

THE HEURISTIC SIGNIFICANCE OF ENACTED VISUALISATION

**A thesis submitted in fulfilment of the
requirements for the degree of**

DOCTOR OF PHILOSOPHY

of

**RHODES UNIVERSITY, GRAHAMSTOWN, SOUTH AFRICA
(Faculty of Education)**

by

DUNCAN ALISTAIR SAMSON

September 2011

ABSTRACT

This study is centred on an analysis of pupils' lived experience while engaged in the generalisation of linear sequences/progressions presented in a pictorial context. The study is oriented within the conceptual framework of qualitative research, and is anchored within an interpretive paradigm. A case study methodological strategy was adopted, the research participants being the members of a mixed gender, high ability Grade 9 class of 23 pupils at an independent school in South Africa. The analytical framework is structured around a combination of complementary multiple perspectives provided by three theoretical ideas, *enactivism*, *figural apprehension*, and *knowledge objectification*. An important aspect of this analytical framework is the sensitivity it shows to the visual, phenomenological and semiotic aspects of figural pattern generalisation. It is the central thesis of this study that the combined complementary multiple perspectives of enactivism, figural apprehension and knowledge objectification provide a powerful depth of analysis to the exploration of the inter-relationship between the embodied processes of pattern generalisation and the visualisation of pictorial cues. The richly textured tapestry of activity captured through a multi-systemic semiotic analysis of participants' generalisation activity stands testament to this central thesis. Insights gleaned from this study are presented as practical strategies which support and encourage a multiple representational approach to pattern generalisation in the pedagogical context of the classroom.

ACKNOWLEDGEMENTS

The author wishes to express his sincere gratitude to the following people:

- Prof Marc Schäfer for his support, encouragement and unwavering good humour, and in particular for knowing just how much freedom to give me.
- Colleagues, reviewers and critical friends for their invaluable comments and suggestions.
- The research participants for so generously giving of their time, spirit, and creativity.
- Family and friends for their steadfast support.

The financial assistance of the National Research Foundation (NRF) towards this research is hereby acknowledged. Opinions expressed and conclusions arrived at, are those of the author and are not necessarily to be attributed to the NRF.

This work is based upon research supported by the FirstRand Foundation Mathematics Education Chairs Initiative of the FirstRand Foundation, Rand Merchant Bank and the Department of Science and Technology. Any opinion, findings, conclusions or recommendations expressed in this material are those of the author and therefore the FirstRand Foundation, Rand Merchant Bank and the Department of Science and Technology do not accept any liability with regard thereto.

TABLE OF CONTENTS

	Page
Abstract	ii
Acknowledgements	iii
Table of contents	iv
List of Abbreviations	vii
A brief note on terminology	vii
List of Figures	viii
List of Tables	xi
Preface	xii

CHAPTER 1 – INTRODUCTION

1.1	Introduction to the study	1
1.2	Goals & objectives	6
1.3	Theoretical framework	8
1.4	Methodology	9
1.5	Significance of the study	10
1.6	Thesis overview	11

CHAPTER 2 – CONTEXTUAL OVERVIEW

2.1	Introduction	13
2.2	Pictorial pattern generalisation	13
2.3	Multiple representations	21
2.4	Visualisation	26
	2.4.1 Introduction	26
	2.4.2 The centrality of visualisation	27
	2.4.3 A working definition of visualisation	28
	2.4.4 Visualisation in mathematics	28
	2.4.5 Visualisation in pictorial pattern generalisation	31
2.5	Concluding comments	37

CHAPTER 3 – THEORETICAL FRAMEWORK

3.1	Introduction	38
3.2	Enactivism	38
3.3	Perception & knowledge objectification	50
3.4	Figural apprehension	55
3.5	Combining three theoretical ideas	67
3.6	The nature of mathematics	69
3.7	Concluding comments	71

CHAPTER 4 – METHODOLOGY

4.1	Introduction & orientation	72
4.2	Participant selection	74
4.3	Ethics	75
4.4	Data generation	76
4.5	Data analysis – Phase 1	82
	4.5.1 Strategy/method classification	82
	4.5.2 Contextual connectivity	90
4.6	Data analysis – Phase 2	90
4.7	Validity	91
4.8	Summary of the methodology	92

CHAPTER 5 – RESULTS, ANALYSIS & DISCUSSION

5.1	Introduction	94
5.2	Phase 1	95
	5.2.1 Strategy/method classification	95
	5.2.2 Contextual Connectivity Rating (CCR)	104
5.3	Phase 2	105
	5.3.1 Broad analysis	105
	5.3.2 Micro-analysis	119
	Vignette1 – Apprehension tension	120

	Vignette 2 – Procedural rigidity, untapped potential & manipulatives	124
	Vignette 3 – Tension between local and global visualisation	131
	Vignette 4 – Objectification through the process of drawing	141
	Vignette 5 – Uneconomical counting and the allure of differencing	143
	Vignette 6 – Unconscious apprehension	148
5.3.3	Discussion of insights gleaned from the micro-analysis	151
5.4	Concluding comments	167

CHAPTER 6 – FINDINGS & CONCLUSION

6.1	Introduction	169
6.2	Review of the objectives	169
6.3	Review of the theoretical framework	171
6.4	Review of the methodology	172
6.5	Findings of this study	173
6.6	Limitations	180
6.7	Significance	181
6.8	Recommendations for further research	182
6.9	Concluding comments	183

	REFERENCES	184
--	-------------------	-----

APPENDICES

Appendix A	Question Response Analysis Sheets	196
------------	-----------------------------------	-----

LIST OF ABBREVIATIONS

CCR	Contextual Connectivity Rating
FET	Further Education and Training
LO	Learning Outcome
MPRS	Mathematical Processing Response Sheets
NCS	National Curriculum Statement
QRAS	Question Response Analysis Sheets

A BRIEF NOTE ON TERMINOLOGY

The expressions *Shape number* and *Term number* are used synonymously and interchangeably throughout the text. Both expressions refer to the independent variable (i.e. the position of the term) in a sequence/progression.

The term *pupil* is used in specific reference to school-going learners, while the terms *learner* and *student* are used in an age-nonspecific generic sense.

LIST OF FIGURES

		Page
Figure 1.1	Pattern generalisation task comprising two non-consecutive terms	5
Figure 2.1	A growing sequence of rectangles	16
Figure 2.2	A growing sequence of tables	16
Figure 2.3	A growing sequence of mountain peaks made from matchsticks	16
Figure 2.4	A growing sequence of dots	20
Figure 2.5	A multi-representational view of pattern generalisation	22
Figure 2.6	Transparent and non-transparent representations, adapted from Sasman et al. (1999)	32
Figure 2.7	Constructive and deconstructive generalisations	33
Figure 2.8	The incorporation of auxiliary constructions	35
Figure 3.1	A simple geometrical figure	56
Figure 3.2	A simple pictorial sequence	56
Figure 3.3	The law of proximity	59
Figure 3.4	The law of similarity (adapted from Katz, 1951, p. 25)	60
Figure 3.5	The law of good continuation (adapted from Katz, 1951, p. 26)	60
Figure 3.6	The law of closed forms (from Spoehr & Lehmkuhle, 1982, p. 66 and Zusne, 1970, p. 130)	61
Figure 3.7	Different modes of figural apprehension	64
Figure 5.1	Grant's apprehension for the expression $T_n = 3n + n - 1$	120
Figure 5.2	Grant's apprehension for the expression $T_n = 3(n - 1) + 4 + n - 2$	120
Figure 5.3	Grant's apprehension for the expression $T_n = 2n + n + n - 1$	121
Figure 5.4	Grant's apprehension for the expression $T_n = 4n - 1$	121
Figure 5.5	Grant's different counting procedures	122
Figure 5.6	Pictorial pattern presented to Brian	124
Figure 5.7	Brian's physical transformation of Term 5 (built from matchsticks)	125
Figure 5.8	Brian's further transformations of Term 5	126
Figure 5.9	Brian's final transformations of Term 5	126

Figure 5.10	A pivotal moment – the incorporation of a “ <i>dead match</i> ”	129
Figure 5.11	Brian’s final transformation of Term 5	130
Figure 5.12	Pictorial pattern presented to Terry	131
Figure 5.13	Terry’s drawing procedure for Term 4	132
Figure 5.14	Two different formats of Terry’s first visualisation	133
Figure 5.15	Terry’s indexical gesturing	134
Figure 5.16	Terry’s drawing procedure for T_4	135
Figure 5.17	Terry’s change in visual apprehension	136
Figure 5.18	Terry’s final apprehension of overlapping squares	138
Figure 5.19	Structural aspects of Terry’s 8 th visualisation	139
Figure 5.20	Terry’s 8 th apprehension ($T_n = 4 + n(5 - 1) + 3n + 5$), giving a 4-match overcount	140
Figure 5.21	A proposed global visualisation: $T_n = 4 + 5n + 3n - (n - 1)$	140
Figure 5.22	Terry’s drawing procedure	141
Figure 5.23	Terry’s change in apprehension	142
Figure 5.24	Kelly’s different counting procedures	143
Figure 5.25	Traced paths of Kelly’s different counting procedures	143
Figure 5.26	Kelly’s verbal commentary upon completion of her counting	144
Figure 5.27	Kelly’s augmentation of Term 3	145
Figure 5.28	Kelly’s transitioning between 3 different apprehensions	146
Figure 5.29	Anthea’s various counting procedures	148
Figure 5.30	“ <i>Oh! Um, ya, „cos there’s 2 times 3 plus 1</i> ”	149
Figure 5.31	Anthea’s <i>economical</i> and <i>uneconomical</i> counting methods	150
Figure 5.32	Anthea’s fourth apprehension	153
Figure 5.33	Philip’s different apprehensions for the expression $T_n = n + 2n + (n - 1)$	154
Figure 5.34	Liza’s progression to a stable awareness of her general formula $T_n = 3n - 3$	156
Figure 5.35	Philip’s visualisation of his formula $T_n = 3 + 4(n - 1)$	158
Figure 5.36	Philip’s visual resolution for even-numbered terms	159
Figure 5.37	Possible visual representations of $T_n = 3 + 4(n - 1)$ for even-numbered terms	160

Figure 5.38	Anthea's subdivision of T_3 and T_5 into non-overlapping triangles	161
Figure 5.39	Anthea's subdivision of T_2 , T_3 , T_4 and T_5 into non-overlapping triangles	161
Figure 5.40	Visual support for Kelly's formula $T_n = 2(n + n - 1) + 1$	163
Figure 5.41	Kelly's visual justification for her formula $T_n = 2(n + n - 1) + 1$	164
Figure 5.42	Lance's visual justification for his formula $T_n = 3(n + 1)$	166

LIST OF TABLES

		Page
Table 2.1	Classification of construction approaches	36
Table 4.1	Strategy/method classification	83
Table 4.2	Contextual Connectivity Rating (CCR) (from Samson, 2007a)	90
Table 5.1	Global overview of numeric versus visual approaches	95
Table 5.2	Overview of visual strategies for V and V/N responses	97
Table 5.3	Overview of numerical strategies for N and V/N responses	99
Table 5.4	Overview of pictorial contexts	100
Table 5.5	Overview of strategy/approach per question	102
Table 5.6	Summary of CCR per question	104
Table 5.7	Grant's nine expressions and apprehensions for T_n	106
Table 5.8	Philip's fifteen expressions and apprehensions for T_n	108
Table 5.9	Brian's seven expressions and apprehensions for T_n	110
Table 5.10	Lance's twelve expressions and apprehensions for T_n	112
Table 5.11	Terry's eight expressions and apprehensions for T_n	114
Table 5.12	Kelly's seven expressions and apprehensions for T_n	115
Table 5.13	Anthea's six expressions and apprehensions for T_n	117
Table 5.14	Liza's six expressions and seven apprehensions for T_n	118

PREFACE

In a way, we are *in* the reality we bring forth. We do not bring forth any reality, we bring forth the one that *we can*, and so it is always dependent on us. As Maturana (1987) says, “everything is said by an observer.” There are no observerless observations or knowerless knowledge. So, as I bring forth a world, I myself is brought forth within my descriptions, I am (in) my descriptions. Both knower and known are part of the world of significance brought forth.

Proulx (2008, p. 22)

CHAPTER ONE

INTRODUCTION

*Patterns are the heart and soul of mathematics*¹

1.1 INTRODUCTION TO THE STUDY

Pattern is prevalent at all levels of mathematical endeavour. Sandefur and Camp (2004) suggest that patterns are “the very essence of mathematics, the language in which it is expressed” (p. 211), while Goldin (2002) describes mathematics succinctly as “the systematic description and study of pattern” (p. 197). Not only is searching for patterns an important strategy for mathematical problem solving (Stacey, 1989, p. 147), but Cuoco, Goldenberg and Mark (1996), in their seminal paper on an organising principle for mathematics curricula, identify the search for pattern as a critical habit of mind.

Intricately connected with the notion of pattern are the fundamental mathematical processes of generalisation and justification. Statements of generality, along with the discovery and investigation of generality, “...are at the very core of mathematical activity” (Lannin, 2005, p. 233). Indeed, Kaput (1999) makes the observation that “generalisation and formalisation are intrinsic to mathematical activity and thinking – they are what make it mathematical” (p. 136).

The use of number patterns, specifically *pictorial* or *figural* number patterns, has been advocated by numerous mathematics educators (de Jager, 2004; Mason,

¹ Zazkis & Liljedahl (2002, p. 379)

Graham, Pimm & Gowar, 1985; Pegg & Redden, 1990b; Walkowiak, 2010) as a didactic approach to the introduction of algebra and as a means of promoting algebraic reasoning. From a pedagogic standpoint, French (2002) comments that the introduction of algebra through what is potentially a wide range of pattern generalisation activities may be effective in assisting pupils to see algebra as both meaningful and purposeful right from the earliest stages.

The initial encounters that students have with algebra are crucially important in establishing both their attitudes towards the subject and the foundations on which to build their subsequent study of the subject together with its links to the rest of mathematics.

(French, 2002, p. 44)

Although this route to the introduction of algebra is not without its problems (see e.g. Warren, 2005), pattern generalisation activities nonetheless present a meaningful way of arriving at algebraically equivalent expressions of generality. This lends itself well to exploring the notion of algebraic equivalence in a practical context where pupils would experience the process of negotiation towards meaning (Mason et al., 1985), a process which in itself has the potential to develop and support pupils' *productive disposition* (Kilpatrick, Swafford & Findell, 2001). Vogel (2005) also suggests that patterning tasks have the potential to develop important metacognitive abilities.

The study of pattern has become an integral component across all Grades of the South African school Mathematics curriculum (Department of Education, 2002; Department of Education, 2003b). In the Intermediate Phase (Grades 4-6) the importance of number pattern activities is in "laying the foundation for the study of formal algebra in the Senior Phase while at the same time developing important mathematical thinking skills" (Department of Education, 2003a, p. 37). As one of the fundamental outcomes of LO1 within the FET band (Grades 10-12) of the National Curriculum Statement (NCS), learners will "explore real-life and purely mathematical number patterns and problems which develop the ability to generalise, justify and prove" (Department of Education, 2003b, p. 12).

Pattern generalisation problems can be presented in a variety of different forms, for example numeric, pictorial, tabular, narrative or contextualised scenarios. Assessment Standards 10.1.3 and 11.1.3 of the NCS assert that within the context of number pattern investigations, learners should be able to "provide explanations and

justifications and attempt to prove conjectures” (Department of Education, 2003b, p. 18). Implicit in this requirement is the necessity that at least some of the pattern questions be set in non-numeric or pictorial contexts. This would appear to be the interpretation of the NCS adopted by textbook developers (e.g. Bennie, Blake & Fitton, 2005; de Waal, McAlister, Müller, Wallace & Williams, 2005; Goba & van der Lith, 2005; Pretorius, Potgieter & Ladewig, 2005). Despite the potential richness of these different contexts, there is a long-standing concern amongst mathematics educators (e.g. Byatt, 1994; de Jager, 1999; Hewitt, 1992; Noss, Healy & Hoyles, 1997) that potentially meaningful pattern generalisation activities carried out in the classroom often become degraded to simple rote exercises in the systematic collection and tabulation of data, from which a generalised formula may be obtained using basic algorithmic methods. While such an approach may well be successful in arriving at the correct algebraic expression for the general formula, the potential for genuine mathematical exploration offered by the context of the question is reduced to a superficial numerical pattern spotting exercise. This activity is often seen as the focal point of the exercise rather than a “means through which insights are gained into the original mathematical situation” (Hewitt, 1992, p. 7). Noss et al. capture the situation as follows:

... attention tends to become focused on the numeric attributes of the output. Worse still, school mathematics becomes constructed – by students and teachers alike – as a stereotypical data-driven „pattern-spotting” activity in which it is acceptable to search for relationships by constructing tables of numeric data without appreciating any need to understand the structures underpinning them.

(Noss et al., 1997, p. 205)

As Thornton (2001) remarks, the danger with such an approach is that the focus becomes “...the development of an algebraic relationship, rather than the development of a sense of generality” (p. 252). Where pictorial patterns are reduced to numeric sequences, this “deprives learners of the benefits of visualisation and the potential insights that may come with it” (Graham & Honey, 2009, p. 41). Expressions of generality arrived at through this approach become divorced from the particular context that gave rise to them, and such generalised statements become “statements about the results rather than the mathematical situation from which they came” (Hewitt, 1992, p. 7). Such disconnected algebraic formulation neither illuminates the original problem nor provides a means for validating the generated functional relationship (Noss et al., 1997). This becomes particularly problematic in

situations where the justification of the general rule assumes significance (Byatt, 1994, p. 25). As Becker and Rivera (2007, p. 135) remark, the act of establishing an invariant property through numerical considerations and then merely fitting this invariant property onto the original sequence of pictorial terms by no means justifies the validity of the algebraic generalisation. Furthermore, Roper (1999) points out that searching for patterns without regard for the underlying contextual structure may in fact be counter-productive in terms of encouraging mathematical problem-solving.

One of the results of such a mechanistic approach to pattern generalisation problems is that pupils tend to foreground recursive aspects by focusing on successive addition rather than using the independent variable in a general formula. Lannin (2004) comments that there seems to be an almost natural tendency for pupils to adopt recursive reasoning when examining number patterns, and a number of researchers have highlighted pupils' fixation on such recursive reasoning as being a critical factor in preventing successful generalisation based on a functional relationship in terms of the independent variable (English & Warren, 1998; Felix, 1998; MacGregor & Stacey, 1993; Orton, 1997). However, Noss et al. (1997) comment that the tendency of pupils to focus on a recursive strategy should not necessarily be interpreted as pupil failure. They make the critical remark that strategies are influenced not only by the nature of the task but also by its presentation, a position that has been echoed by numerous other researchers (e.g. Chua, 2009; Frobisher & Threlfall, 1999; Hadjidemetriou, Pampaka, Petridou, Williams & Wo, 2007; Hershkowitz et al., 2002; Sasman, Olivier & Linchevski, 1999)

This was the departure point for my M.Ed. thesis (Samson, 2007a) which investigated the extent to which question design affects the solution strategies adopted by pupils when solving linear number pattern generalisation tasks presented in pictorial and numeric contexts. The results of the investigation give strong support to the idea that question design can play a key role in influencing which strategies are adopted by pupils when solving pattern generalisation tasks. Pupils' responses gave evidence of the complex interplay between the number pattern itself, the nature of the question design and the *specific* pictorial context chosen.

An unexpected aspect of this earlier study was the rich diversity of visualisation strategies employed by pupils when solving linear pattern generalisation tasks set in

a pictorial context. By way of example, consider the pattern generalisation task comprising two non-consecutive terms as shown in Figure 1.1.

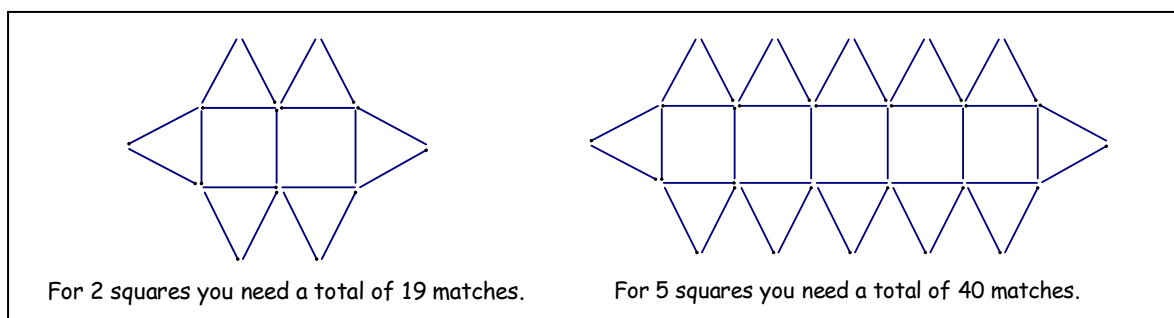


Figure 1.1 Pattern generalisation task comprising two non-consecutive terms

The diagram is characterised by the n^{th} term containing n squares. Among other things, pupils were required to provide an algebraic expression for the n^{th} term as well as a justification for their particular formula. A rich diversity of visually mediated strategies or visual mechanisms was revealed through an analysis of the various algebraic expressions and their associated justifications:

$$T_n = 6n + (n + 1) + 4$$

$$T_n = 4n + 3n + 5$$

$$T_n = 7n + 5$$

$$T_n = [3(n - 1) + 4] + [2(2n + 2)]$$

$$T_n = 3n + 3n + 6 + n - 1$$

$$T_n = 12 + 7(n - 1)$$

$$T_n = (3n + 1) + 4n + 4$$

Such strategies included *dynamic* and *static* visual imagery, *global* observations, *recursive* reasoning, visualisation of overlapping or *nested* structures, visual transformation and re-organisation of the given pictorial structure, the identification of imbedded *macro-structures*, the use of *negative space* to scaffold visual reasoning, the use of *auxiliary constructions*, and the employment of visual strategies prompted by otherwise irrelevant substructure elements. In addition, there was also evidence to suggest that the presence or absence of *specific* terms may well attract or discourage a particular visually motivated strategy (Samson, 2007a).

One of the limitations of this earlier study stems from the data generation protocol which was focused on the *product* of generalisation rather than the *process* of generalisation. Pupils' written responses to a series of exercises based on linear generalisation tasks set in both numeric and 2-dimensional pictorial contexts were analysed with respect to pupils' written justifications of their general formulae. The

process of justification was a critical factor enabling the interpretation of the sub-structure evident in many of the pupils' responses. Within the context of this earlier study, the role of justification is seen as communication of mathematical understanding, and as such was highly successful in providing a window of understanding into each pupil's cognitive reasoning. Notwithstanding some of the successes of this methodology, it nonetheless limited the investigation to the *result* or *final product* of the reasoning process rather than allowing access to the *reasoning process itself*.

Although it has been reported (e.g. Warren, 2000) that individual pupils are capable of generalising pictorial patterns in multiple ways, i.e. by means of a variety of visually mediated strategies, a literature review reveals that little empirical research has been done in this area. This raises an interesting question: To what extent are pupils able to visualise *figural cues*, i.e. objects with both spatial properties and conceptual qualities (Fischbein, 1993), in multiple ways within the context of pattern generalisation? A desire to explore this aspect of pictorial pattern generalisation in greater depth by focusing on the *process* of generalisation was the impetus and motivating factor behind the present study.

1.2 GOALS & OBJECTIVES

Mason et al. (1985) describe three important stages in the process of generalising pictorial patterns – *seeing*, *saying* and *recording*. *Seeing* relates to the grasping of a commonality or relationship in the given terms. *Saying* refers to the verbal description or articulation of this insight, either to oneself or others, while *recording* employs the use of symbols as a means of written communication. Mason et al. (1985) remark that there is a tendency for teachers to rush to the last of these three stages, particularly with respect to symbolic representation, thereby missing the potential benefit that could be extracted from time invested in the first two stages, *seeing* and *saying*. Although there are likely to be obstacles for pupils at all three stages, the perceptual, verbalizing and symbolisation levels, as Lee (1996, p. 94) points out, the key to successful pattern generalisation seems to hinge on the first of these stages where perceptual flexibility is required in order to see mathematically or algebraically *useful* patterns. As Becker and Rivera (2007) highlight, within the

realm of pictorial pattern generalisation there is “a need for research that addresses the issue of recognizing mathematically-valid invariant structures or properties” (p. 135).

A review of pertinent literature relating specifically to figural pattern generalisation identifies four broad categories of focus: (a) descriptions of solution strategies and levels of attainment, (b) the influence of task design and the nature of the pictorial terms, (c) the transition between pupils’ arithmetic and algebraic reasoning, and (d) the affordances offered by technological environments. However, within the context of figural pattern generalisation, little empirical research focusing on the *process of visualisation*, as opposed to the *product of visualisation*, seems to have been carried out.

The central goal of this study is thus to gain insight into the inter-relationship between the embodied processes of pattern generalisation and the visualisation of pictorial cues. In pursuance of this goal, the study will be framed by the following guiding questions:

1. To what extent, if any, do individual pupils favour specific visualisation strategies when generalising figural patterns?
2. To what extent are pupils able to generalise patterning tasks, set in a pictorial context, in multiple ways?
3. What embodied processes are evinced by pupils engaged in figural pattern generalisation tasks?
4. In what ways do these embodied processes either assist or hinder pupils’ ability to visualise figural cues in multiple ways?
5. Finally, in what ways can insights gleaned from the above be meaningfully employed in the pedagogical context of the classroom?

1.3 THEORETICAL FRAMEWORK

One of the dominant theories influencing recent curriculum reform within South African schools, particularly with respect to mathematics education, has been that of constructivism (Graven, 2002; Vithal & Volmink, 2005). However, it has been suggested (see e.g. Begg, 2000) that a more encompassing theory of learning may be desirable. An interest in moving beyond constructivism has contributed, at least in part, to a steadily growing interest in enactivism – a theory which emphasises knowing rather than knowledge, in which the learner is seen to be part of a complex learning context. Enactivism, with its focus on self-organising systems, provides a meaningful perspective of complexity in relation to mathematics teaching and learning.

There are a number of features of enactivism that have an important practical bearing on the learning and teaching of mathematics with specific reference to pictorial pattern generalisation. The first of these is that language and action are not merely outward manifestations of internal workings, but rather visible aspects of embodied understandings (Davis, 1995, p. 4). There is thus a need not only to consider the formal mathematical ideas that emerge from action, but to give close scrutiny to those preceding actions (Davis, Sumara & Kieren, 1996). Secondly, in terms of the co-evolution of knower and known, learning can be seen to occur “at the interstices where the learner meets the environment, stresses particularities within the environment, and generates a response whose viability in the environment is then determined” (Dawson, 1999, p. 154). However, of critical importance is each pupil’s predilection to take notice of the potentialities afforded by a given situation. An appropriate framework is thus needed in order to engage with pupils’ whole-body experience and expression while they explore the potentialities afforded by a given pictorial pattern generalisation task. Radford’s (2008) theoretical construct of *knowledge objectification* and Duval’s (1995, 1998, 1999) notion of *figural apprehension* are explored as a meaningful framework for doing this. Along with enactivism, these theoretical ideas are drawn together into a single framework which has the potential to provide powerful complementary insights into the underlying tensions and subtle complexities of generalisation tasks set in pictorial contexts.

1.4 METHODOLOGY

This study is oriented within the conceptual framework of qualitative research, and is anchored within an interpretive paradigm. The study aims ultimately to gain insights into the embodied processes of pupils' visualisation activity when engaged in figural pattern generalisation tasks through an in-depth analysis of each pupil's lived experience. A case study methodological strategy was adopted and an appropriate group of research participants was identified - the members of a mixed gender, high ability Grade 9 class of 23 pupils at an independent school in South Africa. The data collection and analysis occurred in two phases.

Phase 1 of the data generation process takes the form of a series of pencil and paper exercises based on 10 linear generalisation tasks set in pictorial contexts. For each pattern participants were required to provide a numerical value for the 40th term (along with a written articulation of their reasoning), and an algebraic expression for the n^{th} term (along with a justification/explanation of their expression). The responses to the 10 linear generalisation tasks were classified in terms of the specific method or strategy employed. A coding system was developed to provide a nuanced characterisation of both numeric and visual strategies. In addition, a quasi-quantitative measure was used to characterise the extent to which pupils used the pictorial scenario as a referential context. Phase 1 of the study seeks to identify those pupils who prefer visual as opposed to numeric approaches when solving pictorial generalisation tasks in addition to characterising the extent to which individual pupils favour *specific* visualisation strategies.

Seven research participants who were identified in Phase 1 as preferring visual strategies took part in Phase 2. In addition, an eighth research participant, identified in the pilot study², who similarly showed a preference for visual methods, was also included in Phase 2. These eight research participants were individually provided with a further linear pattern and were required to provide multiple expressions for the n^{th} term. Tools such as paper, pencils and highlighters as well as appropriate manipulatives such as matchsticks and plastic counters were provided. Participants were asked to think aloud while engaged with their particular pattern generalisation

² This was at a point in the pilot study when the data collection and analysis techniques had been refined to a point commensurate with the main study.

task. Each session was audio-visually recorded and field-notes were taken. Audio-visual recordings were analysed with specific reference to participants' use of semiotic means of objectification such as words, linguistic devices, metaphor, gestures, rhythm, graphics and physical artefacts. This analysis process culminates in a series of vignettes which serve to characterise the affordances brought forth by the complementary multiple perspectives of the theoretical framework.

1.5 SIGNIFICANCE OF THE STUDY

It is the central thesis of this study that the combined complementary multiple perspectives of enactivism, figural apprehension and knowledge objectification add a powerful depth of analysis to the exploration of the inter-relationship between the embodied processes of pattern generalisation and the visualisation of pictorial cues. The richly textured tapestry of activity captured through a multi-systemic semiotic analysis of participants' generalisation activity stands testament to this central thesis.

Furthermore, this framework allows for an additional depth of analysis when compared with other frameworks presently employed to analyse the process of pattern generalisation. This extra layer of insight arises from the complementary multiple perspectives that constitute the framework of analysis. Not only does this framework acknowledge perception as being critically related to the *manner* of one's interaction with perceptual objects, but it also remains sensitive to both the phenomenological and semiotic aspects of the generalisation process.

In addition, it is of significance that this study is able to provide practical strategies which support and encourage a multiple representational approach to pattern generalisation in the pedagogical context of the classroom.

1.6 THESIS OVERVIEW

Chapter 2 – Contextual Overview

This chapter provides a contextual background to the study. Firstly, a number of pertinent issues relating specifically to pictorial pattern generalisation are discussed. This leads into a discussion of the relevance and importance of multiple representations. Finally, visualisation and visual reasoning are characterised within the context of figural pattern generalisation.

Chapter 3 – Theoretical Framework

The purpose of this chapter is to establish a theoretical framework for the study. Three theoretical aspects of pictorial pattern generalisation are presented: *enactivism*, *knowledge objectification* and *figural apprehension*. Literature pertinent to each of these topics is reviewed prior to situating each idea within the context of pictorial pattern generalisation. Finally, the three key theoretical ideas are drawn together into a single framework which has the potential to provide powerful complementary insights into the underlying tensions and subtle complexities of generalisation tasks set in pictorial contexts.

Chapter 4 - Methodology

Further theoretical elements pertaining to more practical methodological issues are interrogated in this chapter. The choice of methodology and methodological protocols are justified within the context of the theoretical framework.

Chapter 5 – Results, Analysis & Discussion

A global overview of the results of Phase 1 and Phase 2 is presented in order to investigate the extent to which individual pupils favour specific visualisation strategies as well as the extent to which pupils are able to generalise pictorial patterns in multiple ways. A fine-grained micro-analysis of Phase 2 data is then presented in the form of a series of vignettes which show the rich tapestry of generalisation activity which was evidenced by the research participants. The chapter closes with a discussion of broad insights that gradually emerged during the course of the micro-analysis.

Chapter 6 – Findings & Conclusion

The purpose of this final chapter is to consolidate the findings of the study with reference to the original research question and within the context of the theoretical and methodological framework. In addition, both the limitations and significance of the study are interrogated, and some recommendations for further research are suggested.

CHAPTER TWO

CONTEXTUAL OVERVIEW

*The only real voyage of discovery consists not in seeking
new landscapes but in having new eyes.*

MARCEL PROUST

2.1 INTRODUCTION

The purpose of this chapter is to provide a contextual background to the study. Firstly, a number of pertinent issues relating specifically to pictorial pattern generalisation are discussed. This leads into a discussion of the relevance and importance of multiple representations. Finally, visualisation and visual reasoning are characterised within the context of figural pattern generalisation.

2.2 PICTORIAL PATTERN GENERALISATION

It has been observed that mathematical power lies not only in being able to detect, construct, invent, understand and manipulate patterns, but “in being able to communicate these patterns to others” (Goldin, 2002, p. 213). A critical component of this process of communication involves a meaningful articulation of the essential features of the pattern – features that would enable the determination of a non-specific (i.e. *general*) term in the pattern. Although expressions of generality are not necessarily restricted to the language of algebra, algebraic symbolism does however allow for neat, compact and semantically unambiguous general statements. Indeed, as Kaput (2002) comments, in the sense that algebra embodies generality and is a

systematic expression of generality, it is “an intrinsic way of being mathematical” (p. 122). Mason (1996) takes this sentiment further when he describes generalisation as the “life-blood, the heart of mathematics” (p. 74).

For Becker and Rivera (2008) generalisation is seen as both a process and a concept and is “a critical aspect of algebraic thinking and reasoning” (p. 1). Generalisation can broadly be described as “deliberately extending the range of reasoning or communication beyond the case or cases considered, explicitly identifying and exposing commonality across cases” (Kaput, 1999, p. 136). As such it is “an important aspect in mathematics that permeates all branches of the subject” (Dindyal, 2007, p. 236). Within the arena of figural pattern generalisation, there are numerous pictorial and practical contexts in which questions can be set (Mason et al., 1985; Orton, Orton & Roper, 1999), among the most obvious being dot patterns (Kenney, Zawojewski & Silver, 1998), tiling patterns (Lannin, 2004), matchstick patterns (English & Warren, 1998; Orton, 1997; Pegg & Redden, 1990a; Samson, 2007b) as well as two- and three-dimensional building block patterns (Abbott, 1992; Lannin, 2003; Miller, 1991; Nolder, 1991; Pagni, 1992). Polygonal or figurate numbers (e.g. triangular, square, pentagonal and hexagonal numbers) also make use of simple visual patterns to portray numbers (Andrews, 1990; Crookes, 1988; Malloy, 1997; Miller, 1990).

Number patterns presented in the form of a sequence of pictorial terms are more than simply a visual representation of a given numeric pattern. Clausen (1992) suggests that working directly from a pictorial context is usually preferable to purely numeric arguments since it limits the chance that “irrelevant number patterns will mislead one into assuming the truth of an invalid generalization” (p. 18). The critical difference between numeric and pictorial patterns is that, provided the pictorial context has been meaningfully understood, a pictorial representation is inherently less ambiguous than its isomorphic numeric counterpart. This can be understood by taking cognizance of the fact that a finite numeric sequence can be generated by an infinite number of functions. This can readily be understood by considering a finite number of points plotted in the Cartesian Plane where there would clearly be an infinite number of curves that could be drawn through the specified points (Samson, 2006, p. 8). Thus, no finite sequence of numerical terms uniquely specifies the following term in the sequence (see e.g. Mason, 2002). However, this is not the

case for pictorial sequences, since the pictorial context suggests a deeper underlying structure.

By way of example, consider the numeric sequence 2 ; 6 ; 12 ; ... where only the first three terms have been specified. There are clearly an infinite number of ways to continue the sequence, and for each of these sequences there would be a corresponding general formula. One could argue that some sequences suggest themselves more readily or obviously than others, but even taking this argument into account still leaves the given sequence far from being uniquely specified. The pattern could, for example, continue based on a trivial repetitive cycle such as 2 ; 6 ; 12 ; 2 ; 6 ; 12 ; 2 ; 6 ; 12 ; etc. or 2 ; 6 ; 12 ; 6 ; 2 ; 6 ; 12 ; 6 ; 2 ; etc. If we assume that the sequence is based on a quadratic expression where there is a constant second difference, then we could generate the sequence 2 ; 6 ; 12 ; 20 ; 30 ; 42 ; etc. One could then use any number of standard techniques (Samson, 2008) to arrive at the general term $T_n = n^2 + n$. One could arrive at an equivalent general term by noticing that $T_1 = 1 \times 2$, $T_2 = 2 \times 3$, $T_3 = 3 \times 4$ and thus arguing through inductive reasoning that the n^{th} term must be of the form $T_n = n \times (n + 1)$. A different sequence could be arrived at by noticing that the 3rd term is the product of the two preceding terms, and thus argue that the general term could be expressed using the second-order recursive expression $T_n = T_{n-1} \times T_{n-2}$ where $T_1 = 2$, $T_2 = 6$ and $n \geq 3$. This would yield the sequence 2 ; 6 ; 12 ; 72 ; 864 ; 62 208 ; etc. Yet another possibility would be to assume the sequence is based on a cubic expression where the third difference is constant. Arbitrarily setting this constant third difference to equal 2, one can readily work backwards to give the sequence 2 ; 6 ; 12 ; 22 ; 38 ; 62 ; etc. This yields the general term $T_n = \frac{n^3 - 3n^2 + 14n - 6}{3}$.

However, a less ambiguous situation arises if the same numerical sequence (2 ; 6 ; 12 ; ...) is accompanied by a pictorial representation – a representation which inherently suggests a deeper underlying structure. By way of example, consider the pictorial patterns shown in Figures 2.1 – 2.3.

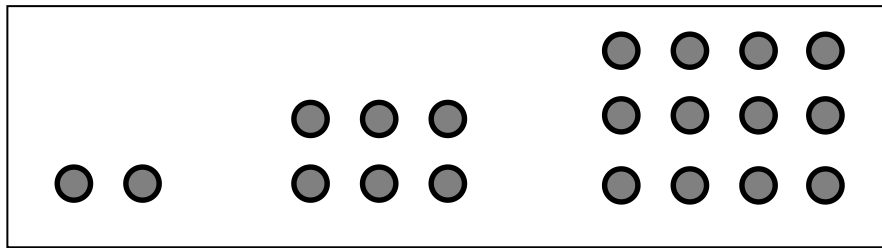


Figure 2.1 A growing sequence of rectangles

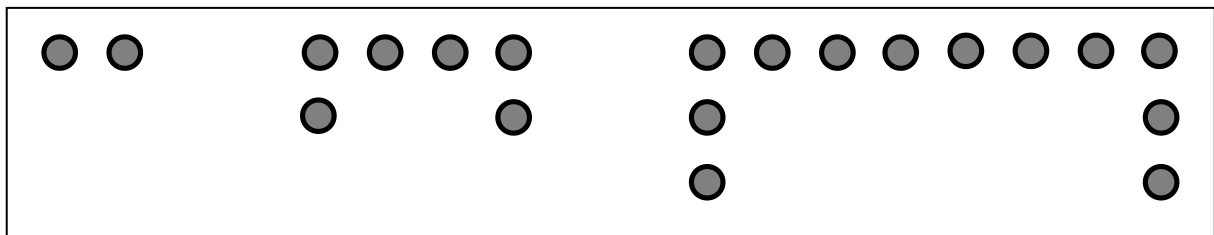


Figure 2.2 A growing sequence of tables

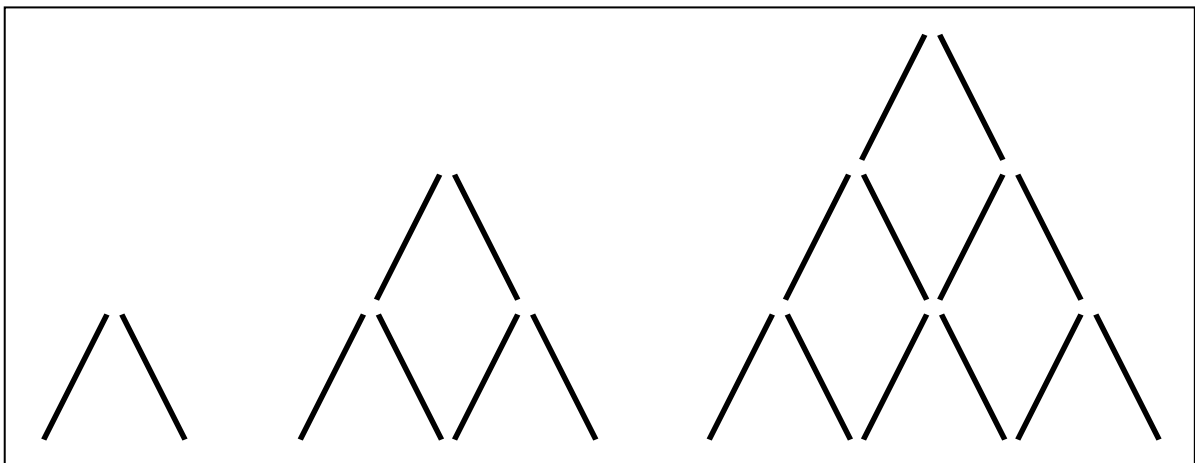


Figure 2.3 A growing sequence of mountain peaks made from matchsticks

For all three patterns the first three terms are numerically equivalent, *viz.* 2 ; 6 ; 12. However, the underlying structure suggested by the various pictorial representations yields sequences based on different expressions for the general term. Continuing the growing pattern of rectangles leads to the sequence 2 ; 6 ; 12 ; 20 ; 30 ; 42 ; etc. Continuing the growing pattern of tables leads to the sequence 2 ; 6 ; 12 ; 22 ; 40 ; 74 ; etc. while a continuation of the growing pattern of mountain peaks once again leads to the sequence 2 ; 6 ; 12 ; 20 ; 30 ; 42 ; etc.

Careful analysis of Figure 2.1 reveals that both the length and breadth of each successive rectangle are one more than the preceding rectangle. The dimensions of the three given rectangles are 1×2 , 2×3 , and 3×4 . The sequence thus continues

with rectangles with dimensions 4×5 , 5×6 and 6×7 . The general expression for the sequence is thus $T_n = n \times (n + 1)$. An analysis of Figure 2.2 suggests that the upper surface of the tables is growing exponentially (2 ; 4 ; 8 ; 16 ; 32 ; etc.) while the legs are increasing linearly, each leg growing according to the sequence 0 ; 1 ; 2 ; 3 ; etc. Combining these two elements yields the formula $T_n = 2^n + 2(n - 1)$. Finally, an analysis of Figure 2.3 suggests that the structure comprises layers of matchsticks. The first term contains 1 pair of matchsticks, the second term contains 1 + 2 pairs of matchsticks, and the third term contains 1 + 2 + 3 pairs of matchsticks. Generalising this structure suggests that the n^{th} term contains 1 + 2 + 3 + ... + n pairs of matchsticks and thus $2 \left(\sum_{i=1}^n i \right)$ matchsticks in total. This yields the general term $T_n = n^2 + n$, which is algebraically equivalent to the formula arrived at for Figure 2.1. Importantly, however, both formulae were structured on different processes of visual reasoning.

Mathematics education journals abound with number pattern activities and investigations making use of pictorial contexts (de Mestre, 2001; Farmer & Neumann, 2004; French, 1990; Lannin, 2004; Malloy, 1997; Onions, 1991; Pagni, 1992; Quinn, 2005; Szetela, 1999; Van de Walle & Holbrook, 1986). In essence, the use of a pictorial context aims to exploit the visual decoding of the pictorial sequence to give meaning to the symbolic expressions constructed. However, critical to this process is the ability not only to grasp in a meaningful way the perceived underlying structure of the pictorial context, but also the ability to use this structure to articulate a direct expression for the general term³.

Dindyal (2007), in a study of high school students engaged in generalisation tasks, identified a sequence of four stages through which successful students seemed to proceed: (i) a direct modelling stage, (ii) pattern identification, (iii) proof-testing of the pattern, and (iv) determining a rule for the general case. The initial direct modelling stage was characterised by students engaging with instantiations of specific terms either by drawing, counting or writing/recording. This process was often carried out in a systematic manner. The second stage, the identification of a specific pattern, seemed to depend on the systematic engagement with the specific terms in the first stage. In the third stage, the proof-testing of the identified pattern, students tested

³ This aspect of pictorial pattern generalisation is discussed in detail in Chapter 3.

their conjectures against particular terms that were too large to be modelled directly. In the final stage, students expressed their identified pattern in the form of a symbolic (algebraic) statement of generality. Dindyal (2007), similar to Lee (1996), identified three types of conceptual obstacles to the generalisation process, those at the *perceptual*, *verbalising*, and *symbolisation* levels. However, the crucial step in the four sequential stages seemed to be the identification of a useful pattern, as this played a significant role in the successful symbolic generalisation. This observation resonates with Lee's (1996) remark that often the problem is not with seeing a pattern per se, but rather in "perceiving an algebraically useful pattern" (p. 95).

Andrews (1990) comments that it is "preferable to offer pupils a situation which can be generalised with reference to the situation itself" (p.13). This notion is supported by Hershkowitz et al. (2002) who observed that presenting sequences in a pictorial context tends to encourage generalisation expressed in terms of the independent variable (as opposed to a step-by-step recursive method), particularly if the pictorial terms are non-consecutive. Pictorial contexts would thus seem to have the potential to support the generalisation process. However, Orton et al. (1999) caution that placing a pattern in a pictorial context must not automatically be assumed to be helpful. In addition, some contexts may be more difficult than others and the perceived relationship between pattern and context may also be problematic. For some pupils a pictorial context may simply obfuscate and create additional complications (Orton et al., 1999).

Hershkowitz et al. (2002) comment that as a reflection of the counting method employed to determine specific terms in the sequence, the use of a pictorial context seems to give strong meaning to the general formula. It is exactly this aspect of pictorial pattern generalisation that a number of specially designed computer software environments or microworlds, what Noss et al. (1997) refer to as *autoexpressive* environments, have attempted to make more explicit. The general rationale behind the development of such microworlds is that their purpose-built functionalities support the expression of generality by providing a domain of situated abstraction within which meaningful construction and analysis of patterns can occur. A number of such microworlds have been developed, for example *ShapeBuilder* (Geraniou, Mavrikis, Hoyles & Noss, 2008) and *Mathsticks* (Healy & Hoyles, 1999; Noss et al., 1997). Healy and Hoyles (1999) suggest that the pedagogical value of

autoexpressive environments lies in the fusion of action, visualisation and symbolic representation which has the potential to provoke cognitive reorganisation and forge connections between the visual and symbolic representations.

Ainley, Wilson and Bills (2003) make a useful distinction between generalising the *context* and generalising the *calculation* in their study of early secondary school pupils articulating expressions of generality. Although their particular context has a practical aspect to it (the arrangement of chairs around a growing sequence of tables placed end to end), their comments are equally valid for pictorial contexts such as those involving matchsticks or dot patterns. Those statements of generality that focused on the context tended to be descriptive statements articulating the manner in which the various elements were arranged relative to one another, while statements of generality that focused on the calculation were distinguished not only by the incorporation of specific operations (e.g. add, double, multiply) but also included phrases that lend themselves to the notion of a variable (e.g. „for however many tables there are...“). From their results, Ainley et al. (2003) suggest that the generalisation of the context is not sufficient to enable pupils to move to a symbolic expression of generality, while generalisations of the calculation seem to support this transition. By way of example, within the context of dot patterns, a statement of generality focusing on the *context* of Figure 2.4 might be that there are two horizontal rows of dots where the dots in the upper horizontal row are placed directly over the space left between each pair of adjacent dots in the bottom row. Such a description of the context doesn't explain how to work out how many dots there are in each term, but is more a general instruction for the process of constructing the pattern. A statement of generality focusing on the *calculation* might be that for however many dots there are in the bottom horizontal row, the upper row will contain one less. In this case it is clear that the latter statement of generality, i.e. the one focusing on the calculation, is far more readily transferrable to a symbolic expression of generality.

Radford (2001, 2006) provides a useful distinction between different layers of generality within the context of pictorial pattern generalisation. Apart from *arithmetic* generalisation, Radford distinguishes between three different layers of algebraic

generalisation – *factual*, *contextual* and *symbolic*⁴. These different layers are discussed here in relation to determining the 50th term of Figure 2.4.

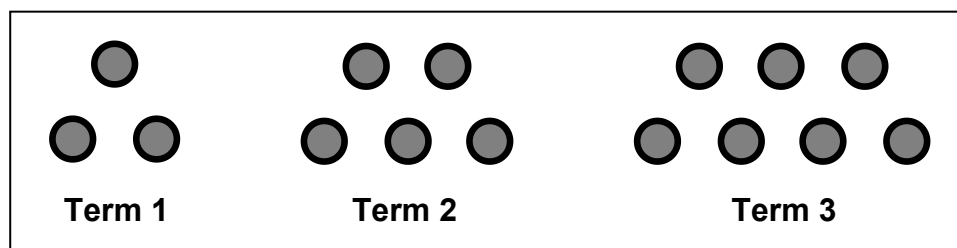


Figure 2.4 A growing sequence of dots

An *arithmetic* generalisation would be characterised, for example, by a pupil noticing that to proceed from one term to the next always requires the addition of two dots, but being unable to use this local commonality to determine the desired 50th term. In Radford’s (2006) nomenclature, such a recursive generalisation is still in the realm of arithmetic. A *factual* generalisation could be characterised by a pupil noticing that the first term contains 1 + 2 dots, while the second and third terms require 2 + 3 and 3 + 4 dots respectively, and thus reasoning that the 50th term would contain 50 + 51 dots. Factual generalisation is a generalisation of actions in the form of an operational schema that is restricted to concrete cases. Here we begin to see the emergence of algebraic reasoning, although such reasoning is still at a level where *indeterminacy*⁵ has not yet reached articulation. *Contextual* generalisations represent the next level of generality, a non-symbolic generalisation performed on spatially and temporally situated conceptual objects (Radford, 2001, p. 85). A pupil noticing that the first term of Figure 2.4 contains 1 + 2 dots, while the second and third terms require 2 + 3 and 3 + 4 dots respectively, and then being able to express this through a situated description, for example, “the sum of the shape and the next shape”, shows a progression from a *factual* to a *contextual* generalisation where the indeterminate is now made linguistically explicit. Although the description “the sum of the shape and the next shape” still appears to be an operational schema, the distinction is that it is no longer bound to concrete cases. Specific concrete terms

⁴ Williams (2005) comments that Radford’s characterisation of *factual*, *contextual* and *symbolic* generalisation resonates with Peirce’s categories of sign: icon, index and symbol (see e.g. Peirce, 1985).

⁵ Radford (2006, p. 3) describes *indeterminacy* (as opposed to numerical determinacy) as one of the critical elements that characterise algebraic thinking. Typical indeterminate objects include unknowns, variables and parameters. The two other elements that Radford uses to characterise algebraic thinking are the *analytic* way in which indeterminate objects are treated, and the *symbolic* mode of designating such objects.

can now be described in a more generic sense through the articulation of indeterminate objects. The final level of generalisation is that of the *symbolic* in which these indeterminate objects and operational schemas are expressed algebraically, e.g. $T_n = n + (n + 1)$.

A critical aspect of pattern generalisation as highlighted by a number of researchers (e.g. Bishop, 2000; English & Warren, 1998; Lee, 1996; Moss & Beatty, 2006) is that of perceptual flexibility – the willingness and ability to move between several perceived patterns and associated representations thereof, and in “being able to see several patterns and willing to abandon those that do not prove useful” (Lee, 1996, p. 95). This aspect of pattern generalisation relates to a broader issue of mathematics education, that of multiple representations. This is the topic of the following section.

2.3 MULTIPLE REPRESENTATIONS

The role of multiple representations is widely acknowledged as a central element in both problem solving and the understanding of mathematical concepts (Goldin, 2002; Greeno & Hall, 1997; Kaput, 1998). Indeed, as Amit and Fried (2005) comment, “the general case for multiple representations in mathematics education hardly needs defending anymore” (p. 57). This position could be summarised by the position that mathematical meanings are developed “by forging connections between different ways of experiencing and expressing the same mathematical ideas” (Healy & Hoyles, 1999, p. 60). Related to this stance, Healy and Hoyles (1999) present the following view of mathematical development of individual students:

...mathematical progress is not characterized by the replacement of one way of knowing by another that supposedly is “higher” or more abstract; rather, it is characterized by the development and interlinking of different forms of reasoning that can develop alongside and in combination with one another. (p. 60)

It has been suggested (Rivera & Becker, 2005) that rather than thinking in terms of hierarchical development – from perceptual to conceptual, from concrete to abstract, and from informal to formal – a more dynamic view is preferable, one in which students are able to oscillate between different approaches or modes, thereby “...enabling them to develop greater flexibility, notational fluency, and

representational competence” (p. 202). The ability to explore and experience multiple representations of a given mathematical situation, and to switch flexibly between different representations, has the potential to enable students to develop deep insights and heightened conceptual understanding of mathematical topics, as well as an enhanced problem solving ability (Even, 1998, p. 105). Critical aspects related to the teaching and learning of mathematics with multiple representations have found voice in contemporary literature (see e.g. Ainsworth, 2006; Özmantar, Akkoç, Bingölbali, Demir & Ergene, 2010).

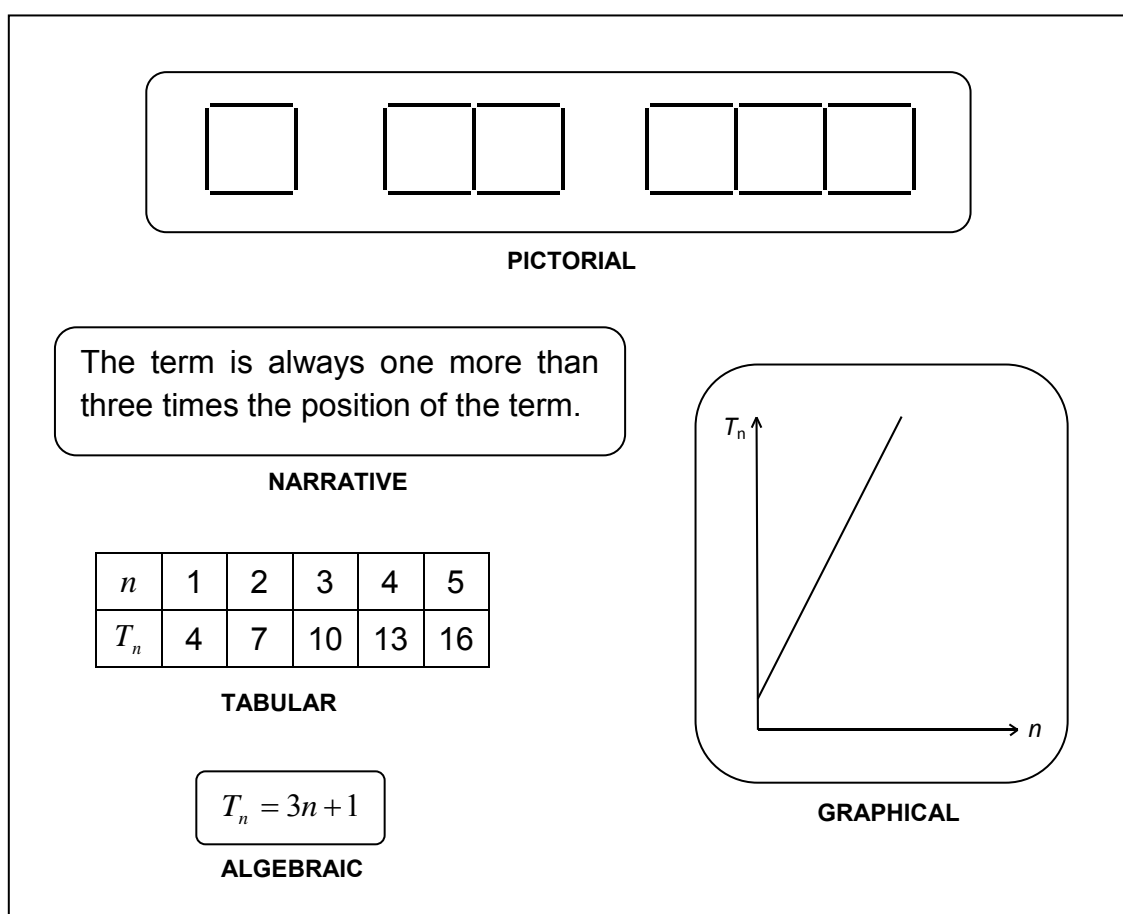


Figure 2.5 A multi-representational view of pattern generalisation

Figure 2.5 gives an example of a multiple representational view within the context of pattern generalisation. The same⁶ mathematical situation is portrayed through five different representational modes, each of which depicts the relationships of the given situation in a different manner. The five forms of representation depicted are

⁶ Although the „same“ mathematical situation is portrayed in these five representational modes, use of the word „same“ is not meant to imply that the representational systems are *themselves* equivalent. Rather, different modes of representation are seen to allow for different ways of knowing and thus support and enhance different reasoning processes (Parnafes & Disessa, 2004).

pictorial (or diagrammatic), *narrative* (or verbal), *tabular*, *graphical*, and *algebraic*. Pictorial representations generally depict one or more specific cases, although appropriately annotated they are also able to represent the general case. Narrative descriptions can either be verbal descriptions of the general case (as in Figure 2.5) or they can be descriptions of an operational schema, for example: “starting from 4, keep adding multiples of 3 to determine subsequent terms”. Tabular representations are an organised display of specific cases. Graphical representations can either be viewed as a general depiction of the mathematical situation (if a continuous curve is displayed) or as a sequence of specific cases (if only discrete points are displayed). Finally, algebraic representations can either be explicit expressions of the general case (e.g. $T_n = 3n + 1$) or recursive expressions showing a general operational schema ($T_{n+1} = T_n + 3$; $T_1 = 4$; $n \geq 1$).

Familiarising students with and exposing students to multiple and alternative representations has the benefit of developing in them a deeper appreciation for the interconnections between different areas of mathematics (Schultz & Waters, 2000, p. 453). This is a sentiment which finds popular voice. However, as Swafford and Langrall (2000) caution, “...the use of multiple representations in and of themselves is not enough” (p. 109). A critical aspect of multiple representations is for pupils not only to establish a meaningful link between the various representations and the mathematical context or problem, but to establish links between parallel representations and to be able to switch flexibly between them (Dreyfus, 1991). However, as Healy and Hoyles (1999) comment, the evidence from research suggests that “the majority of students make these shifts neither spontaneously nor easily” (p.60). Swafford and Langrall (2000, p. 109) remark that mathematical instruction tends to focus on the *use* of multiple representations with an apparent assumption that the links between parallel representations will be established as a serendipitous by-product of activities involving multiple representations. Amit and Fried (2005) suggest two possible reasons why students fail to forge meaningful links between different representations of a given mathematical situation. Firstly, students often view a specific representation (e.g. algebraic or graphical) not as a representation *per se*, but rather as a *solution method*. Secondly, Amit and Fried (2005, p. 63) suggest that establishing links between different representations requires not only the presence of the parallel representations themselves, but

genuine “mediating elements” or “connectors”. Powell and Maher (2003) have shown that students are able to discover such “connectors” or *isomorphisms* through the heuristic strategy of “attending to dynamical links among objects and relations between two systems” (p. 29). However, they point out that there needs to be a motivating force behind such a heuristic action, and present an example where the driving force behind the discovery of isomorphism was the students’ desire to *justify* a conjecture related to transitivity between different problem contexts, a transitivity that would aid in the solution of the original problem.

There are no doubt many other factors - subtle, idiosyncratic or otherwise - that influence the flexibility with which students are able to move between different mathematical representations of a given situation. Even (1998) suggests that factors related to the context in which the original question or mathematical task is framed may well influence students’ facility with respect to employing multiple representational strategies. Herman (2007) cites factors such as student perception of what is most efficient or “mathematically proper”, as well as beliefs about and perceptions of the “value” of different methods or representations, as influencing students choosing one representation over another in problem solving contexts. The situation would thus seem to be somewhat complex, however what is critical to take cognizance of is that mere exposure to multiple representations themselves is in general insufficient for most students to make meaningful interconnections *between* representations.

Another dimension related to the importance of valuing different representations relates to the exhortation by Hiebert et al. (1997) that students “...form their perceptions of what a subject is all about from the kinds of tasks they do” (p. 17). Furthermore, Hiebert et al. (1997) advocate a classroom culture of sharing, analysing and discussing a variety of student-generated solution methods to given tasks or mathematical problems, thereby encouraging reflection on mathematical relationships. It is interesting to note that in Japan, which consistently scores near the top of international comparisons of mathematics achievement, mathematics classes are characterised by the presentation and discussion of student-generated alternative solution methods (Stigler & Hiebert, 1999). Interestingly, in an analysis of 46 research studies investigating co-operative versus competitive problem solving activities, Qin, Johnson and Johnson (1995) suggest that one possible reason for co-

operative settings being more effective, and producing higher quality problem solving than competitive settings, may be related to co-operative groups being exposed to a variety of approaches and solution strategies.

A further consideration to keep in mind in terms of multiple representations is that representations do not exist in isolation, but can only be meaningfully understood within socio-cultural contexts and historical backgrounds. As Font, Godino and D'Amore (2007) elaborate, a representation or representational system "only acquires meaning as part of a larger *system* with established meanings and conventions" (p. 6). Furthermore, different representational systems representing the „same idea“ should not be seen to be redundant. Different representational systems each allow for different *forms* of expressivity and different *ways of knowing* (see e.g. Garcia, Benitez & Ruiz, 2010; Parnafes & Disessa, 2004). Thus, at the heart of multiple representational systems lies the critical notion of *non-redundancy* (Benveniste, 1985, p. 235). As Radford (1999, p. 149) comments, the important pedagogical act lies in the contextualisation that links and aligns conceptualisations arrived at through different semiotic modes or representational systems.

The development of technology environments has resulted in powerful ways to support pupils in forging critical connections between different representations and representational systems (see e.g. Lapp, 1999). As Geraniou et al. (2008) comment:

...a major rationale for designing with digital technologies is allowing students to see different representations, such as symbolic, iconic, numeric or even verbal ones, and realise the relationships and the equivalence of different representations. (p. 38)

An important aspect of technological environments is that different representations can be dynamically linked so that the manipulation of one representation results in an equivalent alteration in other linked representations. This real-time dynamic linking has the potential to draw students' awareness to those critical aspects of each representational system, and thus support conceptual development leading to a deeper appreciation of the underlying interconnections. Abramovich, Fujii and Wilson (1994) for example have demonstrated the usefulness of a multiple-application medium for the study of polygonal numbers (e.g. triangular, square, pentagonal and hexagonal numbers) by using software tools such as dynamic geometry, a relational grapher, and spreadsheets as a means of enhancing

mathematical visualisation by providing a dynamic interplay between geometric, analytical and numerical representations.

Quite apart from any pedagogic or epistemological concerns, I would also argue that there is a strong moral or ethical dimension to acknowledging the importance of multiple representations. This ethical dimension relates to the idea that different pupils have different learning styles, different ways of engaging with or making sense of mathematical situations, and different ways of “seeing” the world. I believe that teachers have a moral obligation not only to value and embrace these different “ways of knowing”, but to provide appropriate classroom environments that support and actively encourage a multi-representational view of mathematics. Furthermore, as Graham and Honey (2009) comment:

An awareness of this multiplicity of possible „ways of seeing“ is important for learners both as an important idea in mathematics and as a way of helping them appreciate that, in maths, there isn’t always one correct way. (p. 41)

Broadly speaking, most forms of mathematical representation fall into two broad categories: *visual* and *analytic*. As Arcavi (1999) asserts, “...flexible and competent translation back and forth between visual and analytic representations of the same situation ... is at the core of understanding much of mathematics” (p. 74). Issues surrounding mathematical visualisation are foregrounded in the following section.

2.4 VISUALISATION

2.4.1 INTRODUCTION

The purpose of this section is to present a number of key issues pertaining to visualisation, and in particular to review pertinent literature relating to how other researchers have incorporated the notion of visualisation into studies of pictorial pattern generalisation.

The section begins with a brief discussion of the central role of visualisation, particularly within the realm of mathematics. A short working definition of the term *visualisation* is then presented. The remainder of the section reviews a number of

visualisation characterisations and classification systems as well as a variety of idiosyncratic terminologies drawn from the relevant literature, firstly relating to mathematics in general and finally in relation specifically to pictorial pattern generalisation.

2.4.2 THE CENTRALITY OF VISUALISATION

Not only is vision "...central to our biological and socio-cultural being" (Arcavi, 1999, p. 55), visualisation is recognised as being a central component in mathematical activity (Arcavi, 2003; Cunningham, 1991; Duval, 1999; Hershkowitz, Arcavi & Bruckheimer, 2001). It has even been suggested that visual thinking may well become "...the primary way of thinking in the future" (Hershkowitz & Markovits, 1992, p. 38) and that graphicacy may well constitute the fourth „R“⁷ (Aldrich & Sheppard, 2000). Furthermore, Cunningham (1991, p. 70) comments that visualisation within the realm of mathematics education not only promotes intuition and understanding, but also allows students to "learn new ways to think about and do their *own* [emphasis mine] mathematics". As Presmeg (1997b) asserts, "...imagery is of vital concern in fostering mathematical creativity in school students and mathematicians alike" (p. 299). Arcavi (1999) ascribes the centrality of visualisation to the fact that "visualization is no longer related to the merely illustrative only, but is also being recognized as a key component of reasoning (deeply engaging with the conceptual and not the merely perceptual)" (p. 74). Notwithstanding the advocated benefits of visualisation and visual reasoning, Eisenberg and Dreyfus (1991) remark on the reluctance of many students to visualise in mathematics, and their preference for algorithmic approaches to visual thinking. Eisenberg (1994) suggests that one of the possible reasons for this reluctance to visualise is that "visualization techniques, which require a gestalt of a situation, are cognitively more demanding of the learner than analytical techniques which are more algorithmic in nature" (p. 109).

Despite the acknowledged central role of visualisation, there is still a widely held opinion that, although pedagogically important, pictures are nonetheless essentially heuristic devices (Brown, 1997) and that visual representation remains "a second-class citizen in both the theory and practice of mathematics" (Barwise &

⁷ A reference to the so-called "three „R“s" of *reading*, *writing*, and *arithmetic*, which are often regarded as the three fundamentals of learning.

Etchemendy, 1991, p. 9), particularly with respect to proofs that make crucial use of visual imagery and representation. However, there is some support for the notion that pictures have "...a legitimate role to play as evidence and justification, well beyond a heuristic role" (Brown, 1997, p. 161). As Barwise and Etchemendy (1991) claim, "visual forms of representation can be important, not just as heuristic and pedagogic tools, but as legitimate elements of mathematical proofs" (p. 9) and as "...essential and legitimate components in valid deductive reasoning" (p. 16). Fischbein (1987) claims that visualisation "not only organizes data at hand in meaningful structures, but ... is also an important factor guiding the analytical development of a solution" (p. 104). Building on from Fischbein is the suggestion by Hershkowitz et al. (2001) that visualisation can even take on the role of "the analytical process itself which concludes with a general formal solution" (p. 262).

2.4.3 A WORKING DEFINITION OF VISUALISATION

Terminologies relating to visualisation are not only extensive but are used rather inconsistently in the research literature (Presmeg, 2006). Gutiérrez (1996, p. 4) suggests that such limited consensus is a result of the wide diversity of backgrounds and theoretical frameworks used in relation to the field of visualisation. Drawing on ideas from various researchers, notably Bishop (1989, p. 7), Hershkowitz et al. (1990, p. 75) and Zimmermann and Cunningham (1991, p. 3), Arcavi (2003) synthesizes the following useful working definition of visualisation:

Visualization is the ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understandings. (p. 217)

2.4.4 VISUALISATION IN MATHEMATICS

Battista (2007, p. 844) identifies five important types of basic objects within the realm of visual reasoning:

- *Physical objects* which are actual physical entities.
- *Sensory objects* which relates to those sensory activations which are evoked when viewing a physical object.

- *Perceptual objects* which are the mental entities perceived when viewing a physical object.
- *Conceptual objects* which relate to the conscious meanings or ways of thinking invoked by perceptual objects.
- *Concept definitions* which are formal mathematical specifications of conceptual objects.

Presmeg (1986a, 1992b) identifies five different kinds of visual imagery in an attempt to “operationalize” (van Garderen & Montague, 2003, p. 246) such visual imagery. The five types of imagery included in Presmeg’s taxonomy (1986a, 1992b) can be summarised as follows^{8,9}.

- *Concrete, pictorial imagery* or mental images
- *Pattern imagery* showing pure relationships depicted in a visual-spatial scheme
- *Memory images* of formulae, involving the visual recall of formulae
- *Kinaesthetic imagery* involving movement and gestures
- *Dynamic imagery*, involving dynamic transformations of geometric figures

Although Presmeg (1986a) acknowledges that all imagery types have the potential to play a functional role in mathematical problem solving, she considers pattern imagery as being the most essential type, as it identifies the relational aspects of a problem and is thus arguably better suited to abstraction and generalisation. As Thornton (2001) points out, the development of such mathematical imagery which focuses on relationships and patterns, “is surely one of the principal goals of mathematics education” (p. 254).

Hegarty and Kozhevnikov (1999) distinguish between two different types of visual-spatial representations: *schematic* imagery which focuses on the spatial relationships between objects or parts of an object, and *pictorial* imagery where the focus lies with the visual appearance of the objects themselves. Hegarty and

⁸ It is interesting to note, as Presmeg (2006, p. 208) discusses, that Dörfler’s (1991) *figurative, operative, relational* and *symbolic* image schemata correspond roughly to four of Presmeg’s (1986a, 1992b) types of imagery: *concrete, kinaesthetic, dynamic* and *memory* images respectively.

⁹ Owens and Clements (1998) modified Presmeg’s (1986a) taxonomy into the five categories of *concrete, dynamic, pattern, action, and procedural* imagery in order to highlight the rich diversity of the uses of visual imagery stemming from their data.

Kozhevnikov (1999) found that the use of schematic representations was positively related to success in mathematical problem solving, whereas use of pictorial representations was negatively correlated with success. This echoes Presmeg's (1986a) ascription of pattern imagery, in which the concrete details are disregarded in favour of pure relationships, as the most essential role in mathematical problem solving. Presmeg (1997b, p. 308) also acknowledges dynamic imagery as being well adapted to abstraction and generalisation.

Kirby and Kosslyn (1992) suggest that inasmuch as image representations are depictive, imagery can be exploited to aid problem solving. This stems from the notion that, unlike propositional representations, spatial relations in imagery are an emergent property of the depicted perceptual units. However, while some apprehensions of a visual stimulus may well be efficacious in terms of a particular solution strategy, other apprehensions may result in obscuring crucial aspects of the visual stimulus and consequently "conceal" possible solution strategies. Imagery is thus not without its complications.

Presmeg (1986a, pp. 44-45; 1992b, p. 42) highlights a number of difficulties associated with visual images that have an important bearing on the present study. Images that represent a specific case, i.e. a specific or concrete instantiation of a general scenario, may result in attention being focused on irrelevant or unhelpful figural details. Furthermore, strong or vivid perceptual images may persist resulting in an inflexibility of thinking. Such inflexibility may prevent the opening up of alternative, and perhaps more fruitful, trains of thought. To complicate matters further, not only is our perception shaped by past experience and established knowledge structures, but in many instances is critically influenced by the context in which the observation is made where "in different contexts, the „same“ visual objects may have different meanings" (Arcavi, 2003, p. 232). A similar remark is made by Battista (2007, pp. 844-845) who highlights a number of complications associated with the perception of visual images, amongst others the notion that experience and prior knowledge influence what aspects of the visual stimulus are attended to and that what one "sees" in a visual stimulus is influenced by what one knows or conceives in terms of the contextual setting.

As Presmeg (1999, p. 152) comments, a diagram by its very nature depicts a *single* instance, an *individual* concrete case. However, despite Presmeg's (1986a, pp. 44-45; 1992b, p. 42) assertion that the concreteness of depictions of specific instances of a general scenario may result in the foregrounding of unhelpful or irrelevant details, Fischbein (1987) makes the pertinent observation that it is this very concreteness that constitutes "...an essential factor for creating the feeling of self-evidence and immediacy" (p. 104). Fischbein (1987) goes on to explain that the word *immediacy* in this sense relates not only to the direct perception of a given reality, but also means that "the individual is directly, personally, somehow emotionally, involved in the given reality" (p. 104). This suggests that there is an important *personal* aspect to one's engagement with imagery and visual representations where the affective domain of human existence may play a critical role.

2.4.5 VISUALISATION IN PICTORIAL PATTERN GENERALISATION

SENSORY & COGNITIVE PERCEPTION

With specific reference to pictorial pattern generalisation tasks, Becker and Rivera (2007) highlight two different modes of visual perception or „ways of seeing“: *sensory* perception and *cognitive* perception. Sensory perception relates to the perception of an object as a mere object-in-itself. However, cognitive perception extends beyond the sensory as a result of associated conceptual aspects or „properties“ of the perceived sensory object.

Cognitive perception necessitates the use of conceptual and other cognitive-related processes, enabling learners to articulate what they choose to recognize as being a fact or a property of a target object. It is mediated in some way through other types of visual knowledge that bear on the object, and such types could be either cognitive or sensory in nature. (Rivera & Becker, 2008, p. 67)

These different modes of perception (sensory and cognitive) resonate with Fischbein's (1993) theory of figural concepts, and the notion that all geometrical figures (or *figural objects*) possess, simultaneously, both conceptual and figural properties¹⁰.

¹⁰ This aspect is explored in more detail in Section 3.4.

TRANSPARENT & NON-TRANSPARENT REPRESENTATIONS

Sasman et al. (1999) make a distinction between what they refer to as “transparent” and “non-transparent” pictorial representations. The defining characteristic of *transparent* representations is that the “function rule is embodied in the structure of the pictures” (p. 162) whereas in *non-transparent* representations the function rule is not easily discerned in the structure of the pictorial representation. This distinction is shown in Figure 2.6 where the function rule for the first sequence ($T_n = n^2 + 1$) can readily be arrived at from the physical structure of the given terms, being perceptually composed of a growing square of dots (containing n^2 dots) and a constant single dot on the right-hand side. The second sequence also contains a discernable regular structure. However, in this instance the function rule ($T_n = n^2$) is not easily „seen” in the structure of the pictorial terms. This is not to say that the function rule is unable to be arrived at through visual strategies, but rather that the expression for the general term is not embodied in the structure of the pictorial representations as overtly as it is in the case of transparent representations.

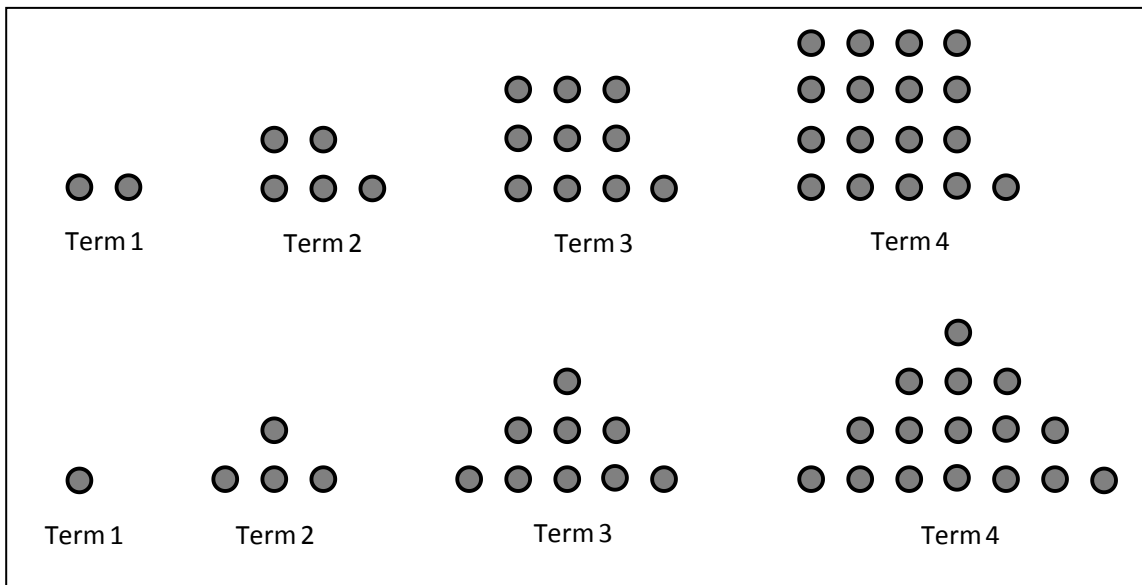


Figure 2.6 Transparent and non-transparent representations, adapted from Sasman et al. (1999)

The two sequences shown in Figure 2.6 have quadratic expressions for the general term. With reference to *linear* sequences, however, Rivera and Becker (2008) suggest that almost all sequential or growing linear patterns are transparent “in the sense that the closed formulas associated with them are somehow visibly embodied in each cue” (p. 72). I would argue that different linear patterns are not equally

transparent, and perhaps a more useful way of thinking of this aspect would be in terms of a *degree of transparency* where the terms *transparent* and *non-transparent* would be the two extremes of a continuum. Quite apart from the nature of the pictorial representation itself, the degree of transparency is also likely to depend on the idiosyncrasies of the individual interacting with the pictorial representation, and their particular „ways of seeing“.

CONSTRUCTIVE & DECONSTRUCTIVE GENERALISATIONS

Rivera and Becker (2008) differentiate between two different types of generalisation within the context of pictorial patterning tasks – *constructive* generalisation and *deconstructive* generalisation. Constructive generalisation arises from the perception of a figural pattern as containing non-overlapping constituent gestalts or structural units. Deconstructive generalisations arise when the perceived sub-configurations of the figural pattern overlap. In such cases the total count would need to be adjusted to take into account the overlapping units. Figure 2.7 illustrates the difference between these two types of generalisation with respect to the given pictorial sequence.

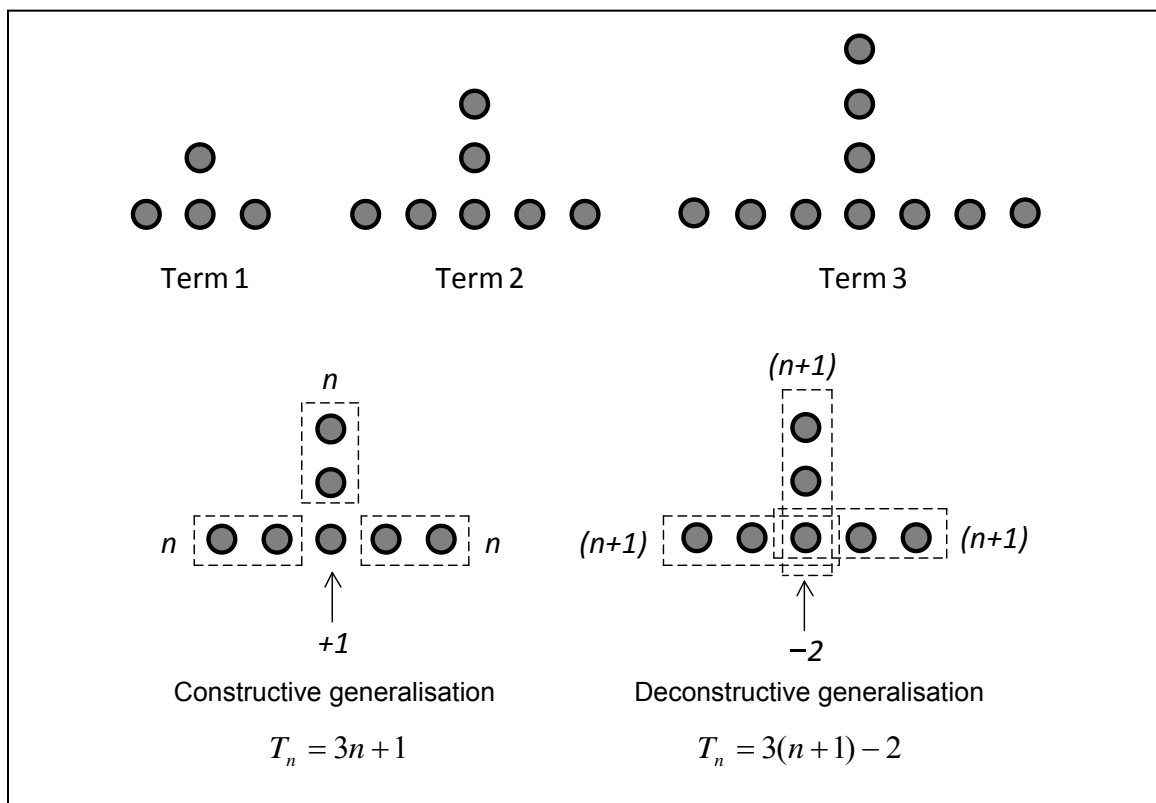


Figure 2.7 Constructive and deconstructive generalisations

Constructive generalisation is arrived at through perceiving the n^{th} term of the sequence to be composed of a central dot with three radial arms extending out from it, each arm containing n dots. This results in the general expression $T_n = 3n + 1$. Deconstructive generalisation could be arrived at through perceiving the n^{th} term of the sequence to be composed of three overlapping radial arms each containing $n + 1$ dots. As a result of the overlapping arms the central dot would in effect have been counted three times. Adjusting for this overlap yields the general expression $T_n = 3(n + 1) - 2$.

MECHANISMS OF VISUALISATION

Within the realm of pattern generalisation, Hershkowitz et al. (2001) uncovered various “mechanisms” of visualisation in the building of a mathematical generalisation in a pictorial context. They distilled the various visual strategies into the following analytical components: (a) decomposition of a structure into smaller substructures and units, (b) creation of auxiliary constructions, (c) transformation of the whole structure into a different configuration, and (d) recomposition and synthesis. Their results led Hershkowitz et al. (2001) to propose that visualisation can be far more than the intuitive support of higher level reasoning, in that it may well constitute “the essence of rigorous mathematics” (p. 255). Although this research was conducted with more mature subjects (in-service teachers), there is evidence to suggest that pupils are also capable of utilising similar visualisation mechanisms (Orton et al., 1999; Waring, Orton & Roper, 1999).

GENERIC EXAMPLE

A study by Samson (2007a), which focused on visualisation strategies employed by Grade 9 pupils during pictorial patterning tasks, also revealed a number of mechanisms of visualisation. These mechanisms were most revealing in terms of the subtlety and complexity of the visual reasoning evident in the generalisation strategies. Most visual strategies began by deconstructing a *generic example*¹¹ into a number of component parts. In some instances these component parts were further subdivided into even smaller parts. The decomposition of the *generic example* was essentially a retro-synthesis of the whole into perceived component

¹¹ What Lannin (2005) describes as “a particular example that embodies the general characteristics of an argument” (p. 236).

parts. Once separated into component parts, the visualisation process became one of reconstruction by means of multiplying the various parts by the frequency of their appearance, and finally summing the various multiples and constants together to arrive at a final general term. There was also evidence to suggest that pupils were able to perceive the visual stimulus in more than one way through a transformation and reorganisation of the whole-part relationship.

AUXILIARY CONSTRUCTIONS

The study by Samson (2007a) also revealed the use of what Hershkowitz et al. (2001, p. 263) refer to as *auxiliary constructions*. For example, in a sequence similar to that shown in Figure 2.7, instead of seeing the structure as three radial arms extending out from a single central dot, one pupil visualised the entire structure as being composed of inverted V-shapes nested inside one another as shown in Figure 2.8. The straight lines between the dots were added by the pupil, and the incorporation of these *auxiliary constructions* assisted in the transformation and reorganisation of the whole-part relationship of the given visual stimulus.

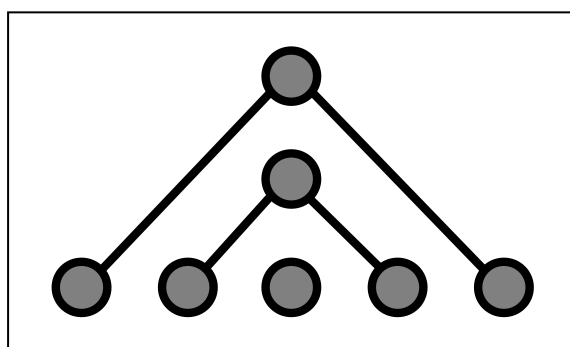
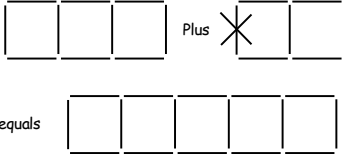
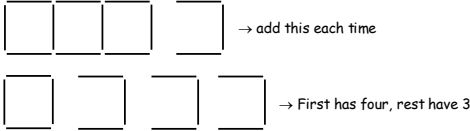
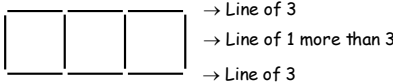


Figure 2.8 The incorporation of auxiliary constructions

ICONIC & SYMBOLIC CHARACTERISATION SYSTEM

Healy and Hoyles (1996, 1999) devised a classification system of strategies (construction approaches) that purposefully distinguished between iconic (visual) approaches and symbolic strategies. This classification system was used to investigate the connections pupils made between visual and symbolic reasoning while generalising number patterns. The classification system is shown in Table 2.1 which highlights the mathematical equivalence of the symbolic and iconic approaches.

Table 2.1 Classification of construction approaches

Symbolic approach	Iconic approach
<p>Counting Counting individual items in an unstructured way</p>	<p>Eidetic Focusing on perceptual rather than mathematical properties of the data</p>
<p>Operating terms Calculations using known terms to obtain a target term</p>	<p>Combining diagrams “Chunking” of known terms to obtain others</p> 
<p>Operating on differences between terms Calculations based on the numerical difference between consecutive terms</p>	<p>Inter-term “chunking” “Chunking” based on a relation between terms</p> 
<p>Operating on a variable Calculations based on a relation between dependent and independent variables</p>	<p>Intra-term “chunking” “Chunking” based on a relation within a term</p> 

Adapted from Healy and Hoyles (1999, p. 67)

PATTERNING STRATEGIES

The literature reveals that there is little consistency in the naming of patterning strategies/approaches. Although the basic procedural descriptions of various strategies are largely similar, nomenclature seems to be somewhat idiosyncratic. By way of example, Lannin (2003) differentiates between a “contextual” strategy and a “rate-adjust” strategy. Both of these strategies result in an explicit formula for determining the numerical value of any term from the independent variable. In the case of the contextual strategy, the general formula is derived from the context of the problem situation, while the “rate-adjust” strategy stems from an essentially numeric or abstract argument. However, the distinction between the two strategies does not take into account those pupils who make use of a blend of both abstract and contextual elements.

Samson (2007a) provides a framework of strategies which was arrived at by distilling and somewhat modifying the various nomenclatures found in the research literature (English & Warren, 1998; Hargreaves, Shorrocks-Taylor & Threlfall, 1998; Hargreaves, Threlfall, Frobisher & Shorrocks-Taylor, 1999; Healy & Hoyles, 1999; Lannin, 2003, 2005; Orton & Orton, 1999; Stacey, 1989; Swafford & Langrall, 2000). The seven strategies identified include counting, chunking, difference product, explicit, whole-object uncorrected, whole-object corrected, and the nature of numerical terms (Samson 2007a). While this is a useful framework, it nonetheless has a critical limitation in that it focuses on the *product* as opposed to the *process* of generalisation.

2.5 CONCLUDING COMMENTS

The purpose of this chapter was to provide a contextual background to the study. Firstly, a number of pertinent issues relating specifically to pictorial pattern generalisation were discussed. This led into a discussion of the relevance and importance of multiple representations. Finally, visualisation and visual reasoning were characterised within the context of both mathematics in general, and figural pattern generalisation in particular.

Key issues highlighted in this chapter will be used to inform both the focus and methodology employed in the present investigation, as well as provide a critical backdrop to the processes of analysis and interpretation.

Having presented the contextual background, the next chapter establishes and interrogates the theoretical framework underpinning the study.

CHAPTER THREE

THEORETICAL FRAMEWORK

*Perception is not some empty “having” of perceived things,
but rather a flowing lived experience...*

EDMUND HUSSERL

3.1 INTRODUCTION

The purpose of this chapter is to establish a theoretical framework for the study. Three theoretical aspects of pictorial pattern generalisation are presented: *enactivism*, *knowledge objectification* and *figural apprehension*. Literature pertinent to each of these topics is reviewed prior to situating each idea within the context of pictorial pattern generalisation. Finally, the three key theoretical ideas are drawn together into a single framework which has the potential to provide powerful complementary insights into the underlying tensions and subtle complexities of generalisation tasks set in pictorial contexts. The implications that these theoretical underpinnings have in terms of the nature of mathematics is also briefly discussed.

3.2 ENACTIVISM

One of the central concerns in Western philosophy, certainly from Descartes onwards, has centred on the so-called mind-body split (Varela, Thompson & Rosch, 1991, p. 28). To what extent are the mind and body two separate entities, both physically and metaphysically speaking, and what is the nature of the ontological relation between them? For rationalists such as Descartes, the path to knowledge

lies in intrinsic logic and the notion that all philosophy must in consequence “begin with the individual mind and self” (Durant, 1947, p. 142). This tendency to elevate rationality above other ways of knowing, as Hamilton (2006, p. 1) points out, represents a long tradition in Western patterns of thought, a tradition stemming from Greek Aristotelian philosophy. Related to this history of thought are a number of dichotomies¹² which are now seen as being problematic (Begg, 2000). Such dichotomies include mind/body, self/other, subject/object and knower/known¹³.

Merleau-Ponty’s (1962) phenomenology affords an alternative to this dichotomous mode of thinking that has come to permeate Western thought. Phenomenology aims at elucidating “both that which appears and the manner in which it appears” and attempts to “get beyond immediately experienced meanings in order to articulate the prereflective level of lived meanings, to make the invisible visible” (Kvale, 1996, p. 53). For Merleau-Ponty (1962) the body is of primary significance: “In order to perceive things, we need to live them” (p. 325). Of critical import, however, is the role of the body in this lived experience: “...I am not in front of my body, I am in it, or rather I *am* it” (Merleau-Ponty, 1962, p. 150, my emphasis). Not only is the body our means of belonging to the world, but the body critically renders the mind and world inseparable. Thus, the body is *simultaneously* both a biological structure as well as a lived-experiential structure. As Davis (1995, p. 4) explains, “Our bodies are shaped by the world that they participate in shaping; they render the mind-and-world, subject-and-object, individual-and-collective, mental-and-physical inseparable.”

For Varela et al. (1991), who build on Merleau-Ponty’s phenomenology, it is critical that one sees one’s body as both a physical (biological) structure as well as a lived, phenomenological structure. This notion of *embodiment* thus has an important double sense: “it encompasses both the body as a lived, experiential structure and the body as the context or milieu of cognitive mechanisms” (Varela et al., 1991, p. xvi). This double embodiment is a critical aspect of enactivism, a theory of cognition that draws on ideas from ecology, complexity theory, phenomenology, neural biology, and post-Darwinian evolutionary thought.

¹² Often termed Cartesian dichotomies and variously referred to as *Descartes’ myth* (Ryle, 1949) and *Descartes’ error* (Damasio, 1994).

¹³ Davis (1995) makes the interesting observation that such dichotomous thinking probably has to do with our historical predisposition to define objects or phenomena in terms of Aristotelian axioms of logic, specifically the *Law of Contradiction* (A cannot be both B and not-B) and the *Law of Excluded Middle* (A must be either B or not-B).

As Nickson (2000, p. 8) discusses, the basic tenet of enactivism is that there is no division between mind and body, and thus no separation between cognition and any other kind of activity. Enactivist theory brings together action, knowledge and identity so that there is a conflation of *doing*, *knowing*, and *being* (Davis, Sumara & Kieren, 1996). Within an enactivist framework there is a purposeful blurring of the line between thought and behaviour, with the focus on the dynamic interdependence of thought and action, knowledge and knower, self and other, individual and collective (Davis, 1997, p. 370).

The Cartesian divide between mind and world, and consequently between mind and body, led to an important split between metaphysics and epistemology, a split that still plagues contemporary philosophy (Lakoff & Johnson, 1999, p. 95). Since the mind and world were not one, they not only had to be different but different *kinds* of entities. And since the body was of the world, the mind was not. The mind, thus separated from the body and world, could no longer be in direct touch with the world (Lakoff & Johnson, 1999). Knowledge, in a sense, tries to bridge this divide. However, we are left with the perplexing question of location – does knowledge and/or truth lie *out there* or *in here*? As Davis et al. (1996, p. 165) phrase it, “...we are not sure if it reflects something that has a prior, real existence, or if it is essentially a mental activity.” In other words, is truth objective (discovered) or subjective (created)?

From an enactivist standpoint this is an irrelevant issue since the starting premise, the mind-world split, is denied. Not only is the body our means of belonging to the world, but it is simultaneously of oneself and of the world. Enactivism thus sees the negotiation of a middle path between the two extremes of cognition seen as the projection of an inner world and cognition seen as the recovery of a pre-given outer world (Varela et al., 1991, p. 172). By reconceptualising cognition as *embodied action*, enactivism elegantly avoids the problematic question of location, not in terms of a compromise between dichotomous viewpoints but rather by “going beyond the conflict by jumping to a metalevel” (Varela, 1987, p. 62). Dualities such as inner versus outer and objective realism versus subjective idealism are no longer critical concerns. Rather, enactivism sees cognition as “the enactment of a world and a mind on the basis of a history of the variety of actions that a being in the world performs” (Varela et al., 1991, p. 9). In other words, cognition is viewed as an

embodied and co-emergent interactive process, “an ongoing bringing forth of a world through the process of living itself” (Maturana & Varela, 1998, p. 11) where the emphasis is on *knowing* as opposed to *knowledge*.

For the enactivist, the act of perceiving something is not a process of recovering properties of an external object, rather, “perception consists of perceptually guided action” (Varela, as quoted in Lozano, 2005, p. 26). Thus, we perceive things in a certain way because of the manner in which we relate to them through our actions (Lozano, 2005). This idea is succinctly stated in Maturana and Varela’s (1998, p. 26) aphorism: “*All doing is knowing, and all knowing is doing.*” Thus, how we make sense of our experiences, and indeed what we are *able* to experience, is dependent on the kinds of bodies that we have and the ways that our bodies afford interactions with the world we inhabit and the various environments in which we find ourselves (Johnson, 1999, p. 81). If we had different bodies and consequently interacted with the world in different kinds of ways, not only would our lived experience be different but this would lead to “a different sense of self and different ways of understanding and reasoning” (Johnson, 1999, p. 99).

In some respects this resonates with situated cognition (e.g. Greeno, 1997; Lave & Wenger, 1991) where learning is seen to be located in the *process* of participation rather than individual minds, and in which there is a “blurring of the distinction between the self and the world” (St. Julien, 1997, p. 267). As Fenwick (2001b) observes, the apparent similarities between enactivism and situated cognition lie in “this primacy granted to environment as integrated with cognition, not simply supplemental to the individual consciousness” (p. 47). Indeed, as Kirshner and Whitson (1997) point out, “the central philosophical assumption against which situated cognition theories struggle is the functionalist belief in mind-body dualism” (p. 4). However, as Fenwick (2000) comments, there are some important distinctions between situated cognition and enactivism. Within the context of situated cognition, learning is seen as a *way of being* in the social world where “...agent, activity, and the world mutually constitute each other” (Lave & Wenger, 1991, p. 33) and in which “learning is viewed as an aspect of all activity” (Lave & Wenger, 1991, p. 38). However, Núñez, Edwards and Matos (1999) argue that the nature of situated cognition cannot be fully understood by attending only to the social and contextual factors. They further claim that the situated cognition perspective doesn’t completely

account for the *grounding* of situated knowing and learning, and leaves open the question of the *basis* of social situatedness. Importantly, learning and thinking are also situated “within biological and experiential contexts, contexts which have shaped, in a non-arbitrary way, our characteristic ways of making sense of the world” (Núñez et al., 1999, p. 46). Núñez et al. (1999) claim that the grounding for situatedness comes from “the nature of shared human bodily experience and action, realized through basic embodied cognitive processes and conceptual systems” (p. 46). As Fenwick (2001a, p. 248) summarises, while situated cognition arose from the discipline of psychology and is anthropocentric in consequence, enactivism takes as its premise *ecological complexity theory*, and as such has strong *biological* roots. This is one of the critical distinctions between enactivism and the perspective of situated cognition.

Enactivist theory rejects the notion of optimal knowledge or ultimate Truth in favour of *effective* or *adequate* action – a “survival of the *fit*” logic (Davis et al., 1996; Maturana & Varela, 1998). In terms of individual cognition this suggests that learning “...is not a process of selecting “correct actions” but of discarding those actions that do not work” (Davis et al. 1996, p. 166). This notion of *viable* action as opposed to *optimal* action – what has been referred to as *satisficing* versus *optimizing* (Campbell & Dawson, 1995, p. 241; Simon, 1956, p. 129) – is thus one of *proscription* rather than *prescription* (Varela et al., 1991). In other words, when an individual interacts with their surroundings, rather than only certain actions being permitted by the surroundings, all actions are allowed other than those that would violate the integrity of the individual’s structure. Actions are not *specified* or *caused* by constraints or features in the environment. Rather, action/learning is *triggered* or *occasioned* by the surroundings, but it remains the structure of the individual that determines what is *able* to be perceived and experienced¹⁴. Succinctly put, “the way that a complex system responds to a situation is determined by the system itself, not the situation” (Davis, as quoted in Proulx, 2008, p. 12). From an enactivist stance, knowing is *effective action*, i.e. an action “that will enable a living being to continue its existence in a definite environment as it brings forth its world” (Maturana & Varela, 1998, p. 30). As Varela et al. (1991, p. 188) comment, this view of cognition as

¹⁴ Maturana and Varela (1998) refer to this notion as *structural determinism* in which any change that occurs as a result of a living being interacting with its surroundings, although occasioned by the surroundings, is *determined* by the *structure* of the living being.

embodied action is the counterpart to biological evolution seen as *natural drift* (Maturana & Varela, 1998; Varela, 1987). Central to the notion of natural drift is the move from optimal selection (survival of the *fittest*) to one of viability (survival of the *fit*), a shift from a pursuit of optimal fit within a given environment to a context in which the satisfaction of viability constraints is the only criterion – a scenario which allows for any number of organisms to exist provided their structure has sufficient integrity to allow their continued existence in the given environment.

Intimately associated with cognition being seen as embodied action are the notions of *co-determination* and *mutual specification*. Richard Lewontin (as quoted in Dupuy & Varela, 1992) eloquently states the position as follows:

...the organism and the environment are not actually separately determined. The environment is not a structure imposed on living beings from the outside but is in fact a creation of those beings. The environment is not an autonomous process but a reflection of the biology of the species. Just as there is no organism without an environment, so there is no environment without an organism. (p.17)

From a biological evolutionary perspective, there is a mutual interaction between organism and environment. This interaction is experienced as a mutual history of transformation and evolutionary change, a process that results in an organism and its environment mutually influencing each other and co-adapting to one another¹⁵. Organism and environment thus emerge together through a process of mutual co-determination, the organism's world being brought forth through sensorimotor coupling with its surroundings (Thompson, 2005, p. 407). Proulx (2008) summarises the situation as follows:

Put bluntly, I need a physical world to make sense of it, and I need a structure to perceive that physical world, which allows the physical world to be perceived by myself. Without a physical world or a subjective knower, there is no meaning that can emerge. (p. 21)

It follows that depending on the structure of the organism – i.e. depending on the way in which our bodies afford interactions with our environment and the nature of the distinctions we are able to make – there are many different worlds of experience that could result.

¹⁵ Maturana and Varela (1998) refer to this co-evolution and co-influence between organism and environment as *structural coupling*. Co-determination through structural coupling also resonates with Luhmann's (1995) intersystem relationship which he refers to as *interpenetration*.

As Dupuy and Varela (1992, p. 19) point out, even if we restrict our focus of embodied cognition to human beings, there are still any number of ways in which the world can be experienced – “...what we do is what we know, and ours is but one of many possible worlds” (Varela, 1987, p. 62). This may initially seem to imply a reduction to the chaos of utter arbitrariness. However, as Varela (as quoted in Breen, 2005) makes clear, although reality is perceiver-dependent this is “...not because the perceiver „constructs“ it as he or she pleases, but because what *counts* as a relevant world is inseparable from the structure of the perceiver” (p.78). Thus, to the extent that all human beings have intrinsically similar bodies and sensorimotor capabilities, our interaction with any given environment is far from arbitrary. Lakoff and Johnson (1999) take this idea further in their exploration of embodied reason:

Philosophically, the embodiment of reason ... is a crucial part of the explanation of why it is possible for our concepts to fit so well with the way we function in the world. They fit so well because they have evolved from our sensorimotor systems, which have in turn evolved to allow us to function well in our physical environment. (pp. 43-44)

Knowing is thus situated within biological and experiential contexts, “contexts which have shaped, in a non-arbitrary way, our characteristic ways of making sense of the world” (Núñez et al., 1999, p. 46). It is the non-arbitrary nature of our interaction with, and perceptual experience of, our world that results in the commonalities of human experience. Lakoff and Johnson (1980, p. 57) take this notion further and even suggest that “we experience our „world“ in such a way that our culture is already present in the very experience itself”, where our physical interaction with and experience of our surroundings is seen to take place within a milieu of cultural presuppositions. Thus, to summarise, knowledge, or rather *knowing*, depends on “being in a world that is inseparable from our bodies, our language, and our social history – in short, from our *embodiment*” (Varela et al., 1991, p. 149).

As Varela et al. (1991, p. 205) comment, “...cognition in its most encompassing sense consists in the enactment or bringing forth of a world by a viable history of structural coupling.” However, what does this mean for teaching and learning in a formal educational context? Discussing mathematics cognition and education, Simmt (2000) draws attention to the distinction between behaviour as *caused* by features or constraints in the environment and the notion of understanding as being *occasioned* by one’s interactions with the environment. For Dawson (1999, p. 154), learning “occurs at the interstices where the learner meets the environment, stresses

particularities within the environment, and generates a response whose viability in the environment is then determined.” However, of critical importance is each pupil’s predilection to take notice of the potentialities afforded by a given situation.

The particular pathway chosen “is but one of many possible ways of *satisficing* the demands of the interaction” (Dawson, 1999, p. 154). Thus, in the absence of ultimate criteria, the enactive view of cognition can be seen as one of transformational potential, of “actualizing our freedom in choosing, in the immediacy of our conscious experience, from the paths that are present and lay open to us” (Campbell & Dawson, 1995, p. 242). Nonetheless, in order for this transformational potential to be realised, these potentialities for change must first be “recognised” in the environment through interaction (Proulx, 2004, p. 116). If one is unable to “see” the triggers in the environment then one cannot be “affected” by them¹⁶. As Proulx (2004, p. 119) succinctly notes, “You get triggered by what you CAN get triggered by.”

Within a formal educational setting it is thus the structure of each individual that, in interaction with the surroundings, is crucial in terms of the world which is experienced and hence brought forth. Intimately associated with this process of bringing forth is the notion of proscription, the idea that any number of “good-enough” understandings may arise (Zack & Reid, 2003, 2004). Thus, students engaged in mathematical activity through interaction with a given mathematical context have the potential to give rise to many different worlds of experience. The only proviso is that these “good-enough” understandings, or “adequate conduct” (Maturana, 1987, p. 66), satisfy viability constraints within the given context – i.e. that these understandings don’t fail in some way (Zack & Reid, 2003).

At this juncture it may seem as though there are some strong parallels between enactivism and constructivism, and this is indeed the case. Both constructivism and enactivism reject the theoretical notion of cognitivism and its “mind as computer” metaphor by discarding the representationist survival-of-the-fittest logic in favour of a post-Darwinian survival-of-the-fit logic, a notion of truth founded on a criterion of adequacy (Sumara & Davis, 1997, p. 409). However, constructivism (both radical and social) has a number of problematic dichotomies, amongst others the separation

¹⁶ From an educational perspective this raises an interesting notion of *access*.

of individual cognizing agent from external world, and mental thoughts from physical action (Begg, 2000; Ernest, 2006; Sumara & Davis, 1997). While enactivism embraces a great deal of constructivism, there are nonetheless a number of important distinctions between the two theories of cognition. Proulx (2008) highlights three important differences between enactivism and constructivism relating to, (a) the biological basis of cognition, (b) the phylogenic¹⁷/ontogenic¹⁸ nature of knowledge, and (c) the nature of reality and knowing. Proulx (2008) asserts that while the first two distinctions could be seen as subtle extensions of constructivist thinking, the third point marks an important conceptual break from the various forms of constructivism. Each of these three distinctions is elaborated on in the following paragraphs.

(A) THE BIOLOGICAL BASIS OF COGNITION

As previously intimated, from an enactivist perspective, cognition in its most encompassing sense is not restricted to human beings, nor is it restricted to brain-bearing living systems, but can be seen as a biological phenomenon. Enactivists see cognition in terms of the process of living itself, an idea which is succinctly stated in Maturana and Varela's (1998) aphorism: "*All doing is knowing, and all knowing is doing*" (p. 26)¹⁹. Seen in these terms, any act of structural coupling whereby a living system undergoes structural change as an adaptive response to interactions with its surroundings is considered an act of cognition. Proulx (2008) points out that there is an important terminological distinction between the use of the word structure in enactivist and constructivist discourses. From a constructivist stance (e.g. von Glasersfeld, 1991), individuals actively build up their understanding of the world through a process whereby the coherence and viability of constructed knowledge is constantly tested against the lived experience as well as previously established knowledge structures (Ernest, 2006). The use of the word structure in this sense is thus a metaphorical one. For the enactivist, structure has a far more literal meaning in the sense that it is the physical structure of the individual/organism that determines the distinctions that are able to be made, and the associated affordances

¹⁷ Phylogeny is defined as the history of evolutionary development of an organism, what Proulx (2008) refers to as "biological inheritance" (p. 19).

¹⁸ Maturana and Varela (1998) describe ontogeny as "...the history of structural change in a unity without loss of organization in that unity" (p. 74). This continuous structural change within an organism is triggered through interactions with its surroundings or through a change in internal dynamics.

¹⁹ This finds resonance with Piaget's (1954) remark that "intelligence organizes the world by organizing itself" (p. 355).

that are able to be realised, during interaction with the surroundings. It is this biological basis of cognition that marks the first important distinction between enactivism and constructivism (Proulx, 2008).

(B) THE PHYLOGENIC/ONTOGENIC NATURE OF KNOWLEDGE

While constructivism focuses on experiential learning, conceptualised within Piagetian notions of adaptation, assimilation and accommodation (Piaget, 1936), Proulx (2008, p. 18) highlights that the notion of innate knowledge is ignored, or simply not taken into account. Nonetheless, cognitive science has shown that humans are indeed born with certain instinctive capacities or innate knowledge (e.g. Lakoff & Núñez, 2000). This observation marks the second important distinction between enactivism and constructivism. By foregrounding the biological basis of cognition, the enactivist stance considers not just the ontogeny of a system on the basis of a history of structural change, but situates cognition within the phylogenic lineage, i.e. the history of evolutionary development, of the species itself. As Maturana (1987) expresses it, "...not only are we here now as a result of our personal histories, but we are here now as a result of the history of our ancestors" (p. 78). A similar phylogenic notion lies behind Lakoff and Johnson's (1980) remark that "we experience our „world" in such a way that our culture is already present in the very experience itself" (p. 57). Thus, not only are we the product of our own individual ontogenies, based on a history of structural coupling through our individual interaction with our surroundings, but we are also the product of the phylogeny of our particular species, based on a historical lineage of evolutionary development. It is the biological basis of cognition that allows the enactivist theory of cognition, as formulated by Maturana and Varela, to take into account both ontogeny and phylogeny. However, since constructivism does not attempt to theorise about phylogeny, this marks a second distinction between constructivist and enactivist discourses (Proulx, 2008).

(C) THE NATURE OF REALITY AND KNOWING

While the previous two distinctions can be seen as subtle extensions of constructivist thinking, the third difference, related to the nature of reality and knowledge, marks an important conceptual break from constructivism (Proulx, 2008). To appreciate this conceptual distinction we need to recall that enactivism is premised on two important ideas or principles. As Fenwick (2001a, p. 247) points out, the first of these is

ontological²⁰ while the second is epistemological. The ontological premise is that a living system is inseparable from the context of its given surroundings. The epistemological premise is that knowledge, or rather *knowing*, is an emergent phenomenon that arises from the interaction of a living system with its surroundings. Of fundamental importance to this embodied notion of cognition is that the word “embodied” has a critical double sense in that it encompasses the body as both a “lived, experiential structure” as well as the “context or milieu of cognitive mechanisms” (Varela et al., 1991, p. xvi) in which cognition is seen as “an ongoing bringing forth of a world through the process of living itself (Maturana & Varela, 1998, p. 11). This double-embodiment results in an important circularity which Proulx (2008) believes represents a conceptual distinction between enactivism and constructivism since “it is within this circularity, of bringing forth and of being brought forth, that knower and known co-determine and mutually influence each other” (p. 22). In constructivism, access to an external reality, if it were to exist, is denied on the premise that all we have access to is our experiential world in which “all knowledge is necessarily a product of our own cognitive acts” (Confrey, 1990, p. 108) built up through our own experiences and vetted in terms of viability against our experiential world and previously established knowledge structures (Ernest, 2006). Constructivism thus foregrounds the subjective realm of the experiential world where cognition is seen as “a process of organising and re-organising one’s own subjective world of experience” (Sumara & Davis, 1997, p. 409). However, from an enactivist perspective, ontology is actively shaped by epistemology in the sense that a living system and its environment are reciprocal and simultaneous co-specifications of one another (Fenwick, 2001a; Proulx, 2008). It is my physical body that allows me to interact with the world. The world that I experience is in turn a result of the *kind* of body that I have and the *type* of distinctions it can make. As a result of these interactions a world of experience is brought forth which at the same time influences my structure, which, in a circular manner, influences the manner in which I am able to experience the world. Thus, it is the nature of knowledge/knowing that affects the nature of my reality which in turn affects the *ways* I can know. It is this co-emergence and co-defining of knower and known that marks the critical distinction between constructivism and enactivism. Unlike constructivists, for whom all one has

²⁰ Proulx (2008) however makes the observation that “ontological” in this context needs to be viewed as metaphorical, as opposed to metaphysical, since knowledge, from an enactivist stance, is constantly emerging, evolving and re-emerging in a “continual flow of emerging interactions between knower and known” (p. 23).

access to is one's experiential world, enactivists avoid the subjectivist/objectivist dichotomy by moving to a *metalevel* (Varela, 1987, p. 62), what Proulx (2008, p. 23) refers to as an *interobjective discourse*.

The stage has now been set to highlight those features of enactivism that have an important practical bearing on the learning and teaching of mathematics with specific reference to pictorial pattern generalisation. As Davis (1995) comments, language and action are not merely outward manifestations of internal workings, but rather they are "visible aspects of ... embodied (enacted) understandings" (p. 4). For Davis et al. (1996), enactivism prompts us not only to consider the formal mathematical ideas that emerge from action, but to give close scrutiny to those preceding actions – "the unformulated exploration, the undirected movement, the unstructured interaction, wherein the body is wholly engaged in mathematical play" (p. 156). This resonates with Mason's (2003, 2004) notion of the *structure of attention* in which attention is seen "...not just as what puts me in touch with the world of my experience, but what creates and maintains that world" (Mason, 2004, p. 3). There are two important components to this notion of structure of attention – *what* one attends to, and *how* one attends to it (Mason, 2004).

The totality of what I experience at any one moment *is* my attention. This is meant to include things of which I am subliminally or covertly aware, sometimes through body awareness, sometimes through social awareness, sometimes through emotional resonance, and sometimes through cognitive awareness. None of these need be conscious. (Mason, 2004, p. 3)

Mason (2003) takes this idea further and suggests that learning can be seen not only as shifts in the structure of attention, but more broadly as "extending sensitivity to notice" (p. 24). In terms of the co-evolution of knower and known, learning can be seen to occur "at the interstices where the learner meets the environment, stresses particularities within the environment, and generates a response whose viability in the environment is then determined" (Dawson, 1999, p. 154). However, of critical importance is each pupil's predilection to take notice of the potentialities afforded by a given situation. The particular pathway chosen "is but one of many possible ways of *satisficing* the demands of the interaction" (Dawson, 1999, p. 154). An appropriate framework is thus needed in order to engage with pupils' whole-body experience and expression while they explore the potentialities afforded by a given pictorial pattern generalisation task. Radford's (2008) theoretical construct of

knowledge objectification is explored in the following section as a meaningful framework for doing just this.

3.3 PERCEPTION & KNOWLEDGE OBJECTIFICATION

To *perceive* something means “to endow it with meaning, to subsume it in a general frame that makes the object of perception *recognizable*” (Sabena, Radford & Bardini, 2005, p. 129). From an enactivist stance perception needs to be considered as a fully *embodied* process - a complex activity related to the *manner* of our acquaintance with the objects of perception, in other words the *activity* that mediates our experience with objects (Radford, Bardini and Sabena, 2007). Radford (2008, p. 87) refers to the process of making the objects of knowledge apparent as *objectification*, a multi-systemic, semiotic-mediated activity during which the perceptual act of noticing progressively unfolds and through which a stable form of awareness is achieved. Use of the word “objectification” in this context needs to be interpreted in a phenomenological sense, a process whereby something is brought to one’s attention or view (Radford, 2002a, p. 14). In terms of its etymology, objectification is related to “those actions aimed at bringing or throwing something in front of somebody or at making something visible to the view” (Radford, 2003, p. 40)²¹.

A fully embodied notion of cognition prompts us to consider the process of “thinking” as not only being *mediated by*, but also *located in*, body, artefacts and signs (Radford, Bardini, Sabena, Diallo & Simbagoye, 2005). Knowledge objectification is thus a multi-semiotic activity which can include the use of, *inter alia*, spoken or written words, gestures, drawings, formulae and artefacts (Radford, 2006, p. 6). It is through this multi-systemic, semiotic-mediated activity that the objects of perception, or rather the objects of knowledge, progressively emerge – a process of “concept-noticing and sense-making” (Radford, 2006, p. 15). Importantly, from an enactivist stance, use of the word “object” by no means suggests that these “objects” are pre-existing properties inherent in the environment. Rather, the “objects” of perception are *brought forth* through the co-determination of knower and known, the co-

²¹ Objectification has as its Latin roots the words *obicio* or *obicere*, meaning “to throw or put before or in the way”, and *facere*, meaning “to carry out”.

evolution of each individual pupil and their surroundings. Radford (2003, p. 41) uses the term *semiotic means of objectification* to refer to the artefacts, tools, linguistic devices and signs used by individuals during the process of objectification.

Words, both written and spoken, are seen to perform an important part of knowledge objectification. This is particularly relevant in terms of how they support the process of generalisation. As Radford (2000) explains, linguistic articulation related to ways of thinking about and expressing generality seems to focus on two different categories of words, each of which has a different but crucial semiotic function. The first category relates to those words that serve a *deictic* function, while the second relates to those that have a *generative action* function. Deictic words, for example “that”, “this”, “here” or “there”, serve a primary function of pointing to objects in the visual field of the speaker, and necessarily take an element of their meaning from the given context or situation. Deixis supports a powerful referential mechanism in the context of pictorial pattern generalisation (e.g. Radford, 2002a). By way of example, terms such as “top” and “bottom” which are indicative of a distinct spatial location serve as *spatial deictics*. A student using such spatial deictics to characterise the n^{th} term (i.e. the *general* term) of a pictorial sequence has the means of giving linguistic shape to a term which is not possible to be physically instantiated by drawing or construction. As Radford (2000, p. 247) remarks, even when talking about a specific but not at that time materially instantiated term (e.g. the 10th term when only the first four terms have been drawn or constructed), students’ use of spatial deictics allows for a metaphorical access of the general *through* the particular. Spatial deictics are thus a means to make apparent *general* contextual structures and as such represent “...pivots located between the particular and the general” (Radford et al., 2007, p. 515). Words such as “vertical” and “horizontal” are also deictic terms, and function as *structural descriptors* (Radford, 2000, p. 249). The second important category of words relates to those that have a *generative action* function, for example the adverb “always”. Radford (2000) describes the generative action function as a linguistic mechanism “expressing an action whose particularity is that of being repeatedly undertaken in thought” (p. 248). In terms of the generalisation process, the generative action function plays an important role in objectifying the generality of a given context through the imaginative conception of iterative potential action. *En route* to the objectification of generality, students may also find *generic* and *locative* terms such as, respectively, “the figure” and “the next figure” to be helpful stepping

stones. As Radford (2003, p. 53) explains, generic and locative terms such as these enable students to refer to terms of a sequence that are not materially present, and in so doing allow the emergence of *new* objects that still contain all the features of their physical genetic referents.

Gestures are seen to be another key element in the process of knowledge objectification, far more than mere communicative devices. Their co-emergent nature is foregrounded in LeBaron and Streeck's (2000) position that "the fabrication of knowledge and the formation of signs [gestures] are not simply dependent upon one another, but are two aspects of the same process" (p.119). Thus, "gesture is not simply an epiphenomenon of speech or thought; gesture can contribute to creating ideas" (Arzarello & Edwards, 2005, p. 125). From a fully embodied stance it is also important to keep in mind that gestures originate from and are grounded in the corporality of our actions, "...the tactile contact that mindful human bodies have with the physical world" (LeBaron & Streeck, 2000, p. 119). In terms of pictorial pattern generalisation there are two important types of gesture that need consideration – *iconic* and *indexical/deictic*²². Iconic gestures mirror the semantic content of speech and are marked by a resemblance to the referent which they symbolise, for example a hand making a corkscrew motion to describe a spiral staircase. Such gestures support the process of generalisation by representing an important means of making apparent *general* contextual structures – for example, a student gesturing a perceived zigzag structure in a pictorial term. The second important category of gestures represents indexical or deictic gestures which are used to indicate objects in the concrete world, for example pointing to a specific part of a pictorial term. Such gestures support the generalisation process as they allow a transition from *existential* signification to an *imaginative* signification (Sabena et al., 2005, p. 134). By way of example, a student presented with the first three terms of a pictorial sequence is able physically to touch or point to specific aspects of those terms, terms which are physically instantiated at that moment. However, the student is still able to use the same gestures to indicate similar aspects or features of terms which are *not* physically present, for example the 4th, 10th or 50th terms. This can be accomplished through *imaginative* signification where such gestures are

²² Other categories of gestures include *metaphoric* gestures, where the content represents an abstract idea without physical form, and *beat* gestures, which are simple repeated gestures used for emphasis (see e.g. McNeill, 1992).

progressively distanced from the existing terms. As Sabena et al. (2005, p. 134) explain, since such gestures mime an existing referent they are a means of making apparent certain features of the new referent, a referent that is not materially present, and thus aid in the process of knowledge objectification.

Rhythm, the “ordered characteristic of succession” (Fraisse, 1982, p. 150), is considered another important element in the process of knowledge objectification. Rhythm, whether in speech or gesture, is not merely the conscious or unconscious perception of order, but crucially it creates a sense of expectation or the “...anticipation for something to come” (You, 1994, p. 363). There is thus an inherent sense of expectancy associated with rhythm, and it is seen as a crucial semiotic device in the process of generalisation (Radford, Bardini & Sabena, 2006; Radford et al., 2007, p. 522). Rhythm is a subtle yet powerful semiotic device since it is able to operate on multiple levels – verbal, aural, kinaesthetic and visual. In the process of generalisation, rhythm aids and supports the move from the particular to the general by enabling pupils to project and make apparent a regularity or perception of order that transcends the specific cases under scrutiny (Radford et al., 2006).

Metaphor and metonymy have both been acknowledged as playing an important role in processes related to mathematical reasoning and understanding (Presmeg, 1992a, 1997a, 1998). Within the context of pictorial pattern generalisation, the use of metaphor and metonymy has the potential to support the process of generalisation by objectifying perceived features of the pictorial context in terms of structures or ideas not materially present but which form part of a student’s ontology – their biological, experiential and cultural history. Nolder (1991), for example, reports on the use of metaphors such as “staircases”, “wings” and “triangles” by pupils presented with number patterns in a 3-dimensional practical context. However, on a cautionary note, she also comments that although these metaphorical images are useful in helping pupils structure and express their perceptual ideas, some metaphors are less helpful than others when it comes to notions of generality and the formulation of an algebraic expression representative of the perceived structure.

In terms of graphics and the use of artefacts, different graphical means of emphasis (e.g. colour), along with the tactile manipulation of artefacts (e.g. matches, counters),

are not mere auxiliary components but each open up different spaces of possible action and have the potential to differently shape enactive processes of coming to know (Lozano, Sandoval & Trigueros, 2006, p. 90; Radford, 2003, p. 41). In addition, symbolic expressions of generality can be viewed as representing a *symbolic narrative* of the objectification process (Radford, 2002b, p. 83). Algebraic expressions for the general term (e.g. $4(n-1)+3$ or $n+3(n-1)+2$) which have not purposely been reduced to simplest algebraic form (e.g. $4n-1$) thus retain elements of the student's objectification process.

Knowledge objectification is a theoretical construct to account for the manner in which students engage or interact with a given scenario or context in order to make sense of it *en route* to a stable form of awareness (Radford, 2006, p. 7). Knowledge objectification is premised on two notions. Firstly, semiotic means such as gestures, rhythm and speech are not simply epiphenomena, but are seen to play a fundamental role in the formation of knowledge (Radford, 2005a, p. 142). Secondly, in order to study the process of knowledge production one needs to pay close scrutiny to *multiple* means of objectification, for example words, linguistic devices, gestures, rhythm, graphics and the use of artefacts, where "...meaning is forged out of the interplay of various semiotic systems" (Radford, 2005b, p. 144)²³. Such a multi-semiotic view takes cognizance of the *principle of non-redundancy* (Benveniste, 1985, p. 235), the notion that different semiotic systems are not "synonymous" or "mutually interchangeable". Rather, different semiotic systems allow for different *forms* of expressivity and hence play different roles in the objectification process.

Within the context of figural pattern generalisation, the processes of visualisation and generalisation are deeply interwoven. Pattern generalisation rests on an ability to *grasp* a commonality from a few elements of a sequence, an awareness that this commonality is applicable to *all* terms of the sequence, and finally being able to use it to articulate a direct *expression* for the general term. Inherent in this notion of generalisation are (a) a *phenomenological* element related to the grasping of the generality, and (b) a *semiotic* element related to the sign-mediated articulation of

²³ Radford, Demers, Guzmán and Cerulli (2003, p. 56) use the term *semiotic node* to refer to pivotal moments of semiotic activity when a variety of semiotic means coalesce to achieve knowledge objectification.

what is noticed in the phenomenological realm (Radford, 2006, p. 5). An interrogation of the *embodied* processes of perception thus needs to focus on the phenomenological realm of students' experience in order to emphasize the subjective dimension of knowing (Radford, 2006).

Visually mediated approaches to pattern generalisation tasks set within a pictorial context provide for an interesting interplay between two different modes of visual perception: sensory perception and cognitive perception (Rivera and Becker, 2008). These different modes resonate with Fischbein's (1993) theory of figural concepts, and the notion that all geometrical figures (or *figural objects*) possess, simultaneously, both conceptual and figural properties. Mariotti (as cited in Jones, 1998, pp. 30-31) stresses the dialectic relationship between figure and concept as an important interaction in the field of geometry, a relationship that can create tension from a student's perspective. A similar tension is likely to underlie visual strategies applied to pattern generalisation tasks set within a pictorial context. The notion of figural *apprehension*, as espoused by Duval (1995, 1998, 1999), is explored in the following section as a framework for exploring this visual tension.

3.4 FIGURAL APPREHENSION

As Duval (2006, p. 116) succinctly notes, there are many ways of *seeing*. Let us consider a simple geometrical figure composed of a number of lines such as that shown in Figure 3.1. Pupils visually interacting with the figure have the potential to perceive it in a number of different ways. In the same vein, a *single* pupil may be able to perceive the figure in multiple ways. If we take Figure 3.1 as an example, the figure could be perceived as comprising two overlapping Hs. Alternatively, it could be seen to comprise four overlapping squares, where the "lids" of the top two squares and the "bases" of the bottom two squares are missing. A third possibility is for the figure to be perceived as comprising three vertical lines with two interconnecting horizontal lines. Yet another possibility is for the figure to be seen as a "+" symbol contained within two sets of vertical lines, one on either side.

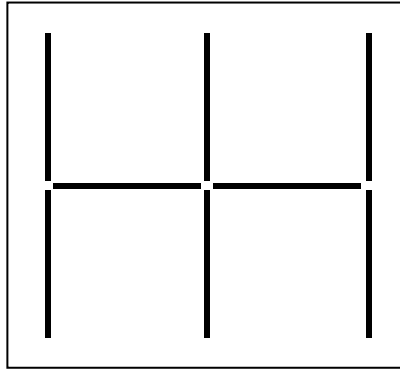


Figure 3.1 A simple geometrical figure

If we now take the same image shown in Figure 3.1 but place it in the context of a pictorial sequence, then the visual scenario is profoundly altered. This situation is shown in Figure 3.2.

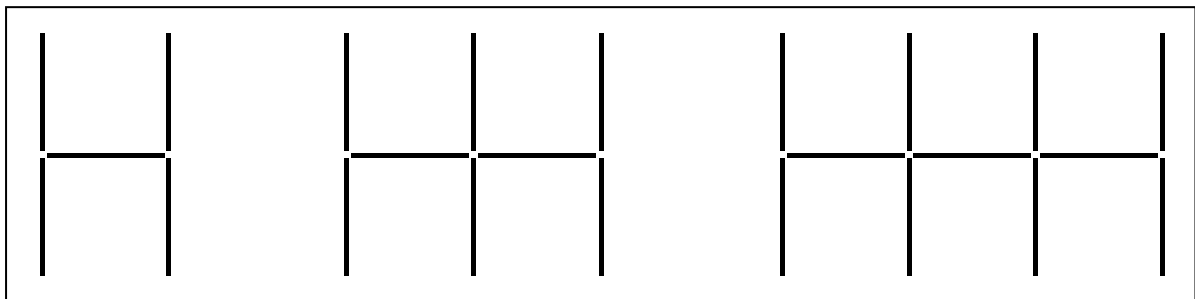


Figure 3.2 A simple pictorial sequence

The middle diagram of Figure 3.2 is identical to the image shown in Figure 3.1. However, the contextual setting is no longer simply that of geometrical figure. There is now a tacit suggestion that the image is part of a larger context, that it is part of a *sequence* of visually/structurally similar images. Given this new context, and the associated yet implicit sense of sequential growth, the middle diagram of Figure 3.2 could be perceived as a vertical line on the left followed by two sideways T-shapes. Alternatively, it could possibly be perceived in terms of a horizontal “backbone” with vertical lines extending off it in two directions, upwards and downwards. The previous four visualisations are of course still possible, but the added context provides additional/alternative perceptual features²⁴ to be made apparent. Thus, by modifying the context, different ways of perceiving the figure are *brought forth*.

²⁴ Use of the word “feature” is not meant to imply that such features (or structures) are intrinsically contained within the image, simply waiting to be extracted or noticed by an observer. Rather, such features are seen to *co-emerge* from the interaction of a perceiver and the given figural context.

One can add a further dimension to the visual image shown in Figure 3.2 by requiring pupils to provide an algebraic expression for the general term of the sequence. Although the context remains the same as that represented in Figure 3.2, the additional requirement provides a further layer of complexity as it necessitates not only the perception of the figure within the context of a sequence of similar figures, but it requires the perception of *generality*, the notion that the figures in the sequence have a *related* structure. Finally, there is a requirement that this perceived generality can be articulated in such a way that it is able to be written in the form of an algebraic expression. A critical aspect of the visualisation process thus relates to the *usefulness* or *meaningfulness* of the perceived structure of the image in terms of the extent to which this perceived structure supports or hinders the process of generalisation.

Drawing on the nomenclature used by Fischbein (1993), figures such as the image shown in Figure 3.1 could be said to contain figural properties. What one sees in the image is a result of the Gestalt laws of figural organisation. Images such as that shown in Figure 3.2 could be said to contain both figural properties and conceptual qualities. What one sees in the image is still based on the Gestalt laws of figural organisation, but this is further influenced by the additional conceptual qualities of the image, qualities that have been added by virtue of the image being contextualised, in this case within a sequence of similar images²⁵. The critical point here is that an underlying tension is likely to pervade visual strategies applied to pictorial pattern generalisation tasks as a result of the relationship between the figural properties and conceptual qualities of the given images.

Spatial relations in imagery are an emergent property of the depicted perceptual units. For a given visual configuration, the perceived emergent structure may thus obscure aspects of the image which may be helpful or efficacious in terms of the generalisation process. In order for a pupil to move beyond this perceived structure a “reorganization of the elements is needed, which then hopefully enables the

²⁵ Perception guided either by context, expectation or past experience is often referred to as the *top-down* (as opposed to bottom-up) approach, or alternatively as *constructive* or *intelligent* perception (Sternberg, 1999, p. 136). See Gal and Linchevski (2002) for an account of how these aspects of perception could be used in an analysis framework to support the teaching and learning of high-school geometry.

individual to comprehend how the elements fit together, thus achieving what Gestaltists termed *structural understanding*" (Orton, 2004, p. 78).

The founding of the Gestalt movement in the early 20th century is considered one of the most important events in the history of perception (Palmer, 1992a, p. 40), and led to the discovery of some of the most interesting phenomena of visual perception. The central concept of Gestalt theory is that of the "gestalt" which can be roughly translated from the German as "form" or "configuration" (Zusne, 1970, p. 108). The essence of Gestalt psychology is that the mind attempts to interpret sensations and experiences (visual, auditory, tactile etc.) not as a collection of individual units of data but rather as an organised whole. The central tenet of Gestalt theory was that of *holism*, the idea that "a perceptual whole is different from – and not reducible to – the sum of its parts" (Palmer, 1992a, p. 40). One of the principal interests of the Gestalt movement, led by Max Wertheimer, Kurt Koffka and Wolfgang Köhler, was how individual stimuli are grouped together during perception into these wholes or "gestalts". Within the area of visual perception, one of the aims of the Gestalt psychologists was to specify the principles by which individual items are combined into larger, organised wholes. They also sought to find the principles by which these wholes are perceptually segregated and separated from other organised wholes (Spoehr & Lehmkuhle, 1982, pp. 63-65; Wertheimer, 1938a, p. 2).

The literature concerning Gestalt laws/principles is substantial (see e.g. Katz, 1951; Wertheimer, 1938b; Zusne, 1970, pp. 111-135). Helson (1933) was able to distil and articulate 114 such "laws" or propositions. Although only a few of these mention visual form specifically, the majority are applicable to visual perception (Zusne, 1970, p. 111). It is important to note, however, that the various Gestalt laws are by no means independent of one another. A number of configurational forces may be in operation at the same time, often in conflict with one another. In such instances the stronger force (dependent on the nature of the visual field) will predominate, leading to the perception of that form in which it is dominant.

The Gestaltists' fundamental principle of perceptual organisation was the law of *Prägnanz*: "Of several geometrically possible organizations that one will actually occur which possesses the best, the most stable shape" (Koffka, 1935, p. 138). Numerous other Gestalt laws/principles were proposed which can be subsumed

under the law of *Prägnanz*. Of particular interest to the field of visualisation are the following four, the laws of *proximity*, *similarity*, *good continuation*, and *closed forms*, as outlined by various authors (Katz, 1951, pp. 24-29; Spoehr & Lehmkuhle, 1982, pp. 65-67; Wertheimer, 1938b, pp. 71-88; Zusne, 1970, pp. 111-135):

- The law of proximity

Other things being equal, the grouping of individual elements occurs on the basis of proximity. In other words, elements which are close to one another tend to be perceived as a unit.

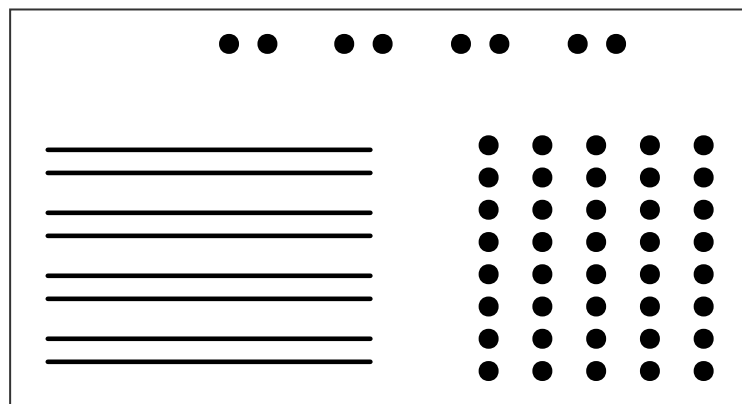


Figure 3.3 The law of proximity

In Figure 3.3 the dots in the upper horizontal row appear to be clustered in pairs. The spatial proximity forces the grouping of the first two dots but the separation of the second and third dot. In the array of horizontal lines, those lines closest to each other also form pairs. The matrix of dots on the right appears to be grouped in columns as opposed to rows, the grouping being induced by the close vertical proximity of the dots.

- The law of similarity

Other things being equal, when more than one type of element is present, elements which are similar in structure tend to be perceived as groups.

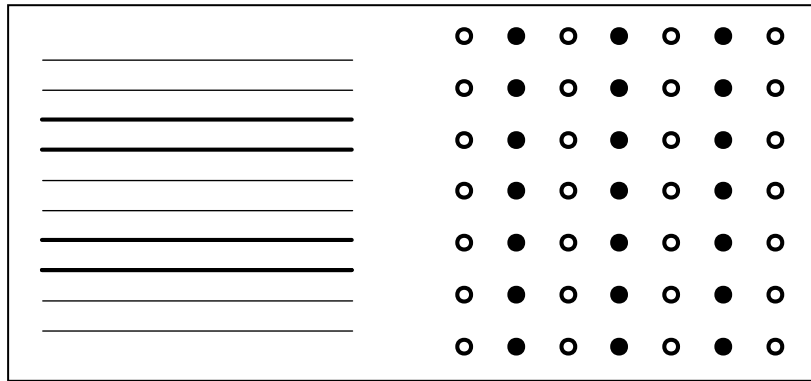


Figure 3.4 The law of similarity (adapted from Katz, 1951, p. 25)

In Figure 3.4 the bold lines combine to form pairs despite the fact that all the lines are equally spaced. In the matrix of dots the vertical and horizontal distances between the dots are equal. Proximity should thus yield no perceived impression of either rows or columns. However, because alternate columns are composed of similar elements, they form good perceptual groups. The perceptual organization into columns is thus the result of similarity of form.

- The law of good continuation

Other things being equal, elements tend to be organised into groups that yield few interruptions or changes in continuity.

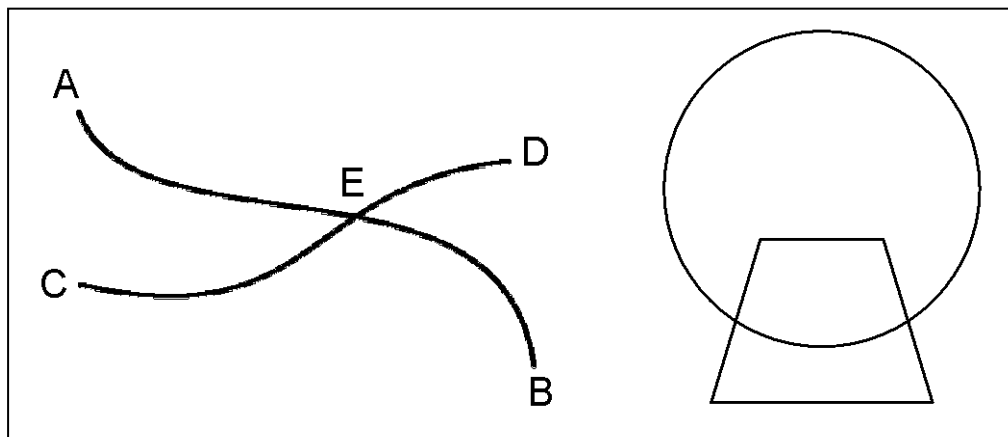


Figure 3.5 The law of good continuation (adapted from Katz, 1951, p. 26)

The left-hand diagram in Figure 3.5 could be described as four curvy line segments radiating out from a central point E. Alternatively, it could be described as two curvy V-shaped line segments, AEC and DEB (or equally

AED and CEB) meeting at point E. However, the seemingly more “accurate” or “natural” description of the grouping is as two continuous wavy line segments, AB and CD, overlapping at E. In the right-hand diagram of Figure 3.5, the figure is perceptually broken up into a circle and a trapezium because the parts of each have good continuation.

- The law of closed forms

The law of closed forms, also known as the “law of closure” or the “law of good form”, is closely related to the law of good continuation. The law of closed forms states that our organisation of elements tends to form them into simple, closed figures, independent of their other continuation, similarity or proximity properties (Spoehr & Lehmkuhle, 1982, p. 66). Where there are several alternate ways in which a complex diagram may be perceived, the simpler and more regular configuration will be chosen.

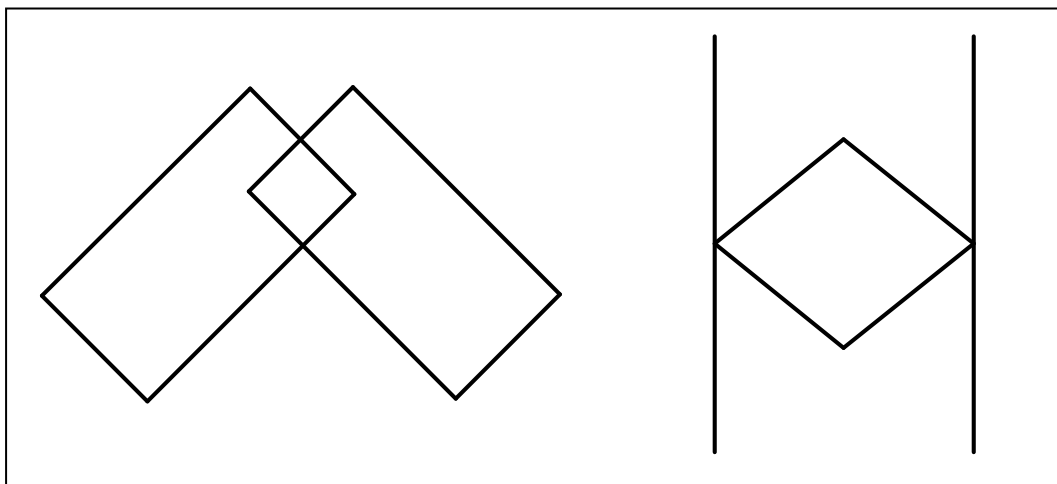


Figure 3.6 The law of closed forms (from Spoehr & Lehmkuhle, 1982, p. 66 and Zusne, 1970, p. 130)

In Figure 3.6, the left-hand figure is perceived as two overlapping rectangles (as opposed to two abutting rectangles each with a corner cut out) since a complete rectangle is both simpler and more regular. The right-hand diagram serves to show the interplay between the laws of good continuation and closed forms. The diagram is perceived as two vertical lines with a diamond shape between them rather than the letters W and M (Zusne, 1970, p. 129).

The Gestalt laws of figural organisation have been remarkably resilient, to the extent that not one of them has been refuted (Rock & Palmer, 1990, p. 50). Nonetheless, as Palmer (1992a, p. 41) remarks, despite their theoretical importance Gestalt phenomena are still relatively poorly understood. However, at least some progress has been made (see e.g. Kellman, 2000; Palmer, 1992a). In addition, two new principles of perceptual organisation have been proposed – that of *common region*²⁶ (Palmer, 1992b) and *uniform connectedness*²⁷ (Palmer & Rock, 1994a, 1994b).

Having briefly reviewed the relevant Gestalt principles of perceptual organisation we can now return our focus to Duval's (1995, 1998, 1999) notion of figural *apprehension*. Although Duval developed his idea of figural *apprehension* within a more classical geometry context, it can readily be adapted to other contexts involving geometrical figures. To begin with, reconsider Figure 3.1. What a pupil sees in the given image is brought about through the Gestalt laws of figural organization. However, Duval (1998, p. 39) comments that in order for such a figure to represent a *mathematical* object, two specific requirements must be fulfilled. Firstly, the figure needs to be a configuration, i.e. a merger of *several* constituent gestalts characterised by the relations between them. Secondly, the figure needs to be anchored in a statement which fixes some of the properties represented by the gestalt. Within Duval's original geometry context this second requirement could be arrived at either *visually* (by annotating a given figure in some way to foreground specific properties) or *discursively* (by accompanying a given figure with a discursive statement which would similarly anchor specific features). In terms of our context of figural pattern generalisation, Figure 3.1 meets Duval's first requirement since it represents a merger of a number of constituent gestalts. As such it could be perceived in a variety of ways. Figure 3.2 on the other hand meets *both* requirements since not only is it a merger of various constituent gestalts, but it has been visually anchored within the context of a *sequence* of images which provides a sense of *regularity*. Figure 3.2 can thus be considered a *mathematical* object.

²⁶ "...all else being equal, elements that are located within the same perceptually defined region will tend to be grouped together" (Palmer, 1992b, p. 439).

²⁷ "...a connected region of uniform visual properties - such as luminance or lightness, color, texture, motion, and possibly other properties as well - strongly tends to be organized as a single perceptual unit" (Palmer & Rock, 1994a, p. 30).

Duval (1995, 1998, 1999) distinguished between four different modes of figural apprehension²⁸ – *perceptual*, *sequential*, *discursive* and *operative*. Although Duval originally described these modes of apprehension in terms of a classical geometry context, they have been somewhat modified and are described here in terms of the adopted context, namely figural pattern generalisation.

- Perceptual Apprehension

This refers to the *initial* apprehension of a figure, i.e. what we see in a perceived figure at first glance as determined by the unconscious integration of Gestalt laws of figural organisation. During perceptual apprehension it is possible to discriminate between component sub-figures of the perceived figure, once again determined by Gestalt laws of figural organisation.

- Sequential Apprehension

This relates to the emergence of sub-figures or elementary figural units which stem from either the construction of the perceived figure, or a description of its construction. Specific sub-figures or elementary units arise not from unconscious laws of figural organisation, but from the physical process of construction²⁹.

- Discursive Apprehension

This is a process of perceptual recognition during which certain gestalt configurations gain prominence due to an association with discursive statements accompanying the geometric figure. Within a classical geometry context this relates to the limitation that it is not possible to determine the *mathematical* properties represented in a figure through perceptual apprehension alone. The provision of initial discursive or pictorially annotated criteria is also necessary in order to render the figure unambiguous³⁰.

²⁸ The word “apprehension” was chosen deliberately by Duval to highlight the fact that there are many ways in which a visual stimulus can be perceived (Duval, 1995, p. 143).

²⁹ This resonates with the heuristic strategy *Watch What You Do* (Mason, 2007; Mason, Graham & Johnston-Wilder, 2005). The underlying principle of this heuristic strategy is that paying attention to the construction of *particular* cases may reveal *general* structure that was not initially apparent.

³⁰ This resonates with Gobert’s (2007) assertion that “*a drawing alone does not possess any geometric signification. It may possess it only if a geometric referent is specified*” (p. 122).

- Operative Apprehension

This mode of apprehension relates to the various ways by which a given figure can be modified while retaining its physical integrity. Figural modification can be accomplished by (i) a reconfiguration of the whole-part relation of the given figure by means of a recombination of various elementary figural units, (ii) size or plane variation (e.g. viewing the image from an oblique angle), and (iii) position or orientation variation (e.g. rotating the image and viewing it from a different perspective). Duval (1995) refers to these three types of figural change as *mereologic*, *optic* and *position* or *place* respectively.

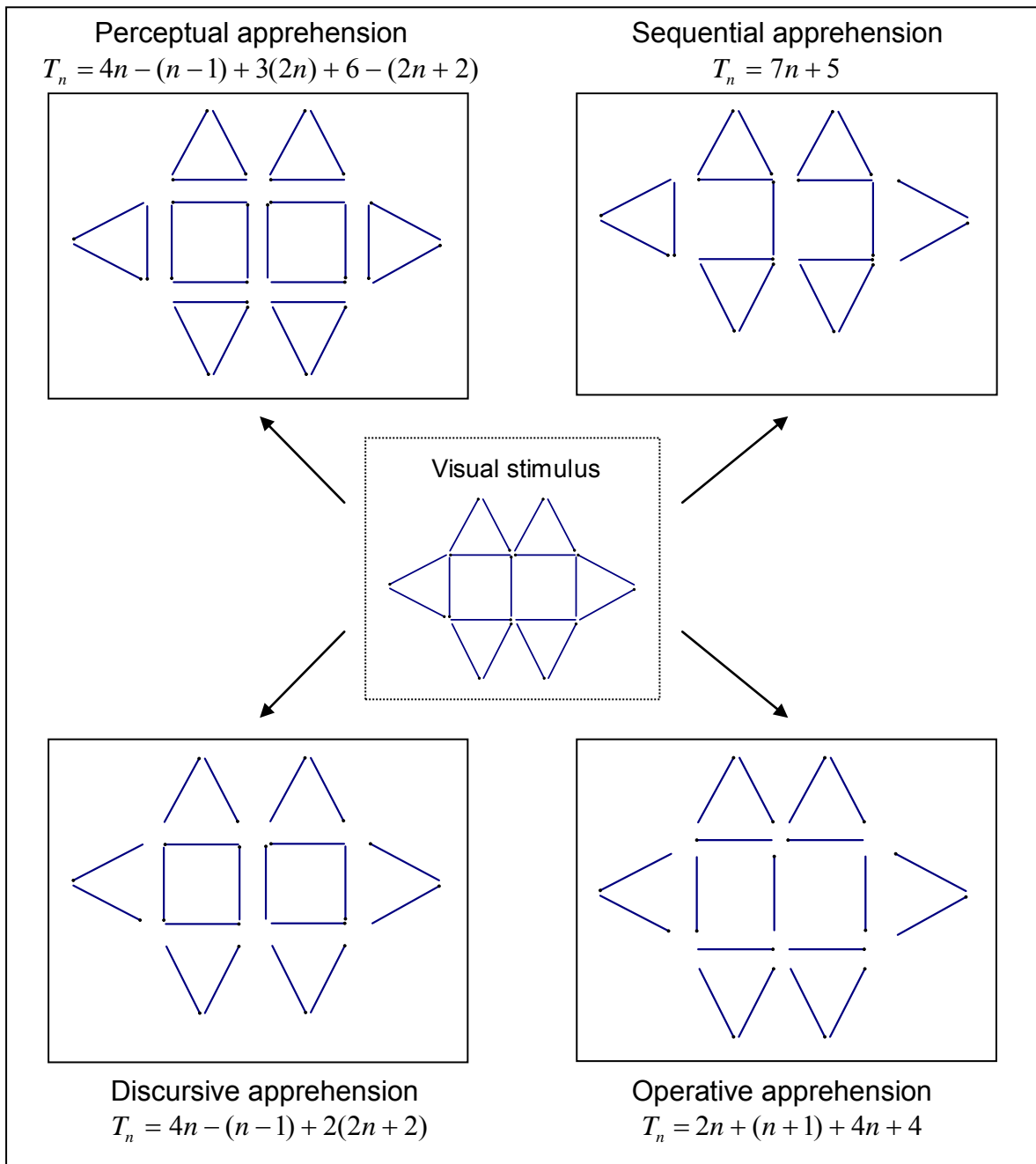


Figure 3.7 Different modes of figural apprehension

Within the context of pictorial pattern generalisation, the different modes of figural apprehension have the potential to lead to different expressions for the general term. This scenario is represented in Figure 3.7 which shows possible outcomes for each of the four modes of figural apprehension based on the given visual stimulus. The visual stimulus, displayed in the centre of Figure 3.7, is the second term of a pictorial sequence.

Perceptual apprehension could possibly subdivide the given figure into squares and triangles based on the Gestalt law of closure or good form. This could potentially lead to a complicated algebraic expression for the general term which would need to take into account overlapping matches. The n^{th} figure in the sequence contains n squares requiring a total of $4n$ matches. However, since there is an overlap between adjoining squares we need to subtract $(n-1)$ matches from this count. Moving on to the triangles, the n^{th} figure contains n triangles at the top and n triangles at the bottom (i.e. $2n$ triangles) requiring $3(2n)$ matches. In addition, we require a further 6 matches for the two triangles positioned at either end. However, since there is an overlap between each triangle and its adjoining square we need to subtract $(2n+2)$ matches from this count. Combining these could yield the general expression: $T_n = 4n - (n-1) + 3(2n) + 6 - (2n+2)$.

Sequential apprehension could arise from noticing that the construction of a term from the previous term in the sequence can be accomplished by the insertion of a 7-match additive unit. This construction process could be accomplished either mentally or physically. The n^{th} figure would thus contain n 7-match additive units, requiring a total of $7n$ matches, plus 5 additional matches for the triangle on the left and the V-shape on the right. This particular apprehension thus has the potential to yield the general expression: $T_n = 7n + 5$.

Discursive apprehension could be invoked, for example, by accompanying the visual stimulus with the wording “for 2 squares you need a total of 19 matches”. This accompanying statement incorporating the word “squares” could potentially foreground the structural unit of the square. This could yield a general expression similar to that arrived at through perceptual apprehension, but with a subtle modification. The n^{th} figure in the sequence is still seen to contain n squares,

requiring a total of $4n$ matches, corrected for overlapping matches by the subtraction of $(n - 1)$ from the count. However, instead of seeing the rest of the image in terms of triangles, it is now seen in terms of V-shapes. This could arise through discursive apprehension in which the structural feature of the square has been foregrounded and in which the matches previously seen as belonging to both squares and triangles are now specifically associated with the structural unit of the squares. The remaining matches are thus seen in terms of V-shapes, and since there are n V-shapes above, n V-shapes below, and one at either end, this requires $2(2n + 2)$ matches, leading to the final expression: $T_n = 4n - (n - 1) + 2(2n + 2)$.

Finally, operative apprehension may bring about a reconfiguration of the whole-part relation allowing the given figure to be seen in terms of horizontal lines, vertical lines and V-shapes. The n^{th} figure could thus be visualized as comprising $2n$ horizontal matches (seen as two rows each containing n matches), $(n + 1)$ vertical matches, n V-shapes at the top and n V-shapes at the bottom requiring a total of $4n$ matches, and finally a constant V-shape at either end requiring an additional 4 matches. This apprehension thus yields the following general expression: $T_n = 2n + (n + 1) + 4n + 4$.

Duval (1998, p. 41) makes the pertinent point that most diagrams contain a great variety of constituent gestalts and sub-configurations – far more than those initially identified through perceptual apprehension, or those made explicit through construction or accompanying discursive statements. Critically, it is this surplus that constitutes the *heuristic power* of a geometrical figure since specific sub-configurations may well trigger alternative solution paths. Within the context of pattern generalisation, perceptual apprehension may on occasion be sufficient to generalise a given figural pattern. However, perceptual apprehension will not necessarily evoke gestalts which are *appropriate* or *useful* to the generalisation process. An inability to move beyond mere perceptual apprehension of a figure can lead to what Duval (1999, p. 17) refers to as *heuristic deficiency*, which is similar in notion to what Hoz (1981) calls *geometrical rigidity*, and in the context of pattern generalisation can vitiate the process by obscuring potentially meaningful gestalts. In order to actualise the heuristic potential of a diagram it is necessary not only to be aware of the scope of the diagram but also to be able to use it flexibly (Rösken &

Rolka, 2006). Thus, being able to see a diagram in multiple ways necessitates a move *beyond* perceptual apprehension.

3.5 COMBINING THREE THEORETICAL IDEAS

Within the context of figural pattern generalisation, the processes of visualisation and generalisation are deeply interwoven. Furthermore, a complex relationship seems to exist between the embodied processes of pattern generalisation and the visualisation of accompanying pictorial images. However, an analysis based on the novel combination of *enactivism*, *figural apprehension*, and *knowledge objectification* has the potential to shed light on this inter-relationship, showing sensitivity to the visual, phenomenological and semiotic aspects of figural pattern generalisation.

An important aspect of figural pattern generalisation lies in the notion that such pictorial cues, or rather visual *triggers*, possess both figural and conceptual qualities, each of which resonates with a different mode of visual perception – sensory and cognitive, respectively. This position perhaps seems somewhat at odds with an enactivist view of perception as being a fully embodied and co-emergent process. However, it is not being suggested that these two modes of perception are independent of one another, or that they are able to occur in isolation: a drawing of a triangle, for example, can only be perceived as such through a combination of both sensory and cognitive perception. Indeed, one could even argue that sensory perception cannot occur without cognitive perception – a view that resonates strongly with the mind-body unity that is the core of enactivism. Nonetheless, the distinction between *figural* and *conceptual* properties provides a useful framework to discuss figural pattern generalisation. Perhaps an appropriate analogy would be that of a Möbius strip, where figural and conceptual properties are simply different aspects of the same phenomenon depending on the stance of the observer.

Although figural cues contain *simultaneously* both figural and conceptual properties, and while it is acknowledged that perception is *at once* both sensory and cognitive, what is important is the *nature* of the figural and conceptual properties of pictorial cues within the context of pattern generalisation. In order to unambiguously present a pictorial sequence, at least two terms of that sequence need to be shown. Such a

visual stimulus or *trigger* can be perceived in any number of different ways suggested by the Gestalt laws of perceptual organisation. However, by being visually anchored within the context of a *sequence* of images which provides a sense of regularity of structure, the pictorial trigger can also be perceived on the basis of conceptual qualities. What is important here is that there will be interplay between the figural and conceptual properties, resulting in different *apprehensions* of the visual stimulus. While some of these apprehensions may evoke gestalts which are appropriate or helpful to the generalisation process, others may simply obfuscate or vitiate the process.

As Rowlands (2006) notes, the notion of exploration is one of activity rather than passivity, “it is something we do, rather than something that happens to us” (p. 12). Whole-bodied exploration of a given pictorial trigger is a crucial aspect of the generalisation process. From an enactivist stance, we need to consider not only the formal mathematical ideas that emerge from action, but to give close scrutiny to those preceding actions – “the unformulated exploration, the undirected movement, the unstructured interaction, wherein the body is wholly engaged in mathematical play” (Davis et al., 1996, p. 156). Radford’s theoretical construct of *knowledge objectification* foregrounds the phenomenological and semiotic aspects of figural pattern generalisation and hence allows us to critically engage with pupils’ whole-body experience and expression while they explore the potentialities afforded by a given pictorial pattern generalisation task. At the same time, Duval’s concept of figural apprehension provides a meaningful means of discussing visual aspects of the phenomenological realm.

The combination of complementary multiple perspectives provided by three theoretical ideas (*enactivism*, *figural apprehension*, and *knowledge objectification*) has the potential to provide meaningful insight into the tensions and complexities underlying pictorial pattern generalisation tasks, particularly in terms of the inter-relationship between the embodied processes of pattern generalisation and the visualisation of pictorial cues. Evidence of this central thesis is provided in Section 5.3.2 in the form of a micro-analysis of a series of vignettes. Practical issues of how this combined theoretical framework is reflected in the methodology of this study are addressed in Chapter 4.

3.6 THE NATURE OF MATHEMATICS

Juxtaposed against the enactivist notion of cognition seen as a fully embodied process, the Platonic tradition of mathematics as being an objective, independent, transcendent and mind-free set of eternal truths is no longer tenable. The allure of this longstanding view of mathematics, what Lakoff and Núñez (2000) refer to as the “Romance of Mathematics” (p. xv), is understandable. The comforting thought of mathematics representing truths transcending beyond human existence, and as representing the language of nature itself, is certainly compelling. However, from an embodied perspective, “the notion of an objective mathematics, independent of human understanding, no longer makes sense” (Núñez et al., 1999, p. 46).

Mathematics, as the product of human ideas, has a number of unique features – amongst others, generalisability, stability, precision and applicability to the real world (Núñez, 2006). Thus, from an embodied stance, the critical question is the following: “How can an embodied view of the mind give an account of an abstract, idealized, precise, sophisticated and powerful domain of ideas if direct bodily experience with the subject matter is not possible?” (Núñez, 2006, p. 161). Lakoff and Núñez (2000) go some way to answering this crucial question by means of *mathematical idea analysis*³¹.

The key to the answer lies in the thesis that although human abstraction is socially constructed, more critically it is “constructed through strong non-arbitrary biological and cognitive constraints that play an essential role in constituting what human abstraction is” (Núñez, 2007, pp. 134-135). In short, it is this species-specific non-arbitrary manner in which we make sense of our lived experience that lies at the heart of the embodied nature of mathematics. As Núñez (2008) elucidates, “...the abstract conceptual systems we develop are possible *because* we are biological beings with specific morphological and anatomical features. In this sense, human abstraction is *embodied* in nature” (p. 351).

The applicability of mathematics to the real world can thus be understood in terms of the manner in which it is brought forth – i.e. through characteristically human forms

³¹ See also Núñez (2000).

of sense-making which are grounded in bodily, lived experience. The stability of mathematics as a domain of abstraction can be understood in the sense that, as human beings, we “naturally experience the world in fundamentally similar ways” (Núñez et al., 1999, p. 62). Thus, it is the commonalities of our human experience, the non-arbitrary nature of our interaction with, and perceptual experience of, our world that are fundamental to socially and culturally constructed human abstraction. As Núñez et al. (1999) summarise: “...our understandings of the world, and of mathematics, may be socially and culturally situated, but it is the commonalities in our physical embodiment and experience that provide the bedrock for this situatedness” (p. 63).

Lakoff and Núñez (2000) claim that the conceptual system of human mathematical reason is based on sensory-motor grounding and metaphorical projection. It is through embodied mechanisms such as conceptual metaphor³², conceptual blends, conceptual metonymy, fictive motion and dynamic schemas that the “inferential patterns drawn from direct bodily experience in the real world get extended in very specific and precise ways to give rise to a new emergent inferential organization in purely imaginary domains” (Núñez, 2007, p. 132).

The decision to believe in a transcendentalist view of mathematics as a “realm of mathematical objects, standing outside of time and history and the experience of any beings” (Lakoff, 1987, p. 356) is thus a decision based on faith or belief rather than science. However, from an enactivist position this is an irrelevant issue since the only mathematics we have access to is that brought forth through our embodied, lived experience. As such, human mathematics *is* embodied mathematics (Lakoff & Núñez, 2000, p. 346). More so, however, the theory of embodied mathematics is a “theory of the only mathematics we know or can know, it is a theory of what mathematics *is* – what it really is!” (Lakoff & Núñez, 2000, p. 346).

³² Technically, conceptual metaphors are cognitive mechanisms, “inference-preserving cross-domain mappings ... that allow us to project the inferential structure from a source domain, which usually is grounded in some form of basic bodily-experience, into another one, the target domain, usually more abstract” (Núñez, 2007, p. 132).

3.7 CONCLUDING COMMENTS

The purpose of this chapter was to establish a theoretical framework for the study. Three theoretical aspects of pictorial pattern generalisation were presented: *enactivism*, *knowledge objectification* and *figural apprehension*. Literature pertinent to each of these topics was reviewed prior to situating each idea within the context of pictorial pattern generalisation. Finally, the three key theoretical ideas were drawn together into a single framework which has the potential to provide powerful complementary insights into the underlying tensions and subtle complexities of generalisation tasks set in pictorial contexts. The implications that these theoretical underpinnings have in terms of the nature of mathematics was also briefly discussed. Practical issues of how this theoretical framework is reflected in the methodology of the present research are addressed in the following chapter.

CHAPTER FOUR

METHODOLOGY

*The range of what we think and do is limited
by what we fail to notice.*

R. D. LAING

4.1 INTRODUCTION & ORIENTATION

This study is oriented within the conceptual framework of qualitative research, and is anchored within an interpretive paradigm. The central endeavour within the context of the interpretive paradigm is “to understand the subjective world of human experience” (Cohen & Manion, 1994, p. 36). In an effort to retain the integrity of the phenomenon under investigation, efforts must be made to “get inside” the research subject in order to “understand from within”. Thus, rather than detachment from, the interpretive paradigm compels a direct interaction with the research subjects (Jackson, 1995, p. 17) and an “intimate relationship between the researcher and what is studied” (Denzin & Lincoln, 2003, p. 13).

This study was undertaken primarily to investigate pupils’ visualisation strategies when engaged in pattern generalisation tasks set within pictorial contexts. In order to gain deeper insights into the embodied processes of pupils’ visualisations, and the manner in which these experiences are manifested in the algebraic generalisation process, an in-depth analysis of each pupil’s lived experience is necessary. Concern for the intimacy of this analysis process, and a sensitivity for the embodied processes under scrutiny, informed and necessitated the choice of an interpretive paradigm.

Attempting to see a situation as perceived by another human being should be imbued “with the assumption that the constructions of others ... have integrity and sensibility within another’s framework” (Confrey, 1990, p. 108). This has particular import within an interpretive research paradigm, and resonates strongly with the enactivist theory which forms a crucial backdrop to this study. A useful guiding tenet is Maturana and Varela’s (1998) aphorism: “Everything said is said by someone” (p. 26). Although a simple statement in itself, these six words are fundamental to the character and spirit of the research process, and provide a constant and vigilant reminder that there are no “observerless” observations, no “experiencerless” experiences. A useful analogy to help characterise this position can be extracted from Maturana’s discussion of errors and mistakes made by children. Maturana makes the pertinent remark that children very rarely commit logical mistakes. Rather, a mistake “is a statement made in a particular domain of reality, which is heard and evaluated in the context of another” (Maturana & Poerksen, 2004, p. 132). In other words, an error does not exist as an error at the time of its occurrence, but its validity is devalued only subsequent to the event, i.e. *a posteriori*, and only in the evaluative and reflective context of other lived experiences (Maturana & Poerksen, 2004, p. 133). Although this study is not focused on student errors, it *is* focused on students’ lived experience – experiences which, from an enactivist perspective, are seen as a process of co-determination between each student and their particular environment. Thus, the essential character underpinning the data acquisition and analysis protocol of this study is the treatment of all responses, particularly those that are unexpected or idiosyncratic, with a genuine interest in understanding their character and origins.

Not only does enactivism form a crucial ontological backdrop to this study, but enactivist notions of epistemology also have important implications for the research process. As Reid (1996) comments, since research is itself an instance of human learning, it is appropriate to incorporate into the research methodology aspects of the epistemological theoretical framework of the study. From an enactivist perspective, researchers are seen as “developing their learning in a particular context” (Lozano et al., 2006, p. 91), a context within which researcher and research environment are seen to co-emerge (Reid, 2002). This interdependence of researcher and context makes the process of investigation both flexible and dynamic (Trigueros & Lozano, 2007). Research is thus not seen as a linear enterprise, but rather as a recursive

process (Lozano et al., 2006). Furthermore, within the context of a qualitative study, “research design should be a reflexive process operating through every stage of a project” (Hammersley & Atkinson, as cited in Maxwell, 1996, p. 2). Thus, this study will be characterised by the use of multiple perspectives and the continuous refinement of methods and data analysis protocols.

4.2 PARTICIPANT SELECTION

The research makes use of a case study approach. The case study is not a methodological choice *per se*, but rather a choice of the specific object to be studied (Stake, 1994). Previous research (Samson, 2007a) suggests that in the present study one is more likely to gain “insight into the [research] question by studying a particular case” (Stake, 1995, p. 3), in this instance a cohort of high ability pupils. Stake (1994, 1995) refers to this type of enquiry as an *instrumental* case study, as opposed to two other broad types of case study which he identifies - *intrinsic* and *collective*. In an *instrumental* case study, the choice of case is made on the basis that it is expected to advance the understanding of the issue under investigation. The choice of participants for this study was thus guided by the adopted case study methodology.

A mixed gender, high ability Grade 9 class of 23 pupils at an independent school in South Africa constituted the research participants for the main study. A second group of 23 high ability Grade 9 pupils took part in a pilot study which served to develop and refine the data collection and analysis techniques. The choice of Grade 9 as the age-group of focus, and in particular the decision to choose only high ability pupils, represents non-probability purposive sampling (Cohen, Manion & Morrison, 2000, p. 102). This methodological decision was based on a number of critical considerations. Firstly, the purpose of purposeful sampling is “to select information-rich cases whose study will illuminate the questions under study” (Patton, 1990, p. 169). Since the data collection protocol requires pupils to provide both written and verbal articulations of their own reasoning, a high ability group of pupils is more suited to this particular methodology. Secondly, experience from previous research (Samson, 2007a) suggests that high ability pupils are also more likely to progress further in the type of pattern generalisation tasks under investigation and thus more

likely to constitute “information-rich cases” (Patton, 1990, p. 169). Finally, as part of the pattern generalisation tasks provided, research participants were required to provide algebraic expressions for the general term. Choice of Grade 9 thus ensures that participants have sufficient algebraic background to attempt this.

4.3 ETHICS

The issue of ethics is recognised as playing an important role in research (Cohen & Manion, 1994, p. 347). Prior to the commencement of the pilot study, formal permission to conduct the research was obtained from the principal of the school in question. Anonymity of both the school as well as the research participants was assured, and appropriate pseudonyms are used throughout the text when referring to research participants. In addition, only those pupils who agreed to participate in the study through voluntary informed consent formed part of the research sample, and participants had the freedom to withdraw from the study at any stage without explanation. In the case of participants who were audio-visually recorded, written consent was obtained from each research participant as well as from each participant’s parents or legal guardians. Specific to the audio-visual recordings, all research participants (along with their parents or legal guardians) were also asked to indicate whether or not they gave their consent to the following³³:

- For two to three additional researchers to view portions of the videos in order to provide input into the analysis process.
- For single frames of the video footage to be included in the final written thesis.
- For single frames of the video footage to be included in conference presentations.
- For short clips of the video footage to be included in conference presentations.
- For short clips of the video footage to be used in teaching seminars for the purposes of teacher development.

From a more philosophical standpoint, there is also an important ethical consideration stemming from the enactivist theoretical underpinnings of this study. In enactivist terms we need to be sensitive to the notion that “...our actions have the

³³ Consent to all five bulleted requests was received from all research participants and all parents and/or legal guardians.

potential to alter the worlds and possibilities of others” (Simmt, 2000, p. 158). Furthermore, an enactivist stance compels us to see each person’s certainty as being “...as legitimate and valid as our own” (Maturana & Varela, 1998, p. 245). Sensitivity to both of these ethical considerations was maintained throughout the study.

4.4 DATA GENERATION

Data generation proceeded in two phases:

PHASE 1

Data was generated from a series of pencil and paper exercises based on linear³⁴ generalisation tasks set in 2-dimensional pictorial contexts. For each pattern, participants were required to provide a numerical value for the 40th term (along with a written articulation of their reasoning), and an algebraic expression for the n^{th} term (along with a justification/explanation of their expression). Patterns were presented as two non-consecutive terms. This was a purposeful decision based on previous research experience (Samson, 2007a) as well as research literature (e.g. Healy & Hoyles, 1996; Hershkowitz et al., 2002) which suggested that non-consecutive terms would be more appropriate with respect to encouraging generalisation by means of the independent variable, i.e. by encouraging attention to be focused on the visual stimulus³⁵. Responses to the pattern generalisation tasks were recorded on Mathematical Processing Response Sheets (MPRS). Ten pattern generalisation tasks were administered during three different lessons spanning a period of one week. On average this allowed approximately 14 minutes per pattern³⁶. In addition, individual participants were also informally interviewed in instances where the written articulation of their reasoning was either ambiguous or required additional explication. The purpose of these interviews, in which field notes were taken, was to provide research participants with the opportunity to further explain or expand on the

³⁴ Linear/arithmetic sequences of the type $ax \pm c$ with $c \neq 0$ were chosen.

³⁵ One could of course present each pictorial pattern as a *single* term. However, such a single pictorial term would need to provide an *unambiguous* explanation of the manner in which the pattern continued. This would be difficult to accomplish in most cases without some form of verbal description or explanation accompanying the figural cue.

³⁶ The pilot study suggested that this was sufficient time for the chosen research participants.

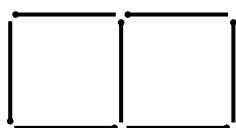
written articulation of their reasoning. This process of member checking thus constitutes a form of external validation (Lewis & Ritchie, 2003, p. 276).

This initial phase of the data collection protocol serves a dual purpose. Firstly, it seeks to investigate the extent to which individual pupils favour specific visualisation strategies when generalising figural patterns. Secondly, it is intended to identify those pupils whose preferred mode of solving figural pattern generalisation tasks is visual as opposed to numeric, and who would thus be appropriate participants for Phase 2 of the study. The data analysis tool used to accomplish this characterisation is discussed in Section 4.5.

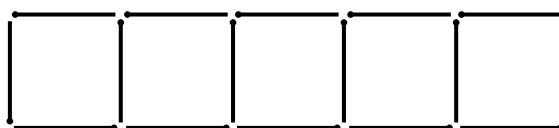
The two non-consecutive terms of the ten pictorial patterns used are summarised hereunder. The specific choice of pictorial patterns as well as the accompanying wording was carefully considered so that there would be a sufficient variety of pictorial contexts that could potentially lead to all four modes of *figural apprehension*³⁷.

Question 1

Look at the diagrams below. 7 matches are needed to make a row of 2 squares while 16 matches are needed to make a row of 5 squares.



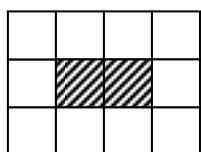
2 squares require 7 matches



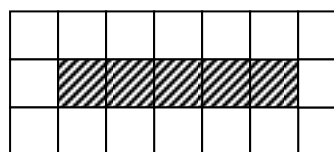
5 squares require 16 matches

Question 2

Look at the diagrams below. White tiles have been used to build a border around a row of striped tiles.



For a row of 2 striped tiles there are 10 white tiles in the border.

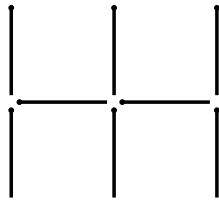


For a row of 5 striped tiles there are 16 white tiles in the border.

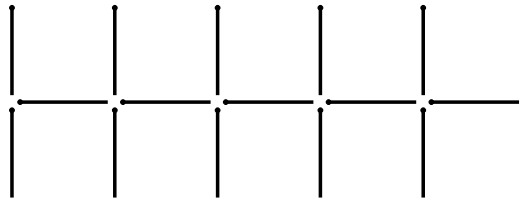
³⁷ See Section 3.4.

Question 3

Look at the diagrams below.



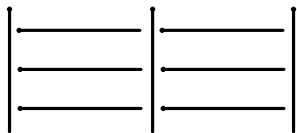
A pattern with 2 **horizontal** matches requires a **total** of 8 matches



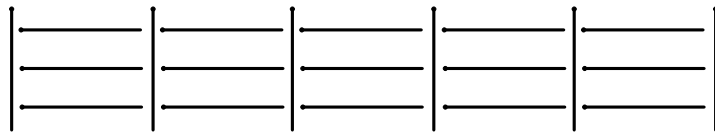
A pattern with 5 **horizontal** matches requires a **total** of 17 matches

Question 4

Look at the following diagrams. A "fence" has been built using matches.



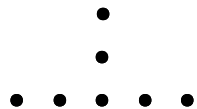
For a fence containing 2 sections you need a total of 9 matches.



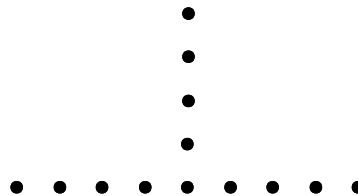
For a fence containing 5 sections you need a total of 21 matches.

Question 5

Look at the following dot patterns.



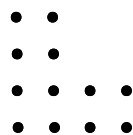
SHAPE 3



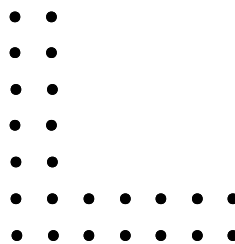
SHAPE 5

Question 6

Look at the following "double L" diagrams made from dots.



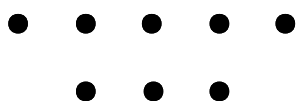
Base is 4 dots long



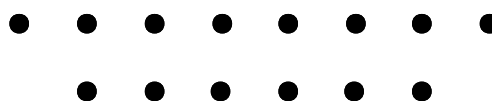
Base is 7 dots long

Question 7

Look at the following diagrams made from dots. Shape 2 needs 8 dots while Shape 5 needs 14 dots.



Shape 2

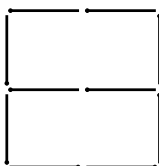


Shape 5

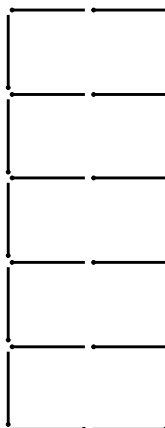
Question 8

Look at the following towers made from matches.

For a tower containing 2 levels, 10 matches are needed.

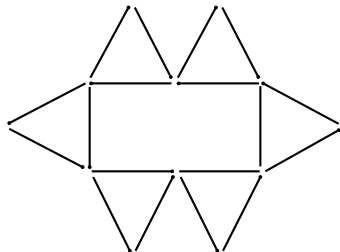


For a tower containing 5 levels, 22 matches are needed.

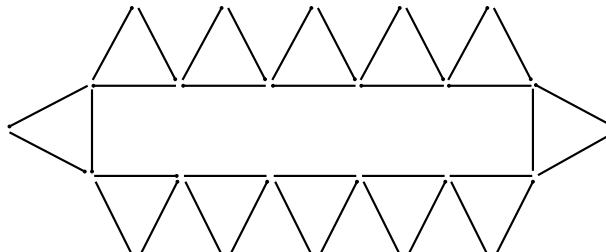


Question 9

Look at the following diagrams made from matches. Shape 2 needs 18 matches while Shape 5 needs 36 matches.



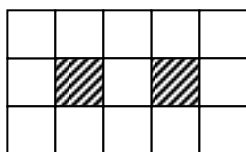
SHAPE 2



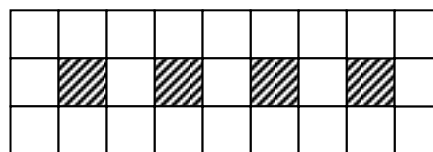
SHAPE 5

Question 10

Look at the patterns below made from striped tiles and white tiles.



For 2 striped tiles you
need 13 white tiles.



For 4 striped tiles you
need 23 white tiles.

PHASE 2

The second phase of the data collection protocol seeks to explore the embodied processes evinced by pupils engaged in figural pattern generalisation tasks, while at the same time investigating the extent to which pupils are able to generalise patterning tasks in multiple ways.

Seven research participants who were identified in Phase 1 as preferring a visual mode when solving pattern generalisation tasks took part in Phase 2. In addition, one of the research participants from the pilot study³⁸, who similarly showed a preference for visual methods, also took part in Phase 2. These eight research participants were individually provided with two pictorial terms of a further linear pattern and were required, in the space of approximately one hour, to provide as many different expressions for the n^{th} term (along with justifications/explanations of their expressions) as they could. Tools such as paper, pencils and highlighters as well as appropriate manipulatives such as matchsticks and plastic counters were provided. The eight research participants were asked to think aloud while engaged with their particular pattern generalisation task, and the researcher also prompted the participants to keep talking or provide further explication as and when necessary. Each session was audio-visually recorded, and field-notes were taken by the researcher.

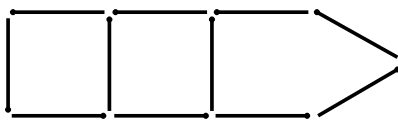
The provision of a variety of tools and manipulatives stems from a sensitivity to the enactivist theoretical framework. Such items are not mere auxiliary components but

³⁸ The pilot study was by this time at a stage when the data collection and analysis techniques had been refined to a point commensurate with the main study.

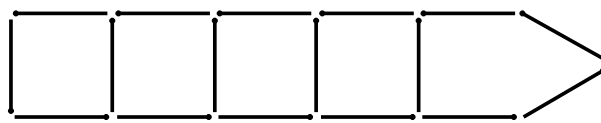
open up spaces of possible action and thus have the potential to shape enactive processes of construction (Lozano et al., 2006, p. 90; Radford, 2003, p. 41).

The eight research participants were supplied with the pictorial patterns as shown below. The decision to use a variety of pictorial contexts was to ensure that a number of different physical characteristics were represented within the selection of pattern tasks. This important issue is discussed further in Section 5.2.1 under Table 5.4.

Brian

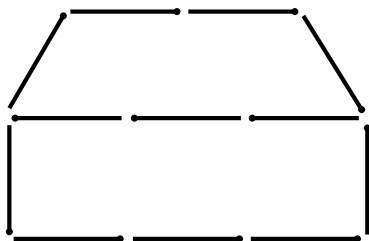


Shape 3

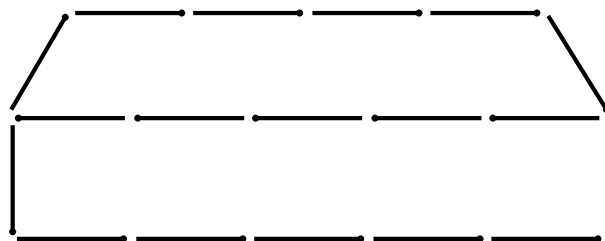


Shape 5

Lance

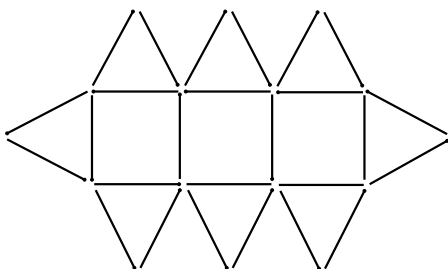


Shape 3

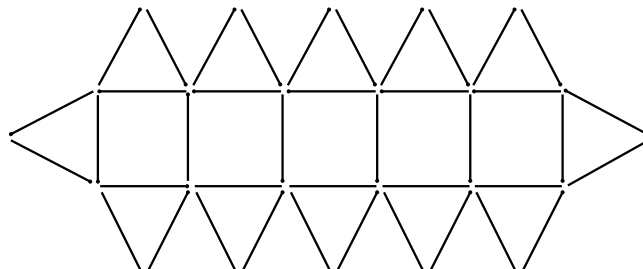


Shape 5

Terry

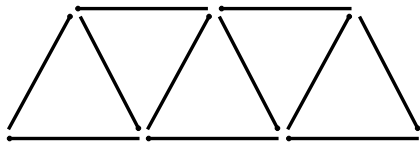


Shape 3

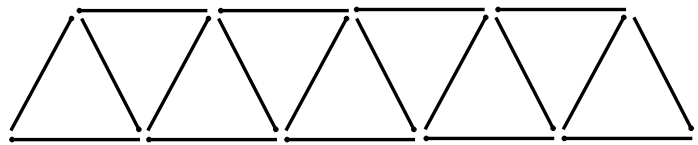


Shape 5

Kelly & Philip

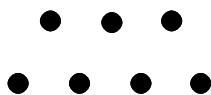


Shape 3

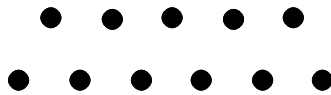


Shape 5

Anthea

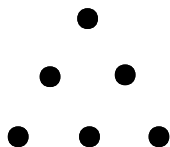


Shape 3

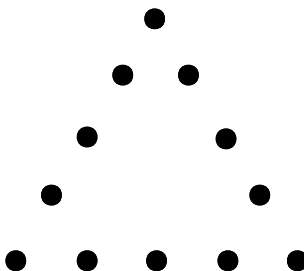


Shape 5

Liza



Shape 3



Shape 5

4.5 DATA ANALYSIS – PHASE 1

4.5.1 STRATEGY/METHOD CLASSIFICATION


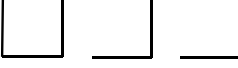





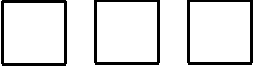
In order to assess the extent to which individual pupils favoured specific visualisation strategies, the justification/explanation of their expression for the n^{th} term, as well as the written articulation of their reasoning for the numerical value of the 40th term, were carefully analysed.

The literature review revealed very little consistency in the naming of patterning strategies. Although the basic procedural descriptions of various strategies are largely similar (see e.g. English & Warren, 1998; Hargreaves et al., 1998, 1999; Healy & Hoyles, 1999; Lannin, 2003, 2005; Orton & Orton, 1999; Stacey, 1989; Swafford & Langrall, 2000), nomenclature seems to be somewhat idiosyncratic. However, very few strategy classifications distinguish specifically between numerical approaches and visually mediated approaches. Phase 1 of this study not only seeks to identify pupils who prefer visually mediated strategies, but also whether they favour *specific* visual strategies. A far more nuanced classification system, specifically with reference to visual strategies, was thus developed. The classification/coding system employed, which was developed and gradually refined during the analysis of the pilot study data, is summarised in Table 4.1.

Table 4.1 Strategy/method classification

CODE	DESCRIPTION						
V	Evidence of only visual reasoning						
N	Evidence of only numeric reasoning						
V/N	Evidence of both visual and numeric reasoning						
If the strategy was classified as N or V/N then it was further classified as follows:							
RA	<i>Rate-Adjust</i> ³⁹ : The constant difference (d) between consecutive terms is determined numerically and used as a multiplying factor to give $T_n = dn$. The formula is then adjusted by the addition or subtraction of a constant to ensure that it gives the correct numerical answer for the given terms.						
DF	<i>Direct Formula</i> : Determining a direct and constant rule to determine the dependent variable (T_n) from the independent variable (n) by trial and adjustment.						
If the strategy was classified as V or V/N, then it was further classified as follows:							
LAU	<i>Local Additive Unit</i> : Very similar to the Rate Adjust method except that in this case specific use is made of the diagrams to identify the constant difference (i.e. the visual additive unit).						
LF1	<i>Local Feature 1</i> : The general formula is determined in the form $T_n = c + n \times d$ where c is a constant and d is the additive unit. e.g. <table style="display: inline-table; border: none;"><tr><td style="border: 1px solid black; width: 20px; height: 20px;"></td><td style="border: 1px solid black; width: 20px; height: 20px;"></td><td style="border: 1px solid black; width: 20px; height: 20px;"></td></tr></table> seen as <table style="display: inline-table; border: none;"><tr><td style="border: 1px solid black; width: 20px; height: 20px;"></td><td style="border: 1px solid black; width: 20px; height: 20px;"></td><td style="border: 1px solid black; width: 20px; height: 20px;"></td></tr></table>						

³⁹ Using Lannin's (2003) nomenclature.

LF2	<p><i>Local Feature 2:</i> The general formula is determined in the form $T_n = T_1 + (n-1) \times d$ or $T_n = T_2 + (n-2) \times d$ etc. where T_1 and T_2 are the first and second terms respectively and d is the additive unit.</p> <p>e.g.  seen as </p>
LF3	<p><i>Local Feature 3:</i> The general formula is determined in the form $T_n = (n+1) \times d - c$ where c is a constant which corrects for the over-count and d is the additive unit.</p> <p>e.g.  seen as </p> <p>Depending on the given pictorial sequence, the general formula for LF3 categorisation can also sometimes be of the form $T_n = nd - c$. The critical feature for LF3 categorisation is the correction of the overcount resulting from an extraneous portion of the local additive unit.</p>
GF1	<p><i>Global Feature 1:</i> A building-up process where there is no overlap of structural units⁴⁰.</p> <p>e.g.  seen as  n horizontal matches n+1 vertical matches n horizontal matches </p>
GF2	<p><i>Global Feature 2:</i> A building-up process where there are overlapping structural units⁴¹.</p> <p>e.g.  seen as </p>

Since the classification system summarised in Table 4.1 is crucial to this study, the various classification codes are discussed in a little more detail hereunder.

- **V, N and V/N**

The classification into V (Visual), N (Numeric) and V/N (Visual-Numeric) relates to the extent to which each research participant's written explanation/justification of their 40th and n^{th} terms make specific reference to the pictorial context. A research participant is classified as V (Visual) if their justification makes explicit reference (either diagrammatically or verbally) to the pictorial context. A classification of N (Numeric) is indicative of a research participant whose

⁴⁰ Similar to Rivera and Becker's (2008) *constructive* generalisation.

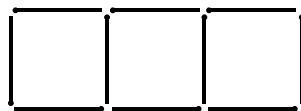
⁴¹ Similar to Rivera and Becker's (2008) *deconstructive* generalisation.

justification is simply based on numeric considerations and contains no reference to the given pictorial context. A research participant whose justification makes use of both pictorial and numerical elements is classified as V/N (Visual-Numeric).

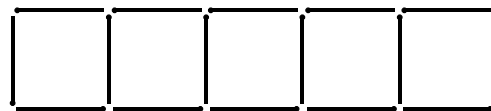
NUMERIC APPROACHES

- **RA**

In the RA (Rate-Adjust) classification, the constant difference (d) between consecutive terms is determined numerically. Using this constant difference as a multiplying factor leads to the partial formula $T_n = dn$. This partial formula is then completed by the addition or subtraction of a constant to ensure that the final formula gives the correct numerical value for the known terms of the sequence. By way of example, consider the following two terms of a linear sequence:



Shape 3



Shape 5

Shape 3 contains 10 matches while Shape 5 contains 16 matches. Assuming that the sequence is linear, one can reason that Shape 4 should contain 13 matches since consecutive terms would need to be numerically evenly spaced. One could then argue that since the common difference is 3 (i.e. the numerical difference between consecutive terms), the general term must be of the form " $3n \pm c$ " where c is a constant. Since $3n \pm c$ must equal 10 for the third term (where $n = 3$), one readily arrives at the general formula $T_n = 3n + 1$. Testing this formula for other known terms confirms its correctness.

- **DF**

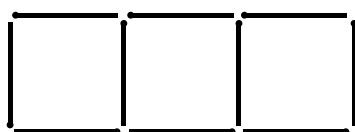
In the DF (Direct Formula) classification, a process of trial-and-adjustment is used to determine an algebraic expression for the n^{th} term. Considering the same two terms as before (Shape 3 containing 10 matches and Shape 5 containing 16 matches), through purely numerical considerations one might strike at the observation that Shape 3 contains nearly as many matches as the

Shape number squared. However, this observation doesn't seem to be true for Shape 5 since 5 squared is 25 and Shape 5 only contains 16 matches. By refining the observation one might notice that Shape 3 has nearly as many matches as three times the Shape number. In this case a similar result holds for Shape 5, since three times 5 is 15, which is very close to the required 16 matches contained in Shape 5. By comparing the numerical results one could, through this process of trial-and-improvement, arrive at a final formula $T_n = 3n + 1$.

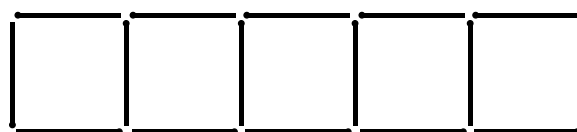
VISUAL APPROACHES

Visual approaches are divided into two broad categories. The first category (LAU, LF1, LF2 and LF3) incorporates those visual strategies that are characterised by the foregrounding of the local additive unit – i.e. the structural unit which is added to (or inserted into) a given pictorial term in order to form the next term in the sequence. This focus on the structural additive unit represents an iterative or recursive process of visual reasoning. The second category (GF1 and GF2) incorporates those visual strategies characterised by a more holistic or global view, where each term of the pictorial context is seen in terms of a generalised building-up process that doesn't make use of the iterative addition of the additive unit.

For ease of discussion, all visual approaches will be discussed in relation to the following pictorial context, which represents Term 3 and Term 5 of a pictorial sequence:



Shape 3



Shape 5

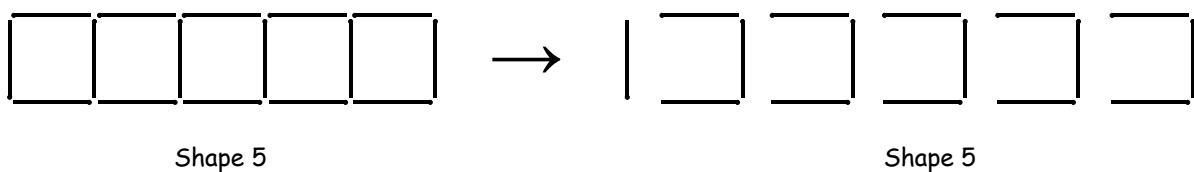
- **LAU**

The LAU (Local Additive Unit) classification is very similar to the visual RA (Rate-Adjust) method, except that specific use is made of the pictorial context to identify the constant difference in the form of a visual additive unit. Through a

process of visual reasoning one could argue that to construct Shape 4 from Shape 3 (and in consequence to construct *any* Shape from the *preceding* Shape) would require the addition of 3 matches in the form of a backwards C-shape. Using this visually identified constant difference as a multiplying factor leads to the partial formula $T_n = dn$. The remainder of the formula is determined numerically for example by using the Rate-Adjust (RA) strategy previously described. Since the LAU (Local Additive Unit) classification contains both numerical as well as visual reasoning, it is only applicable to research participants who were identified as being Visual-Numeric (V/N).

- **LF1**

The LF1 (Local Feature 1) classification applies to visual strategies that yield general formulae in the form $T_n = c + n \times d$ where both c (the constant) and d (the additive unit) are determined through visual reasoning. By comparing the given terms, or by drawing additional terms, one could reason that to get from one term to the next requires the addition of 3 matches in the form of a backwards C-shape, the visual additive unit. A visual deconstruction of the pictorial context based on this visual additive unit will lead to the following general structure, which is shown here for Shape 5:

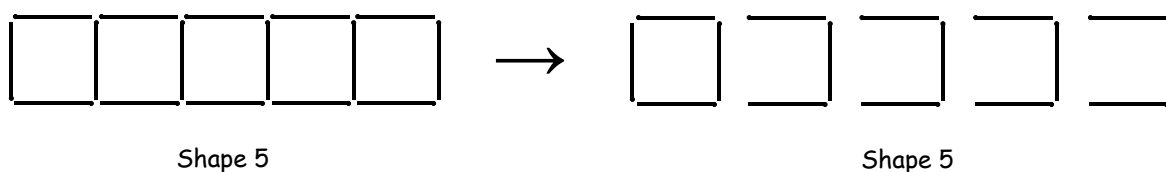


The general visual subdivision is thus based on an initial starting match (the constant c) and n multiples of 3 matches in the form of a backwards C-shape (the additive unit d), thus yielding the general formula $T_n = 1 + 3n$.

- **LF2**

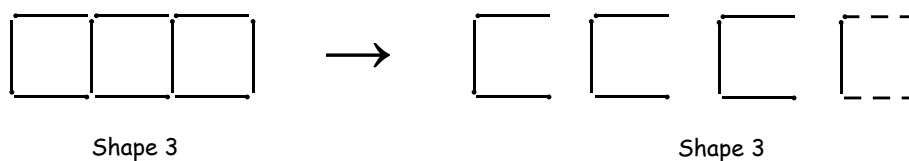
The LF2 (Local Feature 2) classification applies to visual strategies that yield general formulae of the form $T_n = T_1 + (n - 1) \times d$ or $T_n = T_2 + (n - 2) \times d$ etc. where T_1 and T_2 are the first and second terms respectively and d is the additive unit. Once again the local additive unit is identified through visual reasoning, either by

comparing the given pictorial terms, or by drawing additional terms. However, rather than subdividing the pictorial context into an initial constant followed by n multiples of the additive unit, a larger visual constant (typically either Term 1 or Term 2 of the pictorial sequence) is seen to be followed by either $(n-1)$ or $(n-2)$ multiples of the additive unit. A visual deconstruction of the pictorial context based on a 4-match constant followed by $(n-1)$ multiples of the visual additive unit will lead to the general formula $T_n = 4 + (n-1) \times 3$, the general pictorial structure of which is shown here for Shape 5:



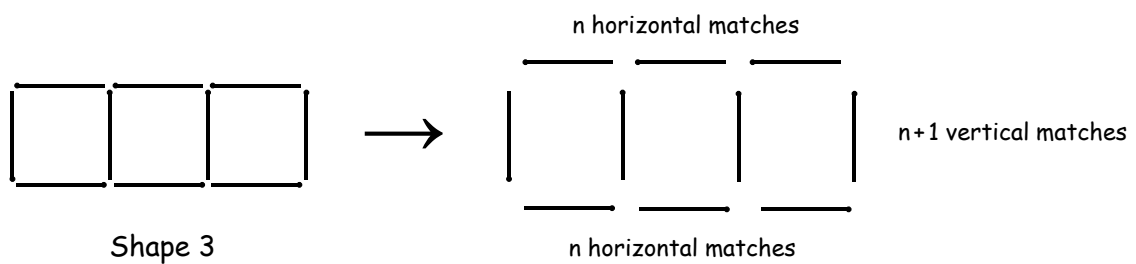
- **LF3**

The LF3 (Local Feature 3) classification applies to visual strategies that yield general formulae in the form $T_n = (n+1) \times d - c$ where d is similar to the additive unit as before, but c is a constant which now corrects for an overcount. In some instances, depending on the specific pictorial sequence, the general formula for LF3 categorisation can also be in the form $T_n = nd - c$. However, the defining characteristic for LF3 categorisation is the correction of the overcount resulting from an extraneous portion of the local additive unit. By way of example, each term of the given pictorial context could be progressively built up using a sequence of C-shapes, each containing 3 matches. However, for each term the final C-shape would contain 2 additional matches that would need to be subtracted from the final tally. This would yield the general formula $T_n = (n+1) \times 3 - 2$, the general pictorial structure of which is shown here for Shape 3:



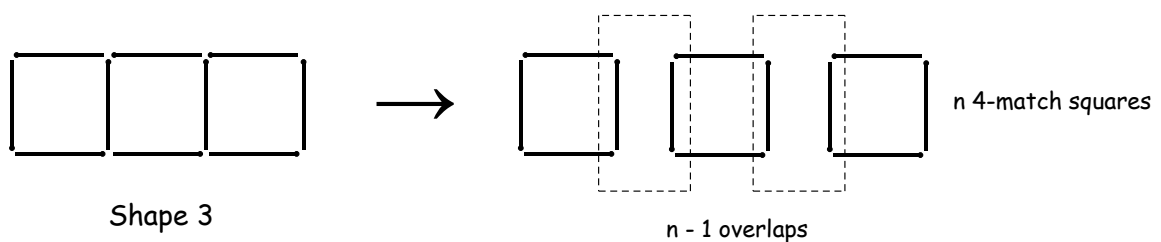
- **GF1**

The GF1 (Global Feature 1) classification applies to visual strategies that are characterised by a more holistic view that doesn't make use of the iterative addition of the additive unit. Each pictorial term is seen in terms of a generalised building-up process where there is no overlap of structural units. By way of example, one could subdivide each term of the given pictorial context into an upper row of n horizontal matches, a lower row of n horizontal matches, and a central row of $n + 1$ vertical matches, thus yielding the following general formula: $T_n = 2n + (n + 1)$. This visual subdivision is shown below for Term 3.



- **GF2**

The GF2 (Global Feature 2) classification also applies to visual strategies that are characterised by a more holistic view. However, in this case the generalised building-up process contains overlapping structural units. By way of example, each term of the given pictorial context could be seen to contain n overlapping squares, each made up of 4 matches, thus giving a count of $4n$ matches. However, this would give an overcount since $n - 1$ of these matches would have been counted twice as a result of the various overlaps. Correcting for this overcount gives the final formula $T_n = 4n - (n - 1)$. This visual subdivision is shown below for Term 3.



4.5.2 CONTEXTUAL CONNECTIVITY

In order to identify those pupils who preferred a visual as opposed to numeric mode of solving figural pattern generalisation tasks, responses were rated in terms of the extent to which the pictorial context featured in the justification/explanation of the 40th and n^{th} terms. A quasi-quantitative measure dubbed the *Contextual Connectivity Rating* (CCR) developed in a previous study (Samson, 2007a) was employed to accomplish this. The CCR ascribes a numeric value to the extent to which pupils used the pictorial scenario as a referential context. Scores of 1, $\frac{1}{2}$ or 0 were awarded depending on the extent to which the pictorial context featured in the justification/explanation of the 40th and n^{th} terms.

Table 4.2 Contextual Connectivity Rating (CCR) (from Samson, 2007a)

CCR	Description
1	Justification makes express reference (either diagrammatically or verbally) to the pictorial context
$\frac{1}{2}$	Justification either makes only partial reference to the pictorial context or makes use of both pictorial and numerical elements
0	Justification is purely numerically based and contains no reference to the pictorial context

4.6 DATA ANALYSIS – PHASE 2

This phase of the data analysis protocol is characterised by a multi-systemic semiotic analysis based on the theoretical construct of knowledge *objectification* (Radford, 2002a, 2008) as discussed in Section 3.3. As the Phase 2 research participants engaged with their particular pattern generalisation task they made use of multiple means of objectification *en route* to a stable form of awareness. These means of objectification included the use of words, linguistic devices, metaphor, gestures, rhythm, graphics and physical artefacts. These processes of “coming to know” were carefully scrutinised through multiple viewings of the audio-visual recordings of each research participant.

The data analysis was guided by an enactivist methodological framework in which researcher and research environment are seen to co-emerge (Reid, 2002). This interdependence of researcher and context was characterised by a flexible and dynamic process of investigation (Trigueros & Lozano, 2007). This iterative and reflexive process of co-emergence was built on over time through the use of multiple perspectives and the continuous refinement of methods and data analysis protocols⁴². Audio-visual data was examined repeatedly in different forms (e.g. video and transcript) and in conjunction with additional data retrieved from field-notes and participants' worksheets. In addition, nodes of activity which seemed particularly interesting were identified and meticulously characterised with reference to the various semiotic means of objectification in the form of descriptive vignettes.

4.7 VALIDITY

In broad terms, validity refers to the extent to which the research findings are sufficiently authentic (Lincoln & Guba, 2003, p. 274).

The main threat to validity in Phase 1 is the possible ambiguity or lack of clarity with respect to participants' written articulation of their own reasoning processes. Individual participants were informally interviewed where written responses were either ambiguous or required illumination by oral explication. Member checking was thus used as a form of external validation (Lewis & Ritchie, 2003, p. 276).

Of critical importance in terms of validity considerations for both Phase 1 and Phase 2 is not only the appropriate choice of the figural pattern generalisation questions themselves, which was guided by pertinent literature, but the nature of their presentation. A literature review as well as previous research experience (Samson, 2007a) suggested that linear sequences of the form $ax \pm c$ ($c \neq 0$) would be most appropriate in terms of eliciting rich data in all phases of the study. This decision is also sensitive to the nature of what Sasman et al. (1999, p. 162) refer to as *transparent* versus *non-transparent* figures⁴³.

⁴² This process was developed during the analysis of the pilot study data.

⁴³ As discussed in Section 2.4.5.

Patterning tasks presented with consecutive terms have been shown to encourage a recursive strategy (Hershkowitz et al., 2002; Samson 2007a) and thus tend to draw attention away from global structural features⁴⁴ of the pictorial context. However, since not all patterning tasks could be presented unambiguously using a single term, it was decided to use two non-consecutive terms for all questions.

The decision to ask pupils to provide both the 40th and n^{th} terms in Phase 1 was to ensure that those pupils who were unable to provide an algebraic expression for the general term were still afforded the opportunity to give evidence of their reasoning process. The specific choice of 40th term, what Stacey (1989, p. 150) calls a *far generalisation* task, is a purposeful attempt to encourage explicit strategies based on the pictorial context rather than recursive numeric approaches. Although the 40th term is what Sasman et al. (1999, p. 161) consider a *seductive number*, our focus on visual as opposed to purely numeric strategies justifies this choice. All of the above considerations relate to identifying the most appropriate participants for Phase 2 of the investigation, i.e. “information-rich cases whose study will illuminate the questions under study” (Patton, 1990, p. 169).

Within Phase 2 the data collection and analysis protocols are sensitive to the enactivist underpinnings of the study, and thus make use of multiple data sources and approaches to data handling (as outlined in Sections 4.4 and 4.6). This in turn is a form of triangulation (Richards, 2005, p. 140) which seeks to ensure validity.

4.8 SUMMARY OF THE METHODOLOGY

This study is oriented within the conceptual framework of qualitative research, and is anchored within an interpretive paradigm. The study aims ultimately to gain insights into the embodied processes of pupils’ visualisation activity when engaged in figural pattern generalisation tasks through an in-depth analysis of each pupil’s lived experience. A case study methodology was adopted, with the research participants for the main study comprising 23 pupils from a mixed gender, high ability Grade 9 class at an independent school in South Africa.

⁴⁴ Use of the word “features” is not meant to imply that such features (or structures) are intrinsically contained within the pictorial image itself, but rather that such features could potentially *co-emerge* through the interaction of a perceiver and the given figural context.

Phase 1 of the data generation process took the form of a series of pencil and paper exercises based on 10 linear generalisation tasks set in 2-dimensional pictorial contexts. Patterns were presented as two non-consecutive terms, and for each pattern participants were required to provide a numerical value for the 40th term (along with a written articulation of their reasoning), and an algebraic expression for the n^{th} term (along with a justification/explanation of their expression). Responses to the pattern generalisation tasks were recorded on Mathematical Processing Response Sheets (MPRS). In addition, individual participants were informally interviewed where the written articulation of their reasoning required additional explication. The responses to the 10 linear generalisation tasks were classified in terms of the specific method or strategy employed. A coding system was developed to provide a nuanced characterisation of both numeric and visual strategies. In addition, a quasi-quantitative measure dubbed the *Contextual Connectivity Rating* (CCR) was used to ascribe a numeric value to the extent to which pupils used the pictorial scenario as a referential context. This methodology was able to identify those pupils who preferred visual as opposed to numeric approaches as well as being able to characterise the extent to which individual pupils favoured *specific* visualisation strategies.

Seven research participants who were identified in Phase 1 as preferring a visual mode when solving pattern generalisation tasks took part in Phase 2. In addition, one of the research participants from the pilot study, who similarly showed a preference for visual methods, also took part in Phase 2. These eight research participants were individually provided with a further linear pattern and in the space of approximately one hour were required to provide multiple expressions for the n^{th} term. Tools such as paper, pencils and highlighters as well as appropriate manipulatives such as matchsticks and plastic counters were provided. Participants were asked to think aloud while engaged with their particular pattern generalisation task. Each session was audio-visually recorded and field-notes were taken. Audio-visual recordings were analysed with specific reference to participants' use of semiotic means of objectification such as words, linguistic devices, metaphor, gestures, rhythm, graphics and physical artefacts. This analysis process culminated in a series of vignettes which serve to characterise the affordances brought forth by the complementary multiple perspectives of the theoretical framework.

CHAPTER FIVE

RESULTS, ANALYSIS & DISCUSSION

*The environment contains no information, the
environment is at it is.*

HEINZ VON FOERSTER

5.1 INTRODUCTION

Responses to all questions in Phase 1 were carefully analysed and categorised in terms of strategy/method employed and the contextual connectivity behind the justification of the 40th and n^{th} terms. The results of this analysis were summarised on Question Response Analysis Sheets (QRAS) for each of the 10 questions, and these appear in Appendix A. These summary sheets were used to give a global view of the results of Phase 1 in order to investigate the extent to which individual pupils favour specific visualisation strategies.

Phase 2 focuses on those research participants who were identified in Phase 1 as preferring visual strategies. Analysis of Phase 2 data seeks to explore the extent to which pupils are able to generalise pictorial patterns in multiple ways. A fine-grained micro-analysis of Phase 2 data is then presented in the form of a series of vignettes which show the rich tapestry of generalisation activity which was evidenced by the research participants.

The chapter closes with a discussion of broad insights that gradually emerged during the course of the micro-analysis. These are discussed in relation to my own experiences regarding possible pedagogical strategies that could be used to support

pictorial pattern generalisation activities and to overcome the difficulties experienced by some of the participants in the course of this study.

5.2 PHASE 1

Analysis of Phase 1 data serves a dual purpose. Firstly, it seeks to answer the first of the guiding research questions outlined in Section 1.2:

1. To what extent, if any, do individual pupils favour specific visualisation strategies when generalising figural patterns?

Secondly, the analysis of Phase 1 data is intended to identify those pupils whose preferred mode of solving figural pattern generalisation tasks is visual as opposed to numeric.

5.2.1 STRATEGY/METHOD CLASSIFICATION

For each of the 10 number patterns, the 23 research participants were each asked to determine a numerical value for the 40th term as well as a general algebraic expression for the n^{th} term. In order to assess the extent to which individual pupils favoured specific visualisation strategies, the justification/explanation of their expression for the n^{th} term, as well as the written articulation of their reasoning for the numerical value of the 40th term, were carefully analysed and categorised according to the classification/coding system previously described (Section 4.5.1). Tables 5.1, 5.2 and 5.3 summarise this process. Table 5.1 categorises pupils' responses as being numerically based, visually based, or containing a blend of both numeric and visual elements.

Table 5.1 Global overview of numeric versus visual approaches

PUPIL	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10
Anna	N	N	N	N	N	N	N	N	N	N
Anthea	V	V	V/N	V	V	V	V	V	V	V
Arthur	N	N	N	N	N	N	N	N	N	N
Barry	N	N	N	N	V	V/N	N	N	N	N
Brian	V	V	V	V/N	V/N	N	V	V/N	V	V/N
Charles	N	V/N	N	N	N	N	N	N	N	N
David	V/N	V	V/N	V/N	V/N	V/N	V/N	V/N	V/N	V/N

Dylan	V/N	V/N	V/N	V/N	V/N	N	N	V	V/N	V/N
Fiona	V/N	V/N	V/N	V/N	N	V/N	V/N	N	V/N	V/N
Harry	N	N	N	V/N	N	N	N	N	N	N
Katie	N	N	N	N	N	N	N	N	N	N
Kelly	N	V	V	V	V	V	V	V	V	V
Lance	N	V	V	V	V	V	V	V	V	V
Liza	V/N	V	V/N	V	V	V/N	V	N	V/N	V/N
Mike	N	N	N	N	N	N	N	N	N	N
Mitch	N	N	N	N	V/N	N	V/N	N	N	N
Philip	V	V	V	V	V	V	V	V	V	V
Rose	V/N	N	N	N	V/N	V/N	V	N	V/N	V
Ryan	N	V/N	V/N	N	N	N	N	N	N	N
Sally	V/N	V/N	V/N	N	V/N	V/N	N	V/N	V/N	V/N
Susan	N	N	N	N	N	N	N	N	N	N
Taylor	V/N	V	V	V	V	V	N	V	V	V
Terry	V	V	V	V	V/N	V	V	V	V	V

Table 5.1 reveals a continuum with respect to numeric versus visual approaches. At one end of this continuum are pupils like Anna, Arthur, Katie, Mike and Susan whose generalisation strategies gave evidence of only numeric considerations. These pupils in effect extracted/divorced the numbers from the given diagrams and worked with them with no further consideration for the pictorial scenario. At the other end of the continuum are pupils like Philip whose generalisation strategies all made explicit use of the given diagrams as a referential context. During the process of justification and explanation, Philip used the given pictorial terms as a *generic example*, what Lannin (2005) describes as “a particular example that embodies the general characteristics of an argument” (p. 236). Centred approximately midway between these two extremes are pupils like David, Fiona and Sally who made use of both visual as well as numeric approaches in most of their patterning activities.

Although this initial strategy/method characterisation was based on a single type of generalisation task, it nonetheless reveals a broad diversity of styles amongst the 23 research participants. These different styles resonate with Krutetskii’s (1973) characterisation of mathematics pupils into four groups based on the relative role of the verbal-logical and visual-pictorial components of their activity:

- *Analytic* – pupils who operate easily with abstract relationships and have no need for visual supports in problem-solving.
- *Geometric* – pupils who find it necessary to give visual expression to abstract mathematical relationships.
- *Abstract-harmonic* – pupils who have equally well developed verbal-logical and visual-pictorial components but who are disinclined to use visual supports.
- *Pictorial-harmonic* – pupils who have equally well developed verbal-logical and visual-pictorial components but who find the use of visual supports helpful.

Table 5.2 Overview of visual strategies for V and V/N responses

PUPIL	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10
Anna	-	-	-	-	-	-	-	-	-	-
Anthea	LF1	GF1	GF1	LF1	GF1	GF1	GF2	LF1	GF1	LF1
Arthur	-	-	-	-	-	-	-	-	-	-
Barry	-	-	-	-	GF1	GF1	-	-	-	-
Brian	LF1	GF1	LF1	LF1	LF1	-	GF1	LF1	GF1	GF1
Charles	-	LAU	-	-	-	-	-	-	-	-
David	LAU	GF1	GF1	LF2	LAU	LAU	LAU	LAU	LAU	LAU
Dylan	LAU	GF1	LF3	LF1	LAU	-	-	LF1	LF1	LAU
Fiona	LAU	GF1	LAU	LAU	-	GF1	GF1	-	GF1	LF1
Harry	-	-	-	LAU	-	-	-	-	-	-
Katie	-	-	-	-	-	-	-	-	-	-
Kelly	-	GF2	LF1	LF1	GF1	GF1	GF1	GF1	GF1	GF2
Lance	-	GF1	GF1	GF1	GF2	GF1	GF1	LF1	GF1	LF1
Liza	LF1	GF1	GF1	LF1	GF2	GF1	GF1	-	GF1	GF1
Mike	-	-	-	-	-	-	-	-	-	-
Mitch	-	-	-	-	LAU	-	GF1	-	-	-
Philip	GF1	GF1	LF1	GF1	GF1	GF2	GF1	LF1	GF1	GF1
Rose	LF1	-	-	-	GF1	GF1	GF1	-	GF1	GF1
Ryan	-	LAU	LAU	-	-	-	-	-	-	-
Sally	LF1	GF1	LF1	-	GF1	GF2	-	LF1	GF1	LF1
Susan	-	-	-	-	-	-	-	-	-	-
Taylor	LF1	GF1	LF1	LF1	GF2	GF1	-	LF1	GF1	LF1
Terry	LF1	GF1	LF1	LF1	GF1	GF1	GF1	LF1	GF1	LF1

Table 5.2 provides an overview of the extent to which pupils favoured specific *visual* strategies/approaches. Only those responses that were categorised as either visual (V) or a blend of visual and numeric (V/N) feature in the table. Blank cells represent questions that were approached using only numerical considerations. Table 5.2 shows a remarkable variety of approaches across individual pupils. Only three pupils (Charles, Harry and Ryan) exhibited use of a single strategy. However, all three of these pupils used numeric strategies throughout the 10 questions. Their featuring in Table 5.2 is a result of one or two of their approaches using a blend of visual and numeric elements. In these instances the strategy in question (LAU) simply utilised the pictorial scenario to identify the numerical value of the constant difference (i.e. the visual additive unit) prior to using a Rate-Adjust technique to determine the general formula. Since these pupils are at the numeric end of the visual-numeric continuum, the fact that they used only a single visual approach is not of significance. A similar argument holds for Barry who, while also positioned towards the numeric end of the visual-numeric continuum, gave evidence of using global features (specifically GF1) to support his generalisation process in Questions 5 and 6. Although Barry's use of only a single type of visual strategy is not of significance, what *is* of interest are the possible reasons behind his shift from a numeric to visual strategy for these two questions. This raises some interesting questions. Although all 10 questions were presented in a pictorial context, do some diagrams support or encourage visual strategies more than others? At a more nuanced level, do some diagrams support or encourage specific *types* of visual strategies over other types? These questions will be returned to later in the discussion.

Other than the four pupils previously mentioned (Barry, Charles, Harry and Ryan), all other pupils made use of two or more different types of visual reasoning. Table 5.2 reveals a continuum with respect to local versus global approaches. Some pupils (e.g. Dylan) seemed to favour local strategies while other pupils (e.g. Lance, Philip and Rose) seemed to favour global strategies. These pupils are at the two ends of the local-global continuum, while others (e.g. Sally and Terry) made use of global and local strategies in equal proportion. However, what is of particular interest is that these pupils all made use of *both* local as well as global visual reasoning. Local strategies make use of identifying a constant starting structure to which is added, in a recursive manner, multiples of the additive unit. Global strategies on the other

hand view the overall structure of the pictorial terms more holistically. This involves the subdivision of the pictorial terms into constituent gestalts whose relationship to the overall structure is then determined⁴⁵. The visual reasoning behind these two broad types of visual strategy is markedly different, so it is intriguing that 19 of the 23 pupils made use of both local and global strategies. This again raises the question of whether certain diagrams support *specific* visual strategies more effectively than others.

Table 5.3 Overview of numerical strategies for N and V/N responses


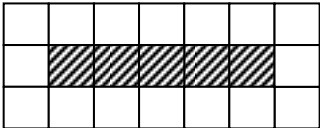
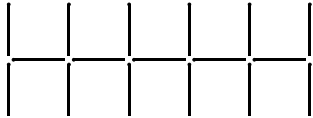
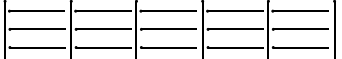
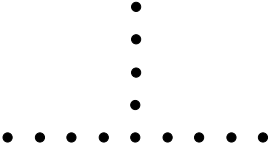
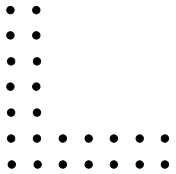
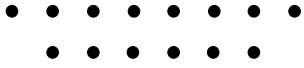
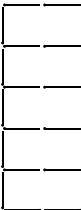
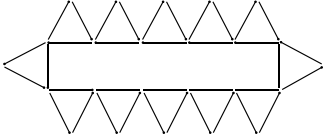
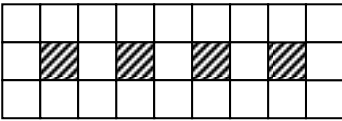
PUPIL	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10
Anna	RA	RA	RA	RA	RA	RA	RA	RA	RA	RA
Anthea	-	-	DF	-	-	-	-	-	-	-
Arthur	RA	RA	DF	DF	RA	RA	RA	DF	RA	DF
Barry	DF	DF	DF	DF	-	DF	DF	DF	DF	DF
Brian	-	-	-	DF	DF	RA	-	RA	-	DF
Charles	RA	RA	RA	RA	RA	RA	RA	RA	RA	RA
David	DF	-	DF	DF	RA	RA	RA	RA	RA	DF
Dylan	DF	DF	DF	RA	DF	RA	RA	-	RA	DF
Fiona	RA	RA	RA	RA	RA	RA	RA	RA	RA	RA
Harry	RA	RA	RA	DF	DF	DF	DF	DF	DF	DF
Katie	RA	RA	RA	RA	RA	RA	RA	RA	RA	RA
Kelly	RA	-	-	-	-	-	-	-	-	-
Lance	DF	-	-	-	-	-	-	-	-	-
Liza	RA	-	RA	-	-	RA	-	DF	DF	DF
Mike	DF	DF	DF	DF	DF	DF	DF	DF	DF	DF
Mitch	RA	RA	RA	RA	RA	RA	RA	RA	RA	RA
Philip	-	-	-	-	-	-	-	-	-	-
Rose	RA	RA	RA	DF	DF	RA	-	DF	RA	-
Ryan	DF	RA	DF	DF	DF	DF	DF	DF	DF	DF
Sally	RA	RA	RA	RA	RA	RA	RA	RA	RA	RA
Susan	DF	RA	DF	DF	DF	DF	DF	DF	RA	DF
Taylor	RA	-	-	-	-	-	DF	-	-	-
Terry	-	-	-	-	DF	-	-	-	-	-

Table 5.3 provides an overview of the extent to which pupils favoured specific *numeric* strategies/approaches. Only those responses that were categorised as either numeric (N) or a blend of visual and numeric (V/N) feature in the table. Blank cells thus represent questions that were approached using strictly visual

⁴⁵ See Table 4.1 in Section 4.5.1 for an overview of *local* and *global* strategies.

considerations. Once again there seems to be a continuum of approaches with some pupils (Anna, Charles, Fiona, Katie and Mitch) using exclusively a *Rate-Adjust* strategy, while other pupils (e.g. Barry and Mike) made exclusive use of a trial-and-adjustment *Direct Formula* approach. Others (e.g. Dylan and Liza) made use of both strategies. The Rate-Adjust method represents a methodical approach based on a structured algorithmic strategy. The constant difference (d) between consecutive terms is determined numerically and used as a multiplying factor to give an initial structure to the general expression in the form $T_n = dn$. The formula is then adjusted by the addition or subtraction of a constant to ensure that it works, i.e. gives the correct numerical answer, for the given terms. This is a highly mechanistic and structured approach, so what is of interest here is not that this strategy was utilised more often than the Direct Formula approach, but that those pupils who exhibited use of both the Rate-Adjust and Direct-Formula approach only made use of the structured and algorithmic Rate-Adjust method for *specific* questions.

Table 5.4 Overview of pictorial contexts

<p>Q1</p> 	<p>Q2</p> 	<p>Q3</p> 	
<p>Q4</p> 	<p>Q5</p> 	<p>Q6</p> 	
<p>Q7</p> 	<p>Q8</p> 	<p>Q9</p> 	
<p>Q10</p> 			

A number of questions were flagged during the preceding discussion. These questions related to whether or not there is evidence to suggest that certain questions play a better role in supporting specific generalisation strategies. In order to investigate this more thoroughly it is important to notice that the 10 pictorial contexts, as summarised in Table 5.4, can be grouped into a number of subtly yet critically different categories based on their physical structure.

- **TYPE A:** Questions 1, 3, 4, 7, 8 and 10 have the defining characteristic that their growth pattern occurs in a single direction. As such, progression from one term to the next can be accomplished relatively simply by the direct attachment of the additive unit, i.e. the visual analogue of the “common difference”, onto the end of the preceding term. By way of example, in Question 4 the next term in the sequence could be arrived at directly by the attachment of 3 horizontal matches and a single vertical match onto the rightmost end of the given diagram.
- **TYPE B:** Questions 5 and 6 have the defining characteristic that their growth pattern occurs in more than one direction. Question 5 displays growth in 3 directions (vertically upwards and horizontally to the left and right) while Question 6 displays growth in 2 directions (vertically upwards and horizontally to the right). For these questions the additive unit is no longer a single entity, but is split (equally in these cases) between each of the growth directions.
- **TYPE C:** Questions 2 and 9 only have a single direction of growth. However, their defining characteristic is that progression from one term to the next cannot be accomplished by direct attachment of the additive unit onto the preceding term. Instead, a certain degree of *dynamic visualisation* is required as the additive unit needs to be *inserted* into the previous term. By way of example, to arrive at the next term in Question 2 would require the insertion of a column of 3 tiles (white, striped, white) into the body of the given structure. If the growth was visualised in a step-by-step process from one term to the next then this insertion would need to take place to the left of the rightmost column of 3 white tiles.

Table 5.5 provides an overview of the different strategies exhibited for each of the 10 pictorial pattern generalisation tasks. Careful analysis of Table 5.5 in conjunction with the above question categorisation system yields some interesting observations

with respect to visual strategies. The second-last row of Table 5.5 (entitled “Total Local”) was arrived at by grouping the four local strategies (LAU, LF1, LF2 and LF3) into a single tally. This row thus gives the *total* number of local strategies evidenced in each of the 10 patterning tasks. The final row of Table 5.5 (entitled “Total Global”) gives the combined tally for global strategies (GF1 and GF2) for each of the 10 questions.

Table 5.5 Overview of strategy/approach per question

	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	TOTAL
General strategy/approach											
V	4	9	6	7	7	6	8	7	7	7	68
N	12	9	10	11	9	11	12	13	10	10	107
V/N	7	5	7	5	7	6	3	3	6	6	55
Numeric approaches											
RA	12	11	9	7	8	12	9	8	11	6	93
DF	7	3	8	9	8	5	6	8	5	10	69
Visual approaches											
LAU	3	2	2	2	3	1	1	1	1	2	18
LF1	7	0	6	7	1	0	0	8	1	6	36
LF2	0	0	0	1	0	0	0	0	0	0	1
LF3	0	0	1	0	0	0	0	0	0	0	1
GF1	1	11	4	2	7	9	9	1	11	4	59
GF2	0	1	0	0	3	2	1	0	0	1	8
Total Local	10	2	9	10	4	1	1	9	2	8	56
Total Global	1	12	4	2	10	11	10	1	11	5	67

For each of the 10 questions either a *local* or *global* strategy dominated in terms of visually mediated generalisation methods. Local strategies clearly dominated in Type A questions. This can readily be understood in terms of the relatively easy identification of the additive unit supporting a local visual strategy. Global strategies dominated in Type B and Type C questions. In terms of Type C structures, which required the insertion of the additive unit along with associated dynamic visualisation, one could argue that this added level of complexity supported global strategies as being a more direct approach when compared with local strategies. With respect to Type B questions, where the pattern growth occurs in multiple directions, the support of global over local strategies can possibly be understood in terms of the local additive unit being spread over a number of physical regions as opposed to constituting a single structural unit. Another contributing factor for Type B structures supporting global strategies may well have to do with the particular

spatial arrangement of potential perceptual gestalts. Question 5 for example gives a strong visual sense of 3 perpendicular beams radiating outwards from a central dot. From an enactivist stance such a description is by no means inherent in the diagram itself. However, the potential for such a perceptual experience to be brought forth through our individual engagement with the diagram is supported by our common ontogenic situatedness and the characteristic ways in which we interact with our surroundings.

It is interesting to note that Question 7, which has a Type A structure, seemed to support global rather than local strategies. Although the local additive unit could be visualised as an oblique 2-dot unit, it is possible that the 2 parallel horizontal rows of dots provided far stronger visual imagery and thus supported a global structural interpretation of the visual stimulus.

Table 5.5 also reveals that local visual strategies LF2 and LF3 had only a single occurrence each. A possible explanation for this observation relates to what Radford (2000) refers to as the *positioning problem*. In LF2 the general formula is typically determined in the form $T_n = T_1 + (n-1) \times d$ where T_1 is the first term and d is the additive unit. For LF3 strategies the general formula takes the form $T_n = (n+1) \times d - c$ where c is a constant which corrects for the over-count and d is the local additive unit. Inherent in the structure of these formulae is the nontrivial requirement of being able to express sentiments such as “one less than the shape number” and “one more than the shape number” using algebraic notation, i.e. $(n-1)$ and $(n+1)$ respectively. Given the fact that the research participants constituted a high ability group of students, it is unlikely that this algebraic consideration would be significantly problematic. However, since the ultimate aim of each of the 10 generalisation tasks was to provide an algebraic expression for the general term, it is possible that this goal influenced the overall visualisation process in favour of more algebraically useful generalisations. A similar explanation could underlie the observation that in terms of global visual strategies, GF2 (8 occurrences) was employed far less regularly than GF1 (59 occurrences).

5.2.2 CONTEXTUAL CONNECTIVITY RATING (CCR)

For each of the 10 questions, responses were also rated in terms of the extent to which the pictorial context featured in the justification/explanation of the 40th and n^{th} terms. CCR⁴⁶ ratings of 1, $\frac{1}{2}$ or 0 were awarded for each question. For a score of 1 to have been awarded, the justification must have made explicit reference (either diagrammatically or verbally) to the pictorial context. A score of 0 indicates that the justification is purely numerically based and contains no reference to the pictorial context. A score of $\frac{1}{2}$ is indicative of those justifications that either made only partial reference to the pictorial context or where both pictorial and numerical elements played a role in the justification.

The purpose of the CCR was to identify appropriate research participants for Phase 2 of the study – i.e. those pupils who preferred a *visual* mode when solving pictorial generalisation tasks. There were nine pupils who scored a CCR value greater than 5. Of this cohort, seven pupils agreed to take part in Phase 2 of the study – Anthea, Brian, Lance, Liza, Kelly, Philip and Terry.

Table 5.6 Summary of CCR per question

	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	TOTAL
Anna	0	0	0	0	0	0	0	0	0	0	0
Anthea	1	1	$\frac{1}{2}$	1	1	1	1	1	1	1	9.5
Arthur	0	0	0	0	0	0	0	0	0	0	0
Barry	0	0	0	0	1	$\frac{1}{2}$	0	0	0	0	1.5
Brian	1	1	1	$\frac{1}{2}$	$\frac{1}{2}$	0	1	$\frac{1}{2}$	1	$\frac{1}{2}$	7
Charles	0	$\frac{1}{2}$	0	0	0	0	0	0	0	0	0.5
David	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	5.5
Dylan	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	1	$\frac{1}{2}$	$\frac{1}{2}$	4.5
Fiona	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	3.5
Harry	0	0	0	$\frac{1}{2}$	0	0	0	0	0	0	0.5
Katie	0	0	0	0	0	0	0	0	0	0	0
Kelly	0	1	1	1	1	1	1	1	1	1	9
Lance	0	1	1	1	1	1	1	1	1	1	9
Liza	$\frac{1}{2}$	1	$\frac{1}{2}$	1	1	$\frac{1}{2}$	1	0	$\frac{1}{2}$	$\frac{1}{2}$	6.5
Mike	0	0	0	0	0	0	0	0	0	0	0
Mitch	0	0	0	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	0	1
Philip	1	1	1	1	1	1	1	1	1	1	10

⁴⁶ The CCR rating system is outlined in Section 4.5.2.

Rose	½	0	0	0	½	½	1	0	½	1	4
Ryan	0	½	½	0	0	0	0	0	0	0	1
Sally	½	½	½	0	½	½	0	½	½	½	4
Susan	0	0	0	0	0	0	0	0	0	0	0
Taylor	½	1	1	1	1	1	0	1	1	1	8.5
Terry	1	1	1	1	½	1	1	1	1	1	9.5
TOTAL	7.5	11.5	9.5	9	10.5	9	9.5	8.5	10	10	

5.3 PHASE 2

Analysis of Phase 2 data seeks to answer the second, third and fourth guiding research questions which frame this study. These guiding questions were originally outlined in Section 1.2:

2. To what extent are pupils able to generalise patterning tasks, set in a pictorial context, in multiple ways?
3. What embodied processes are evinced by pupils engaged in figural pattern generalisation tasks?
4. In what ways do these embodied processes either assist or hinder pupils' ability to visualise figural cues in multiple ways?

Data for Phase 2 emanates from eight research participants - the seven participants identified in Phase 1 as preferring *visual* methods (Anthea, Brian, Lance, Liza, Kelly, Philip and Terry) as well as Grant, a research participant from the pilot study who similarly showed a preference for visual methods as evidenced by a very high CCR score. Grant's participation in the pilot study was at a point when the data collection and analysis techniques had been refined to a point commensurate with the main study.

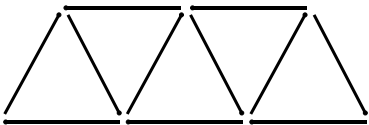
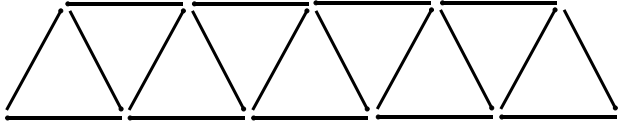
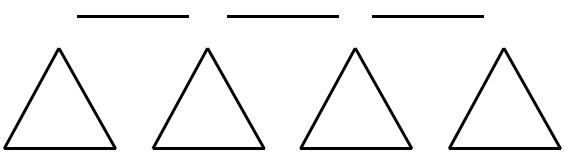
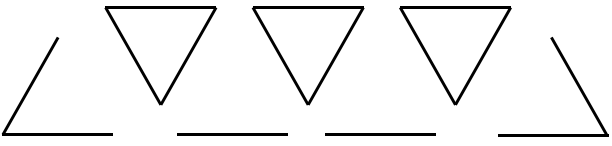
5.3.1 BROAD ANALYSIS

The purpose of this initial overview of Phase 2 data is to explore the extent to which pupils are able to generalise pictorial patterns in multiple ways (i.e. the second of the guiding research questions). The generalisations arrived at for each of the Phase 2 research participants are displayed in Tables 5.7 – 5.14. Each table begins with the visual stimulus (i.e. the two non-consecutive pictorial terms) that was presented to the participant in question. The rest of the table is divided into two columns. The

left-hand column contains the various expressions for T_n as arrived at by the research participant, while the right-hand column represents the associated visual apprehension as evidenced by his or her justification and/or explanation of the general term. The left-hand column also contains the strategy/method classification for each general expression as previously outlined in Table 4.1 of Section 4.5.1. In the right-hand column, where possible, the same specific term has been used in the representation of each different apprehension in order to ease comparison. In Table 5.7, for example, T_4 has been used throughout. However, in some instances it was necessary to use a different term to show the specific apprehension, for example when the general formula was arrived at by making use of specific aspects of a particular term (e.g. certain symmetry elements), or when the visualisation only holds true for a particular term. In all cases the specific term chosen to represent the apprehension is indicated. All expressions and corresponding apprehensions are presented in the same order as they were determined by the research participants.

Table 5.7 shows Grant's various expressions and apprehensions for the given pictorial pattern. All 9 of Grant's algebraic expressions for the general term are correct and consequently algebraically equivalent.

Table 5.7 Grant's nine expressions and apprehensions for T_n

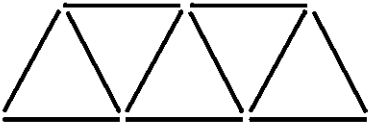

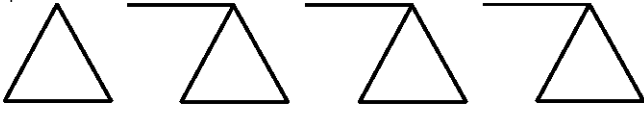
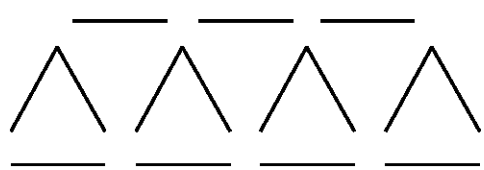
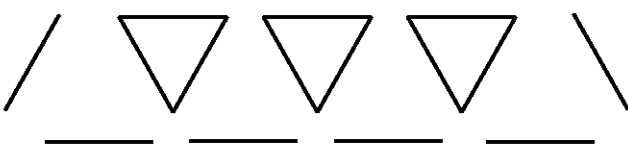
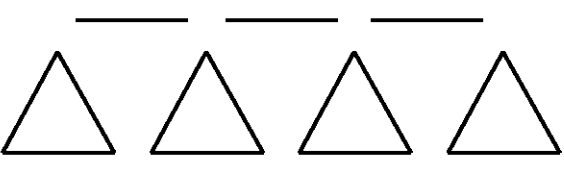
			
Shape 3		Shape 5	
Expression for T_n	Apprehension		
GF1 $3n + n - 1$	T_4		
GF1 $3(n - 1) + 4 + n - 2$	T_4		

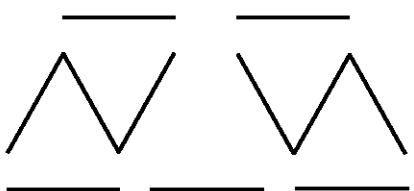
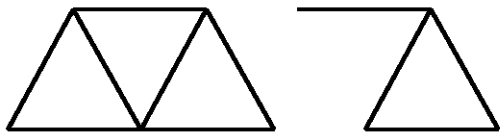

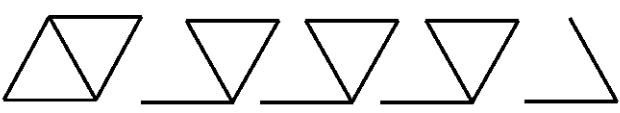

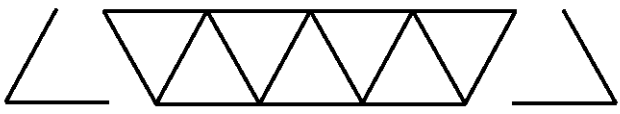

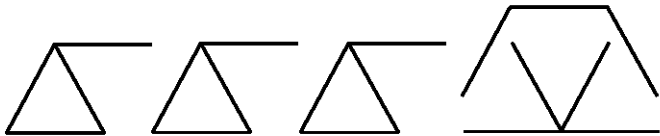
GF1 $2n + n + n - 1$	T_4
LF3 $4n - 1$	T_4
GF1 $n + n + n + [n - 1]$	T_4
LF2/GF1 $2(2[n - 1]) + 3$	T_4
LF2 $4(n - 1) + 3$	T_4
GF1 $n + 3(n - 1) + 2$	T_4
GF2 $3n + 3(n - 1) - [2n - 2]$	T_4

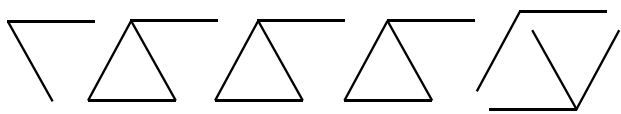
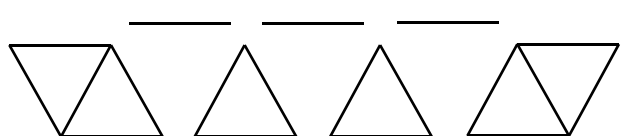
Philip was able to arrive at a remarkable 15 expressions for the general term of his pictorial pattern (shown in Table 5.8). This happened to be the same pattern that was given to Grant. Of the 15 expressions, only the first was justified purely numerically, while the remainder were all justified through the shown apprehension.

An interesting aspect of Philip's various generalisations is that the general expression $3 + 4(n - 1)$ appears twice, but with significantly different visualisations. The same holds for the expression $n + 2n + (n - 1)$. In Philip's final two expressions the apprehensions were arrived at by turning the image upside down. All 15 of Philip's algebraic expressions for the general term are correct and thus algebraically equivalent.

Table 5.8 Philip's fifteen expressions and apprehensions for T_n

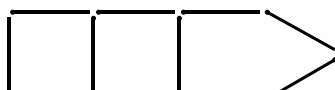
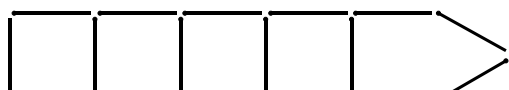
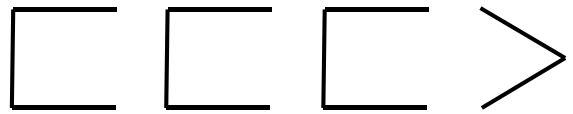
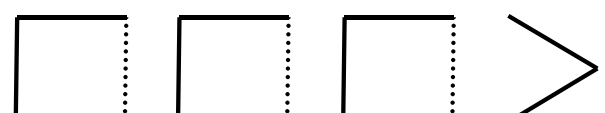
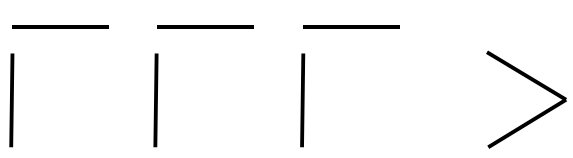
			
Shape 3		Shape 5	
Expression for T_n	Apprehension		
RA $4n - 1$	<i>Arrived at numerically</i>		
LF2 $3 + 4(n - 1)$	T_4		
GF1 $n + 2n + (n - 1)$	T_4		
GF1 $n + 2 + 3(n - 1)$	T_4		
GF1 $3n + n - 1$	T_4		




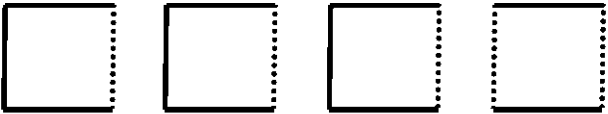
<p>GF1</p> $n + 2n + (n - 1)$	<p>T_3</p> 
<p>DF/GF1</p> $4(n - 1) - 1 + 4$	<p>T_3</p> 
<p>GF1</p> $2(n - 1) + (n - 1) + (n - 2) + 4$	<p>T_5</p> 
<p>GF1</p> $5 + 4(n - 2) + 2$	<p>T_5</p> 
<p>GF1</p> $3 + 4(n - 1)$	<p>T_5</p> 
<p>DF/GF1</p> $(4(n - 1) - 1) + 2 + 2$	<p>T_5</p> 
<p>GF1</p> $n + n + n + (n - 1)$	<p>T_4</p> 
<p>LF2</p> $4(n - 2) + 4 + 3$	<p>T_5</p> 

GF1 $4(n-2)+3+2+2$	T_5 (inverted) 
GF1 $5+5+(n-2)+3(n-3)$	T_5 (inverted) 

Brian was able to arrive at 7 expressions for the general term of his pictorial pattern (shown in Table 5.9). In the case of the 5th and 6th expressions, although the algebraic expressions are the same $(3(n+1)-1)$, the two apprehensions are strikingly different through the use of dynamic visualisation in the case of the 6th expression.

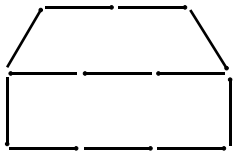
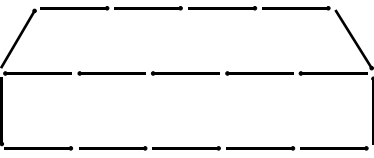
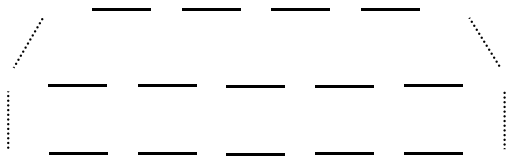
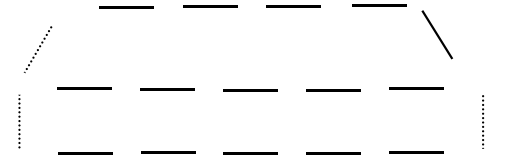
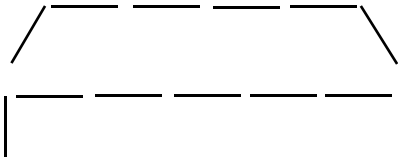
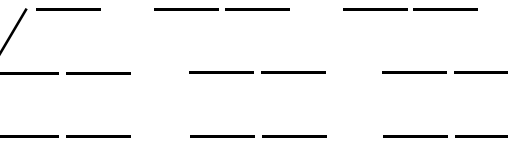
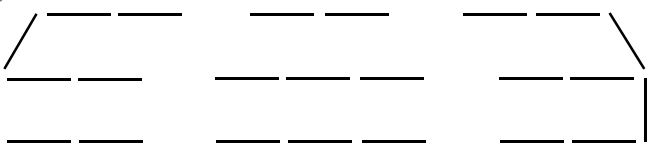
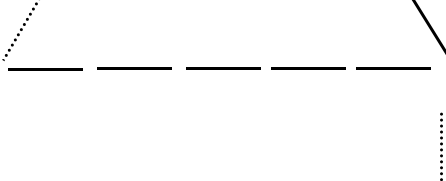
Table 5.9 Brian's seven expressions and apprehensions for T_n

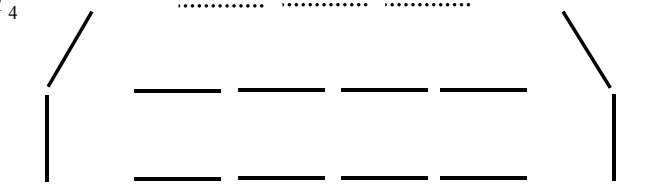
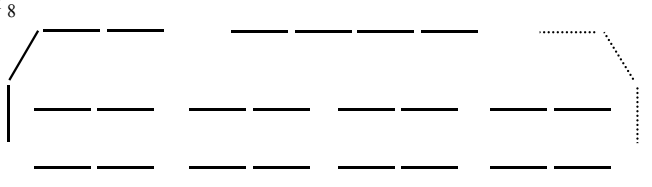
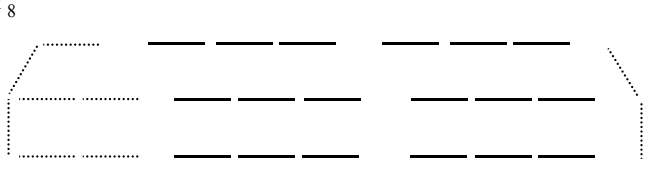
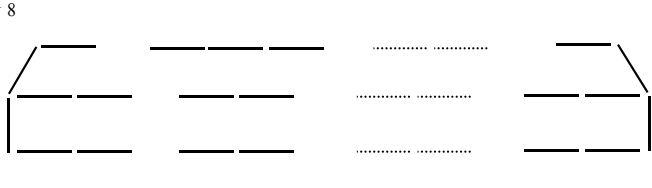
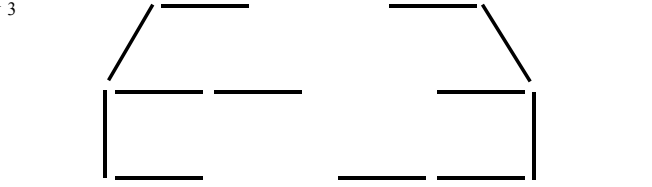
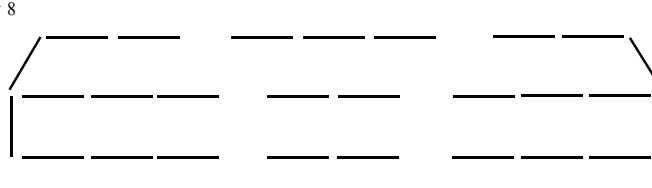
 Shape 3		 Shape 5	
Expression for T_n	Apprehension		
LF1 $3n+2$	T_3		
GF2 $4n+2-n$	T_3		
GF1 $n+2+2n$	T_3		

<p>LF1</p> <p>$3+3n-1$</p>	<p>T_3</p> 
<p>GF1</p> <p>$3(n+1)-1$</p>	<p>T_3</p> 
<p>GF1</p> <p>$3(n+1)-1$</p>	<p>T_3</p> 
<p>GF2</p> <p>$4(n+1)-(n+2)$</p>	<p>T_3</p> 

Lance managed to determine 12 expressions for the general term of his pictorial pattern (shown in Table 5.10). The general expression $3n+3$ was arrived at on three different occasions, each time through a different apprehension, while the expressions $3(n+1)$ and $2n+4+(n-1)$ both occur twice through different apprehensions. All twelve expressions are correct and algebraically equivalent with the single exception of Lance's penultimate expression, $2n \times 2 = 4n$. This particular expression, along with its apprehension, only yields a correct answer for T_3 . Lance's seventh apprehension, while algebraically correct, is nonetheless based on a spurious apprehension that is only applicable to T_4 . Interestingly, all of Lance's general expressions fall into the strategy/method category GF1.

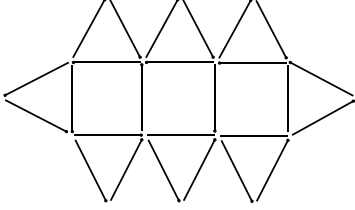
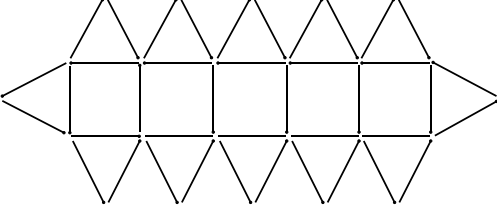
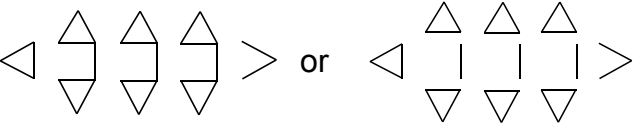
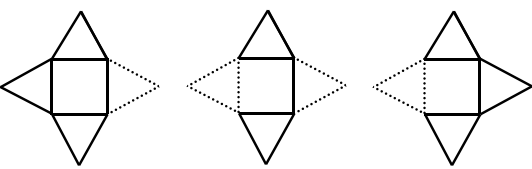
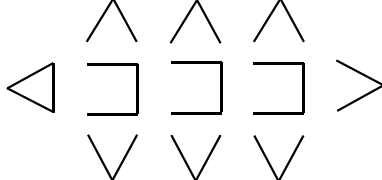
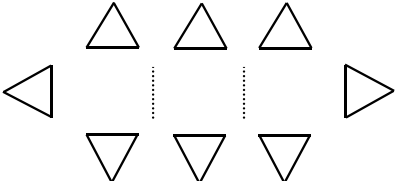
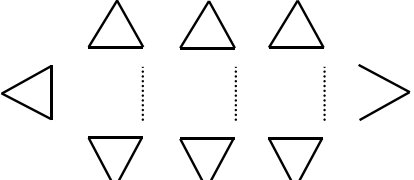
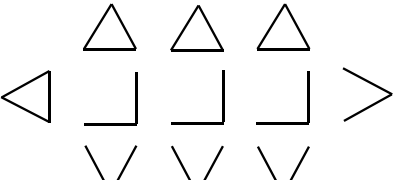
Table 5.10 Lance's twelve expressions and apprehensions for T_n

 	
Shape 3 Shape 5	
Expression for T_n	Apprehension
GF1 $2n + 4 + (n - 1)$	T_5 
GF1 $3n + 3$	T_5 
GF1 $2n + 2 + 1n + 1$	T_5 
GF1 $3n + 3$	T_6 
GF1 $3(n + 1)$	T_7 
GF1 $2n + 1 + 1n + 2$	T_5 

GF1 $3n + 3$	T_4 
GF1 $6\left(\frac{n}{2}\right) + 3$	T_8 
GF1 $3(n + 1)$	T_8 
GF1 $3n - 3 + 6$ [visualised as $3(n - 1) + 6$]	T_8 
GF1 $2n \times 2 = 4n$	T_3 
GF1 $2n + 4 + (n - 1)$ [visualised as $2(n + 2) + (n - 1)$]	T_8 

Terry was able to arrive at 8 expressions for the general term of his pictorial pattern (shown in Table 5.11). All these expressions are correct and algebraically equivalent with the single exception of the final expression $4 + n(5 - 1) + 3n + 5$ which gives an overcount of 4 matches. This final expression was nonetheless arrived at through a fascinating apprehension. Of Terry's 8 expressions, two could not be unambiguously classified into a single strategy category since they contained aspects of different methods. In these instances the blend of strategies is shown in the categorisation.

Table 5.11 Terry's eight expressions and apprehensions for T_n

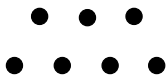
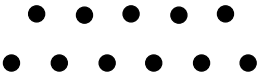
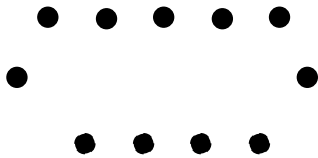
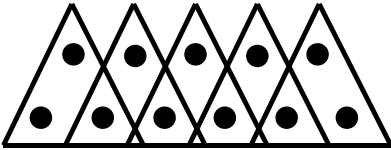
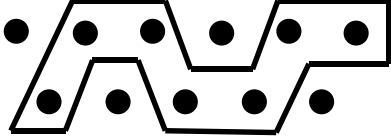

 	
Expression for T_n	Apprehension
LF1/GF1 $7n + 5$ or $3 + n(3 + 3 + 1) + 2$	T_3 
LF2 $12 + (n - 1)(12 - 5)$	T_3 
LF1/GF1 $3 + n(3 + 3 + 2) + 2$	T_3 
GF1 $3 + n(3 + 3) + (n - 1) + 3$	T_3 
GF1 $3 + n + n(6) + 2$	T_3 
GF1 $3 + 2n + n(5) + 2$	T_3 

GF1 $n-1+2n+n$	T_4
GF1 $2(n-1)+2+n-1+n$	T_4
GF2 $3(n+n-1)-(2n-2)$	T_4
DF $2n+2n-1$	<i>Arrived at numerically</i>
DF $2(n+n-1)+1$	<i>Arrived at numerically</i>

Anthea was able to arrive at 6 expressions for the general term of her pictorial pattern (shown in Table 5.13). All 6 expressions are correct and algebraically equivalent. The first two expressions were arrived at through numerical considerations while the last three showed a mixture of both numerical and visual elements. Only the third expression was arrived at through strictly visual reasoning. Her final expression was only arrived at through substantial scaffolding⁴⁷.

⁴⁷ Anthea's apprehension was interesting in that she realised that there was a difference between her visualisation of odd- and even-numbered terms. I was interested to see if she could, with the help of sufficient scaffolding, make use of this realisation in the structuring of what was going to be an algebraically complex general expression.

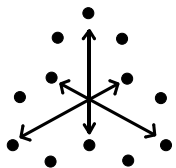
Table 5.13 Anthea's six expressions and apprehensions for T_n

  Shape 3 Shape 5	
Expression for T_n	Apprehension
DF $n + (n + 1)$	<i>Arrived at numerically</i>
DF $2n + 1$	<i>Arrived at numerically</i>
GF1 $(n + 2) + (n - 1)$	T_5 
RA/GF2 $3n - (n - 1)$	T_5 
DF/GF1 $(n + 2) + (n - 1)$	T_5 (inverted) 
DF/GF1 $\left(\frac{n+1}{2}\right) \times 3 + n - \left(\frac{n+1}{2}\right)$	T_5 

Liza managed to determine 6 expressions for the general term of her pictorial pattern (shown in Table 5.14). She was unable to arrive at a general formula for her 7th apprehension, but all 6 of the general expressions she was able to arrive at are correct and thus algebraically equivalent.

Table 5.14 Liza's six expressions and seven apprehensions for T_n

Expression for T_n	Apprehension
GF1 $(n) + (n-1) + (n-2)$	T_5
DF/GF1 $2n + (n-3)$	T_5
GF2 $3n-3$	T_5
GF1 $(n+1) + 2(n-2)$	T_5
GF1 $3(n-2) + 3$	T_5
GF1 $3(n-1)$	T_5

<p>GF1</p> <p>Unable to arrive at formula</p>	<p>T_5</p> 
---	--

With reference to the second guiding research question of this study⁴⁸, i.e. the question of the extent to which pupils are able to generalise pictorial patterns in multiple ways, this initial broad analysis shows that these 8 research participants have a remarkable ability to generalise pictorial patterns in multiple ways. All 8 participants were able to determine at least 6 different general expressions, each with a different associated apprehension, while one participant (Philip) was able to generalise his pictorial pattern in 15 different ways.

During this overview of the Phase 2 data, a number of interesting aspects of various apprehensions and generalisations were briefly remarked on. An unpacking of these observations, along with a number of others, forms the basis of the next section.

5.3.2 MICRO-ANALYSIS

The purpose of this micro-analysis is twofold. Firstly it serves to capture, in the form of a series of vignettes, the rich tapestry of generalisation activity which was evidenced by the research participants. This evidence was accumulated through a multi-systemic semiotic analysis of participants' activity during the process of arriving at a stable form of awareness. These means of objectification included the use of words, linguistic devices, metaphor, gestures, rhythm, graphics and physical artefacts. This analysis seeks to shed light on the third and fourth guiding research questions:

- 3 What embodied processes are evinced by pupils engaged in figural pattern generalisation tasks?
- 4 In what ways do these embodied processes either assist or hinder pupils' ability to visualise figural cues in multiple ways?

⁴⁸ As outlined in Section 1.2.

The second purpose of this micro-analysis is to provide evidence of the central thesis of this study, namely that the combined complementary multiple perspectives of enactivism, figural apprehension and knowledge objectification add a powerful depth of analysis to the exploration of the inter-relationship between the embodied processes of pattern generalisation and the visualisation of pictorial cues.

VIGNETTE 1 – APPREHENSION TENSION

This first vignette describes the 3½ minutes Grant spent arriving at the expression $n+n+n+[n-1]$ through a process of operative apprehension⁴⁹. This was the fifth general expression Grant was able to arrive at for the given visual stimulus and, to place the expression in context, his visual apprehensions for the preceding four expressions are shown in Figures 5.1 – 5.4, all of which are illustrated for T_4 .

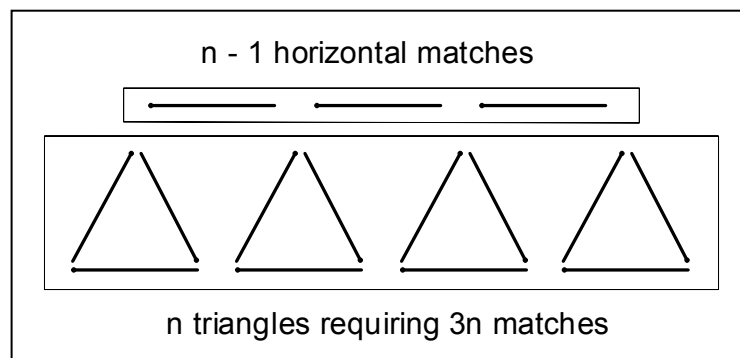


Figure 5.1 Grant's apprehension for the expression $T_n = 3n + n - 1$

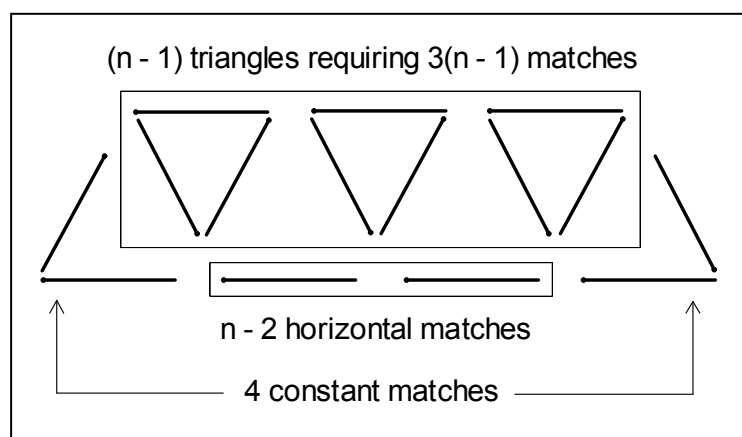


Figure 5.2 Grant's apprehension for the expression $T_n = 3(n - 1) + 4 + n - 2$

⁴⁹ See Section 3.4 for an overview of figural apprehension.

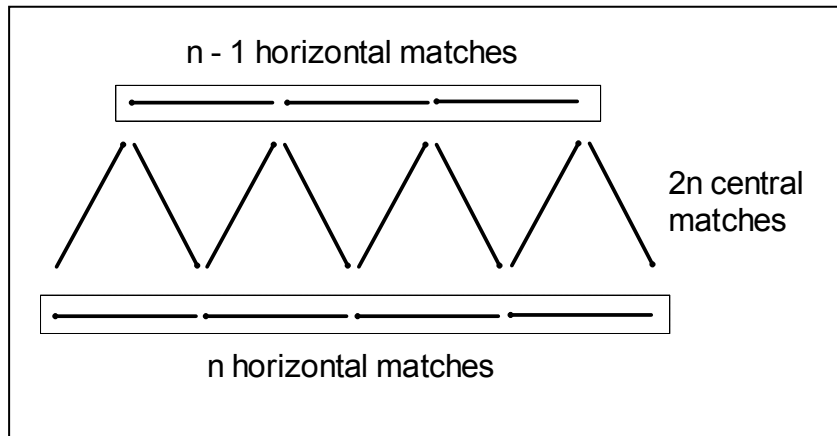


Figure 5.3 Grant's apprehension for the expression $T_n = 2n + n + n - 1$

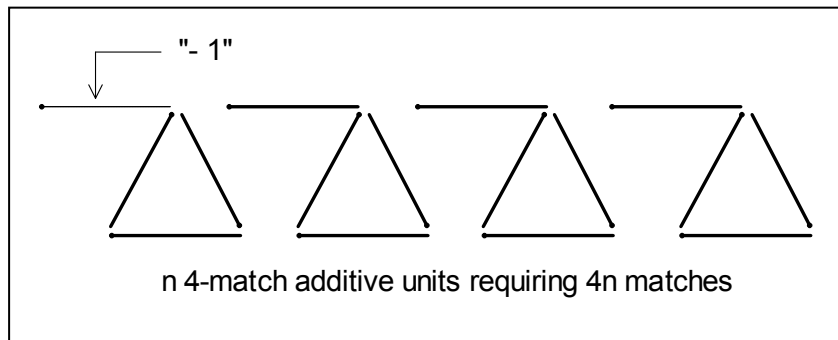


Figure 5.4 Grant's apprehension for the expression $T_n = 4n - 1$

Grant's initial apprehension (i.e. his *perceptual* apprehension) was in terms of overlapping upward-pointing and downward-pointing triangles – Term 3 containing a total of five such triangles and Term 5 containing nine. To arrive at a general expression for the n^{th} term from this apprehension would have required a relatively complicated correction for the overlapping matches. Rather than attempting this, Grant was able to subconsciously reconfigure the whole-part relation of the given pictorial terms into n upward-pointing triangles and an upper horizontal row of $n - 1$ matches (Figure 5.1), arriving at a general expression $3n + n - 1$. His second general expression (Figure 5.2) stemmed from a further reconfiguration of the figural cue into $n - 1$ downward-pointing triangles, $n - 2$ horizontal matches along the bottom, and a constant 2 matches at either end. Figure 5.3 shows yet another reconfiguration into an upper row of $n - 1$ matches, a lower row of n matches, and $2n$ matches for the central zigzag. The reconfigurations of the whole-part relation of the pictorial terms as shown in Figures 5.1 – 5.3 represent *operative* apprehension in Duval's nomenclature. The fourth general expression as shown in Figure 5.4 was arrived at

through a process of physically building the 4th term of the sequence using matchsticks, and thus represents *sequential* apprehension.

In order to arrive at his fifth general expression, Grant began by counting the forward-leaning parallel matches of Shape 5 from left to right. After a brief pause he then worked his way back from right to left counting the backward-leaning parallel matches. He then counted the remaining top and bottom matches in pairs, rhythmically alternating between top and bottom as shown in Figure 5.5: 1,2...3,4...5,6...7,8...9. Rhythm, whether in speech or gesture, is not merely the perception of order, it is “the demand, preparation and anticipation for something to come” (You, 1994, p. 363). There is thus an inherent sense of expectancy associated with rhythm, and it is seen as a crucial semiotic device in the process of generalisation (Radford et al., 2007, p. 522). This counting procedure was thus central in alerting Grant, whether consciously or unconsciously, to the non-paired match in the bottom row. Based on this counting procedure, Grant was able to arrive at the following general expression for the n^{th} term of the sequence: $n + n + 2n - 1$.

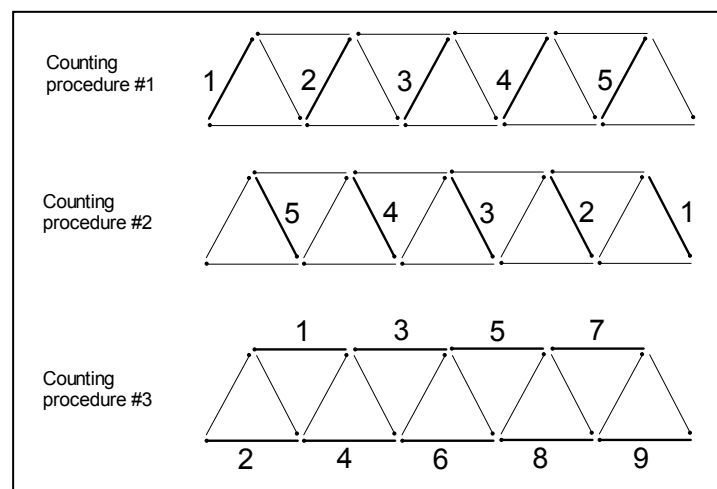


Figure 5.5 Grant's different counting procedures

Grant was able to justify his general expression ($n + n + 2n - 1$) by relating the $n + n$ portion to two sets of “parallel central matches”, while the $2n - 1$ he associated with what he referred to as the “outside matches”. Just prior to writing the $2n - 1$ part of the expression, Grant made use of *indexical* gesturing - he first gestured a horizontal line across the top of Term 5 and then a second horizontal line across the bottom of Term 5. He also made the comment that “it’ll always be one less on top”, use of the

word “always” performing a generative action function (Radford, 2000, p. 248) and thus aiding the notion of generality.

Interestingly, there seems to be a slight mis-match between the $2n-1$ portion of Grant’s expression and his indexical gesturing of the top and bottom rows of matches in Term 5 – the “outside matches”. As Grant wrote down the $2n-1$ expression he commented that he was just simplifying $n+[n-1]$. When asked to articulate how he was “seeing” it, he was insistent that he saw the structure as $n+[n-1]$, i.e., in terms of n matches along the bottom and $n-1$ matches along the top, and that the $2n-1$ portion of his expression was in fact an algebraic simplification of $n+[n-1]$. Grant went further to describe the $[n-1]$ as representing the “*top gap-filling matches*”, a metaphorical visualisation of the spaces between the inverted V-shapes created by the two central series of parallel matches. Grant then re-wrote his expression for the n^{th} term as $n+n+n+[n-1]$ which he explained as being a truer representation of his visual apprehension of the pictorial pattern.

There seems to be an interesting tension between two different modes of operative apprehension with figural modification having been accomplished by means of a recombination of various elementary figural units in two different ways. Although Grant ultimately presented the expression $n+n+n+[n-1]$ as being representative of his apprehension of the figural pattern under investigation, his initial formula was $n+n+2n-1$.

Both formulae suggest a sub-division of the “central” matches into n forward-leaning and n backward-leaning parallel matches. However, the remaining “outside” matches seem to be sub-divided differently in each of the two formulae. The initial formula ($n+n+2n-1$) suggests that the “outside” matches have been split into pairs - one match of each pair forming part of the upper horizontal row with its paired match positioned below it in the bottom row. The $2n-1$ “outside” matches in Grant’s initial formula seem to represent n pairs of matches, making $2n$ matches in total, the “ -1 ” being an adjustment required due to the right-most pair missing a match in the upper row. However, his final expression ($n+n+n+[n-1]$) suggests that the “outside” matches were in fact sub-divided into two distinct horizontal rows, with n matches along the bottom and $n-1$ matches along the top.

Grant maintained that in his initial expression he had written $2n-1$ as an algebraic simplification of $n+[n-1]$ which in turn is likely to have been a remnant of an earlier operative apprehension, that shown in Figure 5.3. However, the $2n-1$ is likely to have been inspired by his counting procedure shown in Figure 5.5, in which the rhythmical pairing of the top and bottom matches was central to alerting him to the non-paired match in the bottom row. Even though Grant's algebraic expression $2n-1$ retains what Radford (2002b) refers to as a *symbolic narrative*, this narrative by no means has a unique interpretation. Seen in isolation, the $2n$ portion of the expression could equally represent either 2 multiples of n matches (top and bottom horizontal rows) or n multiples of 2 matches (matches grouped in pairs). It was only through careful observation of the *activity* that mediated Grant's experience with the figural cues that the $2n$ portion of his general expression could be fully interpreted. Thus, what is important here is that the visual tension inherent in Grant's fifth algebraic expression only became apparent through a combined analysis of multiple semiotic means of objectification (linguistic devices, metaphor, gestures, rhythm, symbolic expressions) using a lens of figural apprehension within the context of figural pattern generalisation being viewed as a fully embodied process.

VIGNETTE 2 – PROCEDURAL RIGIDITY, UNTAPPED POTENTIAL & MANIPULATIVES

PART 1

This next vignette relates to a portion of the time Brian spent arriving at his sixth general expression. He seemed to be struggling to come up with a different way of "seeing" his structure.

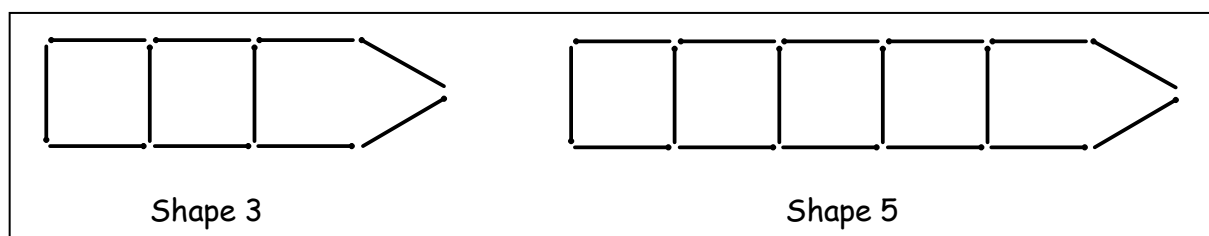


Figure 5.6 Pictorial pattern presented to Brian

Brian had an established strategy that related to taking a pictorial term and trying to identify in the diagram a particular feature or features that occurred the same number of times as the term number itself. So for instance in Term 5 he was intent on searching for structures that occurred 5 times in the diagram, and he found it

quite difficult to move away from this strategy. Brian articulated this feature of his approach to pictorial patterning as follows, in this instance specifically referring to the diagrammatic representation of the 5th term:

It's sort of like into my head that I must look at it to do with the 5 straight away. And then therefore I'll be looking at this 5 [indicating the bottom row of 5 horizontal matches] or this 5 [indicating the middle row of 5 vertical matches] or this 5 [indicating the top row of 5 horizontal matches].

Brian had got to a point where he felt he had exhausted this strategy (4 of his previous 5 apprehensions were based on this approach), but was finding it difficult not to be constantly drawn back to it. I suggested he consider using the physical matchsticks to build one of the pictorial terms. I made this suggestion in the hope that the process of physical construction would assist Brian in coming to see the pictorial terms in other ways. He was sceptical about this approach helping him:

The thing about when I'm drawing it and when I'm writing it down like that [indicating some of his previous written/drawn working], I can write down what I think and then it doesn't lose my mind, because if I'm just working with these matches like this ... and I move a match then it'll be like „hang on, where did that match just come from?“ if you see what I mean.

Despite this scepticism he nonetheless built Term 5 using matchsticks. At this point the process of construction didn't seem to assist him in any way and his focus remained on the 5 squares:

You see look, like, now I'm looking at this structure, right, and I'm seeing a whole lot of squares, and then that as if that was a square there [swivelling the top right oblique match in a clockwise direction so that it created a square out of the irregular pentagon on the right-most end of Term 5, as shown in Figure 5.7].

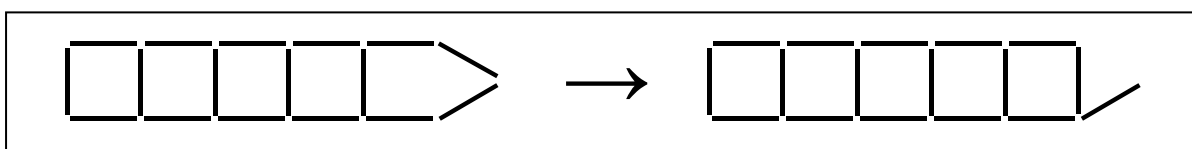


Figure 5.7 Brian's physical transformation of Term 5 (built from matchsticks)

He then swivelled the match back into its original position before deconstructing the pictorial term into the five parts shown in Figure 5.8(a). He then reconstituted the original Term 5 before slightly later splitting it into the two sections shown in Figure 5.8(b). This splitting was then subsequently adjusted by transferring the right-most vertical match of the left-hand section across to the right-hand section, as shown in

Figure 5.8(c). Each of these three arrangements was considered for a short time before being abandoned.

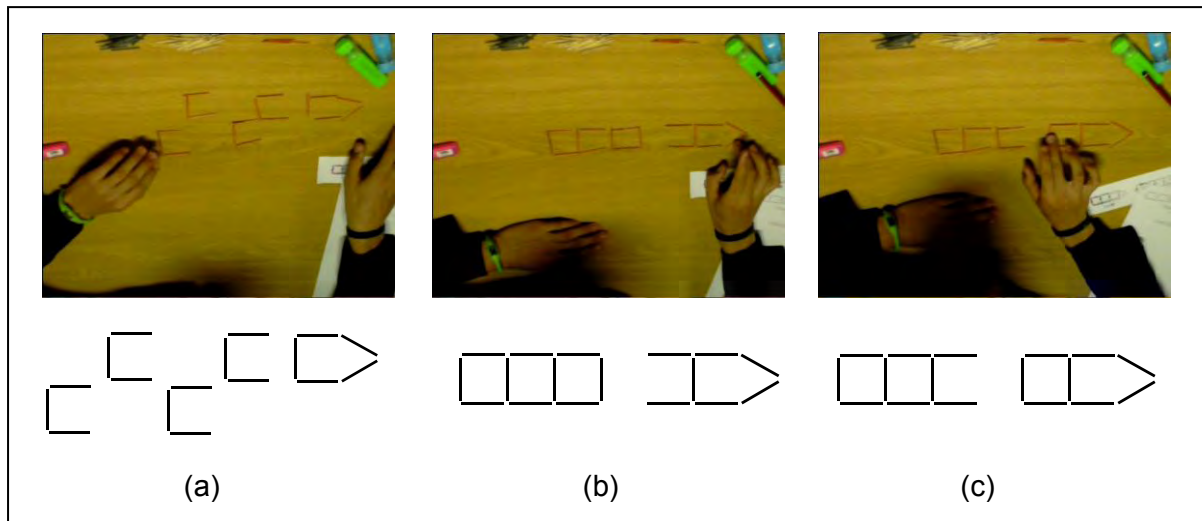


Figure 5.8 Brian's further transformations of Term 5

Brian then restructured his matchstick construction into that shown in Figure 5.9 (a) while commenting: *"Now I'm just having fun – I'm joking!"* A few seconds after he had completed this construction he had his "aha" moment. He suddenly dropped his pencil onto the table, muttered *"hang on!"* to himself and rearranged the matchsticks into the structure shown in Figure 5.9 (b), each triangle being constructed out of the three matches forming C-shapes as depicted in Figure 5.8 (a).

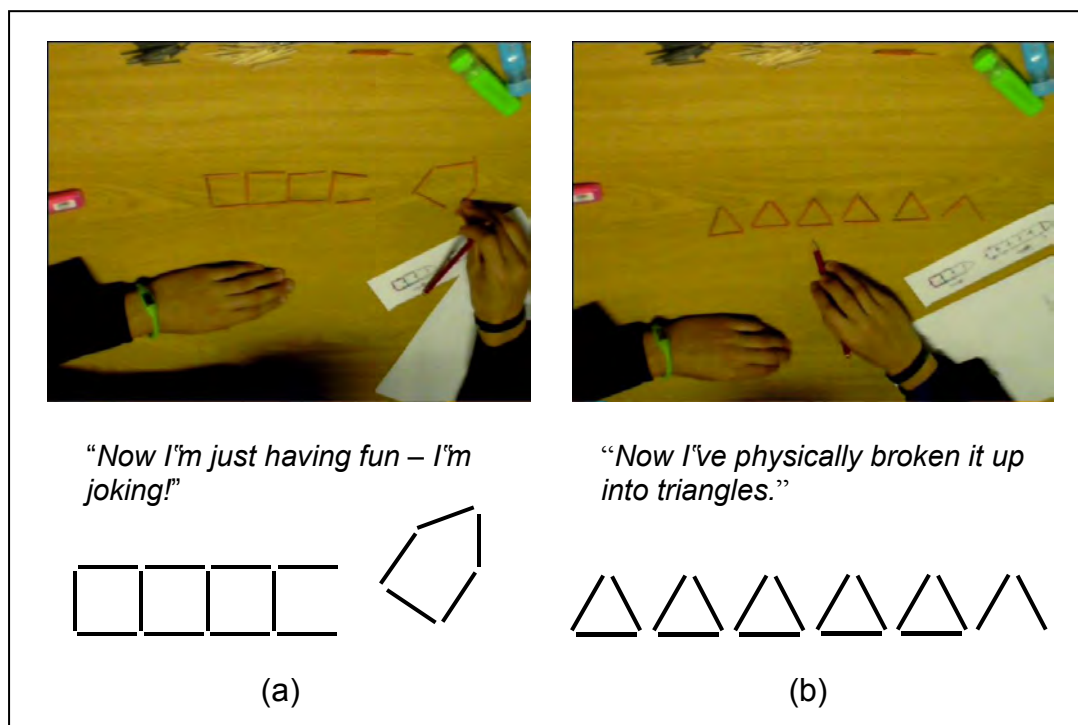


Figure 5.9 Brian's final transformations of Term 5

Now I've physically broken it up into triangles. (...) And then the triangles, they'll be the same amount as this [indicating the Term/Shape number] plus 1 [i.e. $n + 1$], but this triangle won't be completed [indicating the partial triangle at the far right] (...) But now, instead of there being 5 triangles, or 5 things, there's 6 over here, so, but this one isn't completed [indicating the missing base match from the partial triangle on the right]. (...) So now I'm gonna be saying $n + 1$ times 3 minus 1, I think.

Brian was thus able to arrive at the algebraic expression $3(n+1)-1$ as being representative of his particular apprehension. Despite Brian's initial scepticism of the usefulness of building matchstick structures of specific pictorial terms, it seems that it was precisely this process that ultimately led him to the evolution of a "new way of seeing". His arrival at the final apprehension (based on a transformation of the original structure into triangles) was an emergent process arrived at through unformulated exploration and unstructured interaction with the pictorial context, a process in which the body was engaged with mathematical play (Davis et al., 1996, p. 156).

An important aspect of this new apprehension was Brian's understanding of its generality, as this was critical in terms of his arriving at an associated algebraic expression for the general term. This grasping of a commonality and the awareness of its applicability to all terms in the sequence, i.e. its generality, is a crucial first step in the generalisation process. The next important step is the ability to articulate, through the use of signs and symbols, the commonality or generality noticed in the phenomenological realm. An analysis of the intermediate structures that Brian created en route to his final apprehension reveals a wealth of untapped potential. Although each of these structural arrangements was briefly considered, they were all ultimately discarded as being unhelpful to the generalisation process. What this means is that Brian was unable to bring forth any sense of generality from his matchstick structures. In each matchstick arrangement, however, there are nonetheless sub-configurations that could potentially trigger an awareness of generality. Figure 5.7, which represents Term 5, could be seen to contain n squares. Since each square comprises 4 matches, this would require a total of $4n$ matches. However, since there is an overlap between adjoining squares we need to subtract $(n - 1)$ matches from this count. Finally, the constant oblique match on the far right would need to be added to the tally, giving a final algebraic expression of

$T_n = 4n - (n-1) + 1$. Figure 5.8(a) could be seen to contain a constant 5-match irregular pentagon unit on the far right, and $(n-1)$ multiples of 3 matches, thus yielding a final expression $T_n = 3(n-1) + 5$. Figure 5.8(b), in terms of Brian's indexical gesturing of the C-shapes while counting the matches in the left-hand structure, could potentially be seen to contain $(n-2)$ C-shapes, each requiring 3 matches, plus a single match to close off the left-hand structure, plus a constant 7 matches on the right. This could yield the algebraic expression $T_n = [3(n-2) + 1] + 7$. Alternatively, if the structure on the left was seen in terms of $(n-2)$ squares with $(n-3)$ overlaps, then the formula $T_n = [4(n-2) - (n-3)] + 7$ could potentially have been arrived at. Finally, Figure 5.8(c) could have been interpreted as containing $(n-2)$ C-shapes on the left, each requiring 3 matches, and a constant 8 matches on the right. This interpretation could be represented by the expression $T_n = 3(n-2) + 8$.

There is thus a wealth of untapped potential in Brian's various matchstick constructions and deconstructions of Term 5. Although Brian was able to create a host of differently structured constructions, he was unable to bring forth any sense of generality in them. This raises a critical concern: What pedagogical strategies could be employed to assist pupils in becoming aware of structural commonalities and generalities in sequences of pictorial terms? This issue is returned to in the concluding chapter.

PART 2

The second part of the vignette relates to Brian's final apprehension and its associated algebraic expression $T_n = 4(n+1) - (n+2)$. Once again it was the physical use of manipulatives, in this case matchsticks, that played a fundamental role in the evolution of a "new way of seeing".

In Part 1 of this vignette Brian had used orange matchsticks to build his various arrangements of Term 5. In his explanation/justification of his general expression he incorporated a blue match into the diagram shown in Figure 5.9(b), the blue match indicating the extra match required to complete the final triangle and thus the extra match that needed to be subtracted from his final count. It was the use of a second

colour in his matchstick construction that was the pivotal moment that sparked the gradual development of his final apprehension.

From the construction shown in Figure 5.10(a), Brian rebuilt the original Term 5. However, he incorporated the blue match into his construction, commenting as he did so: *“I think maybe I should incorporate this dead match a bit more”*. His description of the blue match as a *“dead match”* is interesting. This description was probably inspired by the dark blue colour of the match giving the appearance of it having been used – i.e. a burnt-out match. However, this metaphorical imagery may well have scaffolded his later reasoning process where the incorporation of a number of blue matches allowed the creation of a different structural arrangement, but where each of these blue matches was in fact a *“dead match”* and thus didn’t count towards the final tally.

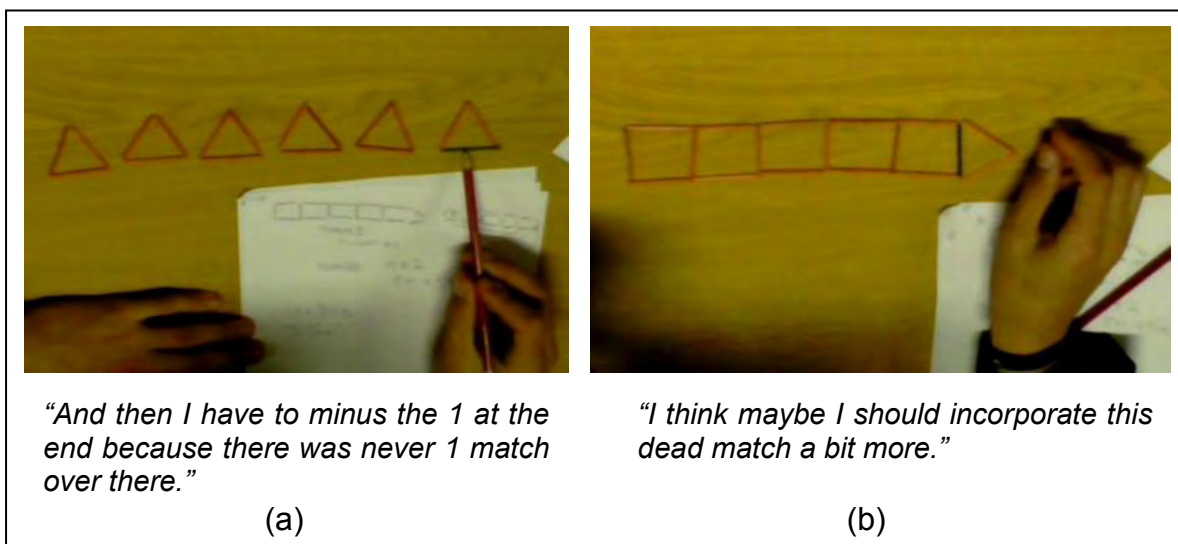


Figure 5.10 A pivotal moment – the incorporation of a *“dead match”*

Brian then began dismantling the structure shown in Figure 5.10(b) into 5 C-shape units similar to those shown in Figure 5.8(a). When he got to the final 3 matches (i.e. the blue match and the 2 oblique matches forming the triangle on the far right) he picked up the blue match, kept it in his fingers, and pushed the 2 oblique matches to one side (*“take that away”*). Instead of rearranging the 3 matches of each of the C-shapes into a triangle, he then added a blue match to each C-shape, thereby creating 5 squares (*“What if I (...) had to make it into squares now instead of triangles?”*). He then realised he still needed to incorporate the 2 oblique matches that he had pushed to one side (*“Okay, these 2 still have to...”*), but in a flash of

inspiration decided to create a square out of them as well (“...but maybe I should make a square out of this one as well”). The final structure thus created is shown in Figure 5.11 where, in the diagrammatic version, the blue “dead” matches are represented by dotted lines. This structural arrangement allowed him to arrive at the algebraic expression $T_n = 4(n+1) - (n+2)$ which he justified in terms of there being $n+1$ squares, each requiring 4 matches. However, of these $4(n+1)$ matches $(n+2)$ were blue and thus needed to be subtracted to give the correct final count. In his explanation of his formula, Brian placed a pencil between the 5 squares on the left (each of which contained a single blue match) and the single square on the right (which contained 2 blue matches). Of the 6 squares in the diagram, 5 are on the left of the pencil while 1 is on the right. Of the 7 blue matches in the diagram, 5 are on the left of the pencil while 2 are on the right. The number of squares on the left of the pencil matches the Term number (Term 5 in this case), as do the number of blue matches. Thus, more generally, of the $n+1$ squares in the diagram, n are on the left of the pencil while “+1” is on the right. Of the $(n+2)$ blue matches in the diagram, n are on the left of the pencil while “+2” are on the right. This physical splitting of Term 5 into two sections is likely to have reinforced his conviction of the *generality* of his apprehension.

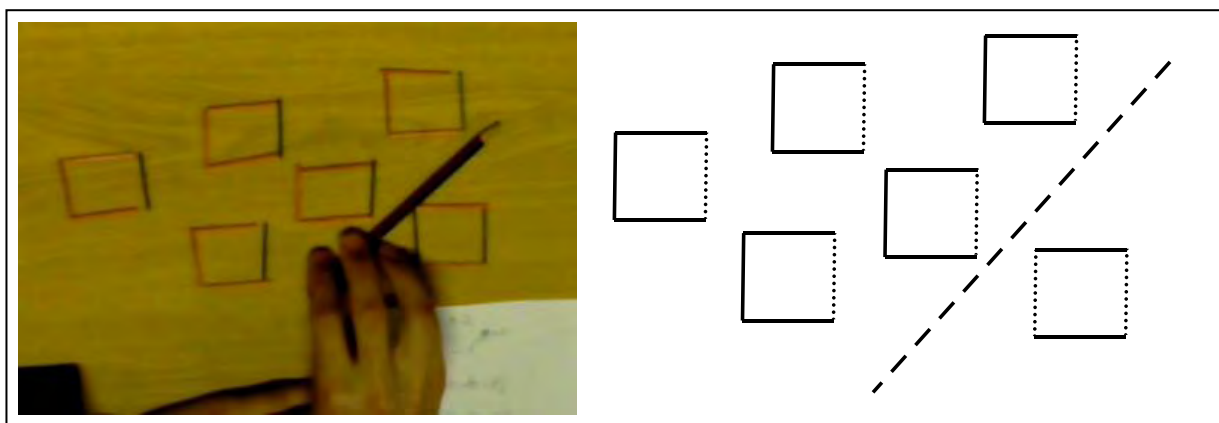


Figure 5.11 Brian's final transformation of Term 5

A particularly interesting aspect of this particular apprehension is that at no point does Brian talk about “overlapping” squares. Thus, his visualisation is not one in which the blue matches need to be subtracted because they represent *overlaps* and have thus in effect been counted twice. Rather, they need to be subtracted by virtue of being additional matches that have been added to the overall structure to create

structural regularity, but which don't count towards the final tally. It is unlikely that this particular apprehension would have been brought forth without the process of arranging and rearranging physical manipulatives of different colours, since not only would the visualisation have required a dynamic element to form the final square from the two oblique matches, but it would have required a visual awareness of the fact that the overlap between the right-most pair of squares required a *double* subtraction.

VIGNETTE 3 – TENSION BETWEEN LOCAL AND GLOBAL VISUALISATION

PART 1

The following vignette attempts to capture the tension between local and global visualisations as evidenced by Terry's generalisation activity.

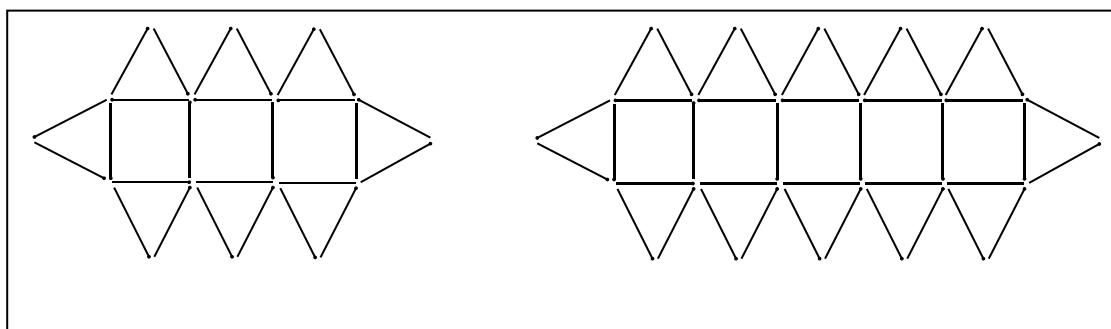


Figure 5.12 Pictorial pattern presented to Terry

Terry was presented with the pictorial terms shown in Figure 5.12. After staring at the two terms for a few seconds he remarked:

It seems like it's basically just adding on the same sort of thing again, every time, and then just finishing it off with that [indicating the right-most > shape].

After making this remark he carefully drew Term 4 in a very structured manner. He began by drawing the 3 matches of the left-most triangle. Thereafter he drew the middle section of the structure in a very rhythmic fashion: 4,5,6 ... 7,8,9 ... 10,11,12 ... 13,14,15. Interestingly, instead of drawing each group of three matches in the flowing form of a backward C-shape (top match, then vertical match, then bottom match) which would have been slightly more economical, he instead very methodically drew each group of 3 matches by first drawing the top horizontal match, then the bottom horizontal match, and then finally the vertical match. To check that

he had drawn the correct number of matches for Term 4 he then carefully counted the four “squares” in the diagram before adding on the two oblique matches on the far right. After completing the middle section of the diagram he then drew a series of inverted V-shapes across the top of the structure (rhythmically drawing them in pairs from left to right) and then finished the diagram by drawing a series of V-shapes along the bottom of the diagram, once again drawn in rhythmic pairs from left to right. This drawing procedure is shown in Figure 5.13.

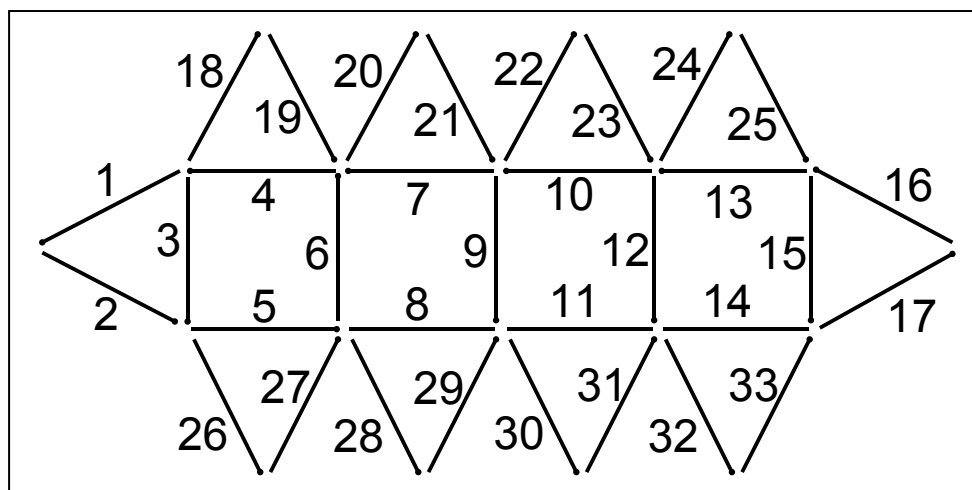


Figure 5.13 Terry's drawing procedure for Term 4

Based on this drawing procedure Terry was able to arrive at the formula $7n + 5$:

So you started off with your little triangle [indicating the left-most < shape] so that's obviously +2, then you finish it off with a little triangle again [indicating the right-most > shape], plus another 2, so it's +4; and then however many things in between, just work out how many it is for that [indicating the 7-match additive unit] (...) well how many it is for the top triangle, bottom triangle and then plus that one [indicating the vertical match connecting the top and bottom triangles in the 7-match additive unit].

At this point Terry wrote down the formula $7n + 4$. However he quickly realised that he had not taken into account the first vertical match from the left:

Shape 3 is Term 3, it's got 3 of these little squares like that [pointing to the 3 central squares of Term 3] and Shape 5 has 5. So then you've got these 2 [indicating the left-most < shape] so you start off, there's +2, there's another 2 [indicating the right-most > shape] plus 4, then you've already got [points to the left-most vertical match and realises he has missed it out] – oi! (...) With the front triangle [indicating the triangle formed from the left-most < shape and the left-most vertical match] it's the full triangle that

you're starting off with, so it's 1, 2, 3 matchsticks. With the end one you've already got the (...) base of the triangle coming from the previous square.

Terry thus gave his final formula as $7n+5$. Interestingly he later altered it to $3+n(3+3+1)+2$ as being more representative of how he was visualising the pictorial context. Both expressions represent sequential apprehension since it was a process of construction (first done mentally and subsequently physically) that brought forth the 7-match additive unit as a recurring structure. In the altered version of the formula the 7-match unit is further sub-divided into a “top triangle”, a “bottom triangle” and a vertical line.

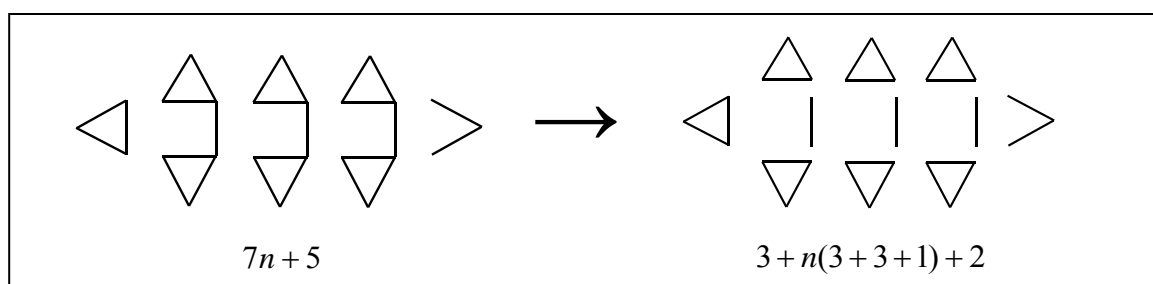


Figure 5.14 Two different formats of Terry's first visualisation

An interesting aspect of Terry's discussion is his frequent reference to *squares* in the pictorial terms. He makes express reference to the fact that the n^{th} term in the sequence would contain n “*little squares*”. In addition, when he initially drew Term 4 he did so by drawing the central structure first and then checking that he had drawn the correct number of matches by quickly counting the 4 squares. However, these squares don't feature anywhere in either of the two versions of his initial formula. In fact, the horizontal matches from these squares are seen to form part of an upper and lower triangle. Thus, after visually deconstructing the diagram into triangles, the squares become negative space as the matches that originally formed them have been apportioned to different component parts. Nonetheless, Terry continued to refer to them as a helpful structural unit. There are two possible reasons for this that are worth considering. Firstly, the ultimate aim of the patterning task from Terry's perspective is to arrive at an algebraic expression for the general term through a process of visualisation. It is possible that this goal had an unconscious influence on the visualisation process since some visualisations would be more algebraically useful than others – e.g. squares would overlap and a correction would thus be necessary for the resulting overcount. A second possibility is that there is a tension between *local* and *global* aspects of the pictorial context. *Local* considerations focus

on the additive unit by virtue of attention being focused on the step-by-step process of constructing the next term from the previous one. It is possible that this local focus obscured a more global outlook where the structural unit of a square could be properly incorporated into the general expression.

PART 2

We now move to Terry's 7th general formula. After silently and motionlessly staring at the two printed pictorial terms for a few minutes he made the following comment, referring to the printed Term 3:

What I'm trying to do now is almost use the squares. So now instead of having that sort of backwards C, actually have a full-on square (...) that gets connected to another square [gesturing to the right with his pencil], that then, just got to take out that one [indicating the overlapping match between two squares], and gets connected to another square [making multiple gestures further to the right with his pencil].

While Terry was explaining his strategy he made a number of crucial semiotic means of objectification. The first of these was his gesturing to the right while saying the words *"that gets connected to another square"*. This indexical or deictic gesturing was specifically related to the printed Term 3 which Terry had in front of him. He thus used the gestures to signify existing physical structures in the particular diagram he was looking at (Figure 5.15(a)).

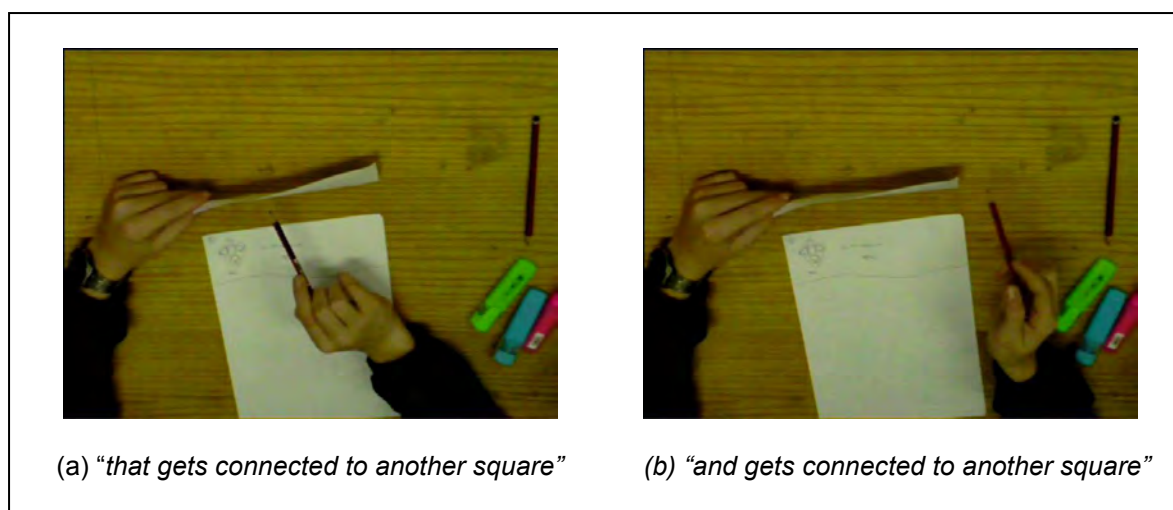


Figure 5.15 Terry's indexical gesturing

His second set of gestures, accompanying the words *"and gets connected to another square"* mark a transition from *existential* signification to what Sabena et al. (2005, p.

134) refer to as *imaginative* signification. This second set of gestures moves from indicating materially instantiated aspects of the pictorial term to miming an ongoing sequence of connected squares, squares that are not yet materially present. We thus see a progressive distancing from the physical referent (Figure 5.15(b)). Another important aspect of Terry's objectification process is his use of the words "*another square*". These words serve an important generative action function in terms of objectifying the generality of the interconnecting squares through an imaginative conception of iterative potential action. This linguistic device supports the process of objectification by allowing the recursive addition of squares to be "...repeatedly undertaken in thought" (Radford, 2000, p. 248).

Terry then went on to draw Term 4. Interestingly, the order in which he drew the various lines (Figure 5.16) didn't seem to correlate to his description of overlapping squares. His drawing process instead seemed to suggest a subdivision into a triangle at either end, two rows of horizontal matches, a row of vertical matches, and V-shapes at the top and bottom.

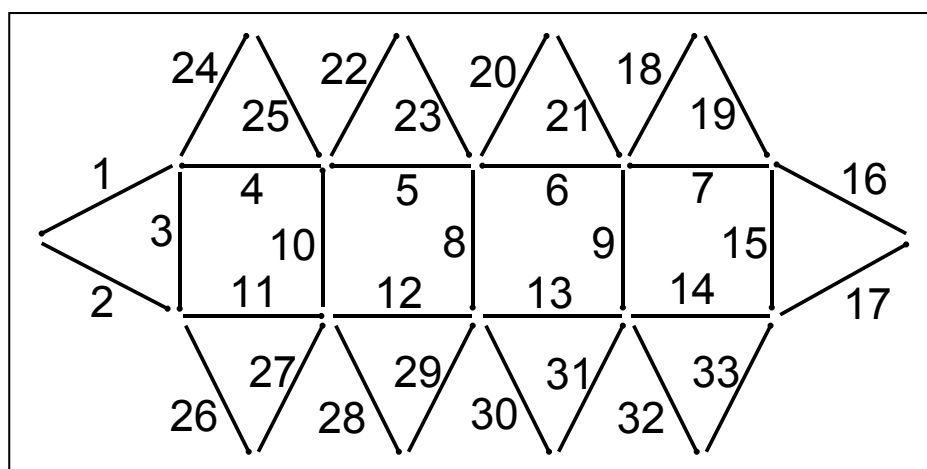


Figure 5.16 Terry's drawing procedure for T_4

After completing his drawing of Term 4, Terry sat staring at it for just over a minute before commenting: "*I had something and now I've, I had something else but now I've lost it*". It thus seems that his initial idea of using overlapping squares came from a flash of insight that has now receded. It is possible that this may have been at least partially precipitated by Terry's drawing of Term 4 in a manner which didn't mimic his initial visualisation of overlapping squares. In this instant it is possible that the drawing process itself obfuscated the visual apprehension. However, another

interpretation of the data could suggest that the drawing process actually reflects a competing, albeit unconscious, visualisation of the pictorial terms, thus suggesting an underlying visual tension between two different apprehensions of the pictorial context.

Terry then came up with the formula $3 + n(4 - 1) + 4n + 2$. The 3 at the beginning of the formula represents the starting triangle on the far left while the +2 at the end of the formula represents the > shape at the extreme right of each term. Terry described the $n(4 - 1)$ portion of his formula as representing “each square minus the one that’s being taken up by either the previous one or the next one” while the $4n$ is required for “the triangles above and below it”. Although he specifically refers to “squares”, these structural features are not reflected in his present visualisation. Although his initial visualisation was suggestive of overlapping squares, he has in essence reverted to a previous visualisation (the third in Table 5.11) in which the central structure is seen not in terms of overlapping squares but rather in terms of a series of backward C-shapes as shown in Figure 5.17.

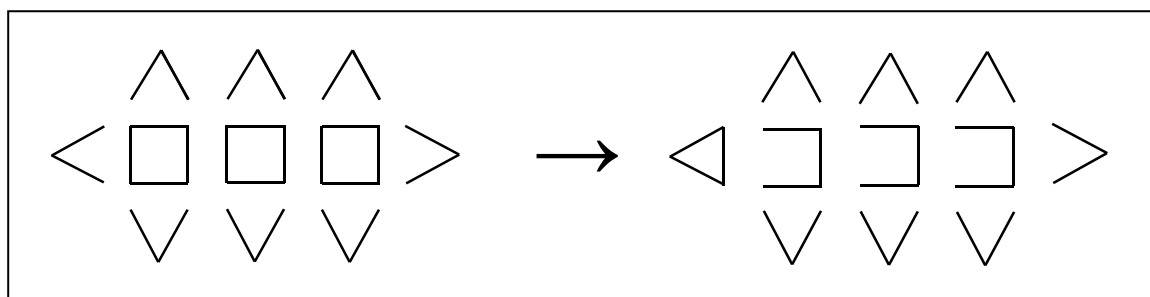


Figure 5.17 Terry’s change in visual apprehension

A possible explanation for this reversion to an earlier visualisation is that Terry’s focus on the recursive nature of the construction process supported a *local* generalisation but not a *global* one. A global generalisation/visualisation would entail seeing the structure in a holistic manner as being composed of a series of n overlapping squares. Since 4 matches are needed for each square, the n squares would require a total of $4n$ matches. However, this would lead to an overcount since overlapping would mean that some matches would in effect have been counted twice. To correct for this we would need to subtract $n - 1$ matches from the tally since n overlapping squares would have $n - 1$ overlaps. However, Terry’s constant

focus on a recursive, step-by-step process of construction is somewhat incompatible with this global view. To proceed from one term to the next would require the addition of a square and the removal of the overlapping match *each time*. The addition of a whole square each time thus becomes a redundant process if the overlapping match is immediately removed since the process could be accomplished in a far simpler manner by just adding on 3 matches in the form of a backward C-shape each time, thereby avoiding the unnecessary removal of the overlapping match. It is this focus on a stepwise process of construction that is likely to have contributed to the initial visualisation of overlapping squares being transformed into a visualisation of backward C-shapes.

At this point I asked Terry what had happened to his initial idea of focusing on the squares:

I don't know, I had something ... I was busy looking at it and something hit me and then I lost it. I noticed something to do with n minus, open brackets n minus 1, and then that in brackets [i.e. $(n - (n - 1))$], that had something to do with it, but I cannot for the life of me remember what it was.

Terry's reference to his noticing something to do with $(n - (n - 1))$ doesn't initially seem to make any sense as the expression simplifies to $+1$. However, it retains an interesting remnant of his initial visualisation in which there are n overlapping squares with $n - 1$ overlaps. At my suggestion he continued to pursue his initial idea. After staring at the diagrams for about half a minute he commented:

I think I might have found it ... So what I'm trying to do is now, is almost separate it so you've got, you just put all the squares together (...) and then take out this extra match right at the end [pointing in turn to each of the 3 overlaps between the 4 squares in Term 4].

This marks the crucial moment when Terry changes from a local to a global visualisation and is thus able to make sense of, and articulate, his initial fleeting visualisation. After trying to incorporate $(n - (n - 1))$ into his general expression he eventually abandoned it and came up with the final formula $2 + 4n - (n - 1) + 4n + 2$:

That works, then you've got your 2 that starts it off [indicating the left-most < shape], your 2 that finishes it off [indicating the right-most > shape], you've got your 4 for each square, then the $-(n - 1)$ (...) for each square there's an extra line except for the first (...) then $+4n$ for each triangle above and below it.

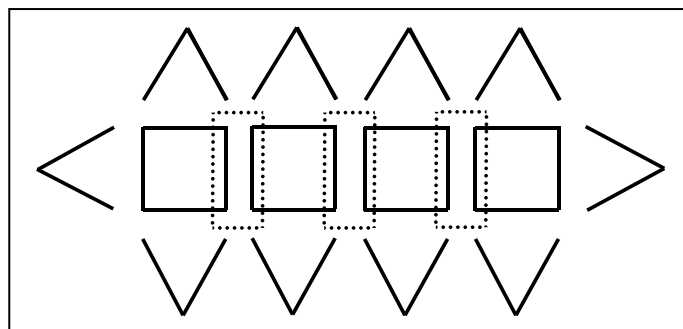


Figure 5.18 Terry's final apprehension of overlapping squares

PART 3

Terry's generalisation process en route to his 8th formula $(4 + n(5 - 1) + 3n + 5)$ also reveals interesting tensions between global visualisation and a more local visualisation based on a stepwise process of construction.

My eye keeps on getting drawn to these little intersections [indicating the gap or space created by the convergence of the five lines shown in Figure 5.19(a)]. (...) all the open almost sort of dots in between [referring to the white dot created at the point of intersection of the five lines] (...) and there's the little space between, where, almost where they all come together. (...) I keep on noticing the same sort of 5-point pattern. I've actually seen it a few times while doing this. (...) „Cos where they come together it makes a shape like that [drawing the 5-line structure shown in Figure 5.19(a)]. So then, actually adding, the idea is adding on one of those [indicating the 5-line structure he had just drawn] every time, and then you just subtract this little overlap there [indicating the right-most horizontal line in Figure 5.19(a)]. „Cos then you've got your, can then just add on little full triangle at the bottom each time. So then you've - but one thing you do is you need to start off with 4 [referring to the 4 solid lines in Figure 5.19(b)] because you need 1 to start off that original triangle at the top [pointing to the highlighted line in Figure 5.19(b)].

Using this reasoning he was able to arrive at the general formula $4 + n(5 - 1) + 3n + 5$. Although the expression is partially incorrect (it gives an overcount of 4 matches since the $+5$ at the end should in fact only be $+1$), Terry was convinced of its correctness⁵⁰. He explained the various parts of the formula as follows:

The original 4 is for, okay, the starting three which you always need, then also adding on that line [pointing to the highlighted line in Figure 5.19(b)], „cos that's the one thing that that bit doesn't have. (...) Then the 5 minus 1 [referring to the $n(5 - 1)$ portion of his formula] comes from, you've got these 5 lines [indicating the 5-line structure shown in Figure 5.19(a)] and

⁵⁰ This conviction may in part have been supported by a minor mental miscalculation which resulted in Terry concluding that the formula gave the correct count for Term 2.

you still want to get rid of that one [indicating the right-most horizontal line of the 5-line structure], the one that comes underneath the triangle from either the previous one or the next one, so it makes almost something like that [draws the 4-line structure shown in Figure 5.19(c)], then plus another one, like that [draws a second 4-line structure to the right of the first one], then continuous [gesturing to the right of the two 4-line structures he had just drawn]. $3n$ is for the „cos with each of these [indicating the 4-line structure shown in Figure 5.19(c)] you need to add on the little bottom triangle, just to finish off the pattern. (...) Then you've got to finish off with actually 5 now instead of 4 [the 5 lines shown in Figure 5.19(d)]. (...) So right at the end it doesn't have a next one to give it that [indicating the right-most horizontal line in Figure 5.19(a)], that line at the bottom, so it needs, you need to give that to it, so it'll be $4 + n(5 - 1) + 3n + 5$.

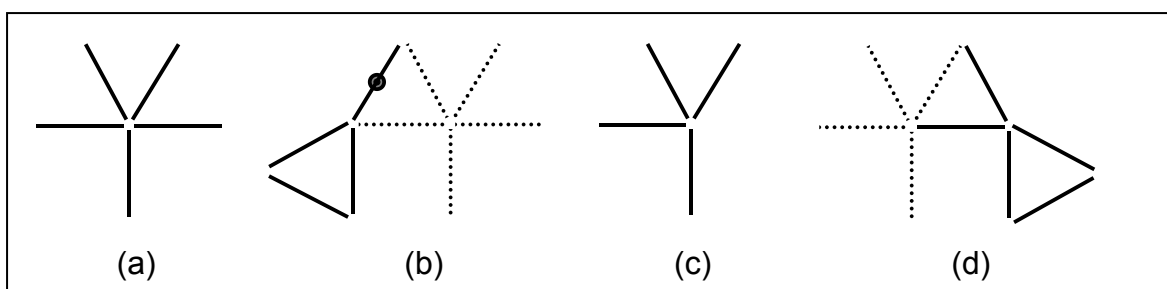


Figure 5.19 Structural aspects of Terry's 8th visualisation

Terry's words as well as actions once again reveal subtle underlying tensions between local and global visualisations. His initial observation is of the white dot created at the interstices of repeating groups of 5 matches. This observation is of a global feature of the pictorial terms which was gradually brought forth by his interaction with the figural cue ("I keep on noticing the same sort of 5-point pattern"). From this global observation he then suddenly moves to a local visualisation by deciding to make use of this structural feature to *construct* the pictorial pattern in an *iterative* manner ("the idea is adding on one of those every time"). The words "every time" serve an important generative action function in terms of objectifying the generality of the structural unit, but they also tend to focus attention on the recursive nature of step-by-step construction, thus drawing attention away from a more holistic view of the overall general structure. This is unfortunate, as Terry is now drawn to the repeated overlap which will need to be corrected for each time (*and you still want to get rid of that one, the one that comes underneath the triangle from either the previous one or the next one, so it makes almost something like that* [draws the 4-line structure shown in Figure 5.19(c)], *then plus another one, like that* [draws a second 4-line structure to the right of the first one], *then continuous* [gesturing to the right of the two 4-line structures he had just drawn]). At this point Terry has shifted

almost entirely from a global to a local visualisation. The original 5-line star structure has now been reduced to a 4-line structure which is added on to each previous term in a recursive manner. Interestingly, his description of the overlapping line as being “the one that comes underneath the triangle from either the previous one or the next one” seems to contain vestiges of a more global visualisation, where the overlapping line is considered to belong partially to both adjacent 5-line structures.

This shift from an initial global observation to a more local treatment of that initial observation may in part have led Terry to miscalculating his final formula, which gives an overcount of 4 matches (Figure 5.20). A more global structural understanding of his initial apprehension may well have been more helpful in arriving at a correct final formula. Viewed from a global perspective one could argue that one would need 4 matches on the far left, n multiples of the 5-match star shape (requiring $5n$ matches), n triangles along the bottom (requiring $3n$ matches), and a final subtraction of $n - 1$ matches to correct for the overlapping 5-match star shapes. The final triangle at the extreme right could then be created from the 2 right-most matches of the final 5-match star shape without the addition of any additional matches. This would yield a correct final formula of $4 + 5n + 3n - (n - 1)$ as shown in Figure 5.21. Thus, although both local and global visualisations can be useful in their own particular way, it is likely that the process of objectifying and articulating an appropriate algebraic expression for the general term is complicated when tension exists between these two modes of visualisation.

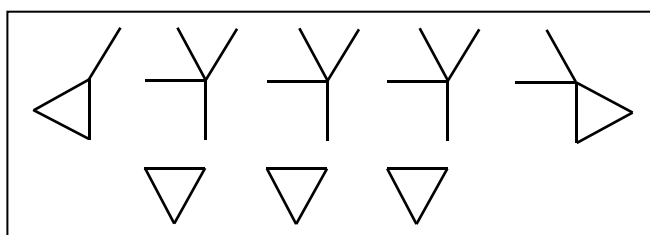


Figure 5.20 Terry's 8th apprehension ($T_n = 4 + n(5 - 1) + 3n + 5$), giving a 4-match overcount

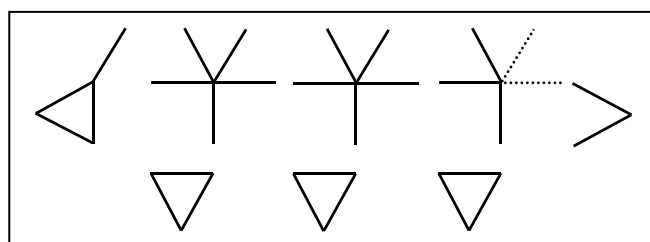


Figure 5.21 A proposed global visualisation: $T_n = 4 + 5n + 3n - (n - 1)$

VIGNETTE 4 – OBJECTIFICATION THROUGH THE PROCESS OF DRAWING

This short vignette describes the genesis of an idea and its development into a stable generalised visualisation. The focus lies on the drawing procedure that Terry employed while drawing Term 4 of his pictorial sequence. Terry began by staring at the two printed pictorial terms for about half a minute without saying or doing anything. He then said:

Okay, I'm gonna try something else. It's probably just a variation of the second one that I found [i.e. his expression $3 + n(3 + 3 + 1) + 2$ as shown in Figure 5.23] but I just wanna see if it takes me somewhere.

Terry then drew Term 4 of the pictorial sequence with each match being drawn in the order shown in Figure 5.22:

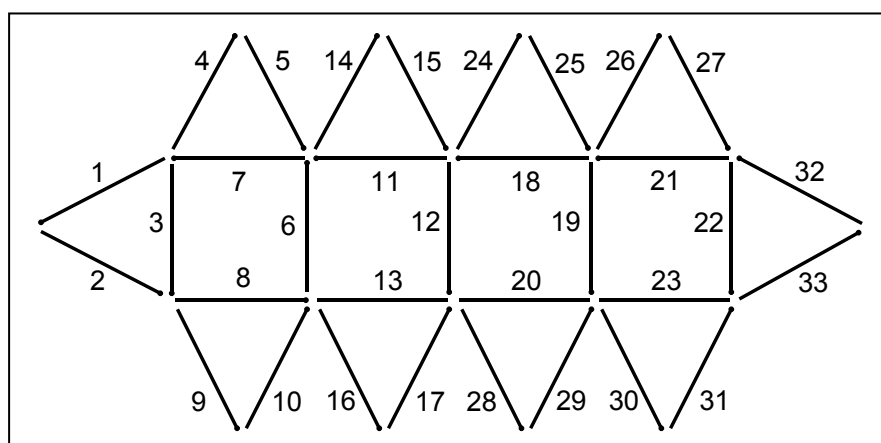


Figure 5.22 Terry's drawing procedure

He began by drawing the first 3 matches, what he had previously referred to as the "front triangle" with which the pictorial term begins. The next 7 matches he drew somewhat haphazardly. He began with the 2 matches for the top inverted V-shape, he then drew in the vertical match followed by the upper and lower horizontal matches, and finally the remaining 2 matches at the bottom. Although a little haphazard, the order in which these 7 matches were drawn retains elements of an earlier visualisation, specifically the single vertical match and the lower triangle. Terry's drawing of the next 7 matches displays the initial development of his idea, and we see the subdivision of the 7 matches into 3 groups. He began by drawing the central 3 matches which form the shape of a backwards C. This was immediately followed by the 2 matches needed for the upper inverted V-shape and the 2 matches needed for the lower V-shape. Rather than repeating this process for

the next unit of 7 matches, Terry instead drew in the next 2 backward C-shapes. He then completed the drawing by returning to the 3rd backward C-shape and adding in the V-shapes above and below it, and then repeating the process with final pair of V-shapes associated with the final backward C-shape. Here we see how Terry's earlier subdivision of the 7-match additive unit into upper and lower triangles and a single vertical match has been subtly rearranged into a backward C-shape with an associated pair of V-shapes, one above and one below. More than this, however, we also see a developing awareness of the generality of this structural rearrangement. By closely attending to the order in which the various matches were drawn we are able to see the development of a general visualisation objectified through the process of Terry's physical drawing of a pictorial term, and the transformation of one apprehension into another as shown in Figure 5.23.

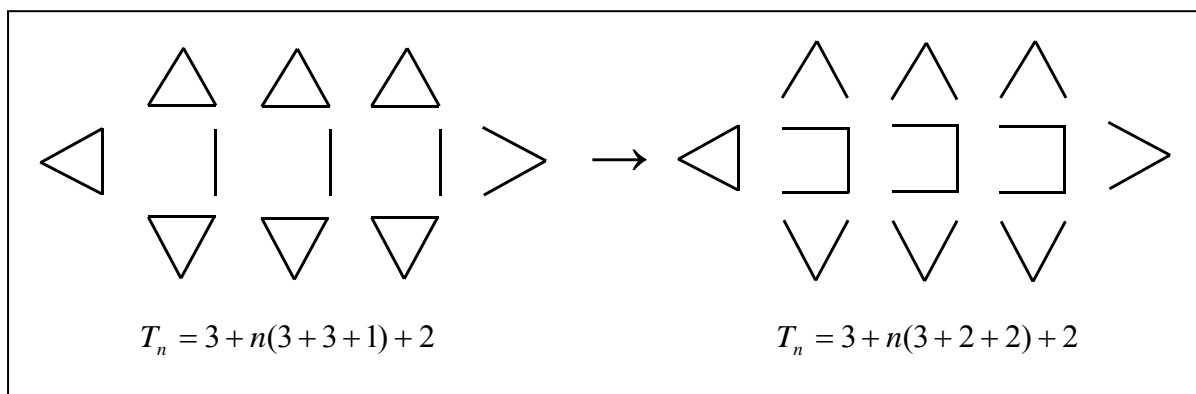


Figure 5.23 Terry's change in apprehension

In terms of Duval's nomenclature, this structural rearrangement of the whole-part relationship represents operative apprehension where the figural modification is *mereologic*. For Terry, the recombination of the various elementary figural units was a conscious decision, purposefully undertaken (*"I'm gonna try something else. It's probably just a variation (...) but I just wanna see if it takes me somewhere"*). There is thus an important distinction to be made here between operative apprehension and sequential apprehension (in which the emergence of sub-figures stems from the process of constructing the perceived figure). In Terry's drawing of the term shown in Figure 5.22 the structural rearrangement had already begun as a conscious effort prior to the commencement of his drawing process. This drawing process did not result in the *emergence* of elementary figural units (e.g. the backward C-shape, the V-shape, and the inverted V-shape) but rather played a crucial role in *stabilizing* the particular visualisation and in *developing an awareness of its generality*.

VIGNETTE 5 – UNECONOMICAL COUNTING AND THE ALLURE OF DIFFERENCING

This short vignette focuses on the idea of uneconomical counting as an indicator of perceived structure, as well as highlighting the allure of the rate-adjust or differencing method.

When presented with her pictorial pattern for the very first time, Kelly counted the matches in Term 3 in the manner shown in Figure 5.24(a). Immediately upon completion of this counting procedure she double-checked her tally by re-counting the matches. However, she now used a very different counting technique (Figure 5.24(b)). In both cases she counted aloud while pointing to each match in turn with her pencil. She then went on to count the total number of matches in Term 5 using the second of these two counting procedures.

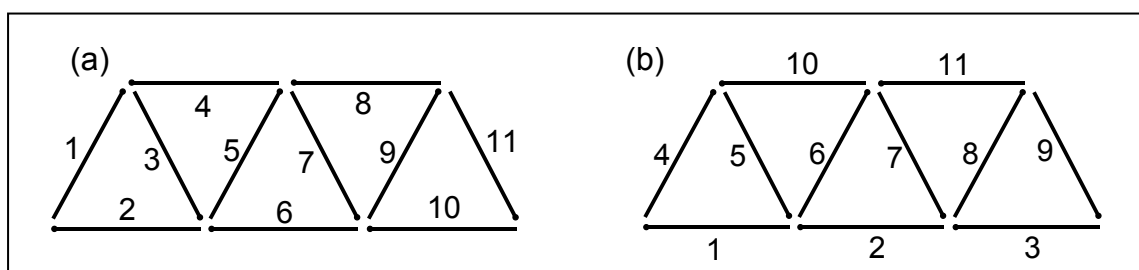


Figure 5.24 Kelly's different counting procedures

The first of these two counting procedures I would characterise as being *economical* in the sense that it utilises the minimum amount of time and energy to individually count each match. The second procedure I would characterise as being *uneconomical* since it requires significantly more time and energy to accomplish. This can readily be understood in terms of the overall path of the pencil as traced in the two counting procedures (Figure 5.25).

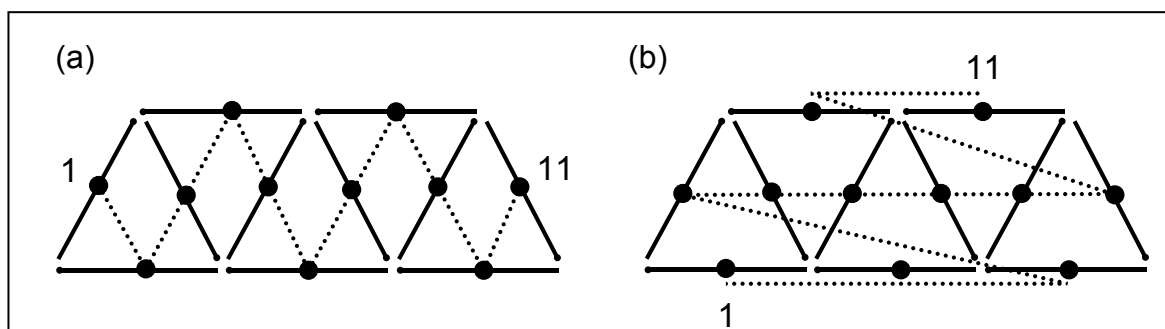


Figure 5.25 Traced paths of Kelly's different counting procedures

The first counting method traces a continuous zigzag path through the 11 matches from left to right. The second counting method requires counting from left to right along the base of the structure, then returning to the far left to count the central matches, and then once again returning to the left to count the top row of matches. Since the second counting procedure is *uneconomical*, I would argue that it must then be *systematic* – i.e. from the counter’s perspective it must represent an *efficient* way to accomplish the task of counting. I would argue further that a necessary condition for a counting method to be systematic and/or efficient is a perceived sense of structure, whether conscious or unconscious, on the part of the person performing the counting operation. It is this perceived sense of structure that then guides the systematic counting procedure. In Kelly’s second counting method the perceived structure seems to be in terms of a bottom row of horizontal matches, a central zigzag of oblique matches, and a top row of horizontal matches. This is confirmed by her later verbal commentary after completing the counting:

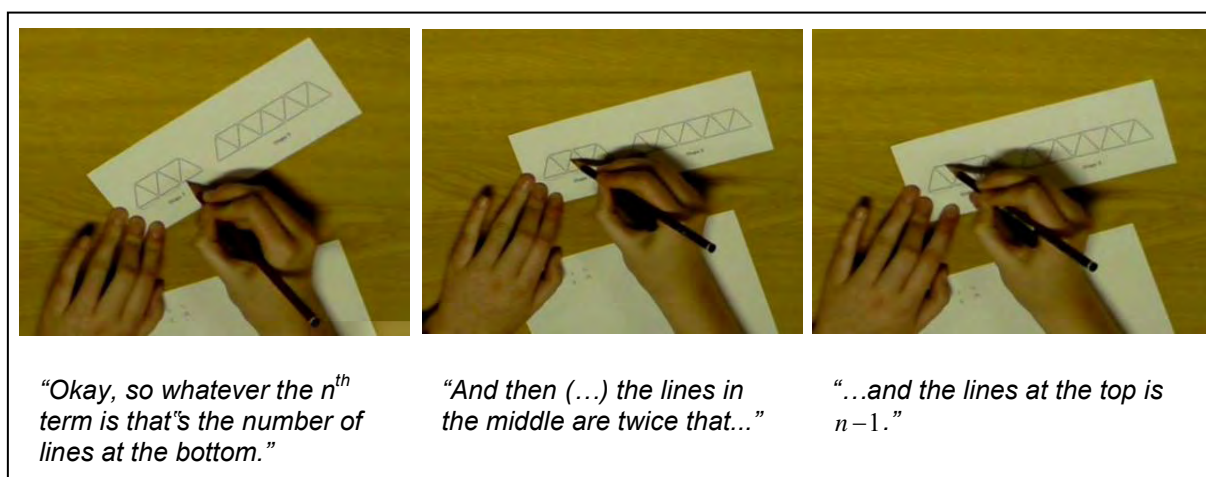


Figure 5.26 Kelly’s verbal commentary upon completion of her counting

At this point, based on her second counting procedure, it would have been possible for Kelly to construct the following expression for the n^{th} term: $T_n = n + 2n + (n - 1)$. However, instead of doing this she continued to interact with the pictorial context with hardly a pause.

Um, 5 triangles in Shape 3 [pointing to each in turn] and 1, 2, 3, 4, 5, 6, 7, 8, 9 triangles in Shape 5 [pointing to each in turn]. Okay, so I’m guessing that you’re adding on 1 there [creates an extra triangle by adding 2 lines onto Term 3 – the first two dashed lines shown in Figure 5.27] which would give you 1, 2, 3, 4, 5, 6 [counting the 6 triangles but then adding on another 2 lines (the second two dashed lines shown in Figure 5.27) to

create a 7th triangle]. Okay, so I'm guessing that that's Term 1 [indicating the 5-unit structure shown by matches a – e in Figure 5.27] and then you're adding on, okay no hang on. (...) Hang on, there're 11 in Term 3 and 19 in Term 5, and 4 [i.e. Term 4] has to come somewhere in between those two numbers [pointing to the numbers 11 and 19 which she had written down earlier]. So you're adding on, you're either adding on 2 matchsticks [indicating the first two dashed lines shown in Figure 5.27] or you're adding on 1, 2, 3, 4 matchsticks [indicating all 4 dashed lines shown in Figure 5.27]. And if you're adding on 4 matchsticks that would make that 15 [referring to Term 4] and then it would go, and then it would plus 4 each time [indicating the jump from 15 to 19, i.e. from Term 4 to Term 5]. Ya, that'll work. Hmm, but, if this is Shape 3 and you're adding on 4, [Kelly then started counting backwards in multiples of 4 matches to arrive ultimately at Term 1], 1, 2, 3, 4 [counting off the right-most multiple of 4 matches in Term 3] 1, 2, 3, triangles in Shape 2 [pointing to each of the 3 triangles], then 1, 2, 3, 4 [counting off the next group of 4 matches from the right in Term 3], and 1 triangle in Shape 1. Okay, that makes sense. Okay, so you're adding on 4 each time. Um, so the difference is 4 so that makes it $4n$, um and $4n$ will give me 12 [indicating Term 3] and $4n$ will give me 20 in Term 5, so I'm gonna minus 1 to get $4n$ is 12 minus 1 is 11 [for Term 3], $4n$ is 16 minus 1 is 15 [for Term 4], $4n$ is 20 minus 1 is 19 [for Term 5], 4 times 6 is 24 minus 1 is 23 [for Term 6] and the difference between 5 [i.e. Term 5] and 6 [i.e. Term 6] is 4. Okay, so the first one is $4n - 1$.

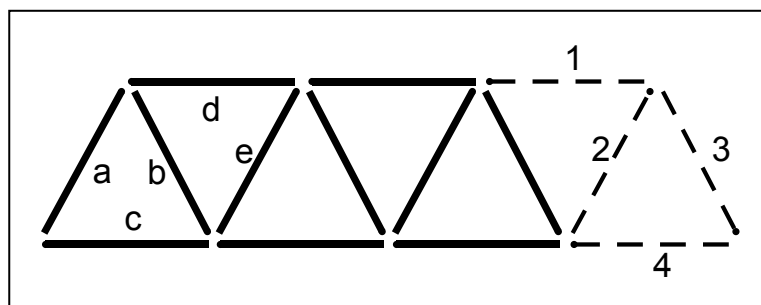


Figure 5.27 Kelly's augmentation of Term 3

From her initial perceptual apprehension of the figural cue – i.e. two horizontal rows of matches with a zigzag of matches between them – Kelly very quickly changed her apprehension by becoming aware of the total number of triangles (upward pointing and downward pointing) in each pictorial term. This new apprehension led her to “guess” the number of matches that one would need to add to Term 3 in order to construct Term 4. Her guess was that it would be either 2 or 4 matches, which would respectively create either 1 or 2 additional triangles. At this point she reverted to a numeric argument. Since Term 3 contained 11 matches and Term 5 contained 19 matches she reasoned that Term 4 had to fit somewhere between these two

terms. Sensing that the addition of 4 matchsticks was more likely to be correct (perhaps because of the difference between 11 and 19) she added 4 to 11 to arrive at 15 (Term 4) and was satisfied with the veracity of her conjecture when she realised that the addition of another 4 would give the 19 matches required for Term 5. She then returned to the pictorial representation of Term 3 and worked backwards in multiples of 4 matches to determine that Term 1 was in fact a single triangle and not a 2-triangle structure as she had initially thought. This visual appreciation of the structure of Term 1 was the final component in the development and ultimate stabilisation of a new apprehension of the pictorial context. Happy that a common difference of 4 matches made sense both visually and numerically she returned to a final numerical argument using a rate-adjust strategy to arrive at a final formula of $T_n = 4n - 1$.

This gradual growing awareness, as different structural aspects of the pictorial terms were brought forth, shows a transition between three different apprehensions (Figure 5.28). Kelly's initial apprehension (two horizontal rows of matches with a zigzag of matches between them), which was on the verge of being stabilised in the form of a general algebraic expression, was rapidly replaced with an apprehension that brought forth the gestalt of the triangle. This triangular feature in turn led to the gradual development of the 4-match unit that represented the constant difference, a process that incorporated both visual and numeric elements. The foregrounding of the visual analogue of the numeric constant difference, along with a retro-synthesis of the growth pattern to determine the visual structure of Term 1, finally led to a new apprehension – a single triangle for Term 1 with multiples of the 4-match additive unit.

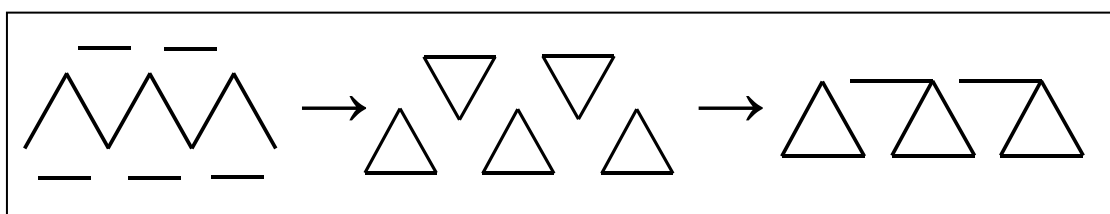


Figure 5.28 Kelly's transitioning between 3 different apprehensions

Kelly's initial apprehension was arrived at through a visually mediated global structural awareness – i.e. the perceptual organisation of the matches into different groups as supported by the Gestalt laws of figural organisation (specifically the laws

of *similarity* and *proximity*). The transition between the first and second apprehension was very rapid and one can only conjecture that the transition was once again supported by the Gestalt laws of figural organisation, in this case the laws of *good continuation* and *closed forms*, which led to the structural unit of the triangle gaining prominence. In spite of these two highly visually mediated apprehensions, it was the gradual foregrounding of the constant difference with its numeric as well as visual recursive allure that led to the final apprehension and the final algebraic expression for T_n .

There are two final points that are worth making in terms of this vignette. Firstly, if one looks back at Kelly's very first counting procedure (as shown in Figure 5.24(a)) then one could perhaps argue that right from the beginning there seems to be fleeting evidence of this final apprehension. Although her counting procedure does seem to have some semblance to this final apprehension, the rhythm in her counting suggests that this similarity is merely coincidental. She counted the first 5 matches slowly and deliberately, as if establishing a counting strategy, after which she counted the remaining 6 matches more rapidly (Figure 5.24(a)). The rhythmic gaps between each count, although shorter in the case of the remaining 6 matches, were nonetheless constant. This rhythm suggests that after the counting strategy had been established, i.e. after counting the first 5 matches, all further matches were seen to be equivalent. This suggests that the counting procedure was used for its economy rather than as a result of an unconscious apprehension based on perception of the 4-match additive unit. The second point which is worth highlighting is that both of the apprehensions that Kelly passed through en route to her final apprehension, i.e. the first two apprehensions shown in Figure 5.28, resurfaced again later. The subdivision into two horizontal rows of matches with a zigzag of matches between them led to her 3rd algebraic expression ($T_n = n - 1 + 2n + n$), while her apprehension of overlapping triangles eventually led to her 5th algebraic expression ($T_n = 3(n + n - 1) - (2n - 2)$). Thus, in this particular case, the potential in these earlier transitional apprehensions was still able to be realised.

VIGNETTE 6 – UNCONSCIOUS APPREHENSION

This vignette describes the approximately 9 minutes that Anthea spent arriving at, and justifying/explaining, her first two expressions for the n^{th} term⁵¹. Although these two expressions seem to have been determined through numerical considerations only, close scrutiny of her embodied counting procedure, juxtaposed with later developments, suggests that the numerical considerations may have been inspired or at least scaffolded by the development of unconscious apprehensions of the pictorial context – i.e. an unconscious structural perception of the figural terms.

Upon initial presentation of her pictorial pattern, Anthea counted the dots in Term 3 and Term 5 in an economical zigzag manner as shown in Figure 5.29 (a) and (b). She then double-checked her two answers. When she re-counted the dots in Term 3 she used the same counting method as she used in the initial count. However, when she re-counted the dots in Term 5 she did so in a slightly modified manner. This new counting procedure is shown in Figure 5.29(c). After a bit of silent thinking she counted the dots in Term 3 one final time, using her *initial* counting method, before writing down the formula $n + (n + 1)$. She then tested her formula mentally (“*let me just check if it works*”) for $n = 3$ and $n = 5$, and was satisfied that the formula worked for these two cases.

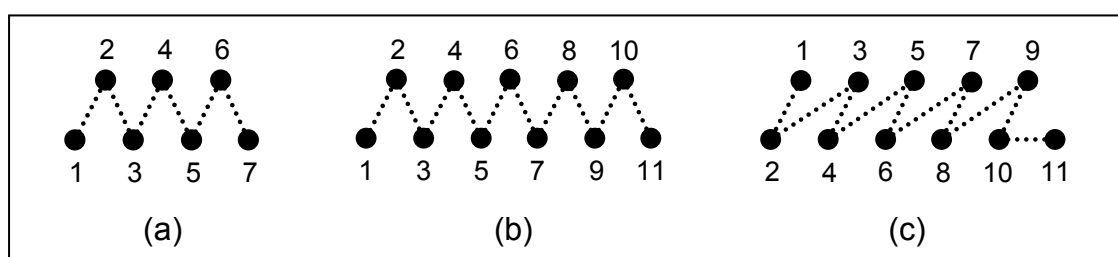


Figure 5.29 Anthea's various counting procedures

At this point I asked her if she was convinced that the formula would always work for any value of n . In response to this she meticulously worked out, writing down all her working, the number of dots in T_2 , T_3 , T_4 and T_5 based on her formula $T_n = n + (n + 1)$. “*Okay, well it worked out for all of them, and every time it’s plus 2. (...) It’s a pattern as well. Every time it adds 2 from the shape before.*” Although she had not drawn any additional terms, she was nonetheless satisfied that the numeric

⁵¹ Approximately 4½ minutes for each expression.

values which her formula produced were correct ($T_2 = 5$, $T_3 = 7$, $T_4 = 9$, $T_5 = 11$). Her remarks seem to suggest that it was the *regularity* of the sequence of answers that was the source of her conviction. Anthea was unable to justify her formula any further, simply saying: *“It just came into my mind”*.

We now move onto the development of Anthea’s second algebraic expression. After sitting silently for about 30 seconds she suddenly wrote down $2n + 1$. During this 30 second period her only movement was to point her pencil at the “3” in the wording “Shape 3” that was printed under the given pictorial representation of Term 3. She similarly pointed to the “5” in the printed wording “Shape 5” under the given Term 5. She explained her reasoning process as follows:

Okay, well what I came up with is, if you times that by 2 [pointing to the “3” in “Shape 3” printed under Term 3] it actually equals something quite close to 7, so then I just plussed 1, and I just checked if it worked on that one too [pointing to Term 5].

Anthea’s explanation of her reasoning process, along with her pointing actions in the 30 seconds of silent contemplation, suggest that her formula $2n + 1$ was arrived at through only numerical considerations. However, what is interesting is that the formula $2n + 1$ seems to resonate with the counting method shown in Figure 5.29(c). In order to explore this seeming connection I asked Anthea if she was able to justify her formula (i.e. $T_n = 2n + 1$) by focusing on the pictorial terms themselves rather than just the numbers. After staring at the pictorial terms for a few seconds she suddenly responded: *“Oh! Um, ya, „cos there’s 2 times 3 plus 1”*, indicating in Term 3 the 3 multiples of 2 dots and the extra single dot (as represented in Figure 5.30). I then asked her if this was part of her reasoning in terms of her arriving at the expression $T_n = 2n + 1$, to which her response was: *“No, I just saw that now”*.

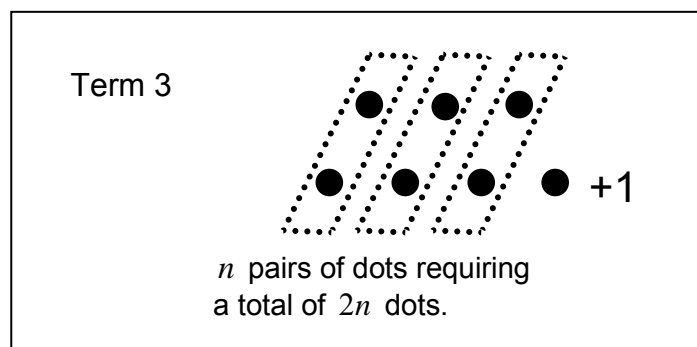


Figure 5.30 *“Oh! Um, ya, „cos there’s 2 times 3 plus 1”*

Although Anthea confirmed that she arrived at her general expression $T_n = 2n + 1$ through a process of numeric rather than visual reasoning, it is likely that the second of her two earlier counting processes unconsciously inspired this algebraic expression. With the first counting method the starting point is the dot furthest to the left on the *bottom*. For the second counting method the starting point is the dot furthest to the left on the *top*. Both methods result in an overall zigzag movement from left to right, but the first method is more economical in terms of total distance traversed during the counting process. The second method is uneconomical since each movement from top to bottom has a small lateral component to the left (e.g. the move from dot 1 to dot 2 in Figure 5.31(b)) with the result that a longer distance needs to be covered when moving from bottom to top (e.g. the move from dot 2 to dot 3 in Figure 5.31(b)). The distance traversed in the second counting method is in fact just over 20% longer than that traversed in the first method. However, what is crucial to appreciate is that this alternating top-to-bottom and bottom-to-top movement, where the top-to-bottom movement is accomplished slightly faster than the bottom-to-top movement as a result of the two different path lengths, creates a critical sense of *rhythm*: 1,2...3,4...5,6...7,8...9,10...11. The critical distinction here is that instead of the rhythm being an artefact of a counting method inspired by a perceived structural regularity, the rhythm is actually an artefact borne out of the counting process itself, an artefact which in turn may *lead* to perceived, albeit possibly unconscious, structural regularity and thus to the development of a new apprehension.

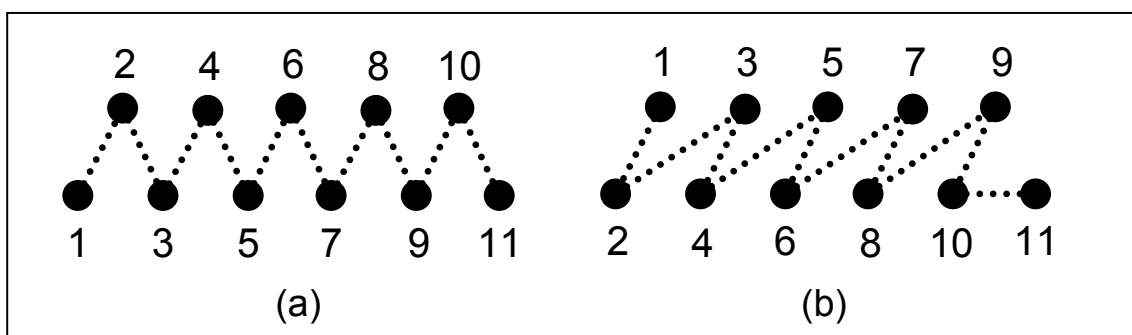


Figure 5.31 Anthea's *economical* and *uneconomical* counting methods

This analysis shows how the second counting procedure may well have brought forth, at an unconscious level, the development of an apprehension that inspired or at least supported the general algebraic expression $T_n = 2n + 1$, an expression which

otherwise seems to have as its origin only numerical reasoning. Interestingly, there is evidence to suggest that Anthea's *first* expression, $T_n = n + (n + 1)$, may also have been influenced by this counting procedure. Although Anthea was unable to explain how she arrived at this particular formula ("*It just came into my mind*"), after she had realised the connection between the second expression and the pictorial context ("*Oh! Um, ya, „cos there's 2 times 3 plus 1*", referring specifically to T_3) I asked her if she was able to explain her *first* expression with specific reference to the diagrams:

Compared to this [pointing to her written expression $n + (n + 1)$] I have n which is 3 [pointing individually to each of the 3 dots on the top row of Term 3], plus n again [pointing individually to all the dots on the bottom row with the exception of the dot furthest to the right], plus 1 [indicating the remaining dot on the bottom row].

She then checked that this reasoning worked for Term 5 as well: "*1, 2, 3, 4, 5, ... 1, 2, 3, 4, 5, ..., plus 1*". Thus, although this structural realisation only came to the fore some time after she had arrived at the expression $n + (n + 1)$, and only after I had prompted her to consciously consider the relationship between the diagram and algebraic expression, what is interesting is that the visualisation she associated with the expression is of n dots plus another n dots plus a single dot rather than an upper horizontal row of n dots and a lower horizontal row of $n + 1$ dots, which is perhaps the more likely interpretation of her expression from an outsider's point of view.

5.3.3 DISCUSSION OF INSIGHTS GLEANED FROM THE MICRO-ANALYSIS

During the course of the micro-analysis a number of broad insights gradually emerged and evolved. These are discussed here in relation to my own thoughts and experience regarding possible pedagogical strategies that could be used to support pictorial pattern generalisation activities and to overcome the difficulties experienced by some of the participants in the course of this study.

An interesting first observation is that a number of research participants were not only consciously aware of how they were engaging with the pictorial patterns in a general sense, but were also able to articulate this general strategy. Brian's broad approach was to look at the pictorial terms and identify specific features that contained as many elements as the term number itself, or that occurred as many times as the term number. By way of example, If he was looking at the pictorial

representation of Term 5, his conscious strategy would be to look for structural features that contained 5 elements (e.g. 5 horizontal matches or 5 vertical matches) or for features that occurred 5 times (e.g. 5 squares or 5 C-shapes). “It’s sort of like into my head that I must look at it to do with the 5 straight away” – referring specifically to Term 5 in this instance. While this is a useful strategy in some respects, it also has its associated limitations. Particularly in cases where an apprehension based on this strategy leads to strong visual imagery, there is a danger that this could invoke a degree of perceptual inflexibility or *geometrical rigidity* (Hoz, 1981) leading to what Duval (1999, p. 17) refers to as *heuristic deficiency*. This perceptual rigidity has the danger of obscuring other potentially useful gestalts that could ultimately lead to alternative solution paths.

Lance’s broad approach, although less overtly articulated than Brian’s, was to subdivide the pictorial terms into groupings containing either exactly or nearly as many elements as the term number itself. More generally, presented with the n^{th} term, Lance would attempt to subdivide the pictorial representation into groups containing either n , $n \pm 1$ or $n \pm 2$ elements. Interestingly, this subdivision was often very symmetrically done – for example by “*working in from the sides*”.

There are two important aspects that one needs to consider with respect to these strategies. The first relates to what Radford (2000, p. 250) refers to as the *positioning problem*, the “nontrivial problem of referring to a nonspecific figure by the position the figure occupies in the sequence” (Radford et al., 2007, p. 517). The critical consideration here is the semiotic complication resulting from “the dramatic changes in the mode of designation that the disembodied algebraic language brings with it” (Radford, 2003, p. 57). The “disembodied” nature of algebraic symbolism is perhaps better thought of in terms of its desubjectified nature. Sentiments such as “the next term”, “one less than the shape number” and “two more than the shape number” require a semiotic contraction into the form $(n+1)$, $(n-1)$ and $(n+2)$ respectively. Such semiotic contractions require the exclusion of linguistic terms that convey important spatial and positional characteristics. In addition, such algebraic symbolism is also desubjectified in the sense that words (such as the personal pronouns “I” and “you”) relating to the person performing the action must also be excluded (Radford, 2003, p. 57). The following short extract, which relates to Anthea’s fourth apprehension of her pictorial sequence, highlights some of these

important aspects. Her specific apprehension, which was based on the visualisation of a series of overlapping triangles, is shown in Figure 5.32 along with some of her working.

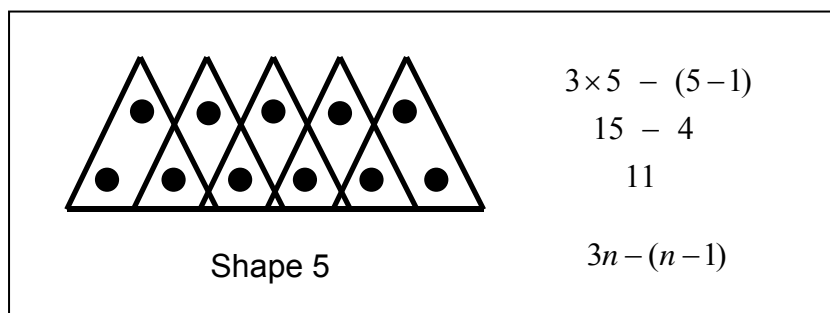


Figure 5.32 Anthea's fourth apprehension

Because there's 3 in the triangle [i.e. 3 dots per triangle], so, um, I timesed it by the number of the shape [pointing to the printed wording "Shape 5"], and then, I minused the number minus 1.

Based on the visualisation of the n^{th} term containing n overlapping triangles, Anthea multiplied 3 by n ("the number of the shape") and then subtracted 1 less than the term number ("I minused the number minus 1") to correct for the overcount created by overlapping triangles. Based on this visualisation, and a calculation that was carried out on Term 5 to verify that the described procedure worked, Anthea was able to arrive at the algebraic expression $T_n = 3n - (n - 1)$. An interesting aspect of the structure of Anthea's sentences is that they already contain something of a semiotic contraction. Rather than saying for example "I subtracted 1 less than the shape number", she instead says, far more succinctly, "I minused the number minus 1". Even her reference to "the number of the shape" gets contracted simply to "the number". The semantic content of the expression "the number minus 1" is much more closely aligned with the algebraic syntax $n - 1$ when compared with the wordier expression "1 less than the shape number", and this verbal contraction may well be a useful transitional stage en route to desubjectified algebraic symbolism.

The second important aspect of these strategies relates to the ambiguity inherent in symbolic expressions such as $3n$ or $2(n+1)$. By way of example let us consider Philip's algebraic expression $n + 2n + (n - 1)$. Philip had two different apprehensions that led to this same general expression, and these are shown in Figure 5.33 for Term 5 of his particular pictorial pattern.

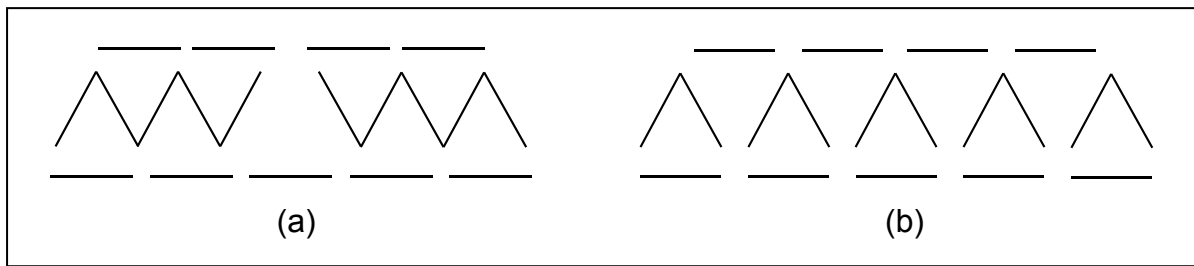


Figure 5.33 Philip's different apprehensions for the expression $T_n = n + 2n + (n - 1)$

Although both visualisations see the pictorial terms as comprising a bottom row of n matches and a top row of $(n - 1)$ matches, there is a subtle distinction in the interpretation of the $2n$ portion of the algebraic expression. In Figure 5.33(a) the $2n$ represents 2 multiples of n matches while in Figure 5.33(b) it represents n multiples of 2 matches. This is an important consideration to bear in mind in conjunction with strategies like that expressed by Brian: "*It's sort of like into my head that I must look at it to do with the 5 [i.e. n , the Shape or Term number] straight away*". There are two distinct "ways of looking" at the diagram with this sort of strategy - identifying specific features that contain *as many elements* as the term number itself, or that occur *as many times* as the term number. This is a crucial distinction to be aware of since it significantly augments, in fact *doubles*, the number of potential apprehensions that could be brought forth. When one extends this approach to identifying specific features that contain, more generally, n , $n \pm 1$, $n \pm 2$ etc. elements, or that occur n , $n \pm 1$, $n \pm 2$ etc. times, then this has the potential to become an incredibly powerful conscious strategy.

Philip's broad strategy was to identify any visually striking feature of the pictorial term and use that particular feature as the basis of his apprehension ("*I just picked a feature and just tried to work from that*"; "*I just picked a feature again*"; "*Find another feature!*"). Sometimes this feature was a recurring element in the diagram, while on other occasions it was a solitary feature which served as a keystone for the rest of his visualisation. The features that he identified included triangles, half-hexagons, trapeziums and diamond shapes. While this too is a useful strategy, it nonetheless has its drawbacks as well. By focusing on visually striking features, strong visual imagery may result which in turn may lead to a degree of perceptual rigidity or inflexibility. In order to move beyond this impasse, a critical ability lies in being able to relinquish particularly vivid imagery with a willingness to explore others. Terry

made use of a similar strategy to that described by Philip. Terry describes his own experience as follows:

So far I've just been looking at, so if I see, I'll see some, one thing, like I'll see a particular shape that stands out, and use that as almost a basis point, like I used the triangles, it stood out a bit, then used the lines, used the squares ... and I've run out of things to use!

A comparison of Philip's and Terry's remarks suggests different levels of conscious engagement with the pictorial context. For Philip there seems to be a conscious and purposeful element of exploration – one of *active engagement* rather than *passivity*. His remarks such as “*I just picked a feature again*” and “*Find another feature!*” suggest a conscious endeavour to search for structure. Terry's remarks on the other hand – “*I'll see a particular shape that stands out*” and “*it stood out*” – suggest a more passive engagement with the pictorial context, one in which the structural elements are hoped to reveal *themselves*. Rowlands (2006) characterises the notion of exploration as one of activity rather than passivity, “it is something we do, rather than something that happens to us” (p. 12). I would agree with this characterisation, and thus acknowledge that Terry's seemingly passive engagement is probably far from passive⁵². However, there nonetheless seems to be a difference in the degree of conscious agency, which is worth reflecting on.

The micro-analysis revealed generalisation methods that evolved out of a conscious search for structure, but it also revealed generalisation methods that emerged and developed serendipitously from unstructured exploration and interaction with the pictorial context. Both conscious and unconscious exploration thus have the potential to lead to expressions of generality, and both have an important role to play in the generalisation process. However, one can enhance the potential success of active exploration by consciously searching for structural features or characteristics that contain, for example, n , $n \pm 1$ or $n \pm 2$ elements, or that occur in multiples of n , $n \pm 1$ or $n \pm 2$.

There is another important aspect to bear in mind when considering this pedagogical strategy, one that relates to the *specific term* chosen to explore. This aspect is highlighted in a brief episode which formed part of Liza's route to her third algebraic

⁵² Indeed, one could argue that sensory perception (e.g. visualisation) cannot occur without cognitive perception – a view that resonates strongly with the mind-body unity that is the core of enactivism.

expression. The dashed box in Figure 5.34 shows the two printed terms of the pictorial pattern from which she was working, while diagrams (a) – (c) represent the various developmental stages of her final visualisation.

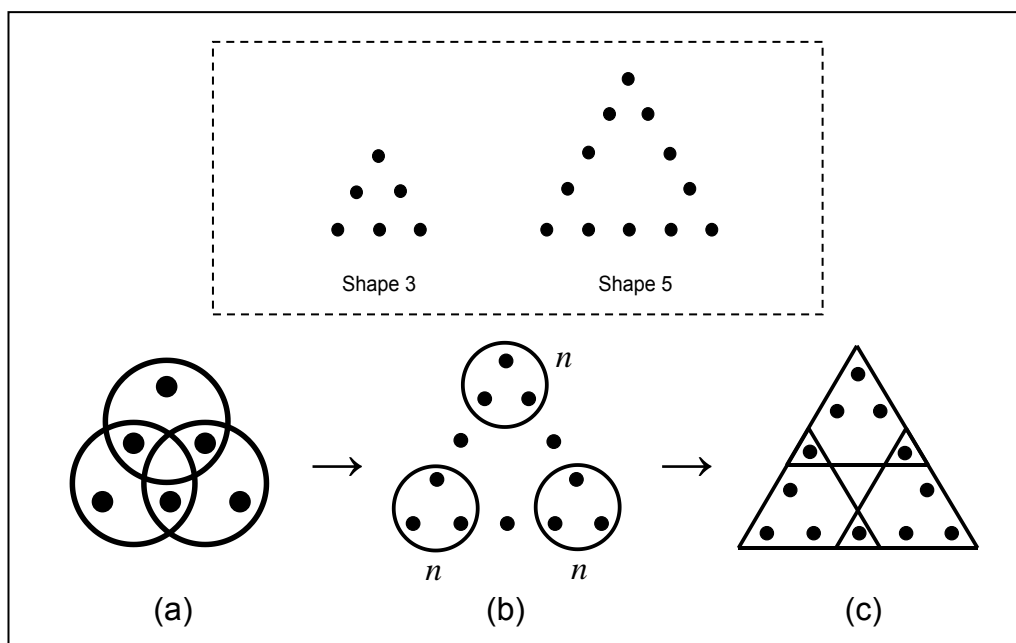


Figure 5.34 Liza's progression to a stable awareness of her general formula $T_n = 3n - 3$

After Liza had silently looked at the two printed terms for almost a minute I asked her what she was thinking. She commented “*I was just looking for patterns in the shape*” before drawing Term 3 and circling the 3 sets of dots as shown in Figure 5.34(a). After a few moments of silent contemplation she wrote down the formula $3n - 3$ and checked that it worked numerically for Term 5, which it did.

Okay, well I've got this one, $3n - 3$. Okay, so I looked at the corners, um, so 3 on each thing like a triangle [highlighting in bright yellow each of the 3 groups of 3 corner dots in Shape 5 (Figure 5.34(b))] so that would be $3n$ already. And then, wait, $3n$ plus, wait. Is it a plus 3 or minus 3? [muttering to herself]. Ya, minus 3, and then, wait. ... Well I've just noticed that there're 3 left over [highlighting the 3 un-circled dots on each side of Shape 5], and ... so you ... right, no I think what I did was, so, made that n and that n and made that n [annotating Shape 5 as shown in Figure 5.34(b)] so it would kind of give you 5, I don't know what happened, but ... „Cos I made it $3n$ [pointing to her formula] and if I did that [circling the bottom-right triangle of 3 dots in Shape 5], no wait [suddenly drawing a triangular outline around the top 5 dots in Shape 5] or did that [making similar triangular markings around the bottom-left and bottom-right groups of 5 dots in Shape 5, giving the final annotation shown in Figure 5.34(c)]. Oh, ja! (...) all the dots must be in a triangle. So like for Shape 3, like that [drawing shape 3 and marking off the top triangle of 3 dots] do a circle, a triangle of 3. (...) this is Shape 5, so it would have 5 dots in the triangle [indicating her annotations as shown in Figure 5.34(c)].

Liza was then able to clearly articulate that the “ -3 ” in her expression was to correct for the overcount caused by the dots in the overlapping corners of the three triangles effectively being counted twice. Although Liza was able to successfully reconcile her visualisation, her struggle to do so went through a number of interesting stages. The journey began with the emergence of a structural feature in the given pictorial terms – triangles of dots within each triangular term. Liza then drew Shape 3 and circled the 3 groups of 3 corner dots (Figure 5.34(a)), and this is where the complications arose. Although she was able to express a correct algebraic formula for the n^{th} term, it is clear that her visual reasoning had not yet reached a stable form. When she transferred her reasoning to Shape 5 (Figure 5.34(b)) the extent of this instability is revealed, particularly in her annotation of the groups of 3 dots as “ n ”, but also in her confusion as to why her formula, which she has tested numerically for Term 5 and thus knew to be correct, suggests the *subtraction* of 3 dots while her annotated diagram revealed 3 dots that still need to be *added* to the total count. She eventually managed to reconcile this apparent contradiction by realising that her 3 groupings should in fact contain 5 dots instead of 3. What is crucial to understand here is not the reconciliation itself, but rather the *source* of the *initial* confusion, which was the specific choice of Shape 3 as a generic reference point. How such confusion or instability arose through the choice of this specific pictorial term can readily be understood by considering the numeric equivalent of her general formula $T_n = 3n - 3$, viz. $T_3 = 3 \times 3 - 3$. Not only does 3 represent the Term number as well as the constant number of dots that need to be subtracted, but crucially it is *also* the number of elements in the identified structural feature *as well as* the number of times this structural feature occurs in the pictorial term itself. Choice of any other Term number would have avoided the ambiguity of the $3n$ portion of her general term, since it was only in Term 3 that this could be ambiguously interpreted as either “ n groups of 3 dots” or as “3 groups of n dots”. Thus, a useful cautionary strategy to keep in mind when consciously searching for structural elements in a pictorial term is to make use of bigger terms (e.g. $n \geq 5$) where there is less chance of such ambiguity obfuscating the generalisation process.

Another interesting insight that arose during the course of the micro-analysis relates to the specific choice of either even- or odd-numbered terms. During the individual pattern generalisation tasks, participants were all given two non-consecutive terms,

specifically T_3 and T_5 . There were a number of instances where perceived visual commonalities between these two odd-numbered terms led to the development of a general formula which, although algebraically correct, did not necessarily make visual sense when applied to even-numbered terms. The crucial trigger that occasioned the particular structural understanding in these instances was the juxtaposition and/or comparison of two odd-numbered terms⁵³. Philip's 10th general formula is a good example. The dashed box in Figure 5.35 shows the two printed terms of the pictorial pattern from which he was working, while diagrams (a) and (b) show his visualisation for T_3 and T_5 respectively. Philip described his visual strategy as “picking the centre triangle as the sort of cornerstone”.

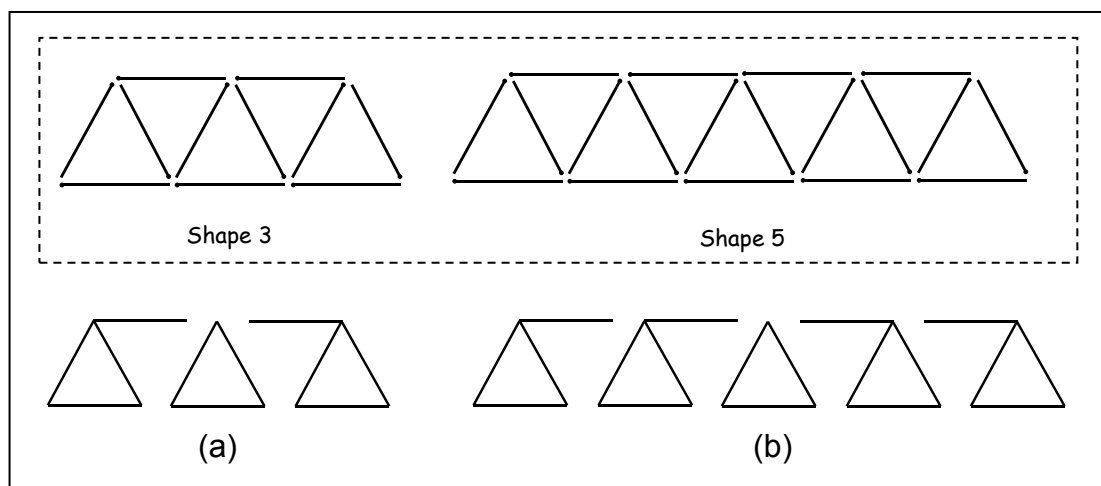


Figure 5.35 Philip's visualisation of his formula $T_n = 3 + 4(n-1)$

A crucial trigger for this particular apprehension to have been brought forth is the commonality in the symmetry of the two juxtaposed odd-numbered terms. As a prompt I asked Philip if he thought his visual reasoning would still hold for even-numbered terms. He was poised to draw Term 2 when he suddenly realised that there was no central triangle as before. He managed to resolve this problem by inverting the two printed terms, placing his left hand over the 4 left-most matches of the inverted Term 3 (thus creating Term 2), and in this manner creating a central triangle to act as a “cornerstone”. Operative apprehension (Duval, 1995, 1998, 1999) was thus invoked by a reconfiguration of the whole-part relation of the given pictorial terms through a variation in their orientation. Philip then drew inverted images for Term 2 and Term 4 to clarify his reasoning, as shown in Figure 5.36 (a)

⁵³ A similar situation could equally have arisen for the case of two even-numbered terms.

and (b). His visualisation for odd-numbered terms, based on his general formula $T_n = 3 + 4(n-1)$, is of a central triangle symmetrically surrounded by 4-match units. The complication for even-numbered terms is that there are now an *odd* number of 4-match units to symmetrically position about the central triangle. Philip's solution to this problem was to split one of the 4-match units into 2 sections, thereby retaining the overall symmetry ("the first group of 4 in the even ones just fills up the gaps").

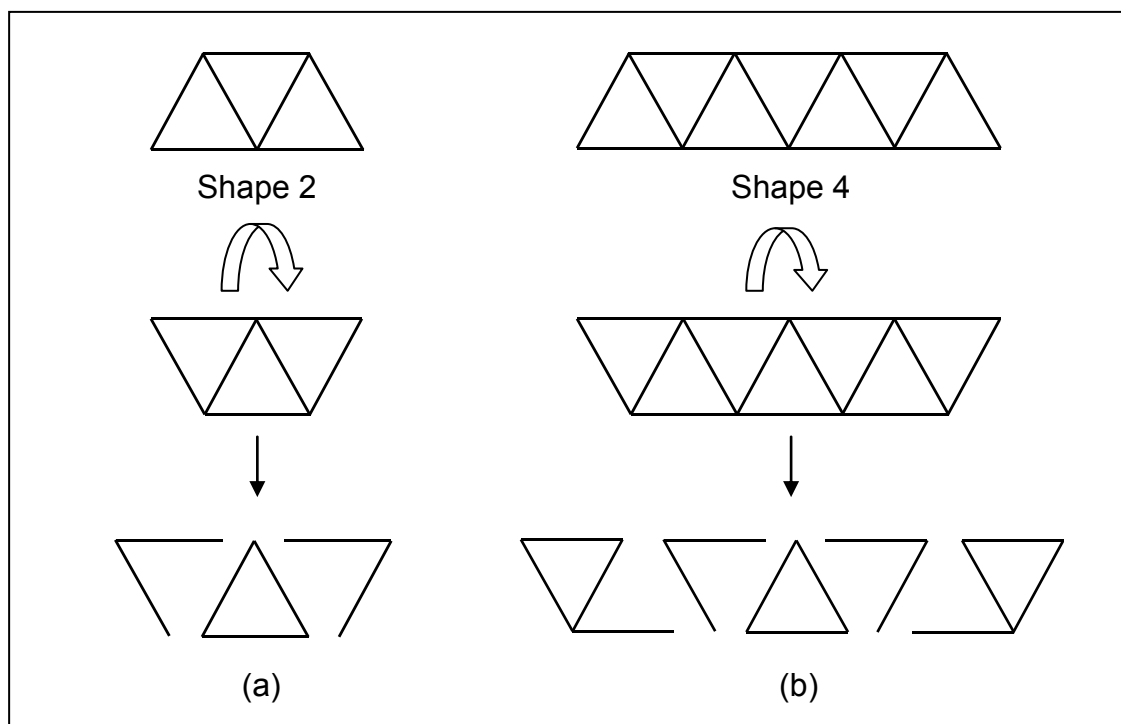


Figure 5.36 Philip's visual resolution for even-numbered terms

The manner in which Philip managed to extend his visualisation of odd-numbered terms to incorporate even-numbered terms is remarkable. Not only did it require a complete reconfiguration of the structural units in the pictorial context, but it did so in a way that retained generality and conformed to the algebraic expression he had arrived at for odd-numbered terms. Notwithstanding the impressive complexity of this process, it is worth contemplating why Philip needed to invert the pictorial images in order to find his cornerstone triangle, since it could have been visualised in the un-inverted image as an *inverted triangle* – i.e. a downward-pointing triangle rather than an upward-pointing one. One possibility, which relates to an issue raised earlier in this section, is that the visual imagery associated with the upward-pointing triangle may have been so strong that the gestalt of the downward-pointing triangle was obscured, and as such was not able to be brought forth. Another interesting

point worth contemplating is the reason for Philip choosing to position the “split” 4-match unit directly adjacent to the central triangle rather than at the extreme left and right of the diagram, since his choice to position it adjacent to the central cornerstone triangle had the knock-on effect of the 4-match units being *inverted* when compared with his visualisation of the original odd-numbered terms. The most likely reason for this is that his drawing of Term 4 was based on an extension of the structure that he established when he first visualised Term 2 (see Figure 5.36). If his initial visualisation had been of Term 4 or Term 6 then it is likely that the two halves of the split 4-match unit would have been positioned at the extreme left and right, thereby retaining the original orientation of the 4-match units. This conjecture gains weight when one considers that it is only after having symmetrically subdivided the matches into 4-match units *by progressing outwards from the central triangle* that Philip would have realised that he was left with 2 matches on either end.

It is thus possible for at least four different apprehensions to have arisen from the process of trying to reconcile the algebraic formula $T_n = 3 + 4(n - 1)$ with even-numbered terms. These four possibilities are shown in Figure 5.37.

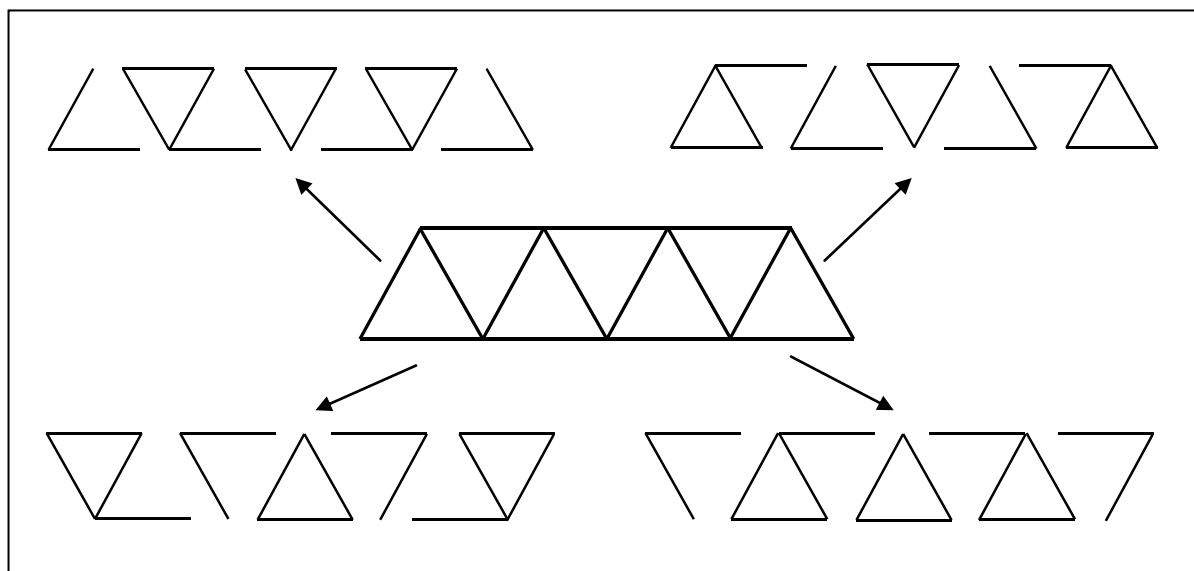


Figure 5.37 Possible visual representations of $T_n = 3 + 4(n - 1)$ for even-numbered terms

To return to the discussion of the juxtaposition of two odd- or even-numbered terms, it is likely that in most instances the combination of an odd-numbered term with an even-numbered term would not have constituted an appropriate trigger for the

emergence of structural commonalities. However, an appropriate combination of more than two terms may well be a suitable trigger for some pupils. By way of example, Anthea noticed that she could subdivide the two given terms (T_3 and T_5) of her pictorial pattern into a series of non-overlapping triangles as shown in Figure 5.38.

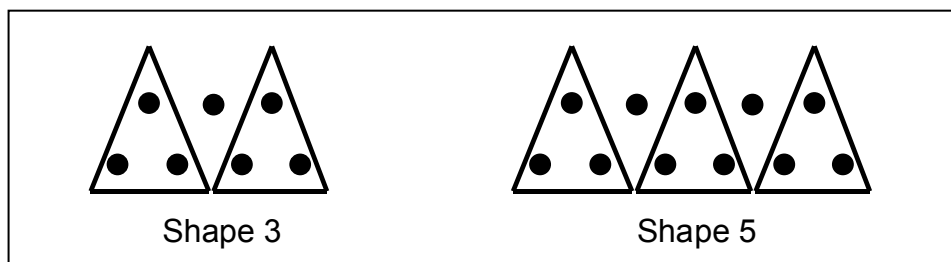


Figure 5.38 Anthea's subdivision of T_3 and T_5 into non-overlapping triangles

She was further able to make the astute observation that the number of triangles plus the number of the remaining dots gave the Shape number in each respective case. Struggling to find a general way to state this observation she drew out four consecutive terms (T_2 , T_3 , T_4 & T_5) and subdivided them into triangles as before (Figure 5.39). It was at this point that she suddenly realised there was a difference between the odd-numbered and even-numbered terms – i.e. she noticed commonalities *within* these two sub-groups.

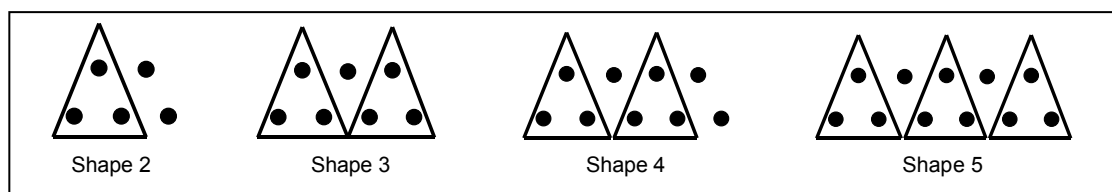


Figure 5.39 Anthea's subdivision of T_2 , T_3 , T_4 and T_5 into non-overlapping triangles

With a substantial amount of scaffolding, and *focusing only on the odd-numbered terms*, she was eventually able to express the number of triangles in each shape as $\frac{n+1}{2}$, the number of remaining dots as $n - \left(\frac{n+1}{2}\right)$, and hence the total number of dots in the n^{th} term as $\left(\frac{n+1}{2}\right) \times 3 + n - \left(\frac{n+1}{2}\right)$. In this particular case it would be very difficult to transfer the visual reasoning that led to the formula onto even-numbered

pictorial terms since the number of triangles in each shape $\left(\frac{n+1}{2}\right)$ would be a fraction. However, since one can show that the expression simplifies to $2n+1$, it will nonetheless provide the correct numerical answer for even-numbered terms. Furthermore, by focusing on only even-numbered terms with a similar sort of reasoning one could potentially arrive at the following general formula: $T_n = \left(\frac{n}{2}\right) \times 3 + n - \left(\frac{n}{2}\right) + 1$. Thus, for certain pictorial patterns, a pedagogical strategy of focussing on only even- or odd-numbered terms may not only be useful with respect to the generalisation process itself, but also in terms of its potential educational value.

A further insight that stemmed from the micro-analysis relates to the dangers of single-case concreteness, a notion that is something of a two-edged sword. Fischbein (1987) makes the pertinent observation that it is this very concreteness that constitutes "...an essential factor for creating the feeling of self-evidence and immediacy" (p. 104). However, Presmeg (1986a, pp. 44-45; 1992b, p. 42) asserts that the concreteness of depictions of specific instances of a general scenario may result in the foregrounding of *unhelpful* or *irrelevant* details. Although I would argue that what is unhelpful or irrelevant for one pupil may well be *helpful* or *relevant* for another pupil, within the context of pattern generalisation where the crux of the enterprise lies in an evolving sense of generality, extended focus on a single case may well obfuscate this central endeavour.

As a case in point, consider Kelly's final algebraic expression $T_n = 2(n+n-1)+1$ which evolved from an initial numerical observation:

I suppose you could go n as 4 ... no you couldn't, because n as 4 plus 3 will give you 7. 1, 2, 3, 4, 5, 6, 7 [counting the 7 triangles in Term 4]. That could work! [triumphantly]. n plus n-1 will give you the amount of triangles, so you have to times that by 2 to get the amount of sticks and then plus the 1 on the end to get the last number [indicating the right-most matchstick in Term 4]. (...) in Term 4 there are 1, 2, 3, 4, 5, 6, 7 visible triangles [counting the 7 triangles in Term 4]. Okay, so I get 7, so I know that there's 7 triangles, but from previous working out [referring to the second general formula she determined] I know that each triangle is made up by 2 sticks, so that's 2 times 7. But I also know that on the end of each pattern I have to add on one last stick to make, to finish off the drawing, so that's where my +1 comes in.

Although she arrived at this formula largely through serendipitous numeric considerations, there is an associated visual element that draws on an earlier visualisation shown in Figure 5.40.

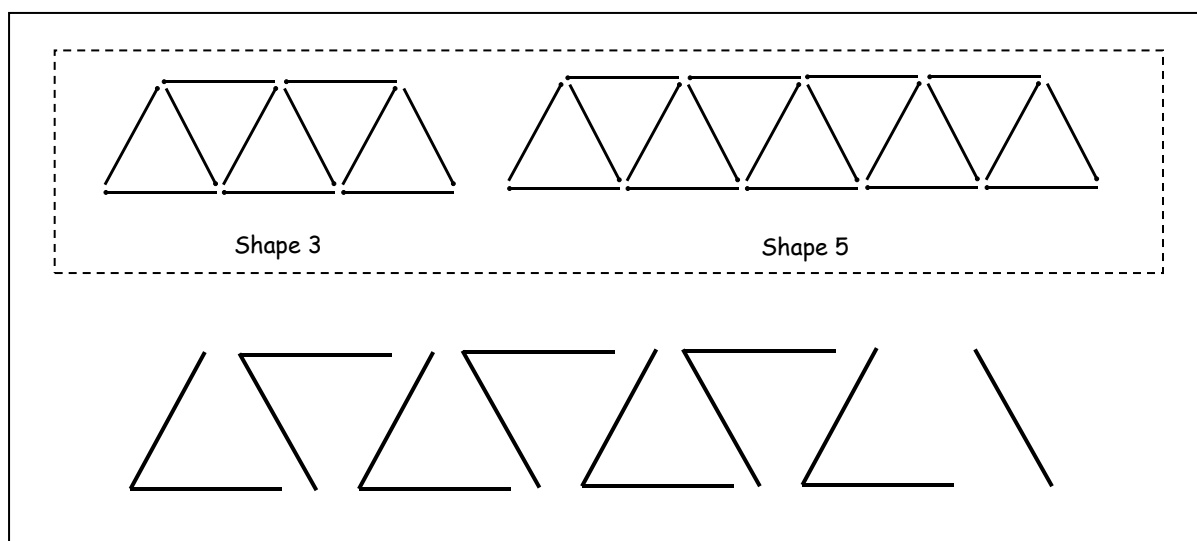


Figure 5.40 Visual support for Kelly's formula $T_n = 2(n + n - 1) + 1$

At this point I was interested in whether Kelly would be able to justify her formula with specific reference to the pictorial context. She chose Shape 4 as her generic reference and subdivided it into five sections as shown in the bottom left image of Figure 5.41 – two groupings of n matches, two groupings of $n - 1$ matches, and the remaining single match. She was satisfied that this subdivision justified the formula visually:

So I'm finding $n - 1$ twice in my diagram, and I'm finding n twice in my diagram, and then whatever's left over is going to be $+ 1$, always.

Despite Kelly's use of the word "always" as suggesting a certain generality to her visual reasoning, her description of the subdivisions suggested that she was assigning the matches in the diagram to the five different groupings in a somewhat arbitrary manner. To investigate this further I asked her if she was able to justify the formula for a different Term number. Choosing Shape 3 she once again subdivided the diagram into two groupings of n matches, two groupings of $n - 1$ matches, and the remaining single match. This somewhat different subdivision is shown in the bottom right image of Figure 5.41. Once again she was satisfied that this subdivision justified the formula visually, even though the nature of the subdivisions was different to those done before. When I pointed this out to her she commented: "Ya, „cos it

actually doesn't matter which sticks you choose to take". Her approach thus seems to be that provided she is able to take a *specific term* and subdivide it, in whatever manner, into the various groupings as required by the algebraic expression, then that subdivision justifies the veracity of the algebraic expression *for that specific term*. The crucial element that is missing from this reasoning process is that of generality – an explanation of how her subdivisions could be performed in the same way for *any term*. This lack of generality stems from one-case concreteness of the individual diagrams Kelly chose, since a diagram by its very nature depicts a *single* instance and is thus unavoidably an *individual* concrete case (Presmeg, 1999, p. 152). An appropriate general visual justification for her general formula $T_n = 2(n + n - 1) + 1$ would be the following (with reference to Figure 5.40): For the n^{th} term there are n matches in the bottom horizontal row and $n - 1$ matches in the top horizontal row. Since these $n + (n - 1)$ matches each form part of a V-shape incorporating 2 matches, this accounts for $2(n + n - 1)$ matches. Since each V-shape forms a triangle by virtue of the following V-shape closing it off, it is only the final V-shape on the far right that needs an additional match to close it off and thus complete the pattern, thus giving a final tally of $2(n + n - 1) + 1$ matches.

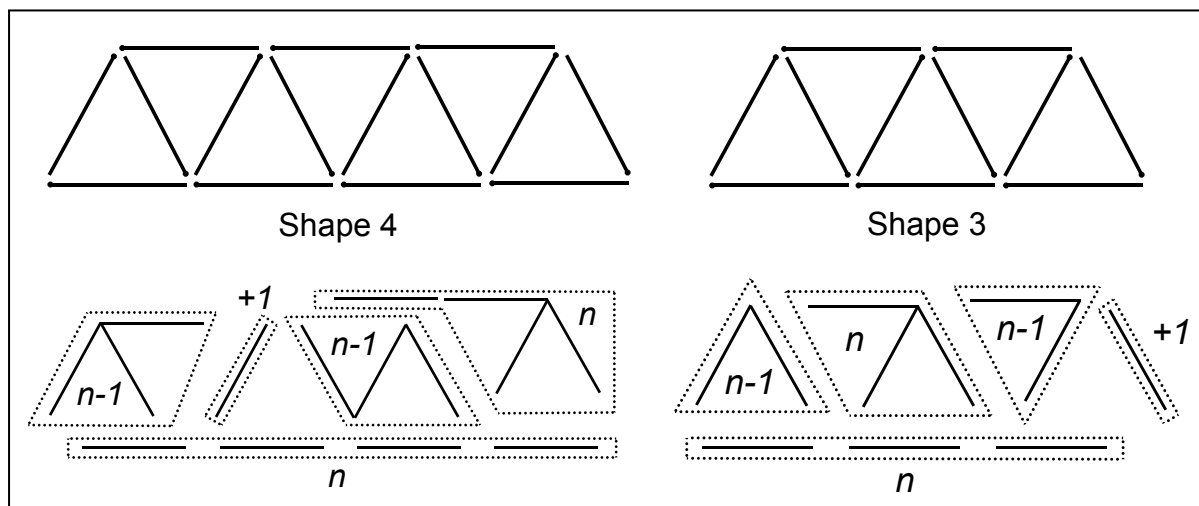


Figure 5.41 Kelly's visual justification for her formula $T_n = 2(n + n - 1) + 1$

Another interesting point regarding this particular episode in Kelly's pictorial pattern generalisation is the manner in which a serendipitous numerical observation led to the development of a different general expression. A similar situation arose for Lance during the evolution of his fifth algebraic expression, $T_n = 3(n + 1)$. Having already arrived at 4 different algebraic expressions based on different apprehensions

of the pictorial patterns, Lance decided to adopt a numerical approach to see if that would yield anything further. He was specifically looking at Term 7 (which contained 24 matches) when he remarked:

I was thinking of working from numbers and somehow getting from n^2 , which is 49, to 24. But that wouldn't have much to do with the shape.

Lance's observation may have been numerically inspired by the fact that in the case of Term 7 the number of matches in the shape, multiplied by 2, is 1 less than n^2 . This is a spurious observation as it only works in the case of Term 7. However, he then noticed, again for Term 7, that $24 = 3 \times 8$ where 8 is 1 more than the term number (i.e. $n + 1$). This numeric observation inspired Lance to attempt to subdivide Term 7 into 3 groupings of $n + 1$ matches. He accomplished this, as shown in Figure 5.42(a), and was thus satisfied with his formula $T_n = 3(n + 1)$. I then asked him if he thought it would be possible to justify his formula visually using a different term. His astute response, after a bit of thought, was:

I see what you mean. Maybe not in the exact same way. (...) You might be able to split it up, but it would not be as even as that has worked out to be [referring to his subdivisions in Term 7].

He then chose Term 3 and subdivided it into 3 groups of $n + 1$ matches as shown in Figure 5.42(b). Although he has used a similar symmetrical approach ("working in from the sides") to that used for Term 7, the subdivisions are necessarily slightly different⁵⁴. The *generality* of Lance's three subdivisions would be difficult to justify since the three groups of $n + 1$ matches are not individually related to any particular structural feature of the pictorial term. For Lance, these subdivisions are simply a convenient and symmetrical splitting of *specific* terms into 3 groups of $n + 1$ matches which, similar to Kelly, he has used to justify the veracity of his algebraic expression for *specific terms*. However, Lance's acknowledgement that his subdivisions wouldn't be as symmetrical for all terms in the sequence would be a crucial first step towards developing a more refined visual justification for his algebraic expression.

⁵⁴ Lance's original subdivision of Term 7, based on the two side groupings containing the single vertical and single oblique match along with 3 equal rows of horizontal matches, would in fact only work for $T_1, T_4, T_7, T_{10}, T_{13}$, etc.

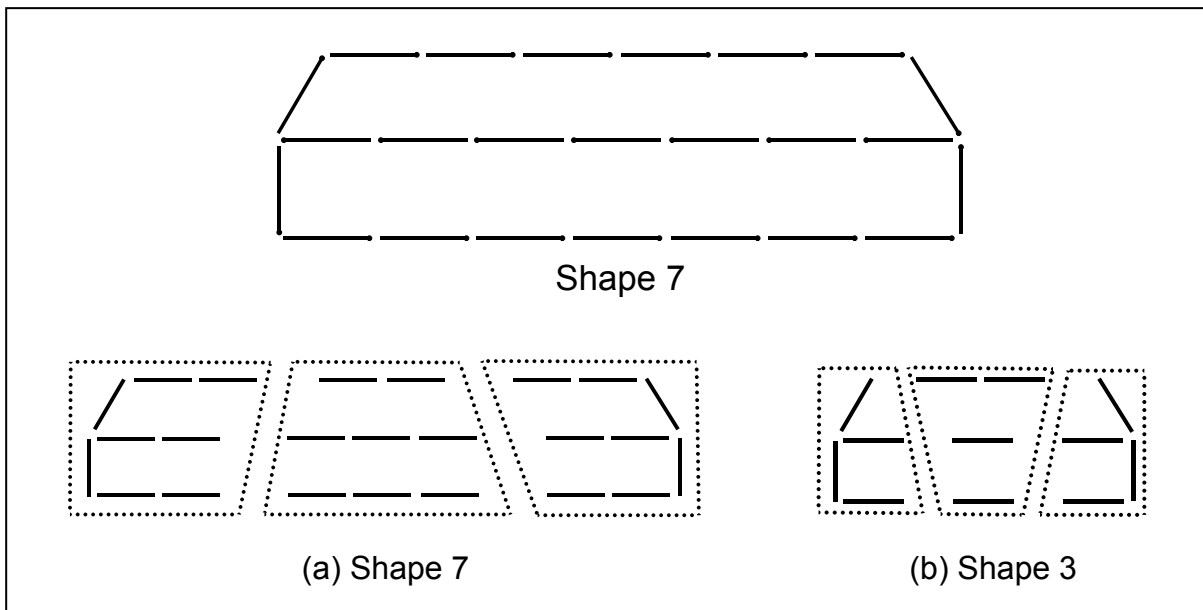


Figure 5.42 Lance's visual justification for his formula $T_n = 3(n+1)$

During the course of the micro-analysis it became increasingly apparent that the chosen research participants were able to visualise pictorial patterns in a remarkable number of ways. However, what also became apparent is that not all visualisations were able to be successfully expressed in the form of an algebraic expression. In some instances this was as a result of the particular visualisation being based on a *specific* term and thus not generally applicable to all terms. In other instances, where a general treatment of the visualisation may potentially have been possible, such endeavours were often abandoned in favour of other routes. What this highlights is that the crucial process is not the multiple visualisation itself (although competing apprehensions have been shown to cause tension), but rather the process of coming to realise *how* the visualisation is regular, and how this regularity can be expressed in an algebraically useful manner.

Research participants exhibited a number of interesting strategies while grappling with this very issue. Grant, for example, tabulated a summary of structural features along with the total number of occurrences of each structural feature for specific terms. By way of example, in one particular instance he listed not only the *total* number of triangles in T_3 , T_4 and T_5 , but he further subdivided this into upward-pointing triangles and downward-pointing triangles.

Shape 3 = 11 $\Delta 5$ [3 up / 2 down]

Shape 4 = 15 $\Delta 7$ [4 up / 3 down]

Shape 5 = 19 $\Delta 9$ [5 up / 4 down]

For Grant this was a powerful way of making sense of the regularity of each of his subdivisions of the pictorial pattern. For other pupils a useful strategy in terms of looking for regularity was to draw or construct much larger terms than those initially given. Lance for example made extensive use of Term 8, while Liza made use of Term 9 on a number of occasions. Interestingly, Philip made extensive use of Term 1. However, rather than using this term to explore regularities in the pictorial patterns, he used it as a sounding board to test the validity of a number of his algebraic formulae where the associated visualisation became problematic for Term 1. Satisfied with his visual reasoning for larger terms, and satisfied that the associated algebraic expression gave the correct numerical answer for cases that were problematic to visualise, he felt comfortable in the general validity of his algebraic formula. The usefulness of this particular strategy was highlighted when Terry was struggling to find the error in his particular visualisation which he thought made visual sense but which gave a constant overcount when tested against known terms. It was only when he tested his formula for Term 1 that he realised the “ n ” in his expression should have been “ $n - 1$ ”.

5.4 CONCLUDING COMMENTS

The analysis of Phase 1 data revealed a continuum with respect to numeric versus visual approaches to pictorial pattern generalisation tasks. A nuanced coding system was developed to characterise this spectrum of numeric and visual strategies. The results of this analysis suggest that very few pupils favour a specific visual strategy, with the majority of pupils being able to draw on a vast repertoire of visual approaches. However, there is evidence to suggest that the *nature of the pictorial patterns themselves* could act as potential triggers in terms of favouring specific visual strategies.

The analysis of Phase 2 data revealed that the research participants in this study have a remarkable ability to generalise pictorial patterns in multiple ways. A fine-grained micro-analysis of Phase 2 data, presented and discussed in the form of a series of vignettes, showed the rich tapestry of generalisation activity which was evidenced by the research participants.

The following final chapter consolidates the findings of this analysis with specific reference to the five guiding research questions as originally outlined in Section 1.2.

CHAPTER SIX

FINDINGS & CONCLUSION

If we are always arriving and departing, it is also true that we are eternally anchored. One's destination is never a place but rather a new way of looking at things.

HENRY MILLER

6.1 INTRODUCTION

The purpose of this final chapter is to consolidate the findings of this study with reference to the original research question and within the context of the theoretical and methodological framework. In addition, both the limitations and significance of the study are interrogated, and some recommendations for further research are suggested.

6.2 REVIEW OF THE OBJECTIVES

This study stems from an earlier study (Samson, 2007a) which investigated the extent to which question design affects the solution strategies adopted by pupils when solving linear number pattern generalisation tasks presented in pictorial and numeric contexts. One of the limitations of this earlier study stems from the data generation and analysis protocol, which was focused on the *product* of the generalisation process. Notwithstanding some of the successes of this methodology, it nonetheless limited the investigation to the *result* or *final product* of the reasoning process rather than allowing access to the *reasoning process itself*.

An unexpected aspect of this earlier study was the rich diversity of visualisation strategies employed by pupils while completing the pictorial generalisation tasks. This overall feature of pupils' responses raised an important but unanswered

question: To what extent are pupils capable of visualising *figural cues*, i.e. objects with both spatial properties and conceptual qualities (Fischbein, 1993), in multiple ways within the context of pattern generalisation? A literature review revealed that little empirical research has been carried out in this area. A desire to explore this aspect of pictorial pattern generalisation in greater depth by specifically focusing on the *process* of generalisation was the impetus and motivating factor behind the present study.

A review of pertinent literature relating specifically to figural pattern generalisation identified four broad categories of focus: (a) descriptions of solution strategies and levels of attainment, (b) the influence of task design and the nature of the pictorial terms, (c) the transition between pupils' arithmetic and algebraic reasoning, and (d) the affordances offered by technological environments. However, within the context of figural pattern generalisation, little empirical research focusing on the *process of visualisation*, as opposed to the *product of visualisation*, seems to have been carried out.

The central goal of this study was thus to gain insight into the inter-relationship between the embodied processes of pattern generalisation and the visualisation of pictorial cues. In pursuance of this goal, the study was framed by the following guiding questions:

1. To what extent, if any, do individual pupils favour specific visualisation strategies when generalising figural patterns?
2. To what extent are pupils able to generalise patterning tasks, set in a pictorial context, in multiple ways?
3. What embodied processes are evinced by pupils engaged in figural pattern generalisation tasks?
4. In what ways do these embodied processes either assist or hinder pupils' ability to visualise figural cues in multiple ways?
5. Finally, in what ways can insights gleaned from the above be meaningfully employed in the pedagogical context of the classroom?

6.3 REVIEW OF THE THEORETICAL FRAMEWORK

The three central theoretical ideas which constituted the theoretical framework were *enactivism*, *figural apprehension*, and *knowledge objectification*.

From an *enactivist* stance, one needs to consider not only the formal mathematical ideas that emerge from action, but to give close scrutiny to those preceding actions – “the unformulated exploration, the undirected movement, the unstructured interaction, wherein the body is wholly engaged in mathematical play” (Davis et al., 1996, p. 156), in which language and action are seen not merely as outward manifestations of internal workings, but rather as “visible aspects of ... embodied (enacted) understandings” (Davis, 1995, p. 4).

Knowledge objectification is a theoretical construct to account for the manner in which learners engage or interact with a given scenario or context in order to make sense of it *en route* to a stable form of awareness (Radford, 2006, p. 7). Knowledge objectification is premised on two notions. Firstly, semiotic means such as gestures, rhythm and speech are not simply epiphenomena, but are seen to play a fundamental role in the formation of knowledge (Radford, 2005a, p. 142). Secondly, in order to study the process of knowledge production one needs to pay close scrutiny to *multiple* means of objectification, for example words, linguistic devices, gestures, rhythm, graphics and the use of artefacts, where “...meaning is forged out of the interplay of various semiotic systems” (Radford, 2005b, p. 144). The theoretical construct of knowledge objectification proved to be an ideal framework to complement the enactivist theoretical stance.

Duval’s concept of *figural apprehension* proved a meaningful means of discussing visual aspects of the phenomenological realm. Duval (1998, p. 41) makes the pertinent point that most diagrams contain a great variety of constituent *gestalts* and sub-configurations – far more than those initially identified through perceptual apprehension. Critically, it is this surplus that constitutes the *heuristic power* of a geometrical figure since specific sub-configurations may well trigger alternative solution paths. In order to actualise the heuristic potential of a diagram it is necessary not only to be aware of the scope of the diagram but also to be able to use it flexibly (Rösken & Rolka, 2006).

Within the context of figural pattern generalisation, the processes of visualisation and generalisation are deeply interwoven, with a complex relationship existing between the embodied processes of pattern generalisation and the visualisation of accompanying pictorial images. A theoretical framework based on a novel combination of three key complementary theoretical ideas (*enactivism*, *figural apprehension*, and *knowledge objectification*) proved effective in terms of shedding light on the tensions and complexities underlying pictorial pattern generalisation tasks.

6.4 REVIEW OF THE METHODOLOGY

This study was oriented within the conceptual framework of qualitative research, and was anchored within an interpretive paradigm. The study aimed ultimately to gain insights into the embodied processes of pupils' visualisation activity when engaged in figural pattern generalisation tasks through an in-depth analysis of each pupil's lived experience. A case study methodological strategy was adopted with the research participants representing the members of a mixed gender, high ability Grade 9 class of 23 pupils at an independent school in South Africa. The data collection and analysis occurred in two phases.

Phase 1 of the data generation process took the form of a series of pencil and paper exercises based on 10 linear generalisation tasks set in pictorial contexts. For each pattern participants were required to provide a numerical value for the 40th term (along with a written articulation of their reasoning), and an algebraic expression for the n^{th} term (along with a justification/explanation of their expression). The responses to the 10 linear generalisation tasks were classified in terms of the specific method or strategy employed. A coding system was developed to provide a nuanced characterisation of both numeric and visual strategies. In addition, a quasi-quantitative measure was used to characterise the extent to which pupils used the pictorial scenario as a referential context. Phase 1 of the study had a dual purpose. Firstly it aimed to identify those pupils who preferred *visual* as opposed to *numeric* approaches when solving pictorial generalisation tasks. Secondly, it characterised the extent to which individual pupils favoured *specific* visualisation strategies.

Eight research participants who were identified as preferring visual strategies took part in Phase 2. These eight research participants were individually provided with a further linear pattern and were required to provide multiple expressions for the n^{th} term. Tools such as paper, pencils and highlighters as well as appropriate manipulatives such as matchsticks and plastic counters were provided. Participants were asked to think aloud while engaged with their particular pattern generalisation task. Each session was audio-visually recorded and field-notes were taken. Audio-visual recordings were analysed with specific reference to participants' use of semiotic means of objectification such as words, linguistic devices, metaphor, gestures, rhythm, graphics and physical artefacts.

6.5 FINDINGS OF THIS STUDY

The findings of this study are presented as responses to each of the five guiding questions.

1) TO WHAT EXTENT, IF ANY, DO INDIVIDUAL PUPILS FAVOUR SPECIFIC VISUALISATION STRATEGIES WHEN GENERALISING FIGURAL PATTERNS?

A nuanced coding system was developed to characterise both numeric and visual strategies. The details of this coding system are provided in Table 4.1, and are briefly summarised here. Strategies were initially characterised as being based on either numeric reasoning (**N**) or visual reasoning (**V**). Strategies that contained a blend of visual and numeric reasoning were coded as **V/N**. Numeric components of reasoning were categorised either as **RA** (a rate-adjust method based on the additive difference between terms) or as **DF** (a direct formula being determined by trial and adjustment). Visual components of reasoning were categorised as either focusing on specific *local* features based on the additive unit (**LAU**, **LF1**, **LF2**, or **LF3**) or specific *global* features involving a building up process characterised by either an absence of overlapping structural units (**GF1**) or a presence of overlapping structural units (**GF2**).

An initial broad characterisation of strategies revealed a continuum with respect to numeric versus visual approaches. At one end of this continuum were pupils who

simply used the diagrams to extract numerical values which they then utilised with no further consideration for the pictorial scenario. At the other end of the continuum were pupils whose generalisation strategies all made explicit use of the given diagrams as a referential context. The majority of pupils, who made use of both visual as well as numeric approaches, were spread between these two extremes.

An analysis of specific *visual* approaches revealed a remarkable diversity of approaches across individual pupils. With the exception of four pupils, all other pupils made use of *at least* two different types of visual reasoning, exhibiting a continuum with respect to local versus global approaches. Local strategies make use of identifying a constant starting structure to which is added, in a recursive manner, multiples of the additive unit. Global strategies on the other hand view the overall structure of the pictorial terms more holistically. This involves the subdivision of the pictorial terms into constituent *gestalts* whose relationship to the overall structure is then determined. The visual reasoning behind these two broad types of visual strategy is markedly different, so what is of particular significance is that 19 of the 23 pupils made use of *both* local as well as global visual reasoning.

The results of this analysis suggest that very few pupils favour a specific visual strategy, with the majority of pupils being able to draw on a vast repertoire of visual approaches. However, there is evidence to suggest that the *nature of the pictorial patterns themselves* could act as potential triggers in terms of favouring specific visual strategies. Local strategies clearly dominated in those questions where the growth pattern occurred in a single direction and where progression from one term to the next could be accomplished by the direct attachment of the additive unit. Global strategies dominated in those questions in which the growth pattern occurred in more than one direction or in which progression from one term to the next could only be accomplished by the *insertion* of the additive unit *into* the previous term as opposed to the *direct attachment* of the additive unit *onto* the previous term.

2) TO WHAT EXTENT ARE PUPILS ABLE TO GENERALISE PATTERNING TASKS, SET IN A PICTORIAL CONTEXT, IN MULTIPLE WAYS?

The 8 research participants who took part in Phase 2 of the study each showed a remarkable ability to generalise their given pictorial pattern in multiple ways. All 8

participants were able to determine *at least* 6 different general expressions for their specific sequence, each with a different associated apprehension and justification, while one participant was able to generalise his pictorial pattern in an impressive 15 different ways. In most instances, other than those expressions arrived at through numerical considerations, the move from one apprehension to another was achieved through a reconfiguration of the whole-part relation of the given pictorial context by means of a recombination of elementary figural units.

3 & 4) WHAT EMBODIED PROCESSES ARE EVINced BY PUPILS ENGAGED IN FIGURAL PATTERN GENERALISATION TASKS, AND IN WHAT WAYS DO THESE EMBODIED PROCESSES EITHER ASSIST OR HINDER PUPILS' ABILITY TO VISUALISE FIGURAL CUES IN MULTIPLE WAYS?

A broad spectrum of embodied processes was exhibited by the research participants. This richly textured tapestry of activity was captured in the form of a series of vignettes through a multi-systemic semiotic analysis of participants' generalisation activity which included the use of words, linguistic devices, metaphor, gestures, rhythm, graphics and physical artefacts.

The physical process of building specific pictorial terms with physical manipulatives such as matchsticks and plastic counters was found to be useful in terms of the evolution of "new ways of seeing". In some instances this was due to the process of construction facilitating *sequential* apprehension as the physical additive unit was gradually brought forth as a recurring regularity. In other instances it was the process of unformulated exploration and unstructured interaction with the physical manipulatives that led to pivotal moments in the gradual emergence of alternative apprehensions.

All pupils made use of indexical gesturing, with either a pencil or with their fingers, while counting the individual elements of their pictorial patterns. Close scrutiny of this counting procedure proved to be most revealing, particularly in relation to its *economy*. Economical counting utilises the minimum amount of time and energy to individually count each element (e.g. each matchstick). Uneconomical counting on the other hand requires more time and energy to accomplish this task. Uneconomical counting methods proved to be a useful indicator of perceived structural regularity, since it is this perceived sense of structure that guides the systematic counting procedure, whether consciously or unconsciously.

There is also evidence to suggest that counting methods are able to unconsciously inspire specific apprehensions. In these instances the physical positioning of the individual elements (e.g. matchsticks) led to a sense of rhythm arising from the counting process itself, an artefact which in turn led to perceived, albeit possibly unconscious, structural regularity and thus to the development of a new apprehension.

The inherent sense of expectancy associated with rhythm was also a crucial element associated with participants' counting methods, particularly where rhythm was an artefact of a counting method inspired by a perceived structural regularity. This inherent sense of expectancy, when suddenly unfulfilled, was key to creating an awareness of unpaired objects or other structural irregularities.

All pupils created physical instantiations of specific terms of their pictorial sequence through the process of drawing. This drawing process was in many ways a two-edged sword. In some instances the physical process of drawing led to the emergence of structural commonalities or regularities. This supported the generalisation process where these regularities were algebraically useful (i.e. where the generality of what was noticed in the phenomenological realm could be readily expressed using algebraic symbolism). However, where the perceived regularity was algebraically unhelpful in terms of expressing generality, this had the danger of obfuscating the generalisation process. Particularly in those instances where the perceived regularity was associated with strong visual imagery, this often became a hindrance to the emergence of other potentially more useful sub-structures or gestalts. Furthermore, the physical process of drawing often led to attention being focused on the recursive nature of the step-by-step process of constructing each term from the previous one, thereby foregrounding *local* considerations rather than allowing for a more holistic or *global* apprehension. However, for some pupils the drawing process played a crucial role in stabilizing the particular visualisation and in developing an awareness of its generality. Careful observation of the drawing process also proved useful from the point of view of analysis in that it was able to reveal underlying visual tension between different apprehensions of the same pictorial context.

Indexical or deictic gesturing was also used to signify existing structures in the particular diagrams under scrutiny. More importantly, however, were those gestures that moved from indicating materially instantiated aspects of the pictorial term to the miming of similar aspects or structures in terms that were not yet materially present. This *imaginative* signification (Sabena et al., 2005, p. 134) played an important role in objectifying the generality of structural aspects of the pictorial terms through a process of progressive distancing from the physical referent.

5) IN WHAT WAYS CAN INSIGHTS GLEANED FROM THIS STUDY BE MEANINGFULLY EMPLOYED IN THE PEDAGOGICAL CONTEXT OF THE CLASSROOM?

During the course of the micro-analysis number of broad insights gradually emerged and evolved. These are synthesised here in relation to possible pedagogical strategies that could be used to support pictorial pattern generalisation activities. Firstly, pupils should be encouraged consciously to engage with pictorial terms by:

- Searching for structural features that contain as many elements as the term number (n), or that occur as many times as the term number.
- Applying the above strategy more generally by searching for features or structural units that contain *nearly* as many elements as the term number itself (e.g. $n \pm 1$ or $n \pm 2$) or that occur nearly as many times as the term number.
- Identifying elements of symmetry such as left-right equivalence or symmetrical structures radiating out from a central point.
- Identifying visually striking geometrical features that could be used as structural keystones for particular apprehensions. These features could either be recurring elements in the diagrams or solitary items.

Pupils should be encouraged to look for comparative regularities between only even-numbered or odd-numbered terms. Unexpected visual commonalities may be perceived in this manner that could serve as crucial triggers to occasion the evolution of new general formulae. Although the general formula thus determined may not necessarily make visual sense with respect to *all* the terms in the pictorial sequence, it would still be algebraically correct. This observation in itself could open up interesting classroom discussion.

Pupils should guard against the pitfalls of single-case concreteness. Within the context of pattern generalisation the crux of the enterprise lies in an evolving sense of generality. Prolonged focus on a single pictorial term may well act against this central endeavour. Pupils should thus be encouraged to look for commonalities between different terms, preferably non-consecutive terms since this is more likely to occasion a more holistic structural perception where attention is not necessarily focused on the additive unit.

Where pupils are able to describe perceived visual regularities but are unable to express this regularity in an algebraically useful manner, the following strategies may be useful:

- Tabulate a summary of structural features along with the total number of occurrences of each structural feature for specific terms.
- Make use of (i.e. draw or construct) pictorial terms further along in the sequence (e.g. $n \geq 6$) to search for structural regularities. Larger terms often act as more efficient triggers than smaller terms.
- Investigate Term 1. There are often structural anomalies or subtle differences with smaller terms that may well trigger structural understanding.

When teachers present pictorial patterns to the class they should take care not to make use of diagrams in which the term number also represents the number of elements in structural features that are likely to be brought forth by pupils. So, for instance, if there is a likelihood of pupils focusing on squares in a particular sequence, avoid Term 4. Similarly avoid Term 3 if it contains triangular structures (or any other potential 3-unit features) that could act as triggers. This should help avoid confusion arising from situations where the same numerical value represents different conceptual aspects of the given pictorial term.

In terms of expressing different visualisations in the form of algebraic expressions, some pupils may find it useful to make use of a stepwise process of semiotic contraction. By way of example, verbal expressions such as “1 multiplied 3 by 1 less than the shape number” could first be expressed in the form “3 times the shape number minus 1” as an interim step en route to the algebraic symbolism $3(n-1)$.

The advantage of this approach is that the interim verbal syntax is far more closely aligned with the desubjectified algebraic symbolism.

Pupils should be encouraged to look out for serendipitous numerical observations that could lead to the development of general algebraic expressions. For example, if $T_6 = 21$ one could make the numerical observation that $21 = 3 \times 7$. Since 7 in this instance is 1 more than the term number this could lead to an investigation to assess whether this situation is always true, in which case $T_n = 3(n+1)$ could be an appropriate algebraic formula for the general term. Having determined this general rule numerically, one could then search for an associated visual justification.

Certain features of pictorial patterns tend to encourage particular generalisation strategies. Since one would want pupils to be able to experience a range of strategies, a range of pictorial patterns should be included in patterning tasks. These should include:

- Questions where the growth pattern occurs in a single direction and where progression from one term to the next can be accomplished by the direct attachment of the additive unit.
- Questions in which the growth pattern occurs in more than one direction.
- Questions in which progression from one term to the next can only be accomplished by the *insertion* of the additive unit *into* the previous term as opposed to the *direct attachment* of the additive unit *onto* the previous term.

Teachers should also be aware of the subtle semantic ambiguity (Samson, 2011) associated with expressions of generality within the context of pictorial patterns. For instance, in a certain matchstick pattern $2n$ could represent either “2 multiples of n matches” or “ n multiples of 2 matches”, both interpretations of which could have an associated visual apprehension. This semantic ambiguity has the potential to open up interesting spaces for classroom discussion.

Finally, while a conscious search for structure is a useful generalisation strategy, so too is unstructured exploration and interaction with the pictorial context - a process which could lead to the serendipitous awareness of structural regularity. Pupils should be encouraged to make use of physical manipulatives (such as matchsticks

and plastic counters) to encourage such unstructured exploration. Even pupils who are sceptical about the use of manipulatives, and who profess to preferring more visual or abstract engagement, should be encouraged to make use of them. It is often the tactile, physical and whole-body engagement of such activity that leads to unconscious moments of mathematical play that could serve as crucial pivots for the evolution of new “ways of seeing”.

6.6 LIMITATIONS

A case study approach was adopted as methodological strategy for this study. Accordingly, the members of a mixed gender, high ability Grade 9 class of 23 pupils were chosen as research participants - “information-rich cases whose study will illuminate the questions under study” (Patton, 1990, p. 169). This purposeful sampling was justified in terms of previous research (Samson, 2007a) as well as a pilot study undertaken prior to the commencement of the main study. Although the emphasis of a case study is to optimise understanding of the specific case under scrutiny rather than generalisation beyond that case, a case study can nonetheless be a useful small step towards a larger generalisation, or an increasingly refined generalisation (Cohen & Manion, 1994; Stake, 1994, 1995). Thus, although any general observations made in the course of this study are only relevant to the group of research participants who took part in the study, such “generalisations” could be broadened or increasingly refined by future research involving further samples from the larger population.

The decision to choose only high-ability pupils was a methodological one based on the data collection protocol which required pupils to provide both written and verbal articulations of their own reasoning as well as to provide algebraic expressions for the general term. Furthermore, only those pupils who preferred visual modes of pattern generalisation took part in the second phase of the data collection process. Thus, two limitations of this study are that the embodied processes evinced by pupils are restricted to those emanating from pupils who are characteristically visual in terms of their generalisation strategies and who are high-ability in terms of their general mathematical capabilities. However, rather than being seen as a limitation,

the purposeful choice of participants, and the richly textured data they provided, can be considered to be a *strength* of this study.

Although a broad range of semiotic activity was scrutinised in the course of this study – words, linguistic devices, gestures, rhythm, graphics and the use of physical artefacts – this list of semiotic means of objectification is by no means exhaustive. Additional elements of the data analysis could have included a prosodic analysis of verbal utterances focusing on temporal distribution of words and word intensity (Radford, 2008, p. 91; Radford et al., 2007, p. 522), as well as an analysis of saccadic eye movement focusing on rapid changes in visual attention. Absence of these elements from the analysis protocol can also be seen as a limitation.

6.7 SIGNIFICANCE

The richly textured tapestry of activity captured through a multi-systemic semiotic analysis of participants' generalisation activity stands testament to the central thesis of this study: that the combined complementary multiple perspectives of enactivism, figural apprehension and knowledge objectification add a powerful depth of analysis to the exploration of the inter-relationship between the embodied processes of pattern generalisation and the visualisation of pictorial cues.

Furthermore, this framework allows for an additional depth of analysis when compared with other frameworks presently employed to analyse the process of pattern generalisation. This extra layer of insight arises from the complementary multiple perspectives that constitute the framework of analysis. Not only does this framework acknowledge perception as being critically related to the *manner* of one's interaction with perceptual objects, but it also remains sensitive to both the phenomenological and semiotic aspects of the generalisation process. This combined lens allows the researcher access to the subtle yet powerful underlying tensions that exist as different modes of figural apprehension jostle for prominence (Samson & Schäfer, 2011).

Of further significance is that this study provides critical insights into interwoven pedagogical issues relating to three key educational topics: pattern generalisation,

visualisation, and the cognitive significance of embodied processes. Hamilton (2006, p. 4) comments that "...learning refers to transformations that expand the learner's potential range of actions." Pedagogical insights gleaned from this study aim to empower pupils with appropriate strategies to interpret figural patterns in multiple ways by moving flexibly between different modes of apprehension, thus creating the potential for just such transformations. It is thus a strength of this study that it is able to provide practical strategies which support and encourage a multiple representational approach to pattern generalisation in the pedagogical context of the classroom.

A review of pertinent literature relating specifically to figural pattern generalisation identified four broad categories of focus: (a) descriptions of solution strategies and levels of attainment, (b) the influence of task design and the nature of the pictorial terms, (c) the transition between pupils' arithmetic and algebraic reasoning, and (d) the affordances offered by technological environments. However, within the context of figural pattern generalisation, little empirical research focusing on the *process* of visualisation, as opposed to the *product* of visualisation, seems to have been carried out. Thus, amongst other things, this study has also gone some way to addressing a lacuna in the research literature. In addition, this study supports and strongly resonates with Rivera's (2007) view of visualisation being an important and meaningful approach to the contemporary treatment of algebra in schools:

...fostering visualization in school algebra articulates the most important description we have about algebra in contemporary times – that is, algebra as the symbolic medium that provides the systematic means to establishing, constructing, and justifying invariant structures and relationships among mathematical objects. Such a medium, by institutional practice and as a consequence of its historical evolution, has been narrowly interpreted in our classrooms as being primarily about manipulating variables and expressions. Visualization in algebra offers an alternative way to understand structures and relationships that necessitate the use of variables. (p. 75)

6.8 RECOMMENDATIONS FOR FURTHER RESEARCH

The central portion of this study focused on a high ability group of pupils who exhibited a preference for visual over numeric modes of pictorial pattern generalisation. It would be interesting to repeat this study using (i) a lower ability

group of research participants, and (ii) participants whose preference was *not* for visual methods. The former of these considerations would in all likelihood require a modification of the data collection protocol where the emphasis would need to lie on *numerical* as opposed to *algebraic* expressions of generality. The latter consideration may also require a methodological modification such as the incorporation of *collaborative* rather than *individual* generalisation tasks.

It would be equally interesting to repeat the present study with other high ability groups of pupils, possibly with an augmented selection of patterning questions. This would add further insight into the complex underlying tensions that exist as different modes of figural apprehension jostle for prominence.

6.9 CONCLUDING COMMENTS

The connections between mathematics and the concepts of pattern and generalisation permeate all levels of mathematical endeavour. Furthermore, the cognitive significance of the body has become one of the major topics in current psychology (Radford et al., 2005, p. 113). In addition, the use of multiple representations has been acknowledged as playing a central role in problem solving, the learning and understanding of mathematical ideas, and the development of a deeper appreciation for the interconnections between mathematical concepts (Goldin, 2002; Greeno & Hall, 1997; Kaput, 1998). Not only are these fundamental mathematical concepts, but they are critically contextualised within contemporary curriculum reform in South Africa.

As Adler (2005, p. 2) succinctly comments, “at the most basic level, we have yet to understand how to make mathematics learnable by all children.” By focusing on issues of visualisation and pattern generalisation, central components of mathematical activity, this study has afforded me with deeper insight not only into the pedagogical issues relating to these fundamental concepts, but also into the critical notion of mathematical accessibility. Furthermore, an awareness of and appreciation for a diversity of visualisation processes considered from an enactivist theoretical framework has significantly contributed to my own pedagogical discourse as a practising mathematics educator and researcher.

REFERENCES

- Abbott, S.** (1992). Algebra with multilink cubes. *Mathematics in School*, 21(2), 12-13.
- Abramovich, S., Fujii, T., & Wilson, J.** (1994). Exploring and visualizing properties of polygonal numbers in a multiple-application computer-enhanced environment. In G.H. Marks (Ed.), *Proceedings of the 1994 International Symposium on Mathematics/Science Education and Technology*. Charlottesville, VA: AACE.
- Adler, J.** (2005). Mathematics for teaching: What is it and why is it important that we talk about it? *Pythagoras*, 62, 2-11.
- Ainley, J., Wilson, K., & Bills, L.** (2003). Generalising the context and generalising the calculation. In N.A. Pateman, B.J. Dougherty & J.T. Zilliox (Eds.), *Proceedings of the 27th Conference of the International Group for the Psychology of Mathematics Education held jointly with the 25th Conference of PME-NA, Volume 2* (pp. 9-16). Honolulu, HI: PME.
- Ainsworth, S.** (2006). DeFT: A conceptual framework for considering learning with multiple representations. *Learning and Instruction*, 16(3), 183-198.
- Aldrich, F., & Sheppard, L.** (2000). „Graphicacy“: The fourth „R“? *Primary Science Review*, 64, 8-11.
- Amit, M., & Fried, M.N.** (2005). Multiple representations in 8th grade algebra lessons: Are learners really getting it? In H.L. Chick & J.L. Vincent (Eds.), *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education, Volume 2* (pp. 57-64). Melbourne, Australia: PME.
- Andrews, P.** (1990). Generalising number patterns. *Mathematics in School*, 19(4), 9-13.
- Arcavi, A.** (1999). The role of visual representations in the learning of mathematics. In F. Hitt & M. Santos (Eds.), *Proceedings of the 21st Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Volume 1* (pp. 55-80). Cuernavaca, Morelos, Mexico.
- Arcavi, A.** (2003). The role of visual representations in the learning of mathematics. *Educational Studies in Mathematics*, 52, 215-241.
- Arzarello, F., & Edwards, L.** (2005). Gesture and the construction of mathematical meaning. In H.L. Chick & J.L. Vincent (Eds.), *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education, Volume 1* (pp. 123-127). Melbourne: PME.
- Barwise, J., & Etchemendy, J.** (1991). Visual information and valid reasoning. In W. Zimmermann & S. Cunningham (Eds.), *Visualization in teaching and learning mathematics* (pp. 9-24). Washington: Mathematical Association of America.
- Battista, M.T.** (2007). The development of geometric and spatial thinking. In F.K. Lester, Jr. (Ed.), *Second handbook of research on mathematics teaching and learning: A project of the National Council of Teachers of Mathematics, Volume 2* (pp. 843-908). Charlotte, NC: Information Age Publishing.
- Becker, J.R., & Rivera, F.D.** (2007). Factors affecting seventh graders' cognitive perceptions of patterns involving constructive and deconstructive generalizations. In J.H. Woo, H.C. Lew, K.S. Park & D.Y. Seo (Eds.), *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education, Volume 4* (pp. 129-136). Seoul: PME.
- Becker, J.R., & Rivera, F.D.** (2008). Generalization in algebra: the foundation of algebraic thinking and reasoning across the grades. *ZDM Mathematics Education*, 40, 1.
- Begg, A.** (2000). *Enactivism: A personal interpretation*. Retrieved April 11, 2008, from http://www.emr.vic.au/Downloads/enactivism_andybegg.doc
- Bennie, K., Blake, P., & Fitton, S.** (2005). *On track with maths: Grade 10 learner's book*. Cape Town: Phumelela.
- Benveniste, É.** (1985). The semiology of language. In R.E. Innis (Ed.), *Semiotics: An introductory anthology* (pp. 228-246). Bloomington: Indiana University Press.
- Bishop, A.J.** (1989). Review of research in visualization in mathematics education. *Focus on Learning Problems in Mathematics*, 11(1), 7-16.

- Bishop, J.** (2000). Linear geometric number patterns: Middle school students' strategies. *Mathematics Education Research Journal*, 12(2), 107-126.
- Breen, C.** (2005). Promising practices in teaching and learning. In C. Kasanda, L. Muhammed, S. Akpo & E. Ngololo (Eds.), *Proceedings of the 13th Annual SAARMSTE Conference*. Windhoek, Namibia.
- Brown, J.R.** (1997). Proofs and pictures. *The British Journal for the Philosophy of Science*, 48(2), 161-180.
- Byatt, S.** (1994). Blindfolding the train-spotters! *Micromath*, 10(1), 25-26.
- Campbell, S., & Dawson, A.J.** (1995). Learning as embodied action. In R. Sutherland & J. Mason (Eds.), *Exploiting mental imagery with computers in mathematics education* (pp. 233-249). Berlin: Springer.
- Chua, B.L.** (2009). Features of generalising tasks: Help or hurdle to expressing generality? *The Australian Mathematics Teacher*, 65(2), 18-24.
- Clausen, T.** (1992). Say it with pictures. *Mathematics in School*, 21(4), 17-19.
- Cohen, L., & Manion, L.** (1994). *Research methods in education* (4th ed.). London: Routledge.
- Cohen, L., Manion, L., & Morrison, K.** (2000). *Research methods in education* (5th ed.). London: RoutledgeFalmer.
- Confrey, J.** (1990). What constructivism implies for teaching. In R. Davis, C. Maher & N. Noddings (Eds.), *Constructivist views on the teaching and learning of mathematics. Journal for Research in Mathematics Education. Monograph 4.* (pp. 107-122). Reston, VA: NCTM.
- Crookes, D.Z.** (1988). De pulchritudine numerorum figuratorum (On the beauty of figurate numbers). *Mathematics in School*, 17(3), 38-39.
- Cunningham, S.** (1991). The visualization environment for mathematics education. In W. Zimmermann & S. Cunningham (Eds.), *Visualization in teaching and learning mathematics* (pp. 67-76). Washington: Mathematical Association of America.
- Cuoco, A., Goldenberg, E.P., & Mark, J.** (1996). Habits of mind: An organizing principle for mathematics curricula. *Journal of Mathematical Behaviour*, 15, 375-402.
- Damasio, A.R.** (1994). *Descartes' error: Emotion, reason, and the human brain*. New York, NY: Grosset/Putnam.
- Davis, B.** (1995). Why teach mathematics? Mathematics education and enactivist theory. *For the Learning of Mathematics*, 15(2), 2-9.
- Davis, B.** (1997). Listening for differences: An evolving conception of mathematics teaching. *Journal for Research in Mathematics Education*, 28(3), 355-376.
- Davis, B., Sumara, D.J., & Kieren, T.E.** (1996). Cognition, co-emergence, curriculum. *Journal of Curriculum Studies*, 28(2), 151-169.
- Dawson, S.** (1999). The enactive perspective on teacher development: "A path laid while walking". In B. Jaworski, T. Wood & S. Dawson (Eds.), *Mathematics Teacher Education: Critical International Perspectives* (pp. 148-162). London: Falmer.
- De Jager, T.** (1999). Do we encourage pupils to search for pattern incorrectly? *Pythagoras*, 50, 20-24.
- De Jager, T.** (2004). Introducing and teaching first-degree equations. *Learning and Teaching Mathematics*, 1, 8-11.
- De Mestre, N.** (2001). Discovery: Mathematical patterns. *The Australian Mathematics Teacher*, 57(1), 34-35.
- Denzin, N.K., & Lincoln, Y.S.** (2003). Introduction: The discipline and practice of qualitative research. In N.K. Denzin & Y.S. Lincoln (Eds.), *The landscape of qualitative research: Theories and issues* (2nd ed.) (pp. 1-45). Thousand Oaks, CA: Sage.
- Department of Education** (2002). *Revised National Curriculum Statement Grades R-9 (Schools): Mathematics*. Pretoria: Government Printer.
- Department of Education** (2003a). *Revised National Curriculum Statement Grades R-9 (Schools): Teacher's guide for the development of learning programmes: Mathematics*. Pretoria: Government Printer.
- Department of Education** (2003b). *National Curriculum Statement Grades 10-12 (General): Mathematics*. Pretoria: Government Printer.

- De Waal, L., McAlister, S., Müller, A., Wallace, S., & Williams, G.** (Eds.) (2005). *Shuters mathematics: Grade 10 learner's book*. Pietermaritzburg: Shuter & Shooter.
- Dindyal, J.** (2007). High school students' use of patterns and generalisations. In J. Watson & K. Beswick (Eds.), *Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia, Volume 1* (pp. 236-245). MERGA.
- Dörfler, W.** (1991). Meaning: Image schemata and protocols. In F. Furinghetti (Ed.), *Proceedings of the 15th Conference of the International Group for the Psychology of Mathematics Education, Volume 1* (pp. 17-32). Assisi, Italy.
- Dreyfus, T.** (1991). Advanced mathematical thinking processes. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 25-41). Dordrecht: Kluwer.
- Dupuy, J-P., & Varela, F.J.** (1992). Understanding origins: An introduction. In F.J. Varela & J-P. Dupuy (Eds.), *Understanding origins: Contemporary views on the origin of life, mind and society* (pp. 1-25). Dordrecht: Kluwer.
- Durant, W.** (1947). *The story of philosophy*. London: Ernest Benn Limited.
- Duval, R.** (1995). Geometrical pictures: Kinds of representation and specific processings. In R. Sutherland & J. Mason (Eds.), *Exploiting mental imagery with computers in mathematics education* (pp. 142-157). Berlin: Springer.
- Duval, R.** (1998). Geometry from a cognitive point of view. In C. Mammana & V. Villani (Eds.), *Perspectives in the teaching of geometry for the 21st century* (pp. 37-52). Boston: Kluwer.
- Duval, R.** (1999). Representation, vision and visualization: Cognitive functions in mathematical thinking. Basic issues for learning. In F. Hitt & M. Santos (Eds.), *Proceedings of the 21st Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Volume 1* (pp. 3-26). Cuernavaca, Morelos, Mexico.
- Duval, R.** (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics*, 61(1-2), 103-131.
- Eisenberg, T.** (1994). On understanding the reluctance to visualize. *Zentralblatt für Didaktik der Mathematik (ZDM)*, 26(4), 109-113.
- Eisenberg, T., & Dreyfus, T.** (1991). On the reluctance to visualise in mathematics. In W. Zimmermann & S. Cunningham (Eds.), *Visualization in teaching and learning mathematics* (pp. 25-37). Washington: Mathematical Association of America.
- English, L.D., & Warren, E.A.** (1998). Introducing the variable through pattern exploration. *The Mathematics Teacher*, 91(2), 166-170.
- Ernest, P.** (2006). Reflections on theories of learning. *ZDM Mathematics Education*, 38(1), 3-7.
- Even, R.** (1998). Factors involved in linking representations of functions. *Journal of mathematical behavior*, 17(1), 105-121.
- Farmer, J.D., & Neumann, A.M.** (2004). Patterns in perfect squares: An activity for exploring mathematical connections. *Mathematics Teacher*, 98(4), 260-265.
- Felix, C.B.A.** (1998). From number patterns to algebra: A cognitive reflection on a Cape Flats experience. In A. Olivier & K. Newstead (Eds.), *Proceedings of the 22nd Conference of the International Group for the Psychology of Mathematics Education, Volume 2* (pp. 288-295). Stellenbosch, South Africa.
- Fenwick, T.J.** (2000). Expanding conceptions of experiential learning: A review of the five contemporary perspectives on cognition. *Adult Education Quarterly*, 50(4), 243-272.
- Fenwick, T.J.** (2001a). Work knowing "on the fly": Enterprise cultures and co-emergent epistemology. *Studies in Continuing Education*, 23(2), 243-259.
- Fenwick, T.J.** (2001b). *Experiential learning: A theoretical critique from five perspectives. Information series no. 385*. Columbus, OH: ERIC Clearinghouse on Adult, Career, and Vocational Education.
- Fischbein, E.** (1987). *Intuition in science and mathematics: An educational approach*. Dordrecht: Kluwer.
- Fischbein, E.** (1993). The theory of figural concepts. *Educational Studies in Mathematics*, 24(2), 139-162.
- Font, V., Godino, J.D., & D'Amore, D.** (2007). An onto-semiotic approach to representations in mathematics education. *For the Learning of Mathematics*, 27(2), 2-7 & 14.

- Fraisse, P.** (1982). Rhythm and tempo. In D. Deutsch (Ed.), *The psychology of music* (pp. 149-180). Orlando, FL: Academic Press.
- French, D.** (1990). Sums of squares and cubes. *Mathematics in School*, 19(3), 34-37.
- French, D.** (2002). Beginnings. In D. French (Ed.), *Teaching and learning algebra* (pp. 26-45). London: Continuum.
- Frobisher, L., & Threlfall, J.** (1999). Teaching and assessing patterns in number in the primary years. In A. Orton (Ed.), *Pattern in the teaching and learning of mathematics* (pp. 84-103). London: Cassell.
- Gal, H., & Linchevski, L.** (2002). Situations by theory of perception. In A.D. Cockburn & E. Nardi (Eds.), *Proceedings of the 26th Conference of the International Group for the Psychology of Mathematics Education, Volume 4* (pp. 81-88). Norwich, UK: PME.
- Garcia, M.L., Benitez, A.A., & Ruiz, E.F.** (2010). Using multiple representations to make and verify conjectures. In P. Brosnan, D.B. Erchick & L. Flevaris (Eds.), *Proceedings of the 32nd annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Volume VI* (pp. 270-278). Columbus, OH: The Ohio State University.
- Geraniou, E., Mavrikis, M., Hoyles, C., & Noss, R.** (2008). A constructionist approach to mathematical generalisation. In M. Joubert (Ed.), *Proceedings of the British Society for Research into Learning Mathematics*, 28(2), 37-42. Retrieved December 30, 2008, from <http://www.bsrlm.org.uk/IPs/ip28-2/BSRLM-IP-28-2-07.pdf>
- Goba, B., & van der Lith, D.** (2005). *Study & master mathematics: Grade 10 learner's book*. Cambridge: Cambridge University Press.
- Gobert, S.** (2007). Ways of thinking about the uses of images in learning and teaching geometry: A more thorough investigation of the links between drawings and figures. In D. Pitta-Pantazi & G. Philippou (Eds.), *Proceedings of the Fifth Congress of the European Society for Research in Mathematics Education* (pp. 121-130). Larnaca, Cyprus.
- Goldin, G.A.** (2002). Representation in mathematical learning and problem solving. In L.D. English (Ed.), *Handbook of international research in mathematics education* (pp. 197-218). Mahwah, NJ: Erlbaum.
- Graham, A., & Honey, S.** (2009). Algebra – I just don't get it. *Mathematics Teaching*, 212, 38-41.
- Graven, M.** (2002). The effect of the South African curriculum change process on mathematics teacher roles. In P. Valero & O. Skovsmose (Eds.), *Proceedings of the 3rd International MES Conference* (pp. 1-10). Copenhagen: Centre for Research in Learning Mathematics. Retrieved May 27, 2011, from <http://www.mes3.learning.aau.dk/Papers/Graven.pdf>
- Greeno, J.G.** (1997). On claims that answer the wrong questions. *Educational Researcher*, 26(1), 5-17.
- Greeno, J.G., & Hall, R.P.** (1997). Practicing representation. *Phi Delta Kappan*, 78(5), 361-367.
- Gutiérrez, A.** (1996). Visualization in 3-dimensional geometry: In search of a framework. In L. Puig & A. Gutiérrez (Eds.), *Proceedings of the Twentieth International Conference for the Psychology of Mathematics Education, Volume 1* (pp. 3-19). Valencia: Universitat de València.
- Hadjidemetriou, C., Pampaka, M., Petridou, A., Williams, J., & Wo, L.** (2007). Developmental assessment of algebraic performance. In D. Pitta-Pantazi & G. Philippou (Eds.), *Proceedings of the Fifth Congress of the European Society for Research in Mathematics Education* (pp. 893-902). Larnaca, Cyprus.
- Hamilton, M.** (2006). *Enactivism: a theory of learning for the 21st century*. Retrieved April 11, 2008, from http://www.emr.vic.edu.au/Downloads/Enactivism_theory_learning.doc
- Hargreaves, M., Shorrocks-Taylor, D., & Threlfall, J.** (1998). Children's strategies with number patterns. *Educational Studies*, 24(3), 315-331.
- Hargreaves, M., Threlfall, J., Frobisher, L., & Shorrocks-Taylor, D.** (1999). Children's strategies with linear and quadratic sequences. In A. Orton (Ed.), *Pattern in the teaching and learning of mathematics* (pp. 67-83). London: Cassell.

- Healy, L., & Hoyles, C.** (1996). Seeing, doing and expressing: An evaluation of task sequences for supporting algebraic thinking. In L. Puig & A. Gutiérrez (Eds.), *Proceedings of the Twentieth International Conference for the Psychology of Mathematics Education, Volume 3* (pp. 67-74). Valencia: Universitat de València.
- Healy, L., & Hoyles, C.** (1999). Visual and symbolic reasoning in mathematics: Making connections with computers? *Mathematical Thinking and Learning*, 1(1), 59-84.
- Hegarty, M., & Kozhevnikov, M.** (1999). Types of visual-spatial representations and mathematical problem solving. *Journal of Educational Psychology*, 91(4), 684-689.
- Helson, H.** (1933). The fundamental propositions of Gestalt psychology. *Psychological Review*, 40, 13-32.
- Herman, M.** (2007). What students choose to do and have to say about use of multiple representations in college algebra. *Journal of Computers in Mathematics and Science Teaching*, 26(1), 27-54.
- Hershkowitz, R., Arcavi, A., & Bruckheimer, M.** (2001). Reflections on the status and nature of visual reasoning – the case of the matches. *International Journal of Mathematical Education in Science and Technology*, 32(2), 255-265.
- Hershkowitz, R., Ben-Chaim, D., Hoyles, C., Lappan, G., Mitchelmore, M., & Vinner, S.** (1990). Psychological aspects of learning geometry. In P. Nesher & J. Kilpatrick (Eds.), *Mathematics and Cognition: A Research synthesis by the International Group for the Psychology of Mathematics Education* (pp. 70-95). Cambridge: Cambridge University Press.
- Hershkowitz, R., Dreyfus, T., Ben-Zvi, D., Friedlander, A., Hadas, N., Resnick, T., et al.** (2002). Mathematics curriculum development for computerised environments: A designer-researcher-teacher-learner activity. In L.D. English (Ed.), *Handbook of international research in mathematics education* (pp. 657-694). Mahwah, NJ: Lawrence Erlbaum Associates.
- Hershkowitz, R., & Markovits, Z.** (1992). Conquer mathematics concepts by developing visual thinking. *Arithmetic Teacher*, 39(9), 38-41.
- Hewitt, D.** (1992). Train spotters" paradise. *Mathematics Teaching*, 140, 6-8.
- Hiebert, J., Carpenter, T.P., Fennema, E., Fuson, K.C., Wearne, D., Murray, H., et al.** (Eds.). (1997). *Making sense: Teaching and learning mathematics with understanding*. Portsmouth, NH: Heinemann.
- Hoz, R.** (1981). The effects of rigidity on school geometry learning. *Educational Studies in Mathematics*, 12(2), 171-190.
- Jackson, W.** (1995). *Methods: Doing social research*. Scarborough, Ontario: Prentice-Hall Canada.
- Johnson, M.L.** (1999). Embodied reason. In G. Weiss & H.F. Haber (Eds.), *Perspectives on embodiment: The intersections of nature and culture* (pp. 81-102). New York, NY: Routledge.
- Jones, K.** (1998). Theoretical frameworks for the learning of geometrical reasoning. *Proceedings of the British Society for Research into Learning Mathematics*, 18(1&2), 29-34.
- Kaput, J.J.** (1998). Representations, inscriptions, descriptions and learning: A kaleidoscope of windows. *Journal of Mathematical Behavior*, 17(2), 265-281.
- Kaput, J.J.** (1999). Teaching and learning a new algebra. In E. Fennema & T.A. Romberg (Eds.), *Mathematics classrooms that promote understanding* (pp. 133-155). Mahwah, NJ: Lawrence Erlbaum.
- Kaput, J.** (2002). Research on the development of algebraic reasoning in the context of elementary mathematics: A brief historical overview. In D.S. Mewborn, P. Sztajn, D.Y. White, H.G. Wiegel, R.L. Bryant & K. Nooney (Eds.), *Proceedings of the 24th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Volume 1* (pp. 120-122). Athens, GA: PME-NA.
- Katz, D.** (1951). *Gestalt psychology: Its nature and significance*. London: Methuen.
- Kellman, P.J.** (2000). An update on Gestalt psychology. In B. Landau, J. Sabini, J. Jonides & E. Newport (Eds.), *Perception, cognition, and language: Essays in honour of Henry and Lila Gleitman*, (pp. 157-190). Cambridge, MA: MIT Press.

- Kenney, P.A., Zawojewski, J.S., & Silver, E.A.** (1998). Marcy's dot pattern. *Mathematics Teaching in the Middle School*, 3(7), 474-477.
- Kilpatrick, J., Swafford, J., & Findell, B.** (Eds.) (2001). *Adding it up: Helping children learn mathematics*. Mathematics Learning Study Committee. Washington, DC: National Academy Press.
- Kirby, K.N., & Kosslyn, S.M.** (1992). Thinking visually. In G.W. Humphreys (Ed.), *Understanding vision: An interdisciplinary perspective* (pp. 71-86). Oxford: Blackwell
- Kirshner, D., & Whitson, J.A.** (1997). Editors' introduction to Situated Cognition: Social, Semiotic, and Psychological Perspectives. In D. Kirshner & J.A. Whitson (Eds.), *Situated cognition: Social, semiotic, and psychological perspectives* (pp. 1-16). Mahwah, NJ: Lawrence Erlbaum.
- Koffka, K.** (1935). *Principles of Gestalt psychology*. London: Routledge & Kegan Paul.
- Krutetskii, V.A.** (1973). *The psychology of mathematical abilities in schoolchildren*. Chicago: University of Chicago Press.
- Kvale, S.** (1996). *InterViews: An introduction to qualitative interviewing*. Thousand Oaks, CA: Sage.
- Lakoff, G.** (1987). *Women, fire, and dangerous things: What categories reveal about the mind*. Chicago: The University of Chicago Press.
- Lakoff, G., & Johnson, M.** (1980). *Metaphors we live by*. Chicago: The University of Chicago Press.
- Lakoff, G., & Johnson, M.** (1999). *Philosophy in the flesh: The embodied mind and its challenge to western thought*. New York, NY: Basic Books.
- Lakoff, G., & Núñez, R.E.** (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*. New York, NY: Basic Books.
- Lannin, J.K.** (2003). Developing algebraic reasoning through generalization. *Mathematics Teaching in the Middle School*, 8(7), 342-348.
- Lannin, J.K.** (2004). Developing mathematical power by using explicit and recursive reasoning. *Mathematics Teacher*, 98(4), 216-223.
- Lannin, J.K.** (2005). Generalization and justification: The challenge of introducing algebraic reasoning through patterning activities. *Mathematical Thinking and Learning*, 7(3), 231-258.
- Lapp, D.A.** (1999). Multiple representations for pattern exploration with the graphing calculator and manipulatives. *The Mathematics Teacher*, 92(2), 109-113.
- Lave, J., & Wenger, E.** (1991). *Situated learning: Legitimate peripheral participation*. Cambridge: Cambridge University Press.
- LeBaron, C., & Streeck, J.** (2000). Gestures, knowledge, and the world. In D. McNeill (Ed.), *Language and gesture* (pp. 118-138). Cambridge: Cambridge University Press.
- Lee, L.** (1996). An initiation into algebraic culture through generalization activities. In N. Bednarz, C. Kieran & L. Lee (Eds.), *Approaches to algebra: Perspectives for research and teaching* (pp. 87-106). Dordrecht: Kluwer.
- Lewis, J., & Ritchie, J.** (2003). Generalising from qualitative research. In J. Ritchie & J. Lewis (Eds.), *Qualitative research practice: A guide for social science students and researchers* (pp. 263-286). London: Sage.
- Lincoln, Y.S., & Guba, E.G.** (2003). Paradigmatic controversies, contradictions, and emerging confluences. In N.K. Denzin & Y.S. Lincoln (Eds.), *The landscape of qualitative research: Theories and issues* (2nd ed.) (pp. 253-291). Thousand Oaks, CA: Sage.
- Lozano, M.D.** (2005). Mathematics learning: Ideas from neuroscience and the enactivist approach to cognition. *For the Learning of Mathematics*, 25(3), 24-27.
- Lozano, M.D., Sandoval, I.T., & Trigueros, M.** (2006). Investigating mathematics learning with the use of computer programmes in primary schools. In J. Novotná, H. Moraová, M. Krátká & N. Stehliková (Eds.), *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education, Volume 4* (pp. 89-96). Prague: PME.
- Luhmann, N.** (1995). *Social systems*. Stanford, CA: Stanford University Press.
- MacGregor, M., & Stacey, K.** (1993). Seeing a pattern and writing a rule. In I. Hirabayashi (Ed.), *Proceedings of the Seventeenth Conference for the Psychology of Mathematics Education* (pp. 181-188). Tsukuba, Japan.

- Malloy, C.E.** (1997). Mathematics projects promote students' algebraic thinking. *Mathematics Teaching in the Middle School*, 2(4), 282-288.
- Mason, J.** (1996). Expressing generality and roots of algebra. In N. Bednarz, C. Kieran & L. Lee (Eds.), *Approaches to algebra: Perspectives for research and teaching* (pp. 65-86). Dordrecht: Kluwer.
- Mason, J.** (2002). What makes an example exemplary?: Pedagogical & research issues in transitions from numbers to number theory. In A.D. Cockburn & E. Nardi (Eds.), *Proceedings of the 26th Conference of the International Group for the Psychology of Mathematics Education, Volume 1* (pp. 224-229). Norwich, UK: PME.
- Mason, J.** (2003). On the structure of attention in the learning of mathematics. *The Australian Mathematics Teacher*, 59(4), 17-25.
- Mason, J.** (2004). *Doing ≠ construing and doing + discussing ≠ learning: The importance of the structure of attention*. Retrieved April 16, 2009, from http://www.icme10.dk/proceedings/pages/regular_pdf/RL_John_Mason.pdf
- Mason, J.** (2007). Research and practice in algebra: Interwoven influences. In D. Pitta-Pantazi & G. Philippou (Eds.), *Proceedings of the Fifth Congress of the European Society for Research in Mathematics Education* (pp. 913-923). Larnaca, Cyprus.
- Mason, J., Graham, A., & Johnston-Wilder, S.** (2005). *Developing thinking in algebra*. London: Paul Chapman Publishing.
- Mason, J., Graham, A., Pimm, D., & Gower, N.** (1985). *Routes to / Roots of algebra*. Milton Keynes: Open University Press.
- Maturana, H.** (1987). Everything is said by an observer. In W.I. Thompson (Ed.), *Gaia: A way of knowing. Political implications of the new biology* (pp. 65-82). Hudson, NY: Lindisfarne Press.
- Maturana, H.R., & Poerksen, B.** (2004). *From being to doing: The origins of the biology of cognition*. Heidelberg: Carl-Auer.
- Maturana, H.R., & Varela, F.J.** (1998). *The tree of knowledge: The biological roots of human understanding* (Rev. ed.). Boston, MA: Shambhala.
- Maxwell, J.A.** (1996). *Qualitative research design: An interactive approach*. Thousand Oaks, CA: Sage.
- McNeill, D.** (1992) *Hand and mind: What gestures reveal about thought*. Chicago: Chicago University Press.
- Merleau-Ponty, M.** (1962). *Phenomenology of perception*. London: Routledge & Kegan Paul.
- Miller, W.A.** (1990). Polygonal numbers and recursion. *Mathematics Teacher*, 83(7), 555-558.
- Miller, W.A.** (1991). Physical representations for the sums of some polygonal numbers. *Mathematics in School*, 20(4), 44-45.
- Moss, J., & Beatty, R.** (2006). Knowledge building in mathematics: Supporting collaborative learning in pattern problems. *Computer-Supported Collaborative Learning*, 1, 441-465.
- Nickson, M.** (2000). *Teaching and learning mathematics: A guide to recent research and its applications* (2nd ed.). London: Continuum.
- Nolder, R.** (1991). Mixing metaphor and mathematics in the secondary classroom. In K. Durkin & B. Shire (Eds.), *Language in mathematical education: Research and practice* (pp. 105-113). Milton Keynes: Open University Press.
- Noss, R., Healy, L., & Hoyles, C.** (1997). The construction of mathematical meanings: Connecting the visual with the symbolic. *Educational Studies in Mathematics*, 33(2), 203-233.
- Núñez, R.E.** (2000). Mathematical idea analysis: What embodied cognitive science can say about the human nature of mathematics. In T. Nakahara & M. Koyama (Eds.), *Proceedings of the 24th Conference of the International Group for the Psychology of Mathematics Education, Volume 1* (pp. 3-22). Hiroshima, Japan: PME.
- Núñez, R.** (2006). Do real numbers really move? Language, thought, and gesture: The embodied cognitive foundations of mathematics. In R. Hersh (Ed.), *18 unconventional essays on the nature of mathematics* (pp. 160-181). New York, NY: Springer.

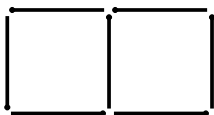
- Núñez, R.** (2007). The cognitive science of mathematics: Why is it relevant for mathematics education? In R. Lesh, E. Hamilton & J.J. Kaput (Eds.), *Foundations for the future in mathematics education* (pp. 127-154). Mahway, NJ: Lawrence Erlbaum.
- Núñez, R.E.** (2008). Mathematics, the ultimate challenge to embodiment: Truth and the grounding of axiomatic systems. In P. Calvo & A. Gomila (Eds.), *Handbook of cognitive science: An embodied approach* (pp. 333-353). Amsterdam: Academic Press.
- Núñez, R.E., Edwards, L.D., & Matos, J.F.** (1999). Embodied cognition as grounding for situatedness and context in mathematics education. *Educational Studies in Mathematics*, 39, 45-65.
- Onions, C.** (1991). The pyramid problem. *Mathematics in School*, 20(3), 40-41.
- Orton, A.** (2004). *Learning mathematics: Issues, theory and classroom practice* (3rd ed.). London: Continuum.
- Orton, J.** (1997). Matchsticks, pattern and generalisation. *Education 3-13*, 25(1), 61-65.
- Orton, J., & Orton, A.** (1999). Pattern and the approach to algebra. In A. Orton (Ed.), *Pattern in the teaching and learning of mathematics* (pp. 104-120). London: Cassell.
- Orton, J., Orton, A., & Roper, T.** (1999). Pictorial and practical contexts and the perception of pattern. In A. Orton (Ed.), *Pattern in the teaching and learning of mathematics* (pp. 121-136). London: Cassell.
- Owens, K.D., & Clements, M.A.** (1998). Representations in spatial problem solving in the classroom. *Journal of Mathematical Behavior*, 17(2), 197-218.
- Özmantar, M.F., & Akkoç, H., Bingölbali, E., Demir, S., & Ergene, B.** (2010). Pre-service mathematics teachers' use of multiple representations in technology-rich environments. *Eurasia Journal of Mathematics, Science & Technology Education*, 6(1), 19-36.
- Pagni, D.L.** (1992). Extending investigations from two dimensions to three dimensions. *Mathematics in School*, 21(2), 6-11.
- Palmer, S.E.** (1992a). Modern theories of Gestalt perception. In G.W. Humphreys (Ed.), *Understanding vision: An interdisciplinary perspective* (pp. 39-70). Oxford: Blackwell.
- Palmer, S.E.** (1992b). Common region: A new principle of perceptual grouping. *Cognitive Psychology*, 24, 436-441.
- Palmer, S., & Rock, I.** (1994a). Rethinking perceptual organization: The role of uniform connectedness. *Psychonomic Bulletin & Review*, 1(1), 29-55.
- Palmer, S., & Rock, I.** (1994b). On the nature and order of organizational processing: A reply to Peterson. *Psychonomic Bulletin & Review*, 1(4), 515-519.
- Parnafes, O., & Disessa, A.** (2004). Relations between types of reasoning and computational representations. *International Journal of Computers for Mathematical Learning*, 9, 251-280.
- Patton, M.Q.** (1990). *Qualitative evaluation and research methods* (2nd ed.). Newbury Park, CA: Sage.
- Pegg, J., & Redden, E.** (1990a). Procedures for, and experiences in, introducing algebra in New South Wales. *Mathematics Teacher*, 84(5), 386-391.
- Pegg, J., & Redden, E.** (1990b). From number patterns to algebra: The important link. *The Australian Mathematics Teacher*, 46(2), 19-22.
- Peirce, C.S.** (1985). Logic as semiotic: The theory of signs. In R.E. Innis (Ed.), *Semiotics: An introductory anthology* (pp. 4-23). Bloomington: Indiana University Press.
- Piaget, J.** (1936). *The origin of intelligence in the child*. Harmondsworth: Penguin.
- Piaget, J.** (1954). *The construction of reality in the child*. New York, NY: Basic Books.
- Powell, A.B., & Maher, C.A.** (2003). Heuristics of twelfth graders' building isomorphism. In N.A. Pateman, B.J. Dougherty & J.T. Zilliox (Eds.), *Proceedings of the 27th Conference of the International Group for the Psychology of Mathematics Education held jointly with the 25th Conference of PME-NA, Volume 4* (pp. 23-30). Honolulu, HI: PME.
- Presmeg, N.C.** (1986a). Visualisation in high school mathematics. *For the Learning of Mathematics*, 6(3), 42-46.
- Presmeg, N.C.** (1992a). Prototypes, metaphors, metonymies and imaginative rationality in high school mathematics. *Educational Studies in Mathematics*, 23, 595-610.
- Presmeg, N.C.** (1992b). Different thinking styles. In M. Moodley, R.A. Njisane & N.C. Presmeg (Eds.), *Mathematics Education for In-service and Pre-service Teachers* (pp. 38-44). Pietermaritzburg: Shuter & Shooter.

- Presmeg, N.C.** (1997a). Reasoning with metaphors and metonymies in mathematical learning. In L.D. English (Ed.), *Mathematical reasoning: Analogies, metaphor and images* (pp. 267-279). Mahwah, NJ: Lawrence Erlbaum.
- Presmeg, N.C.** (1997b). Generalization using imagery in mathematics. In L.D. English (Ed.), *Mathematical reasoning: Analogies, metaphors and images* (pp. 299-312). Mahwah, NJ: Lawrence Erlbaum.
- Presmeg, N.C.** (1998). Metaphoric and metonymic signification in mathematics. *Journal of Mathematical Behavior*, 17(1), 25-32.
- Presmeg, N.C.** (1999). On visualization and generalization in mathematics. In F. Hitt & M. Santos (Eds.), *Proceedings of the 21st Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Volume 1* (pp. 151-155). Cuernavaca, Morelos, Mexico.
- Presmeg, N.C.** (2006). Research on visualization in learning and teaching mathematics: Emergence from psychology. In A. Gutiérrez & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education: Past present and future* (pp. 205-235). Rotterdam: Sense.
- Pretorius, J., Potgieter, R., & Ladewig, W.** (2005). *Oxford mathematics plus grade 10 learner's book*. Cape Town: Oxford University Press Southern Africa.
- Proulx, J.** (2004). The enactivist theory of cognition and behaviorism: An account of the processes of individual sense-making. In B. Davis, R. Luce-Kapler & R. Uptis (Eds.), *Proceedings of the Second Conference on Complexity Science and Educational Research* (pp. 115-120). Chaffey's Locks, Ontario.
- Proulx, J.** (2008). Some differences between Maturana and Varela's theory of cognition and constructivism. *Complicity: An international journal of complexity and education*, 5(1), 11-26.
- Qin, Z., Johnson, D.W., & Johnson, R.T.** (1995). Cooperative versus competitive efforts and problem solving. *Review of Educational Research*, 65(2), 129-143.
- Quinn, R.J.** (2005). A constructivist lesson to introduce arithmetic sequences with patterns. *The Australian Mathematics Teacher*, 61(4), 18-21.
- Radford, L.** (1999). Rethinking representations. In F. Hitt & M. Santos (Eds.), *Proceedings of the 21st Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Volume 1* (pp. 147-150). Cuernavaca, Morelos, Mexico.
- Radford, L.** (2000). Signs and meanings in students' emergent algebraic thinking: A semiotic analysis. *Educational Studies in Mathematics*, 42(3), 237-268.
- Radford, L.** (2001). Factual, contextual and symbolic generalizations in algebra. In M. van den Heuvel-Panhuizen (Ed.), *Proceedings of the 25th Conference of the International Group for the Psychology of Mathematics Education, Volume 4* (pp. 81-88). Utrecht, The Netherlands.
- Radford, L.** (2002a). The seen, the spoken and the written: A semiotic approach to the problem of objectification of mathematical knowledge. *For the Learning of Mathematics*, 22(2), 14-23.
- Radford, L.** (2002b). On heroes and the collapse of narratives: A contribution to the study of symbolic thinking. In A.D. Cockburn & E. Nardi (Eds.), *Proceedings of the 26th Conference of the International Group for the Psychology of Mathematics Education, Volume 4* (pp. 81-88). Norwich, UK: PME.
- Radford, L.** (2003). Gestures, speech, and the sprouting of signs: A semiotic-cultural approach to students' types of generalization. *Mathematical Thinking and Learning*, 5(1), 37-70.
- Radford, L.** (2005a). The semiotics of the schema: Kant, Piaget, and the calculator. In M.H.G. Hoffmann, J. Lenhard & F. Seeger (Eds.), *Activity and sign – grounding mathematics education* (pp. 137-152). New York: Springer.
- Radford, L.** (2005b). Why do gestures matter? Gestures as semiotic means of objectification. In H.L. Chick & J.L. Vincent (Eds.), *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education, Volume 1* (pp. 143-145). Melbourne: PME.

- Radford, L.** (2006). Algebraic thinking and the generalization of patterns: A semiotic perspective. In S. Alatorre, J.L. Cortina, M. Sáiz & A. Méndez (Eds.), *Proceedings of the 28th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Volume 1* (pp. 2-21). Mérida, Mexico: Universidad Pedagógica Nacional.
- Radford, L.** (2008). Iconicity and contraction: a semiotic investigation of forms of algebraic generalizations of patterns in different contexts. *ZDM Mathematics Education*, 40(2), 83-96.
- Radford, L., Bardini, C., & Sabena, C.** (2006). Rhythm and the grasping of the general. In J. Novotná, H. Moraová, M. Krátká & N. Stehliková (Eds.), *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education, Volume 4* (pp. 393-400). Prague: PME.
- Radford, L., Bardini, C., & Sabena, C.** (2007). Perceiving the general: The multisemiotic dimension of students' algebraic activity. *Journal for Research in Mathematics Education*, 38(5), 507-530.
- Radford, L., Bardini, C., Sabena, C., Diallo, P., & Simbagoye, A.** (2005). On embodiment, artifacts, and signs: A semiotic-cultural perspective on mathematical thinking. In H.L. Chick & J.L. Vincent (Eds.), *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education, Volume 4* (pp. 113-120). Melbourne: PME.
- Radford, L., Demers, S., Guzmán, J., & Cerulli, M.** (2003). Calculators, graphs, gestures and the production of meaning. In N.A. Pateman, B.J. Dougherty & J.T. Zilliox (Eds.), *Proceedings of the 27th Conference of the International Group for the Psychology of Mathematics Education held jointly with the 25th Conference of PME-NA, Volume 4* (pp. 55-62). Honolulu, HI: PME.
- Reid, D.A.** (1996). Enactivism as a methodology. In L. Puig & A. Gutiérrez (Eds.), *Proceedings of the 20th Conference of the International Group for the Psychology of Mathematics Education, Volume 4* (pp. 203-209). Valencia: Universitat de València.
- Reid, D.A.** (2002). Conjectures and refutations in grade 5 mathematics. *Journal for Research in Mathematics Education*, 33(1), 5-29.
- Richards, L.** (2005). *Handling qualitative data: A practical guide*. London: Sage.
- Rivera, F.D.** (2007) Visualizing as a mathematical way of knowing: Understanding figural generalization. *Mathematics Teacher*, 101(1), 69-75.
- Rivera, F.D., & Becker, J.R.** (2005). Figural and numerical modes of generalizing in algebra. *Mathematics Teaching in the Middle School*, 11(4), 198-203.
- Rivera, F.D., & Becker, J.R.** (2008). Middle school children's cognitive perceptions of constructive and deconstructive generalizations involving linear figural patterns. *ZDM Mathematics Education*, 40(1), 65-82.
- Rock, I., & Palmer, S.** (1990). The legacy of Gestalt psychology. *Scientific American*, 263(6), 48-52 & 60-61.
- Roper, T.** (1999). Pattern and the assessment of mathematical investigations. In A. Orton (Ed.), *Pattern in the teaching and learning of mathematics* (pp. 178-191). London: Cassell.
- Rösken, B., & Rolka, K.** (2006). A picture is worth a 1000 words – the role of visualization in mathematics learning. In J. Novotná, H. Moraová, M. Krátká & N. Stehliková (Eds.), *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education, Volume 4* (pp. 457-464). Prague: PME.
- Rowlands, M.** (2006). *Body language: Representation in action*. Cambridge, MA: Bradford.
- Ryle, G.** (1949). *The concept of mind*. Harmondsworth: Penguin.
- Sabena, C., Radford, L., & Bardini, C.** (2005). Synchronizing gestures, words and actions in pattern generalizations. In H.L. Chick & J.L. Vincent (Eds.), *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education, Volume 4* (pp. 129-136). Melbourne: PME.
- Samson, D.** (2006). Number patterns, cautionary tales and the calculus of finite differences. *Learning and Teaching Mathematics*, 3, 3-8.

- Samson, D.A.** (2007a). *An analysis of the influence of question design on pupils' approaches to number pattern generalisation tasks*. Unpublished master's thesis. Rhodes University, Grahamstown.
- Samson, D.** (2007b). Patterns of Visualisation. *Learning and Teaching Mathematics*, 5, 4-9.
- Samson, D.** (2008). Approaches to generalising quadratic sequences. *Learning and Teaching Mathematics*, 6, 9-15.
- Samson, D.** (2011). Capitalising on inherent ambiguities in symbolic expressions of generality. *The Australian Mathematics Teacher*, 67(1), 28-32.
- Samson, D., & Schäfer, M.** (2011). Enactivism, figural apprehension and knowledge objectification: An exploration of figural pattern generalisation. *For The Learning of Mathematics*, 31(1), 37-43.
- Sandefur, J., & Camp, D.** (2004). Patterns: Revitalizing recurring themes in school mathematics. *Mathematics Teacher*, 98(4), 211.
- Sasman, M.C., Olivier, A., & Linchevski, L.** (1999). Factors influencing student's generalisation thinking processes. In O. Zaslavski (Ed.), *Proceedings of the 23rd Conference of the International Group for the Psychology of Mathematics Education, Volume 4* (pp. 161-168). Haifa, Israel.
- Schultz, J.E., & Waters, M.S.** (2000). Why representations? *Mathematics Teacher*, 93(6), 448-453.
- Simmt, E.** (2000). Mathematics knowing in action: A fully embodied interpretation. In E. Simmt, B. Davis & J.G. McLoughlin (Eds.), *Proceedings of the 2000 Annual Meeting of the Canadian Mathematics Education Study Group (CMESG/GCEDM Proceedings 2000)* (pp. 153-159). Université du Québec à Montréal.
- Simon, H.A.** (1956). Rational choice and the structure of the environment. *Psychological Review*, 63(2), 129-138.
- Spoehr, K.T., & Lehmkuhle, S.W.** (1982). *Visual information processing*. San Francisco: Freeman.
- Stacey, K.** (1989). Finding and using patterns in linear generalising problems. *Educational Studies in Mathematics*, 20, 147-164.
- Stake, R.E.** (1994). Case studies. In N.K. Denzin & Y.S. Lincoln (Eds.), *Handbook of qualitative research* (pp. 236-247). Thousand Oaks, CA: Sage.
- Stake, R.E.** (1995). *The art of case study research*. Thousand Oaks, CA: Sage.
- Sternberg, R.J.** (1999). *Cognitive psychology* (2nd ed.). Fort Worth, TX: Harcourt Brace.
- Stigler, J.W., & Hiebert, J.** (1999). *The teaching gap: Best ideas from the world's teachers for improving education in the classroom*. New York, NY: The Free Press.
- St. Julien, J.** (1997). Explaining learning: The research trajectory of situated cognition and the implications of connectionism. In D Kirshner & J.A. Whitson (Eds.), *Situated cognition: Social, semiotic, and psychological perspectives* (pp. 261-279). Mahwah, NJ: Lawrence Erlbaum.
- Sumara, D.J., & Davis, B.** (1997). Enactivist theory and community learning: Towards a complexified understanding of action research. *Educational Action Research*, 5(3), 403-422.
- Swafford, J.O., & Langrall, C.W.** (2000). Grade 6 students' preinstructional use of equations to describe and represent problem situations. *Journal for Research in Mathematics Education*, 31(1), 89-112.
- Szetela, W.** (1999). Triangular numbers in problem solving. *The Mathematics Teacher*, 92(9), 820-824.
- Thompson, E.** (2005). Sensorimotor subjectivity and the enactive approach to experience. *Phenomenology and the Cognitive Sciences*, 4(4), 407-427.
- Thornton, S.** (2001). A picture is worth a thousand words. In A. Rogerson (Ed.), *New Ideas in Mathematics Education: Proceedings of the International Conference of the Mathematics Education into the 21st Century Project* (pp. 251-256), Palm Cove, Queensland, Australia.
- Trigueros, M., & Lozano, M.D.** (2007). Developing resources for teaching and learning mathematics with digital technologies: An enactivist approach. *For the Learning of Mathematics*, 27(2), 45-51.

- Van de Walle, J.A., & Holbrook, H.** (1986). Patterns, thinking, and problem solving. *Arithmetic Teacher*, 34(1), 6-12.
- Van Garderen, D., & Montague, M.** (2003). Visual-spatial representation, mathematical problem solving, and students of varying abilities. *Learning Disabilities Research & Practice*, 18(4), 246-254.
- Varela, F.** (1987). Laying down a path in walking. In W.I. Thompson (Ed.), *Gaia: A way of knowing. Political implications of the new biology* (pp. 48-64). Hudson, NY: Lindisfarne Press.
- Varela, F.J., Thompson, E., & Rosch, E.** (1991). *The embodied mind: Cognitive science and human experience*. Cambridge, MA: MIT Press.
- Vithal, R., & Volmink, J.** (2005). Mathematics curriculum research: roots, reforms, reconciliation and relevance. In R. Vithal, J. Adler & C. Keitel (Eds.), *Researching Mathematics Education in South Africa: Perspectives, practices and possibilities* (pp. 3-27). Cape Town: HSRC Press.
- Vogel, R.** (2005). Patterns – a fundamental idea of mathematical thinking and learning. *Zentralblatt für Didaktik der Mathematik (ZDM)*, 37(5), 445-449.
- Von Glasersfeld, E.** (1991). Introduction. In E. von Glasersfeld (Ed.), *Radical constructivism in mathematics education* (pp. xiii-xx). Dordrecht: Kluwer Academic Publishers.
- Walkowiak, T.A.** (2010). An examination of algebraic reasoning: Elementary and middle school students' analyses of pictorial growth patterns. In P. Brosnan, D.B. Erchick & L. Flevares (Eds.), *Proceedings of the 32nd annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Volume VI* (pp. 194-201). Columbus, OH: The Ohio State University.
- Waring, S., Orton, A., & Roper, T.** (1999). Pattern and proof. In A. Orton (Ed.), *Pattern in the teaching and learning of mathematics* (pp. 192-206). London: Cassell.
- Warren, E.** (2000). Visualisation and the development of early understanding in algebra. In T. Nakahara & M. Koyama (Eds.), *Proceedings of the 24th Conference of the International Group for the Psychology of Mathematics Education, Volume 4* (pp. 273-280). Hiroshima, Japan.
- Warren, E.** (2005). Young children's ability to generalise the pattern rule for growing patterns. In H.L. Chick & J.L. Vincent (Eds.), *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education, Volume 4* (pp. 305-312). Melbourne, Australia: PME.
- Wertheimer, M.** (1938a). Gestalt theory. Abstract of Wertheimer, M. (1925) "Über Gestalttheorie" [an address before the Kant Society, Berlin, 17th December, 1924], Erlangen. In W.D. Ellis (Ed.), *A source book of Gestalt psychology* (pp. 1-11). London: Kegan Paul.
- Wertheimer, M.** (1938b). Laws of organization in perceptual forms. Summary of Wertheimer, M. (1923) "Untersuchungen zur Lehre von der Gestalt," II, *Psychol. Forsch.*, 4: 301-350. In W.D. Ellis (Ed.), *A source book of gestalt psychology* (pp.71-88). London: Kegan Paul.
- Williams, J.** (2005). Gestures, signs and mathematisation. In H.L. Chick & J.L. Vincent (Eds.), *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education, Volume 1* (pp. 146-148). Melbourne, Australia: PME.
- You, H.** (1994). Defining rhythm: Aspects of an anthropology of rhythm. *Culture, Medicine and Psychiatry*, 18(3), 361-384.
- Zack, V., & Reid, D.A.** (2003). Good-enough understanding: Theorising about the learning of complex ideas (Part 1). *For the Learning of Mathematics*, 23(3), 43-50.
- Zack, V., & Reid, D.A.** (2004). Good-enough understanding: Theorising about the learning of complex ideas (Part 2). *For the Learning of Mathematics*, 24(1), 25-28.
- Zazkis, R., & Liljedahl, P.** (2002). Generalization of patterns: the tension between algebraic thinking and algebraic notation. *Educational Studies in Mathematics*, 49, 379-402.
- Zimmermann, W., & Cunningham, S.** (1991). Editors' introduction: What is mathematical visualization? In W. Zimmermann & S. Cunningham (Eds.), *Visualization in teaching and learning mathematics* (pp. 1-8). Washington: Mathematical Association of America.
- Zusne, L.** (1970). *Visual perception of form*. New York: Academic Press.

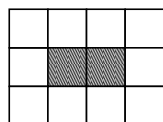


2 squares require 7 matches

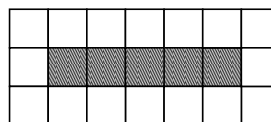


5 squares require 16 matches

Pupil	4 th term correct?	n^{th} term correct?	Formula for n^{th} term	Visual vs. Numeric	Numeric Strategy	Visual strategy	Contextual Connectivity Rating
Anna	✓	✓	$3n + 1$	N	RA	-	0
Anthea	✓	✓	$(n \times 3) + 1$	V	-	LF1	1
Arthur	✓	✓	$3n + 1$	N	RA	-	0
Barry	✓	✓	$3n + 1$	N	DF	-	0
Brian	✓	✓	$3n + 1$	V	-	LF1	1
Charles	✓	✓	$3n + 1$	N	RA	-	0
David	✓	✓	$n \times 3 + 1$	V/N	DF	LAU	$\frac{1}{2}$
Dylan	✓	✓	$3n + 1$	V/N	DF	LAU	$\frac{1}{2}$
Fiona	✓	✓	$3n + 1$	V/N	RA	LAU	$\frac{1}{2}$
Harry	✓	✓	$3n + 1$	N	RA	-	0
Katie	✓	✓	$n \times 3 + 1$	N	RA	-	0
Kelly	✓	✓	$3n + 1$	N	RA	-	0
Lance	✓	✓	$4n - (n - 1)$	N	DF	-	0
Liza	✓	✓	$3n + 1$	V/N	RA	LF1	$\frac{1}{2}$
Mike	✓	✓	$3n + 1$	N	DF	-	0
Mitch	✓	✓	$3n + 1$	N	RA	-	0
Philip	✓	✓	$n \times 2 + n + 1$	V	-	GF1	1
Rose	✓	✓	$3n + 1$	V/N	RA	LF1	$\frac{1}{2}$
Ryan	✓	✓	$3n + 1$	N	DF	-	0
Sally	✓	✓	$3n + 1$	V/N	RA	LF1	$\frac{1}{2}$
Susan	✓	✓	$3n + 1$	N	DF	-	0
Taylor	✓	✓	$3n + 1$	V/N	RA	LF1	$\frac{1}{2}$
Terry	✓	✓	$3n + 1$	V	-	LF1	1

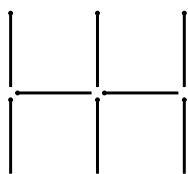


For a row of 2 striped tiles there are 10 white tiles in the border.

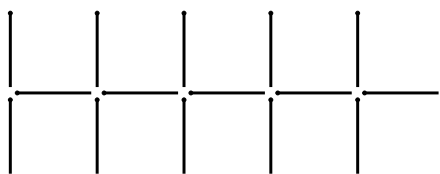


For a row of 5 striped tiles there are 16 white tiles in the border.

Pupil	40 th term correct?	n^{th} term correct?	Formula for n^{th} term	Visual vs. Numeric	Numeric Strategy	Visual strategy	Contextual Connectivity Rating
Anna	✓	✓	$2n + 6$	N	RA	-	0
Anthea	✓	✓	$(n \times 2) + 6$	V	-	GF1	1
Arthur	✓	✓	$n2 + 6$	N	RA	-	0
Barry	✓	✓	$2n + 6$	N	DF	-	0
Brian	✓	✓	$2n + 6$	V	-	GF1	1
Charles	✓	✓	$2n + 6$	V/N	RA	LAU	$\frac{1}{2}$
David	✓	✓	$(n \times 2) + 6$	V	-	GF1	1
Dylan	✓	✓	$2(n + 3)$	V/N	DF	GF1	$\frac{1}{2}$
Fiona	✓	✓	$2n + 6$	V/N	RA	GF1	$\frac{1}{2}$
Harry	✓	✓	$n \times 2 + 6$	N	RA	-	0
Katie	✓	✓	$n \times 2 + 6$	N	RA	-	0
Kelly	✓	✓	$3n + 6 - n$	V	-	GF2	1
Lance	✓	✓	$2n + 6$	V	-	GF1	1
Liza	✓	✓	$2n + 6$	V	-	GF1	1
Mike	✓	✓	$2n + 6$	N	DF	-	0
Mitch	✓	✓	$2n + 6$	N	RA	-	0
Philip	✓	✓	$6 + n \times 2$	V	-	GF1	1
Rose	✓	✓	$2n + 6$	N	RA	-	0
Ryan	✗	✗	You add 2 white tiles every time	V/N	RA	LAU	$\frac{1}{2}$
Sally	✓	✓	$2n + 6$	V/N	RA	GF1	$\frac{1}{2}$
Susan	✓	✓	$2n + 6$	N	RA	-	0
Taylor	✓	✓	$2n + 6$	V	-	GF1	1
Terry	✓	✓	$2n + 6$	V	-	GF1	1

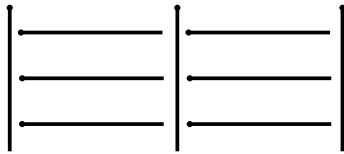


A pattern with 2 horizontal matches

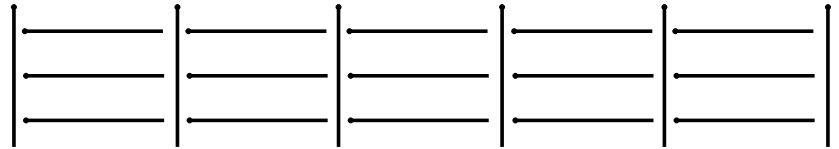


A pattern with 5 horizontal matches

Pupil	40 th term correct?	n^{th} term correct?	Formula for n^{th} term	Visual vs. Numeric	Numeric Strategy	Visual strategy	Contextual Connectivity Rating
Anna	✓	✓	$3n + 2$	N	RA	-	0
Anthea	✓	✓	$(n + 1) \times 2 + n$	V/N	DF	GF1	$\frac{1}{2}$
Arthur	✓	✓	$n^3 + 2$	N	DF	-	0
Barry	✓	✓	$3n + 2$	N	DF	-	0
Brian	✓	✓	$3n + 2$	V	-	LF1	1
Charles	✓	✓	$3n + 2$	N	RA	-	0
David	✓	✓	$(n \times 2) + (n + 2)$	V/N	DF	GF1	$\frac{1}{2}$
Dylan	✓	✓	$3n + 2$	V/N	DF	LF3	$\frac{1}{2}$
Fiona	✓	✓	$3n + 2$	V/N	RA	LAU	$\frac{1}{2}$
Harry	✓	✓	$3n + 2$	N	RA	-	0
Katie	✓	✓	$n \times 3 + 2$	N	RA	-	0
Kelly	✓	✓	$3n + 2$	V	-	LF1	1
Lance	✓	✓	$n + 2 \times (n + 1)$	V	-	GF1	1
Liza	✓	✓	$3n + 2$	V/N	RA	GF1	$\frac{1}{2}$
Mike	✓	✓	$3n + 2$	N	DF	-	0
Mitch	✓	✓	$3n + 2$	N	RA	-	0
Philip	✓	✓	$2 + n \times 3$	V	-	LF1	1
Rose	✓	✓	$3n + 2$	N	RA	-	0
Ryan	✓	✓	$3n + 2$	V/N	DF	LAU	$\frac{1}{2}$
Sally	✓	✓	$3n + 2$	V/N	RA	LF1	$\frac{1}{2}$
Susan	✓	✓	$3n + 2$	N	DF	-	0
Taylor	✓	✓	$3n + 2$	V	-	LF1	1
Terry	✓	✓	$3n + 2$	V	-	LF1	1

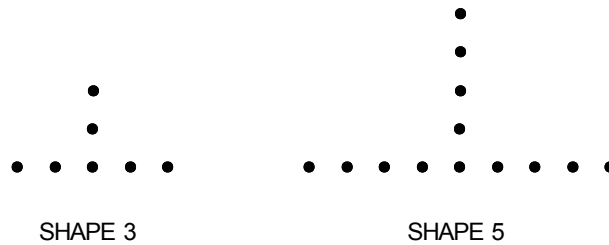


For a fence containing 2 sections
you need a total of 9 matches.

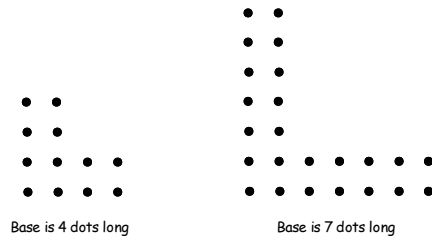


For a fence containing 5 sections
you need a total of 21 matches.

Pupil	40 th term correct?	n^{th} term correct?	Formula for n^{th} term	Visual vs. Numeric	Numeric Strategy	Visual strategy	Contextual Connectivity Rating
Anna	✓	✓	$4n + 1$	N	RA	-	0
Anthea	✓	✓	$(3+1) \times n + 1$	V	-	LF1	1
Arthur	✓	✓	$n4 + 1$	N	DF	-	0
Barry	✓	✓	$4n + 1$	N	DF	-	0
Brian	✓	✓	$5n - (n - 1)$ or $4n + 1$	V/N	DF	LF1	$\frac{1}{2}$
Charles	✓	✓	$4n + 1$	N	RA	-	0
David	✓	✓	$(n \times 4) + 1$	V/N	DF	LF2	$\frac{1}{2}$
Dylan	✓	✓	$4n + 1$	V/N	RA	LF1	$\frac{1}{2}$
Fiona	✓	✓	$4n + 1$	V/N	RA	LAU	0
Harry	✓	✓	$4n + 1$	V/N	DF	LAU	$\frac{1}{2}$
Katie	✓	✓	$n \times 4 + 1$	N	RA	-	0
Kelly	✓	✓	$4n + 1$	V	-	LF1	1
Lance	✓	✓	$(n \times 3) + n + 1$	V	-	GF1	1
Liza	✓	✓	$4n + 1$	V	-	LF1	1
Mike	✓	✓	$4n + 1$	N	DF	-	0
Mitch	✓	✓	$4n + 1$	N	RA	-	0
Philip	✓	✓	$3n + n + 1$	V	-	GF1	1
Rose	✓	✓	$4n + 1$	N	DF	-	0
Ryan	✓	✓	$4n + 1$	N	DF	-	0
Sally	✓	✓	$4n + 1$	N	RA	-	0
Susan	✓	✓	$2n + 2n + 1$	N	DF	-	0
Taylor	✓	✓	$4n + 1$	V	-	LF1	1
Terry	✓	✓	$4n + 1$	V	-	LF1	1

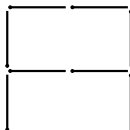


Pupil	40 th term correct?	n^{th} term correct?	Formula for n^{th} term	Visual vs. Numeric	Numeric Strategy	Visual strategy	Contextual Connectivity Rating
Anna	✓	✓	$3n - 2$	N	RA	-	0
Anthea	✓	✓	$n + (n - 1) \times 2$	V	-	GF1	1
Arthur	✓	✓	$n^3 - 2$	N	RA	-	0
Barry	✓	✓	$n - 1 = x$ you then go $3x + 1$	V	-	GF1	1
Brian	✓	✓	$3n - 2$	V/N	DF	LF1	$\frac{1}{2}$
Charles	✓	✓	$3n - 2$	N	RA	-	0
David	✓	✓	$(n \times 3) - 2$	V/N	RA	LAU	$\frac{1}{2}$
Dylan	✓	✓	$3n - 2$	V/N	DF	LAU	$\frac{1}{2}$
Fiona	✓	✓	$3n - 2$	N	RA	-	0
Harry	✓	✓	$3n - 2$	N	DF	-	0
Katie	✓	✓	$3n - 2$	N	RA	-	0
Kelly	✓	✓	$2n - 1 + n - 1$	V	-	GF1	1
Lance	✓	✓	$3n - 2$	V	-	GF2	1
Liza	✓	✓	$3n - 2$	V	-	GF2	1
Mike	✓	✓	$3n - 2$	N	DF	-	0
Mitch	✓	✓	$3n - 2$	V/N	RA	LAU	$\frac{1}{2}$
Philip	✓	✓	$n + 2n - 2$	V	-	GF1	1
Rose	✓	✓	$(n - 1) + (2n - 1)$	V/N	DF	GF1	$\frac{1}{2}$
Ryan	✓	✓	$3n - 2$	N	DF	-	0
Sally	✓	✓	$3n - 2$ or $n + (n - 1) \times 2$	V/N	RA	GF1	$\frac{1}{2}$
Susan	✓	✓	$2n + (n - 2)$	N	DF	-	0
Taylor	✓	✓	$3n - 2$	V	-	GF2	1
Terry	✓	✓	$2n + (n - 2)$ or $n + (n - 1) + (n - 1)$	V/N	DF	GF1	$\frac{1}{2}$

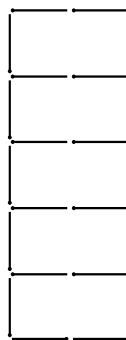


Pupil	40 th term correct?	n^{th} term correct?	Formula for n^{th} term	Visual vs. Numeric	Numeric Strategy	Visual strategy	Contextual Connectivity Rating
Anna	✓	✓	$4n - 4$	N	RA	-	0
Anthea	✓	✓	$n + (n-1) \times 2 + n - 2$	V	-	GF1	1
Arthur	✓	✓	$4n - 4$	N	RA	-	0
Barry	✓	✓	$n-1 = x \therefore 4x = \text{overall amount}$	V/N	DF	GF1	$\frac{1}{2}$
Brian	✓	✓	$4n - 4$	N	RA	-	0
Charles	✓	✓	$4n - 4$	N	RA	-	0
David	✓	✓	$(n \times 4) - 4$	V/N	RA	LAU	$\frac{1}{2}$
Dylan	✓	✓	$4n - 4$	N	RA	-	0
Fiona	✓	✓	$2n + 2(n-2)$	V/N	RA	GF1	$\frac{1}{2}$
Harry	✓	✓	$4n - 4$	N	DF	-	0
Katie	✓	✓	$n \times 4 - 4$	N	RA	-	0
Kelly	✗	✗	$(n+2)4$	V	-	GF1	1
Lance	✓	✓	$2n + (n-2) \times 2$	V	-	GF1	1
Liza	✓	✓	$4n - 4$	V/N	RA	GF1	$\frac{1}{2}$
Mike	✓	✓	$4n - 4$	N	DF	-	0
Mitch	✓	✓	$4n - 4$	N	RA	-	0
Philip	✓	✓	$2n + 2n - 4$	V	-	GF2	1
Rose	✓	✓	$4n - 4$	V/N	RA	GF1	$\frac{1}{2}$
Ryan	✓	✓	$4n - 4$	N	DF	-	0
Sally	✓	✓	$4n - 4$	V/N	RA	GF2	$\frac{1}{2}$
Susan	✓	✓	$4n - 4$	N	DF	-	0
Taylor	✓	✓	$(2n-1) + (Ans-2)$	V	-	GF1	1
Terry	✓	✓	$n + 2(n-1) + (n-2)$	V	-	GF1	1

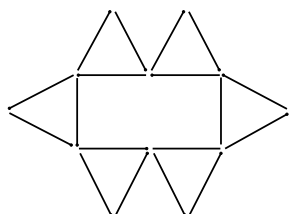
For a tower containing 2 levels, 10 matches are needed.



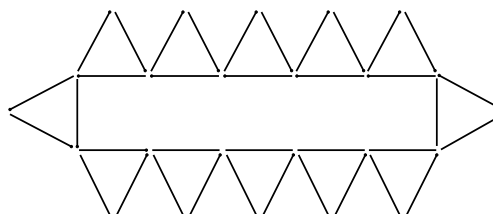
For a tower containing 5 levels, 22 matches are needed.



Pupil	40 th term correct?	n^{th} term correct?	Formula for n^{th} term	Visual vs. Numeric	Numeric Strategy	Visual strategy	Contextual Connectivity Rating
Anna	✓	✓	$4n + 2$	N	RA	-	0
Anthea	✓	✓	$(n \times 4) + 2$	V	-	LF1	1
Arthur	✓	✓	$n4 + 2$	N	DF	-	0
Barry	✓	✓	$4n + 2$	N	DF	-	0
Brian	✓	✓	$4n + 2$	V/N	RA	LF1	½
Charles	✓	✓	$4n + 2$	N	RA	-	0
David	✓	✓	$(4 \times n) + 2$	V/N	RA	LAU	½
Dylan	✓	✓	$4n + 2$	V	-	LF1	1
Fiona	✓	✓	$4n + 2$	N	RA	-	0
Harry	✓	✓	$4n + 2$	N	DF	-	0
Katie	✓	✓	$n \times 4 + 2$	N	RA	-	0
Kelly	✓	✓	$4n + 2$	V	-	GF1	1
Lance	✓	✓	$4n + 2$	V	-	LF1	1
Liza	✓	✓	$4n + 2$	N	DF	-	0
Mike	✓	✓	$4n + 2$	N	DF	-	0
Mitch	✓	✓	$4n + 2$	N	RA	-	0
Philip	✓	✓	$4n + 2$	V	-	LF1	1
Rose	✓	✓	$4n + 2$	N	DF	-	0
Ryan	✓	✗	$4n + 4$	N	DF	-	0
Sally	✓	✓	$4n + 2$	V/N	RA	LF1	½
Susan	✓	✓	$4n + 2$	N	DF	-	0
Taylor	✓	✓	$4n + 2$	V	-	LF1	1
Terry	✓	✓	$4n + 2$	V	-	LF1	1

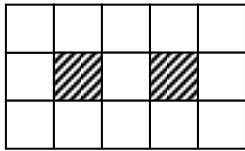


SHAPE 2

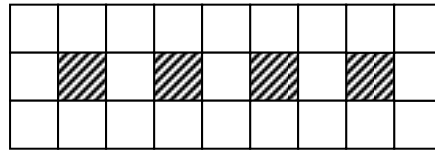


SHAPE 5

Pupil	4 th term correct?	n^{th} term correct?	Formula for n^{th} term	Visual vs. Numeric	Numeric Strategy	Visual strategy	Contextual Connectivity Rating
Anna	✓	✓	$6n + 6$	N	RA	-	0
Anthea	✓	✓	$(n \times 3) \times 2 + 6$	V	-	GF1	1
Arthur	✓	✓	$n6 + 6$	N	RA	-	0
Barry	✓	✓	$6n + 6$	N	DF	-	0
Brian	✓	✓	$6n + 6$	V	-	GF1	1
Charles	✓	✓	$6n + 6$	N	RA	-	0
David	✓	✓	$6n + 6$	V/N	RA	LAU	$\frac{1}{2}$
Dylan	✓	✓	$6n + 6$	V/N	RA	LF1	$\frac{1}{2}$
Fiona	✓	✓	$2(3n) + 6$	V/N	RA	GF1	$\frac{1}{2}$
Harry	✓	✓	$6n + 6$	N	DF	-	0
Katie	✓	✓	$n \times 6 + 6$	N	RA	-	0
Kelly	✓	✓	$2n + n + n + 2n + 6$ or $6n + 6$	V	-	GF1	1
Lance	✓	✓	$3 \times (n \times 2 + 2)$	V	-	GF1	1
Liza	✓	✓	$6n + 6$	V/N	DF	GF1	$\frac{1}{2}$
Mike	✓	✓	$6n + 6$	N	DF	-	0
Mitch	✓	✓	$6n + 6$	N	RA	-	0
Philip	✓	✓	$6n + 6$	V	-	GF1	1
Rose	✓	✓	$6n + 6$	V/N	RA	GF1	$\frac{1}{2}$
Ryan	✓	✓	$6n + 6$	N	DF	-	0
Sally	✓	✓	$6n + 6$	V/N	RA	GF1	$\frac{1}{2}$
Susan	✓	✓	$6n + 6$	N	RA	-	0
Taylor	✓	✓	$6n + 6$	V	-	GF1	1
Terry	✓	✓	$6n + 6$	V	-	GF1	1



For 2 striped tiles you need 13 white tiles.



For 4 striped tiles you need 23 white tiles.

Pupil	40 th term correct?	n^{th} term correct?	Formula for n^{th} term	Visual vs. Numeric	Numeric Strategy	Visual strategy	Contextual Connectivity Rating
Anna	✓	✓	$5n + 3$	N	RA	-	0
Anthea	✓	✓	$(n \times 5) + 3$	V	-	LF1	1
Arthur	✓	✓	$n5 + 3$	N	DF	-	0
Barry	✓	✓	$5n + 3$	N	DF	-	0
Brian	✓	✓	$5n + 3$	V/N	DF	GF1	$\frac{1}{2}$
Charles	✓	✓	$5n + 3$	N	RA	-	0
David	✓	✓	$5n + 3$	V/N	DF	LAU	$\frac{1}{2}$
Dylan	✓	✓	$5n + 3$	V/N	DF	LAU	$\frac{1}{2}$
Fiona	✓	✓	$5n + 3$	V/N	RA	LF1	$\frac{1}{2}$
Harry	✓	✓	$5n + 3$	N	DF	-	0
Katie	✓	✓	$n \times 5 + 3$	N	RA	-	0
Kelly	✗	✗	$3(2n + 3) - (n + 1)$	V	-	GF2	1
Lance	✓	✓	$5n + 3$	V	-	LF1	1
Liza	✓	✓	$5n + 3$	V/N	DF	GF1	$\frac{1}{2}$
Mike	✓	✓	$5n + 3$	N	DF	-	0
Mitch	✓	✓	$5n + 3$	N	RA	-	0
Philip	✓	✓	$3n + 3 + 2n$	V	-	GF1	1
Rose	✓	✓	$(3n + 3) + (2n)$	V	-	GF1	1
Ryan	✓	✓	$5n + 3$	N	DF	-	0
Sally	✓	✓	$5n + 3$	V/N	RA	LF1	$\frac{1}{2}$
Susan	✓	✓	$5n + 3$	N	DF	-	0
Taylor	✓	✓	$5n + 3$	V	-	LF1	1
Terry	✓	✓	$5n + 3$	V	-	LF1	1