a_n) + \cdots + f(a_1; a_2; \cdot; a_n - \frac{1}{m-1})\,) + (-1)^2(\, f(a_1 - \frac{1}{m-1}; a_2 - \frac{1}{m-1}; a_3; \cdots; a_n) + \cdots, + f(a_1; a_2, \cdots, a_{n-1} - \frac{1}{m-1}; a_n - \frac{1}{m-1}) + \cdots$

$+(-1)^n f(a_1 - \frac{1}{m-1}; a_2 - \frac{1}{m-1}, a_3 - \frac{1}{m-1}, \cdots, a_n - \frac{1}{m-1})$.

We illustrate the above theorem with the following example.

Example 5.4.6 . We refer to diagrams 1, 2, 3, 4 attached at the end of the thesis. Consider $X = \{x_1, x_2, x_3\}$, $M = \{0, \frac{1}{2}, 1\}$

Let us define in $\mathcal{F}(\mathcal{X})$ two functions f and g such that

$f(\lambda_j) = \sum_{\lambda_i \leq \lambda_j} g(\lambda_j)$

Then:

$f(000) = g(000)$ so that $g(000) = f(000)$ or $g(000) = (-1)^0 f(000)$

$f(\frac{1}{2}00) = g(000) + g(\frac{1}{2}00)$ so that

$g(\frac{1}{2}00) = f(\frac{1}{2}00) - f(000)$ or $g(\frac{1}{2}00) = (-1)^0 f(\frac{1}{2}00) + (-1)^1 f(000)$

$f(0\frac{1}{2}0) = g(0\frac{1}{2}0) + g(000)$ so that

$g(0\frac{1}{2}0) = f(0\frac{1}{2}0) - f(000)$ or

$g(0\frac{1}{2}0) = (-1)^0 f(0\frac{1}{2}0) + (-1)^1 f(000)$

$f(00\frac{1}{2}) = g(00\frac{1}{2}) + g(000)$ so that

$g(00\frac{1}{2}) = f(00\frac{1}{2}) - f(000)$or

$g(00\frac{1}{2}) = (-1)^0 f(00\frac{1}{2}) + (-1)^1 f(000)$

$\vdots$ $f(\frac{1}{2}\frac{1}{2}0) = g(\frac{1}{2}\frac{1}{2}0) + g(\frac{1}{2}00) + g(0\frac{1}{2}0) + g(000)$ so that

$g(\frac{1}{2}\frac{1}{2}0) = f((\frac{1}{2}\frac{1}{2}0) - [\, f(\frac{1}{2}00) + f(0\frac{1}{2}0)\,] + f(000)$ or

$g(\frac{1}{2}\frac{1}{2}0) = (-1)^0 f((\frac{1}{2}\frac{1}{2}0) + (-1)^1[\, f(\frac{1}{2}00) + f(0\frac{1}{2}0)\,] + (-1)^2 f(000)$

$\vdots$

$g(1\frac{1}{2}1) = (-1)^0 f(1\frac{1}{2}1) + (-1)^1[\, f(\frac{1}{2}\frac{1}{2}1) + f(101) + f(1\frac{1}{2}\frac{1}{2})] + (-1)^2[f(\frac{1}{2}01) + f(10\frac{1}{2}) + f(\frac{1}{2}\frac{1}{2}\frac{1}{2})\,] + (-1)^3 f(\frac{1}{2}0\frac{1}{2})$(See diagram 3 attached at the end of this thesis).

$\vdots$

$g(11\frac{1}{2}) = (-1)^0 f(11\frac{1}{2}) + (-1)^1[\, f(110) + f(1\frac{1}{2}\frac{1}{2}) + f(\frac{1}{2}1\frac{1}{2})\,] + (-1)^2[f(1\frac{1}{2}0) + f(\frac{1}{2}\frac{1}{2}\frac{1}{2}) + f(\frac{1}{2}10)\,] + (-1)^3 f(\frac{1}{2}\frac{1}{2}0)$ (See diagram 2 attached at the end of this thesis).

$\vdots$

$g(111) = (-1)^0 f(111) + (-1)^1[f(\frac{1}{2}11) + f(1\frac{1}{2}1) + f(11\frac{1}{2})] + (-1)^2[f(\frac{1}{2}\frac{1}{2}1) + f(\frac{1}{2}1\frac{1}{2}) + f(1\frac{1}{2}\frac{1}{2})] + (-1)^3 f(\frac{1}{2}\frac{1}{2}\frac{1}{2})$ (See diagram 4 attached at the end of this thesis).

5.4.1 REMARKS

1. Now solving in this fashion each $g(\lambda_j)$ of $\mathcal{F}(\mathcal{X})$ in terms of $f(\lambda_i)$, we have established a table of μ matrix of the lattice $\mathcal{F}(\mathcal{X})$ with $X = \{x_1, x_2, x_3\}$ and $M = \{0, \frac{1}{2}, 1\}$. See table 1. This table could have been obtained by inverting the ζ matrix of the lattice $\mathcal{F}(\mathcal{X})$. See table 2. As a matter of fact Table 1 and Table 2 are inverses of each other.

2. The function μ does not depend on the order in which we list the elements of $\mathcal{F}(\mathcal{X})$ even though the μ matrix of the lattice $\mathcal{F}(\mathcal{X})$ is obtained by listing elements of $\mathcal{F}(\mathcal{X})$ in a certain order.

3. For any λ_j, if $t = 0$, then $\frac{t}{m-1} = 0$ and $\lambda_j = \lambda_i$. this implies that $\mu(\lambda_j, \lambda_j) = (-1)^0 = 1$.

4. For any λ_j, and $\lambda_i \leq \lambda_j$ $\sum\limits_{\lambda;\lambda_i \leq \lambda \leq \lambda_j} \mu(\lambda_i, \lambda) = 0$.

5. It is clear that if $\lambda_i \not\leq \lambda_j$, then there is no t with $0 \leq t \leq\mid X \mid$ and

$\mid \lambda_j \mid = (\mid \lambda_i \mid + \frac{t}{m-1})$. In which case $\mu(\lambda_i, \lambda_j) = 0$.

To establish the tables of μ matrix and ζ matrix for the lattice $\mathcal{F}(\mathcal{X})$ with $X = \{x_1, x_2, x_3\}$ and $M = \{0, \frac{1}{2}, 1\}$ we let $A = 000; B = \frac{1}{2}00; C = 0\frac{1}{2}0; D = 00\frac{1}{2}; E = 100; F = \frac{1}{2}\frac{1}{2}0; G = \frac{1}{2}0\frac{1}{2};\ H = 010; I = 0\frac{1}{2}\frac{1}{2}; J = 001; K = 1\frac{1}{2}0; L = 10\frac{1}{2}; M = \frac{1}{2}10; N = \frac{1}{2}\frac{1}{2}\frac{1}{2}; O = \frac{1}{2}01;$

$P = 01\frac{1}{2}; Q = 0\frac{1}{2}1; R = 110; S = 1\frac{1}{2}\frac{1}{2}; T = 101; U = \frac{1}{2}1\frac{1}{2}; V = \frac{1}{2}\frac{1}{2}1;$

$W = 011; X = 11\frac{1}{2}; Y = 1\frac{1}{2}1; Z = \frac{1}{2}11; Z_1 = 111$

The fuzzy subsets of X are arranged in such a way that the ζ matrix and its inverse the μ matrix are triangular with the diagonal constituted by the 1's as one would expect.

The sum of the entries in each line of the μ matrix satisfies the relation

$$\sum_{z \in [x,y]} \mu(x, z) = \begin{cases} 1, & \text{if } x \leq y \\ 0, & \text{otherwise.} \end{cases}$$

which allows the Möbius function of a poset to be calculated recursively.

TABLE 1

ζ	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	$Z1$
A	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
B	0	1	0	0	1	1	1	0	0	0	1	1	1	1	1	0	0	1	1	1	1	1	0	1	1	1	1
C	0	0	1	0	0	1	0	1	1	0	1	0	1	1	0	1	1	1	1	0	1	1	1	1	1	1	1
D	0	0	0	1	0	0	1	0	1	1	0	1	0	1	1	1	1	0	1	1	1	1	1	1	1	1	1
E	0	0	0	0	1	0	0	0	0	0	1	1	0	0	0	0	0	1	1	1	0	0	0	1	1	0	1
F	0	0	0	0	0	1	0	0	0	0	1	0	1	1	0	0	0	1	1	0	1	1	0	1	1	1	1
G	0	0	0	0	0	0	1	0	0	0	0	1	0	1	1	0	0	0	1	1	1	1	0	1	1	1	1
H	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0	1	0	1	0	0	1	0	1	1	0	1	1
I	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	1	1	0	1	0	1	1	1	1	1	1	1
J	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	1	0	0	1	0	1	1	0	1	1	1
K	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1	1	0	0	0	0	1	1	0	1
L	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1	1	0	0	0	1	1	0	1
M	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0	1	0	0	1	0	1	1
N	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	1	1	0	0	1	1	1
O	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	1	1
P	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0	1	0	1	1
Q	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	1	0	1	1	1
R	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	1
S	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	1	0	1
T	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	1	0	1
U	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	0	1	1
V	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	1	1
W	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	1
X	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1
Y	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1
Z	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1
$Z1$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

TABLE 2

μ	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	$Z1$
A	1	−1	−1	−1	0	1	1	0	1	0	0	0	0	−1	0	0	0	0	0	0	0	0	0	0	0	0	0
B	0	1	0	0	−1	−1	−1	0	0	0	1	1	0	1	0	0	0	0	−1	0	0	0	0	0	0	0	0
C	0	0	1	0	0	−1	0	−1	−1	0	0	0	1	1	0	1	0	0	0	0	−1	0	0	0	0	0	0
D	0	0	0	1	0	0	−1	0	−1	−1	0	0	0	1	1	0	1	0	0	0	0	−1	0	0	0	0	0
E	0	0	0	0	1	0	0	0	0	0	−1	−1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
F	0	0	0	0	0	1	0	0	0	0	−1	0	−1	−1	0	0	0	1	1	0	1	0	0	−1	0	0	0
G	0	0	0	0	0	0	1	0	0	0	0	−1	0	−1	−1	0	0	0	1	1	0	1	0	0	−1	0	0
H	0	0	0	0	0	0	0	1	0	0	0	0	−1	0	0	−1	0	0	0	0	1	0	0	0	0	0	0
I	0	0	0	0	0	0	0	0	1	0	0	0	0	−1	0	−1	−1	0	0	0	1	1	1	0	0	−1	0
J	0	0	0	0	0	0	0	0	0	1	0	0	0	0	−1	0	−1	0	0	0	0	1	0	0	0	0	0
K	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	−1	−1	0	0	0	0	1	0	0	0
L	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	−1	−1	0	0	0	0	1	0	0
M	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	−1	0	0	−1	0	0	1	0	0	0
N	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	−1	0	−1	−1	0	1	1	1	−1
O	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	−1	0	−1	0	0	1	0	0
P	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	−1	0	−1	0	0	1	0
Q	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	−1	−1	0	0	1	0
R	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	−1	0	0	0
S	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	−1	−1	0	1
T	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	−1	0	0
U	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	−1	0	−1	1
V	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	−1	−1	1
W	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	−1	0
X	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	−1
Y	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	−1
Z	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	−1
$Z1$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

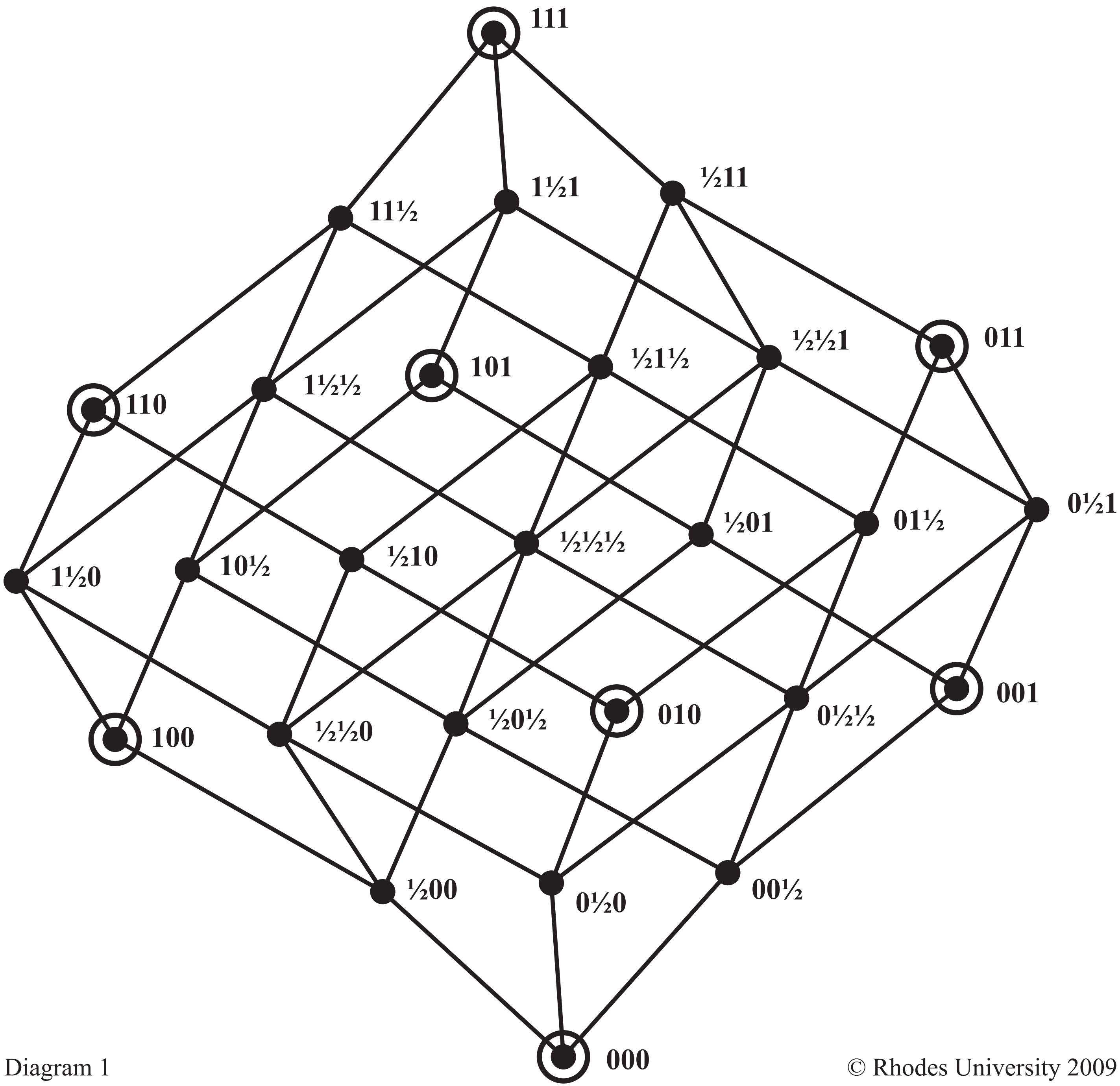

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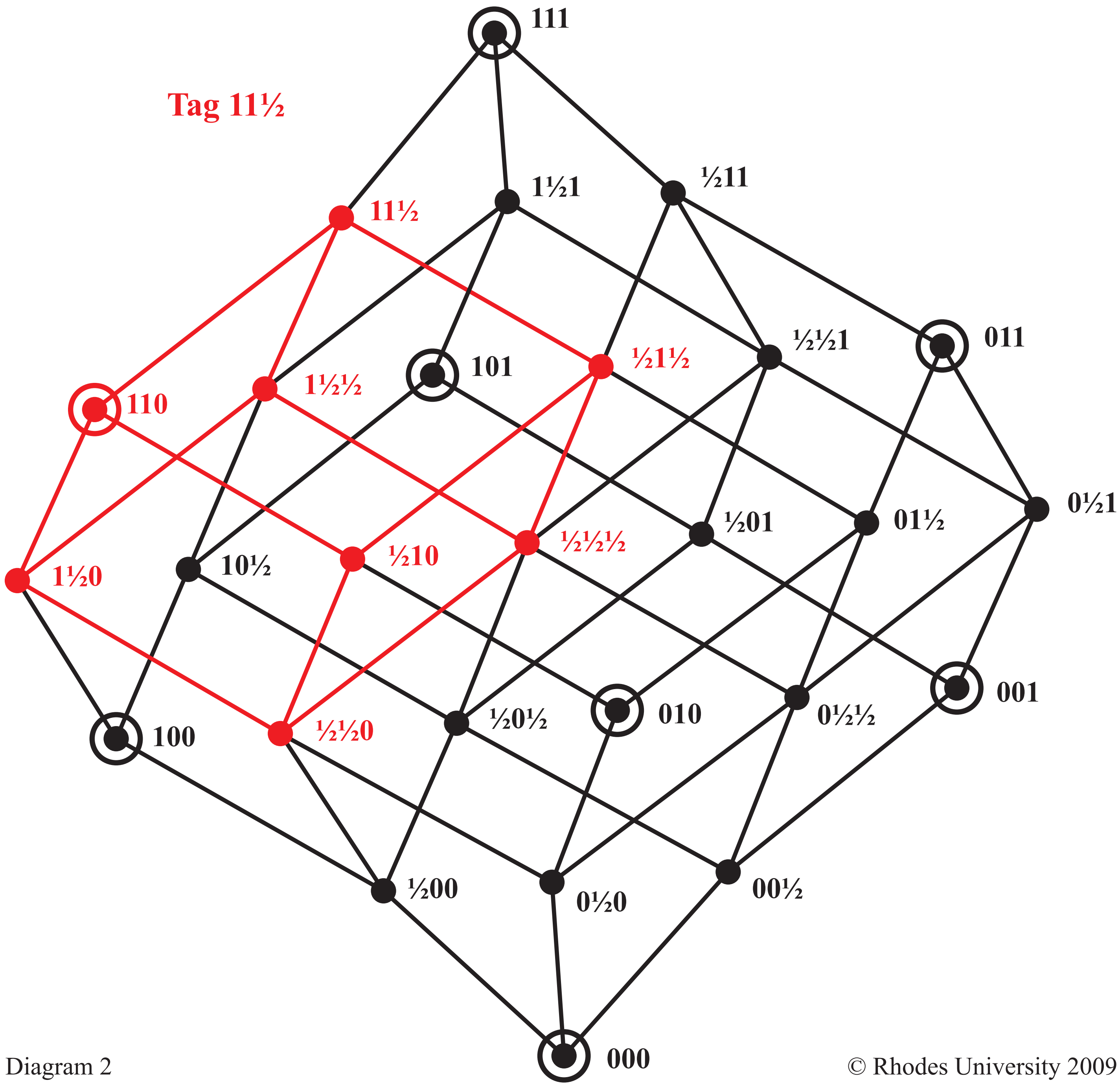
Diagram 1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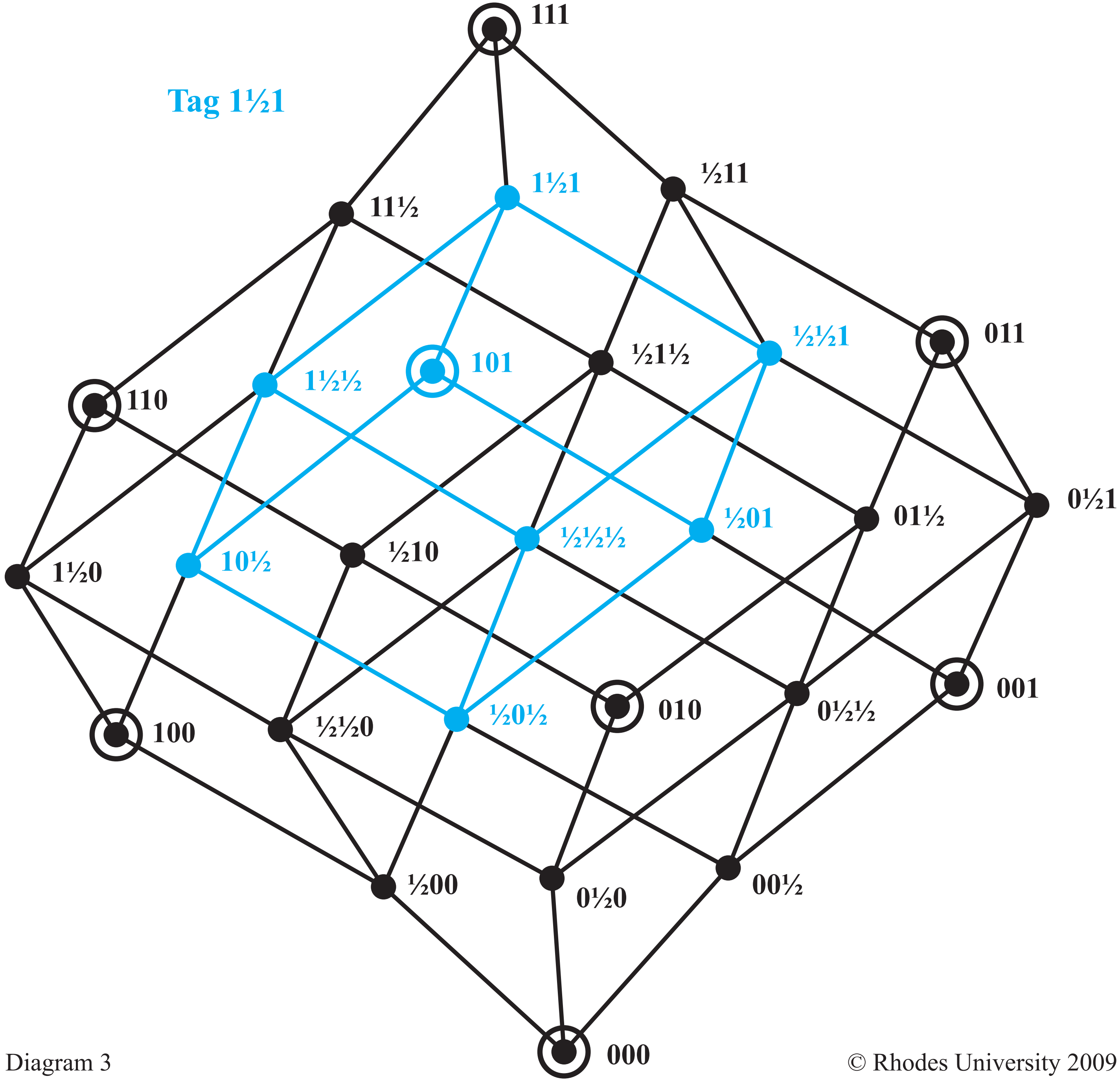

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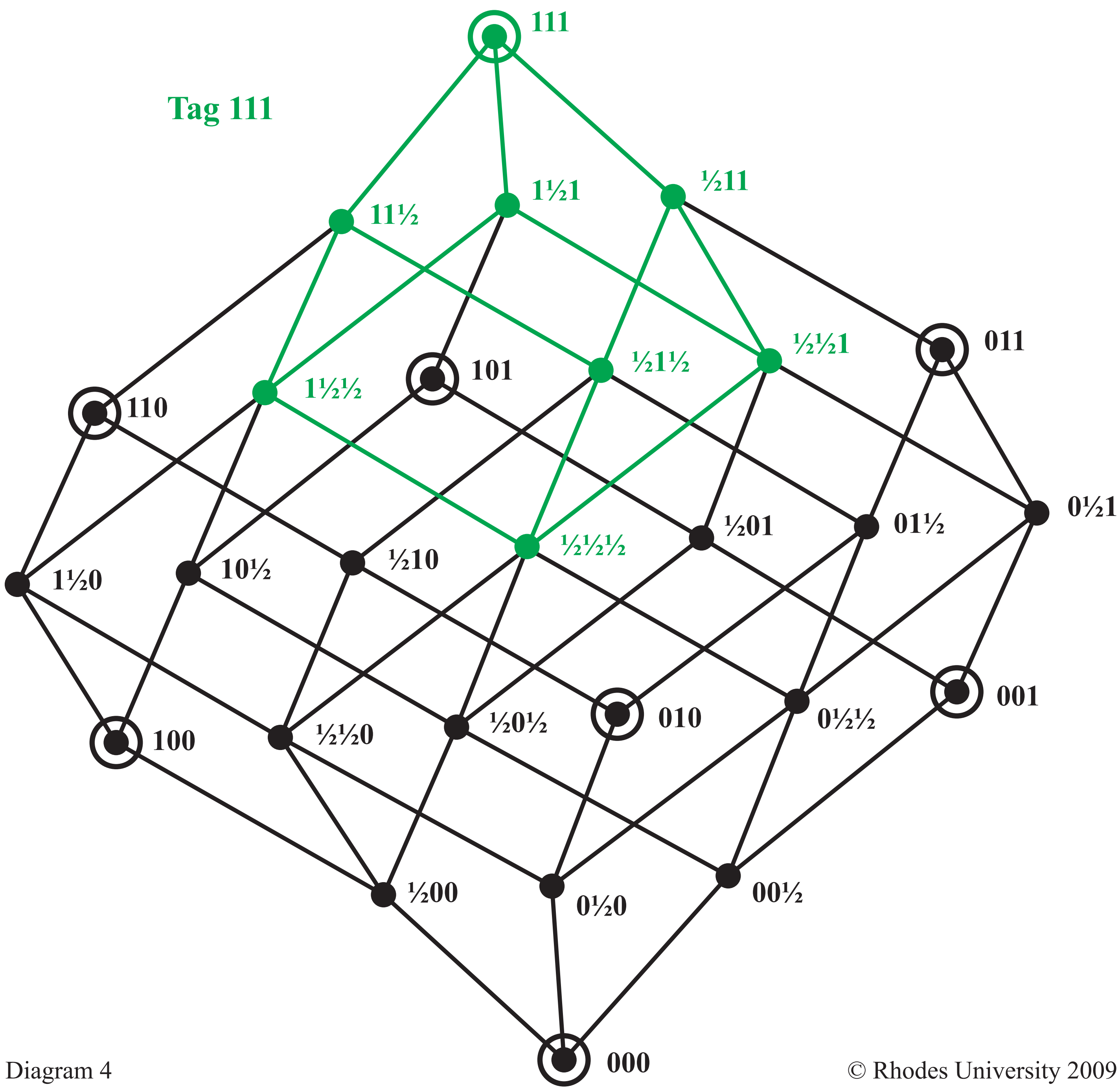

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10½
½10
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001
010
100
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½00
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00½
000

Diagram 4

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