An analysis of how visualisation capabilities in dynamic geometric software develop meaning-making of mathematical concepts in selected Grade 11 learners.

A thesis submitted in fulfilment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

of

RHODES UNIVERSITY, SOUTH AFRICA

(Faculty of Education)

by

DEEPAK PRAVIN MAVANI

MAY 2019

ABSTRACT

Visualisation plays a central role in developing mathematical ideas because it can be used to make these ideas explicit and visible, and thus has the potential to advance understanding. This study centred around the *GeoGebra* Literacy Initiative Project (GLIP), a teacher development initiative in Mthatha that aimed to grow and develop appropriate Information and Communication Technology (ICT) skills in teachers, to harness the teaching and learning potential of *GeoGebra*. *GeoGebra* is a dynamic geometric software package that is very interactive and makes use of powerful features to create images which can be moved around the computer screen for mathematical exploration. This research project was located within GLIP and analysed how *GeoGebra* applets develop conceptual and procedural understanding in selected Grade 11 learners. One aspect of GLIP was for teachers to use *GeoGebra* applets that they had developed themselves and implemented in their classrooms in pre-determined cycles that were aligned to the curriculum. My study specifically focused on how the selected learners made use of these applets and explored how learning had taken place in terms of developing mathematical meaning-making. This interpretive research study was designed as a case study. The case was a cohort of selected Grade 11 learners who had been taught by GLIP teachers, and my unit of analysis was the learners' interaction with the applets. A screen capturing software package was used to capture learners' interactions with the *GeoGebra.* My data consisted mainly of recorded observations and interviews. An analytical framework derived from the works of Kilpatrick, Swafford, and Findell (2001) and Carter et al. (2009) guided and informed the data analysis of the learners' activities with the *GeoGebra*. The theoretical orientation of this study was constructivist learning.

An in-depth analysis and detailed descriptions of the participants' interactions enabled me to gain a comprehensive understanding of their meaning-making processes in a technological classroom context. An analysis across the participants identified distinguishable patterns or differences in the development of the learners' mathematical proficiency and making sense of mathematical ideas. The research argued that technology enabled visualisation was a powerful tool to not only enrich mathematically activities, but to also enrich conceptual and procedural understanding. The findings recognised that exploration of, or manipulation on mathematical objects in *GeoGebra* was a key activity in the participants' meaning-making process. It also enabled learners to offer self-proclaimed theories.

ACKNOWLEDGEMENTS

ajñāna-timirāndhasya jñānāñjana-śalākayā

cakṣur unmīlitaṁ yena tasmai śrī-gurave namaḥ

(Sanskrit verse from an ancient sacred text)

Meaning : I offer my respectful salutations unto my teachers, who with the torchlight of knowledge have opened my eyes, which were blinded by the darkness of ignorance.

I express my sincerest gratitude to the following people who contributed towards the completion of this thesis:

I begin with my deepest gratitude to Prof. Marc Schäfer, 'my guide and philosopher' and my supervisor for this research project for his timely support and guidance. Prof. Marc was always keen to listen to my ideas and critically respond to them. He engaged with my work and me (a novice researcher) with patience.

I also extend my gratitude to Prof. Fred Lubben, chief editor of the African Journal of Research in Mathematics, Science and Technology Education, who gave me valuable advice on academic writing skills.

I express my sincere gratitude to research colleagues and critical reviewers for their invaluable comments and suggestions.

I thank the research participants, both teachers and learners, for their generosity in giving their time and sharing their ideas. I also thank the principals for giving me access to their schools and resources.

Last but not the least, I recognise the unwavering support from my family and friends.

DECLARATION OF ORIGINALITY

I, Deepak Pravin Mavani (Student number 615P0005), declare that this doctoral thesis entitled: "A critical analysis of how visualisation capabilities in dynamic geometric software develop meaning-making in terms of conceptual and procedural understanding of mathematical concepts in selected Grade 11 learners", is my own work written in my own words. It has not been submitted in any form for another qualification or any assessment to another university or institution. Where I have drawn on the words or ideas of others, these have been fully acknowledged and referenced in the manner required by the Rhodes University Department of Education Guide to Referencing.

Deepak Pravin Mavani

Date: June 10, 2019

INDEX OF CONTENTS

INDEX OF FIGURES

INDEX OF SCREENSHOTS

INDEX OF TABLES

INDEX OF FRAMES

LIST OF ABBREVIATIONS

CAS	Computer Algebra System
CU	Conceptual Understanding
DBE	Department of Basic Education, South Africa
DGS	Dynamic Geometric Software
GLIP	GeoGebra Literacy Initiative Programme
ICT	Information and Communications Technology
MM	Meaning Making
MP	Mathematical Proficiency
NCTM	National Council of Teachers of Mathematics, USA
PF	Procedural Fluency
VL	Visual Literacy

CHAPTER 1

INTRODUCTION

The purpose of this first chapter is to introduce the reader to the background, context and goals of the research, which focuses on how visualisation capabilities inherent in computer technology can develop meaning-making in terms of conceptual and procedural understanding of mathematical concepts. I briefly introduce the theoretical underpinnings, methodology and significance of the research. The chapter concludes with an outline of each chapter of the research thesis.

1.1 INTRODUCTION TO THE RESEARCH

Mathematics is one of the key learning areas in schools across the globe. It helps to develop mental processes that enhance logical and critical thinking and problem solving skills that can contribute to decision-making. Mathematics enables us to understand the world around us and teaches us to think creatively (Dept of Basic Education, 2011).

1.1.1 Visualisation

Mathematical ability requires generalised and abstract thought (Krutetskii, 1976). Krutetskii suggests that it is productive and strategic to capitalise on the visual aspect of an abstract mathematical idea, as it makes the mathematical idea accessible and real. Acknowledging the centrality of visualisation in mathematics classrooms, Arcavi (2003) affirms that "[v]isualization is no longer related to the illustrative purposes only, but is also being recognised as a key component of reasoning."

The scholarly works of Arcavi (2003), Presmeg (1986b, 2014), Duval (1999, 2013) and other researchers have provided compelling empirical evidence of the important role of visualisation in developing understanding of mathematical concepts and knowledge. Their research found strong evidence that learners do employ visual strategies in conjunction with computer technology to construct meaningful conceptual ideas.

Presmeg (1986b), commending the use of visual imagery, claims that the embodiment of abstract ideas in a concrete image can be effective in learning mathematics, i.e. the use of visualisation can facilitate the learning of mathematical content with understanding. Arcavi (2003) has foregrounded the importance of visualisation and suggested that in mathematics, visualisation "as a way to re-engage and recover conceptual underpinnings which may easily bypassed by formal solution" (p. 224). This is significant in mathematics as many mathematical concepts are abstract, and in my experience are rather 'invisible' to young learners. Duval (1999) asserts that visualisation in mathematics is indeed needed because it can display the organisation of relations and enables learners to form images of complex mathematical ideas. In my experience in the classroom, for those who prefer to think visually, the most complicated of problems are readily solvable when mediated with visual aids such as appropriate software packages.

1.2 THE CONTEXT OF THE STUDY

ICT (Information Communication Technology) has spread its wings across the globe and is accessible to many individuals, including teachers and learners. For many, it has become an essential component in their daily lives. The entry of ICT into some classrooms has thus become inevitable. The partnership between ICT and mathematics has only recently opened up exciting teaching, learning and research opportunities.

1.2.1 Mathematics and technology

Given the widespread use of computer technology, many mathematics educators and research institutions highlight that new technology offers exciting opportunities in mathematics education when used appropriately. Kaput (1992) asserts that "[t]echnologies based on dynamic interactive electronic media embody fundamental attributes that distinguish them from traditional static media in ways likely to have tremendous long-term impact on mathematics education." (p. 525). Digital technologies afford new ways of learning mathematics. The National Council of Teachers of Mathematics (NCTM) (2000) also acknowledges the positive impact of ICT in the teaching and learning of mathematics. Hence, the NCTM identified the "technology principle", as one of the six principles of high-quality teaching of mathematics in schools. The technology principle states that "[t]echnology is essential in teaching and learning mathematics; it influences the mathematics that is taught and enhances students' learning" (p. 24). The use of mathematical tools and technology helps students learn and make sense of mathematical ideas, and when these resources are used appropriately, they support effective teaching and meaningful learning. I wish to add the caveat that for ICT to have the impact indicated above, it needs to be employed appropriately and strategically.

Arcavi (2003) emphasises the potential role of the dynamic capabilities of technology in developing visual means to better understand mathematical concepts. Furthermore, research by

Mudaly (2010, p. 74) on using Dynamic Geometric Software (DGS) shows that "[u]nderstanding generally was attained *after* 'seeing' the solution and the visual impact of the dynamic diagrams allowed the learners to quickly draw adequate conclusions." Thus, technology as a visual aid creates an entry point for learning a mathematical concept. When combined with visualisation processes, ICT in my experience can bring about an exciting paradigm shift in mathematics education which allows for powerful multiple representations of mathematical concepts as visual objects.

1.2.2 **Dynamic Geometric Software – *GeoGebra***

There are many educational software resources available on the market, but many of them are regrettably expensive and require annual licence fees. The form of educational technology used as a focus in this study is DGS. Ruthven, Hennessy, & Deaney (2008) reflect that DGS was developed with educational purposes in mind. The interface of DGS provides an opportunity for learners to conjecture and generalise by clicking and dragging objects, while the software dynamically re-draws and updates information on the screen. My research project is firmly rooted in the free open-source software community and actively supports the use of software like *GeoGebra* that offers an "excellent classroom-teaching tool" (Dept of Basic Education, 2013) for mathematics. *GeoGebra* is one of the DGS technology resources that is available to anyone for the teaching and learning of mathematics. "*GeoGebra* is a multiplatform mathematics software package that gives everyone the chance to experience the extraordinary insights that maths makes possible" (*GeoGebra*, n.d.).

The possibilities afforded by *GeoGebra* as a tool for learning mathematics are enormous. Many advocates of *GeoGebra* such as Jones et al., 2009 and Sherman, 2010, have argued that it is the ability to move between multiple representations of mathematical objects that renders power to the tool. *GeoGebra* is a useful teaching and learning tool that can strengthen learners' conceptual understanding. It can help learners to learn mathematics with understanding. For example, different perspectives enabled in *GeoGebra* allows access to visual modes that are powerful and enables exploration of many situations or examples that are often not possible with only paper and pencil techniques. It enables learners to make and verify their conjectures very quickly and accurately.

1.2.2.1 ***GeoGebra* Applets**

An integral component of *GeoGebra* is the use of applets. The Merriam-Webster dictionary (Merriam-Webster, n.d.) defines an applet as "a small program designed to be executed from within the application." Applets are usually short, user-friendly applications. Applets can be used to provide interactive features enabling the user to change the dimension and the characteristics of an entire graphic or parts thereof. Thus, it is very suitable for demonstration, visualisation and teaching. *GeoGebra* applets are representations of mathematical ideas which anyone can interact with. Often these are small programs made to serve a specific purpose.

1.2.3 **Identifying the key problem of my research**

The Department of Basic Education (DBE) of South Africa (2013) acknowledges the significance of ICT in education as a teaching and learning resource, but the Ministerial Committee Report of 2013 found that there is "poor evidence of educational technology" in many schools (Dept of Basic Education, 2013, p. 8). Much of the technology equipment such as the computers and tablets provided are underutilised because many teachers are not sufficiently equipped for the effective use of this technology in the classroom (Dept of Basic Education, 2013).

As a teacher, I have noticed that despite the availability of technological resources in schools, these are not employed sufficiently and adequately for effective teaching and learning purposes. Despite technologies becoming increasingly available to teachers, teachers themselves often do not make optimal use of these technologies for educational purposes (Stols et al., 2015). In particular, Stols et al. (2015) found that the use of *GeoGebra* in South African classrooms is very limited. This has inspired me to carefully research these available technological resources, and in the process provide meaningful learning opportunities for learners to grasp the mathematical concepts. Thus, the emphasis and aim of this research project is to explore in depth how the effective use of DGS and its embedded visualisation characteristics can develop the learning of mathematics.

1.2.4 **The empirical field of the study - The GeoGebra Literacy Initiative Programme (GLIP)**

The NCTM (2000) states that good mathematics teaching calls for the effective use of technology in the classroom. It has a positive influence on how mathematics is perceived, and it enhances students' learning. Thus, it is important to critically look at ways of integrating the use of computers within the school practice, and then analyse how learning has taken place.

The GLIP is a teacher development project in Mthatha in the Eastern Cape that aims to equip teachers and students with the necessary skills to use technological tools, particularly

GeoGebra, for teaching and learning mathematics. The project was launched in November 2015 with the initial participation of twelve mathematics teachers from a secondary school in Mthatha, Eastern Cape. My fellow researcher and I coordinated the project with the support of the National Research Foundation (NRF) South African Research Chairs Initiative (SARChI) Chair in Mathematics Education at Rhodes University. It is within GLIP that my research project took place. GLIP was thus my empirical field.

There were two phases of the GLIP intervention programme. The first phase of GLIP comprised of the training of teachers and their learners in the basic use of *GeoGebra*. In the second phase of GLIP, the participating teachers started using *GeoGebra* applets that they had collaboratively developed in the GLIP sessions and implemented in their mathematics classrooms. My specific research focus was on the learners as they engaged with these applets, whereas my fellow researcher focused on the teachers implementing these applets. Both studies foregrounded the harnessing of the visualisation opportunities of these applets to develop meaning-making and enhance conceptual understanding, firstly in the context of teaching (my fellow researcher's study) and, secondly in the context of learning (the focus of my study).

1.2.4.1 **Relevance of GLIP**

Ndlovu, Wessels and Villiers (2013) observe that the integration of dynamic mathematics technology into the classroom in South Africa still remains very tentative. They argue that unfortunately most of the research relevant to computer-based instruction for meaningful learning is undertaken in and relates to developed countries. Although *GeoGebra* is implemented in South Africa in isolated cases, its general implementation remains a challenge because so many teachers themselves have little or no prior experience with computers. Hence, Ndlovu et al. (2013) conclude that there is a desperate "need to develop teacher competencies to speed up the pace of integration of Information and Communication Technology (ICT) into mathematics classrooms" (p. 232). They also suggest that if professional teachers work collaboratively and constantly engage in reflective practice, this could help to overcome the technological barrier. Hence, the GLIP project was specifically designed to promote collaborative engagement and reflection among teachers adopting technological tools for effective use in mathematics classrooms.

The technological innovation aimed to improve teaching techniques entails classroom reforms. For these reforms to be implemented, Ndlovu et al. (2013, p. 241) suggest that "[t]eachers working in professional learning communities can develop common identities and overcome their lack of confidence if they work collaboratively together and constantly engage in reflective practice." GLIP envisages the collaborative engagement among teachers to be a way forward in closing the gap between having access to technology and adapting it for effective use in mathematics classrooms.

1.3 **RESEARCH SETTING**

There is little evidence in South Africa, especially in the Eastern Cape, that *GeoGebra* is being used by mathematics teachers in their routine classes. The GeoGebra Institute in Pretoria (at the University of Pretoria) and Port Elizabeth (at the Nelson Mandela Metropolitan University) are active with targeted workshops being conducted at regular intervals (GeoGebra Institute Support Project (GIS), n.d.). The website www.school-maths.com, maintained by the University of Pretoria contains more than 200 *GeoGebra* applets related to the high school curriculum of South Africa. In spite of the open-source programs available, there is little evidence in the literature that *GeoGebra* is used on a regular basis by South African learners in classrooms. Also, there is little evidence of its efficiency in teaching and learning mathematics.

Of the few research-based studies on using DGS, Mudaly's (2010) research in KwaZulu-Natal involved an intervention programme with Grade 9 learners using *Sketchpad* (another DGS application software) to solve two specific tasks in geometry. Mudaly (2010) indicates that DGS can be used as a visual tool to develop analogical reasoning skills, as in his research, *Sketchpad* provided the visual means for the learners to quickly draw mathematical conclusions. Another recent study by Naidoo and Govender (2014) explored the implications of using *GeoGebra* online when teaching Grade10 trigonometric graphs. Their study involved 25 learners from a school in Durban, KwaZulu-Natal. Naidoo and Govender (2014) observed that learners had a meaningful learning experience within the online environment as it "offered a self-explanatory learning process aided by visualisation" (p.7). There are certain advantages of learning within an online environment such as self-regulation of learning, and learning is not restricted by the boundaries of classrooms. However, it entails increased parental involvement.

Sketchpad is a commercially available application software and given the poor economic conditions in the Eastern Cape province, especially in Mthatha, not many educational institutions can afford to buy it. The DBE have already spent heavily in procuring computer hardware in schools. Therefore, the emphasis in my research project is on using free and open sources of DGS in mathematics and shedding light on harnessing technological resources for educational purposes.

1.3.1 Research questions

My research goals and aims have been inspired by questions asked and gaps identified in the literature:

> The power of visualization of computers seems unlimited and has become pedagogically unavoidable. The question is not how to use them but what their impact is on the development of thinking and ability to use spontaneous mathematical knowledge for solving problems (Duval, 2014, p. 169).
>
> How do visual aspects of computer technology change the dynamics of the learning of mathematics? …. the dynamic possibilities of interactive computer technology is another field in which recent research would require a special issue in its own right (Presmeg, 2014, p. 155).

It is thus evident from the above, that there is a need to study learners' development of mathematical understanding and proficiency using visual aspects of DGS.

The research questions that frame this study are:

- How are *GeoGebra* applets used as a learning tool to make mathematical meaning and develop understanding by selected Grade 11 learners?
- What visualisation role can *GeoGebra* play in the learning of Grade 11 mathematics?

1.4 THEORETICAL FRAMEWORK

My research study makes use of an adapted and amalgamated model of 1) mathematical proficiency proposed by Kilpatrick, Swafford, & Findell (2001) and 2) meaning-making proposed by Carter et al. (2009). This analytical tool was used to interrogate various aspects of learning meaningful mathematics. Specifically, I focused on two strands of Kilpatrick et al.'s (2001) mathematical proficiency (MP), namely conceptual understanding (CU) and procedural fluency (PF).

1.4.1 Strands of mathematical proficiency

Developing a more comprehensive framework for mathematical learning and proficiency, Kilpatrick et al. (2001) concur that students become more proficient when they understand the underlying concepts of mathematics and when they are skilled at computational procedures. Kilpatrick et al. (2001) identified a set of five intertwined strands, called strands of "mathematical proficiency" that are important in developing this proficiency. These strands are not independent, but interwoven, representing different aspects of the whole. Due to the scope of this research, only two strands of mathematical proficiency are considered, viz. **conceptual understanding** and **procedural fluency**.

According to Kilpatrick et al. (2001) **conceptual understanding** is the comprehension of mathematical concepts, operations and relations. It refers to the grasp of fundamental mathematical ideas more so than isolated facts and procedures. When mathematical ideas are represented mentally in an organised manner (connected and structured), retrieval of knowledge is easy. Thus, learning with understanding is more powerful than simply memorising, because the organisation improves retention, promotes fluency, and facilitates learning related material. **Procedural fluency** refers to the skills in carrying out mathematical procedures flexibly, accurately, efficiently, and appropriately. It refers not only to the competent execution of procedures, but also to the awareness of various computational tools and selection of the appropriate tools in a given situation. Conceptual understanding and procedural fluency are interwoven. Understanding makes learning skills easier, while procedures help strengthen and develop that mathematical understanding. The knowledge of procedures with understanding provides a foundation for generating new knowledge. Thus, both strands are critical to developing mathematical proficiency.

1.4.2 **Conceptual knowledge and procedural fluency**

Much has been researched and written about the conceptual and procedural knowledge dichotomy (Hiebert & Lefevre, 1986; Van De Walle, 2004). Van De Walle (2004) succinctly comments that "understanding is a measure of the quality of connections that a new idea has with existing ideas." Rittle-Johnson and Alibal (1999) suggest that efficient and accurate procedures help learners to maintain fluency.

However, Silver (1986) argues that these two types of knowledge are inextricably linked. Procedural knowledge must be based on conceptual understanding and procedural fluency is necessary for developing conceptual knowledge. There is a constant interplay between conceptual and procedural knowledge when learners are engaged in tasks that require some level of understanding. His research analysis led him to conclude that the concepts of conceptual and procedural knowledge are intertwined (ibid.).

1.4.3 **Meaning-making habits**

Mathematical meaning-making is inherent in developing mathematical proficiency (Carter et al., 2009, p. 12). Making meaning and conceptual understanding are closely and tightly interrelated. Carter et al. (2009) argue that in the absence of reasoning and sense making, "students carry out procedures correctly but may also capriciously invoke incorrect or baseless rules" (p. 13). For example, when calculating the 'mean' of two numbers, say 21 and 32, most Grade 11 students tend to enter '21+32÷2' into their calculators and obtain 37, not realising that a mistake has been made. In this case the students did not appreciate the hierarchy of operations nor did they realise that the average of two numbers should lie midway between the numbers. According to Carter et al. (2009, p. 13), mathematical proficiency can only be achieved "by engaging in mathematical reasoning and meaning-making as they learn mathematical content." By using technology such as *GeoGebra* as an integral part of the mathematics classroom, they observe that "technology allows multiple representations to be linked dynamically, it can provide new opportunities for students to take mathematically meaningful actions." (ibid.).

1.4.4 **Theoretical perspective: Constructivism**

This study is informed by a constructivist theoretical perspective. The learner's new knowledge draws on prior knowledge and experiences – learners are not blank slates. Constructivism is a theory of knowing that attempts to show that knowledge can only be generated by experience (Jaworski, 1994; Van De Walle, 2004). This implies that learners construct their own knowledge by actively participating in a particular situation, in this case a classroom situation. From a constructivist point of view, mathematical concepts that 'fit' or are viable with our real-world experiences are constructed. Giving credit to Piaget, von Glasersfeld (1981, 2000) argues that knowledge is constructed by an individual as an adaptation to his subjective experience. Knowledge construction and adaptation are the result of cognitive structuring, acknowledged by Piaget's genetic epistemology. Knowledge results from individual construction by the modification of experiences or ideas.

The students' engagement in learning involves cognitive tasks and manipulation of conceptual ideas. Jonassen and Strobel (2006) argue that ICT can and should become the toolkit for meaningful learning. Meaningful learning implies that learners are actively manipulating objects and tools, and during these activities they construct their own meanings and interpretations of their actions. Technology tools like *GeoGebra* can provide opportunities for learners to wrestle with objects, in this case, applets. These opportunities not only develop understanding, but they also facilitate the adaptation of prior knowledge and experiences to new knowledge and meaning. When technological tools are used effectively, students can authenticate their understandings. Thus, *GeoGebra* as used in this study, with its inherent exploratory possibilities, may provide opportunities for an investigative approach to constructing mathematical meaning. The emerging technological tools like DGS, allow students to engage in active learning processes in which they themselves build their own meanings and develop understanding (Khine, 2003). Hence technology-enabled active learning aligns with the principles of constructivism.

Within the context of using DGS for learning mathematics, visualisation processes and mathematical proficiency are deeply intertwined. Presmeg (1986b) claims that the embodiment of abstract ideas in a concrete image can be effective in learning mathematics, i.e. the use of visualisation can facilitate the learning of mathematical content with understanding. Arcavi (2003) sees visualisation as a process of constructing knowledge that promotes understanding. Therefore, visualisation 'as a process' provokes intellectual activity and construction of knowledge.

1.5 **METHODOLOGY**

This research study is oriented within a qualitative research framework. Cohen, Manion and Morrison (2007, p. 21) assert that "to understand the subjective world of human experience" is fundamental in the context of an interpretative paradigm. Hence this research study involves making sense of learners' interactions with the dynamic software as they construct meaning of mathematical ideas. Furthermore, Denzin and Lincoln (2005) assert that qualitative research involves interpretive approaches in natural conditions, attempting to gain a better understanding of the participants and their actions – learners in this context. Thus, the interpretative research paradigm that underpinned this study coheres with the nature of the research enquiry.

I adopted a qualitative case study approach in this research. This interpretive research is designed as a case study of selected learners' engagement with the *GeoGebra* applets when learning mathematical concepts. The case here was a group of selected Grade 11 learners from two schools, who had been taught by GLIP teachers and my unit of analysis was the learners' interaction with the applets.

In order to capture learners' interactions with the computers, I used a novel method of data capturing, namely screen recording software. Screen recording software captures everything on the computer screen including media files and other activities on the desktop, as an audio-video file. It is commonly used to broadcast webinars, games, etc. It is similar to tape-recording or video recording voices and actions during interviews.

My data consisted mainly of screen recordings of learners' computers, and video recorded observations and interviews. The learners' engagement with the applets was analysed with the developed and piloted analytical instrument, namely the Visualisation Applets Meaning-Making Mathematical Proficiency Analysis (VAMMPA) framework.

1.6 SIGNIFICANCE OF THE STUDY

The goal of this research study was to investigate and explore how *GeoGebra* visualisation applets develop conceptual and procedural understanding of mathematical concepts in selected Grade 11 learners.

In the GLIP project, the specifically designed *GeoGebra* applets were prepared by the participating teachers and used for teaching and learning to conceptually understand mathematical ideas and concepts. The development of these applets was a significant contribution to the teaching and learning community at large. In the process of developing and using these applets and participating in this study many teachers and learners were empowered with technology-based teaching and learning and became familiar with DGS in particular.

Furthermore, the research study contributes to knowledge in the field of visualisation. Presmeg (2014) points out that more empirical studies are required to understand the power and pitfalls of visualisation capabilities of interactive technology. Thus, this research study is particularly significant as it has also gone some way in studying the impact of visualisation inherent in computer technology in developing learners' thinking and learning of mathematics.

Screen capturing as a data gathering mechanism is rarely used in the field of education. There is little evidence in educational research done in South Africa that has employed this technology of screen-capturing computer applications. Hence, this research also highlights the use of modern methods of data gathering in the emerging era of employing technological tools for the purpose of teaching and learning.

1.7 THE STRUCTURE OF THE THESIS

The remaining part of the thesis comprises six chapters.

Chapter 2 Literature Review

In this chapter, I discuss the conceptual and theoretical frameworks that underpin my research study. Firstly, the conceptual background of visualisation and its relevance in the teaching and learning of mathematics is examined. Secondly, the role of technology and its effective use in mathematics classrooms is discussed. Thirdly, the key concepts of Kilpatrick et al.'s (2001) framework of mathematical proficiency that form the analytical framework of my study are deliberated. I also address the learners' reasoning and meaning-making habits which are the core of mathematical learning and understanding. Finally, the theory of constructivism is discussed within the context of visualisation in a technological classroom.

Chapter 3 Methodology

Chapter 3 provides a detailed discussion of the case study method that was used in this study. Methods of data collection such as screen recording, classroom observation and interviews are described. I then examine the methods of analysing data and discuss strategies to enhance the validity and reliability of the research study. I address ethical considerations that pertain to this research. The chapter concludes by examining some of the challenges that arose during the data collection process.

Chapter 4, 5 and 6 Analysis of learners' engagement with the applets

The three chapters constitute my analysis of the data. In these chapters, I analyse the participating learners' engagements with the applets developed by their teachers, by employing the analytical framework which I adapted from the literature. I present the findings in relation to the research questions for each mathematical topic taught. Each chapter concludes with an explanation and discussion of the results.

Chapter 7 Conclusion

In this concluding chapter, I summarise the findings, bringing the discussions from the previous three chapters together. I also explore issues and limitations and propose recommendations from the research. I also discuss the contributions to the field and suggest directions for future research.

CHAPTER 2

LITERATURE REVIEW

2.1 INTRODUCTION

In this chapter, I will discuss the conceptual and theoretical frameworks that underpin my research study. Firstly, the conceptual background of visualisation and its relevance in teaching and learning are examined. Secondly, the role of technology and its effective use in mathematics classrooms is discussed. Thirdly, the key concepts of Kilpatrick et al.'s (2001) framework of mathematical proficiency that form the analytical framework of my study are deliberated upon. Finally, the theory of constructivism is characterised within the context of visualisation in a technological classroom, as shown in Figure 2.1.

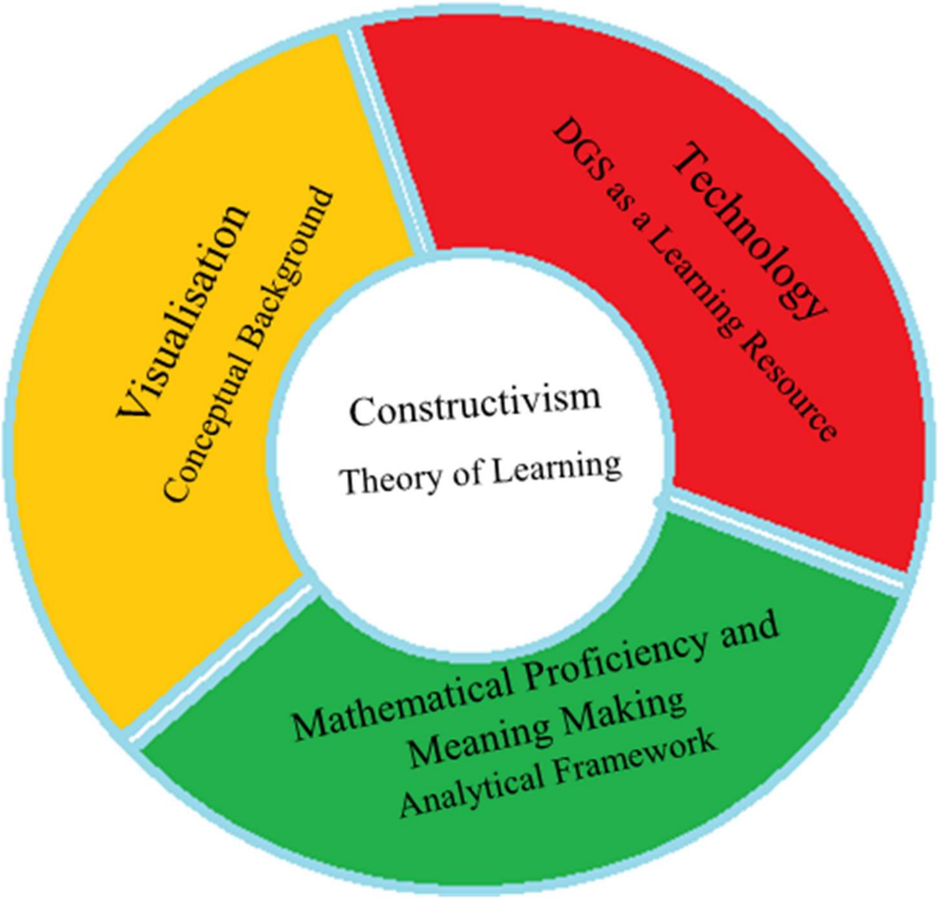


Figure 2.1: Four aspects of my research study

2.2 CONCEPTUAL FOUNDATION – VISUALISATION

Mathematical ability requires generalised and abstract thought (Krutetskii, 1976) and to establish a "mathematical advantage", one should perceive and use clear mental pictures. Krutetskii suggests that it is productive to capitalise on the visual aspect of an abstract mathematical idea. Clements (1982) wrote that many highly original and significant creations of the human mind have been largely the result of non-verbal mental representations. A visual-pictorial approach to solving problems is a powerful tool for pupils to use when engaging with mathematical ideas and reasoning. Presmeg (2006) asserts that the use of imagery is a significant way to generalise mathematical concepts. Acknowledging the centrality of visualisation in mathematics classrooms, Arcavi (2003) affirms that "[v]isualization is no longer related to the illustrative purposes only, but is also being recognised as a key component of reasoning" (p. 235). In my experience in the classroom, those who prefer to think visually, can quickly and accurately solve the most complicated problems through their mental capacity. But what is visualisation?

2.2.1 Visualisation defined

In literature, we find elaborate and wide-ranging descriptions of visualisation. Giaquinto (2007, p. 35) observes that visualisation is an effortless way of acquiring new information. He finds that visualisation is an apt means of "discovery and explanation". Duval (2014, p. 160) emphasises that "visualization consists of grasping directly the whole configuration of relations and in discriminating what is relevant in it." Tall (1994) suggests that visualisation is a powerful process to convey a large amount of data holistically and enables an individual to change focus from one part of the picture to another so that relationships may be observed visually and pictorially. Guzmán's (2002, para. 2) version of mathematical visualisation is "…a way of acting with explicit attention to the possible concrete representations of the objects that one is manipulating in order to have a more efficient approach to the abstract relationships one is handling." Visuality is not merely focused on perceiving images, it is about the process of attributing meaning to an image (Vermeersch & Vandenbroucke, 2015). Zimmermann and Cunningham (1991) define visualisation as images formed in the mind, with pencil and paper, or with the aid of technology.

Arcavi (2003) attempts to provide a comprehensive definition of visualisation and reflects upon the rich role it can play in the teaching and learning of mathematics. He proposes that:

> Visualization is the ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understandings. (p. 217)

The chief characteristic of the above definition that is germane to learning, is the emphasis on 'ability' and 'process', which resonates with Bishop (1988), who claims that visualisation is both a product (phenomena) and a process (activity). For the purpose of this study, Arcavi's (2003) definition is central. It effectively synthesises the ideas of many researchers (Bishop, 1980; Duval, 1999; Zimmermann & Cunningham, 1991) and applies to the central theme of this study.

2.2.2 **Visualisation in learning**

Hilbert (1952) asserts that "[w]ith the aid of visual imagination, we can illuminate the manifold facts and problems of geometry." Arcavi (2003) and Zimmermann and Cunningham (1991) foreground the importance of visualisation and suggest that in mathematics, visualisation enables us to 'see the unseen'. This is significant in mathematics as many mathematical concepts are abstract and, in my experience rather 'obscure to young learners'. Duval (1999) asserts that visualisation can show how relations are organised and enables learners to form images of complex mathematical ideas. When learners encounter complicated problems that involve complex procedures, visual representations can simulate a whole solution (Arcavi, 2003; Krutetskii, 1976).

Visualisation may consist of the construction on some external medium such as paper, chalkboard or computer screen, of objects or events that one identifies with the object or process in one's mind (Zazkis, Dubinsky, & Dautermann, 1996). Thus, drawing a rectangle or paper cut in a rectangular shape alone does not necessarily entail visualisation. Visualisation lies in corresponding known mathematical properties with the constructed object. When we fail to make connections with the representations produced, we succumb to what Duval (1999) refers as a "blind spot of many didactical studies" (p. 20). Similarly, the construction of a graph, for instance — a straight line on a coordinate plane using data tables — does not imply visualisation. Visualisation is required to identify the characteristic features of the equation, like slope and intercepts. Further, "an act of visualisation" (Zazkis et al., 1996, p. 441) takes place when the solution of a linear equation is interpreted as the intersection of a straight line with the x-axis. Indeed, a picture is worth a thousand words, but this assumes that the picture has been correctly interpreted and understood, otherwise the picture is nothing but a non-mathematical diagram (Guzmán, 2002).

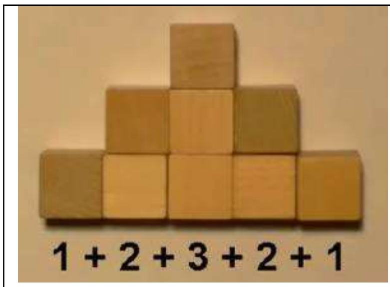
However, Guzmán (2002) considers the graphical representation as a modelling of mathematical problems, and he classifies it as a case of "isomorphic visualisation" (p.5). Though its usefulness lies rather on demonstrating a concept, Guzmán contends "[t]he manipulation of the objects that we perceive with our senses or with our imagination is normally easier and more direct than the handling of abstract objects" (p. 5). Such concrete manipulations and visual approaches very often give students "powerful intuitions"' (Tall, 1994, p. 11) to conjecture abstract and symbolic objects. An important part of our visualisation in mathematics lies in generalisations.

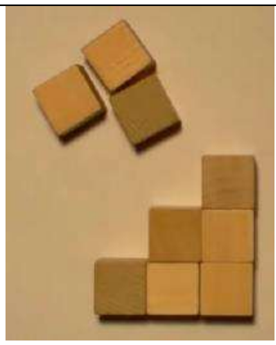
Another example of generalisations is the sum of palindromic numbers (Linneweber, Schäfer, & Samson, 2010,). If we extend the example of the data table, refer to Table 2.1 to find a general rule for the given data values $y = x^2$. Our visualisation helps us to solve the particular problem, in this case to find 'y' when the palindromic sum involves the first five natural numbers. But in mathematics, we are often more interested in generalising the palindromic sum of the first 'n' natural numbers which are greater than 1.

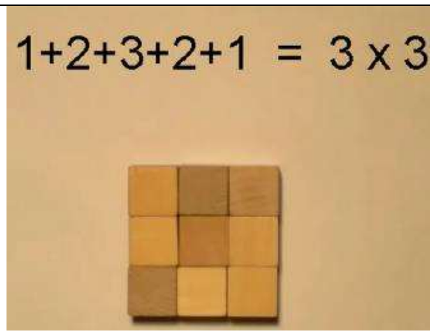
Table 2.1: Sum of palindromic numbers

x	1+2+1	1+2+3+2+1	1+2+3+4+3+2+1	1+2+3+4+5+4+3+2+1
y	4	9	16	?

We can see from the above, that data values can be modelled into a graph or generalised to a rule where there is a pattern. The following screenshots (in the form of figures) in Frame 2.1, are static — hence may not fully convey the effect of movement. I encourage the reader to refer to the video clip on www.vitalmaths.com. The quadratic pattern $y = x^2$ where x is the natural number greater than 1, may seem to be abstract, but , the screenshots of the video in Frame 2.1 concretises the pattern. Guzmán (2002) classifies this type of visualisation as "analogical visualisation" (p. 7) . For the palindromic sum 1+2+3+2+1, the squares (of unit sizes) are arranged in that order in scene 1. Then they are re-arranged as seen in scene 2. In the third scene, the nine-unit squares form a square of size 3×3.







Screenshot 2.1: Arranging cubes	*Screenshot 2.2: Moving the cubes*	*Screenshot 2.3: Becoming a square*
Frame 2.1: Visualising a palindromic sum as a square		

Tall (1994, p. 11) asserts that, “A definition derived from a picture is more likely to be conceived as an object defined (with powerful concept image) rather than a defined object (with formal definition to be used in formal proof).” It requires considerable effort for the learners to switch between concrete objects and symbolic concepts. Hence visualisation poses a challenge to students (Krutetskii, 1976; Presmeg, 2006). Presmeg identifies that these difficulties arise due to the low status accorded to visualisation in the mathematics community. Guzmán (2002) accords these challenges in visualisation to “… the textbooks that our students use is, basically, the written word, a statically vehicle that is not well adapted to the needs of the visualization *dynamic* processes.” [‘dynamic’ added] (p. 14). Arcavi (2003) asserts that one of the reasons for these challenges concerning visualisation is due to “the cognitive demand that is certainly high” (p. 235).

Presmeg (1986a) notes that the inflexibility in thinking prevents learners from recognising a concept in a non-standard diagram. Learners face difficulty in recognising the *angle at the centre theorem* when it does not conform to the standard diagram. Bishop (1988) calls this inflexibility of thinking as "geometrical rigidity" caused by a child being unable to 'see' a diagram in a different way. In their study of learners’ responses in a geometric task, Mhlolo & Schäfer (2013, p. 83) observe “the rather idiosyncratic manner in which learners were using mathematical signs and words” in response to tasks in geometry. For example, learners would measure the length of a side of a triangle in degrees, employing Pythagoras’ theorem to find the third angle of a triangle. They argue that “the learners were making connections based on their subjective impressions” (p. 86) . Hence, conceptual development in geometrical problems require both visual and conceptual understanding (Mhlolo & Schäfer, 2013).

Fischbein (1993) clarifies that a geometrical figure is not a mere concept. All geometric figures possess conceptual properties (like abstractness and generality) and figural properties – like shape and magnitude. Fischbein (1993) defines figural concepts as "mental entities" in the investigation and manipulation of objects. I concur with Fischbein (1993) that visualisation is a fusion of conceptual and figural properties of geometric shapes. Consider the following diagram in Figure 2.2. *In a circle with centre O, the diameters are drawn perpendicular to each other. P is any point on the circle and PQ and PS are drawn perpendicular to the diameter. What is the length of QS?*

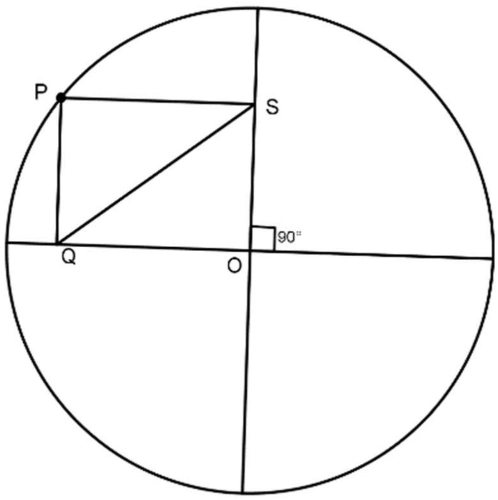


Figure 2.2 Fusion between concept and figure

At first glance, it appears that the data is insufficient to solve this problem. The radius is not given, and the position of *P* is not known, hence the lengths of *PQ* and *PS* are unknown. The moment we visualise $PQOS$ as a rectangle, we can 'see' that *PO* = *QS* because they are diagonals of a rectangle, hence we can conclude that *QS* = *OP* = radius. Fischbein (1993, p. 143) then declares that "*the fusion between the concept and the figure tends to be, in this case, complete*" [author's italics]. Here, visualisation becomes very useful to recognise the configuration of a rectangle and thus the relationships of the other lines in the problem.

While logical reasoning is a step by step procedure, visualisation is the grasp of the whole (Duval, 2013). For Duval, visualisation refers to a cognitive activity and he proposes three kinds of cognitive activities in geometry: firstly, seeing and recognising shapes; secondly, measuring, calculating and comparing magnitude; and thirdly, inferring from properties. This is illustrated in Figure 2.3 below.

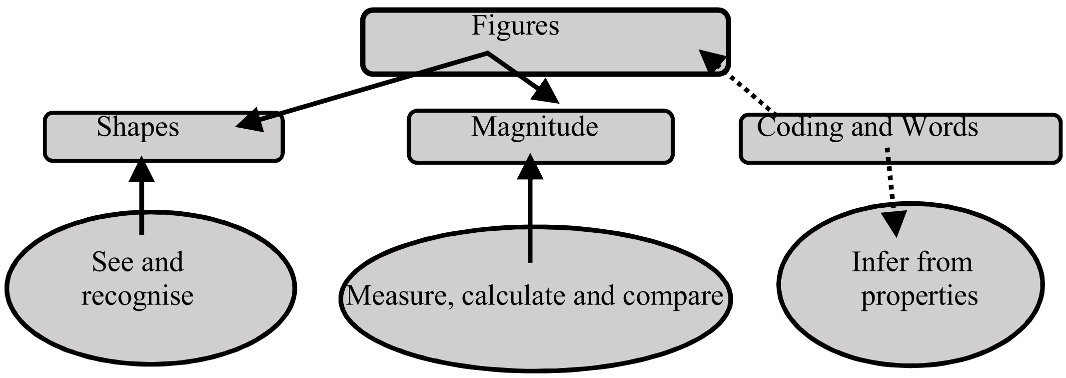


Figure 2.3: The three faces of figure in geometry (Duval, 2013)

Seeing is recognising a shape and object. Duval (2013, p. 25) identifies this as "perceptual recognition". He considers perceptual recognition as an "out of mathematics" (p. 27) visual representation as the given properties of the object are not considered. Many students have difficulty in understanding geometric figures by themselves because the perceptual recognition of shapes and figures is so predominant and spontaneous that it impedes mathematical ways of visualising the figures. Many students, for example, fail to recognise a square when it is tilted. The process of visualisation can help students to realise that a square is not only an image, but also a shape controlled by its definition (or properties).Mathematical visualisation lies in the implicit selection of visual units within the configuration that are relevant, rather than ones that are not. Visualisation requires one to move from the whole, to some of the visual units that are characteristic features of the represented phenomena (Duval, 1999). Geometrical properties correspond to what remains invariant when the drawing is changed, by moving either one of its points or its segments. Visualisation in geometry is specifically related to the cognitive activity of seeing shapes within a figure. In order to achieve this, the notion of *figural unit* is introduced which enables an individual to decompose and recompose a given figure. Duval (2013) defines *figural units* as

> All the elements, which can be visually discriminated in any constructed figure. These elements are not characterised by their shape but by the number of their dimension. (p. 32)

He further argues that a "mathematical property cannot be visualised by a single figural unit, but only by the visual relation between two or more figural units" (p. 33). We can promote processes of visualisation among learners by making them identify the various line segments and points of intersection in any geometrical figure. Therefore, in any geometrical activity visualisation and the corresponding problem solving discourse ought to be linked, and technical words are to be matched with figural units. "The mathematical recognition of objects represented in figures depends on the given properties and not only on the perceptual recognition of shapes" (Duval, 2013, p. 28).

2.2.3 **Individual differences**

The results of the study by Krutetskii (1976) proposes that the ability to visualise is not a necessary component to be mathematically able. He traced some basic typological differences in the structure of mathematical talent and identified three types of different mathematical casts of mind. He suggests that the 'analaytic type' of student prefers to use verbal-logical methods and has no need for visual support. The 'geometric type' prefers to use visual-pictorial methods even when the problem could easily be solved by reasoning. For the 'harmonic type' of student there appears to be a relative equilibrium of well-developed verbal-logic and visual-pictorial processes evident when solving mathematical problems. Kruteski (1976) maintains that there needs to be a balance between emphasising verbal-logical and visual-pictorial schemes as the dominance of one can hinder the generalisation process.

Lean and Clements (1981) classify mathematics students into three broad groups, namely 'visualisers', 'verbalisers' and 'mixers'. Visualisers are individuals who habitaully employ visual imagery to solve problems. Verbalisers are those who tend to use verbal codes in problem solving. Mixers do not have a preference for the verbal or visual mode. Presmeg (1986b) in her study, also classified learners (and teachers) into visualisers and non-visualisers. As with the study of Lean and Clements (1981), Presmeg found that the majority of her participants that performed well in mathematics had a tendency to use non-visual methods in their problem solving strategies. Presmeg identified the following possible factors to explain her findings: 1) The non-visualisers are able to generalise ideas rapidly and thus curtail their procedures when solving problems. They do not feel the need for concrete images. 2) The school mathematics curriculum favours the non-visual thinkers as if achievement is measured in time-bound tests. Visualisers tend to use time-consuming procedures, hence struggle to perform well in tests. 3) The teaching emphasis is usually on non-visual representations and very often visual representations are dispensed with whenever possible. From the above literature, it is evident that there exist individual differences in learners in how they use visualisation.

2.2.4 **Visual literacy and Visualisation**

The notion of visual literacy (VL) is gaining momentum among researchers (Avgerinou & Pettersson, 2011; Vermeersch & Vandenbroucke, 2015). Visual literacy is more than seeing and making images. It is a thinking process and requires cognitive actions like interpretation and reflection, understanding and comprehension (Vermeersch & Vandenbroucke, 2015). From a theoretical perspective, visual literacy "incorporates the philosophical, psychological, and physiological aspects of learning" (Avgerinou & Pettersson, 2011, p. 4). A VL approach is learner-oriented and it helps the learners to effectively relate to visual stimuli. VL encompasses influences and facts from many established academic disciplines and areas of knowledge and research for example mathematics, physics and geography. Acknowledging VL as a skill or an ability, Avgerinou and Pettersson (2011) argue that critical viewing and thinking, imaging, visualising, inferring, constructing meaning and communicating must be learned for true comprehension. Thus, we can infer from the above arguments that VL in a broader sense resonates with Arcavi's (2003) definition of visualisation in mathematics, as mentioned in Section 2.2.1 p.14.

I find further cohesion between VL and visualisation. For example, Presmeg and Duval, academics in visualisation, put forward that visualisation can be developed and taught in classrooms. Avgerinou and Pettersson (2011) state "The VL skills are (a) learnable, (b) teachable, and (c) capable of development and improvement." Duval (1999) argues that in mathematics, "visualization consists of grasping directly the whole configuration of relations and in discriminating what is relevant in it." (p. 13). Vermeersch and Vandenbroucke (2015) consider VL as "the ability to discriminate and decipher what one (wants to) see(s)." (p. 112).

Mudaly (2010) considers visual literacy as an internal process that the mind engages in after seeing the diagram externally or internally. Mudaly draws a distinction between visualisation and visual literacy and argues that "visual literacy is visualisation combined with logical thought" (p. 67). This distinction may be subtle and invoke dichotomy. Nevertheless, I concur with Presmeg (1992) who argues that logic and reasoning are essential to move from concrete images to the abstraction of mathematical concepts. Logical reasoning appears to be an integral component in visualisation.

VL is concerned with the intention to make something new through perceived images or reconstructing existing images. Vermeersch and Vandenbroucke (2015) underscore the importance of learning VL as a 'meaning-making act' and 'higher order thinking', that entails analogical thinking. Visual thinking takes one step further from preception, to understand and add meaning to the images or concrete objects. In other words, in mathematics the focus is on the transition from concrete situations to abstract ideas through a process of 'internalisation' (Rivera, 2011). A child is able to manipulate numerical operations with currencies or counters but struggles with the same process in the absence of these external tangible objects. For the transition from explicit knowledge to implicit knowledge to occur, Rivera (2011) prescribes that dynamic features of computer-generated concrete objects can help "obtain fresh insights concerning complex and poorly understood mathematical objects" (p. 80).

2.3 TECHNOLOGY

2.3.1 ICT as a learning resource

Technology in mathematics education is gaining traction among researchers and curriculum planners. ICT has been highly regarded as a potentially powerful resource for learning (Hollebrands, 2007; Ruthven et al., 2008; Stols & Kriek, 2011). The term ICT implies a range of hardware (computers, laptops, projectors), software applications (simulation and dynamic software, spreadsheets and multimedia), and information systems (internet, intranet and wireless networks). ICT can enable access to a repository which includes textbooks, audio-video files, online learning websites, simulation software, dynamic software, and many more in classrooms. When used appropriately, ICT can shift the focus of learning from computational skills to evaluating conjectures. When combined with visualisation processes, ICT technologies in my experience can bring about an exciting paradigm shift in mathematics education which allows for powerful multiple representations of mathematical concepts as visual objects.

The National Curriculum Statement (2011) aims to produce learners that are able to use science and technology effectively and communicate effectively using the visual and symbolic in various modes. The National Curriculum Statement (2011) recognises the potential that ICT offers for curriculum delivery, hence provides teaching guidelines as follows:

> Generate as many graphs as necessary, initially by means of point-by-point plotting, supported by available technology, to make and test conjectures and hence generalise the effects of the parameter which results in a horizontal shift and that which results in a horizontal stretch and/or reflection about the *y*-axis (p. 12).

Cuban, Kirkpatrick, and Peck (2001, p. 823) observe that "students whose lives changed with increased access to technology …. their computer competence enhanced their self-confidence and motivation to do well in school, hence opened doors to learning." Other researchers

(Bhagat & Chang, 2015; Mariotti, 2000) also conclude that the use of technology can significantly improve learning.

Hollenbrands (2007) found in his empirical study that a technological environment provides more opportunities for students to engage in abstract mathematical concepts. Therefore, it is possible that technology, with its feedback and visual features, will facilitate students' development of deep understanding. In addition to building knowledge in mathematics, by learning using technology, students are provided with necessary knowledge and skills required in the digital world (Zulnaidi & Zakaria, 2012). In the following sections, I will discuss how technology-aided learning promotes conceptual understanding.

2.3.2 **Potentials of Dynamic Geometry Software**

The form of educational technology which is the focus of this study is Dynamic Geometry Software (DGS). I will first look into the evolution from static to dynamic figures.

2.3.2.1 **The evolution of dynamic geometry systems**

Ruthven et al. (Ruthven et al., 2008, p. 298) reflects that "dynamism which has become its defining characteristic was not such from the start". The early generation of geometry software, like *Sketchpad* and *Cabri*, were developed to draw accurate, static figures from Euclidean geometry resembling conventional practices with compasses and rulers (Ruthven et al., 2008). It is important to know how static and dynamic diagrams and operations on diagrams – "stretching and shrinking them, rotating them, seeing them interact" (Goldenberg, 1992, p. 203) influence understanding of mathematical concepts (Dreyfus, 1992). Goldenberg (1992) argues that many students do not develop these abilities "perhaps, most people who lack (or, for whatever reason, fail to use) the sand-box (extracurricular, ambient) opportunities to develop dynamic mental imaging abilities, could nevertheless acquire the skills if they were given optimal experiences" (p. 203). I concur with Goldenburg that technology can support dynamic visual reasoning. Thus, DGS was developed with educational purposes in mind (Ruthven et al., 2008). Kaput (1992, p. 525) argues "[t]echnologies based on dynamic interactive electronic media embody fundamental attributes that distinguish them from traditional static media in ways likely to have tremendous long-term impact on mathematics education."

2.3.2.2 **DGS in classrooms**

DGS can be used as a means of constructing and manipulating objects. Key to every DGS is an interface that affords direct manipulation of geometrical figures, particularly by dragging parts of them with the mouse (Hoyles & Noss, 2003). Initially, the visual artefacts constructed (via programming or tools) in DGS may appear analogous to an instance of Euclidean geometry. But these visual artefacts represent "a class of diagrams" (Hoyles & Noss, 2003, p. 334) that can be dragged around the screen with their constructed properties and underlying geometric relationships preserved. The 'drag mode' (Hölzl, 2001) is a basic feature of DGS. It allows a continuous reconstruction of figures on the computer screen by direct manipulation with the mouse (or touch on a touchpad). The interface of DGS, which dynamically re-draws and updates information on the screen, provides an opportunity for students to conjecture and generalise by clicking and dragging objects.. This feature provides a kind of feedback that further engages students to efficiently test large iterations of the mathematical construction (Moreno-armella, Hegedus, & Kaput, 2008). Thus many researchers suggest that students switch between figures and concepts, between empirical and theoretical considerations while dragging. The drag mode assists in the observation of invariance, and mathematics lies in the generalisation of invariance. As Kaput (1992, p. 525) observes, "One important aspect of mathematical thinking is the abstraction of invariance. But of course to recognise invariance to see what stays the same one must have variation. *Dynamic media inherently makes variation easier to achieve*." The tools of DGS like dragging enable learners to move shapes around the computer screen and thus discover their inherent characteristics.

Ruthven et al. (2008) consider DGS as oriented towards 'inductive reasoning and exploration of hypotheses'. An important feature of DGS is that it can be designed around mathematical principles and inspire students to interact with it. When the emphasis of the lesson is on promoting students' broad understanding, then DGS can develop awareness of mathematical ideas through exploring dynamic figures. DGS can support the heuristic method of problem solving. Nevertheless, Hölzl (2001, p. 65) affirms, "DGS is used only in a *verifying* manner" [author's italics]. In a dynamic environment, learners can change geometric configurations and confirm empirically the axioms and theorems. For example, the learners can verify the axiom that the *sum of the angles in a triangle are supplementary*. In contrast, Sherman (2014) argues that DGS is intended for the purpose of observing relationships and to make and test conjectures about mathematical properties. In my experience, DGS can be a powerful and effective tool to guide students to discover and verify mathematical properties by themselves if used appropriately.

Ruthven et al. (2008) argue that dynamic geometry enables students to work with figures "*easier, faster and more accurately*" [authors' italics] (p. 304). Accordingly, it offloads the tedious skills required for accurate drawing and measuring, and enables students to focus attention on the mathematical properties of the constructed objects (Tall, 1993). Constructing a rectangle in a dynamic environment is done with mathematical rules — what Hollenbrands (2007) and Hölzl (2001) refer to as 'figures', characterised by internal relationships. Casually drawn line segments on the grid of the DGS may be placed in an order which take on the appearance of a rectangle, but that object is destroyed under the 'drag mode'. In contrast, when a rectangle is constructed using properties (perpendicular or parallel lines intersected with circle tools), the object retains its properties under the drag mode. This entails interweaving knowledge of the tools (software functionalities) and knowledge of mathematics (Hollebrands, 2007). Another pedagogical consideration pertinent here is that the former activity of constructing a rectangle resembles that of construction in a static medium, which may actually obscure the mathematical properties and significance in the latter activity.

2.3.3 ***GeoGebra* as a DGS Tool – An overview**

There are many DGS tools available, like *Cabri Geometry*, *Geometer's Sketchpad* and *GeoGebra*. For the purpose of this study, I will be employing the DGS *GeoGebra*[1]. "*GeoGebra* is a multiplatform mathematics software package that gives everyone the chance to experience the extraordinary insights that maths makes possible" (*GeoGebra*, n.d.). According to Hohenwarter, the founder of *GeoGebra*, "the software was designed for use in mathematics education in secondary schools and higher educational institutions" (*GeoGebra*, n.d.). *GeoGebra* is an open-source *dynamic mathematics software* package that affords a bidirectional combination of **geo**metry and al**gebra**[2] (Hohenwarter & Preiner, 2007). The bidirectional combination implies that by typing an equation in the algebra window, the graph (or geometric object like a circle, ellipse etc.) of the equation will appear in the graphics window. Similarly, while dragging the graph (or the geometric object), the equation in the algebra window changes accordingly.

The software runs on virtually any operating system, like *Windows*, *Macintosh*, *Linux* or *Android*, as it requires only a Java plug-in and, unlike commercial products, students and teachers are not constrained by licences to run the software on a limited number of computers (Hohenwarter & Hohenwarter, 2008). Researchers who have begun using *GeoGebra* in their projects and textbooks are being published with *GeoGebra* supplements. Furthermore,

[1] Marucs Hohenwarter the founder of *GeoGebra* refers to it as a dynamic mathematics software.

[2] Descartes (1596-1650) and Fermat merged geometry with algebra, providing equations to geometric objects and Descartes envisaged a machine to do tedious computations of analysis of geometry (Goldman, 2006) – e.g. to find *n* mean proportional (geometric means) between two given lengths *a* and *b* (Fauvel & Maanen, 2002, p. 227)

volunteers have translated *GeoGebra* into 35 languages (Hohenwarter & Lavicza, 2007)[3]. As observed earlier, the DBE, South Africa, also recommends the use of *GeoGebra* as a technological tool for the teaching and learning of mathematics.

By incorporating different disciplines in mathematics, *GeoGebra* has now evolved into multiple perspectives – Geometry, Algebra, Computer Algebra System (CAS), 3D graphics, Spreadsheet and Probability. *GeoGebra* helps the users to create activities incorporating multiple representations of mathematical concepts that are dynamically linked. Thus, *GeoGebra* integrates dynamic multiple representations in a conceptually rich learning environment that supports the exploration, construction, and evaluation of mathematical models and simulations (Jones et al., 2009). Hasek (2011) considers *GeoGebra* as a way of producing geometrical and mathematical models of various real-world phenomena. He notes, "*GeoGebra* especially, thanks to its unique combination of spreadsheet, drawing pad and algebraic representation of objects, represents a powerful tool that enables us to draw graphs and curves, investigate their qualities and simply perform numerical iterative computation and to plot its results" (p. 217). With *GeoGebra*, abstract ideas may be reified not only by making connections between the real and abstract world, but also between different mathematical concepts.

GeoGebra is more of a representational tool than a computational tool (Sherman, 2010), and in tandem Goldman (2006) articulates that "*the main contribution of algebraic geometry to geometric modeling is insight not computation*" [author's italics] (p. 2). When engaging with *GeoGebra*, students have the ability to move between representations fluently with understanding. Researchers (Naidoo & Govender, 2014; Ruthven et al., 2008; Sherman, 2010) concur that *GeoGebra* is a useful teaching and learning tool that can strengthen learner conceptual understanding and improve pedagogy. Based on research reports on *GeoGebra*, Stols and Kriek (2011, p. 137) support the benefits of the software by saying that:

> Dynamic mathematics software is a powerful teaching and learning medium and it has been reported to (a) enhance mathematics teaching; (b) help with conceptual development; (c) enrich visualisation of geometry; (d) lay a foundation for analysis and deductive proof; and (e) create opportunities for creative thinking"

[3] *GeoGebra* is also translated into Afrikaans and IsiXhosa by Rika Grobler and Vuyani Matsha respectively.

The literature implies that *GeoGebra* can be effectively applied to many of the mathematical topics, if not all topics, of any school curriculum. For the purpose of this study though, I will be focusing upon the capabilities of DGS of *GeoGebra* in the context of Grade 11 mathematics.

2.3.4 ***GeoGebra*** **applets as manipulatives**

In mathematics, a manipulative is an object or artefact which is designed so that the learner can explore and interact with any mathematical concept by manipulating it. Just as a picture can be worth a thousand words, manipulatives can provide visual representations of ideas, helping students to learn and to understand mathematics. In my experience as a mathematics teacher, encouraging learners to play with manipulatives can stimulate multisensory experiences such as touch and vision, provide immediate access to ideas and concepts, and offer multiple entry points for discussions and reasoning.

When considering virtual manipulatives, Moyer (2002) writes that they offer learners powerful opportunities to manipulate a virtual object on the computer screen. Virtual manipulatives can be used to make meaning of mathematical concepts and see relationships between mathematical properties as a result of one's own action and input. Virtual manipulatives can be very interactive. They can visually represent an object that can be dynamically manipulated and be used for constructing shapes. This deep engagement with an object, can in my view enhance conceptual understanding of that object.

In DGS, these dynamic artefacts allow learners to engage and control the inherent actions of the objects by pointing, clicking and dragging aspects of the shape on the computer screen. The strength of DGS visualisations is that the movement of the representation preserves the invariant relationships in the visual representation when dragging and moving elements of the representation. This allows for making conjectures and verifying hypotheses (Kaput, 1992). An integral component of *GeoGebra* is the use of applets. The Merriam-Webster dictionary (Merriam-Webster, n.d.) defines an applet as "a small program designed to be executed from within the application." Applets are usually short, user-friendly applications. Applets can be used to provide interactive features enabling the user to change the dimension and the characteristics of the entire graphic or parts thereof. Thus it is very suitable for demonstration, visualisation and teaching (Java Platform, n.d.). *GeoGebra* applets are representations of mathematical ideas which anyone can interact with. Often, these are small programs made to serve a specific purpose. For example Figures 2.4 and Figure 2.5 are screen shots of an applet

designed specifically for learners to explore the relationship between the x-intercepts and the turning points of the graphs of cubic functions (Mavani & Mavani, 2014).
In

Figure 2.4, the intercepts of a cubic function are at points A, B and C and the turning points are D_1 and D_2. This applet allows the learners to drag the intercepts along the appropriate axes. They are encouraged to make conjectures, for example: what would happen to the turning points when the intercepts approach each other? They can explore and test their conjectures dynamically and discover that, as in Figure 2.5, the point of coincidence of the two intercepts must necessarily be a turning point. The meaningful part of a dynamic action is that the intermediate states are visible, hence Kaput (1992, p. 526) argues "*[c]ontinuous transition* of intermediate states is likely to be a cognitively important feature of dynamic systems" [author's italics]. Ruthven (2003) also concurs that "dynamic geometry system provides tools which assist the progressive appropriation of the initially informal actions and relations involved in the task to a disciplined framework of geometrical reasoning" (p. 14). Thus, *GeoGebra* applets are virtual manipulatives providing an opportunity to explore a variety of mathematical concepts through hands-on manipulation of functions, graphs, formulae, shapes and figures. The applet discussed above can also be accessed online at http://ggbm.at/XXMem7fU.

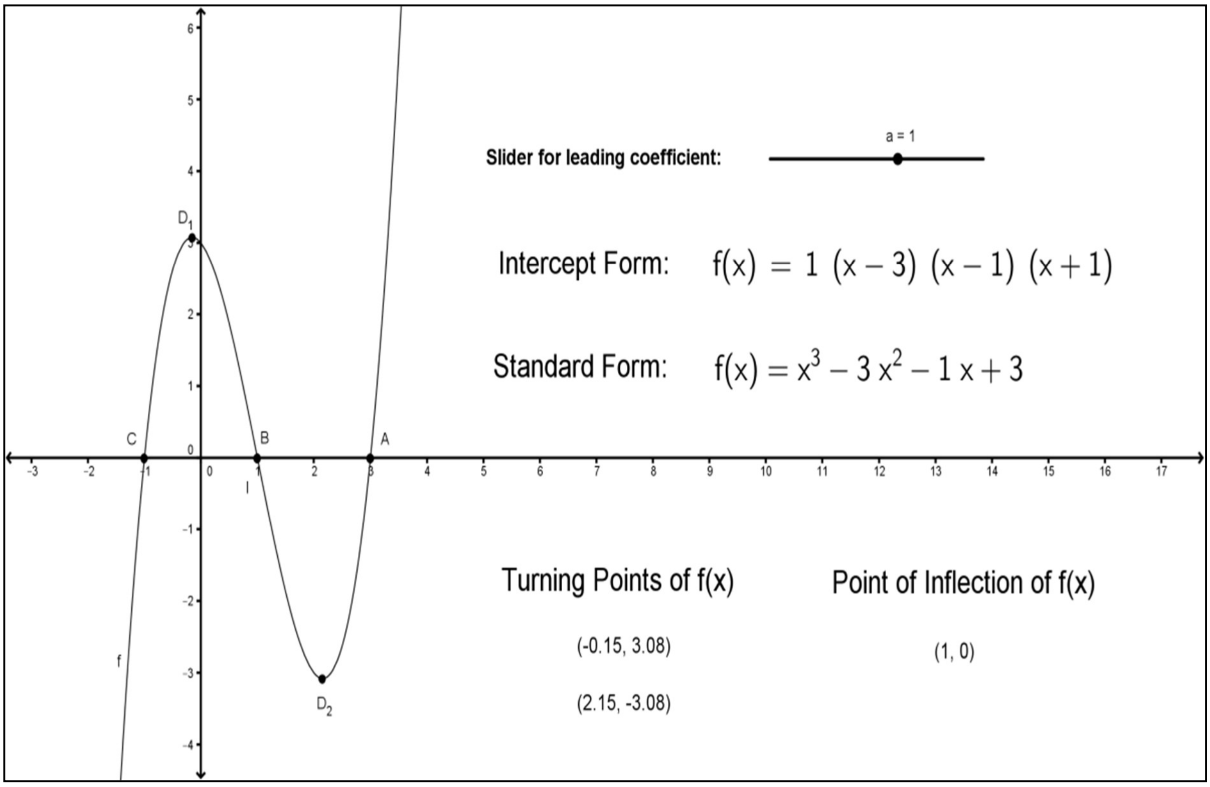


Figure 2.4 Cubic graph with three different roots

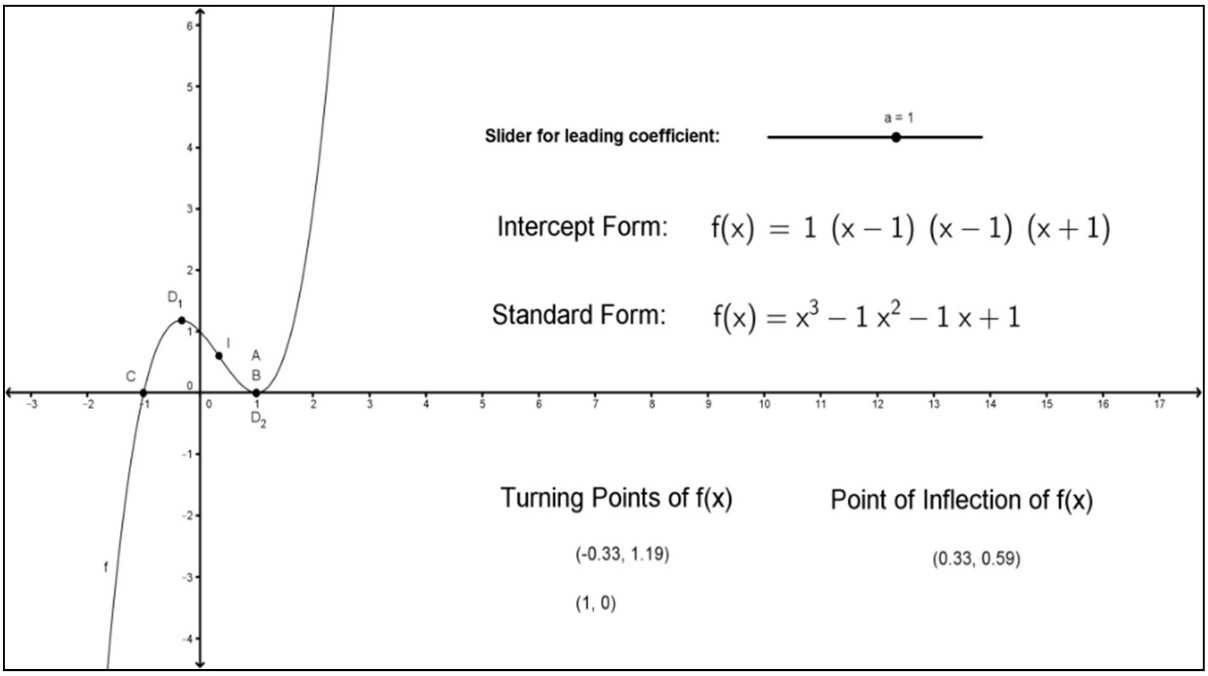


Figure 2.5: Cubic graph with two equal roots

According to Duffin (2010) and Gage (2010) there are at least four strengths of using applets. Firstly, they can complement a discourse with visual demonstration, which would have been difficult to convey in a traditional manner. Secondly, these short applications enable dynamic visualisation of mathematical concepts, allowing the learner to explore relationships. *GeoGebra* applets provide the visual aspect that is missing from the text alone, thereby helping visual learners. Thirdly, representations can be linked to draw attention to the relationships between two representations, and thus deepen understanding. Lastly, the technical advantage is that these can be easily embedded in a web-based application, hence these can be accessed by anyone from anywhere. *GeoGebra* applets can facilitate experimentation and hence create an interactive environment of "learning by doing".

2.3.5 **Visualisation through technology**

Arcavi (2003) emphasises the potential role of technology in visualisation. The dynamic features of technology provide an opportunity for learners to see what is not necessarily obvious to the usual. For example, an image of a graph enables learners to 'see' the relationships between quantities. It may also sharpen our understanding and serve as a springboard for questions which we were not able to formulate before. The visual representations of algebraic expressions favour the conceptualisation process. Computerised environments aid learners to generalise by manipulating "generic objects" (Tall, 1993, p. 400). Thus, technology as a visual aid creates an entry point for learning a mathematical concept.

For instance, graphical representations help to visualise and interpret the behaviour of a function. To illustrate this fact, the symbolic technique may be a meaningless recipe to get the axis of symmetry of a parabola function. The dynamic visualisation with *GeoGebra* helps learners to develop understanding, as they touch and see that the axis of symmetry passes through the turning point of a parabola. The visual feedback of their actions on virtual objects develops new understanding. Yerushalmy (1997) formulates that "technology makes mental objects tangibles by offering them as virtual objects." Technology-enabled visualisation, thus, is not considered as making learning easier but as a basis for a mathematically richer activity.

We have discussed earlier that technology enables one to draw precise geometrical objects. Balacheff and Kaput (1996) argue that the direct manipulation of objects allows the student to check the 'correctness' between expected properties and observed properties in the figure. In a similar view, Pea (1985, as cited in Sherman, 2014) refers to this manipulation as the *amplifier and reorganizer* metaphor of technological use. The relevance here is the dual role of visualisation in a dynamic geometric environment (Laborde, 1993). The first role is the construction or manipulation of virtual objects (mathematical fidelity or amplifier) and second is the reflection of mathematical properties in the object or figure as the learners' test and verify their conjectures. However, Laborde (1993) argues that it remains obscure in traditional environments because of the very low level of reliability of the drawings (with pencil on paper) and the very small number of possible experiments.

When we make connections between mathematical topics we employ visualisation (Presmeg, 2006). Making connections in the mathematics classroom is one of the cornerstones of mathematical proficiency. Schäfer (2016) emphasises that connection making entails linking to prior knowledge, connecting between different topics and making use of multiple representations. The classical algebra-table-graph approach to functions is an illustration of multiple representations. With the aid of multiple perspectives, like algebra, geometry, spreadsheet etc., in GeoGebra, the above multiple representations of functions can be visualised and connected, as shown in Figure 2.6. The line can be dragged along the screen, and the algebraic notation and the table values will also change dynamically in

Figure 2.7.

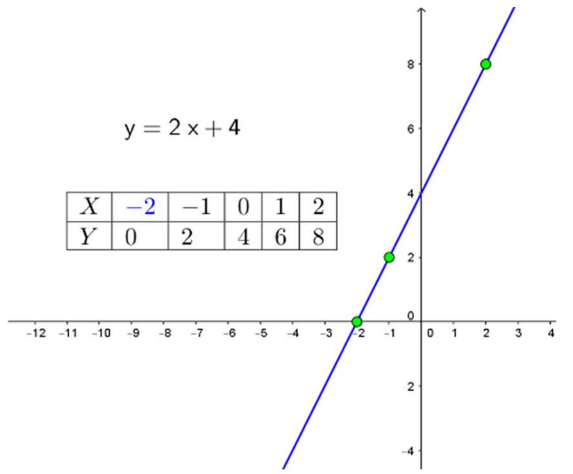


Figure 2.6: Multiple representations of functions in GeoGebra

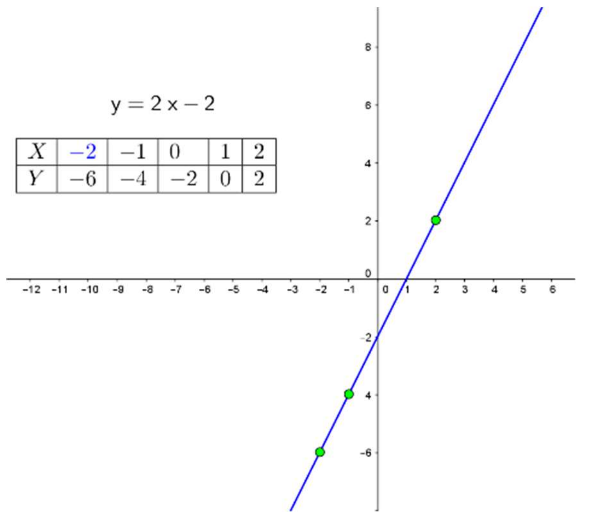


Figure 2.7: Dynamic straight-line function

GeoGebra is an easy-to-use, versatile tool. The *GeoGebra* applets can be effortlessly embedded into webpages for creating interactive teaching materials. These applets can support both classroom demonstration for dynamic visualisations and active student participation through dynamic worksheets (Hohenwarter & Preiner, 2007). I concur with Stols et al. (2015) that the perceived benefits that learners could experience when using technology like DGS in classrooms are enhanced understanding, visualisation and interest in mathematics.

2.3.6 **Limitations of DGS**

As with most of the DGS, *GeoGebra* has certain shortcomings. Ruthven (2003) finds 'approximation' as a shortcoming in DGS, as he contends "while it is possible to specify the degree of rounding with which measures are displayed, this rarely coincides exactly with the screen representation of objects." (p. 11) When calculating the area of a triangle, the lengths of the altitude and sides are displayed on the screen rounded off to two decimals, but the original values are preserved in the system. Consequently, when calculating the area of the triangle the user value varies from that of the system.

DGS offers students an open world in which they can explore, but Balacheff and Kaput (1996), argue that this "does not guarantee that specific learning will occur" (p. 483). Inevitably, the student may focus on constructed objects or representations irrelevant to mathematics (Hoyles & Noss, 2003) and overlook the teacher's intention. Hölzl (2001) shares his experience with learners where he provided a worksheet designed to help construct the altitudes of a triangle and subsequently encouraged " 'dragging' the vertices – 'to see what happens' " (p. 63). It did not transpire to the learners as something interesting that all three altitudes meet at a point for any triangle. Guided discovery learning classroom environments can be a part of the answer, but Hoyles (1995, p. 217 as cited in Balacheff & Kaput, 1996) recommends that “to understand better how learning takes place in such contexts, which requires new research, including new forms of research”.

It can be argued that the software does not provide “symbolic techniques” to solve an equation or how to sketch a parabola. Tall (1993) agrees that the computer environment favours a more generic level of thinking than formal definitions. On the other hand, students acquire conceptual understanding supported by a reflection of the representations, hence the use of software support learning (Tall, 1993).

2.4 MATHEMATICAL PROFICIENCY

Much has been researched and written about the conceptual and procedural knowledge dichotomy (Hiebert & Lefevre, 1986; Van De Walle, 2004). Skemp (1989) makes a distinction between two ways of thinking about mathematical learning, namely “instrumental understanding” and “relational understanding” (p. 2). Skemp argues that instrumental learning mainly involves remembering rules. An example of this is to remember the rule that a negative number times a negative number is a positive number. This, however, does not guarantee deep understanding and engagement with the underlying reasons why the rule is true. In instrumental learning, a learner has a certain set of fixed rules from the question to the required answer and is dependent on outside guidance for learning each new ‘way to get there’. In contrast, relational understanding is characterised by a knowledge base for the procedure that is connected to a rich supply of knowledge about related procedures and concepts. Skemp (1989) expounds that in mathematical learning ‘relational understanding’ would lead to a better transfer and wider application of the procedural knowledge. Van De Walle (2004) succinctly comments that “understanding is a measure of the quality of connections that a new idea has with existing ideas” (p. 25).

However, Silver (1986) argues that these two types of knowledge are inextricably linked. Procedural knowledge must be based on conceptual understanding and procedural fluency is necessary for developing conceptual knowledge. There is a constant interplay between conceptual and procedural knowledge when engaged in tasks that require some level of understanding. His research analysis led him to conclude that the concepts of conceptual and procedural knowledge are intertwined.

Developing a more comprehensive framework for mathematical learning and proficiency, Kilpatrick et al. (2001) concur that students become more proficient when they understand the underlying concepts of mathematics and when they are skilled at computational procedures. Kilpatrick et al. (2001) identify a set of five intertwined strands, called strands of "mathematical proficiency" that are important in developing this proficiency. These strands are not independent, but interwoven, representing different aspects of the whole (p. 16). Due to the scope of this study, only two strands of mathematical proficiency are considered viz. **conceptual understanding** and **procedural fluency**.

2.4.1 Conceptual Understanding (CU)

According to Kilpatrick et al. (2001), **conceptual understanding** is the comprehension of mathematical concepts, operations and relations. It refers to the grasp of fundamental mathematical ideas, much more than the isolated facts and procedures. When mathematical ideas are represented mentally in an organised manner (connected and structured), retrieval of knowledge is easy. Thus, learning with understanding is more powerful than simply memorising, because the organisation improves retention, promotes fluency, and facilitates learning related material (p.118).

The development of conceptual knowledge is achieved by the connections among different pieces of information. Hiebert and Lefevre (1986) stipulate conceptual knowledge "as a connected web of knowledge, a network in which the linking relationships are as prominent as the discrete pieces of information" (p. 4) Conceptual knowledge cannot exist in isolation and therefore exists only if related to other pieces of information. For example, to understand the fact that x^2 can never be negative, is to relate to the existing knowledge that minus times minus is positive. Thus, knowledge grows when isolated facts and propositions are linked to some network. Conceptual understanding is conceived as relating new ideas to an existing knowledge structure. When individual facts and methods are learned with understanding and connected, Kilpatrick et al. suggest that knowledge can be reconstructed even when forgotten.

Skemp (1989) observes that conceptual learning is important for learners because "Understanding extends our powers of adaptation to the new situation" (p. 48). He further argues that conceptual learning develops the learner's confidence in his own ability to deal with any situation. Meaningful learning happens when making use of existing knowledge to arrive at solutions. Learners with conceptual learning also develop different techniques. When the emphasis is on procedures only, it is unlikely to encapsulate the concept (Kilpatrick et al., 2001), as Gray and Tall (1994) observe "to compute the result without necessarily linking input and output in a form that will be remembered as new fact " (p. 124).

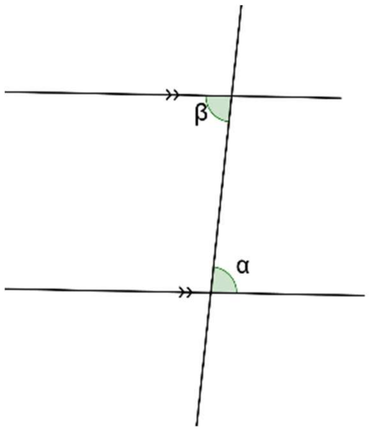
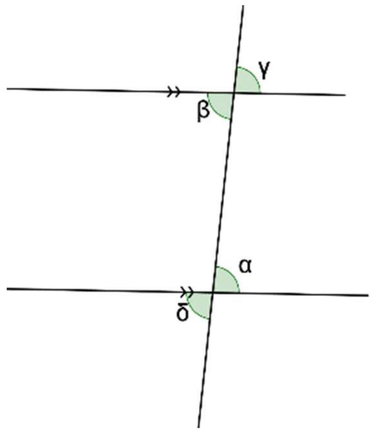
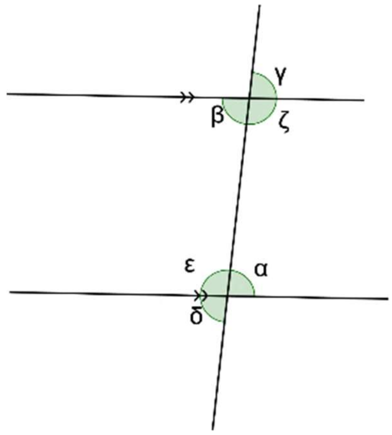
In Euclidean geometry, we find rich connections among concepts and theorems. In the case of parallel lines cut by a transversal line, corresponding angles are equal, alternate angles are equal and co-interior angles are supplementary.[4] However, these angles are again related or linked to other properties – vertically opposite angles are equal; angles on a straight line add up to 180° and the angle around a point is 360°. If we take any one of these angle properties of parallel lines as a basic definition, then the other properties will follow, as shown in the Figures 2.8 to 2.10 in Table 2.2. In Figure 2.8, alternate angles between two parallel lines are given to be equal, $\alpha = \beta$. Following Figure 2.8 and properties of vertically opposite angles, $\alpha = \delta$ and $\beta = \gamma$, and using the transitive property of equality $\alpha = \beta$ and $\beta = \gamma$, we obtain $\alpha = \gamma$, and in a similar fashion, $\beta = \delta$, thus establishing that corresponding angles are also equal (Figure 2.9). In order to identify the co-interior angle property between parallel lines in Figure 2.10), the straight line angle axiom and the alternate angles between two parallel lines are given to be equal theorem from Figure 2.8 is used. It is worth probing further when we focus on a pair of intersecting lines, for instance angles involving β, ζ and γ: for convenience let us assume the value of $\zeta = 95°$, and using the straight-line property, we have: $\beta + 95° = 180°$ and $\gamma + 95° = 180°$. To avoid the algebraic notations this is presented as:

$\square + 95° = 180°$ and $\square + 95° = 180°$

Employing basic arithmetic (again for those who have sound knowledge of the concept of 'sum'), it can be easily seen that there exists only one number that when added to $95°$ yields $180°$. The *vertically opposite angles are equal* based on the known facts of straight line and basic arithmetic equivalence (or using algebraic properties). Tall (2011, p. 4) argues that, "this leads to a relational view of these concepts as the various properties are seen as related to each other" For this purpose, Tall defines *crystalline concepts* with rich interconnections between their properties that "has an internal structure of constrained relationships that causes it to have necessary properties as a consequence of its context" (p. 6). Students who can think flexibly in terms of *concepts* have much more powerful means of relating mathematical ideas than those who are fixated on carrying out complicated procedures without understanding. It is the compression of structural properties of defined formal concepts into crystalline concepts that

[4] When I refer to mathematical theories, properties, axioms and statements, I have demarcated them in italics.

gives gifted mathematicians a simplicity of thought that is beyond the mere proving of theorems (Tall, 2014).

Table 2.2: Rich connections between concepts and theorems		
 *Figure 2.8: Alternate opposite angles*	 *Figure 2.9: Corresponding angles*	 *Figure 2.10: Co-interior angles*
Two parallel lines cut by a line: alternate opposite angles $\alpha = \beta$ (only one pair is shown in the figure for better understanding)	$\alpha = \delta$ and $\beta = \gamma$, from (a) and transitive property of equality we obtain corresponding angles $\alpha = \gamma$ and $\beta = \delta$	$\alpha + \varepsilon = 180°$ and $\beta + \zeta = 180°$; we have (a) $\alpha = \beta$ $\therefore$ co-interior angles $\beta + \varepsilon = 180°$ and $\alpha + \zeta = 180°$

However, Kilpatrick et al. (2001) warn that connections are useful only when linked to related facts and methods in appropriate ways. Extending the above example of parallel lines, the use of mnemonic techniques like, FUN (for further explanation refer to ANNEXURE VIII – Mnemonic of properties of angles formed in parallel lines), aid to perform mathematical operations but do not necessarily lead to conceptual understanding. Such techniques learned by rote do not promote mathematical proficiency.

2.4.2 **Procedural Fluency (PF)**

Procedural fluency refers to knowledge of procedures, knowledge of when and how to use them appropriately, and skills in performing them flexibly, accurately, and efficiently (Kilpatrick et al., 2001, p. 121). It refers not only to the competent execution of procedures, but also to the awareness of various computational tools and the selection of the appropriate tools in a given situation. It is important that computational techniques are efficient and lead to accurate answers. Efficiency and accuracy help students to maintain fluency.

Learners who are fluent in procedures are also able to estimate the result of a procedure. In the calculation of an (annual) effective rate of interest of an investment scheme that offers 9,25% p.a. compounded quarterly, it can be estimated for example that the answer may be just a little above 9,25%. Hence when obtaining a result of 119 39,92%, a proficient learner would revisit the procedure and realise that he/she has overlooked entering the percentage and then obtain the correct answer — 9.58%. But there are many learners who prefer to use a formula to calculate this effective rate of interest — $1 + i_{eff} = \left(1 + \frac{i^m}{m}\right)^m$ where i^m represents the nominal interest rate with m the compounding frequency and i_{eff} represents the effective rate of interest. The effective rate of interest is the actual percentage increase at the end of the year. So, in the above-mentioned situation, we can say that $i^4 = 9.25\%$ p.a. with $m = 4$ and $i_{eff} = 9.58\%$ p.a. This is not a new formula, but in fact a reorganised form of the compounding growth formula $F = P(1 + i)^n$ where F is the future value, P the present value, i the nominal rate of interest per annum and n the period of investment.

Conceptual understanding and procedural fluency are interwoven. Understanding makes learning skills easier, while procedures help strengthen and develop that mathematical understanding. The knowledge of procedures with understanding provides a foundation for generating new knowledge. The findings of Rittle-Johnson and Alibal (1999) suggest that conceptual and procedural knowledge develop iteratively and not independently, "increase in one type of knowledge can lead to gains in the other type of knowledge, which in turn may lead to further increases in the first." (p. 185). Conceptual knowledge influences the generation of procedures in two ways. Firstly, the increase in conceptual understanding helps learners to realise their inconsistent procedures, thus leading them to change and try new methods to solve the problem. Secondly, conceptual knowledge helps learners to identify critical components of a correct procedure thereby preventing them from following incorrect procedures. Next, there are two possibilities for fluency in procedures to influence conceptual knowledge. One possibility is that accuracy and efficiency of procedures help learners to identify the mistake, and this can lead to eliminating possible misconceptions. The other possibility is that the use of newly learned procedures may surprise them and lead them to think of the underlying concepts.

Both strands are critical to developing mathematical proficiency. Gray and Tall (1994, p. 122) prefer to refer to it as "proceptual thinking", an amalgam of conceptual and procedural thinking. This *proceptual thinking* allows the learners to decompose and recompose in different ways, reflecting the variety of procedures that lead to the same solution. Students must have both conceptual and procedural understanding if they wish to understand mathematics in depth. When students persist in memorising procedures rather than knowing how these procedures are derived, it inevitably leads to a cul-de-sac from which there is little hope of future development (Gray & Tall, 1994).

Discussing the issue of what makes a learner proficient in mathematics, Schoenfeld (2007) resolves that there is much more than being able to reproduce standard content. It entails the grasp of flexible properties of a mathematical idea and knowledge of various solution strategies. Learners can therefore identify the appropriate solution strategy in a given situation. Schoenfeld (2007) underscores that "They have many ways to think about problems — alternative approaches if they get stuck, ways of making progress when they hit roadblocks, of being efficient with (and making use of) what they know" (p. 65). Proficient learners are not tied to one computational tool and look for additional methods even after the problem is solved. Moreover, they are able to justify their solution methods and provide rationales for their solving techniques.

Let us look at a practical scenario in a department store where there are offers for peanut crunch, cooking oil and pineapples as shown below. Which is the better buy? (adapted from Kilpatrick et al., 2001)

A: i) 40g of peanut crunch at R4,50; ii) 100g of peanut crunch at R9,00

B: i) 1,4l of oil at R7,90; ii) 1,8l of oil at R8,10

C: i) 3 bags of oranges for R30; ii) 4 bags of oranges for R44

For solving the peanut crunch offer (A) the ratio approach is easier to achieve — the second offer costs twice the first offer but has an excess of 20g hence the second offer is a better buy. While considering savings for oil (B), the larger can (the second offer) costs only 20 cents extra for an excess of 400 ml, hence the larger can saves more. In this case the difference strategy becomes handy. For the oranges in offer (C), it is rather straight-forward to apply the unit-cost method – the first offer is R10 per bag while the second is for R11 per bag. The students with proficiency in procedures would not only come up with several approaches in problem solving but also choose flexibly to suit the demands of the given situation. If the unit-cost strategy is applied for solving the oil offer, it leads to cumbersome calculations. Monitoring of procedures improves the efficiency of problem solving, by virtue of not persisting with unproductive approaches when there is clear evidence of not arriving at the anticipated outcome (Schoenfeld, 2007).

Reflection is another manifestation of proficiency in mathematics (Kilpatrick et al., 2001; Schoenfeld, 2007). Reflecting on the answer, as discussed in the above paragraph, or reflection on progress while engaging in solving problems, is another aspect of mathematical proficiency. Carpenter and Lehrer (1999) also concur that students stand a better chance of developing understanding of mathematical ideas if reflection is a regular feature in teaching and learning. Nevertheless, there is little reflection in instrumental learning where familiar and routine procedures are executed without understanding (Skemp, 1989). Reflection entails the conscious examination of one's own knowledge. It implies that students are consciously aware of the concept that they are learning and the way it is related to existing concepts. Learning does not necessarily always happen with the addition of new concepts and skills, but through the reorganisation of what one already knows (Carpenter & Lehrer, 1999).

2.4.3 **Meaning Making (MM)**

Carter et al (2009) ascertain that mathematical meaning-making is inherent in developing mathematical proficiency. There is no understanding of abstract ideas without making meaning. Carter et al. (2009, p. 13) argue that in the absence of reasoning and sense making, "students carry out procedures correctly but may also capriciously invoke incorrect or baseless rules." As an example, I very often encounter learners, who simplify $\sqrt{a^2 + b^2}$ as a sum $(a + b)$. Here the learners have not developed an understanding rooted in making sense of mathematical procedures but instead consider mathematical procedures as a trick 'the square root of a sum is the sum of square roots.' According to Carter et al. (2009, p. 13), mathematical proficiency can be accomplished "by engaging in mathematical reasoning and meaning-making as they learn mathematical content." Reasoning and sense making are the foundations of mathematical proficiency. These habits have to be integrated in teaching and learning mathematics, and not in isolation, for the learners to understand and apply what they have learnt.

A focus on meaning-making in carrying out mathematical procedures will ensure that the students not only know how to use them but also interpret their results. In the case of elementary multiplication of two decimals, say $3{,}25 \times 0.75$, one of the procedures may be to compute the numerical part and then place the decimal in the result and obtain $24{,}375$. The estimation of the result — that the number has to be less than $3{,}25$ since it is multiplied by a positive number less than 1, may prompt the learner to revisit the problem. In this way the learner would make sense of the results of multiplying different numbers.

In the context of cutting wooden blocks from a twelve -metre wooden log: which would produce more blocks— blocks of one-third of a metre or blocks of two-thirds of a metre? Without making any calculations, the students must be able to display their reasoning ability.

When the blocks are bigger, it is not feasible that the log would produce more blocks than if they were smaller. Kilpatrick et al. (2001) argue that students with a sufficient knowledge base are able to make sense of things. We need procedures to compute the actual number of blocks of one-third or two-thirds that can be made. Learners use these procedures, in this case to invert and multiply for the division of numbers involving fractions (Ma, 2010). When asked to find the number of blocks of two-thirds of a metre which can be cut out of a thirteen metre long wooden log, I have observed that some come up with the answer of 19,5 and I then query that by asking them how the number of blocks can be a decimal number. Then they round off to nearest integer and say the answer is twenty. The learners who understand the question make sense of their answers and come up with the correct answer as nineteen. Only half a metre of wood is available after the nineteenth block has been cut and it is not possible to make a block of two-thirds of a metre from that. Schoenfeld (2007, p. 69) discusses further a seemingly straight-forward arithmetic problem: an army bus holds 36 soldiers. If 1128 soldiers are being bussed to their training site, how many buses are needed? The quantitative analysis of his research shows that although 70% of the students (out of about 45 000) did the computations correctly, only 23% of the students provided the correct final answer of 32 buses. I wonder how the number of buses can be a decimal or a fraction, and what about the remaining twelve soldiers if the number of buses is 31? Here, in contrast to the previous question the number has to be rounded up to the next integer. One of the reasons could be that the students consider mathematical operations as meaningless, hence produce nonsensical answers.

In the case of the following question involving quadratic functions and patterns:

A quadratic pattern: 27, 16, 7, 0, … …. is represented as $f(n) = n^2 - 14n + 40$.

Find the value of:

$$\sum_{n=1}^{7} f(n) - \sum_{n=8}^{13} f(n)$$

The straight-forward response, though cumbersome, would be to substitute the values of n from $n = 1\ to\ n = 13$ and put them into the formula to arrive at the answer 9. But when we closely look at $f(n)$, we understand that it is a quadratic function and its graph takes a parabola shape $f(n) = 0$ when $n = 4\ or\ n = 10\ (or\ x = 4\ and\ x = 10)$. The axis of symmetry (or the turning point of the parabola) is at $n = 7$, (obtained as the midpoint of 4 and 10, $\frac{4+10}{2}$) , which implies that $f(6) = f(8), f(5) = f(9) \ldots f(1) = f(13)$ and thus $f(1) - f(13) = 0$,

$f(2) - f(12) = 0$ and likewise, until we are left with only $f(7)$. Thus, we arrive at the solution $f(7) = -9$. The graphical representation of the pattern is given below, Figure 2.11.

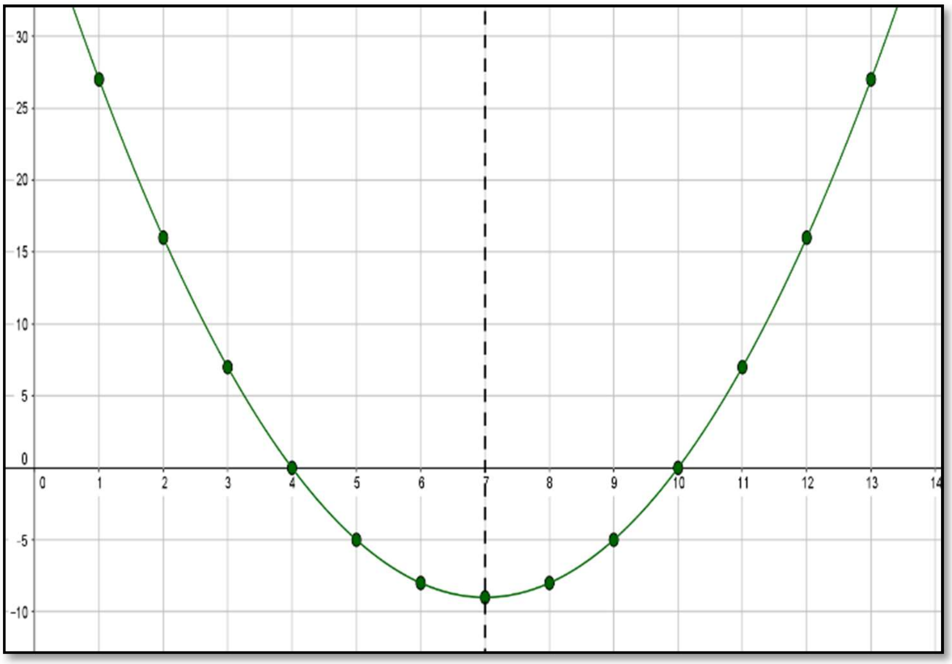


Figure 2.11: Visualising implications of the axis of symmetry

The given number pattern is linked to a quadratic function and its parabola graph, and through applying the properties of transformational geometry we can solve the problem. This interpretation of the solution highlights the relevance of understanding the meaning of concepts — symmetry in this case, and their tacit link to other mathematical ideas. Carter et al. (2009, p. 12) rationalise " Without developing an understanding of procedures rooted in reasoning and sense making, students may be able to correctly perform those procedures but may think of them only as a list of 'tricks'." By using technology such as DGS as an integral part of the mathematics classroom, they observe that "technology allows multiple representations to be linked dynamically, it can provide new opportunities for students to take mathematically meaningful actions."(ibid.)

2.4.4 **Proficiency in learning**

The traditional teaching and learning of mathematics that emphasises the acquisition and mastery of techniques does not enable a student to learn with understanding (Romberg & Kaput, 1999). They argue that the mechanical approach to learning some skills isolates mathematics from its uses and application. The rote deployment of procedures often interferes with students' ingenuity and chance of discovery. After all, as Carpenter and Lehrer (1999) argue, "the ability to extend and apply knowledge is a hallmark of understanding." (p. 23-24). Schoenfeld (2007) concurs that knowledge alone is not sufficient, but "being able to use it in the appropriate circumstances is an essential component of proficiency" (p. 59).

Mathematics is a discipline of logical thinking. In the modern world dominated by advanced technology, it is all-important that young students understand the mathematics that they are doing because in the workplace, as Kilpatrick et al. (2001) observe, "people will be called on more and more to evaluate the relevance and validity of calculations done by calculators and more sophisticated machines. Public policy issues of critical importance hinge on mathematical analyses" (p. 16). Mathematical reasoning is essential for competence in everyday tasks. I would paraphrase Kilpatrick et al. and say, "*All young people should learn to think mathematically, and they should think mathematically to learn*."

Howson (2005) observes that a key aspect of mathematics teaching is to provide meaning to mathematical ideas through real life contextualisation and to develop students' mathematical thinking. When students are exposed to different activities, different meanings are ascribed to mathematical objects. Therefore, as a teacher, a variety of contexts need to be planned so that students' construction of meaning does not become deficient. The teacher may eventually move towards a more abstract view. Howson (2005) argues that as educators we are not only interested in mathematical 'end products' but also in the educational journey and the insights gained en route.

Askew (2016) argues that "knowing and doing mathematics is a blend of proficiencies: fluency, reasoning and problem solving" (p. 37). Mathematical reasoning is the key to helping learners develop deeper understanding. While learning mathematics requires being meaningful, Askew observes that there are certain aspects in which learners are required to be fluent. Reasonable fluency in basic methods frees up working memory, then the door is opened allowing learners to explore a bigger mathematical picture. Significantly, Askew draws up a list of 'elements of fluency', the lack of which hampers learners' development through the years of primary mathematics.

I concur with Haapasalo (2007) and argue that in this study technology-based learning provides a link between conceptual and procedural knowledge. The tools inherent in *GeoGebra* such as sliding and dragging, allow the learner to manipulate mathematical objects, illustrating different forms of mathematical representation. With the aid of multiple perspectives, like algebra, geometry, calculus etc. in *GeoGebra*, multiple forms of representation can be visualised and connected, thus these two types of knowledge develop iteratively. By utilising the dynamic visualisation power of technology, Haapasalo (2007) found with empirical evidence that "a mathematical concept can be built through simultaneous activation of conceptual and procedural knowledge" (p. 2).

2.5 THEORETICAL BACKGROUND - CONSTRUCTIVISM

This study is informed by a constructivist theoretical perspective. Boaler (2009) argues that a traditional approach to teaching is characterised by the 'chalk-and talk method' where the teacher typically explains and demonstrates the mathematical content of a lesson and the students simply watch, consume and practice the problems, mostly in silence. However, Jaworski (1994) argues that this method of direct instruction develops only 'lower level mathematical skills' like computation, but deep conceptual understanding requires 'higher order skills'. The key idea of constructivism is that children construct their own knowledge. Jaworski (1994) asserts that learners do not only absorb ideas presented by their teacher but that they rather create their own knowledge. We ourselves construct knowledge from our own experiences. Thus, each individual's knowledge is unique and contextually dependent. Habibi (2012) argues that substituting traditional teaching methods with active approaches, which is inherent in a constructivist approach, exploits the intuitive and applied activities related to life. This active approach aligns well with teaching with ICT technologies as it requires hands-on teaching and learning strategies, where the teacher and learner are actively engaged in constructing knowledge.

Goldin (1990) states that in a constructivist classroom, learning involves constructive processes. A constructive process is one in which an individual organises and restructures his/her own experience through his/her own construction of knowledge. The constructivist approach recognises that learners have a certain amount of knowledge – they do not come to class as empty vessels to be filled. The transfer of knowledge is therefore not one of filling empty vessels, but one of individual knowledge construction. According to Romberg and Kaput (1999), when mathematics is viewed as a human activity, students make sense of experience arising outside of mathematics instruction and mathematics itself. In an activity-based classroom, exploration of problems is a way of learning mathematics. Understanding is acquired through the creation of knowledge and learners take ownership of their learning. The process of construction demands student responsibility and student autonomy that may lead to a significant improvement in student learning (Confrey, 1990).

2.5.1 Principles of constructivism

The literature on the theory of constructivism differentiates between radical constructivism and social constructivism (Hennessey, Higley, & Chesnut, 2012; Jaworski, 1994).

Based on Piaget's cognitive adaptation, von Glasersfeld (1990) proposes two principles of radical constructivism:

1) Knowledge is not passively received either through the senses or by way of communication. Knowledge is actively built up by the cognising subject.

2) The function of cognition is adaptive, tending towards fit or viability. (p. 22-23).

In simpler terms, the first principle suggests that we all construct our own knowledge (Jaworski, 1994, p. 16; Van De Walle, 2004, p. 22). The learner's new knowledge draws on prior knowledge and experiences. Constructivism is a theory of knowing that attempts to show that knowledge can only be generated by experience. This implies that learners construct their own knowledge by actively participating in a classroom situation. Wright (2000, p. 139) agrees that knowledge is the result of learners' activities rather than of the passive reception of instruction.

The second principle says that an individual learns by adapting. From a constructivist point of view, mathematical concepts that 'fit' or are viable with our real-world experiences are constructed . Giving credit to Piaget, von Glasersfeld (1981, 2000) argues that knowledge is constructed by an individual as an adaptation to one's subjective experience. Knowledge construction and adaptation are results of cognitive structuring, acknowledged by Piaget's genetic epistemology. Knowledge results from individual construction by modification of an experience or idea. This principle emphasises that it is only possible to know the world through experiences.

However, Goldin (1990) disagrees with the above view of constructivism. He argues "The hypothesis that the construction of knowledge by human beings takes place possibly involving several developmental stages - when offered as part of a scientific theory of competence acquisition is *logically independent* of radical constructivist epistemology" [author's italics] (p. 42). The above two principles do not take into account the contextual influences on learning or recognise the individual differences in student understanding. Hardy and Taylor (1997) also argue that tenets of radical constructivism do not offer an adequate explanation for the influence of socio-cultural aspects on learning and the role of language in learning. Identifying social interaction as an important component in the process of learning, von Glasersfeld (1990) observes that

> Every individual's abstraction of experiential items is constrained (and thus guided) by social interaction and the need of collaboration and communication with other members of the group in which he or she grows up. No individual can afford not to establish a relative fit with the consensual domain of the social environment. (p. 26)

Thus, Taylor and Williams (1993) evolve a constructivist related theory of learning based on three principles where the third principle derives from the social perspectives and linguistic influences on learning. They consider "This principle acknowledges the socio-cultural and socioemotional contexts of learning, highlights the central role of language in learning, and identifies the learner as an interactive co-constructor of knowledge" (p. 135). The negotiation of concepts influences the students' mathematical development and the knowledge is constructed intersubjectively. Elaborating on intersubjectivity, Jaworski (1994, p. 211) considers that "Intersubjective or 'taken-as-shared' knowledge can be seen as a product of such interaction where participants seem to agree on certain interpretations represented through discourse and non-verbal communication." Through negotiation and social interaction, individual knowledge may be challenged, and new knowledge constructed. Following on Vygotsky's ideas of the use of tools, Noss and Hoyles (1996) state that meaning in mathematics can be constructed and shared through interaction with computer tools. They argue "The computer is not only a set of tools which defines allowable actions and their expression; it plays a role in communicating the actions, sharing and re-negotiating mathematical expression and facilitating the (co-)construction of mathematical meanings." (p. 228)

I concur with Hardy and Taylor (1997, p. 140) that "it is worth realising that neither social nor the individual components of learning necessarily supersedes the other", and as Jaworski (1994, p. 25) perceives "individual construction within the social environment", the constructivist perspective on learning mathematics envisions "both the teacher and students in an ongoing collaborative endeavour" (Taylor & Williams, 1993, p. 136) that "contribute significantly to individual student's construction of meaning" (Jaworski, 1994, p. 211). As learners interact among themselves and with the computer tools, they are involved in a mathematical experience and may make an important contribution to individual knowledge construction.

2.5.2 **Aligning theory with technological tools**

Sutherland (2007) states that many educational technologies such as *GeoGebra* have been developed for learning mathematics as they enable students to develop 'symbolic tools' for solving mathematical problems. Such tools become 'internalised' to the extent that they can be drawn upon later in other problem solving situations. The process of internalisation is central to constructivism (Marti, 1996, p. 60). Internalisation is the development of knowledge involving simultaneous reorganisation of internal mental spaces and external forms of knowledge.

Hoyles (2005), in the context of technological learning, suggests that mathematical concepts need to be inextricably interwoven with applets. When planning a task for the learner that facilitates construction of mathematical ideas, careful attention is required in designing the applets to include the two points below. Firstly, the tools embedded must be 'just enough' to illuminate structures and relationships while not solving the task completely. Secondly, the tasks should be developed so that they foster students' engagement with mathematical ideas. Drawing on constructivist ideas, Hoyles observes that the student-computer interchange engages learners "in a dialectical relationship of action on the objects and thought" (p. 141). Thus, dynamic software tools can provide insight into mathematical structures and relationships. When learners are active participants in learning using computer tools, they learn better and retain information longer. *GeoGebra* applets offer students an opportunity to do more than just listen to the teacher during class. Hoyles (2005) also observes that virtual manipulatives are mediators and can form a crucial part of internalising knowledge. Mathematical meanings are inextricably interwoven with the computer tools, which enable learners to identify mathematical invariants and make connections. Noss and Hoyles (1996, p. 150) found in their work that "effective learning of conceptually-based material involving the appropriation of mathematical relationships occurred where there was a synergy of interdependence and autonomy through active construction at the computer". Thus, from a constructivist perspective, activities designed in applets should involve students in doing things as well as making them think about what they are doing.

Similarly, Askew (2016) observes that these tools and artefacts are only useful when someone is using them in a meaningful way. Virtual manipulatives and images that are lively and enticing can help children engage with the mathematics. Nevertheless, Askew warns that too much variation in the images may mask some critical mathematical aspects. Therefore, computer tools serve a purpose and meaning when they are used in conjunction with activities.

As learners use the tools jointly with their teachers, they begin to solve problems in their own way and devise their own solution strategies. Acknowledging Vygotsky's theories about tool use, Askew (2016) argues that tools and artefacts mediate "through which the learning is assumed to be enabled" (p. 19).

In a technology-based constructivist classroom, there is less emphasis on transmitting information, but more on developing students' understanding of concepts and their skills. Recent research of Jaworski et al. (2015), confirms that DGS can engage students in deeper understanding of mathematical meanings. Here, students provide mixed responses to *GeoGebra*, however. One group of students for example asserted that in-depth mathematical meaning is unnecessary as 'just because I understand maths better doesn't mean I'll pass the exam'. Another group of students reflected that using *GeoGebra* provided a dynamic visual representation helping them to 'spot patterns and trends' that would have been otherwise missed. Promoting active learning promotes mastery of content with reasoning and understanding Jaworski et al. (2015). In my study, the learners are provided with selected applets created through *GeoGebra* and are encouraged to engage with them to 'construct' their own meaning and verify their own acquired knowledge. When learners are given some control over the tasks they engage in and are allowed to reflect on their activity, Carpenter and Lehrer (1999) ascertain that learners develop a sense of ownership and have control over their learning.

Dynamic mathematics software is a powerful teaching and learning medium and appears to create opportunities for creative thinking (Stols & Kriek, 2011). Stols (2007) further argues that the new technologies can help learners to visualise difficult-to-understand concepts and help teachers to create an active learning environment. In such environments, students do more than just listen; they are encouraged to explore. When such technology is used in the classroom, the focus is on developing higher order thinking skills where students are actively engaged in analysing and synthesising their own knowledge.

The students' engagement in learning involves cognitive tasks and manipulation of conceptual ideas. Jonassen and Strobel (2006) argue that ICT can and should become the toolkit for meaning-making and meaningful learning. Meaningful learning implies that learners are actively manipulating objects and tools, and during these activities they construct their own meanings and interpretations of their actions. Technology tools like DGS can provide opportunities for learners to wrestle with objects — applets in this case. These opportunities not only develop understanding, but they also facilitate the adaptation of prior knowledge and experiences to new knowledge and meaning. When technological tools are used effectively, students can authenticate their understandings. Thus, *GeoGebra* as used in this study with its inherent exploratory possibilities, may provide opportunities for a heuristic approach to constructing mathematical meaning. The emerging technological tools, like DGS, allow students to engage in active learning processes in which they themselves build their own meanings and develop understanding (Khine, 2003). Hence technology-enabled active learning aligns with the principles of constructivism as mentioned above.

2.5.3 **Aligning theory with visualisation**

Arcavi (2003) sees visualisation as a process of constructing knowledge that promotes understanding. Therefore, visualisation 'as a process' provokes intellectual activity and construction of knowledge. Presmeg (2006) stresses the need for visual representations in mathematics education. She considers multiple representations of mathematical concepts as an integral part of a visualisation strategy of teaching and learning mathematics. In their research study on allowing learners to generate their own representations, Watson and Mason (2005) emphasise that learners' active engagement in constructing their own examples enables them to probe and generalise mathematical ideas more deeply than providing facts. In a constructivist classroom, abstract ideas are built upon students' representations that involve examples and counterexamples. The construction of multiple examples by learners themselves, as Watson and Mason argue, not only facilitates conceptual learning but also enables learners to make links among concepts.

However, Duval (2013) asserts that as understanding involves grasping the whole structure, there is no understanding without visualisation. While a single activity focusses on one or some units and properties of a mathematical concept, visualisation leads to "grasping directly the whole configuration of relation and in discriminating what is relevant in it" (Duval, 1999, p. 13). For example, when an activity is confined to computation only, the learner may not look at the whole mathematical structure, but only at one aspect thereof. It is therefore important to provide learning opportunities for learners to make connections and experience the whole mathematical structure of a particular problem. As an example, Duval argues that the instruction to draw the graphs $y = 2x + 2$ and $y = x + 2$ by computing (using a table) may not lead learners to understand the representation as a whole, because the two graphs only differ from each other by virtue of their gradients. The use of visualisation enables learners to discriminate between the equations conceptually.

The scholarly works of Arcavi (2003), Presmeg (1986b, 2014), Duval (1999, 2013) and other researchers have provided compelling empirical evidence of the important role of visualisation in developing understanding of mathematical concepts and knowledge. Their research found strong evidence that learners do employ visual strategies (though different) to construct meaningful conceptual ideas. In particular, Presmeg (1985), commending the use of visual imagery, claims that the embodiment of abstract ideas in a concrete image can be effective in learning mathematics, i.e. the use of visualisation can facilitate the learning of mathematical content with understanding. The applets designed in GLIP engage learners to explore and discover mathematical ideas and subtleties. Thus, technological tools like *GeoGebra* can provide visual learning experiences and allow learners to take control of their own learning (Bransford, Brown, & Cocking, 2000, p. 216). This aligns well with the principles of constructivism.

2.6 **CONCLUSION OF THE CHAPTER**

In this chapter, I have provided the conceptual and theoretical framework that underpin the study. Firstly, several relevant issues related to visualisation within the context of learning of mathematics were discussed. Secondly, the pros and cons of using computer tools, and DGS in particular as visualisation and learning tools were discussed. This led to the discussion of mathematical proficiency focusing on Kilpatrick et al.'s (2001) model that informs the study. Finally, I discussed common perspectives and features of the constructivist theory of learning and also highlighted its potential relationships with the other elements that inform the study. Key issues established in this chapter will be employed to gather and analyse the data, which will be discussed in the following chapter on methodology.

CHAPTER 3

METHODOLOGY

3.1 INTRODUCTION

The study foregrounded the harnessing of visualisation opportunities of DGS packages by learners to develop conceptual and procedural understanding in mathematics. My research aim was specifically to explore how learning takes place in relation to selected learners' interactions with *GeoGebra,* with a special focus on making meaning and growing mathematical proficiency. The two specific research questions that framed this study were:

- How are *GeoGebra* applets used as a learning tool to make mathematical meaning and develop understanding by selected Grade 11 learners?
- What visualisation role can *GeoGebra* play in the learning of Grade 11 mathematics?

This chapter presents the orientation of the research and the methods used. A description of the analytical framework based on Kilpatrick et al.'s (2001) mathematical proficiency is presented. The sampling techniques as well as the participant selection procedures are articulated, along with an account of procedures for the collection and analysis of the data. Pertinent strategies to enhance the validity and reliability of the research study are also discussed. Ethical considerations that pertain to this study are elaborated upon as well as some of the challenges and tensions that arose during the data collection process.

3.2 RESEARCH ORIENTATION

The study is situated within an interpretive research paradigm and oriented within the qualitative research framework. The interpretative research paradigm that underpinned this study coheres with the nature of the research enquiry. Cohen et al. (2007, p. 21) assert that "to understand the subjective world of human experience" is fundamental in the context of an interpretative paradigm. This research study involved describing and making sense of learners' interactions with the DGS as they constructed meaning of mathematical ideas in selected mathematics lessons. The intention of this interpretative research study was to understand the mathematical learning experiences of selected Grade 11 learners when engaged in mathematical tasks in their classrooms, using technological tools, in this case *GeoGebra* applets.

Merriam (2009) suggests that in qualitative research the emphasis is on "understanding how people interpret their experiences, how they construct their worlds, and what meaning they attribute to their experiences" (p. 5). The study focused on learners' mathematical experiences. Their classroom interactions with the *GeoGebra* applets were therefore observed and analysed to understand how they made sense of mathematical concepts. The study was primarily undertaken to investigate and explain how learners' sense making and the possible processes that they employed enhanced their conceptual understanding and procedural fluency. Their engagement with the mathematical content through their interactions with applets was analysed in detail against the analytical tool VAMMPA, which is discussed in Section 3.5.2 on p.60.

Furthermore, Denzin and Lincoln (2005) assert that qualitative research involves interpretive approaches in natural conditions, attempting to gain a better understanding of the participants (selected learners in this case) and their actions. The preferred paradigm allowed me to interpret and understand participants' experiences in the context of integrating *GeoGebra* in teaching and learning in actual classrooms. Through observations and interactions with the selected learners, I aimed to comprehend how they interpreted or made conjectures of mathematical ideas when engaging with *GeoGebra*. For given mathematical tasks, I probed and researched the meanings they made. Stake (2010, p. 63) observes that qualitative research is about how things happen and "happenings are experiences, and the researcher needs to probe the assertions until the experience is credible."

3.3 RESEARCH METHODOLOGY - CASE STUDY

This study involved an intensive and long-term period of interaction with learners from two different schools where the GLIP project was running. The data collection occurred at different points over a period of one year from September 2016 to August 2017. I adopted a qualitative case study approach in this research. Yin (2009) ascertains that the case study method allows investigators to retain the holistic and meaningful characteristics of real life events. A case study approach afforded me to gather rich and holistic information from the participants' perspectives in a natural classroom environment where technological tools were used. The distinct need for a case study arose as the implementation of DGS in classrooms is a relatively new phenomenon and requires rigorous and empirical reflection in order to understand the complexities of learning with these technological resources. Newby (2010, p. 51) considers a case study as a "detailed analysis of individual circumstance or event" that is chosen because something new is in operation. Contextually, as the DGS was introduced in selected classrooms, this study aimed to make sense of possible processes that learners employed to understand mathematical objects.

The study resonated well with several elements typical of a case study approach (Newby, 2010; Yin, 2009) as explained below:

Real life context and natural situations: Primarily, the data was from classroom-based observations. Teachers and learners employed the DGS as a classroom resource. The research aimed to explore the learning context as understood by the participants in their own environment.

Participant perspective: The focus on the participants' (selected learners) perspectives of mathematics meanings attached to their experiences was emphasised. This research explored the classroom learning processes from the perspectives of the participants, that demonstrated characteristics of mathematical proficiency.

Multiple data sources: A variety of sources of data is often considered more compelling in a case study approach. Consequently, I collected data from different sources, which in turn facilitated to enhance the validity of my research through triangulation.

Richness of data and description: One of the strengths of the case study approach is that it allows one to report on the findings of the study through detailed descriptions of the context, participants and events. I sought to provide a narrative analysis of classroom events, unpacking and understanding actual learning experiences. However, I have drawn inferences from a wider perspective that would possibly assist the reader in experiencing the classroom events and make an informed judgement about the findings of the study.

3.3.1 **The unit of analysis**

The units of analysis in a case study are usually the units of observation (Yin, 2009). Hence, the units of analysis need to be clearly determined so as to set boundaries for the research project. This case study involved intensive, long-term interaction with selected learners, carefully recording the events in their classroom and to "catch the unique features" (Cohen et al., 2007, p. 255), if any, of their interactions with *GeoGebra* applets. The case here is thus a cohort of Grade 11 learners.

The unit of analysis is the individual learner's interaction with *GeoGebra*. Yin (2009) states that the unit of analysis is where the researcher obtains the data for the case study.

3.3.2 **Research site**

The GLIP project (as described on page 54) operated in two schools, namely School A and School B, in the Mthatha region. The learners I selected for my research project were from these two schools. The empirical field of the research is thus the GLIP programme. The two schools are equipped with functional computer laboratories.

Once the Higher Degrees Committee of my university approved my proposal for the research study, I applied to the Eastern Cape DBE to conduct research in the two selected secondary schools in the Mthatha region. The two schools were participants of the GLIP programme which was coordinated by my fellow researcher and myself. The GLIP programme was already operational by the time my application to conduct the research in the two schools was considered. The principals of these schools granted my fellow researcher and I permission to conduct training and workshops with the teachers and the learners, and to conduct research on using technology in their schools for teaching and learning purposes. The principal of School B commented "*We are glad that these computers are now used for learning purposes.*" However, the data collection commenced only after approval was formally accorded by the DBE.

School A is an independent mixed gender school, comprising of about 1350 leaners in 2017. The school offers classes from Grade 8 to 12, with a 'first come, first served' admission policy. The school is relatively well resourced and well maintained. The school has three functional computer laboratories with interactive whiteboard facilities and projectors. The regular maintenance of these computers — about 250 — is outsourced. However, there are glitches in the efficacious use of interactive whiteboards. The school offers two computer- related courses, namely CAT (Computer Applications and Technology) and IT (Information Technology), and the computer laboratories are used mainly for these courses. Apart from the designated subjects, the school also provides computer literacy lessons for an hour a week for all learners from Grade 8 to 10. In 2017 there were twelve mathematics teachers in the school, six of whom were teaching Grade 11. The training of teachers in the school on using *GeoGebra* was started in November 2015 and was extended to March 2016. Grade 10 and Grade 11 learners were trained during the second half of 2016.

School B is a public secondary school. During 2017, the school population was about 1200 learners of mixed gender. The school offers classes from Grade 8 to 12. It has one computer laboratory with eighty computers and twenty laptops. It is also equipped with a data projector and a makeshift white screen and a chalkboard. The school offered only one computer-related course, namely CAT. The training of teachers in this school in using *GeoGebra* took place in August 2016. Only Grade 10 learners were trained during the September and October 2016 period.

The classes of two teachers, Antony and George, from school A, and the classes of two teachers, Paul and John, from school B volunteered to participate in my research project.

3.3.3 The participants and selection criteria

My participants were the learners of the four GLIP teachers mentioned above. The focus of my research was on how these learners interacted with selected *GeoGebra* applets that the four teachers collaboratively developed in the GLIP and then individually implemented in their classes. In parallel, the four teachers were participants in my fellow researcher's study whose specific focus was on their teaching practice. I selected three participating learners from each of the four classes that were taught by the four GLIP teachers. These twelve learners were purposefully selected. Patton (2002, p. 47) argues that the "logic and power of purposeful sampling derives from the emphasis on in-depth understanding." These learners were selected on the basis of their good participation during their GLIP training and in consultation with their mathematics teachers. In collaboration with their teachers I selected learners who were able to express themselves well, particularly on their own thinking and understanding. The identification of appropriate learners enabled me to gather a great deal of very specific information about their interactions with the applets. Sampling in my case was thus aimed at gaining rich insight about a specific phenomenon (learning with applets) and not for empirical generalisations from a sample of a large population (Patton, 2002). These learners were also willing to share their learning experiences using *GeoGebra.*

3.4 RESEARCH DESIGN

It was important for me to establish a good and mutually respectful rapport with the learners prior to my data collecting process. This was achieved during the learner training sessions in which I interacted with the learners in and outside the classrooms. This enabled me to generate familiarity with the learners. The research took place in the classrooms when the GLIP teachers implemented their applets on various topics. In this thesis, I report on three topics in Grade 11 that the teachers used with *GeoGebra*. Thus, the data collection schedule was determined by the GLIP cycles, which is discussed in the following section.

3.4.1 **The GLIP cycles**

The GLIP is a teacher development project designed specifically for interested teachers to use *GeoGebra*, a DGS, as a teaching and learning tool of mathematics. The entire GLIP project plan can be accessed on the Rhodes University website https://www.ru.ac.za/mathsedchair/community development/. The project was launched in November 2015 with the participation of twelve mathematics teachers from School A. The project grew as teachers from other schools joined in, while a few teachers from School A left the project after the initial training phase.

There were two phases in the GLIP programme. The first phase comprised the training of teachers and learners. The teacher training in GLIP consisted of introductory training in *GeoGebra* and helping the participating teachers to familiarise themselves with the software. During this phase, teachers were trained to work and explore with *GeoGebra* and to think about how they could use *GeoGebra* to realise its potential in the mathematics classroom. This training enabled the teachers to generate their own *GeoGebra* applets (applets are short programs discussed in Chapter 2). Once the teachers were trained, the learner training commenced. The learner training sessions were conducted along with their mathematics teachers. This training helped learners to familiarise themselves with the software. Learners were encouraged to explore and play with different options available in *GeoGebra*. The materials for the first phase of this workshop were adapted from those of Hohenwarter (2009). They were downloaded from the website on the 26th August 2015 with his consent to adapt and use them for our training purposes.

In the second phase of GLIP, the teachers started using *GeoGebra* applets in their mathematics classrooms. The research is located in this second phase of GLIP. Figure 3.1 illustrates two of the cycles of the GLIP project. It also shows where my research project is located within each cycle. The empirical field of the research coincides with three of these GLIP cycles in Grade 11. The GLIP teachers collaboratively developed applets after a brainstorming session. The applets were directly aligned with their teaching plan and relevant to the curriculum.

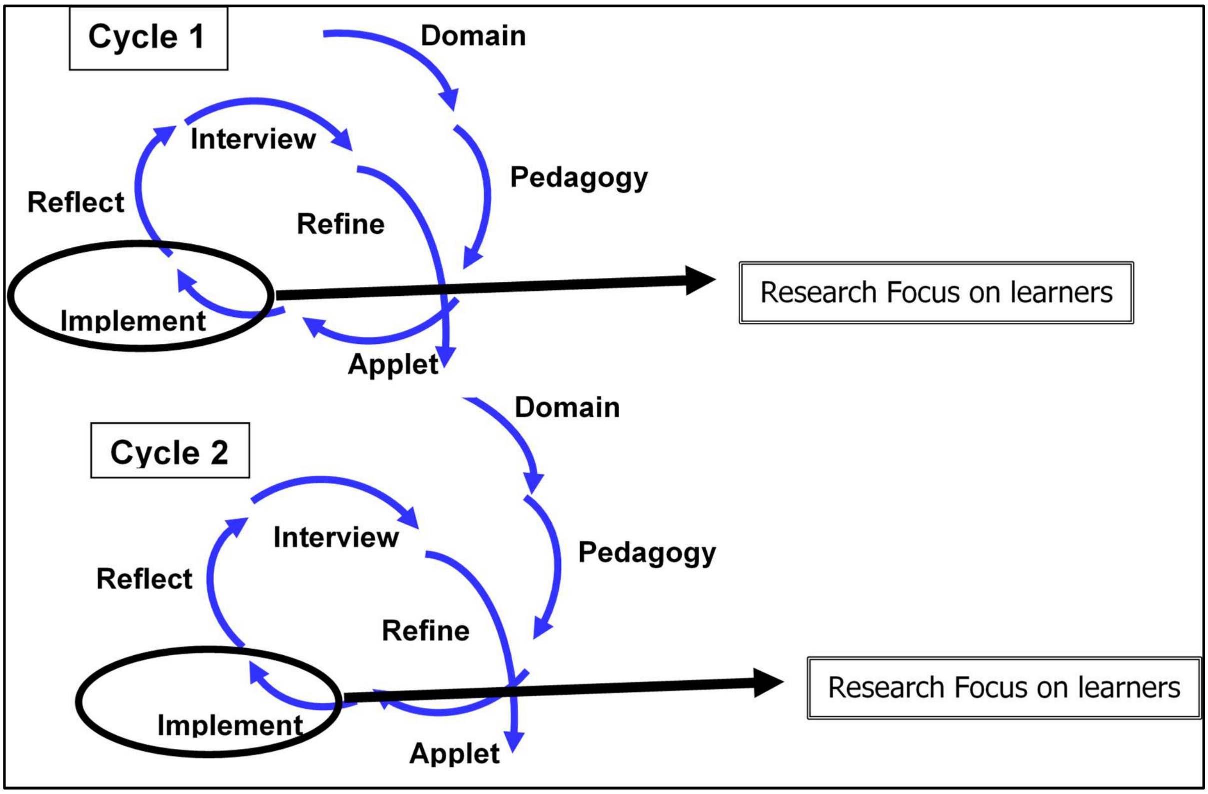


Figure 3.1: The empirical field of research coinciding with GLIP cycles

3.4.2 **The pilot study**

After the first phase of the GLIP workshop with the teachers and learners, the feedback GLIP received was very encouraging and positive. The teachers wished to use *GeoGebra* as a teaching and learning resource in their mathematics classes more frequently and rigorously. Two teachers from School A used the pre-designed applets in their Grade 9 classes to teach the area of a triangle. At the time, the analytical framework was not yet fully developed. The trial runs of using *GeoGebra* guided me to identify the responses and actions of learners when they used the technological tools. These responses are further discussed in Section 3.5.2 on p. 60, where I deal with the full development of the analytical framework.

Initially, the screen recording software posed a challenge in the data collection procedure. After many trials, I finally succeeded in engaging a hassle-free application. Furthermore, after the approval of the framework by my university, I piloted a lesson by George in Grade 10 where the screen recording also worked flawlessly. It helped me to refine the analytical framework and sharpened my data gathering and analysis protocol.

3.4.3 **Research instruments**

My field work started once the four participant teachers agreed to participate in my research project. I engaged with their learners during the *GeoGebra* training sessions. The data collection schedule, however, was determined by the GLIP cycles. Multiple techniques were used to gather data from the participants. As the teachers implemented their applets in the three cycles, I gathered data in two stages from the participating learners. Firstly, I observed them as they interacted with the applets, and secondly, I interviewed each participant after each cycle.

3.4.3.1 **First stage – Classroom Setting**

The first stage of data collection consisted of two methods: a) Classroom Observation (Data A); and b) Screen Capture (Data B).

3.4.3.1.1 *Classroom observation (Data A)*

The research aimed to investigate the learning processes and mathematical meaning-making as understood and constructed by the participants in a classroom setting, while engaging with the *GeoGebra* applets. I used two audio-video recording devices. One was used to capture the entire classroom. This helped me to obtain an overview of the classroom, and to capture the movements of the participants, if any. During the pilot study, I observed learners moving around enthusiastically and sharing their discoveries. Another camera focussed on the participating learners only. I negotiated with the teacher to allow them to sit adjacent to each other. This made for easier observations of my participants and also helped me to obtain data on their actions, such as what they were doing when the computer system was idle, and how and with whom they were interacting during the lesson. It also captured their comments when they interacted with their peers. The specific focus of the video recordings was to capture how the learners responded to the tasks that the teachers gave them and their subsequent interaction with the applets. My role was thus one of a non-participant observer (Cohen et al., 2007).

3.4.3.1.2 *Screen capture (Data B)*

Screen recording software captures everything that a user does on a computer screen, including using media files and any other activities on the desktop such as the use of an audio-video file, social media, or e-learning courses. It is commonly used in webinars, games, etc. Pedagogically, it is commonly used to create tutorials that extend classroom lectures and online courses. Completed videos may be uploaded to YouTube or hosted on a webpage to demonstrate step by step processes. It is a relatively new method of data gathering in the field of education. I found little evidence in educational research done in South Africa that employed this technology of screen-capturing computer applications.

I trialled several video screen recording software programs, particularly the free or open-source versions such as *Cam Studio* software, but they proved to be rather disappointing and limited for my purposes, and did not record the remarks of the user. They also had a recording time limit of 30 minutes. This was problematic for me as I wished to record episodes that lasted in excess of 30 minutes. I also found the software support not very helpful and effective. Thereafter, I attempted some trial versions of software that were on the market. Some of them slowed down the computer processes and thus affected the learners' interactions. They also consumed high memory of the computer. As a result of my pilot study and personal use of several applications, I finally decided on using a paid software called *Ice-cream software* (https://icecreamapps.com). It had hassle-free installation and was user-friendly. It was compatible with the computer systems in both the schools and used little computer memory. It is an intuitive and easy-to-use software that offers a complete suite of tools and options for screen capture with audio facilities.

All the participants were informed about the screen capture of their performance on the computer. I asked for their consent to record all their workings for the purpose of analysis. I briefed all the participants on how the screen recording software worked. I also encouraged them to converse in English wherever possible. The selected learners were fluent in English according to their teachers and my interactions with them prior to their selection. Furthermore, I also introduced them to the *Casio* calculator simulator software and motivated them to use it for their numerical calculations instead of a physical calculator.

Thus, the screen-capturing software enabled the recordings of all their computer screen activities. The purpose here was to gain insight and interpret learners' understandings as they *constructed* mathematical ideas.

3.4.3.2 **Second stage - Stimulated Recall Reflective Interview (Data C)**

After reviewing and analysing the video recorded lessons and the screen captured data, I began my second phase of data collection. I conducted personal interviews with each of the selected learners a few days after the observed lesson. In practice, this was done within seven to eight days after the observed lesson. There were instances when I had to approach them again to seek clarification of their ideas.

I watched the video recordings of the lessons and I jotted down those sections which I thought needed clarification from them. This process in fact already constituted some form of preliminary analysis, as Merriam (2009, p. 131) observes "[T]he joint collection and analysis of data is essential in qualitative research." I prompted the participant learners to reflect on and recollect the strategies used for solving the tasks that they attempted in class. For this I used the 'stimulated recall' technique by reviewing the individual videos with each particular learner in order to facilitate a reflective process (O'Brien, 1993). My specific focus in these interviews was to reflect on how the visualisation processes facilitated constructive engagement with the applets and whether these contributed to making sense in developing mathematical concepts.

Merriam (2009) argues that obtaining good data in an interview is dependent on asking well-chosen questions that can be followed up with probes and requests for more detail. Therefore, during these interviews, I used questions such as those listed in Table 3.1. Further questions were identified for each topic. The objective of the reflective interviews was to gather complementary information from Data A and Data B on the learners' engagement with the dynamic software and their understandings of mathematical concepts. It enabled me to elicit learner perceptions as well as reasons for their methods and actions while engaging with the applets. These individual interviews were conducted after school hours, after obtaining consent from their parents. They each took 45 — 60 minutes. Essentially, those actions and ideas that were used by the learners to explore their assumptions and meanings were recorded for evidence and analysis.

Table 3.1: Examples of my semi-structured questions

Questions	**Objective**
What do we mean when we say 'function'?	Meaning-making and Conceptual Understanding
How did you interact with the applets during the class?	Visualisation Role of *GeoGebra*
Why did you do use this method? Why did this work?	Meaning and Understanding
Can you guess what will be …?	Understanding, Procedural Knowledge and Visualisation
Are you able to relate this problem or solution strategy to any previous learning or any other topic?	Meaning and Understanding
Can you do this in a different way?	Procedural Knowledge and Visualisation

3.4.3.3 **Learning artefacts (Data D)**

Learning in a technological classroom is not necessarily confined only to interactions between learner and computer. Written artefacts in the form of worksheets were also used by the teachers during the lessons to develop mathematical proficiency. The written responses of the learners were collected, photocopied for analysis and returned to them. During the reflective interviews, participants used paper and pencil to expand on their ideas. Such physical objects, emanating from both the stages, provided additional information for analysing their mathematical proficiency and meaning making of concepts. Figure 3.2 is a summary of my data types collected in the research.

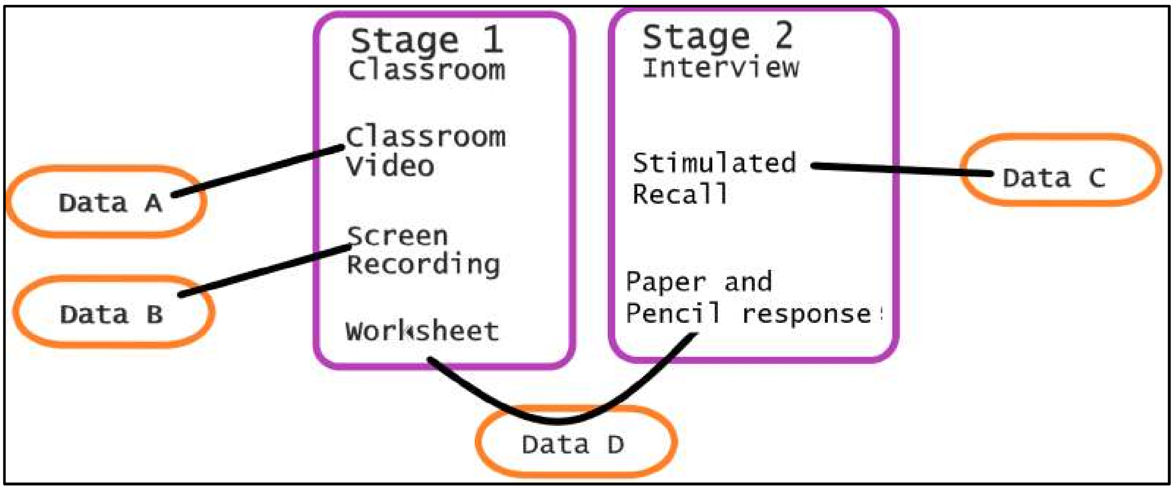


Figure 3.2: Summary of my data types

3.5 DATA ANALYSIS

Newby (2010) considers data analysis as an exercise to "get our data to release the information we need to answer our research question" (p. 395). It is a process through which we make sense of data and communicate the essence of what it reveals. This process involves organising and explaining the data in terms of the participants' perspective, accounting for categories, noting patterns and regularities and themes (Cohen et al., 2007).

Newby (2010) recommends a generic protocol of qualitative analysis. This process has four stages: 1) preparing the data, 2) identifying basic units of data, 3) organising data and 4) interpreting data. I adopted the above process in this research. The following four sections expand on this protocol.

3.5.1 **Preparing the data**

All forms of data that were collected in multimedia format, i.e. Data A (Classroom observation), Data B (Screen recording) and Data C (Stimulated Recall) were transcribed into text form and picture form. I personally transcribed all the data which allowed me to engage with the data as early as possible. Newby (2010) argues that engagement with the data leads to insight into data structure and meaning. I first transcribed the screen capture of participants (Data B). I extracted the data from the screen recording videos in the form of screenshots (pictures) and embedded it into relevant sections of the transcription. Additionally, I embedded the transcribed classroom videos (Data A) into relevant sections of the transcription of Data B. This enabled me to obtain a comprehensive understanding of the participants' interactions in the lessons, including other aspects such as moving around the classroom, interacting with peers, referring to the textbooks, and so on. Thus, I was able to generate "units of meaning" (Cohen et al., 2007, p. 368) of all their actions and responses during the observed lessons. Stimulated recall (Data C) was also transcribed and appended to the above transcriptions.

3.5.2 **Analytical framework**

I identified the following basic units or codes of meaning concerning the research: meaning-making, conceptual understanding, procedural fluency and visualisation processes. Essentially, this meant that I classified the sampled learners' observed actions and responses into the units or codes described above. For the purpose of analysing data in conjunction with the literature, I developed a coding structure, namely the Visualisation Applets Meaning Making Mathematical Proficiency Analysis (VAMMPA) framework illustrated in Table 3.2. This enabled me to reorganise my data and extract information to address the research objectives and answer my research questions.

Based on literature on meaning-making habits propounded by Carter et al. (2009) and the characteristics of mathematical proficiency proposed by Kilpatrick et al. (2001), I identified seven broad indicators, BI1 to BI7, of Meaning Making (MM), Conceptual Understanding (CU) and Procedural Fluency (PF). For each of the broad indicators, from my experiences I then identified different observable or traceable learning activities. The VAMMPA framework in Table 3.2 was used to analyse the use of *GeoGebra* in the learners' behaviours.

The broad indicators BI1 to BI3 indicate the attainment of MM and CU, BI4 and BI5 indicate MM and PF, and BI6 and BI7 indicate the attainment of all the three, MM, CU and PF. For example, within the broad indicator BI7, 'reflecting on solution – considering the reasonableness of a solution', I identified the following learner responses and activities: 1) estimate the results of computation, 2) verify the final answer and 3) carry out the procedure by a different method. Similarly, I identified eighteen such observable activities and categorised them under seven broad indicators, BI1 to BI7. The VAMMPA was the guiding instrument for analysing my data to answer my research questions. The developed coding system was gradually refined during the analysis of the pilot study data.

Thus, the theoretical framework of meaning-making and mathematical proficiency guided my case study analysis. Yin (2009) observes that the theoretical orientation in a case study helps to focus attention on certain data. Furthermore, I took care in the data presentation to use the codes and indicators.

Table 3.2: Visualisation Applets Meaning Making Mathematical Proficiency Analysis (VAMMPA)

Broad Indicators	**Learners' responses or activities and their corresponding code (in colour)**
BI-1: Defining and identifying relevant mathematical concepts, relations and notations. MM and CU (yellow)	Explains or demonstrates concepts- 1EC
	Able to represent mathematical ideas in multiple ways- 1MR
	Provides examples- 1EX
BI-2: Patterns and relationships. MM and CU (green)	Identifies relationships among concepts and makes connections - 2MC
	Understands variant and invariant properties- 2UVI
BI-3: Looking for hidden structures. MM and CU (pink)	Explores and discovers concepts- 3DC
	Makes and verifies conjectures based on explorations - 3CNJ
	Able to think in a curtailed manner - 3CRT
BI-4: Purposeful use of procedures. MM and PF (red)	Shows appropriate, efficient and accurate execution of mathematical procedures and algorithms- 4AP
	Able to explain the procedures- 4EP
	Able to select appropriate tool for a given problem - 4TOL
BI-5: Different approaches to solving a problem. MM and PF (grey)	Carries out the procedure in a different way-5PMW
	Able to reverse a mental process (reverse train of thought) (Krutetskii, 1976, p. 88)- 5REV
BI-6: Applying previously learnt concepts. MM, CU and PF (blue)	Able to link to prior learning- 6LPK
	Employs basic procedures and formulae accurately and efficiently- 6BP
BI-7: Reflecting on a solution – considering the reasonableness of a solution. MM, CU and PF (turquoise)	Estimates the results of computation- 7EST
	Verifies the final answer - 7ANS
	Knowledge of multiple ways to estimate the solution- 7KMW

Since the analytical framework VAMMPA in Table 3.2 is central to this study, the various learner activities codes are discussed briefly hereunder.

- **Explains or demonstrates concepts** (1EC)

  A learner explains or demonstrates knowledge of mathematical concepts, relations or notations and uses mathematically appropriate language in defining or identifying relevant mathematical concepts. For example, identifying the $'y$-intercept of $y = 2x + 5$ without calculation or defining a theorem in geometry.

- **Able to represent mathematical ideas in multiple ways** (1MR)

  A learner uses a multiplicity of representations such as symbolic numbers and/or a number line. This also includes multiple ways of communicating a mathematical idea. Furthermore, technology allows for multiple representations of mathematical concepts such as visual objects.

- **Provides examples** (1EX)

  A learner finds analogical correspondence to a mathematical concept such as comparing the angle at the centre to an analogue clock. Carter et al. (2009) consider seeing "special cases or simpler analogs" as a meaning-making habit.

- **Identifies relationships among concepts and makes connections** (2MC)

  A learner makes conceptual links to other areas of mathematics or considers connections between different concepts. When a learner makes connections between mathematical topics he/she employs visualisation (Presmeg, 2006). For example, identifying an isosceles triangle when a chord subtends an angle at the centre of a circle.

- **Understands variant and invariant properties** (2UVI)

  Mathematics is all about making generalisations of invariance. Kaput (1992) observes that "to recognise invariance to see what stays the same one must have variation" (p. 525). When a learner understands the variant and invariant of a mathematical concept, there is a progress in meaning-making and reasoning habits (Carter et al., 2009).

- **Explores and discovers concepts** (3DC)

  While using DGS, a learner moves points, figures or graphs around the computer screen and thus discovers inherent characteristics of mathematical concepts, for example, changing the parameters of straight lines to discover the gradient of a line.

- **Makes and verifies conjectures based on explorations** (3CNJ)

  Making conjectures is a fundamental reasoning habit in mathematics (Carter et al., 2009). A learner wonders 'what might be happening' through systematically examining several cases. More importantly, he seeks to understand 'why it is happening'. However, while using DGS, it is relatively easy to verify the conjectures e.g. as in the case of the *angle at the centre theorem*, learners conjecturing the relationship between the *angle subtended at the centre* and the angle subtended inside, outside or on the circle by the same arc.

- **Able to think in curtailed manner** (3CRT)

  Krutetskii (1976) argues that able learners are distinguished by a tendency to curtail mathematical operations. As a matter of fact, this shortening of thought processes is not procedural. Instead Krutetskii considers it as a single act of "an immediate judgement of the relations given in the material" (p. 231). For example, the third angle of a right-angled triangle is determined by using $90° - \alpha$, clearly dropping off the links to the sum of the angles of a triangle, i.e. $180° - 90° - \alpha$. Similarly, the expansion of $(a+b)^2$ is directly achieved as $a^2 + 2ab + b^2$ without resorting to the product of $(a+b)(a+b)$.

- **Shows appropriate, efficient and accurate execution of mathematical procedures and algorithms** (4AP)

  The skill in executing mathematical procedures efficiently and accurately is essentially needed to support conceptual understanding. The procedural knowledge of the distributive law $a \times (b+c)$ is essential in developing the algebraic identity of $(a+b)^2$ as discussed in the above example.

- **Able to explain the procedures** (4EP)

  Learners need to be able to explain their ideas and procedures in order to be proficient. For example, it is not sufficient for learners to do only practice questions on determining the equations of parallel or perpendicular lines after a procedure has been learned. If learners are to understand the algorithm, Kilpatrick et al. (2001) observe that they also need experience in explaining and justifying it themselves.

- **Able to select appropriate tool for a given problem** (4TOL)

  Purposeful selection of a procedure engages learners in reasoning and meaning-making (Carter et al., 2009). The contexts may differ; hence the required computational skills may also differ. For instance, there are two approaches in determining the exterior angle of a triangle. Firstly, the *exterior angle of triangle theorem*: the *exterior angle is equal to the sum of its interior opposite angles*. The second approach involves two geometrical concepts, namely the *sum of interior angles of a triangle*, and *adjacent angles on a straight line*. Certainly, for the first option only the addition skill is necessary, while in the second process, several skills such as addition, subtraction and rearranging of an equation are required.

- **Carries out the procedure in a different way** (5PMW)

  A learner may be aware of different procedures to solve a problem, and then use similar procedure for similar problems e.g. when solving a quadratic equation, there are different methods such as factorisation, completing the square and quadratic formula.

- **Able to reverse a mental process** (5REV)

  From his empirical research, Krutetskii (1976) characterises different abilities for mathematically capable learners. He ascertains that the formation of mathematical concepts is related to the mastery of ability to switch from a direct to a reverse train of mathematical operations. In elementary algebra, factoring a quadratic polynomial is the reverse procedure of a product of two binomials. In my experience, I have noticed that proving the converse of a theorem evokes major difficulties for most of the learners.

- **Able to link to prior learning** (6LPK)

  Carter et al. (2009) argue that mathematical content is built on learners' reasoning and meaning-making habits, including making links to prior knowledge and skills. In the instance of developing the concept of gradient of a line, the prerequisite knowledge of properties of equal fractions such as $\frac{2}{3} = \frac{4}{6}$ is essential. In essence, mathematical concepts and skills ought to be linked to learners' prior experience and knowledge, otherwise, as Kilpatrick et al. (2001, p. 146) succinctly comment "they are learned as isolated bits of knowledge".

- **Employs basic procedures and formulae accurately and efficiently** (6BP)

  Learners apply previously acquired skills while they are learning new material. In discovering the relationship between the *angle at the centre* and the *angle on the circumference*, fluency in doubling and halving is necessary. This prerequisite skill is crucial in appreciating and developing the concept of the *angle at the centre theorem*.

- **Estimate the results of computation** (7EST)

  Kilpatrick e al. (2001) consider that the ability to estimate the results of calculations has an impact on mathematical understanding. For example, in sketching the line $y = 2x + 3$, it is good practice to estimate the direction of a line even before it is plotted. A learner should be aware of the relationship between the direction of lines and their gradient.

- **Verify the final answer** (7ANS)

The meaning-making habits of a learner stimulates reflection on a solution. In the above example of sketching a line, should an error occur while calculating the points on the line, a learner may consider an incorrect solution e.g. a line sloping in the opposite direction. Reflection is an important ability as it can guide learners to identify implausible solutions.

- **Knowledge of multiple ways to estimate the solution** (7KMW)

  The learner exhibits different methods of estimating the solution. In other words, the learner is aware of multiple methods for the solution. For example, for solving a quadratic equation, a learner may attempt to use the method of factorisation or the quadratic formula.

3.5.3 **Structuring narratives, horizontal analysis and emerging themes**

The basic units of data were identified and coded. I tagged the colour codes in the transcriptions and the artefacts. The data analysis was done in three phases and is thus presented in three sections in the following chapters. The summary of different phases of data collection is illustrated in Table 3.3.

Table 3.3: Summary of the three phases of analysis

Phase	**Level of analysis**	**Analytical Instrument**	**Analysis Approach**
Phase 1	Level 1 Data A, B and D	VAMMPA	Narrative Analysis
	Level 2 Primarily Data C and Data D	Clarify and sharpen Level 1 analysis	
Phase 2	Consolidation of Stage 1 across participants	VAMMPA	Horizontal Analysis
Phase 3	All Data	Patterns of interactions that surfaced in the lessons from the data	Emerging themes

3.5.3.1 **Phase 1**

In the first phase of analysis there were two interconnected levels. In fact, these levels of analysis were iterative.

3.5.3.1.1 *Level 1*

In this level, classroom observation videos and worksheets were analysed. The videos and the transcriptions were reviewed to describe and document the unique or common patterns of understanding. The screen capture videos comprised of the computer interactions of the selected learners and provided an idea of how the learners engaged with *GeoGebra*. The kinds of questions I asked were: Is the system idle for long? Do the learners start exploring straight away or do they wait for instructions from the teacher? Do they access any other applications like Paint or the Internet? More importantly, the analysis of screen captured videos was used to uncover what happened during the lesson and interpret the learner's activities borne out of their mathematical thinking. The data were categorised, coded and tagged by linking the learners' responses and activities with the broad indicators mentioned in VAMMPA — see Table 3.2. This analysis level allowed me to gain an in-depth understanding of their actions and motives.

3.5.3.1.2 *Level 2*

The reflective interviews and written responses by the participants helped me to clarify my preliminary analysis. This level sharpened my Level 1 analysis of the other two data sets. The analysis of the learners' articulation and thinking processes illuminated their mathematical thinking and meaning-making skills.

3.5.3.2 **The narrative analysis**

Newby (2010) considers narrative analysis as an approach to get insights into the participants' worlds. Thus, narrative accounts are a distinct way of conveying events and actions. As I wanted to get insights into the participants' mathematical thinking, I adopted an approach of narrative analysis. The narrative approach empowered my research to gain an understanding of participant learners' activities and responses in a DGS classroom.

Cohen et al. (2007, p. 461) suggest that " a case study may be most suitably written as a descriptive narrative, often chronologically, with issues raised throughout". I thus present the qualitative data chronologically and holistically, essentially using a story style of reporting "to catch the wholeness" (Cohen et al., 2007, p. 468) of my participants' actions. I thus ensured that the coherence and integrity of a participant's response was preserved. For a given topic, the responses and interactions (with DGS) of an individual participant learner were analysed, and then the analysis would move on to the next participant. This enabled me to present a detailed and comprehensive picture of each participant learner.

3.5.3.3 **Phase 2 - Horizontal Analysis**

However, a second level of analysis was necessary in order to look for patterns of interactions, differences and similarities and emerging themes (Cohen et al., 2007). I essentially summarised the data in my horizontal analysis as I put together all the main issues and themes that arose out of and across the participants learners. Narrative accounts cannot be seen in isolation as these stories shed light on the research questions. Thus, cross-referencing of what transpired from the narrative analysis enabled me to make sense of the learners' responses while they used DGS for learning mathematics. Thus, a horizontal analysis allowed me to identify apparent patterns or differences in the development of the learners' mathematical proficiency.

3.5.3.4 **Phase 3 - Emerging Themes**

In developing the VAMMPA analytical framework, I also considered using a broader level of learner responses, however, categories or codes emerged from the data, as Cohen et al. (2007) warn "the codes themselves derive from the data responsively rather than being created pre-ordinately" (p. 478). As I became more and more involved in and with the data, some interesting themes emerged inductively from the data "as if waiting to be discovered" (Cohen et al., 2007, p. 491). Through comparison, recurring patterns emerged from the data. My analysis protocol concurred with Merriam's (2009) contention that although pre-existing categories initially guide the study, at later stages other categories are expected to emerge.

3.5.4 **Interpreting the data**

The above structure allowed me to interpret the data in order to answer the research questions. This analysis protocol afforded me the opportunity to discover how the participant learners positioned themselves in relation to the harnessing of the visualisation capabilities of *GeoGebra* in solving and representing mathematical ideas. In my analysis, the notion of visualisation, both as a process as well as a product, was used to analyse the mathematical actions that shaped learners' meaning-making and mathematical proficiency. It should be remembered that Arcavi (2003) puts an emphasis on visualisation as an ability as well as a process in learning.

Newby (2010) affirms that at one level all of what we know is the product of our interpretations of data. Specifically, in social science research, Newby argues that only by examining the lives, actions and statements of people can we understand the world from their perspective. Thus, interpretation of data underpins the qualitative approach to research. Essentially, from the perspective of constructivist learning, learners interpret knowledge in their own way. Learners do not merely absorb what the teacher instructs them to do, they rather interpret and generate meanings in their own way (Boaler, 2009; Jaworski, 1994). Contextually, *GeoGebra* offered opportunities for the participants to manipulate and explore with the applets. Thus, in terms of research, the theory of constructivism aligns well with the interpretive paradigm of research. The interconnectedness of the theory of constructivism in mathematical learning provided the context within which I analysed learners' interactions with *GeoGebra* and how it developed their mathematical proficiency.

3.6 **VALIDITY AND RELIABILITY**

Validity in qualitative research implies that the explanation of a particular event can actually be sustained by the data (Cohen et al., 2007, p. 135). We, as human beings, cannot be completely objective about what we are observing and researching, and other people's interpretations and perspectives are equally valid. Validity thus accounts for the meaning that subjects give to data and the inferences drawn from the data that are pertinent to a specific research project (Cohen et al., 2007).

Merriam (2009) argues that validity and reliability in qualitative research is derived from several principles – i) the nature of interaction between the researcher and the participants; ii) triangulation of data; iii) the interpretation of perceptions, and iv) rich, thick descriptions to make the researcher's conclusions sensible. I adhered to all these principles in order to ensure validity and reliability.

I adopted different strategies to present the research study in an authentic manner. I interacted with the participating learners of both the schools for quite a long time prior to the actual data collection. Also, my interactions with the participating learners during the training in *GeoGebra* was useful and thus increased the validity of the research findings. Maxwell (2008) echoes Becker and Geer (1957) when he states that the sustained presence of the researcher in a natural setting can help rule out spurious associations. I was thus 'close' to reality. The long-term interaction also enabled me to collect 'rich' data which was detailed and varied enough to provide a full and revealing picture of the events.

Usually associated with qualitative data, is 'triangulation' that improves the internal validity of the research (Merriam, 2009). I used the triangulation method of using multiple sources of data – screen recordings, artefacts (in the form of worksheets and written responses) and reflective interviews. I also employed another strategy in enhancing the validity of the research — peer review. My fellow researcher reviewed my analysis chapters (Chapter 4 to Chapter 6) thoroughly. Her comments were taken into consideration after consultations, and where a difference of opinion persisted, it was discussed during the GLIP meetings with the teachers. My supervisor also assessed and commented on whether the findings were congruent with the data. Triangulation was used when analysing the learners' mathematical meaning-making and visualisation processes by applying multiple data sources such as classroom observations, screen capture videos and stimulated recalls. The use of multiple sources of evidence in case studies allows an investigator to address the issues under study through a process of data triangulation (Yin, 2009).

I looked at a range of similar and contrasting activities to understand the participants' engagement with *GeoGebra.* The inclusion of multiple cases is a common strategy for enhancing the external validity of the research (Merriam, 2009, p. 50).

Maxwell (2008, p. 243) identifies that 'bias' and 'reactivity' are two common threats to validity. Bias refers to ways in which data collection or analysis is distorted by the researcher's theory, values, or preconceptions (Maxwell, 2008, p. 243). Cohen et al. (2007, p. 144) state that the "reactivity effect" is that respondents behave differently when subjected to scrutiny or observation. I recognise that as a teacher I am in a position of power. I am also aware of the reactions of learners owing to my presence in classrooms during data collection. However, my role as a trainer during the training sessions in *GeoGebra* created opportunities for free and comfortable situations for learners through informal interactions. The learners became familiar with my presence in their classrooms prior to actual data collection. An 'intensive, long-term involvement' with the learners was one of the strategies to deal with these validity threats. This research was spread over an entire academic year hence the above-mentioned threats were minimised. I acknowledge that the 'bias' and 'reactivity effect' in my research might have compromised the participants' actions and responses. Although these threats to validity can never be erased completely, they can be reduced by attention to validity (Maxwell, 2008).

Furthermore, my role as a researcher was to prompt learners to reveal their thoughts and understandings without exhibiting any sort of authority. During stimulated recall interviews, I highlighted the importance of freedom of expression and encouraged the learners to talk and express themselves.

In order to increase validity in a stimulated recall technique situation, researchers should minimise the time delay between event and recall (Lyle, 2003). The reflective interview thus immediately followed the classroom observation.

The analytical framework VAMMPA that was adapted from the scholarly works of Kilpatrick et al. (2001) and Carter et al. (2009) was tested during the pilot study before the actual data collection. Although the analytical instrument already had an inherent level of validity, it was refined as a result of the pilot study.

3.7 THE ETHICAL CONSIDERATIONS

All Rhodes University ethical protocols for conducting research were observed. Ensuring validity and reliability in qualitative research involves conducting the investigation in an ethical manner (Merriam, 2009). As a first step I obtained permission from the principals and the school governing bodies to conduct the case study. I explained to the school management and the teachers the possible benefits accruing from participating in the research, such as a deeper understanding of how to integrate technology in mathematics classrooms. I then obtained the permission from the DBE, Eastern Cape Province, to conduct the research in Mthatha District — refer to the Annexure I.

O' Leary (2004) emphasises that informed consent implies that the participants (learners in this case) are voluntarily involved, have a full understanding of the commitment and activities involved and are fully aware that they are under no obligation to continue their involvement to the end of the project. In order to ensure that the learners' consent was voluntary and informed, a signed consent from every learner of each class and his/ her guardian (Cohen et al., 2007) was obtained — refer to Annexure II. Cohen et al. (2007) emphasise that the principle of informed consent implies the participant's right to freedom and withdrawal from the study. The participants were informed that they had the right to withdraw their participation at any time without providing reasons. During one of the *GeoGebra* training session, the learners were provided with a detailed explanation of the research goals (Cohen et al., 2007) and what the research wished to achieve. In particular I was very explicit in explaining that their engagement with the software would be captured for the sole purpose of analysis of the research. I assured them that they would remain anonymous in the write-up of the results and that pseudonyms would be used. Confidentiality and anonymity was thus assured and maintained.

In Paul's class a learner did not provide the signed consent letter and he therefore did not attend the classes that I observed. This was unfortunate as the learner had undergone the training in *GeoGebra* and had also attended the training interlude between the cycles.

I also ensured that the names of the participating teachers and their schools remained anonymous. The audio-visual media, screen capture videos, the recordings of stimulated recall interviews, the transcripts and/or any other data collected are stored safely and securely on my password-secured computer. The data will be stored there for a period of five years after which they will be destroyed.

3.8 PROBLEMS ENCOUNTERED

A problem that we encountered time and again in this research project was the inconsistent functionality of the computer laboratories in both schools. The available school ICT resources were not maintained regularly. The learners faced intermittent technical problems in using computers, such as blurred screens, faulty keyboards and dysfunctional mice. Both the participating schools had maintenance plans in place, but they were not effective. Hence, much time was lost during the lessons in identifying and fixing technical problems that arose. This caused disruptions in our research design and required flexibility on my part to rearrange the planned lessons and interviews.

A final limitation of the study was the fact that embedding DGS in normal classrooms was a rather new phenomenon in the participating schools. The processes of integrating DGS evolved during the process of the study. As a result, not all activities that were supposed to take place during the research, happened as planned. The teachers and I drew up a schedule of lessons to be observed that aligned with their annual teaching plans, well ahead of time. After each GLIP cycle, the participating teachers and I reflected upon the lessons in the GLIP meetings and workshopped ideas for improving the lessons. This, at times, led to further training for the learners. This training was often difficult to squeeze in between the already packed curriculum and the research plan.

The quantity of data was huge, hence I analysed only two participants per class, instead of the originally anticipated three participants.

3.9 CONCLUSION OF THE CHAPTER

In this concluding section of the chapter, I summarise the methods of study that were used to address the main research questions. The study was situated within a broad interpretive paradigm and adopted a qualitative analysis approach. A case study methodology was adopted for its focus on gathering rich and holistic information from the participants' perspectives in a natural classroom environment where technological tools were used. This study involved an intensive, long-term period of interaction with the selected participant learners. The data collection commenced when the collaboratively developed applets by the teachers in the GLIP were implemented in classrooms. In fact, several trial runs of data collection were conducted in order to sharpen the data gathering and analysis protocol. Multiple techniques were used to gather data from the participants at different stages. An analytical instrument, the VAMMPA, was developed to analyse my data. The trial runs also helped me to refine the analytical framework. I adopted different strategies to conduct the study and presented the analysis in an authentic and ethical manner. Finally, I dealt with various methodological tensions and problems encountered during the course of the study.

In the next chapter, I present the analysis of the collected data in accordance with the analytical framework.

CHAPTER 4

ANALYSIS OF LEARNERS' ENGAGEMENT WITH *GEOGEBRA* – THE *ANGLE AT THE CENTRE THEOREM*

4.1 INTRODUCTION

The goal of this research study was to investigate and explore how *GeoGebra* applets develop conceptual and procedural understanding of mathematical concepts in selected Grade 11 learners. In the GLIP project, the participant teachers designed *GeoGebra* applets and used them in their lessons for teaching and learning. Based on the annual teaching plan for Grade 11, the GLIP members decided amongst themselves which topics warranted the use of applets, and the design of the applets took place during these meetings. Each applet discussed in this thesis is referred to as 'Applet n.m' where n and m denote the m^{th} applet of the n^{th} GLIP cycle, for example Applet 2.3 refers to the third applet in the second cycle.

Each cycle of the GLIP was framed by a mathematical topic selected by the teachers. In this thesis I present each cycle of the GLIP by analysing the classroom observations of the selected participant learners. These observations include both video recordings and screen capture recordings. This is then followed by an analysis of the reflective interviews with each of them.

For each topic taught by the teachers (in this chapter, the *angle at the centre theorem*), I present firstly a vertical analysis and secondly a horizontal analysis. The former is an analysis of how each learner interacted with the applet and the latter is an analysis across the learners. Given the huge amount of data and lengthy analysis and accounts of the narratives, I present the vertical analysis of only three learners in this chapter. However, when answering the research questions, I take into account the data from all the learners.

This chapter is structured as follows:

Section 4.2 – I briefly describe the discussion of the GLIP meeting and the applets that were designed and used in the observed classroom lessons.

Section 4.3 – Vertical analysis. I present data in the form of narratives of three participants – Rosy and Daisy whose teacher is Antony, and Aster whose teacher is Paul.

Section 4.4 – Horizontal analysis. The narratives are examined by pulling together the indicators across the participants, complementing the vertical analysis.

Section 4.5 – Emerging themes – further insights into the research questions.

Section 4.6 – Seeking answers to the research questions.

Section 4.7 – Concluding remarks.

4.1.1 **Recapping of my methodology**

My research design was framed by a case study methodology. The narrative analysis that follows, details how each participating learner experienced, used and interpreted the mathematical content by engaging with *GeoGebra*. The narratives for each learner are in the form of prose account, interspersed with relevant figures, screenshots, artefacts, quotations and analyses. The narratives unfold in chronological order, i.e. as events developed in real time.

Chapter 2 and 3 described how I identified eighteen different observable or traceable indicators or learning activities (Broad Indicators). These are anchored in the works by Carter et al.(2009) on meaning-making, and by Kilpatrick et al. (2001) on mathematical proficiency. I only focused on two of Kilpatrick et al.'s (2001) mathematical proficiency strands, namely conceptual understanding and procedural fluency. Acronyms were then introduced for each of these observable actions, hereafter referred to as codes. These words or numeral codes were further categorised into seven broad indicators, namely BI1, BI2 etc…. These formed my analytical framework — VAMMPA, tabulated in Chapter 3. In order to operationalise the framework, I introduced colour schemes for each of the seven broad categories of indicators. For example, three codes under the broad category BI3, namely 3DC, 3CNJ and 3CRT were identified in pink. I analysed and coded each lesson transcript using the appropriate colour codes.

The VAMMPA allowed me to deconstruct each learner's interpretation, actions and engagement with the applets into a rich and detailed description. Occasionally, I used the reflective interview data to support or contradict the learners' communication during the lesson. The colour codes are embedded into the narrative accounts of these descriptions to provide evidence of meaning-making and characteristics of mathematical proficiency. Additionally, the visualisation processes, described in the literature review in Chapter 2, and in the methods of study in Chapter 3 of this thesis, complement the VAMMPA model to facilitate a deeper analysis of learners' engagement with the applets.

The relevant portions of screen capture video clips have been captured as a sequence of pictures and referred to as screenshots, while other diagrams and pictures including the applets generated by the teachers are referred to as figures. In order to facilitate a flow in reading and comprehending an episode of classroom interaction, I have consolidated screenshots, figures or transcripts into frames. In transcripts, I have used the following abbreviations: 'T' refers to the teacher, 'L' refers to a learner whose name is not known, and 'R' refers to the researcher. When I refer to mathematical theories, properties, axioms and statements, I have demarcated them in italics for ease of understanding.

4.2 OVERVIEW OF THE APPLETS USED IN THE *ANGLE AT THE CENTRE THEOREM*

In the discussion on the *angle at the centre theorem*, the teachers collaboratively designed five applets on the conceptual development of different aspects of the theorem. Three other applets, one of which is a formal proof of the theorem and another two that were for learners' practice, were developed by the researchers as these involved the use of advanced Java Scripting.

In applet 1.1 see Figure 4.1 learners were expected to identify angles at the centre and the angle on the circumference, and then explore the relationship between them. The points on the circumference of the circle could be dragged around but the ratio of angles was invariant.

Applet 1.2, in Figure 4.2, the teachers wanted to reinforce the fact that the theorem only applies to the angle at the centre of the circle and not anywhere else in the circle.

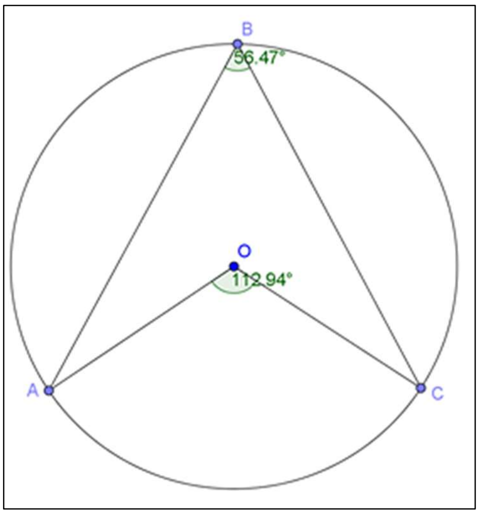


Figure 4.1: Applet 1.1

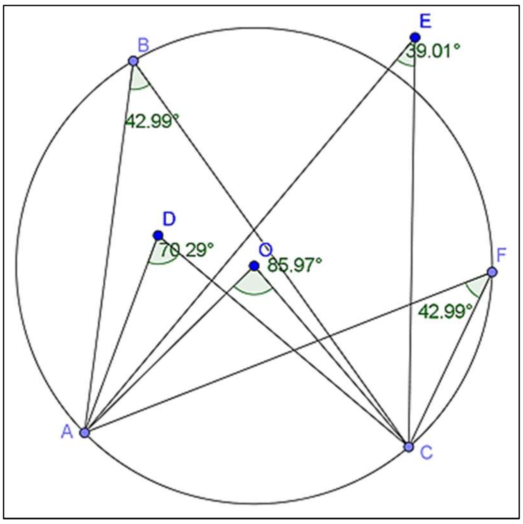


Figure 4.2: Applet 1.2

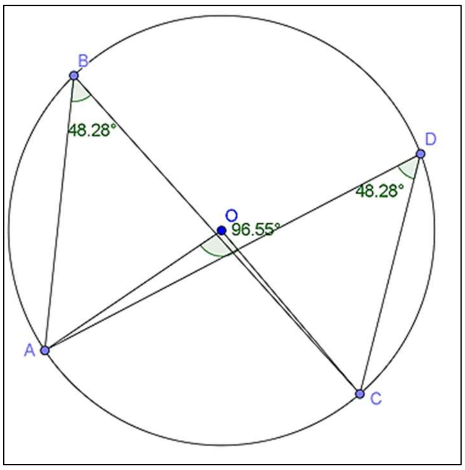


Figure 4.3: Applet 1.3

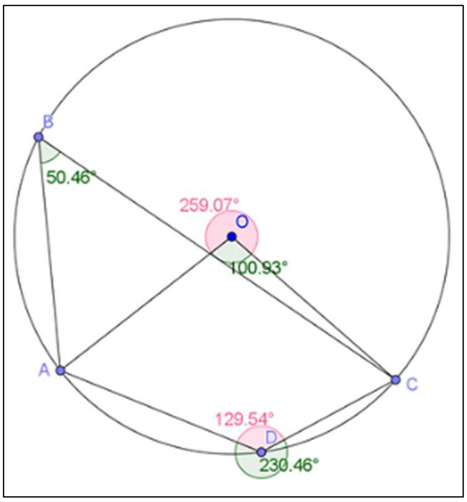


Figure 4.4: Applet 1.4

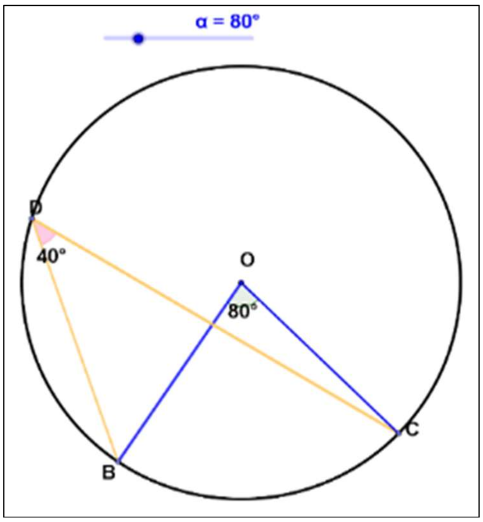


Figure 4.5: Applet 1.5

Applet 1.1 was modified with two additional points – i) a point outside the circle and ii) a point inside the circle, to generate Applet 1.2. These free points could be moved around the screen in order to confirm whether the new angle could be related to the *angle at the centre* or an angle on the circumference.

In Applet 1.3 in Figure 4.3, the *angle at the centre theorem* was linked to the *angles subtended by the same chord are equal theorem*.

In the fourth Applet 1.4 in Figure 4.4, two aspects of the theorem were highlighted. Firstly, that the angle on the circumference should be on the same side of the arc; and secondly, that the *angle at the centre theorem* was linked to the *opposite angles of a cyclic quadrilateral theorem*. Since in the above applets the angle measures were rounded off to two digits, the teachers felt that they would not be able to convince the learners about the accuracy of the *angle at the centre theorem* due to rounding-off errors. Further, the angle at the centre could not be dragged exactly through 180° in order to highlight a special case of the theorem, which is the *angle on a semi-circle theorem*. Hence, a fifth applet was specially designed in which the angle at the centre could be changed (using a slider tool) by unit degrees.

Applet 1.5 in Figure 4.5 has multiple uses such as exploring the ratio between the angles at the centre and on the circumference, highlighting the *same side of the arc concept* and the *angle subtended by a diameter*.

The teachers wanted to provide learners with questions to solve, specifically applying the *angle at the centre theorem*, but not limited to it. Two questions and their sub-questions were selected from a textbook. As these required learners to capture their answers and verify them against the correct answers, the applet demanded some advanced scripting language, hence these were handled by the researchers themselves. The applet for disclosing the formal proof of the theorem was also designed by the researchers in consultation with the teachers in GLIP.

In the practice question applets, the learners were expected to record their answers. The applet would indicate whether the correct answers were typed in or not. On clicking the 'next' button, a similar question would appear with a different value of the given angle. In the first practice question in applet 1.6, (see Figure 4.6), the angle at the centre was given, as presented in the figure. The learners had to calculate three different angles – one was for the unknown reflex angle at the centre and the other, for the two angles on the circumference. Angle a on the circumference could be solved by directly applying the *angle at the centre theorem*. Angle b at the centre could be found using prior knowledge on the *angles around a point*. And angle c

on the circumference could be solved either by employing the *opposite angles of a cyclic quadrilateral theorem* or the *angle at the centre theorem*.

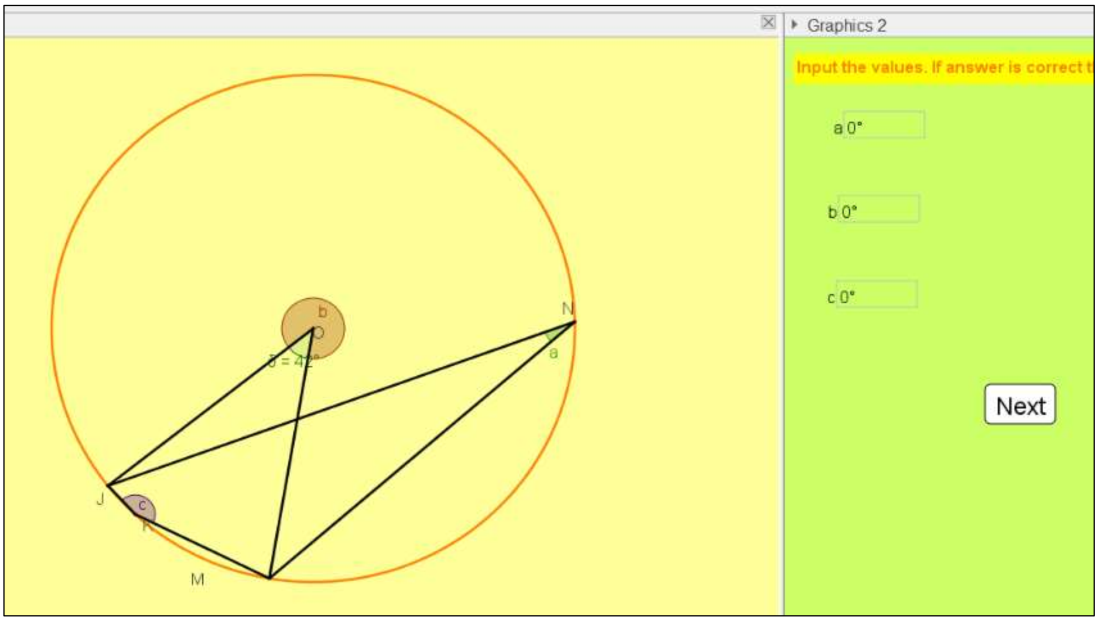


Figure 4.6: Practice question Applet 1.6

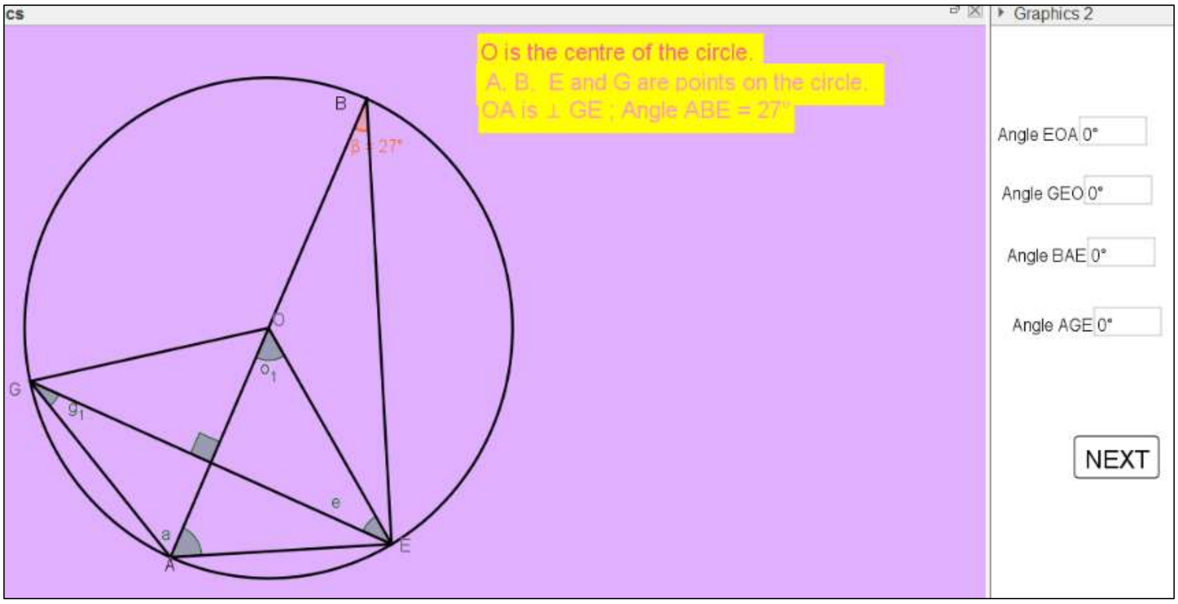


Figure 4.7: Practice Question Applet 1.7

In the second practice question Applet 1.7 (see Figure 4.7), the angle on the circumference was given and the learners had to find four different angles. In this question, learners were expected to form various strategies to solve the required angles. Firstly, the learners had to identify the relevant arc or chord in order to apply the *angle at the centre theorem* and solve for $A\hat{O}E$. Secondly, prior knowledge of the properties of triangles like the *sum of interior angles* or the *exterior angles of a triangle* was required to be applied to find GÊO. Thirdly, the learners were obliged to see the bigger triangle formed by the diameter of the circle, apply the *angle subtended by a diameter theorem* and then solve BÂE using the *sum of interior angles of the triangle*. However, BÂE could be solved in a different manner using the *properties of an isosceles triangle*. Lastly, the learners had to recognise the *angles subtended by the same chord or arc* to find AĜE.

The Applet 1.8 in Figure 4.8 was intended to provide proof for the *angle at the centre theorem*. Two graphics windows were used for better readability. The *angle at the centre theorem* was used as shown in the standard diagram, and the theorem was proved by using the *exterior angle property of a triangle theorem*, with the *properties of isosceles triangles theorem*. Angles were shown in two different colours, red for x and green for y, so that learners could easily identify the angles.

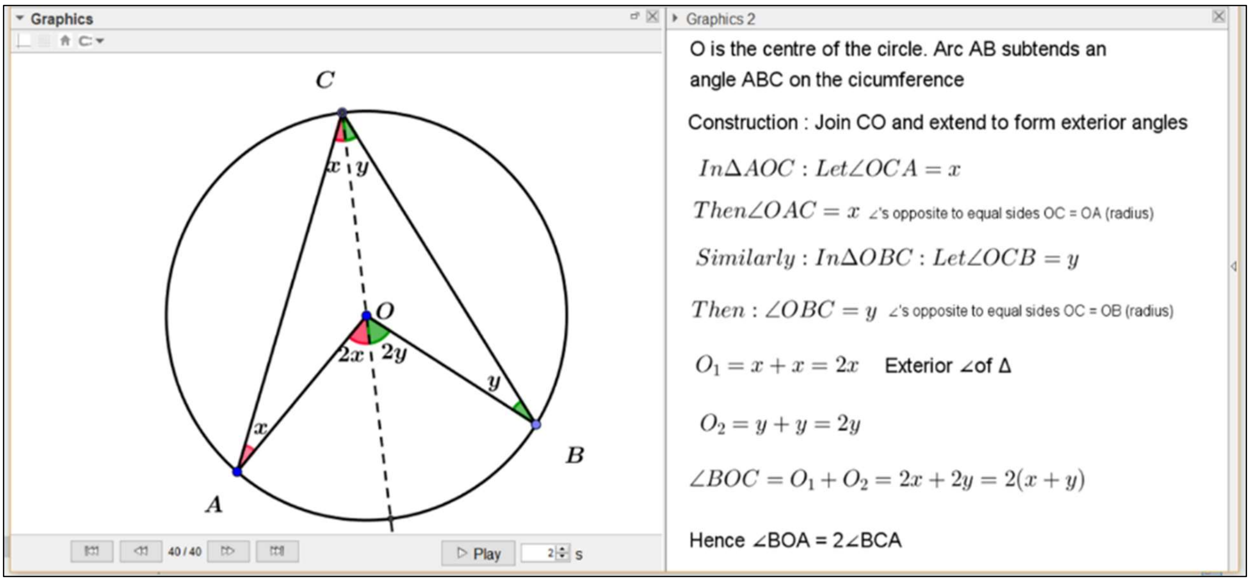


Figure 4.8: Proof of angle at the centre theorem Applet 1.8

4.3 DATA PRESENTATION AND VERTICAL ANALYSIS OF INDIVIDUAL PARTICIPANTS

In this section, I analyse the learners' interactions with the applets during their recorded lessons.

4.3.1 Antony's lesson

4.3.1.1 Overview of the lesson

Mr Antony's class comprised of 34 learners. Although the lesson was scheduled for two hours, almost thirty minutes of the lesson was spent in overcoming initial technical glitches with the desktop computers. Antony had given an introduction of the circle theorems in his previous lesson. The teacher wanted learners to explore the *angle at the centre theorem* using *GeoGebra*. The teacher provided step by step instructions to construct geometric shapes and simultaneously displayed his construction using the projector. They initially struggled to construct circles, segments and measure the angles, but they acquired the relevant skills as the class progressed. In the beginning, many learners had strange figures on the screen and many technical mistakes were made. Some learners were not able to plot points exactly on the circumference of the circle. Instead of choosing the line segment tool, a learner chose a line tool and had lines all around the screen. Measuring angles posed a challenge to many learners. The incorrect order of selecting the appropriate points was a major reason for the failure to obtain the required angle. Some learners measured the reflex angles, and others measured the angles at the arms of the angle at the centre as shown in Figure 4.9. Notwithstanding this, as the lesson progressed each learner in the class succeeded in constructing a diagram on his/her desktop similar to that of Antony.

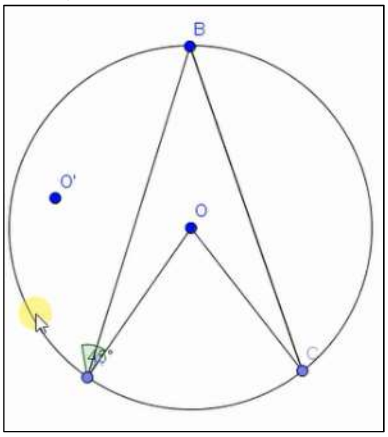


Figure 4.9: Bizarre Angles

The facial expressions and the body language of the learners suggested that they were excited when they played around with their own constructions. Some learners informally 'high-fived' each other. They congratulated one another. When questioned by the teacher, they responded with radiant smiles on their faces. They provided values of the angles and asserted that the "*angle on the circumference is half the angle at the centre*" (IEC and 1MR). The learners found it intriguing when they saw that the angles changed by dragging the arc and inferred (3CNJ) that the angle at the centre remained double the angle on the circumference. Equally fascinating for the learners, was the understanding that the *opposite* *angles of a cyclic quadrilateral property* was linked to the *angle at the centre theorem*. When the teacher linked and referred to the special case of the theorem — that the angle subtended by a diameter on the circumference was 90°, some curious learners were seen dragging the arc to verify if the angle subtended by a diameter was indeed 90° (3DC). It was evident that they had fun when attempting practice questions. One learner was clapping and humming in excitement as she found out that her answers were correct. Throughout the lesson, the learners appeared focussed on exploring the relationships between the angles and applying the concepts in practice questions. None of learners used any other programs or played computer games. Instead, they often drew the attention of the teacher to show off their correct answers in practice questions.

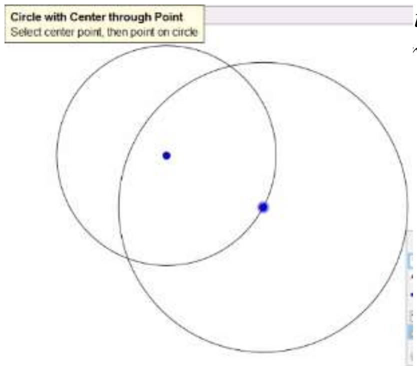
The analysis that follows is the selected learners' interactions with *GeoGebra* in class and their responses during the personal interviews after class.

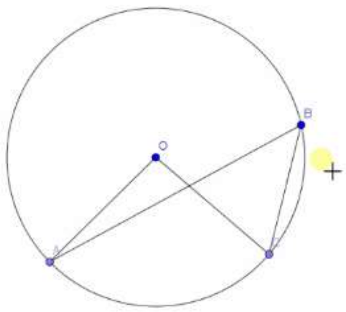
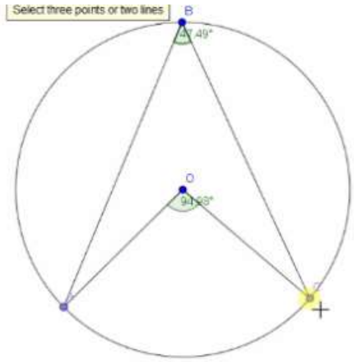
The two participants in the class were Rosy and Daisy. Both of them are females aged sixteen. Although I advised Rosy and Daisy to use the 'Casio Simulator' calculator on the desktops for their calculation purposes, they preferred to use their own physical calculators.

4.3.1.2 **The story of Rosy**

4.3.1.2.1 *Rosy identifies relevant mathematical concepts*

Rosy demonstrated her conceptual understanding of the angle at the centre theorem twice. Firstly, when the teacher asked her to state the theorem at the beginning of the class and secondly during the interview (1EC). She initially placed the angle at the centre outside the circle, and the angle on the circumference inside the circle (1MR and 4AP), as reflected in her interview, (see the transcript in Frame 4.1). Screenshot 4.1 in Frame 4.1 shows Rosy's initial efforts in constructing circles. She learnt quickly until she produced the illustrated Screenshot 4.2 in Frame 4.1. After the construction of the circle with the angle at the centre and angle at the circumference, Rosy was asked to measure the angles. As her figure did not match that of the teacher's, she dragged the point on the circumference to produce the standard diagram as shown in Screenshot 4.3 in Frame 4.1. This was similar to Applet 1.1 discussed above. Without waiting for the teacher's instruction, Rosy explored the theorem (3DC). Rosy was observed dragging the point B on the circumference between the arc AC (2MC). I could see that she frequently used her calculator to verify the relationship between the angles. Like many in class, she affirmed that the *angle at the centre is double the angle on the circumference*.

<table>
<tr><td colspan="3">When asked by the teacher, Rosy plotted another point inside the circle and measured its angle from the arc. The learners were asked to determine any relationship between the new angle inside the</td></tr>
<tr>
<td>

Screenshot 4.1: Initial constructions</td>
<td>
Screenshot 4.2: Getting there</td>
<td>

Screenshot 4.3: Aligning</td>
</tr>
</table>

Classroom Observation	*T: Here we want to investigate theorem 3. You remember theorem 3? Can anyone please state theorem 3 for us?* ***Rosy**: Theorem 3 Sir?* *T: Yes Theorem 3. Angle at the centre* ***Rosy**: The angle at the centre is double the angle at the circumference provided they are subtended by the same chord or arc. 1EC*
Reflective Interview	*R: How was your lesson using GeoGebra? What did you learn?* ***Rosy**: Ok, the centre is double the angle at the circumference provided that they are subtended by the same arc 1EC.*
	***Rosy:** As much as this angle, as much as this angle are subtended by the same chord but the textbook didn't mention that. if the angle at centre is outside then the angle at the circumference should be inside 1EC. Yes sir*

Frame 4.1: Rosy's initial construction and knowledge of the theorem

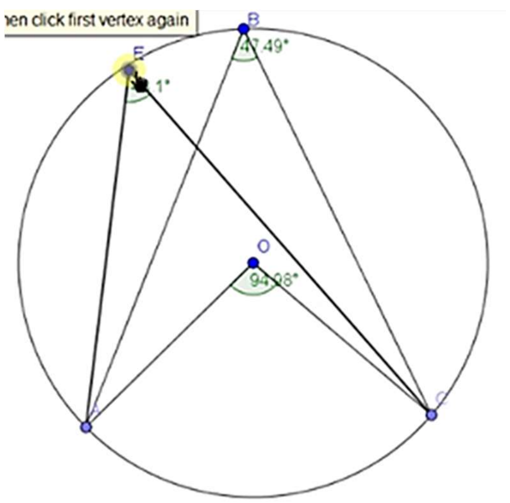
After that there was not much activity on her computer screen for a few minutes. Rosy was busy helping her friend to construct circles, line segments and measure angles (4EP). She understood which tools to select and how to use them. She could also explain her ideas and knowledge to others.

4.3.1.2.2 *Rosy explores and makes conjectures*

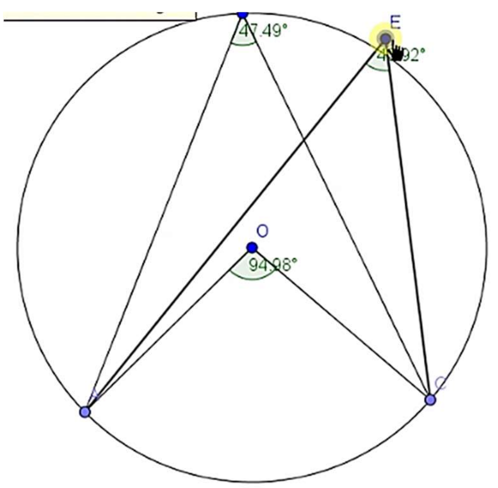
Frame 4.3. She moved the point *E* all around the screen, inside the circle, at the centre, on the circumference and outside the circle (see screenshots in Frame 4.3). Based on her exploration (3DC), Rosy made a conjecture (3CNJ) that the angle inside the circle was not half the angle at the centre of the circle, because the angle was not on the circumference(see transcript in Frame 4.2). Rosy's engagement with the manipulative led her to conclude convincingly that the point must be on the circumference for the theorem to hold. Later in her reflective interview, (transcript in Frame 4.2), she stated that the textbook did not provide any information on angles formed by an arc inside or outside the circle. She paraphrased the theorem, replacing double with 'half', thus displaying her expertise in representing mathematical ideas in multiple ways, (1MR) in a given situation. This again highlighted that Rosy could identify the relationship between angles (2MC) and understood the variant and invariant properties (2UVI). The other learners seemed to be convinced by her argument. She was heard explaining again to her friend about her discovery (1EC).

Classroom Observation	*T: Ok, now is there any relationship now between the new angle and the angle at the centre? Can you relate now the new angle and the angle at the centre? Is there any relationship between those two? Is there any relationship between those two angles and the new angle? the angle at the centre or the angle at the circumference? yes* ***Rosy*** is dragging the point 'E' (E is a point inside the circle) around the screen. Observing the change of value of angle at E while dragging point E 3DC within the circle, on the circumference on both sides of arc AC, at the centre and outside the circle. ***Rosy**: not audible* *T: Can you say that to everyone who is here? Aaha!* ***Rosy**: I am saying the angle [pause] the angle in the circle is not half the angle at the centre of the circle 1MR because the angle is not in the circumference 3CNJ.*
Reflective Interview	***Rosy**: Even if they are subtended by the same chord, the centre and circumference if that is not at the circumference and around the circle or outside the circle then it will not be half the angle at the centre. 1MR and 2UVI I observed that the textbook did not really point out this. 2MC*

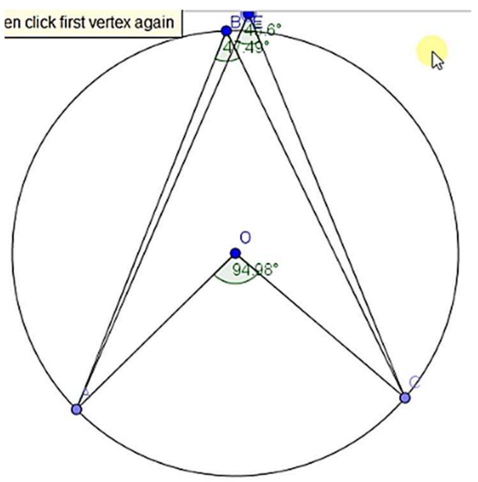
Frame 4.2: Rosy's conjecture for angles not on the circumference of the circle.



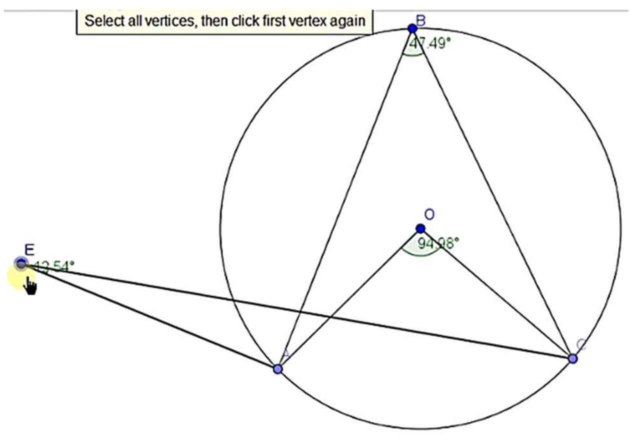
Screenshot 4.4: Drags E to the circumference



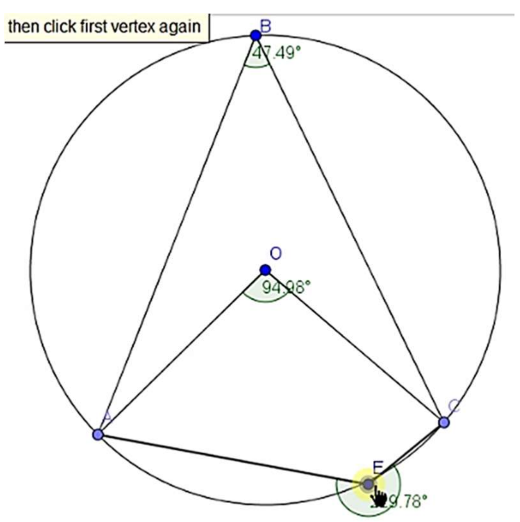
Screenshot 4.5: Drags E outside the circle



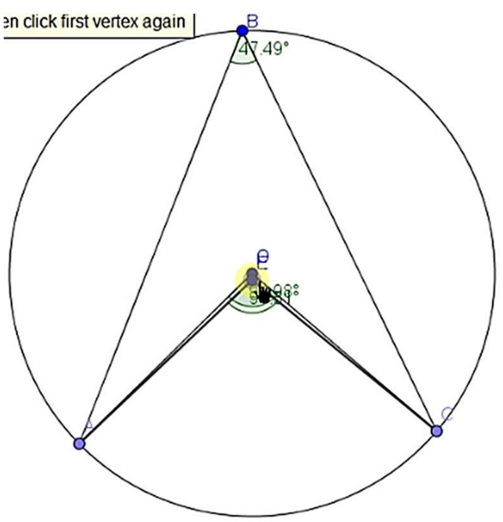
Screenshot 4.6: Drags E above B



Screenshot 4.7: Drags E very far from the circle



Screenshot 4.8: Drags E on the other side of arc



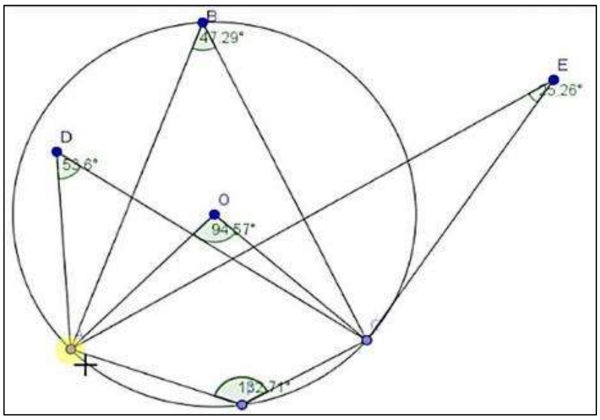
Screenshot 4.9: Drags E to the centre

Frame 4.3: Rosy dragging the point around the screen

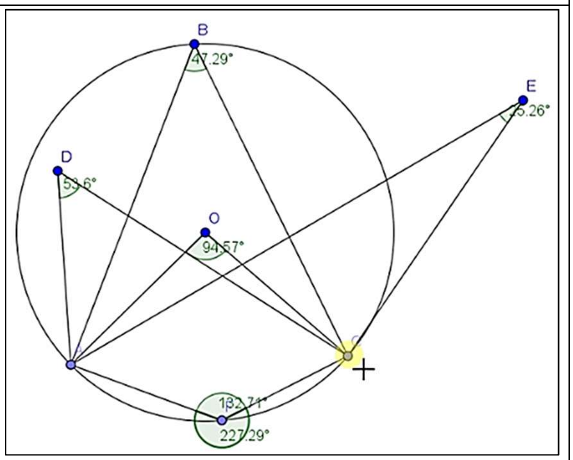
4.3.1.2.3 *Rosy discovers the same side of the arc concept*

Rosy plotted point F on the other side of the arc and measured its angle correctly using the appropriate *GeoGebra* tools (Screenshot 4.10 to Screenshot 4.12 in Frame 4.4). She agreed with the teacher that $C\hat{F}A$ was not double the (measured) angle at the centre (1EC). When asked by the teacher to find the *angle at the centre* that could be doubled to angle $\hat{F}$, Rosy measured the exterior angle $C\hat{F}A$, (3DC). After thinking, however, she measured the appropriate angle at the centre (2MC). She double-checked her measurements with a calculator (7ANS). Her friend sought her help to identify and measure the required angle, and Rosy obliged and explained her observations (4EP). But her friend wanted further clarification from her as to why she drew the outside angle $C\hat{F}A$ if it was the other angle at the centre that required to be measured. Rosy said that she was just checking(see transcript in Frame 4.4).

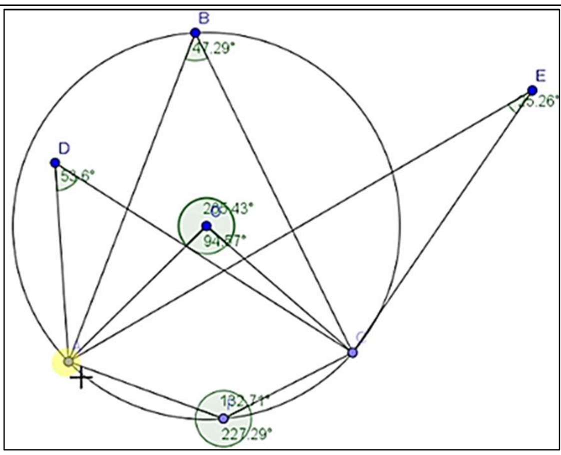
Classroom Observation



Screenshot 4.10: Rosy plotting F on the other side of the arc



Screenshot 4.11: Rosy measuring the reflex angle F



Screenshot 4.12: Rosy measuring reflex angle O

After a moment, the measured the other angle at the centre 2MC, using a calculator to check her answer. (7ANS)

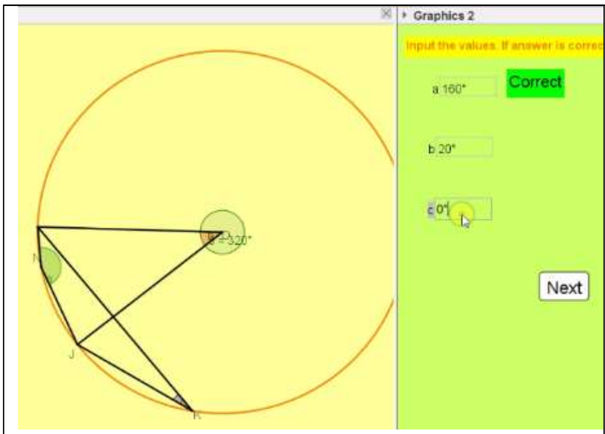
Zinnia: You got the answer.
***Rosy**: Yes. You did not measure.*
Zinnia: No, [isiXhosa]
***Rosy**: The angle F is inside, circumference the angle is inside. Angle at the centre must outside. Measure that angle (isiXhosa)* [gesture using a pen to indicate to measure the other angle at the centre] *4EP*
Zinnia: why did you find the other angle F?
***Rosy**: don't worry I was just checking.*

Frame 4.4: Rosy measuring the angle at the centre for the other side of the arc

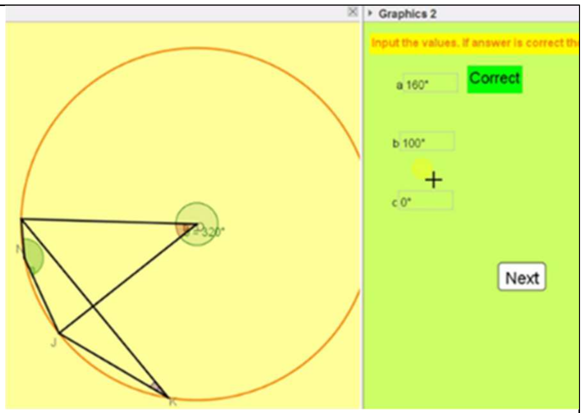
Antony provided his learners with two applets to practise the *angle at the centre theorem*. Prior knowledge on geometric properties and finding unknown angles were essential in solving these tasks. Learners were also required to link the theorem to other theorems. These practice questions were discussed above in Section 4.2 (p.76)

4.3.1.2.4 *Rosy analyses and solves questions applying previously learned concepts*

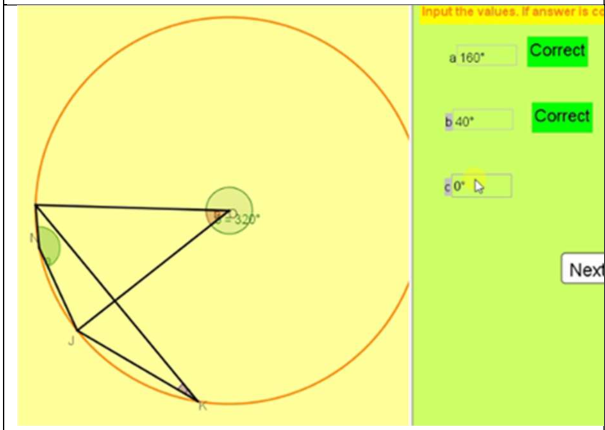
Initially, Rosy attempted Question 1 with some uneasiness and misconceptions, but later she did not find any difficulty in solving it; (see screenshots in Frame 4.5). In solving the first sub-question, to find a, she correctly applied the *angle at the centre theorem* (1EC) and solved it using the appropriate tool (4TOL) and algorithm (4AP). But she made two unsuccessful attempts before she solved the value of b (see Screenshot 4.13 and Screenshot 4.14). In the first instance, she made an incorrect assumption about a cyclic quadrilateral and applied its properties of supplementary opposite angles. In the second instance, she incorrectly executed the revolutionary angle method (6LPK). In her third attempt, she realised the correct values and carried out the procedure accurately (6BP) (see Screenshot 4.15). In solving the third angle she made mistakes before solving it correctly. She first had a misconception that opposite angles of cyclic quadrilateral were equal (in Screenshot 4.16), but she corrected herself by finding the appropriate arcs and angles (2UVI; 1EC) and applied the *angle at the centre theorem* (4TOL and 4AP) in Screenshot 4.17. She continued to solve similar questions of the Question1 type with different given values. She completed two other such questions correctly on her first attempt seen in Screenshot 4.18).



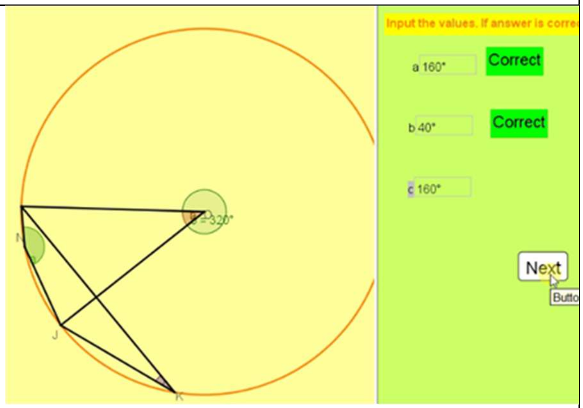
Screenshot 4.13: Rosy: 'a' correct but 'b' incorrect



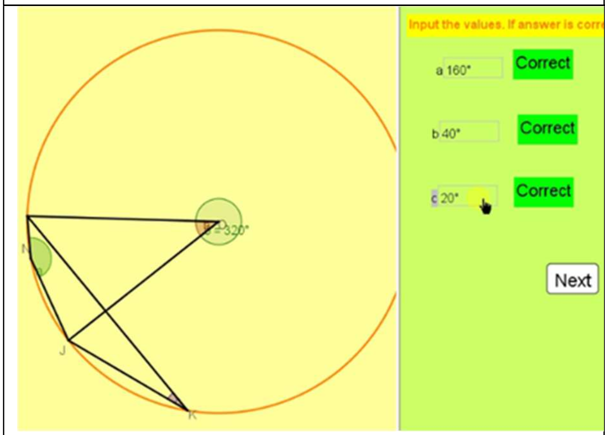
Screenshot 4.14: 'b' wrong at second attempt



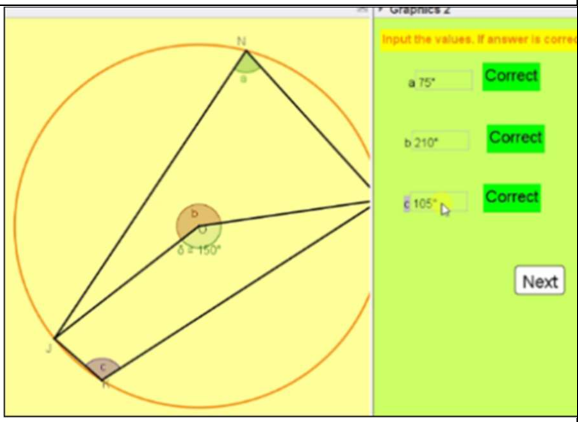
Screenshot 4.15: 'b' correct at third attempt



Screenshot 4.16: 'c' wrong at first attempt



Screenshot 4.17: 'c' correct at second attempt



Screenshot 4.18: Next questions all correct on first attempts

Frame 4.5: Rosy's answers to practice question Applet 1.6

In my interview with her later, she could correctly explain her procedures by talking about 'revolutionary angles' and 'angle at the centre', indicating her skill in presenting mathematical ideas explicitly (4EP). See the transcript in Frame 4.6. She used the technique of 'inside-outside' to identify and relate to the *angle at the centre* and the *angle on the circumference*. However, she could not find c without the value of b — in other words she was unable to apply different methods in different ways when the opportunity arose. She could not relate the figure to a cyclic quadrilateral; hence she was not able to apply the *opposite angles of a cyclic quadrilateral property*.

Reflective Interview

R: What challenge did you face while answering this question?
***Rosy**: Challenges I faced? This one?*
R: This question one.
***Rosy**: This one*
R: This question. right from the beginning, I do not know where the beginning here (pointing to the screen recording). Yeah this is the first position. Any challenge you faced while answering this question?

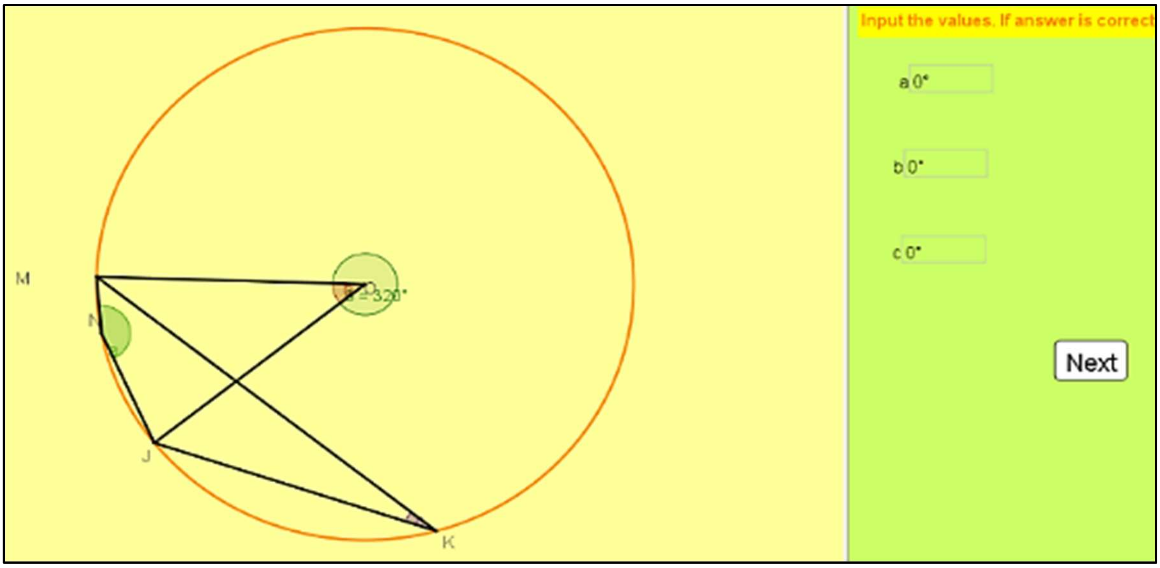


Figure 4.10: Answering during Interview

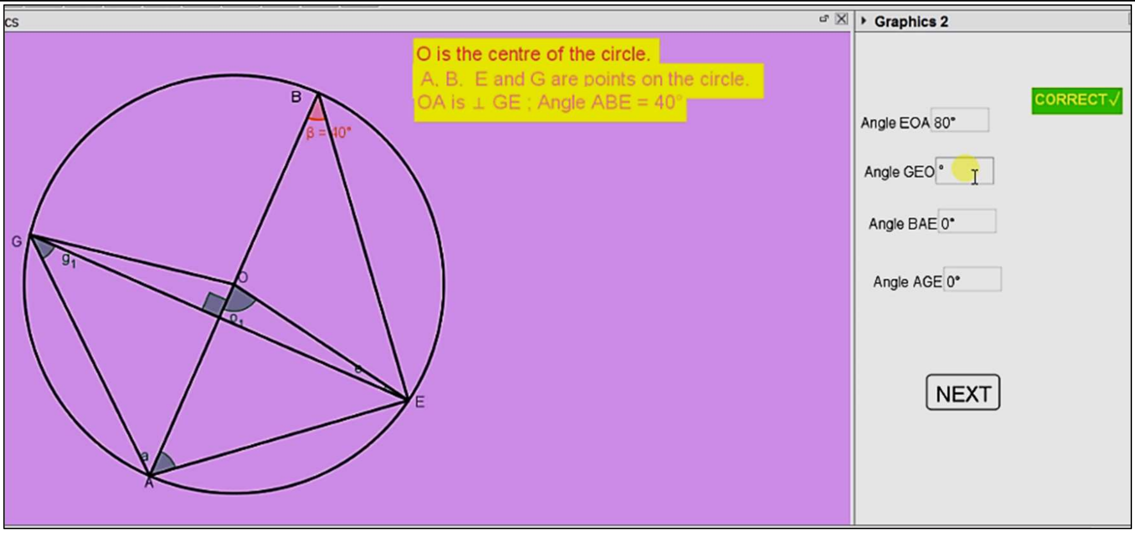
***Rosy**: Oh yes sir, I think it is finding 'a' sir…was it finding 'a' it was not finding 'a'. challenge challenge I know that 'a' is inside and 'b' is inside. This is one side to get this one oh ok this is 'a' and this is 'b' right?*
R: Let me take the question in GeoGebra itself. (Refer to above figure). It's better to see. Yes this is 'a' and this is 'b'. Is there relation with 'a' and 320 is there any relation?
***Rosy**: Ok yes sir…oh oh yes. Asked to find 'b' we know that this is a point revolutionary angle 6LPK. Therefore, it has to add up to 360 6BP. Yes, sir so the first thing I did I first divided this 320 by two 6BP to get this a so as this one is outside and this one is inside. Then we need to get this I said 360 – 320 and I got the angle which is 40 4EP. And then divided by 2 to get c as 20. 4TOL 4AP*
R: Reason
***Rosy**: Angle on the circumference is half the angle at the centre 1EC*
R: So can you find 'c' without finding 'b'?
***Rosy**: 'c' to find 'c'?... meaning only 320 is known.*
R: You first find 'a' and then jump to find c
***Rosy**: Ok … I think so can I use paper* (But she uses a calculator… murmurs to herself) …
R: Can you please be louder?
***Rosy**: It is twenty sir?*
R: How?
***Rosy**: I don't know*
R: can I see your calculator? (By pressing the previous answers in the calculator, I can see that she first found the reflex angle and then divided by 2). *Ok..* (not a different method from previous one, hence not coded)

Frame 4.6: Transcript of Rosy's response on practice question Applet 1.6

4.3.1.2.5 *Rosy implements strategies to solve questions*

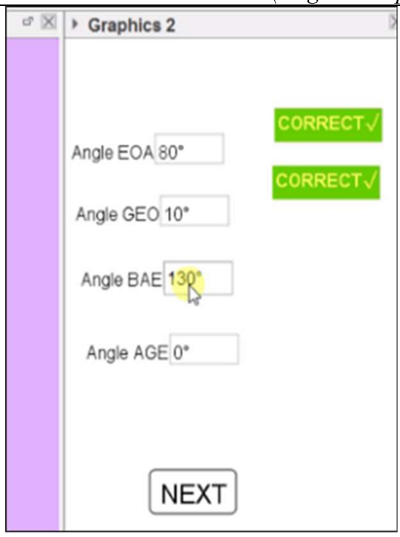
The teacher provided learners with the second practice question(Applet 1.7). Rosy started off with a correct solution (4AP), identifying the arc (2MC) and the arms and the angles (1EC), and applied the angle at the centre theorem appropriately (4TOL) as in the screenshots in Frame 4.7. She doubled the given angle on the circumference and obtained the required angle at the centre — O_1 in Screenshot 4.19. In order to solve the second sub-question, Rosy identified and processed the notations of perpendicular lines, segregated the right-angled triangle (2MC) from the figure, and applied her prior knowledge about the *sum of angles of a triangle* to solve the unknown angle G$\hat{E}$O (6BP) in Screenshot 4.20. In her reflective interview, she recalled it as using the *sum of interior angles of a triangle*. However, her calculator showed that she used the basic procedure $90° - 54°$, (6BP) to find the angle, hence exhibiting her ability to select an appropriate tool in a given situation (4TOL). She could not solve the third sub-question without seeking assistance from the teacher. In her own first attempt, she assumed that angle B$\hat{E}$A was angle G$\hat{E}$O. She correctly recognised the ΔBAE and carried out the procedure accurately, but misconstrued B$\hat{E}$A. She was reminded by the teacher about the *angle subtended by a diameter* property and thus realised that B$\hat{E}$A was therefore 90° as it was subtended by diameter $\overline{BA}$ (2MC). She applied the basic procedures as before and executed it accurately (6BP)as in the Screenshot 4.22. However, the lesson ended, and she did not continue with the remaining question. When I asked her to redo the question during the reflective interview, she correctly completed these questions again. She solved the angle A$\hat{G}$E, which is equal to the given angle, stating that the *angles were subtended by the same arc* AE (2MC and 4TOL). However, when I asked her to solve for B$\hat{A}$E using a different method, she was not able to solve it.

Classroom Observation

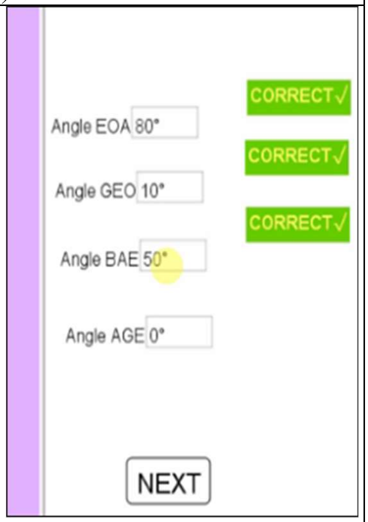


Screenshot 4.19: $E\hat{O}A$ correct (single attempt)

Screenshot 4.20: $G\hat{E}O$ correct (single attempt)



Screenshot 4.21: $B\hat{A}E$ wrong (first attempt)



Screenshot 4.22: $B\hat{A}E$ correct (second attempt)

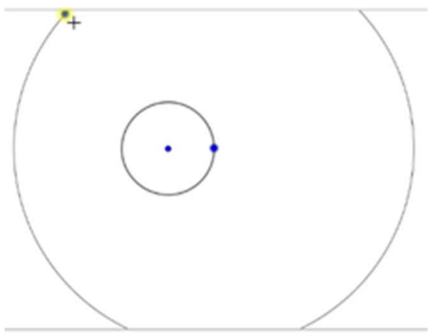
Rosy: *Sir ?*
T: Yes, what's your problem?
Rosy: *How can we find this angle BAE ?*
T: Angle BAE ? Ok. Hmm… the angle subtended by a diameter is ?
Rosy: *90°. 1EC*
T: do you see the diameter here?
Rosy: *Yes. It's BA. 6LPK*
T: Now you can solve it?
Rosy: *Yes, I can.*

Frame 4.7: Rosy answers practice question Applet 1.7

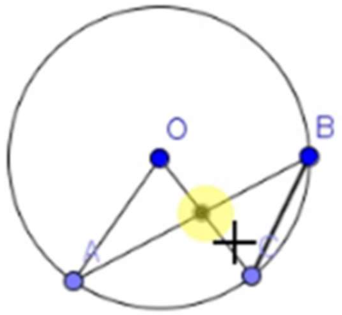
4.3.1.3 **The story of Daisy**

4.3.1.3.1 *Daisy's initial irregular constructions*

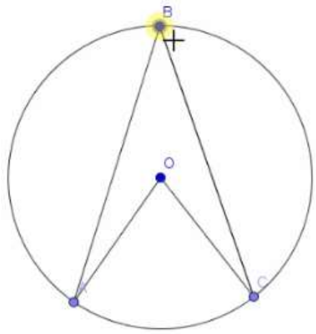
Daisy initially started with irregular constructions, struggling to measure the angles, but proceeded to construct figures as instructed. See Screenshot 4.23 to 4.26 in Frame 4.8.



Screenshot 4.23: Weird constructions



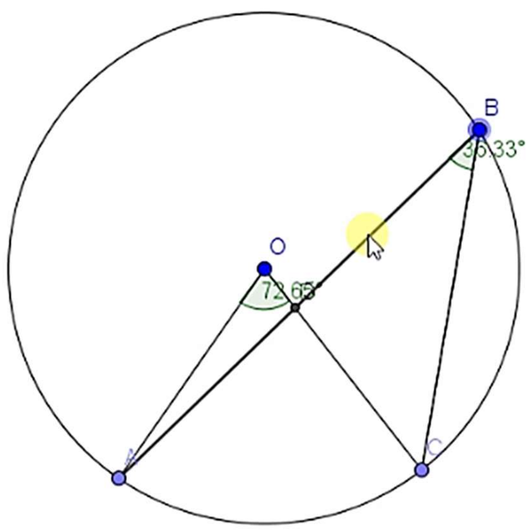
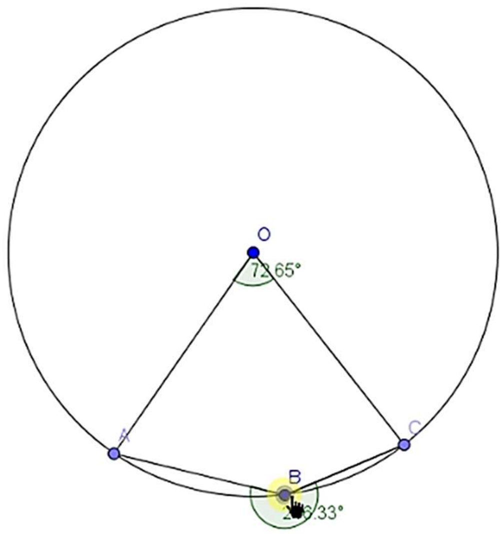
Screenshot 4.24: Irrelevant point of intersection



Screenshot 4.25: Getting there

Frame 4.8: Daisy's initial constructions

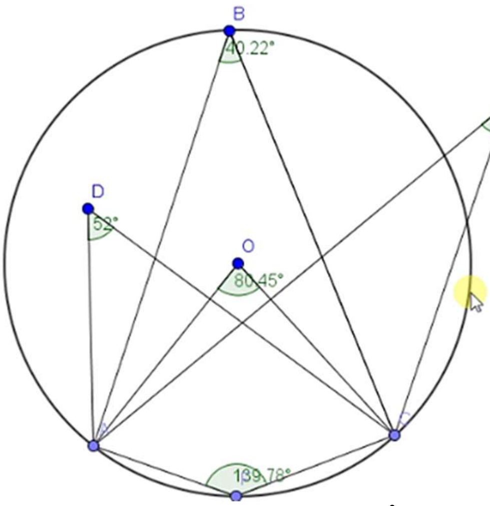
Daisy remained silent and did not interact with other learners during the lesson. However, she was observed dragging the point all around the circumference of the circle in Screenshot 4.26 and Screenshot 4.27 in Frame 4.9. In the reflective interview, when she spoke about the *angle at the centre theorem*, she emphasised that both angles must be on the same side of the arc (1EC). According to Daisy, the angle on the circumference should be above the angle at the centre (1MR and 4AP). Further, she clarified that when the angle at the circumference was dragged down to the minor arc, it would not be half the angle at the centre.

Classroom Observation	
 *Screenshot 4.26: Drags B on the same side*	 *Screenshot 4.27: Drags B on the other side*
Reflective Interview	***Daisy****: For the angle in the circumference to be half the angle at the centre 1EC, it has to be on circumference. And that when the angle is outside the circle, it will not be half the angle of the circumference 1EX.* *... another thing sir, the angles on the circumference must be above the angle at the centre 1MR. If I dragged that angle on the circumference down below the arc then it won't be half [inaudible]. I mean they should be on the same side of arc or chord 1EC*

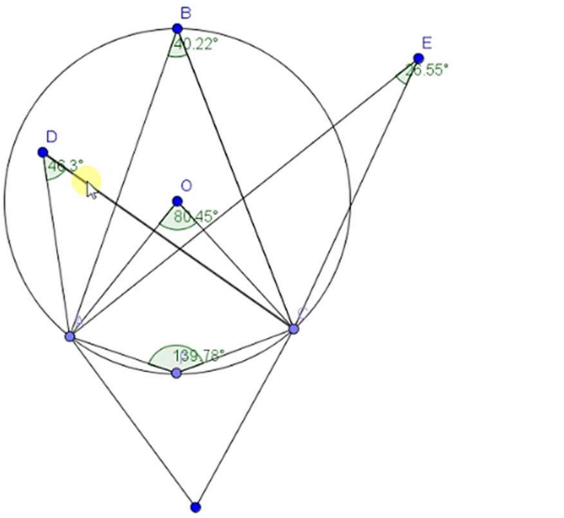
Frame 4.9: Daisy dragging point B

4.3.1.3.2 *Daisy discovers the same side of the arc concept through construction and measurement*

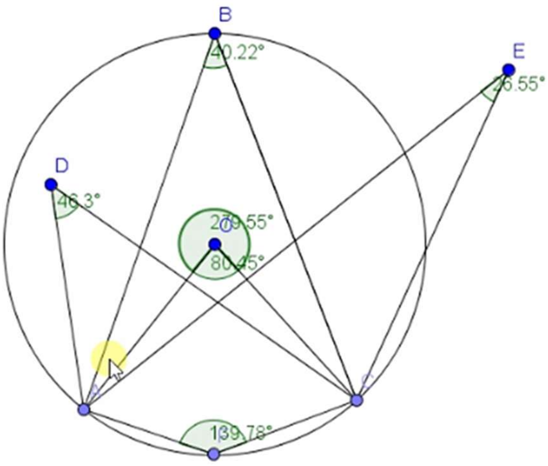
Daisy followed the instructions of the teacher and did the constructions on the computer. The teacher asked the learners to construct a point, named F by Daisy, on the other side of the arc. After measuring the inside angle at point F, Daisy was asked to identify and measure the angle that would be double $C\hat{F}A$. But Daisy started incorrectly by constructing an angle outside the circle, below $C\hat{F}A$. She adjusted the figure to accommodate a new point outside the circle. Soon she realised that the angle should be inside the circle and at the centre of the same circle (2MC and 3CNJ). She then measured the angle at the centre and related it to the angle $C\hat{F}A$ appropriately. She then dragged the point F around the minor arc AC continueing to verify if it was still half the angle at the centre (3DC). Screenshot 4.28 to Screenshot 4.31 in Frame 4.10 indicate the sequences of her measuring the angle at the centre related to $C\hat{F}A$.



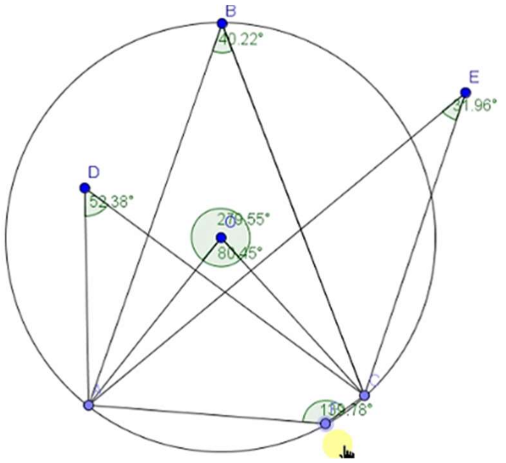
Screenshot 4.28: Measuring $\hat{F}$ on the circumference of minor arc



Screenshot 4.29: Measuring angle at the centre by plotting a point outside the circle



Screenshot 4.30: Reflects her approach and measuring of the angle at the centre on the minor arc



Screenshot 4.31: Drags F on the minor arc.

Classroom Observation	*T: Ok now can you please measure for me the outside angle, the correct angle we can relate to angle F, the correct angle measure that angle.* *T: I am saying you must measure the angle that you are going to relate to angle F. That is the angle we are talking about is the angle at the centre Measure the correct angle at the centre.* ***Daisy**: She plots a point outside the circle and joins them with a line segment. She deleted her recent construction. She pauses, looks at the diagram* 6BP and indicates the correct angle 3CNJ using the mouse pointer. After a little technical struggle, she measures the angle. *2MC* *Teacher: You managed ne? Ok…then now what is the relationship between the new angle that you measure and that angle F? So, what is the relationship.* *L: Angle at the centre is double the angle at the circumference 1EC.*
Reflective Interview	*R: Let me take you to this file of yours. You are joining these points…now you are going to draw the other angle related to this angle F. This is actually your working. At this point what made you to draw?* ***Daisy**: Hmm … first is that I forgot that it has to be in the circle 4EP. It must be a line from the centre. And then I drew that without the circle I know I was wrong there 3DC but the angle here is double the angle here 3CNJ.* *R: what you did was wrong?* ***Daisy**: I think I am wrong because the angle that I draw was outside the circle. It needs to start inside the circle. And the point wasn't on the center1EC and 3CNJ.*

Frame 4.10: Daisy measuring the angle at the centre related to $\hat{F}$.

4.3.1.3.3 *Daisy makes logical deductions on cyclic quadrilaterals*

The participating learners could link the diagram and identity BAFC as a cyclic quadrilateral (1EC), as shown in Screenshot 4.30 in Frame 4.10 an indication of conceptual understanding. Daisy supplemented this with the properties of a cyclic quadrilateral (1EC), providing evidence for her explanations of the concepts. She told the class that the interior opposite angles in a cyclic quadrilateral added up to 180°. Again, the use of alternate phrases for supplementary angles displayed her ability to express mathematical ideas in multiple ways (1MR), as shown in the transcript in Frame 4.11. During the reflective interview, she argued and gave reasons the *interior opposite angles of a cyclic quadrilateral being supplementary*. She rationalised that since the angles at the centre were double the angles on the circumference and that angles at the centre added up to 360°, (6LPK), the *sum of the angles on the circumference would be half of sum of angles at the centre*. She was thus able to link the theorem to the *sum of angles of cyclic quadrilaterals properties*, hence coded as (2MC).

Classroom observation	*T: I want us to relate now these two angles that is angle F and angle B. What did we say about cyclic quadrilaterals?* *L: All four points are on the circumference 1EC* ***Daisy**: two opposite angles are supplementary 1EC* *T: yeeha! All four points are on the circumference. and if a quadrilateral is having all four vertices on the circumference then we say that quadrilateral is cyclic. Then what about the angle? What did we say about the angles? We said?* ***Daisy**: Two opposite interior angles are, supplementary* ***Daisy** : Opposite angles add up to 180 1MR opposite angles they add up to 180°. We can add these two angles, angle F and angle B and see they give you 180°.*
Reflective Interview	*R: One more question. In the class, you confidently said that the opposite angles of a cyclic quadrilateral are supplementary. Why you think so?* ***Daisy**: I think sir because the angles on the circumference are half the angle at the centre and angles at the centre is 360°, I mean revolutionary angle.2MC Therefore its half is 180° 6LPK*

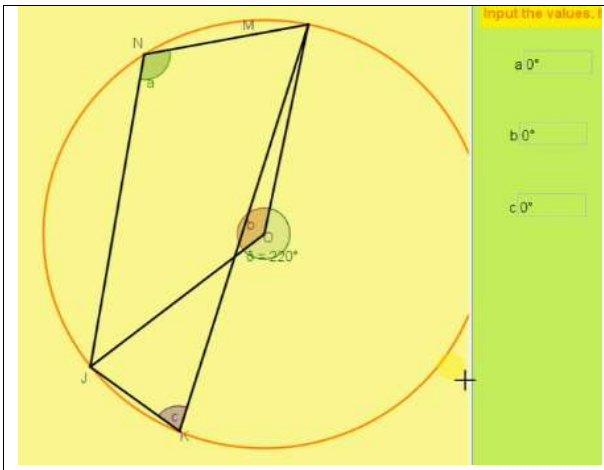
Frame 4.11: Daisy identifying cyclic quadrilaterals

Daisy still seemed to be fascinated by the discoveries that she was making. She now started to once again drag *B* around the circle. Then she moved the point *E* towards the circumference (3DC). She kept on moving the point outside the circle and inside the circle, trying to place it on the circumference of the circle. She reflected that she had dragged the point outside the circle for two reasons (3CNJ). Firstly, to verify whether $A\hat{E}C$ was half the angle at the centre; and secondly, to ensure that the interior opposite angles were supplementary in a cyclic quadrilateral. She concluded that when the opposite angles did not add up to 180°, then the points could not make a cyclic quadrilateral. The points *BACE* in Screenshot 4.31 in Frame

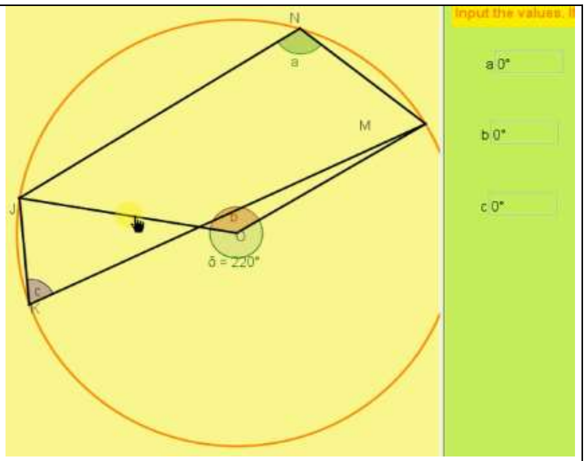
4.10 did not lie on the circumference of any circle because the opposite angles were not supplementary (5REV).

4.3.1.3.4 *Daisy analyses a problem seeking hidden structures*

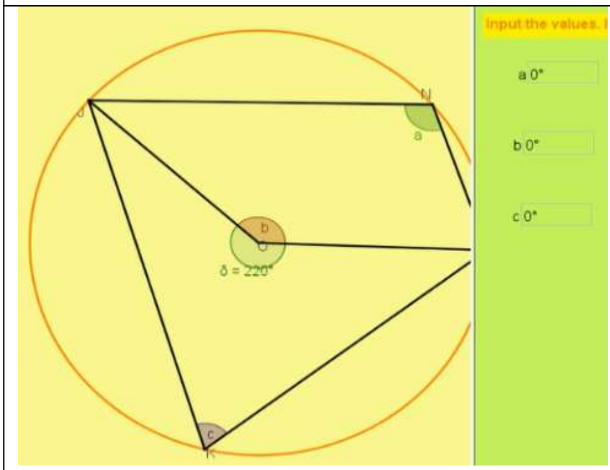
Daisy had a tumultuous start in answering the practice questions. This task seemed challenging for Daisy, and she began to drag all the points on the circle (3DC). One of the points on the circumference was deleted by mistake, and at one point the figure that she dragged became rather haphazard. She settled down when she dragged the points on the circumference to match the standard diagram of the *angle at the centre theorem* as shown in the sequence Screenshot 4.32 to Screenshot 4.35 in Frame 4.12. She was then able to relate the interior angle on the circumference to the angle at the centre (1EC), indicating that she was able to identify and apply the appropriate procedure (4TOL) and execute the algorithm correctly (4AP). When questioned, she could explain the procedure confidently (4EP). In solving *b*, she could relate her prior knowledge on angles around a point (6LPK) and apply basic procedures accurately (6BP). Further, in solving c, she could relate the angle at the centre to the angle on the circumference (2UVI, 1EC) and apply the *angle at the centre* theorem appropriately (4TOL and 4AP). In her subsequent attempt, she again dragged the angle *c*, this time very cautiously, to get a better idea of the angles (3DC). She then solved them in Screenshot 4.37 in Frame 4.13. It was observed that she relied heavily on the calculator to find the answers.



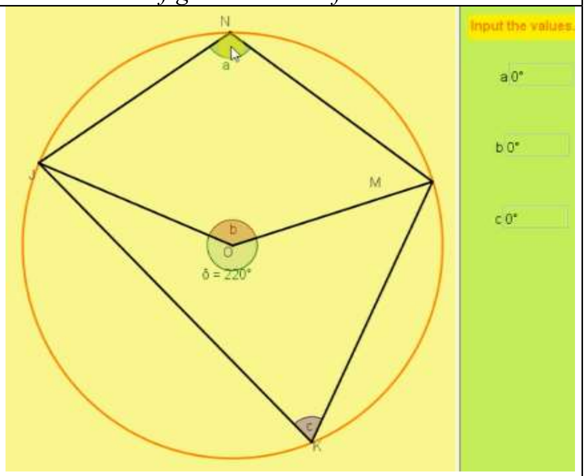
Screenshot 4.32: Daisy baffled by the diagram



Screenshot 4.33: Daisy drags a point but the entire figure was transformed



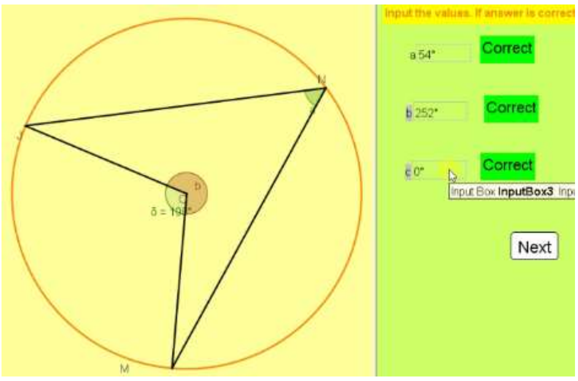
Screenshot 4.34: Daisy drags point K

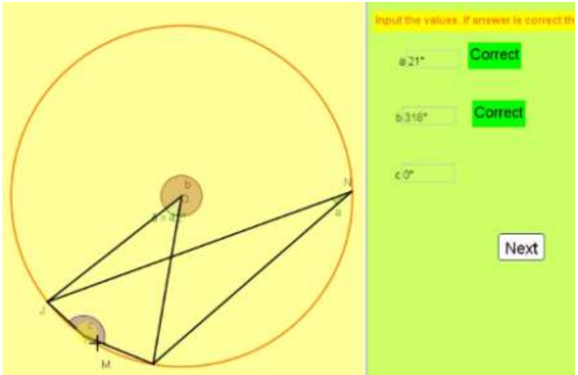


Screenshot 4.35: Daisy drags point N

Frame 4.12: Daisy's tumultuous start in practising questions in Applet 1.6

Thereafter, I no longer saw her dragging the points as she solved the problems. After a few successful answers, she misinterpreted the diagram gave an incorrect answer for angle a. Despite this, she did not drag the points. Instead, she skipped a without solving it correctly and solved for angle b, then angle c and then finally came back and solved for a. During the reflective interview, I challenged her to solve for c without the value of b. She accepted the challenge and answered fluently using the *properties of a cyclic quadrilateral theorem*. Once she understood the question, she was quick to grasp the cyclic quadrilateral (2MC) and applied the *opposite angles of a cyclic quadrilateral theorem* (4AP and 4TOL). However, she shortened the word quadrilateral to 'quad'. This, according to my experience, appears to be common practice amongst many learners and teachers.

<table>
<tr>
<td colspan="2">

Screenshot 4.36: Daisy's correct answers with a deleted point</td>
<td>

Screenshot 4.37: Daisy again drags angle 'c' to solve another question</td>
</tr>
<tr>
<td>Reflective Interview</td>
<td colspan="2">R: Let us look at the question of your recording? Here's the question. You see that angle given is 108°, and found a is equal to 54°. Let me stop here. Skip 'b' and do the next question i.e. find 'c'
Daisy : we first find 'b' then half of it is 'c' angle at the circumference is half at the angle at the centre.
R: I mean another method
Daisy: 'a' plus 'c' is 180° opposite angles are supplementary of cyclic quad. So it's 180-54. 2MC 4AP 4TOL</td>
</tr>
</table>

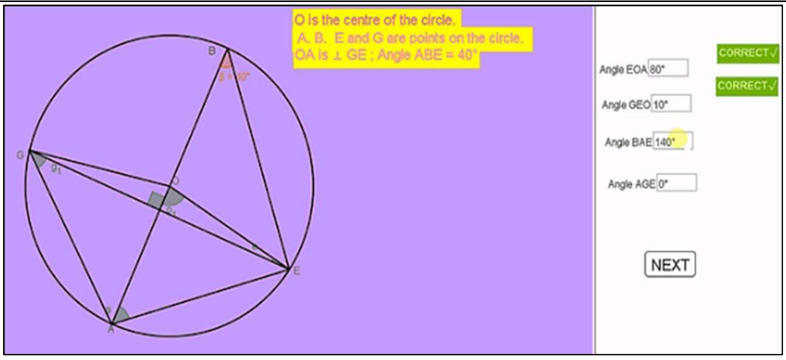
Frame 4.13: Daisy's correct answers

4.3.1.3.5 *Daisy implements different strategies in solving question Applet 1.7*

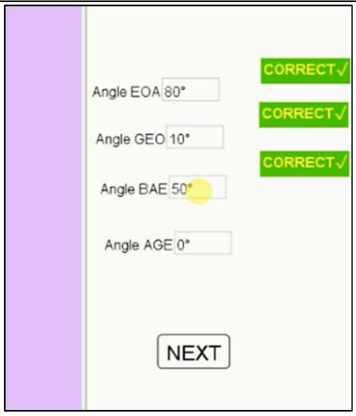
For the second practice question, Applet 1.7, Daisy recollected that initially it was challenging to relate the angles. One of the arms of the angle at the centre and that of the angle on the circumference is shared by a line segment $\overline{AOB}$ which is also a diameter. Hence this posed the challenge of recognising that arc AE and the angles are subtended. Nevertheless, observing them carefully, she could see that the angle at the centre $A\hat{O}E$ and the other on the circumference $A\hat{B}E$ were subtended by the same chord AE (2MC). She doubled the given angle (1EC and 4TOL) to obtain the correct answer (4AP). In the second sub-question, she made use of previous results and her prior knowledge of the *sum of interior angles of a triangle property* (6BP). She could identify and translate symbols (6LPK) and mathematical language into an equation (2MC) in order to determine the required angle, $G\hat{E}O$ accurately. While explaining her procedure to me, she employed the *exterior angle of a triangle property* (3CRT) for solving the required angle, showing her ability to select an efficient procedure for a given problem, as shown in the transcript in Frame 4.14.

In solving the angle for $B\hat{A}E$ in the classroom, she started with a wrong answer as shown in Screenshot 4.38 in Frame 4.14. Clearly, she either misinterpreted the *sum of angles of a triangle theorem* or mistakenly applied the *opposite angles of a cyclic quadrilateral theorem*. She moved away from her seat to have a discussion with one of her classmates. (That was the only instance she moved from her seat). She returned to her seat and then solved it correctly. When she solved the problem correctly, her friend came to her and sought an explanation. She could narrate the procedure (4EP) that she followed to find the solution. She employed her prior knowledge on radii, (6LPK) and identified ΔOAE as an isosceles triangle (2MC). She applied two properties of triangles to solve it (6BP), namely the *sum of interior angles of a triangle* and the *equal angles opposite to equal sides property.*

Later, she explained to me another way to solve the problem, i.e. using the angle subtended by a diameter theorem, indicating versatile methods of problem solving (5PMW) in the transcript in Frame 4.14. In her explanation, she said that the angle at the centre subtended by a diameter was 180° and hence the angle subtended by the diameter on the circumference was 90°, which was 180° when doubled. Here, she expanded on her concept of the *angle subtended by a diameter property*, providing an appropriate rationale for the *angle subtended by diameter theorem* (2MC). Though she mentioned that she applied the *sum of angles of a triangle property* (6BP), in her calculator, she punched in 90 - 40, indicating that she was able to efficiently consolidate her knowledge. (4TOL).



Screenshot 4.38: Daisy's first incorrect attempt at the third sub-question



Screenshot 4.39: Daisy's second correct attempt

Classroom observation	**Daisy**: [to friend] *Come here. Let me show you this.* Her friend approaches her and sits down beside her. *Friend: Ok can you explain this me?* ***Daisy**: Since O is 80° and OAE is an isosceles triangle [isiXhosa]. 180 minus 80 is 100 therefore A and E is 50 and 50 [IsiXhosa]* *L: [IsiXhosa] – is this another one?* ***Daisy**: [IsiXhosa] – no it's the same. Look OA and OE is same, radii, 6LPK and OAE is isosceles. 2MC Sum of angles is 180. O is 80° 180 minus 80 is 100 [types in calculator] look its 100 and divided by 2 is 50. Look A is 50. 6BP Angles opposite to equal sides of isosceles triangle. 4EP*
Reflective Interview	*R: Of course O I got you understood that when you realise the arc. What about angle $\hat{E}$?. What about angle $G\hat{E}O$?* ***Daisy**: I used the rule that says that the interior angles of a triangle should add upto 180°. So I saw that here $\hat{O}_1$ and here should be 90°, because EA is perpendicular to GE 6LPK So I used angle $\hat{O}$ that I got then I said 40 times 2 I got angle $\hat{O}$ and then $\hat{O}$ plus $\hat{E}$ is 90°, 3CRT and 2MC exterior angle is equal to sum of its opposite angles in a triangle 6BP*
	R: What about angle BAE? ***Daisy**: Ohh…I think I said that since this is a line. Angle A BAE, since it is a line and centre O since it is the diameter. Obviously the angle is 180°. So the angle at the centre is 180°. It is double the angle at the circumference E. So that is how I got it as 90°, the whole E 2MC. [her reasoning in class was using isosceles triangle] 5PMW and 4AP* *R: How do you get A?* ***Daisy**: A? Hmmm … aahh I said 90 ° here* *R: Angle BAE* ***Daisy**: I said it is 40°, right? Oh sorry, I say it is 40 °here and 90 °here so I calculated it using the rules of a triangle – sum of angles in a triangle is 180° 2MC and 4AP [she typed in calculator as 90-4] 4TOL*

Frame 4.14: Daisy's solution for practice question Applet 1.7

4.3.2 **Paul's lesson**

4.3.2.1 **Overview of the lesson**

The lesson was scheduled for a Saturday morning in order to use the computer lab for an hour and a half. Mr Paul faced several challenges during his lesson. Due to glitches in the computer hardware, there was a delay in starting the class. Although they were enough computers to accommodate all of the 48 learners, a number of desktops and monitors were not working. Some learners, therefore, had to share the computers. Almost 25 minutes were lost before he started to focus on the *angle at the centre theorem* using *GeoGebra*. Prior to this lesson he had asked the learners to do an investigative pen and paper task on circle theorems. Paul wanted to provide an overview of the theorems that were linked to the *angle at the centre theorem*. Employing his pre-designed applets, he spent the first twenty minutes of the lesson explaining and exploring different aspects of the theorem. He started with the Applet 1.5 shown in Figure 4.5 (p. 77). He then allowed the learners to do the practice questions for the rest of the lesson. Paul was with the learners guiding and supporting them to solve the questions. During the last five minutes of the lesson, using the pre-designed applet, he explained to the learners why the angle at the centre was double the angle on the circumference.

The introduction of the applets received mixed reactions from the learners. Some struggled to open the file and some could not hold the applet on their screens due to poor mouse control. Very often, particularly during the initial part of the lesson, Paul interjected and told them to close and re-open the file. Many learners were excited to drag the points. The learners found it interesting when they saw that the angles were changing as they moved the slider. As the teacher dragged the slider (Applet 1.5), to fix the angle at the centre at 180°, one or two learners were quick to identify the *right-angled triangle subtended on the circumference of the circle*. As the lesson progressed, especially while doing practice questions, learners cheered when they saw that their answers were correct. Paul moved amongst the learners guiding them to solve the problems. Often two particular learners would stand up and dance when they completed answering the question correctly, seeking the attention of the class. But most of them were either busy with solving the angles or exchanging words over their computers. Learners sought help from all around the classroom to solve the problems successfully.

Although I focussed on two learners from this class, Aster and Jasmine, the in-depth analysis that follows is only that of Aster's interactions with *GeoGebra* and his responses during the reflective interview. Jasmine's classwork was similar to that of Aster and other learners, however, due to the limited scope of this thesis, I will not fully report on the story of Jasmine. When pulling together all the indicators and seeking answers to the research questions, I will include her interactions with the DGS and her words in the interview.

4.3.2.2 **The story of Aster**

4.3.2.2.1 *Aster considers various connections of the theorem through exploration*

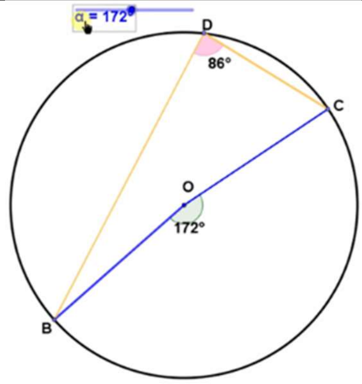
Aster started to explore the theorem as instructed by the teacher. He dragged point D on the circumference of the arc CB very rapidly. From the screen recording, it appeared as if he was having fun playing around with the points. When asked to move 'α' to get different values for the angle at the centre, Aster moved the text $\alpha = 172°$, shown in Screenshot 4.40 in Frame 4.15. The teacher had to intervene to help him to drag the slider. The teacher asked the learners to explore using different values of α and did not mention any specific values. Aster dragged the slider to $\alpha = 180°$ and stopped there (3DC). The movement of the points slowed down. Then he moved the slider for different values of α closer to 10° and to the maximum of 300°, as in Screenshot 4.42 in Frame 4.15 below. He always stopped at $\alpha = 180°$ (3DC). As the teacher asked the class to drag the angle at the centre to 180°, Aster anticipated and spoke to himself that it was about the angle in the semi-circle (IEC and 3CNJ) shown in Screenshot 4.42 in Frame 4.15. When asked by the teacher, he said that $\overline{BC}$ was a hypotenuse. Aster could identify a right-angled triangle in the figure (2MC and IEC). Again, he started to drag point D around the circle, but this time more cautiously (3DC). When encouraged by the teacher, Aster and his classmates could see $\overline{BC}$ as the diameter of the circle (2MC and IEC).

In the Applet 1.2, Aster was seen dragging the points all around the screen, exploring and verifying his answer. The teacher provided explanations about the applets and common misconceptions about angles that were not on the circle. Aster was busy exploring and confirming his own conjectures (3DC) in the screenshots in

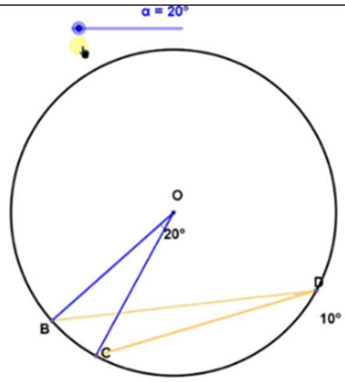
Frame 4.16. He first dragged point E and placed it over the circle at point C as he wanted to verify that the angles were of equal distance on the circumference shown in Screenshot 4.44. In the reflective interview, he said that he wanted to check if $A\hat{E}B$ would still be half the angle at the centre if it was on the circumference (3DC). After placing E outside the circle, he dragged point F to the other side of the arc in Screenshot 4.45. He was aware that he might not have achieved the desired results, therefore he dragged point D and placed it over point F. He said that he dragged point D over point F to measure the angle inside the circle, but $A\hat{D}B$ also gave only the reflex angle. His purpose was to verify the *opposite angle of a cyclic quadrilateral theorem* (3CNJ). Aster could not recollect the techniques of measuring angles in the software. *GeoGebra,* measures the angles only in one direction. $A\hat{D}B$ was measured by clicking on point A, point D and then point B shown in the second and third figures in

Frame 4.16. So, when point *D* was above point *A* and point *B*, arc *ADB* appeared clockwise, but when point *D* was below point *A* and point *B*, arc *ADB* appeared anticlockwise, hence the reflex angle was measured when dragged on the other side of the arc.

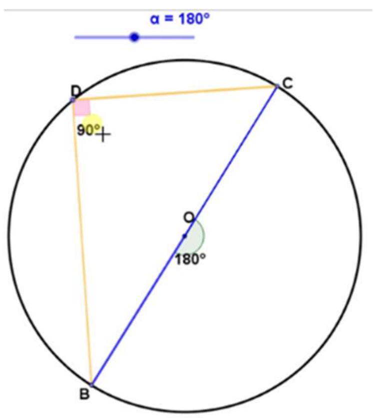
Classroom observation



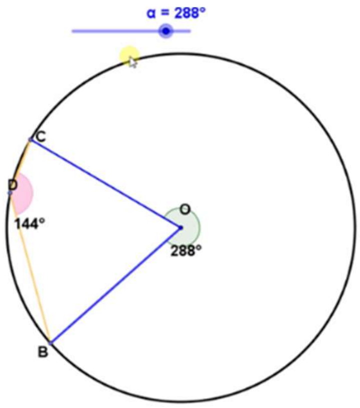
Screenshot 4.40: Moves text α instead of slider



Screenshot 4.41: Slides α to 20°



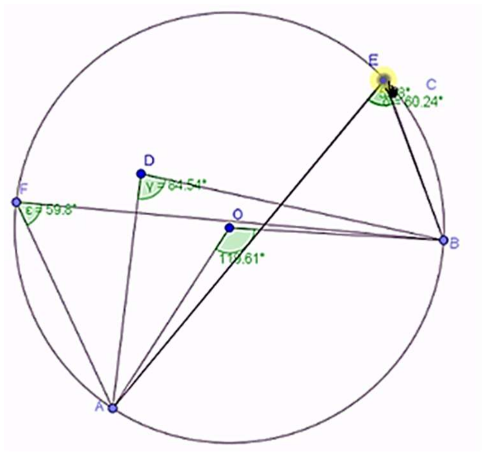
Screenshot 4.42: Slides again α to 180° and drags D



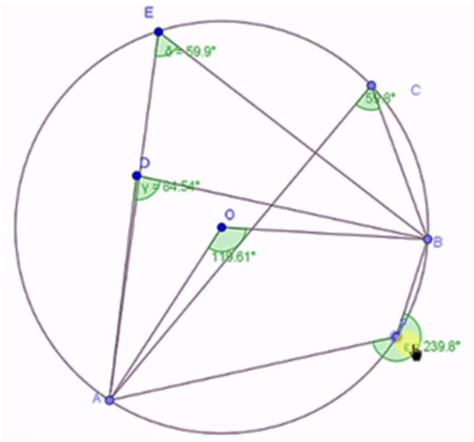
Screenshot 4.43: Slides α to 288°

T: There's a special theorem that you must know. What will be angle D if our alpha is at 180° ? I want you to calculate.
***Aster**: It's 90° 4AP*
T: Yeah Now move alpha to 180
***Aster**: Angle in a semi-circle theorem – drags the point D around the arc   IEC and 3CNJ*
T: What's the angle at the centre?
***Aster** and other Learners: 180 °*
T: And what's angle at the circumference?
***Aster**: 90 °*
T: So, there would be so many questions that will link to angle 90°. You can see that angle at the centre is now 180. Therefore, BC is called, what?
***Aster**: Hypotenuse   2MC and IEC*
T: Yeah its hypotenuse when you see the triangle. What is BC called when you see the circle?
***Aster** and other Learners: Diameter. 2MC and IEC*
T: Yes, BC is also a diameter. Are we together?
Learners: Yes sir

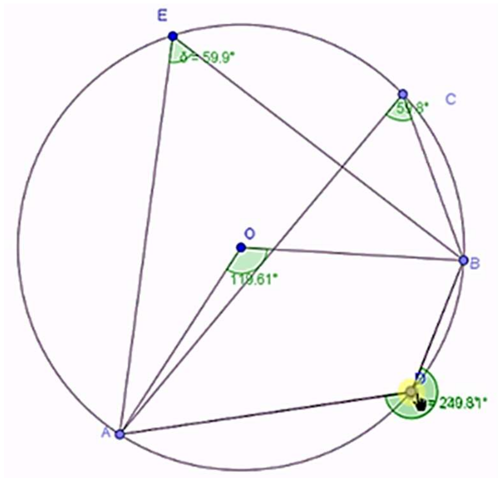
Frame 4.15: Aster's exploration in Applet 1.5



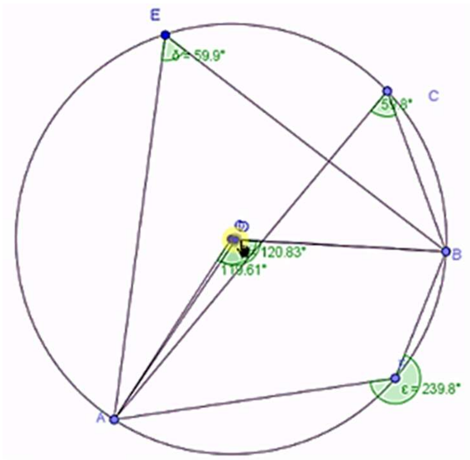
Screenshot 4.44: Aster drags E over C



Screenshot 4.45: Aster drags F on the other side



Screenshot 4.46: Aster drags D over F



Screenshot 4.47: Aster drags D over centre

Frame 4.16: Aster exploring and verifying

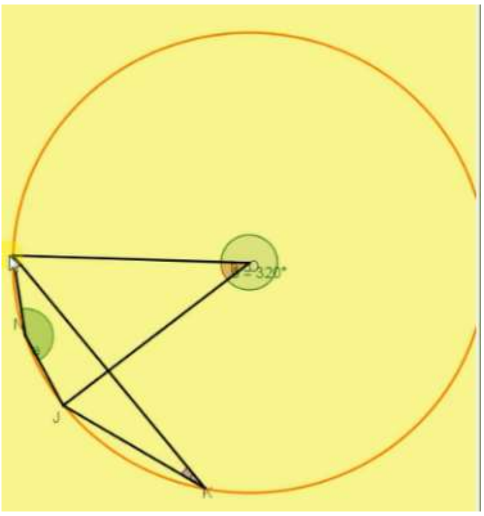
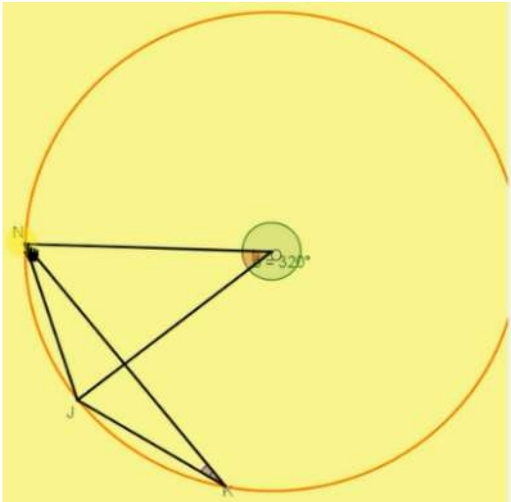
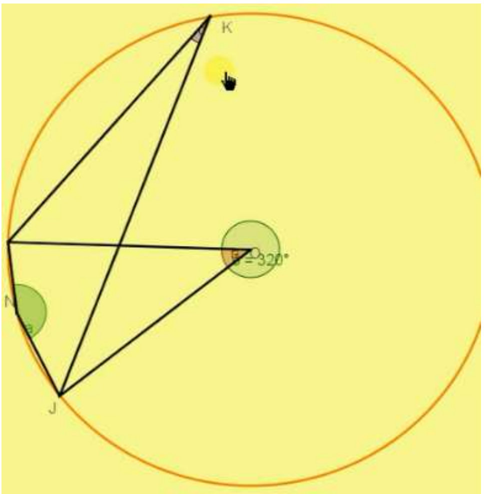
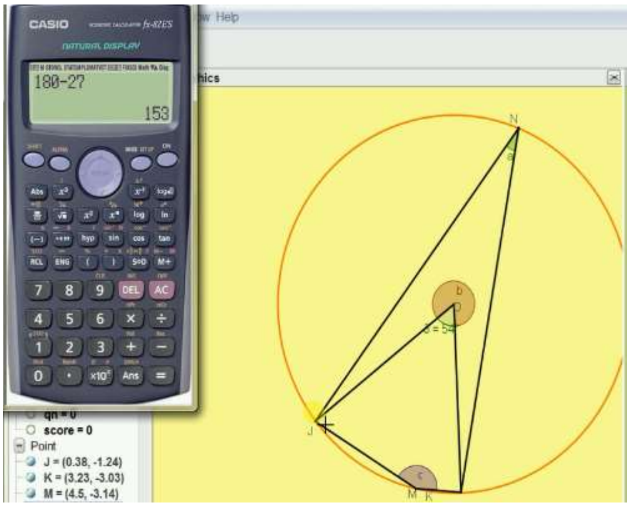
4.3.2.2.2 *Aster analyses the problem and solves it, applying relevant mathematical concepts*

Aster did the practice questions in the class and most of his answers were correct. However, for the purpose of this analysis, I present his answers to the practice questions during the reflective interview as it provided me with a good opportunity to make sense of his understanding. More often than not, Aster did mental calculations, but occasionally he used the calculator. I advised him to use the 'Casio Simulator' on the computer whenever he wanted to use the calculator.

For the first set of practice questions (on Applet 1.6), because he wanted to obtain a better view of the figure, he dragged the points on the circumference as is shown in the Screenshot 4.48 to Screenshot 4.50 in Frame 4.17. He correctly answered the first question i.e. to find a. He could identify the relationship between the angles and then execute the calculations accurately (1EC

4TOL and 4AP). In answering b, he linked the angles to his prior knowledge on angles around a point, (6LPK) applied the basic procedure (6BP) using the calculator and obtained the correct answer. He dragged the point K to his advantageous position, identified the angle to relate with the theorem and performed the calculations accurately (1EC, 4TOL and 4AP). In the subsequent question, again he dragged the point N to a vantage point and then solved the required angles. Thereafter he no longer needed to drag the points until he encountered a problem. Nevertheless, he tilted his head, moved his fingers across the lines and formed angles as he solved the problems.

As he was solving the question, I asked him to find c without solving for b. He immediately dragged the point N along the circumference as seen in Screenshot 4.51 in Frame 4.17. He gave up and said that it was a 'cyclic quad' and did not know of any method of finding the c angle on the circumference. He moved on to next question, and after solving a he told me that he had found a way out to solve c directly. He had already identified the cyclic quadrilateral (2MC), but now he applied the *opposite angle of a cyclic quadrilateral theorem* (4TOL) and did the calculations accurately (4AP). I further probed whether he could find b without using the *angle at a revolution property*. He appropriately identified the angles at the centre and on the circumference (1EC). He then reversed the procedure (5REV), i.e. by doubling c, and performed the calculations proficiently (4AP). He continued with the similar questions and his procedure was to solve for a first, then c and lastly b.

 *Screenshot 4.48: Aster drags M and the figure moves*	 *Screenshot 4.49: Aster then Drags N but does not move beyond M*
 *Screenshot 4.50: Aster drags K before solving*	 *Screenshot 4.51: Aster solving 'c' without finding 'b'*

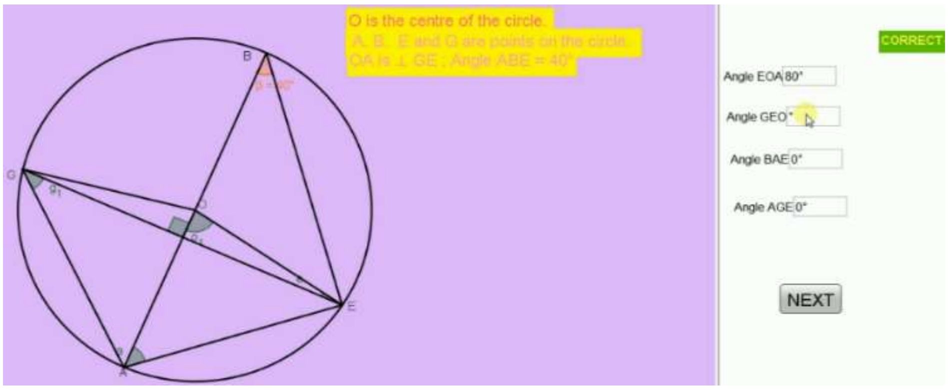
Reflective Interview	*R: now find the angle c without finding the value of b.* ***Aster**: a is 160. Hmm…. c is on the circumference. (He does not complete the question and clicks on 'next'.)* ***Aster**: I think it's a cyclic quad. (But doesn't calculate.) No I cannot find c without b because b angle at the centre and c is on the circumference.* *R: So how can you find c* ***Aster**: No sir we cannot find c without b. Is there a method?* *R: I don't know but there should be a way.* ***Aster**: I don't think so. But immediately thereafter he keys in 140 and clicks on 'next' (without waiting to see if its correct or not)* *Takes on the new question. Gets 'a' correct. . 1EC , 4TOL and 4AP* *R: You can use the calculator if you want to.* ***Aster**: I found a way to find 'c'. Clicks on the calculator. 180-27 is 153. I found a way sir, 153 is opposite side of a cyclic quad. 2MC , 4TOL , 4AP, 5PMW and 5REV*

Frame 4.17: Aster's answers to practice question Applet 1.6

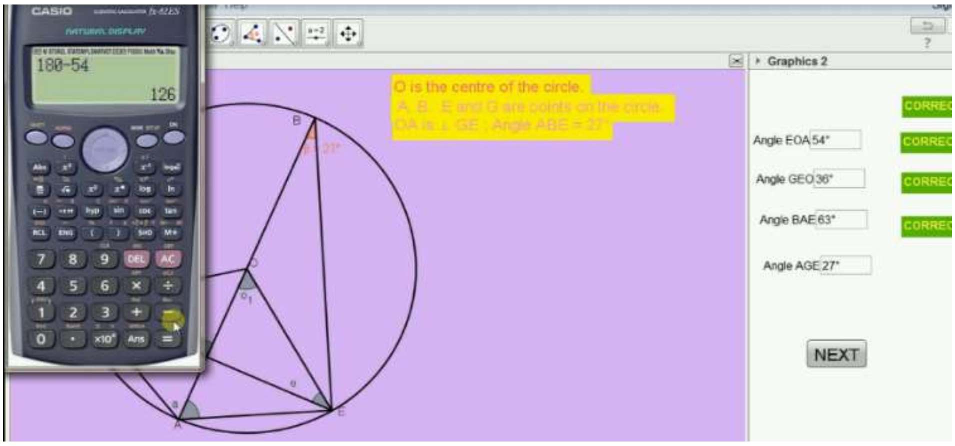
4.3.2.2.3 *Aster reconciles different strategies and makes purposeful use of procedures*

In the classroom, Aster answered the second practice question for Applet 1.7 quite easily. He did not make any attempt to drag the points. Occasionally he would gesticulate while attempting to draw on the screen. Only in his first attempt did I see an incorrect answer. This was due to a mistake he made with his mental calculations. He was sure of his procedure therefore he used the calculator, and the answer was correct. One of his friends, Lily, sought help from him to find $A\hat{O}E$. Aster explained in very clear terms (shown in Screenshot 4.52 and the transcript in Frame 4.18), about the *angles subtended by the arc* and the application of the *angle at the centre theorem*, (2MC, 1EC, 4TOL and 4AP), demonstrating not only his skill in problem solving but also in explaining mathematical ideas to others (4EP).

During the reflective interview, I asked him to talk about the procedure that he employed. He explained an entirely different method from his procedure in class, shown in the transcript in Frame 4.18. He identified ΔBOE as an isosceles triangle (2MC), because the $\overline{BO}$ and $\overline{EO}$ formed the radius of the same circle (6LPK). He further applied the *properties of an isosceles triangle* in conjunction with the *exterior angle of a triangle property* (4TOL) and solved E$\hat{O}$A (6BP and 4AE). He further exhibited multiple strategies in problem solving, (5PMW). In the second sub-question to solve for G$\hat{E}$O, Aster identified the basic mathematical symbols of representing perpendicular lines (2MC). He applied the exterior angle property of a triangle to solve G$\hat{E}$O (4TOL and 6BP). This time I did not prompt him to use an alternate method, but he voluntarily employed this innovative technique. He also clarified that one of his friends did not understand how to apply the theorem to solve for the angle. He thus thought of this approach to solve the problem.



Screenshot 4.52: Aster explains to Lilly after getting the correct answer during the lesson



Screenshot 4.53: Aster uses Casio Simulator to solve for angle BAE during reflective interview

Classroom Observation	*Lily: Aster, can you help me here. How did you find EOA.?* ***Aster**: Angle at the centre is twice the angle of the circumference and hence EOA = 80.* *Lily: What is it?* ***Aster**: It's the theorem what we discussed. Ok. Look at arc AE Can you see that?* *Lily: Hmm…* ***Aster**: Now see that angle ABE it's on the circumference which is given as 40 in my computer.* *Lily: Ok I can see that* ***Aster**: Then look at AOE it's on the centre. We discussed in the class that angle at the centre is twice the angle on the circumference. Therefore, 40 times 2 is 80. 80 is the answer. 4EP* *Lily: Hmm… I can see that now.*
Reflective Interview	***Aster**: 27 multiplied by 2* (uses a calculator). *Look at triangle OBE an isosceles triangle. Why? Because OB = OE remember O is the centre of the circle. OB and OE are radii. Then EOA is the exterior angle of triangle OBE. Given beta equals 27, therefore OEB is also 27, angles opposite to equal sides. Angle O1 i.e. angle EOA is the exterior angle of isosceles triangle OBE. And exterior angle is equal to the sum of the interior opposite angels. Its 27 plus 27 i.e. 27 times 2. 6LPK – 2MC – 4TOL – 6BF – 4AE 5PMW* *R: How you were able to think this method? What is in your mind? I heard you explaining (to Lily) using angle at the centre theorem.* ***Aster**: My friend asked me to explain him to solve for that angle, but I think he did not get what I was telling him. So I thought of an alternate method since I saw OB and OE as radius.*

Frame 4.18: Aster answers practice question Applet 1.7

For the third sub-question to solve for $B\hat{A}E$, Aster could identify the line segment $\overline{AB}$ as a diameter and connect it to the *angle subtended on a semi-circle theorem* (2MC), to obtain $B\hat{E}A$

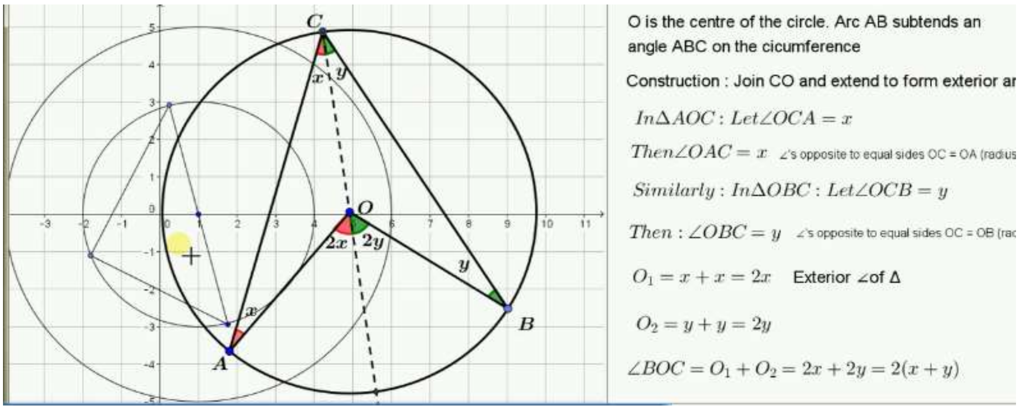
as 90°. Using a condensed procedure ($90° - 27°$), he solved the angle (3CRT and 4AP). As I probed further, he provided another method from his earlier procedure, similar to the procedure employed by Daisy (discussed above) (5PMW). He considered the isosceles triangle ΔAOE with $\overline{AO}$ and $\overline{EO}$ as the radius, applied the *property of isosceles triangles*, coming up with the *sum of interior angles of triangle* justification, and then solved $B\hat{A}E$. See Screenshot 4.53 in Frame 4.18. The last question was straight-forward, and he used the *angles subtended by the same chord theorem* (2MC and 4TOL).

4.3.2.2.4 *Aster seeks patterns and relationships and unpacks hidden structures*

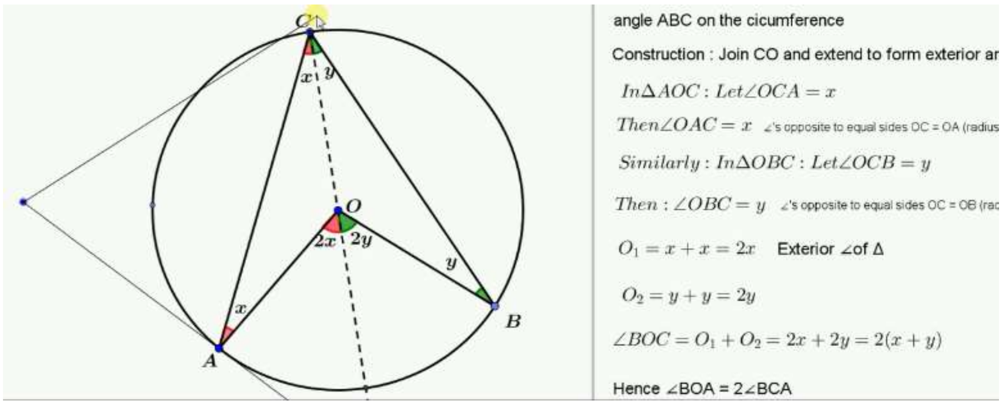
During class, as the teacher was explaining the proof of the theorem, Aster was busy exploring — as shown in Screenshot 4.54 and Screenshot 4.55 in Frame 4.19. He clarified that he was trying to draw and verify whether the angle subtended by a diameter was a right angle on the circumference (3DC). He constructed lines outside the circle to establish whether the tangents from a point outside the circle were equal in distance (3DC). He could not, however, complete his investigation as he did not know how to measure lengths and angles in the software. (Later, after the reflective interview, I showed him how to measure the angle on the circumference and the length of the tangent lines).

Initially, Aster could not answer why the angle subtended by a diameter was 90°. Interestingly, he claimed that they were previously taught the theorem without being given the proof or reasons. I played that section of the screen recording where he was dragging the angle at the centre to 180° (Screenshot 4.42 in Frame 4.15). He then joked that it was because the *angle subtended by a diameter at the centre was 180°*, and therefore *the angle on the circumference would be its half*, hence 90°. He then connected both the theorems (2MC).

When asked about the reasons why the *angle at the centre was twice that of the angle on the circumference*, he was cautious in his reply. He carefully observed the figure (below) as in Frame 4.19 and gesticulated before answering. After almost a minute, he was able to see how the radii formed an isosceles triangle (2MC). He then linked this to his previous understanding of the exterior angle of a triangle (4TOL). He had already solved a similar problem using these geometric properties. See the transcript in the reflective interview in Frame 4.18.



Screenshot 4.54: Draws circles, a 'diameter' and subtends an angle on the circumference.



Screenshot 4.55: Draws tangents to a circle from a point a outside

Reflective Interview	*R: Why angle subtended by a diameter is 90°* ***Aster**: We were never taught sir. It's a theorem sir.* *R: You said you ever never taught about angle subtended by a diameter? Look at this screen recording of yours. Here you are dragging alpha to 180 many times and see what angle is formed on the circumference.* ***Aster**: Oh yeah yes now I remember. I can see angle formed by a diameter at the centre is 180° therefore its half will be on the circumference, therefore angle subtended by a diameter on the circumference is 90° because it forms a straight line with the centre. (2MC)*
	R: Why you think that the angle at the centre is double the angle on the circumference. ***Aster**: Why it's double?* *R: I mean why its double and not equal or triple?* ***Aster**: I think it's because… hmmm …* *R: Take your time.* ***Aster**:* (The screen recording was paused, and he was carefully observed carefully the figure on the screen. Draws lines and angles over the screen with his fingers. He spoke after a minute.) *Because radius makes an isosceles triangle (draws a triangle over the screen). Exterior angle O is twice the sum of the interior opposite angles. (2MC and 4TOL)*

Frame 4.19: Aster explaining the reasons behind the theorems

4.4 DISCUSSIONS AND INSIGHTS – HORIZONTAL ANALYSIS ACROSS PARTICIPANTS

In this section, I analyse my data across the participants to obtain deeper insights into how they made mathematical meaning and developed mathematical proficiency. I do this by comparing and contrasting different perspectives of learners' interactions with the DGS. Based on the work of Carter et al. (Carter et al., 2009) on meaning-making and Kilpatrick et al. (2001) on mathematical proficiency, I identified eighteen such observable activities in total and categorised them under the seven broad indicators, BI-1 to BI-7 (refer to VAMMPA framework Table 3.2, chapter 3 on p. 62). The horizontal analysis of the data, category-wise, showed that the participants demonstrated meaning-making (MM) traits and evidenced their conceptual understanding (CU) and procedural fluency (PF). Nonetheless, there was sparse evidence of actions indicating different approaches to solving problems (BI-5) and reflections on their solutions (BI-7) .

4.4.1 Essence of interactions - BI-1 (Defining and identifying relevant mathematical concepts, relations and notations)

4.4.1.1 Synopsis

Daisy:

- defined the *angle at the centre* theorem (CU);
- used flexibly and accurately the *half the angle* or *double the angle properties* appropriately (CU);
- considered angles as a pair of angles with one of them above the centre (MM);
- established the special case of the theorem (MM and CU);
- defined the *opposite angles of a cyclic quadrilateral theorem* (CU); and
- displayed knowledge of one of corollaries of the theorem *angles subtended by the same chord or arc are equal* (CU).

Jasmine:

- defined the *angle at the centre* theorem (CU);
- defined the *angles in a semi-circle* theorem (CU);
- defined the *opposite angles of a cyclic quadrilateral* theorem (CU); and
- displayed knowledge of the theorem *angles subtended by the same chord or arc are equal* (CU).

Rosy:

- defined the *angle at the centre* theorem (CU);
- used flexibly and accurately the *half the angle* or *double the angle* properties appropriately (CU);
- considered angles as inside and outside the polygon (MM);

- demonstrated knowledge of the special case of the *angle at the centre theorem* (CU); and
- displayed knowledge of one of corollaries of the theorem *angles subtended by the same chord or arc are equal* (CU).

Aster:

- defined the *angle at the centre* theorem (CU);
- deducted the special case of the *angle at the centre theorem* (MM and CU);
- established the *opposite angles of a cyclic quadrilateral theorem* (CU); and
- displayed knowledge of one of corollaries of the theorem *angles subtended by the same chord or arc are equal* (CU).

4.4.1.2 **Discussion**

All participant learners demonstrated their acquisition of knowledge in explaining and defining the relevant mathematical concepts. Rosy and Daisy flexibly used ‘half’ or ‘double’ as they explained the angle at the centre. Jasmine referred to the *angle at the centre* as ‘theorem 3’. The textbook used in schools referred to theorems by numbers, such as ‘Theorem 3’ followed by the statement of the theorem, thus Jasmine tended to use this term and referred to one of the corollaries as ‘Theorem 4’. Aster, however, preferred to use “*angle at the centre is double the angle at the circumference”*, as a full sentence, often omitting the phrase “*subtended by the same arc”*.

> ***Rosy:*** *I am saying the angle the angle at the centre the angle in the circle is not half the angle of the circle because the angle is not in the circumference.*
>
> *…*
>
> *The angle at the centre is double the angle at the circumference provided they are subtended by the same chord or arc.*
>
> ***Daisy:*** *I learned that in order for the like see the angle in the circle, in order for the angle in the circumference to be half the angle at the centre, it has to be on circumference.*
>
> *…*
>
> *See when angle on circumference is dragged down the angle is, I mean new pair of reflex angles which is half the angle at the centre.*
>
> ***Jasmine****: you mean theorem 3. It states that the angle at the centre is twice the angle at the circumference subtended by the same arc or chord.*

Rosy and Daisy demonstrated their meaning-making habits as they interpreted the *angle at the centre theorem* in their own ways. Carter et al. (2009) observe that reasoning and sense making habits provide learners with a context for mathematical ideas and the application of them. Rosy considered the angle at the circumference as an angle inside the polygon and the angle at the centre as an angle outside the polygon. An arc subtending an angle at the centre and at the circumference thus forms a polygon (a quadrilateral) inside the circle. For Daisy, the angle at the circumference was above the angle at the centre and she considered these angles as a pair of angles that demonstrate the *angle at the centre theorem*. Both chose a visual reference that conveyed the important aspects of the theorem. For Jasmine it was just another theorem number as she was influenced by the captions in the textbooks that used numbers for the theorems.

Rosy and Jasmine demonstrated their knowledge of the other theorems related to the *angle at the centre theorem* such as the *angles in a semi-circle theorem*, the *opposite angles of a cyclic quadrilateral theorem* and *angles subtended by the same chord or arc are equal* theorem. As Daisy justified her reasoning in solving $B\hat{E}A$ in Applet 1.7, she established the semi-circle theorem as a special case of the *angle at the centre theorem:*

> ***Daisy****: Since it is a line and centre O since it is the diameter. Obviously, the angle is 180°. So the angle at the centre is 180°. It is double the angle at the circumference E so that is how I got it as 90°*.

Aster deducted the *special case of the angle at the centre theorem* from his interactions with the Applet 1.5, shown in Screenshot 4.42 in Frame 4.15 (p.103). With Applet 1.4, see Screenshot 4.58 and Screenshot 4.59 in Frame 4.21 (p. 125). All participant learners identified and defined relevant mathematical concepts and applied them fluently while solving the angles.

4.4.2 **Essence of interactions - BI-2 (Patterns and relationships)**

4.4.2.1 **Synopsis**

Daisy:

- identified relevant geometric properties (CU and MM) – i) the *diameter is split into two radii*; ii) the *chord subtending an angle at the centre makes an isosceles triangle*;
- understood variant and invariant properties (CU) – non-examples of the *angle at the centre* theorem.

Jasmine:

- identified relevant geometric properties (CU and MM), such as that the *diameter is split into two radii*;
- understood variant and invariant properties (CU) – non-examples of the *angle at the centre theorem.*

Rosy:

- identified relevant geometric properties (CU and MM) such as that the *diameter is split into two radii*;
- understood variant and invariant properties (CU) – non-examples of the *angle at the centre theorem.*

Aster:

- identified relevant geometric properties (CU and MM) – i) the *diameter is split into two radii*; ii) the *chord subtending an angle at the centre makes an isosceles triangle*; iii) the *diameter as a hypotenuse*.

- understood variant and invariant properties (CU) – non-examples of the *angle at the centre theorem*.

4.4.2.2 **Discussion**

In Euclidean geometry, we find rich connections among concepts and theorems. Taking cognisance of the importance of visual literacy (VL) in education, Vermeersch and Vandenbroucke (2015) consider VL as an ability to discriminate and decipher what one wants to see. Duval (1999, p. 13) supports this by saying that "visualization (in mathematics) consists in grasping directly the whole configuration of relations and in discriminating what is relevant in it." My participants sought to identify relevant geometric concepts and recognised variant and invariant properties, but in varying degrees. All of them visualised the line segment passing through the given centre of the circle into relevant segments (radii, in fact), as they solved $A\hat{O}E$ in Applet 1.7, applying the theorem appropriately. Referring to Figure 4.7 (p. 79) of Applet 1.7, the line segment $\overline{AOB}$ was differentiated into two-line segments, namely segment $\overline{AB}$ inscribing $A\hat{B}E$ on the circumference, and segment $\overline{AO}$ inscribing an angle at the centre. Furthermore, the same line segment $\overline{AOB}$ was considered as a diameter and connected with the *angles in a semi-circle theorem* in order to solve $B\hat{A}E$ in Applet 1.7. In fact, Rosy was prompted by the teacher to identify a diameter in the figure. Daisy and Aster further recognised that the *angle formed at the centre by a chord resulted in an isosceles triangle, owing to equal radii.*

4.4.3 Essence of interactions - BI-3 (Looking for hidden structures)

4.4.3.1 **Synopsis**

Daisy:

- made preliminary conjectures and deductions through construction and dragging (MM);
- discovered concepts, such as pair of angles that demonstrate the *angle at the centre* theorem (CU); and
- dragged points to make the angle at the circumference above the angle at the centre (CU).

Jasmine:

- dragged the point not on the circle (MM), made deductions and verified conjectures (CU).

Rosy:

- discovered concepts such as the angles on other side of the arc (CU);
- dragged the point not on the circle (MM); and
- made deductions and verified conjectures (CU).

Aster:

- made preliminary conjectures and deductions through dragging (MM);
- discovered concepts such as *the angles in a semi-circle theorem* and *opposite angles of cyclic quadrilateral theorem*(CU);
- dragged points to coincide with the standard diagram of *angle at the centre theorem* (CU); and
- justified the theorems (MM) as he discovered the reasons behind them.

4.4.3.2 **Discussion**

Researchers suggests (Hölzl, 2001; Hoyles & Noss, 2003) that the interface of DGS provides an opportunity for students to conjecture and generalise by clicking and dragging objects, because it dynamically re-draws and updates information on the screen. In my research, it was evident that the participants switched between figures and concepts, and between empirical and theoretical considerations while dragging the objects. Initially, Rosy consciously dragged the free point inside the circle, shown in Screenshot 4.4 and Screenshot 4.9 in Frame 4.3 (p. 85), and inferred that the angle inside or outside the circle was not half the angle at the centre of the circle because the angle was not on the circumference (see the transcript in Frame 4.2 on p. 84). Like Rosy, Daisy dragged the points on the figure. In another instance, when prompted by the teacher, Through active construction and measurement of angles, Rosy and Daisy discovered the finer nuances of the theorem, such as the fact that *both angles should be on the same side of the arc* in order for the *angle at the* centre theorem to hold.

Rosy, however, did not explore the theorem further to look for the hidden structures as Daisy and Aster did. Both Jasmine and Rosy did not undergo the process of seeking and making conjectures. Carter et al. (2009, p. 56) suggest that conjecturing and exploring "gives students the opportunity to become immersed in, and deepen their understanding of, the mathematical relationships involved". Throughout, Daisy and Aster continued to drag the points during the practice applets. More often than not, Aster made discoveries which verified his conjectures. In the Applet 1.2, he dragged the free point at the centre onto the circumference. In the interview he reflected that he wanted to verify that the angle at the centre and the angle on the circumference were equal. Later, when I prompted him, he justified the ratio in the theorem. He visualised the isosceles triangle as shown in the transcript in Frame 4.19 (p.110), formed by radii and applied the *exterior angles of a triangle theorem*, thus rationalising the ratio in the theorem. Significantly, earlier he applied a similar method in solving $A\hat{O}E$ in Applet 1.7.

4.4.4 Essence of interactions - BI-4 (Purposeful use of procedures)

4.4.4.1 Synopsis

Daisy:

- executed procedures accurately (PF);
- justified the procedures (MM and PF);
- applied the theorem correctly (MM) including the *exterior angle of a triangle property*; and
- competently used the procedure to find the third angle in a right-angled triangle (PF).

Jasmine:

- accurately executed procedures (PF);
- justified the procedures (MM and PF);
- appropriately applied the theorem (MM); and
- competently used the procedure to find the third angle in a right-angled triangle (PF).

Rosy:

- executed the procedures accurately (PF);
- justified the procedures (MM and PF);
- applied the theorem correctly (MM); and
- competently used the procedure to find the third angle in a right-angled triangle (PF).

Aster:

- accurately executed the procedures (PF);
- justified the procedures (MM and PF);
- appropriately applied the theorem (MM), including the *exterior angle of a triangle property*; and
- competently used the procedure to find the third angle in a right-angled triangle (PF).

4.4.4.2 Discussion

All participant learners indicated accurately and purposefully the procedures they used, and they enthusiastically completed the practice questions. In other words, the correct answers inspired them to undertake more practice questions. It was intriguing to observe that at the beginning, Daisy and Aster, took their time before solving the first practice question. Both were in the exploratory mode, consciously dragging the points around, seemingly discovering and understanding the concepts before endeavouring to solve the problems. Once they conceptualised the idea, they could swiftly solve the questions. Discussing the typological differences in mathematical abilities, Krutetskii (1976) describes 'geometric type' as a learner who may start slowly but can progress quickly in learning a task and attaining a stable formation of ideas. Daisy and Aster developed their reasoning slowly, step by step, "omitting no flight of thought" (Krutetskii, 1976, p. 192). It is not that time is an essential factor in higher achievements, but the quality of time spent in the educational journey and the insights gained en route (Howson, 2005). From a constructivist point of view, exploration of problems is a way of learning mathematics. According to Romberg and Kaput (1999), learners make sense of the experience as understanding is acquired through the creation of knowledge, and they take ownership of their learning.

Nevertheless, Rosy and Jasmine made mistakes while solving the angles, not because they did not have the conceptual knowledge, but because they were seemingly overcome by fatigue. Rittle-Johnson, Star, and Durkin (2009) warn that when learners have garnered new information at once, "tasks can easily overload their working memory" (p. 837). Cognitive overload with a multitude of concepts and procedures can divert the attention when engaging with interesting mathematics.

4.4.5 Essence of interactions - BI-5 (Different approaches to solving a problem)

4.4.5.1 Synopsis

Daisy:

- solved problems using different methods where necessary (PF); and
- switched from a theorem to its converse properties (MM and PF) such as establishing the converse properties of the *opposite angles of a cyclic quadrilateral*.

Jasmine:

- made little effort in using different methods.

Rosy:

- made little effort in using different methods.

Aster:

- reconciled different approaches to solve problems when necessary (MM and PF); and
- switched from a theorem to its converse properties (MM and PF), such as establishing the converse properties of the *angles of a semi-circle theorem*.

4.4.5.2 Discussion

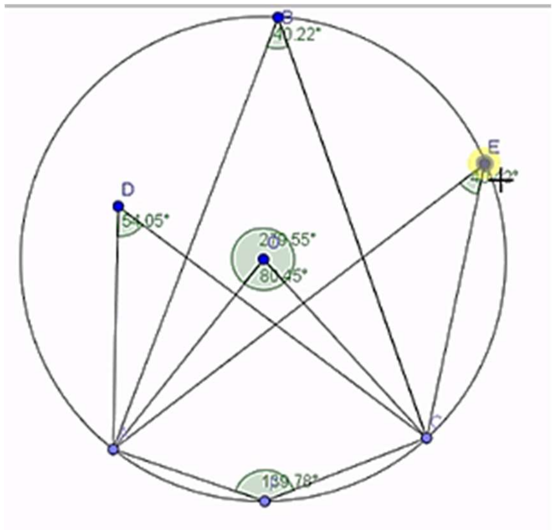
Jasmine and Rosy could not solve angles using different approaches, even when prompted. There were few instances of seeking for hidden structures, as Carter et al. (2009) discuss — that making and validating conjectures allows learners to employ mathematical ideas in new situations. Owing to a lack of developing mathematical conjectures, Rosy and Jasmine could not offer alternate methods for their solutions. Rittle-Johnson et al. (2009) affirm that learners may need some familiarity with one of the methods before teachers suggest alternative solution methods.

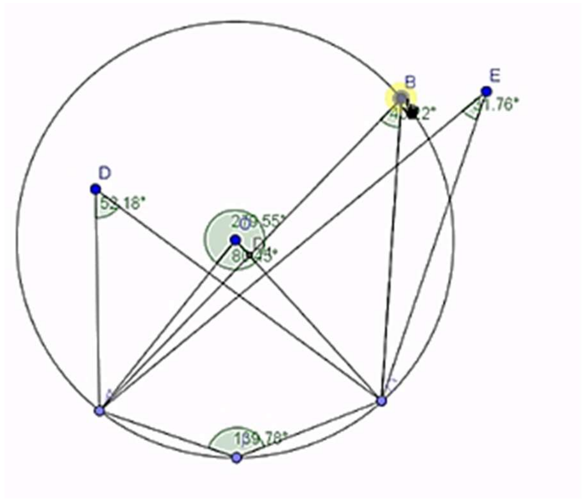
Daisy and Aster however employed diverse strategies in solving unknown angles. Kilpatrick et al. (2001) argue that learning to choose among different procedures is an important part in developing mathematical proficiency. Daisy continued her engagement with the applet as she dragged the independent point inside the circle. During the discussion on cyclic quadrilaterals in class, Daisy dragged point *E* over to the circumference and outside the circle in Screenshot 4.56 and Screenshot 4.57 in Frame 4.20 (p. 119). She explained that she wanted to check whether the point outside the circle made a cyclic quadrilateral and that the opposite angles were supplementary. Refer to the transcript in Frame 4.20 (p. 119).

Daisy not only established the *opposite angles of a cyclic quadrilateral* but also its converse — *when the opposite angles of quadrilateral are supplementary then it is a cyclic quadrilateral*. Thus, she was able to disprove that the quadrilateral *OGAE* in Applet 1.7 was a cyclic quadrilateral, justifying the procedure for using the *converse of opposite angles of cyclic quadrilaterals theorem*. Aster engaged with Applet 1.5 and it appeared that he was interested in the *angle subtended on the circumference at* $90°$. He considered the *diameter as a hypotenuse of the right-angled triangle*. Refer to the classroom interactions in Frame 4.15 (p. 103). Hence, during the reflective interview, I asked him to identify the diameter of the circle passing through the right-angled triangle in Applet 1.7. He was able to find it by appropriately applying the *converse of the angles of the semi-circle theorem* shown in the transcript in Table 4.1

Table 4.1: Aster applying converse of the angles in a semi-circle theorem

Reflective Interview	*R: Look at the figure (applet 1.7), Let me call this point of intersection between perpendicular line from the centre and the chord as D. Can you tell me if there exists a circle passing through G, A and D.* ***Aster**: A circle through G, A and D.* (He gestures a circle over the relevant points). *I think so yes.* *R: What will be the diameter of that circle?* ***Aster**: yes. It's GA.* *R: Why?* ***Aster**: Hypotenuse is the diameter. Angle subtended by a diameter is 90°.*

<table>
<tr><td colspan="3">Classroom Observation</td></tr>
<tr><td colspan="2">

Screenshot 4.56: Daisy dragging E to the circumference</td><td>

Screenshot 4.57: Daisy dragging E outside the circle</td></tr>
<tr><td>Reflective interview</td><td colspan="2">I replayed the relevant screen recording.
R: In class the teacher was discussing on cyclic quadrilateral. What was in your mind? Can you recollect it?
Daisy: Yes I can recollect it. I wanted E to be on the top of the circle to verify two things. One is angle E half the angle when on circumference? And two if the point outside E was making a cyclic quadrilateral? But it made a cyclic quad when on the circumference and two interior opposite angles were supplementary.</td></tr>
</table>

Frame 4.20: Daisy dragging the free point E during the discussion on cyclic quadrilaterals

Jasmine and Rosy could not indentify or justify the non-existence of cyclic quadrilateral *OGAE* in Applet 1.7 (shown in Figure 4.7 on p.79. They remained fixed to the definitions and theroems of the cyclic quadrilateral. The following transcript during the reflective interview with Rosy shows this:

> *R: Is it (OGAE) cyclic quadrilateral? Why?*
> ***Rosy****: its cyclic because all the points on the quadrilateral …. No … its not cyclic all the points are not on the circle.*
> *R: Can we draw another circle through OGAE so that it becomes a cyclic quadrilateral?*
> ***Rosy****: OGAE is a kite. We cannot draw a circle through O, G , A and E*
> *R: Why?*
> ***Rosy****: The points are not the circle.*

Krutetskii (1976) characterises different abilities in mathematically capable learners from his empirical research. He notes the reversibility of thought operations as a distinct ability to use direct and reverse instructions . Apparently, Daisy and Aster transitioned from a theorem to its converse fairly easily, but most learners found this transition difficult.

4.4.6 **Essence of interactions - BI-6 (Applying previously learnt concepts)**

4.4.6.1 **Synopsis**

All participants:

- drew on the prerequisite knowledge that *perpendicular lines are at 90°*, the *revolutionary angle* theorem, the *sum of interior angles of a triangle* theorem (CU and MM) and also drew on their computational fluency (PF).

4.4.6.2 **Discussion**

All the participant learners demonstrated their prior knowledge in most of the basic concepts of geometry, in varying degrees. I concur with Askew (2016) who argues that reasonable fluency in basic methods such as doubling and halving, frees up working memory, allowing learners to process bigger mathematical themes. Thus, the previously learned concepts indicated under BI-6, become crucial for enhancing access to the concepts in circle geometry. I argue that conceptual understanding was made possible because of strong computational fluency, such as doubling and halving and determining the exterior angle of an isosceles triangle. Thus, PF is a prerequisite for appreciating and developing concepts in the *angle at the centre theorem* and other connected theorems.

Askew (2016) compiled a list of elements of fluency in basic number operations (p.68 — 69) and argues that learners' lack of elements of fluency in basic calculations causes them to lose sight of mathematical ideas. The above observations allowed me to add on to Askew's list of elements of fluency essential in geometry – such as the *angles at a revolution* and *adjacent angles on a straight line;* the *properties of triangles* such as the *sum of angles in a triangle*, the *exterior angle of a triangle theorem* and *properties of an isosceles triangle*, a lack of which hampers conceptual development in circle geometry. Similarly, visualising the line from a point on the circumference of a circle to its centre as a radius, and *understanding that these radii are equidistant in a given circle*, is crucial to meaning-making and developing proficiency in circle geometry. Hiebert and Lefevre (1986) stipulate that the development of conceptual knowledge is achieved by making connections between different pieces of already acquired knowledge.

4.4.7 **Essence of interactions - BI-7 (Reflecting on a solution – considering the reasonableness of a solution)**

4.4.7.1 **Synopsis**

Daisy:

- No evidence of reflecting over solution.

Jasmine:

- No evidence of reflecting over solution.

Rosy:

- Revisited her initial assumptions as she verified the reflex angle (MM).

Aster:

- Interpreted the solution and considered the unreasonable value of a reflex angle (MM).

4.4.7.2 **Discussion**

The participant learners did not reflect on their solutions, hence there was not much evidence of BI-7. One of the few notable instances when the participants considered the reasonableness of their solutions was when Rosy reflected on her approach in determining that the *angle at the centre would be twice the angle subtended on the minor arc*. Rosy measured the reflex angle on the circumference and verified her solution in Screenshot 4.11 and Screenshot 4.12 in Frame 4.4 (p. 86). In another instance, Aster wrongly calculated the reflex angle b in practice question Applet 1.6. He seemed to realise that the reflex angle for a given angle of 80° could not be more than 100°, so he stopped entering his answer and recalculated it, using the simulator of Casio calculator. It could be argued that in a computerised environment, it might be difficult to capture the estimation and reflection of the results of computation. But there were instances such as the one faced by Aster, as discussed above, for all the other participant learners. Rosy made errors in her calculations in determining the value of b and misapplied the concepts one after the other, for the reflex angle in the practice question Applet 1.6. She reconsidered her method only after it was incorrect, thus losing an opportunity to reflect on the solution. Similarly, both Daisy and Jasmine did not consider their solution for $B\hat{A}E$ in practice Applet 1.7, when they determined it as a supplementary angle to $A\hat{B}E$. Obviously, they did not consider ΔBAE while arriving at the value of $B\hat{A}E$. Thus, there were missed opportunities to exhibit their ability to reflect or make sense of their answers. A possible reason for the thin evidence of BI-7 is that, as Schoenfeld (2007) argues, "if you [learners] believe that mathematics is not supposed to make sense, and that working mathematics problems involves rather meaningless operations on symbols, you will produce nonsensical responses such as these" (p. 70). I concur with Schoenfeld that typically learners have beliefs that engaging in mathematics is to simply memorise and apply mechanical algorithms without understanding.

4.5 EMERGING THEMES

This stage of analysis is still part and parcel of my horizontal analysis. The themes I discuss below emerged in my analysis across all the participants – They are grounded in concrete incidents that were observed when the leaners engaged with *GeoGebra.* These themes might have been unacknowledged in the vignettes discussed above yet they are pertinent to the meaning-making process of my participants. The themes identified are: 1) gaining knowledge beyond print and 2) internalising the theorem.

4.5.1 Gaining knowledge beyond print

As the learners interacted with *GeoGebra*, they showed evidence of how technology as a visual aid developed meaning-making and conceptual understanding.

4.5.1.1 Privileging dynamic visualisation processes

DGS, with its dynamic capabilities provided an opportunity for Rosy and her classmates to learn beyond textbooks. In her reflective interview she described and explained her experiences with dragging the independent point E around the screen, shown in Screenshot 4.4 to Screenshot 4.9 in Frame 4.3 (p. 85). She observed that the textbooks talked about the points on the circumference in an immobile way. The textbook could not consider multiple points on the circumference or points inside or outside the circle.

> *Rosy: Even if they are subtended by the same chord, the centre and circumference if that is not at the circumference and around the circle or outside the circle then it will not be half the angle at the centre. I observed that the textbook did not really point this out.*

Jasmine also observed something similar in her reflective account of the lesson. She said that a teacher cannot draw multiple diagrams and drag those angles for learners to see and understand the theorem. However, she was critical that *GeoGebra* did not provide an option for stating and providing reasons such as the learners are required to provide in their examinations. She was concerned that in an examination situation, marks were awarded for step by step procedures, but for the purpose of understanding, she preferred computers to the chalkboard.

> *Critical Comments:*
> *It's better but worse at the same time because when writing an exam, I need to know those steps and reasons to get marks.*

Preference:
For circle geometry, I prefer computer one. It's because the angles, line segments, chords, circle are actually displayed on the screen rather than teacher's preferences. In chalkboard you cannot move angles, but in computers you can. ... It's cool, it's really nice because it got me to understand things much better and put them into context than in the classroom, I mean actual chalkboard classroom.

(Jasmine was a learner in Paul's class, but her vignette was purposefully omitted. Her actions and responses in the classroom were very similar to those of Daisy and Rosy).

Guzmán (2002, p. 14) considered textbooks as " the written word, a statically vehicle that is not well adapted to the needs of the visualization processes." For Guzmán, visualisation is a dynamic process that is difficult to replace with the written word.

4.5.1.2 **Examples of 'out-of-the-box thinking'**

In class, I could not collect any evidence of the teachers making connections from the theorems to real life situations. So, in the reflective interview, I asked the participant learners to provide real life examples of circles or angles at the centre. I encouraged them to think beyond the textbook scenarios. All the participants listed similar examples — items such as bottle caps, buttons, pizzas and other circles. Daisy said that the computer was the only other source outside the textbook where she had encountered an angle at the centre of a circle. Jasmine provided two examples that were striking, as evident in the following transcript. She likened the angle at the centre of a circle to the school surveillance camera. She said that the lenses of the camera could capture and focus only within the limits of its radius. She was also reminded of a clock, where the hour hand and minute hand formed angles at the centre of the clock face. She even wondered if the rapidly moving second hand had anything to do with the moving of the arc, she encountered on the computer screen.

Jasmine*: I know I will find it (moving her eyes all around the room)... in this school cameras are actually at angle at the centre it is the point where it actually capture. Is the point where angle at the centre that captures. It can capture only within a particular radius….*
... I can see the angle formed by the hands of hour hand and minutes hand. When I am moving the arc when teacher asked to do it, I think of seconds hand and minutes hand moving forwards and backwards.

4.5.1.3 **Inversely proportional relationship**

Similar to Rosy's observation, Jasmine came up with her own theory of points inside and outside the circle. She maintained that there was no obvious relation between the angle outside and the angle 'hanging' inside, only that the angle at the centre was twice as big as the angle at the circumference. But she came up with a theory of her own – "*The farther your angle from the centre the smaller it becomes.*" *The angle subtended by an arc becomes smaller in size as it moves away from the centre*. This example may not be relevant in the context of the curriculum, but it is a legitimate mathematical observation and illustrates that the experience with the *GeoGebra* software enabled learners to visualise phenomena that their textbook illustrations did not achieve.

4.5.2 **Internalising the theorem**

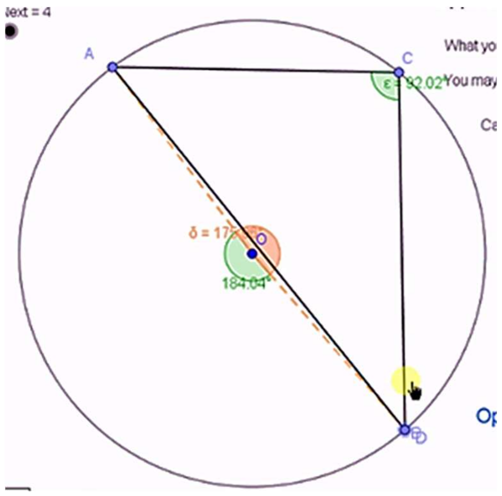
As the learners worked with the DGS, the participants showed evidence of meaning-making and conceptual understanding.

4.5.2.1 **Developing own stances**

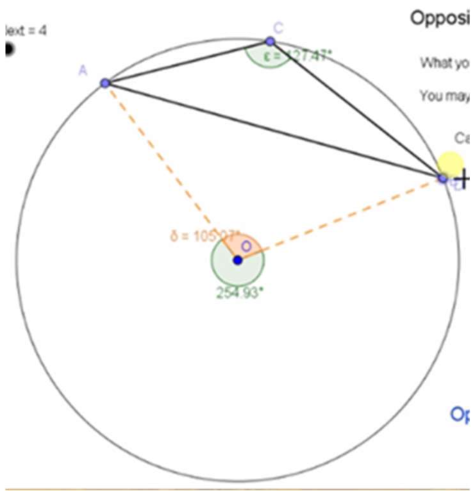
Working with DGS, Jasmine felt that it helped her to internalise those particular theorems. She could understand how the theorems linked to one another. This aligns well with notions of constructivism, which acknowledges learners as knowledge creators. Jasmine created her own knowledge with little reliance on her teacher.

> ***Jasmine****: If I summarise, I would say it's informative. It was nice. It helped me to internalise most of those theorems. ... It also helped me to independently think and not always ask questions to the teacher.*

In many instances, learners came up with their own ideas and theories and solved problems by themselves. While interacting with Applet 1.4 in Figure 4.4 (p. 77), Aster dragged the points to make a cyclic quadrilateral into a triangle. The applet was intended to allow visualisation and generate the property of the sum of interior opposite angles of a cyclic quadrilateral. Instead of visualising a cyclic quadrilateral, Aster dragged a point over the other at various positions to form triangles, including a right-angled triangle, as shown in Screenshot 4.58 and Screenshot 4.59 in Frame 4.21. He appeared to establish whether other triangles formed on the circumference of the circle were right-angled.



Screenshot 4.58: Aster drags to make a right-angled triangle a side passing through centre



Screenshot 4.59: Aster drags to make another triangle side not passing through centre

Frame 4.21: Aster dragging points to make triangles from the cyclic quadrilateral in Applet 1.4

Meanwhile, Daisy's initial failed attempt to solve $B\hat{A}E$ in Applet 1.7 offered another opportunity for her to solve it in an elegant manner using previously learned concepts — see the classroom observation transcript in Frame 4.14 (p. 100). I expected learners to use the *angles in the semi-circle theorem* to solve the angle, but here the participant learners devised approaches of their own to solve the angles. The participating learners developed their own opinions and strived to internalise different forms of the *angle at the centre theorem*. I concur with Mariotti (2000) that the process of internalisation takes place when the tools in DGS transform into psychological tools that assimilate and direct learners' geometrical thinking in constructing mathematical meanings.

4.5.2.2 **Unpacking unseen structures**

Aster's innovative solution strategies were of particular interest. He was aware of geometric principles or axioms, especially in circle geometry, prior to this lesson. We have seen in previous sections that Aster could apply geometric ideas when problem solving, but he knew little about how these ideas were connected. Aster's ideas were isolated facts – "It's a theorem sir. It's like that and we were never told why." However, as he engaged with the software, the invisible structures became apparent to him. Let us look at two such instances: Firstly, he initially maintained that it was impossible to find *c* without solving *b* (in Screenshot 4.51, Frame 4.17 on p.106). He could not identify the cyclic quadrilateral until he dragged the points. When speaking about the proof of the theorem of *sum of interior opposite angles of cyclic* *quadrilateral*, he did not put it in formal language, but he managed to generate the 'proof' from the figure on the computer screen in front of him.

> *Aster:* (on the cyclic quadrilateral) *They are supplementary it's obvious sir from the figure. b and other b is 360° . c and a are half of b and reflex b. half of 360° is 180°. So, a and c are supplementary.*

Secondly, he initially maintained that the *angle formed by the diameter theorem* did not have proof. But as soon as he saw the screen recording of his own work in the classroom, he was quick to link the theorem to the *angle at the centre theorem*.

> ***Aster**:* (on the *angles in a semi-circle* theorem) *we were never taught sir. It's a theorem sir. It's like that we never told why?*
> (After reviewing screen recording) *: Oh yeah yes now I remember. The angle formed by a diameter at the centre is 180° therefore its half will be on the circumference; therefore, the angle subtended by a diameter on the circumference is 90° because it forms a straight line with the centre.*

Similarly, Daisy unpacked the *angle subtended in a semi-circle theorem* as a special case of the *angle at the centre theorem*. While she was clarifying her procedure to find $B\hat{A}E$ (in Applet 1.7), she realised that she first had to solve angle B$\hat{E}$A. She could explain how she applied the *angle at the centre theorem*, and therefore that the *angle subtended by a diameter is 90°*.

> ***Daisy**: The angles on the circumference must be above the angle at the centre. If I dragged that angle on the circumference down below the arc, then it won't be half.* (inaudible) *I mean they should be on the same side of arc or chord.*
> ***Daisy**:* (explaining her solution for angle B$\hat{A}$E in practice question 2)*. I think I said that since this is a line B$\hat{A}$E, since it is a line and centre O since it is the diameter. Obviously, the angle is 180°. So, the angle at the centre is 180°. It is double the angle at the circumference E. So that is how I got it as 90°, the whole E*

In the sense of the theoretical framework of constructivism, learners authored their own learning, as postulated by Carpenter and Lehrer (1999). "Understanding involves the construction of knowledge by individuals through their own activities so that they develop a personal investment in building knowledge" (p. 23). Furthermore, this aligns well with Arcavi (2003) who considers visualisation as the seeing of the hidden structures behind a mathematical concept.

These above episodes provide further evidence of the overall development of the *angle at the centre theorem* and the concepts connected to it. The use of applets offered learners an opportunity to do more than just listen to the teacher during class. I argue that the dynamic features of technology provided an opportunity for learners to see the relationships between geometric concepts. My observations align with Tall (1993) who maintains that technology as a visual aid makes an entry point for learning a mathematical concept. The learners' responses and actions during the lesson can be mapped onto Vygotsky's theories about tool use, and as Askew (2016) argues, tools and artefacts (applets in this case) mediate "through which the learning is assumed to be enabled" (p.19). Additionally in a Vygotskian perspective, Hoyles (2005) observes that virtual manipulatives are mediators and form a crucial part of internalising knowledge.

4.6 ANSWERING THE RESEARCH QUESTIONS FOR THE FIRST CYCLE

As a reminder, the first research questions that framed this study as discussed in Chapter 1, was:

- How are *GeoGebra* applets used as a learning tool to make mathematical meaning and develop understanding by selected Grade 11 learners?

Kilpatrick et al.'s model of MP and Carter et al.'s meaning-making habits formed the basis for the analysis of the various datasets. I presented the lesson observations in the form of vignettes of three learners.

Rosy's exploration as she dragged the independent point around the screen convinced her that the theorem held only for angles at the centre and angles on the circumference, as shown in the sequences of screenshots from Screenshot 4.4 to Screenshot 4.9 in Frame 4.3 (p. 85). Her exploration and subsequent conjecture had an effect on the entire class. Daisy (Rosy's classmate) also spoke in her reflective interview about the points not on the circle or centre and it was a significant discovery for her that only *the points at the centre and on the circumference* maintained the *angle at the centre* theorem. Jasmine made her own theory of inversely proportional relationships between the *angle subtended by an arc at the centre* and the *angle subtended by the same arc away from the centre*, as discussed earlier in 4.5.1.3 (p. 123). The constructivist approach considers learners as having a certain amount of knowledge – they do not come to school as empty vessels to be filled. Learners actions illustrate Goldin's (1990) views that a constructive process is one in which an individual organises and restructures his/her own experiences through his/her own construction of knowledge.

Rosy could understand the relationship and measure the angle at the centre when the angle on the circumference was on the minor arc, using her 'inside-outside' notion. She referred to the position of the angles as 'inside-outside'; whereas for Daisy the angles were at 'above-below' positions. However, Rosy found it necessary to measure the counterpart angle (the reflex angle $C\hat{F}A$) on the circumference, shown in Screenshot 4.11 in Frame 4.4 (p.86),. She convinced herself that the reflex angle $C\hat{F}A$ was not twice the angle at the centre. Daisy had a different conception of the angle at the centre, hence she drew a point outside the circle in

Screenshot 4.29, Frame 4.10 on p.94. As she was engrossed in constructing her figures, reflecting and experimenting with her conjectures, she was able to overcome her disequilibrium. Her manipulations with the geometric objects lead her to visualise and conceptualise the appropriate angle at the centre. Locating constructivism as a learning process, Hoyles (2005) argues that constructing mathematical meanings are inextricably interwoven with the computer tools which enable learners to identify mathematical invariants and make

connections.

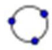
When I asked Daisy to identify whether the quadrilateral *OGAE* in Applet 1.7 (second practice question) was a cyclic quadrilateral, her first preference was to draw on the computer, but she did not know how to manipulate the program (she only knew how to draw a circle with a centre). Persisting in solving the problem another way, she drew the figure and a circle on a sheet of paper and then came up with the converse of the theorem. She used the numerical properties of the *sum of interior opposite angles in a cyclic quadrilateral* and concluded that it was not cyclic. She wanted to see if a circle passed through the points, and I therefore guided her in *GeoGebra* to select the appropriate tool 'circle through three points' , convincing her that a circle cannot be drawn through four points. In Figure 4.11 to Figure 4.13 in Table 4.2, I drew all the alternatives for a circle with the points *O*, *G*, *A* and *E*. When we drew a circle passing through *OGA*, it did not touch *E*. A circle through *OAE* did not pass through the point *G*. The point *A* was inside the circle passing though *OGE*. The circle *GAE* is the given circle with *O* as the centre. Daisy knew the procedure to determine the quadrilateral as cyclic, but she felt it more convincing to see the non-existence of a circle. This is in concurrence with Hölzl's (2001) argument that DGS is best used for verifying purposes and "learners are just supposed

Table 4.2: Non-existence of a circle passing through O, A G and E in Applet 1.7

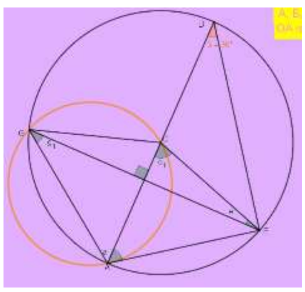
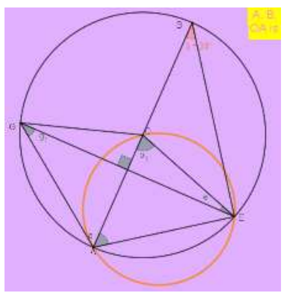
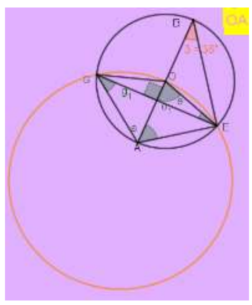
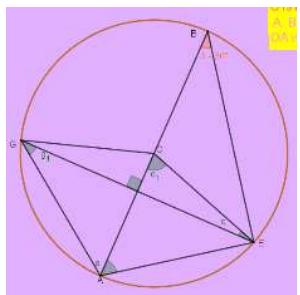
			

Figure 4.11: Circle through OGA but does not touch E	*Figure 4.12: Circle through OAE but does not touch G*	*Figure 4.13: Circle through OEG but does not touch A*	*Figure 4.14: Circle through GAE but O is the centre (the given circle)*

to vary geometric configurations and confirm empirically more or less explicitly stated facts" (p. 65).

Proponents of DGS (Hölzl, 2001; Hoyles & Noss, 2003; Sherman, 2010) encourage a guided learning environment. A guided environment is one where teachers guide the learners to recognise the situations conforming to the mathematical norms. Aster may be an outlier in this case. His teacher had little intention of allowing learners to construct lines and angles in DGS, as he had pre-constructed his applets and brought them to class. But Aster constructed his own circles regardless, and explored the angles subtended by a diameter and 'tangents' in his own way. He found some abstract facts such as the properties of the tangent, but he found it necessary to verify his ideas and made use of the opportunity. Gawlick (2002) supports the use of DGS in the mathematics classroom as it "motivates the students to develop individual interests" (p.86). The significance of DGS enabled learning cannot, therefore, be overemphasised in its potential to direct and promote conceptual development in the learner.

Now I attempt to answer the second research question.

- What visualisation role can *GeoGebra* play in the learning of Grade 11 mathematics?

Considering Duval's (2013) observation that construction of geometrical figures by the learners is crucial in visualisation, DGS here proved to be an efficient visualisation tool for learners to enable their understanding.

GeoGebra with its dynamic possibilities, enabled the learners to focus their attention on continuous variations of the angles as they dragged the vertex to different positions on the computer screen. This dynamic way of exploring would be an uneconomical strategy of constructing multiple static figures to make inferences in a pen and paper environment. As discussed earlier under emerging themes in Section 4.5.1.1 (p. 122), Rosy and Jasmine observed that neither the textbook nor a teacher could consider multiple points on the circumference and drag those angles for learners to see and understand the theorem. The dynamic visualisation capabilities of DGS enabled the participant learners to move from concrete instances to conjecturing abstract ideas, echoing Guzmán (2002) who observed visualisation as an important aspect in mathematical activity, through which one can explore different structures of concrete reality. Dragging of the points highlights the variant and invariant properties of the angles, supported by visual recognition of the relations. This is consistent with Laborde (2001) who claims that in a DGS environment "students must themselves achieve these abstraction processes by manipulating, experimenting and observing." (p. 284). Hence, the role of DGS in this study was to support learners make conjectures about the relations using the drag facility in *GeoGebra*. Furthermore, these instances provided evidence that the interactions with the *GeoGebra* software enabled learners to visualise phenomena that their textbook illustrations did not provide.

Arcavi (2003) considered visualisation as the seeing of the hidden structures behind a mathematical concept. Daisy and Aster could also identify and link the opposite interior angle of cyclic quadrilateral ltheorem with the angle at the centre theorem. When I asked Aster to find the value of c (in practice question 1), he initially said that it could not be solved. But as soon as he dragged the point around the circle, he could visualise the cyclic quadrilateral (shown in Screenshot 4.48 through to Screenshot 4.50 in Frame 4.17 p.106). Daisy also recognised the cyclic quadrilateral in practice question 1, but without dragging the point. She had already abstracted the concept that the *sum of the interior angles of a cyclic quadrilateral theorem* was borne out of the *angle at the centre theorem*. Similarly, Daisy and Aster unpacked the *angle subtended in a semi-circle theorem* as a special case of the *angle at the centre theorem* as shown in the discussions in Section 4.5.2 'Internalising the theorem'. This reiterates what Duval (2013) asserts, that visualisation is to identify the relevant and irrelevant parts in a given figure.

Daisy and Aster had similar concerns about the angles. They wanted to establish whether the angle subtended outside the circle could become equal to the angle subtended on the circumference when dragged towards the circumference. Secondly, they wanted to see if the quadrilateral would be cyclic and its interior opposite angles supplementary. Daisy verified two of her conjectures by dragging a single point *E*, as shown in Screenshot 4.31, Frame 4.10 on p.94. Aster used the drag mode twice for similar conjectures, once by moving *E* and then by moving point *D* (Screenshot 4.44 through to Screenshot 4.47 in Frame 4.16, p.104). I contend that dynamic visualisation capabilities in *GeoGebra* help learners to develop understanding, as the visual feedback of their actions on applets develops new understanding.

Learners explored the properties of angles by dragging the points on the circle until they could conceptualise the theorem. When they faced difficulties, they could drag the points in their minds and thus overcome the diagram block as described by Presmeg (1992), who describes several cases, including the standard diagram of the *angle at the centre* where vivid imagery is actually a hindrance in generalising mathematical concepts.

The participant learners were able to see and connect theorems rather than regarding them as isolated facts and statements. The DGS, with its drag mode capability enabled them to visualise the *angle subtended by a diameter* as a special case of the *angle at the centre theorem* and not as a discrete theorem. Working with DGS, learners could understand how the theorems were linked to one another. Significantly, Jasmine mentioned that it helped her to internalise those theorems as shown in the discussion in 4.5.2.1 (p.124). For Presmeg (2014), visualisation is employed in classrooms when we make connections between topics in mathematics. Technology enabled visualisation, thus, is not considered as making learning easier but as a basis for a mathematically richer activity.

4.7 CONCLUSION OF THE CHAPTER

The main purpose of this chapter was to provide an in-depth narrative analysis of the participating learners' meaning-making from their interactions with the DGS on the topic ***angle at the centre theorem***. The horizontal analysis of data indicated that there were similarities and differences across the participants in their development of conceptual understanding and procedural fluency. As the analysis unfolded, different characteristics of learning proficiency in relation to visualisation were evident. Furthermore, a detailed analysis of learners' actions and responses revealed that DGS facilitated participants to make deductions and verify conjectures and thus created a setting of learning by doing.

In the following chapter, I present an in-depth narrative analysis on the next topic i.e. Quadratic Equations and Inequalities.

CHAPTER 5

ANALYSIS OF LEARNERS' ENGAGEMENT WITH *GEOGEBRA* – QUADRATIC INEQUALITIES AND NATURE OF ROOTS

5.1 INTRODUCTION

In the previous chapter, I narrated and analysed participant learners' interactions with the DGS during the lessons on circle geometry. The aim of this chapter is to analyse and answer the research questions on the topic 'Equations and Inequalities'. I recommend that the reader browse through the recap of the methodology section (Section 4.1.1 on p. 75) of the previous chapter.

This chapter follows a similar structure to Chapter 4, (p. 74).

5.2 TOPIC – EQUATIONS AND INEQUALITIES

The next topic in the sequence of the teaching plan was quadratic equations and inequalities. The GLIP teachers discussed the sub-topics and decided on the following concepts: 1) roots of equations, 2) quadratic inequalities and 3) the nature of roots of quadratic equations. The topics would be introduced in class and *GeoGebra* would serve to emphasise the interconnectedness of concepts through visualising the underlying mathematical ideas. We found that the CAS perspective of *GeoGebra* could help in the teaching and learning of quadratic equations and inequalities.

Primarily, the CAS perspective of *GeoGebra* is used for symbolic computations. Nonetheless, equations and functions are always shared between the CAS perspective and graphics windows. For example, if a function $f(x) = x^2 - 4$ is defined in the CAS perspective to solve for x, then $f(x)$ can also be used to display its graph in the graphics window.

In the GLIP meeting, teachers designed a worksheet (WS-1 in Annexure V). Learners would follow the instructions of the teachers to enter the quadratic equations, solve the equations and then construct the corresponding graphs in CAS. The learners were expected to fill in the columns of the worksheet, based on their observations in GeoGebra. They were also expected to visualise inequalities, as in Figure 5.1 and 5.2.

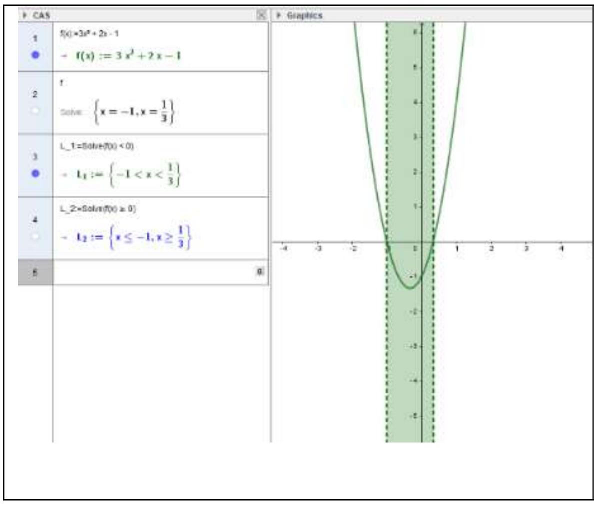


Figure 5.1: Inequality less than zero

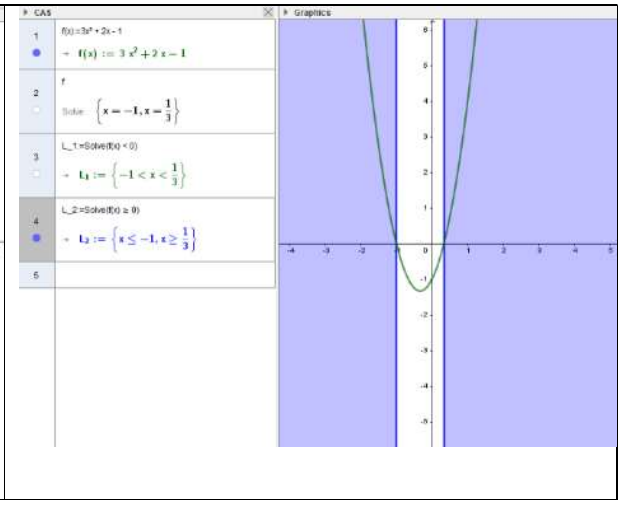


Figure 5.2: Inequality greater than or equal to zero

Only two applets were designed in order to visualise and practise the nature of quadratic roots. Applet 2.1 was designed to engage the learners in visualising the nature of quadratic roots. (refer to Figure 5.3 and the answer in columns 5 and 6 in worksheet WS-1). It was expected that learners would be able to relate the values of the discriminant with the numerical values of the x-intercepts of the graph.

When there are no intercepts, the roots are non-real, as shown in Figure 5.4 of Table 5.1, and when there are x-intercepts the roots are real. There are three sub-classifications of real roots: equal roots (Figure 5.5), irrational and unequal roots (Figure 5.6) and rational and unequal roots Figure 5.7) shown in Table 5.1.

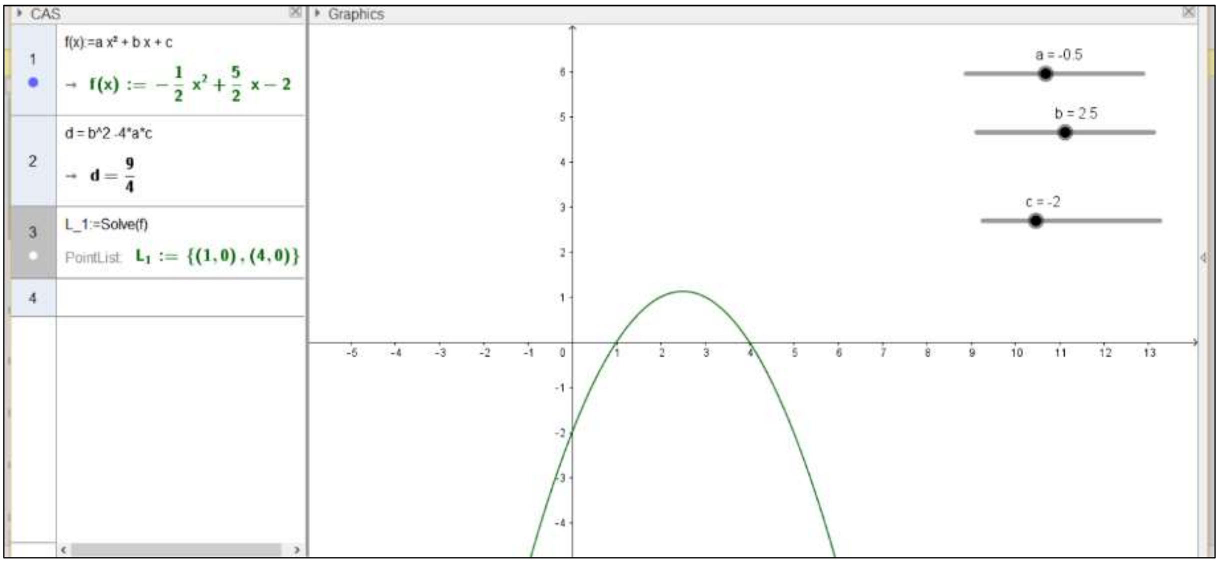


Figure 5.3: Exploring the nature of quadratic roots — Applet 2.1

Table 5.1: 'Visualising the nature of roots' classification

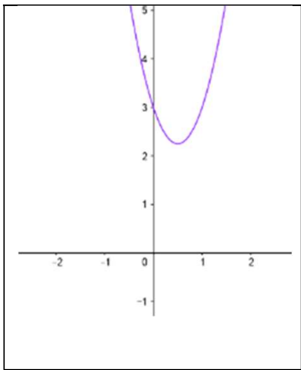
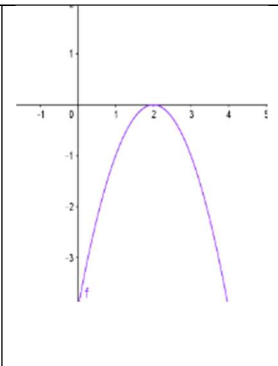
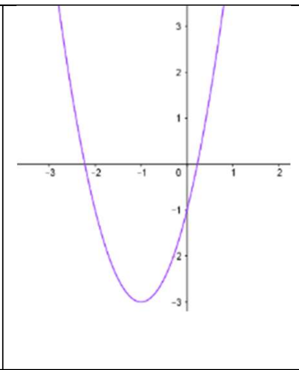
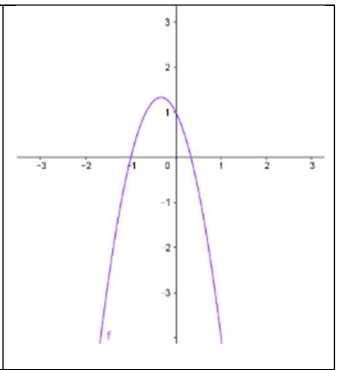
			

Figure 5.4: Non-real roots	*Figure 5.5: Real and equal roots*	*Figure 5.6: Real, unequal and irrational roots*	*Figure 5.7: Real, unequal and rational roots*

An applet for practising the nature of roots (Applet 2.2) was developed as shown in Figure 5.8. It would be used after completion of the worksheet. This applet was designed to classify the roots through visualising the graph. The premise behind this applet was that the absolute value of the leading term of the quadratic expression is a prime number. The quadratic expression is also displayed on the graphics window.

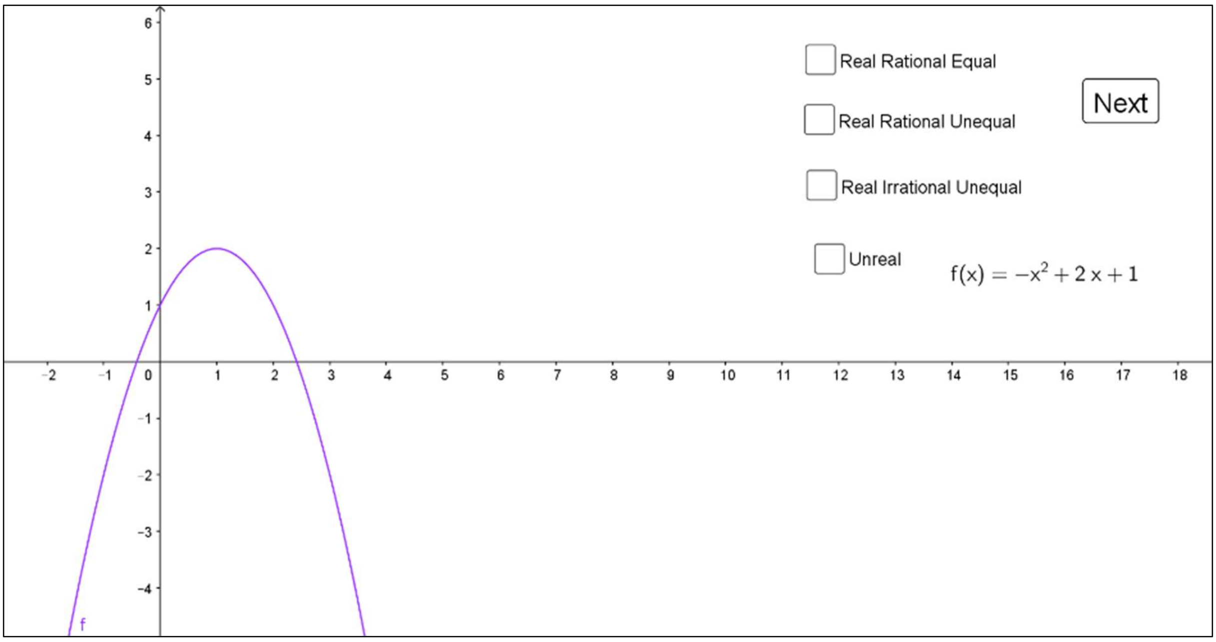


Figure 5.8: Practice question on the nature of roots in Applet 2.2

One of the recommendations of the GLIP teachers after the first cycle was to provide training to the learners on using *GeoGebra* specific to the topic. My co-researcher and I designed an hour-long training programme on using CAS within *GeoGebra*. The CAS handout for training the learners can be accessed in the Annexure IV under the heading 'Training Interlude - CAS'. However, the training could not be accommodated in both schools because of the full timetable of the learners and teachers and the non-availability of the computer laboratory. In school A, (Antony and George's school), there are two computer labs, but the learners could not be scheduled for training due to other priorities like sports, cultural activities and additional lessons. In school B, there was an incongruity between the learners and the teacher Paul's schedules, and when both had a free period , the computer lab was occupied. Our observation calendar was planned at the beginning of the year. Given both the crammed schedule and the dynamics of the participant schools, we decided to proceed with the lesson observation as scheduled. The CAS training handout was used during the lesson to facilitate the use of *GeoGebra*. In the discussion that follows, the quadratic equation is represented by: $ax^2 + bx + c = 0$ and its roots are given by the quadratic formula: $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$ .The learners were aware of the procedures to solve quadratic equations and inequality prior to the observed lessons. They had learnt the parabola of the form $ax^2 + b$ in previous grades.

5.3 DATA PRESENTATION AND VERTICAL ANALYSIS OF INDIVIDUAL PARTICIPANTS

In this section, I analyse the learners' interactions with the above applets during their recorded lessons.

5.3.1 Antony's lesson

5.3.1.1 Overview of the lesson

Antony issued worksheet WS-1 to the learners a day ahead of this lesson. He had briefed them on what was expected of them. He also mentioned what was meant by the value of $b^2 - 4ac$, a surd in the quadratic formula, as the determining factor for roots of a quadratic equation.

There were 34 learners in the class and there were fewer computer hardware related glitches compared to his first lesson. There was an atypical sense of anticipation amongst the learners at the beginning of the lesson, as they were very excited to be in the computer laboratory. The lesson was scheduled for one hour, but the it lasted for about 70 minutes. The handouts were distributed, and Antony went through the 'commands' in the CAS as laid out in the handouts. He then discussed inequalities and the nature of roots. He closed the lesson after using the practice applet on the nature of roots.

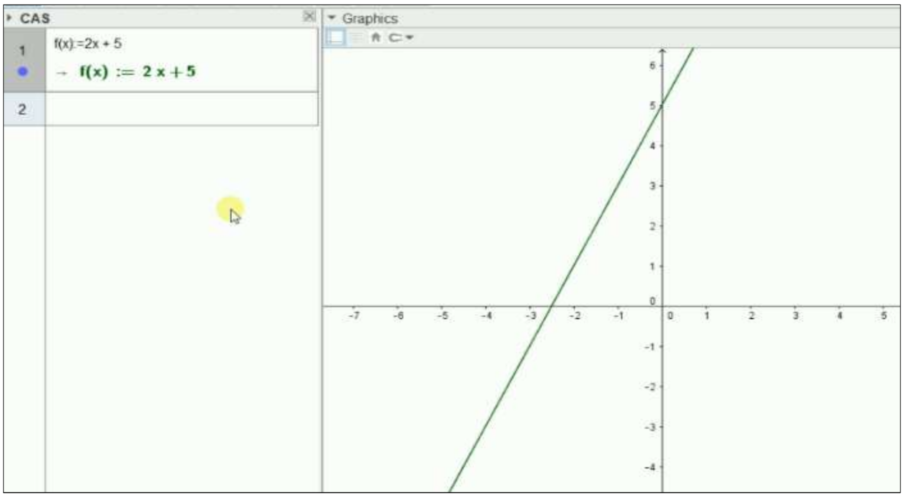
Again, many of the learners struggled with the keyboards. As the teacher instructed and projected onto the screen, many were unable to follow. For instance, in order to write $2x + 5$ for the straight-line function, Antony's instructions were, "*Press 2 then shift 8. The multiplication sign is asterisk (*) on top of 8. Therefore press 2 and shift 8".* Confused, some of the learners pressed '2 shift then 8', making $28 + 5$, others pressed '+ and then 8' making $'2 + 8 + 5'$, while another set of learners had $'2 * +5'$. In another instance, while entering a quadratic equation, a learner used the capital letter X instead of small letter x and did not produce any graph on the screen. Nevertheless, once the technicalities of the keyboard were sorted out, the learners were able to associate the solution of the equations with the x-intercepts of the parabola graph (2MC).

There was little enthusiasm when the teacher embarked on discussing the quadratic inequality. This prompted Antony's comment, "I don't see anyone sleeping. Why you are silent?" However, once the learners were able to link the critical values of inequality to the x-intercepts of the graph, it led to a discussion. The learners appeared to be persuaded by the teacher's explanations on correlating the solution in the CAS window and its graphical representation. They could see and then explain the difference between the relational operators like $\leq$ and $<$ (1EC). The teacher then proceeded to discuss the nature of roots. A few learners were still not proficient in handling the keyboard and his became particularly evident when the teacher wanted them to create sliders and equations in order to determine the nature of roots of a quadratic equation, as seen in the Figure 5.3. The teacher provided step by step instructions on how to construct the sliders, enter the equation and then determine the value of the discriminant. Some learners could not key in the quadratic equation correctly i.e. they used values and not the variables of the sliders. A few others could not key in $b^2 - 4ac$ correctly. The sliders were not linked to the quadratic equation nor to the discriminant, hence this generated confusion in class. Nevertheless, some learners succeeded in entering the quadratic equations using the variables of the sliders and helped others who were struggling to achieve similar results. When all of them had their sliders and the equations and the discriminant were synchronised, calmness reigned in the class. They started responding to the teacher's questions with fervour. These questions included determining if the roots of the quadratic equation were real or imaginary (1EC and 2MC). Most of the learners attempted to complete the worksheet. By then the lesson had already encroached into the 'interval time' of the school. Learners only had seven to eight minutes to answer the applet question on the nature of roots.

5.3.1.2 **The story of Rosy**

5.3.1.2.1 *Rosy's insight into linking equations and graphs*

Rosy was very competent in using the keyboard, unlike most of her classmates. She could understand how the solution $2x + 5 = 0$ as in Screenshot 5.1, was related to the straight line $2x + 5$ (1EC and 2MC). Her conversation with her friend Daisy during the lesson, revealed that she grasped the important features in the figure as she identified points like x-intercepts and y-intercepts (1EC and 2MC) as shown in the transcripts in Frame 5.1. She also explained to Daisy how the *x*-intercept was linked to the *x*-value of the linear equation (4EP, 2MC and 1EC).

Classroom observation	 *Screenshot 5.1: Rosy's effortless sketch of the straight line using GeoGebra tools*
	***Rosy** and Daisy :* (On clicking the white dot as instructed) *"Wow a straight line"… 2MC* *Daisy: isiXhosa (How do we make head of these things?) …* ***Rosy**: Let me explain this … isiXhosa (how to get rid of this line)…isiXhosa (this 5 is the constant 'c' 1EC on the y-axis can you see that?) 2MC the line is clearly about 2x+5.* *Daisy: Let's use calculator…* ***Rosy**: Right. Let 2x+5 = 0 , x equal to is this point, x-intercept y = 0 . isiXhosa (when we draw the line x-intercept will be the solution of this equation) 2MC, 1EC and 4EP. … When they are asking for x, then this is the point we find (teacher giving instructions to the class, they continue to discuss on 2x+5), this point between -3 and -2… isiXhosa (and 5 is this point you see)* *Daisy : on the y-axis the y-intercept* ***Rosy**: … 2 is the coefficient of x. 1EC And what I was say neah, when we solve 2x+5= 0 the point between -3 and -2 is -2,5. 1EC and 2MC*
Reflective Interview	***Rosy**: Ooh that sir we were talking that, once we are in front of computers, we suddenly become serious in learning. We three, other two friends (their names) were discussing on that. My friend was telling that we are becoming serious in front of computers and another friend said that it was obvious. Sir, it is obvious that we are serious working in computers as in class you have only books. I was telling them I was exited using computers. The learning becomes easier when using GeoGebra that's what we were talking.* *R: How you were able to identify that that point on the graph?* ***Rosy**: Sir, you mean solution x=-5/2. I was telling that after finding the solution you see the x-intercept 2MC* *R: How you were able to tell that its x-intercept the answer before it was asked by the teacher?* ***Rosy**: I knew I was going to plot the graph 7EST. After finding the solution, I got the answer -5/2, I knew that it was x-intercept of the graph 1EC*

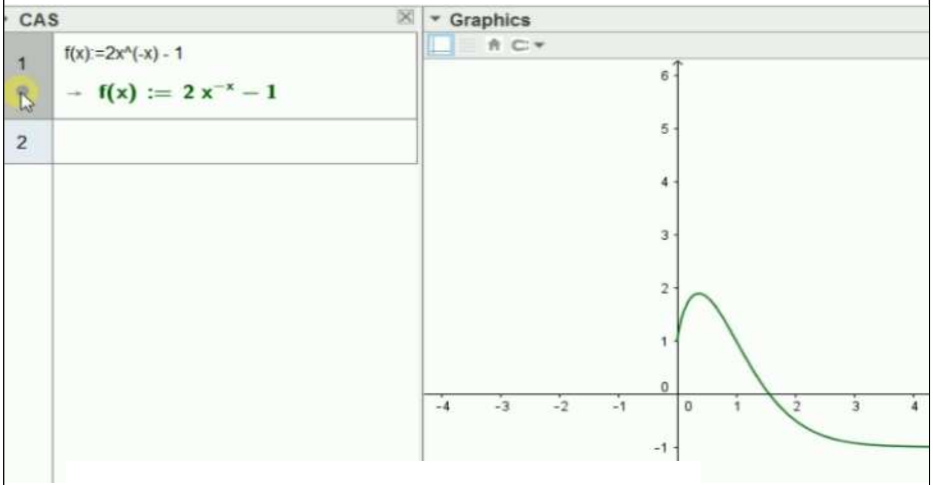
Frame 5.1: Rosy sketching straight lines and related conversation with Daisy and the researcher

When asked to comment on the conversation with Daisy about the straight line, Rosy mentioned that while attending computer classes they were keen to learn and enjoyed the lessons. She claimed that she conjectured (7EST) that the x-intercept of the graph would be the solution to the equation. Her argument was backed by her intuitive thinking of linking the expression $y = 2x + 5$ with the equation $2x + 5 = 0$.

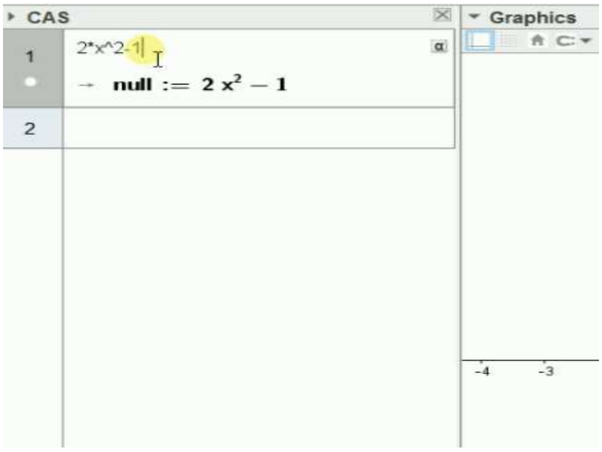
5.3.1.2.2 *Rosy's claim on repeated roots*

Rosy struggled to key in the quadratic equation $2x^2 - x - 1$ as shown in Screenshot 5.2 and Screenshot 5.3 in Frame 5.2, but she knew that she was wrong as it did not match with her conjectures (7EST). She keyed in $2x^{-x} - 1$ instead of the quadratic equation and exclaimed, "It's not a parabola" (shown in the transcript in Frame 5.3). However, as soon as she sketched the quadratic graph using the computer, she proclaimed that the roots of the equation were the *x*-intercepts of the graph (1EC, 2MC and 6LPK). When the teacher asked her to verify the answers with the worksheet, she demonstrated only two of them as per the screen recording. She did, nonetheless, complete the fourth column of the worksheet. Rosy claimed that she estimated the other intercepts, because she knew that they have to be equivalent to the roots. She wrote two intercepts for equal roots as shown in Screenshot 5.4. In her reflective interview, she maintained that there were two intercepts, but that they were equal (1EC, 1MR and 2UVI). Her ideas were different from that of her teacher who said that there was only one intercept when roots were equal. In the conversation during the reflective interview (in the transcript in Frame 5.3, her conclusion on the topic was interesting, but I will discuss this in section 5.4.2 (p.165).

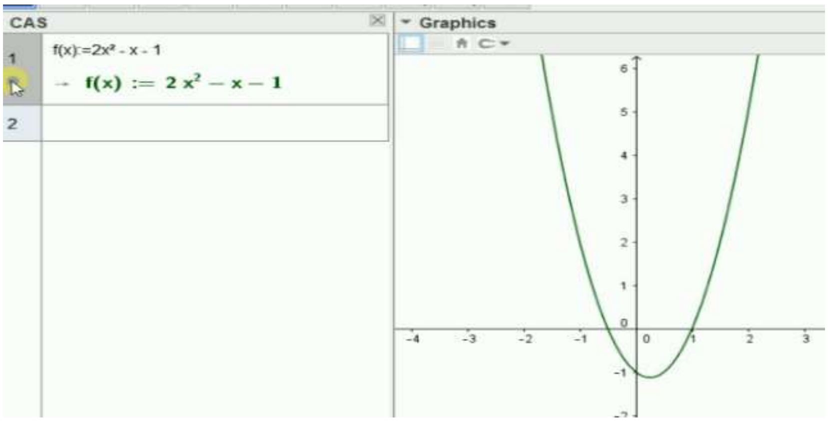
Classroom Observation	



Screenshot 5.2: The initial attempt



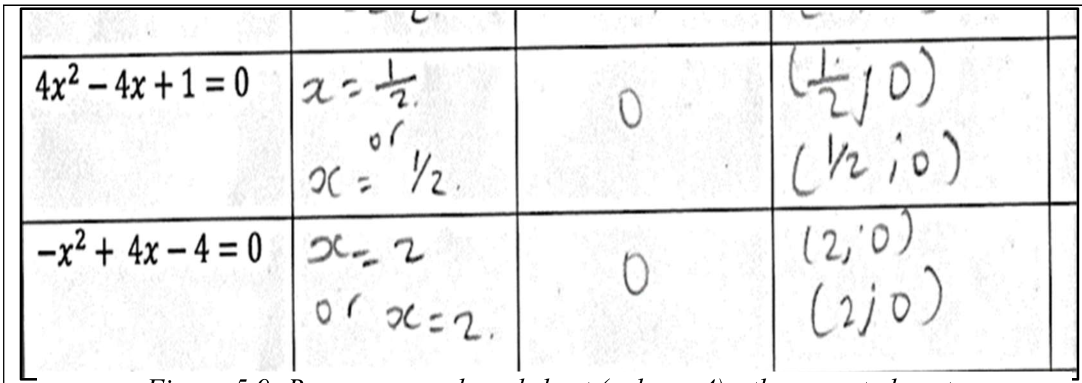
Screenshot 5.3: Another wrong attempt



Screenshot 5.4: Finally there

Rosy: Instead of entering $2x^2 - x - 1$, she keyed in 2*x^-x-1 and obtained the function $2x^{-x} - 1$ and the subsequent graph. She realised that here was something wrong, *It's not a parabola 7EST*
She retyped same function. Then she entered the function $2x^2 - 1$. She eventually keyed in the function $2x^2 - x - 1$ and clicked on the 'show' button to obtain the parabola graph on the graphics window.
And the x-values are the x-intercepts of the parabola graph. 2MC and 1EC

Frame 5.2: Rosy verifying and constructing a quadratic function in CAS



$4x^2 - 4x + 1 = 0$	$x = \frac{1}{2}$ or $x = 1/2$.	0	$(\frac{1}{2}; 0)$ $(1/2; 0)$	
$-x^2 + 4x - 4 = 0$	$x = 2$ or $x = 2$.	0	$(2; 0)$ $(2; 0)$	

Figure 5.9: Rosy answered worksheet (column 4) - the repeated roots

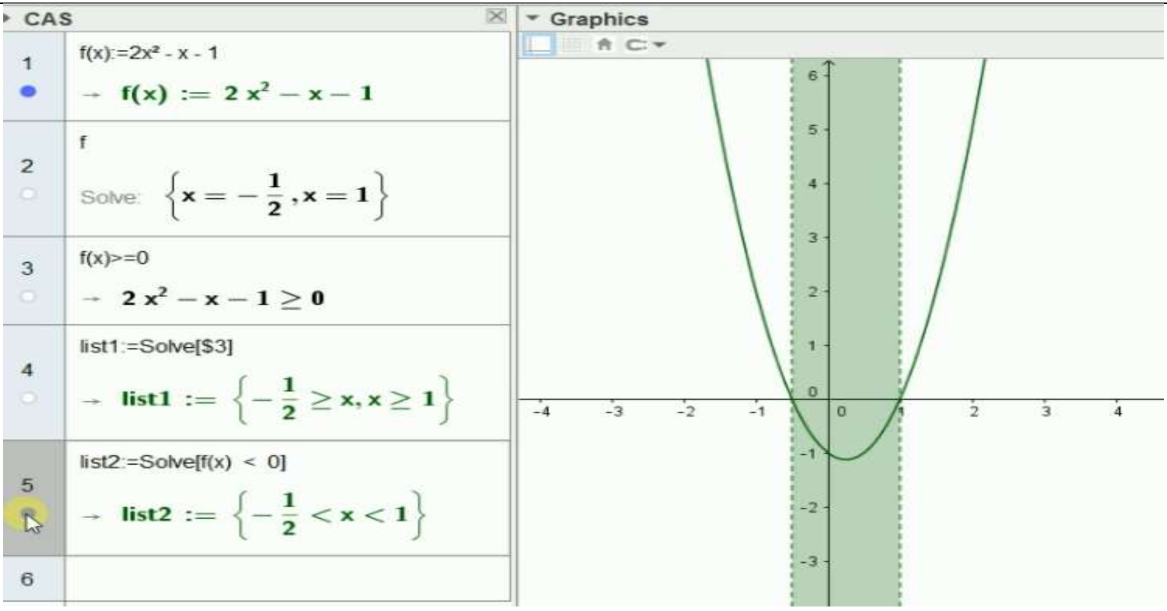
Reflective Interview	***Rosy**:I guessed the remaining values. They have to be equal to the roots.... there is one root that is x=2 1EC... no there are two values for x-intercepts, but they are same x=2 1EC and 1MR. They coincide at the same point. And if they were not equal say for example one of them is negative -3 then the parabola will turn between them (pointing to a below x-axis between -3 and 2) 2UVI and 1EC. ...* *R: What did you understand from this? You can summarise in your own words.* ***Rosy**: Ok sir. My understanding is that the x-values are the x-intercepts when you solve. The turning point will be between them. (Pauses) ... that my understanding sir. 3DC and 1EC.*

Frame 5.3: Rosy's ideas relating to roots and x-intercepts

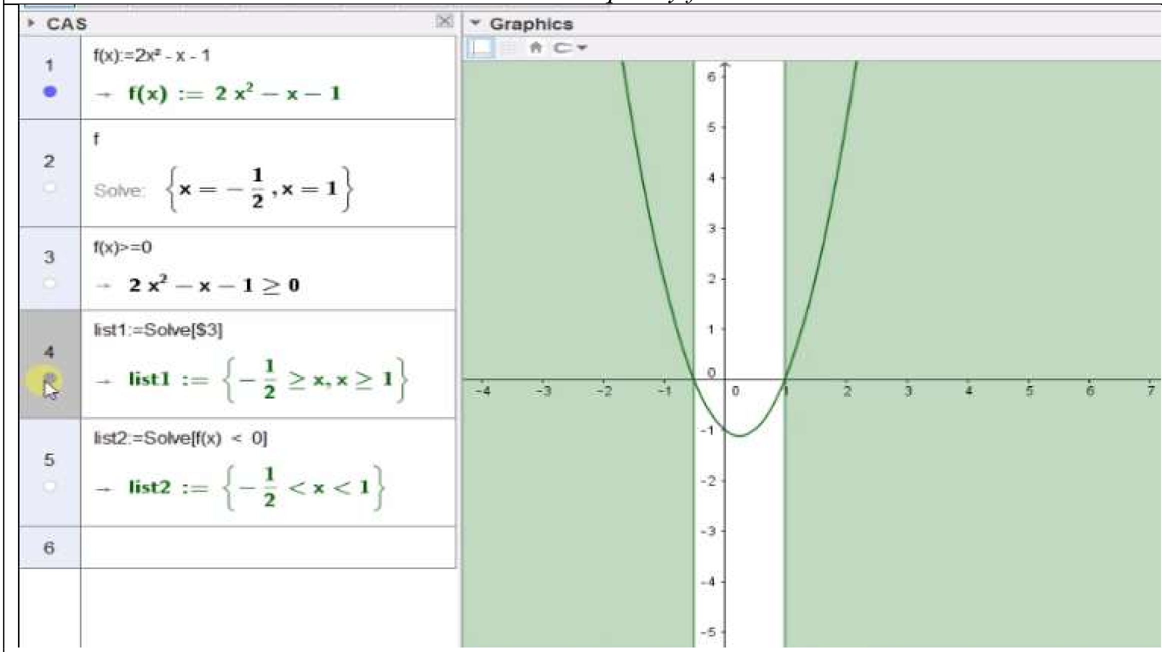
5.3.1.2.3 *Rosy takes on quadratic inequality*

Following the teacher's instructions, Rosy did not have any difficulty in solving the inequality in DGS. Her conversation with her friend Daisy revealed her engagement and understanding of the topic. Her first response to the first inequality (greater than or equal to) under discussion, was that there was a disconnection (see Screenshot 5.5 in Frame 5.4), in the solution (2UVI), so she considered $-\frac{1}{2} \geq x$ as left of $-\frac{1}{2}$ and $x \geq 1$ as right of 1 (2UVI). Rosy regarded the highlighted part in the second solution as a common area between the intercepts. However, she did not talk about the algebraic solution and neither did she understand when Daisy mentioned the solutions involving infinity. Although she used terms like '*from one onwards*' she could not relate this to the mathematical terminology '*from one to infinity'* as shown in the transcript in Frame 5.4. When prompted by Daisy, Rosy engaged in another inequality: $-3x^2 + 4x + 4$ (3DC and 3CNJ) but solved it with Daisy's computer. The class had meanwhile proceeded onto another topic, as these two learners were still clarifying (or struggling with) the inequalities. They explored and verified their conjectures. Though Rosy identified the x-intercepts and the shape of the parabola (1EC), she did not respond to the algebraic solution mentioned by her friend. She often toggled between the two inequalities $\geq$ and $<$ that the teacher was discussing as shown in the screenshots in Frame 5.4.

Classroom observation



Screenshot 5.5: Inequality for less than



Screenshot 5.6: Inequality for greater than or equal to …. and then she toggles between the inequalities

***Rosy**: Rosy: f(x) >=0 is that first solution… separation 1MR…. IsiXhosa… left side of -1/2 and right side of 1 1MR and in second solution it's a common area 1MR.*
Daisy: still discussing with her friend on f(x) < 0 and > 0 (the class moved on to the nature of roots). … .. x is greater than 1 and less than negative half for greater than 0.
***Rosy**: Greater than 0 the solution is up to negative half 1MR and then from 1 onwards.*
…
***Rosy**: Negative half to positive one for less than 0 1EC (the teacher is giving instructions to make sliders) …*
Daisy : For greater than its from there to negative half and from there to positive one…that is from infinity to negative half, and one to infinity 1EC meanwhile she draws points A, B and C instead of sliders a, b and c
***Rosy**: for greater than zero you mean its from infinity to negative … I don't know*

Frame 5.4: Rosy's engagement in DGS on quadratic inequalities

In her response during the reflective interview, she did not use algebraic notations for her solutions on inequalities. Her answers reflected that she attempted to imitate the figures as seen on the computer screen as she calculated the critical values of the inequality. She preferred to use trial and error when attempting to solve the problem, and she spoke about finding the 'x-values' when factorising (6BP), the quadratic inequality. She drew the parabola with only the horizontal axis, marked the x-intercepts on the axis, and then shaded the relevant part, as shown in (4AP). Providing reasons for her answers, Rosy said that she considered $f(x) > 0$ as lying above the x −axis and $f(x) < 0$ as lying below the x-axis. Rosy referred to the x-values of the inequality as 'edges' and that either appeared as a left or right separation from the critical values or lay in a common area between them. See the transcript in Frame 5.5. My co-researcher and I debated at length whether we should consider the figure as the final answer. In our subsequent GLIP meeting with the other teachers, we presented the point of discord, and it was decided that the figure alone could not be treated as the final answer although the learner correctly translated algebraic expressions into graphical representations.

Reflective Interview

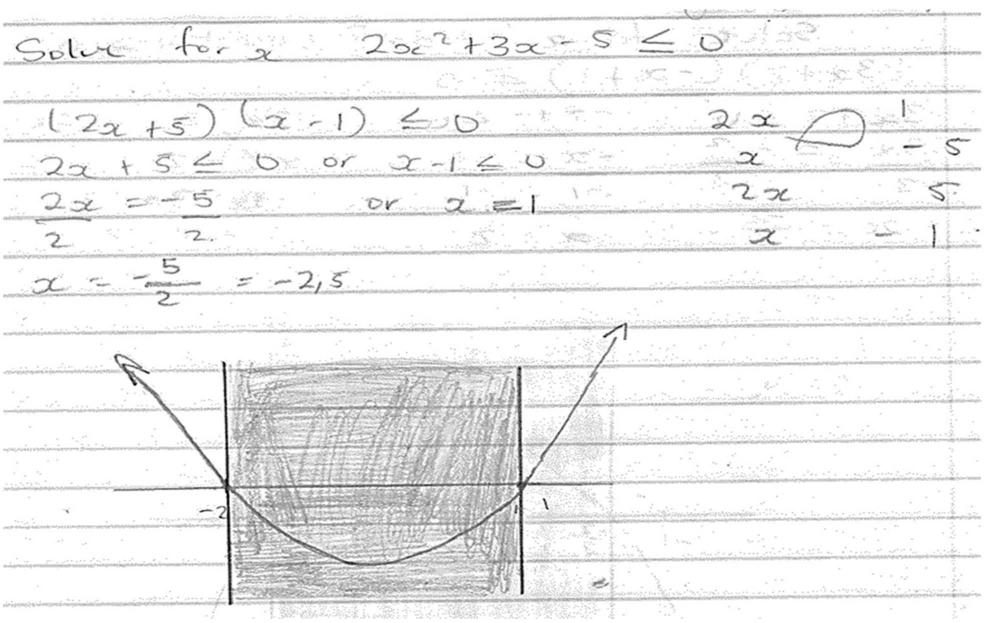


Figure 5.10: Rosy's solution to inequality

R: How did you solve the inequality $2x^2 + 3x - 5 \leq 0$*?* (showing her answer script)
Rosy*: first I found the x-values* 6BP *Then I drew the parabola smile* 4AP *. The graph less than zero lies below the x-axis because its less than zero. So, my answer starts from negative 2,5 and end at one.* 2MC
R: What was the challenge in the second inequality question: $3x^2 - 5x - 2 > 0$*?*
Rosy*: Challenge. After drawing the parabola, I realised that the for greater than 0,* 6BP *I have to see for graph beyond x-axis, and they were on the sides. So, I look at the edges* 4AP *and then I drew lines on opposite directions, left of -1/3 and right of 2.* 2MC

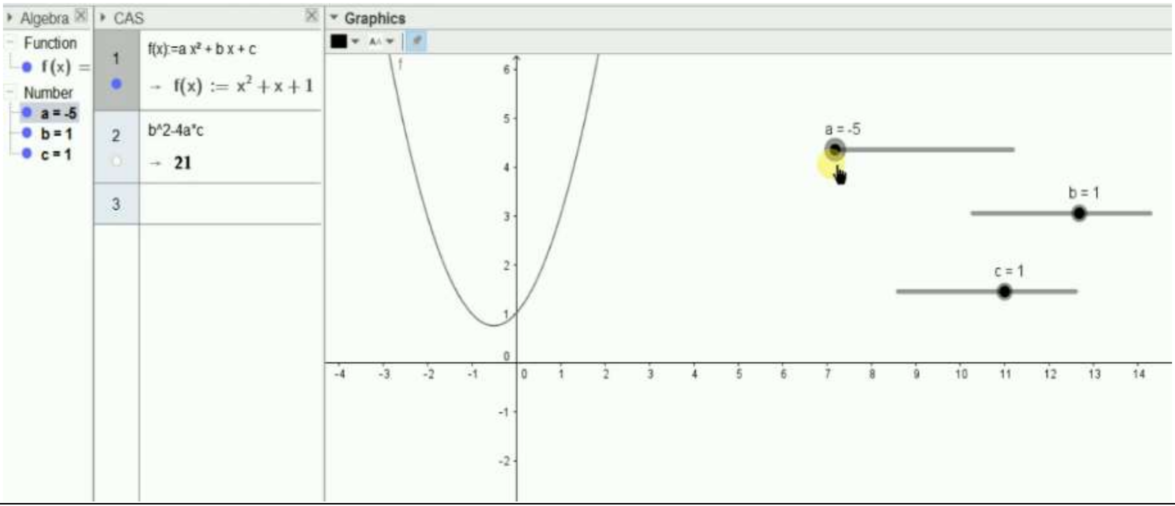
Frame 5.5: Rosy's understanding of inequality

5.3.1.2.4 *Rosy battles with the nature of roots*

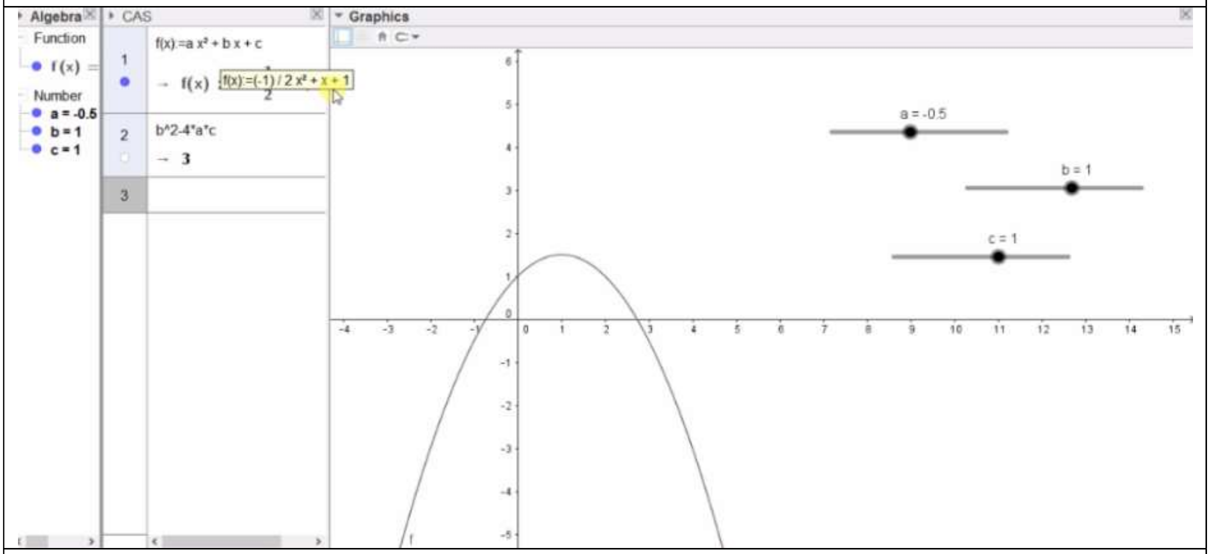
As mentioned earlier, Rosy and Daisy were working at their own pace, trying to make sense of the inequalities. The teacher started to provide instructions for the next applet on the classification of quadratic roots. Rosy missed some of the instructions and thus struggled to coordinate the sliders — the function and the value of the discriminant $b^2 - 4ac$. In the top figure of Frame 5.6, the function in the CAS window did not match with the values of sliders in the graphics window, but the discriminant was aligned. However, Rosy entered her responses incorrectly and started exploring (3DC). In the Screenshot 5.8 of Frame 5.6, both the function and the discriminant were then synchronised with the sliders. When prompted by the teacher, Rosy moved the sliders to obtain the functions $4x^2 - 4x + 1$ and $4x^2 - 4x + 3$. When asked by the teacher she said that the graph in the figure was 'beyond the x-axis and did not touch the x −axis' (1EC and 1MR). Rosy identified that the graph did not intersect with the x-axis, but that it was above the x-axis. She could not relate the figure to principles of the nature of roots, and whether the roots were non-real or imaginary.

Daisy was told by the teacher to share the desktop computer with Rosy. Even while Rosy shared the computer with Daisy for a brief period, there was little conversation between them. Rosy moved the sliders to align them with the quadratic equations provided in the worksheet, for example $3x^2 + 5x + 1$, as shown in Screenshot 5.9 in Frame 5.7. She filled in the worksheet as instructed, but it was incomplete when I took it from her after the lesson. She could not complete it as she had been asked by the teacher to start practising the question applet. Her interaction with the practice applet showed that she did not comprehend the classification of roots of a quadratic equation. The responses in her worksheet also indicated that Rosy did not relate the x-intercepts with the concept of the nature of roots.

Classroom Observations



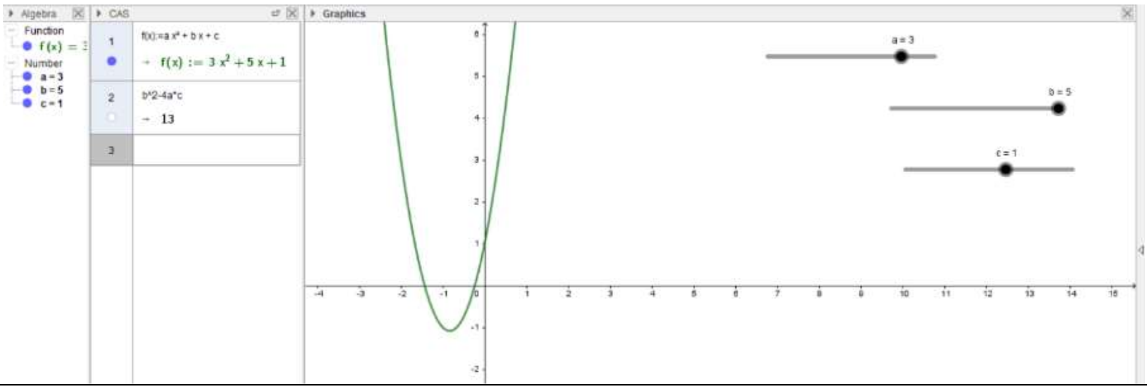
Screenshot 5.7: The function in CAS did not match with the values of the slider but discriminant matched



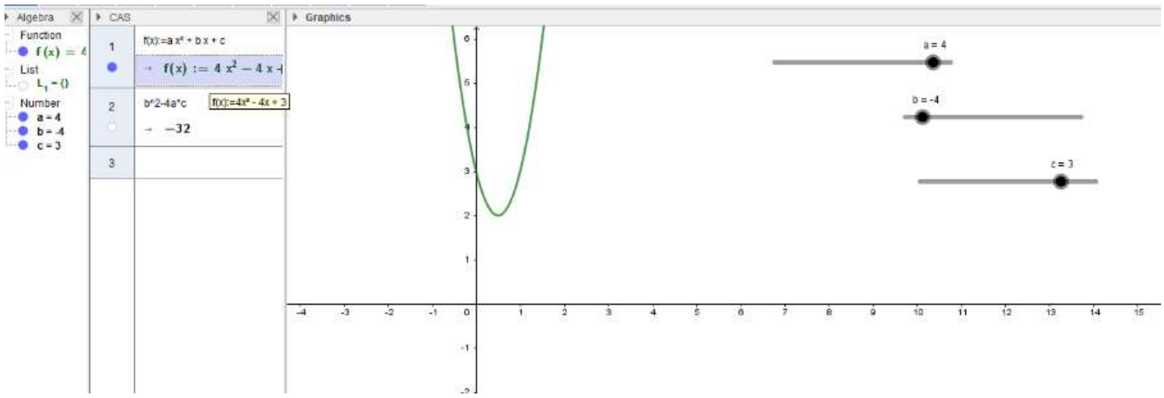
Screenshot 5.8: Both function and discriminant now matched

Frame 5.6: Rosy's construction of sliders and functions in an applet similar to Applet 2.1

Classroom Observations



Screenshot 5.9: Aligning with the second equation in the worksheet



Screenshot 5.10: Responding to the teacher's instruction

Frame 5.7: Rosy's minimal exploration with sliders on the nature of roots in Applet 2.1

During the reflective interview, she struggled to reply to the questions based on the nature of roots, as in the transcript in Frame 5.8. However, for the question that required one to choose the best option as in Figure 5.11, she maintained that the roots were 'unreal' because the *"value was -11"*. Rosy resorted to a numerical approach in classifying the nature of roots (4AP). As mentioned earlier, she did not link the non-intersection of the graph with the *x*-axis to the non-existence of real roots of the equation. She also had an incomplete concept of real and non-real numbers. The transcript in Frame 5.8 exhibits her lack of comprehensive knowledge of rational and irrational numbers. She was confused between negative numbers and non-real numbers. As a teacher, I have experienced more often than not, that learners misconstrue non-real numbers and consider the negative 'discriminant' as a non-real number. They rarely use the term 'discriminant'. Indeed, Rosy had correctly identified the coefficients of the leading, middle and constant term and substituted correctly for the expression $b^2 - 4ac$, but she could not make mathematical inferences. More significantly, her prerequisite knowledge on properties and the definitions of real and non-real numbers were rather vague.

Reflective Interview

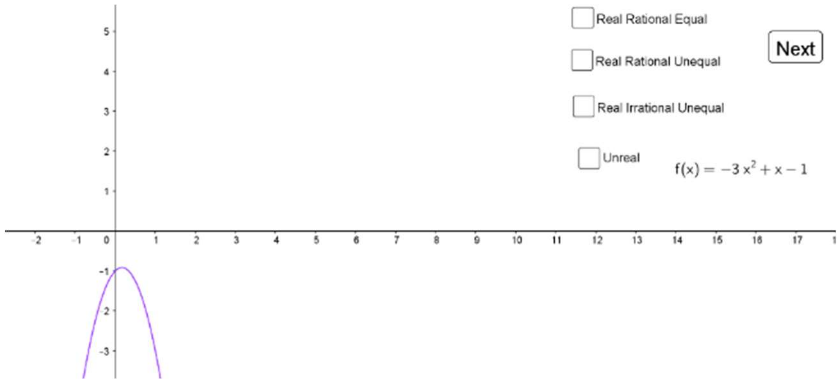


Figure 5.11: Questions asked during the reflective interview

R: Ok which one of the four options you choose.
***Rosy**: I don't think I will be able to answer without using calculator*
R: Ok… try …
***Rosy**: No sir*
R: What?
***Rosy**: It's unreal sir,*
R: Ok… why you say so?
***Rosy**: The value here is -11 4AP*
……..
R: Now you said that when the value was -11 it was non-real. What according to you is a non-real number?
***Rosy**: When the value of (pointing to $b^2 - 4ac$) is negative, then we say its non-real.*
R: Is -4 non-real number?
***Rosy**:-4 is non-real*
R: So, all negative integers are non-real?
***Rosy**: No.*
R: You said – 4 is non-real
***Rosy**: I mean when the value is -4 its non-real.*
R: What you mean by a rational number?
***Rosy** : A rational number that can be written as a fraction*
R: Say an example.
***Rosy** : Half*
R: Is three a rational number
***Rosy** : Yes, I can write it as $\frac{3}{1}$*
R: Ok. ….Ok. Now what is an irrational number?
***Rosy** : A square root is an irrational number $\sqrt{2}$, $\sqrt{5}$ etc.*
R: Ok. Another example that is not a root.
***Rosy** : Cube root of 7.*

Frame 5.8: Rosy's response on the nature of the root in Applet 2.2

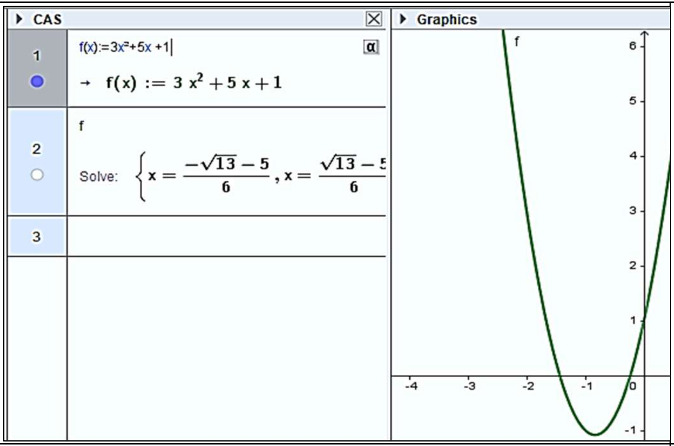
5.3.1.3 **The story of Daisy**

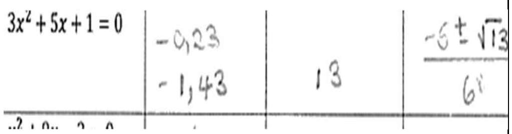
5.3.1.3.1 *Daisy makes sense of equations and graphs*

Like Rosy, Daisy was not self-assured when using the computer keyboards. As Rosy exclaimed when correctly sketching a line, Daisy was also excited to see the straight line $2x + 5$ in the graphics window. However, her excitement was short-lived as it seemed everything passed over her head. Confused, she queried, "How do we make head [sic] of these things?"— see the transcript in Frame 5.1 on p.138. However, she made sense of the line when Rosy allowed her to look at the solution and the *x*-intercept of the straight line. When a quadratic equation was keyed in, she quickly identified the *y*-intercept and subsequently connected the *x*-intercepts for its solution. She then related the figure with her previous knowledge on graphs (1EC, 2MC and 6LPK).

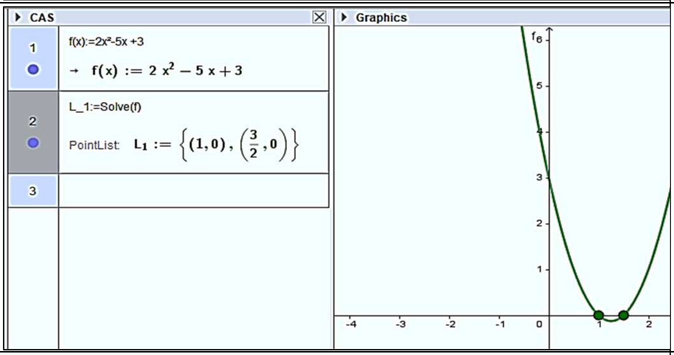
Daisy verified the roots for all of the quadratic equations on the worksheet. A few of such confirmations in the DGS are captured in Frame 5.9, along with her answers on the worksheet. For the equation having irrational roots, she wrote in the numerical form as shown in Screenshot 5.11, Frame 5.9. For the others, she wrote in the coordinate form, seen in Screenshot 5.12 and Screenshot 5.13 in Frame 5.9. Initially, the CAS solves an equation and displays it in numerical form, shown in 'row 2' of Screenshot 5.11. Once the 'show' button is activated, CAS displays the coordinate form — see 'row 2' in Screenshot 5.12 . For the equal roots, Daisy only wrote one intercept, as shown in Screenshot 5.13, and CAS therefore calculated only one intercept. And for non-real roots, she interpreted the concept as non-intersection with the *x*-axis as shown in Screenshot 5.14 of Frame 5.9 and her response on the worksheet.

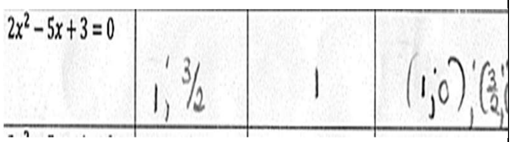
Classroom Observation



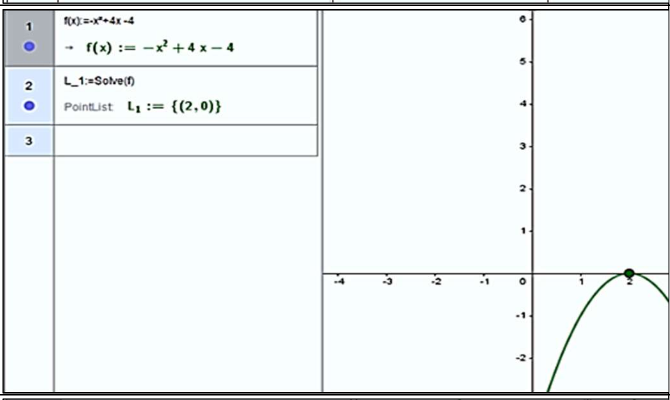


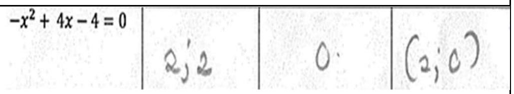
Screenshot 5.11: Daisy began with the second equation, with her response on the worksheet in numerical form



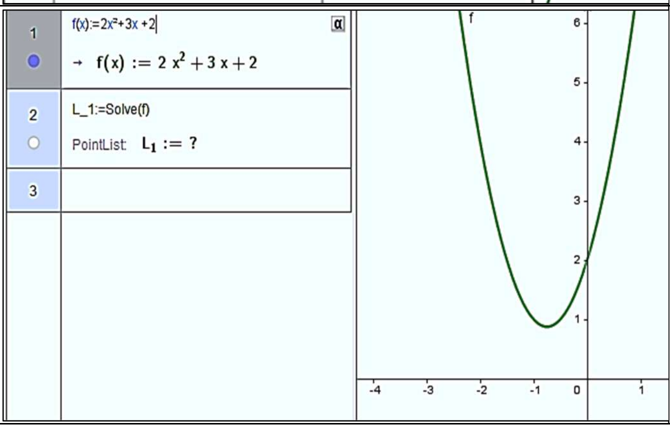


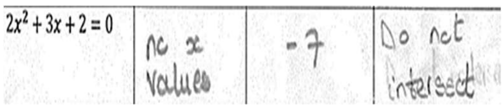
Screenshot 5.12: Daisy writes in coordinate form.





Screenshot 5.13: Daisy's response of one intercept





Screenshot 5.14: Daisy's response on non-real roots

Frame 5.9: Daisy verifying her answers (left hand side) in order to fill the fourth column of the worksheet (right hand side)

5.3.1.3.2 *Daisy makes sense of inequalities*

When the teacher resumed with the function $2x^2 - x - 1$ to discuss inequality, Daisy opened a new file, keyed in the function and followed the instructions of the teacher. Daisy initially understood this as inequality i.e. "*x is greater than 1 and less than negative half for* ≥ 0" (1EC), as she saw the highlighted part of the graph. Ostensibly, she compared the graphical solution in the graphics window and the inequality notation in the algebraic window. She was quick to insert another inequality — $f(x) < 0$ — and highlighted the solution in the graphics window without waiting for instructions from the teacher. She toggled between the solutions of $f(x) \geq 0$ and $f(x) < 0$, in the graphics window. Meanwhile, the teacher moved onto the next topic. Daisy, however, continued to work on the inequality applet a little longer. With interest, she interacted with Rosy on inequalities even as the class was already constructing 'sliders' for discussing the nature of roots (see the transcript in Frame 5.10). As Daisy was engaged in interpreting inequality, she modified from her earlier version of *'negative infinity to negative half',* to that of '*one to infinity'* (1EC). Daisy and Rosy engaged in identifying and understanding concepts in solving inequalities. For instance, Daisy typed in the new function, $3x^2 + 4x + 4$, (3DC and 3CNJ) which was purposefully chosen from the worksheet to have a concave down parabola graph as in Screenshot 5.15 in Frame 5.10. Daisy and Rosy immediately identified the intercepts of the graph. Daisy became concerned when she saw that the inequality for ≥ 0 did not change as expected (2UVI and 7EST) as seen in row 4 in Screenshot 5.15 in Frame 5.10. (Inequality in row 4 was independent of the expression in row 1; shown in the red oval marked in Screenshot 5.15). Nevertheless, Rosy indicated that the inequality < 0 had changed according to the new function. Daisy convinced herself of the solution for the inequality (1EC) in the transcript in Frame 5.10.

Classroom observation

***Daisy**:* (still discussing with her friend about f(x) < 0 and > 0, in spite of the class having moved on to the nature of roots. … (Her equation is now $2x^2 - x - 1$. x is greater than 1 and less than negative half for greater than 0.)

Rosy: (makes her equation to $2x^2 - x - 1$) *Greater than 0 the solution is up to negative half and then from 1 onwards.*

***Daisy**: Less than 0 is from here to here* 2MC

Rosy: Negative half to positive one (the teacher is giving instructions to make sliders) …

***Daisy** : For greater than, its from there to negative half and from there to positive one…i.e. from infinity to negative half, and one to infinity 1EC* (meanwhile Daisy draws points A, B and C instead of sliders a, b and c).

Rosy: For greater than zero you mean it's from infinity to negative …. IsiXhosa… I don't know

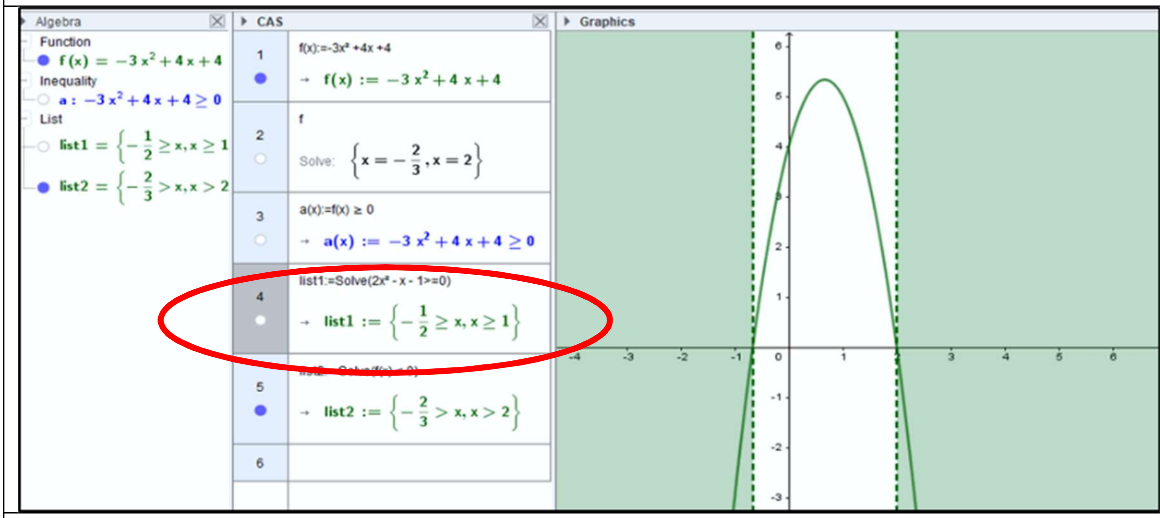
***Daisy**: It's from negative infinity to negative half and from one to infinity. IsiXhosa …. (Look its x element of between) - negative infinity and negative half and one to positive infinity. let us clarify this once again…* (she toggles back to the inequality applet). *What if I use a negative parabola? Here we have* $-3x^2 + 4x + 4$. She changes the first equation to have 3CNJ and 3DC

Rosy: Look at x-intercepts— they are -$\frac{2}{3}$ and 2. The y-intercept at 4. And it's a frown parabola. What is the turning point?

***Daisy**: Yeah what is the turning point? Something wrong. Look here* (clicks on 'show' button at 4). *It's still negative half and one. It's not right.* Laughs ….*2UVI and 7EST*

Rosy: laughs …. *But look below. Check at the last one, I can see that its -$\frac{2}{3}$ and 2*

***Daisy**:* (clicks on show button*)… isiXhosa … Yeah correct. For less than zero, it's from negative infinity to negative 2 over 3, and from two to infinity 1EC*. She quickly toggles back to 'nature of roots' file.

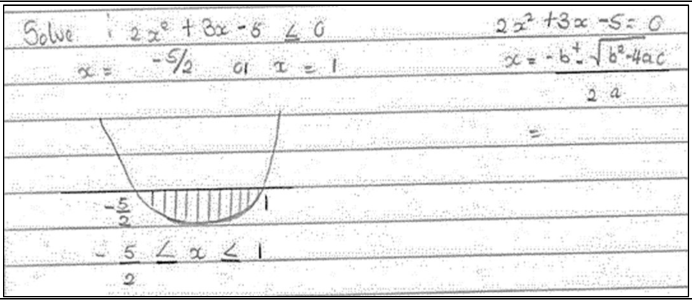
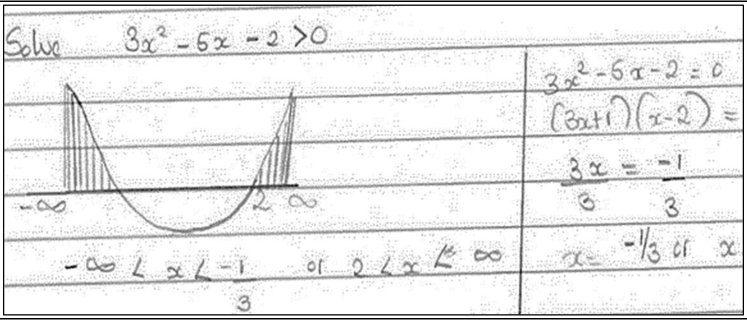
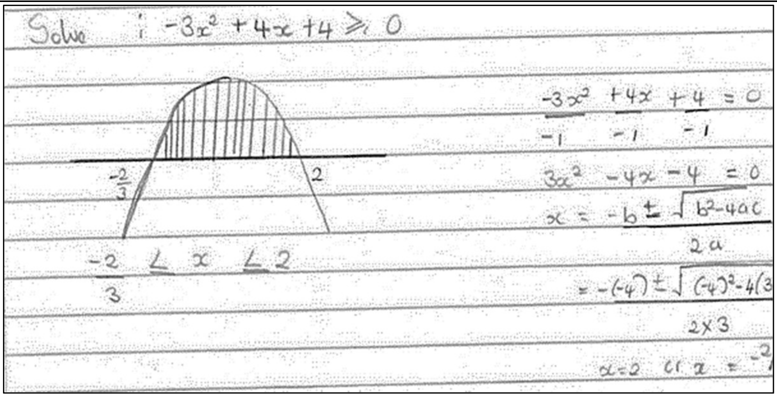


Screenshot 5.15: Daisy verifying with another function

Frame 5.10: Continuation of Daisy's and Rosy's interaction on inequality applets

During the reflective interview, Daisy employed different approaches to determine the critical values of the inequality (4AP and 5PMW) in the figures in

Frame 5.11. In one problem she chose to use "*difference between two squares*" — her terminology for factorisation, and in another question, she used the quadratic formula. However, the critical values and final solution were accurate (4AP). She interpreted the relevant graph in Figure 5.12 in Frame 5.11 as "*dip*" (1EC and 1MR) , Figure 5.13 in Frame 5.11 as "*hanging up*" (1EC and 1MR), and Figure 5.14 in Frame 5.11 as "*hump*" (1EC and 1MR) (see the transcript in Frame 5.11). Daisy also exhibited a clear understanding of the algebraic notation of the solution (1EC), although she wrote all inequality notations with the 'less than' symbol (<) only. Instead of simply establishing connections between equations and their graph, Daisy demonstrated her ability to progress from visual and graphical solutions to a symbolic notation for solving quadratic inequalities. Furthermore, her algebraic solution did not mirror the solution in the DGS. She generated her own solution, owing to her additional engagement with the applet and supported by prerequisite knowledge on linear inequalities.

Reflective Interview	
	*Figure 5.12: Graph of the inequality and its algebraic notation – considered as a "dip"*
	*Figure 5.13: Factorised to determine the critical values, and graphed the inequality as "hanging up"*
	*Figure 5.14: Used the quadratic formula for critical values, interpreting the graph as "a hump"*
Daisy's solution to inequalities	
***Daisy**: I used to find difference between two squares first…. I have the solutions $x = -\frac{5}{2}$ or $x = 1$. I drew a parabola and the question was less than. So I know the answer was the dip in the graph. 4AP, 1EC and 1MR* *R: dip in the graph?* ***Daisy**: the required graph below x-axis. 1EC* *R: what do you call here in the second one here where you have marked above the x-axis* ***Daisy**: its hanging up from the x-axis. 4AP, 1EC and 1MR* *R: what is the third one the bulged one?* ***Daisy**: Yeah its bulged or humped 4AP, 1EC and 1MR*	

Frame 5.11: Daisy's interpretation of quadratic inequality

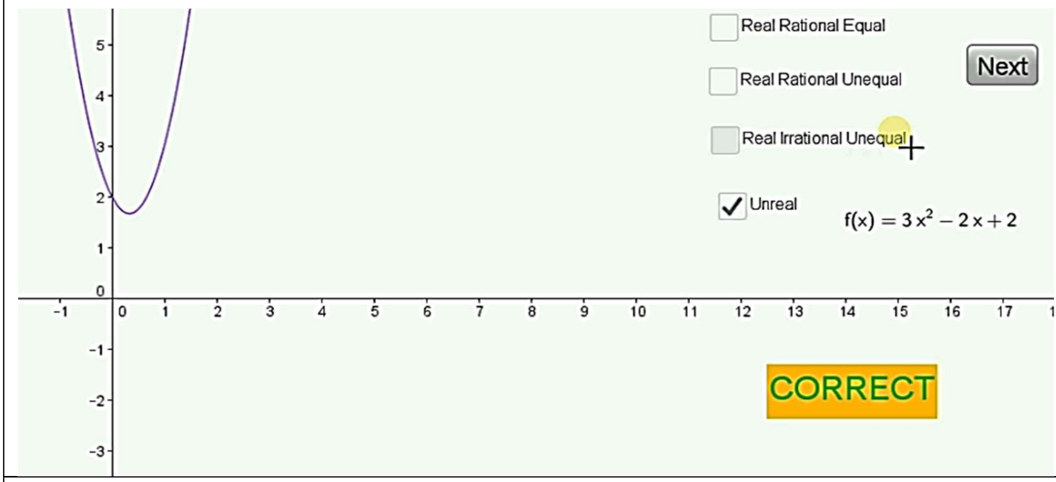
5.3.1.3.3 *Daisy tussles with the nature of roots*

Daisy struggled to construct the sliders, functions and the value of the discriminant. She began with the points *A*, *B* and *C*, instead of the variables (sliders) as the parameters of a quadratic function. She appeared to be influenced by the geometric constructions in DGS that she had experienced before. She managed to overcome these errors but synchronising the slider function and the value of the discriminant remained a challenge to Daisy. The graph did not move dynamically with the change in the values of sliders *a*, *b* or *c*. Her mind was apparently pre-occupied with inequalities, and consequently she could not follow the instructions in *GeoGebra*. Seeing her awkward sliders, Antony asked Daisy to share the computer with Rosy until the practice session ended. There was little interaction between them during this time, and Rosy was dragging the sliders in order to answer the worksheet. Nonetheless, Daisy made relevant and accurate responses to the teacher's questions. When the teacher encouraged the learners to provide reasons as to why the roots were irrational for the equation $3x^2 + 5x + 1 = 0$, Daisy identified *d* as not being a square number (1EC, 2MC and 6LPK). She recollected properties of numbers and linked it to the current scenario, as shown in the transcript below in Table 5.2.

Table 5.2: Daisy identifies rational and irrational roots

Classroom observation	*T: For real roots we further classify equal unequal and rational/irrational. ... look at the second one on the worksheet -* $3x^2 + 5x + 1 = 0$ *the roots:* *L: ... real unequal* *T: The roots are real and that is the roots exists and the graph intersects the x-axis. Here roots are real, unequal and irrational. Why irrational?* ***Daisy****: d is not a square number. 1EC* *T: Yes… listen to her Can you say it aloud?* ***Daisy****: I am saying that the roots are irrational because d =13 is not a square number. 1EC, 2MC and 6LPK*

In the practice Applet 2.2, she took her time to answer the question, and consequently only responded to one question. She claimed that she used the value of *d* in deciding the correct option (1EC and 4AP) Screenshot 5.16 of Frame 5.12. In her reflective interview, she also relied on calculating the value of the discriminant to determine whether the roots were real or non-real (1EC and 4AP). She admitted that she could interpret it from the graph, but she was not confident enough. In the subsequent question (Figure 5.15 in Frame 5.12), she said that the roots were real (1EC) but could not respond as to whether they were rational or irrational from the graph. She could not proceed without seeing the equation of the graph and insisted, "I want to calculate if b squared minus 4ac is a square number or not" This is shown in the transcript in Frame 5.12). Daisy failed to discriminate between rational and irrational numbers directly from the graph. She did not realise that the values of the x-intercepts were equivalent to the roots of the equation. It is important for learners to establish a connection between points of intersection and the roots of an equation. Furthermore, Daisy did not understand that if one of the roots is rational then the other one ought to be rational as well.



Screenshot 5.16: Daisy's response to the nature of roots in Applet 2.2 during the class

Reflective Interview

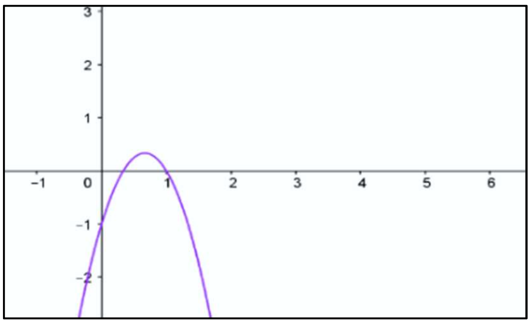


Figure 5.15: Daisy's response on nature of roots during reflective interview.

R: Tell me which option you will choose. (it's a graph that does not intersect the x-axis)
Daisy: *Can I use a calculator?*
R: Yes, you can
Daisy: (uses a calculator). *The roots are non-real. 4AP and IEC*
R: Why?
Daisy: *Because the solution in square root is negative. b squared minus 4ac is a negative number*
R: Can you tell from the graph?
Daisy: *Yes, but I wasn't sure about it.*
R: Ok. Next equation (purposefully hiding the equation) (refer to in *Figure 5.15*)
Daisy: *Where is the equation?*
R: Why you want to see the equation?
Daisy: *How can we tell without calculating?*
R: Look at the graph and tell.
Daisy: *Ok. I think its real IEC*
R: What makes you think?
Daisy: *Because sir …sir, what is the equation?*
R: Can you tell if its rational or irrational from the graph?
Daisy: *I want to calculate if b squared minus 4ac is a square number or not*

Frame 5.12: Daisy's response on the nature of roots

5.3.2 **Paul's lesson**

5.3.2.1 **Overview of the lesson**

Antony advised Paul to confine himself to only one topic in the class, as he felt learners struggled to understand the latter part of the lesson when he had discussed both inequalities and the nature of roots. Therefore, the recorded lesson of Paul included only quadratic inequalities. Although the lesson on the nature of roots was scheduled for the following day, it was not recorded.

There were computers which were not working, and these were indicated accordingly. This saved time, unlike Paul's previous lesson. Out of 48 learners, a few still had to share the desktop computers. Paul had deliberated on solving quadratic equations and inequalities prior to this lesson. The lesson was scheduled for 40 minutes outside normal school hours and lasted for 35 minutes. We changed the handouts to include a caution: 'Use small letter 'x'', but the new handouts could not be printed in time due to technical problems at the school. Many learners faced the challenge of using keyboards, as was the case in Antony's class. The teacher started off with the quadratic equation $x^2 - x - 6$, giving clear instructions as how to key in the equation, "*You press your x ne then you press shift and 6 then you press 2*". However, a few learners attempted to enter '6' on the number pad. Another learner used the 'shift 6' for the number '6' as well. The teacher often found that learners were using capital letter *X*, instead of small letter x. *GeoGebra* distinguishes between x and X, and ignoring this may not provide the desired results. Another reason for learners struggling was the inadvertent pressing of keys like 'caps lock' and 'num lock'. Once the learners keyed in the correct equation in the correct format, they became excited to see a parabola graph appearing on the screen. Those who could not get the graph on their screens attracted the teacher's attention for help. Once the learners had the required equation, Paul handed out the worksheet WS-1. He encouraged them to fill in the fourth column of the worksheet — the intercepts of the equation. A few of them faced problems in *GeoGebra* since they used the equation and not the function in determining its solution. Consequently, the intercepts would not change accordingly, and this caused confusion among the learners. Despite technical difficulties, however, the learners eagerly wrote down the intercepts on their worksheets.

Paul then moved to the quadratic inequalities. Entering inequality symbols like $<$, $>$ and $\leq$ also posed a problem for the learners as a few of them incorrectly thought of using the 'shift' key for activating the keys on the top rows of the keyboard. However, the enthusiasm of the learners continued throughout the lesson. They remained engaged with the computer tools and some of them tried out the inequalities with the equations on the worksheet.

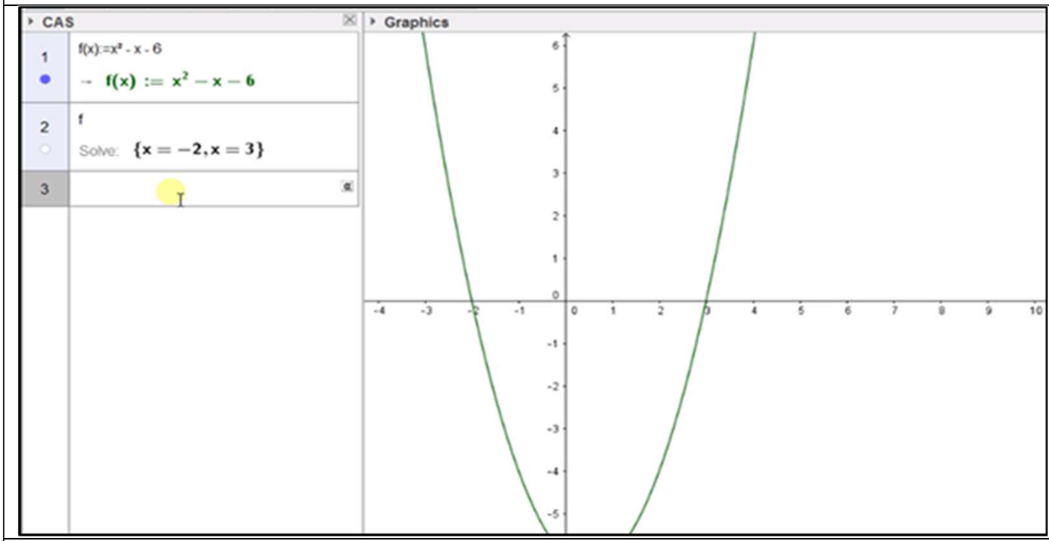
In the following narrative, I have included that of Jasmine only. Aster's screen recording was corrupted during the lesson and could not be recovered. However, his reflections are discussed in the subsequent analysis.

5.3.2.2 **The story of Jasmine**

5.3.2.2.1 *Jasmine engages with quadratic equations and graphs*

Jasmine did not face many problems when entering the quadratic equation on her keyboard. She was delighted to see a parabola. "*Wow it's a smile parabola*", she exclaimed (1EC) in Screenshot 5.17 of Frame 5.13. She frequently engaged her friend in conversation and raised her concern that the software did not provide step by step procedures for the solutions, in the transcript in Frame 5.13. She identified the solutions of the equation with the points where the "*graph cutting or cut? across the x-axis*," (1EC, 1MR and 2MC).

Classroom Observation

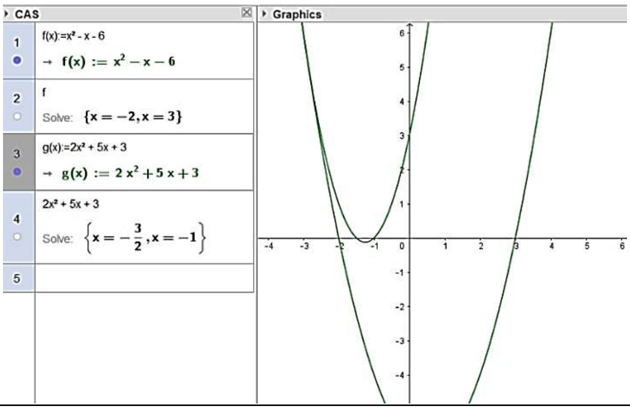
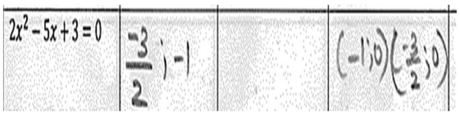
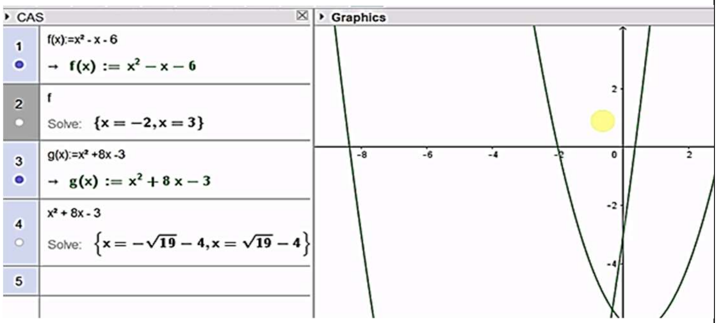
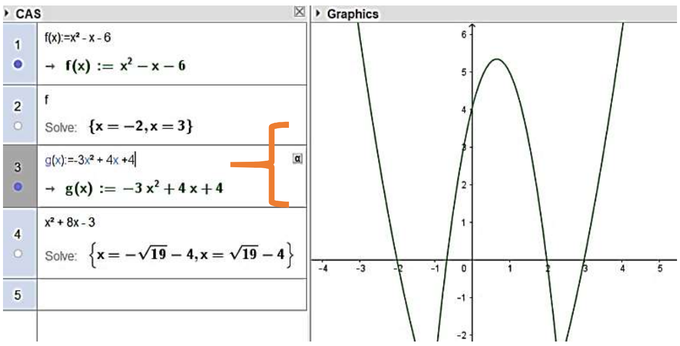


Screenshot 5.17: Jamsine's first equation with solution and graph

Jasmine: *I solved it wow. I wish I got this software while writing exams.*
Lilly: Yeah true.. we will all be level 7.
Jasmine: *But [isiXhosa] I didn't like this. It doesn't show you the steps. We will not get marks for writing alone the answer.*
Lilly: That's also true.
Jasmine: *Where are the steps? Ne.... Do you see that those values of x are the x-intercepts*
Lilly: What?
Jasmine: *look here at number 2 ne ..isiXhosa ... x is -2 and positive 3. 2MC Then see the graph*
Lilly: What? I didn't see
Jasmine: *the x-intercepts 1EC the graph cutting across 1MR the x-axis those points are at -2 and +3.*

Frame 5.13: Jasmine's interaction with CAS and in conversation with Lilly

Jasmine attempted to find the intercepts of the equations as instructed. The first attempt did not match with the standard solution due to her incorrect input into the CAS — $2x^2 + 5x + 3$ instead of $2x^2 - 5x + 3$. She wrote only the values of the solutions of the equation in column two of the worksheet as shown in Screenshot 5.18, Frame 5.14. However, under column four, she wrote the equation in the Cartesian coordinates form (6LPK and 1EC), employing her prior knowledge on analytical geometry. She was not happy with the algebraic notation of the solutions involving surds as shown in Screenshot 5.19, Frame 5.14. Apparently, Jasmine was used to writing numbers involving surds only, in the form $a + b\sqrt{c}$, "*first rational then irrational*" (1EC). She adjusted the screen to see the values of the *x*-intercepts (she knew exactly how to drag the edges of the CAS window), and then restored it to the preceding view. She had typed the equations in rows three and four of CAS, hence she had to type those equations twice as shown in refer to Screenshot 5.20 of Frame 5.14. She further expressed her dissatisfaction over it. No sooner had she seen the concave down graph than she quipped, "*After a few smileys comes the frowning one.*"

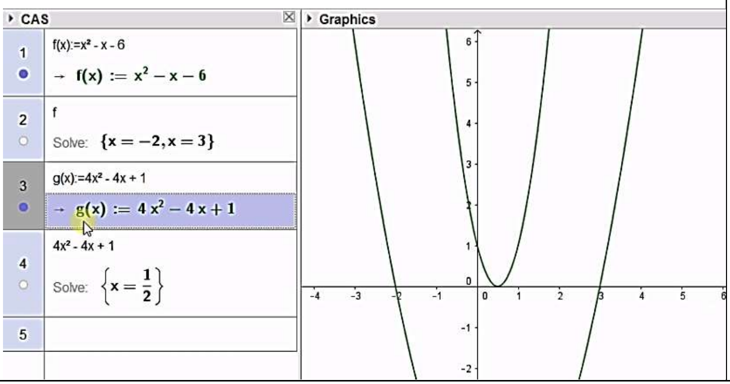
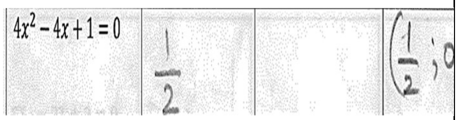
Classroom Observation	
	 *Screenshot 5.18: Jasmine's* response *to the first questions of the worksheet*
	 *Screenshot 5.19: Jasmine drags the screen to view the x-intercepts*
	*Screenshot 5.20: Row 3 and row 4 are not linked - Jasmine was forced to key in twice.*

Frame 5.14: Jasmine's exploration with different equations

5.3.2.2.2 *Jasmine explores 'special' equations*

The next equation in her exploration was $4x^2 - 4x + 1$, (3DC), Screenshot 5.21, Frame 5.15. She remained motionless for a while with the equation on the computer screen, as she considered it as something 'special', and it stayed on her screen throughout the lesson. In conversation with her friend, she said that it was special because the turning point of the graph was on the x-axis (6LPK and IEC) (see the transcript in Frame 5.15). It was also peculiar to her because it had only one solution and one x-intercept. In her reflective interview, she claimed that she worked out the solution (6BP) and found that it had only one solution (IEC) which was

$\frac{1}{2}$ (4AP). She observed that the graph had only one x-intercepts (2MC) and that it was 'resting on the x −axis' (1MR). On further prompting, she concluded that when there was only one x-intercept for a parabola then that would be its turning point (2UVI and 1EC).

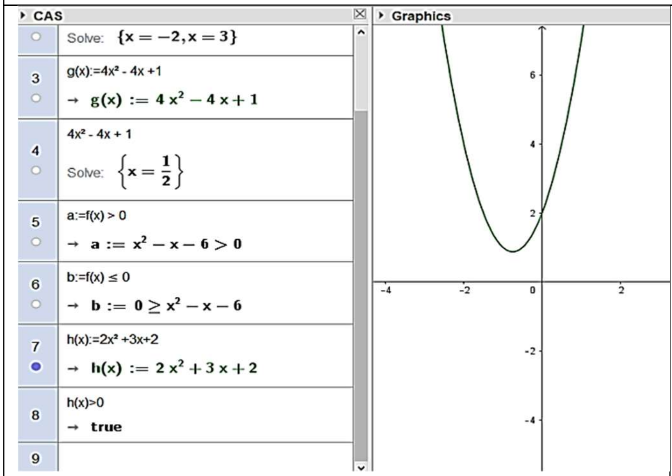
<table>
<tr><td colspan="2">Classroom Observation</td></tr>
<tr><td>
</td><td>

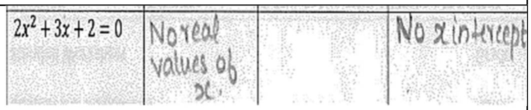
Screenshot 5.21 Jasmine ponders overs the graph of equal roots for a while</td></tr>
<tr><td colspan="2">Jasmine: I was just thinking why there are only smileys. After a few smileys, here comes the frowning one. 1EC
…
Jasmine This one is special.
Lilly: Why so special?
Jasmine: don't worry. It's because the turning point is on the x-axis? 6LPK and IEC
Lilly: let me see. Yeah.   IsiXhosa (But so what?)
Jasmine: look look… there is only one solution ½ and there is only x-intercept. 2MC
Lilly: I can see that. Now you are talking.
Jasmine: I told you that I am genius.</td></tr>
<tr><td>Reflective interview</td><td>R: we see that you spend at least two minutes here, at $4x^2 - 4x + 1$ what you were thinking?
Jasmine: I remember that I found the solution which was $x = \frac{1}{2}$, (6BP and 4AP) and it was the only solution. The graph also had only one x-intercept makes sense so I liked the graph. I saw the graph is resting on the x-axis. 1MR …
R: yeah true. Let's make it mathematically; you are right when you say it's resting on the x-axis. What can you something else about the x-intercept in this context.
Jasmine: x-intercept here is x= (½ ,0) only x-intercept. Yeah it's the turning point of the graph. 2UVI and 1EC</td></tr>
</table>

Frame 5.15 Transcript – Jasmine's exploration with the equations

She did not attempt all the equations on the worksheet, skipping the second last question. The last one, $2x^2 + 3x + 2$, was attempted towards the end of the lesson, as shown in the figure on the left in Frame 5.16 and her answer in the worksheet in Screenshot 5.22 of Frame 5.16. She solved the inequality and wrote on the worksheet '*no real values of x*'. She sought help from the teacher to explain the inequality, since the graph was not highlighted, and then verified her answer on the equality with him. During the personal interview, she revealed that once she realised that the graph had no x-intercepts, she inferred that the equation had no solutions, leading to the answer, '*No real values of x*' (see the transcript in Frame 5.16). This answer provided evidence of her proficiency in a reverse train of thought — linking the x-intercepts to the solutions (5REV). She associated the graph with the properties of real numbers (2MC), thus exhibiting her understanding of invariant properties (2UVI).

Classroom Observation





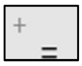
Screenshot 5.22: Jasmine's response on the worksheet as in the figure above

Jasmine: *Teacher, why doesn't it highlight?*
T: What you notice here, neah ?
Jasmine: *No x-intercepts*
T: So you see the graph is always positive, so for all values of x.
Jasmine: *So am I ok here 'no real values of x' because there are no solutions. 5REV and 2MC*

Reflective interview	*R: How did you calculate the for the last one where you have written 'no real values of x'.* ***Jasmine***: *I have seen that there are no x-intercepts of the graph. So, I thought there are no solutions 5REV. I remembered that when there are no solutions we write as no real values of x 2MC.*

Frame 5.16: Jasmine's encounter with the last equation

5.3.2.2.3 *Jasmine tackles the quadratic inequality*

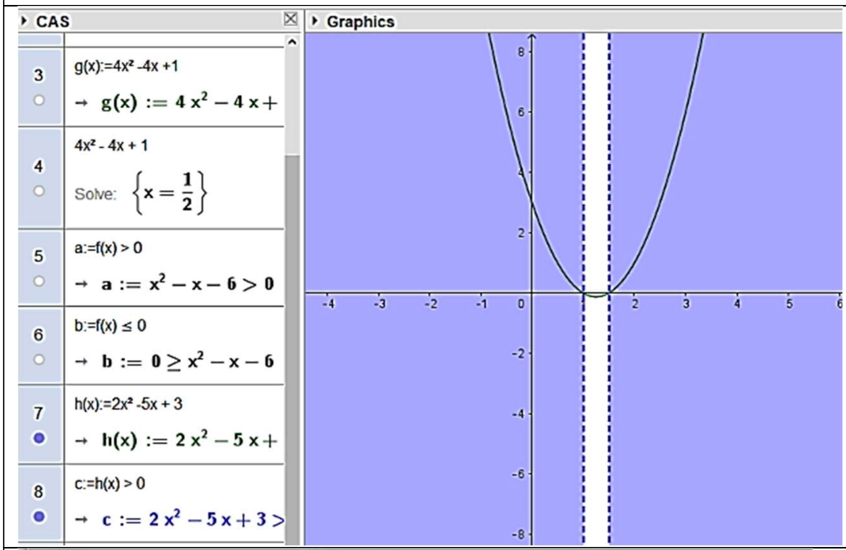
As instructed, Jasmine keyed in the first inequality $x^2 - x - 6 > 0$, but initially, she could not make sense of the figure. She proceeded to the next inequality $x^2 - x - 6 \leq 0$. Jasmine could not find the correct key required to type the equal to (=) symbol. She repeatedly entered '+' instead of '='. (The symbols '+' and '=' share a common key on the computer keyboard. , with '=' being activated by pressing the key while '+' is activated by pressing 'shift' and the key [5]. She was assisted by the teacher to correct her error.

Soon she gained confidence and began to explore further. Without any instructions from the teacher, she began to enter more equations and verify their inequalities (shown in Screenshot 5.23, Frame 5.17). The next equation in her exploration was $2x^2 - 5x + 3$ (the first equation on the worksheet). Her input was correct this time, but to her astonishment the DGS highlighted

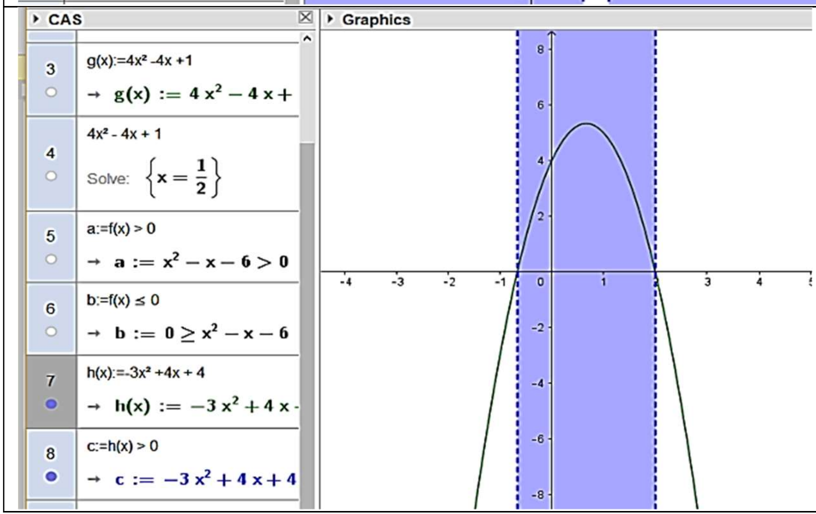
[5] Image taken from Windows ® on-screen keyboard.

the inequality in the graphics window and this did not coincide with her conjecture (7EST). She evidently became confused because her x-intercepts on the worksheet were different because she had calculated them incorrectly. She estimated, expecting that the x-intercepts that she had written down would be the critical values of the inequality. She hid the irrelevant graphs in the given context as she did not realise that they were different owing to the incorrect input of the equation and continued with other equations in the worksheet. She could not attempt all the equations including her favourite, because of time constraints. She jumped to the last equation in the worksheet, where she toggled between the inequalities, apparently attempting to understand the solution related to the inequality (3DC).

Classroom Observation

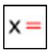


Screenshot 5.23: The first equation in the worksheet that confused Jasmine with x-intercepts



Screenshot 5.24: A concave down equation

Frame 5.17: Jasmine exploring further inequalities

During the reflective interview, when asked to solve the inequalities, Jasmine first determined the 'x-intercepts' using the quadratic formula (6BP). She drew the concave up parabola graph, including the last one, which had a negative coefficient of x^2. She correctly related the concave down graph as a mirror image of the concave up graph when reflected over the *x*-axis, and hence changed the inequality (2MC and 2UVI) from $\geq$ to $\leq$. She employed her prior knowledge on reflection (or was influenced by her manipulation of equations in *GeoGebra*) of objects over the *x*-axis. She interpreted 'greater than zero' as positive and 'less than zero' as negative, as is evident from her answers (1EC) in the response to the question in Figure 5.17, Frame 5.18. However, she concluded her solutions in algebraic notations shown in Figure 5.16 of Frame 5.18. She relied on procedural knowledge as she said, "*Less than the inequalities is 'less than', 'greater than' then the inequalities is greater than*". What she meant was that when the question involved the 'less than' inequality symbol, then the final answer would also have the 'less then' inequality symbol. She could not translate the critical values of the graph into a full understanding of quadratic inequality and its representations. Significantly, during classroom interactions, she did not utilise the tool  available in the CAS perspective of *GeoGebra* that would have solved the inequality algebraically.

Reflective interview

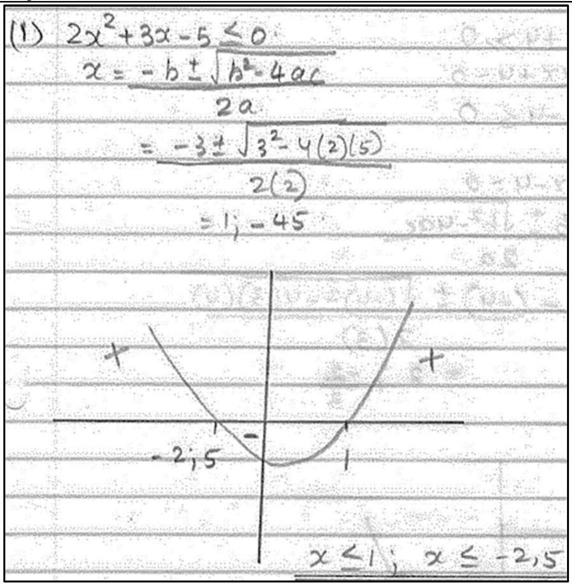


Figure 5.16: Solving inequality using her own rules such as 'less than', then inequality is less than'

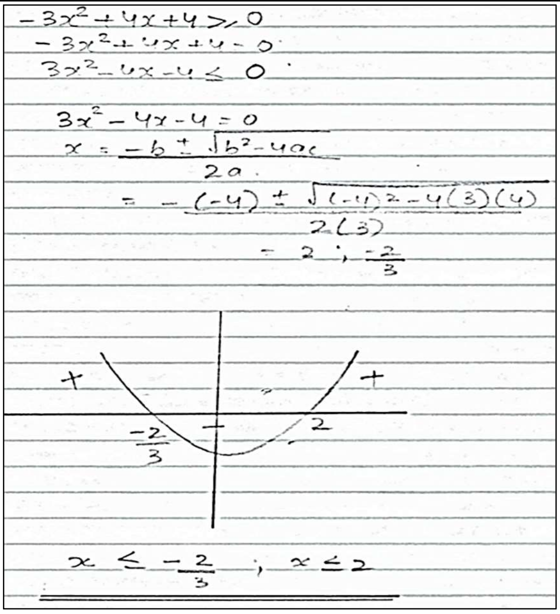


Figure 5.17: Jasmine changes to positive parabola and then solved the inequality

R: Can you explain how did you solve the inequality $3x^2 - 5x - 2 > 0$*?*

Jasmine*: I found the x-intercepts of the equation 6BP, which were -1/3 and 2 4AP. I had to sketch the smile parabola graph, 1EC I was not sure if I had to plot y-intercept or turning point. Then I remembered that we did not discuss anything about y-intercept and turning point. So, I just plotted the graph with x-intercepts. 4TOL*

R: You said this is smile graph, what about this one $-3x^2 + 4x + 4 \geq 0$*? Can you use a mathematical term for smile?*

Jasmine*: It's a frown graph, it's called concave down 1EC and 1MR. But I understand from the graphs that smile and frown are basically same expect that one is up and other is down 2MC and 2UVI. So, when I changed from concave down to concave up the upper portion will go to the lower portion and lower portion to the upper portion 4AP. So, I changed the inequality. I know for a concave up the lower portion is negative and upper portion is positive. 1EC*

R: I understand your graphs, but let's look at your final answer. How did you arrive there?

Jasmine*: Less than then inequalities in less than, greater than then the inequalities in greater than.*

Frame 5.18: Jasmine's solutions to the inequality

5.4 DISCUSSION AND INSIGHTS - HORIZONTAL ANALYSIS ACROSS PARTICIPANTS

The participating learners' responses and activities were collated under each broad indicator to determine the development of their meaning-making and mathematical proficiency in quadratic inequalities. Since Paul did not include the nature of roots in his lesson, Jasmine did not have the opportunity to interact with Applets 2.1 and 2.2. Therefore, indicators relating to the discussion of the nature of roots include Rosy's and Daisy's responses only.

5.4.1 Essence of interactions –BI-1 (Defining and identifying relevant mathematical concepts, relations and notations)

5.4.1.1 Synopsis

Daisy:

- identified relevant mathematical concepts in a quadratic inequality (CU);
- used relevant mathematical notations to represent inequality (MM and CU);
- identified relevant mathematical relationships between the discriminant and the roots of a quadratic equation (CU).

Jasmine:

- identified relevant mathematical concepts in a quadratic inequality (CU);

Rosy:

- identified relevant mathematical concepts in a quadratic inequality (CU);
- exhibited partial (or confused) understanding of the relationship between the discriminant and the roots of the equation.

5.4.1.2 Discussion

The learners exhibited their conceptual understanding in slightly varying degrees. Alsina and Nelsen (2006) acknowledge the principle that every inequality should come from an equality which makes the inequality obvious. All the participants' responses supported this principle of finding equalities associated with the inequalities. The solutions during the reflective interviews exhibited their ability to analyse the behaviour of the polynomial function of degree two. They accurately sketched the parabola graph in the direction appropriate to the nature of the leading coefficient and plotted the roots of the equation as the points of intersection with the x-axis. Despite this, their deductions from the graph differed. Daisy's solution was appropriate, as she used the relevant mathematical notations and she also employed the mathematical conjunction 'or' meaningfully. Rosy and Jasmine made incorrect interpretations from their own graphical representations. Rosy and Jasmine did not exhibit the correct notation, symbolic or visual, of representing an inequality. Jasmine simply replaced the equality; sign with the given inequality; sign in the question, showing no relevance to her diagram, while Rosy did not come to a valid conclusion using mathematical terms or notations.

Indeed, Rosy and Daisy revealed their understanding of classifying the roots of a quadratic equation accurately, based on the calculated value of the discriminant. However, in the discussion of the nature of roots, the participants did not use the appropriate terminology for the value of the expression $b^2 - 4ac$ as a discriminant. During the reflective interview, Rosy appeared confused while responding to a question that required her to discuss the nature of roots. Initially, she considered the negative coefficient of x, i.e. the value of b in the standard form of the quadratic equation that determined the roots to be real or non-real. After one or two equations and graphs, she wrote down $b^2 - 4ac$ in her book and calculated the value of the expression for a given equation. However, the concepts of number properties were rather vague as is further discussed under BI-6, Section 5.4.6 (p.171).

5.4.2 Essence of interactions –BI-2 (Patterns and relationships)

5.4.2.1 Synopsis

Daisy:

- connected and recognised equivalence of x-intercepts and solutions to equations (MM and CU);
- related the position of the turning point of the parabola, (MM) such as 'equal roots have TP on the x −axis'.

Jasmine:

- connected and recognised the equivalence of x-intercepts and solutions to the equations (MM and CU);
- related the position of the turning point of the parabola (MM) such as 'equal roots have TP on the x −axis'.

Rosy:

- connected and recognised the equivalence of x-intercepts and the solutions to equations (MM and CU);
- related the position of the turning point of the parabola (MM) such as 'equal roots have the turning point on the x −axis' and 'the turning point is in between the x-intercepts'.

5.4.2.2 Discussion

Carter et al. (2009) illustrate the power of using technology to link functions and expressions in an abstract mathematical context. Vignettes of the learners' interactions with the *GeoGebra* applets provide evidence of learners establishing connections between equations and their related graphical representation. More significantly, all the participants identified relationships and understood the invariant approaches to solutions to a quadratic equality and the critical values of its quadratic inequality. This identification of the connection between an equation and its graphical representation proved to be crucial in their sense making of quadratic inequality. All the participants, except Aster, sketched the graph of the parabola in order to solve the inequality. Figure 5.10 (p. 143) by Rosy, Figure 5.12 (p. 152) by Daisy and Figure 5.16 (p. 163) by Jasmine show their sketches and how their understanding of graphs differed when attempting to arrive at the final solution. These differences are discussed under BI-4, section 5.4.4 (p.167).

Furthermore, the participants connected the repeated roots of an equation with the position of the turning point of parabola on the x-axis. Jasmine expressed her excitement in seeing a parabola 'resting' on the x-axis for a quadratic polynomial having repeated roots in the transcript in Frame 5.15 (p.159). She theorised from her experiences to identify that the equal x-intercepts of a parabola would also be its turning point. Rosy's meaning- making of repeated roots was equally interesting. The transcript, illustrated in Frame 5.3 (p.141) reflects Rosy's understanding of repeated roots and the turning point. For Rosy, there were two roots for the equation $x^2 - 4x + 4 = 0$, but they were equal; hence she wrote her answer, $x = 2 \; or \; x = 2$ for solutions to the equation shown in Figure 5.9 (p. 141). She appeared to understand that for a quadratic equation there were either two roots or no roots at all. Meanwhile, Daisy and Jasmine chose to write only one root for the repeated roots of the equation. Furthermore, Rosy specifically identified that the turning point of a parabola was placed between the x-intercepts.

5.4.3 **Essence of interactions –BI-3 (Looking for hidden structures)**

5.4.3.1 **Synopsis**

Daisy:

- explored and made conjectures on commonalties between $f(x) > 0$ and $f(x) < 0$ (MM)

Jasmine:

- explored and made conjectures on commonalties between $f(x) > 0$ and $f(x) < 0$ (MM)

Rosy:

- explored and made conjectures on commonalties between $f(x) > 0$ and $f(x) < 0$ (MM)

5.4.3.2 **Discussion**

Carter et al. (2009) suggest that conjecturing and exploring "gives students the opportunity to become immersed in, and deepen their understanding of, the mathematical relationships involved" (p. 56). The participant learners explored with different functions to grasp the idea of inequalities. In particular, learners purposefully used positive and negative leading coefficients of quadratic functions to establish their conjectures on the critical values of the quadratic inequalities. Daisy and Rosy both deliberated on the inequality f(x) > 0 for the function $2x^2 - x - 1$ as instructed by their teacher. In order to validate their assumptions, however, they chose to use different quadratic equations such as $-3x^2 + 4x + 4$. They evidently concluded that the x-intercepts of the graph were the critical values of the inequality;

however, none of them used the term ‘critical values’. The following transcript during classroom interactions between Daisy and Rosy demonstrated the kinds of conjecturing and conclusions they had.

Rosy: (makes her equation to $2x^2 - x - 1$) *Greater than 0 the solution is up to negative half and then from 1 onwards.*
Daisy: Less than 0 is from here to there
Rosy: Negative half to positive one …
Daisy : For greater than it's from there to negative half and from there to positive one…that is from infinity to negative half, and one to infinity …
Rosy: For greater than zero you mean it's from infinity to negative …. [IsiXhosa]… I don't know
Daisy: It's from negative infinity to negative half and from one to infinity. [IsiXhosa] Look its x element of between negative infinity and negative half and one to positive infinity. let us clarify this once again… What if I use a negative parabola? Here we have $-3x^2 + 4x + 4$. (She changes the first equation).
Rosy: Look at x-intercepts they are -2/3 and 2. y-intercept at 4. And it's a frown parabola. What is the turning point?
Daisy: Yeah what is the turning point? Something wrong. Look here at 4, (clicks on ‘show’ button) *it's still negative half and one. It's not right.* Laughs ….
Rosy: laughs …. *But look below. Check at the last one, I can see that its* $-\frac{2}{3}$ *and 2*
Daisy: (clicks on show button)… *[IsiXhosa] Yeah correct. For less than zero, it's from negative infinity to negative 2 over 3, and from two to infinity.*

Furthermore, Jasmine frequently switched between the inequalities ‘greater than zero’ and ‘less than zero’.

Despite the fact that the learners took special note of repeated roots, none of them used those equations to explore inequality, neither did they attempt to explore inequalities with quadratic equations with no real roots. It would have been intriguing to observe their discussion and conclusions on inequalities of perfect squares.

In the discussion of the nature of roots of a quadratic equation, Daisy and Rosy did not engage with the applet. They lost the opportunity to explore further and discover the links between the values of the roots of an equation and its coefficients.

5.4.4 **Essence of interactions –BI-4 (Purposeful use of procedures)**

5.4.4.1 **Synopsis**

Daisy:

- organised the solution of a quadratic inequality based on exploration and conjectures (MM and PF);
- accurately executed a mathematical algorithm in determining the nature of roots (PF); and
- considered a simpler analogy of the graphical procedure while solving the quadratic inequalities (MM).

Jasmine:

- accurately executed the graphical procedure while solving the quadratic inequalities (PF).

Rosy:

- accurately executed the graphical procedure while solving the quadratic inequalities (PF); and
- accurately executed a mathematical algorithm in determining the nature of roots (PF).

5.4.4.2 **Discussion**

BI-4 indicates that these learners were able to carry out the procedures accurately and efficiently in determining the roots or critical values of the quadratic expressions. This was again crucial as it allowed the learners to identify the processes involved in solving the quadratic inequalities. It may appear that the applets did not directly help them to solve the equations, but the operations and procedures adopted by the participants provide evidence that they treated the roots of the equation identically to the critical values of the inequality. Initially, Jasmine complained that the DGS did not provide step by step procedures in solving the quadratic equations, however, when she made sense of the *x*-intercepts of the graph and the roots of the equation and she further engaged enthusiastically with the DGS. Thus, DGS provided opportunities for the development of understanding. However, computation of the solutions of equations is necessary to develop understanding of inequalities.

Rosy and Jasmine displayed varying aspects of conceptual and procedural understanding of the topic. They were backed by a foundation of quadratic equations and efficient calculations of the roots of the equation, but they could not conclude the solutions of the inequality in a mathematically relevant manner. Rosy and Jasmine could translate algebraic equations into their graphic equivalent but could not translate vice versa i.e. a graphical representation to its algebraic inequality. Rosy's answer in Figure 5.10 (p. 143) demonstrates this, where she drew the parabola accurately and shaded the inequality but did not orchestrate her conclusion, algebraically or graphically. Jasmine also drew the parabola in Figure 5.17 (p.163), but her inferences were flawed.

Meanwhile, Daisy realised the relevance of connections between algebraic equations and their graphic representation, and she concluded the inequalities accurately (Figure 5.14 on p.152 reveals her answers). She appeared to rely on her prerequisite knowledge and was supported by her interactions with *GeoGebra* and her peers in representing an inequality accurately. Daisy exemplified the constructivist perspective of learning mathematics (Hoyles, 2005; Jaworski, 1994). As learners interacted with the computer tools and among themselves, they were involved in a mathematical experience and this made an important contribution to individual knowledge construction.

The participants did the computations correctly, but only Daisy and Aster came up with the accurate final answer. They demonstrated that they had learnt to solve quadratic equations and inequalities in an integrated manner using the algebraic and graphical representation afforded by DGS. I concur with the views of Howson (2005), who argues that as educators we are not only interested in mathematical end products but also in the educational journey and the insights gained en route. Consistent with Haapasalo (2007), this research study also finds that modern technological tools like DGS allow learners to manipulate and relate different representations and thus support the development of procedural and conceptual knowledge simultaneously.

In discussing the nature of roots of a quadratic equation, Daisy and Rosy exhibited their knowledge of the algorithm of determining the value of the discriminant, i.e. $b^2 - 4ac$. In spite of this accomplishment , they rarely used the term 'discriminant' and preferred to use the simple term 'value'. Furthermore, they could not convincingly determine the nature of roots. I argue that they could not comprehend the nature of roots because they lacked conceptual understanding of the properties of number. This is further discussed under BI-6, Section 5.4.6 (p. 171).

5.4.5 **Essence of interactions – BI-5 Different approaches to solving a problem)**

5.4.5.1 **Synopsis**

Daisy:

- exhibited knowledge of three different methods of solving a quadratic equation (PF).

Jasmine:

- exhibited knowledge of two different methods of solving a quadratic equation (PF); and
- reversed the connection between the solutions and *x*-intercepts (MM and PF).

Rosy:

- exhibited knowledge of three different methods of solving a quadratic equation (PF).

5.4.5.2 **Discussion**

Kilpatrick et al. (2001) argue that learning to choose among different procedures is an important part of developing mathematical proficiency. In a few, BI-5 , indicated that the learners were also aware of multiple approaches in solving the equations such as using the factorisation method, completing the square method and the quadratic formula. The terminologies used for these methods were inconsistent with the mathematical terms, such as Rosy's 'trial and error' method for factorisation and Daisy's 'square' method for the quadratic formula. They all showed preference in using one particular method, except when they were asked to use a different method.

Krutetskii (1976) characterises the reversibility of thought operations as a distinctive association of the mind in direct and reverse directions. There was only evidence of one instance of exercising the ability to reverse a mathematical process demonstrated in the transcript below: — when Jasmine identified a two-way correspondence between the x-intercepts of a graph and the solutions to an equation. Jasmine was confused when she entered the equation $2x^2 + 3x + 2$, which had no x-intercepts. She sought clarification from the teacher who in turn helped her to identify the salient features of the graph.

T: What you notice here, neah ?
***Jasmine**: No x-intercepts*
T: So, you see the graph is always positive, so for all values of x.
***Jasmine**: So am I ok here no real values of 'x' because there are no solutions?*

In fact, Daisy also observed on her worksheet that there were no solutions for the equation $2x^2 + 3x + 2$. Her conclusion was, however, backed by calculations using the quadratic formula, while Jasmine's conclusion was justified by the nature of its corresponding graph.

In classifying roots of a quadratic equation, Daisy and Rosy could not relate the parabola graph with no x-intercepts to the non-existence of real value solutions of its equation. They associated the roots of the equation with the x-intercepts of the graph and it was a noticeable connection to them. Nonetheless, the inverse relationship of a graph to its equation was not an obvious connection to Daisy and Rosy. Possibly they were fixated on the formula that prevented them from noticing the reverse correspondence from the x-intercepts of a graph to the solutions of its equation. In particular, Krutetskii (1976, p. 85) ascertains that the formation of mathematical concepts is related to "the ability to *switch from a direct to a reverse train of thought*" [author's italics]. It was evidently difficult for these participant learners form reverse bonds, and they perhaps need special intervention.

5.4.6 **Essence of interactions –BI-6 (Applying previously learnt concepts)**

5.4.6.1 **Synopsis**

Daisy:

- identified expressions as graphs (CU);
- identified constants as *y*-intercepts (CU); and
- employed procedures of solving a linear and quadratic equations (PF).

Jasmine:

- identified expressions as graphs (CU);
- identified constants as *y*-intercepts (CU);
- employed procedures of solving a linear and quadratic equations (PF); and
- applied previously learned concepts on the reflection of graphs (CU and MM).

Rosy:

- identified equations as graphs (CU);
- identified constants as *y*-intercepts (CU); and
- employed procedures of solving a linear and quadratic equations (PF).

5.4.6.2 **Discussion**

All the participating learners significantly used previously learned material while interacting with the applets. This suggests that the learners had the knowledge of the concepts of variables and quadratic expressions, and they were able to correctly recognise constants and coefficients of variables in a quadratic equation. As discussed earlier, the participant learners were aware of different procedures of solving a quadratic equation and had a reasonable understanding of graphical representation of a quadratic function and its significance when solving quadratic inequalities. The participant learners applied their previously acquired knowledge and concepts and adapted them to the new set of circumstances in solving the quadratic inequality.

Furthermore, Jasmine applied the properties of transformation of a function in solving the quadratic inequality with a leading negative coefficient of x, as in $-3x^2 + 4x + 4 \geq 0$. She flipped the function upside down i.e. the concave down parabola to a concave up parabola, shown in Figure 5.17. Ostensibly, she was able to visualise a portion of the original graph upside down in an (imaginary) mirror. In simpler terms, she changed the function from $f(x)$ to $-f(x)$. This meant that she had to substitute a 'minus' for all the positive values in the function. and vice versa. Accordingly, she changed the inequality from '*greater than equal to'* to '*less than equal to'*. The *x*-intercepts stayed where they were originally, and only off-axis points were mirrored across the axis. She thus applied her prior knowledge on preserving or not preserving properties in a reflection. Despite using this mathematically accurate approach, her conclusions of the inequality of the graph were incorrect.

As argued earlier, Daisy and Rosy could not convincingly conclude the nature of roots owing to lack of conceptual understanding of number properties. It might not be possible to discuss the nature of roots of an equation without a knowledge of properties of i) rational and irrational numbers and ii) real and non-real numbers. The participants gave mixed responses for these properties of numbers, giving rather superficial or procedural replies to questions. Rosy identified a rational number as a fraction, and also identified a whole number as a rational number but did not provide the mathematical definition of a rational number. Daisy was closer to the definition except that she omitted the conditional part of the denominator. Both participants identified non-real numbers as a negative value of $b^2 - 4ac$. The transcript below of the reflective interview exhibits Daisy's lack of full understanding of the properties of numbers. Daisy appreciated the differences between rational and irrational numbers, but the deeper concepts of real and imaginary numbers was however vague.

> *R: What do you understand by rational number and an irrational number?*
> ***Daisy*** *Rational number can be written in the form of a/ b where a and b are integers. Irrational numbers are surds.*
> *R: Let us be more specific on rational numbers, you said a/b but what happens when b is zero?*
> ***Daisy:*** *When* $\frac{a}{b}$ *and b is zero that is divided by zero its undefined.*
> *R: So, when we say a rational number, we must also include the condition of b. Ok. Can you give an example of an irrational number other than surds?*
> ***Daisy:*** *Irrational number other than surd. I don't think of any.*
> *R: We have learnt in geometry what is the area of a circle?*
> ***Daisy:*** *It's pi r squared* πr^2.
> *R: Pi is an irrational number; it cannot be expressed in a/b form.* $\frac{22}{7}$ *is an approximation of* π. *Ok. Now tell me what is a non-real number?*
> ***Daisy:*** *Non-real number is when I get the value as negative.*
> *R: Which value?*
> ***Daisy:*** $b^2 - 4ac$ *is negative it is non-real.*

The analysis of the data concur with Hiebert and Lefevre (1986) who stipulate that the development of conceptual knowledge is achieved by making connections between different pieces of already acquired knowledge.

5.4.7 Essence of interactions –BI-7 (Reflecting on a solution – considering the reasonableness of a solution)

5.4.7.1 Synopsis

Daisy:

- interpreted or estimated the solution and disagreed with the solution provided by the computer owing to wrong input (MM and PF).

Jasmine: None

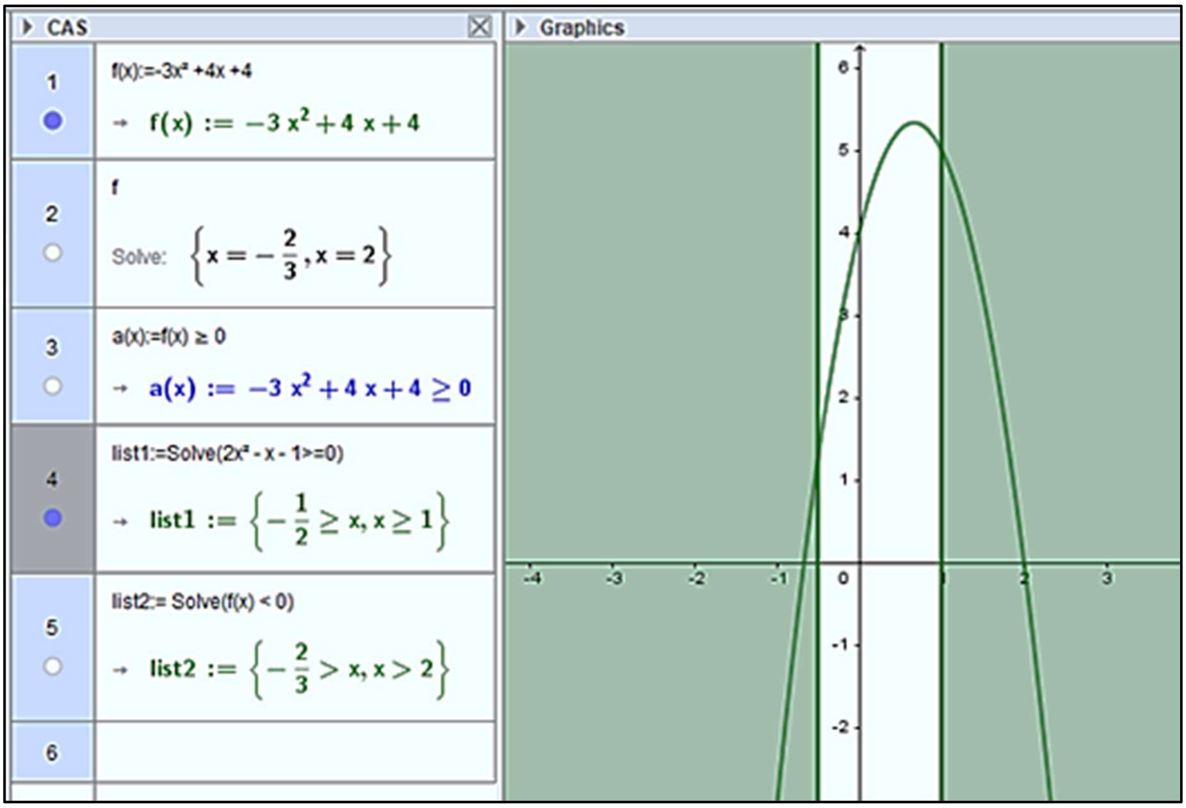
Rosy:

- interpreted how the graph was linked to the equation (MM and CU); and
- estimated the graph of a quadratic equation and hence disagreed with the graph in DGS owing to wrong input (MM and PF).

5.4.7.2 **Discussion**

As in the previous cycle, there was not much evidence of indicator BI-7. Seemingly, the participant learners did not frequently interpret or reflect on their actions. Jasmine had an opportunity to reflect over her solutions when she had incorrectly entered the equation. While determining the solutions of the first equation of the worksheet, her input into the DGS was $2x^2 + 5x + 3$ instead of $2x^2 - 5x + 3$. However, when engaging with the inequalities, her input was accurate and therefore the x-intercepts were distinct from the earlier ones. It did not bother her to look for reasons behind the different pair of x-values.

Meanwhile, Daisy's responses exemplified the literature on learning for fluency wherein Kilpatrick et al. (2001) accentuate that mathematical proficiency is connected to knowledge of ways to predict the result of a mathematical activity. She made a logical deduction based on her initial findings that the roots of a quadratic equation were equivalent to the critical values of the corresponding inequality. She wanted to verify for other quadratic equations, but the DGS did not update the inequality. She typed in the inequality $-3x^2 + 4x + 4 \geq 0$ to validate her conjectures on the correspondence of its x-intercepts with the critical values of the inequality. Daisy, by oversight, looked at the solution that she had provided earlier, shown in Screenshot 5.25 below, wherein the inequality was highlighted for a different graph. Row four was independent of row one where Daisy typed in the new equation and owing to her previous interaction the inequality in row four was highlighted. Unaware of these intricacies of the DGS, Daisy reconsidered her procedure and reflected on the graph. Therefore, she disagreed with the inequality provided by the DGS.



Screenshot 5.25: Daisy reflecting on the inequality

Apparently, Daisy anticipated that the DGS would highlight the region with edges at $x = -\frac{2}{3}$ and $x = 2$, which were the x-intercepts of the graph that she had determined earlier for the function, $-3x^2 + 4x + 4$. When the computer gave an unreasonable result, owing to her incorrect selection of buttons, Daisy's inability to reflect on the solution was very much evident.

When Rosy made an input error in entering the quadratic equation, she knew that the graph generated by the DGS did not correspond with a quadratic function. She reflected on the unreasonable shape of the graph that the computer had produced. She corrected herself again and again until she saw a parabola graph on the screen shown in Screenshot 5.2 to Screenshot 5.4, in Frame 5.2 (on p. 140). Earlier too, Rosy expressed her excitement to see a straight-line graph and its x-intercepts when a linear equation was solved in DGS. It is clear that Rosy made a connection between a function and its corresponding graph.

Furthermore, these observations also indicated that the learners made sense of quadratic equations and inequality, aligning with the meaning- making habits propounded by Carter et al. (2009).

5.5 **EMERGING THEMES**

Two themes were identified that are pertinent to the meaning-making process of my participants: 1) Extended knowledge and 2) Visualising through similes of a parabola graph.

5.5.1 **Extended knowledge**

There were two instances that suffice in proving that the learners generated knowledge that was not intended. Firstly, Rosy identified the turning point of a quadratic equation. She made a very important observation and learned that when there are two intercepts, it follows that the turning point will be between them. She could also sketch the parabola quite effortlessly as the learners had been taught the parabola of $ax^2 + b$ form in the previous grade. In the lesson it was not expected that they learn this aspect of the parabola, but for Rosy the understanding of the graph turning between the x-intercepts was one of the key lessons that she had that day.

R: What did you understand from this lesson? How will you can summarise.?
Rosy*: Ok sir. My understanding is that the x-values are the x-intercepts when you solve. The turning point will be between them.* (Pauses) *... that my understanding sir.*

Secondly, while explaining her strategy on solving inequalities, Jasmine considered the part of the parabola between the x-intercepts as the middle portion of the graph. The transcript below and the figure in Frame 5.17 demonstrate this:

R: What is difference between 'greater than' or 'greater than or equal to'? say for e.g. $2x^2 - 5x + 3 \leq 0$ *and* $2x^2 - 5x + 3 < 0$
Jasmine*: In the first inequality, its less or equal to then its x-intercepts are included. If it's less than, then its x-intercepts are excluded. My understanding is that when it touches the x-axis it's zero and when not it's + or -. When we look at the graph here the 'less than' is in the middle portion of the graph the lower portion is negative, but its x-values are zero therefore it would exclude the intercepts.*

These kinds of observations help learners to better visualise the concept of the turning point as an axis of symmetry of a parabola graph.

Jasmine exhibited another understanding that was beyond the scope of the lesson. As mentioned earlier, she retained the graph of $4x^2 - 4x + 1$ until the end of the lesson. During the reflective interview, she observed that its x-intercept was its turning point. This knowledge of two equal x-intercepts as a turning point is primarily encountered either in problem solving of graphs in Calculus, or in a discussion about general behaviour of a polynomial. I argue that engagement with the different algebraic forms, afforded by software, allow learners to build extended knowledge of inter-locking blocks that can assist them in other related topics.

5.5.2 **Visualising through similes of a parabola graph**

More often than not, learners consider the 'concave up' parabola graph as a '*smile*' and concave down as a '*frown*'. Teachers and textbooks also mention them as '*smiles*' and '*frowns*. Frequently, learners use '*smile*' or '*frown*' terminology assuming them to be mathematical terms. Except for Jasmine, other participants used similes for the shape of graphs, like Rosy using "*facing up*" and "*facing down*". However, Jasmine did not use accurate mathematical terms until prompted to use mathematical vocabulary.

Daisy's processing of inequalities revealed interesting primitive comparisons, oblivious of mathematics shown in the transcript in Frame 5.11 (p.152). She considered the inequality $f(x) < 0$ (the shaded part between the x-intercepts and the graph) in the case of a 'concave up' parabola the middle portion as a "*dip*" (Figure 5.12, Frame 5.11 on p.152) and in the case of a 'concave down' parabola, the middle portion was considered as a "*hump*"(perhaps she regarded the x-axis as a road) The shaded part of the inequality $f(x) > 0$, was considered as "*hanging up*" (Figure 5.13 in Frame 5.11, p.152). Jasmine was fascinated by a parabola graph with a turning point on the x-axis, she had a sound knowledge of parabola graphs such as intercepts and turning points. At the same time, she joked, 'resting on the x-axis' (Screenshot 5.21 and transcript in Frame 5.15, (p.159).

Such similes apparently have little mathematical basis, but from the learners' perspectives such interpretations help them to relate to mathematical situations and make learning worthwhile. The translation of mathematical graphs or jargon to things around us, and the use of non-scientific language allowed the participants to think about the quadratic inequalities visually. Rosy and Jasmine may however require further intervention to switch from their visual ideas to abstract ideas related to inequalities. Furthermore, insightful teachers can draw upon these abbreviated, cursory but quality thoughts from learners. I concur with Blais (1995, para. 8) who asserts that "[t]his heightened awareness is critical to helping students use their minds effectively in mathematical situations."

5.6 **ANSWERING THE RESEARCH QUESTIONS FOR THE SECOND CYCLE**

- How are *GeoGebra* applets used as a learning tool to make mathematical meaning and develop understanding by selected Grade 11 learners?

The evidence from the above lessons illustrates that using DGS, facilitated by tools of the software does provide students with access to the world of equations and functions, although in a limited way. I first discussed where *GeoGebra* proved to be a useful tool for the learners. The *GeoGebra* applets paved a way for the participant learners to understand the equivalence between the roots of an equation and the x-intercepts of its graph. The graphics window of DGS played an important role in developing understanding, and the CAS window essentially served as a computational tool for symbolic representations. The learners' activities during the lesson and in the reflective interviews indicated that they inferred from the mathematical properties without relying on routine calculations in the case of equations and inequalities. The applets did not, however, provide a step by step procedure to either solve the equation or sketch the graph. The learners were however able to perform the routine calculations for solving the x-values of the equation.

Rosy rationalised that there were two x-intercepts for equal roots as the two x-intercepts 'coincided'. She might have been influenced by the dynamic movement of the x-intercepts coming closer to one another when the slider c was moved as in Applet 2.2, while the other participants maintained there would be only one x-intercept for equal roots. Jasmine mentioned that the equation having only one solution and thus the graph having only one x-intercept made sense to her. Screenshot 5.21 in Frame 5.15 (p.159) shows how the graph displayed only one x-intercept and the CAS also computed only one solution.

Furthermore, I have discussed how Rosy and Jasmine extended their knowledge on parabola graphs by linking their x-intercepts and their turning points discussed in Emerging Themes in 5.5.1 under the heading Extended knowledge. I argue that *GeoGebra* is a powerful tool for teaching and learning that can strengthen learner conceptual understanding. These observations align with the research reports by Stols and Kriek (2011) who weigh the benefits of the software to create opportunities for creative thinking.

The participants did not show any inconsistency in relating the critical values of the inequalities with the x-intercepts of the equation. All the participants had figures or graphs in their solutions for inequalities, except for Aster who did not rely on any images, but rather solved the problems using logical methods. Rosy's solutions shed light on the degree of influence of DGS on her thinking. She highlighted the entire region as displayed in the DGS, without arriving at the final solution. Daisy too was influenced by the DGS as she also shaded the region between the graph and x-axis. Nevertheless, her algebraic notations of inequality were distinct from the solutions provided by the DGS. Jasmine though, drew the parabola graph and didn't shade any portion of the graph. Instead she marked the portions below and above the *x*-axis as negative and positive respectively. In her sketches, unlike Rosy and Daisy, she drew the *x*-axis and *y*-axis. Thus the learners' thinking was shaped by the artefact (Hoyles & Noss, 2003). Aster is an exception to the above statement. He did not draw any graphs during the reflective interview and only solved equations using numerical methods, applying the multiplication properties of positive and negative real numbers.

GeoGebra provided an opportunity to satisfy their conjectures as they attempted equations and inequations. Rosy and Daisy were determined to unlock the solutions of the inequalities, even as the class advanced to the next section of the lesson. Although not required by the teacher Jasmine attempted all the equations on the worksheet to gain a better understanding on the concept of inequalities. The graphical representations produced by the CAS inspired participants to reflect in ways that would have been difficult to achieve with traditional techniques alone. Jasmine and Rosy were more focused on the graphs, although Jasmine did not even use the CAS window and Daisy made an attempt to make sense of the algebraic notation of the inequality with the graph. As Kieran and Drijvers (2006) point out, the benefit of multiple representations enabled in CAS, is in making algebraic objects such as expressions and equations more meaningful to students.

Nevertheless, DGS also negatively impacted the learners. DGS provided the algebraic solution without using appropriate conjunctions in the mathematical context, such as 'and' and 'or'. Rosy was inconsistent in using these conjunctions in her solutions. In one equation she used the word 'and' to denote two solutions, in another equation she used 'or', and in yet another she did not use any conjunction. Jasmine, too, consistently in ignored the use of appropriate conjunctions in her solutions. Thus, DGS does not address the difficulties that students face in the usage of conjunctions.

Furthermore, DGS failed to link algebraic notation of inequality to its equivalent number line notation. The understanding of equality and equations may not suffice in the complete understanding required for solving inequations. The equivalence of the critical values of an inequality and roots of an equation is one part and interpreting is another part. I find that inferring from inequalities involves a deeper understanding of a real number line or properties of real number that DGS does not directly address. The blame could perhaps lie in the lacunae of prior knowledge of linear inequalities like reading from a number line. Apparently, the shaded region of the graph was Rosy's final answer, not really conforming to mathematical practice. Jasmine's answers on the graph were not justifiable and exposed her conflict between rote rules and understanding inequalities. This is typical of the context emphasised by Artigue (2002), that tends to reduce the 'epistemic value' of CAS.

For the nature of roots of a quadratic equation, the participant learners displayed their tendency to use the formula $b^2 - 4ac$ in determining if the roots were real or unreal. Only when prompted to interpret from the graph whether the roots were real or unreal, they attempted to look at the graph. The participants linked the roots of the equation, but the reverse link of connecting the x-intercepts with the roots of the quadratic equation was however not made. Although Daisy said the roots were real since the graph intersected with the x-axis, she could not interpret if the roots were rational until she had calculated the value of the discriminant This is shown in the transcript in Frame 5.12 (p.154) where Daisy calculated the discriminant during the reflective interview. The learners were tied to the value of the discriminant and could not relate the actual value of the x-intercept as rational or irrational.

The above vignettes on dealing with the nature of roots applets reveal the difficulties faced by the learners in using tools or commands of DGS appropriately. The complexities of creating the sliders might have bogged them down in focusing on the graph and its relation to its zeros. Another reason could be that Rosy and Daisy were still working at their own pace, discussing and inferring from the inequalities, while the teacher proceeded to discuss the nature of roots. *GeoGebra* applets used in the class did not provide symbolic procedures to solve an equation and neither did they provide mathematical definitions of rational and irrational numbers. Tall (1993) agrees that the computer environment favours a more generic level of thinking than formal definitions. It seemed that the challenges had ended in unmitigated failure of CAS as a learning tool. But Berger (2011) observes that even the more technologically advanced learners have problems in converting CAS into a tool which effectively mediates learning. The learners focussed on the numerical value of the discriminant in the CAS window and did not correspond and infer from the graph of the nature of roots of a quadratic equation.

I suggest that when teaching and learning at the practical level, prerequisite knowledge is addressed through activities that engage learners with the basic concepts already acquired in earlier grades. The participant learners also lacked the understanding of the multiplicative property of rational and irrational numbers. (The multiplicative property of rational and irrational numbers is: $a.b = c$, if a and b are rational then c is also rational number. If either

a or b is rational and the other irrational, then their product c is irrational. If a and b are both irrationals, then their product c may be rational or irrational). Thus, in factors of a quadratic expression of rational coefficients, say $(x - x_1)$ and $(x - x_2)$, if x_1 is rational, then by multiplicative property, x_2 must be a rational number. Furthermore, it requires one to explore the feasibility of constructing an applet that would include the properties of real numbers, such as marking off rational and irrational numbers on a real number line. As Skemp (1989) observes, new concepts need to be related to the learners' existing schema for meaningful learning to take place.

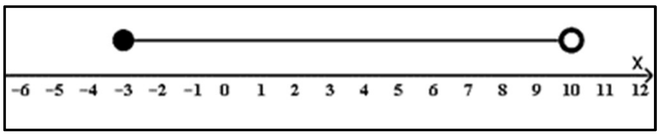
- What visualisation role can GeoGebra play in the learning of Grade 11 mathematics?

As mentioned earlier, the multiple representations enabled in DGS make algebraic expressions more meaningful to learners. As soon the graph was sketched in the DGS, Rosy manifested an insight visualising essential relations between the equation and the graph. The straight-line graph might have provoked her previously acquired knowledge on solving x-intercepts, as Krutetskii (1976) considers insight as a "result of previous protracted thinking" (p. 305). Jasmine initially wondered why *GeoGebra* solved the equations without procedures, but the quadratic equation, its solutions and its graph all in one screen induced Jasmine to grasp the fundamentals of equations and the graph without much intervention from teachers or peers. Daisy, on the other hand, prompted by her friend to see the obvious link, was not convinced until she calculated on her own, the roots of the equation. Thereafter, the solutions and the graphs afforded by DGS were sufficient for Daisy to develop an understanding of the connections between them. I argue that the basic mathematical ideas are borne from concrete and visualisable situations, as Guzmán (2002, p. 3) observes, "mathematical concepts, ideas, methods, have a great richness of visual relationships that are intuitively representable in a variety of ways". Furthermore, Presmeg (2006) notes that effective use of visualisation in mathematics classrooms is in 'making connections'.

We have already discussed how graphs in *GeoGebra* influenced the learners in their procedures while answering the inequalities. Let us look from the conceptual perspective of visualisation. The three participants visualised the critical values of the inequality and hence solved the roots of the equation, but the commonality ends there. Rosy moved horizontally left or right from the critical values. This is also evident from her conversations both in the classroom and reflective sessions. Daisy moved horizontally but sequentially starting from the left of the number line i.e. negative infinity until the critical value. Her conversation with her friend in the transcript in Frame 5.10 (p. 151), provides evidence of the evolution of her thinking, after observing and analysing the graph, from ‘*x less than* -$\frac{1}{2}$’ *to ‘from negative infinity to* -$\frac{1}{2}$’. Furthermore, Daisy compared, though loosely, parabola graphs of inequality to road signs, such as in the case of a ‘concave up’ parabola inequality $f(x) < 0$. The shaded part between the *x*-intercepts and the graph was compared to a “*dip*” in Figure 5.12 of Frame 5.11 (p.152). This is also discussed under Emerging Themes in 5.5.2 Visualising through similes of a parabola graph. The efficacy of “concrete imagery” (Presmeg, 1986b) in learning mathematics was evident in the thinking of Daisy. Presmeg (1986b) emphasises that the imagery may be used to serve the function of generalisation in mathematics.

Meanwhile, Jasmine moved vertically and horizontally and then marked positive and negative values of the graphs Figure 5.16 and Figure 5.17 in Frame 5.18 (p. 163). The portion of the graph below the *x*-axis and the critical value was marked negative and the part above the *x*-axis and the critical values as positive. As Guzmán (2002) underscores, a visual-pictorial approach to solving problems is a powerful tool when engaging with mathematical ideas and reasoning. Notwithstanding this, her predominant faulty rules led her to wrong conclusions. A further intervention may be required to improve her understanding.

Aster solved equations using numerical methods, applying the multiplication properties of positive and negative real numbers. He admitted that he understood the quadratic expression as a parabola graph but did not find it necessary for solving the given inequalities. Furthermore, the participant learners had challenges in considering solutions of inequalities on a number line. They lacked the ability to translate the algebraic inequality onto a basic number line. In order to minimise the learners’ difficulties in solving quadratic inequality, it would be beneficial if they were encouraged to establish different notations of expressing an inequality. As a prerequisite resource, an applet may be designed to introduce different notations of inequalities such as algebraic notations, number lines and interval notations. For instance, for the inequality between -3 (inclusive) and 10 (exclusive), we can denote it in three different mathematical ways:

1) number line representation: 

2) algebraic notation: $-3 \leq x < 10$ and

3) interval notation: $x \in [-3,10)$.

In DGS, the solution of the inequality was highlighted vertically, by default in the graphics window. It could have been better if the solution was highlighted on the x-axis. Rivera (2011) prescribes that the use of number line diagrams is very important for understanding and visualising the intervals that form solutions to various inequalities.

The learners missed the opportunity to visualise the connection between the x-intercepts and the value of the discriminant because they disconnected the numerical value of the discriminant and the zeros of the parabola. The numerical value was predominant and masked the connection between the graph and the roots, which hampered the visualisation of 1) real and unreal roots and 2) equal and unequal real roots. The participant learners connected the equal roots to having only one x-intercept of the parabola, but the values of the x-intercepts of the graph were not translated to the values of the roots. More research data is required to analyse and compare whether the learners used a pre-designed applet for the nature of roots. Clearly, the lesson design and planning of using DGS in the context of equations and inequalities requires to be reconsidered.

5.7 CONCLUSION OF THE CHAPTER

This chapter provided an in-depth analysis of the participating learners' meaning-making out of their interactions with the DGS on the topic of quadratic equations and inequalities. The additional dynamic experience with the DGS allowed learners to develop their understanding of what it means to solve a quadratic equation and inequality. Notwithstanding this, the analysis revealed that some of the participant learners had difficulties in organising their solutions in a mathematically relevant manner.

In discussing the nature of roots of quadratic equations, the learners displayed a disciplined sequence of finding $b^2 - 4ac$, although there was not much evidence of deep understanding of the discriminant and what the value of $b^2 - 4ac$ meant for the discriminant. The analysis showed that learners faced hurdles in meaning-making and developing proficiency in classifying the roots of a quadratic equation. These hurdles arose out of a lack of understanding of prerequisite knowledge of the properties of real numbers.

Nevertheless, a detailed analysis of the learners' interactions revealed different visualisation processes that learners developed, thus enhancing their knowledge. The benefit of multiple representations afforded by DGS was evident, making algebraic objects such as expressions and equations more meaningful to learners.

In the following chapter, I present an in-depth narrative analysis on the next topic i.e. equations of parallel and perpendicular lines.

CHAPTER 6

ANALYSIS OF LEARNERS' ENGAGEMENT WITH *GEOGEBRA* – EQUATIONS OF PARALLEL AND PERPENDICULAR LINES

6.1 INTRODUCTION

In the previous chapters, I narrated and analysed the participant learners' interactions with the DGS during the lessons on (i) circle geometry and (ii) equations and inequalities. The aim of this chapter is to analyse and answer the research questions on the topic of equations of parallel and perpendicular lines. This is the last topic that I analysed. I recommend that the reader browses Section 4.1.1 of my research methodology on p. 75.

This chapter follows similar structure as dealt with in Chapter 4, (p. 74).

6.2 ANALYTICAL GEOMETRY – EQUATIONS OF PARALLEL AND PERPENDICULAR LINES

The next topic that the participating teachers taught (according to the pacesetter established by the DoE) was analytical geometry. This included how to determine the equations of parallel and perpendicular lines, and the angle of inclination of a straight line. During earlier phases of schooling, the learners had sketched straight lines of the form $y = mx + c$ and $ax + by = c$. The terms 'slope' and 'intercepts' were thus familiar to them. Throughout this thesis, I use slope and gradient interchangeably. Experiences of the GLIP teachers showed that learners find it difficult to conceptually understand the concept of the slope of a line as a ratio of vertical and horizontal changes. They prefer to simply use the formula $m = \frac{y_2 - y_1}{x_2 - x_1}$ where (x_1, y_1) and (x_2, y_2) are points on a Cartesian plane, and substitute these into the formula to determine the gradient between two points. For this topic, the teachers intended to visualise the concept of a gradient of a straight line and interpret it from a given line graph. The learners were asked to investigate the gradients of parallel and perpendicular lines of a given line using a predesigned applet along with a paper worksheet, WS-2, shown in Annexure VII. Finally, the relationship between the gradient and the tangent of the angle of inclination of a line was also explored. The angle of inclination is defined as the angle that a line makes with the positive *x*-axis. Prior to the lessons, the teachers managed to engage learners with *GeoGebra* for an hour to plot and identify points on a Cartesian plane.

Together, we developed six applets, and one applet (Applet 3.1) was adapted from https://www.geogebra.org./Pkb2Z9z5. Applet 3.1 is shown in Figure 6.1 and Figure 6.2, and was adapted to suit our needs. It would engage learners in dragging a line using two points, to match the given slope. We purposefully hid the axes and the coordinates so that the slope would be considered as a constant ratio irrespective of the pair of points chosen. We expected to reinforce the concept ‘rise over run’ (simile for vertical change over horizontal change). Applet 3.2 which is displayed in Figure 6.3 and Figure 6.4, allowed learners to drag a line using two points, labelled A and B, for a given equation of a straight line. With Applet 3.3, learners were required to find and interpret the slope of a given line that was sketched. Though the equation of the line was not provided, the intercepts were clearly indicated on the axes: Figure 6.6.

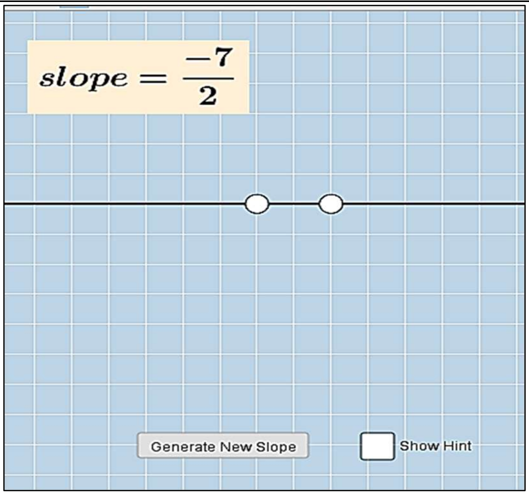


Figure 6.1: Applet 3.1

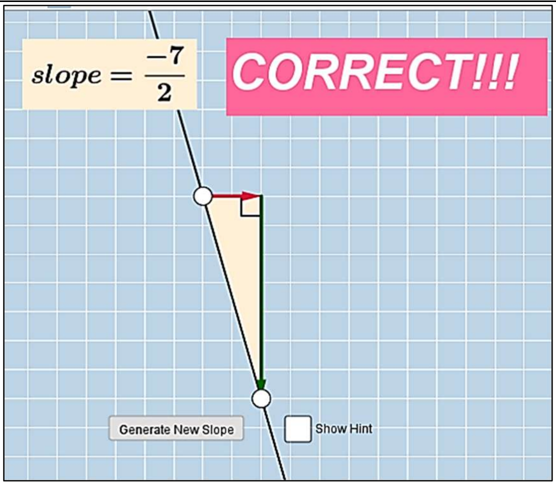


Figure 6.2: Sketching the line in Applet 3.1

Applet 3.1: Sketching the line for a given slope

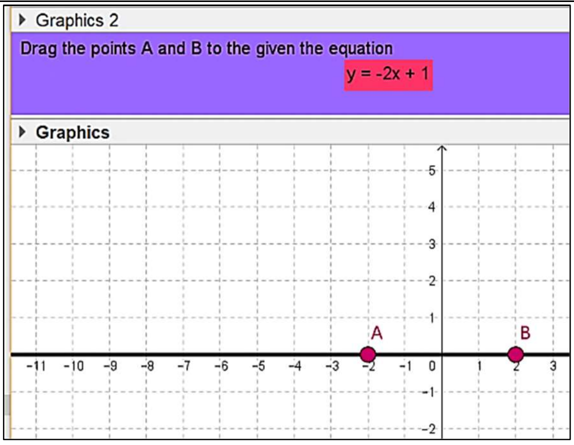


Figure 6.3: Applet 3.2

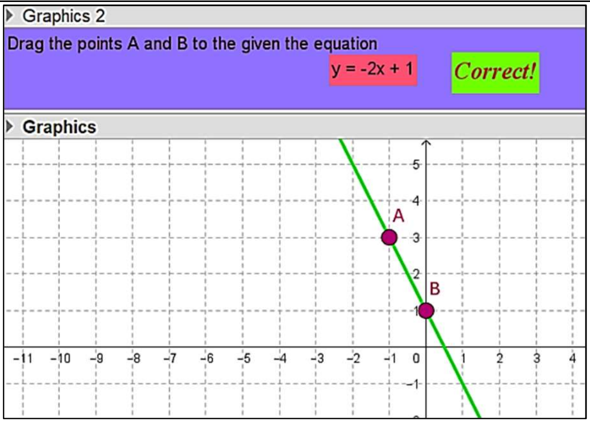


Figure 6.4: Sketching the line in Applet 3.2

Applet 3.2: Sketching the line for a given equation

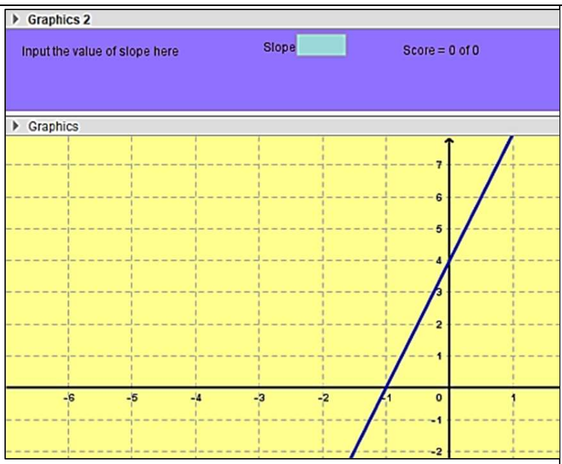


Figure 6.5: Applet 3.3

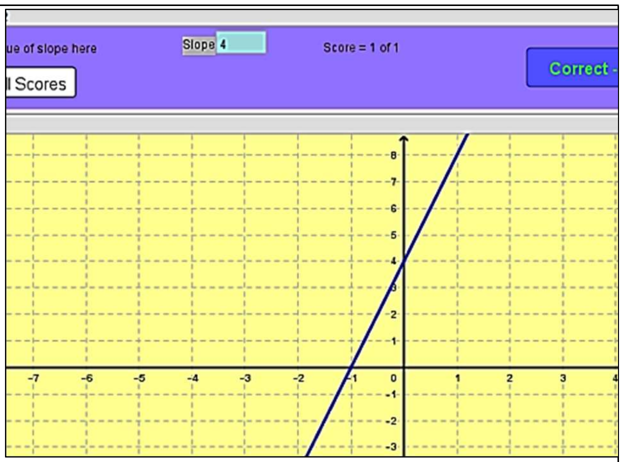


Figure 6.6: Interpreting slope in Applet 3.3

Applet 3.3: Interpreting the slope from the line (equation not provided)

Applet 3.4 in

Figure 6.7 was intended to investigate the relationships between the slope of a given line with the slope of its parallel and perpendicular line. The learners would work in conjunction with a worksheet (WS-2), in which they were expected to fill in the answers while interpreting from the applet. Applet 3.5 in Figure 6.8 provided procedures for finding the equations of parallel or perpendicular lines. The applet also displayed parallel or perpendicular lines after solving the equation. Applet 3.6 (Figure 6.9) was used to sketch parallel and perpendicular lines for a given line. The learners were expected to determine the slope of the given line and then sketch parallel or perpendicular lines accordingly. Lastly, Applet 3.7 (Figure 6.9, Figure 6.10) was about visualising the angle of inclination of a straight line with the slope of a line and establishing connections with trigonometric ratio. It could also provide an explanation for gradients for horizontal and vertical straight lines.

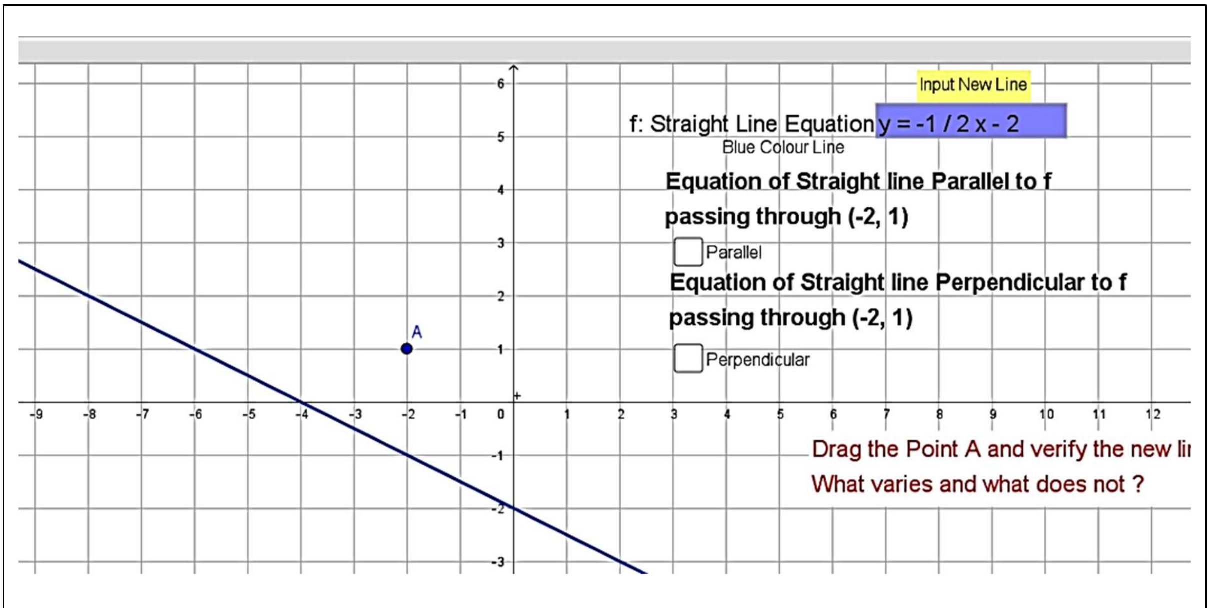


Figure 6.7: Applet 3.4 — Investigating slopes of parallel line and perpendicular line.

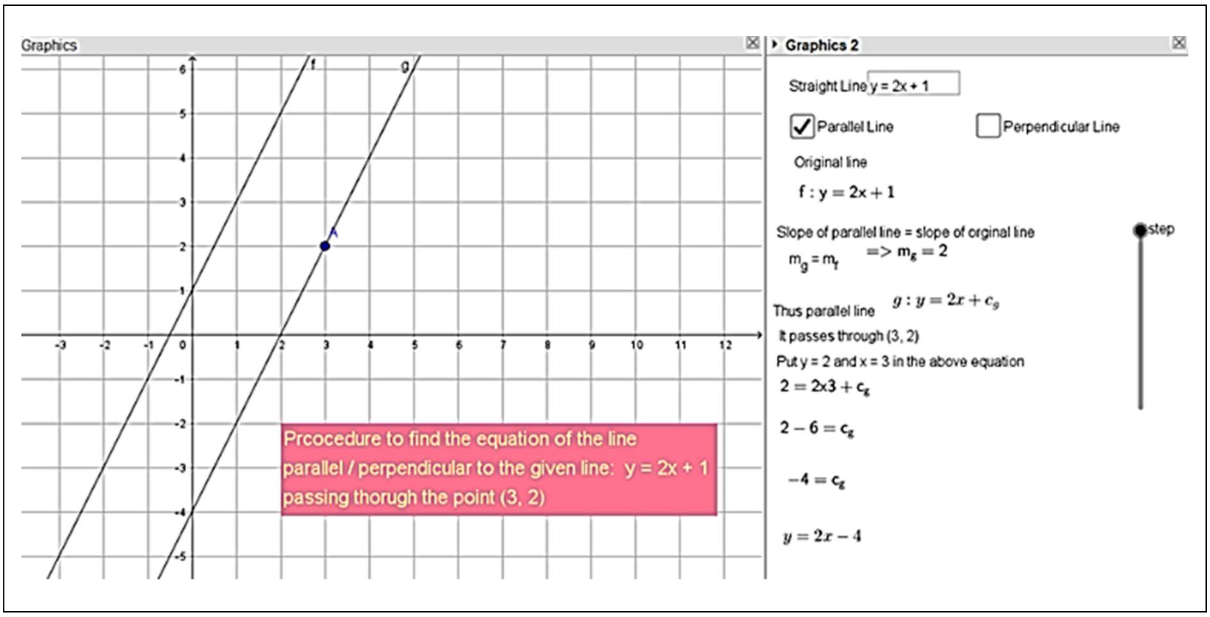


Figure 6.8: Applet 3.5 — Procedures for determining the equations of parallel and perpendicular lines.

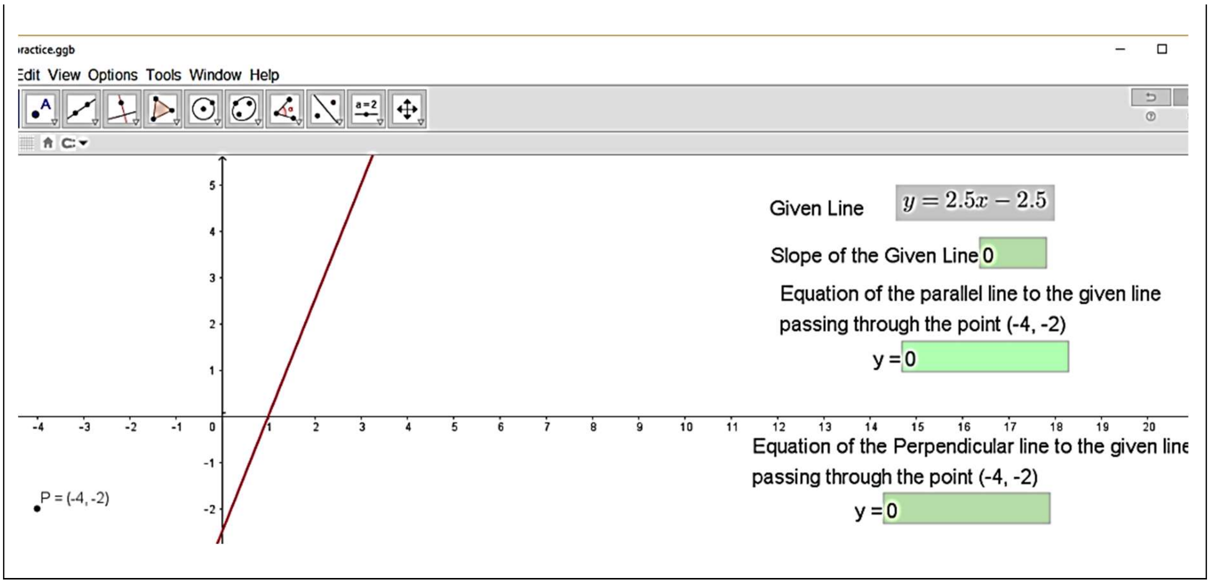


Figure 6.9: Applet 3.6 — Identifying the slope from the equation and sketching the parallel and perpendicular lines

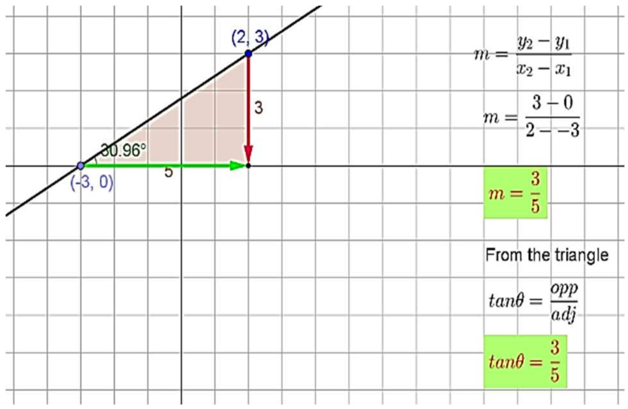


Figure 6.10: Applet 3.7

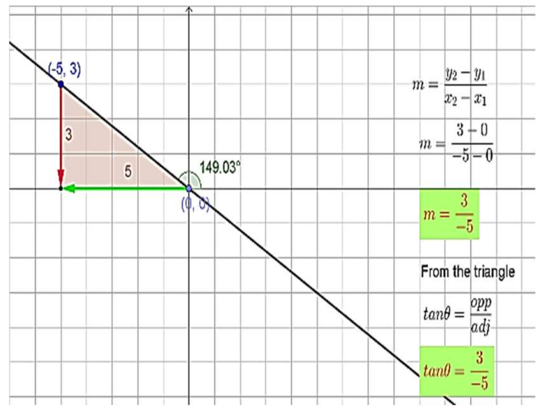


Figure 6.11: Exploring slope-tangent relationship

The Applet 3.6 intended for the learners to put into practice their understanding of slopes and equations of parallel and perpendicular lines did not run properly on the desktop computers of both the schools. It was first identified in Paul's lesson, and we could only rectify the error after both lessons were recorded. In fact, it was not an error but a compatibility issue, as the teachers' version of *GeoGebra* was updated while the schools' version was not.

In this chapter, I analyse four learners, two from Paul's class and the other two from George's class. Prior to the observed lessons, both the teachers conducted training for the learners for an hour on coordinates of the graph and straight lines using *GeoGebra*. The training resource can be accessed in Annexure VI.

As the scope of this thesis is limited, and for the sake of brevity, I only present the horizontal analysis after providing a brief overview of each of the lessons.

6.2.1.1 **Paul's lesson - an overview**

Paul planned two lessons around these applets. The first three applets were used during the first day and the last three on the second day. The second lesson was held three days after the first lesson. The learners by then, knew which computers were working, although some of them still had to share them. On both days, there were 46 learners who attended the lessons (two learners were absent). There were no delays and once the learners found their seats and desktops the lesson started promptly. It lasted for 30 minutes. Most of the computer systems were up and running prior to the learners entering the laboratory. The learners simply had to log in to the computers and access the required files. Paul started the lesson with the Applet 3.1. He encouraged learners to work with the first three applets — Applets 3.1 to Applet 3.3, at their own pace, and he l provided instructions on what was expected of them. The learners were enthusiastic about working with the applets as Paul was moving around the computer laboratory and helping those who needed assistance. At the beginning of the second lesson, Paul handed out the worksheet for investigating parallel and perpendicular lines. With the fourth applet of the topic, he encouraged learners to drag point *A* and asked them to write their observations about the gradients of parallel and perpendicular lines. Learners seemed to be more inclined to play around with the points rather than write down their observations, so the teacher reminded them to complete the worksheets. Most of the learners identified the gradients of parallel lines as identical. The conclusion on the relationship of gradients of perpendicular lines was done by the teacher. Paul proceeded to the next applet, Applet 3.5, demonstrating, step by step, the procedure in determining the equation of parallel and perpendicular lines of a given line. The learners found this applet interesting as it displayed procedures to determine the equation of a straight line. As mentioned, Applet 3.6 could not be used as the learners' desktops were not compatible with the new version of *GeoGebra* in which this applet was designed. Paul linked the tangent of the angle of inclination with the gradient of a line in Applet 3.7. The learners were evidently excited to see the changing angles as they dragged the points around. They identified and rationalised slopes for vertical lines as undefined for a 90° angle of inclination and horizontal lines as zero for a 0° angle of inclination (6LPK, 2MC and 1EC).

6.2.1.2 **George's lesson – an overview**

George's class comprised of 35 learners. His lesson lasted for almost two hours. Almost all the desktop computers were up and functioning before the class started, and the only delay was the time it took for the learners to settle down with the computers as the computer laboratory that was used for this lesson was more compact when compared to other computer laboratories. There was an issue with the lighting in the laboratory, as the learners complained that the images on the projector screen were vague and blurred, and consequently George had to frequently switch the lights on and off or ask a learner to do so.

He started the class by specifically mentioning that he was dealing with previously learned concepts like the gradient of a line as in Applet 3.1. However, learners appeared confused when he used terms like 'rise over run' and 'slope'. They only associated this new terminology with gradient when he used the term 'gradient' and supported this by providing the formula for the gradient i.e. 'change in y over change in x'. Some recognised the formula as $\frac{y1-y2}{x1-x2}$. He challenged the learners with questions like, "*From your observation when a slope has a negative sign and when it is positive what is difference do you notice between the two lines?*", "*What other English or mathematical term can I use instead of 90 degrees?*" Furthermore, he encouraged learners to share their observations and he often summarised the concepts in his discussions.

For each applet, he informed the learners about what was expected of them. With Applet 3.2, learners identified the gradient and the y-intercepts when prompted by the teacher. Thereafter, the teacher demonstrated by sketching the line using the *slope-intercept method* onto the projector screen. He moved around the lab ensuring that all the learners were engaged and helped them. He allotted very little time to Applet 3.3 (determining the gradient of a line from its graph). With Applet 3.4, he motivated learners to fill in the worksheets. He repeatedly directed learners to share their observations and checked what they had written in their worksheets. Only one learner, Iris, came forward and shared her viewpoints on parallel lines and perpendicular lines. George scaffolded learners to generalise on the gradients of perpendicular lines. Only a few minutes were left at the end of the scheduled lesson as he concluded with Applet 3.4. With a few minutes available, some of the learners were keen to engage with the Applet 3.5 whilst others remained idle.

6.3 DISCUSSIONS AND INSIGHTS – HORIZONTAL ANALYSIS ACROSS PARTICIPANTS

6.3.1 Essence of interactions – BI-1 (Defining and identifying relevant mathematical concepts, relations and notations)

6.3.1.1 Synopsis

Aster:

- identified the gradient and y-intercept from the equation of a line (CU) including horizontal and vertical lines;
- employed the ‘rise over run’ concept in determining the gradient of a line (MM and CU);
- concluded that gradients of parallel lines are equal with reasons (MM and CU);
- considered the analogy of positive and negative straight lines (CU); and
- identified the angles of inclination of horizontal and vertical lines (CU).

Cypress:

- employed the ‘rise over run’ concept in determining the gradient of a line (MM and CU);
- concluded that gradients of parallel lines are equal with reasons (MM and CU);
- considered the analogy of straight lines (CU).

Jasmine:

- identified the gradient and y-intercept from the equation of a line (CU) including horizontal and vertical lines;
- concluded that gradients of parallel lines are equal with reasons (MM and CU); and
- identified the angles of inclination of horizontal and vertical lines (CU).

Iris:

- identified the gradient and y-intercept from the equation of a line (CU) including horizontal lines;
- concluded that gradients of parallel lines are equal (CU), but could not provide reasons; and
- considered the analogy of straight lines (MM and CU) as rays of light.

6.3.1.2 **Discussion**

In this section I consolidate the different relevant mathematical concepts demonstrated by the participants under the following headings:

- Identifying the gradient and the y-intercept
- Formulating variant and invariant properties of parallel and perpendicular lines
- Identifying gradients of vertical and horizontal lines
- Recognising the angle of inclination of vertical and horizontal lines

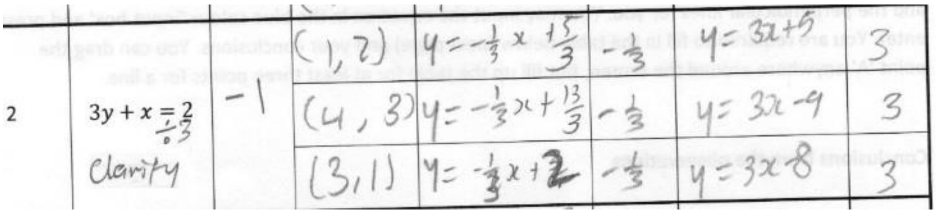
6.3.1.2.1*Identifiying the gradient and the y-intercept*

Pierce, Stacey, and Bardini (2010) suggest certain key features with respect to linear functions — $y = mx + c$ to be recognised by the learners, namely

- m as a rate of change or as the gradient of a straight-line graph and
- c as a constant or as the y-intercept of a straight-line graph.

Participant learners showed conceptual understanding by recognising key components as mentioned above. Only Cypress found the conventions of notations difficult. This might be the reason for him inadequately identifying the gradient of a straight line in worksheet WS-2. Cypress' response suggested that he was unaware of the concept of other algebraic forms of a straight line i.e. $ax + by = c$. For him "the coefficient of x in the equation", always represented the gradient of the line, irrespective of the form of the equation. Iris initially had a vague idea of rearranging the equation when it was not given in slope-intercept form, but only after she was guided by her teacher to make y the subject of the formula, she thereafter identified the gradient accurately as seen in Figure 6.12 of Frame 6.1. Aster and Jasmine did not have any confusion in identifying the key features of straight lines given in the worksheet.

Classroom Observation



2	$3y + x = 2$ $\div 3$	-1	$(1, 2)$	$y = -\frac{1}{3}x + \frac{5}{3}$	$-\frac{1}{3}$	$y = 3x + 5$	3
	Clarity		$(4, 3)$	$y = -\frac{1}{3}x + \frac{13}{3}$	$-\frac{1}{3}$	$y = 3x - 9$	3
			$(3, 1)$	$y = -\frac{1}{3}x + 2$	$-\frac{1}{3}$	$y = 3x - 8$	3

Figure 6.12: Iris's response on the worksheet for the second equation

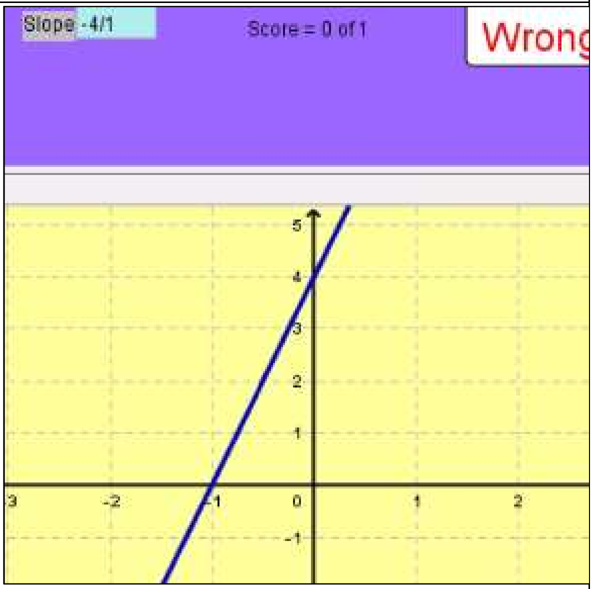
Teacher: you have made 3y as the subject and not y
***Iris**: So we have to rearrange the equation.*
Teacher: Make y as the subject of the formula, the standard form of a straight line function.
***Iris** :* (calculates) *y = -x + 2 whole over 3'.* *4AP*
…
Teacher: So now this can be rewritten as y equals negative one over three x plus two over three. What's the gradient?
***Iris** : It's negative one over three.* *1EC*

Frame 6.1: Iris's understanding of the gradient from an equation

Furthermore, in Applet 3.3, all the participants except Cypress identified the gradient of a given straight line graph. Jasmine and Iris used the formula method in determining the gradient. Cypress struggled to determine the slope of the line using the ratio $\frac{y_2-y_1}{x_2-x_1}$. In fact, he attempted several times to get a correct answer. Screenshot 6.2 of Frame 6.2. show some of his attempts. Cypress was supported by Iris with the formula $m = \frac{y_2-y_1}{x_2-x_1}$ but he could not substitute the values accurately. He admitted to Iris that he initially used the formula $\frac{x_2-x_1}{y_2-y_1}$. In the Screenshot 6.2 in Frame 6.2, Cypress not only employed the wrong formula, but his substitution was equally wrong. He did not attempt other points on the line. It appeared that he was certain that the intercepts would lead to the solution. With the correct formula, his answer $-\frac{4}{1}$ was again wrong owing to the incorrect substitution of values into the formula — see Screenshot 6.2 in Frame 6.2.

Classsroom Observation

Screenshot 6.1: Cypress' incorrect substitution of incorrect values into the formula



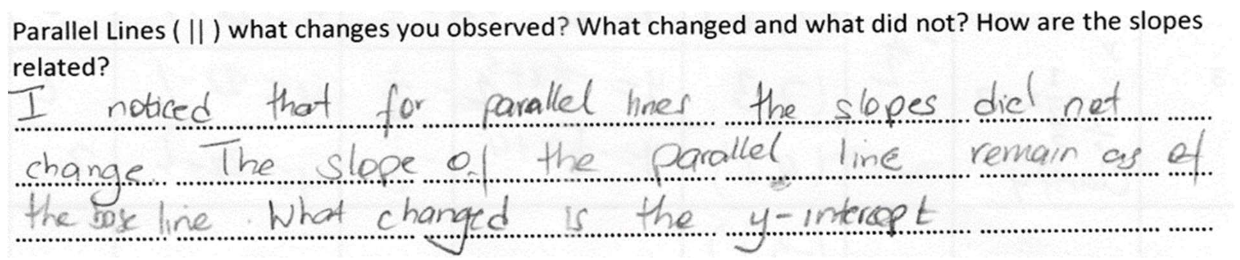
Screenshot 6.2: Cypress again has a wrong answer due to substituting the incorrect values and using the incorrect formula

***Cypress**:* [to Iris] *What are you doing here? Which formula?*
Iris: The formula here is y2-y1 over x2-x1
***Cypress** : Oh I think I used x2-x1 over y2-y1*

Frame 6.2: Cypress' brief engagement with the Applet 3.3

6.3.1.2.2 *Formulating variant and invariant properties of parallel and perpendicular lines*

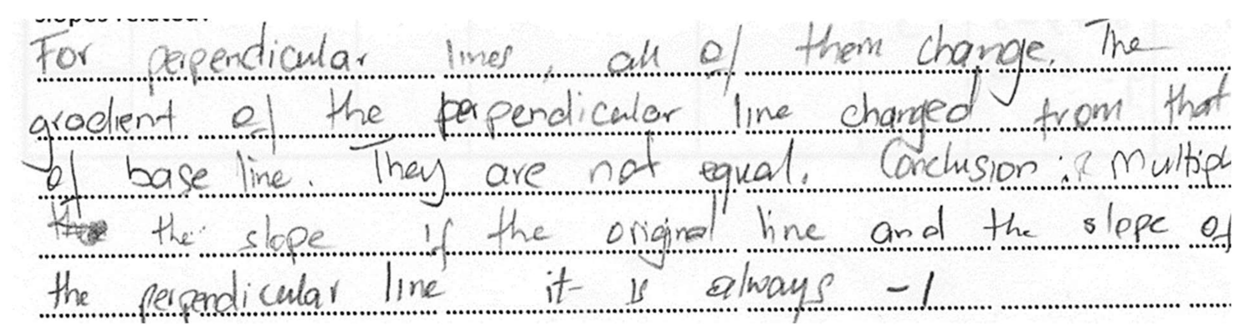
Recognising the parameters of a straight line was a vital step in comprehending the variant and invariant properties of parallel and perpendicular lines. All the participants accurately formulated the properties of parallel lines. They inferred from their interactions with the Applet 3.4 that the gradients of parallel lines were equal. However, their formulations about the properties of perpendicular lines were not analogous to the intended conceptual understanding of negative reciprocals of the gradients. Iris's conclusion in Figure 6.14, Frame 6.3, may be considered as the numerical perspective of relationships between the gradients of perpendicular lines.



Parallel Lines (||) what changes you observed? What changed and what did not? How are the slopes related?

I noticed that for parallel lines the slopes did not change. The slope of the parallel line remain as of the base line. What changed is the y-intercept

Figure 6.13: Iris's conclusion on parallel lines



For perpendicular lines, all of them change. The gradient of the perpendicular line changed from that of base line. They are not equal. Conclusion: Multipl the slope if the original line and the slope of the perpendicular line it is always -1

Figure 6.14: Iris's conclusion on perpendicular lines

Frame 6.3: Iris's response on the worksheet

Carter et al. (2009) observes that reasoning and sense making habits provide learners with a context to mathematical ideas and their application. Some participants demonstrated their understanding as they explained their reasons behind the variant and invariant properties of parallel lines. Among them, Iris and Cypress could not provide a rationale for the equality of gradients of parallel lines. Jasmine justified that the numerical ratio remained constant for parallel lines, hence the behaviour of their gradients, as shown in following transcript below:

Reflective Interview

Jasmine*: Why? Their gradients are equal because the change in y-values and change in x-values will be same.*
R: Can you explain again?
Jasmine*: If the change in x-values is 4 then its parallel line will also have change in x-values as 4 and change in y-values is 2 and parallel line will also have change in values as 2. Therefore, when we divide y2-y1 over x2-x1, the ratio will be equal 2/4.*

I will discuss the reasons given by Aster for the equality of gradients of parallel lines in the following section, 6.4.1.2, p. 230. However, none of the learners could provide reasons for the relationship of gradients of perpendicular lines.

6.3.1.2.3*Identifying gradients of vertical and horizontal lines*

Despite recognising the parameters, Cypress and Iris could not identify gradients of vertical lines because they were not expressed in standard form. Cypress considered the gradient of the horizontal line as one, while Iris considered the gradient of the vertical line as one. The transcript below provides evidence of Iris's knowledge of the gradients of vertical and horizontal lines.

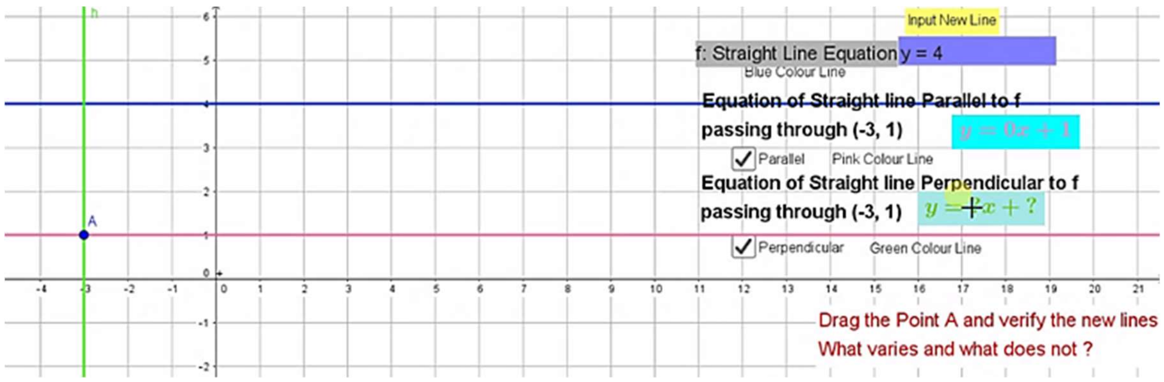
Reflective interview

R: Now tell me how the line y=4 look like?
***Iris**: It will be a straight line at 4.*
R: Which 4?
***Iris**: Positive 4 on the y-axis.*
R: ... and the line?
***Iris**: It will be horizontal.*
R: What will be the gradient of the line y = 4
***Iris**: It's zero. 1EC*
R: Do know why it's zero?
***Iris**: I think it's because parallel to x-axis. 1EC*
R: What is the gradient of line x = 2
***Iris**: I think it's one. It is a vertical line therefore it is one.*

Aster and Jasmine on the other hand, identified the gradients of vertical and horizontal lines. Aster and Jasmine exploited the opportunity to explore the lines of the form $y = k$ and $x = k$ in Applet 3.4.

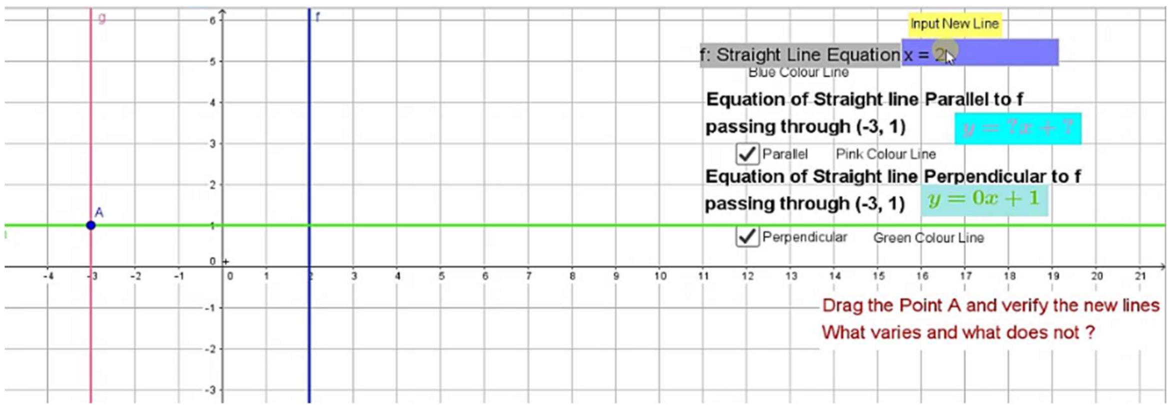
Screenshot 6.3 in Frame 6.4 captures Jasmine's interactions with the applet (in the applet the undefined values of gradient or non-existence of intercepts were represented by the symbol '?'). They exchanged ideas and engaged in meaningful dialogue as they interacted with the Applet 3.4. In transcript of the conversation in Frame 6.4, Jasmine asked Aster what the symbol '?' represented, Aster had little difficulty in answering her question and stated that it implied 'undefined'. Jasmine clarified with her friend and shared her thoughts, summarising her understanding of vertical and horizontal lines. In the transcript of conversation with Lilly in Frame 6.4, Jasmine provides two different reasons for the gradients of vertical and horizontal lines: i) the numerical concept of dividing by zero and divided by zero; ii) the non-existence of the triangle in either case. The following transcript in Frame 6.4 of the reflective interview demonstrates Aster's deep understanding of vertical and horizontal lines. He identified that these lines were perpendicular to one another. His meaning-making trait was further evident when he provided reasons for the gradient of the line parallel to the$y$-axis.

Classroom Observations



Screenshot 6.3: Jasmine wondering about the question mark symbol

Jasmine: *The blue line is horizontal and the pink the parallel line. ....*
Jasmine : *What is this question mark here?* *3DC*
Aster : In my computer name says its 'undefined'
Jasmine: *Undefined? Ok what about that 0x for that pink line? I mean parallel line* *3DC*
Aster : Horizontal line is having zero slope and vertical line slope is undefined.
Jasmine: *Vertical line is undefined is that so? Let me try with horizontal line ...*



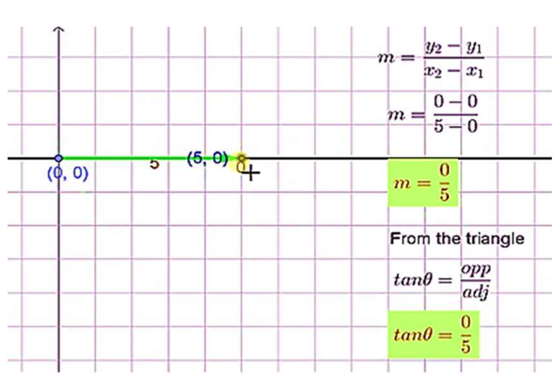
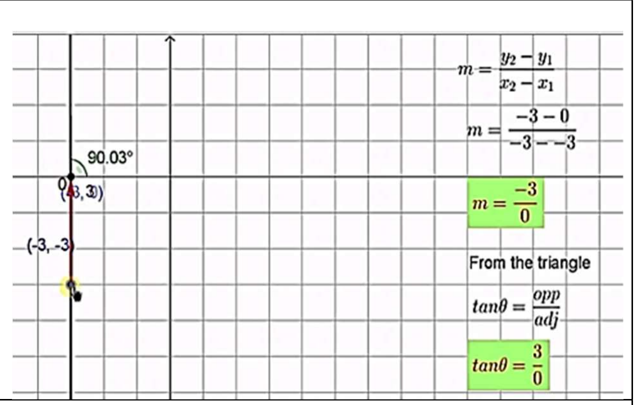
Screenshot 6.4: Unsolicited, Aster keys in x= 2

Jasmine: *Now the parallel line undefined* *1EC*
Lilly: Parallel line undefined I don't understand.
Jasmine: *Slope of parallel line. Pink line is vertical line the slope of pink line is undefined.* *1EC*
Lilly: Where?
Jasmine: *Look here at my screen. Its question mark there. It means undefined.*
Lilly: And that 0x+1. Means slope is zero.
Jasmine: *A line having a slope undefined is a vertical line then its perpendicular line is a horizontal line and it's slope is zero.* *1EC*

Reflective Interview with Aster	*R: What's the gradient of line y = 9.* ***Aster***: *It's a horizontal line and its slope is zero.* *R: What will be the gradient of its perpendicular line?* ***Aster***: *You mean perpendicular to y =* *R: Sorry yes, gradient of perpendicular line y = 9?* ***Aster***: *It's a vertical line and gradient is undefined.* *R: Why you think the gradient of a vertical line is undefined?* ***Aster***: *Why it is undefined?* *R: Once again why the line parallel to y-axis is having gradient undefined?* ***Aster***: *Line parallel to y-axis ... (pause)... in vertical line change in x2-x1 is zero, and any number divided by 0 is undefined*

Frame 6.4: Jasmine and Aster further their understanding on gradients of vertical and horizontal lines

As mentioned, Cypress and Iris did not have the opportunity to interact with Applet 3.7 during the recorded lesson. While interacting with the Applet 3.7, Aster and Jasmine verified the gradient of vertical lines in Screenshot 6.5 and Screenshot 6.6 in Frame 6.5, to convince themselves that gradients of all vertical lines are undefined. Significantly, Aster also wanted to verify if the gradient of horizontal lines were zero in that applet. Aster and Jasmine extended their understanding of vertical and horizontal lines from the sketching of lines in Applet 3.2 to determining their gradients in Applet 3.4 and 3.7.

<table>
<tr><td>Classroom Observation</td><td></td></tr>
<tr><td>
</td><td>
</td></tr>
<tr><td>Screenshot 6.5: Aster makes a straight line with the x-axis with no angle formed</td><td>Screenshot 6.6: Aster drags to have line almost parallel to the y-axis for an undefined gradient.</td></tr>
</table>

Frame 6.5: Aster's engagement with Applet 3.7

6.3.1.2.4 *Recognising the angle of inclination of vertical and horizontal lines*

Aster and Jasmine implicitly and accurately identified the angle of inclination of lines parallel to the *x*-axis and the *y*-axis as they interacted with Applet 3.7. The transcript below of classroom conversation shows how Jasmine clearly stated that the vertical line made an angle of 90° with the *x*-axis. Their meaningful interaction and exploration with the applets proved to be efficacious in developing their proficiency and meaning-making. It was evident that they generalised the relationship between the gradient of a line and its angle of inclination with the *x*-axis.

Classroom Observation

***Jasmine**: I think the horizontal line slope is zero. Vertical line is perpendicular to x-axis. You see it [Applet 3.7] was showing 90° for vertical line and horizontal line is parallel to x-axis.*
Lilly: (interrupting) *Excuse me guys can you explain once again that undefined gradient.*
***Jasmine** (isiXhosa) Parallel to y-axis gradient is divided by 0 therefore undefined. When the line is vertical there is no triangle. Can you see any triangle?*
Lilly: Triangle?
***Jasmine**: Look here. inaudible Vertical line is perpendicular, you can see 90° and there is no triangle. When the line is on the axis again there is no triangle therefore gradient is 0.*
(Jasmine moves the point backwards and forwards from y-axis to x-axis).

In the following section, I discuss how the participant learners further enhanced their knowledge of straight lines by integrating the property of *the angle of inclination of lines* during their interaction with Applet 3.7.

6.3.2 Essence of interactions –BI-2 (Patterns and relationships)

6.3.2.1 Synopsis

Aster:

- interpreted the positive or negative gradient of a line and thus made it an increasing or decreasing function (MM and CU);
- recognised variant and invariant properties of parallel and perpendicular lines (CU); and
- demonstrated the relationship between the gradient and the angle of inclination (CU).

Cypress:

- interpreted the positive or negative gradients of a line and how they made increasing lines (MM); and
- recognised variant and invariant properties of parallel and perpendicular lines (CU).

Jasmine:

- interpreted the positive or negative gradients of a line and how they made increasing or decreasing functions (MM and CU);
- recognised variant and invariant properties of parallel and perpendicular lines (CU); and
- demonstrated the relationship between the gradient and the angle of inclination (CU), though not comprehensively.

Iris:

- interpreted the positive or negative gradients of a line and how they made increasing lines (MM); and
- recognised variant and invariant properties of parallel and perpendicular lines (CU).

6.3.2.2 Discussion

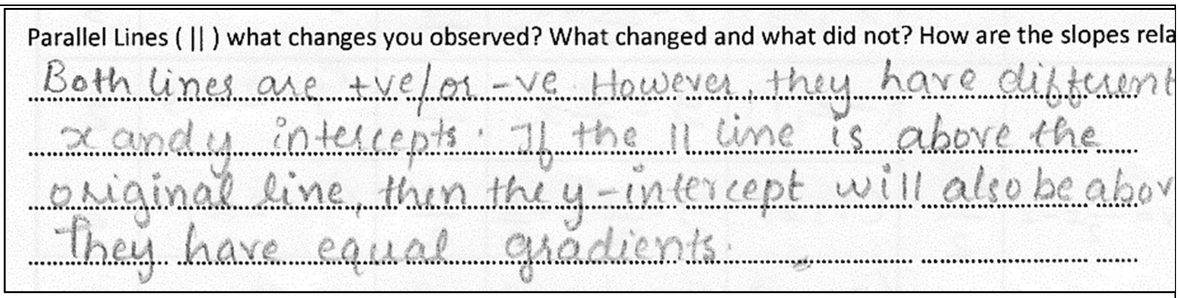
In this section I consolidate different instances of the BI-2 indicator, identifying patterns and relationships demonstrated by the participants, under the following headings:

- Identifying variant and invariant properties of parallel and perpendicular lines
- Recognising the relationship between gradient and angle of inclination
- Interpreting positive and negative gradients of a line

6.3.2.2.1 *Identifying variant and invariant properties of parallel and perpendicular lines*

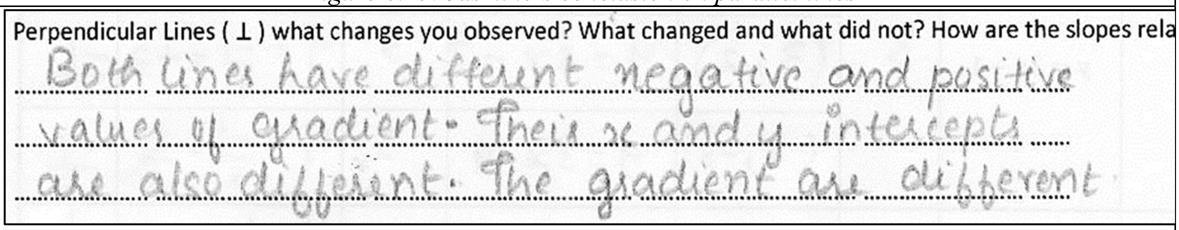
Section 6.3.1.2 discussed formulating the relationship between gradients of parallel and perpendicular lines and how the learners did not conclude their observations in a single sentence. For instance, Jasmine observed that for parallel lines the gradients were of the same sign (i.e. positive or negative) and for perpendicular lines, the gradients were of different signs (Figure 6.15 and Figure 6.16 in Frame 6.6 demonstrate this). Aster also considered that parallel lines were either decreasing or increasing. Evidently, they overlooked special cases of parallel lines which coincided with the *x*-axis or the *y*-axis, which are neither positive or negative, nor monotonic (increasing or decreasing).

Iris and Aster noted the change in the y of parallel lines. Whilst Jasmine perceived a sort of a pattern in the change of the of parallel lines, her responses in WS-2, Figure 6.15 and Figure 6.16, of Frame 6.6 show how she observed that if the parallel line was above the original line, then its *y*-intercepts would also be above, irrespective of the positive or negative gradient. A concept of comparisons of graphs was generated in her mind. The idea of the *y*-intercept above or below the original line could be a useful tool to estimate the equations of a straight line.


Parallel Lines (||) what changes you observed? What changed and what did not? How are the slopes rela

Both lines are +ve/or -ve However, they have different x and y intercepts. If the || line is above the original line, then the y-intercept will also be abov They have equal gradients.

Figure 6.15: Jasmine's conclusion on parallel lines


Perpendicular Lines (⊥) what changes you observed? What changed and what did not? How are the slopes rela

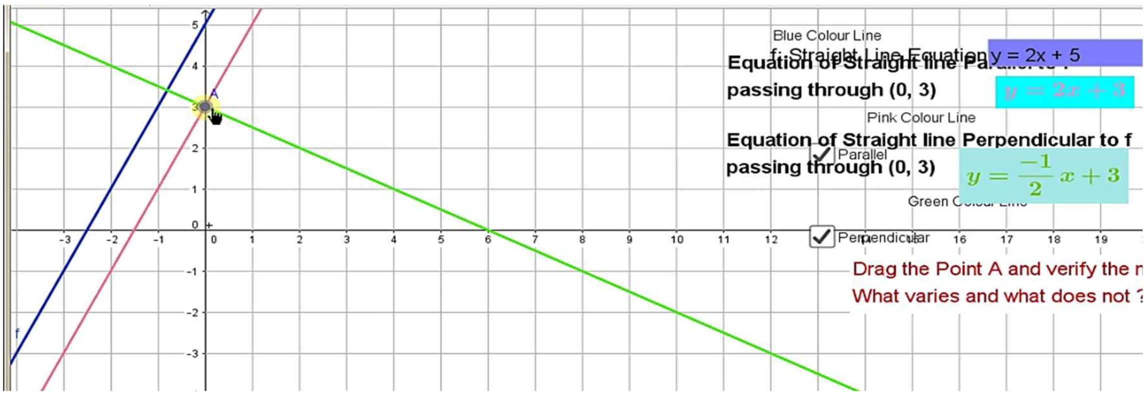
Both lines have different negative and positive values of gradient. Their x and y intercepts are also different. The gradient are different

Figure 6.16: Jasmine's conclusion on perpendicular lines

Frame 6.6: Jasmine's response on the worksheet following interaction with Applet 3.4

Aster made an interesting observation on the worksheet on intercepts of perpendicular lines. While interacting with the Applet 3.4, he dragged the point all over the axes in Screenshot 6.7 and Screenshot 6.8 in Frame 6.7. He observed that when the point of intersection of perpendicular lines was on the *y*-axis, their *x*-intercepts were of opposite signs, and when the point of intersection was on the *x*-axis, then their *y*-intercepts were of opposite signs (see Figure 6.17 in Frame 6.7). These kinds of generalisations were made possible from observations under different points of intersection, owing to the dynamic and interactive nature of the applet. Aster noted an important mathematical fact on intercepts of perpendicular lines.

Classroom Observations



Screenshot 6.7: Aster dragging to the y-intercepts

Screenshot 6.8: Aster dragging to x-intercepts

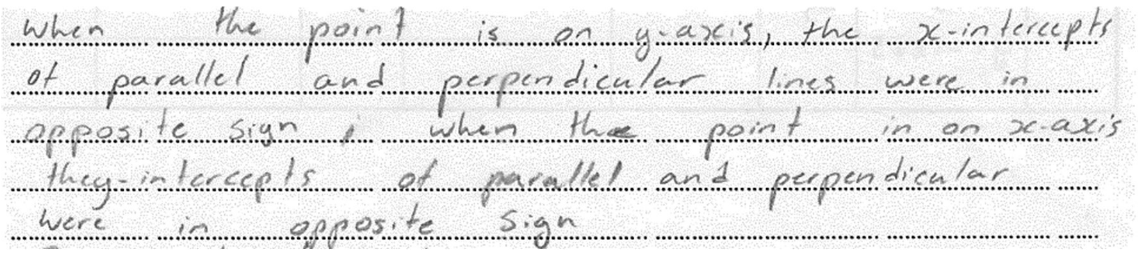


Figure 6.17: Aster's observation on x and y-intercepts of perpendicular lines

Frame 6.7: Aster's explanations on intercepts during his interaction with Applet 3.4

Cypress took an entirely different perspective on variant and invariant properties of parallel and perpendicular lines and this is discussed in the section 6.4.1.2. on p. 230.

When allowed to engage with mathematical activities, learners perceive with their senses. Such concrete manipulations very often give students "powerful intuitions" (Tall, 1994, p. 11) to identify patterns and relationships. This analysis concurs with the assertion made by Carpenter and Lehrer (1999) that mathematical understanding emerges when making mathematical knowledge one's own. Learners constructed meaning for relatively new processes of parallel and perpendicular lines by relating it to ideas that they already understood.

6.3.2.2.2 *Recognising the relationship between gradient and angle of inclination*

Owing to interactions with the Applet 3.7, Aster and Jasmine were able to recognise the relationship between the gradient of a line and the tangent of the angle of inclination, i.e. $m = tan\theta$. Their recognition of angles of inclination of vertical and horizontal lines has already been discussed in 6.3.1.2.4 on p. 198.

During the reflective interview, Aster and Jasmine exhibited their abstract understanding of the relevant mathematical relationship between the gradient and the angle of inclination. I provided four options to identify an angle of inclination that would make a line with a positive or negative gradient — see the transcript in Frame 6.8. Jasmine could not estimate an appropriate angle of inclination for a positive line. She initially managed to determine the value of θ, as $\theta = \tan^{-1}(m)$, but could not proceed further. Of the four options that I gave her, she eliminated two options i.e. $0°$ and $90°$. She had valid reasons for the rejection, demonstrating her knowledge of gradients of vertical and horizontal lines and the angles that they made with the *x*-axis. However, she did not demonstrate her understanding of gradients of lines that inclined with the positive *x*-axis other than 0° and 90°, showing that she did not fully understand the concept. Aster considered the options in sequential order and shortlisted the choices given to him. Initially, he tried to apply the ratios of special angles and eliminated the two choices — 0° and 45°, that did not yield negative gradients. Thereafter, he realised that 180° formed a straight line with the *x*-axis making a gradient of zero and applied his understanding that the angles above 90° would render a negative gradient.

Reflective interview
R: Is there is any relationship between slope and angle of inclination? ***Jasmine**: Yes m = tanθ. ...* *R: Can I ask you what will be the angle of inclination if the slope is positive?* ***Jasmine**: I didn't hear you what is the slope sir?* *R: Gradient is positive. What would be the angle of inclination? I will give you choices,* *60° 2) 150° 3) 90° and 4)* 0° (I write down these values on the pad). ***Jasmine**: Gradient is positive. Then $\theta = tan^{-1}(m)$ I don't know. It should be between 60° and 150°.* *R: Why you think so?* (she wanted to use calculator, but I restricted her). ***Jasmine**: 90° means gradient is undefined. 0° means slope is zero. So, it's either 60° or 150°.*
R: What would be the angle of inclination for a negative gradient? I will give you choices. *45° 2) 120° 3) 180° and 4)* 0° (I write down these values in the pad). ***Aster**: Gradient is negative. That is m is negative, implies decreasing line.* (pauses looking down) *R: Ok. Decreasing therefore what could be angle from these choices? Speak aloud as you think* ***Aster**: I was thinking it cannot be 45 because then gradient is 1 and for -1 angle is 135° its 120° I do not know tan120° 180° is negative side of x-axis and* 0° *is zero.* *R: What's your final answer? how many choices you shortlisted?* ***Aster**: Two and three… above 90°, the gradient is -ve it cannot be 180° it zero it's a line on the x-axis its 120° .* *R: Is that your final answer?* ***Aster**: It's 120°* *R: How did you know that angles above 90° will be negative?* ***Aster**:* (He makes a gesture, places his hand over the table and starts rotating anticlockwise direction his hand from vertical to horizontal). *I was trying to recollect that when I moved the point from the first quadrant to the second quadrant the gradient changed to negative.*

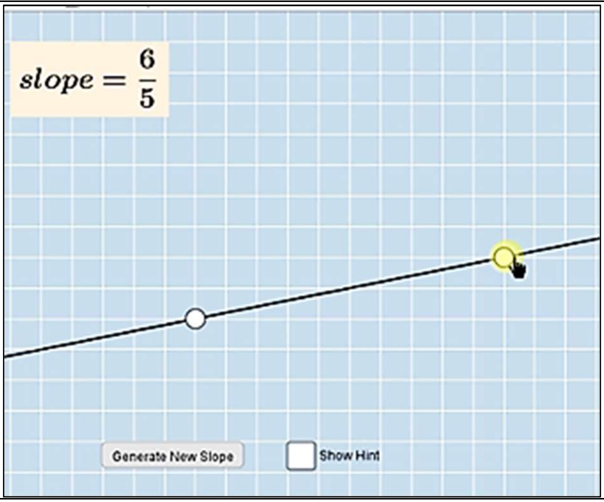
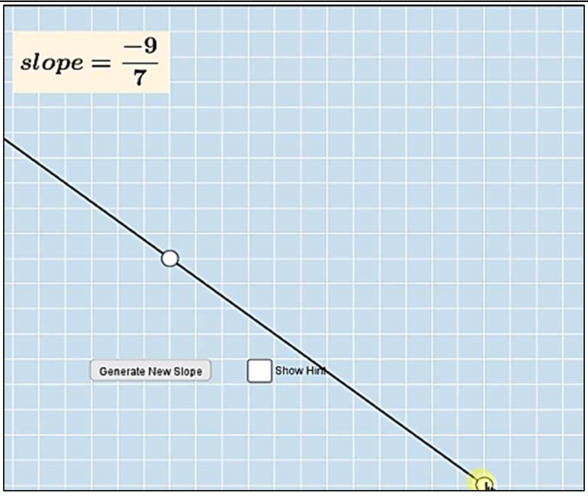
Frame 6.8: Jasmine and Aster conjecturing the angle of inclination

Aster and Jasmine dragged the point on all the quadrants, and more importantly they moved backwards and forwards between the quadrants. Nonetheless, Jasmine focused on the numerical perspective of the gradient in the applet rather than on the values of the angle of inclination. Jasmine abstracted the concept of positive slope as an increasing straight line but did not comprehend that the positive slope inevitably rendered an acute angle with the positive *x*-axis. More significantly, Aster made sense of the interactions with the Applet 3.7 thus demonstrating that he developed proficiency by recognising relationships among the different mathematical concepts.

6.3.2.2.3 *Interpreting positive and negative gradients of a line*

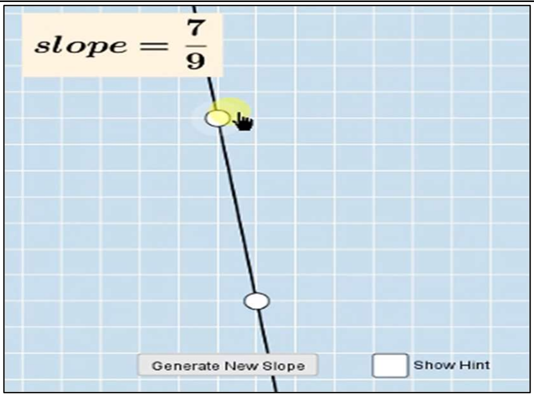
All participant learners identified a pattern for positive or negative gradients of straight lines. In his interaction with Applet 3.1, Aster initially struggled to sketch the line for a given gradient, nonetheless, his unsuccessful attempts had the direction of the line aligned with the positive or negative value of the slope. See Screenshot 6.9 and Screenshot 6.10 in Frame 6.9. During the reflective interview, Aster observed that positive slopes imply that a graph is looking up, hence he moved the point right and then upwards in transcript in Frame 6.9. He considered the negative slope as looking down and therefore he moved the point left and then upwards.

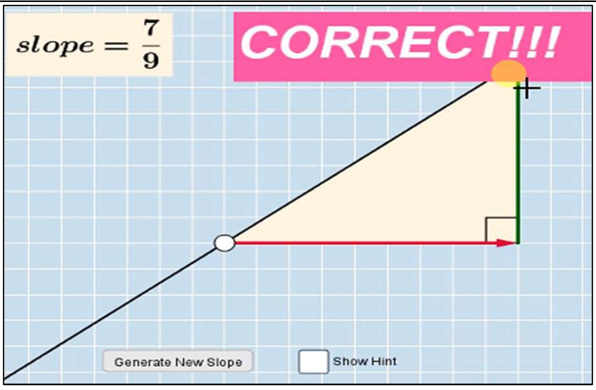
Furthermore, when I asked him to establish the angle of inclination of a line with a negative gradient, Aster immediately related it to a decreasing line '*looking down'*. Significantly, participants established a connection between positive and negative gradients of straight lines and monotonic functions.

<table>
<tr><td colspan="2">Classroom Observation</td></tr>
<tr><td>
</td><td>
</td></tr>
<tr><td>Screenshot 6.9: Aster's random dragging proved unfruitful</td><td>Screenshot 6.10: Aster still dragging the points but with no result. Note the direction and gradient</td></tr>
<tr><td>Reflective interview</td><td>R: When we look at the first applet to sketch a line for a given slope, you struggled a bit in the beginning?
…
R: What strategy you used if you remember? You can see that you moved in either direction? What was in your mind?
Aster: Positive slope is looking up, so I moved right then I have to move up.
R: What is negative slope of a line?
Aster: It's looking down.</td></tr>
</table>

Frame 6.9: Aster's interaction with Applet 3.1

While sketching the line in Applet 3.1, Cypress often moved the point left or right, and then he would correct himself, as if he already knew the direction of the line for a given gradient. In Screenshot 6.11 and Screenshot 6.12 in Frame 6.10 he dragged the point seven units up and then dragged one unit left. Immediately he then changed direction nine units right to sketch the line for the slope $\frac{7}{9}$. Apparently, he interpreted the direction of the line but could not express it unless he saw the line in the required direction. On their worksheets, Aster and Cypress also mentioned parallel lines moving in the same direction and perpendicular lines moving in opposite directions.

<table>
<tr><td colspan="2">Classroom Observations</td></tr>
<tr><td>

Screenshot 6.11: Cypress confused by horizontal movements</td><td>

Screenshot 6.12: Cypress realised his mistake, then corrected it</td></tr>
<tr><td></td><td>Iris: (interrupts) How you do this?
Cypress: Slope is -2. So I see it as $-\frac{2}{1}$. So move 2 units up and then 1 unit left or right. 6LPK
Iris: Left or right? Tell me
Cypress: Ok. First let us move 2 units up, now move one unit left. 1EC
Iris: Why did you say right then?</td></tr>
</table>

Frame 6.10: Cypress' engagement with Applet 3.1

Iris understood the variant patterns of the positive and negative gradients and this is further discussed in section 6.4.2.1. on p. 232, whilst Jasmine explicitly linked positive and negative gradients to increasing and decreasing functions. Jasmine's first observation about the parallel lines in the worksheet (in Figure 6.15, Frame 6.6 on p.200) was that both the lines were either positive or negative. She clarified this during the reflective interview:

> *Positive line I meant as increasing line going up. An increasing line will also have parallel lines that are also increasing.*

In another instance during the reflective interview, she again linked the gradient of a line to a monotonic function.

> *Slope is positive or negative. And function is either decreasing or decreasing. I saw increasing straight line moving up and positive gradient also is moving up. So, the positive gradient is increasing.*

My analysis led me to the findings of Hollenbrands (2007) that a technological environment facilitates learners' development of making sense of abstract mathematical concepts. Furthermore, applets provided opportunities for learners to explore mathematical relationships and develop deep understanding, which is a major tenet of the constructivist theory of learning (Jaworski, 1994; von Glasersfeld, 1991).

6.3.3 Essence of interactions –BI-3 (Looking for hidden structures)

6.3.3.1 Synopsis

Aster:

- explored different strategies of sketching a line for a given gradient in Applet 3.1 (MM);
- conjectured and explored to validate whether 'Is the line parallel to itself?' (MM);
- implemented a strategy in Applet 3.5 to verify his inferences in *determining the equation of a parallel line*, through sketching a line using the *gradient-intercept concept* (MM and CU); and
- explored in Applet 3.7 to verify *angles of inclination in positive and negative gradients* (CU).

Cypress:

- implemented a strategy in Applet 3.2 to verify his inferences in sketching a line using the *gradient-point concept* (MM and CU); and
- conjectured a strategy in determining the *equation of a straight line*, however, he did not validate it (CU).

Jasmine:

- explored to validate the *variant and invariant of parallel and perpendicular lines* (CU); and
- explored Applet 3.7 to verify *angles of inclination in positive and negative gradients* (MM and CU) including horizontal and vertical lines.

Iris:

- explored different strategies of *sketching a line for a given gradient* in Applet 3.1, but failed; and
- explored to validate the *variant and invariant of parallel and perpendicular lines* (CU).

6.3.3.2 Discussion

In this section I consolidate different relevant indicators of BI-3 demonstrated by the participants under the following headings:

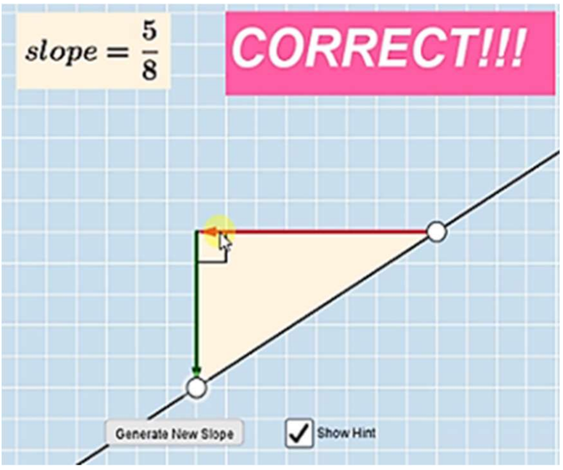
- Exploring different strategies in sketching a line for a given gradient
- Exploring and conjecturing variant and invariant properties of parallel and perpendicular lines
- Exploring alternate methods of sketching a line for a given equation
- Exploring and sketching parallel lines

6.3.3.2.1 *Exploring different strategies in sketching a line for a given gradient*

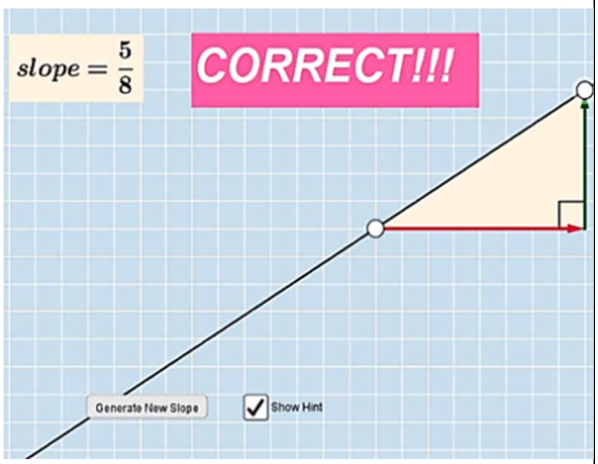
The participants explored different aspects of the basic concepts of a gradient of a line, looking for hidden structures. Aster initially struggled to sketch the line for a given gradient when interacting with Applet 3.1. He spent a few minutes attempting quite a number of slopes. Finally, by chance, he managed a correct line in Screenshot 6.13 in Frame 6.11. Screenshot

6.14 in Frame 6.11 shows how, instead of focussing on this feat, he quickly attempted to sketch the line for the same slope ($\frac{5}{8}$). This time he moved eight units right then five units up. Aster, although initially struggling to get to grips with the concept of slopes, then started to show understanding by arranging different permutations and combinations of horizontal and vertical changes. He did not confine himself to moving the points in one direction (vertical or horizontal). He knew that irrespective of the first movement he had to adjust the second movement in accordance with the given gradient, as seen in Screenshot 6.15 and Screenshot 6.16 in Frame 6.11. Aster, through exploration and conjecturing, discovered different methods of sketching a line for a given gradient.

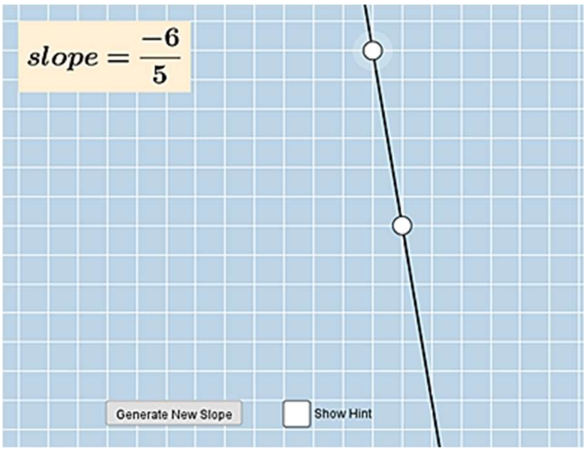
Classroom Observation



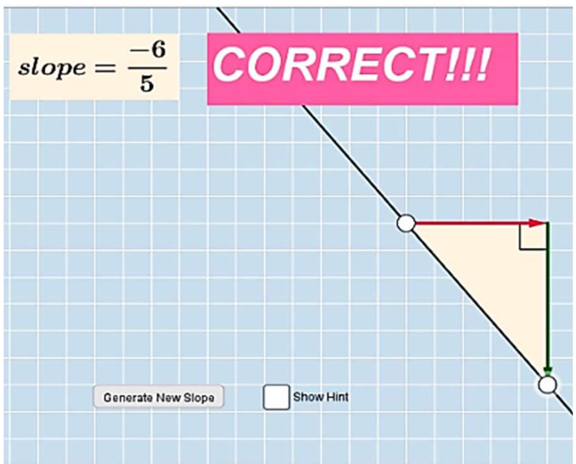
Screenshot 6.13: Aster moved left eight then five down. At last he is correct



Screenshot 6.14: Aster's alternate method: movement right eight then five up



Screenshot 6.15: First vertical up and then horizontal left

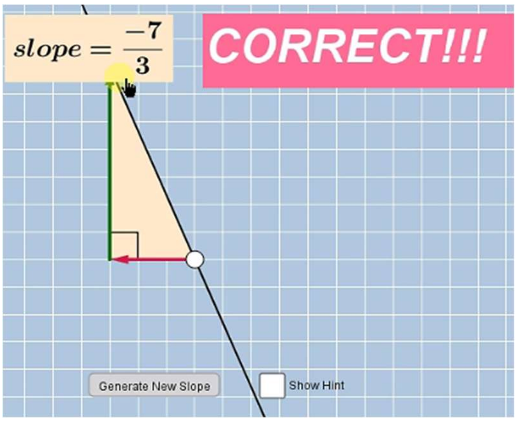


Screenshot 6.16: Aster solves using two methods

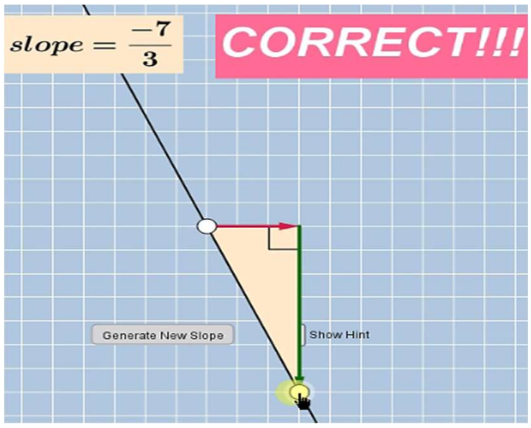
Frame 6.11: Aster's interaction with Applet 3.1

It appeared that Cypress followed the footsteps of his teacher in sketching a straight line. Throughout, in Applet 3.1, his approach remained identical. He brought the points together then moved one point vertically upwards and then moved the point horizontally. However, after about eighteen successful attempts, he thought of sketching in different directions. In sketching the slope of $-\frac{7}{3}$, Cypress used his previous approach as captured in Screenshot 6.17 of Frame 6.12. For the last question the slope of $-\frac{7}{3}$, Cypress moved vertically downwards and then moved the point towards the right in Screenshot 6.18, Frame 6.12. Cypress attempted different methods, suggesting an approach of sketching and exploring.

Classroom Observations



Screenshot 6.17: Cypress first moved vertically up and then horizontally left



Screenshot 6.18: Cypress first moved vertically down and then horizontally right

Frame 6.12: Cypress explored two methods in Applet 3.1

Jasmine made no attempt to sketch lines using different methods. Iris attempted a different approach in her solutions after a few correct answers; however, this was not successful. The following Table 6.1 is a summary of the learners' strategies in sketching the line of a given gradient, for instance the gradient $\frac{3}{4}$.

Table 6.1: A summary of the strategies employed by learners in Applet 3.1

1st movement	2nd movement	Participant learners
Vertically Up – three units	Horizontal right – four units	Aster, Cypress and Iris
Vertically Down – three units	Horizontal left – four units	Aster and Cypress
Horizontal right – four units	Vertically Up – three units	Aster, Jasmine and Iris (failed)
Horizontal left – four units	Vertically Down – three units	Aster

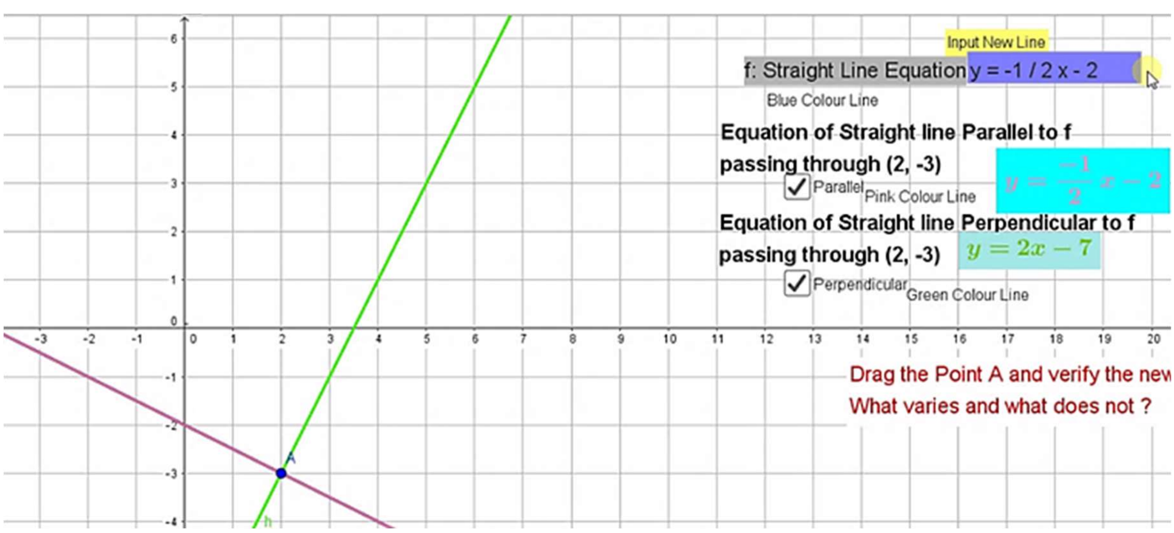
6.3.3.2.2 *Exploring and conjecturing variant and invariant properties of parallel and perpendicular lines*

Participant learners formulated different mathematical arguments as they made sense of their interactions with DGS. Constructivist teaching and learning, as Gordon (2009) envisions, can be summed up succinctly: "[T]he underlying goal in such lessons is not to feed students a list of essential facts or formulae, but rather to help them construct a deeper understanding of an issue that is significant to them." (p. 51). It was evident that applets provided an opportunity for learners to think critically and reflect on what they were learning by asking questions.

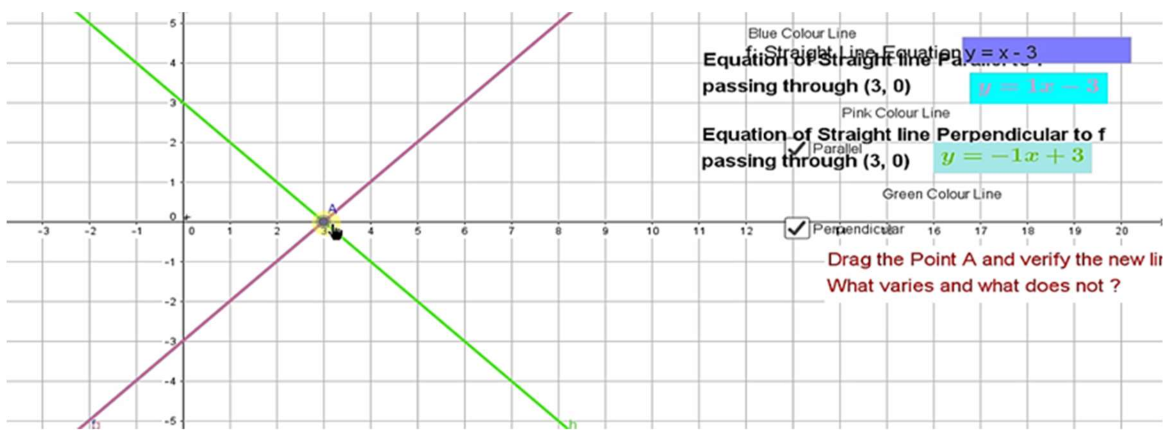
In Applet 3.4, Aster was exploring, moving the point around the screen rather than filling in the worksheet WS-2. Aster's interaction with Applet 3.4 was not confined to answering the worksheet. He went beyond the stipulated worksheet, making and verifying conjectures, and thus exploring and discovering concepts. In an instance of conjecture and exploration, Aster purposefully dragged the point making a pink line that overlapped the original blue' line shown in Screenshot 6.20 of

Frame 6.13. He raised as a concern in a conversation with his friend Jasmine, "*Is a line parallel to itself?*" Meanwhile, Jasmine also encountered a similar situation of coincident lines shown in Screenshot 6.19, Frame 6.13. Initially, Aster argued with Jasmine that a line was parallel to itself as the gradients were equal. The transcript of Jasmine's conversation In Frame 6.13 shows that she maintained that all the points of the lines were common to each another and parallel lines did not share points by virtue of their definition. In her reflective interview, Jasmine maintained that a line was not parallel to itself. Conflicting thoughts on parallel and intersecting lines provoked Aster to drag the point, again and again, making the lines overlap. However, even at the time of the reflective interview he remained inconclusive on the issue of a line being parallel to itself as seen in the reflective interview in Frame 6.13. He was aware of parallel and intersecting lines (perpendicular lines as a special case of intersecting lines), but not about 'coincident lines.'

Classroom Observation



Screenshot 6.19: Jasmine considered a parallel line overlapping with the original equation as unusual.



Screenshot 6.20: Aster purposefully dragging to make lines overlap

Jasmine: Look at this. The pink line is above the blue one, isn't it so?
Aster: I don't know.
Jasmine: Ok I will change the points
Aster: Tell me is the line parallel to itself?
Jasmine: Now what's that?
Aster: Look if I place the parallel line over the blue line, it means it should be parallel to itself. Their gradients are equal.
Jasmine: But all the points of the line are common to the parallel line. They share points and they cannot be parallel.
Aster: Yeah true. They have common points. But one is over the other they cannot be intersecting. Will it be a tangent?
Jasmine: Your thoughts do not make sense to me. I changed the point. But it's something very unusual to have line over itself.
Aster: I think a line is parallel to itself.

Reflective Interview	*R: So what do you think a line is parallel to itself* ***Aster**: The gradients are equal, but all the points are common. I don't know.*
	R: Is a line a parallel to itself? I heard something like that in your conversation with [Aster] ***Jasmine**: No, a line is not parallel to itself.*

Frame 6.13: Aster and Jasmine argue about the overlapping lines in Applet 3.4

Applet 3.4 provoked Cypress to raise an important and mathematically relevant question with his friends while comparing the equations of straight-line functions. He and Iris agreed that the gradients of parallel lines did not change, but that the *y*-intercepts changed. Cypress wanted to know the reason behind the change in the *y*-intercepts of parallel lines. Iris accepted the situation as it was and did not probe further. Cypress appeared to reflect on the situation and came up with a rationale for the change of a parameter in parallel lines. This transcript is shown below in in Frame 6.14.

Classroom Observation	*Cypress:* interrupting *[Iris] you say slopes do not change and what changes?* *Iris: y coordinates changes* *Cypress: Why the y-intercepts change? Why did it happen?3DC* *Iris: I don't know.*
Reflective Interview	*R: In class you asked [Iris] why the y-intercepts changed for parallel lines and gradients did not change? I am asking you the same question* ***Cypress**: For parallel lines, they have a constant distance throughout. Therefore, the y-intercepts of parallel lines also maintained the same distance, and hence they changed. 2UVI 1EC*

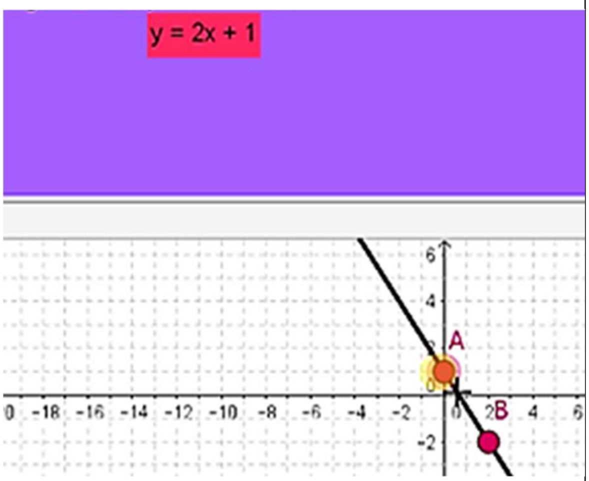
Frame 6.14: Cypress questions variant properties of parallel lines

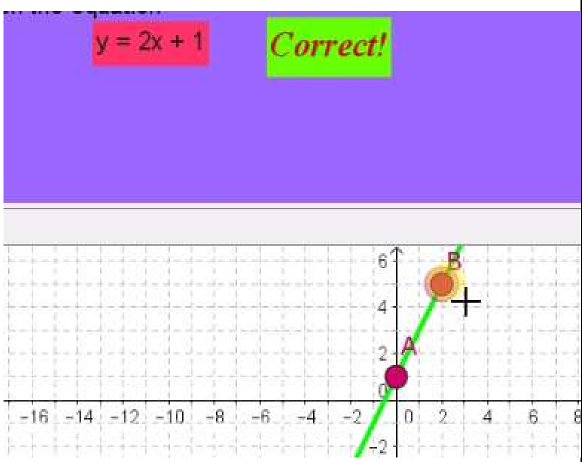
6.3.3.2.3 *Exploring alternate methods of sketching a line for a given equation*

All participant learners were well informed about the procedures to sketch straight lines, such as the *dual-intercept method* of sketching a straight-line equation in Applet 3.2. However, when the preferred method failed, owing to the fractional *x*-intercept, all the participants except Jasmine, explored and developed alternate strategies to sketch the line.

Iris encountered problems when she could not place a point for a non-integer *x*-intercept value of $x = -\frac{1}{2}$ for the equation $y = 2x + 1$. She placed point *A* on the *y*-axis at $y = 1$ but could not place point B between 0 and 1 on the *x*-axis (Screenshot 6.21 in Frame 6.15). As usual, she asked for help from Pansy and Cypress. Cypress suggested bringing the points together and then using the *gradient of the line technique*. Cypress then took control of the mouse and dragged point B to sketch the line successfully in Screenshot 6.22, Frame 6.15. Iris said that she did not understand what 'gradient' meant. She understood 'gradient' only to mean the direction of the line. In the transcript in Frame 6.15, she pondered over this particular situation and concluded that the other point would satisfy the equation of the line by substituting $x = 2$ into the equation to get $y = 5$. Cypress and Iris encountered an impasse in sketching a straight line using the *dual-intercept method* and they brainstormed with each other to find a solution. They developed different strategies in order to sketch the line. It was intriguing to observe that

Iris did not understand the *gradient-intercept method* suggested by Cypress, while Cypress in turn, did not understand the *gradient-point method* for sketching a line discovered by Iris.

<table>
<tr><td>Classroom Observation</td><td></td></tr>
<tr><td>

Screenshot 6.21: Iris faced challenges when sketching straight lines</td><td>

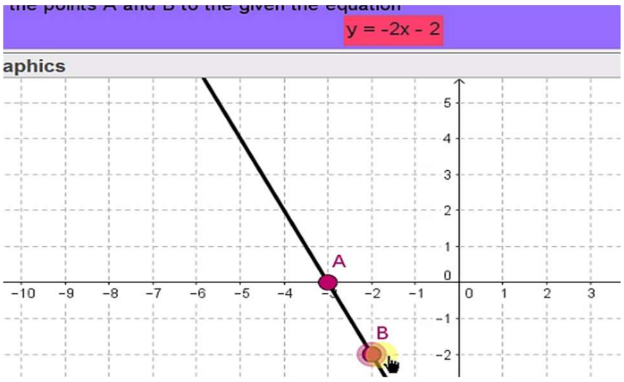
Screenshot 6.22: Cypress discovered the intercept-gradient method of sketching lines</td></tr>
<tr><td colspan="2">
Iris: Yeah, it's x= $-\frac{1}{2}$. How do we sketch this line? [Cypress]
Cypress: Ok we have to bring that point to between 0 and – 1.
Iris: But there isn't any point with x= $-\frac{1}{2}$.
Pansy: That's true.
Cypress: I am also having the same problem.
Iris: But guys do you see that our gradient is 2 so we must have a line to the right of A(1EC)(7EST)
Pansy: Yes, you are correct, so our point x= -1/2 must be correct.
Cypress: The point must be on the right-hand side of A.
Iris: But which point [cypress]?
Cypress: Bring the points together and use the gradient theory.
Iris: What? Use the gradient theory? How?
Pansy: Gradient is 2. So, it should be on the right of A. but which point [Cypress]?
Cypress: Let me help you. (Cypress now handles the mouse; the required line is sketched)
Pansy: You did it. Just explain this. I skipped this kind of question.
Cypress: I use gradient theory.
Iris: What is the gradient theory? I never heard of.
Pansy: I too never heard of it.
Cypress: Look we have to move to right from A. So, when we move to point B the gradient is 2. Can you see that?
Pansy: I understand that right of A, so you say that is your theory.
Iris: I don't understand what your trying to tell. But about x-intercept? We must understand this. (3DC)
Cypress: Look there at that point. The line is passing through 0 and -1 and it is $-\frac{1}{2}$
Pansy: I see that but are you sure that point $-\frac{1}{2}$.
Iris: it's between 0 and -1. So, it has to be if our calculations are correct. But I have noticed something else of B.
Pansy: Yeah, the coordinates are 2 and 5 isn't it?
Iris: [isiXhosa] … (you are correct, we are thinking at the same time) if we put x=2 into the equation its 2 times 2 plus 1 which is 5. So, we must use intercept and a point. (6BP)
</td></tr>
</table>

Frame 6.15: Three friends discussing sketching the line with Applet 3.2

In sketching the line $y = -2x - 2$, Aster initially attempted to employ the *gradient-intercept method*. Point A was by default at $x = -3$, he dragged point B to $x = -3$ and both points overlapped. He then moved B one unit right and then two units down. He was sketching a gradient of -2 but the line only synchronised with the gradient and not with the y-intercept.

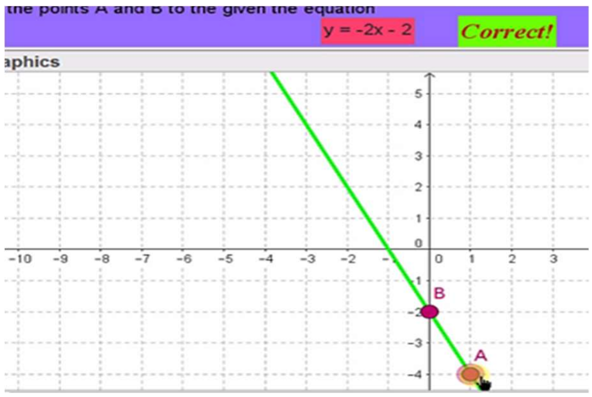
This is shown in Screenshot 6.23, Frame 6.16. Since his intercepts were flawed, he could not successfully graph the required line. He then employed the *dual-intercept method* to sketch the line successfully. As in the case with the Applet 3.1, he did not proceed to the next line. He was dragging A (with B still at $y = -2$) to other points of the line such as $(-2;\ 2)$, $(1;-4)$ in Screenshot 6.24 in Frame 6.16 . Apparently, he was verifying the other points on the line.

<table>
<tr><td colspan="2">Classsroom Observation</td></tr>
<tr><td>



Screenshot 6.23: Aster's futile attempt at using the slope-intercept method to sketch the line

</td><td>



Screenshot 6.24: Aster verifying other points on the line after successfully sketching using the dual-intercept method

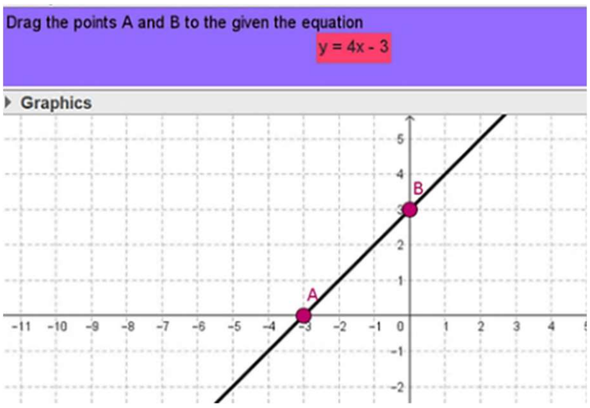
</td></tr>
</table>

Frame 6.16: Aster graphing straight lines of the slope-intercept form in Applet 3.2

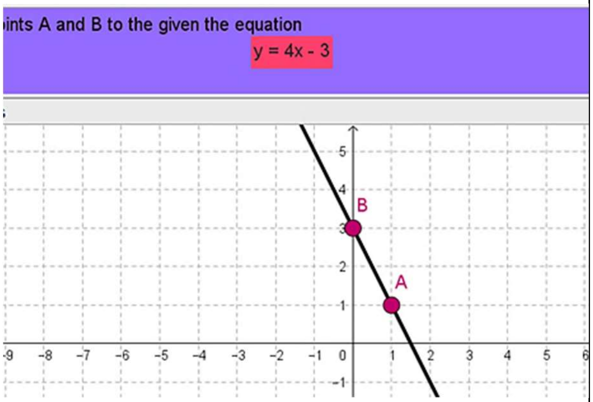
Not all learners developed alternate methods to successful sketch the lines. In her attempt to sketch the line $y = 4x - 3$, Jasmine dragged B to the y-intercept at $y = 3$ and then moved A to the x-intercept at $x = -3$ (in Screenshot 6.25, in Frame 6.17). But her line was incorrect. She apparently substituted $x = 1$ into the equation and thus obtained the point $(1; 1)$ She dragged point A to $(1; 1)$ shown in Screenshot 6.26 in Frame 6.17. The line was again flawed. Evidently , she didn't realise her error; the y-intercept ought to be at $y = -3$ instead of at $y = +3$.

Aster, Cypress and Iris indulged in exploring and developed alternate methods of sketching straight lines. A constructivist perspective suggests that when learners are constructing their own meaning it "enables students to follow routes which are their own" (Boaler, 1993, p. 17).

Classroom Observations



Screenshot 6.25: Jasmine drags B to y-intercept y= 3



Screenshot 6.26: She substitutes x=1 to obtain y=1 and then drags A to (1:1).

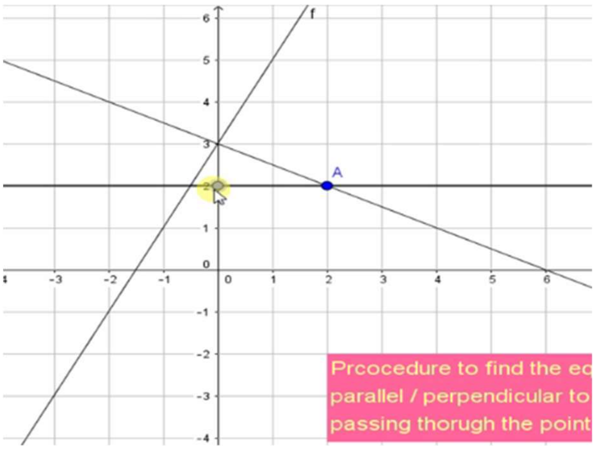
Frame 6.17: Jasmine's failed attempt to sketch lines of the form $y = mx + c$ *in Applet 3.2*

6.3.3.2.4 *Exploring and sketching parallel lines*

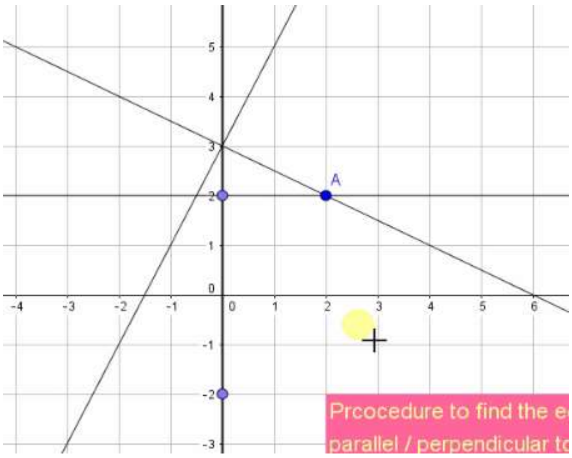
In Applet 3.5, Aster actively explored constructions. He first changed the equation in the input box to $y = 2x + 3$, the first equation in the worksheet WS-2. He dragged point *A* at random and then placed it at $(2; 2)$ and he searched for additional tools available in the toolbar of the applet. He first selected 'line segment' to draw a line segment from *A* to the *y*-axis at $y = 2$. He undid the construction and then selected the 'line passing through two points' tool and again sketched a horizontal line passing through *A* and the *y*-intercept at $y = 2$. He then sketched another line over the *y*-axis with points at $y = 2$ and $y = -2$. (Screenshot 6.28 in Frame 6.18). He then cancelled all these lines. He made several attempts to sketch a line, and finally he sketched a line passing through *A* in Screenshot 6.30 of Frame 6.18. Significantly, he sketched a line $y = 2x - 2$ parallel to $y = 2x + 3$ passing through $(2; 2)$.

Based on my analysis, I concur with the view that exploration is a way of learning mathematics in a technological environment of teaching and learning. Furthermore, my analysis is consistent with Romberg and Kaput (1999) who emphasise the exploration of problems as a way of understanding mathematics.

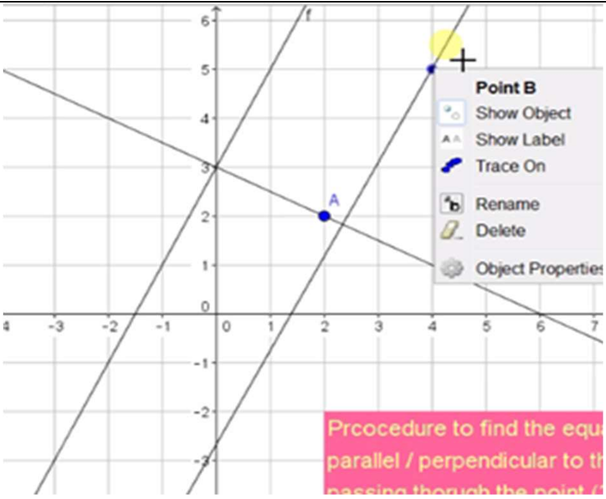
Classroom Observation



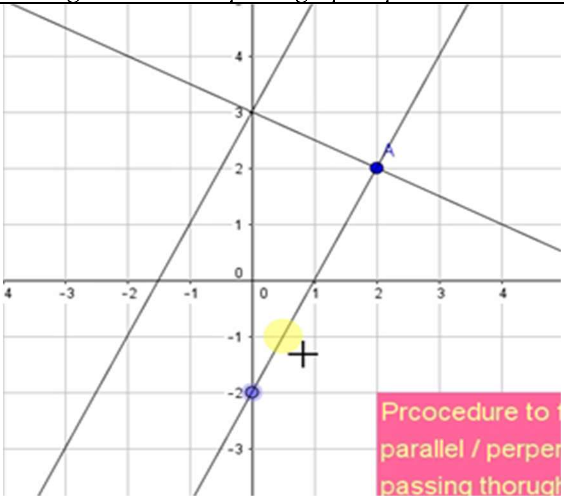
Screenshot 6.27: Aster's initial attempts to draw a parallel line



Screenshot 6.28: Aster continued his exploration, making several attempts to graph a parallel line.



Screenshot 6.29: One among the several failed attempts



Screenshot 6.30: Aster getting there

Frame 6.18: Aster's exploration in graphing a parallel line in Applet 3.5

6.3.4 **Essence of interactions –BI-4 (Purposeful use of procedures)**

6.3.4.1 **Synopsis**

Aster:

- made purposeful use of formulae and algorithms in finding the equations of parallel and perpendicular lines (PF); and
- justified the procedure chosen in solving a line (MM and PF).

Cypress:

- showed little evidence of accurate use of formulae or procedures across the lesson.

Jasmine:

- made purposeful use of formulae and algorithms in finding the equations of parallel and perpendicular lines (PF).

Iris:

- made purposeful use of formulae and algorithms in finding the equations of parallel and perpendicular lines (PF).

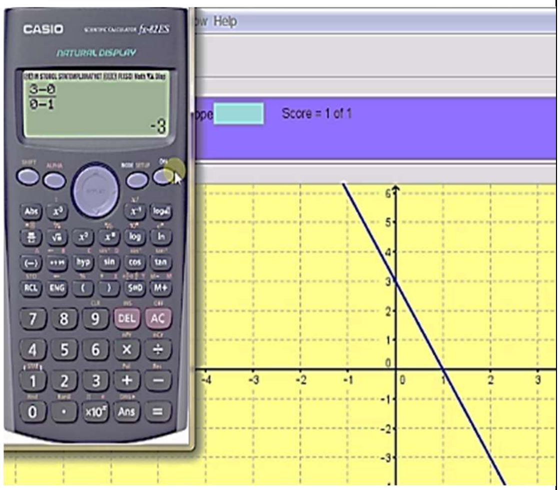
6.3.4.2 **Discussion**

In all the participants, except for Cypress, I found evidence of the BI-4 indicator which is efficient and accurate use of algorithms and procedures. In the following section, I discuss the evident indicator.

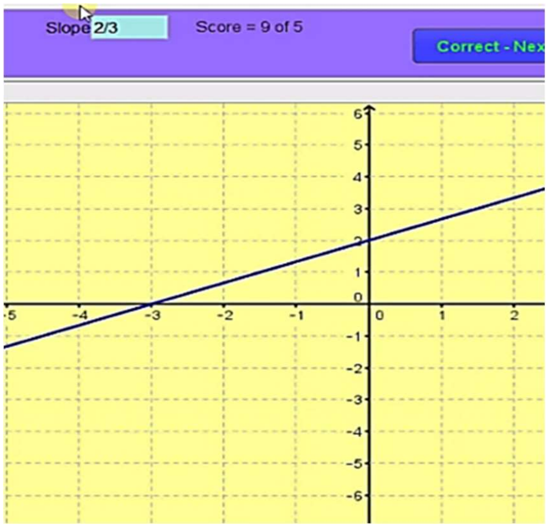
6.3.4.2.1 *Using the formula and the procedures*

While interacting with the Applet 3.3, all participant learners employed the gradient formula. The aim of Applet 3.3 was to determine the slope of a given line. Jasmine determined the slope of the line as $(-1;-2)$ as shown in Screenshot 6.31 in Frame 6.19. She sought help from the teacher who instructed her to use the formula $\frac{y_2-y_1}{x_2-x_1}$. Screenshot 6.32 in Frame 6.19 demonstrates how she resorted to using a calculator to find the correct slope. Jasmine was motivated to proceed with other questions, and interestingly, she generalised a technique on her own: She keyed in the y coordinate of the y-intercept over the negative x-coordinate of the x-intercept as the slope, hence the negative sign in the denominator in Screenshot 6.34, Frame 6.19.

<table>
<tr><td colspan="2">Classroom observations</td></tr>
<tr><td>


Screenshot 6.31: Jasmine recorded the slope as a coordinate, prior to the teacher's instructions
</td><td>


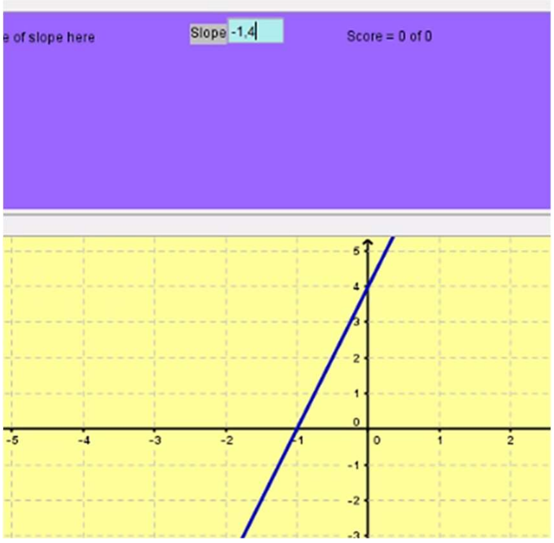
Screenshot 6.32: Jasmine using the formula on the calculator to determine the slope
</td></tr>
<tr><td>

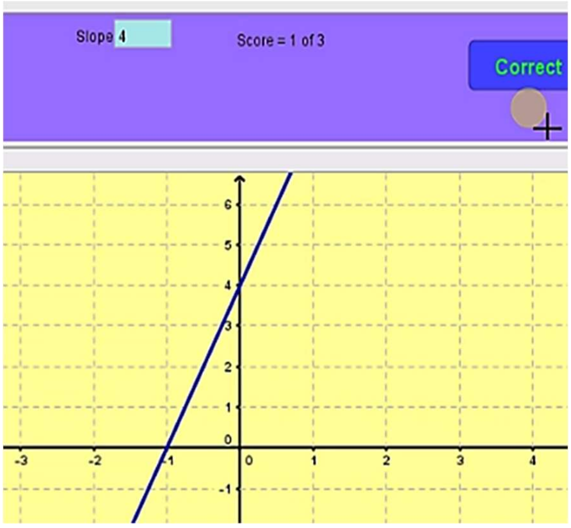

Screenshot 6.33: Jasmine generalised a formula: y-intercept over negative x-intercept.
</td><td>


Screenshot 6.34: She changed the sign of x-intercept.
</td></tr>
</table>

Frame 6.19: Jasmine determining the gradient of a given line in Applet 3.3

Interestingly, Iris also started with a misunderstanding of the question asked in Applet 3.3. Similarly, to Jasmine, Iris recorded the slope in coordinate forms, such as $'-1,4'$ as shown in Screenshot 6.35, Frame 6.20. It appeared that she entered the intercepts of the line. However, her friend Pansy asked her to apply the gradient formula and she then used a calculator to determine the gradient of the line. During the reflective interview, she acknowledged this applet as she realised that "the two intercepts determine the gradient" in the transcript in Frame 6.20.

<table>
<tr><td colspan="2">Classroom Observations</td><td></td></tr>
<tr><td colspan="2">

Screenshot 6.35: Iris considering gradient as a coordinate form</td><td>

Screenshot 6.36: Iris applies the formula to get a correct answer</td></tr>
<tr><td>Reflective interview</td><td colspan="2">R: How you think GeoGebra helped you in learning straight lines, if it helped you?
Iris: It helped me to construct lines. It requires two points to construct a straight line.(1EC) To determine the gradient has become easier. I now know how gradient can be found whether it's right or left (referring to the applet where she moved the points horizontally depending on the negative or positive slope). And how the two intercepts determine the gradient.</td></tr>
</table>

Frame 6.20: Iris's engagement with the Applet 3.3

Iris and Jasmine showed evidence of their understanding of procedures and formulae. During the reflective interview, they demonstrated their knowledge of the procedure in determining the equation of the line parallel or perpendicular to a given line passing through a point. Significantly, Jasmine considered that Applet 3.5 was the most useful one for learning. The following transcript demonstrates the method (the substitution method in Applet 3.5) used to determine the equation of a parallel line. In determining the equation, Jasmine first identified the slope of the parallel line by equating it with the slope of the given line. She wrote the equation in *slope-intercept form* with the *y*-intercept as an unknown. The given point was substituted in the new equation to solve for the *y*-intercept. However, she could not solve it by any other method.

R: What you think you learnt most on that day or both the days. Which applet appealed you most?
***Jasmine**: The one that explained procedures in determining the equation of parallel or perpendicular lines. And it showed both the lines and it tells us the points through which the line passes.*
R: Ok. There is a line 2x+3. How will you determine the equation of parallel line passing through (2,-2)? (I write down on my pad)
***Jasmine**: Gradients are equal (1EC) m1 = m2. m1 = 2 (4AP) . Therefore, it's y = 2x+c (1EC) we have to find y-intercept. It passes through (2,-2)neh].* (scribbles on the pad - she substitutes into the equation) *(4AP)....*
*R: (*I interrupt her)*. Any other method?*
***Jasmine**: Not that I know of.*

In determining the gradient of a perpendicular line, Iris accurately used the formula to calculate the product of gradients. She then used the substitution method to determine the y-intercept of the required line as shown in the following transcript. Her conclusion on perpendicular lines in the worksheet in Figure 6.14 in Frame 6.3 (p. 195), also corroborated the numerical relationship of gradients.

R: Ok. How you will sketch a line perpendicular to $y = 2x - 3$ *passing through say (2,3)?*
***Iris**: The slope of original line and multiply the slope of the perpendicular line is negative -1, so* (uses calculator) *slope is -1 over 2 and put (2,3) into the equation. 1EC*
R: (Interrupting) *How did you calculate the gradient ? Explain once more*
***Iris**: I said that 2 times m equals -1. Therefore, m is -1 divided by 2. 4AP*

Furthermore, when asked to conjecture the angle of inclination of a positive gradient, Jasmine's immediate response was to determine the angle in the formula $m = tan\theta$ as shown in the transcript in Frame 6.8 on p. 203. She even made θ the subject of the formula, revealing her efficient and accurate knowledge of mathematical algorithms. Nonetheless, it was not effective in the given context, proving that techniques learned by rote do not promote mathematical proficiency (Kilpatrick et al., 2001). Procedural fluency is not confined to the competent execution of procedures; it also encompasses the awareness and selection of appropriate procedures in a given situation.

Aster, Jasmine and Iris used the formula in determining the gradient of a perpendicular line; Cypress, however, could not recollect procedures and formulae and revealed his inefficiency in performing even when the formula was known. Furthermore, he felt the burden of memorising the algebraic formula of a gradient between two points, as discussed in section 6.3.1.2.1 p. 192. His failed attempts are shown in Screenshots 6.1 and 6.2 on p. 194. Seemingly, he lost confidence when he could not successfully determine the gradient in Applet 3.3. Incorrect procedures and formulae did not allow Cypress to enhance his mathematical understanding of straight lines. My observation concurs with Kilpatrick et al. (2001) who maintain that "[w]ithout sufficient procedural fluency, students have trouble deepening their understanding of mathematical ideas or solving mathematics problems" (p. 122).

6.3.5 **Essence of interactions –BI-5 (Different approaches to solving a problem)**

6.3.5.1 **Synopsis**

Aster:

- reversed the processes of determining the *gradient of line* (MM and PF);
- attempted diverse methods of *sketching a line* in Applet 3.2 (MM and PF); and
- devised different approaches of determining *equations of parallel lines* (MM and PF).

Cypress:

- attempted diverse methods of *sketching a line* in Applet 3.2 (MM and PF).

Jasmine:

- reversed the processes of *determining the angle of inclination* (PF).

Iris:

- attempted diverse methods of *sketching a line* in Applet 3.2 (MM and PF).

6.3.5.2 **Discussion**

The BI-5 indicator was less prominent among learners. Generally, learners were not keen to solve problems using different approaches. In this section I consolidate different relevant mathematical concepts demonstrated by the participants under the following headings:

- Attempting different approaches to sketching a straight line
- Isolated instances of reverse train of thought

6.3.5.2.1 *Attempting different approaches to sketching a straight line*

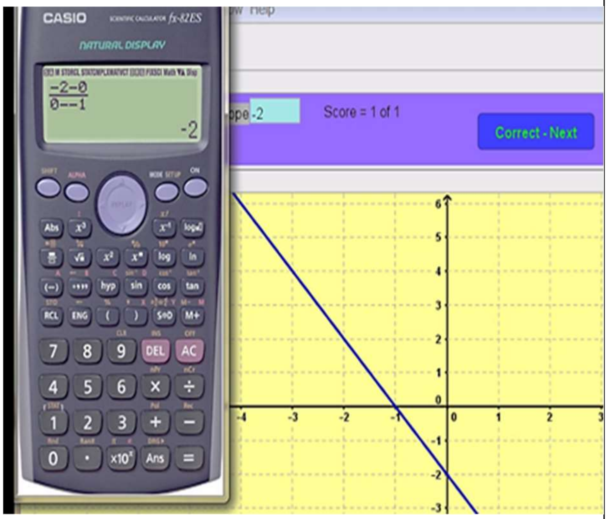
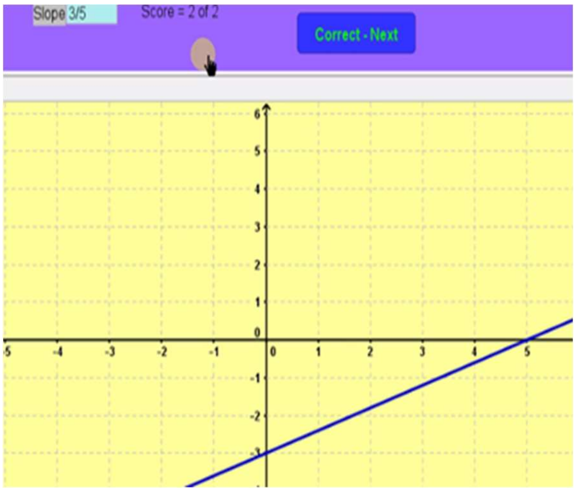
Carpenter and Lehrer (1999) ascertain that learning mathematics with understanding requires flexibility and involves the creation of rich, integrated knowledge structures. I discussed in section 6.3.3.2.3 (p.211) that learners explored and then discovered alternate methods of sketching straight lines while interacting with the Applet 3.2. When their favourite approach of the *dual-intercept method* posed problems in dragging points to non-integer values on the axis, they searched for alternate methods. Aster, Cypress and Iris in this context were flexible in their ability to discover new things and apply their previously acquired knowledge. Meanwhile, Jasmine adhered to the *dual-intercept procedure* and other irrelevant details and was thus not able to adopt alternate methods of sketching a straight line.

Furthermore, Aster's and Cypress' distinct and innovative methods of finding the *y*-intercepts of parallel lines passing through a point (discussed in the following section 6.4.2 on p. 232), demonstrated diverse trajectories in determining the equation of a parallel line.

Procedural flexibility encompasses knowledge of multiple ways to solve problems (Kilpatrick et al., 2001; Rittle-Johnson & Star, 2007), and is an integral part of mathematical proficiency.

6.3.5.2.2 *Isolated instances of reverse train of thought*

There was evidence of two isolated incidents of employing a reverse train of thought. Firstly, Aster used the '*rise over run'* concept in determining the gradient of a line in Applet 3.3 evidence of a reverse train of thought. Only in the first instance in Applet 3.3, Aster used the formula of gradient when given two points, and his answer was correct, shown in Screenshot 6.37 of Frame 6.21. He attempted the next question without using a calculator. He moved the mouse pointer over the *y*-axis to -1, -2 and -3, , and then moved over the *x*-axis 1 , 2, 3, 4 and 5; thereafter he entered the ratio $\frac{3}{5}$ as in Screenshot 6.38, Frame 6.21. It appeared that Aster identified the concept of *rise over run* to reveal the slope of the line. During the reflective interview, he disclosed that he first considered the positive or negative slope of the line and considered intercepts as changes from the origin in absolute terms, and then wrote the ratio (transcript in Frame 6.21). It shed light of the learner's proficiency in identifying the gradient of a line without recourse to formula. Significantly, the *rise over run concept* made sense to Aster. Procedural knowledge leads to greater conceptual understanding, especially when the relation between the formula and the underlying concepts becomes visible (Rittle-Johnson & Alibali, 1999).

<table>
<tr><td colspan="2">Classroom Observation</td></tr>
<tr><td>
</td><td>
</td></tr>
<tr><td>Screenshot 6.37: Aster using the formula on the calculator</td><td>Screenshot 6.38: Aster determining the slope as a change in y over x</td></tr>
<tr><td>Reflective Interview</td><td>R: How did you calculate the slope here? (showing the clipping of his interaction with Applet 3.3) You dropped using calculator after the first question?
Aster: I look at the line and see if it's positive gradient or negative gradient. I thought like y-intercepts and x-intercepts are changes from (0,0) so I counted from the origin. Then I wrote the ratio y over x and negative if negative gradient.</td></tr>
</table>

Frame 6.21: Aster's interaction with the Applet 3.3

Aster demonstrated another instance of a reverse train of thought when he determined the *y*-intercept of a line from a given point using the converse *gradient-intercept method* of sketching lines. In the direct method, a point on the line is determined from the intercept using the gradient concept, and in converse, the *y*-intercept is determined from a point using the gradient concept discussed in detail in section 6.4.2.3 on p. 233. Krutetskii (1976) characterised reversibility of thought operations as a distinctive ability for mathematically proficient learners.

6.3.6 **Essence of interactions –BI-6 (Applying previously learnt concepts)**

6.3.6.1 **Synopsis**

Aster:

- sketched *horizontal and vertical lines* (MM and PF);
- applied the *dual-intercept procedure* in sketching the line in Applet 3.2 (PF);
- applied the *properties of rational numbers* such as gradient of 2 is equivalent to $\frac{2}{1}$ and *a number divided by 0 is undefined* (PF);
- linked the *properties of equal fractions* such as $\frac{4}{3} = \frac{8}{6}$ (MM and CU); and
- employed the prerequisite knowledge of *making y the subject of the formula* (MM and PF).

Cypress:

- applied the *properties of rational numbers* such as gradient of 2 is equivalent to $\frac{2}{1}$ (PF);
- applied the *dual-intercept procedure* in sketching the line in applet 3.2 (PF); and
- identified the geometrical relationship of the e*quidistant property of parallel lines* (MM and CU).

Jasmine:

- sketched *horizontal and vertical lines* (MM and PF);
- applied the *dual-intercept procedure* in sketching the line in applet 3.2 (PF);
- applied the *properties of rational numbers* such as gradient of 2 is equivalent to $\frac{2}{1}$ and *a number divided by 0 is undefined* (PF);
- linked the *properties of equal fractions* such as $\frac{4}{3} = \frac{8}{6}$ (MM and CU); and
- employed the prerequisite knowledge of *making y the subject of the formula* (MM and PF).

Iris:

- sketched *horizontal and vertical lines* (MM and PF);
- applied the *dual-intercept procedure* in sketching the line in applet 3.2 (PF);
- employed the prerequisite knowledge of *making y the subject of the formula* (MM and PF).

6.3.6.2 **Discussion**

The participant learners relied heavily on previously learned concepts such as the sketching of a line, parallel lines, horizontal lines etc., while developing relevant mathematical concepts and procedures in determining equations of parallel and perpendicular lines. Carter et al. (2009) observe that learning new topics becomes easier when there is a network of previously learned concepts.

However, there was little evidence of linking the concept of reciprocals to the gradients of perpendicular lines.

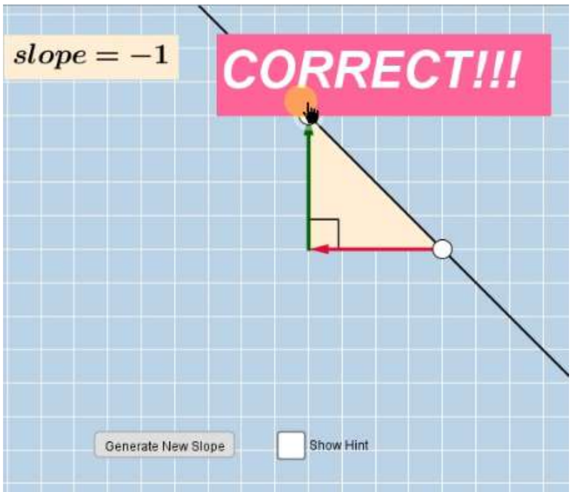
6.3.6.2.1 *Sketching and determining the equation of straight lines*

There are numerous basic concepts that the learners demonstrated while interacting with the applets in the third cycle. I discuss these concepts and procedures case by case in order to analyse those interactions where these concepts promoted or lacked mathematical proficiency in sketching straight lines and determining the equations of parallel and perpendicular lines.

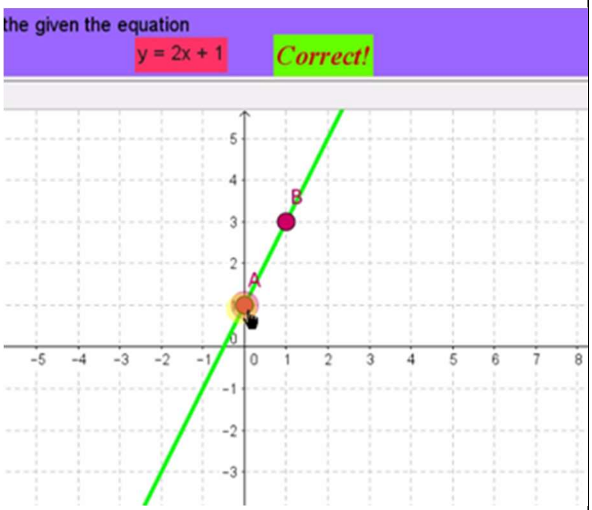
The case of Aster

Aster initially struggled to sketch the line in Applet 3.1, however, on his own he developed an understanding of the concept required to sketch the line. Aster's understanding of this concept was evident when he reversed the process of determining the gradient from a given line in Applet 3.3, which was discussed in Section 6.3.5.2.2 above. He also demonstrated his prior knowledge of number properties of rational numbers when he sketched the line of the gradient -1 in Screenshot 6.39 of Frame 6.22. It was evident that he could express an integer as a rational number and he then employed the idea of equivalent ratios in effectively sketching the line. Furthermore, Aster relied on properties of fractions (i.e. *a number divided by zero is undefined* and *zero divided by a number is zero*), in providing reasons for the gradients of lines parallel to the x-axis and the y-axis as shown in the transcript in Frame 6.22. In Applet 3.2, Aster employed the *dual-intercept method* and the *intercept-point method* of sketching the line as evident in Screenshot 6.40 in Frame 6.22. He was aware of angle properties of parallel lines and this paved the way for him in finding reasons behind invariant properties of parallel lines. Aster also flexibly switched between different forms of linear equations in order to determine the gradient of the line of $ax + by = c$ by translating it into $y = mx + c$ form. This prior knowledge was necessary for comparing the gradients of parallel and perpendicular lines with the given line.

Classroom observation



Screenshot 6.39: Aster applying equivalent fractions in Applet 3.1



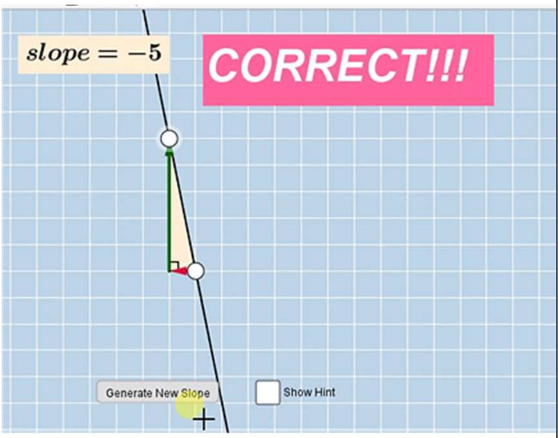
Screenshot 6.40: Aster employing the intercept-point method in sketching the line in Applet 3.2

Reflective Interview	*R: What's the gradient of line y = 9?* *Aster: It's a horizontal line and its slope is zero.* 6BP 1EC *R: What will be the gradient of its perpendicular line?* *Aster : You mean perpendicular to y =* *R: Sorry yes, gradient of perpendicular line y = 9?* *Aster: It's a vertical line x = 9 and gradient is undefined.* 6BP 1EC *R: Why you think the gradient of a vertical line is undefined* *Aster: Why it is undefined?.* *R: Why the line parallel to y-axis is having gradient undefined?* *Aster: Line parallel to y-axis …* (pause)… *in vertical line change in x2-x1 is zero, and any number divided by 0 is undefined* 6BP 2UVI

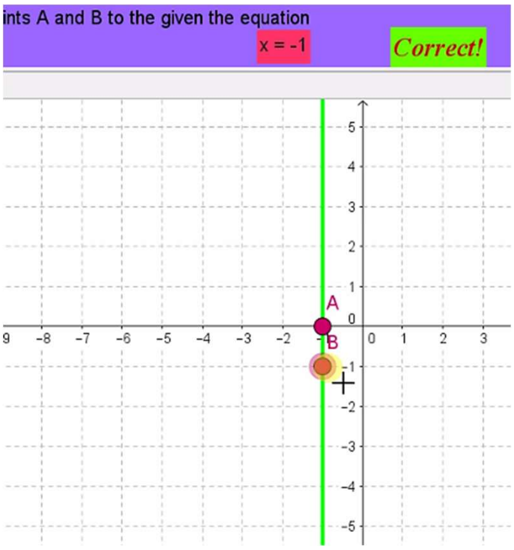
Frame 6.22: Aster applying prior knowledge

The case of Cypress

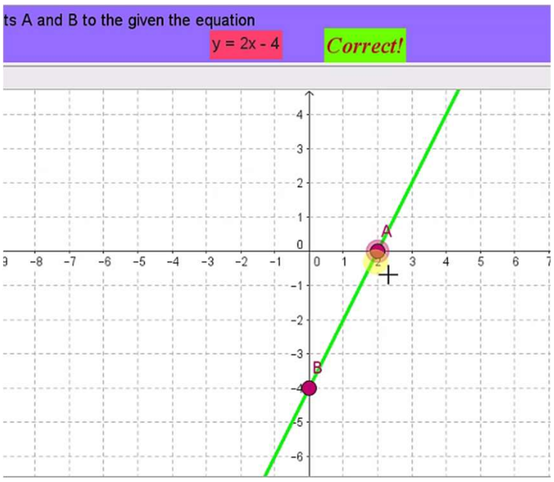
Cypress expressed his grasp of basic concepts and procedures in sketching a line, using the *rise over run* concept. As mentioned previously, he sketched almost eighteen straight lines in Applet 3.1, applying the basic concept of a gradient i.e. *rise over run*. In Screenshot 6.41 of Frame 6.23 he also demonstrates his knowledge of *turning an integer into a rational number* such as the integer '-5' expressed as a rational number. This knowledge was essential in developing the procedure for sketching a straight line using the *gradient-intercept method* in Applet 3.2, as shown in Screenshot 6.44, Frame 6.23. He displayed his knowledge and fluency in determining the intercepts of a straight line, and hence sketched a straight line using the *dual-intercept method* in Screenshot 6.43 of Frame 6.23. Nonetheless, he could not apply the gradient formula efficiently nor the *rise over run* concept in finding the gradient of a line in Applet 3.3. He was aware of the procedures to make a variable the subject of a formula (evident from determining the intercepts of a straight line equation), but he lacked awareness of the different forms of representing a linear equation. Subsequently he did not determine relationships that linked to gradients of perpendicular lines, because of the flawed gradient of the original line. Similarly, he could not identify the *gradient of vertical or horizontal lines*, even though he successfully sketched both horizontal and vertical lines (Screenshot 6.42 in Frame 6.23). Furthermore, he did not recognise the concept of a point on a given line. This prior knowledge was an important connection to finding the equation of a parallel line and Cypress' lack of this knowledge proved to be detrimental.



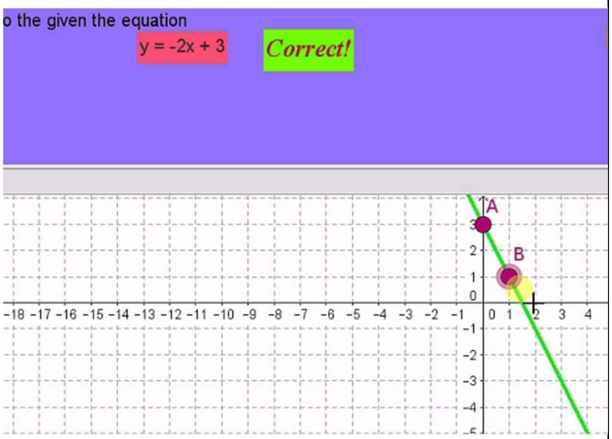
Screenshot 6.41: Cypress considered integer -5 as a fraction $-\frac{5}{1}$



Screenshot 6.42: Cypress sketched a vertical line



Screenshot 6.43: Cypress sketched a line using the dual-intercept method



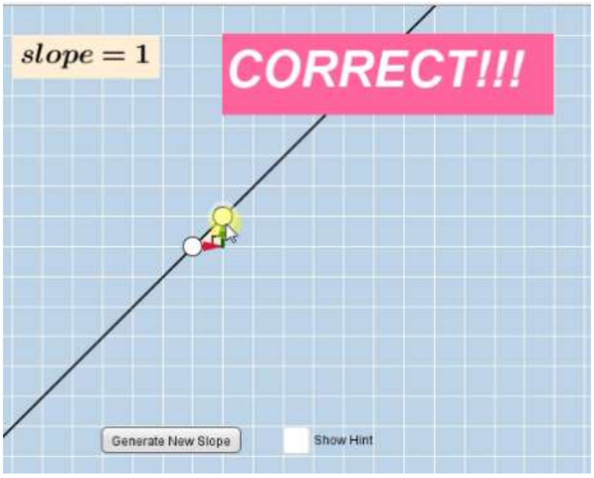
Screenshot 6.44: Cypress sketched a line using gradient-intercept method

Frame 6.23: Cypress' basic concepts and procedures he used to sketch a straight line

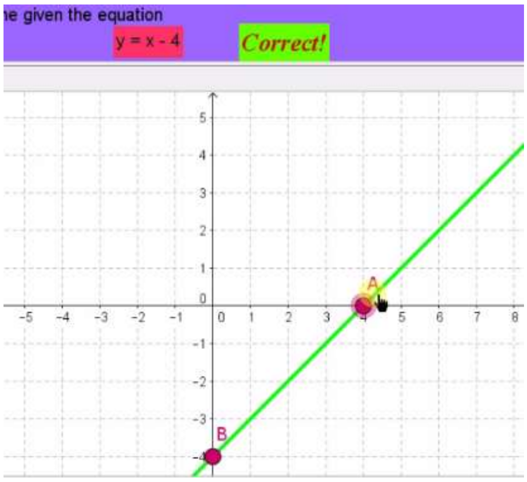
The case of Jasmine

Jasmine expressed her grasp of basic concepts and procedures for sketching a line, using the *rise over run* method. Screenshot 6.45 in Frame 6.24 demonstrates Jasmine using her prior knowledge of *representing an integer as a rational number*. She used her knowledge of sketching vertical and horizontal lines in Applet 3.2 using the *dual-intercept method* shown in Screenshot 6.46 in Frame 6.24, and displayed her proficiency in finding a point on the line using the substitution method, but she was not able to plot the point accurately in Applet 3.2. Instead of plotting a point at $(1; 4)$, she plotted two points, one on the x-axis at 1 and the other on the y-axis at 4. Her understanding of the concept of monotonic functions was evident when she linked positive and negative gradients to increasing or decreasing lines respectively. Jasmine displayed her proficient prior knowledge of *substituting a point of a line*, and this was crucial in her procedure for determining *equations of parallel and perpendicular lines*. Her prerequisite knowledge was also evident when she found reasons for the gradient of parallel lines to be equal, as shown in the discussion in section 6.3.1.2.2 on p. 194. She applied the idea that the ratio could be written in different ways while keeping the same value.

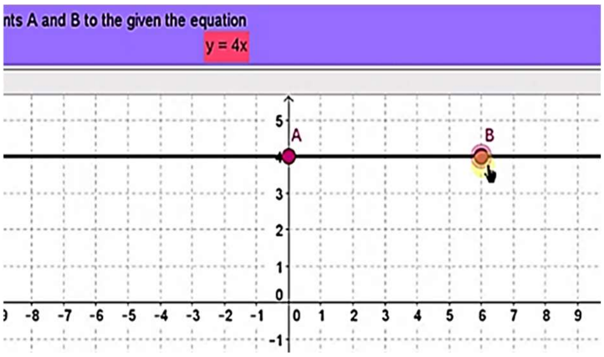
Classroom Observation



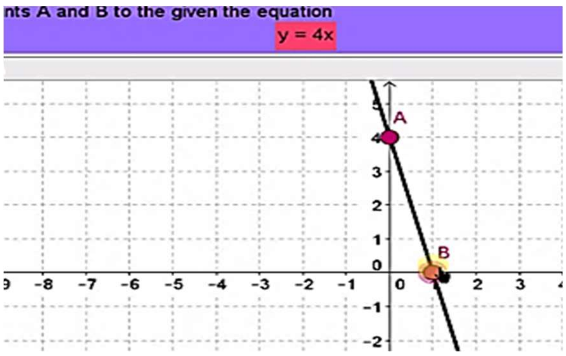
Screenshot 6.45: Jasmine sketching line of gradient 1



Screenshot 6.46: Jasmine using the dual-intercept method



Screenshot 6.47: Jasmine's unsuccessful attempt



Screenshot 6.48: Jasmine misinterpreted points on the axes

Jasmine draws the attention of Lilly:
Jasmine: (to Lilly) *It's correct. I am correct. I am correct. But the system is not showing correct.*
Lilly: Why you think that it's correct?
Jasmine: *Look when x= 1 y =4, the gradient of the line is 4. Yeah, I see it's not correct— it's negative.*
She then skips and clicks on 'new line'.

Frame 6.24: Jasmine's display of prior knowledge or lack thereof

The case of Iris

While interacting with Applet 3.1, Iris initially thought of using her previously acquired knowledge of gradient formula, but once she realised the irrelevance of gradient formula, she used the *rise over run* method to sketch the line in Applet 3.1. When Iris struggled to sketch a line of gradient '2', Cypress helped her to sketch it. Possibly, Iris did not understand the gradient method of sketching a straight line in Applet 3.2 because of her lack of fluency in *writing an integer as a rational number* in her prior knowledge. Nevertheless, Iris discovered another method — the *gradient-point method* in sketching a straight line — which she was evidently familiar with, from previous grades. She was able to employ her knowledge on the *dual-intercept method* in sketching the line, and also displayed fluency in graphing vertical and horizontal lines in Applet 3.2. She relied on her prior knowledge of the *gradient formula* in Applet 3.3 in order to find the gradient of a line. Apparently, Iris's understanding of various forms of straight line equations was incomplete, as evident from her response in the worksheet WS-2. However, it did not deter her from making deductions on gradients of parallel and perpendicular lines. Prior knowledge of concepts such as *a point of a line* was evident when she substituted the given point in order to determine the *y*-intercept of a parallel line or the perpendicular line of a given line shown in the discussion in Section 6.3.4.2.1 p. 216.

6.3.7 Essence of interactions –BI-7 (Reflecting on a solution – considering the reasonableness of a solution)

6.3.7.1 Synopsis

Aster:

- estimated the result of the calculation of the formula as in the case of determining the gradient of a given line (MM and PF).

Cypress:

- displayed little evidence of interpreting or estimating the solution.

Jasmine:

- displayed little evidence of interpreting or estimating the solution.

Iris:

- reflected on the solution provided by the DGS leading to the discovery of a procedure (*dual points*) in sketching a straight line in Applet 3.2

6.3.7.2 Discussion

There were fewer instances of the indicators BI-7, illustrating that the learners rarely reflected on their solutions.

6.3.7.2.1 *Estimating direction of line based on its gradient*

Iris encountered problems with Applet 3.2 when she could not place a point for the non-integer x-intercept value of $x = -\frac{1}{2}$ for the equation $y = 2x + 1$, as discussed in Section 6.3.3.2.3. Prior to her friend Cypress solving the problem, Iris estimated the direction of the line and told her friends that the line ought to be on the right hand side of the y-intercept at $y = 1$, shown in Screenshot 6.21 and the transcript in Frame 6.15. Apparently, her meaningful engagement with the previous Applet 3.1 in sketching the line for a given gradient enabled her to estimate the direction of the required line, if not the complete solution. Her own deduction on linking the positive or negative gradient of a line to its direction allowed her to reflect on her procedure while sketching a line.

Similarly, Jasmine sketched the line $y = 4x$ in Applet 3.2. She was confident about her solution (though flawed) and drew the attention of her friend Lilly. As she was talking with Lilly, she realised that the line that she had graphed was a decreasing line instead of an increasing line (see the transcript of her conversation with Lilly in Frame 6.24). She estimated the direction of the required line, but did not, however, further explore in sketching the line.

Aster also recognised the line in Applet 3.3 as having a positive or negative gradient based on its direction. Significantly, he estimated the positive or negative sign of the required gradient as depending upon the direction of the line. This estimation helped him to solve the gradient for a line without relying on the gradient formula, when two points were given. Kilpatrick et al. (2001) argue that learners with a sufficient knowledge base are able to make sense of things. I concur with Carter et al. (2009) that the ability to reason and being a good estimator are vital to becoming proficient in sketching straight lines.

6.4 EMERGING THEMES

Two major themes emerged from the learners' interaction in this third cycle of sketching straight-line functions, namely geometry of linear equations and self-proclaimed theories.

6.4.1 Geometry of linear equations

6.4.1.1 The reality of straight lines

Cypress and Iris visualised a line as an intersection of two faces of a box. There could not be a better way of linking line segments with real life examples. Cypress frequently brought geometrical ideas into the straight-line functions. He said that a straight line in the middle of a soccer field divided the field into two equal halves. He recognised a close relationship between geometry and reality as he saw a symmetrical line across a rectangular field. As Duval (2013) puts forward, the ability to structure real life situations is crucial for developing mathematical knowledge.

6.4.1.2 **The geometry of parallel lines**

Aster and Cypress considered the geometrical aspects when they were asked to provide variant and invariant properties of parallel lines, and Aster provided geometrical reasons for the invariance of gradients of parallel lines. He explained that parallel lines made equal angles of inclination with the x-axis, owing to the equality of corresponding angles of parallel lines. He then linked the gradient to the trigonometric ratio of the angle of inclination of the line. Aster evidently relied on his previous knowledge to explain the phenomena. The transcript in Frame 6.25 shows his explanation.

During the reflective interview, Cypress provided a rationale for the change of the y-intercepts. He concluded in worksheet WS-2 that parallel lines maintained a constant distance between them (Figure 6.18 in Frame 6.25), therefore, the y-intercepts of parallel lines also preserved the same distance (transcript in Frame 6.25). Cypress exemplified Arcavi's (2003) notion of vertical displacement of parallel lines resulting in a difference in y-intercepts.

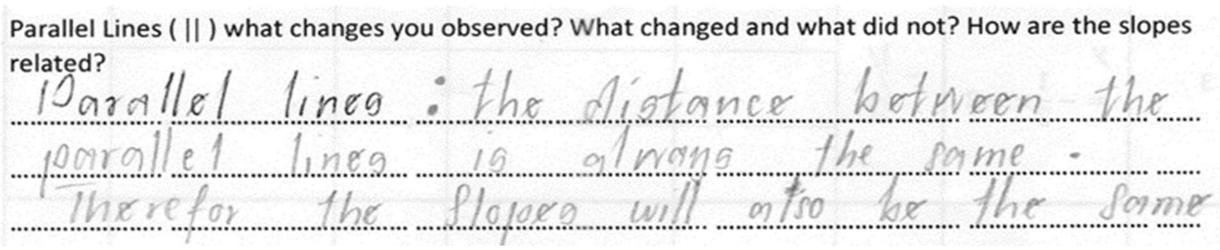


Figure 6.18: Cypress' conclusion on parallel lines

Reflective Interview with Cypress	*R: In class you asked [Iris] why the y-intercepts changed for parallel lines and gradients did not change? I am asking you the same question.* ***Cypress**: For parallel lines, they have a constant distance throughout. Therefore, the y-intercepts of parallel lines also maintained the same distance, and hence they changed. 2UVI 1EC*
Reflective Interview with Aster	*R: In your worksheet you have written that 'c' values changes but 'm' does not change for parallel lines. Why you think so?* *Aster:* (he draws axes and two straight lines). *I think they make equal angles with the x-axis. We know m = tanθ. 2MC These angles will be equal because the lines are parallel, 2MC 1EC and therefore gradients will be equal. 1MR* *R: what are these angles called, can you recollect?* *Aster: Yeah, corresponding angles are equal. 6LPK*

Frame 6.25: Reasoning variant and invariant properties of parallel lines

6.4.1.3 **The triangle syndrome - a bug**

In her reflective interview, Jasmine verbalised her notion of seeing a triangle when a point was plotted. In fact, Jasmine was overwhelmed by the image of a triangle even when it was not required. I have chosen the phrase '*triangle syndrome'*, to describe the challenges faced by learners. Jasmine reflected that she was fascinated by the triangle that formed when sketching the correct line in Applet 3.1. The significance of the triangle in Applet 3.1 (refer to Figure 6.1 on p.186), was to identify the horizontal and vertical changes and Jasmine discovered that these triangles formed below and above the line for increasing and decreasing lines respectively. As Jasmine first moved horizontally left, she observed that the triangle formed below the increasing line (see the transcript below); however, she did not realise that if she first moved horizontally right, because these horizontal movements towards the right were being made first the triangles would have been formed above the line.

She often encountered triangles as she interacted with the applets. When a straight line is sketched it forms a triangle with the axes, so in Applet 3.2, as Jasmine dragged the points, the straight line and the triangle changed dynamically, and her attention was focused on the triangle. The image of the triangle formed by the straight line with the axes dominated over other concepts, leading to wrong conclusions. The 'triangle syndrome' took its toll as Jasmine attempted to sketch the line $y = 4x$ in Screenshot 6.48 of Frame 6.24 (p. 227). For a point on the line $(1; 4)$, she plotted A at the y-intercept $(0; 4)$ and B at the x-intercept $(1; 0)$; however, she could plot a point on a graph paper as shown in the transcript below. Apparently, the visual rigidity of the 'triangle syndrome' hampered her from successfully sketching a straight line of the form $y = mx + c$ in DGS. This difficulty exemplifies Presmeg's (1992) description of vivid images relating to 'one-case concreteness' that prevents learners from finding solutions and generalisations. Consider that Jasmine experienced little difficulty in sketching the vertical and horizontal lines in Applet 3.2, since no triangles were formed.

R: What do you understand by increasing straight line? How did GeoGebra helped you to understand?
Jasmine*: I have noticed another thing in GeoGebra other day that for increasing line the right-angled triangle 2MC formed is below the line and for decreasing line the triangle will be above the line 2UVI*
….
R: How do you plot a point say (2;5)?
Jasmine*:* Attempts to draw a graph on the paper.
R: Don't draw just tell me the process that you use to plot I said (2;5)?
Jasmine*: Here x is 2 and y is 5 so I move 2 units right then five units up. 4EP A triangle is formed.*
R: Now draw the point (2;5)
Jasmine*:* (she plots the point accurately).
R: Where you see the triangle?
Jasmine*:* She now draws the triangle. One side is on the x-axis for 2 units, the second side is parallel to the *y*-axis; from *x*= 2 for 5 units and the third side point is joined with the origin.

6.4.2 **Self-proclaimed theories**

6.4.2.1 **All lines are increasing**

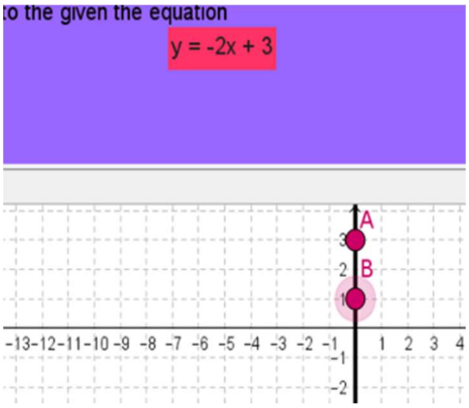
We have discussed interpretations of increasing and decreasing lines in Section 6.3.2.2.3 (p. 203). When the teacher asked the learners to distinguish between the lines with positive and negative slopes, Iris made an interesting observation on the lines, distinguishing positive and negative slopes. As per the teacher's instructions, she addressed the whole class and revealed her findings: "When the slope of the line is negative, it increases on the left. When the slope of the line is positive, it increases on the right" (see the transcript in Frame 6.26). For Iris, all lines were increasing irrespective of the positive or negative sign of the slopes. For her, a line with a positive slope (upwards) increases to the right of the *y*-axis (i.e. the graph increases from left to right), whilst a line with a negative slope (downwards) increases to the left of the *y*-axis(i.e. the graph increases from right to left). Furthermore, her observation had an impact on other learners, as we have seen earlier when Cypress made similar comments on increasing and decreasing lines.

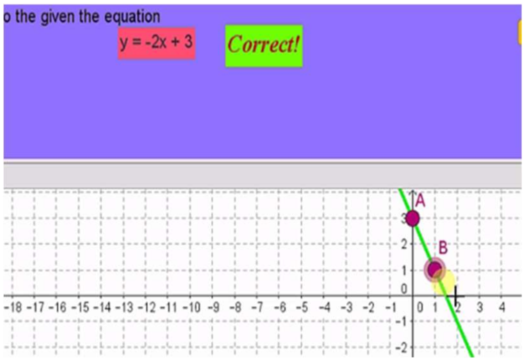
Classroom Observations	*Teacher: Now that there is something that I want you to tell me. What you observe when a gradient is negative and when a gradient positive throughout. When a slope is negative. Compare that with one with slope positive. What is the difference between those two lines?...* **Iris*:*** After a while, skips a few questions until she comes across a negative gradient. *When the slope of the line is negative, it increases on the left. When the slope of the line is positive, it increases on the right.2UVI*
Reflective interview	*R: What do you understand from positive and negative slopes?* ***Iris**: A positive slope is increasing to the right and negative slope is increasing to the left. Yes, I am listening to you now. When the line is increasing to the right then it's called increasing line. And for decreasing line, the line is increasing to the left. 2MC*
	R: Why you say it's an increasing line? ***Cypress**: Its slope is positive* *R: What kind of line is it?* ***Cypress**: What sort of line ...* (pauses) *a line increases to the right or to the left. Positive slope increases to the right, moving in upward direction 2UVI.*

Frame 6.26: Further interpretation of positive or negative gradients

6.4.2.2 **Cypress' 'gradient theory'**

It was during a discussion with his friends Iris and Pansy, that Cypress propounded his 'gradient theory' to end the stalemate they were in when the dual-intercept method failed in sketching a line in Applet 3.2, shown in the transcript in Frame 6.15, p. 212. When sketching the line $y = -2x + 3$, Cypress explained his thoughts and the method he employed, naming it as 'gradient theory' (see the transcript in Frame 6.27). Cypress brought the points A and B together at the y-intercept $y = 3$, moved point B two units down along the y-axis (for negative) and moved one unit right, to have B at $(1; 1)$ thus sketching the line successfully in Screenshot 6.49 and 6.50. In other words, Cypress' 'gradient theory' was the gradient-intercept method of sketching a straight line.

<table>
<tr>
<td>Classroom Observation</td>
<td>

Screenshot 6.49: Cypress attempting to sketch the line</td>
<td>

Screenshot 6.50: Cypress sketching a line using the gradient-intercept method</td>
</tr>
<tr>
<td>Reflective Interview Cypress</td>
<td colspan="2">R: Can you explain how did you sketch the line? [middle row figures above] and what is your gradient theory that you were talking about [with Iris]?
Cypress: I moved the point to 3 — the y-intercept. I was calculating x-intercept, but I know I could not plot between one and two. Then I remember of the slope that I did just before this one… I was a bit confused, but I thought slope has to be -2, … (pause)… a in the equation of a straight line (y= ax + b) It is about using gradient of the line to sketch the line.</td>
</tr>
</table>

Frame 6.27: Cypress' gradient theory

6.4.2.3 **Aster extending the 'gradient theory'**

Aster extended the *gradient theory* in sketching the line to determine equations of parallel or perpendicular lines. Indeed, he never used the phrase 'gradient theory' but applied the method to determine the y-intercept of a parallel line, as discussed in Section 6.3.3.2.4 on p.214. For instance (in Screenshot 6.30, p.215), he sketched the parallel line for $y = 2x + 3$ passing through A (2; 2) then during the reflective interview in the transcript in Table 6.2, his ideas were clearer and more straight-forward as he demonstrated his approach in sketching and then determining the equation of a line parallel to a given line $y = 3x - 4$, passing through $(2; 1)$. He first equated the gradient of the given line using the gradient properties of parallel lines, which was three in this case. From the given point $(2; 1)$ he moved left towards the y-axis by two units to touch the y-axis at $(0; 1)$; the number of units of horizontal movement being determined by the x-coordinate of the given point. The number of units of vertical movement along the y-axis was determined by the ratio of vertical movement over horizontal movement, i.e. the gradient. Therefore, in order to have the ratio of three, he moved six units down — a vertical movement along the y-axis from $(0; 1)$ to $(0; -5)$. Apparently, he applied his basic knowledge of equivalent ratios $\frac{6}{3} = \frac{2}{1}$. The terminating point on the y-axis would be the y-intercept of the required line transcript below. Thus, the equation of the required parallel line was $y = 3x - 5$. Nevertheless, this approach can be extended to sketch and determine the equation of a perpendicular line.

Table 6.2: Aster reflecting on sketching parallel lines

Reflective Interview with Aster
R: Let me show you this? (Applet 3.5 – attempt to sketch a line) *Can you recollect how you sketched those lines? What was in your mind?* ***Aster**: I wanted to draw a line passing though y-intercept at y =-2 and see that it passed through A(2;2)* (The video is paused with the parallel line and he refers to the graph on the screen as he speaks). *R: Why through point on the y-axis at y =-2. ?* ***Aster**: From A, I moved two units to the y-axis and then four units down which was y =-2. Now I realise that the parallel line at y =-2 will pass through 'A'.* *R: Can you determine the equation of a straight line parallel to y = 3x-4 passing through (2;1)?* ***Aster**: The gradients are equal, so I have y = 3x+c and I put (2;1) into the equation and c is -5.* *R: I want you to repeat the logic that you used now while drawing the parallel line?* ***Aster**: Can I use the graph?* (I opened Applet 3.5 for him. He types in 3x-4 and drags the point to (2;1) he uses the mouse to make movements – two units left at y-axis (1;0) and then six units down arrive at -5*It's y =3x-5.* *R: Which method you prefer?* ***Aster**: If there's graph this method is easy, but if there is no graph then it becomes complicated, so I prefer substitution method.*

6.4.2.4 **Equidistant method**

As discussed earlier, Cypress often mentioned the geometrical perspective of straight-line equations. He made an inadequate attempt to determine the equation of a parallel line. During reflective interview in , when I asked him to determine the equation of a parallel line to $y = 2x + 3$ passing through (2;2), he identified and applied the property of *equal gradients of parallel lines* as shown in the transcript in Table 6.3. Like Aster, Cypress also wanted to sketch the line passing through the given point. The geometrical concept of equal distance between parallel lines was dominant in his thoughts; therefore, he wanted to draw a line that preserved the equal distances between them. Remember, Cypress made similar conclusions on parallel lines; refer to Figure 6.18 p.230. However, he was not successful in sketching the line.

Significantly, Cypress applied the concept of *equidistant property of parallel lines*. If we calculate the point on the line at $x = 2$, it is $(2; 7)$ (A point on the line satisfies the equation, therefore $y = 2 \times 2 + 3 = 7$). The vertical distance between the point (2;2) and the line is five units (the difference in y-coordinates). Since the lines are parallel the distance in y-

intercepts is also five units, hence five units down from 3 is -2. Thus, the equation of the required line is $y = 2x - 2$. Seemingly, he could conceive the idea of displacement of the *y*-intercepts for parallel lines, but he lacked the concept or procedure in determining a point on a line. He required further intervention to complete the procedure in sketching the line and the equation of the parallel line.

Table 6.3: Cypress explaining the equidistant method in sketching parallel lines

Reflective Interview
R: We have a line y = 2x+3. Find the equation of a line parallel to the given line passing through (2;2) ***Cypress**: Equation is* $y = 2x + 3$. *For parallel lines, the distances between the parallel lines are equal throughout. ....* [long pause] *1EC* *R: A straight line has a gradient and a y-intercept so What's the gradient of the parallel line?* ***Cypress**: For parallel line the gradients are equal so it will be 2 1EC* *R: And how will you determine the y-intercept* ***Cypress**: y-intercept ...* *R: Ok my original line is y = 2x+ 3 I want the equation of line passing through (2;2).* ***Cypress**: I want to sketch a line.* *R: Ok doesn't matter. You can draw a line.* ***Cypress**: I want to draw a parallel line so that the distances are equal throughout. if I can draw such a line -through (2; 2) then it will intersect at y-axis (he gestures the line parallel line through the point).*

6.5 ANSWERING THE RESEARCH QUESTIONS FOR THE THIRD CYCLE

- How are *GeoGebra* applets used as a learning tool to make mathematical meaning and develop understanding by selected Grade 11 learners?

In Applet 3.1, participants employed different approaches to sketch the line for the given gradient, and the summary of these approaches is captured in on Table 6.1 (p.208). Despite all the participant learners being able to identify the directions of lines as having positive and negative gradients, their mathematical definitions, although conceptually appropriate, were different. Jasmine her continuous engagement with the applet, led her to internalise the concept of increasing and decreasing straight line functions as she linked the concept with positive or negative gradients. Meanwhile, Iris considered the '*all lines are increasing*' maxim either to the right or left, depending on the positive and negative gradients of the line. The inquisitive Aster attempted every permutation and combination of points in order to sketch the given gradient of a line and leading to a comprehensive idea of gradient. Cypress only attempted an alternate method after eighteen correct attempts using his initial method of sketching lines. The learners also exhibited an interplay of conceptual and procedural knowledge as they reflected and interpreted the reasonableness of their solutions under uncertain situations. Iris distinguished between 'increasing to the right' and 'increasing to the left' lines and this allowed her to identify the direction of the required line when she was unable to sketch it in Applet 3.2 recorded in the discussion in Section 6.3.3.2.3 on p. 212. Similarly, while sketching the line $y = 4x$ in Applet 3.2, Jasmine realised her mistake of relying on her understanding of the direction of a line of a positive gradient (see the discussion in section 6.3.6.2.1 on p. 228). My findings corroborate the recent research work of Jaworski et al. (2015) who confirm that DGS can engage students in deeper understanding of mathematical meanings.

In sketching the line of standard form in Applet 3.2, all participant learners exhibited a preference to use the *dual-intercept method*. However, when the method failed, Jasmine was at loss and could not progress, whereas Iris and Aster discovered the *substitution method* in determining the second point of the line (the other point being the *y*-intercept) and Cypress explored and discovered the technique of sketching the line using the *gradient-intercept method*, or the so called '*gradient theory*'. He could not, however, explain to his peers his self-proclaimed '*gradient theory*', as discussed in Section 6.4.2.2 on p. 233. Thus, Applet 3.2 provided an opportunity for the participant learners to enhance their conceptual and procedural knowledge of sketching straight lines using different methods. Mathematical knowledge is enhanced when learners encounter multiple procedures, as Kilpatrick et al. (2001, p. 110) observe "students should have experience with other useful interpretations and representations, which also are important parts of the content."

Notwithstanding this enhanced learning, Jasmine was bogged down by the dynamic movement of lines and triangles in the applet described as '*triangle syndrome*' and discussed in Section 6.4.1.3 on p. 231. She could nevertheless successfully sketch a line using paper and pencil.

Interestingly, all the participants except for Aster started with determining the gradient in coordinate form and the intercepts of the line graph in Applet 3.3. Owing to feedback capability of the applet, they realised their mistakes, , and proceeded to use the *gradient formula*. As she progressed from one question to other, Jasmine appeared to employ her own generalised and succinct methods to determine the gradient without using the formula, demonstrating an instance of procedural fluency leading to conceptual understanding. Meanwhile, Aster switched from the formula to use the *rise over run* method to identify the gradient of the line. This discussion is in Section 6.3.5.2.2 on p. 221. Therefore, DGS with its feedback and visual features facilitated the participants' development of deep understanding. I argue that learning mathematics using technology builds knowledge.

Cypress grappled with the formula while determining the gradient in Applet 3.3 because he incorrectly applied the formula of gradient as $\frac{x_2-x_1}{y_2-y_1}$; and in another instance he mixed up the substitution of coordinates. Skemp (1989) argues that instrumental learning involves the remembering of rules and formula as isolated facts and procedures. When the emphasis is on procedures only, it is unlikely to encapsulate the concept (Kilpatrick et al., 2001).

Applet 3.5 explained the step by step procedure to determine the value of c in equations of parallel and perpendicular lines passing through a given point, of the form $y = mx + c$.. All participants, except Cypress, determined m by using the gradient properties of parallel and perpendicular lines. To find c and the *y*-intercept, Iris and Jasmine employed the *substitution method* of substituting the point into the equation, as adopted in the applet. Nonetheless, Aster and Cypress came up with their own methods of sketching the required line. In Applet 3.5, Aster made numerous attempts before he successfully implemented the '*extended gradient theory'* for sketching and determining the *y*-intercept of a parallel line; Cypress, in the meantime, made geometrical links to real life examples of vertical displacement in order to determine the *y*-intercept of a parallel line. . The dynamic movement of parallel lines in Applet 3.4 possibly suggest the idea of vertical displacement, thus providing reasons for the change of the *y*-intercepts in parallel lines. However, he was not successful in determining the equations of a parallel line. Cypress was, in fact hampered by another basic concept — *a point satisfying the equation of a line*. His approach cannot be extended to perpendicular lines.

I argue that when the participant learners critically engaged with the applet, they constructed new meanings and approaches in problem solving. *GeoGebra* provided a dynamic visual representation of straight lines, enabling the learners to construct their own meaning and verify their self-proclaimed theories. The applets provided an opportunity for the learners to deepen their conceptual understanding of the concept of gradients as they applied their knowledge in various contexts. My findings agree with Hollenbrands (2007) who found in his empirical study that technological environments like DGS provide more opportunities for students to engage in abstract mathematical concepts. Furthermore, I concur with the constructivist perspective that when learners take ownership of their learning, making sense and understanding is developed through the creation of knowledge.

Interactions with Applet 3.4 instigated thought-provoking conversation between learners. For instance, Aster and Jasmine debated the issue of overlapping lines in Frame 6.8 on pg. 210, with Aster using the term 'overlapping' instead of the mathematical term 'coincident'.

Significantly, Aster discovered the third category of a pair of straight lines that are neither parallel nor intersecting all by himself. Meanwhile, Cypress searched for reasons for variance of the *y*-intercepts of parallel lines and asked Iris, "*Why did it happen?*" He reflected over the situation and came up with an answer to his own question during the reflective interview in Section 6.4.1.2 on p. 230. He associated this with his prior knowledge on geometrical characteristics of equal distances of parallel lines. Researchers (Kilpatrick et al., 2001; Hiebert & Lefevre, 1986) have stipulated that knowledge grows with understanding when individual facts and propositions are connected with existing knowledge.

- What visualisation role can *GeoGebra* play in the learning of Grade 11 mathematics?

The learners frequently visualised gradients as having an increasing or decreasing function. Jasmine visualised a decreasing line as the line moving downward, for the horizontal movement from left to right. However, Iris viewed the decreasing line as increasing to the left of the *y*-axis. As discussed earlier in Section 6.4.2 on p. 232, Iris and Cypress considered all lines to be increasing. In order to determine the direction of the line for positive and negative gradients, Iris's starting point was the origin of the axes. She then moved right or left of the *y*-axis. Jasmine's starting point was from the left of the axes and moved to the right. Iris visualised the *y*-axis as the line of reference of change while Jasmine visualised the *x*-axis as the line of reference of change, hence the distinction between their interpretations on the lines with positive or negative gradients. Presmeg (2014) documents that visualisation enhances learning as connections are made to related facts and methods in appropriate ways.

Duval (1999) observed that the sketching of straight lines using a method of *table of points* does not allow learners to visualise the properties of a straight line. Instead, he suggests that the *dual-intercept method* of sketching a line enables learners to visualise its salient characteristics such as gradient and intercepts. However, the *dual-intercept* procedure was meaningless for the participant learners. It was simply reduced to a sub-set of the *table of points procedure*. All participants expressed their knowledge of *dual-intercepts*, apparently not derived from visualisation processes but rather considered as a procedural process in sketching a straight line.

The participants were inclined to use the *gradient formula* to determine the gradient in Applet 3.3. They could fluently sketch the line for a given gradient in Applet 3.1, however, they (except Aster) could not visualise the gradient of a given line in Applet 3.3. Aster was able to reverse the *rise over run* concept in determining the gradient of a line. Significantly, he visualised the triangle of slope with changes from the original. This is evident in the discussion in section 6.3.5.2.2 on p. 221.

It was intriguing to observe that Cypress rationalised the change in the y-intercepts of parallel lines as being dependent on the *equidistant property of parallel lines*, as discussed in Section 6.4.2.4. on p. 234. Here, Cypress exhibited the characteristics of Krutetskii's (1976) 'geometric thinker', as he utilised visual techniques for problem solving. In determining the equation of a parallel line passing through a point, Cypress applied the concept of *equal distances between parallel lines* but could nonetheless not completely sketch the line as he did not know how to determine a point on the line. Apparently, the visual image of equal distances between the parallel lines was overwhelming for Cypress. His lack of knowledge in determining a point on the line hampered his ability to establish the equation of a parallel line. In this instance, Cypress was disposed to use visualisation skills, but this alone was insufficient to find the equation of a parallel line. The basic idea of the *point of a line* proved to be equally important in this context.

Jasmine, as mentioned, was afflicted by 'triangle syndrome', as discussed in Section 6.4.1.3 on p. 231. When sketching a straight line in Applet 3.2, she misinterpreted and plotted the point $(a; b)$ as the x-intercept at $x = a$ and the y-intercept at $y = b$. Jasmine evidently visualised a triangle formed by the axes and the line joining the intercepts — quite irrelevant in the given context. Jasmine and Cypress exhibited what Presmeg (1986b) identifies as an inflexibility in thinking that prevents visual learners from successful problem solving. Their minds were " "riveted" to visual-pictorial schemes", as put forward by Krutetskii (1976, p. 325), possibly creating an obstacle to developing conceptual understanding. Bishop (1988) also recognises this inflexibility in spatial thinking and called it 'geometrical rigidity'. This sort of rigidity is often encountered when dealing with visual strategies in problem solving.

The DGS applets allowed learners to engage with and control the inherent actions of the objects by plotting, clicking and drag lines and points on the computer screen. The learners utilised the opportunity of enhancing their knowledge as they interacted with the applets. The strength of DGS visualisations i.e. preserving the invariant relationships in the visual representation when dragging the representation, allowed learners to experience the invariance when exploring the dynamic representations. As Kaput (1992) maintains, continuous transitions inherent in dynamic systems provide opportunities to make conjectures and to verify hypotheses. Furthermore, the learners exhibited what Rivera (2011) prescribes — that dynamic features of computer-generated concrete objects can help "obtain fresh insights concerning complex and poorly understood mathematical objects" (p. 80).

When we make connections between mathematical topics we employ visualisation (Presmeg, 2006). Making connections in mathematics classroom is one of the cornerstones of mathematical proficiency. Schäfer (2016) emphasises that connection making entails linking to prior knowledge and connecting between different topics. All the learners had identified that for parallel lines, gradients remained invariant and the *y*-intercepts changed. As they simultaneously manipulated the applets and jotted down their observations, they could visualise that the gradients of parallel lines did not change. It was relatively easy to visualise the obvious equality of gradients of parallel lines, but to identify the negative reciprocal of a rational number was a difficult task for learners. No participant learner identified the visual or functional relationship between the gradients of perpendicular lines. However, Iris, when prompted by teacher, identified the numerical and constant product of gradients as a negative one, although this was a algebraic, formula-driven deduction rather than a visual, intuitive one. Applet 3.4 provided graphical representation of perpendicular lines that changed dynamically as the point was dragged or the original line was changed. Therefore, the learners could only identify the directions of perpendicular lines. On his worksheet, Aster noted "*I have also noticed that if a line is decreasing then its perpendicular line will be increasing*'". Despite this, the learners were aware of the mathematical formula for gradients of perpendicular lines. One could argue that the learners had identified the relevant mathematical idea that the product of gradients of perpendicular lines is -1. But my contention here is that they did not visualise that relationship between the gradients while interacting with applet 3.4. It seemed to be difficult for the learners to visualise the product of two numbers as -1. They could not see on their worksheets how gradients of perpendicular lines were related as a rational number and its negative reciprocal. There could be another possibility — that there was a lack of fluency in converting a rational number to its negative reciprocal. Apparently, the analytical method of association of gradients was predominant in their minds and deterred them from visualising the functional relationship.

6.6 CONCLUSION OF THE CHAPTER

This chapter dealt with the data presentation and an in-depth analysis of the participant learners' classroom interactions with the applets on equations of parallel and perpendicular lines, as well as insights gleaned from the reflective interviews with the individual learners. In general, a detailed analysis of their screen recordings, individually and in relation to each other, indicated that there were resemblances and differences in their conceptual and procedural knowledge. It is evident that the participant learners engaged in higher order thinking in conjecturing, analysing and synthesising their own theories as they interacted with the applets. Furthermore, as the analysis unfolded, it showed that the *GeoGebra* enabled learning. It is an effective strategy to improve the understanding of concepts and resolving problems using the visualisation capabilities inherent in this software.

Notwithstanding the above-mentioned important implications, the results also show that the learners' prerequisite knowledge was substantial in developing further concepts in mathematics. Significantly, a lack of proficiency in number systems such as equivalent ratios caused hindrances in adopting multiple methods of determining the equations of lines. Learners faced difficulties with this topic owing to a visual rigidity of images which were irrelevant and inappropriate in the context. This hindered the understanding of concepts.

The dynamic representation of perpendicular lines and the worksheet proved to be insufficient to aid visualisation of the numerical association of gradients of perpendicular lines. A feasible way to develop understanding of the relationship of gradients of lines could be the development of an applet with a dynamic line and its rotated line at 90° as the line is dynamically changed for different gradients. From the perspective of teaching, and with the intention of improving and incorporating visual methods, I recommend that teachers should teach the concept of reciprocals of rational numbers before they engage learners in exploring and discovering relationships between the gradients of parallel and perpendicular lines. More significantly, the designing of applets and planning of lessons require an overhaul in order to investigate whether learners can make sense of the change of gradients of perpendicular lines.

Furthermore, the analysis of learners' interactions with the applets and their responses during the reflective sessions identified broad themes of interesting facets that are important to consider in relation to the use of DGS in mathematics classes. In the following chapter, I recommend some ways of using DGS in mathematics classrooms as I summarise and reflect on the all the GLIP cycles.

CHAPTER 7

CONCLUSION AND IMPLICATIONS

7.1 INTRODUCTION

In this chapter, I consolidate the findings of this research project with special reference to my research questions. I briefly summarise key findings from the analysis of data, emphasise the significance of this study based on the research perspective, examine limitations of this research and finally explore future research opportunities.

7.2 REVIEW OF THE RESEARCH OBJECTIVES

In chapter 2, I unpacked the four integral components that pertain to this study as illustrated in Figure 7.1. The fundamental conceptual foundation that underpinned the research was *visualisation*. I examined the role of technology, in particular DGS and its effective use in mathematics. The key concepts of Kilpatrick et al.'s *mathematical proficiency* and Carter et al.'s *meaning-making* habits were discussed in parallel. This informed and led to the development of my analytical tool with which I interrogated my data. Finally, the theory of *constructivism* was discussed within the context of visualisation in a technological classroom and how it framed the entire study. Thus, the four central pillars that constituted the conceptual framing of this research were: visualisation, mathematical proficiency and meaning-making, and integration of technology. The theoretical framing was constructivism.

Furthermore, I established the inter-relationships between and among these perspectives. A review of the relevant literature revealed the implications of ICT in the learning of mathematics. However, within the context of DGS, little empirical research focusing on actual classrooms integrating technological tools in the teaching and learning of mathematics, seems to have been carried out. The emphasis was thus to gain insight into how the visualisation capabilities of DGS develop mathematical proficiency and how learners make sense of mathematics.

In the context of learning mathematics in Grade 11 and using technology in two selected schools in Mthatha, the research addressed the following:

- how *GeoGebra* applets were used as a learning tool by selected Grade 11 learners, and
- the role that visualisation inherent in *GeoGebra* played in the learning of mathematics by these learners.

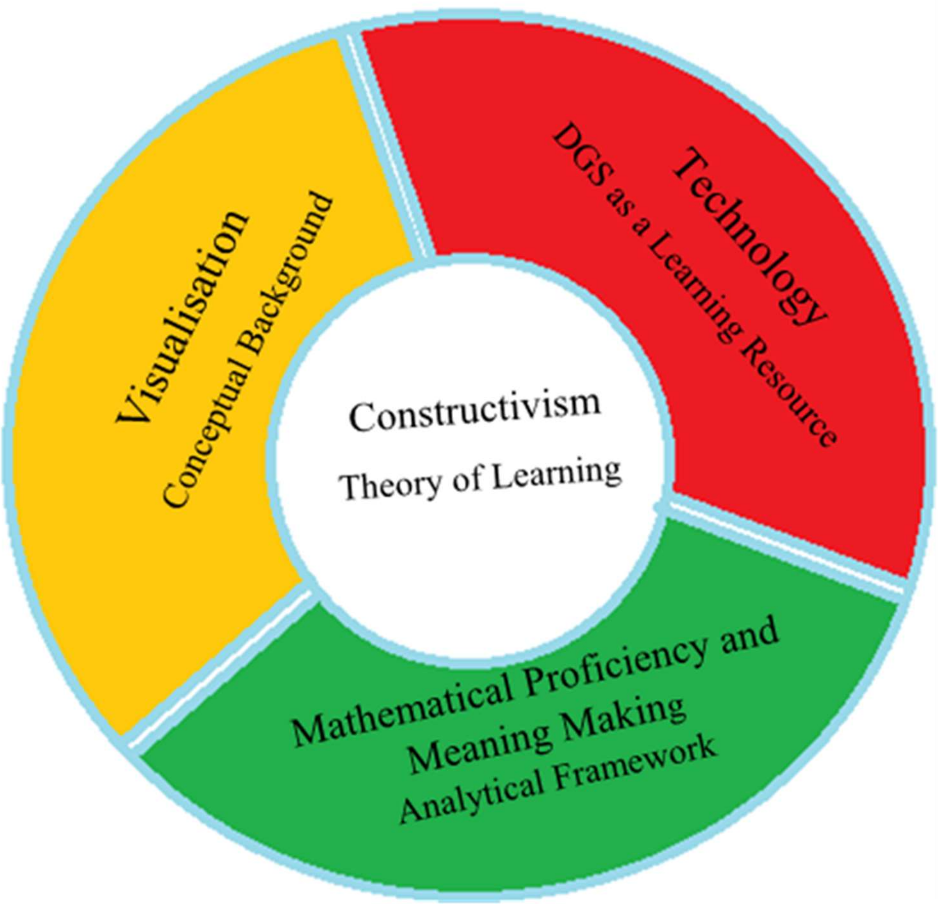


Figure 7.1: Four perspectives of my research — revisited

7.2.1 **My research method revisited**

A qualitative research approach was adopted in this research project. The interpretative research paradigm underpinned this study as it cohered well with the nature of the research enquiry. This interpretive research project was designed as a case study of selected learners' engagement with specific *GeoGebra* applets in learning mathematical concepts. The case here was a group of selected Grade 11 learners from two schools, who had been taught by GLIP teachers. My unit of analysis was the learners' interaction with the said applets. I gathered data in two stages from the participating learners. In total I had four sets of data. A variety of data collection techniques or tools were utilised during the process, namely audio-video recorded classroom observations, screen recordings of learners' computers, stimulated recall reflective interviews and learning artefacts. The data collection schedule was however, determined by the GLIP cycles. GLIP was the intervention programme that the teachers of my participants participated in. It was in the GLIP sessions that the teachers developed the applets that my participants engaged with.

7.3 SUMMARY OF THE FINDINGS

An analytical framework derived from the works of Kilpatrick et al. (2001) and Carter et al. (2009) guided and informed the data analysis of the various datasets. The details of the coding structure, VAMMPA, was discussed in Section 3.5.2. VAMMPA enabled me to reorganise my data and extract information to address the research questions. I presented vertical (participant by participant) and horizontal (across the participants) analyses of the three cycles in GLIP in Chapter 4, 5 and 6. In these three chapters, I organised and discussed the findings from the three cycles of GLIP with regard the research questions.

This summary of the findings is presented to respond to the research questions *how learning took place using GeoGebra applets* and the *role of visualisation using GeoGebra.*

7.3.1 How learning took place using *GeoGebra* applets

In this respect, I discussed how prior knowledge was recalled by the learners as they interacted with the applets. I discovered that exploration of, or manipulation on an object in the form of applets was a key activity in the lessons. This exploration prompted the learners to offer self-proclaimed theories and to develop mathematical concepts and meaning-making. I discussed how *GeoGebra* aided learners to develop procedures to solve problems.

Prior and connected knowledge

A key finding of the research was that, across the topics, relevant prior knowledge was brought to mind by the learners as they interacted with the mathematical activities during the lessons. As I focussed on individual participants, I found that learners made sense of mathematics and developed an understanding of concepts and procedures based on their previous mathematical knowledge and experiences. This research unpacked the specific mathematical knowledge required for the specific Grade 11 topics, namely *circle geometry*, *quadratic equations and inequalities*, and *equations of straight lines*.

In **circle geometry**, the participants exhibited their understanding and application of the fundamental concepts in geometry. For instance, Aster and Daisy combined the *exterior angle of a triangle theorem* and the *properties of an isosceles triangle* to deconstruct the *angle at the centre theorem*. I found that the learners applied the following properties and theorems during the lesson: *angles at a revolution*; *adjacent angles on a straight line*; and properties of triangles such as the *sum of angles in a triangle*, the *exterior angle of a triangle theorem* and *properties of an isosceles triangle*. I suggest that fluency in the above list of properties is crucial to moving up through the grades of high school geometry. When prior knowledge is applied strategically and effectively, it enhances conceptual development in circle geometry.

Whilst solving **quadratic inequalities**, the participating learners used previously learned concepts, such as *identifying constants and coefficients of variables* in quadratic expressions, procedures of *solving a quadratic equation* and an awareness of *graphical representation of a quadratic function*. In spite of their mathematically accurate approach to problem solving, the learners' conclusions on inequality, especially from the graph, were inappropriate. It seemed that the learners lacked the knowledge of different notations of inequalities, such as *algebraic notation*, *number line notation* and *interval notation*. I found that the applets used in the lessons did not directly address their lack of understanding of the *real number line* or *properties of real numbers*. Furthermore, there were difficulties in understanding basic definitions and distinctions of different types of number systems. Learners could thus not understand and convincingly conclude the *nature of roots of quadratic equations*.

In **equations of straight lines**, prior knowledge of the following was a prerequisite: the *number properties of rational numbers;* different forms of *linear equations;* fluency in *rearranging formulae of equations;* and the salient properties of a straight line such as *intercepts with the axes* of a straight line graph, the concept of *correspondence or assignment between two variables* in a straight line equation, *angle properties of parallel lines* and the *gradient of a line* as a formula or as a concept of change in y over change in x. It was evident from the analysis of the interactions with the applets that the learners enhanced their knowledge of *gradient of a line* from a routine formula to the concept of *'rise over run'*. Nonetheless, not all participants applied the concept of *rise over run* in sketching the straight lines. In particular, Cypress could not successfully apply the *gradient-point method* of determining the equations of parallel lines, owing to his inability to connect to the concept of the *correspondence between* x *and* y in a function. This linking of concepts was a necessity in finding the equation of a parallel line.

In sum, the research findings reiterate other researchers' (Kilpatrick et al., 2001; Hiebert & Lefevre,1986) observations that knowledge grows with understanding when individual facts and propositions are connected with existing knowledge.

Exploration

The participants generally engaged enthusiastically and critically in exploring and manipulating the given applets.

In **circle geometry**, *GeoGebra* with its dynamic possibilities enabled the learners to focus their attention on continuous variations of angles as they dragged the vertex of the angle to different positions on the computer screen. This dynamic way of exploring and making inferences would not have been possible in a static pen and paper environment. The participants were engrossed in their own constructions, reflecting and experimenting with their conjectures. Their engagement and exploration with *GeoGebra* allowed them, for instance, to make different conjectures about the relationships between *angles subtended by an arc at the centre and outside the circle*. Rosy argued that the *angle at the centre* was not double the angle subtended inside or outside the circle, while Jasmine observed an inverse proportional relationship between angles inside and outside the circle as the *angle subtended by an arc* becomes smaller in size as it moves away from the centre. Their manipulations with the geometric objects therefore lead them to conceptualise the circle geometry theorems.

In **quadratic inequalities**, *GeoGebra* provided an opportunity for the participants to conjecture whether their conjectures about different equations and inequalities were valid or not. I observed that the dynamic graphic representations inherent in the software motivated participants to reflect in ways that would have been difficult to achieve with traditional static techniques alone. Thus, while solving inequalities, it was evident that the participants' solutions were influenced by the dynamic nature of the software, thus reinforcing Hoyles and Noss' (2003) observation that the learners' thinking was shaped by the artefacts they used. The findings also reiterate Kieran and Drijvers' (2006) view that the benefit of dynamic multiple representations is in making algebraic objects such as expressions and equations more real and meaningful to students.

In **sketching straight lines**, the findings suggest that the applets provided an opportunity for the participant learners to enhance their conceptual and procedural knowledge. I argue that when the participant learners critically engaged with the applets, they constructed new meanings and approaches to problem solving for themselves and by themselves. The learners deepened their conceptual understanding of the *concept of gradients* as they applied their knowledge in various contexts. Significantly, after numerous failed attempts, Aster successfully came up with his own method of determining the equation of a parallel line. Iris and Cypress, when faced with challenges using the *dual-intercept method,* also devised their own distinct methods of sketching straight lines. Cypress discovered the *slope-intercept method,* while Iris came up with the *intercept-point method* to sketch lines by herself. These observations align with the research reports by Stols and Kriek (2011) who identify the benefits of the software as creating opportunities for creative thinking. My participants also employed different approaches as they indulged in exploring and developing alternate methods of sketching straight lines.

In general, the exploration possibilities in *GeoGebra* promoted self-learning opportunities as the participant learners often offered their own theories, examples and procedures. For instance, by herself Jasmine came up with the inverse proportional relationship between the *angle at the centre of a circle* and the *angles subtended by the same arc inside and outside the circle*. Daisy produced similes by translating mathematical graphs to objects around us. Thus, based on my analysis, I concur with the view that a technological environment can promote exploration as a way of learning, provided that it is facilitated appropriately and correctly. The findings align with the constructivist perspective that when they are constructing their own meaning, it "enables students to follow routes which are their own" (Boaler, 1993, p. 17). When learners take ownership of their learning, meaning-making and understanding is developed through the creation of knowledge.

GeoGebra thus provided a dynamic and visual environment of working and exploring mathematical concepts, enabling learners to construct their own meaning and verify their own theories.

Procedures

GeoGebra did not necessarily offer step by step procedures or geometrical reasons. It did not provide a method on how to solve a linear or a quadratic equation, rather the participants were usually left to their own devices to develop their procedures and articulate their own reasoning. The practice applets were criticised for not providing geometrical reasons which are required for answering examination questions. Applet 3.5 was the only applet that provided a procedure of determining the *equation of parallel lines and perpendicular lines*. However, the learners did develop alternative methods of arriving at solutions and were enthusiastic while solving the questions in the applets. Most important, the feedback facility of the applets allowed the learners to reflect on their incorrect mathematical approaches.

7.3.2 **The role of visualisation using *GeoGebra***

The dynamic visualisation capabilities of *GeoGebra* enabled the participant learners to move from concrete instances to conjecturing abstract ideas, echoing Guzmán (2002) who observes that visualisation is an important aspect in mathematical activity through which one can explore different structures of concrete reality.

In circle geometry, the dragging of various points emphasised the *variant and invariant properties of the angles.* This was supported by the visual recognition of the *relationship of angles*. The multiple representations enabled in *GeoGebra,* made algebraic expressions more meaningful to the learners, as they visualised the essential relations between the equation and the graph. In dealing with straight lines, the participant learners visualised gradients as increasing or decreasing lines, however, their axis of reference of change differed. Iris and Cypress considered a line with positive slope as increasing to the right of the vertical axis and a line with a negative slope as increasing to the left of the vertical axis. On the other hand, Jasmine visualised the decreasing line as the line moving downward for the horizontal movement from left to right.

Analysis of the data provided evidence that interactions with the *GeoGebra* software enabled the participant learners to visualise phenomena and devise visual strategies that their textbook illustrations did not provide. Through the manipulation of objects in the applets, learners visualised and conceptualised appropriate mathematical ideas. Presmeg (1992) describes several cases where vivid imagery was a hindrance in generalising mathematical concepts, but in the instance of the *angle at the centre* applets, the participant learners however repeatedly manipulated the geometrical objects until they conceptualised the theorem correctly.

Furthermore, considering Duval's (2013) observation that construction of geometrical figures by the learners was absolutely vital for visualisation, *GeoGebra* proved to be an efficient visualisation tool for learners to facilitate meaning-making and understanding.

Participant learners exhibited the characteristics of Krutetskii's (1976) 'geometric thinker', by utilising visual techniques for problem solving. In determining the equation of a parallel line passing through a point, for example, Aster devised a method of sketching a parallel line using the *gradient method*. Therefore, visualisation "as a process" provoked intellectual activity and the construction of knowledge (Arcavi, 2003, p. 227). Technology as a visual aid created an opportunity for developing mathematical proficiency.

Furthermore, working with *GeoGebra*, learners could discover how the theorems were linked to one another. For Presmeg (2014), visualisation should be employed when we make connections between topics in mathematics.

GeoGebra allowed learners to engage and control the inherent actions of the objects by plotting, clicking and dragging lines and points on the computer screen. The strength of *GeoGebra* visualisations i.e. preserving the invariant relationships in the visual representation when dragging the representation, allowed learners to experience the invariance obeyed when exploring the dynamic representations. I contend that the dynamic visualisation capabilities in *GeoGebra* helped learners to develop understanding, as it provided opportunities to make conjectures and verify hypotheses. Thus, *GeoGebra* illuminates Arcavi's (2003) observation that visualisation is a key component of reasoning, not only for illustrative purposes. Technology enables visualisation and is thus not only considered as making learning easier but as a basis for a mathematically richer activity.

Nonetheless, visualisation in *GeoGebra* was not without challenges. Presmeg (1986b) puts forward that inflexibility in thinking prevents visual learners from successful problem solving. A case in point was Cypress who applied the concept of *equal distances between parallel lines* but could not completely sketch the line. Evidently the visual image of equal distances between the parallel lines was overwhelming for Cypress. Occasionally the participant learners exhibited a tendency to attach irrelevant details to a given context, obscuring mathematical meaning and concepts.

7.4 LIMITATIONS OF THE RESEARCH STUDY

This research was designed as a case study of selected learners' engagement with various *GeoGebra* applets in learning mathematical concepts. The limitations of a case study research pertain to the generalisation of its results, although Yin (2009) raises concerns that case studies provide little basis for generalisation. It is inappropriate to generalise the main findings of this research to a broader context of South African population as the socio-economic conditions and background of data collection sites are different from that of the other regions. The emphasis of this case study was to understand and interpret learning in terms of its selected participants. Thus, the interpretative research paradigm that underpinned this research study, cohered with the nature of the research enquiry. Therefore, generalisability was not the purpose of this research. Nonetheless, generalisability in a case study is interpreted as transferability (Cohen et al., 2007). In this respect, Schofield (1990, as cited in Cohen et al., 2007, p. 137) suggests that transferability requires thick description. Thus, by in-depth analysis and providing detailed and rich descriptions of the learner's interactions, the findings may be applicable to other situations. The limitation of generalisability could therefore be resolved by engaging in further research on learners' actions and responses while using technology in classrooms in different regions. The strength of the research design is that it can be replicated in different regions of the country.

Furthermore, the research is also limited by the duration of data collection. The small number of classrooms, relatively few lessons and limited time period of eight months may also have impacted the findings of the research. In the analysis, I considered only three cycles of GLIP due to the vast amount of data gathered from the four cycles of GLIP which involved nine classroom observations. Also, the use of the isiXhosa language by the learners may have influenced the data, in spite of encouraging learners to use English during classroom observations. I do not understand isiXhosa and relied on other learners to help translate their comments into English. This could have led to some superficial analysis on my part.

In addition, there were challenges in organising training sessions with the learners as mentioned in Section 3.8. Integrating *GeoGebra* in classrooms was a rather novel phenomenon in the participating schools. After the first cycle in GLIP, the participating teachers and I felt the need for providing more training sessions to the learners. However, in both the schools, the availability of computer laboratories posed a problem for us to the extent that we had to drop the idea of providing further training in *GeoGebra*. In my view, if I could have provided more focused training on using different tools in *GeoGebra,* this would have possibly enabled participants to explore their conjectures even further, thus enriching my data even more.

7.5 **SIGNIFICANCE AND CONTRIBUTIONS OF THE STUDY**

Whilst this research study has provided a rich account of how learners harnessed the visualisation capabilities of *GeoGebra* in developing mathematical proficiency and making sense of mathematical ideas, it also has contributed significantly to i) mathematics education in general, and ii) research methodology.

7.5.1 **Significance to mathematics education**

A review of pertinent literature relating specifically to the integration of technology in mathematics classrooms identified that: a) available technological resources are not used to the fullest for teaching and learning purposes; and b) much relevant research has been conducted in the context of advanced or developed cities. However, within the context of teaching and learning using DGS, little empirical research seems to have been carried out focussing on learners in previously deprived communities such as in Mthatha, Eastern Cape. Thus, this research study is particularly significant as it made use of an empirical field and context that has been the victim of under-development, inequality and woeful distribution of educational resources.

In addition, the research study contributes to new knowledge in the field of visualisation. Researchers (Duval, 2014; Presmeg, 2014) have raised concerns about the dearth of research in studying the impact of the power of visualisation in computer technology, of developing learners' thinking and learning of mathematics. This research study found that computer technology can positively influence the meaning-making and problem solving ability of learners. Aster, for instance, developed his own visual method in determining the *equations of parallel lines* by sketching the required lines.

Visualisation is inherent in *GeoGebra*, nonetheless, learners may not 'see' the concept. For instance, in the case of the applet determining if the *roots of a quadratic function were real or unreal*, Rosy and Daisy insisted on using the formula $b^2 - 4ac$ rather than interpreting it from the displayed graph. Hence, this study emphasises that the integration of technology requires proper and appropriate planning, support and intervention for mathematics learning to take place. I make suggestions on intervention plans in Section 7.6.2 (p. 253).

Furthermore, I hope that the study will be of interest to all the stakeholders in mathematics education, especially teachers and researchers who should be aware of the significance of the visual and dynamic capabilities of DGS in the teaching and learning of mathematics. The findings of the study testify that technological tools such as *GeoGebra* have the potential to change the dynamics of learning mathematics. From the findings, using *GeoGebra* in classrooms provides exciting opportunities for learners to develop dynamic visual reasoning abilities.

7.5.2 **Methodological considerations**

Screen recording software captures everything that occurs on a computer screen, including voice media files. It is commonly used to broadcast webinars, games, etc. Screen capturing as a data gathering mechanism is a relatively novel idea in the field of education. There is little evidence in educational research in South Africa that employs this digital technological application. Hence, this research also highlights the use of modern methods of data gathering in the emerging era of employing technological tools for the purpose of research, teaching and learning.

In my research, the screen recording method worked almost flawlessly, except for one lesson, where the learner shut down the computer without closing the recording. The learners' activities on the computer, captured with the help of the screen recording application, provided the precise and accurate information required for my research. Learners were shown their workings from their own computer recordings during the stimulated recall interviews, to clarify and describe their thinking and meaning-making. This method enabled me to gain accurate insights into their mathematical thinking and visualisation processes. It is evident from the in-depth analysis, narrated in the earlier chapters, that I relied heavily on this form of data to address my research objectives. From the review of the methodology, I propose that studies addressing integrating technology into classrooms should be conducted using a screen recording software. Screen recording as a research instrument to deconstruct learners meaning-making habits and thoughts is, in my experience, very powerful. I do recognise the ethical implications of this methodology which I have alluded to in Chapter 3.

7.6 **IMPLICATIONS**

This research study informs many stakeholders in mathematics education such as teachers in schools, policy makers and curriculum designers and researchers. In the light of the discussion and findings in Chapters 4, 5 and 6, I suggest the following for the integration of technology in mathematics classrooms.

7.6.1 **Implications for schools and teachers**

Many schools have invested heavily in computer laboratories. As discussed in earlier chapters, much of the teaching time was lost in rearranging learners and schedules according to the functioning of computers and their peripherals. As I write this report, teachers of both the schools still complain that there is a lack of continuous professional technical support in their computer laboratories. It is vital that computer laboratories are functional and properly maintained.

Fortunately, during our data collection schedule, load shedding and the resulting power outages of electricity did not pose a challenge to us. However, during the current times of load shedding the teaching with technology and running the GLIP has been seriously compromised. Schools and the DBE need to find alternate sources of electricity that can supplement the power supply from the national grid.

This research study was designed around an intervention programme which meant that the teachers implemented what they learned in the intervention directly in their classrooms. This was a very effective strategy and ensured that my research project remained grounded and had an impact on my immediate community. I thus suggest that the integration of technology in other mathematics classrooms be accompanied by a parallel intervention programme to ensure that on the basis of the teachers' and leaners' reflections, appropriate and immediate support can be actioned to ensure a smooth and coherent integration process. This intervention should be infused into the routine school life, and not only for research purposes. The computer laboratories ought to be considered as a resource for teaching and learning mathematics.

The findings of this imply and reinforce the notion that technological resources, such as *GeoGebra,* help learners to extend their prior knowledge and allow them to proceed gradually to develop new mathematical ideas or see the same mathematical structure from a new perspective. It is important to note that learners who had mastered the basic mathematical concepts and skills before my intervention were able to make immediate use of *GeoGebra* in growing their mathematical proficiency. However, those learners who lacked basic knowledge and fluency, struggled to make sense of mathematical ideas when they interacted with the *GeoGebra* tools. Hence, teachers require proper planning in addressing learners with diverse mathematical experiences and knowledge while they integrate technology in classrooms. As learners use computers in the classroom, the teacher has an opportunity to examine learners' thinking processes and thereby adapt instructions accordingly.

7.6.2 **Implications for policy makers and curriculum designers**

This research study would not have taken place without the collaboration of teachers in the GLIP. These teachers were willing to learn skills in *GeoGebra* and were eager to implement them in their classrooms. Workshops and teacher development activities hosted by the DBE and other stakeholders need to incorporate technological tools such as *GeoGebra* with clear and explicit goals of implementing them in classrooms. In such teacher development activities, teachers learn and develop their skills in integrating technological tools in mathematics classrooms. Furthermore, learning technological tools such as *GeoGebra* would enhance their own knowledge not only in technology, but also in mathematics. Professional development in this area should be a key strategy in the DBE's in-service offerings.

I had tacitly assumed that the learners in this millennium were well informed and proficient in the use of computers. However, I found that the learners struggled to make use of seemingly straight-forward keyboard and mouse controls. For instance, Iris struggled to use the fraction or division symbol, Jasmine struggled to use the 'equal to' symbol, and more significantly , Daisy could not successfully manipulate a quadratic equation by changing its parameters. In general, the learners struggled to control graphs and figures on the screen. It is important to note that computer use *per se* does not necessarily improve learners' knowledge. It needs to be supported by guidance of an expert or a knowledgeable person. As Klien, Nir-Gal and Darom (1998) conclude, interaction with computers alone does not necessarily benefit the development of young children's higher thinking processes. Hence, I recommend that the DBE and the curriculum should also provide end-user computer training for all learners. This training may include the use of various functions of the keyboard and the mouse, the correct handling and controlling of which is vitally important when using computers and it requires training and practice. Training and mediation in the use of computers at an early stage of schooling is likely to benefit learners significantly in visual and abstract t in later schooling.

For my teachers participating in GLIP the integration of technology was not part of their teacher training. They had to learn how to use *GeoGebra* and the computer. For teachers to use compute technology effectively they should be trained on how to use these devices in their pedagogy at University. I thus concur with the argument of Vrasidas and Mcissac (2001) that "if we except teachers to teach in a constructivist way of using technology, we need to teach them in constructive ways using technology." (p. 130). This of course implies that schools then need to be appropriately equipped and resourced with the appropriate technology that the teachers were trained for.

7.6.3 **Implications for further research**

Some of my participants missed the opportunity to correctly visualise the connection between the x-intercepts and the value of the discriminant because of their lack of knowledge of the properties of real numbers. Further, while determining the *equation of a parallel line passing through a point*, Cypress could not properly sketch a parallel line to a given line, since he did not know how to determine a point on the line. The two participants thus in this instance struggled to strategically use the *GeoGebra* technology to consolidate their knowledge. Kilpatrick et al. (2001) argue that mathematical ideas are interconnected and build on one another to produce a new coherent mathematical idea. This is significant in the context of this research project as these learners were never exposed to any sort of technology-based learning of mathematics in their earlier grades, and thus did not know how to use these devices strategically in this particular instance. My research was focused on Grade 11 learners, whilst the basic concepts that were essential to develop new ideas should have been dealt with in earlier grades. Hence, there is also a need to research how learning takes place when technological tools are used in earlier grades such as in Grade 7 or 8. It would be intriguing to understand how young learners would respond in a mathematics classroom where DGS is integrated at an early phase of learning.

Research should also be undertaken that critically evaluates the DBE policy and its implementation of ICT in schools in the whole region, of which my empirical scrutiny is but a small sub-set.

I also recommend that longitudinal research should be conducted using much larger samples in South Africa to interrogate the effectiveness of using DGS in mathematics classrooms.

7.7 PERSONAL REFLECTION ON GLIP

As stated in Chapter 1, GLIP is a teacher development project that aims to equip teachers and students with the necessary skills to integrate technological tools, particularly *GeoGebra*, into classrooms, for teaching and learning of mathematics. GLIP envisages that collaborative engagement among teachers is a way forward in closing the gap between having access to technology and adapting it for effective use in mathematics classrooms.

For the duration of this research, my co-researcher and myself ran the programme and worked directly with the teachers who were willing to integrate *GeoGebra* into their lessons. We held GLIP meetings to design, discuss and implement different applets on a given topic before using them in the classrooms. At the beginning of the research, there were only four teachers who wished to participate. Currently, there are nine teachers including my co-researcher and myself, who are regular and active members. Antony, a teacher in GLIP, once reflected, "*You pick up something new, whenever we meet*". Recently, a research paper about the GLIP teachers integrating *GeoGebra* was published by Mavani, Mavani and Schäfer (2018, p. 11) who observe that "[t]he collaborative teamwork among the [GLIP] teachers is a feasible way to bridge the gap between having access to technology and adapting it for effective use in the classrooms." My study supported the notion that teaching using DGS, provided teachers with some additional teaching strategies that were difficult to implement in traditional classrooms. Thus, the GLIP provided a space for discussion and experimenting with new pedagogical approaches among practising mathematics teachers and researchers, and I am thankful that GLIP was willing for me to conduct my research project in its space.

7.8 FINAL REMARKS

My engagement with *GeoGebra* in this research project has enhanced my own ability in solving mathematical problems. I have always shown a tendency to use visual methods in mathematical problem solving, and reflecting on my teaching career, my use of *GeoGebra* in my own practice over the past five years has made me a better, more innovative and more effective teacher.

Personally, I committed to this research journey when I noticed that many of the learners in my class could not use calculators and electronic tablets effectively for learning purposes. I thus focussed on how I could integrate technological tools into my mathematics lessons in order for my learners to make sense of mathematics. As I progressed with my research project, it afforded me deeper insight into the visualisation processes that were utilised by the learners, that often went unnoticed on my part in my mathematics lessons. Furthermore, from a constructivist theory of learning perspective, my study made me cognisant of diverse meaning-making approaches that learners develop. This has significantly contributed to the growth and development of my own teaching and learning practice.

Some mathematics becomes *more important* because technology *requires* it.
Some mathematics becomes *less important* because technology *replaces* it.
Some mathematics becomes *possible* because technology *allows* it.
(NCTM, 1991, Transparency 34)

REFERENCES

Alsina, C., & Nelsen, R. B. (2006). *Math Made Visual: Creating images for understanding mathematics*. Mathematical Association of America.

Arcavi, A. (2003). The Role of Visual Representations in the Learning of Mathematics. *Educational Studies in Mathematics*, *52*(3), 215–241.

Artigue, M. (2002). Learning Mathematics in a CAS Environment: The Genesis of a Reflection about Instrumentation and the Dialectics between Technical and Conceptual work. *International Journal of Computers for Mathematical Learning*, (7), 245–274.

Askew, M. (2016). *Transforming Primary Mathematics Understanding classroom tasks, tools and talk* (Updated an). London: Routledge.

Avgerinou, M. D., & Pettersson, R. (2011). Toward a Cohesive Theory of Visual Literacy Toward a Cohesive Theory of Visual Literacy. *Journal of Visual Literacy*, *30*(January), Number 2, 1-29. https://doi.org/10.1080/23796529.2011.11674687

Balacheff, N., & Kaput, J. J. (1996). Computer-Based Learning Environments in Mathematics. In A. J. Bishop, K. Clements, C. Keitel, J. Kilpatrick, & C. Laborde (Eds.), *International Handbook of Mathematics Education* (First Edit, pp. 469–504). Dordrecht, The Netherlands.: Kluwer Academic Publishers. https://doi.org/10.1007/978-94-009-1465-0

Berger, M. (2011). A framework for examining characteristics of computer-based mathematical tasks. *African Journal of Research in MST Education*, *15*(2), 111–123.

Bhagat, K. K., & Chang, C. (2015). Incorporating GeoGebra into Geometry learning-A lesson from, *11*(1), 77–86. https://doi.org/10.12973/eurasia.2015.1307a

Bishop, A. J. (1980). Spatial Abilities and Mathematics Education : A Review. *Educational Studies in Mathematics*, *11*(3), 257–269.

Bishop, A. J. (1988). A review of research on visualisation in mathematics education. In *Proceedings of the 12th Annual conference of the International Group for the Psycology of Mathematics Education, Hungary* (pp. 170–177).

Blais, D. M. (1995). Elemental Psychology and the Tèaching of Mathematics. *Journal of Developmental Education*, *18*(3), 2–8. Retrieved from http://www.jstor.org/stable/42775558

Boaler, J. (1993). The Role of Contexts in the Mathematics Classroom: Do They Make Mathematics More “Real”? *For the Learning of Mathematics*, *13*(2), 12–17.

Boaler, J. (2009). *The Elephant in the Classroom: Helping Children Learn and Love Maths*. Souvenir Press Ltd.

Bransford, J. D., Brown, A. L., & Cocking, R. R. (2000). *How People learn: Brain, Mind, Experience*. Washington: National Academy Press.

Carpenter, T. P., & Lehrer, R. (1999). Teaching and Learning Mathematics With Understanding. In E. Fennema & T. A. Romberg (Eds.), *Mathematics Classroom that Promote Understanding* (pp. 19–32). Mahwah, New Jersey: Lawrence Erlbaum Associates, Publishers.

Carter, J., Forster, S., Howe, R., Kader, G., Kepner, H., Quander, J. R., & Valdez, P. (2009). *Focus on High School Mathematics: Reasoning and Sense Making*. Reston, VA: National Council of Teachers of Mathematics.

Clements, M. A. (Ken). (1982). Visual Imagery and School Mathematics. *For the Learning of Mathematics*, *2*(3), 33–39.

Cohen, L., Manion, L., & Morrison, K. (2007). *Research Methods in Education* (6th Editio). Routledge.

Confrey, J. (1990). What Constructivism Implies for Teaching. *Journal for Research in Mathematics Education Monograph*, *4*, 107–122.

Cuban, L., Kirkpatrick, H., & Peck, C. (2001). High Access and Low Use of Technologies in High School Classrooms: Explaining an Apparent Paradox. *American Educational Research Journal*, *38*(4), 813–834. https://doi.org/10.3102/00028312038004813

Denzin, N. K., & Lincoln, Y. S. (2005). Introduction: The Discipline and Pratice of Qualitative research. In N. K. Denzin & Y. S. Lincoln (Eds.), *The Sage Handbook of Qualitative Research* (Third, pp. 1–32). Thousand Oaks: Sage Publications.

Dept of Basic Education. (2011). *National Curriculum Statement - CAPS*. Pretoria, South Africa.

Dept of Basic Education. (2013). *Investigation into the Implementation of Maths, Science And Technology*. Pretoria, South Africa.

Dreyfus, T. (1992). Imagery for diagrams. In R. Sutherland & J. Mason (Eds.), *Exploiting Mental Imagery with Computers in Mathematics Education* (pp. 3–19).

Duffin, J. (2010). Using Virtual Manipulatives To Support Teaching And Learning Mathematics. In *Proceedings of the First North American GeoGebra Conference*. Ithaca NY.

Duval, R. (1999). Representation, Vision and Visualization: Cognitive Functions in Mathematical Thinking. Basic Issues for Learning. In *Twenty First Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 25, pp. 3–26). https://doi.org/10.1076/noph.25.1.3.7140

Duval, R. (2013). The First Crucial Point in Geometry Learning: Visualization. *Mediterranean Journal for Research in Mathematics Education*, *12*(1–2), 23–37.

Duval, R. (2014). Commentary: Linking epistemology and semio-cognitive modeling in visualization. *ZDM - International Journal on Mathematics Education*, *46*(1), 159–170.

Fischbein, E. (1993). The Theory of Figural Concepts. *Educational Studies in Mathematics*, *24*(2), 139–162.

Gage, M. E. (2010). Embedding GeoGebra Applets in WebWork Homework. In *Proceedings of the First North American GeoGebra Conference*. Ithaca NY.

Gawlick, T. (2002). On Dynamic Geometry Software in the Regular Classroom. *ZDM - International Journal on Mathematics Education*, *34*(3), 85–92.

GeoGebra. (n.d.). Retrieved July 9, 2015, from http://www.geogebra.org/

GeoGebra Institute Support Project (GIS). (n.d.). Retrieved February 21, 2016, from http://mbeki-maths-dev.nmmu.ac.za/Projects/GeoGebra-Institute-Support-Project-(GIS)

Giaquinto, M. (2007). *Visual Thinking in Mathematics* (Online). Oxford Scholarship Online. https://doi.org/10.1093/acprof

Goldenberg, E. P. (1992). Ruminations about dynamic imagery (and a strong plea for research). In R. Sutherland & J. Mason (Eds.), *Exploiting Mental Imagery with Computers in Mathematics Education* (pp. 202–224). NATO ASI Series. https://doi.org/10.1007/978-3-642-57771-0

Goldin, G. a. (1990). Epistemology, Constructivism, and Discovery Learning in Mathematics. *Journal for Research in Mathematics Education.*, *4*(1990), 31–210. https://doi.org/10.2307/749911

Goldman, R. (2006). Algebraic geometry and geometric modelling: insight and computation. In M. Elkadi, B. Mourrain, & R. Piene (Eds.), *Algebraic Geometry and Geometric Modelling - Mathematics and Visualization* (pp. 1–20). Springer.

Gordon, M. (2009). Toward A Pragmatic Discourse of Constructivism : Reflections on Lessons from Practice. *Educational Studies*, *45*(752), 39–58. https://doi.org/10.1080/00131940802546894

Gray, E. M., & Tall, D. (1994). Duality, Ambiguity , and Flexibility : A “Proceptual” View of Simple Arithmetic. *Journal for Research in Mathematics Education*, *25*(2), 116–140.

Guzmán, M. De. (2002). The role of visualization in the teaching and learning of mathematical analysis. In *Proceedings of the International Conference on the Teaching of Mathematics* (Vol. 7, pp. 1–25). Crete, Greece. https://doi.org/10.1109/IEMBS.1992.5761740

Haapasalo, L. (2007). Adapting Mathematics Education to the Needs of ICT. *The Electronic Journal of Mathematics and Technology*, *1*(1), 1–10.

Habibi, M. (2012). The Role of Intuitive and Life-related Activities in Improving Teaching of Geometry in Primary School, *8*(6), 359–366.

Hardy, M. D., & Taylor, P. C. (1997). Von Glaserfeld’s Radical Constructivism: A Critical Review. *Science and Education*, *6*, 135–150.

Hasek, R. (2011). Investigation of logarithmic spirals in nature by means of dynamic geometry and computer algebra systems. *The Electronic Journal of Mathematics and Technology*, *6*(3).

Hennessey, M. N., Higley, K., & Chesnut, S. R. (2012). Persuasive Pedagogy: A New Paradigm for Mathematics Education. *Educational Psychology Review*, *24*, 187–204. https://doi.org/10.1007/s10648-011-9190-7

Hiebert, J., & Lefevre, P. (1986). Conceptual and Procedural Knowledge in Mathematics: An Introductory Analysis. In J. Hiebert (Ed.), *Conceptual and Procedural Knowledge: The case of Mathematics* (pp. 1–26). Hillside, New Jersey: Lawrence Erlbaum Associates.

Hohenwarter, M., & Hohenwarter, J. (2008). Teaching and Learning Calculus with Free Dynamic Ma- thematics Software GeoGebra Calculus with GeoGebra. In *Proceedings of the International Conference on the Teaching of Mathematics - TSG 16* (pp. 1–9). Mexico.

Hohenwarter, M., & Lavicza, Z. (2007). Mathematics Teacher Development With ICT : Towards An International GeoGebra Institute. In *Proceedings of British Society for research in learning mathematics* (pp. 49–54).

Hohenwarter, M., & Preiner, J. (2007). Dynamic Mathematics with GeoGebra. *Journal for Online Mathematics and Its Applications*, *7*, 1–7.

Hollebrands, K. F. (2007). The Role of a Dynamic Software Program for Geometry in the Strategies High School Mathematics Students Employ. *Journal for Research in Mathematics Education*, *38*(2), 164–192.

Hölzl, R. (2001). Using dynamic geometry software to add contrast to geometric situations – a case study. *International Journal of Computers for Mathematical Learning*, *6*, 63–86.

Howson, G. (2005). Meaning and School Mathematics. In J. Kilpatrick, C. Hoyles, & Ole Skovsmose (Eds.), *Meaning in Mathematics Education* (pp. 17–38). Springer Science+Business Media.

Hoyles, C. (2005). Making Mathematics and Sharing Mathematics: Two Paths to Co-constructing Meaning? In J. Kilpatrick, C. Hoyles, O. Skovsmose, & P. Valero (Eds.), *Meaning in Mathematics Education* (pp. 139–158). New York: Springer Science+Business Media.

Hoyles, C., & Noss, R. (2003). Digital Technologies in Mathematics Education. In A. J. Bishop, M. A. Clements, C. Keitel, J. Kilpatrick, & F. K. S. Leung (Eds.), *Second International Handbook of Mathematics Education* (pp. 323–350). Dordrecht: Kluwer Academic Publishers. https://doi.org/10.1007/978-94-010-0273-8

Java Platform. (n.d.). Retrieved July 16, 2015, from http://docs.oracle.com/javase/7/docs/api/java/applet/Applet.html

Jaworski, B. (1994). *Investigating Mathematics Teaching : A Constructivist Enquiry*. London: The Falmer Press.

Jaworski, B. (2015). Mathematical Meaning-Making And Its Relation To Design of Teaching. *PNA*, *9*(4), 261–272.

Jonassen, D. H., & Strobel, J. (2006). Modeling For Meaningful Learning. In D. Hung & M. S. Khine (Eds.), *Engaged Learning With Emerging Technologies* (pp. 29–48). Dordrecht, The Netherlands: Springer.

Jones, K., Lavicza, Z., Hohenwarter, M., Lu, A., Dawes, M., & Borcherds, M. (2009). BSRLM Geometry working group: Establishing a professional development network to support teachers using dynamic mathematics software. In M. Joubert (Ed.), *British Society for Research into Learning Mathematics* (Vol. 29, pp. 97–102).

Kaput, J. J. (1992). Technology and Mathematics Education. In D. A. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning* (pp. 515–556). New York: Macmillan Publishing Company.

Khine, M. S. (2003). Creating a Technology Rich Constructivist Learning Environment in a Clasroom Management Module. In M. S. Khine & D. Fisher (Eds.), *Technology-Rich Learning Environments A Future Prespective* (pp. 21–39). Singapore: World Scientific Publishing Co. Pte Ltd.

Kieran, C., & Drijvers, P. (2006). The Co-Emergence of Machine Techniques, Paper-And-Pencil Techniques, And Theoretical Reflection: A Study of CAS Use in Secondary School Algebra. *International Journal of Computers for Mathematical Learning*, (11), 205–263. https://doi.org/10.1007/s10758-006-0006-7

Kilpatrick, J., Swafford, J., & Findell, B. (2001). *Adding it Up: Helping children learn Mathematics*. Washington: National Academy Press.

Klien, S. P., Nir-Gal, O., & Darom, E. (1998). The use of Computers in Kindergarten, With or Without Adult Mediation: Effects on Children's Cognitive Peformance and Behaviour. *Computers in Human Behaviour*, *16*, 591–608.

Krutetskii, V. A. (1976). *The Psychology of Mathematical Abilities in Schoolchidren*. (Transalated from the Russian by Joan Teller, Ed.). Chicago: Univeristy of Chicago.

Laborde, C. (1993). The Computer as Part of the Learning Environment: The Case of Geometry. In C. Keitel & K. Ruthven (Eds.), *Learning Through Computers: Mathematics and Educational Technology* (pp. 48–67). Berlin: Springer Verlag.

Laborde, C. (2001). Integration of technology in the design of geometry tasks with cabri-geometry. *International Journal of Computers for Mathematical Learning*, *6*, 283–317.

Lean, G., & Clements, M. A. (Ken). (1981). Spatial Ability, Visual Imagery, and Mathematical Performance. *Educational Studies in Mathematics*, *12*(3), 267–299.

Linneweber, H., Schäfer, M., & Samson, D. A. (2010). Vital Maths - Palindromic sums. Retrieved July 8, 2017, from http://www.vitalmaths.com/videos/all-videos/item/196-palindromic-sums

Ma, L. (2010). *Knowing and Teaching Elementary Mathematics*. Taylor & Francis e-Library.

Mariotti, M. A. (2000). Introduction to Proof : The Mediation of a Dynamic Software Environment. *Educational Studies in Mathematics*, *44*(1), 25–53.

Marti, E. (1996). Mechanisms of Internalisation and Externalisation of knowledge in Piaget's and Vygotsky's theories. In A. Tryphon & J. Voneche (Eds.), *Piaget-Vygotsky the social genesis of thought* (pp. 57–80). UK: Psychologycal Press.

Mavani, D., & Mavani, B. (2014). Investigating Cubic Functions Using GeoGebra Dynamic Geometry Software. *Learning and Teaching Mathematics*, (Dec).

Mavani, D., Mavani, B., & Schäfer, M. (2018). A Case Study of Two Selected Teachers as they Integrated Dynamic Geometry Software as a Visualisation Tool in Teaching Geometry. *African Journal of Research in Mathematics, Science and Technology Education*, *0*(0), 1–11. https://doi.org/10.1080/18117295.2018.1522716

Maxwell, J. A. (2008). Designing a Qualitative Study. In L. Bickman; & D. J. Rog (Eds.), *The SAGE Handbook of Applied Research* (2nd Editio, pp. 214–253). California: Sage Publications.

Merriam-Webster. (n.d.). Retrieved July 17, 2015, from http://www.merriam-webster.com/dictionary/applet

Merriam, S. B. (2009). *Qualitative research : a guide to design and implementation*. San Francisco: Jossey-Bass.

Mhlolo, M. K., & Schäfer, M. (2013). ' The Area of a Triangle is 180 0 C '— An Analysis of Learners ' Idiosyncratic Geometry Responses through the Lenses of Vygotsky ' s Theory of Concept Formation. *African Journal of Research in Mathematics, Science and Technology Education*, *17*(1–2), 83–93. https://doi.org/10.1080/10288457.2013.826973

Moreno-armella, L., Hegedus, S. J., & Kaput, J. J. (2008). From static to dynamic mathematics : historical and representational perspectives. *Educational Studies in Mathematics*, (68), 99–111. https://doi.org/10.1007/s10649-008-9116-6

Moyer, S. P. (2002). What Are Virtual Manipulatives. *Teaching Children Mathematics (NCTM)*, 372–377.

Mudaly, V. (2010). Thinking With Diagrams Whilst Writing With Words. *Pythagoras*, *75*(July), 65–75.

Naidoo, J., & Govender, R. (2014). Exploring the use of a dynamic online software programme in the teaching and learning of trigonometric graphs. *Pythagoras*, *35*(2). https://doi.org/10.4102/pythagoras.v35i2.260

NCTM. (2000). Principles and Standards for School Mathematics. National Council of Teachers of Mathematics.

Ndlovu, M., Wessels, D., & Villiers, M. De. (2013). Competencies in using Sketchpad in

Geometry Teaching and Learning : Experiences of preservice teachers Competencies in using Sketchpad in Geometry Teaching and Learning : Experiences of preservice teachers. *African Journal of Research in Mathematics, Science and Technology Education*, *17*(3), 231–243. https://doi.org/10.1080/10288457.2013.848536

Newby, P. (2010). *Research Methods for Education*. London: Pearson Education Limited.

Noss, R., & Hoyles, C. (1996). *Windows on mathematical meanings: learning cultures and computers*. Dordrecht: Kluwer Academic Publishers.

O'Brien, J. (1993). Action Research through stimulated Recall. *Research and Science Education*, *23*, 214–221.

O'Leary, Z. (2004). *The Essential Guide to doing research*. Sage Publications.

Patton, M. Q. (2002). *Qualitative Research & Evaluation Methods* (3rd Editio). Thousand Oaks: Sage Publications.

Pierce, R., Stacey, K., & Bardini, C. (2010). Linear functions : teaching strategies and students ' conceptions associated with y = mx + c. *Pedagogies: An International Journal*, *5*(3), 202–215. https://doi.org/10.1080/1554480X.2010.486151

Presmeg, N. (1986a). Visualisation and Mathematical Giftednessss. *Educational Studies in Mathematics*, *17*(3), 297–311.

Presmeg, N. (1986b). Visualisation in High School Mathematics. *For the Learning of Mathematics*, *6*(3), 42–46.

Presmeg, N. (1992). Prototypes, Metaphors, Metonymies and Imaginative Rationality in High School Mathematics. *Educational Studies in Mathematics*, *23*(6), 595–610.

Presmeg, N. (2006). Research on Visualisation in Mathematics. In A. Gutierrez & P. Boero (Eds.), *Handbook of Research on the Psychology of Mathematics Education Past, Present and Future* (pp. 205–235). Sense Publishers.

Presmeg, N. (2014). Contemplating visualization as an epistemological learning tool in mathematics. *ZDM - International Journal on Mathematics Education*, *46*(1), 151–157.

Rittle-Johnson, B., & Alibali, M. W. (1999). Conceptual and Procedural Knowledge of Mathematics : Does One Lead to the Other ? *Journal of Educational Psychology*, *91*(1), 175–189.

Rittle-Johnson, B., & Star, J. R. (2007). Does Comparing Solution Methods Facilitate Conceptual and Procedural Knowledge ? An Experimental Study on Learning to Solve Equations. *Journal of Educational Psychology*, *99*(3), 561–574. https://doi.org/10.1037/0022-0663.99.3.561

Rittle-Johnson, B., Star, J. R., & Durkin, K. (2009). The Importance of Prior Knowledge When Comparing Examples : Influences on Conceptual and Procedural Knowledge of Equation Solving. *Journal of Educational Psychology*, *101*(4), 836–852.

https://doi.org/10.1037/a0016026

Rivera, F. D. (2011). *Towards a Visually Orientend School Mathematics Curriculum*. (A. J. Bishop, Ed.). Springer.

Rivera, Ferdinand D. (2011). *Toward a Visually-Oriented School Mathematics Curriculum*. Springer Science+Business Media. https://doi.org/10.1007/978-94-007-0014-7

Romberg, T. A., & Kaput, J. J. (1999). Mathematics Worth Teaching, Mathematics Worth Understanding. In E. Fennema & T. A. Romberg (Eds.), *Mathematics classrooms that promote understanding* (pp. 1–18). Mahwah, New Jersey: Lawrence Erlbaum Associates, Inc.

Ruthven, K. (2003). Linking algebraic and geometric reasoning with dynamic geometry software, (March).

Ruthven, K., Hennessy, S., & Deaney, R. (2008). Constructions of dynamic geometry: A study of the interpretative flexibility of educational software in classroom practice. *Computers and Education*, *51*(1), 297–317. https://doi.org/10.1016/j.compedu.2007.05.013

Schäfer, M. (2016). Making Connections in the Mathematics Classroom. In *SAARMSTE Namibia Chapter*.

Schoenfeld, A. (2007). What is Mathematical Proficiency and How Can It Be Assessed? In A. Schoenfeld (Ed.), *Assessing Mathematical Proficiency* (pp. 59–73). Cambridge University Press, New York: MSRI Publications.

Sherman, M. (2010). A Conceptual Framework For Using GeoGebra With Teachers and Students. In *Proceedings of the First North American GeoGebra Conference* (pp. 205–218).

Sherman, M. (2014). The Role of Technology in Supporting Students ' Mathematical Thinking : Extending the Metaphors of Amplifier and Reorganizer. *Contemporary Issues in Technology and Teacher Education*, *14*(3), 220–246.

Silver, E. A. (1986). The Focus on Relationships. In J. Hiebert (Ed.), *Conceptual and Procedural Knowledge: The case of Mathematics* (pp. 181–197). Hillside, New Jersey: LAWRENCE ERLBAUM ASSOCIATES, PUBLISHERS.

Skemp, R. R. (1989). *Mathematics in the primary school*. London: Routledge. https://doi.org/10.4324/9780203403891

Stake, R. E. (2010). *Qualitative Research Studying How Things Work*. New York: The Guilford Press.

Stols, G. (2007). Designing mathematical-technological activities for teachers using the Technology Acceptance Model. *Pythagoras*, *65*, 10–17.

Stols, G., Ferreira, R., Pelser, A., Olivier, W. A., Van der Merwe, A., De Villiers, C., & Venter, S. (2015). Perceptions and needs of South African Mathematics teachers concerning their use of technology for instruction. *South African Journal of Education*, *35*(4), 1–13. https://doi.org/10.15700/saje.v35n4a1209

Stols, G., & Kriek, J. (2011). Why dont all maths teachers use dynamic geometry software in their classrooms? *Astralasian Journal of Educational Technology*, (27(!)), 137–151.

Sutherland, R. (2007). *Teaching for Learning Mathematics*. London: Open University Press.

Tall, D. (1993). Interrelationships, between mind and computer: processes, images, symbols. In *Advanced Educational Technologies for Mathematics and Science* (NATO ASI S, pp. 385–414). Springer Verlag.

Tall, D. (1994). A Versatile Theory of Visualisation and Symbolisation in Mathematics. In *Proceedings of the 46th Conference of CIEAEM* (Vol. 1). Toulouse, France.

Tall, D. (2011). Crystalline Concepts in Long-Term Mathematical Invention and Discovery. *For the Learning of Mathematics*, *31*(1), 3–8.

Tall, D. (2014). Making Sense of Mathematical Reasoning and Proof. In M. N. Fried & T. Dreyfus (Eds.), *Advances in Mathematics Education Mathematics & Mathematics Education : Searching for Common Ground* (pp. 223–236). Dordrecht: Springer Science+Business Media. https://doi.org/10.1007/978-94-007-7473-5

Taylor, P. C., & Williams, M. C. (1993). Discourse Towards Balanced Rationality in the High School Mathematics Classroom: Ideas from Habermas's Critical Theory. In J. A. Malone & P. C. Talyor (Eds.), *Constructivist Interpretations of Teaching and Learning Mathematics* (pp. 135–148). Perth Australia: National Key Center for School Science and Mathematics.

Van De Walle, J. A. (2004). *Elementary and Middle School Mathematics Teaching Developmentally* (Fifth). Pearson Education Inc.

Vermeersch, L., & Vandenbroucke, A. (2015). Kids , take a look at this ! Visual Literacy Skills in the School Curriculum. *Journal of Visual Literacy*, *34*(1), 106–130.

von Glasersfeld, E. (1981). An Attentional Model for the Conceptual Construction of Units and Number. *Journal for Research in Mathematics Education*, *12*(2), 83–94. Retrieved from http://www.jstor.org/stable/748704

von Glasersfeld, E. (1990). An Exposition of Constructivism : Why Some Like It Radical. *Journal for Research in Mathematics Education. Monograph*, *4*, 19–29. Retrieved from http://www.jstor.org/stable/749910

von Glasersfeld, E. (1991). Introduction. In E. Von Glasersfeld (Ed.), *Radical Constructivism in Mathematics* (2002nd ed., pp. xiii–xx). USA: Kluwer Academic Publishers.

von Glasersfeld, E. (2000). Problems of Constructivism. In Leslie P. Steffe & P. W. Thompson (Eds.), *Radical Constructivism in Action* (pp. 3–9). London: RoutledgeFalmer.

Vrasidas, C., & Mcissac, M. S. (2001). Integrating Technology in Teaching and Teacher

Education: Implications for Policy and Curriculum Reform. *Educational Media International*, *38*(2–3), 127–132. https://doi.org/10.1080/09523980110041944

Watson, A., & Mason, J. (2005). *Mathematics as a Constructive Activity : Learners Generating Examples*. Mahwah, New Jersey: Lawrence Erlbaum Associates, Inc.

Wright, R. (2000). Development in Recovery Education. In Lesile P. Steffe & P. W. Thompson (Eds.), *Radical Constructivism in Action* (pp. 134–150). London.

Yerushalmy, M. (1997). Reaching the unreachable: technology and the semantics of asymptotes. *International Journal of Computers for Mathematical Learning*, *2*, 1–25.

Yin, R. K. (2009). *Case Study Research Design and Methods* (Fourth Edi). Thousand Oaks: Sage Publications.

Zazkis, R., Dubinsky, E., & Dautermann, J. (1996). Visual and Analytic Coordinating A Study of Students ' Strategies : Understanding of the Group D4. *Journal for Research in Mathematics Education*, *27*(4), 435–457. https://doi.org/10.2307/749876

Zimmermann, W., & Cunningham, S. (1991). Editors' Introduction : What Is Mathematical Visualization? *MAA*, (Notes 19), 1–7.

Zulnaidi, H., & Zakaria, E. (2012). The Effect of Using GeoGebra on Conceptual and Procedural Knowledge of High School Mathematics Students. *Asian Social Science;*, *8*(11). https://doi.org/10.5539/ass.v8n11p102

ANNEXURE I – Letter of permission to undertake research in secondary schools in Mthatha District

Province of the
EASTERN CAPE
EDUCATION

STRATEGIC PLANNING POLICY RESEARCH AND SECRETARIAT SERVICES
Steve Vukile Tshwete Complex • Zone 6 • Zwelitsha • Eastern Cape
Private Bag X0032 • Bhisho • 5605 • REPUBLIC OF SOUTH AFRICA
Tel: +27 (0)40 608 4773/4035/4537 • Fax: +27 (0)40 608 4574 • Website: www.ecdoe.gov.za

Enquiries: NY Kanjana   Email: nykanjana@live.co.za   Date: 26 January 2017

Mr. Deepak Pravin Mavani

P.O. Box 205

Mthatha

5099

Dear Mr. Pravin Mavani

PERMISSION TO UNDERTAKE A DOCTORAL THESIS: A CRITICAL ANALYSIS OF WHETHER VISUALISATION CAPABILITIES IN DYNAMIC GEOMETRIC SOFTWARE DEVELOP MEANING IN TERMS OF CONCEPTUAL AND PROCEDURAL UNDERSTANDING OF MATHEMATICAL CONCEPTS IN SELECTED GRADE 11 LEARNERS

1. Thank you for your application to conduct research.

2. Your application to conduct the abovementioned research in four Secondary Schools under the jurisdiction of the Mthatha District of the Eastern Cape Department of Education (ECDoE) is hereby approved based on the following conditions:

    a. there will be no financial implications for the Department;

    b. institutions and respondents must not be identifiable in any way from the results of the investigation;

    c. you present a copy of the written approval letter of the Eastern Cape Department of Education (ECDoE) to the Cluster and District Directors before any research is undertaken at any institutions within that particular district;

    d. you will make all the arrangements concerning your research;

    e. the research may not be conducted during official contact time;

    f. should you wish to extend the period of research after approval has been granted, an application to do this must be directed to Chief Director: Strategic Management Monitoring and Evaluation;

building blocks for growth   *Ilamva eliqaqambileyo!*

Page 1 of 2

g. your research will be limited to those institutions for which approval has been granted, should changes be effected written permission must be obtained from the Chief Director: Strategic Management Monitoring and Evaluation;

h. you present the Department with a copy of your final paper/report/dissertation/thesis free of charge in hard copy and electronic format. This must be accompanied by a separate synopsis (maximum 2 – 3 typed pages) of the most important findings and recommendations if it does not already contain a synopsis.

i. you present the findings to the Research Committee and/or Senior Management of the Department when and/or where necessary.

j. you are requested to provide the above to the Chief Director: Strategic Management Monitoring and Evaluation upon completion of your research.

k. you comply with all the requirements as completed in the Terms and Conditions to conduct Research in the ECDoE document duly completed by you.

l. you comply with your ethical undertaking (commitment form).

m. You submit on a six monthly basis, from the date of permission of the research, concise reports to the Chief Director: Strategic Management Monitoring and Evaluation.

3. The Department reserves a right to withdraw the permission should there not be compliance to the approval letter and contract signed in the Terms and Conditions to conduct Research in the ECDoE.

4. The Department will publish the completed Research on its website.

5. The Department wishes you well in your undertaking. You can contact the Director, Ms. NY Kanjana on the numbers indicated in the letterhead or email nykanjana@live.co.za should you need any assistance.

NY KANJANA
DIRECTOR: STRATEGIC PLANNING POLICY RESEARCH & SECRETARIAT SERVICES

FOR SUPERINTENDENT-GENERAL: EDUCATION

ANNEXURE I – Continued Approval Letter from School A

Date: 02 Feb 2017

To,
Deepak Pravin Mavani
Rhodes University
Grahamstown

Ref: <u>Yourself : Permission to conduct empirical research study</u>

Dear Mr D.P. Mavani,

We refer to the above: -

The school management team has approved your request to conduct a research study in our school. Kindly ensure that your research activities do not affect the normal school activities.

You have to follow and adhere to a set of computer lab rules of the school.

Yours faithfully,

P

N

ANNEXURE I – Continued approval letter from school B

MRS BEENA MAVANI & MR DEEPAK MAVANI
RHODES UNIVERSITY
GRAHAMSTOWN

DEAR SIR/ MADAM

Re: PERMISSION TO CONDUCT EMPIRICAL RESEARCH STUDY- YOURSELVES

The above matter refers:

The Management of the above named school has considered your request to conduct research at the school. Our understanding is that this will require a teacher and learners as participants. In principle the school has no objection to permit you for this purpose as it will also benefit teachers in the teaching of Mathematics as well as enhance learners' attainment in this subjects.

Wishing you well in your studies.

Yours faithfully

ANNEXURE II – Letter to Parents and Learner Participant Consent Form

Deepak Mavani
P O Box 205
Mthatha

email: deepakmavanip@gmail.com
Cell: 0720278333

Date: 28-07-2016

Dear Parent,

Request for permission to allow your child to participate in a study

I am a mathematics teacher at -------------------------------------undertaking a research study as a PhD student of the Rhodes University, Grahamstown. I request you to give consent and allow your child to participate in the research project entitled 'A critical analysis of whether visualisation capabilities in dynamic geometric software develop meaning making in terms of conceptual and procedural understanding of mathematical concepts in selected Grade 11 learners.' Hereafter, I refer to the project title as 'Integrating Technology in Mathematics Classroom'.

The aim of this project is to grow and develop appropriate ICT skills in participating teachers and learners to harness the teaching and learning potential of dynamic geometry software. For the purpose of this study, we will be using an open source software GeoGebra. My study will focus on the learners and explore whether learning has taken place in terms of developing mathematical meaning making. This PhD study, with a focus on learning, is paired with another PhD study, that focuses on aspects of teaching with GeoGebra applets in mathematics education. The study results will be used for informing and improving mathematics teaching and learning using technological resources like GeoGebra within the South African education system.

I do not anticipate any risk greater than normal life. Your child may enjoy while learning more productively about using technological tools in learning. Your child's participation in this project is completely voluntary. In addition to your permission, your child will also be asked if he or she would like to take part in this project. Any child may stop taking part at any time. The choice to participate or not will not impact your child's grades or status at school. Pseudonyms or codes will be substituted for the names of children and the school. This helps protect confidentiality.

The research study does not intend to disturb normal teaching and learning programme of the school. The mathematics lessons will be audio-video recorded. The audio-video recordings and all other documented information that are obtained during this research project will be strictly confidential and will not become a part of your child's school record. On completion of the project, these video-recordings will be archived and securely stored accessible only to project personnel.

The results of this study will be used for my dissertation. If you have any questions about this research project, please feel free to contact me or my research project supervisor, Prof Marc Schäfer at M.Schafer@ru.ac.za.

Please sign and return the attached consent form indicating whether you do or do not want your child to participate in this project. Your cooperation will be highly appreciated.

Yours Sincerely,

Deepak Mavani

PARENT CONSENT – Research Study - Integrating Technology in Mathematics Classroom.

I, Mr/Mrs ____

parent of ____________________________________ in grade _______

1) do agree ☐ / do not agree ☐ to allow my child to participate in this project
2) do agree ☐ / do not agree ☐ to allow my child to be video-recorded.
3) do agree ☐ / do not agree ☐ to one-one discussion my child.

Parent's signature: __________________________ Date ____________

If you have any questions about your rights as a research participant, please contact Rhodes University Ethical Standards Committee: email ethics-committe@ru.ac.za , phone 046-603 8055

Participant Information sheet (learner applications)

Researcher: Deepak Pravin Mavani

Project title: ***Integrating Technology in Mathematics Classroom***

Purpose: to critical analyse whether visualisation capabilities of technological tools develop meaning making in terms of conceptual and procedural understanding of mathematical concepts.

The research will take place for the rest of 2016 and 2017. This research project focusing on learning is paired with another research project focusing on teaching. At least four of your lessons using computer will be observed. The research may be carried a follows:

i. **you will be required to engage with GeoGebra, a dynamic geometry software, with the aid of the computer, and to record your findings in a word processor or in your answer book;**
ii. **your engagement with this software may be captured using scree-capture software and your lesson will audio-video recorded for the sole purpose of analysis and**
iii. **a discussion will be held with you regarding your use of GeoGebra and also your understanding and meaning of mathematical concepts involved in this section in mathematics.**

You are under no obligation to participate and may withdraw at any time. Only aggregated results will be reported. If you participate in this study, the information will not be linked back to you as an individual. The information will be stored in a secure environment and access to the data will be made available only to the members of the research team. Your comments will be kept confidential and any information provided will only be used for the purposes of this research.

You are welcome to discuss your participation in this study with the researcher or his/her academic advisor or to impose conditions, or withdraw from the study at any time.

If you would like to speak to an officer of the University not involved in this study, you may contact the Rhodes University Ethical Standards Committee: email ethics-committe@ru.ac.za, phone 046-603 8055

I, ________________________________ *(name of learner)*, read the above and have been explained by the researcher on the above.

I will □ / will not □participate in the above study and am fully aware that I can withdraw anytime from the study.

__________________________ ____________________

Student signature Date

ANNEXURE III – Letter to Teachers and Participation Consent

Subject: Information for participation in the PhD research project and request for participation consent

Dear colleague,

1. Introduction
I am a PhD. student in mathematics education in the Faculty of Education at Rhodes University. I humbly request your participation in an exciting research project titled 'integrating technology in mathematics classroom'. The research is a critical analysis of how the potential of technological tools may enhance teaching of mathematics.

2. Background
This study is part of the requirements for the completion of my PhD degree in mathematics education. One of the objectives of the research is to develop the appropriate ICT skills and use it effectively and strategically as both a teaching and learning tool of mathematics. For the purpose of research, we will be using dynamic geometry software, GeoGebra, an open source software.
The purpose of this research study is to:

- *analyse how teachers make use of ICT technology-aided visualisation for proficient teaching,*
- *analyse the advantages and weaknesses of using Dynamic Geometry Software (DGS) in teaching and learning of mathematics,*
- to interpret, the pedagogical practices and instructional fluencies when visual aspects of technological tools are employed in a classroom,
- contribute to the growth of a community of practice collaborating in building IT resources (applets) aligned with the South African curriculum content.

More specifically, I am looking at teachers' teaching for mathematical proficiency in relation to their classroom instructional practices using DGS. To accomplish the objectives of the research I need to observe you in practice. The focus will be on classroom teaching using GeoGebra applets in grade 10 and 11 classes. Further, I will video-record your lessons, since this will provide us with a more comprehensive recording of the lessons for a detailed analysis. This PhD study, with a focus on teaching, is paired with another PhD study, that focuses on aspects of learning with GeoGebra applets in mathematics education.

3. Description and Invitation
The study will take place during the rest of 2016 and 2017. The data collection involves meetings among teachers deliberating on how to use DGS. At least three (3) of your lessons will be audio-video recorded as you go about teaching integrating technology. You will be also engaged in a post lesson reflection interview, during which we will reflect and evaluate on each of your lessons in order to gain insights into your lessons and discuss in-depth evidences of your effective instructional practices. The ultimate research objective is not to evaluate your teaching or your compliance to the curriculum as a mathematics teacher, but rather to analyse and discuss enabling and constraining factors that you encountered when using DGS as a teaching resource to teach mathematics. Your participation in this study will not affect your regular teaching in any way as observations are being planned in during normal class times, and interviews will be conducted at conveniently arranged times. The data will be strictly confidential and only you, the researchers, my supervisor and other participating teachers will have access to it.

4. Risks and Benefits
There are no foreseeable risks involved in participating in the study. Participating in this study will give you an opportunity to study and contribute to knowledge and understanding of effective teaching practices, and share your valuable experience with others. The study results will be used for informing and improving mathematics teachers' teaching practices integrating technology, GeoGebra in particular, within the South African education system.

5. Time involvement
There will a short induction programme on introducing the key concepts of the research study for about thirty minutes. There will be short meetings to discuss on how to use the GeoGebra applets. The lesson observations will be conducted during normal class times. I also anticipate that the reflective interviews might last for about an hour, and the timings will be negotiated with you. Due to complexity of the study, your involvement and commitment is very essential. You may also be contacted over phone or email to discuss any matter related to this study.

6. Participants' rights
Participating in this research is absolutely voluntary. That is, your participation is strictly optional and at your personal discretion. Hence, after acquainting yourself with this information, you may decide whether or not to take part in the study and give consent to that effect. You also have the right to withdraw from any part of the study at any time. Your identity will be confidential and pseudonyms or codes will be substituted for the names of the participants and the school.

7. Audio-video recordings
All aspects of the study will be strictly confidential. On completion of the project, all data collected will be archived and securely stored with only research team having access to it. The findings of my study may be communicated to you upon completion of my study. However, should you have any concerns about your participation or the conduct of this research project, please feel free to contact me (see my contact details above) or my research project supervisor Prof Marc Schäfer at M.Schafer@ru.ac.za.

8. Consent

Please complete, sign the attached consent form and return it to me at your earliest convenience. I will be happy to answer any questions or queries that you might have. Hoping for a favourable response!

Yours Sincerely,

Appendix: Participant consent form

I, ………………………………, mathematics teacher of …………….…………., agree to participate in the PhD research project entitled: Integrating Technology in mathematical classroom.

In giving my consent I acknowledge that:

1. The procedures required for the project and the time involved have been spelled out to me, and any questions I had about this project have been answered to my satisfaction and expectation.

2. I have read the Information and Participant Consent Forms and have been given the opportunity to discuss the information and my involvement and level of participation in the research project with the researcher and other participating mathematics teachers who will analyse the recorded lesson videos with me.

3. I understand that I can withdraw from the study at any time at no cost without affecting my relationship with the researcher now and/or in the future.

4. I understand that the researcher will observe and record at least three of my lessons using GeoGebra applets, and involve me in post lesson reflective session, and that only the research team, their supervisor and other teachers involved in this study will have access to these data.I agree to:

- *I agree to participate in meetings for discussing the pedagogical practices using GeoGebra applets.*

- *I give consent to observe and audio-video record them my lessons using DGS.*

- *I agree to be engaged in a reflective interview, after each lesson in order to have an in depth discussion and analysis on my teaching practice.*

- *I also give consent to audio-video record such interviews conducted, for the purpose of providing an accurate record of the interviews for later analysis and interpretations.*

I understand that my involvement is strictly confidential and that no information about me or my school will be used in any way that reveals my identity or that of my school.

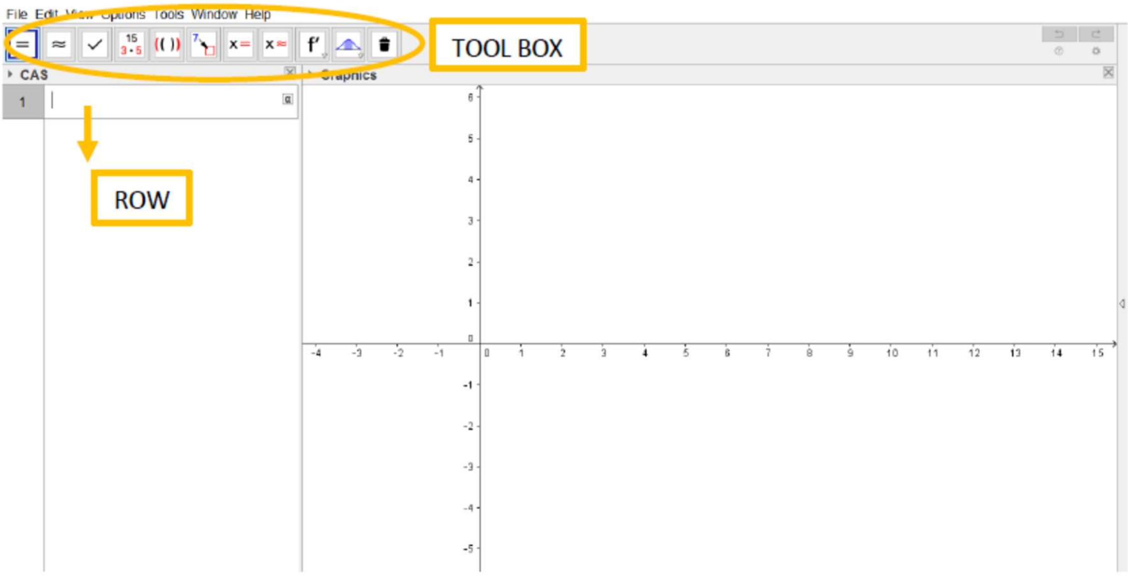
Name (participating teacher): ………………………………

Signed (participating teacher): …………………………… Date: …………………..

ANNEXURE IV – Training Interlude: CAS

Introducing CAS

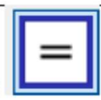
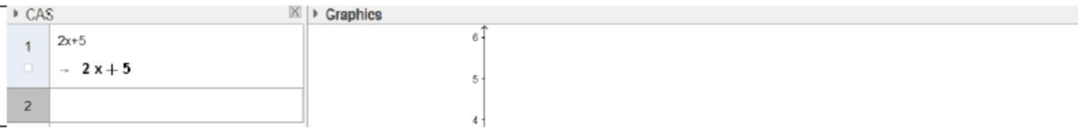
CAS is the acronym for Computer Algebra System. Choose CAS perspective by clicking on File and then click 'CAS', the following screen will appear. A CAS is a software that is used to manipulate mathematical expressions and symbols like manual computations.

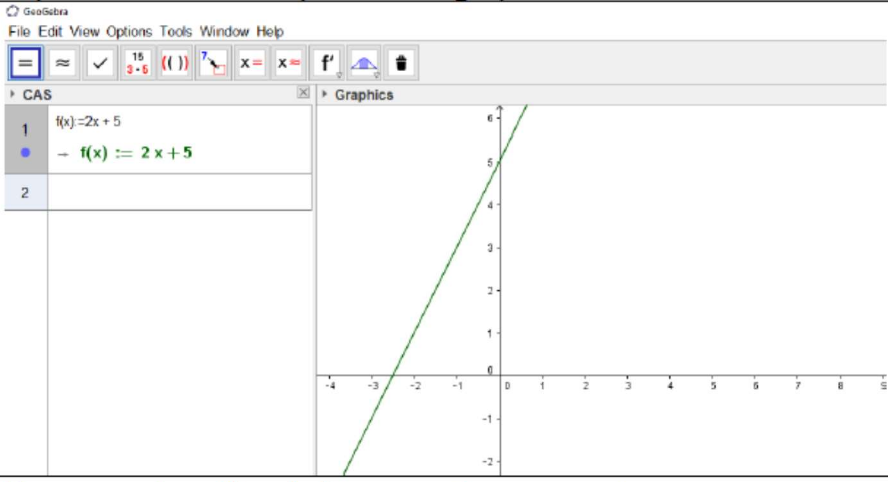
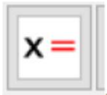
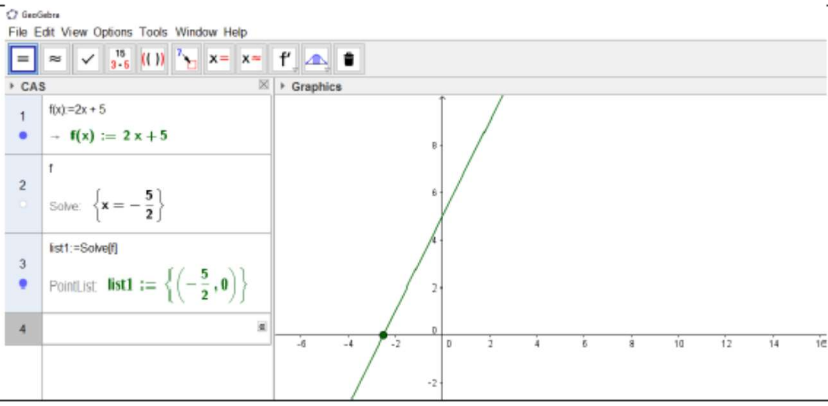


TOOLS

	Symbolic Evaluation - It evaluates the expression in the CAS row.
	Solve for x - it solves one or more equations in CAS row.

Instructions

1	Type an expression in Row: like $2x + 5$ then click 
	

2	Then click on the white button, called 'show button', below the row number – it implies to show the expression in 'graphics'.
	
3	You may retype the whole expression once again **or** copy and paste the expression **or** simply type 'f' (since the software allots it as f(x)" then select  . It will solve the equation.
	Solve: $\{x = -\frac{5}{2}\}$ will be dispalyed in row 2
4	In row 3 again type as typed in row 2 and then solve Then click on show button on row 3
	

You can now try using a different expression or equation say quadratic. When typing exponents, we use '^' which is obtained by using a combination of keys [Shift][6]. Thus to input $2x^2 - 4x + 3$, type as : **2x^2 - 4x + 3**

ANNEXURE V - WS-1 worksheet equations, inequalities and nature of roots

In GeoGebra file, drag a , b , and c so as to match the following expression / equation

Equation	Solve for x	Calculate the value of $b^2 - 4ac$	Observe the graph – x-intercepts	Observe the value of d in CAS	Comment on the nature of roots of the equation
$x^2 + 2x - 3 = 0$					
$3x^2 + 5x + 1 = 0$					
$x^2 - 4x = 0$					
$2x^2 - 5x + 3 = 0$					
$-3x^2 + 4x + 4 = 0$					
$x^2 + 2x + 1 = 0$					
$-x^2 + 4x - 4 = 0$					
$2x^2 + 3x + 2 = 0$					

ANNEXURE VI — Training interlude: Coordinates and straight lines

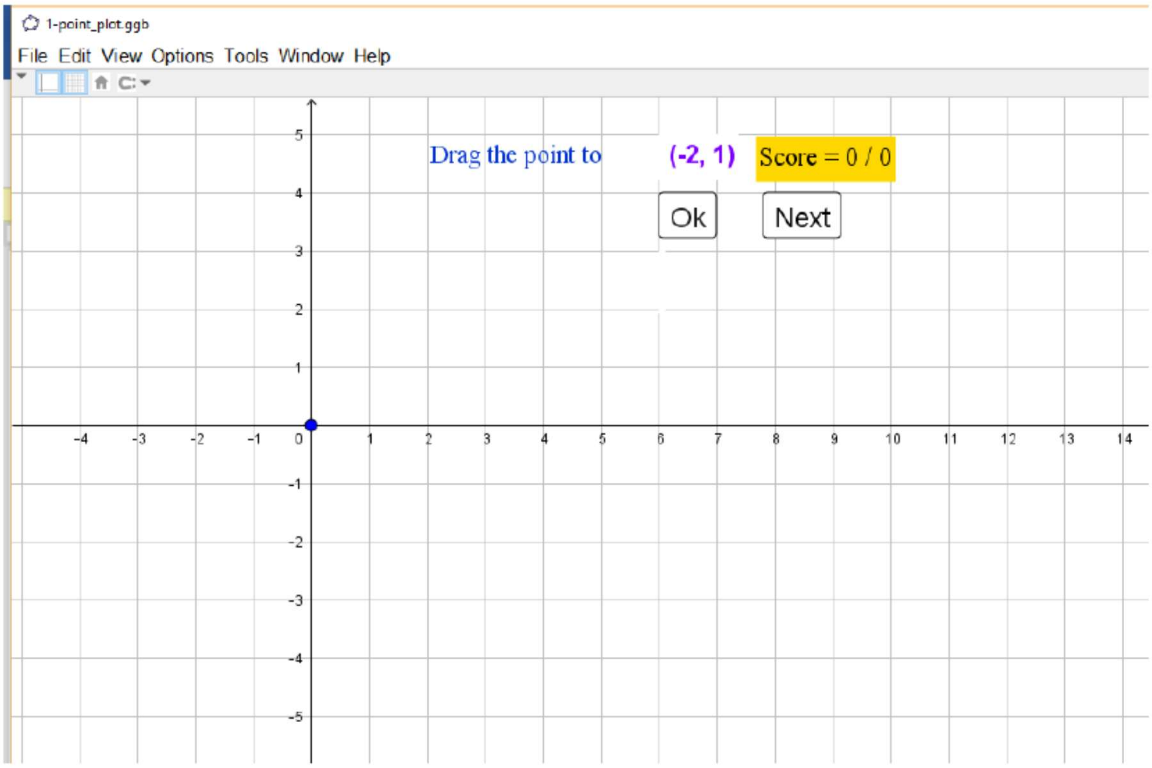
Familiarise with the coordinates

Choose Graphics perspective by clicking on File and then click Graphics.

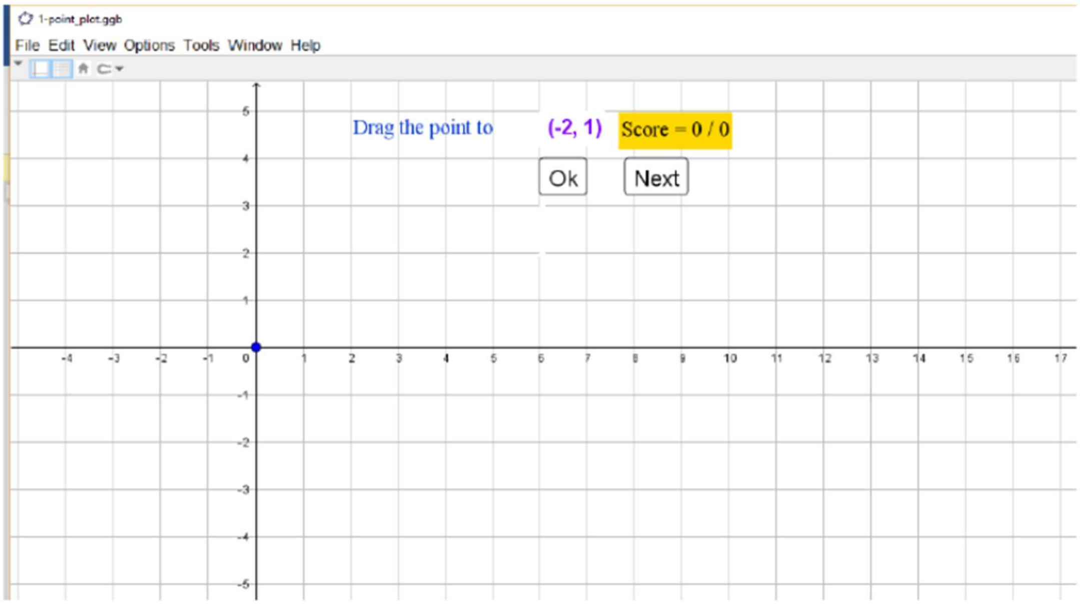
Select a point 'A' on and click on the graph. Observe carefully the values of x and y. The coordinate form (x, y) is displayed. If not displayed, right click the point, scroll down to object properties, a pop-up window will appear, tick the second check box if not 'ticked' and from the drop-down options select 'value', then click ok button and close 'object properties' dialog box, the coordinates form will appear.

Drag this point around the screen and note down the coordinates of the point. Observe carefully where 'x' values are negative and positive and similarly for 'y' values. Also note in which quadrant where both values are positive , negative and one negative and other negative.

Open the applet 1 to drag the points as mentioned. Drag the blue colour point to the required point on the graph, eg, as in the figure drag the point to (-2,1). After placing the point as required, click n the ;Ok' button and click on 'next' to move to next point to be plotted.

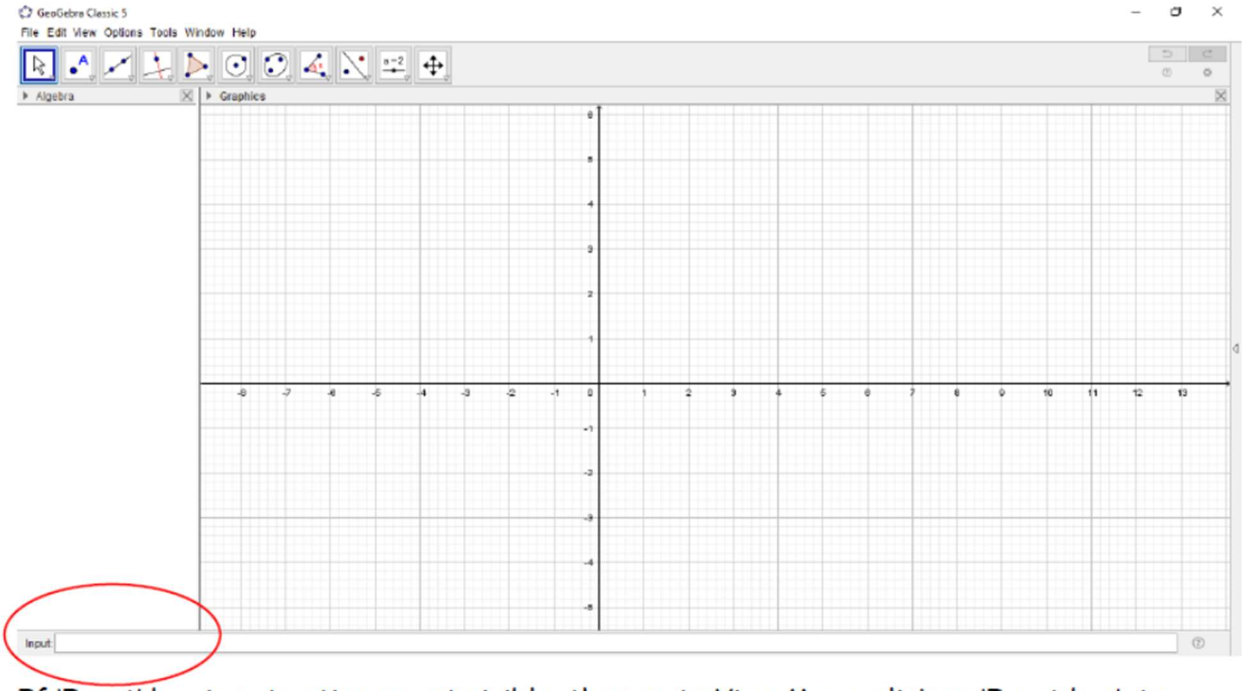


In the second applet, you have to interpret the point, and type into the space provided in coordinate form, eg. You are expected to type the coordinates of 'A'. After entering the ordered pair, click on 'Ok button, if the answer is put the software will provide feedback accordingly.



Parameters of a Linear Equation

Choose a new GeoGebra file. Then choose Graphics perspective by clicking on File and then click Graphics, the following screen will appear.



If 'Input' bar is not active or not visible, then go to View Menu, click on 'Input bar', to activate it.

Some important tips

Multiplication needs to be entered using an asterisk. Example: a*x
Asterisk symbol in keyboard is obtained by pressing shift and 8 together

GeoGebra is case sensitive! Uppercase and lowercase letters must not be mixed up.

The variable x or y within a function always need to be lower case. (e.g., f(x) = 3*x + 2 or g: y = 2*x + 4)

Confirm an expression you entered into the Input Bar by pressing the *Enter* key.

In input bar type: y = x+1. Then continuing the following equations :

$y = 2x + 1;\ y = 3x + 1;\ y = 5x + 1;\ y = -2x + 1;\ y = -4x + 1;\ \ y = -5x + 1$

Observe the lines carefully. Take note of the slant of the line, here the only chang is on the coefficient of x and the constant remains unchanged.

Now let's delete all the lines and type new set of straight lines. This time the coefficient of x remain unchanged and the constant are changed.

$y = 2x + 1$; $y = 2x + 3$; $y = 2x - 4$; $y = 2x - 2$; $y = 2x - 5$; $y = 2x$

Observe the lines carefully. Take note of the y-intercepts of the line.

Delete these lines. You may now enter more lines on your own and observe the slope and intercepts of the lines.

ANNEXURE VII – WS-2 Equations of parallel and perpendicular lines

Investigating Parallel and Perpendicular lines

Equation of a straight line : $y = mx + c$; m is the slope and c is the y-intercept. A line of the form $ax + by = c$, its slope will be determined by making y as the subject of the formula ie $y = -\frac{a}{b}x + \frac{c}{b}$

Open the applet '6_parallel_prependiclar_lines_investigate'. Note: This applet will generate the parallel and the perpendicular lines for you. You may input the equation in the blue colour 'input box' and press enter. You are required to fill in the table below (next page) and your conclusions. You can drag the point 'A' anywhere around the screen, but fill up the table for at least three points for a line.

Conclusions from the observations

Parallel Lines (||) what changes you observed? What changed and what did not? How are the slopes related?

....

....

....

....

....

Perpendicular Lines (⊥) what changes you observed? What changed and what did not? How are the slopes related?

....

....

....

....

....

Sl No	Input Line	Slope	Point (A)	Equation of || line	Slope	Equation of ⊥ line	Slope
1	$y = 2x + 3$						
2	$3y + x = 2$						
3	$y = -\frac{1}{2}x - 2$						
4	$y = 3x - 1$						
5	$5x + y = 3$						

ANNEXURE VIII – Mnemonic of properties of angles formed in parallel lines

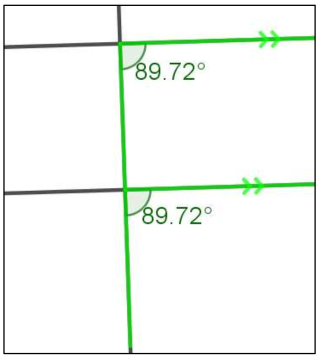


Figure 8.1: F corresponding ∠'s =

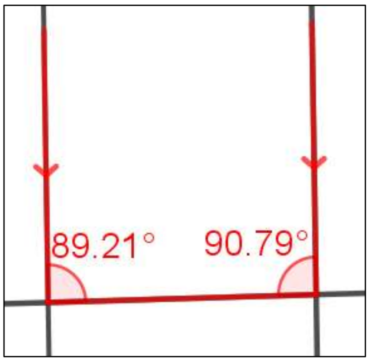


Figure 8.2: U co-interior ∠'s 180° ;

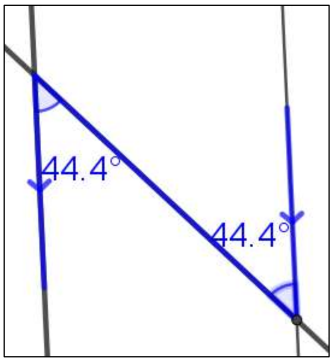


Figure 8.3: N alternate opp. ∠'s =

Representation of mnemonic F U N - properties of angles in parallel lines

There are lots of mnemonics for remembering the properties of angles in parallel lines. In south Africa, learners and teachers use the mnemonic FUN. The 'F' letter helps learners to identify the corresponding angles, illustrated in Figure 8.1 'U' letter is used to identify the co-interior angles illustrated Figure 8.2 and 'N' is used to identify the alternate opposite angles illustrated Figure 8.3.