The Emergence of Classical Worlds from a Quantum Universe

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Abstract

How does a classical world emerge from a quantum world? Can this emergence occur without invoking non-unitary processes such as measurements? Recently, an approach that makes use of just a Hilbert space and the associated Hamiltonian to explain the emergence of a classical world has been proposed. To understand this approach, we will require a clear understanding of the nature of measurements in quantum theory and the different interpretations of it. We then progress onto discussions regarding quantum Darwinism and related fields of knowledge and how they "bypass" the problem of measurement in quantum theory. Then, we discuss how, using the appropriate choice of factorization of a Hilbert space into a system and an environment and using an acceptable basis observable, we can obtain a quasi-classical state of a system. This approach has previously been applied to study one limit (when interactions dominate the Hamiltonian), but we generalize by applying it to the opposite limit (when interactions are minimal) and suggest a method for the general case (when interactions are neither minimal nor dominant).

We then look at Hilbert space fundamentalism, which is the idea that a vector in Hilbert space is the fundamental nature of reality. Hilbert space fundamentalism is a generalized application that takes the idea of the emergence of a classical world from a quantum one and applies it to the Universe as a whole. This leads to the question: could Hilbert space fundamentalism be a candidate for the fundamental theory? Before we evaluate Hilbert space fundamentalism as a candidate fundamental theory, we analyze the theory and inquire as to what makes something a fundamental theory. To understand Hilbert space fundamentalism, we see what a model of the world it predicts looks like. This is done by proposing a mapping from a fundamental Hilbert space to emergent space times utilizing entanglement and the aforementioned recently proposed approach that makes use of Hilbert spaces and Hamiltonians to explain the emergence of classical worlds. To determine if Hilbert space fundamentalism could be a fundamental theory, a set of criteria (completeness in all domains, self-contained, and that specific theories emerge from it) is noted. We find that Hilbert space fundamentalism, when viewed through these criteria, cannot be the fundamental theory.

Declaration

I declare that I know what plagiarism is and the consequences of that most serious of offenses. I further declare that all this work is my own based on my understanding of the source materials which are acknowledged and referenced. Further, this thesis has not been submitted to any other institutions for degree purposes.

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Introduction

1.1 Contextualization

Questions about certain aspects of nature, and the theories we use to describe it, have many points of contention. These include the ontological aspects, reversibility of processes in a theory, and ideas pertaining to fundamentality. In this thesis, we shall examine these questions within quantum theory[1] by looking at the emergence of classical worlds from it and analyze an attempt at generalizing quantum theory in the form of Hilbert space fundamentalism as a candidate for the fundamental theory.

1.1.1 Measurements

Quantum physics has always faced a challenge: on one hand, we have a deterministic theory when states evolve according to the Schrödinger equation. On the other hand, when measurements occur, a superposition of states collapses into a single state in an indeterminate manner [1, 2, 3, 4, 5]. Evolutions according to the Schrödinger equation are deterministic since they are evolutions in time by a unitary operator and can be reversed. (A unitary operator is one whose inverse is also the adjoint of the operator.) This is not true when external measurements are

[1]Quantum theory refers to conventional undergraduate-textbook quantum mechanics unless stated otherwise.

introduced. When we measure something, we terminate the Schrödinger evolution and obtain the measured outcome in the form of an eigenstate of the observable we measured. We can then restart the Schrödinger evolution but it is discontinuous from the previous one [6, 7, 8]. Measurements are something we add to the theory to explain how we, as classical observers, obtain information about the quantum realm. A good way to understand the measurement problem is through the use of Schrödinger's cat [9, 10]. In this example, we have a closed box with three things in it. 1) An atom which has the eigenstates of decay and no decay. 2) A vial of cat poison that will either break if the atom decays or else it will not break. 3) A cat that can have the eigenstates of either dead or alive. When sealed in the box, the eigenstates of the cat are entangled with the eigenstates of atom because of the poison. As a result: if the atom is in the eigenstate of decay, then the cat is in the eigenstate of dead and, if the atom is in the do-not-decay state, then the cat is in the alive eigenstate. Once the cat is sealed in the box, we do not know its eigenstates nor the eigenstates of the entangled atom. Our understanding of quantum theory is that the cat in the box is a superposition of both of its eigenstates when it cannot be observed. Further the same is true for the atom being in a superposition of both its eigenstates. When we measure the cat in the box (by opening the box and observing its state), we "break" the quantum superposition state of both dead and alive states and force the cat to have an eigenstate of either dead or alive and the atom to have the corresponding eigenstates of decay or no decay. This serves as a transfer of the information from the entangled state within the box to the observer who records the classical correlations. It must be acknowledged that, while quantum theory does not contradict classical theories in the classical domain,[2] it also does not explain why a classical world exists.

The most common interpretation of quantum theory is the Copenhagen interpretation [1, 3, 11, 12]. In this interpretation, when a measurement is performed,

[2]What we mean by domain is the parameters in which a theory is applicable. For example, the quantum domain is that of $\frac{1}{c} = 0$, $G = 0$, and $\hbar \neq 0$ (with c being the speed of light in a vacuum, G being Newton's gravitational constant, and $\hbar$ being Plank's constant). The classical domain for Newtonian mechanics is that of $\frac{1}{c} = 0$, $\hbar = 0$, and $G \neq 0$.

we collapse the wave-function. But what does that mean? In the example of Schrödinger's cat, we do not know if the cat is dead or alive when it is in the box. We say it is a superposition of both dead and alive states; when, however, we open the box and measure the cat, we can see if it is dead or alive and hence force it to have a single eigenstate. This is what is meant by collapsing the wave function: when something goes from being a superposition of many eigenstates (both dead and alive) to having a single eigenstate (either dead or alive). The collapse of the wave-function cannot be observed – meaning we do not have any way of knowing what occurs during the collapse of the wave function. The Copenhagen interpretation has often been called the "shut up and calculate" approach [3], since proponents of this interpretation tend not to care about the measurement problem. However, there are other ways of interpreting what occurs when we measure the cat in the box.

Apart from the measurement problem, another reason we have other interpretations of quantum theory is the infamous Einstein–Podolsky–Rosen (EPR) paradox in which we have the conclusion of "spooky" action at a distance [13, 14][3]. Consider two electrons in a spin-singlet state (two electrons that have opposing spins and are entangled). We shall label the electrons A and B. This allows us to illustrate the electrons with a wave-function:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|\hat{c}\uparrow\rangle_A|\hat{c}\downarrow\rangle_B + |\hat{c}\downarrow\rangle_A|\hat{c}\uparrow\rangle_B), \tag{1.1}$$

with $\hat{c}$ being a direction that defines the basis. For example, $|\hat{c}\uparrow\rangle_A$ is an eigenstate describing electron A with spin in an upwards orientation in the $\hat{c}$ direction. If two people named Alice and Bob each take an electron (Alice electron A, and Bob electron B), and the electrons are spatially separated by a substantial distance (let us say the one electron accompanies one of them to Paris while the other person–electron pair goes to Tokyo), both of them will have all the information

[3]The version we shall present is not the original version in the EPR paper but rather the Aharonov–Bohm version that has become standard.

about the two-electron system. For instance, Alice can measure her electron's orientation of spin in some random direction and will, through entanglement, know the orientation of the spin of Bob's electron in that same direction no matter the spatial separation between the electrons or the observers. The information Bob potentially has about electron B is dependent on Alice's measurement of electron A's spin [14]. Further, if Alice were to communicate her result to Bob by some causal means, he will then have the same information as Alice concerning the electrons. In the EPR setup, Alice's electron's spin is non-locally correlated with the spin of Bob's electron due to the fact that they are entangled. This was argued to be a result of quantum theory being indeterministic, hence, it was hypothesized that there are hidden parameters that would lead to a deterministic theory[4] [15, 16]. These parameters give us the hidden-variable interpretations of quantum theory[5]. However, using the idea of hidden parameters, we can make predictions that are known as Bell's inequalities [17, 18]. Bell considered the Wigner model, in which the outcome of the measurement of the spin of an electron is predetermined by some hidden variables. If an ensemble of electrons' spin were measured, we would (despite the fact that we cannot know the spin orientation of an electron in more than one direction at a given time), be able to calculate the expected probabilities for all the spin orientations in all possible directions (since they are predetermined). This leads to a set of inequalities that are based on assuming the validity of this model. For instance, one of the Bell inequalities is:

$$P_A(\hat{a}\uparrow;\hat{b}\uparrow) \leq P_A(\hat{a}\uparrow;\hat{c}\uparrow) + P_A(\hat{c}\uparrow;\hat{b}\uparrow), \tag{1.2}$$

where $P_A(\hat{a}\uparrow;\hat{b}\uparrow)$ is the probability that Alice finds her electron to spin "up" in the specified directions. Likewise for $P_A(\hat{a}\uparrow;\hat{c}\uparrow)$ and $P_A(\hat{b}\uparrow;\hat{c}\uparrow)$. The three directions ($\hat{a}$, $\hat{b}$, and $\hat{c}$) might not necessarily be perpendicular but are mutually non-parallel. Quantum theory violates these inequalities while hidden-variable interpretations are supposed to conform to them. This means that locally hidden-

[4] Or, in the language of EPR, a complete theory.

[5] This discussion only applies to locally hidden-variable interpretations, we will discuss non-locally hidden-variable interpretations when we discuss Bohmian mechanics.

variable interpretations are not the same as quantum theory. However, experiments confirm quantum theory and not hidden-variables [19, 20], and hence we can conclude that hidden-variable interpretations of quantum theory as incorrect [21, 22]. Furthermore, even though quantum theory allows for non-local entanglement, this is not necessarily an issue since causality is not violated. Causality is that no faster-than-light communication may occur. In the example, Bob does not know what Alice has measured until he gets a message from her that has either traveled at the speed of light or by some slower means. This means that quantum theory does not violate any consensual physical laws.

Another interpretation that makes use of hidden variables (but not local ones) is Bohmian mechanics which claims that the wave function is only a partial description of a quantum object [5, 24]. To have a complete description of a quantum object, the actual positions of the particles are required. The actual positions evolve according to the "pilot wave" which expresses the velocities in terms of a wave equation. The pilot wave uses hidden variables to describe the evolution of the positions, but these are variables that are not localized: they are Universal. This interpretation is deterministic in the sense that, if we knew the pilot wave equation, we (in theory) could predict every object's behaviour in the universe.

Next, we have interpretations such as the Everettian interpretation that claims that wave-functions are "real" objects (more to follow in section 1.1.2) that do not collapse and evolve according to the Schrödinger equation. Over time, the eigenstates of the systems and eigenstates of the environment become entangled due to interactions. This leads to a one-to-one correspondence between eigenstates of the system and those of the environment. This causes the off-diagonal terms of the density matrix to become less and less influential until they finally decay to zero. This is a reversible process as are all interactions in the Everretian interpretation. However, decoherence is generally considered to be irreversible. This paradox is resolved in the Everettian interpretation because there is a proliferation of systems that interact with the environment in the same way and hence share in this one-to-one correspondence between their own eigenstates and those of the environment. This results in observers having memories of the interactions between the system and environment (observers in this context can also mean environmental records in a manner similar to rocks recording the tide through erosion). Since so many observers share similar memories, they consider the interactions as being irreversible, even though this irreversibility is only apparent. This apparent loss of coherence causes a suppression of interference between different states (as illustrated in Appendix A). As a result, wave-functions appear to split into different branches. Each branch corresponds to a particular potential outcome for a given event [26, 27, 28, 29, 30]. Everettianism insists that the wave-function is an element of reality, so all possible outcomes become actual realities in different branches. Branches that are close to each other will have the highest level of overlap in their states, however their inner-product will still be zero as long as they differ in as little as one quantum number. In the example of Schrödinger's cat, when we open the box, we observe a branching event. In one branch, the cat is dead, while in a different branch, the cat is alive. Some people refer to the branches as worlds, which is why the interpretation is sometimes referred to as "many worlds". We will use branches and worlds interchangeably. It must be noted that different people have different ways of viewing the Everettian interpretation, so that within the interpretation there is no consensus [31].

Of interest to us is as an epistemic interpretation (more to follow in section 1.1.2) known as relational quantum mechanics [10, 11]. Relational quantum mechanics treats all systems and observers as equal (as first proposed in the Ithaca interpretation). In this interpretation, events are ontological and between quantum systems, with the wave-function serving as a book-keeping device to help us understand the relationships between events. Furthermore, there is no collapse of the wavefunction, merely quantum interactions between quantum systems [10, 32, 33, 34]. As a result, it can be viewed as the epistemic counterweight to the Everettian interpretation.

We even have consistent and decoherent histories where the order of events is important [35, 36, 37], solipsist interpretations [38, 39, 40, 41] where the observer's consciousness plays an active role in the collapse of the wave-function and Penrose-like collapse interpretations [43, 44, 45] where gravity plays an active role in the collapse of the wave-function. The list of interpretations of quantum theory is long and vast, and new ones are somewhat common. For further discussions and an extensive catalog, see [5, 46, 47]. Perhaps we could overcome the need for interpretations if we could "bypass" the measurement problem.

1.1.2 Another Nuance Regarding Interpretations of Quantum Theory

Before we discuss a means to "bypass" the measurement problem, we must discuss ontological models and epistemological models [18, 48, 49, 50, 51, 52]. An ontological model strives to be a complete model of reality (more to follow in Chapters 3, 5, 6, and 7), while an epistemological model is one that reflects the understanding of reality (in other words, a representation of reality). With this in mind, we can now discuss two aspects of quantum theory that can be epistemological or ontological, namely the wave-function and what happens when a measurement is performed.

In quantum theory, there is no consensus concerning the ontology of the wave-function amongst physicists or philosophers. Different interpretations have different views concerning the ontological characteristic of the wave-function. The Copenhagen interpretation is agnostic as to whether or not the wave-function is ontological or epistemological. Bohmian mechanics holds the wave-function as well as the "pilot wave" as being ontological. The Everettian interpretation (in general) holds the wave-function to be ontological: it exists and is "hard coded" as part of the universe. An advantage of the Everettian approach is that it allows all possible outcomes in the Universe to be physically realized, just in different branches/worlds. For relational quantum mechanics, by contrast, the only thing that is ontological is events and the relations between them with the wave-function being a "bookkeeping" device to allow us to understand events[6].

We do not know what the ultimate ontology of reality is. This is another reason for different interpretations of quantum theory, since some believe the wave-function must be ontological while others think it is just a tool to understand events (and then there are physicists and philosophers who have no opinion concerning the ontology of the wave-function). For our purposes, a discussion around the ontology processes associated with measurement is as important as the ontology of the wave function.

Measurement is something that brings in significant complications, as previously mentioned, and different interpretations invoke different ways of explaining what occurs when a measurement is performed. In general, we have two models of what occurs: the "collapse" interpretations that view the "collapse" as the means of reflecting our knowledge of what occurs (examples included Copenhagen and consistent histories) and "event" interpretations that view measurements to be ontological and real events (examples include Everettian and collapse interpretations). In the Everettian interpretation, when a measurement is performed, there is a branching event. While in Penrose-like collapse interpretations, parts of the wave-function get removed from reality. One interpretation that does not play nicely into "event" or "collapse" dichotomy is relational quantum mechanics [53, 54]. Proponents of relational quantum mechanics do not hold the measurement to be anything more than an interaction between two or more quantum systems. This is not as drastic as the branching event in Everettian physics but also a rejection of the "collapse" that we have in Copenhagen where we transition from a quantum world to a classical world when performing a measurement.

[6] A recent interpretation that, for our purposes, has a high level of similarities with relational quantum theory is the everything–is–a–quantum–wave interpretation [53].

It is also pertinent that we talk about a special group of interpretations where the measurement is reversible; we can take the wave-function after the measurement and using unitary transformations re-obtain the states before the measurement. Relational quantum mechanics, Bohmian mechanics, and Everettian interpretations fall into this category. In all other interpretations, once the wave function "collapses", the states before the measurement are no longer obtainable.

Interpretation	Wave-function	"Collapse"	Reversibility
Copenhagen (in general)	Agnostic	Epistemological	No
Many worlds	Ontological	Ontological	Yes
Bohmian Mechanics	Ontological	Ontological	Yes
Relational Quantum Mechanics	Epistemological	Controversial	Yes
Penrose Interpretations	Ontological	Ontological	No
Solipsist	Epistemological	Epistemological	No
Consistent Histories	Agnostic	Epistemological	No

Of interest, only three interpretations are reversible; all the others are not. Further, we can see that normally the ontology of the wave-function and that of the "collapse" are highly correlated. However, with three interpretations that does not hold true, namely Copenhagen (which is agnostic about the wave-function and uses the "collapse" as a tool explain the quantum-to-classic transitions), relational quantum mechanics, and consistent histories (which is Copenhagen in formulation with the addition that events act as constraints on the evolution of the wave-function). Relational quantum mechanics is controversial with regards to the "collapse" since it does not view measurement as anything more than the interaction between two quantum systems.

1.1.3 A Way Around the Measurement Problem?

What if, however, we were able to "bypass" the measurement problem as is proposed with the Everettian interpretation? This is what is proposed by quantum Darwinism [2, 30, 55, 56, 57, 58]. It is helpful to start a discussion on quantum Darwinism with a discussion concerning the model of measurement provided by von Neumann. The von Neumann model is that of a quantum system and an apparatus that measures a property of the system. A mathematical description is available in Appendix A. A full model is that of a system and an apparatus immersed in an environment. This has become the standard model of measurements for quantum theory.

Decoherence, in the framework of quantum Darwinism, uses the model of a system, environment, and an apparatus and takes it a step further. Decoherence holds that, through interactions, the system gradually becomes more entangled with the environment over time as discussed previously[7] [1, 8]. The idea is that the system and environment are initially in a pure product state (by assumption), in which case we can obtain all the information about the system by tracing out the environment. As the system and environment become more entangled, tracing out the environment will provide us with less information about the system. In extreme cases, we will obtain a maximally entangled state in which case we would obtain zero quantum information[8] after tracing but will still obtain information in the form of classical correlations. Mathematically, this pattern is reflected by the von Neumann entropy of the system after tracing out the environment, which will be zero when the state is unentangled (the initial state). As the system gets entangled with the environment, the von Neumann entropy will increase. In the time between the state being pure and the state getting maximally entangled, we know that the von Neumann entropy after tracing will monotonically increase

[7] This process produces entropy and hence is correlated with the second law of thermodynamics and related laws of information [59].

[8] By quantum information, we mean mutual information as defined in Appendix B.

from 0 to $\ln N$ [9] (where N is the dimensionality of the system)[10]. It must be stressed that the von Neumann entropy for the total system (the combination of system and environment) is zero at all times given that it starts as a pure state (as shown in Appendix B).

One of the things quantum theory cannot do, because of the measurement problem, is to explain the emergence of the classical world. Quantum Darwinism takes the picture of decoherence (as previously discussed) and uses it to explain the emergence of the classical world. For this, it is useful to introduce the notion of a pointer observable, with which we will explain the emergence of the classical world. A pointer observable is a Hermitian operator whose eigenstates form a complete basis in which the state of the system looks quasi-classical [61]. These states are called pointer states. We can discuss classicality in terms of three qualitative features [57]; a classical state must have a robust pointer observable, be predictable, and have a high level of redundancy.

Robustness: suppose the system is, initially, approximately unentangled with the environment (i.e. the von Neumann entropy is much less than $\ln N$ as previously discussed). Then, there would be a minimal amount of eigenstates that share the one-to-one correspondence between system and the environment. Then, the pointer states of the system will be restricted to this minimal number of eigenstates that share the aforementioned correspondence. This leads to locality (as will be established throughout this section and has been established in the literature as can be seen in [2, 30, 56]) if the pointer states are initially localized in phase space (by being peaked together in some basis). Further, suppose that the rate of entanglement growth is sufficiently gradual so that the system does not become more entangled with the environment during the time scales of interest. These time scales could correspond to the characteristic time scale of the system of interest. For example, for a simple harmonic oscillator, the characteristic time

[9] With the Boltzmann's constant set to 1.

[10] Assuming the system to be the smaller of the system and environment pair.

scale is the inverse of the frequency. Then, the number of eigenstates sharing the one-to-one correspondence between the system and environment will only slowly increase. This means that the pointer states will remain localized in phase space if they started localized. Since we want a gradual growth in entanglement to maintain locality, we can measure robustness with the rate of change of the von Neumann entropy.

Predictability: consider the scenario where we have a basis in which the Hamiltonian's interactions are mostly short-range (for instance, nearest neighbor and/or next nearest neighbor). Furthermore, suppose that the eigenstates of a pointer observable for a system are represented by a probability distribution that, for example, is approximately Gaussian and remains approximately Gaussian (effectively no change in our aforementioned time scale). Any highly peaked distribution with minimal variance would do, but we are using Gaussian distributions for illustrative purposes. We can say that our system is predictable. This means that the trajectory of the system's state will be effectively localized in phase space with respect to the pointer basis. To measure this, we will use a metric called the pointer entropy (to be formally defined at a later point) which will measure a quantity similar to the variance of a pointer observable. Figures 1.1, 1.2, and 1.3 are all distributions, with 1.2 being the predictable distribution.

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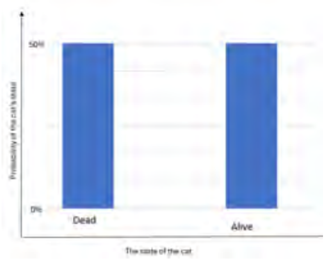

Figure 1.1: Probability distribution for Schrödinger's cat. It is not possible to obtain predictability from this distribution.

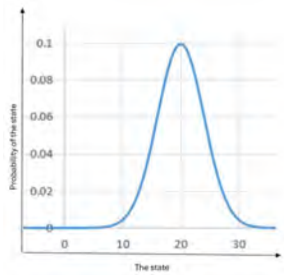

Figure 1.2: An ideal distribution for a predictable system. All the eigenvalues are localized on a basis.

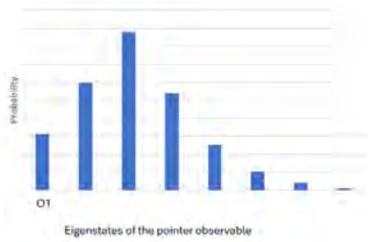

Figure 1.3: A distribution that is less predictable.

Redundancy: suppose we have information about the system classically correlated with observables in the environment. Further, the aforementioned information is stored with abundance in the environment. For instance, it is duplicated on the scale of $R \gg 1$ with R being the number of replicas. Then, the information is said to be redundant, and the system has a high level of redundancy. Unlike the previous qualitative features of classicality, redundancy does not have a measure in the current framework.

Quantum Darwinism holds that the reason we have a classical world is that some states are “fitter” than others [56, 57, 60]. These states have a high level of robustness and predictability. Further, these states are more redundant, meaning there is a proliferation of the information about these states in the universe.

Do decoherence and the related idea of quantum Darwinism get rid of the measurement problem? The answer is yes, but only in interpretations (such as the

Everettian interpretation) where measurements are reversible. However, quantum Darwinism does help us model how a classical world emerges from a quantum one.

To illustrate the ideas of decoherence and quantum Darwinism, we will use the example of a brick wall. The brick wall, as a system, maintains a reasonably constant defined location (within the uncertainty limits). Therefore, when we observe a brick wall and change our observation to another object, we can anticipate that our second observation of the wall will not be different to the first unless some physical action was performed on the wall. This is because the information about the wall is available in the environment, since the information was taken from the wall by photons reflecting off the wall and hence is stored in the environment. When we look at the wall, the information about the wall is in the photons that enter our eyes and allow us to obtain information about the wall without us measuring it. If we call one thousand more people to look at the wall, they will all agree with what we see concerning the wall because there are so many photons that the information is highly redundant. In this example, the photons are classically correlated with the pointer observable which is the position of the wall; the photons are not a part of the wall (the system) but a part of the environment that the wall is in. The position of the brick wall is predictable (it will behave how we expect it to), and the position of the wall is robust (as it does not get entangled with the photons or the remainder of the environment).

However, in this example, locality is clearly in terms of position. This is because the Hamiltonian of the wall has short range interactions in the position basis. However, not all Hamiltonians will behave in that manner. Consider the example of a black body's radiation. In this example, the Hamiltonian's interactions are not short range in position space but rather in momentum space because the frequency of radiation is peaked highly around a certain value. We as observers only think of locality in terms of position because the main way that we observe reality is with our eyes leading to pointer basis in position. However, some hypothetical observer who uses momentum, for example, to view reality will observe locality as the Hamiltonian's interactions being short-ranged in momentum space.

1.1.4 A Different Approach to Bypassing the Measurement Problem

We want a way to mathematically model how a pointer observable can give us a quasi-classical picture of reality. Our starting point is a Hilbert space and a Hamiltonian. What we shall do is factorize the Hilbert space and Hamiltonian into a system and environment, with the best possible choice of pointer observable and factorization leading to a potentially classical picture of the system of interest. How do we choose the correct pointer observable and factorization for a given Hilbert space and Hamiltonian? For this, we can use an algorithm as described below.

We start by defining candidate pointer observables and candidate factorizations which are suitable for the given Hilbert space and Hamiltonian. We look at the rates of change of the two entropies for each candidate pair. These entropies are the von Neumann entropy (which will measure robustness) and the pointer entropy (which is a measure of predictability). The candidate pair with the lowest entropies is the choice of factorization and pointer observable that will provide the most classical state of the system.

1.1.5 An Introduction to "Hilbert Space Fundamentalism"

Another challenge faced by physics concerns the nature of a fundamental theory. Firstly, we do not have a clear idea of what is fundamental. We will often make circular definitions such as: 'What is fundamental? Something is fundamental when it is the most basic explanation. What is the most basic explanation? It is the one that is fundamental!' Apart from the circularity of the example just provided, another question is also raised: what is meant by basic? Basic can mean a host of things ranging from easy to fundamental! Everyone will agree that Newton's laws are amongst the most basic physics to understand in the sense of the basic meaning 'simple to understand'. In fact, we often use the word fundamental to describe quantum field theory and general relativity even though neither can cover all physics and hence require a more fundamental theory. Quantum field theory cannot describe cosmological scales, and general relativity cannot explain subatomic behaviours. However, both are more fundamental than classical mechanics. This brings the important caveat: something can be more fundamental than something else while itself not actually being fundamental. While string theory can have both standard quantum field theory and general relativity emerge from it in specific limits, it no longer is believed to be the most fundamental theory [63]. In fact, string theory is believed to represent five special limiting cases of some more fundamental theory. For instance, certain variants of 10-dimensional string theory are considered to be limiting cases of a particular 11-dimensional theory [64]. A proposed hierarchy of theories can be found in figure 1.4 with the higher a theory is in the figure, the more fundamental it is meant to be. Furthermore, while the fundamental theory will have all the features of a theory of everything, not all theories of everything would necessarily be the fundamental theory.

But what if we were to think that all of reality could be described as a vector in a Hilbert space? This has been argued in *Reality as a Vector in Hilbert Space* [65]. This approach has and will be called "Hilbert space fundamentalism"[11][66]. Hilbert space fundamentalism posits two things. The first is that the complete state of the Universe can be described as a vector in a complex Hilbert space. This is mathematically the same as quantum theory but conceptually different since there is no assumed classical structure nor a specifying algebra of observables (such as position and momentum) having physical meaning. Hilbert space fundamentalism is not quantum theory; it just happens to use the same type of Hilbert space that quantum theory uses (namely a complex space with an inner product rule). The second postulate is that this aforementioned vector is an on-

[11]It has also been called "Mad Dog Everettianism".

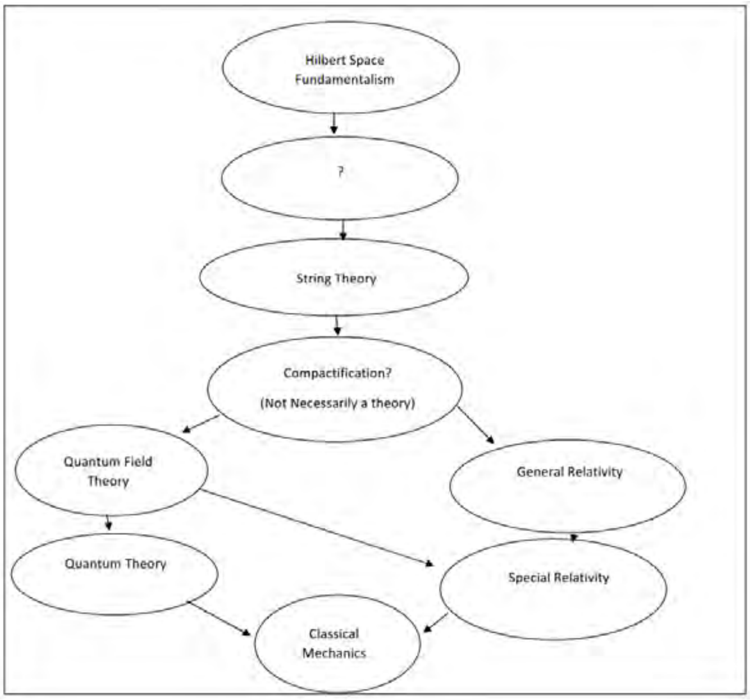

Figure 1.4: The proposed hierarchy of theories with the bottom theories being less fundamental and as we go to the top of the figure, we get the most fundamental.

tological object. This vector can be viewed as analogous to the wave-function in quantum theory. We, however, want to add an additional postulate: the "collapse" of the vector (which will be analogous to the "collapse" of the wave-function) must be reversible. This results in only one viable framework, namely the formulation of Hilbert space fundamentalism uses the Everettian world view (which will be critiqued later with an alternative being examined). If this approach can give a working explanation of the Universe on a fundamental level, can it be considered to be the fundamental theory?

We shall be examining two things with Hilbert space fundamentalism. The first is to use it as a "test theory" to discuss what is required for a theory to be the fundamental theory. The reason we use Hilbert space fundamentalism is that it uses the same maths and hence intuition as quantum mechanics. The second reason we will use Hilbert space fundamentalism is to see if it can be the fundamental theory and then discuss the implications that brings.

If Hilbert space fundamentalism is the fundamental theory, how will specific theories[12] emerge from it? How general relativity and quantum field theory emerge will not be a straightforward process as we do not know the steps between them and Hilbert space fundamentalism. However, we assume that specific theories are contained within the vectors that are contained in accessible parts of Hilbert space (which will be elaborated on shortly). What we can now discuss is how, according to Hilbert space fundamentalism, our Universe is the way it is and could be stable. The Universe is constantly changing provided we have a non-trivial Hamiltonian. However, the types of change available to the Universe are suitably restricted, provided there is a basis in which the Hamiltonian acts locally, and the states of the Universe start in a region for which it has a quasi-classical description. This affords us the ability to observe it as being predictable and robust, as previously described. If there are classical correlations between states with a high redundancy between accessible and inaccessible regions of Hilbert space (analogous to the system and environment), then the analogy to quantum Darwinism will be complete.

This can be framed in terms of accessible and inaccessible regions of Hilbert space. Inaccessible regions are ones we do not know about or have any idea of what occurs in them. What we can, however, describe is what cannot be in the accessible regions. Accessible regions support life because they have to be similar to what we experience. Each choice and/or observation in the accessible world within the Hilbert space corresponds to a jump to adjacent branches of the many-worlds vector. What follows are examples of what we cannot have in the accessible regions: "Schrödinger's cat" jumps where the Universe changes between two nearly orthogonal states; negative curvature of space-time which would result in the observable Universe having very short life spans [70]. Other inaccessible possibilities might be Boltzmann brain dominated [71] (which cause stellar systems to appear and vanish at random) or even worlds where space-time has no classical description

[12] Specific theories are ones that explain a certain set of phenomena but not all.

(sometimes called the "swamp land" of theory space) [72].

1.2 Primary Source Material

The work within this thesis is in response to the work of Carroll and Singh as a collective in [61, 66] and Carroll alone in [65]. However, in [61], the algorithm presented was only examined in the quantum measurement limit (when the interaction Hamiltonian dominates the Hamiltonian). We will be extending on it by exploring the algorithm in the general case and decoherence limit (when the self-Hamiltonian dominates the Hamiltonian). Furthermore, despite claims of the fundamentality of the Hilbert space in [65] and [66], neither have looked at the validity of "Hilbert space fundamentalism" as a fundamental theory, which we shall do.

1.3 Structure and Summary of Contents

In this thesis, we will analyze how to use pointer observables and factorizations to obtain a quasi-classical description of a system. Further, we will examine "Hilbert space fundamentalism" as a generalized application of emergent quasi-classical descriptions. In Chapters 2, 3, and 4, we will discuss obtaining quasi-classical descriptions. To assist in the discussion regarding quasi-classical descriptions, we will illustrate certain tools that can be used to help identify these descriptions. Namely, collimation (which measures the propensity of the Hamiltonian to evolve the state classically) and the Carroll and Singh algorithm. Chapters 5, 6, and 7, we will look at Hilbert space fundamentalism as a theory applicable to all domains and examine if it can be considered a fundamental theory.

1.3.1 Chapter 2

The aim of Chapter 2 is to lay the foundations required for obtaining a quasi-classical description. The first section is concerned with how a Hilbert space can be factorized, what the wave-function is, the canonical conjugation pairs, and the Hamiltonian. These topics will mostly cover standard textbook quantum mechanics with special emphasis on ideas that will be used as a starting point for the emergence of quasi-classical worlds.

The second section of this chapter will explain the content that is specific to pointer observables, with the exception of linear entropy (which appears quite often in other branches of quantum theory). We will give a full definition of pointer observables, the special relationships they can have with the Hamiltonian, and how those relationships can be used to define limits. Thereafter, we will discuss how we will use entropies to measure how predictable and robust something is.

1.3.2 Chapter 3

In the third chapter, we will discuss Everettian quantum theory. Then, we will discuss the dimensionality of the Hilbert space in quantum theory. Lastly, we will discuss the Everettian interpretation's world view.

1.3.3 Chapter 4

In this chapter, we will first formulate a general form for possible candidate pointer observables. We will need to create an expression that works when: 1) the self-term of the Hamiltonian dominates, 2) the case when the interaction terms of the Hamiltonian dominate, and 3) when neither the interaction nor self terms dominate. Then, we will discuss the algorithm and state each step of the algorithm. We will then discuss what collimation is, and how it informs us of Hamiltonians that might have quasi-classical states of the system; furthermore, how collimation could be used as an alternative method to find predictable states of the system. Lastly, we will examine classicality with the example of the coupled oscillator.

1.3.4 Chapter 5

In Chapter 5, we will discuss Hilbert space fundamentalism: what it entails, as well as its potential utility and potential problems. This will entail discussing the emergence of space-times from the Hilbert space. We will also discuss elements such as its dimensionality, the reason multi-worlds do not mean infinite degrees of freedom, and why we have a stable world. Lastly, we will discuss the shortcomings of this approach.

1.3.5 Chapter 6

Chapter 6 will explore what it entails for something to be fundamental and to be a theory. We will look at properties of theories and the different subsets that theories can fall into. Further, we look at both what physics literature and literature of metaphysics tell us about fundamentalism. With this, we will then present a set of criteria to determine if a theory is fundamental or not. These criteria are a complete theory in all domains, that specific theories must emerge from the fundamental theory and that the theory must be self-contained.

1.3.6 Chapter 7

We will use the criteria to examine if Hilbert space fundamentalism is the fundamental theory. This will be done by going through each criterion devised in Chapter 6. We will further discuss possible changes to Hilbert space fundamentalism and see if they could have better success in meeting the criteria.

1.3.7 Chapter 8

In this chapter, we will discuss the merits and flaws of all the analysis within this thesis. This will be done by first looking at the proposed emergence of classicality from a quantum world, what we can conclude from it and the possible flaws in our analysis. Then, we shall turn our attention to Hilbert space fundamentalism and discuss its strengths, limitations, and implications.

1.3.8 Appendices

The appendices will document any long derivations and calculations that are used. The main text will generally present starting points and results but not the full calculations. Further, it will contain mathematical models and examples that are supplementary to the main text. Additionally, the appendices will include discussions of philosophical concepts that are used (particularly in Chapter 6).

In Appendix A, we will present the von Neumann model for measurement in a formal way. In Appendix B, we will present the example of two coupled electrons and how the entanglement of them leads to an increase in mutual information.

Appendix D will record the calculations concerning the linear entropy for the setup of a system in an environment. Appendix E will concern itself with calculating the general pointer entropy.

In Appendix F, we will present a derivation for the collimation functional which tells us which operators are collimated.

In Appendix G, we will present the 4 schools of thought concerning fundamentalism in metaphysics definitions. The first one we will define is the absolute independence. We will then define a similar one, restricted independence. Then, complete minimal basis will be defined. Lastly, we will discuss primitivism.

1.4 Conventions Used

Capital Latin letter subscripts refer to regions, while lower case Latin letters will be indices when in the subscript. Symbols with a "hat" or circumflex are operators. A dot above a symbol means we have differentiated the symbol by time, for example $\dot{X} = \frac{dX}{dt}$. With differentiation with respect to time, we will be using the Schrödinger picture.

Throughout this thesis, we have set K_B, c and $\hbar$ to 1.

In the table that follows, we will refer to $\hat{H}_A$ as the Hamiltonian for region A, but in general it will be the system and $\hat{H}_B$ will be the Hamiltonian for the environment.

Additionally, in the table we refer to $\hat{A}$ and $\hat{B}$ as A interactions and B interactions respectively. This is shorthand for $\hat{A}$ and $\hat{B}$ being the hermitian operators that we have used to diagonalize the interaction terms of the Hamiltonian.

Symbol	Meaning
$\mathcal{H}$	Hilbert space
$\mathcal{H}_A$	Hilbert space for region A
$\hat{H}$	Hamiltonian
$\hat{H}_A$	Hamiltonian for region A
D	Dimensions of the Hilbert space
d_A	Dimensions for region A of the Hilbert space
$\|\Psi\rangle$	Wave-function / state vector
ρ	Density matrix
$\hat{\phi}_i, \hat{\Pi}_i$	Pair of canonical conjugates
$\hat{I}$	The identity operator
δ_{ij}	The Kronecker delta symbol
λ_i	Dimensional coupling strength
$_A$	The system
$_B$	The environment
Ω	Elements of the hermitian generators
$\hat{A}_i$	A interactions
$\hat{B}_i$	B interactions
$\hat{V}(\hat{\phi})$	The potential part of the Hamiltonian
$\hat{D}_i(\hat{\phi})$	Coefficient of a power series in $\hat{\Pi}$
$\hat{T}(\Pi)$	The dynamical/kinetic part of the Hamiltonian
$\hat{U}$	Generic unitary operator
$\hat{O}$	A generic observable
$\hat{O}_A$	Pointer observable for A

Symbol	Meaning
$\|o_i\rangle$	Eigenstates of $\hat{O}_A$
$\hat{\Lambda}_i$	The projection operator
$\hbar$	Planck's constant
c	Speed of light
S_{lin}	Linear entropy
$\mathcal{O}()$	The order of
$\triangle$	Variance of
$S_{pointer}$	Pointer entropy
A	Area
G	Gravitational constant
r	Radius
$\hat{O}_{CPO}$	Candidate pointer observable
$\hat{x}$	The position operator
$\hat{p}$	The momentum operator
K_B	Boltzmann constant
$p_j(t)$	Probability
$C_{\hat{\phi}}(\hat{M})$	Collimation in $\hat{\phi}$
$C_{\hat{\Pi}}(\hat{M})$	Collimation in $\hat{\Pi}$
N	Number of nodes
H_0	Hubble constant
$a(t)$	Scale factor
t_H	Hubble time

Chapter 2

A Mathematical Prelude

2.1 Textbook Quantum Mechanics

This section provides a limited literature review of standard textbook quantum mechanics, as can be seen in [1].

2.1.1 The Space

A Hilbert space is a complex vector space along with an associated product rule which will be explained in section 1.1.2. If the vectors within a Hilbert space are consistently separable into different parts, the Hilbert space can be decomposed into corresponding subsystems. Suppose we have a system with two degrees of freedom (for instance, two particles) that can be separated into two subsystems, then we can say:

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2, \tag{2.1}$$

where $\mathcal{H}_1$ is the first subsystem and $\mathcal{H}_2$ is the second subsystem (with subscripts of 1,2,3... for generic subsystems). In general, a subsystem can have more than one degree of freedom associated with it and typically has many degrees of freedom. For a large but finite number of degrees of freedom, we can decompose a separable

Hilbert space into multiple subsystems such that:

$$\mathcal{H} = \bigotimes_{n=1}^{N} \mathcal{H}_n, \tag{2.2}$$

where N is the number of subsystems. We then can express the dimensionality of the separable Hilbert space as:

$$\dim(\mathcal{H}) = \prod_{n=1}^{N} \dim(\mathcal{H}_n). \tag{2.3}$$

For this thesis, we will not be working in such general terms but will rather reduce our analysis to the picture of two subsystems, namely a system (the region of the Hilbert space we are interested in) and an environment (the remainder of the Hilbert space). These are different from the subsystems provided in equation 2.1 since the system and environment will each have much more than one degree of freedom. The system will be indicated by $\mathcal{H}_A$, and the environment by $\mathcal{H}_B$ (see figure 2.1). Whenever an operator or a wave-function has the subscript A, it is associated with the system, likewise for the environment with the subscript B.
Equation 2.2 will now reduce into:

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B. \tag{2.4}$$

Now equation 2.3 becomes:

$$\dim(\mathcal{H}) = \dim(\mathcal{H}_A) \dim(\mathcal{H}_B). \tag{2.5}$$

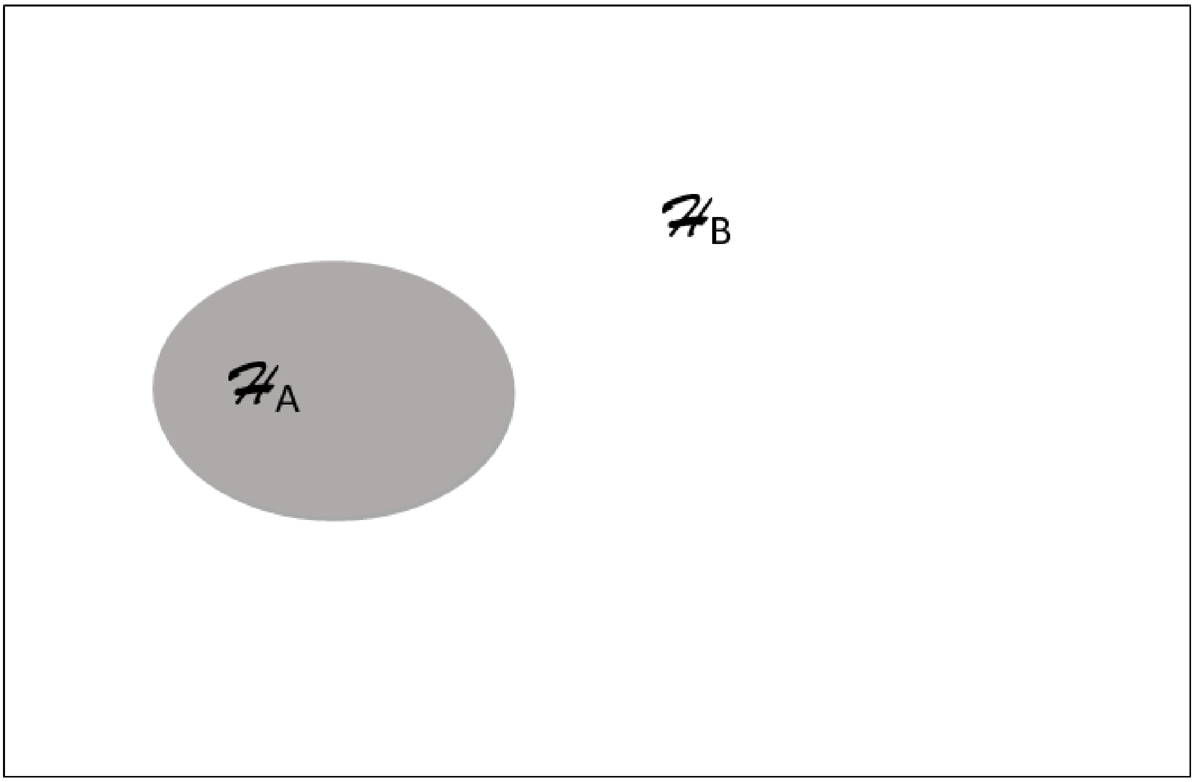

Figure 2.1: A system in an environment.

For the sake of convenience, we define the following quantities, as well as assume the inequalities on the far right:

$$\begin{aligned} D &= \dim(\mathcal{H}) \gg 1, \\ d_A &= \dim(\mathcal{H}_A) \gg 1, \\ d_B &= \dim(\mathcal{H}_B) \gg 1. \end{aligned} \tag{2.6}$$

A feature of Hilbert spaces is that two spaces with the same dimensionality are isomorphic [69]. This means the structures are identical and they have a one-to-one reversible mapping from one space to another with the same dimensionality. This results in observers unable to use just a system's Hilbert space to distinguish it from another. Rather, we will use an operator; in particular, the Hamiltonian since it is the only Universal operator.

2.1.2 Quantum States

We can express vectors as quantum states by denoting them as kets: $|\Psi\rangle$. Further, each ket has a corresponding bra[1]: $\langle\Psi|$, which is a member of the dual Hilbert space. For the sake of convenience, we will treat all kets and bras as being normalized going forward. Kets and bras can be multiplied to form inner products ($\langle\Psi_1|\Psi_2\rangle$) and outer products ($|\Psi_1\rangle\langle\Psi_2|$). Inner products are maps from pairs of states to complex numbers with magnitudes in the range between zero and one. Further there is the postulate: $\langle\Psi_2|\Psi_1\rangle^* = \langle\Psi_1|\Psi_2\rangle$ (where * denotes the conjugate). The inner-product mapping plus the aforementioned postulate forms the product rule for the Hilbert space in quantum mechanics (see section 2.1.1). There are two special cases for the inner product; namely parallel states (states that are the same): $\langle\Psi_1|\Psi_1\rangle = 1$, and orthogonal states (states that have no overlap): $\langle\Psi_1|\Psi_2\rangle = 0$.

The outer product forms an operator: $\hat{O}_{12} = |\Psi_1\rangle\langle\Psi_2|$. Operators transform states in quantum theory: $\hat{O}_{12}|\Psi_3\rangle = |\Psi_1\rangle\langle\Psi_2|\Psi_3\rangle$. An important class of operators is Hermitian operators: $\hat{O} = \hat{O}^\dagger$ (in other words, the operator is equal to its own adjoint). Hermitian operators are associated with physical observables. The bases of Hilbert spaces can be expressed by the eigenstates of a Hermitian operator. Bases of an Hermitian operator are complete if there are no degeneracies. (There are no degeneracies if no two eigenstates share an eigenvalue). If there are degeneracies, the eigenstates of at least one other Hermitian operator will be required to form a complete basis for a Hilbert space. Any operator can be expressed as a linear combination of outer products. A mathematical definition for an observable in terms of its eigenstates is:

$$\hat{O} = \sum_{i=1}^{D} o_i|\psi_i\rangle\langle\psi_i| = \sum_{i=1}^{D} o_i\hat{\Lambda}_i, \tag{2.7}$$

[1] One can think of kets as column vectors and bras as row vectors.

where $|\psi_i\rangle$ is an eigenstate, o_i is its corresponding eigenvalue and $\hat{\Lambda}_i$ is a projection operator.

Due to the linearity of quantum theory, we can express states in the previously discussed two-subsystem Hilbert space:

$$|\Psi\rangle = \sum_{i=1}^{d_A}\sum_{j=1}^{d_B} c_{ij}|\Psi_{A_i}\rangle|\Psi_{B_j}\rangle. \tag{2.8}$$

Here and going forward subscripts A_i, B_j and so forth are indices corresponding to the complete set of bases states for the respective subsystems. The complex constants c_{ij} satisfy $\sum_{i,j}|c_{ij}|^2 = 1$ due to the normalization condition.

Equation 2.8 includes two limiting cases, namely a pure product state:

$$|\Psi\rangle = |\Psi_A\rangle|\Psi_B\rangle \tag{2.9}$$

and a maximally entangled state:

$$|\Psi\rangle = \sum_{i=1}^{d} c_i|\Psi_{Ai}\rangle|\Psi_{Bi}\rangle, \tag{2.10}$$

where $d = \min[d_A, d_B]$.

An important relationship to note is the inner product: $\langle\Psi_{A_i}|\Psi_{A_j}\rangle = \delta_{ij}$ and similarly: $\langle\Psi_{B_i}|\Psi_{B_j}\rangle = \delta_{ij}$.

In most calculations, we will rather use the density matrix. It is defined as:

$$\hat{\rho} = |\Psi\rangle\langle\Psi|. \tag{2.11}$$

This however can be expanded by using equation 2.8:

$$\hat{\rho} = \sum_{i=1}^{d_A}\sum_{j=1}^{d_B}\sum_{k=1}^{d_A}\sum_{l=1}^{d_B} c_{ij}c_{kl}^{*}|\Psi_{A_i}\rangle|\Psi_{B_j}\rangle\langle\Psi_{A_k}|\langle\Psi_{B_l}|, \tag{2.12}$$

with which we can define the trace and the partial trace over a single subsystem of a density matrix. Respectively:

$$\begin{aligned}
\mathrm{Tr}(\hat{\rho}) &= \mathrm{Tr}\left(\sum_{i=1}^{d_A}\sum_{j=1}^{d_B}\sum_{k=1}^{d_A}\sum_{l=1}^{d_B} c_{ij}c_{kl}^{*}|\Psi_{A_i}\rangle|\Psi_{B_j}\rangle\langle\Psi_{A_k}|\langle\Psi_{B_l}|\right), \qquad (2.13)\\
&= \sum_{m=1}^{d_A}\sum_{n=1}^{d_B}\sum_{i=1}^{d_A}\sum_{j=1}^{d_B}\sum_{k=1}^{d_A}\sum_{l=1}^{d_B} c_{ij}c_{kl}^{*}\langle\Psi_{A_m}|\Psi_{A_i}\rangle\langle\Psi_{B_n}|\Psi_{B_j}\rangle\langle\Psi_{A_k}|\Psi_{A_m}\rangle\langle\Psi_{B_l}|\Psi_{B_n}\rangle,\\
&= \sum_{m=1}^{d_A}\sum_{n=1}^{d_B}\sum_{i=1}^{d_A}\sum_{j=1}^{d_B}\sum_{k=1}^{d_A}\sum_{l=1}^{d_B} c_{ij}c_{kl}^{*}\delta_{mi}\delta_{nj}\delta_{km}\delta_{ln} = \sum_{m=1}^{d_A}\sum_{n=1}^{d_B}|c_{mn}|^2.
\end{aligned}$$

The partial trace is defined as:

$$\begin{aligned}
\mathrm{Tr}_A(\hat{\rho}) &= \mathrm{Tr}_A\left(\sum_{i=1}^{d_A}\sum_{j=1}^{d_B}\sum_{k=1}^{d_A}\sum_{l=1}^{d_B} c_{ij}c_{kl}^{*}|\Psi_{A_i}\rangle|\Psi_{B_j}\rangle\langle\Psi_{A_k}|\langle\Psi_{B_l}|\right), \qquad (2.14)\\
&= \sum_{m=1}^{d_A}\sum_{i=1}^{d_A}\sum_{j=1}^{d_B}\sum_{k=1}^{d_A}\sum_{l=1}^{d_B} c_{ij}c_{kl}^{*}\langle\Psi_{A_m}|\Psi_{A_i}\rangle|\Psi_{B_j}\rangle\langle\Psi_{B_l}|\langle\Psi_{A_k}|\Psi_{A_m}\rangle,\\
&= \sum_{m=1}^{d_A}\sum_{j=1}^{d_B}\sum_{l=1}^{d_B} c_{jm}c_{ml}^{*}|\Psi_{B_j}\rangle\langle\Psi_{B_l}| = \sum_{j=1}^{d_B}\sum_{l=1}^{d_B} c_j c_l^{*}|\Psi_{B_j}\rangle\langle\Psi_{B_l}|,
\end{aligned}$$

where: $c_j c_l^{*} = \sum_{m=1}^{d_A} c_{jm}c_{ml}^{*}$.

2.1.3 Canonically Conjugate Pairs

For this thesis, we are interested in canonically conjugate pairs. This is because canonically conjugate pairs illustrate classicality through the familiar case of position and momentum. We know that the classicality of the state of the system is related in part to locality, which itself depends on the Hamiltonian being expressible in bases such that it is short-ranged in interactions. It should be possible to describe such bases as eigenstates of operators that are members of canonically conjugate pairs[2], which we will call position-like operators ($\hat{\phi}_i$). The canonically conjugate partners ($\hat{\Pi}_j$) will generate translations in the $\hat{\phi}_i$ operators' bases. These operators ($\hat{\Pi}_j$) will be called the momentum-like operators. The canonically conjugate pairs have the following relations[3]:

$$\begin{aligned} \left[\hat{\phi}_i, \hat{\Pi}_j\right] &= i\delta_{ij}, \\ \left[\hat{\phi}_i, \hat{\phi}_j\right] &= 0, \\ \left[\hat{\Pi}_i, \hat{\Pi}_j\right] &= 0. \end{aligned} \tag{2.15}$$

From this point forward, however, we will sometimes assume just one pair for the system and environment respectively ($\left[\hat{\phi}_A, \hat{\Pi}_A\right] = i$ and $\left[\hat{\phi}_B, \hat{\Pi}_B\right] = i$), for the sake of mathematical convenience.

2.1.4 The Hamiltonian

The Hamiltonian of a Hilbert space, $\hat{H}$, is a bounded operator that is the generator of time evolution of states in the Schrödinger picture of quantum theory: $\hat{U}(\hat{H})|\Psi(t_o)\rangle = |\Psi(t_1)\rangle$, where $\hat{U}(\hat{H}) = e^{-i\hat{H}(t_1-t_0)}$ is the evolution operator. (In the Heisenberg picture of quantum mechanics, the Hamiltonian generates time evolution in operators.) Bounded operators are Hermitian operators whose eigenvalues cannot be lower and/or higher than a certain value. (In the case of a Hamiltonian, we have at least a lower bound.) A Hamiltonian can be decomposed such that each subsystem of the Hilbert space has an associated self-Hamiltonian ($\hat{H}_{self}$); additionally, there can be interaction terms coupling the subsystems ($\hat{H}_{int}$) [74]. This can be expressed as:

$$\hat{H} = \hat{H}_{self} + \hat{H}_{int}. \tag{2.16}$$

[2] We can express Hermitian operators in terms of canonically conjugate pairs [73].

[3] These expressions are technically correct only for infinite-dimensional Hilbert spaces. However, even if finite, the Hilbert space and each of its subsystems has a sufficiently large dimensionality (see equation 2.6) so that the corrections will be negligible.

We consider the two-subsystem situation of a system and environment. Hence the self term will be split into $\hat{H}_A$ and $\hat{H}_B$. Further, we can express the interaction Hamiltonian by expanding it in terms of an operator basis to obtain the following [74]:

$$\hat{H} = \hat{H}_A \otimes \hat{I}_B + \hat{I}_A \otimes \hat{H}_B + \sum_{a=1}^{d_A^2-1} \sum_{b=1}^{d_B^2-1} h_{ab} \left(\hat{\Omega}_a \otimes \hat{\Omega}_b \right), \tag{2.17}$$

where $\hat{\Omega}_a$ are elements of the Hermitian-generator basis for the system and $\hat{\Omega}_b$ are elements of the Hermitian-generator basis for the environment. These can be viewed as arbitrary-dimensional generalizations of the Pauli spin matrices [75]. (The Pauli spin matrices apply to subsystems or systems with a dimensionality of 4.) Following from section 1.1.1, d_A^2 and d_B^2 serve as the dimensionality of the operator basis of the respective subsystems. Further, we have to subtract one from the upper limit of the summation to exclude the identity operators from the sum. The identity operators are given by $\hat{I}_A$ and $\hat{I}_B$ for the system and environment respectively. The tensors denoted by h_{ab} are real constants. Assuming we can diagonalize $\hat{H}_{int}$ in a conventional sense, our expression is:

$$\hat{H} = \hat{H}_A \otimes \hat{I}_B + \hat{I}_A \otimes \hat{H}_B + \sum_{\nu=1}^{P} \beta_\nu \left(\hat{\hat{A}}_\nu \otimes \hat{\hat{B}}_\nu \right), \tag{2.18}$$

with $\hat{\hat{A}}_\nu$ and $\hat{\hat{B}}_\nu$ being linear combinations of Hermitian generators of the system and environment's basis respectively. Further, $P = \min[d_A^2 - 1, d_B^2 - 1]$ (this makes sure that the sum has the form of a diagonalized square matrix) and λ_ν can be viewed as the coupling strength. For later convenience, we assume β_ν to have a dimensionality of energy with $\hat{\hat{A}}$ and $\hat{\hat{B}}$ being scaled to be dimensionless.

We assume the state of a system to behave quasi-classically with the correct choice of basis. When the Hamiltonian has a basis where it acts locally, it can be assumed to be diagonalized in canonically conjugate pairs:

$$\hat{H} = \hat{H}_A \otimes \hat{I}_B + \hat{I}_A \otimes \hat{H}_B + \sum_{\nu=1}^{P} \lambda_\nu \left(\hat{A}_\nu(\hat{\phi}_{A_\nu}, \hat{\Pi}_{A_\nu}) \otimes \hat{B}_\nu(\hat{\phi}_{B_\nu}, \hat{\Pi}_{B_\nu}) \right), \tag{2.19}$$

with $\hat{A}_\nu$, $\hat{B}_\nu$, and λ_ν being different to the ones in equation 2.18 and ν serving as an index that specifies the basis. Operators of the form $\hat{\phi}_{A_\nu}$ means that the reduced $\hat{\phi}$ operator for the subsystem A for the basis specified by the index ν.

We will be working under the assumption that the interaction term of the Hamiltonian will only contain a single operator of the pair $\left[\hat{\phi}, \hat{\Pi}\right]$, for example $\lambda\hat{\phi}_A \otimes \hat{\phi}_B$. This assumption stems from the fact that we want the evolutions caused by the interaction terms to be unitary ($\hat{U} = e^{i\hat{H}_{int}t}$). If the interaction term of the Hamiltonian contains both members of a pair, we will have a von Neumann measurement ($e^{i\hat{\phi}\otimes\hat{\Pi}}$). For instance, if we have $\hat{x}_A$ for the system and $\hat{p}_B$ for the environment (with $\hat{x}$ being position and $\hat{p}$ being momentum), we will have $e^{i\hat{x}_A\hat{p}_B}$. This means a von Neumann measurement of the system's position will occur (as illustrated in Appendix A). By insisting we have an interaction Hamiltonian basically of the form $\lambda\hat{\phi}_A \otimes \hat{\phi}_B$ (see equation 2.20 below) will have a unitary evolution operator and thus a von Neumann measurement. As a result, we will have:

$$\hat{H} = \hat{H}_A \otimes \hat{I}_B + \hat{I}_A \otimes \hat{H}_B + \sum_{\nu=1}^{P} \lambda_\nu \left(\hat{A}_\nu(\hat{\phi}_{A_\nu}) \otimes \hat{B}_\nu(\hat{\phi}_{B_\nu}) \right). \tag{2.20}$$

In this thesis, it is convenient to express the self-Hamiltonian terms as potential and dynamic terms. This allows us to write with $\hat{\phi}_{A,B}$ as the preferred basis[4]:

$$\begin{aligned} \hat{H}_A &= \hat{V}(\hat{\phi}_A) + \sum_{i=1}^{\infty} \hat{D}_i(\hat{\phi}_A)(\hat{\Pi}_A)^i, \\ \hat{H}_B &= \hat{V}(\hat{\phi}_B) + \sum_{i=1}^{\infty} \hat{D}_i(\hat{\phi}_B)(\hat{\Pi}_B)^i, \end{aligned} \tag{2.21}$$

with each $\hat{D}_i(\hat{\phi})$ being the coefficients of a power-series expansion in $\hat{\Pi}$.

Another type of Hamiltonian we need to talk about is the effective Hamiltonian[5]. This is obtained by tracing out the subsystem we are not interested in. Since we

[4]In D_i, the i is an index. In $\hat{\Pi}^i$, the i is a power.

[5]As used by Carrol and Singh in [61] on page 35 in equation D6.

are in the Schrödinger picture, we will not have time dependence in the Hamiltonian or any operator. However, we will have time dependence in the reduced Hamiltonian since "expectation values" have time dependence:

$$\begin{aligned}\hat{H}_A^{eff}(t) &= \mathrm{Tr}_B(\hat{H}) = \hat{H}_A + \sum_{\nu=1}^{P} \lambda_\nu \langle \hat{B}_\nu \rangle_t \hat{A}, \\ \hat{H}_B^{eff}(t) &= \mathrm{Tr}_A(\hat{H}) = \hat{H}_B + \sum_{\nu=1}^{P} \lambda_\nu \langle \hat{A}_\nu \rangle_t \hat{B}.\end{aligned} \tag{2.22}$$

Note that $\langle \hat{A}_\nu \rangle_t$ and $\langle \hat{B}_\nu \rangle_t$ are not traditional expectation values but the numerical values of $\hat{A}$ and $\hat{B}$ in the ν basis at time t after tracing.

2.2 Classical Behavior

In this section and the chapters to follow, it will be assumed that the state of the system is initially unentangled with the environment. Further, we assume (for illustrative purposes) that the probability distribution of the system's eigenstates are initially a Gaussian distribution in a pointer basis.

2.2.1 Pointer Observables

Let us recall the definition of a pointer observable from section 1.1.3. The eigenstates of a pointer observable form a complete basis in which the state of the system appears quasi-classical. A more complementary discussion on the pointer observable with respect to the predictability sieve can be found in Appendix C. A mathematical definition of pointer observable ($\hat{O}_A$), like any observable whose eigenstates form a complete basis, is:

$$\hat{O}_A = \sum_{i=1}^{d_A} o_i |\psi_{Ai}\rangle\langle\psi_{Ai}| = \sum_{i=1}^{d_A} o_i \hat{\Lambda}_i, \tag{2.23}$$

where Λ_i is a projection operator, $|\psi_{Ai}\rangle$ is the eigenstates of the operator, and o_i is the eigenvalues of the operator.

We expect pointer observables to be position-like. This is due to the assumptions made in sections 2.1.3 and 2.1.4 concerning classicality and locality. However, there are exceptions as will be seen with the decoherence limit, which will be described shortly.

The pointer observable will have certain relationships with the Hamiltonian in different limits as will be discussed below.

Pointer Observables' Limits

The first limit for us to discuss is the quantum measurement limit (QML). In this limit, $\hat{H} \approx \hat{H}_{int}$. However, despite having a minimal effect on the total Hamiltonian, the self-term of the Hamiltonian will still contribute to the systems predictability (as will be seen later in this section) and hence cannot be neglected. This limit is called the QML since measurements (which are generally non-unitary in most interpretations) are associated with the interactions of the system with the environment. However, since we assume our interaction Hamiltonian to contain a single member of the pair $\left[\hat{\phi}, \hat{\Pi}\right]$ (as explained in the previous section), we will not actually have a measurement in the Copenhagen sense in any case including in this limit. This means that the choice of name is a slight misnomer. We expect to have the following relation: $\left[\hat{O}_A, \hat{H}_{int}\right] = 0$ due to the assumptions we made in section 2.1.4, in particular equation 2.20. Furthermore, $\left[\hat{O}_A, \hat{H}\right] \approx 0$.

The second limit is the decoherence limit (DCL). In this limit, $\hat{H} \approx \hat{H}_{self}$. The name (which is a misnomer, however, we shall stick to using for historical consistency [76]) stems from the idea that it is the limit that decoherence does not occur. As a result the initial state does not decohere as it is prevented from doing so as it would with a more influential interaction term[6]. The natural choice for a pointer

[6]This is mathematically seen with the coherent density matrix being $|\psi_A\rangle\langle\psi_A| \otimes |\psi_B\rangle\langle\psi_B|$ with $|\psi_A\rangle = \sum_j |i\rangle\langle i|\psi_A\rangle$, and through time evolution it becomes the decoherent density matrix: $\mathrm{Tr}_B(\hat{\rho}) = \sum_{i,j} \psi_{A_i}\psi^*_{A_j}|i\rangle\langle j|$ with $\psi_{A_i}\psi_{A_j} = c_i\delta_{ij}$ following from the time scales of interest [76].

observable in the DCL is $\left[\hat{H}_A, \hat{O}_A\right] = 0$, which means that a quasi-classical state remains quasi-classical since it will be one that does not decohere a significant amount. Furthermore, $\left[\hat{O}_A, \hat{H}\right] \approx 0$.

2.2.2 Linear Entropy

In section 1.1.3, we discussed robustness. Using the assumptions at the start of section 2.2, robustness means that there is little growth in entanglement during the time scales of interest. This will be measured with the rate of change of the linear entropy (S_{lin}). The linear entropy (given by equation 2.25) is obtained from the Von Neumann entropy (given by equation 2.24) by expanding the logarithm to the first order about $ln(1)$.

$$S_{von}(t) = -\mathrm{Tr}(\hat{\rho}(t) ln \hat{\rho}(t)) \tag{2.24}$$

$$S_{lin}(t) = 1 - \mathrm{Tr}(\hat{\rho}^2(t)) \tag{2.25}$$

Since we assume our system to be initially unentangled with the environment and the probability for the pointer observable's eigenstates, for example, to be in a Gaussian distribution[7] – we can expect the state of the system to be in an equilibrium. Since our time scales are short, if we expand S_{lin} about $S_{lin}(t=0)$ such that it is: $S_{lin} = S_{lin}(0) + \dot{S}_{lin}(0)t + \frac{1}{2}\ddot{S}_{lin}(0)t^2$, of these terms, only $\ddot{S}_{lin}(0)$ will be non-zero. Hence, we will use it as our measure for robustness.

A full derivation of linear entropy can be found in Appendix D, but the important

[7] As we did in section 1.1.3.

result is:

$$\ddot{S}_{lin}\left(\hat{\rho}(t)\right) = -\frac{2}{\hbar^2}\sum_{\alpha}^{P}\sum_{\beta}^{P}\lambda_\alpha\lambda_\beta\langle\hat{B}_\alpha\rangle_0\langle\hat{B}_\beta\rangle_0\left(2\langle\hat{A}_\alpha\rangle_0\langle\hat{A}_\beta\rangle_0 - \langle\hat{A}_\alpha\hat{A}_\beta + \hat{A}_\beta\hat{A}_\alpha\rangle_0\right) \tag{2.26}$$

$$- \frac{1}{\hbar^2}\sum_{\alpha}^{P}\sum_{\beta}^{P}\lambda_\alpha\lambda_\beta\langle\hat{B}_\alpha\hat{B}_\beta\rangle_0\langle\hat{A}_\alpha\hat{A}_\beta\rangle_0 + \frac{1}{\hbar^2}\sum_{\alpha}^{P}\sum_{\beta}^{P}\lambda_\alpha\lambda_\beta\langle\hat{B}_\alpha\hat{B}_\beta\rangle_0\langle\hat{A}_\alpha\rangle_0\langle\hat{A}_\beta\rangle_0$$
$$+ \frac{1}{\hbar^2}\sum_{\alpha}^{P}\sum_{\beta}^{P}\lambda_\alpha\lambda_\beta\langle\hat{B}_\beta\hat{B}_\alpha\rangle_0\langle\hat{A}_\alpha\rangle_0\langle\hat{A}_\beta\rangle_0 - \frac{1}{\hbar^2}\sum_{\alpha}^{P}\sum_{\beta}^{P}\lambda_\alpha\lambda_\beta\langle\hat{B}_\beta\hat{B}_\alpha\rangle_0\langle\hat{A}_\alpha\hat{A}_\beta\rangle_0$$
$$+ \mathcal{O}(t) \quad .$$

However, it is important to note that when we are in the decoherence limit, $\hat{H}_{int}$ acting on the state is minuscule relative to $\hat{H}$ acting on the state. As a result, $\ddot{S}_{lin} \approx 0$.

In the special case of $\hat{H}_{int} = \lambda(\hat{A} \otimes \hat{B})$, our entropy has the simpler form of:

$$\ddot{S}_{lin}\left(\hat{\rho}_A(t)\right) = \frac{4}{\hbar^2}\lambda^2\left(\langle\hat{A}^2\rangle_0 - \langle\hat{A}\rangle_0^2\right)\left(\langle\hat{B}^2\rangle_0 - \langle\hat{B}\rangle_0^2\right). \tag{2.27}$$

2.2.3 The Pointer Entropy

As discussed in section 1.1.3, predictability means that the state of the system's probability distribution starts, for example, in a Gaussian distribution in the pointer basis (as assumed) and remains approximately Gaussian over the time scales of interest. The pointer entropy is based on the variance of the pointer observable.

Variance

To be able to say if an object is predictable, we will need to have a quantity that we can calculate/measure. One of the measures we could use is the variance probability distribution in the pointer basis. The variance is defined as:

$$\triangle^2 \hat{O}_A(t) = \mathrm{Tr}(\hat{\rho}_A(t)\hat{O}_A^2) - \mathrm{Tr}^2(\hat{\rho}_A(t)\hat{O}_A). \tag{2.28}$$

This is not, however, a practical expression to use in conjunction with linear entropy since it is not generally a dimensionless measure. As a result, we need to use a quantity that allows a comparison with the linear entropy (and ideally this measure should be an entropy). Using the variance, we can justify the pointer entropy, which is an appropriate measure.

For future reference, by expanding the operator, completeness, and using the cyclical property of the trace, we can write 2.28 as:

$$\triangle^2 \hat{O}_A(t) = \sum_{j=1}^{d_A} o_j^2 |\langle o_j|\Psi_A\rangle|^2 - \left(\sum_{j=1}^{d_A} o_j |\langle o_j|\Psi_A\rangle|^2\right)^2. \tag{2.29}$$

Pointer Entropy

To find a measure similar to variance, let us define p_j as the probability of the state of the system in the j-th eigenstate of the pointer observable. This probability is given by: $p_j = \mathrm{Tr}_A(\hat{\rho}_A \Lambda_j)$.
The first order moment of the probability distribution is:

$$\sum_{j=1}^{d_A} p_j = 1 = \sum_{j=1}^{d_A} |\langle o_j|\Psi_A\rangle|^2. \tag{2.30}$$

The second order is given by:

$$\sum_{j=1}^{d_A} |p_j|^2 = \sum_{j=1}^{d_A} \left(|\langle o_j|\Psi_A\rangle|^2\right)^2, \tag{2.31}$$

which can be combined into:

$$\sum_{j=1}^{d_A} |p_j| - \sum_{j=1}^{d_A} |p_j|^2 = \sum_{j=1}^{d_A} |\langle o_j|\Psi_A\rangle|^2 - \sum_{j=1}^{d_A} \left(|\langle o_j|\Psi_A\rangle|^2\right)^2. \tag{2.32}$$

This is similar to the equation of variance (see equation 2.29). However, the variance has an undesirable weighting of the eigenvalues o_j.

This can be combined into the pointer entropy[8]:

$$S_{pointer}(t) = 1 - \sum_{j=1}^{d_A} \left(\mathrm{Tr}_A\left(\hat{\rho}_A(t)|o_j\rangle\langle o_j|\right)\right) = 1 - \sum_{j=1}^{d_A} \mathrm{Tr}_A\left(\hat{\rho}_A(t)\hat{\Lambda}_j\right). \tag{2.33}$$

Now, we will not just be using the pointer entropy but the second order derivative, for the same reasons as for the linear entropy. Hence, we can express the result as:

$$\begin{aligned}
\ddot{S}_{pointer}(0) =& \frac{2}{\hbar^2}\sum_{j=1}^{d_a}\langle[\hat{\Lambda}_j, \hat{H}_A]\rangle_0^2 + \frac{2}{\hbar^2}\sum_{j=1}^{d_a}\left(\sum_{\alpha}\lambda_\alpha\langle\hat{B}_\alpha\rangle_0\langle[\hat{\Lambda}_j, \hat{A}_\alpha]\rangle_0\right)^2 \\
&+ \frac{4}{\hbar^2}\sum_{j=1}^{d_a}\sum_{\alpha}\lambda_\alpha\langle\hat{B}_\alpha\rangle_0\langle[\hat{\Lambda}_j, \hat{A}_\alpha]\rangle_0\langle[\hat{\Lambda}_j, \hat{H}_A]\rangle_0 \\
&+ \frac{2}{\hbar^2}\sum_{j=1}^{d_a}\left(p_j(0)\langle\hat{\Lambda}_j\hat{H}_A^2 + \hat{H}_A^2\hat{\Lambda}_j - 2\hat{H}_A\hat{\Lambda}_j\hat{H}_A\rangle_0\right) \\
&+ \frac{2}{\hbar^2}\sum_{j=1}^{d_a}\left(p_j(0)\sum_{\alpha}\lambda_\alpha\langle\hat{B}_\alpha\rangle_0\langle\left[\hat{\Lambda}_j, [\hat{H}_A, \hat{A}_\alpha]\right]\rangle_0\right) \\
&+ \frac{2}{\hbar^2}\sum_{j=1}^{d_a}\left(p_j(0)\sum_{\alpha}\lambda_\alpha\langle\hat{B}_\alpha\rangle_0\langle\left[\hat{\Lambda}_j, [\hat{A}_\alpha, \hat{H}_A]\right]\rangle_0\right) \\
&+ \frac{2}{\hbar^2}\sum_{j=1}^{d_a}\left(p_j(0)\sum_{\alpha}\lambda_\alpha\langle\hat{B}_\alpha\rangle_0\sum_{\beta}\lambda_\beta\langle\hat{B}_\beta\rangle_0\langle\left[\hat{\Lambda}_j, [\hat{A}_\alpha, \hat{A}_\beta]\right]\rangle_0\right).
\end{aligned} \tag{2.34}$$

(We have derived it fully in Appendix E.) Grouping the terms such that they

[8]Pointer entropy is equivalent to the second order Tsillis Entropy [77].

correspond to the two limits, following is obtained:

$$\ddot{S}_{pointer}(0) = \ddot{S}^{QML}_{pointer} + \ddot{S}^{DCL}_{pointer} + \frac{4}{\hbar^2}\sum_{j=1}^{d_a}\sum_{\tau} \lambda_\tau \langle \hat{B}_\tau \rangle_0 \langle [\hat{\Lambda}_j, \hat{A}_\tau] \rangle_0 \langle [\hat{\Lambda}_j, \hat{H}_A] \rangle_0. \quad (2.35)$$

Chapter 3

Everettian Quantum Theory

In this chapter, we will discuss Everettian quantum theory so that we can understand the world it postulates. Further, this chapter will inform us, in part, of Hilbert space fundamentalism as the generalization of Everettian quantum theory. Everettian quantum theory considers the wave function as ontological rather than epistemological. Furthermore, it views the "collapse"[1] of the wave function as both ontological and reversible.

3.1 The Model

Everettian quantum theory deploys a proposed model in which standard quantum theory is stripped to its most basic form, namely that the state of the Universe (Ψ) evolves within a Hilbert space. This vector evolves according to the Schrödinger equation in the Schrödinger picture [66]:

$$\hat{H}|\Psi\rangle = i\frac{\partial}{\partial t}|\Psi\rangle. \tag{3.1}$$

This requires us to view Ψ as an ontological object [66]. However, for us to have this theory work as quantum theory, we will need to have a finite-dimensional

[1] "Collapse" is now being used as a synonym for measurement which has a different meaning in different interpretations of quantum theory.

Hilbert space. This is to account for the fact that, if the Hilbert space had uncountably infinite dimensions, we would have unitarily inequivalent representations of canonical-commutation relations (according to the Haag Theorem [78]). Another possibility that provides inequivalent representations of canonical-commutation relations are unseparable Hilbert spaces, however, we shall not concern ourselves with that possibility since we have already assumed that our Hilbert space is separable as discussed in section 2.1.1. With canonical-commutation relations: normally, a vacuum state is defined as being annihilated by every applicable annihilation operator. However, if we have inequivalent representations of canonical commutation relations, this no longer holds true. Suppose that we have two vacuum states, $|0\rangle$ and $|0^{\star}\rangle$, which we are illustrating with the notation of quantum field theory. Each vacuum will have annihilation operators with $\hat{a}_k$ for $|0\rangle$ and $\hat{a}_k^{\star}$ for $|0^{\star}\rangle$. If we have inequivalent representations of canonical commutation relations, then $\hat{a}_k^{\star}|0\rangle \neq 0$ and $\hat{a}_k|0^{\star}\rangle \neq 0$. This means that $|0^{\star}\rangle$ lies in a Hilbert space outside that of $|0\rangle$. Hence, the canonical relations used to define $\hat{a}_k$ are not equivalent to those used to define $\hat{a}_k^{\star}$.

A justification given for the finite dimensionality of the Hilbert space is that of gravity since the number of degrees of freedom correspond to the dimensionality of a system's Hilbert space (if one is finite, the other will be as well). The reason we can use gravity in this quantum argument is that, as we probe smaller and smaller distances – due to the uncertainty principle – we will need more and more energy. Eventually, we will probe a length scale so small that the energy needed to probe it will be so great as to cause the measuring device to collapse into a black hole [79, 80, 81]. The highest entropy configuration for a spherical region R with a radius r is that of a black hole of the same size[2], as prescribed by the holographic principle [83, 84, 85, 86]:

$$S_{BH} = \frac{A}{4G} = \frac{\pi r^2}{G}, \tag{3.2}$$

[2]For a strongly gravitating system, r can be replaced with a suitably defined length scale [82].

where G is Newton's constant and all other fundamental constants have been set to one.

This will correspond to the following bound for the number of degrees of freedom in a region R:

$$dim\mathcal{H}_R \lesssim exp(\frac{\pi r^2}{G}). \tag{3.3}$$

While this will be a large number, it is clearly not infinite, which means finite dimensionality. An alternative reason for the lack of infinities is that they do not exist in nature [87, 88]. This is something we will look at in more detail at a later point.

3.2 Everettian Interpretation

Everettianism, also known as many worlds (which has stolen the popular imagination), does not invoke the wave function collapsing when a measurement is performed (such as the Copenhagen Interpretation does). Rather, it introduces the idea of branching. As previously discussed (see pages 5 and 6 concerning the Everettian interpretation), the system and environment become more and more entangled over time because of interactions. This causes the eigenstates of the system and eigenstates of the environment to share a one-to-one correspondence. This leads to the influence of the off-diagonal terms of the density matrix to eventually decay to zero. This is a reversible process as are all interactions in the Everretian interpretation (i.e. there is no collapse in the Copenhagen sense). However, decoherence is generally considered to be irreversible due to the role of memories and records. Since many observers share similar memories (and many objects share similar records) due to a proliferation of subsystems in the environment that share in this one-to-one correspondence, they consider the interactions as being irreversible, even though this irreversibility is only apparent. As a result,

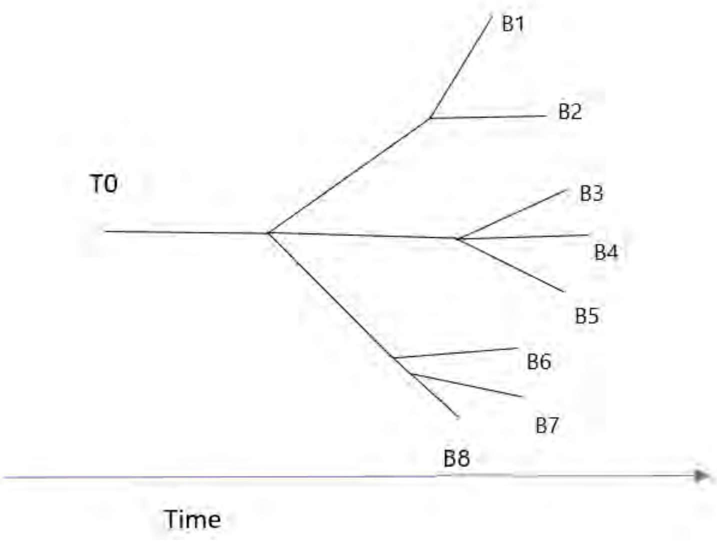

Figure 3.1: A schematic showing how different branches become more numerous over time. Additionally when branching occurs, an event occurred.

wave-functions appear to split into different branches, with each branch representing a potential outcome. Branches that are close to each other in an appropriate phase space (as can be seen in figure 3.1 with branches B3 and B4 being close together) will have the highest level of overlap in their states even though their inner product is zero. In figure 3.1, a schematic is presented showing how, as more and more interactions occur over time, more and more branches form. This means that the ever-growing number of branches can serve as an arrow of time for the Everettian interpretation. The branches that are the closest together have the most in common (they share the most history and quantum numbers), but due to interactions, have split from one another.

The Everettian Interpretation will be essential for the formalism of Hilbert space fundamentalism. This is because none of the other interpretations view the wave-function and its "collapse" as ontological. Additionally, other interpretations do not view measurement as being reversible without any additional nuances, as fully discussed in section 1.1.2. For example, Penrose collapse interpretations do not allow measurements to be reversed. Copenhagen is agnostic to the wave-functions' ontology and views measurement as epistemic and irreversible.

An interesting interpretation is that of Relational Quantum Mechanics (RQM), which holds the wave-function as epistemic. However, RQM does not view the collapse the same way as other interpretations of quantum theory, but it does hold that measurements are real events. Furthermore, events are reversible. This can be used to create an alternative to Hilbert space fundamentalism which we will call "Hilbert space representationalism". The idea is that the Universe can be represented by a vector in Hilbert space.

Before we can evaluate these more nuanced aspects of Hilbert space fundamentalism, however, let us explore the emergence of quasi-classical states of the state of the system according to Hilbert space fundamentalism.

Chapter 4

Quasi-Classical States of Systems

In this chapter, we will be analyzing the emergence of quasi-classical worlds from quantum theory. While we will be applying the method to Everettian quantum theory, it is applicable to any quantum theory. This could be done by means of an algorithm. However, before we can discuss any algorithm, we need to discuss the candidate pointer observables we could be using with the algorithm.

4.1 Candidate Pointer Observables

For a candidate pointer observable, we expect two properties [61], namely that it should be factorisable:

$$\hat{O}_{CPO} = \hat{O}_A \otimes \hat{O}_B, \tag{4.1}$$

and that Frobenius norm of the operator commuting with the Hamiltonian is minimized:

$$\Big|\Big|[\hat{O}_{CPO}, \hat{H}]\Big|\Big|_F \text{ is minimized with respect to a real parameter } \alpha \text{ (to be defined below).} \tag{4.2}$$

The Frobenius norm is a matrix norm. It is defined as $||A||_F = \sqrt{Tr(A^\dagger A)}$ and has some useful properties, such as being invariant under unitary operations.

Through the predictability sieve (see appendix C) and the nature of the pointer observable (see section 2.2.1), it is established that $\left[\hat{O}_{OCP}, \hat{H}\right] = 0$ for the ideal pointer observable. The Hamiltonian under consideration is one of the form: $\hat{H} = \hat{H}_{int} + \hat{H}_{self}$ as described in section 2.1.4. It has been established that $\hat{H}_{self} = \hat{H}_A \otimes \hat{I}_B + \hat{I}_A \otimes \hat{H}_B$, with $\hat{H}_A$ having a dependence of $\hat{\phi}_A$ and $\hat{\Pi}_A$ and $\hat{H}_B$ having a dependence of $\hat{\phi}_B$ and $\hat{\Pi}_B$. Note that $\hat{\phi}_A$ and $\hat{\Pi}_A$ will have a finite dimensionality of d_A, while $\hat{\phi}_B$ and $\hat{\Pi}_B$ will have a finite dimensionality of d_B (as set out in chapter 2).

The interaction term of the Hamiltonian will (as stated in section 2.1.3 and 2.1.4, and further expanded upon in section 4.3 when we discuss collimation) only consist of one member of the pair of conjugates $\hat{\phi}$ and $\hat{\Pi}$ (typically, $\hat{\phi}$) in order for the state to exhibit robustness. To model the Hamiltonian under consideration, the pointer observable is modeled as the functional of the interaction term ($\hat{f}(\hat{\phi}_A, \hat{\phi}_B)$) plus the functional of the self-terms ($\hat{g}(\hat{\phi}_A, \hat{\phi}_B, \hat{\Pi}_A, \hat{\Pi}_B)$). Hence:

$$\hat{O}_{CPO} = \alpha \hat{f}(\hat{\phi}_A, \hat{\phi}_B) + (1-\alpha)\hat{g}\left(\hat{H}_{self}(\hat{\phi}_A, \hat{\phi}_B, \hat{\Pi}_A, \hat{\Pi}_B)\right). \tag{4.3}$$

Here α is the real parameter which we will use to minimize our Frobenius norm above. It will range between 1 and 0 with $\alpha = 1$ corresponding to the QML, while $\alpha = 0$ will correspond to the DCL. This means that α will map to the extreme limits (the QML and DCL) and all the cases between the QML and DCL. Further, when $\alpha = 0$, we will have $[\hat{g}(\hat{\phi}_A, \hat{\phi}_B, \hat{\Pi}_A, \hat{\Pi}_B), \hat{H}_{self}] = 0$; and when $\alpha = 1$, we will have $[\hat{f}(\hat{\phi}_A, \hat{\phi}_B), \hat{H}_{int}] = 0$ assuming that $\hat{H}_{int}$ is suitable for the state to exhibit robustness. The factorization of the ansatz is $\hat{O}_{CPO} = \alpha\left(\hat{f}(\hat{\phi}_A) \otimes \hat{f}(\hat{\phi}_B)\right) + (1-\alpha)\left(\hat{g}\left(\hat{H}_A(\hat{\phi}_A, \hat{\Pi}_A\right) \otimes \hat{g}\left(\hat{H}_B(\hat{\phi}_B, \hat{\Pi}_B)\right)\right)$. This does not comply with the algorithm to be discussed in the next section nor equation 4.1. There is a possibility that the algorithm could be modified to accommodate the ansatz. This, however, is left for future work.

We can now evaluate the commutation pair that the matrix norm is being taken of (equation 4.2). This we shall do in the Heisenberg picture for this section only (all other sections assume the Schrödinger picture). The usage of the Heisenberg picture is so that we can rewrite equation 4.3 in a differential form:

$$\left[\hat{O}_{CPO}, \hat{H}\right] = \left[\alpha \hat{f}(\hat{\phi}_A, \hat{\phi}_B) + (1-\alpha)\hat{g}\left(\hat{H}_{self}(\hat{\phi}_A, \hat{\phi}_B, \hat{\Pi}_A, \hat{\Pi}_B)\right), \hat{H}\right]. \tag{4.4}$$

$$= i\alpha \frac{d\hat{f}(\hat{\phi}_A, \hat{\phi}_B)}{dt} + \left[(1-\alpha)\hat{g}\left(\hat{H}_{self}(\hat{\phi}_A, \hat{\phi}_B, \hat{\Pi}_A, \hat{\Pi}_B)\right), \hat{H}_{int} + \hat{H}_{self}\right]. \tag{4.5}$$

$$= i\alpha \frac{d\hat{f}(\hat{\phi}_A, \hat{\phi}_B)}{dt} + \left[(1-\alpha)\hat{g}\left(\hat{H}_{self}(\hat{\phi}_A, \hat{\phi}_B, \hat{\Pi}_A, \hat{\Pi}_B)\right), \hat{H}_{int}\right]. \tag{4.6}$$

We used the simplification of $[x, f(x)] = 0$ between equations 4.5 and 4.6 for the self-Hamiltonian.

The quantum state is assumed to be highly peaked in some basis that serves as a pointer basis. Further the operator whose eigenstates define this pointer basis will have minimal variance as described at the start of section 2.2.

Now, a discussion of $\hat{H}_{int}$ is required. In section 2.1.4, it was established that $\hat{H}_{int} = \sum_\gamma \lambda_\gamma \left(\hat{A}_\gamma(\hat{\phi}_{A_\gamma}) \otimes \hat{B}_\gamma(\hat{\phi}_{B_\gamma})\right)$. Furthermore, $\hat{H}_{self} = \hat{H}_A(\hat{\phi}_A, \hat{\Pi}_A) + \hat{H}_B(\hat{\phi}_B, \hat{\Pi}_B)$. This means that we can write 4.6 as:

$$\left[\hat{O}_{CPO}, \hat{H}\right] = \left[(1-\alpha)\hat{g}\left(\hat{H}_{self}(\hat{\phi}_A, \hat{\phi}_B, \hat{\Pi}_A, \hat{\Pi}_B)\right), \sum_\gamma \lambda_\gamma \left(\hat{A}_\gamma(\hat{\phi}_{A_\gamma}) \otimes \hat{B}_\gamma(\hat{\phi}_{B_\gamma})\right)\right] \tag{4.7}$$

$$+ i\alpha \frac{d\hat{f}(\hat{\phi}_A, \hat{\phi}_B)}{dt}.$$

This serves as an end point to the calculation unless we make further assumptions. The simplest but non-trivial form of g would be that it is linear in its argument. This would allow us to think of this assumption as the leading non-constant term in the Taylor expansion of g. Hence we would write:

$$\left[\hat{O}_{CPO}, \hat{H}\right] =(1-\alpha)\left[\hat{g}(\hat{H}_A(\hat{\phi}_A, \hat{\Pi}_A) + g(\hat{H}_B(\hat{\phi}_B, \hat{\Pi}_B)), \sum_\gamma \lambda_\gamma \left(\hat{A}_\gamma(\hat{\phi}_{A_\gamma}) \otimes \hat{B}_\gamma(\hat{\phi}_{B_\gamma})\right)\right] \tag{4.8}$$
$$+ i\alpha \frac{d\hat{f}(\hat{\phi}_A, \hat{\phi}_B)}{dt}.$$

We can expand our terms such that $(H_A + H_B)(K_A \otimes K_B) = H_A K_A \otimes K_B + K_A \otimes H_B K_B$:

$$\begin{aligned}\left[\hat{O}_{CPO}, \hat{H}\right] =&(1-\alpha)\sum_\gamma \lambda_\gamma \left[\hat{A}_\gamma(\hat{\phi}_{A_\gamma}), \hat{g}\left(\hat{H}_A(\hat{\Pi}_A, \hat{\phi}_A)\right)\right] \otimes \hat{B}_\gamma(\hat{\phi}_{B_\gamma}) \\ &+ (1-\alpha)\sum_\gamma \lambda_\gamma \hat{A}(\hat{\phi}_{A_\gamma}) \otimes \left[\hat{B}_\gamma(\hat{\phi}_{B_\gamma}), \hat{g}\left(\hat{H}_B(\hat{\Pi}_B, \hat{\phi}_B)\right)\right] \\ &+ i\alpha \frac{d\hat{f}(\hat{\phi}_A, \hat{\phi}_B)}{dt}.\end{aligned} \tag{4.9}$$

This will be most useful if we were to put it into a differential form. This will be done using: $[\hat{\phi}, y(\hat{\Pi})] = \frac{\partial y(\hat{\Pi})}{\partial \hat{\Pi}}$. However, we have $\left[\hat{z}(\hat{\phi}), y\left(\hat{Z}(\hat{\Pi}, \hat{\phi})\right)\right]$. As a result, we invoke the chain rule $\left(\frac{\partial \hat{z}(\hat{\phi})}{\partial \hat{\phi}} \frac{\partial y\left(\hat{U}(\hat{\Pi}, \hat{\phi})\right)}{\partial \hat{\Pi}} - \frac{\partial \hat{z}(\hat{\phi})}{\partial \hat{\Pi}} \frac{\partial y\left(\hat{U}(\hat{\Pi}, \hat{\phi})\right)}{\partial \hat{\phi}}\right)\left[\hat{\phi}, \hat{\Pi}\right]$ which leads to the expression becoming:

$$\begin{aligned}\left[\hat{O}_{CPO}, \hat{H}\right] =&i(1-\alpha)\sum_\gamma \lambda_\gamma \frac{\partial \hat{A}_\gamma(\hat{\phi}_{A_\gamma})}{\partial}\hat{\phi}_{A_\gamma} \frac{\partial \hat{g}\left(\hat{H}_A(\hat{\Pi}_A, \hat{\phi}_A)\right)}{\partial_\gamma \hat{\Pi}_A} \otimes \hat{B}_\gamma(\hat{\phi}_{B_\gamma}) \\ &+ i(1-\alpha)\sum_\gamma \lambda_\gamma \hat{A}_\gamma(\hat{\phi}_{A_\gamma}) \otimes \frac{\partial \hat{B}_\gamma(\hat{\phi}_{B_\gamma})}{\partial \hat{\phi}_{B_\gamma}} \frac{\partial \hat{g}\left(\hat{H}_B(\hat{\Pi}_B, \hat{\phi}_B)\right)}{\partial_\gamma \hat{\Pi}_{B_\gamma}} \\ &+ i\alpha \frac{d\hat{f}(\hat{\phi}_A, \hat{\phi}_B)}{dt}.\end{aligned} \tag{4.10}$$

This means our condition for a candidate pointer observable is one that could meet the criteria of equations 4.1 and 4.3 such that:

$$\begin{aligned} &||i(1-\alpha)\sum_{\gamma}\lambda_{\gamma}\frac{\partial\hat{A}_{\gamma}(\hat{\phi}_{A_{\gamma}})}{\partial}\hat{\phi}_{A_{\gamma}}\frac{\partial\hat{g}\left(\hat{H}_{A}(\hat{\Pi}_{A},\hat{\phi}_{A})\right)}{\partial_{\gamma}\hat{\Pi}_{A}}\otimes\hat{B}_{\gamma}(\hat{\phi}_{B_{\gamma}}) \\ &+i(1-\alpha)\sum_{\gamma}\lambda_{\gamma}\hat{A}_{\gamma}(\hat{\phi}_{A_{\gamma}})\otimes\frac{\partial\hat{B}_{\gamma}(\hat{\phi}_{B_{\gamma}})}{\partial\hat{\phi}_{B_{\gamma}}}\frac{\partial\hat{g}\left(\hat{H}_{B}(\hat{\Pi}_{B},\hat{\phi}_{B})\right)}{\partial_{\gamma}\hat{\Pi}_{B_{\gamma}}} \\ &+i\alpha\frac{d\hat{f}(\hat{\phi}_{A},\hat{\phi}_{B})}{dt}||_{F} \quad \text{is minimized with respect to } \alpha. \end{aligned} \tag{4.11}$$

4.2 The Algorithm

We would like a tool that is optimal for finding the quasi-classical factorization of a system and environment for a given candidate pointer observable. One of these tools is the Carroll and Singh algorithm [61], which has a procedure that we will illustrate and that works in the extreme limits. Unfortunately (as stated in the previous section), this algorithm is not applicable in the general case without modification.

The Carroll and Singh algorithm starts by defining all possible pointer observables that meet our conditions as set in the previous section and finds the compatible factorizations. Then, the algorithm uses the measures of predictability and robustness to find the most suitable pointer observables and their respective factorizations of the system. What follows is a step-by-step description of the algorithm.

1) Find a suitable candidate pointer observable such that $\hat{O}_{CPO} = \hat{O}_A \otimes \hat{O}_B$. Our ansatz as defined in equation 4.3 will not be separable outside of the extreme limits, however, this step of the algorithm might be amendable in future work (for instance, by rather making the step's criteria be: $\hat{O}_{CPO} = \sum_{i=1}\alpha_i(\hat{F}_{A_i}\otimes\hat{F}_{B_i})$).

This will require the best choice in values for α depending on the nature of the Hamiltonian for the given system.

2) Construct a set of initial states that represents the basis of $\hat{O}_A$. This will correspond to the following pure product states: $|\Psi_j(0)\rangle_{CPO} = |\Psi_j(0)\rangle_A \otimes |\Psi_j(0)\rangle_B$.

3) For $|\Psi(0)\rangle_{CPO}$, we will calculate $\ddot{S}_{lin}(0)$ and $\ddot{S}_{pointer}(0)$ as outlined in Chapter 2, section 2 and in appendices C and E.

4) Find the measure $\mathcal{S}$ [1]: $\mathcal{S} = max\left(\ddot{S}_{lin}(0); \ddot{S}_{pointer}(0)\right)$.

5) Find the factorization that minimizes $\mathcal{S}$.

6) Repeat the following for every choice of candidate pointer observable until we find the $\mathcal{S}$ that is the most minimal.

Using this method, we can sieve through different factorizations to find the closest state to the classical state of the system.

4.3 Collimation

It has been established that locality is required for us to have a notion of quasi-classicality. We can assess locality or the lack thereof with the notion of collimation. Collimation is defined in [61] as the lack of spreading of the state of a system in a given basis when acted on by a particular operator (i.e., does the state of the system remain localized in this circumstance). To be able to measure collimation, we will use the collimation functional as defined in [61]. Consider a Hermitian operator $\hat{\phi}$ with its eigenvalues denoted by ϕ and let us use them to define the basis. Further, consider a Hermitian operator $\hat{\Pi}$ that is the canonical conjugate of $\hat{\phi}$ so that $\hat{\Pi}$ acts as the generator of transformation with respect to the $\hat{\phi}$ basis.

[1] It has been called the Schwinger entropy.

Now consider a more general Hermitian operator $\hat{M}(\hat{\phi}, \hat{\Pi})$ that is a functional of both $\hat{\phi}$ and $\hat{\Pi}$. We then can measure the collimation of $\hat{M}$ in the basis of $\hat{\phi}$ using the functional $C_\phi(\hat{M})$ (a full presentation of which can be found in Appendix F) to perform a calculation which will map $\hat{M}$ to real numbers in the range 0 to 1, with 0 for not collimated and 1 for perfectly collimated.

To assist us in identifying the Hamiltonians that provide us with quasi-classical states, we will make use of collimation. In sections 2.1.3 and 2.1.4, we discussed expressing the Hamiltonian in terms of canonical pairs. Using the collimation functional, we can determine if a Hamiltonian is well or poorly collimated in a basis. This will be done by considering $C_{\phi_i}(\hat{H})$. By convention, $C_{\phi_i}(\hat{H})$ will be better collimated then $C_{\Pi_i}(\hat{H})$ due to ϕ being considered the position-like basis. Note, if $C_{\phi_i}(\hat{H}) \sim 1$, then $C_{\Pi_i}(\hat{H}) \sim 0$ as well as the converse, due to the uncertainty principle.

Consider a Hamiltonian of the form of a potential plus a power series in Π. Let us now see what a collimated Hamiltonian will look like. The least collimated terms of a Hamiltonian are the higher power ones. This is because they do not act in a manner that is short-ranged, as discussed in section 1.1.3. Ideally, a Hamiltonian will have a potential term. The first Hamiltonian which has a single-step operator that comes to mind is one of the form: $V_0\hat{\Pi}^0 + q\hat{\Pi}$ (with q being a non-zero real number). However, this Hamiltonian has the problem that it does not have definite parity, which is generally expected. Rather, we will consider a Hamiltonian of the form: $\hat{H} = V(\hat{\phi})\hat{\Pi}^0 + D(\hat{\phi})\hat{\Pi}^2$. This has definite parity and a potential term and is the "traditional" Hamiltonian of classical and quantum physics when $D(\hat{\phi}) = \frac{1}{2m}$. The higher order terms that have the same parity (such as Π^4) are not as collimated. Hence the best model is that of the traditional Hamiltonian. It has been argued that $\cos(\hat{\Pi})$ has better collimation [61]. This has a Taylor expansion of: $1 - \frac{1}{2}\hat{\Pi}^2 + ...$ However, the presence of $-\frac{1}{2}\hat{\Pi}^2$ rules it out for quantum theory since it is not physical in either classical nor quantum theory [89] (although it might be allowed for Hilbert space fundamentalism in general).

As has been discussed in section 2.1.3 and 2.1.4, we expect the interaction term of the Hamiltonian to only consist of a single member of each of the canonical-conjugate pairs, which we choose to be the position-like operators: $\hat{H}_{int} \sim \lambda\hat{\phi}_A \otimes \hat{\phi}_B$. This form of $\hat{H}_{int}$ is required if there is to be a self-consistent pointer observable for the system of interest. If the pointer observable has the form $\hat{O}_A = f(\hat{H}_A) + g(\hat{\phi}_A)$ as motivated in section 2.1. and we calculate $\frac{d^2}{dt^2}\Delta^2\hat{O}_A$, we then find: $\frac{d^2}{dt^2}\Delta^2\hat{O}_A \propto [\hat{H}_A, \hat{\phi}_A]$. Now $\frac{d^2}{dt^2}[\hat{H}_A, \hat{\phi}_A]$ is a direct measure of the absence of collimation of $\hat{H}_A$ in the $\hat{\phi}_A$ basis since $\frac{d^2}{dt^2}[\hat{H}_A, \hat{\phi}_A] \neq 0$ means spreading in the $\hat{\phi}_A$ basis as the state of the system evolves in time due to the action of the self-Hamiltonian. Further, it is a measure of the lack of robustness (as discussed in section 1.1.3). This means that there is a one-to-one correspondence between robustness and high collimation.

4.4 An example

We are now going to look (in principle) at the pointer observable when analyzed in the QML case, the DCL case, and general case. This we will do with the use of the well understood example of the coupled harmonic oscillator.

The coupled oscillator has the following Hamiltonian:

$$\hat{H} = \hat{H}_A + \hat{H}_B + \hat{H}_{int}, \tag{4.12}$$

with:

$$\hat{H}_A = \mu_A\frac{\hat{\Pi}_A^2}{2} + \nu_A\frac{\hat{\phi}_A^2}{2}, \hat{H}_B = \mu_B\frac{\hat{\Pi}_B^2}{2} + \nu_B\frac{\hat{\phi}_B^2}{2}, \hat{H}_{int} = \lambda\left(\hat{\phi}_A \otimes \hat{\phi}_B\right), \tag{4.13}$$

with μ_A, μ_B, ν_A, ν_B, and λ all having dimensions of energy and ϕ and Π being dimensionless.

For the sake of simplicity, we shall hold that $\nu_A = \nu_B$ as well as $\mu_A = \mu_B$. For the sake of only having one quantity to change in our analysis, we shall hold the aforementioned quantities' value to be fixed with only λ as being variable. The characteristic energy is defined for this example as: $E_c = \frac{\nu_A \mu_A}{\nu_A + \mu_A}$. We will use λ and E_c to define the limits:

- $\frac{\lambda}{E_c} \ll 1$ for the DCL ($[\hat{O}_A, \hat{H}_A] \approx 0$),
- $\frac{\lambda}{E_c} \gg 1$ for the QML ($[\hat{O}_A, \hat{\phi}_A] \approx 0$), and
- $\frac{\lambda}{E_c} \sim 1$ for the general case.

Since we have a means to define the limits, we shall not use α to define the limits. This means our candidate pointer observables will have the form: $\hat{O}_A = f(\hat{\phi}_A) + g(\hat{H}_A)$, which is in line with step one of the algorithm in section 4.2, namely to have separability and linearity.

This is an example where the interaction term of the Hamiltonian has the ideal form of only consisting of $\hat{\phi}_A$ and $\hat{\phi}_B$. As a result:

$$\ddot{S}_{lin}\left(\hat{\rho}_A(t)\right) = \frac{4}{\hbar^2}\lambda^2 \left(\langle \hat{\phi}_A^2 \rangle_0 - \langle \hat{\phi}_A \rangle_0^2\right)\left(\langle \hat{\phi}_B^2 \rangle_0 - \langle \hat{\phi}_B \rangle_0^2\right). \tag{4.14}$$

Since we assume our states to initially be coherent (and hence have minimal uncertainty), and $\ddot{S}_{lin}$ measures the variance of the minimal uncertainty case, we expect equation 4.14 to be minimized for all three cases and it cannot be minimized further. Hence, our system will be robust in all three cases. All that is left for the algorithm to minimize is $\ddot{S}_{pointer}$ for the three cases.

The equation for $\ddot{S}_{pointer}$ will have a form:

$$\begin{aligned}\ddot{S}_{pointer}(0) =& \frac{2}{\hbar^2}\sum_{j=1}^{d_a}\langle[\hat{\Lambda}_j,\hat{H}_A]\rangle_0^2 + \frac{2}{\hbar^2}\sum_{j=1}^{d_a}\Big(\langle\hat{\phi}_B\rangle_0\langle[\hat{\Lambda}_j,\hat{\phi}_A]\rangle_0\Big)^2 \\ &+ \frac{4}{\hbar^2}\sum_{j=1}^{d_a}\lambda\langle\hat{\phi}_B\rangle_0\langle[\hat{\Lambda}_j,\hat{\phi}_A]\rangle_0\langle[\hat{\Lambda}_j,\hat{H}_A]\rangle_0 \\ &+ \frac{2}{\hbar^2}\sum_{j=1}^{d_a}\Big(p_j(0)\langle\hat{\Lambda}_j\hat{H}_A^2+\hat{H}_A^2\hat{\Lambda}_j-2\hat{H}_A\hat{\Lambda}_j\hat{H}_A\rangle_0\Big)\,,\end{aligned} \tag{4.15}$$

with Λ_j's being projection operators of $\hat{O}_A$.

4.4.1 In the DCL Case

In this case: $\frac{\lambda}{E_c} \ll 1$. This has the effect that the candidate pointer observable will be of a form: $\hat{O}_A \approx g(\hat{H}_A)$ such that $\left[\hat{O}_A, \hat{H}_A\right] \approx 0$. For example, our candidate pointer observable could be as simple as: $\hat{O}_A = \hat{H}_A$.

If we look at equation 4.15 and apply the parameters of this case, we will see that: $\hat{H}_{int}$ has approximately no effect (~ 0) due to its relative weakness, since $\hat{H}_A \approx \hat{H}$. Further, $[\hat{O}_A, \hat{H}_A] \approx 0$. As a result, $\ddot{S}^{DCL}_{pointer} \sim 0$.

The DCL is the case where extracting the classical description of the system is trivial. This makes sense as the environment is affecting the system the least. As a result, the system is more predictable and robust. However, a system that does not interact with the environment is not consistent with most cases of interest.

4.4.2 In the QML Case

In this case: $\frac{\lambda}{E_c} \gg 1$. This has the effect that our candidate pointer observable will be: $\hat{O}_A = f(\hat{\phi}_A)$ such that $\left[\hat{O}_A, \hat{H}_{int}\right] \approx 0$. For example, our pointer observable could be as simple as: $\hat{O}_A = \hat{\phi}_A$.

If we look at equation 4.15 and apply the parameters of this case, we will see that: $\hat{H}_{self}$ has approximately no effect (~ 0) due to its relative weakness, since $\hat{H}_{int} \approx \hat{H}$. Further, $[\hat{O}_A, \hat{H}_{int}] \approx 0$. As a result, $\ddot{S}^{QML}_{pointer} \sim 0$.

The QML has the same predictable and robust properties as the DCL. However, this is also not consistent with most cases of interest similar to the DCL.

4.4.3 In the $\hat{H}_{int} \approx \hat{H}_{sef}$ Case

In this case: $\frac{\lambda}{E_c} \sim 1$. This means our candidate pointer observable will have: $f(\phi_A) \sim g(H_A)$. For example, our pointer observable could be as simple: $\hat{O}_A = \text{Tr}_B(\hat{H})$.

This means that our expression for the $\ddot{S}_{pointer}$ will be the same as equation 4.15 and will not ~ 0. This will be the largest pointer entropy amongst the cases and allows for us to conclude that it is the most unpredictable case for this example. However, it will be predictable compared to the evolution of most quantum states. This, coupled with robustness, makes this the quasi-classical state of the system.

Collimation allows us to understand why this case will have the quasi-classical state of the system. Since $\hat{H}_A$ is highly collimated (with $\hat{H}_A$ meeting the form of the "traditional" Hamiltonian) and $\hat{H}_{int}$ is perfectly collimated (H_{int} being of a form that contains only one operator of a conjugate pair), the effective Hamiltonian is collimated and hence the state's evolution will be predictable. However, as this is the ideal case, a different interaction Hamiltonian will yield different results.

Chapter 5

Hilbert Space Fundamentalism

Now, let us turn our attention to Hilbert space fundamentalism as considered beyond contemporary quantum theory. The main idea of Hilbert space fundamentalism is that the fundamental description of the Universe is a vector in the fundamental Hilbert space which has finite dimensionality (the dimensionality will be explained in section 5.2). This requires the vector in the Hilbert space to be a fundamental as well as an ontological object. If we rather held the Universe to be represented by a vector in the Hilbert space, we would assume the vector to be epistemic and not fundamental but a representation of what is fundamental. Further, events are ontological and reversible (and hence behave analogously to branching in the Everettian interpretation as discussed in section 3.2); this is to avoid any "collapse" of the vector (which would be analogous to the collapse of the wave-function in quantum theory).

Hilbert space fundamentalism holds that the Universe, at a fundamental level, has no classical structure (and hence there does not exist one to collapse into, as needed by the Copenhagen interpretation, for instance) or a specifying algebra of observables (such as $\hat{\phi}$ and $\hat{\Pi}$ having physical meaning). No physical structures exist other than the Hamiltonian. All we have, at a fundamental level, is the Hilbert space (all possible states of the Universal vector correspond to linear combinations of a basis that spans this Hilbert space) and the Hamiltonian. All other valid specific theories, as well as the specifying algebras, will be encoded within the Hilbert space fundamentalism. How to decode them from the Hilbert space fundamentalism is something that is unknown at this time. However, the emergence of theories is something that has been established as physically and philosophically sound, as is shown in [90, 91].

In the discussion to follow, we will continue to follow the convention established in Figure 1.4. In that figure, the more fundamental a concept is, the higher it is positioned.

The vector in the Hilbert space is posited by Hilbert space fundamentalism as the fundamental entity that encapsulates the true nature of reality. Reality (for our purposes) is a multi-layered hierarchy where each level emerges from a more fundamental level positioned above it. This means that lower-level elements of reality, such as consciousness and our understanding of the laws of physics, emerge from a more fundamental level (the level immediately above it). We expect there to be a certain but currently unknown number of levels intervening between the lowest and highest levels. However, the elements of each level emerge from the level above it until we eventually (at the top of the hierarchy) reach the fundamental level, which contains the fundamental elements. According to Hilbert space fundamentalism, these elements are the vector in the Hilbert space and the Hamiltonian. By taking these elements as fundamental, rather than relying on classical interpretations of physical entities and processes, we adopt a minimalist view of the structure of reality. However, there is a level of abstraction since perceived reality (the level of the hierarchy that we inhibit) has specifying algebras which must emerge from this vector. This emergence will be discussed next.

5.1 Emergence of Space-time

With the starting details Hilbert space fundamentalism provides, namely a Hilbert space with a vector and a Hamiltonian, how does space-time emerge? Further, how do the inaccessible and accessible branches that were described in section 1.1.5 emerge from these starting details? What follows is the proposed description of the emergence of space-time.

Consider the decomposition of the Hilbert space into some number $N \gg 1$ nodes. The Hilbert space has some dimensionality D; hence we expect $N \ll D$ such that:

$$\mathcal{H} = \bigotimes_{i=1}^{N} \mathcal{H}_i. \tag{5.1}$$

With these nodes, we can create a nodal network, and we can then use the semi-familiar language of networks as used in field of network sciences [92, 93, 94]. If we think of each node as being connected to all the others by edges (analogous to a link in internet infrastructure), we can develop a topological network (a network of nodes connected to each other by edges).

We shall now find a means to map the topological network to space-time manifolds. Since Hilbert space fundamentalism is technically a quantum theory (and maps directly to textbook quantum theory), we can use concepts unique to that theory. We shall use entanglement. If we assign weightings to edges using entanglement, we can develop a notion of "closeness" and "farness" with nodes that are close together in the network being ones with high entanglement and ones that are far apart having low entanglement. A useful measure of entanglement is the mutual information $I(a : b)$. Consider the nodes $\mathcal{H}_{i_1}$ to $\mathcal{H}_{i_2}$, the mutual information $I(i_1 : i_2)$ will be:

$$I(i_1 : i_2) = S_{i_1} + S_{i_2} - S_{i_1 i_2}. \tag{5.2}$$

This results in: $0 \leq I(i_1 : i_2) \leq S_{i_1} + S_{i_2}$, with 0 for no entanglement and $S_{i_1} + S_{i_2}$ for maximally entangled states. With these possibilities for the weightings of the edge, we can map mutual information to space-time metrics using established ways in quantum information theory and thermodynamics [95, 96, 97, 98]. Hence, space-time emerges from the fundamental Hilbert space. If these space-times are not flat, they will have space-time dependent metrics which will allow for the possibility of gravity. A special case to consider is that of subsystems that represent vacuum states. These will be be mapped to Minkowski space-time (namely that their edges map to the Minkowski metric). From such a flat space-time, we can directly obtain the appropriate time parameter from the metric for that of quantum theory and classical mechanics [99, 100]. While the same applies for curved space-times which are flat enough to apply a Newtonian approximation. For general relativity, time will be a more subtle feature of the space-time.

This leads to a proposed structure of three levels of “space”. The first and most fundamental is the fundamental Hilbert space. The second and less fundamental (and a construction within the Hilbert space) is the topological network. The third level, and least fundamental (in the framework of Hilbert space fundamentalism), is the emergent space-times.

5.1.1 The Emergence of Accessible Regions

Not all of the space-time metrics that we can recover will have quasi-classical descriptions. These will be the swamp land space-times that are theorized to be in the inaccessible regions (as explored in section 1.1.5). How then do accessible regions emerge? This is where the given Hamiltonian will be used. At the level of the fundamental Hilbert space, we have four things. 1) There will be the given Hamiltonian. 2) Some notion of time embedded in the topographical network's emergent space-times. 3) There will be a basis in which the Hamiltonian will act over short-range (the basis will be formed by eigenvalues of some operator $\hat{\phi}$). 4) Lastly, the operators whose eigenvalues form the basis will have a conjugate operator.

To have classicality, we need to have a notion of locality and meaningful time evolutions. If the Universal vector starts in regions where nodes have high entanglement with each other in the topological network and remains in those regions when evolved by the Hamiltonian, we will have a sense of locality at the topological level. We can then map the sense of locality in the topological level to locality within space-time. Then, the basis that the Hamiltonian acts locally will map to the position basis. Further, the conjugate of position will map to momentum.

5.2 The Dimensionality of the Hilbert Space

Now that we have an emergent space-time, a discussion of dimensionality of the fundamental Hilbert space is prudent. However, a discussion of the fundamental Hilbert space is one with abstraction. Since the Hilbert space maps to the emergent space-times, we can apply discussions of the emergent space-time's degree of freedom to its dimensions. Note that the fundamental Hilbert space has an associated dimensionality (i.e., the number of basis vectors) while an emergent space-time is endowed with degrees of freedom (i.e., the number of independent parameters). An emergent space time will, due to the nature of the mapping, either have a total number of degrees of freedom that is the same as the dimensionality of the Hilbert space or it will be larger due to the possibility of gauge redundancy as will be discussed below.

What follows is a presentation of three arguments regarding the degrees of freedom in theories that utilize multiverses such as inflationary cosmology. However, these mechanisms have to apply to the fundamental Hilbert space by means of some "bootstrap" method. A bootstrap is when a higher-level theory uses mechanisms that emerge in lower-level theories to set certain parameters and features of the theory. If they apply without using a bootstrap (but by external means), this will impact this theory's candidacy as the fundamental theory (to be discussed in Chapter 7).

It has been shown that the Everettian interpretation of quantum theory is the same thing as the many worlds in the multiverse [27]. This maps to a model of eternal inflation (in other words inflation that has no upper bound). In this scheme, the number of Hubble volumes[1] increases by a factor of e each Hubble time $(t_H)^2$ interval forever [101]. However, it has been argued that there is a cap on eternal inflation provided by some form of gauge transformation that allows Hubble volumes that are related through gauge transformations to be considered redundant, since they contain identical physical information [67, 68, 71]. This is not unlike classical electromagnetism, where we can modify gauge fields while having the physical fields remain the same. This is a bootstrapping argument as it is using physical gauges and inflation (which are emergent) to explain the restriction of the dimensionality of the Hilbert space.

Another argument provided is similar to the one presented in Chapter 3, section 1 concerning gravity. For a spherical region R with a radius r, the maximal entropy it can have is that of a black-hole which is $S_{BH} = \frac{\pi r^2}{G}$. Region R will have $\leq exp(\frac{\pi r^2}{G})$ degrees of freedom. This is a bootstrap as it uses black-holes and the finiteness of their degrees of freedom, which is a phenomenon which must emerge from the fundamental theory to justify the finiteness in the dimensionality of Hilbert space.

All of these arguments for finiteness have been argued to be unnecessary in [87] where it is shown that all infinities in physics are actually just large numbers. If infinities do not exist, then the Hilbert space must be finite but have a large number of dimensions due to the mapping. This is a bootstrap argument since it applies what is true for physics as it is currently known (and as it emerges) to the fundamental level.

[1] A volume of $(c/H_0)^3$, with H_0 being the Hubble constant. The Hubble constant being defined as $H(t) = \frac{\dot{a}(t)}{a(t)}$ with $a(t)$ being the scale factor. The scale factor determines the extent of the spatial directions relative to their original size. In an inflating universe, $H(t)$ is approximately constant, hence for our purposes H_0.

[2] $t_H = \frac{1}{H_0}$.

5.3 Hilbert Space Fundamentalism's Possible Criticisms

The first area of dispute for Hilbert space fundamentalism is how closely related it is to Everettianism. The many worlds interpretation of quantum theory has a host of refutations facing it, including (but not limited to): it is unfalsifiable and hence not scientific [103, 104], but this is refuted by some physicists and philosophers [105]. Further, some view many worlds as inconceivable since it allows for all possibilities imaginable and unimaginable [106]. The last has to do with what many worlds' motivation is. Many worlds attempts to explain what occurs when a measurement is performed in quantum theory, something the theory fails to do. (Proponents of the Copenhagen interpretation famously have the attitude of "nothing to see here".) To resolve this issue, we introduce the idea of worlds being formed, which itself is as vague and arbitrary as the collapse of the wave function that it is trying to replace [107]. This is a "sky hook": we cannot explain something with physics, so we introduce a metaphysical construction to explain away the problem, but the solution itself has some "hand wavy" mechanism. If any of these arguments are true, they have heavy implications on Hilbert space fundamentalism due to the central role the Everettian formalism plays in its "inner workings".

Unfortunately (or fortunately for the Everettian proponents), we cannot use other ontological interpretations of quantum theory to develop a fundamental theory as they all require some collapse mechanism (which cannot occur in the fundamental Hilbert space) or the pilot wave (which we would need to add to our formalism of the Hilbert space with the Universal vector and a Hamiltonian) as discussed in section 1.1.2 and Chapter 3. The other approach would be to take a relational approach (analogous to what relational quantum mechanics provides to quantum theory).

What if we become agnostic to the ontology of the vector? This would be equivalent to viewing reality as represented by a vector in Hilbert space. This would mean replacing the Everettian ideas with those of relational quantum mechanics. Let us for the sake of conciseness call this alternative theory Hilbert space representationalism. This difference in viewpoints can be illustrated by thinking of the Hilbert-space-fundamentalism view of the Universal vector in the same way that classical electromagnetism views physical fields (something that can be directly measured in a laboratory), while Hilbert space reprentationalism views the Universal vector analogously to the gauge fields in classical electromagnetism (mathematical devices that cannot be directly measured). Note, in the same way that the Everettian interpretation of quantum theory is not the same as Hilbert space fundamentalism (rather it is an application of ideas from quantum theory to the Universe at a fundamental level), Hilbert space representationalism is not the same as relational quantum mechanics.

We will not criticize Hilbert space fundamentalism for being in a state of "work in progress". This would be an unfair criticism. However, we do note it since it does make determining if Hilbert Space fundamentalism could be a fundamental theory more difficult.

5.3.1 The Problem of "Times"

In physics, we have the problem of time [102, 108, 109, 110]. For instance: why does the arrow of time point in the direction it does? (In other words: where does causality come from?) Further, why does time function as a parameter in quantum theory and classical mechanics, but it is a part of space-time in general relativity? Hilbert space fundamentalism must provide answers to these questions if it is to be considered a fundamental theory. The answers have to be encoded in the vector since we know it is not emergent with space-time. Furthermore, Hilbert space fundamentalism faces a new problem: where does the given Hamiltonian from?

The working assumption will be that the Hamiltonian is from some form of bootstrap since the alternative would violate the criteria for the fundamental theory (as will be seen in the next chapter, section 6.3). If it does stem from a bootstrap, we need some way of knowing which Hamiltonian is going to be chosen by the bootstrap. A candidate means would be the Hamiltonian that maximizes accessible regions (or in other words: maximizes reality). We could call the Hamiltonian that provides the most accessible regions the pointer Hamiltonian since, on the scale of the universe, it serves a role similar to what the pointer observable does for the emergence of classical states of the system in quantum mechanics.

One guide to finding this pointer Hamiltonian is through finding the correct time parameter for the quantum theory. While we have the Minkowski space-time metric to obtain time, it lies beyond the domain of quantum theory (as defined in section 1.1.1). Consider a subsystem within the environment ($\mathcal{H}_C$) that is designated to play the role of the clock. A clock which satisfies certain properties can be regarded as a good clock [102], and we propose that the time kept by such a clock can be mapped to the time coordinate in the Minkowski line element [109]. These properties are: 1) The clock does not interact with the environment. 2) The effective Hamiltonian of the clock (the Hamiltonian with the complement of the clock traced out) has a conjugate observable[3] whose eigenvalues will determine the clock's time parameter. 3) The clock is maximally entangled with the environment.

[3]Hamiltonians cannot have a conjugate since they are bounded from below due to the existence of a ground state. However, an effective Hamiltonian can have a conjugate because it is not necessarily bounded in this way.

Chapter 6

What is a Fundamental Theory?

Before we can assess if Hilbert space fundamentalism is a fundamental theory, we first have to know when a theory is fundamental.

In the literature, the word fundamental is used liberally and often misses sight of what is actually fundamental. For example, it is correct to say that general relativity is more fundamental than special relativity, but it is not correct to say general relativity is the most fundamental theory since there is clearly a more fundamental theory in the form of some unknown theory of quantum gravity[1]. This becomes a lot more nuanced when we look at the definition of what is a fundamental theory. While the definition of theory and what makes something a theory is well established from the responses of Popper to Hume [118, 119, 120], a fundamental theory is less clearly described in the literature of physics. In this chapter, we shall look at what constitutes a theory, what makes something fundamental, and then present a clear set of criteria for the fundamental theory.

[1]Competing thoughts do exist that hold gravity to be classical, and that it interacts with quantum systems in some hybrid manner [116]. However, the only consistent way for gravity to co-exist with quantum field theory is in string theory [117].

6.1 What Is A Theory?

Theories form the heart of physics. They are crucial in science generally, but we shall restrict ourselves to physics, even though what is being stated is applicable to all science. This is because of the major ways knowledge can be confidently developed in physics. Knowledge can be formed by a bottom-up manner, namely we observe something, develop a theory to explain the observation, and then test the theory through new observations or experiments. Another method that is often applicable is to develop a theory and then see how much it can explain, and then test it (the top-down manner). This brings us to the three key properties of a theory: 1) A theory must be falsifiable. Falsifiability means that the theory must make unique predictions that can be tested, which will prove or disprove it. If a theory is proven beyond doubt or by definition, it is not a theory but a law. If a theory is falsified, it is then no longer an accepted theory but a dis-proven theory. 2) A theory must, furthermore, be supported by many different sets of evidence. For example, classical mechanics has the evidence in the form of Kepler's laws and the Coriolis effect. 3) Lastly, a theory must be consistent with pre-existing experimental results and observations [111, 112].

This leads us to a discussion concerning the parts of a theory. The first part of a theory we need to discuss is the starting assumptions. For example, Einstein, when he formulated the theory of special relativity, assumed that the speed of light is constant in a vacuum (perhaps in different wording) and all inertial observers are equal [121]. Both assumptions are prescient and were based on the best science at the time the theory was introduced. Assumptions also can (often inadvertently or in some cases implicitly) set the domain of a theory. The domain is the parameters in which the theory is relevant. The parameters that can be used are the fundamental parameters: $\hbar$,c and G [122, 123]. With the example of classical mechanics, it was inadvertently chosen that $\hbar \to 0$, since $\hbar$ was not known at the time that the formulation of classical mechanics occurred. Sometimes, the assumptions are known to be inaccurate in the general domain but are perfectly fine for what the theory is explaining without needing extra layers of complications. (For example, gravity is not considered in most of classical thermodynamics, since it has minimal effect in explaining heat transfer.)

A second element of theories is that they often build on applicable preceding theories. For example, quantum field theory uses the knowledge from quantum theory and the theory of special relativity. This can be used to test new theories, since when we restrict them to old theories' domains, they must replicate the older theories. For example, when we model quantum theory to the classical domain, it will replicate classical mechanics.

With a firm understanding of theories and their domains, we can discuss a special class of theories: best available theories. The best available theory is one that gives the best explanation of a phenomenon or a set of phenomena. These theories are often based on preceding theories but explain what preceding theories could not. For example, quantum theory is precise when explaining the results of the double slit experiment but will come up short when explaining subatomic matter's interactions. However, quantum field theory can replicate quantum theory and can explain subatomic interactions. The current best available theories are quantum field theory and the theory of general relativity. However, they are not applicable to all domains and cannot be unified without working around the Weinberg–Witten theorem [113]. The Weinberg–Witten theorem shows that it is not possible to incorporate gravity into quantum field theory by means of some composite particle with a spin of 2 and zero mass because such a particle cannot couple to the energy–momentum tensor. The Weinberg–Witten theorem does not apply to fundamental particles with spin 2 and zero mass (i.e. a hypothetical quantum excitation of the background metric or "graviton"), but such particles are beyond the realm of standard quantum field theory.

There is another class of theories that we need to discuss, namely theories that could become best available theories. We will call these theories candidate theories. What is preventing these theories from being considered best available theories is that they are either not fully formulated (have pieces missing) or that they need experimental verification that is unique to them (a result that they alone can explain). Theories that fall in this category include string theories[2] which make predictions in principle that have not yet been tested in practice.

6.1.1 When is a Theory Complete?

Completeness is a very important feature for a theory to meet. The issue of completeness was raised in the famous Einstein–Podolsky–Rosen paper (EPR) [13]. While this paper is mostly thought of as showing the concept of entanglement, we will be looking at the notion of completeness introduced (since the papers actual title is: "Can quantum-mechanical description of physical reality be considered complete?").

The main crux of completeness in EPR is the notion that – for a theory to be complete – we must be able to, in principle, extract information about a system without changing it. This, however, is not possible in quantum theory due to the uncertainty relationship and the collapse of the wave-function (or, in the case of Everettian quantum theory and relational quantum mechanics, the extraction will lead to entanglement with the system of interest). It was believed by some, at the time of EPR, that a deterministic theory which is complete according to EPR existed through locally hidden variables. However, as stated previously in subsection 1.1.1, we cannot have a locally hidden variable theory that makes the same predictions as quantum theory (due to the Bell inequalities). This leads to a situation where the notion of completeness from EPR is not compatible with accepted theories. Further, we now consider quantum theory to be a complete theory since it is fully applicable to the quantum domain [114], regardless of what EPR claims about it. Hence, despite first being raised in EPR, modern completeness does

[2] String theory is not falsifiable at this time, but for illustrative purposes, we will suppose it could be.

not follow the criteria of that paper. Rather, the ideas of EPR are now viewed as being more about the determinism of a theory [115] which is beyond this project.

We will rather define completeness by seeing if the theory makes accurate predictions within the theory's domain, with domain being the limits set by $\hbar$, $\frac{1}{c}$ and G. Clearly, quantum theory makes some predictions that are incompatible with the classical world (such as Bell inequalities). However, both theories are applicable in their own domains. Hence, we shall define a theory as complete when it is fully consistent within the domain it is restricted to. In this sense, quantum mechanics is a complete theory in the domain of $\hbar \neq 0$, $\frac{1}{c} \to 0$ and $G \to 0$. Classical mechanics, on the other hand, is complete when $\hbar \to 0$, $\frac{1}{c} \neq 0$ and $G \neq 0$.

6.2 What Is Fundamental?

Perhaps because the notion of what is fundamental is so vague, it is difficult to discuss what it means. This can be seen in section 1.1.5 and the beginning of this chapter. In the physics literature, we have not found any suitable definitions for what is fundamental, although some attempts have been made, as seen in [124, 125, 126]. There is one clear element of fundamental that must be described: the fundamental theory must map to the accepted theories when applied to their domains (for instance, the fundamental theory must agree with quantum theory when applied to the quantum domain), but this is not enough to obtain a clear definition or even sense of what is fundamental. As a result, we will look to philosophy for aid, in particular, metaphysics.

In metaphysics, there are four different schools of thought on what makes something fundamental [127, 128, 129, 131] and hence no clear conceptual agreement. As a result, we will rather look at the key properties of the various schools of thought and not choose to follow any particular school of thought. (A brief look at each of the four different schools is available in Appendix G.)

The first property (which is used in defining two schools of thought) is that of independence. Independence is when an item's existence (be it substantive, conceptual, or metaphysical) is fully independent of all other items. In our case, an idea is fundamental when its existence does not depend on another idea but other ideas use it and other independent ideas to be formulated. An example is space and time, with which we develop the notions of distance, velocity, and acceleration.

Some in the metaphysics community follow a different school of thought for what is fundamental. This is the complete minimal basis school of thought. An item in this approach is fundamental when it belongs to a set of entities and that set of entities forms a complete basis that determines everything else. These entities would be things such as space, time, and certain symmetries. Using these concepts, we can build the different concepts.

The last philosophical school concerning what is fundamental holds that nothing can be fundamental. We shall not be using that view as it would be trivial. (If nothing can be fundamental, a fundamental theory could not exist, and we have a no-go.) It could be that nothing is fundamental. However, arriving at this conclusion using the other schools of thought is more convincing than holding it as a starting premise.

6.3 Criteria for the Fundamental Theory

Before we can discuss what the fundamental theory is, the following disclaimer is important: a theory of everything is not necessarily the same as the fundamental theory. While some physicists called their attempts to formulate a theory of everything the fundamental theory (notably Eddington [132]), this is in conflict with modern ideas of theory space (such as the swamp land) as discussed in section 1.1.5 and in the literature as presented in: [133, 134, 135].

With a clear definition and concept of what is fundamental and what constitutes a theory, we can start formulating a set of criteria to capture all the features we have mentioned. The criteria are as follows: 1) it replicates specific theories when in their domains, 2) it is a complete theory in all domains, and 3) it is self-contained.

Completeness in all domains and replication of specific theories emerging from the fundamental theory are criterions that are pervasive in literature, as can be seen in some of the unsuccessful attempts at formulating the fundamental theory [132, 136, 137, 138]. The fundamental theory must be a theory that is complete in all physical domains. As it stands, we do not have a theory that is widely accepted that can be viewed as the fundamental theory because no current theory is applicable in all domains (quantum field theory is inapplicable in the domain of general relativity and general relativity is inapplicable to quantum field theory's domain[3]). Further, all accepted specific theories must emerge from the fundamental theory (for example, how special relativity emerges from general relativity when the space-time metric becomes the flat Minkowski metric). This leads to a prospective fundamental theory being more readily found to be incomplete or incorrect than a specific theory.

Self-containedness is the synthesis of the ideas from metaphysics concerning what is fundamental (as presented in section 6.2). We define a theory to be self-contained when all elements of the theory are organic to it. For example, quantum theory is generally self-contained; however, if it were to require the bending of space-time to explain the emission spectrum of a hydrogen atom, it could no longer be viewed as self-contained since it is using a concept that is not organic to the theory to explain something within the theory[4]. Within this criterion, there is a source of contention, namely: are bootstraps, as defined in section 5.1, admissible? On the one side of the contention, bootstraps should be admissible since they do

[3]However, general relativity and quantum field theory are fundamental to a degree as they cover the bedrock on which all other contemporary physics rests.

[4]The nature of time in quantum theory is controversial since, on one hand, it is an external parameter but, on the other, it is evolved by the Hamiltonian which is contained within quantum theory.

not use inorganic elements for the theory, but emergent information to set parameters in the fundamental theory. (In the case of Hilbert space fundamentalism, for instance, the bootstrap sets the dimensionality and the Hamiltonian.) The other side of contention is that bootstraps lead to circular arguments (an argument of: if A then B, if B then A), which was shown to be a logical fallacy in section 1.1.5 and in agreement with [139]. Using the example of Hilbert space fundamentalism's space's dimensionality, we see: the space is finite dimensional since the emergent theories of cosmology have finite dimensionality. The emergent theories are finite dimensional since they map to a Hilbert space with finite dimensionality.

Using the clarity as to what the fundamental theory entails, we can start to look at which theories could be possible fundamental theories. We can also ask if it is possible for a fundamental theory to exist.

Chapter 7

Is Hilbert Space Fundamentalism a Fundamental Theory?

Can Hilbert space fundamentalism be considered a candidate for the fundamental theory? Hilbert space fundamentalism holds the Hilbert space and the vectors that define it as fundamental. However, Hilbert space fundamentalism cannot be considered a theory, let alone a fundamental one. This stems from the fact that Hilbert space fundamentalism is not falsifiable (as defined in the previous chapter). There are no ways to test Hilbert space fundamentalism nor any unique experiments for it[1]. We have no way of reversing the mapping from space-time to the topological network as described in section 5.1. We can determine what the semi-classical space-time will look like but not what the topological network would look like. This is because we have a notion of what semi-classical worlds correspond to since we reside in a classical world. However, the same is not true for the topological network of which we know very few defining properties. This is comparative to having an unsorted set with no defining criteria, and then sorting it to correspond to a set of criteria. If we were to try to obtain the original set by way of some form of shuffling, we would have to randomly shuffle until we randomly obtain the set that corresponds to the original. However, the number of possible shuffles will depend on the number of configurations in the set [141].

[1]This is not necessarily true for Everettian quantum theory which has a few proposed experiments that are alleged to prove or disprove it. See, for example, [140].

On the scale of the universe, this will correspond to a number much greater than one, which makes deducing the topological network from the emergent space-times with any certainty an impossibility.

The Fundamental theory	
Complete theory in all domains	Yes
Specific theories emerge	Yes
Self-contained	Yes

However, what if Hilbert space fundamentalism could become a theory? What then are its prospects?

7.1 Hilbert Space Fundamentalism as a Theory

Let us, for the sake of argument, hold Hilbert space fundamentalism to be a theory (either by finding a way to test it or some other means). Will it then be able to meet the criteria for the fundamental theory?

7.1.1 Is Hilbert Space Fundamentalism Complete?

If we were to hold that reality is a vector in Hilbert space, then from a conceptual point of view, the theory is complete at a Universal level. This is one of the motivations for Hilbert space fundamentalism.

7.1.2 Would Specific Theories Emerge from Hilbert Space Fundamentalism?

It is postulated that specific theories emerge from Hilbert space fundamentalism. However, this postulate cannot be tested since we cannot reverse the mapping as described in Chapter 5.

7.1.3 Is Hilbert Space Fundamentalism a Self-Contained Theory?

This is where the contention regarding bootstraps plays a role. If Hilbert space fundamentalism is to be fully organic (as required for self-containedness), it will require the use of bootstrap arguments for the dimensionality of the Hilbert space and for the source of the Hamiltonian. As discussed in the previous chapter, this is contentious. If Hilbert space fundamentalism does not use any bootstraps, it is not self-contained; if it does use bootstraps, it contains the logical fallacy of circular reasoning. This contention could become an entire thesis unto itself, it does raise similar questions as to what was briefly mentioned in Chapter 6, namely, "why a theory is a theory". It was argued by some (Hume most notably, but he was not alone) that inductive reasoning is inherently circular (a component of the philosophical problem of induction) [118, 119]. Inductive reasoning is when we generalize an observation. (For instance: if someone touches a hot stove, they will conclude that a hot stove will burn them.) If this is so, then theories will inherently include circular reasoning since they use inductive reasoning to be tested. We can go a step further and contend that the fundamental theory should be circular in its testing. However, this view is not widely accepted as seen with Popper's solutions to the problem of induction which use the abundance of evidence to overcome the flaws of induction [130]. As has been discussed, Hilbert space fundamentalism does not have any evidence to invoke when trying to solve this problem, adding to the contention.

Hilbert Space Fundamentalism	
Complete theory in all domains	Yes
Specific theories emerge	Yes
Self-contained	Contentious

7.2 Prospects for Hilbert Space Representationalism?

If Hilbert space fundamentalism is inspired by many worlds and does not meet the criteria, how would a model that is inspired by relational quantum mechanics fair? Hilbert space representationalism holds that some vector in Hilbert space represents reality. It is a tool for us to understand reality. This theory, however, could not be considered a fundamental theory but rather a representation of what the fundamental theory could be. This is because it is not self-contained, but has a dependence on what it is a representation of.

Despite not being able to be the fundamental theory, Hilbert space representationalism has some key advantages. Since Hilbert space representationalism is agnostic to the nature of the ultimate ontology of the Universe, the level of physics it can be applied to is more vast than that of Hilbert space fundamentalism. Hilbert space fundamentalism, on the other hand, is constrained to the fundamental level where it cannot (as argued at the start of this section) be the fundamental theory. Further, due to Hilbert space representationalism's epistemic nature, it is a convenient tool to model our understanding in the pursuit of the fundamental theory.

<u>This leads us to two questions.</u>
1) Can an epistemological theory ever be a fundamental theory? The suspected answer is no. The reason for that is that epistemological theories are ones that represent our understanding of nature, not of nature itself. As a result, they are representations of reality, not the ontological descriptions, and hence their elements are not of a fundamental nature [142, 143]. However, people with a more empirical world view will refute this position by arguing that, while it is a representation, it could be the fundamental representation [144]. This is the representation that provides the best possible understanding, and meets all the criteria, and hence is fundamental.

2) Can a theory be the fundamental theory? This is a question that has some tie in to the fact that we do not have an accepted theory of everything at this time, let alone the fundamental theory. Perhaps this is because a fundamental theory could violate the Turing proof which states that: if we have an arbitrary computer with a given input, a second computer that "knows" the construction of the first computer and its input cannot consistently predict the output of the first computer [145]. This is strongly connected to Gödel's incompleteness theorems, which shows that a system cannot demonstrate its own consistency [146]. Further it has long been argued that the mind cannot account for its own existence[2][144, 147]. It has further been contended that the Universe maps to a computer [148, 149], since both are systems that store information and have internal rules that govern processes within them. If a computer cannot demonstrate its own consistency, how can it account for itself? The answer is that it cannot and, by extension, the Universe cannot. If the Universe cannot account for itself, how can we devise a fundamental theory? This brings up the primitivism school of thought that holds: nothing can be defined as fundamental. Perhaps primitivism is the correct school of thought in this context.

Hilbert Space Representationalism	
Complete in All domains	Yes
Specific Theories emerge	Yes
Self-contained	No

[2]This is the cause of the distinction between mind and brain. A single person cannot show the mind is contained in the brain nor can the mind provide a reason for the person's existence.

Chapter 8

Discussion and Conclusion

8.1 Emergence of Classicality

In Chapters 1, 2, and 4, we discussed how to extract a quasi-classical state of the system from a Hilbert space with a specified Hamiltonian. This is done by finding a suitable factorization of the Hilbert space and an acceptable pointer observable that meets the criteria of robustness and predictability. To assist in our understanding of how the correct choice of factorization and pointer observable can emerge, we used the Carroll and Singh algorithm and the notion of collimation of the Hamiltonian to illustrate how the procedure could work.

Furthermore, we have seen how pointer observables behave in the general case and decoherence limits in addition to the quantum measurement limit as was seen in [61, 62]. This was done with the use of the coupled oscillator example, with the coupling strength being varied for each case.

8.1.1 Possible Problems with the Emergence of Classical Worlds

The first issue could be candidate pointer observables that we formulated. We made the assumption that a candidate pointer observable is a linear combination of the expected pointer observable for the decoherence limit and the expected pointer observable for the quantum measurement limit. It could be that candidate pointer observables outside the limits are not a linear combination of pointer observables for the quantum measurement limit and decoherence limit, but some other relationship. However, our assumed candidate pointer observable is applicable to the aforementioned limits, so the issue would have to be with the general case.

Another issue faced with our quasi-classical factorization of the state of the system from the environment is that the algorithm presented does not have a way of including redundancy. As was discussed, we can use three qualities to define classicality, namely robustness, predictability, and redundancy. While the presented algorithm takes into account robustness and predictability, we must ask: why did we not take redundancy into account? The answer is straightforward; we do not have a way of calculating it. This is because, to take into account redundancy, we would need to count classical correlations. However, we have a pure quantum system, and hence there cannot be any presupposition of any classical correlations.

8.2 Hilbert Space Fundamentalism as the Fundamental Theory

As illustrated in Chapters 1, 5, and 7, Hilbert space fundamentalism is the idea that we can describe the Universe as a vector in Hilbert space. This has the benefit that it gives a possible explanation for how all accepted physical theories emerge from a single vector in Hilbert space. This is further supported by the fact that it does so with very few "ingredients" with the only requirements being a given Hamiltonian and Hilbert space. This is ideal from an Occam's razor point of view. Despite the benefit, we conclude that Hilbert space fundamentalism cannot currently be considered the fundamental theory. This is because Hilbert space fundamentalism is not a theory. If we consider Hilbert space fundamentalism a theory, for the sake of argument, it might then be a complete theory in all domains and specific theories are postulated emerge from it. However, we have the issue that the theory is not necessarily self contained. This is because it uses a bootstrap, and bootstraps are contentious within the self-contained criterion.

If we rather use the Hilbert space representation theory as the fundamental theory, we no longer have an ontological theory. This can lead us to the question: could any epistemological theories ever be considered to be the fundamental theory? As discussed in the previous chapter, the suspected answer is no. Further, must the fundamental theory then be ontological? Can a theory be ontological? Knowing what is real and not real is a cause of lots of contention in philosophy that often slips into physics (as can be seen through solipsistic interpretations of quantum theory). If we cannot for certain say that a theory is describing nature itself, but rather how we understand it, then how do we describe nature itself fundamentally? This brings us to the question: can a theory ever be considered fundamental?

8.2.1 Problems with our Analysis Concerning Fundamental Theories

Potentially, the biggest issue with the analysis we have provided is that people might conclude that our criteria are too restrictive or not restrictive enough. However, the criteria provided is a synthesis of metaphysical fundamentalism, what constitutes a theory, and what is a complete theory. Some might also prefer criteria that has a mathematical backing (by means of some theorem(s)); however, this might be impractical when the theories being discussed do not yet fit into the existing structure of accepted physics.

Another potential problem could be with the source of our criteria. Some people might feel that obtaining the criteria with the assistance of metaphysics is not a solution, but rather that the question is what is fundamental to physics as a body of knowledge. The response to this is that the criteria from metaphysics are Universal to all bodies of knowledge.

8.3 Future Work

It appears, based on our analysis, that no further work can be done with Hilbert space fundamentalism despite the interesting notions it introduces and utilizes. This stems from Hilbert space fundamentalism having too many abstract ideas in it that are untestable.

However, further work can be done on the emergence of quasi-classical states as described in Chapter 4. Firstly, someone could create a computer program that implements the algorithm presented in section 4.2. This has been done in part for the quantum measurement limit in [150], but a full analysis has not been done. Further work could be done on the notions of locality raised in [61] and [74] by providing more details and looking at locality in non-position spaces such as momentum space.

Alternatively, someone could try and obtain a better candidate pointer observable by not assuming its linearity and rather modeling it to be a more accurate reflection of what the pointer observable's behaviour actually is. However, this would require a far better understanding of the behaviour of pointer observables when not in the special cases of $\hat{H}_{int} \sim \hat{H}$ and $\hat{H}_{self} \sim \hat{H}$.

Areas of work beyond physics could include developing mathematical descriptions for each criterion and seeing if they are able to be met or are no-go theorems preventing them from being met. Furthermore, work can be done to answer the questions raised in Chapter 7 concerning the existence of an epistemological fundamental theory. Alternatively, going even further, can a fundamental theory exist?

Appendix A

Measurements

In sections 1.1.1, 1.1.2, and 2.1.4, we discuss measurements and how they cause non-unitary transformations. In this appendix, we will illustrate the von Neumann model of measurement to support the statements in the aforementioned sections.

Suppose some system S is about to be measured by some apparatus A that couples the system to the environment E. Before measurement, A will be in an initial state. We will notate A's state before measurement as 0. If we are to measure the position x with the associated operator of $\hat{X}_S$ for the system, then we write:

$$|\Psi\rangle_{before} = |x\rangle_S |0\rangle_A. \tag{A.1}$$

When the measurement is performed, an operator will set the apparatus to be able to record what the system state is. This is conceptualized with a translation operator that operates on the apparatus: $e^{i\hat{X}_s\hat{P}_A}$ that will set the apparatus to agree with the system:

$$|\Psi\rangle_{after} = e^{i\hat{X}_s\hat{P}_A}|x\rangle_S|0\rangle_A = e^{ix_S\hat{P}_A}|x\rangle_S|0\rangle_A = |x\rangle_S|x\rangle_A. \tag{A.2}$$

The previous equation is unitary, but measurements are not unitary. This is where the environment comes into the model since the observer can be a part of the environment. Before measurement, we actually have:

$$|\Psi\rangle_{before} = |x\rangle_S |0\rangle_A \left(\sum_{i=1}^{d_E} |y_i\rangle_E \right), \tag{A.3}$$

with d_E being a sufficiently large number. After measurement, the state of the environment must be able to agree with the measured result. This will collapse the state of the environment and "set" it to agree with the system and apparatus:

$$|\Psi\rangle_{after} = |x\rangle_S |x\rangle_A |x\rangle_E. \tag{A.4}$$

Appendix B

Growth of Entanglement with Time Evolutions

In sections 1.1.3 and 1.1.4, we discuss the growth entanglement between the system and environment. In this appendix, we will illustrate the process. This will be done by showing the relationship between entropy, information, and entanglement.

As time evolves, entropy increases due to the second law of thermodynamics. Further, information is strongly related to entropy. This can be understood by the fact that, when storing information, you are replacing previous information with the new information which causes the generation of heat [151]. For example, consider saving a file on a computer: this will result in the changing of bits (either from 1 to zero or zero to one), which uses electricity and causes the generation of heat and hence increases entropy. Further, mutual information is often used to measure entanglement. Mutual information between two systems is given by [8]:

$$I(A:B) = S(A) + S(B) - S(AB), \tag{B.1}$$

with $I(A:B)$ being the mutual information, $S(A)$ being the entropy of system A, $S(B)$ the entropy of system B, and $S(AB)$ is the joint entropy of the two systems.

Consider the example of an electron couplet with one electron labeled as 1 and the other labeled as 2. When they are the product of singlet states, we have the following description for the couplet:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_1 + |\downarrow\rangle_1) \otimes \frac{1}{\sqrt{2}}(|\downarrow\rangle_2 + |\uparrow\rangle_2). \tag{B.2}$$

The von Neumann entropies are $S(1) = 0$, $S(2) = 0$, and $S(\Psi) = 0$ and the mutual information is $I(1:2) = 0$.

Now consider the electrons getting entangled. For the entangled system:

$$\Psi = \frac{1}{\sqrt{2}}(|\uparrow\rangle_1 |\downarrow\rangle_2) + \frac{1}{\sqrt{2}}(|\downarrow\rangle_1 \| \uparrow\rangle_2), \tag{B.3}$$

which has the von Neumann entropies: $S(1) = 2\ln 2$, $S(2) = 2\ln 2$, but $S(\Psi) = 0$ and the mutual information is $I(1:2) = 2\ln 2 + 2\ln 2 + 0 = 4\ln 4$.

Hence as time passes, entropy and information will increase. When information increases, the entanglement will increase.

Appendix C

Predictability Sieve and Pointer Observables

In the main text we presented the pointer observable in a manner that was ignorant of how it was originally defined by Zurek [55, 56, 57, 58]. In this appendix, we shall discuss the historical mechanism that gave rise to this definition of the pointer observable. This mechanism is the predictability sieve.

C.1 Predictability Sieve and Preferred states

Suppose that we want to identify states that are suitably localized in some preferred basis and thus exhibit classicality. These classically preferred states will be the ones that are least affected by the environment [30] as was observed by Zurek [57, 156]. The mechanism that we can employ to identify these state is Zurek's predictability sieve.

The process is as follows: first, we identify all the pure states of the system. Then we evolve each of those states over some fixed time interval that is longer than the decoherence time (the minimal time interval that is required before the effects of decoherence occur) but otherwise arbitrary. Then we calculate the entropy of each of these states using one of the following entropies: the von Neumann entropy or the linear entropy. Finally, based on their entropies, we list the states beginning with those having lowest entropy, as these are expected to be the most classical. To determine the classical states on the list will depend on a threshold which separates the classical states from the non-classical ones. To determine the threshold is a somewhat subjective process in the framework of decoherence that will depend on the system of interest.

C.2 Pointer Observables from the Preferred States

As observed by Zurek, the classically preferred states tend to be mixtures of the eigenstates of a few selected observables. These selected observables were referred to by Zurek as pointer observables [30, 57]. The algorithm presented in section 4.2 could be viewed as a more sophisticated version of the predictability sieve as it uses two entropies (linear and pointer entropy) rather than just the one.

Appendix D

Linear Entropy

In the calculations in this appendix, and the ones to follow, we will treat $\hbar = 1$. However, all previous conventions and notations will still be observed.

In this appendix, we will calculate the linear entropy which is defined in section 2.2.2 and used in section 4.4.

First, we define the unitary evolution of our density matrix:

$$\hat{\rho}(t) = \hat{U}(t)\hat{\rho}(0)\hat{U}^{\dagger}(t). \tag{D.1}$$

The unitary operator $\hat{U}(t)$ and its complex conjugate depend on the Hamiltonian. We will have self and interactions terms in the Hamiltonian as discussed. The self terms correspond to each Hilbert space and the interactions across them. This means that $\hat{H} = \hat{H}_{int} + H_{self}$ and $\hat{H}_{self} = \hat{H}_A + \hat{H}_B$. Furthermore, $\hat{U}(t) = exp(-i\hat{H}t)$. Using this information, the unitary operator can be expanded to its higher order terms such that:

$$\hat{U}(t) = exp\left(-i\hat{H}t + \frac{(-it)^2}{2}[\hat{H}_{int}, \hat{H}_{self}] + \mathcal{O}(t^3)\right). \tag{D.2}$$

Applying equation D.2 to equation D.1 will give us a "messy" result, so let's define our factorization of $\hat{\rho}$ as $\hat{\rho} = \hat{\rho}_A \otimes \hat{\rho}_B$. Further, we define: $\hat{\sigma}_A(t) =$

$exp(-i\hat{H}_A t)\hat{\rho}_A(0)exp(i\hat{H}_A t)$ and $\hat{\sigma}_B(t) = exp(-i\hat{H}_B t)\hat{\rho}_B(0)exp(i\hat{H}_B t)$ (identifying the terms that do not change with the interaction terms). Hence, we can write:

$$\begin{aligned}\hat{\rho}(t) =& exp\left(-i\left(\hat{H}_{int} + \frac{it}{2}\left[\hat{H}_{int}, \hat{H}_{self}\right]\right)t\right)(\sigma_A(t)\otimes\sigma_B(t)) \\ &\cdot exp\left(-i\left(\hat{H}_{int} + \frac{it}{2}\left[\hat{H}_{int}, \hat{H}_{self}\right]\right)t\right).\end{aligned} \tag{D.3}$$

Now, if we were to configure it into an expanded form, we would get:

$$\begin{aligned}\hat{\rho}(t) =& (\hat{\sigma}_A(t)\otimes\hat{\sigma}_b(t)) - it\left[\hat{H}_{int} + \frac{it}{2}\left[\hat{H}_{int}, \hat{H}_{self}\right], (\hat{\sigma}_A(t)\otimes\hat{\sigma}_b(t))\right] \\ &+ \frac{(-it)^2}{2}\left[\hat{H}_{int} + \frac{it}{2}\left[\hat{H}_{int}, \hat{H}_{self}\right], \left[\hat{H}_{int} + \frac{it}{2}\left[\hat{H}_{int}, \hat{H}_{self}\right], (\hat{\sigma}_A(t)\otimes\hat{\sigma}_b(t))\right]\right] \\ &+ \mathcal{O}(t^3).\end{aligned} \tag{D.4}$$

Let us perform our factorization. We choose to calculate the A component of our factorization. To do this, we will trace out the B component. This gives the expression:

$$\begin{aligned}\hat{\rho}_A(t) =& \mathrm{Tr}_B\left(\hat{\rho}(t)\right) = \hat{\sigma}_A(t) - it\mathrm{Tr}_B\left[\hat{H}_{int} + \frac{it}{2}\left[\hat{H}_{int}, \hat{H}_{self}\right], (\hat{\sigma}_A(t)\otimes\hat{\sigma}_b(t))\right] \\ &- \frac{t^2}{2}\mathrm{Tr}_B\left[\hat{H}_{int}, \left[\hat{H}_{int}, \hat{\rho}(0)\right]\right] + \mathcal{O}(t^3).\end{aligned} \tag{D.5}$$

The inclusion of the expanded forms of $\hat{H}_{int}$ and $\hat{H}_{self}$ gives the bulky expression:

$$\begin{aligned}\hat{\rho}_A(t) =& \hat{\sigma}_A(t) - it\sum_{\alpha}^{P}\lambda_\alpha\mathrm{Tr}_B\left(\hat{A}_\alpha\hat{\sigma}_A(t)\otimes\hat{B}_\alpha\hat{\sigma}_B(t) - \hat{\sigma}_A(t)\hat{A}_\alpha\otimes\hat{\sigma}_B(t)\hat{B}_\alpha\right) \\ &+ \frac{t^2}{2}\sum_{\alpha}^{P}\lambda_\alpha\mathrm{Tr}_B\left[\left[\hat{A}_\alpha\otimes\hat{B}_\alpha, \hat{H}_A + \hat{H}_B\right], \hat{\sigma}_A(t)\otimes\hat{\sigma}_B(t)\right] \\ &- \frac{t^2}{2}\sum_{\alpha}^{P}\sum_{\beta}^{P}\lambda_\alpha\lambda_\beta\mathrm{Tr}_B\left(\left[\hat{A}_\alpha\otimes\hat{B}_\alpha, \left[\hat{A}_\beta\otimes\hat{B}_\beta, \hat{\rho}(0)\right]\right]\right).\end{aligned} \tag{D.6}$$

From now on, P will be the upper bound for all the sums in this appendix. As a result, it will be dropped from the sums. We can split this up into additional portions: $\hat{\rho}_A(t) = \sigma_A(t)_{\mathcal{O}(t^2)} + T_1 + T_2 + T_3$. Thus:

$$T_1 = -it\sum_\alpha \lambda_\alpha \mathrm{Tr}_B\left(\hat{A}_\alpha\hat{\sigma}_A(t)\otimes\hat{B}_\alpha\hat{\sigma}_B(t) - \hat{\sigma}_A(t)\hat{A}_\alpha\otimes\hat{\sigma}_B(t)\hat{B}_\alpha\right) + \mathcal{O}(t^3), \quad \text{(D.7)}$$

$$T_2 = \frac{t^2}{2}\sum_\alpha \lambda_\alpha\left(\left[\left[\hat{A}_\alpha,\hat{H}_A\right],\hat{\rho}_A(0)\right]\langle\hat{B}_\alpha\rangle_0 + \left[\hat{A}_\alpha,\hat{\rho}_A(0)\right]\langle\left[\hat{B}_\alpha,\hat{H}_B\right]\rangle\right),$$

$$T_3 = \frac{-t^2}{2}\sum_\alpha\sum_\beta \lambda_\alpha\lambda_\beta\hat{A}_\alpha\hat{A}_\beta\hat{\rho}_A(0)\langle\hat{B}_\alpha\hat{B}_\beta\rangle_0 - \frac{-t^2}{2}\sum_\alpha\sum_\beta\lambda_\alpha\lambda_\beta\hat{A}_\beta\hat{\rho}_A(0)\hat{A}_\alpha\langle\hat{B}_\alpha\hat{B}_\beta\rangle_0$$
$$-\frac{-t^2}{2}\sum_\alpha\sum_\beta\lambda_\alpha\lambda_\beta\hat{A}_\alpha\hat{\rho}_A(0)\hat{A}_\beta\langle\hat{B}_\beta\hat{B}_\alpha\rangle_0 + \frac{-t^2}{2}\sum_\alpha\sum_\beta\lambda_\alpha\lambda_\beta\hat{\rho}_A(0)\hat{A}_\beta\hat{A}_\alpha\langle\hat{B}_\beta\hat{B}_\alpha\rangle_0.$$

Now, we want the linear entropy:

$$S_{lin}(\hat{\rho}) = (1 - \mathrm{Tr}\hat{\rho}^2). \quad \text{(D.8)}$$

If $\hat{\rho}_A(0)$ is pure, then $\mathrm{Tr}(\hat{\sigma}_A(t)) = 1$. Further, $\mathrm{Tr}(\hat{\sigma}_A(t)T_1) = \mathrm{Tr}(\hat{\sigma}_A(t)T_2) = 0$ to the order of $\mathcal{O}(t^3)$ using the cyclic property of traces. This means that we can write S_{lin}:

$$S_{lin}(\hat{\rho}_A(t)) = -\mathrm{Tr}(T_1^2) - \mathrm{Tr}(\hat{\sigma}_A(t)T_3) + \mathcal{O}(t^3). \quad \text{(D.9)}$$

Looking at T_1 and using $\mathrm{Tr}\left(\hat{A}_\alpha\hat{\rho}_A(0)\hat{A}_\beta\hat{\rho}_A(0)\right) = \langle\hat{A}_\alpha\rangle_0\langle\hat{A}_\beta\rangle_0$ along with the cyclical property of traces:

$$\mathrm{Tr}(T_1^2) = (-it)^2\sum_\alpha\sum_\beta\lambda_\alpha\lambda_\beta\langle\hat{B}_\alpha\rangle_0\langle\hat{B}_\beta\rangle_0\mathrm{Tr}\left(\left[\hat{A}_\alpha,\hat{\rho}_A(0)\right]\left[\hat{A}_\beta,\hat{\rho}_A(0)\right]\right) \quad \text{(D.10)}$$
$$= -t^2\sum_\alpha\sum_\beta\lambda_\alpha\lambda_\beta\langle\hat{B}_\alpha\rangle_0\langle\hat{B}_\beta\rangle_0\left(2\langle\hat{A}_\alpha\rangle_0\langle\hat{A}_\beta\rangle_0 - \langle\hat{A}_\alpha\hat{A}_\beta + \hat{A}_\beta\hat{A}_\alpha\rangle_0\right).$$

Further, with the same procedure as before, the $\mathrm{Tr}(\hat{\sigma}_A(0)T_3)$ term can be written as:

$$\mathrm{Tr}(\hat{\sigma}_A(t)T_3) = -\frac{t^2}{2}\sum_\alpha\sum_\beta\langle\hat{B}_\alpha\hat{B}_\beta\rangle_0\langle\hat{A}_\alpha\hat{A}_\beta\rangle_0 + \frac{t^2}{2}\sum_\alpha\sum_\beta\langle\hat{B}_\alpha\hat{B}_\beta\rangle_0\langle\hat{A}_\alpha\rangle_0\langle\hat{A}_\beta\rangle_0$$
$$+\frac{t^2}{2}\sum_\alpha\sum_\beta\langle\hat{B}_\beta\hat{B}_\alpha\rangle_0\langle\hat{A}_\alpha\rangle_0\langle\hat{A}_\beta\rangle_0 - \frac{t^2}{2}\sum_\alpha\sum_\beta\langle\hat{B}_\beta\hat{B}_\alpha\rangle_0\langle\hat{A}_\beta\hat{A}_\alpha\rangle_0.$$
(D.11)

This allows us to see:

$$S_{lin}\left(\hat{\rho}(t)\right) = -\, t^2 \sum_{\alpha}\sum_{\beta} \lambda_\alpha \lambda_\beta \langle \hat{B}_\alpha \rangle_0 \langle \hat{B}_\beta \rangle_0 \Big(2\langle \hat{A}_\alpha \rangle_0 \langle \hat{A}_\beta \rangle_0 - \langle \hat{A}_\alpha \hat{A}_\beta + \hat{A}_\beta \hat{A}_\alpha \rangle_0 \Big) \tag{D.12}$$

$$- \frac{t^2}{2} \sum_{\alpha}\sum_{\beta} \lambda_\alpha \lambda_\beta \langle \hat{B}_\alpha \hat{B}_\beta \rangle_0 \langle \hat{A}_\alpha \hat{A}_\beta \rangle_0 + \frac{t^2}{2} \sum_{\alpha}\sum_{\beta} \lambda_\alpha \lambda_\beta \langle \hat{B}_\alpha \hat{B}_\beta \rangle_0 \langle \hat{A}_\alpha \rangle_0 \langle \hat{A}_\beta \rangle_0$$
$$+ \frac{t^2}{2} \sum_{\alpha}\sum_{\beta} \lambda_\alpha \lambda_\beta \langle \hat{B}_\beta \hat{B}_\alpha \rangle_0 \langle \hat{A}_\alpha \rangle_0 \langle \hat{A}_\beta \rangle_0 - \frac{t^2}{2} \sum_{\alpha}\sum_{\beta} \lambda_\alpha \lambda_\beta \langle \hat{B}_\beta \hat{B}_\alpha \rangle_0 \langle \hat{A}_\beta \hat{A}_\alpha \rangle_0 + \mathcal{O}(t^3).$$

Appendix E

Pointer Entropy

In section 2.2.3 and section 4.4, we defined and used the second order derivative of pointer entropy. In this appendix, we will take the definition of $\ddot{S}_{pointer}$ and calculate $\ddot{S}_{pointer}(0)$.

The second order derivative of pointer entropy at t=0 is given by:

$$\ddot{S}_{pointer}(t=0) = -2\sum_{j=1}^{d_a}\left(\dot{p}_j^2(0) + p_j(0)\ddot{p}_j(0)\right). \tag{E.1}$$

Now, $p_j(t)$ is defined as:

$$p_j(t) = \mathrm{Tr}_A(\hat{\rho}_A(t)\hat{\Lambda}_j). \tag{E.2}$$

Defining $\hat{\rho}_A(t)$ as a zeroth order term plus a first order term and higher order terms:

$$\hat{\rho}_A(t) = \hat{\rho}_A(0) - it[\hat{H}_A^{eff}, \hat{\rho}_A(0)] + \mathcal{O}(t^2), \tag{E.3}$$

where

$$\hat{H}_A^{eff} = \hat{H}_A + \sum_{\alpha}^{P}\lambda_\alpha\langle\hat{B}_\alpha\rangle_0\hat{A}_\alpha. \tag{E.4}$$

Combining equations E.2, E.3, and E.4 and taking the trace, we obtain:

$$p_j(t) = p_j(0) - it\langle[\hat{\Lambda}_j, \hat{H}_A]\rangle - it\langle[\hat{\Lambda}_j, \sum_{\alpha}^{P}\lambda_\alpha\langle\hat{B}_\alpha\rangle_0\hat{A}_\alpha]\rangle + \mathcal{O}(t^2). \tag{E.5}$$

E.1 Obtaining the Derivative of $p_j(t)$

The derivative of $p_j(t)$ is given by:

$$\dot{p}_j(t) = -i\mathrm{Tr}\left(\hat{\Lambda}_j \frac{d}{dt}\hat{\rho}_A(t)\right). \tag{E.6}$$

Using the quantum Liouville equation:

$$\frac{d\hat{\rho}_A(t)}{dt} = \left(-i[\hat{H}, \hat{\rho}_A(t)]\right) + \mathcal{O}(t^2), \tag{E.7}$$

for this case, $\hat{H}$ will be $\hat{H}_A^{eff}$ as defined in section 2.1.4. Combining E.6 and E.7 produces:

$$\frac{dp_j(t)}{dt} = -i\mathrm{Tr}\left([\hat{H}_A^{eff}, \hat{\rho}_A(t)]\hat{\Lambda}_j\right) + \mathcal{O}(t^2). \tag{E.8}$$

We can expand out H_A^{eff} and $\hat{\rho}(t)$ to obtain:

$$\begin{aligned}
\dot{p}_j(t) = &- i\mathrm{Tr}\left([\hat{H}_A, \hat{\rho}_A(0)]\hat{\Lambda}_j\right) + t\mathrm{Tr}\left(\left[\hat{H}_A, [\hat{H}_A, \hat{\rho}_A(0)]\right]\hat{\Lambda}_j\right) \\
&+ t\mathrm{Tr}\left(\left[\hat{H}_A, \sum_{\alpha}^{P}\lambda_\alpha\langle\hat{B}_\alpha\rangle_0[\hat{A}_\alpha, \hat{\rho}_A(0)]\right]\right) - i\mathrm{Tr}\left(\sum_{\alpha}^{P}\lambda_\alpha\langle\hat{B}_\alpha\rangle_0[\hat{A}_\alpha, \hat{\rho}_A(0)]\hat{\Lambda}_j\right) \\
&+ t\mathrm{Tr}\left(\sum_{\alpha}^{P}\lambda_\alpha\langle\hat{B}_\alpha\rangle_0\left[\hat{A}_\alpha, [\hat{H}_A, \hat{\rho}_A(0)]\right]\hat{\Lambda}_j\right) \\
&+ t\mathrm{Tr}\left(\sum_{\alpha}^{P}\lambda_\alpha\langle\hat{B}_\alpha\rangle_0\left[\hat{A}_\alpha, \sum_{\beta}^{P}\lambda_\beta\langle\hat{B}_\beta\rangle_0[\hat{A}_\beta, \hat{\rho}_A(o)]\right]\hat{\Lambda}_j\right) + \mathcal{O}(t^2).
\end{aligned} \tag{E.9}$$

Then, expanding out the brackets and taking the trace, the following can be computed:

$$\begin{aligned}
\dot{p}_j(t) = &- i\langle[\hat{\Lambda}_j, \hat{H}_A]\rangle_0 - t\langle\hat{\Lambda}_j\hat{H}_A^2 + \hat{H}_A^2\hat{\Lambda}_j - 2\hat{H}_A\hat{\Lambda}_j\hat{H}_A\rangle_0 - t\sum_{\alpha}^{P}\lambda_\alpha\langle\hat{B}_\alpha\rangle_0\langle\left[\hat{\Lambda}_j, [\hat{H}_A, \hat{A}_\alpha]\right]\rangle_0 \\
&- i\sum_{\alpha}^{P}\lambda_\alpha\langle\hat{B}_\alpha\rangle_0\langle[\hat{\Lambda}_j, \hat{A}_\alpha]\rangle_0 - t\sum_{\alpha}^{P}\lambda_\alpha\langle\hat{B}_\alpha\rangle_0\langle\left[\hat{\Lambda}_j, [\hat{A}_\alpha, \hat{H}_A]\right]\rangle_0 \\
&- t\sum_{\alpha}^{P}\lambda_\alpha\langle\hat{B}_\alpha\rangle_0\sum_{\beta}^{P}\lambda_\beta\langle\hat{B}_\beta\rangle_0\langle\left[\hat{\Lambda}_j, [\hat{A}_\alpha, \hat{A}_\beta]\right]\rangle_0 + \mathcal{O}(t^2).
\end{aligned} \tag{E.10}$$

When t=0, the expression becomes:

$$\dot{p}_j(0) = -i\langle[\hat{\Lambda}_j, \hat{H}_A]\rangle_0 - i\sum_{\alpha}^{P} \lambda_\alpha \langle \hat{B}_\alpha \rangle_0 \langle [\hat{\Lambda}_j, \hat{A}_\alpha]\rangle_0. \tag{E.11}$$

E.2 The Second Order Derivatives of $p_j(0)$

The second order derivative will be:

$$\ddot{p}_j(t) = \frac{d}{dt}(\dot{p}_j(t)) = -i[\hat{H}, \dot{p}_j(t)] = -i\left[\hat{H}, -i\mathrm{Tr}\left([\hat{H}_A^{eff}, \hat{\rho}_A(t)]\hat{\Lambda}_j\right)\right]. \tag{E.12}$$

Taking the trace after the commutation, we obtain:

$$\ddot{p}_j(t) = -\mathrm{Tr}\left(\left[\hat{H}_{eff}, [\hat{H}_{eff}, \hat{\rho}_A(t)]\right]\hat{\Lambda}_j\right). \tag{E.13}$$

Using equations E.3 and E.4, we obtain the expression[1]:

$$\begin{aligned}
\ddot{p}_j(t) = &- \mathrm{Tr}\left(\left[\hat{H}_A, [\hat{H}_A, \hat{\rho}_A(0)]\right]\hat{\Lambda}_j\right) - \mathrm{Tr}\left(\left[\hat{H}_A, [\sum_{\alpha}^{P} \lambda_\alpha \langle \hat{B}_\alpha\rangle \hat{A}_\alpha, \hat{\rho}_A(0)]\right]\hat{\Lambda}_j\right) \qquad \text{(E.14)}\\
&- \mathrm{Tr}\left(\left[\sum_{\alpha}^{P} \lambda_\alpha \langle \hat{B}_\alpha\rangle \hat{A}_\alpha, [\hat{H}_A, \hat{\rho}_A(0)]\right]\hat{\Lambda}_j\right)\\
&- \mathrm{Tr}\left(\left[\sum_{\alpha}^{P} \lambda_\alpha \langle \hat{B}_\alpha\rangle \hat{A}_\alpha, [\sum_{\alpha}^{P} \lambda_\alpha \langle \hat{B}_\alpha\rangle \hat{A}_\alpha, \hat{\rho}_A(0)]\right]\hat{\Lambda}_j\right)\\
&- \mathrm{Tr}\left(\left[\hat{H}_A, \left[\hat{H}_A, -it[\hat{H}_A^{eff}, \hat{\rho}_A(0)]\right]\right]\hat{\Lambda}_j\right)\\
&- \mathrm{Tr}\left(\left[H_A, \left[\sum_{\alpha}^{P} \lambda_\alpha \langle \hat{B}_\alpha\rangle \hat{A}_\alpha, -it[\hat{H}_A^{eff}, \hat{\rho}_A(0)]\right]\right]\hat{\Lambda}_j\right)\\
&- \mathrm{Tr}\left(\left[\sum_{\alpha}^{P} \lambda_\alpha \langle \hat{B}_\alpha\rangle \hat{A}_\alpha, \left[\hat{H}_A, -it[\hat{H}_A^{eff}, \hat{\rho}_A(0)]\right]\right]\hat{\Lambda}_j\right)\\
&- \mathrm{Tr}\left(\left[\sum_{\alpha}^{P} \lambda_\alpha \langle \hat{B}_\alpha\rangle \hat{A}_\alpha, \left[\sum_{\alpha}^{P} \lambda_\alpha \langle \hat{B}_\alpha\rangle \hat{A}_\alpha, -it[\hat{H}_A^{eff}, \hat{\rho}_A(0)]\right]\right]\hat{\Lambda}_j\right) + \mathcal{O}(t^2).
\end{aligned}$$

[1] Alternatively, we can take the time derivative of equation E.10 and jump to equation E.15.

This is more bulky than necessary. Using t=0 (as it will be for $\ddot{p}_j(0)$) will get rid of $-it[\hat{H}_A^{eff}, \hat{\rho}_A(0)]$ and any higher order terms. This leads to a more manageable expression:

$$\begin{aligned}\ddot{p}_j(0) = &- \mathrm{Tr}\left(\left[\hat{H}_A, [\hat{H}_A, \hat{\rho}_A(0)]\right]\hat{\Lambda}_j\right) - \mathrm{Tr}\left(\left[\hat{H}_A, [\sum_\alpha^P \lambda_\alpha \langle \hat{B}_\alpha \rangle \hat{A}_\alpha, \hat{\rho}_A(0)]\right]\hat{\Lambda}_j\right) \\ &- \mathrm{Tr}\left(\left[\sum_\alpha^P \lambda_\alpha \langle \hat{B}_\alpha \rangle \hat{A}_\alpha, [\hat{H}_A, \hat{\rho}_A(0)]\right]\hat{\Lambda}_j\right) \\ &- \mathrm{Tr}\left(\left[\sum_\alpha^P \lambda_\alpha \langle \hat{B}_\alpha \rangle \hat{A}_\alpha, [\sum_\alpha^P \lambda_\alpha \langle \hat{B}_\alpha \rangle \hat{A}_\alpha, \hat{\rho}_A(0)]\right]\hat{\Lambda}_j\right).\end{aligned} \tag{E.15}$$

Working through this expression and taking the trace will give:

$$\begin{aligned}\ddot{p}_j(0) = &- \langle \hat{\Lambda}_j \hat{H}_A^2 + \hat{H}_A^2 \hat{\Lambda}_j - 2\hat{H}_A \hat{\Lambda}_j \hat{H}_A \rangle_0 - \sum_\alpha^P \lambda_\alpha \langle \hat{B}_\alpha \rangle_0 \langle \left[\hat{\Lambda}_j, [\hat{H}_A, \hat{A}_\alpha]\right] \rangle_0 \\ &- \sum_\alpha^P \lambda_\alpha \langle \hat{B}_\alpha \rangle_0 \langle \left[\hat{\Lambda}_j, [\hat{A}_\alpha, \hat{H}_A]\right] \rangle_0 \\ &- \sum_\alpha^P \lambda_\alpha \langle \hat{B}_\alpha \rangle_0 \sum_\beta^P \lambda_\beta \langle \hat{B}_\beta \rangle_0 \langle \left[\hat{\Lambda}_j, [\hat{A}_\alpha, \hat{A}_\beta]\right] \rangle_0.\end{aligned} \tag{E.16}$$

E.2.1 The Pointer Entropy's Second Order Time Derivative

In this section, we will use the previous sections in this chapter to derive $\ddot{S}_{pointer}$ as it was used in Chapter 4 after being defined in section 2.2.3.

If equations E.5, E.11, and E.16 are all inserted into equation E.1, the following is derived:

$$
\begin{aligned}
\ddot{S}_{pointer}(0) =& 2\sum_{j=1}^{d_a}\langle[\hat{\Lambda}_j,\hat{H}_A]\rangle_0^2 + 2\sum_{j=1}^{d_a}\left(\sum_{\alpha}^{P}\lambda_\alpha\langle\hat{B}_\alpha\rangle_0\langle[\hat{\Lambda}_j,\hat{A}_\alpha]\rangle_0\right)^2 \\
&+ 4\sum_{j=1}^{d_a}\sum_{\alpha}^{P}\lambda_\alpha\langle\hat{B}_\alpha\rangle_0\langle[\hat{\Lambda}_j,\hat{A}_\alpha]\rangle_0\langle[\hat{\Lambda}_j,\hat{H}_A]\rangle_0 \\
&+ 2\sum_{j=1}^{d_a}\left(p_j(0)\langle\hat{\Lambda}_j\hat{H}_A^2+\hat{H}_A^2\hat{\Lambda}_j-2\hat{H}_A\hat{\Lambda}_j\hat{H}_A\rangle_0\right) \\
&+ 2\sum_{j=1}^{d_a}\left(p_j(0)\sum_{\alpha}^{P}\lambda_\alpha\langle\hat{B}_\alpha\rangle_0\langle\Big[\hat{\Lambda}_j,[\hat{H}_A,\hat{A}_\alpha]\Big]\rangle_0\right) \\
&+ 2\sum_{j=1}^{d_a}\left(p_j(0)\sum_{\alpha}^{P}\lambda_\alpha\langle\hat{B}_\alpha\rangle_0\langle\Big[\hat{\Lambda}_j,[\hat{A}_\alpha,\hat{H}_A]\Big]\rangle_0\right) \\
&+ 2\sum_{j=1}^{d_a}\left(p_j(0)\sum_{\alpha}^{P}\lambda_\alpha\langle\hat{B}_\alpha\rangle_0\sum_{\beta}^{P}\lambda_\beta\langle\hat{B}_\beta\rangle_0\langle\Big[\hat{\Lambda}_j,[\hat{A}_\alpha,\hat{A}_\beta]\Big]\rangle_0\right).
\end{aligned}
\tag{E.17}
$$

While this is quite a long expression, if analyzed in the quantum measurement limit ($[\hat{\Lambda}_j,\hat{A}_\alpha]=0$), the equation would read:

$$
\begin{aligned}
\ddot{S}^{QML}_{pointer}(0) =& 2\sum_{j=1}^{d_a}\langle[\hat{\Lambda}_j,\hat{H}_A]\rangle_0^2 + 2\sum_{j=1}^{d_a}\left(p_j(0)\langle\hat{\Lambda}_j\hat{H}_A^2+\hat{H}_A^2\hat{\Lambda}_j-2\hat{H}_A\hat{\Lambda}_j\hat{H}_A\rangle_0\right) \\
&+ 2\sum_{j=1}^{d_a}\left(p_j(0)\sum_{\alpha}^{P}\lambda_\alpha\langle\hat{B}_\alpha\rangle_0\langle\Big[\hat{\Lambda}_j,[\hat{H}_A,\hat{A}_\alpha]\Big]\rangle_0\right).
\end{aligned}
\tag{E.18}
$$

If, however, it was in the decoherence limit ($[\hat{\Lambda}_j,\hat{H}_A]=0$), it would rather read:

$$
\begin{aligned}
\ddot{S}^{DCL}_{pointer}(0) =& 2\sum_{j=1}^{d_a}\left(\sum_{\alpha}^{P}\lambda_\alpha\langle\hat{B}_\alpha\rangle_0\langle[\hat{\Lambda}_j,\hat{A}_\alpha]\rangle_0\right)^2 + 2\sum_{j=1}^{d_a}\left(p_j(0)\sum_{\alpha}^{P}\lambda_\alpha\langle\hat{B}_\alpha\rangle_0\langle\Big[\hat{\Lambda}_j,[\hat{A}_\alpha,\hat{H}_A]\Big]\rangle_0\right) \\
&+ 2\sum_{j=1}^{d_a}\left(p_j(0)\sum_{\alpha}^{P}\lambda_\alpha\langle\hat{B}_\alpha\rangle_0\sum_{\beta}^{P}\lambda_\beta\langle\hat{B}_\beta\rangle_0\langle\Big[\hat{\Lambda}_j,[\hat{A}_\alpha,\hat{A}_\beta]\Big]\rangle_0\right).
\end{aligned}
\tag{E.19}
$$

Thus, equation E.17 can (using equations E.18 and E.19) be written as:

$$\ddot{S}_{pointer}(0) = \ddot{S}^{QML}_{pointer} + \ddot{S}^{DCL}_{pointer} + 4\sum_{j=1}^{d_a}\sum_{\alpha}^{P}\lambda_\alpha\langle\hat{B}_\alpha\rangle_0\langle[\hat{\Lambda}_j,\hat{A}_\alpha]\rangle_0\langle[\hat{\Lambda}_j,\hat{H}_A]\rangle_0. \quad \text{(E.20)}$$

Appendix F

Collimation

In section 4.3, we defined the concept of collimation. In this appendix, we will give a mathematical formulation for it using canonical commutation relations.

Starting with equation 2.15, we can define two unitary operators [73]. Let them be called $\hat{X}$ and $\hat{Y}$ such that:

$$\hat{X} = e^{i\alpha\hat{\Pi}} \text{ and } \hat{Y} = e^{i\beta\hat{\phi}} \tag{F.1}$$

where α and β are non-zero parameters that set the scale for the set of eigenvalues (the eigenspectrum) of $\hat{\phi}$ and $\hat{\Pi}$. We shall notate $|\phi_j\rangle$ as eigenstates of $\hat{\phi}$ and $|\Pi_k\rangle$ as eigenstates of $\hat{\Pi}$. With this, we can start defining how the operators translate through the different eigenstates: $\hat{X}|\phi_j\rangle = |\phi_{j+1}\rangle$ and $\hat{Y}|\Pi_k\rangle = |\Pi_{k+1}\rangle$. Further, for some l, we have the periodicity condition: $|\phi_{k+l}\rangle = |\phi_k\rangle$ and $|\Phi_{j+l}\rangle = |\Phi_j\rangle$.

Using the method outlined in [61], we can define a unitary operator $\hat{M}$ that has the following expansion:

$$\hat{M} = \sum_{p,q=-l}^{l} m_{pq}\hat{Y}^p\hat{X}^q. \tag{F.2}$$

In equation F.2, $\hat{Y}^p\hat{X}^q$ generate evolutions of eigenstates caused by operations by the operator $\hat{M}$, as well as p and q being the powers for the $\hat{Y}$ and $\hat{X}$ operators respectively. It must be noted, p and q are indices of m. Further, if we have the relationship[1]:

$$\mathrm{Tr}\left[\left(\hat{Y}^{p'}\hat{X}^{q'}\right)^{\dagger}\left(\hat{Y}^{p}\hat{X}^{q}\right)\right] = d\delta_{pp'}\delta_{qq'}, \tag{F.3}$$

then we are able to derive m_{pq} as:

$$m_{pq} = \frac{1}{d}\mathrm{Tr}\left[\hat{X}^{-q}\hat{Y}^{-p}\hat{M}\right], \tag{F.4}$$

with $d = 2l + 1$. Now we are interested in the spreading of $\hat{M}$ in the basis (ϕ) and its conjugate (Π). We hence shall define relative contributions of $\hat{X}$ and $\hat{Y}$ as:

$$m_q^{(\phi)} = \frac{\sum_{p=-l}^{l} |m_{pq}|}{\sum_{p',q'=-l}^{l} |m_{p'q'}|}, \tag{F.5}$$

$$m_p^{(\Pi)} = \frac{\sum_{q=-l}^{l} |m_{pq}|}{\sum_{p',q'=-l}^{l} |m_{p'q'}|}. \tag{F.6}$$

Since collimation is a measure of spreading, it shall be a synthesis of the relative contributions immediately above and contribution of the powers the operators $\hat{X}$ and $\hat{Y}$ introduce to $\hat{M}$ as seen in equation F.2. The collimation can hence be defined as:

$$C_\phi(\hat{M}) = \sum_{q=-l}^{l} m_q^{(\phi)} exp\left(-\frac{|q|}{2l+1}\right), \tag{F.7}$$

and

$$C_\Pi(\hat{M}) = \sum_{p=-l}^{l} m_p^{(\Pi)} exp\left(-\frac{|p|}{2l+1}\right). \tag{F.8}$$

Minimal spreading will occur with small values of p and q which means that $C(\hat{M}) \to 1$, since exponential terms will then tend to 1. Maximal spreading will occur for large values of q and p which means $C(\hat{M}) \to 0$, since the exponential will tend to zero.

[1]The full story is that, if we start with the Weyl Braiding [73]: $\hat{X}\hat{Y} = \omega\hat{Y}\hat{X}$, ($\omega$ is a phase) we then would be able to say F.3.

Appendix G

The Schools of Thought with Regards to Fundamentalism

In this appendix, we will define the four schools of thought regarding what is fundamental as the properties that come up in the definitions are used in sections 6.2 and 6.3.

Before we can discuss the four schools of philosophical thought regarding what is fundamental, we should first define what is meant by dependence. Dependence means that something has a reliance on something else. Fundamental items must be irreducible. This means we cannot split them nor have them depend on any other items.

G.1 Absolute Independence

Item x can only be fundamental if at least one other item depends on it alone, but x depends on no other items (it is irreducible) [127, 128, 129].

G.2 Restricted Independence

The set X is fundamental if and only if at least one other set depends on X alone, but X depends on no other items or set of items [127, 130, 131].

G.3 Complete Minimal Basis

The set of items X is fundamental if it is a complete basis that all other items build from [127, 152, 153].

G.4 Primitivism

This school holds that we cannot define what is fundamental; we can only characterize it [127, 154, 155]. These characterizations will include the properties mentioned previously. However, these characterizations will not be able to be used to define fundamentalism. We will not hold the position that fundamentalism cannot be defined in physics apriori.

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