

**RHODES UNIVERSITY**

**DEPARTMENT OF MATHEMATICS**

**FINITE FUZZY SETS, KEYCHAINS AND THEIR APPLICATIONS**

by

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## **Abstract**

The idea of keychains, an  $(n + 1)$ -tuple of non-increasing real numbers in the unit interval always including 1, naturally arises in study of finite fuzzy set theory. They are a useful concept in modeling ideas of uncertainty especially those that arise in Economics, Social Sciences, Statistics and other subjects. In this thesis we define and study some basic properties of keychains with reference to Partially Ordered Sets, Lattices, Chains and Finite Fuzzy Sets. We then examine the role of keychains and their lattice diagrams in representing un certainties that arise in such problems as in preferential voting patterns, outcomes of competitions and in Economics - Preference Relations.

### **KEYWORDS:**

Fuzzy sets, Posets, Chains, Lattices, Equivalence, Keychains, Voting, Race, Preference relation.

### **A.M.S SUBJECT CLASSIFICATION:**

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## PREFACE

In 1965, Lofti Zadeh introduced to the world the word *fuzzy* as a formalization of uncertainty or vagueness in complex systems. So in this study the interest is on the Mathematical models of fuzzy situations using the idea of *keychains*, an idea that is one of research areas of both Professors V. Murali and B.B Makamba, [19], [23], [27]. The build up to the applications of keychains is discussed in this thesis.

Chapter 1 provides some basic concepts of ordinary set and fuzzy set theories as discussed in [1], [4], [35], [5], [13], [14], [15], [16], [33], [29]. Characteristics and properties of both ordinary and fuzzy sets are displayed. This chapter also gives some examples that model some few fuzzy concepts. Later in the chapter we show how ordered sets like posets, [19], [8] lattices can be represented diagrammatically with the help of [2], [7], [10], [11], [12]. In all this chapter serves as an introduction of some concepts that will be used in the build up of the next chapters of this thesis.

Chapter 2 looks at the definition keychains according to [27], [28], [22]. We also look at the representation of keychains using pins in the form of symbols and binary digits. The keychain lattice diagrams formed by either keychains made of pins of same or different symbols and binary digits are looked at and compared. The illustrations of keychain diagrams are also well articulated in this chapter.

Chapter 3 provides the connection between keychains and fuzzy sets ac-

according to [24], [25], [26], [28] whereby the notions of keychains and flags are used in the representation of finite fuzzy sets. This chapter also provides a discussion of the equivalence of fuzzy sets, [28], [24], [38] and pinned flags. This equivalence is called the preferential equality of fuzzy sets. Later we look at the enumerations of fuzzy sets using the idea of keychains where pins of keychains will be of more help.

Chapter 4 gives us an opportunity to come with the modeling of real life situations using the idea of keychains. These life situations involve voting and competitions where there will be choices that may lead to difficulty in a preferred choice that is in voting situation. The idea of keychains is going to be utilized to show how the vagueness can be reduced to one of preferentiality. Later we show how this idea can work in a race competition.

Chapter 5 concentrates only on the applications of keychains in Economics, as read in [31], [34], [32], [18], [37]. Since the interaction of humans in the socio economic world necessitates one to make choices, each of which can be uncertain, as in [6], [9], one is expected to make a choice out of more than one at one's disposal. The choices made can be modeled as a keychain. So in this chapter we will look at how keychains can be applied in economics in preference relations and choices with the help of [3], [17].

# Chapter 1

## Review of set theory

In this chapter we look at some basic ideas of ordinary set theory and those of fuzzy set theory that are going to be useful in the formulations of this thesis. The discussion includes some explanations of partially ordered sets and lattices. Set theory is the most basic concept of modern mathematics.

### 1.1 Ordinary Set theory

By description [40], a set is a collection of objects called members or elements. These elements can be countable or uncountable.

A set is an abstract object as it does not necessarily have to contain a physical collection of elements to be a set.

In ordinary set theory the membership criteria of elements in a set must be well defined. Each single element in a collection can either belong to or not belong to a given set.

Now, let us consider a set  $X$  and an element  $x$ .

$x \in X$  indicates that the element  $x$  is a member of the set  $X$  and  $x \notin X$  indicates that  $x$  is not a member of  $X$ .

The belonging or not belonging of elements in a set is characterized by the features and properties of elements. Thus according to [13] elements in or not in a set can be determined by means of a characteristic function. For a set  $X$  and element  $x$ , its characteristic function,  $\mu_X(x)$  is denoted by

$$\mu_X(x) = \begin{cases} 0 & \text{if } x \text{ is not a member of } X \\ 1 & \text{if } x \text{ is a member of } X. \end{cases}$$

This function describes elements that are not in a set unfairly as there are elements that could differ slightly but because they do not have almost the same features may be discarded from one list to the other. In some events elements that are slightly different can be treated to be the same or having the same properties. Hence the notion of ‘*preference*’ is better suited for certain complex problems.

## Operations on sets

The operations on sets include union, intersection and complementation.

Let  $X$  and  $Y$  be two sets.

The union of  $X$  and  $Y$  denoted by  $X \cup Y$  is a set consisting of the elements of  $X$  or  $Y$  or both, which can also be interpreted as

$$x \in X \cup Y \text{ if and only if } x \in X \text{ or } x \in Y$$

The intersection of  $X$  and  $Y$  denoted by  $X \cap Y$  is a set consisting of the

elements of both  $X$  and  $Y$ . This can also be interpreted as

$$x \in X \cap Y \text{ if and only if } x \in X \text{ and } x \in Y$$

The difference or complement of the set  $Y$  in  $X$  is a set made up of elements of  $X$  that are not elements of  $Y$  denoted by  $X \setminus Y$ .

## 1.2 Fuzzy set theory

The concept of fuzzy subset was first introduced by L.A Zadeh in the year 1965. The theory of fuzzy sets is a very useful concept in representing uncertainty . It also brings a reasonable perspective when analyzing some vague concepts.

In ordinary set theory the method of looking at the collection of objects using the idea of a crisp set discriminates the elements that are not completely in the set and also those that are not completely out of the set, i.e elements that are partially in or out are not taken into consideration.

For an example the concept of "Tall people" is vague as the boundaries of tallness are uncertain. Using the concept of ordinary set theory it would mean that many people will be discarded in the set of tall people without being given a chance of belonging to that set.

It is at this point that the theory of fuzzy subsets comes in handy. It is a way of handling the uncertainty in describing the collection of tall people. This is done by grading all the people or members of the set of tall people with all possible values from 0 to 1 that give the elements a chance to have degree of membership in the said fuzzy set.

If the grading of a given element is 1, then that particular element belongs absolutely to the set. On the other hand if the grading of an element is 0 then that element is absolutely not a member of that set. The values between 0 and 1 strictly indicate the relative degrees of membership of elements to the given fuzzy subset.

The concept of fuzzy subset provides a remarkably simple way of drawing definite conclusions from vague, ambiguous or imprecise information. The theory of fuzzy sets allow linguistic variables to capture numerical values between zero (0 - absolute non-membership) and one (1 - absolute membership). These numerical values in the unit interval,  $I = [0, 1]$  can be regarded as a measure of fuzziness since they come up with precise information.

Linguistic variables can be modeled by the concept of fuzzy sets so as to come up with precise information. Some fuzzy concepts include; young, small, large, big, old, very cold, high speed and many more. The following is the definition of a fuzzy set in [40] from which the author shows that its characteristic function takes various membership values in the interval  $I = [0, 1]$ .

**Definition 1.2.1 .**

If  $X$  is a collection of objects denoted generically by  $x$  then a *fuzzy set*  $A$  in  $X$  is a set of ordered pairs:

$$A = \{(x, \mu_A(x)) | x \in X\}$$

where  $\mu_A(x)$  is the membership value of  $x$  in  $A$ .

From this definition we see that fuzzy sets are sets that allow elements to be partially in a set by assigning a value or a degree of membership to each element of a given fuzzy subset within an interval of  $I = [0, 1]$ .

Ronald R. Yager et al in [39] and [13] comes up with the following definition of a fuzzy subset. In general a fuzzy subset is always defined in the framework of some crisp set called the universe of discourse. Thus the following definition of a fuzzy subset was proposed by them.

**Definition 1.2.2 .**

Assume  $X = \{x_1, x_2, \dots, x_n\}$  is a set corresponding to the universal set. A *fuzzy set*  $\mu \in X$  is a subset in which the membership grade of any element  $x \in X$  is drawn from a unit interval,  $I$ .

Yager [39] goes on to show the framework of fuzzy set theory can be formulated as  $a_i/x_i$  where  $x_i$  represents the  $i^{th}$  element of  $X$  and the value  $a_i$  is the membership value of the  $i^{th}$  element to the fuzzy subset of  $X$ . From Yager's definition above, a fuzzy subset  $\mu$  of  $X$  can be represented as follows,

$$\mu = \{a_1/x_1, a_2/x_2, \dots, a_n/x_n\}$$

If  $a_i = 1$  then, the element  $x_i$  is an absolute member of  $\mu$ . For  $0 < a_i < 1$ , then the element  $x_i$  is a relative member of  $\mu$  to a degree  $a_i$ . The membership value,  $a_i = 0$  means that the element is totally not a member of  $\mu$ .

From this definition, one notices that as  $a_i \rightarrow 0$  the degree of membership of element,  $x_i$  gets weaker and weaker to the fuzzy subset,  $\mu$  while the degree

of membership gets stronger and stronger as  $a_i \rightarrow 1$ .

Another useful representation of fuzzy sets by [27] is as follows

$$X_0^1 \subset X_1^{\lambda_1} \subset \dots \subset X_n^{\lambda_n},$$

where  $X_0 \subset X_1 \subset \dots \subset X_n$  is a maximal chain of subsets of  $X = X_n$  and  $1 \geq \lambda_1 \geq \dots \geq \lambda_n \geq 0$  with  $\mu(x) = \lambda_i$  if and only if  $x \in X_i \setminus X_{i-1}$ , for  $i = 1, 2, \dots, n$ . This method of representation will be further discussed in chapter 3.

## Membership Values of Fuzzy Sets

A useful representation in addition to the two discussed in last section is by means of a mapping. A fuzzy subset  $\mu$ , of a non empty set  $X$ , is defined as a mapping from  $X$  into the unit interval  $I = [0, 1]$  written as follows:

$$\mu : X \rightarrow [0, 1]$$

The value  $\mu(x)$  is called the membership value of  $x$  to the fuzzy subset  $\mu$  of  $X$ . Right at this point we look at an example that models a fuzzy concept, like a young person among all people, so as to capture mathematically the vagueness using the definition of fuzzy sets.

**Example 1.2.3** *A young person model.*

This concept could be modeled differently by teenagers and other people.

Let  $X$  denote the ages of people in years. Then  $x_i$  is a positive integer. The following are the two models as viewed by the two groups.

**The model of a young person as viewed by teenagers.**

Let us consider  $Y$  to be a fuzzy subset representing a young person;

$$Y(x) = \begin{cases} 1 & \text{if } x < 25 \\ \frac{40-x}{15} & \text{if } 24 < x < 40 \\ 0 & \text{if } x \geq 40 \end{cases}$$

The following table shows degree of membership as taken from the model above.

**Table 1.2.4** *The membership values of different ages.*

Name of a Person	Age	Membership value of being young
Abel	40	0.00
Natasha	20	1.00
Lunga	30	0.67
Nceba	35	0.33
Monde	57	0.00

From the above table we see that the membership values are in the range  $[0, 1]$  where value 1 and value 0 indicate that the persons are absolutely a young person or not a young person at all, respectively.

Using this model in analyzing the way the youth of South Africa view their sports persons. South Africans judge their sports people (especially in the soccer industry) as old once one reaches the age of thirty and above. At an

age of 40 then the sport person is regarded as an old person who can no longer play a competitive sport with people less than his/ her age.

Why?

This is the way young people look at a young person in a South African context when it comes to national/ local sports people that wear the national jersey.

**The model of a young person as viewed by middle aged people.**

This model shows how the middle aged people view a young person;

$$Y(x) = \begin{cases} 1 & \text{if } x < 40 \\ \frac{60-x}{20} & \text{if } 40 \leq x < 60 \\ \frac{65-x}{200} & \text{if } 60 \leq x < 65 \\ 0 & \text{if } x \geq 65 \end{cases}$$

This model looks at a young person in a South African perspective as someone who has not reached 40 years of age and anybody who is 65 or above is old.

## Operations on fuzzy subsets

Suppose  $\mu$  and  $\nu$  are two fuzzy subsets of  $U$ .

- **The inclusion** of  $\mu$  and  $\nu$

is denoted by  $\mu \subseteq \nu$  and it means that  $\mu(x) \leq \nu(x)$  for all  $x \in U$  and at least for one  $x \in U$ ,  $\mu(x) < \nu(x)$ .

- **The Equality** of  $\mu$  and  $\nu$

denoted by  $\mu = \nu$ , this means that  $\mu$  is equal to  $\nu$  if and only if  $\mu(x) = \nu(x)$  for all  $x \in U$ .

- **The Union** of  $\mu$  and  $\nu$

denoted by  $\mu \cup \nu$  is defined as a maximum of the two individual membership functions. i.e

$$\mu \cup \nu = \text{Max}(\mu, \nu) = \mu \vee \nu,$$

and for  $x \in U$ ,

$$(\mu \vee \nu)(x) = \text{Max}(\mu(x), \nu(x)) = \mu(x) \vee \nu(x).$$

### Example 1.2.5 .

For two fuzzy subset  $\mu = \{1/x_1, 0/x_2, 0.4/x_3, 0/x_4, 0.2/x_5\}$  and

$\nu = \{0/x_1, 0.7/x_2, 0.3/x_3, 1/x_4, 0/x_5\}$  in  $X = \{x_1, x_2, x_3, x_4, x_5\}$ . (using the idea of [13] and [39]).

Their union is  $\mu \cup \nu = \{1/x_1, 0.7/x_2, 0.4/x_3, 1/x_4, 0.2/x_5\}$

- **The intersection** of  $\mu$  and  $\nu$

denoted by  $\mu \cap \nu$  is defined as the **minimum** value of the membership function of the two fuzzy subsets, i.e

$$\mu \cap \nu = \text{Min}(\mu, \nu) = \mu \wedge \nu,$$

and for  $x \in U$

$$(\mu \wedge \nu)(x) = \text{Min}(\mu(x), \nu(x)) = \mu(x) \wedge \nu(x).$$

**Example 1.2.6 .**

Using the same values from the above example, then

$$\mu \cap \nu = \text{Min}(\mu, \nu) = \{0/x_1, 0/x_2, 0.3/x_3, 0/x_4, 0/x_5\}.$$

This could also be written as

$$\mu \cap \nu = \text{Min}(\mu, \nu) = \{0.3/x_3\}$$

where 0.3 is the membership value of  $x_3$  in  $\mu \cap \nu$ .

R. R Yager, [39] proved the following theorem about union and intersection of two or more fuzzy sets.

**Theorem 1.2.7 .**

Assume that  $\mu$  and  $\nu$  are two fuzzy subsets of  $X$ . If  $\alpha(x) = (\mu \cup \nu)(x)$  and  $\beta(x) = (\mu \cap \nu)(x)$  for  $x \in X$ , then:

- (a)  $\beta \subset \alpha$ .
- (b)  $\mu \subset \alpha$  and  $\nu \subset \alpha$ .

(c)  $\beta \subset \mu$  and  $\beta \subset \nu$ .

**Proof:**

For each  $x \in X$ ,  $\beta(x) = \text{Min}[\mu(x), \nu(x)]$  and  $\alpha(x) = \text{Max}[\mu(x), \nu(x)]$  hence  $\beta(x) \leq \alpha(x)$  and thus  $\beta \subset \alpha$ .

Since  $\alpha(x) = \text{Max}[\mu(x), \nu(x)]$ , then  $\alpha(x) \geq \mu(x)$  and  $\alpha(x) \geq \nu(x)$  for all  $x \in X$  hence  $\mu \subset \alpha$  and  $\nu \subset \alpha$ .

Since  $\beta(x) = \text{Min}[\mu(x), \nu(x)]$  then  $\beta(x) \leq \mu(x)$  and  $\beta(x) \leq \nu(x)$  thus  $\beta \subset \mu$  and  $\beta \subset \nu$ .

This completes the proof.

- **The Complement**

The complement of  $\mu$ , denoted by  $\mu^C$ , is a fuzzy subset with membership function defined by

$$\mu^C(x) = 1 - \mu(x), \quad \text{for } x \in X.$$

**Example 1.2.8 .**

If  $\mu = \{1/x_1, 0.5/x_2, 0/x_3, 0.2/x_4, 0.8/x_5\}$

The complement of  $\mu$  will be

$$\mu^C = \{0/x_1, 0.5/x_2, 1/x_3, 0.8/x_4, 0.2/x_5\}.$$

## Representation of Fuzzy Subsets through $\alpha$ -cuts

Now let us consider the set of all fuzzy subsets of  $X$ , denoted by  $I^X$ . The Greek letters  $\mu$ ,  $\nu$ , etc denote the fuzzy subsets of  $X$  and  $0 \leq \alpha \leq 1$  is a real number.

### Definition 1.2.9 .

The *weak  $\alpha$ -cut* of  $\mu$  denoted by  $\mu_\alpha$  is the crisp subset given by

$$\mu_\alpha = \{x \in X : \mu(x) \geq \alpha\}.$$

### Definition 1.2.10 .

The *strong  $\alpha$ -cut* of  $\mu$  denoted by  $\mu^\alpha$  is the crisp subset of  $X$  given by

$$\mu^\alpha = \{x \in X : \mu(x) > \alpha\}.$$

### Note:

If  $\alpha = 1$  then  $\mu^\alpha = \emptyset$ ;

If  $\alpha = 0$  then  $\mu_\alpha = X$  ;

Depending on the fuzzy set,  $\mu^\alpha$  or  $\mu_\alpha$  may be empty for certain  $\alpha < 1$  and  $\mu^\alpha$  or  $\mu_\alpha$  may be  $X$  for  $\alpha > 0$ .

### Proposition 1.2.11 .

If  $\mu, \nu \in I^X$  then

1.  $\mu = \nu$  if and only if  $\mu^\alpha = \nu^\alpha$  for all  $\alpha \in I$ .
2.  $\mu = \nu$  if and only if  $\mu_\alpha = \nu_\alpha$  for all  $0 < \alpha < 1$ .

## Proof

Assume  $\mu = \nu$ . This means

$$\mu(x) = \nu(x) \text{ for all } x \in X.$$

Therefore for any  $\alpha \in I$ ,

$$\mu^\alpha = \{x : \mu(x) > \alpha\}$$

and

$$\nu^\alpha = \{x : \nu(x) > \alpha\} \text{ by definition.}$$

So  $x \in \mu^\alpha$  if and only if  $x \in \nu^\alpha$ .

This implies that  $\mu^\alpha = \nu^\alpha$ .

## Conversely

$\mu^\alpha = \nu^\alpha$  for all  $\alpha \in I$  for any two fuzzy sets,

$$\mu : X \longrightarrow I$$

$$\nu : X \longrightarrow I.$$

To show that  $\mu(x) = \nu(x)$  for all  $x \in X$ , let  $\mu(x) = \alpha$  then

$$x \notin \mu^\alpha \Leftrightarrow \alpha(x) \not> \alpha \text{ and}$$

$$x \in \nu^\alpha \Leftrightarrow \nu(x) > \alpha.$$

Suppose not, then

$$\mu(x_0) \neq \nu(x_0) \text{ for some } x_0 \in X,$$

then there is an  $\alpha$  such that

$$\mu(x_0) < \alpha < \nu(x_0), \text{ for } x_0 \notin \mu^\alpha \text{ and } x_0 \in \nu^\alpha.$$

This is a contradiction because  $x_0 \in \nu^\alpha$  implies that  $x_0 \in \mu^\alpha$ .

**Definition 1.2.12 .**

The *core* of  $\mu$  denoted by  $core(\mu)$ , is the crisp subset such that

$$core(\mu) = \{x \in X : \mu(x) = 1\}.$$

We note that the core is the subset of all elements that belong to the fuzzy subset absolutely.

**Example 1.2.13 .**

For  $X = \{x_1, x_2, x_3, x_4, x_5\}$

$$\text{and } \mu = \{1/x_1, 0/x_2, 0.3/x_3, 0/x_4, 0.7/x_5\}$$

the core is  $core(\mu) = \{x_1\}$ .

**Definition 1.2.14 .**

The *support* of  $\mu$  denoted by  $supp(\mu)$  is the crisp subset such that

$$supp(\mu) = \{x \in X : \mu(x) > 0\}.$$

If  $supp(\mu) = X$  then every element belongs to  $\mu$  with some degree  $\mu(x) > 0$ .

It is possible that  $supp(\mu) = X$  without  $\mu^\alpha = X$  or  $\mu_\alpha = X$  for any  $\alpha > 0$ .

**Example 1.2.15 .**

For a fuzzy subset,  $\mu = \{1/x_1, 0/x_2, 0.3/x_3, 0/x_4, 0.7/x_5\}$ ,

the support will be  $supp(\mu) = \{x_1; x_3; x_5\}$ .

**Definition 1.2.16 .**

The *co-support* of  $\mu$  denoted by  $cosupp(\mu)$ , is the crisp set such that

$$cosupp(\mu) = \{x \in X : \mu(x) = 0\},$$

which consists of elements that do not belong to the given fuzzy subset absolutely.

**Example 1.2.17 .**

For  $X = \{x_1, x_2, x_3, x_4, x_5\}$

and  $\mu = \{1/x_1, 0/x_2, 0.3/x_3, 0/x_4, 0.7/x_5\}$

the core is  $\text{cosupp}(\mu) = \{x_2, x_4\}$ .

The characteristic function of the alpha cut of  $\mu$  is denoted by  $\chi_{\mu^\alpha}$ .

**Proposition 1.2.18 .**

Any fuzzy subset  $\mu$  of  $X$  can be represented by its  $\alpha$ -cuts as follows:

$$\mu = \bigvee \{ \alpha \chi_{\mu^\alpha} : 0 < \alpha < 1 \}.$$

**Proof 1.2.19 .**

For any fixed  $x \in X$ , we need to prove that

$$\mu(x) = \bigvee_{\alpha} (\alpha \chi_{\mu^\alpha})(x) \quad *$$

The R.H.S. of \* is equal to  $(\bigvee_{\alpha} \alpha \chi_{\mu^\alpha})(x) = \bigvee_{\alpha} (\alpha \chi_{\mu^\alpha})(x)$ .

If  $x \in \mu^\alpha$  then  $\alpha \chi_{\mu^\alpha}(x) = \alpha$ , otherwise it is 0.

Suppose  $x \in \mu^\alpha$  then  $\mu(x) > \alpha$ . Also if  $\mu(x) > \alpha$  then  $x \in \mu^\alpha$ .

The above imply that  $x \in \mu^\alpha$  if and only if  $\mu(x) > \alpha$ .

Also  $\mu(x) > \alpha$  implies that  $\alpha \chi_{\mu^\alpha}(x) = \alpha$ .

Therefore  $\mu(x) > \alpha \chi_{\mu^\alpha}(x)$  for every  $\alpha$  such that  $\mu(x) > \alpha$ .

Thus  $\mu(x) > \bigvee_{\alpha} \alpha \chi_{\mu^\alpha}$ . The L.H.S of \* > R.H.S of \*.

Conversely for a fixed  $x \in X$ ,  $\mu(x) = \beta$ . If  $\mu(x) = \beta \leq \alpha < 1$  then

$\alpha < 1$ ,  $\alpha \chi_{\mu^\alpha}(x) = 0$ . Also if  $0 < \alpha < \beta = \mu(x)$  then  $\alpha \chi_{\mu^\alpha}(x) = \alpha$ .

Therefore  $\bigvee_{\alpha} \alpha \chi_{\mu^\alpha}(x) = \bigvee_{\alpha} \alpha \leq \beta = \mu(x)$ . We claim the equality holds.

Suppose not. Then  $\bigvee_{\alpha} \alpha \chi_{\mu^\alpha}(x) < \beta = \mu(x)$ .

If we choose  $\varphi$  such that  $\forall \alpha \chi_{\mu^\alpha} < \varphi < \beta = \mu(x)$  then  $\varphi \chi_{\mu^\varphi}(x) = \varphi$   
 Thus  $\forall \alpha \chi_{\mu^\alpha} < \varphi$  is a contradiction. Therefore  $\forall \alpha \chi_{\mu^\alpha} = \mu$ .

## 1.3 Posets

The term poset is an abbreviation for partially ordered set. That is a set whose elements are ordered but not all pairs of elements are required to be comparable in that the order. The partial order is generally denoted by ' $\leq$ '. We say  $x < y$  when  $x \leq y$  and  $x \neq y$ . The order in its usual sense may be strict as in  $<$  or non-strict as in  $\leq$ .

### Definition 1.3.1 .

let  $X$  be a non-empty set. A *strict partial order*  $R$  on  $X$  is a subset of  $X \times X$  and will satisfy the following conditions:

1. Irreflexivity:- for no  $x \in X$  does  $(x, x) \in R$  hold;
2. Antisymmetry:- if  $(x, y) \in R$ , then  $(y, x) \notin R$ ;
3. Transitivity:- if  $(x, y) \in R$  and  $(y, z) \in R$ , then  $(x, z) \in R$ .

### Definition 1.3.2 .

A *non-strict partial order*  $R$  on  $X$  satisfies the following conditions.

1. Reflexivity:- for all  $x \in X$  does  $(x, x) \in R$ ;
2. Antisymmetry:- if  $(x, y) \in R$ , and  $(y, x) \in R$  then  $x = y$ ;
3. Transitivity:- if  $(x, y) \in R$  and  $(y, z) \in R$ , then  $(x, z) \in R$ .

**Note:**

The set  $X$  together with a non strict partial order as defined above can be represented by  $(X, \leq)$ , where  $\leq$  is the non-strict partial order,  $R$ . So in short a poset is a set  $X$  carrying a partial order.

**Example 1.3.3 .**

Let  $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$  be the set of the first twelve consecutive positive integers. We define a relation  $R$  on  $X$  by  $a R b$  if and only if  $a$  divides  $b$  without a remainder for  $a, b \in X$ . It could be checked easily that the relation satisfies the three properties of a partially order, that is reflexivity, anti-symmetry and transitivity.

## Some properties of posets

Let us consider a poset  $X$ . An element  $x \in X$  is called a maximal element if there is no element  $y \in X$  such that  $x < y$ . On the other hand,  $x$  is minimal if no element  $y$  satisfies  $y < x$ .

If in a poset  $X$ , every two elements are comparable, for an example  $x_1 \geq x_2$  then the partial order is said to be a total order in  $X$ . A totally ordered set is also called a *chain*.

The set of real numbers with the usual ordering  $\leq$  defined on it is an example of a chain, because any two numbers are comparable with respect to  $\leq$ .

## Representation of posets

Here we look at a representation of a poset  $X$  in which elements in the given set can be arranged with respect to the partial order. This is shown by means of *weight diagrams* which are drawn according to *levels* as we explain below.

The elements of the poset are represented by black dots called nodes. The ordering relation in a poset gives rise to the notion of weight.

For instance if  $a \leq b$  then  $b$  is of higher weight than  $a$  or  $a$  is of lower weight than  $b$ . We also associate levels with weights. Suppose  $a < b < c$  then  $a$  is said to be on level 1,  $b$  on level 2 and  $c$  on level 3. Thus poset diagrams can be considered as weight diagrams with separating levels.

The line that joins two or more nodes shows that there is a relationship between the two or more elements in that connecting line of the weight diagram.

**A.** A *one element* poset.

Here the poset  $X$  has only one element which is only comparable to itself, i.e if  $a$  is the only element in poset  $X$  then  $a \leq a$ . This shows that there is only one way of ordering one element in a set represented by,



**B.** A *two element* poset.

Let  $X = \{a, b\}$ . From these elements we have three possibilities;

The elements can be

1. comparable to themselves i.e.  $a \leq a$  and  $b \leq b$ , but not with each other.
2. comparable in the following manner  $a \leq a$ ,  $b \leq b$  and  $a \leq b$ .
3. comparable in the following manner  $a \leq a$ ,  $b \leq b$  and  $b \leq a$ .

The last two cases are referred to as the duals of each other. Further, since the diagrams representing case(2) and case(3) are identical, we call each case self dual.

The above three possibilities can be shown diagrammatically where case (2) and (3) will resemble the same diagram.

The number of nodes that we see in each diagram represents the number of elements in the poset. The diagrams represent the possible arrangements of elements in a given poset according to their weights. We also refer to this diagram as a weight diagram of the relation  $\leq$ :

**Case 1**



There is only one way of getting this kind of arrangement.

**Case 2 and 3**



There are two ways of getting this kind of arrangement. Looking at this order one may view it as a meeting between any two members of an institution with respect to the seniority ordering. That is, one member senior to the other or vice versa.

**C.** A *three element* poset.

Let  $X = \{a, b, c\}$ . In this situation we will have 19 possibilities of which we get 5 non-isomorphic types of posets which are represented in the following five weight diagrams:

**Case (i)**



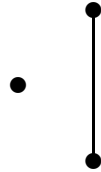
This diagram could represent the gathering of any three members of different organisation. There is only one way of getting this kind of arrangement.

**Case (ii)**



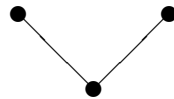
This diagram shows three levels of seniority. If it represents a gathering of members of an association, it could be one of the three senior members of that association. This is a self dual poset. There are six ways of getting this kind of arrangement.

**Case (iii)**



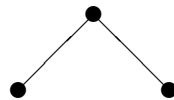
This weight diagram can be viewed as gathering of three members of an organization in a meeting situation, for an example, a disciplinary hearing in which two members are related by seniority and the third person is just an observer. There are six ways of getting this kind of arrangement.

**Case (iv)**



This can be viewed as a gathering of three members from two organizations. For instance, two doctors sharing a secretary. There are three ways of getting this kind of arrangement.

**Case (v)**



This is the opposite of the weight diagram in case iv. It can be viewed as a gathering of three people from two different organizations where one is the chairperson of both organizations. There are three ways of getting this kind of arrangement. Case(iv) and case(v) are dual to each other.

**D.** A *four element* poset.

If we go further and look at all four element posets where  $a, b, c, d \in X$ , then we have 16 non-isomorphic posets which are represented in the following diagrams. The nodes that appear in the diagrams indicate the positions occupied by the elements of a given poset. Each position can be occupied by anyone of the four elements belonging to the given poset and each one of them leads to a distinct diagram.

We list below the sixteen non-isomorphic four-element posets.

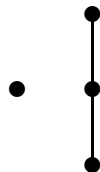
**Case (i)**



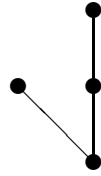
**Case (ii)**



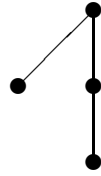
**Case (iii)**



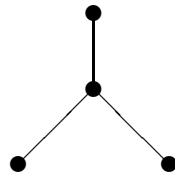
Case (iv)



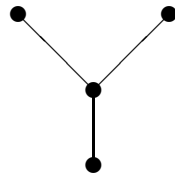
Case (v)



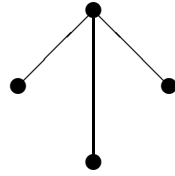
Case (vi)



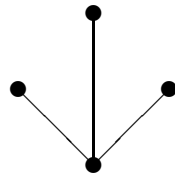
Case (vii)



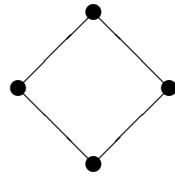
Case (viii)



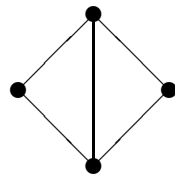
Case (ix)



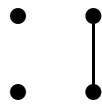
Case (x)



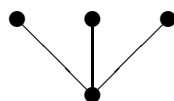
Case (xi)



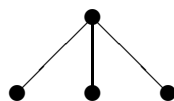
Case (xii)



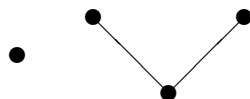
Case (xiii)



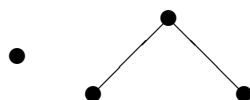
Case (xiv)



Case (xv)



Case (xvi)



The diagrams in cases (i), (ii), (iii), (x), (xi) and (xii) represent self-dual posets, whereas the five pairs namely, (iv) & (v), (vi) & (vii), (viii) & (ix), (xiii) & (xiv) and (xv) & (xvi) are all dual to each other as posets.

## 1.4 Lattices

Lattices are posets in which any two elements possess a least upper bound and greatest lower bound. In this section we develop briefly this idea. These are well-known in the literature, see for instance, [2], [10] and [12].

Suppose  $X$  is a poset. An element  $u \in X$  is said to be an upper bound for  $(x, y)$  in  $X$  if  $x \leq u$  and  $y \leq u$ . The least upper bound for  $(x, y)$  is an element  $u_0 \in X$  such that  $u_0$  is an upper bound for  $(x, y)$  in  $X$  and whenever  $u$  is an upper bound for  $(x, y)$  in  $X$  then  $u_0 \leq u$ .

Clearly the least upper bound is unique if it exists in  $X$ . It is denoted by  $x \vee y$ .

An element  $u \in X$  is said to be a lower bound for  $(x, y)$  in  $X$  if  $x \geq u$  and  $y \geq u$ . The greatest lower bound for  $(x, y)$  is an element  $u_0 \in X$  such that  $u_0$  is a lower bound for  $(x, y)$  in  $X$  and whenever  $u$  is a lower bound for  $(x, y)$  in  $X$  then  $u_0 \geq u$  and is unique if it exists in  $X$ . It is denoted by  $x \wedge y$ .

### Definition 1.4.1 .

A poset  $X$ , is said to be a *lattice* if for  $x, y \in X$  there exist a greatest lower bound of  $(x, y)$  and a least upper bound of  $(x, y)$ . If every subset of poset  $X$  has a greatest lower bound and a least upper bound then  $X$  is said to be a *complete lattice*.

The following table summarizes the above discussion of the greatest lower bound and a least upper bound respectively and sets the notation.

**Table 1.4.2** *The lub and glb notations.*

Name	Notation	Synonyms	Symbol
least upper bound	lub	join, supremum	$\vee$
greatest lower bound	glb	meet, infimum	$\wedge$

For,  $x, y \in X$  where  $X$  is a lattice,

$$x \vee y = \text{lub}(x, y)$$

is a least upper bound and

$$x \wedge y = \text{glb}(x, y)$$

is a greatest lower bound.

If  $X = \{x_1, \dots, x_n\}$  is a lattice  $(X, \leq)$  then the least upper bound of all elements in  $X$  will be denoted by

$$1 = x_1 \vee \dots \vee x_n.$$

The greatest lower bound of elements of  $X$  is denoted by

$$0 = x_1 \wedge \dots \wedge x_n.$$

The 1 and 0 are called the top and bottom elements of  $X$ . The partial order  $\leq$  is related to  $\wedge$  and  $\vee$  in  $X$  in either of the follows ways:

$$x_1 \geq x_2 \text{ if and only if } x_1 \wedge x_2 = x_2$$

and

$$x_1 \geq x_2 \text{ if and only if } x_1 \vee x_2 = x_1.$$

## Lattice Diagrams

Just as in posets, we can represent lattices by diagrams. Here we look at few diagrams that are lattices and those that are not.

Now let us consider a lattice  $(X, \leq, \wedge, \vee)$ , where there are four elements,  $a, b, c, d \in X$ .

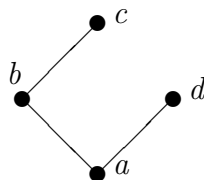
The following are diagrams of posets some of which are lattices and others not lattices.

**Figure 1.4.3** .



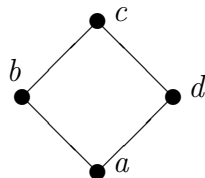
The above shows a lattice diagram where any two elements clearly possess a supremum and an infimum. In general, a totally ordered set as the one above is a lattice.

**Figure 1.4.4** .



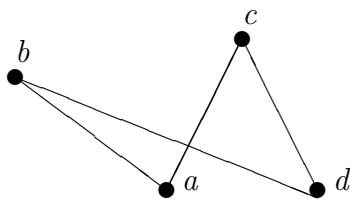
The diagram in the above figure is not a lattice diagram as there is no least upper bound for  $b$  &  $d$ .

**Figure 1.4.5 .**



This is a lattice diagram. This called a diamond lattice and is associated with two element Boolean Algebra.

**Figure 1.4.6 .**



This is not a lattice diagram as there is no least upper bound and greatest lower bound for  $a$  and  $d$ .

# Chapter 2

## Keychain lattices

### 2.1 Keychains

In this chapter we look at subsets of chains with respect to the total order of the chains. These are called keychains and they arise naturally in the study of finite fuzzy sets. We develop a novel way of representing keychains as diagrams with nodes and edges, and code keychains using binary digits. Some of the work in this chapter may be found in [23], [27] and [28].

**Definition 2.1.1** .

Let  $I = [0, 1]$  be the unit interval. Consider a chain of the form  $1 = \lambda_0 > \lambda_1 > \lambda_2 > \lambda_3 > \cdots > \lambda_n$  in  $I$  where  $\lambda_i \in I$  for each  $i = 0, 1, 2, \dots, n$ .

The finite chain or  $n$ -chain, for a natural number,  $n$  is called a *keychain* if

$$1 = \lambda_0 \geq \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n \geq 0$$

denoted by  $\ell$ . We represent the above keychain as  $\ell = 1\lambda_1\lambda_2\cdots\lambda_n$ . The numbers  $1, \lambda_1, \lambda_2, \dots, \lambda_n$  in a keychain are called *pins*. These pins are ar-

ranged and positioned in a descending order of magnitude. So naturally in a keychain 1 always occupies the first position and therefore the first pin has the value 1. The length of the keychain  $\ell$  is  $n + 1$ .

The pins in a keychain can either be distinct or not. The keychains with only pins 1 and 0 may be regarded as keychains of crisp set.

The following example illustrates the association of keychains with membership values of elements to a fuzzy subset.

**Example 2.1.2 .**

We consider five fuzzy subsets  $\mu_1, \mu_2, \mu_3, \mu_4$  and  $\mu_5$  on a three element set  $X = \{a, b, c\}$  as follows:

$$\begin{aligned} \mu_1 &= \{a/1, b/1, c/1\} \leftrightarrow 1111 \\ \mu_2 &= \{a/1, b/\frac{1}{2}, c/0\} \leftrightarrow 11\frac{1}{2}0 \\ \mu_3 &= \{a/\frac{3}{4}, b/\frac{1}{2}, c/\frac{1}{3}\} \leftrightarrow 1\frac{3}{4}\frac{1}{2}\frac{1}{3} \\ \mu_4 &= \{a/\frac{3}{4}, b/0, c/0\} \leftrightarrow 1\frac{3}{4}00 \\ \mu_5 &= \{a/0, b/0, c/0\} \leftrightarrow 1000. \end{aligned}$$

One can see that the elements of  $X$  are comparable through their membership values. Each of the membership values indicate the degree of belonging of  $a, b$  and  $c$  to a fuzzy subset of  $X$ . These membership values form a collection of numbers from the unit interval,  $I = [0, 1]$  giving rise to a keychain.

## 2.2 Keychains Diagrams.

Here we will start by looking at the way keychains are represented using their pins. If we consider keychains of length 5 where all the pins are distinct, then

we expect the following four keychains

$$1 = \lambda_1 > \lambda_2 > \lambda_3 > \lambda_4 > 0,$$

$$1 = \lambda_1 > \lambda_2 > \lambda_3 > \lambda_4 = 0,$$

$$1 > \lambda_1 > \lambda_2 > \lambda_3 > \lambda_4 > 0$$

or

$$1 > \lambda_1 > \lambda_2 > \lambda_3 > \lambda_4 = 0$$

These keychains can be denoted using different symbols for pins as follows;

$$11\lambda\beta\gamma, 11\lambda\beta 0, 1\lambda\beta\gamma\zeta \text{ and } 1\lambda\beta\gamma 0$$

respectively where  $1 > \lambda > \beta > \gamma > 0$ .

In the case of repeated pins in a keychain, we use the notation of same symbol for repetition, that is if we have the following keychain of length 6

$$1 > \lambda_1 = \lambda_2 = \lambda_3 > \lambda_4 > \lambda_5 > 0$$

can be denoted by  $1\lambda\lambda\lambda\beta\gamma$ .

According to [26] the consecutive occurrences of equality signs (pins which are the same) is said to be in an interlocking position of pins where the block of interlocked pins are called *components*.

The first pin 1 in any keychain may or may not interlock with any other pins. This pin is not considered as a part of component of any keychain. In keychains of length  $(n + 1)$ , there could be  $k$  distinct components, for  $1 \leq k \leq n$ . We refer to these types of keychains as  $k$ -pads.

The following example shows keychains with pins in interlocked positions.

**Example 2.2.1 .**

Let us consider the following keychains of the same length, 13.

$$\ell_1 = 111\lambda\beta\beta\beta\gamma\gamma 0000, \quad \ell_2 = 11\lambda\lambda\beta\gamma\gamma\gamma\gamma\zeta\zeta 0$$

$$\ell_3 = 1\lambda\lambda\beta\gamma\gamma\gamma\zeta\zeta 0000 \text{ and } \ell_4 = 1\lambda\beta\beta\gamma\zeta\zeta\zeta\zeta\delta\delta 0$$

$\ell_1$  and  $\ell_3$  are 5-pad keychains whilst  $\ell_2$  and  $\ell_4$  are 6-pad ones.

A  $k$ -pad keychain has  $k$  distinct pins. The number of the pins found in any component of a keychain is called the *padidity* of the component.

From the above example,  $\ell_1$  has the following padidity of its 5 components; 2, 1, 3, 2 and 4 respectively whilst  $\ell_2$  has 1, 2, 1, 5, 2, and 1 respectively.

Now we look at the way keychains are ordered by means of their *weight*.

By weight of a keychain of length  $n + 1$  we mean the arrangement and comparison of all their pins. Distinct keychains have weights attached to them according to the values of pins in any given position. We make this idea clear with following definition.

**Definition 2.2.2 .**

Let us consider the following two keychains of the same length,  $1\lambda_1\lambda_2 \cdots \lambda_n$  and  $1\lambda'_1\lambda'_2 \cdots \lambda'_n$ . We define a partial order  $\leq$  on the set of all keychains of the same length by

$$1\lambda_1\lambda_2 \cdots \lambda_n \leq 1\lambda'_1\lambda'_2 \cdots \lambda'_n$$

if and only if  $\lambda_i \leq \lambda'_i$  for all  $i = 1, 2, \dots, n$ . The above order is called the *weight order*.

**Proposition 2.2.3 .**

The weight order is a partial order on the set of all keychains of the same

length.

Proof: It is clear that the three properties, reflexivity, anti-symmetry and transitivity all satisfied.

For an example, the keychains  $11111$  and  $1111\lambda$  of length 5, are distinct and can be compared because of the last pin of the second one. By this observation we may say the weight of  $11111$  is bigger than that of  $1111\lambda$ . However the keychains  $11\lambda\beta 0$  and  $11100$  are not comparable. They could relatively have the same weight or not.

Here we look at some weight diagrams of keychains of different lengths.

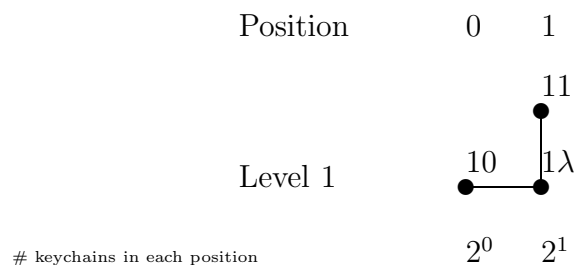
**(i) Keychain of length 1.**

This is a keychain with one pin and that pin is 1.

**(ii) Keychains of length 2.**

The total number of possible keychains of length 2 is three. They are  $11, 1\lambda, 10$ . The weight diagram of keychains of length 2 is illustrated as follows.

**Figure 2.2.4** *Keychains of length 2.*



This is a one-step (level) diagram with one path and two positions, where the movements from left to right and bottom to top show an increase in weight.

The nodes shown in these diagrams represent the keychains and are arranged in such a way that they form rows and columns. The connection of nodes by lines is indicated according to weight order. As one moves from *bottom* to *top* the degree of membership of elements increases and also as one moves from *left* to the *right* the degree of membership increases.

In all we can summarize this by saying that the line connecting the lower node to the upper node means that the lower node is less than the upper node. From this diagram the upper node represents a keychain 11 whilst the bottom two nodes from left to right are represented by  $1\lambda$  and 10 respectively. From the keychain diagram of length 2, the number of keychains is

$$3 = 1 + 2 = 2^0 + 2^1 = \sum_{i=0}^1 2^i$$

The number of terms in the summation represents the number of positions of the weight diagram and each term indicates the number of keychains in each position.

The keychains in the weight diagram are arranged in different positions, where the keychain on top possesses a bigger membership value than the one below it. For this set, in the second position we have the keychains 11 and  $1\lambda$  and

$$11 > 1\lambda$$

and this is indicated in the diagram.

From the level 1 it has been indicated that the keychain on the right of a

particular keychain possesses a bigger membership value as

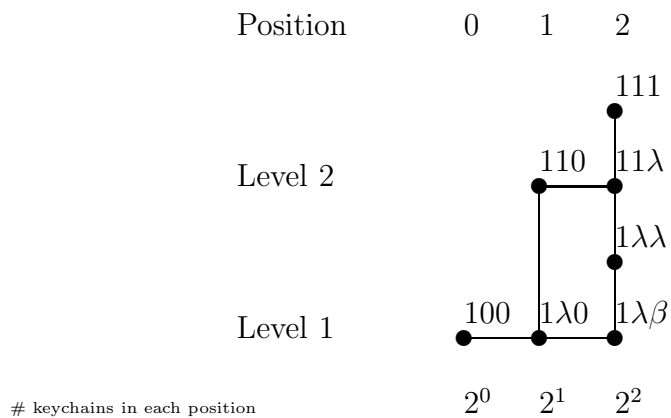
$$1\lambda > 10$$

**(iii) Keychain of length 3.**

There are seven distinct keychains of this length 3 which are:

111, 11 $\lambda$ , 110, 1 $\lambda\lambda$ , 1 $\lambda\beta$ , 1 $\lambda 0$ , 100 The weight diagram of the above keychains is depicted as follows:

**Figure 2.2.5** *Keychain diagram of length 3.*



This weight diagram is a two-step (2 level) diagram with 3 positions where the terms in the following summation represent the number of keychains in each position;

$$2^0 + 2^1 + 2^2.$$

The number of these terms represents the number of positions which are 0, 1 and 2 respectively.

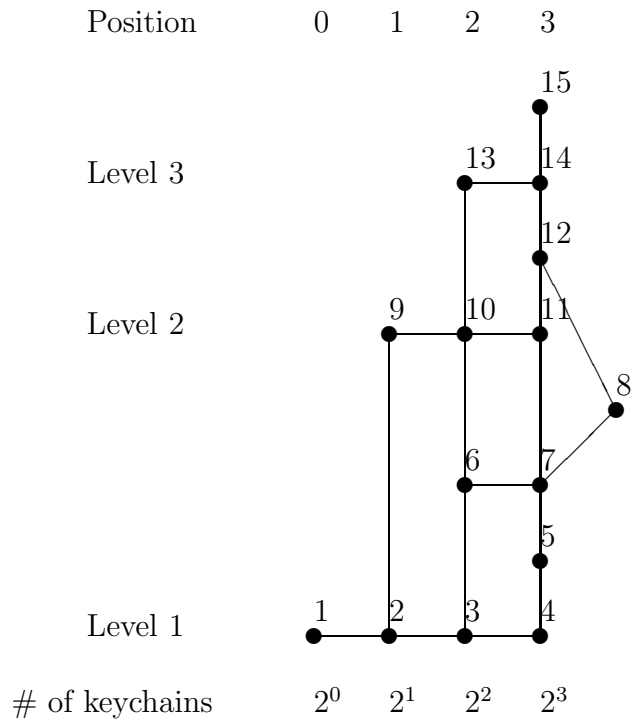
(iv) **Keychain of length 4.**

There are 15 keychains of length 4 and are

1 = 1000, 2 = 1λ00, 3 = 1λβ0, 4 = 1λβγ, 5 = 1λββ, 6 = 1λλ0, 7 = 1λλβ,  
 8 = 1λλλ, 9 = 1100, 10 = 11λ0, 11 = 11λβ, 12 = 11λλ, 13 = 1110,  
 14 = 111λ, 15 = 1111,

The weight diagram of the above keychains is as follows;

**Figure 2.2.6** *Keychain diagram of length 4.*



This is a three-step diagram with four positions.

From this diagram we see that the keychain numbered 11 is not comparable to keychain numbered 8. So there is no line connecting them.

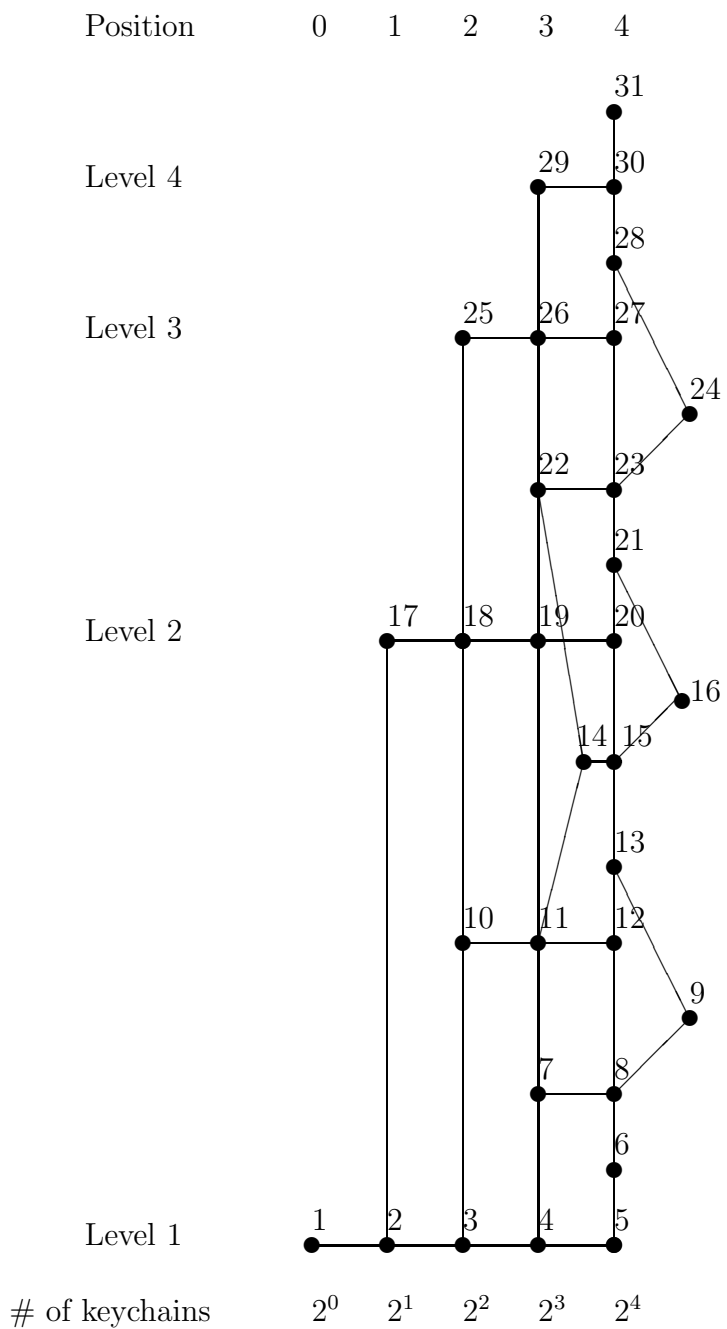
**(v) Keychain of length 5.**

There are 31 keychains of this length 5, they are listed below:

1. 10000, 2. 1 $\lambda$ 000, 3. 1 $\lambda\beta$ 00, 4. 1 $\lambda\beta\gamma$ 0, 5. 1 $\lambda\beta\gamma\tau$ , 6. 1 $\lambda\beta\gamma\gamma$ , 7. 1 $\lambda\beta\beta$ 0,
8. 1 $\lambda\beta\beta\gamma$ , 9. 1 $\lambda\beta\beta\beta$ , 10. 1 $\lambda\lambda$ 00, 11. 1 $\lambda\lambda\beta$ 0, 12. 1 $\lambda\lambda\beta\gamma$ , 13. 1 $\lambda\lambda\beta\beta$ , 14. 1 $\lambda\lambda\lambda$ 0, 15. 1 $\lambda\lambda\lambda\beta$ , 16. 1 $\lambda\lambda\lambda\lambda$ , 17. 11000,
18. 11 $\lambda$ 00, 19. 11 $\lambda\beta$ 0, 20. 11 $\lambda\beta\gamma$ , 21. 11 $\lambda\beta\beta$ , 22. 11 $\lambda\lambda$ 0,
23. 11 $\lambda\lambda\beta$ , 24. 11 $\lambda\lambda\lambda$ , 25. 11100, 26. 111 $\lambda$ 0, 27. 111 $\lambda\beta$ ,
28. 111 $\lambda\lambda$ , 29. 11110, 30. 1111 $\lambda$ , 31. 11111,

The weight diagram of the set of keychains above is shown on the following page due to insufficient space in this page.

**Figure 2.2.7** *Keychain diagram of length 5.*



The above diagram is a four-step diagram with 24 paths. The numbers 1

through to 31 from the diagram represent the 31 keychains and are arranged according to their values in an increasing order. If we look at the above figure 2.2.7, we find out that the smaller keychain diagrams are contained in the bigger keychain diagrams, that is, in a 4-element keychain diagram a 3-element keychain diagram is found embedded into it.

**Proposition 2.2.8 .**

An  $n$ -element set has  $2^{n+1} - 1$  distinct keychains of length  $n + 1$ .

**Note:** Firstly, from our observation of the keychain diagrams we note that a group of distinct keychains is located in different positions. The number of these keychains are in consecutive powers of two, starting with  $2^0$ . Thus the total number of keychains of length  $(n + 1)$  is  $2^0 + 2^1 + 2^2 + \dots + 2^n = \sum_{i=0}^n 2^i = 2^{n+1} - 1$ .

**Proof:**

This is easy to prove by mathematical induction.

- For  $n = 1$  we have  $2^{1+1} - 1 = 3$  keychains; 11,  $1\lambda$  and 10,  
for  $n = 2$  we have  $2^{2+1} - 1 = 7$  keychains; 111,  $11\lambda$ , 110,  $1\lambda\lambda$ ,  $1\lambda\beta$ ,  $1\lambda 0$   
and 100.
- we assume that for  $n = k \geq 1$  there are  $2^0 + 2^1 + 2^2 + \dots + 2^k = 2^{k+1} - 1$   
keychains.
- we want to show that the statement is true for  $n = k + 1$ .  
Thus  $\sum_{i=0}^{k+1} 2^i = 2^0 + 2^1 + 2^2 + \dots + 2^{k+1}$

$$\begin{aligned} \sum_{i=0}^{k+1} 2^i &= 2^0 + 2^1 + 2^2 + \dots + 2^k + 2^{k+1} \\ \text{But } 2^0 + 2^1 + 2^2 + \dots + 2^k &= 2^{k+1} - 1 \\ \text{Therefore } \sum_{i=0}^{k+1} 2^i &= 2^{k+1} - 1 + 2^{k+1} \\ &= 2 \cdot 2^{k+1} - 1 = 2^{(k+1)+1} - 1 \end{aligned}$$

Since the statement is true for  $n = 1$  and it is true for  $n = k + 1$  whenever it is true for  $n = k$ , we conclude the proof by Mathematical induction.

**Note:** As the number of elements increase in a set, the number of keychains also increase but exponentially. Further this leads to an increase in the number of keychains that are not comparable.

From the above keychain diagrams of keychains of length  $n+1$ , we observe that in each keychain diagram there are

- (i)  $n$ -levels or steps,
- (ii) a certain number of keychains under each step.

Counting the number of nodes below each step and arranging the resulting numbers corresponding to all steps in a horizontal line gives rise to triangular array of numbers shown in figure 2.2.9. This triangle shows the number of keychains of length  $n + 1$  under each step of keychain diagram. The first row of numbers at the top of the triangle represents the number of keychains of length 2. Similarly the entries of numbers in the second row represents the keychains of length 3 and so on. From this triangle, see figure 2.2.9, each row has a sequence of numbers where the number of terms in that sequence represents the number of levels formed in the diagram.

If exclude the last term in each row which is 1 then the number of remaining terms in that row indicates the number of steps that are formed in

the corresponding diagram. The numbers themselves indicate the number of keychains under each step. As the keychain diagrams show that any weight diagram of length  $n + 1$  will contain the keychain diagram of length  $n$ . The same fact can be inferred by going through row by row of the following triangle. For an example, all entries of row 4 are contained in row 5 whose entries are themselves contained in row 6.

**Figure 2.2.9 .**

$$\begin{array}{ccccccc}
 & & & & & & 2, 1 \\
 & & & & & & \\
 & & & & & & 2, 5, 1 \\
 & & & & & & \\
 & & & & & & 2, 5, 11, 1 \\
 & & & & & & \\
 & & & & & & 2, 5, 11, 23, 1 \\
 & & & & & & \\
 & & & & & & 2, 5, 11, 23, 47, 1 \\
 & & & & & & \\
 & & & & & & 2, 5, 11, 23, 47, 95, 1 \\
 & & & & & & \\
 & & & & & & 2, 5, 11, 23 \quad \cdots \quad \vdots \quad \cdots \quad 1 \\
 & & & & & & \\
 & & & & & & \vdots
 \end{array}$$

If we exclude the last term which is 1 in each of the patterns formed in the above triangle we get a recurrence relation of the  $a_n = 2a_{n-1} + 1$  also written as  $a(n) = 2a(n - 1) + 1$ , we have followed the notation in the integer sequences, [30], where the initial value  $a_0$  or  $a(0) = 2$ .

The numbers  $0, 1, 2, \dots, n$  represents the steps: ground step,  $1^{st}$  step,  $2^{nd}$  step up to  $n^{th}$  step respectively.

To solve the recurrence relation  $a_n = 2a_{n-1} + 1$

take  $a_h(n) = c_1 r_1^n + c_2 r_2^n + \dots + c_k r_k^n$  and  $a_p(n) = A$  as the homogeneous and particular solutions to the recurrence relation respectively.

The solution of this recurrence relation is  $a(n) = a_h(n) + a_p(n)$ .

We can write the above recurrence relation as  $r^n = 2r^{n-1} + 1$ .

The homogeneous part can be written as,

$$r^n - 2r^{n-1} = r^{n-1}(r - 2) = 0 \text{ and}$$

$$r^{n-1} \neq 0 \text{ so } r = 2.$$

Thus  $a_h(n) = c2^n$ .

Using the particular solution on the recurrence relation we find that  $a(n) = A$  and  $a(n - 1) = A$ , so

$$a(n) = 2a(n - 1) + 1$$

$$A = 2A + 1 \text{ this implies that } A = -1$$

The complete solution to the recurrence relation becomes  $a(n) = c2^n - 1$ , but

$$a(0) = 2 = c - 1 \text{ implies } c = 3.$$

Thus  $a(n) = 3 \cdot 2^n - 1$ . This sequence is referenced as A055010 (the starting point in this sequence is 0 which has no meaning in our set-up). It has many interesting properties, see [30].

The representation of the keychain diagrams brings interesting patterns that may be used to develop the study of keychains along the lines of positions, steps and levels. These can be explored further by bringing the mathematical meaning to the applications of finite fuzzy sets.

## 2.3 Lattices of Keychains.

Here we will look at the notion of keychain lattices by first introducing the two lattice operations; (a) the meet operation of weight and (b) the join operation of weight.

(a) The meet ( $\wedge$ ) operation of weight is defined as

$$(1\lambda_1\lambda_2\cdots\lambda_n) \wedge (1\lambda'_1\lambda'_2\cdots\lambda'_n) = 1\beta_1\beta_2\cdots\beta_n$$

where  $\beta_i = \lambda_i \wedge \lambda'_i$  for all  $i = 1, 2, \dots, n$ .

(b) The join ( $\vee$ ) operation of weight is defined as

$$(1\lambda_1\lambda_2\cdots\lambda_n) \vee (1\lambda'_1\lambda'_2\cdots\lambda'_n) = 1\beta_1\beta_2\cdots\beta_n$$

where  $\beta_i = \lambda_i \vee \lambda'_i$  for all  $i = 1, 2, \dots, n$ .

The meet and join operations of weight are compatible with the weight order.

## 2.4 Notion of Binary digits in Keychains.

We have another way of representing keychains. If  $\beta_1$  and  $\beta_2$  are two pins of a keychain then either  $\beta_1 = \beta_2$  or  $\beta_1 > \beta_2$ . We will denote the

equality  $\text{sign}(=)$  by 0 and

the greater than  $\text{sign}(>)$  by 1.

The two numbers, 0 and 1 are called the binary digits of the keychain. Now let us illustrate the idea of binary digits in keychains with the following two examples.

**Example 2.4.1 .**

Let us consider the following keychain of 6 pins.

$$111\lambda\beta\gamma\gamma$$

which is derived from

$$1 = \lambda_0 = \lambda_1 = \lambda_2 > \lambda_3 > \lambda_4 > \lambda_5 = \lambda_6 > 0.$$

Using the idea of binary digits in the above keychain, it will be represented as follows:

$$0011101$$

**Example 2.4.2 .**

Here we look at four keychains which are in some sense similar to each other. Let us consider the following keychains of length 4,  $11\lambda\beta$ ,  $11\lambda 0$ ,  $1\lambda\beta\gamma$ , and  $1\lambda\beta 0$ . These keychains are derived from

$$[1 = \lambda_1 > \lambda_2 > \lambda_3 > 0]$$

$$[1 = \lambda_1 > \lambda_2 > \lambda_3 = 0]$$

$$[1 > \lambda_1 > \lambda_2 > \lambda_3 > 0]$$

and

$$[1 > \lambda_1 > \lambda_2 > \lambda_3 = 0]$$

respectively. The similarity of the keychains here is that each one of them contains distinct pins.

If we represent these keychains by binary digits, then they could be written as follows 0111, 0110, 1111 and 1110 respectively.

The following tables represent keychains of distinct lengths in symbols and in binary digits.

**Table 2.4.3** *Keychains of length two.*

Symbol representation of key chains	Binary digits
$11 = [1 = \lambda > 0]$	01
$1\lambda = [1 > \lambda > 0]$	11
$10 = [1 > \lambda = 0]$	10

**Table 2.4.4** *Keychains of length three.*

Symbol representation of key chains	Binary digits
$111 = [1 = \lambda_1 = \lambda_2 > 0]$	001
$11\lambda = [1 = \lambda_1 > \lambda_2 > 0]$	011
$110 = [1 = \lambda_1 > \lambda_2 = 0]$	010
$1\lambda\lambda = [1 > \lambda_1 = \lambda_2 > 0]$	101
$1\lambda\beta = [1 > \lambda_1 > \lambda_2 > 0]$	111
$1\lambda 0 = [1 > \lambda_1 > \lambda_2 = 0]$	110
$100 = [1 > \lambda_1 = \lambda_2 = 0]$	100

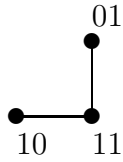
**Table 2.4.5** *Keychains of length four.*

Symbol representation of key chains	Binary digits
$1111 = [1 = \lambda_1 = \lambda_2 = \lambda_3 > 0]$	0001
$111\lambda = [1 = \lambda_1 = \lambda_2 > \lambda_3 > 0]$	0011
$1110 = [1 = \lambda_1 = \lambda_2 > \lambda_3 = 0]$	0010
$11\lambda\lambda = [1 = \lambda_1 > \lambda_2 = \lambda_3 > 0]$	0101
$11\lambda\beta = [1 = \lambda_1 > \lambda_2 > \lambda_3 > 0]$	0111
$11\lambda 0 = [1 = \lambda_1 > \lambda_2 > \lambda_3 = 0]$	0110
$1100 = [1 = \lambda_1 > \lambda_2 = \lambda_3 = 0]$	0100
$1\lambda\lambda\lambda = [1 > \lambda_1 = \lambda_2 = \lambda_3 > 0]$	1001
$1\lambda\lambda\beta = [1 > \lambda_1 = \lambda_2 > \lambda_3 > 0]$	1011
$1\lambda\lambda 0 = [1 > \lambda_1 = \lambda_2 > \lambda_3 = 0]$	1010
$1\lambda\beta\beta = [1 > \lambda_1 > \lambda_2 = \lambda_3 > 0]$	1101
$1\lambda\beta\gamma = [1 > \lambda_1 > \lambda_2 > \lambda_3 > 0]$	1111
$1\lambda\beta 0 = [1 > \lambda_1 > \lambda_2 > \lambda_3 = 0]$	1110
$1\lambda 00 = [1 > \lambda_1 > \lambda_2 = \lambda_3 = 0]$	1100
$1000 = [1 > \lambda_1 = \lambda_2 = \lambda_3 = 0]$	1000

Looking at the binary digit representation of the keychains above we find out that they are not in a numeric order. That means if we make the numeric order of the binary digits, then the weight diagram of these keychains will be totally different to those of symbolic representation.

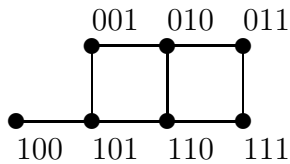
The diagrams formed in this way have two levels, a zero (0) level and a unit (1) level, with the exception of the keychain of length one which represent an empty set.

**Figure 2.4.6** *Binary Digit Keychain Diagram of length 2.*



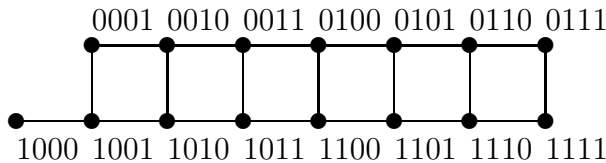
This keychain diagram has no significant difference in the shape of this one.

**Figure 2.4.7** *Binary Digit Keychain Diagram of length 3.*



This diagram shows two boxes connected to each other forming a horizontal strip.

**Figure 2.4.8** *Binary Digit Keychain Diagram of length 4.*



There are six connecting boxes which are formed from its weight diagram.

From the above figures we observe that the number of boxes strictly increase as the number of elements in the keychains increase. The following table summarizes the number of boxes that can be generated by keychains of given length. It also shows the number of paths connecting  $100 \dots 0$  and  $111 \dots 1$ .

**Table 2.4.9 .**

Elements in a set	Number of boxes	Number of paths
1	$0 = 2 - 2$	$1 = 2 - 1$
2	$2 = 4 - 2$	$3 = 4 - 1$
3	$6 = 8 - 2$	$7 = 8 - 1$
4	$14 = 16 - 2$	$15 = 16 - 1$
$\vdots$	$\vdots$	$\vdots$
$n$	$2(2^{n-1} - 1)$ for $n \geq 2$	$2^n - 1$

**Proposition 2.4.10 .**

A  $n$ -element set generates  $2^n - 2$  boxes and  $2^n - 1$  paths.

The proof to the above proposition can be done by means of mathematical induction. It follows the same steps as in the proof of proposition 2.2.8.

# Chapter 3

## Keychains and fuzzy subsets

### 3.1 Flags

Let us consider a non-empty finite set,  $X$  with  $n$ -elements such that

$$X = \{x_1, x_2, \dots, x_n\}.$$

A fuzzy subset  $\mu$  of  $X$  is a mapping  $X \longrightarrow [0, 1]$ .

A fuzzy subset with membership value zero on all of  $X$  is called an empty fuzzy subset of  $X$ .

If  $X$  is empty, then a fuzzy subset of  $X$  does not have a conventional meaning. So we assume  $X$  is always non-empty. We denote  $X_0$  to be the empty subset of  $X$ . Further we assume  $\mu : \emptyset \longrightarrow I$  takes value 1. Also any fuzzy subset  $\mu$  from a non-empty set  $X$  restricted to  $X_0$  takes the value 1.

If  $X_1$  denotes the one element set then there are three possible fuzzy sets.

These three possible fuzzy subsets on  $X_1$  arise from the keychains 11,  $1\lambda$  and 10 where  $1 > \lambda > 0$ .

(a) The keychain 11 represents a fuzzy subset  $\mu_1 : X_1 \rightarrow I$  with  $\mu_1(x) = 1$  for  $x \in X_1$ .

(b) The keychain  $1\lambda$  represents another fuzzy subset  $\mu_2 : X_1 \rightarrow I$  with  $\mu_2(x) = \lambda$  for  $x \in X_1$ .

(c) The keychain 10 represents yet another fuzzy subset  $\mu_3 : X_1 \rightarrow I$  with  $\mu_3(x) = 0$  for  $x \in X_1$ .

We can loosely say that for a one element set there are only three different fuzzy subsets corresponding to the three possibilities, namely;  $x$  may belong to  $X_1$  absolutely or may belong to  $X_1$  relatively to a degree  $0 < \lambda < 1$  or does not belong to  $X_1$  absolutely.

Similarly for two element sets and so on.

Generally when  $X = \{x_1, x_2, \dots, x_n\}$ , the chain under inclusion

$$X_0 \subset X_1 \subset X_2 \subset \dots \subset X_n = X$$

is a maximal chain of subsets of  $X$ , which is referred to as a *flag*.

The way flags and keychains are related to express fuzzy subset will be discussed below.

## 3.2 Pinned flags

**Definition 3.2.1** .

Let  $X$  be a non-empty set. Now let us consider a maximal chain,  $C$  on  $X$

and let  $\ell$  be a keychain, either in binary digits or in symbolic form. The pair  $(C, \ell)$  denoted by

$$X_0^1 \subset X_1^{\lambda_1} \subset \dots \subset X_n^{\lambda_n}$$

is called a *pinned flag* on  $X$ . According to [20] a fuzzy set,  $\mu$  associated with the above pinned flag  $(C, \ell)$  will be

$$\mu(x) = \begin{cases} 1 & \text{if } x \in X_0 \\ \lambda_1 & \text{if } x \in X_1 \setminus X_0 \\ \vdots & \vdots \\ \lambda_n & \text{if } x \in X_n \setminus X_{n-1} \end{cases}$$

where the component  $X_n$  represents the whole set of  $X$ .

**Note:**

From the above fuzzy subset  $\mu$ ,  $\mu(x)$  may or may not take the value zero. We recall that the support of a fuzzy subset  $\mu$  is

$$Supp \mu = \{x \in X : \mu(x) > 0\}.$$

If  $\lambda_n = 0$  then  $Supp \mu$  is strictly contained in  $X$  but if  $\lambda_n$  is different from zero then every element belongs to  $\mu$  with some degree  $\mu(x) > 0$ . So in this case  $Supp \mu = X$ . Therefore the first  $\lambda_i$  not equal to zero determines the support.

### 3.3 Equivalence of fuzzy subsets

Let us consider a non-empty finite set  $X$ . The set of all fuzzy subsets of  $X$  is denoted by  $I^X$  or  $[0, 1]^X$ . According to [21] the equivalence of fuzzy sets

is based on the membership degrees of elements with respect to each other rather than the absolute membership degrees of each element to the fuzzy set under consideration. These membership degrees can be clearly seen in keychains representing the said fuzzy sets.

The definition the equivalence relation  $\sim$  on  $I^X$  is as follows:

**Definition 3.3.1 .**

Let  $\mu$  and  $\nu$  be any two fuzzy subsets on  $X$ . We say  $\mu$  is *equivalent* to  $\nu$  denoted by  $\mu \sim \nu$  if and only if

- (i) for all  $x, y \in X$ ,  $\mu(x) > \mu(y)$  if and only if  $\nu(x) > \nu(y)$ ,
- (ii)  $\mu(x) = 1$  if and only if  $\nu(x) = 1$ ,
- (iii)  $\mu(x) = 0$  if and only if  $\nu(x) = 0$ .

**Note:**

The condition  $\mu(x) = 0$  if and only if  $\nu(x) = 0$  implies that the support of  $\mu$  is equal to the support of  $\nu$ . The strict inequality in condition (i) from definition can be replace by  $\geq$ . This will not affect the results as the two are equivalent.

Now we look at the following proposition of equivalence of two fuzzy subsets as proposed by [27]. The following proposition captures the notion of equivalence of two fuzzy subsets in terms of pinned-flags.

**Proposition 3.3.2 .**

Suppose two fuzzy subsets  $\mu$  and  $\nu$  corresponding to the following pinned flags

$$(C_\mu, \ell_\mu) : X_0^1 \subset X_1^{\lambda_1} \subset \dots \subset X_n^{\lambda_n}$$

and

$$(C_\nu, \ell_\nu) : X_0^1 \subset X_1^{\beta_1} \subset \cdots \subset X_m^{\beta_m}$$

are given. Then  $\mu \sim \nu$  on  $X$  if and only if,

- (i)  $n = m$ ,
- (ii)  $X_i = Y_i$  for  $i = 0, 1, \dots, n$  provided the  $\lambda_i^s$  and  $\beta_i^s$  are distinct,
- (iii)  $\lambda_i > \lambda_j$  if and only if  $\beta_i > \beta_j$  for  $1 \leq i, j \leq n$  and  $\lambda_k = 0$  if and only if  $\beta_k = 0$  for some  $k$  between 1 and  $n$ .

**Proof:**

Let  $\mu \sim \nu$ .

- (i) Define a function  $f : \mu(X) \rightarrow \nu(X)$  from the subset  $\mu(X)$  of  $I$  to the subset  $\nu(X)$  of  $I$  by  $f(\mu(x)) = \nu(x)$  for  $x \in X$ .  
It is easy to check that  $f$  is firstly well defined and secondly bijective since  $\mu \sim \nu$ . Thus  $|\mu(X)| = |\nu(X)|$ . Therefore  $m = n$ .

- (ii) We prove by induction on  $n$ .

- for  $n = 0$ ,  $X_0 = Y_0$  since each set is the empty set.

- We assume that  $X_k = Y_k$  for  $k \geq 0$ .

- We show that  $X_{k+1} = Y_{k+1}$ . Let  $g \in X_{k+1}$ .

If  $g \in X_k$ , then  $g \in Y_k$  which is contained in  $Y_{k+1}$ .

On the other hand, suppose  $g \notin X_k$ , then  $\mu(g) = \lambda_{k+1}$ . We claim that  $g \in Y_{k+1}$ .

Now suppose  $g \notin Y_{k+1}$ . Then  $\nu(g) < \beta_{k+1}$ .

Choose  $x \in Y_{k+1}$  but  $x \notin Y_k = X_k$ . Then  $\nu(g) < \nu(x)$  which implies  $\mu(g) < \mu(x)$  by equivalence. Hence  $\lambda_{k+1} = \mu(g) < \mu(x) = \alpha$  where

$$\lambda_k > \alpha > \lambda_{k+1}.$$

This implies  $x \in \mu^\alpha \subseteq X_k = Y_k$ , a contradiction. Therefore  $X_{k+1} \subseteq Y_{k+1}$ . Similarly we can show that  $Y_{k+1} \subseteq X_{k+1}$ . This completes the induction.

(iii) The property (iii) now follows from (i) and (ii) and from the definition of equivalence pertaining to support.

Conversely

Suppose  $\mu$  and  $\nu$  are two fuzzy subsets as defined in the proposition with pinned flags satisfying (i), (ii) and (iii). Then from (iii)  $\lambda_k = 0$  if and only if  $\beta_k = 0$  for some  $k$  between 1 and  $n$  is true.

By (i)  $n = m$ . Therefore  $\text{supp } \mu = \text{supp } \nu$ .

For  $x, y \in \text{supp } \mu$ , suppose  $\mu(x) > \mu(y)$ . Then  $\mu(x) = \lambda_i$  and  $\mu(y) = \lambda_j$  for some  $i, j$  in 1 to  $n$ .

But from (iii)  $\lambda_i > \lambda_j$  if and only if  $\beta_i > \beta_j$  for  $1 \leq i, j \leq n$ .

Now, (ii) and (iii) together imply  $\nu(x) > \nu(y)$ . From (iii) it is clear that  $\mu(x) = 1$  if and only if  $\nu(x) = 1$ . Thus  $\mu \sim \nu$  as required.

Now let us consider the following example of a three element set given by

$$X_3 = \{x_1, x_2, x_3\}$$

The equivalent fuzzy subsets of  $X_3$  can be handled by keychains. For this set there are 15 keychains of length 4. We illustrate the way keychains give rise to equivalent fuzzy subsets with  $\ell_1 = 111\lambda$  and  $\ell_2 = 1\lambda\lambda 0$ .

For  $\ell_1$  and  $\ell_2$  we describe equivalent fuzzy subsets  $\mu$  and  $\nu$  belonging in  $I^{X_3}$

respectively. The two fuzzy sets are as follows:

$$\mu(x) = \begin{cases} 1 & \text{if } x = x_1, x_2, \\ \lambda & \text{if } x = x_3. \end{cases}$$

$$\nu(x) = \begin{cases} \lambda & \text{if } x = x_1, x_2, \\ 0 & \text{if } x = x_3. \end{cases}$$

It is clear that condition (iii) of definition 3.3.1 is not satisfied as  $\mu(x_3) = \lambda$  and  $\nu(x_3) = 0$ . Therefore  $\mu$  and  $\nu$  are inequivalent. On the other hand the following two fuzzy sets are equivalent. Thus there is no contradiction to the definition.

$$\mu(x) = \begin{cases} 1 & \text{if } x = x_1 \\ \lambda & \text{if } x = x_2 \\ \beta & \text{if } x = x_3 \end{cases}$$

$$\nu(x) = \begin{cases} 1 & \text{if } x = x_1 \\ \lambda & \text{if } x = x_2 \\ \zeta & \text{if } x = x_3 \end{cases}$$

The above fuzzy sets,  $\mu$  and  $\nu$  arise from the keychains,  $\ell_3 = 11\lambda\beta$  and  $\ell_4 = 11\lambda\zeta$  respectively, where  $1 > \lambda > \beta > \zeta > 0$ .

## 3.4 Fuzzy subset enumerations

### Example 3.4.1 .

Now let us consider the following keychain of length  $n + 1$ ,

$$1 = \lambda_0 \geq \lambda_1 \geq \dots \geq \lambda_n \geq 0$$

where

$$\lambda_i \in [0, 1] \text{ for } i = 1, 2, \dots, n.$$

By just looking at the above keychain we see that the pin  $\lambda_0$  is always equal to 1. So there is no keychain such that  $1 = \lambda_0 = \lambda_1 = \dots = \lambda_n = 0$ . Hence the keychain of the form  $\underbrace{00 \dots 0}_{n+1 \text{ times}}$  cannot exist as this will mean that we say that 1 is equal to 0. A situation that is close to this can just be regarded as a keychain representing an empty set. An empty set of any set is represented by the keychain  $100 \dots 0$ .

**Example 3.4.2 .**

Let us consider the following distinct keychains of length 4 namely,  $111\lambda$ ,  $1110$ ,  $1\lambda\lambda\beta$ , and  $1\lambda\lambda 0$

These four distinct keychains have the same ordering in their pins in some sense where the leading pin is always 1.

If we omit the first pin in each of the given keychains then 3 pins will be left in each keychain. Looking at these four keychains we observe that a certain order or pattern into which the pins are arranged. This order can be read as “2-same pins followed by 1-different pin”. We denote this order by  $(2, 1)$ .

**Definition 3.4.3 .**

Let  $\ell$  be a keychain of length  $n + 1$ . By *order* of the keychain  $\ell$ , we mean a set of positive integers  $(n_1, n_2, \dots, n_t)$  for  $1 \leq t \leq n$  such that  $n_1 + n_2 + \dots + n_t = n$ .

**Note:**

This is the arrangement of pins in the  $n$ -positions of a keychain of length  $n + 1$  when the first pin is omitted. The numbers  $n_1, n_2, \dots, n_t$  represent the number of pins in the interlocking position called a component (this was explained in the previous chapter) and are the compositions of the integer  $n$ .

An order  $(n_1, n_2, \dots, n_t)$  may contain one or more positive integers. For example take a keychain of length 5; the possible orders of such keychains are: (4), (31), (22), (211), (13), (121), (112) and (1111). The components in the order can be distinct, the same or a combination of distinct and same, e.g (31), (22) and (112).

Distinct keychains associated with a given  $n$ -element set may share the same order.

For an example the following keychains of length 4,  $11\lambda\beta$ ,  $11\lambda 0$ ,  $1\lambda\beta\gamma$  and  $1\lambda\beta 0$  share the order, (111). There is no other keychain in this set other than these four with this order.

The following tables show orders of the  $n$ -element sets for first few values when first pin of each of the keychains is neglected.

**Table 3.4.4** *Order in a 1-element set.*

				Order
	11	$1\lambda$	10	1

**Table 3.4.5** *Orders in a 2-element set.*

				Order
$11\lambda$	110	$1\lambda\beta$	$1\lambda 0$	11
	111	$1\lambda\lambda$	100	2

**Table 3.4.6** *Orders in a 3-element set.*

				Order
$11\lambda\beta$	$11\lambda 0$	$1\lambda\beta\gamma$	$1\lambda\beta 0$	111
$11\lambda\lambda$	1100	$1\lambda\beta\beta$	$1\lambda 00$	12
$111\lambda$	1110	$1\lambda\lambda\beta$	$1\lambda\lambda 0$	21
	1111	$1\lambda\lambda\lambda$	1000	3

**Table 3.4.7** *Orders in a 4-element set.*

				Order
$11\lambda\beta\gamma$	$11\lambda\beta 0$	$1\lambda\beta\gamma\zeta$	$1\lambda\beta\gamma 0$	1111
$11\lambda\beta\beta$	$11\lambda 00$	$1\lambda\beta\gamma\gamma$	$1\lambda\beta 00$	112
$11\lambda\lambda\beta$	$11\lambda\lambda 0$	$1\lambda\beta\beta\gamma$	$1\lambda\beta\beta 0$	121
$11\lambda\lambda\lambda$	11000	$1\lambda\beta\beta\beta$	$1\lambda 000$	13
$111\lambda\lambda$	11100	$1\lambda\lambda\beta\beta$	$1\lambda\lambda 00$	22
$111\lambda\beta$	$111\lambda 0$	$1\lambda\lambda\beta\gamma$	$1\lambda\lambda\beta 0$	211
$1111\lambda$	11110	$1\lambda\lambda\lambda\beta$	$1\lambda\lambda\lambda 0$	31
	11111	$1\lambda\lambda\lambda\lambda$	10000	4

If we observe the above tables, one will notice that a group of four keychains of any given  $n$ -element set share an order. We conjecture that any order,  $(n_1, n_2, \dots, n_t)$  will always be shared by four distinct keychains. There is only one exception. For an example the following keychains of length 4, i.e. 1111,  $1\lambda\lambda\lambda$  and 1000 are the only keychains that have the order **(3)** from all the possible 15 keychains of length 4. We similarly make a conjecture that the order  $(n)$  will always be shared by three distinct keychains.

The following table shows the generation of orders of keychains from 1-element set to an  $n$ -element set. It also shows the number of orders that will be formed.

**Table 3.4.8 .**

# elements	# orders	Collection of orders
1	1	1
2	2	2,11
3	4	3,21,12,1111
4	8	4,31,22,211,13,121,112,1111
$\vdots$	$\vdots$	$\vdots$
$n$	$2^{n-1}$	$(n), (n-1,1), (n-2,2), (n-2,1,1), \dots, (11 \dots 1)$

From this table we see that the orders of each set form group or blocks of orders which are in the form of  $s_1, s_2, s_3, \dots, s_k$  for  $s_i$  is an order where  $i = 1, \dots, k$  where  $k$  is a positive integer. The value  $k$  corresponds to the number of blocks of orders.

This table further suggests that, with  $2^{n+1} - 1$  keychains of length  $n + 1$ , there will always be  $2^{n-1}$  orders into which the keychains are classified. Each order generates four different keychains of the same length with the exception of the order such that  $(n)$  where each will generate three distinct keychains.

**Proposition 3.4.9 .**

There are  $2^{n-1}$  orders associated with keychains of length  $n + 1$ .

The following proof is a modified version of the proof given in [36].

**Proof:**

Consider an infinite set with an order  $(x + x^2 + x^3 + \dots)$  where  $x$  represents a component of the order in a block and powers of components showing the number of components in a block such that

$$\begin{aligned}(x + x^2 + x^3 + \dots) &\text{ is the first block.} \\ (x + x^2 + x^3 + \dots)^2 &\text{ the second block.} \\ &\vdots \\ (x + x^2 + x^3 + \dots)^n &\text{ the } n^{\text{th}} \text{ block.} \\ &\vdots\end{aligned}$$

We want to show that a keychain of length  $n + 1$  will have  $2^{n-1}$  orders.

Let the sum of blocks be

$$(x + x^2 + x^3 + \dots) + (x + x^2 + x^3 + \dots)^2 + \dots + (x + x^2 + x^3 + \dots)^n + \dots$$

This is an infinite geometric series, so

$$\begin{aligned}s_\infty &= (x + x^2 + x^3 + \dots) + (x + x^2 + x^3 + \dots)^2 + \dots + (x + x^2 + x^3 + \dots)^n + \dots \\ s_\infty &= \sum_1^\infty x^i + (\sum_1^\infty x^i)^2 + \dots + (\sum_1^\infty x^i)^{n-1} + (\sum_1^\infty x^i)^n + \dots\end{aligned}$$

We let  $U = \sum_1^\infty x^i$ , and the above becomes

$$\begin{aligned}&= x + x^2 + x^3 + \dots \\ &= \frac{x}{1-x}\end{aligned}$$

Now  $s_\infty = U + U^2 + U^3 + \dots + U^{n-1} + U^n + \dots$

$$\begin{aligned}&= \frac{U}{1-U} \\ &= \frac{\frac{x}{1-x}}{1-\frac{x}{1-x}} \\ &= \frac{x}{1-2x}\end{aligned}$$

Thus using the binomial expansion we find

$$\begin{aligned}s_\infty &= \frac{x}{1-2x} = x(1 + 2x + 2^2x^2 + \dots + 2^{n-1}x^{n-1} + 2^n x^n + \dots) \\ s_\infty &= \frac{x}{1-2x} = x + 2x^2 + 2^2x^3 + \dots + 2^{n-1}x^n + 2^n x^{n+1} + \dots\end{aligned}$$

Thus the number  $2^{n-1}$  which is the coefficient of  $x^n$ , shows the number of orders for a keychain of length  $n + 1$ . This completes the proof.

## Chapter 4

# Applications of Keychains in Elections and Competitions.

The subject matter of this chapter is to look at the role of keychains in real life problems. Keychains naturally arise in situations that involve choices, for example in the context of voting in elections, in competitions where prizes are awarded, grading of assessments or decision-making on projects.

### 4.1 Preferential Voting

Let us consider a situation where preferential voting is to take place to choose one person from a list of  $n$ -candidates using 3 preferences of choice denoted by non-zero numbers in unit interval and zero for absolute rejection.

We determine the number of possible ways into which the three preferences can be exercised on the list of  $n$  candidates. The idea of keychains will be implemented to explain the expected number of possibilities. We could

also consider choosing, say, three candidates from a list of  $n$ -candidates using preferences. Again the idea of keychains can be applied in such a situation to determine the number of possible ways.

The voting is to be done on ballot papers by means of putting symbols of preferences next to the names of the  $n$ -candidates. These symbols will be real numbers of the form  $1 > \lambda > \beta > 0$  where 1 represents an absolute choice,  $\lambda$  and  $\beta$  are choices with two relative degrees and 0 is absolute rejection.

The voters are allowed to make their own preferences limited to three symbols in choosing their preferred candidates. The following **rules of voting** apply:

1. A ballot paper is filled by putting one of the 3 non-zero symbols next to names of each of the candidates.
2. A voter can either use one, two or all the three symbols in his or her ballot paper, where each symbol is written next to the name of the preferred candidate for that preference. **N.B**, if a voter prefers all the candidates to have the same preference, then the voter can just write the preferred symbol next to the candidate in his ballot paper, but
3. A ballot paper without any of the  $n$  candidate being given a rating, is taken as a spoilt vote. Also ballot papers with more than one symbol next to one name of a candidate are also taken to be spoilt votes.
4. There is a choice of not ranking candidates one wishes not to rank, but at least one candidate should be given a rank out of the three ranks for the ballot paper to be allowed.

From this kind of voting, if we assume that there will be no illegal votes according to the rules, then we expect to have three kinds of ballot papers:

- One is that all the three symbols in a single ballot paper have been used.
- Another is that any two of the three symbols in the ballot paper have been used.
- The third possibility is a situation where a voter decides to use only one of the three symbol in the ballot paper.

In each of the three cases above we want to find the number of possibilities. The idea of the keychains will be brought to find these possibilities into which three symbols can be used on the list of  $n$ -candidates in a voting situation.

Since there are  $n$ -candidates listed in each ballot paper, the choice made by an individual voter can be represented by means of a keychain. This keychain will be of length  $(n + 1)$ . It will be a keychain that will have at most four distinct of pins. The pins will either be 1,  $\lambda$ ,  $\beta$  or 0 with  $1 > \lambda > \beta > 0$ , where 1 shows a choice with the full confidence, and  $\lambda$  and  $\beta$  show a choice with reserved confidences with the first one greater than the other and 0 for total rejection of candidate.

In each keychain the padidity of pins show the number of candidates that are chosen for that specific choice of preference. The padidity of the pin 1 shows the number of candidates that have been voted with full confidence in that ballot paper. The padidity of pin 0 shows the number of candidates that have been rejected in that ballot paper. The same applies to the padidity

of pins  $\lambda$  or  $\beta$  for the reserved confidence with which candidates have been chosen.

### Case 1

A situation where one or two symbols are used including 0: 1; 1&0; 1& $\lambda$  and  $\lambda$ &0.

The following illustrates the generalization of the number of possible keychains that are formed when a voting with full confidence is done on  $n$ -candidates.

$$\begin{array}{c}
 1 \underbrace{11 \dots 1}_{(n)\text{times}} \\
 1 \underbrace{11 \dots 1}_{(n-1)\text{times}} 0 \\
 1 \underbrace{11 \dots 1}_{(n-2)\text{times}} 00 \\
 \vdots \\
 111 \underbrace{000 \dots 0}_{(n-2)\text{times}} \\
 11 \underbrace{000 \dots 0}_{(n-1)\text{times}}
 \end{array}
 \left. \vphantom{\begin{array}{c} 1 \dots 1 \\ 1 \dots 1 \\ 1 \dots 1 \\ \vdots \\ 111 \dots 0 \\ 11 \dots 0 \end{array}} \right\} n \text{ keychains}$$

From above the first pin has no role to play in preferential choice.

The above illustration shows that there will be  $n$  keychains indicating a vote with full confidence. A keychain of the form  $1000 \dots 0$  will always indicate a ballot paper where a voter did not make any choice and therefore such a ballot paper is taken to be a spoilt ballot paper.

In the above illustration we can replace 1 by  $\lambda$  and get  $n$  keychains again. Similarly replacing 0 by  $\lambda$  we get another set of  $n$  keychains. Thus in this



From the above illustration we find out that the number of keychains where only two ranks are used in a list of  $n$ -candidates will be in the following format

$$1 + 2 + 3 + \cdots + (n - 1) = \frac{1}{2}n(n - 1)$$

If we pick three distinct symbols from the symbols  $1, \lambda, \beta, 0$ , then we will have the following three combinations:  $(1, \lambda, \beta)$ ,  $(1, \lambda, 0)$  and  $(\lambda, \beta, 0)$ . These three combinations will give us the sum of

$$3 \sum_{i=1}^{n-1} i = \frac{3}{2}n(n - 1)$$

### Case 3

This is the case where three or four symbols have been used.

$$\begin{array}{l}
 1 \underbrace{111 \cdots 1}_{(n-2)\text{-times}} \lambda\beta \quad \left. \vphantom{\begin{array}{l} 1 \underbrace{111 \cdots 1}_{(n-2)\text{-times}} \lambda\beta \\ 1 \underbrace{111 \cdots 1}_{(n-3)\text{-times}} \lambda\lambda\beta \\ 1 \underbrace{111 \cdots 1}_{(n-3)\text{-times}} \lambda\beta\beta \\ 1 \underbrace{111 \cdots 1}_{(n-3)\text{-times}} \lambda\beta 0 \end{array}} \right\} 1 \text{ keychains} \\
 \\
 \left. \begin{array}{l}
 1 \underbrace{111 \cdots 1}_{(n-3)\text{-times}} \lambda\lambda\beta \\
 1 \underbrace{111 \cdots 1}_{(n-3)\text{-times}} \lambda\beta\beta \\
 1 \underbrace{111 \cdots 1}_{(n-3)\text{-times}} \lambda\beta 0
 \end{array} \right\} 3 \text{ keychains}
 \end{array}$$

$$\begin{array}{l}
1 \underbrace{111 \dots 1}_{(n-4)\text{-times}} \lambda\lambda\lambda\beta \\
1 \underbrace{111 \dots 1}_{(n-4)\text{-times}} \lambda\lambda\beta\beta \\
1 \underbrace{111 \dots 1}_{(n-4)\text{-times}} \lambda\lambda\beta 0 \\
1 \underbrace{111 \dots 1}_{(n-4)\text{-times}} \lambda\beta\beta\beta \\
1 \underbrace{111 \dots 1}_{(n-4)\text{-times}} \lambda\beta\beta 0 \\
1 \underbrace{111 \dots 1}_{(n-4)\text{-times}} \lambda\beta 00 \\
\vdots \\
11 \underbrace{\lambda\lambda\lambda \dots \lambda}_{(n-2)\text{-times}} \beta \\
\vdots \\
11\lambda\beta \underbrace{000 \dots 0}_{(n-3)\text{-times}}
\end{array}
\left. \vphantom{\begin{array}{l} 1 \dots \\ 11 \dots \end{array}} \right\} 6 \text{ keychains}$$

$$\left. \vphantom{\begin{array}{l} 11 \dots \\ 11\lambda\beta \dots \end{array}} \right\} \frac{2(n-2)^2}{\text{keychains}}$$

By inspection if look at the number of keychains by coming up with number of candidates we find the following table

Number of Candidates	Number Keychains
3	1
4	$4 = 1 + 3$
5	$10 = 1 + 3 + 2 \times 3$
6	$20 = 1 + 3 + 2 \times 3 + 2 \times 5$
7	$35 = 1 + 3 + 2 \times 3 + 2 \times 5 + 3 \times 5$
8	$56 = 1 + 3 + 2 \times 3 + 2 \times 5 + 3 \times 5 + 3 \times 7$
$\vdots$	$\vdots$

From this illustration we find out that

$$\sum_{i=1}^{\frac{n}{2}-1} i((2i-1) + (2i+1))$$

will be the number of keychains found when  $n$ -candidates are given, for  $n$  is an even and  $n$  is greater than 3 and that there will be

$$\sum_{i=1}^{\frac{n-3}{2}+1} i((2i-1) + (2i+1)) - \frac{1}{2}n(n-1)$$

keychains when  $n$  is an odd positive integer greater than or equal to 3. The interesting thing here is that the above two summation add and give the same general formula viz:

$$\frac{1}{6}n(n-1)(n-2)$$

when using the theory of Stirling numbers by obtaining  $i^2$  using finite integration.

**Proposition 4.1.1 .**

If 3 ratings are used on  $n$  candidates for  $n$ -even and  $n > 3$  then there will be

$$\sum_{i=1}^{\frac{n}{2}-1} i((2i-1) + (2i+1)) = \frac{1}{6}n(n-1)(n-2)$$

keychains.

**Proof:**

Let the statement be  $P(n)$ .

1. Show that  $P(n)$  is true for  $n = 4, 6$   
 from the LHS of  $P(n)$ ,  $P(4) = 4 = \text{RHS}$ .  
 also  $\text{LHS} = \text{RHS} = 20$  when  $n = 6$ , So  $P(n)$  is true for  $n = 4, 6$ .
2. Now using the inductive hypothesis, we assume that  $P(n)$  is true for  $n = k$  i.e.  $P(k)$  is

$$1(1+3) + 2(3+5) + \cdots + \frac{1}{2}(k-2)((k-3) + (k-1)) = \frac{1}{6}k(k-1)(k-2)$$

3. Now we want to show that  $P(n)$  is true for  $n = k + 2$ , the next positive integer after  $k$ . Thus  $P(k + 2)$  becomes

$$P(k+2) = 1(1+3) + 2(3+5) + \cdots + \frac{1}{2}(k-2)((k-3) + (k-1)) + \frac{1}{2}(k)((k-1) + (k+1))$$

From above we see that

$$\begin{aligned} P(k+2) &= P(k) + \frac{1}{2}(k)((k-1) + (k+1)) \\ &= \frac{1}{6}(k+2)(k+1)k = \frac{1}{6}(k+2)(k+2-1)(k+2-2) \end{aligned}$$

Since the statement is true for  $n = 1$  and it is true for  $n = k + 1$  whenever it is true for  $n = k$ , this completes the proof by mathematical induction.

## 4.2 A prize giving competition

Let us consider a race of  $n$  runners where 3 top finishers get prizes in the form of:

- 1<sup>st</sup> position  $\cdots$  Gold medal
- 2<sup>nd</sup> position  $\cdots$  Silver medal

- 3<sup>rd</sup> position ... Bronze medal.

The **first question** is, in how many ways can the medals be distributed if ties are not allowed.

We answer this question by considering three positions 1<sup>st</sup>, 2<sup>nd</sup> and 3<sup>rd</sup> of the race. We denote these positions by **1**, **2** and **3** respectively. The zero (**0**) indicates the race was not finished by the runner.

We further consider that previous recorded time can be broken and cut-off time can be escaped by the runners to qualify to get a medal. We also take into consideration that no tie will be allowed as a photo finish will be considered to distinguish and check the winners that seems to be finishing at the same time. The finishers who do not beat the cut-off time will not be considered for prizes. The following table models the way into which the medals can be distributed after the race. There are eleven possibilities which are numbered 1 to 11 horizontally on the top row of the table.

**Table 4.2.1** *Different ways of finishing the race.*

	1	2	3	4	5	6	7	8	9	10	11
New Record	1 2 3	1 2	1 2	1	1	1					
Qualifying time		3		2 3	2		1 2 3	1 2	1		
Dis-qualifying time			3		3	2 3		3	2 3	1 2 3	
Did not finish the race											1 2 3

The above model shows that it is necessary for top finishers to reach the qualifying time in order to claim prizes. The task here is to distribute the three prizes to those who finished the race within the time specified as qualifying time. There is practically no difference between columns 10 and 11, each of which is represented by the same keychain.

## Prize Distribution using Keychains.

From the above model, since three medals are to be distributed that will lead to the formation of 15 keychains of length 4. Out of the 15 keychains

the following keychains will best represent the table above.

1111, 111 $\lambda$ , 1110, 11 $\lambda\lambda$ , 11 $\lambda$ 0, 1100, 1 $\lambda\lambda\lambda$ , 1 $\lambda\lambda$ 0, 1 $\lambda$ 00, 1000.

These keychains represent the different ways of completing the race and the subsequent allocation of prizes. The omitted keychains, that is, 11 $\lambda\beta$ , 1 $\lambda\lambda\beta$ , 1 $\lambda\beta\beta$ , 1 $\lambda\beta\gamma$ , and 1 $\lambda\beta$ 0 have the same meaning as 11 $\lambda$ 0, 1 $\lambda\lambda$ 0, 1 $\lambda$ 00 and 1 $\lambda$ 00 respectively according to the rules governing the race.

Pin 1 in the keychains is chosen for the award of the prizes to those who have broken the previous record.  $\lambda$  indicates that the qualifying time has been met by the runner. The other pins ( $\beta, \gamma, 0$ ) indicate that it is either the qualifying time is not met or the race is not finished at all. The following are the possibilities that can be expected in this competition followed by their keychain representation:

1. All three runners break the previously set record and each get a prize. This situation is represented by the keychain 1111.
2. Two runners break the previously set record and the third one qualifies to finish in the qualifying time. So all three runners get prizes and this will be represented by the keychain of the form 111 $\lambda$ .
3. Two runners break the previously set record and the third one does not finish in the qualifying time. So two runners get the prizes, whilst the 3<sup>rd</sup> runner does not, is associated with the keychain 1110.
4. Only one runner breaks the previously set record and the two other runners manage to finish in the qualifying time, so that all three runners get the prizes. This will be represented by the following keychain 11 $\lambda\lambda$ .

5. Only one runner breaks the previously set record and the one other runner manages to finish in the qualifying time, with the third one not getting any prize at all, is represented by  $11\lambda 0$ .
6. Only one runner breaks the previously set record and the two other runners do not manage to finish in the qualifying time, , so the  $1^{st}$  runner gets the prize, is represented as follows  $1100$ .
7. All three top runners make the qualifying time and get the prizes, is represented as follows  $1\lambda\lambda\lambda$ .
8. only two runners make the qualifying time whilst the third one does nit, so that the prizes go to the  $1^{st}$  and  $2^{nd}$  runners, is represented by  $1\lambda\lambda 0$ .
9. only one top runner make the qualifying time and getting the prize, is captured by  $1\lambda 00$ .
10. not a single runner manages to qualify or no one finishes the race so as to get a prize , represented as follows  $1000$

From the above explanation we see that the medals can be distributed to one, two or three of the top finishers if there exist a qualifying top finisher. So far we have treated 1,2 and 3 on equal footing. If 1, 2, 3 are treated as three different runners we could permute these three among themselves so that there would be  $3 \times 2 \times 1 = 6$  ways of distributing the prizes.

For  $n$  runners we can use the idea of keychains in determining the top finishers for the prizes. If we consider the  $n$  participants we may represent them

by keychains of length  $n + 1$ . Excluding the first pin in each keychain, the  $n$  pins in each the keychain will represent the number of runners participating in the race.

In a case where pins are the same, we assume that the runners are on the same category (i.e either reached qualifying time, broke previously recorded time, or could not make the qualifying time). A photo finish may be made to decide the first, second, third, etc in each category.

In general if there are  $n$  runners in a race we have calculated from the table that there will be  $\frac{7n(n-1)(n-2)}{2} - 1$  ways distributing prizes to the top three finishers of the race. In this calculation we have excluded the case of no one completing the race.

If we take into account the runners finish but do not meet the qualifying time and if there is an added prize to the new record holder, then there are  $3n(n - 1)(n - 2)$  ways of distributing the 3 prizes.

## **Keychains of Competitors getting the Prizes.**

Here we look at the ways of representing the prize winners in a race of  $n$  runners without tie. If we represent runners by pins in a keychain neglecting the first pin, the length of the keychain will be  $n + 1$ . Thus using the idea of keychains, from out of  $n$  runners there will be  $2^{n+1} - 1$  keychains that can be formed.

As we did in prize distribution we will concentrate on three pins, i.e. 1,  $\lambda$  and 0, where 1 indicates that a new record is broken,  $\lambda$  indicates a race is

finished in qualifying time and 0 indicates that the candidate could not make the qualifying time.

In this situation the keychain of the form  $1\underbrace{000\dots0}_{n\text{-times}}$  will indicate a prize is not won. This may be due to the runner either not finishing the race or not meeting the qualifying time.

Since the prizes can be won by three top runners, two top runners or by one top runner and in each instant either a previous record is broken or not, there will be six ways of winning a prize (see table 4.2.1 columns 1 to 6), if there is an added bonus to break a previously set record. Now by looking at the restriction of three pins only in any keychain we have three cases to consider as follows;

### Case 1

If all the prizes are claimed, then out of  $n$  participants there will be  $2n(n - 1)(n - 2)$  ways of getting them. The number 2 in front indicate the two situations .i.e a new record or not. The following four keychains represent the situation;

$$1111\underbrace{000\dots0}_{(n-3)\text{times}}$$

- This means the three runners broke the previous recorded time, but to get runners for first, second and third position a photo finish is to be used.

$$111\lambda\underbrace{000\dots0}_{(n-3)\text{times}}$$

-Two of the three candidates broke the previous record and the other finished on qualifying time. For the first two a photo finish can be made if their finishing times seem to be the same for the first and second prize.

$$11\lambda\lambda \underbrace{000 \dots 0}_{(n-3)\text{times}}$$

- Here the first prize is claimed with the old record broken by one person whilst the other two finishes in qualifying time.

$$1\lambda\lambda\lambda \underbrace{000 \dots 0}_{(n-3)\text{times}}$$

- No new record made, all three winners made the qualifying time..

### Case 2

If only two of the prizes are won then there will be  $2n(n-1)$  ways of distributing two prizes to the two top winners. This situation can be represented by the following keychains

$$111 \underbrace{000 \dots 0}_{(n-2)\text{times}}, 11\lambda \underbrace{000 \dots 0}_{(n-2)\text{times}} \text{ or } 1\lambda\lambda \underbrace{000 \dots 0}_{(n-2)\text{times}}.$$

### Case 3

If only one prize is to be claimed and there is no other runner qualifying for the  $2^{nd}$  and  $3^{rd}$  prizes, then there will be  $2n$  ways of getting the top finisher. So in this case we expect the keychains to be of the form

$$11 \underbrace{000 \dots 0}_{(n-1)\text{times}} \text{ or } 1\lambda \underbrace{000 \dots 0}_{(n-1)\text{times}}.$$

Here we were showing how keychains can be used to model practical situations such as that of winning of a race with the restriction of a qualifying time. Other real life situations can be similarly modeled using the idea of keychains. They arise in subjects like Economics, Management or Social Sciences where there are concepts based on uncertainty.

# Chapter 5

## Keychains in Economics

### Introduction

In this chapter we are going to look at some applications of keychain to some ideas in Economics. The fuzzy set theory and keychains have been explained in the previous chapters. We are going to link the ideas of fuzzy sets and keychains, with some concepts of Economics.

We know that some definitions or terms in the theory of Economics are ambiguous and vague and to some extent are based on linguistic connectives or language oriented. Some mathematical tools such as calculus, linear algebra and matrices are of great use in bringing a precise explanation or understanding of Economics concepts. According to A, Pfeilsticker[32], these mathematical tools do not contribute as much as expected to the better understanding of economics concepts, however he agrees that there are advantages of using the above mentioned mathematical tools.

## 5.1 Choice and preference in Economics

Here we look at the behavior of the consumers when making decisions about their preferential choices in choosing a preferred object from a given list of different products.

The aim here is to link the idea of preferences and choices made by the consumer using the concept of keychains developed earlier. When one exercises a choice from a list of alternatives described by linguistic wedges, one is confronted with fuzzy connectives.

The route of arriving to decisions has lots of choices, that is why decision-makings are fuzzy concepts. Hence decisions can be formulated by means of keychains. As one is making a decision she compares a list of alternatives in arriving at the decision. Since comparisons entail partial orders the use of keychains enter here. We may use the values in the unit interval for ranking all the decisions that might have to be made. Partial orders imply these alternatives in Economic theory can be captured by a weight diagram.

According to Anton, P Barten and V Bohm [3], the main objective of consumer theory is to determine the impact on observable demands for commodities of alternative assumptions on the objectives, on the behavioral rules of the consumer and on the constraints which he faces when making decisions.

Commodities (products) can be divided into goods and services. Each commodity is completely specified by its physical characteristics, its location and date at which it is available or produced.

**Definition 5.1.1 .**

Let us assume that there exist a finite number of commodities with finite specification of aspects mentioned above. Let us also consider a *commodity bundle*, a list of real numbers which indicates the quantity of each *commodity*. The commodity bundle is also referred as a vector and :can be represented as follows

$$x = (x_1, x_2, \dots, x_l) \in R^l,$$

where

$x_1$  is the amount of the 1<sup>st</sup> good

$x_l$  is the amount of the  $l^{th}$  good.

These commodities can be ranked by means keychains according to the preferences or choices of the consumer. If we are having  $l$ -commodities then a keychain of length  $l + 1$  may be formed as follows

$$1 = \lambda_0 \geq \lambda_1 \geq \lambda_2, \geq \dots, \geq \lambda_l \geq 0.$$

A keychains  $1 \underbrace{11 \dots 1}_{l - times}$  means that all the  $l$ -goods are equally chosen absolutely.

Among the  $l$ -goods there is a possibility that under certain circumstances some goods are not ranked or chosen at all by the consumer. So in this case the pins representing the corresponding goods are zero. So the number of zero pins in each keychain indicates the number of goods which are out of the consumer's preferences, this means the consumer has no intension of choosing such good.

The following keychain will represent the above situation,

$$111 \underbrace{00 \dots 0}_{(n-2) \text{ times}} .$$

Otherwise if one is uncertain about her choice then the pins different from 1 and 0 may be used to represent the situation. As before we use symbols for numbers in unit interval for the degree of choice, for example,

$$1 \underbrace{\lambda \lambda \cdots \lambda}_{(n-1) \text{ times}} \beta.$$

In all the preferences of an individual can be ranked using the idea of key-chains.

For illustration, suppose a consumer goes to a shop to buy a certain brand of soft drink and come across different kinds of soft drinks. If one does not find one's favourite, then one has to make a choice on the available set of soft drinks using some kind of preferences. It is in this situation that the use of keychains can be effective to rank all the possible choices of an individual. If there are  $l$  number of goods to choose from then there will be

$$2^{l+1} - 1$$

possible preferences of a single individual in choosing one of the  $l$  goods.

In the following we discuss briefly the idea of *consumption bundles*.

Let  $X$  be a set of possible consumption bundles. We assume

1. a consumption set is non-empty,
2. inputs in consumption sets are positive quantities,
3. outputs in consumption sets are negative quantities.

Let  $x \in X$  and  $X \subset R^l$ .

The value of a commodity bundle  $P.x$  is a net outlay i.e. expenses - receipts. The value of a consumption of a consumer should not exceed his/her initial wealth (income). Let the initial wealth given by the price vector  $P$  be  $w$ , so that

$$w = P.\omega.$$

Let the set of possible consumption bundles whose values do not exceed  $w$ , of the consumer be called a budget set,  $\beta$ , so

$$\beta(P.\omega) = \{x \in X | P.x \leq \omega\}.$$

For a consumer to make a decision to choose a good from a consumption set, a preference has to take place. If for any two consumption bundles i.e.  $x$  and  $y \in X$ , it is likely that  $x$  is to be least preferred as to  $y$  or the vice versa. If there are bundles that are most preferred than the others then we let the most preferred ones as her demand.

So the demand is denoted as follows;

$$\varphi(P.\omega) = \{x \in \beta(P.\omega) | x^l \in \beta(P.\omega)\} \implies x \geq x^l \text{ not } x^l \geq x.$$

## 5.2 How fuzzy sets can be applied in Economics

Economists are known of talking about economic concepts that are imprecise, uncertain or fuzzy. Some of these concepts include, inflation, investments, unemployment, preferences and choices and so on.

The concepts are fuzzy because to most of them there is a disagreement or uncertainty over the property or properties that govern or define them. If we may take the case of unemployment: we tend to say if someone is not employed by any institution or is not having a place to work is unemployed. It is difficult to say that someone is unemployed if she is trying to make a living for herself by selling some sweets in the streets, that is a hawker. Yes of course this person is not employed by any institution, but she is unemployed to a certain extent and employed to a certain extent. It is in this sense that the notion of fuzzy sets can be used.

In the following topics the uncertainty or fuzziness of some of the above economic concepts will be explained.

As explained in the previous chapters, the notion of a fuzzy subset is a concept that is well suited for studying vaguely defined ideas.

This notion was conceptualized by Lotfi Zadeh (1965)[13]. The application of this notion is worthwhile in dealing with the uncertainty and ambiguity of fuzzy concepts, where Zadeh defined fuzzy sets as a class of objects with grades of membership. These degrees of membership are expressed by numbers belonging to the interval,  $\mathbf{I} = [0, 1]$ .

Now we look at the following examples of the fuzzy concepts, " Unemployment" and "Employability".

### **Example 5.2.1 .**

So in the case of the fuzzy economic concept "unemployment" mentioned above, if we are trying to explain some one who is regarded as being unemployed, a model using the idea of fuzzy set theory can be made to illustrate

how an unemployed person can be represented.

We can look at this by using the normal working hours of an average worker, that is someone who is working for a government funded institution. Normally one works for 8 hours per day, this makes one to work for almost 40 hrs per week. Using the total time a normal worker in a week we may have the following model which represents the fuzzy concept unemployment. For this model  $t$  represents a person who works  $t$  hours per week. The fuzzy subset  $\mu(t)$  is defined as

$$\mu(t) = \begin{cases} 1 & \text{if } t = 0 \\ \frac{40 - t}{40} & \text{if } 0 \leq t \leq 40 \\ 0 & \text{if } t \geq 40. \end{cases}$$

From the above model  $\mu(t) = 1$  means that the the person is totally unemployed whilst  $\mu(t) = 0$  represents a fully employed person, that is someone who is working for 40 hours or more per week.

**Example 5.2.2 .**

Here we look at the fuzzy concept employability using the idea of keychains. For anyone to qualify for a job, one has to meet certain qualifications. Sometimes the qualifications of candidates for a job differ from one person to another, or they may all have the same qualifications to do a certain job. In the case of candidates who are for a certain job but are having the same qualifications, then a look at their experience and competence will be an advantage.

A keychain model that will explain the concept employability of the candidates that are fully employable are going to be represented by a pin 1, whilst those that are not employable at all will be represented by pin 0. The pins whose values lie within the interval,  $I = [0, 1]$  will indicate the degrees of employability of each candidate relative to the job. This will make a very good platform to compare candidates, so the idea of keychains will be of great help in making selections of a suitable candidate for a given vacancy.

Now let us look at a situation where three people have applied for the same vacancy. Using the idea of the keychains the fuzzy concept "employable" will be represented by 15 keychains of length,  $3 + 1 = 4$ . The length of each keychain indicates the number of pins in a keychain. These keychains are as follows;

1111; 111 $\lambda$ ; 1110; 11 $\lambda\lambda$ ; 11 $\lambda\beta$ ; 11 $\lambda 0$ ; 1 $\lambda\lambda\lambda$ ; 1 $\lambda\lambda\beta$ ; 1 $\lambda\beta\beta$ ; 1 $\lambda\beta\gamma$ ; 1100; 1 $\lambda\lambda 0$ ; 1 $\lambda\beta 0$ ; 1 $\lambda 0 0$ ; 1000.

In discussing the above keychains, let us neglect the first pin of each of the above keychains. The three remaining pins represent the degrees of employability of the three candidates. From keychain 1000 we see that none of the three candidates is employable for the given vacancy.

The following table explains the possible outcomes that may be encountered in selecting one out of three candidates.

**Table 5.2.3 .**

Keychain	Interpretation of the keychain
1111	<p>This shows that all the three candidates are fully employable, they meet all the requirements for the job with their qualifications.</p> <p>A toss of coin can be used to choose one candidate from the three candidates or experience, age, enthusiasm can have a big impact in choosing that one candidate.</p>
111 $\lambda$	Two candidates are fully employable and one is reasonable employable.
1110	Only two are fully employable and one is not employable.
11 $\lambda\lambda$	Only one is fully employable and the other two are equally reasonable employable.
11 $\lambda\beta$	Only one is fully employable and the other two are reasonable employable but one is better than the other
11 $\lambda 0$	Only one candidate is full employable with one of the other two reasonable employable whilst the third one is not employable at all.

**Table 5.2.4 .**

Keychain	Interpret of the keychain (continuation)
1100	Only one candidate is full employable the others are not employable at all
1 $\lambda\lambda\lambda$	All the three candidates are equally reasonable employable
1 $\lambda\lambda\beta$	Two of the three candidates are equally better employable than the third one
1 $\lambda\lambda 0$	Two of the three candidates are equally better employable whilst the third one is not employable at all.
1 $\lambda\beta\beta$	One of the three candidates is better employable than the other two who are reasonable equally employable.
1 $\lambda\beta\gamma$	All the three are reasonable employable but they differ in degrees of employability.
1 $\lambda\beta 0$	Two of the three candidates are reasonable employable but they differ in degrees of employability whilst the third one is not employable at all.
1 $\lambda 00$	One of the three candidates is reasonable employable whilst the others are not employable at all.
1000	All the three candidates are not employable at all.

The general norm that is taken by most institutions when looking for candidates for a certain post is to advertise a position and short list 5 or fewer candidates for possible employment. If there are  $n$  applicants then there will be  $2^{n+1} - 1$  possible ways of employing suitably qualified candidates based on the degrees of suitability.

It is at the state of short listing that the idea of keychains can be fruitful,

whereby the applicants can be ordered according to the possible keychains. Using this idea means that suitable candidates can be short listed for the given position. In the case of unavailability of one or more of the fully qualified candidates then the next set of candidates to be considered will be those with higher degree of membership in the keychain.

So in general, a total of  $2^{n+1} - 1$  keychains of length  $n+1$  will be formed where  $n$  will denote the number of elements in the set on which a fuzzy concept is modelled. The pins that are found in the keychains are graded with respect to degree of membership to the fuzzy subset. Hence the method of keychains can have a very big impact in selecting or looking for a well suited candidate for a job if qualification is a priority or in any other context in which a fuzzy concept is modeled.

### **5.3 Fuzzy sets and keychains in preference relations and choices**

Let  $X$  be a consumption set of all commodity bundles.

A consumer is assumed to make a choice and to have some preferences among commodity bundles in the consumption set  $X$ . The consumer choices from a universal set are always absolute. When one is looking at his or her choices from the commodity bundles, you may find out that the chosen commodities have each a preferential ranking which is totally ordered on  $X$ .

These consumer preferences can be represented by a binary relation,  $\succeq$ , on  $X$  called a preference relation.

**Definition 5.3.1** .

Let  $X$  be a set of alternatives (e.g shares, stocks, etc.) and  $\succeq$  be any relation on  $X$ . Now we consider any 2 commodity bundles  $x, y \in X$ . The statement

$$x \succeq y$$

is read as 'x is at least as good as y'.

The three basic axioms that are satisfied by "  $\succeq$  " are

**Axiom 1** Reflexivity,

$\forall x \in X, x \succeq x$  i.e. any bundle is as good as itself.

**Axiom 2** Transitivity,

for any  $x, y, z \in X, x \succeq y \ \& \ y \succeq z \Rightarrow x \succeq z$

**Axiom 3** Completeness,

for any two bundles  $x, y \in X, x \succeq y$  or  $y \succeq x$ .

We call the relation  $\succeq$  a *preference relation*.

From the above explanation we see that any two preferences must be comparable. The comparability of preferences means that preferences can be ranked using the idea of keychains.

(a) A preference relation that satisfies all the above 3 axioms is a complete preordering on  $X$  known as preference order. From this preference order two relations can be immediately derived, one is a strict preference relation and the other is a relation of *indifference*.

- The strict preference relation ( $\succ$ ).

A commodity bundle  $x$  is said to be strictly preferred to  $y$  if

$$x \succ y \iff x \succeq y \text{ but definitely not } y \succeq x.$$

- The indifference relation ( $\sim$ ) is defined by

$$x \sim y \iff x \succeq y \text{ and } y \succeq x$$

where  $x \sim y$  is read as  $x$  is indifferent to  $y$ .

(b) Individual preferences are assumed to be rational. That is if it possesses the following two properties:

- Completeness i.e. for all  $x, y \in X$  we have  $x \succeq y$  or  $y \succeq x$  (or both).
- Transitivity i.e. for all  $x, y, z \in X, x \succeq y \ \& \ y \succeq z \implies x \succeq z$ .

So from the assumption  $\succeq$  is complete, it means that an individual has a well defined preference between any two possible alternatives in this case.

If the relation,  $\succeq$  is rational, then the

1. Strict preference relation, ( $\succ$ ) is both irreflexive and transitive i.e.
  - $x \succ y$  never holds,
  - $x \succ y \ \& \ y \succ z$  always holds.
2. The indifference relation, ( $\sim$ ) will be,
  - reflexive, that is  $x \sim x$  for all  $x$ ,

- transitivity, that is  $x \sim y$  &  $y \sim z$  then  $x \sim z$ ,
- complete, that is  $x \sim y$  then  $y \sim x$ .

(c) A preference relation,  $\succeq$  on  $X$  is said to be monotone if  $x \in X$  and  $y \succeq x$  imply that  $y > x$ .

Preference relations are often described by means of utility functions which assign a value to each element in the set of alternatives indicating the degree of ones' preferences.

Any preference relation can be represented by utility function only if it is rational. The results on the connections between utility functions and preferential relations can be found in [18].

The behavior of consumers is reflected in their preferences or choice of goods. Hence the idea of a choice structure arises which we discuss below. First we start with an example.

Suppose there are a number of brands of coffee available in a shop. We assume that a customer writes down a list of brands of coffee subject to affordability. This list is called a budget  $\beta$ .

From the budget a consumer can make a choice of taking one or more brands or choose none at all. This leads the consumer to have different alternatives.

If  $X$  is the set of all choices, then  $\beta \subseteq X$ . Now we consider the chosen brands in  $\beta$  to be  $C(\beta)$  called the choice rule. We observe that

$$C(\beta) \subseteq \beta \subseteq X.$$

According to [9] a fuzzy subset is defined in a standard way as

$$C(\beta) \subseteq \beta \Leftrightarrow \mu_{C(\beta)}(x) \leq \mu_{\beta}(x) \quad \forall x \in X,$$

where  $\mu_\beta$  indicates the membership value of a set of alternative choices  $x$  in  $X$ .

## 5.4 Applications to Economics using Keychains

When we talk of a fuzzy preference relation among a set of alternatives we associate a value in  $I = [0, 1]$  to the objects of preference or choice. This allows us to compare any two objects through the associated real numbers. Thus keychains of real numbers arise in the context of alternative preferences.

### **Example 5.4.1** .

If one is in a shop and want to buy coffee then he/ she is confronted with a host of different brands of coffee  $\{x_1, x_2 \cdots x_n\}$ . Thus one is led to make choices. Hence arises a preference relation.

Now if we consider two brands of coffee, say  $x$  and  $y$ , the fuzzy preference relation will indicate a degree to which  $x$  preferred at least as much as  $y$  or vice versa. This comparison can be done with associated numbers in the unit interval  $I = [0, 1]$ . This situation can be represented by the 7 keychains of length 3, which are 111, 11 $\lambda$ , 110, 1 $\lambda\lambda$ , 1 $\lambda\beta$ , 1 $\lambda 0$  and 100. These keychains can be interpreted as follows in relation to the choices and preferences that can be made by a consumer;

**Table 5.4.2 .**

Keychain	Interpretation
111	Both brands of coffee are equally preferred, and one can buy them both depending on one's budget otherwise the cheaper one would do.
11 $\lambda$	One of the brands is preferred absolutely and the other is preferred to a lesser degree. It can be bought if one has sufficient money at disposal.
110	From the two brands available only one is fully preferred by the consumer. The other is not a choice at all.
1 $\lambda\lambda$	The 2 brands available are both preferred relative to the same degree.
1 $\lambda\beta$	The 2 brands are preferred one more than the other.
1 $\lambda 0$	One of the brand is preferred relative to some degree and the other is not preferred at all.
100	Not a single brand of coffee is preferred by the consumer.

As one can see from the above example, the idea of keychains helps in modeling or formulating some decisions that can be taken by consumers when they are facing choices based on their preferences. This is done by interpreting the pins found in each keychain.

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