THE ROLE OF VISUALISATION IN REDEFINING THE PEDAGOGY OF FRACTIONS IN MATHEMATICS CLASSROOMS AMONG SENIOR PRIMARY SCHOOL TEACHERS

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ABSTRACT

This mixed methods study explored the impact of the use of a visualisation approach on the pedagogy of eight teacher participants who were involved in the Rundu Campus Fraction Project (RCFP). The aim of this study was to determine the extent to which participants incorporated visualisation processes in the pedagogy of fractions, in view of their exposure to visualisation activities in the RCFP. Since fractions are difficult to teach and learn, visualisation was considered as an alternative approach to the pedagogy of fractions because it presents learners with opportunities to improve their proportional and spatial reasoning. This study was founded on the premise that the incorporation of both verbal and nonverbal cues can enhance the teaching and learning of fractions rather than the use of a single cue. Hence, the two theories underpinning this study are the Dual Coding Theory and the Constructivist Theory. While the Dual Coding Theory advocates for the use of verbal and nonverbal codes, the Constructivist theory states that meaningful learning occurs when learners are presented with opportunities to construct their own knowledge. Thus, the two codes are intertwined. In other words, the active construction of knowledge among learners is aided by using constructivist teaching approaches through the incorporation of both verbal and nonverbal codes. Although this study was predominantly qualitative, quantitative methods were also used in the data collection process. A questionnaire was administered to identify teacher participants for this study, based on their teaching orientations. Their views on best practices in mathematics classrooms in general and the incorporation of visualisation processes in particular, were instrumental in the selection of participants for this study. In addition, observations and semi-structured interviews were also used as research methods. Twenty-five lesson samples were video recorded, transcribed and analysed using both qualitative and quantitative methods. Moreover, each of the eight participants was exposed to a set of pre- and post-observation interviews during which they were expected to express their views on the selection, incorporation and impact of visualisation processes on the teaching of fractions. Data sets from all three instruments were analysed using both quantitative and qualitative methods. The findings indicate that the RCFP had an impact on the teaching strategies employed by the participants as they all incorporated visualisation processes into their teaching to some extent. In some lessons, the visual code was effectively blended into the verbal code while in others, the purpose of and connection between the two codes was not evident. Hence, although all the participants embraced the incorporation of visualisation in the pedagogy of mathematics, some of them struggled to find its rightful position in the teaching of fractions. The findings suggest that despite the participants' eagerness to use visualisation in their fraction lessons, some of them did not have adequate knowledge to successfully merge it with the conventional verbal code. Thus, for the integration of visuals to be impactful, it should be carefully merged in the teaching of fractions by taking into account various factors.

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DEDICATION

This thesis is dedicated to my children Samson and Emily for allowing me to pursue my PhD which sometimes prevented me fulfilling my parental duties.

and

their late brother Samuel, whose short-lived life impacted our lives profoundly.

DECLARATION OF ORIGINALITY

I, Charity Makwiliro Ausiku (Student number, 05A5503), declare that this doctoral thesis entitled: “Analysing the role of visualisation in redefining the pedagogy of fractions in mathematics classrooms among senior primary school teachers,” is my own work written in my own words. Where I have drawn on the words or ideas of others, these have been fully acknowledged and referenced in the manner required by the Rhodes University Department of Education Guide to referencing.

January 2022

Charity Ausiku (Signature) (Date)

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ACRONYMS

BETD	Basic Education Teachers Diploma
BEd	Bachelor of Education
CK	Content Knowledge
COVID-19	Corona Virus Disease of 2019
CT	Constructivist Theory
CCR	Center for Curriculum Redesign
CCT	Cognitive Constructivist Theory
DCT	Dual Coding Theory
FMs	Fraction Models
EQS	Effective Questioning Strategies
HCF	Highest Common Factor
LCA	Learner Centred Approach
LCD	Least Common Denominator
LCE	Learner Centred Education
ME	Ministry of Education
MEC	Ministry of Education and Culture
MoHSS	Ministry of Health and Social Services
NCTM	National Council of Teachers of Mathematics
NIED	National Institute for Education Development
PCK	Pedagogical Content Knowledge
QFK	Quality Fraction Knowledge
QT	Quantitative Thinking
RCFP	Rundu Campus Fraction Project
SACMEQ	Southern and Eastern African Consortium for Monitoring Educational Quality
SCT	Social Constructivist Theory
UNAM	University of Namibia
ViL	Visual Language
VL	Visuality of Lessons
VNVC	Verbal and Nonverbal Code
WHO	World Health Organisation

CHAPTER ONE
INTRODUCTION

1.1 INTRODUCTION

Considering the abstract nature of mathematics, particularly fractions, alternative methods in the pedagogy of mathematics are imperative. Thus, this study was conducted to determine the impact of visualisation processes on the teaching of fractions for grades four to seven which is senior primary in the Namibian context. This study was located in the Rundu Campus Fraction Project (RCFP); a project that was established to equip teachers and student teachers with effective, visual pedagogical skills, for teaching fractions. Details about the RCFP and the positionality of the researcher in this project are discussed in chapter two (see section 2.7). Generally, fractions is one of the mathematics concepts that both teachers and learners struggle with in terms of teaching and learning. Based on their nature and the way they are presented, fractions seem too foreign for learners and this is usually the first time that they tend to give up on mathematics (Barnett, 2016; Ubah, 2021). This study therefore sought to introduce pedagogical strategies that are responsive to the challenges that both teachers and learners experience in terms of fractions.

Thus, visualisation was identified as a suitable teaching strategy to mitigate the challenges that teachers encounter in presenting fraction content in a more comprehensible and learner-friendly manner. Makina (2010, p. 24) defines mathematical visualisation as "the process of forming images or constructing mental representations and using such images effectively for mathematical discovery and understanding." Based on this definition, the role of visualisation in the teaching and learning of mathematical concepts cannot be overemphasised. Through visualisation, learners are able to create mental images of fractions and make sense of the rules and procedures that are dominant in teaching fractions (Cutting, 2019). The incorporation of visualisation processes in the pedagogy of fractions helps to bring out the true meaning of fractions and what it means to do operations with fractions. Learners' misconceptions about fractions often originate from the rule-based, abstract teaching methods used by teachers (Boaler et al., 2016). Visualisation was therefore considered as a complementary approach to the existing abstract methods. Since visualisation plays a significant role in making the abstract mathematics content more comprehensible, it can be described as a sense- making tool, integral to the pedagogy of mathematics in general. Consequently, visualisation was a key concept in this study.

1.2 BACKGROUND AND CONTEXT OF THE STUDY

My interest in this study emanated from my experience in teaching mathematics to groups of junior and senior primary school teachers enrolled at the University of Namibia (UNAM), Rundu Campus. These students were registered on both fulltime and distance modes and both categories of students struggled to understand many mathematics concepts, however, this problem was more prevalent among the distance students. Since most of the student teachers in this category were practising teachers who already possessed the Basic Education Teacher's Diploma (BETD), I contemplated the impact that the teachers' shortcomings in terms of fraction content and pedagogy, could have on their learners. Considering the fact that mathematics is perceived as a difficult subject (Siegler, 2017; Molina, 2014; Anthony & Walshaw, 2009), it is important to ensure that this subject is taught by teachers who are adept in terms of content and pedagogy. It was against this background that I undertook this study, focusing on one of the critical concepts in mathematics at the senior primary phase (grades 4 – 7), that is, fractions.

Despite the challenges experienced by my students in mathematics in general, fractions emerged as a concept that they struggled with the most. Hence, my interest in fractions. Fractions are generally difficult to teach and learn (Getenet & Callingham, 2017; Richardson, 2019; Ubah & Bansila, 2018). Siegler (2017) attributes the difficulties associated with fractions to inherent and cultural factors. He describes inherent factors as those that are concerned with the nature of fractions, for instance, fraction notation and the procedures involved in doing operations with fractions. He further explains that these difficulties indiscriminately affect all learners everywhere.

On the other hand, cultural factors refer to external factors that can either perpetuate or reduce the impact of the inherent factors in terms of understanding fractions. Siegler (2017) refers to the teachers' understanding of fractions, their fraction language, textbook content and cultural attitudes as cultural contingent factors. Thus, teachers' ability to use appropriate teaching strategies is associated with cultural factors. This study, with its emphasis on determining the role of visualisation on the pedagogy of fractions, therefore falls under cultural factors.

This study aims to ascertain the role of visualisation in enhancing the teaching of fractions in the senior primary phase. According to Yilmaz and Argun (2018), visualisation is an important tool in the teaching of mathematics because it facilitates the process of mathematical abstraction. They further suggest that mathematics thought processes are linked to perceptions or representations. “Therefore, having rich representations that include a large number of related pictures of concepts increases success in mathematics” (Yilmaz & Argun, 2018, p. 41). In addition, Arcavi (2003, p. 26) explains the role of visualisation in “seeing the unseen,” where the unseen in this context refers to abstract fraction concepts. Since the concept of fractions is one of the most abstract concepts at the senior primary phase, the role of visualisation in reducing the abstract nature of fractions was considered. Despite the imminent tensions in the role of visualisation in mathematics, in this context, the focus of this study was on establishing how the use of visual representations can evoke abstract forms of mathematics and in turn, also establish how the abstract forms of mathematics can evoke visual representations. Hence, the use of the Dual Coding Theory (see section 1.5). Furthermore, Boaler et al. (2016) argue that the exclusion of visuals from the mathematics curriculum deprives learners of alternative ways of seeing, understanding and extending mathematical ideas.

The difficulties that learners experience in understanding fractions have been observed at all levels of education, that is, primary, secondary and tertiary levels (Bruce et al., 2013). Therefore, a visual approach to the teaching of fractions is an important consideration because the traditional, abstract, verbal approach has proven to be a challenge over the years (Van de Walle et al., 2013). In many instances, fractions are taught in a manner that does not promote learners’ conceptual understanding (Bruce et al., 2013; Cramer et al., 2002). It is important to be mindful that the topic of fractions is not an isolated concept in mathematics and if not well mastered, it can affect learners’ understanding of other related mathematics concepts such as ratio, proportion, decimals, percentages, algebra and measurement (Van de Walle et al., 2013; Cramer & Whitney, 2010). It was against this background that the concept of fractions was selected for this study since an improvement in the pedagogy of fractions can yield positive results in the understanding of other mathematics concepts.

By their very nature, fractions are abstract, thus, this calls for alternative teaching approaches to fractions. Van de Walle et al. (2013, p. 311) suggest that the difficulties experienced by learners in understanding fractions can be attributed to the following factors: 1) the many meanings of fractions; 2) the unusual way in which fractions are written; 3) the teaching of fractions does not focus on enhancing learners’ conceptual understanding; and 4) learners’ overgeneralisation of their whole number knowledge. In this study, attention was drawn to the third factor because if the teaching aspect is addressed, the other aspects can begin to fall into place. Moreover, the way in which fractions are taught has a major impact on the understanding that is developed. Since the predominantly verbal teaching strategies have not been beneficial in enhancing learners' understanding of fractions, visual strategies should be considered as an alternative way of teaching fractions.

The first National Institute for Educational Development (NIED) report on the performance of learners in mathematics at the senior primary phase presents fractions as one of topics that learners in the Okahandja district struggled with in the fifth, sixth and seventh grades and teachers' pedagogical knowledge was identified as one of the contributing factors to the status quo (NIED, 2009). Moreover, the overuse of the verbal code in Albin and Bruce's (2019) study on Namibian eighth graders' conceptions of fractions using visual models reveals that "10 of the 12 participant learners showed systematic misconceptions and misrepresentations of fraction quantities indicated using these visual models" (p. 211). Therefore, this study seeks to establish the use of the verbal and nonverbal cues in fraction lessons.

Generally, the pedagogy of mathematics in the Namibian has been characterised by the verbal code, with particular emphasis on the application of rules and procedures rather than the conceptual understanding of learners (Hamukwaya & Haser, 2021; Amupolo, 2014, Vatilifa, 2012 & Ausiku, 2008). Furthermore, the limited use of visuals in mathematics classrooms is echoed by Shoba's (2020) findings in KwaZulu-Natal, South Africa where the use of visualisation tools was reported to be infrequent.

Thus, I argue that visualisation is instrumental in bringing about change to the current status of fraction instruction in Namibia. Change in the way learners perceive fractions can only be attained if the way fractions are taught is reformed. There is a need to reconsider the way in which fractions are taught, because the teaching aspect has contributed immensely to the current status quo. Through visualisation, learners are able to make sense of the standard algorithms that usually characterise the teaching of fractions. For Rosken and Rolka (2006, p. 458), "visualisation can be a powerful tool to explore mathematical problems and to give meaning to mathematical concepts and the relationship between them." For instance, with regard to fractions, visualisation can enhance learners' understanding of the procedures involved in determining fraction size, comparing fractions, addition, subtraction, multiplication and division of fractions. All too often, learners apply the rules and procedures associated with these fraction concepts without understanding.

Although visualisation is often regarded as an inferior approach in the pedagogy of mathematics, Boaler et al. (2016, p. 7) argue that visualisation should occupy an important place in the pedagogy of mathematics and should not be viewed as a prelude to more important mathematics. The use of visualisation processes has the potential to enhance learners' ability to represent mathematical objects, generalise mathematical processes as its application focuses on representing, manipulating and constructing physical and mental images of mathematical concepts and reflecting upon these images. A visual approach to working with fractions can facilitate a deep, comprehensive understanding of fractions since learners are encouraged to manipulate, reason and visualise fractions, as opposed to responding to traditional teaching methods which place emphasis on the memorisation of rules and procedures. This can be achieved by deliberately engaging learners in practical, visual processes.

Therefore, the importance of conducting research on pedagogy of fractions cannot be overemphasised. Bruce et al. (2013, p. 7) assert that challenges associated with the learning and teaching of fractions are broad and deep, "affecting foundational understandings that help or hinder the learning of other areas of mathematics." In order to address this problem, it is important to revisit the pedagogy of fractions. Ahmed et al. (2004, p. 316) affirm the importance of pursuing research in pedagogy-related issues in mathematics. They argue that constant research in the teaching of mathematics is vital. Hence, the way we teach, that is, the examples, demonstrations, the explanations and teaching aids that we expose our learners to has a major impact on the way learners learn and the mental images that they construct.

1.3 PURPOSE OF THE STUDY

This study was aligned with Siegler et al.'s (2010) five recommendations on developing effective fraction instruction for kindergarten through to the eighth grade. According to Siegler et al. (2010, p. 1), effective fraction instruction should: 1) be given a high priority by professional development programmes on improving teachers' understanding of fractions and how to teach them. 2) build on students' informal understanding of sharing and proportionality to develop initial fraction concepts; 3) help students recognise that fractions are numbers and that they expand the number system beyond whole numbers. Number lines should be used as a central representational tool in teaching this and other fraction concepts from the early grades onwards; 4) help students understand why procedures for computations with fractions make sense; 5) develop students' conceptual understanding of strategies for solving ratio, rate and proportion problems before exposing them to cross-multiplication as a procedure to use to solve such problems; and 6) Since these are very important recommendations in improving the pedagogy of fractions, this study was informed by them.

Although all the five recommendations were instrumental in the design of the RCFP activities, the focus was more on the fifth recommendation. This study was founded on the premise that the involvement of participants in the visually-based RCFP activities has the potential to improve their understanding of Content Knowledge (CK) and Pedagogical Content Knowledge (PCK) of fractions. Thus, the rationale of this study was *inter alia* to establish the impact of the RCFP on the pedagogy of fractions, based on the understanding that the pedagogy of fractions can only improve if teachers understand the subject matter very well and are able to present it in different ways.

1.4 RESEARCH GOAL AND QUESTIONS

The goal of this study was to determine the impact of visualisation processes on the teaching of fractions at the senior primary phase as a result of the teachers' participation in the RCFP. This goal was addressed through four research questions as stipulated in the section below.

The research questions were composed of one overarching research question and four sub questions.

Main research question: How does the incorporation of visualisation processes in mathematics lessons by the RCFP participants, enhance the teaching of fractions, if at all?

1. What type of visualisation processes do senior primary school teachers incorporate in their mathematics lessons?
2. How do senior primary school teachers incorporate visualisation processes in their mathematics lessons?
3. What significance do senior primary school teachers attach to the incorporation of visualisation processes in mathematics lessons?
4. What are the enabling and constraining factors in teaching fractions in an explicitly visual way at the senior primary phase?

1.5 THEORETICAL FRAMEWORK

The incorporation of visualisation processes into the pedagogy of fractions was informed by two theories, that is, the Dual Coding Theory (DCT) and Constructivist Theory (CT). These are the two theories underpinning this study. The DCT predicts that the use of verbal and the nonverbal codes is more likely to enhance learners' understanding of the subject matter rather than the use of only one code (Mayer & Anderson, 1991). Primarily, the DCT is founded on the notion that verbal and nonverbal codes present learners with two pathways of understanding mathematics concepts. While the verbal code has been adopted as a conventional way of presenting mathematics content, the nonverbal code has been perceived as less mathematical and a prelude to the real mathematics (Boaler et al., 2016). As suggested by Van De Walle, et al (2013), teachers' over dependency on the use of one code (the verbal code) in their teaching has proved to be problematic because mathematics concepts – particularly fractions – continue to be difficult to teach and learn.

In light of the above, it is important to consider the place of the nonverbal code in the pedagogy of mathematics. Hence, I argue that the reintegration of the nonverbal code into the pedagogy of mathematics can improve the teaching of fractions because the use of one code could deprive some learners from alternative methods of learning fractions. Presmeg et al. (2016, p. 3) assert that the nonverbal code is instrumental in the construction of knowledge and the interpretation of data and equations as this code serves as a valuable tool to understand the interplay between words, pictures and formulas. Therefore, research in the incorporation of visualisation in abstract mathematics concepts such as fractions is imperative.

The CT complemented the DCT because the focus of this study was on the nonverbal (visual) code which required participants to be actively involved in their selection, preparation of and use of visuals. Although the CT is usually associated with learning, it can also apply to teaching, especially if the teaching process requires the active involvement of the teacher, as was the case in this study. Moreover, the manner in which learners learn, depends on the way in which they are taught.

In this context, both the Cognitive Constructivist Theory (CCT) and Social Constructivist Theory (SCT) were instrumental in the design and execution of the RCFP activities. Kalina and Powell (2009) asserts that although these two theories are "fundamentally different, both types will ultimately form overall constructivism or constructed learning elements for students to easily grasp." While the CCT in this context, is concerned with the participants' individual understanding, selection and use of visualisation materials in their teaching, the SCT refers to the collaboration demonstrated in the RCFP activities as participants worked collaboratively in the development of visual fraction materials. In addition, Rowell and Palmer (2007) also concur that "cognitive constructivists and social constructivists have much in common, but they differ noticeably in one key area—the extent and type of involvement of both students and teachers."

Thus, the teaching strategies (verbal or nonverbal) used by the participants in their classrooms was influenced by a combination of the CCT and SCT.

The nonverbal code or visualisation is usually associated with the constructivist theory because it promotes learners' conceptual understanding through the formation of concept images (Rosken & Rolka, 2006). Bada and Olesgun (2015) further assert that "constructivism is an approach to teaching and learning based on the premise that cognition (learning) is the result of mental construction." Thus, teachers cannot continue to use the traditional, verbal approaches in their teaching and expect different results in terms of learning. It was therefore against this background that visualisation was considered a way of improving the pedagogy of fractions.

Change in the way fractions are generally perceived and understood can only be brought about by reflecting on the current teaching methods employed by teachers. Bada and Olesgun (2015, p. 67) suggest that "[i]f we accept that constructivist theory is the best way to define learning, then it follows that in order to promote student learning it is necessary to create learning environments that directly expose the learner to the material being studied." In this context, direct exposure to the concept of fractions entails the incorporation of visualisation processes in the teaching of fractions. Through direct exposure, the active construction of knowledge is facilitated. Hence, there is a direct relationship between the DCT and the CT.

1.6 METHODOLOGY

Based on the objectives of this study, a mixed methods approach was adopted. Although this study was predominantly qualitative, its nature and design dictated the use of both qualitative and quantitative data collection methods. The quantitative and qualitative sets of data were complementary, hence, both data sets were required to comprehensively address the research questions of this study. Despite its qualitative nature, quantitative methods were incorporated in the design and analysis of this study.

A sequential mixed methods approach was employed as the qualitative method was preceded by the quantitative method. Quantitative data collection methods were used in the initial stages of this study to select the participants for this study using a questionnaire. Since this study was informed by the DCT and CT, the questionnaire was designed in such a way that it made provision for respondents to indicate their teaching preferences. Consequently, ten participants were selected based on their responses to questions pertaining to the use of visualisation in the pedagogy of fractions. The two main criteria that were considered in the selection of the participants were 1) the use of the verbal and nonverbal codes in teaching fractions and 2) the phase they taught (senior primary). In other words, my intention was to select participants who indicated that they preferred to use both verbal and nonverbal codes in their teaching. Moreover, since the RCFP was comprised of teachers who taught in different phases, I took a keen interest in senior primary school teachers because it is during this phase that a decline in the use of visuals begins to occur (Boaler et al., 2016). Hence, it was necessary to establish the factors contributing to this trend and how this gap affects the pedagogy of fractions.

1.7 SIGNIFICANCE OF THE STUDY

The challenges associated with the teaching and learning of fractions can be attributed to what Wu (2002, p. 2) refers to as the "mystical and mathematically incoherent manner" in which they have been presented to the learners. It is therefore important to conduct research that seeks to understand how visualisation can help to redress "mathematical defects in the presentation of fractions in school texts and professional development materials". This study underscores the importance of visualisation in the teaching and learning of fractions as it could contribute to the success of learners in advanced mathematics.

The significance of this study lay in its potential to help teachers revisit fraction pedagogy at the primary phase as they discover the position, relevance and significance of visualisation in their teaching. The study further sought to establish the significance attached to the use of text (numbers) and visuals in the teaching and learning of fractions because these two modes could enhance learners' understanding of fractions if properly integrated into the pedagogy of fractions. In light of the challenges experienced by learners in understanding fractions, pedagogical reform is inevitable. To address the challenges associated with the pedagogy of mathematics, Brown and Quinn (2007, p. 5) suggest that:

> a change in emphasis from the development of algorithms to perform operations to the development of quantitative understanding based on students' experiences with physical models that emphasise meaning rather than procedure may be warranted.

This study therefore argues that creating a balance between the verbal and nonverbal approach seems to be a more viable solution since both approaches are important in the teaching and learning of fractions. A paradigm shift is required to reduce the prevalence of the dominant, traditional teaching approaches with their emphasis on the verbal code in order.

Although only eight teachers were involved in this study (after the withdrawal of two), there were about forty teachers participating in the Rundu Campus Fraction Project (RCFP). Since all the members of the RCFP were exposed to similar visualisation activities, they were expected to not only use the knowledge and skills acquired in their classrooms but to extend it to the school, cluster, circuit and regional levels. The main objective of this intervention was to empower the participants so that they could become resource persons in the pedagogy of fractions. Mathematics is a broad discipline, thus teachers may not be experts in all the branches of mathematics but helping them to master at least one branch of mathematics effectively can help to mitigate the challenges experienced in those specific areas. Through this intervention, the pedagogy of mathematics and the performance of learners from Rundu circuit is expected to improve.

1.8 STRUCTURE OF THE THESIS

This thesis is structured as follows:

Chapter Two

This chapter discusses the literature sources that informed this study, with particular emphasis on the concept of visualisation and how it relates to the conceptual understanding of fractions and the CK and PCK of fractions. This chapter also explores the significance of visualisation in 21st century mathematics education. An in-depth discussion of visualisation is portrayed because of its key role in this study. Despite the ongoing debates regarding the use of visualisation in the pedagogy of mathematics, the focus of this chapter was on identifying literature sources that interrogate the incorporation of visualisation processes in mathematics classrooms. Other crucial aspects discussed in this chapter include; the concept of fractions, operations with fractions, fraction sense, fraction models, conceptual understanding of fractions, CK and PCK, the gap in research, the Namibian context, the theoretical frameworks and the analytical framework

Chapter Three

This chapter outlines the research methodology used in the design of this study. Based on the research questions of this study, a mixed-methods approach was adopted. This method was particularly useful in addressing the overarching question and sub questions of this study. Thus, this chapter presents a layout of the research methodology, depicting the integration of the qualitative and quantitative methods in the design, the research methods, orientation, sampling methods, data collection methods and the analytical framework of this study. In addition, the data analysis process, validity and reliability, ethical considerations and challenges encountered during the research process are also discussed in this chapter.

Chapter Four

This chapter provides a summary of the findings from the three research instruments, that is, the questionnaire, observations and interviews. As alluded to earlier, the data sets were analysed using both qualitative and quantitative methods, hence the prevalence of tables and case descriptions of individual participants in this chapter. Emerging themes from the three data sets were useful in clustering the findings. For instance, based on the topics taught by the participants, seven themes (clusters) were identified, that is, multiplication of fractions, division of fractions, fractional parts and quantities, fraction size, comparing and ordering fractions, addition and subtraction of fractions, fraction representation and notation, word problems and assessment activities; while the importance of fractions, visualisation, factors contributing to learners' poor performance in fractions, RCFP and pedagogical considerations are the broad themes under interviews. This chapter is concluded with the synthesis of findings from all data sets.

Chapter Five

This chapter provides a synopsis of the entire dissertation by focusing on how the four sub-questions and the overarching research question were addressed. In addition, this study's contribution to new knowledge, recommendations and the limitations of the study are also discussed.

1.9 DEFINITION OF TERMS

This section provides the definitions of terms as depicted in this study.

Abstract is an idea or thought of a mathematics concept that is not represented in a real, visual or concrete form.

Abstraction is the process by which learners derive the underlying principles, rules and concepts of mathematics from the verbal or nonverbal codes. "It is a process of distinguishing an object"s common property (or properties) from the object itself and naming this property" (Yilmaz & Argun, 2018, p. 42)

Concept image as adopted from Tall and Vinner's 1981 definition refers to the cognitive structure in the individual's mind that is associated with a given concept.

Fraction models are different ways of representing fractions to make them comprehensible.

Fraction sense is the ability to comprehensively and flexibly understand fractions and do operations with them.

Reasoning tool refers to any tool or strategy that can enhance learners' understanding of mathematics. Visualisation is an example of a reasoning tool.

Nonverbal code refers to the use of visual representations including pictures, diagrams, concrete objects and manipulatives in mathematics. In this study the terms nonverbal code, visuals and visualisation were used interchangeably.

Verbal code refers to use of symbols, numbers, letters, standard algorithms, formulas and rules in the teaching and learning of mathematics. In some instances, the verbal code and the symbolic code were used interchangeably.

The incorporation of visualisation processes refers to the inclusion of visuals in the teaching of fractions.

Traditional teaching methods refer to teaching methods that focus on the verbal code and emphasise the recall of facts rather than understanding.

Visualisation as adopted from Arcavi's 2003 definition refers to the ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understandings.

Visual representations visual materials or objects that can evoke visualisation

Visualisation processes are visual methods or strategies that help explain fraction concepts.

Visual mathematics is the kind of mathematics that focuses on the use of pictures, diagrams, concrete objects, manipulatives, etcetera, to make mathematics concepts less abstract.

CHAPTER TWO
LITERATURE REVIEW

2.1 INTRODUCTION

This chapter presents literature that informs the use (or lack) of visualisation processes in the pedagogy of fractions. The carefully selected literature sources are based on the predetermined concepts of this study. Consequently, a conceptual framework was derived from these concepts. With visualisation at the centre of this study, other aspects such as how visualisation relates to mathematics, conceptual understanding, fraction sense and Pedagogical Content Knowledge (PCK) were considered. The significance of visualisation in the 21st century was also explored. Since this study was conducted in Namibia, it was deemed necessary to determine the role of visualisation in the Namibian context.

2.2 VISUALISATION

Visualisation is a concept that has not received adequate attention in mathematics mainly because of the view that visual thinking in mathematics can be deceptive (Brating & Peljare, 2008 & Rosken & Rolka, 2006) mathematics is a subject that requires abstract reasoning. This view of mathematics has the potential of disadvantaging learners whose reasoning is visually-oriented. Presmeg (2006, p. 207) explains that "a visual image is taken to be a mental construct depicting visual or spatial information, and a visualizer is a person who prefers to use visual methods when there is a choice." The question that is yet to be answered is whether learners are accorded enough opportunities to visualise mathematical content or not. Zimmermann and Cunningham (1991) point out two tendencies in mathematics, that is, logic and intuition.

While the verbal code is frequently associated with logic, the nonverbal code is often associated with intuition. However, Yilmaz and Argun (2018, p. 43) argue that visualisation is instrumental in logical reasoning because it "not only organizes knowledge as meaningful constructions but is also a significant factor that guides the analytical development of a solution." Hence, visualisation is not limited to intuitive reasoning. On the contrary, it aids logic reasoning in most instances.

Consequently, Zimmerman and Cunningham (1991, p. 2) argue that there is a need to strike a balance between these two tendencies because "to place undue emphasis on one element or group of elements upsets a balance." However, the integration of visualisation in mathematics is not properly understood. Hence, the perception (Boaler, et al, 2016) that visualisation is a misplaced concept in mathematics.

Although visualisation as a mathematical teaching strategy is not common in Namibia, its application in the pedagogy of mathematics has been gaining momentum (Chikiwa & Schafer, 2019; Miranda & Alder, 2010; Nghifimule, 2017). Recent studies on the incorporation of visualisation processes in Namibian mathematics classrooms point to the view that these processes do not only arouse learners' interest in mathematics but most importantly, they enhance learners' understanding of different mathematics concepts (Chikiwa and Schafer, 2019; Katenda, 2018; Muhembo. 2017; Dongwi, 2018) Despite its benefits, visualisation, if not properly incorporated in the pedagogy of mathematics could result in some unanticipated results (Jensen, et al., 2012 & Rieber, 1994). It is therefore vital to consider the teachers' role in the success or failure of visualisation in mathematics classrooms in the Namibian context by analysing the different strategies that teachers use to incorporate visualisation processes in their teaching and this is exactly what this study intended to do.

As reported in the fourth Southern and Eastern African Consortium for Monitoring Educational Quality (SACMEQ IV) (Shigwedha, et al., 2017), learners' performance in mathematics remains a concern. Evidently, the fourth SACMEQ report shows that the performance of learners in fractions from the fifth to the seventh grade proved to be problematic (Shigwedha, et al., 2017). Hence, teachers are urged to re-examine their teaching strategies in order to mitigate the challenges that learners encounter in mastering concepts such as fractions. Since visualisation can make mathematics accessible and less complex to learners (Yilmaz &Argun, 2018 & Brating & Pejlare, 2008), it is an alternative instructional strategy that is worth considering.

In the context of this study, visualisation is perceived as an approach to teaching that could enhance learners' understanding of the concepts taught in mathematics, particularly those that appear to be more abstract and difficult to learn. Understanding the definitions of visualisation in mathematics is thus an important starting point. In light of this, the need to adopt a comprehensive definition of visualisation is imperative.

2.2.1 Definition of visualisation

Visualisation refers to a fundamental reasoning tool in the learning and teaching of mathematics which has the potential to improve the pedagogy of mathematics if well implemented. It refers to the strategies that a teacher employs to make mathematics concepts more visual, imagery based and less abstract. Visualisation is, for example defined by Arcavi (2003, p. 217) as:

> the ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understandings.

For this study, Arcavi's definition was adopted due to its comprehensiveness and applicability to both the learning and teaching of fractions. This definition is appropriate because the study focuses on examining teachers' ability to create, use, interpret and reflect on visuals used in their fraction lessons and the processes involved thereof. Visualisation in this context is described both as a product and a process. According to Arcavi (2003), the visual mode is the dominant mode in which information is presented and it serves as a source of information about the universe. He further states that visualisation should not be confined to what we can see but also to what we do not see with our own eyes.

Arcavi's definition is extended to refer to the visual mental processes (usually unseen) that occur when one is thinking, teaching or learning about mathematical concepts and solving mathematical problems. These processes all involve visual mental processes that may not be explicitly articulated. Presmeg (2006, p. 207) clarifies that "visualization is taken to include processes of constructing and transforming both visual mental imagery and all of the inscriptions of a spatial nature that may be implicated in doing mathematics." Evidently, visualisation is implicitly involved in almost everything that teachers and learners do in mathematics, a claim that is supported by Avgerinou and Petterson (2011, p. 5-6) who maintain that Visual Language (ViL) exists and it must be learned. Although ViL is not universal, it is regarded as vital and may enhance learning. Since ViL must be learnt, by implication it means it must be taught.

Visualisation, in this thesis, is also defined as a mode of thinking which encompasses all the strategies that could be evoked by carefully selected visuals or words that are incorporated in the teaching of fractions. Weber (2014) affirms that "words can evoke images in our minds" (p. 125) and this is where visualisation and verbalisation intersect. In other words, visualisation can be achieved through the use of both visuals and words while verbalisation refers to the use of verbal descriptions to explain visual or abstract mathematics concepts. According to Wertheim and Movshovitz-Hadar (2003), visualisation and verbalisation are complementary and crucial in bringing about meaningful conceptual understanding. Thus, this study focused on determining the combined effect of visualisation and verbalisation on the pedagogy of fractions, with particular emphasis on visualisation.

Contrary to Presmeg's (1986) findings which revealed that gifted learners do not use visualisation in finding solutions to mathematics problems, Phillips et al. (2010, p. 48) present evidence to suggest that visualisation can enhance the ability of gifted learners to "imagine the rotation of a depicted object, to visualise its configuration, to transform it into a different form, and to manipulate it in their imagination." This conception of visualisation is relevant to the pedagogy of fractions. However, the use of visualisation in mathematics classrooms has not been encouraged in all countries, including the United States (US). Boaler et al. (2016, p. 1) explain that

> the provision of ways to *see, understand and extend* mathematical ideas has been underdeveloped or missed in most curriculum and standards in the US, that continue to present mathematics as an almost entirely numerical and abstract subject.

Boaler et al. (2016, p. 2) further acknowledge that good mathematics teachers are expected to use "visuals, manipulatives and motions to enhance learners' understanding of mathematical concepts." Therefore, a paradigm shift in the way that mathematics is taught is worth researching as learners still experience difficulties in mathematics. Since a fraction is often defined as a part of a whole, this concept lends itself towards the concept of size which is visual by nature. It is almost impossible to imagine the concept of size without visualising it. Van de Walle et al. (2013, p. 310) affirm that the difficulties that learners experience with fractions can be mitigated by affording learners opportunities "to experience fractions across many constructs" which include visual constructs. The use of visuals in teaching fractions is vital, a view supported by Dreher et al. (2016). However, it is one thing to incorporate visuals in the teaching of fractions and quite another to understand the role of these visuals in enhancing learners' understanding of mathematical concepts.

2.2.2 The role of visualisation in mathematics

While visualisation and mathematics are perceived to be two distinct concepts by some (Rahim et al., 2011), others see these concepts as intertwined, related and compatible. Hence, the exclusion, overuse or underuse of any of these concepts could lead to negative effects on the teaching and learning of mathematics. According to Swan and Marshall (2010, p. 14), visuals, if well designed and objective-driven, should stimulate learners' thinking and "lead to an awareness and development of concepts and ideas linked to mathematics." This claim is supported by Stokes (2002) who suggests that letters, numbers, words and symbols all begin with visualisation. In other words, visualisation is the beginning of conceptual understanding. Moreover, Turgut and Turgut's (2018) findings on the effect of visualisation on mathematics achievement reveal that visualisation has a strong positive effect on mathematics achievement.

Kosslyn's (1996) definition of visualisation as the construction of a mental image justifies its importance in the teaching and learning of mathematics from primary, through secondary, to tertiary education. Rahim et al. (2011, p. 1) explain that the creation of these mental images is

> not a self-evident, innate process, but something created and learned… we learn to see, we create what we see, visual reasoning or 'seeing to think' is learned, it can also be taught and it is important to teach it.

Therefore, visualisation is an important skill that should be promoted in mathematics classrooms by teachers who have acquired this skill and are concerned with the importance of facilitating their students' construction of knowledge. Yilmaz and Argun (2018, p. 2) explain that "mathematics is a field that is done with describing and objectivizing the concepts abstracted from real cases, and most of the descriptions, which are regarded as meaningful according to experiences, emerge visually." A common misconception regarding the nature of mathematics is that it is often perceived as an abstract subject with little room for visualisation. However, literature (Cramer et al., 2008; Dreher et al., 2016; Kadunz & Yerushalmy; 2015 & Yilmaz & Argun, 2018) reveals that visualisation has an important place in the teaching and learning of mathematics.

According to Vale and Barbosa (2018), visualisation can help learners understand the rules and formulas that are used in mathematics calculations because it helps them "go beyond the mere use of formulas, stimulating the development of intuition and the ability to see new relations, producing the cut with mental fixations contributing to a broader view of mathematics" (p. 24). Concept images facilitate learners' conceptual understanding of different mathematical concepts because "to understand a formal mathematical concept requires of the learner to generate a concept image for it". Foreman and Bennet (2016, p. vii) share their experience from their visual mathematics course 1:

> Teaching visual mathematics ourselves has affirmed our beliefs in the potential within each student, enriched our views of mathematics and the art of teaching and reinforced our commitment to support teachers in their efforts to change the way mathematics is learned and taught.

Although mathematics is regarded by many as "the model of a discipline based on rational thought, clear concise language, and attention to the assumptions and decision-making techniques that are used to draw conclusions" (Makina 2010, p. 4), a visual approach to working with fractions can facilitate a deep, comprehensive understanding of fractions. Through the incorporation of visuals in mathematics lessons, learners are encouraged to manipulate, reason and visualise fractions as opposed to responding to traditional teaching methods which place emphasis on the memorisation of rules and procedures. This can be achieved by engaging learners in practical, visual lessons. All too often, understanding in mathematics classrooms is measured by how well learners can produce correct answers regardless of the meaning that learners attach to the procedures or rules used to arrive at these answers (Bruce, et al., 2013 & Van De Walle, et al., 2013). When concepts are poorly understood, learners' future learning of advanced or related concepts is compromised. In her support for visual mathematics, Boaler et al. (2016, p. 7) explains the impact of visualisation across different levels:

> Visual mathematics is widely thought of as being appropriate for younger or struggling students and as a prelude to the "more important" abstract mathematics. It is true that abstract ideas can come from and be aided by visual mathematics, but visual ideas can also come from abstract mathematics and extend them to much higher levels. They can also inspire students and teachers, to see mathematics differently, to see the creativity and beauty in mathematics and to understand mathematical ideas.

Generally, traditional teaching methods have proved to be largely ineffective in teaching fractions because of the dominant use of rules as opposed to visuals (Bruce et al., 2013). For instance, Fazio and Siegler's (2011) findings reveal that although learners may memorise and use rules to solve fraction computations such as addition and subtraction without conceptual understanding, they tend to forget these procedures easily. Cramer et al. (2008, p. 494) assert that "students need to experience acting out addition and subtraction concretely with an appropriate model before operating with symbols." Therefore, in order to ensure that learners have a functional and conceptual grasp of fractions and are able to apply their fraction knowledge to other mathematical concepts, for example algebra, the overuse of traditional teaching methods should be critically reviewed amongst mathematics teachers. As stated by Boaler et al. (2016, p. 5), "we gesture because we see, experience and remember mathematics physically and visually, and greater emphasis on visual and physical mathematics will help students understand mathematics." This explains why mentally, there is a strong relationship between mathematics and visualisation and why we often unconsciously think about mathematics in a visual way.

I, therefore, argue that the incorporation of visualisation in mathematics lessons can improve the pedagogy of mathematics, since visualisation is an impactful aspect in the teaching and learning of mathematics. According to Zimmermann and Cunningham (1991, p. 2) "visualization offers a method of seeing the unseen. It enriches the process of scientific discovery and fosters profound and unexpected insights." This suggests that visualisation can unlock learners' mathematical world, a view supported by Piggott and Woodham (2008) who acknowledge that visualisation plays an important role in problem solving, concept formation, communication and the understanding of concepts. As stated by Anthony and Walshaw (2009, p. 15), "providing students with multiple representations helps develop both their conceptual understandings and their computational flexibility." This is every mathematics teacher's goal. However, it is rarely achieved because visualisation is sometimes overlooked in the design and execution of mathematics lessons.

The application of visualisation in mathematical calculations, for instance fractions, "debugs our intuitions, so that the symbolic solution is not only regarded as correct, but also natural and intuitively convincing," (Arcavi, 2003, p. 221). In other words, graphical representations complement abstract mathematical computations or arguments. Makina (2010, p. 2-3) attempts to narrow the gap between formal and visual instruction as she elucidates that "in the representational view of [the] mind, the overall goal of instruction is to help learners construct mental representations that correctly or accurately mirror mathematical relationships located outside the mind in instructional representations." This is an important consideration in the use of visualisation in order to ensure that mathematics concepts are not misconstrued.

The nature of fractions lends itself towards visualisation as an alternative approach to teaching fractions. According to Dreher et al. (2016, p. 3) fractions have a “high relevance of multiple representations.” Guzmán (2002) explains that through visualisation, we are “capable of relating, in a versatile manner the constellations of facts and results of the theory that are frequently too complex.” Fractions can, for example, be represented using the area, length and set models. The use of different fraction models, which are visual in nature, can reduce learners’ misconceptions about fractions because “different representations offer different opportunities to learn,” (Van de Walle et al., 2013, p. 312). Consequently, as learners are exposed to these different representations, they acquire a deeper understanding of fractions in different contexts. However, the area model is limited in particular because it only represents the part-whole meaning of a fraction, not its other meanings such as the number line or length model and the set model.

Some learners are visualisers and the incorporation of visualisation processes in mathematics lessons can enhance their understanding of different mathematics concepts. Presmeg (2006, p. 3) defines a visualiser as “a person who prefers to use visual methods when there is a choice.” Admittedly, learners come to mathematics classrooms with different expectations and different learning styles. Anthony and Walshaw (2009, p. 7) concur that “teachers who truly care about the development of their students’ mathematical proficiency show interest in the ideas they construct and express, no matter how unexpected or unorthodox.” Therefore, presenting learners with different opportunities to learn and enhance their understanding of mathematical concepts should be every mathematics teacher’s goal. Rahim et al. (2011, p. 497) concur that “visually based pedagogy opens mathematics to students who are otherwise excluded.” According to Presmeg (2014), visual teachers are characterised by their ability to make connections between the mathematics content and other aspects such as the syllabus, other subjects, prior knowledge, aspects beyond the syllabus and most importantly, the real world.

From a constructivist point of view, meaningful learning is determined by the extent to which learners can construct their own knowledge. Teachers have an important role to play in this process because the way they teach can either enhance the knowledge construction process or impede it. Anthony and Walshaw (2009, p. 11) assert that “effective teachers plan mathematics learning experiences that enable students to build on their existing proficiencies, interests and experiences.” A responsive approach to teaching is highly recommended as it enables teachers to adjust "their instruction to meet students learning needs". Knowledge construction occurs in different ways but at the centre of this very important process is visualisation. According to Klerkx et al. (2014, p. 5), the purpose of visualisation is to "represent an abstract information space in a dynamic way, so as to facilitate human interaction for exploration and understanding." In other words, visualisation can enhance learners' understanding of abstract concepts, fractions being one of them. However, since visuals are often presented in a static manner, Boucheix and Schneider (2009) suggest that it is important to consider the pros and cons of using static or dynamic visuals in any given context.

Although Bråting and Pejlare (2008, p. 2) argue that the use of visualisation in mathematics classrooms could derail learners from "the formal mathematical development", the focus of this study is not on replacing the formal mathematical procedures with visualisation. On the contrary, it intends to determine the complementary role of visualisation on the formal mathematics procedures. Moreover, Rivera (2011, p. 4) argues that "recent classroom research knowledge in mathematics education drawn from purposeful design experiments provides validatory, empirical and complementary proof of the central role of gestural, tactile and visual activity (i.e., sensuous) in rational mathematical activity." One of the common misconceptions about visualisation is that it does not include verbalisation. However, Arcavi (2003, p.227) explains that "visualisation as a process is not intended to exclude verbalisation or symbol", Rather, it is meant to complement it.

Visualisation is now recognised as an important reasoning tool employed in problem solving and mathematical proofs, "deeply engaged with the conceptual not the merely perceptual," (Arcavi, 2003, p. 226). Difficulties associated with visualisation in mathematics classrooms are classified by Arcavi (2003) into three key areas, namely cultural, cognitive and sociological. Cultural difficulties refer to how mathematics is perceived and what it means to do mathematics. Visualisation is sometimes perceived as less mathematical and Presmeg (1997, p. 310) refers to this as the 'devaluation' of visualisation. Such views do not promote classroom practices that value visualisation as an integral part of doing mathematics. Cognitive difficulties refer to the challenges people face in employing visual reasoning.

Often formal symbolic approaches are preferred as opposed to visualisation. Sociological difficulty refers to the transformation which knowledge undergoes in our attempt to avoid analytical representations and resort to visualisation. Generally, analytic representations are perceived as sequential, pedagogically appropriate and efficient (Arcavi, 2003, p. 236). Hence, teachers and learners often prefer to use symbols rather than visuals in mathematics classrooms. Presmeg (1986) asserts that students from visually rich cultures are often disadvantaged in conventional classrooms. A visually rich culture in mathematics is portrayed in classrooms where teachers take into account the cultural diversity, artefacts, interests and experiences of the learners (d'Entremont, 2015). This makes mathematics content accessible to learners.

When learners engage in activities involving fraction size or fraction computation, they often employ procedures which they do not understand. Clarke (2011, p. 24) affirms that "there appears to be a gap between students' procedural and conceptual understanding of fractions." Although formal mathematics strategies are promoted in mathematics classrooms, there is no evidence that these strategies are effective and mastered by all learners. Similarly, visualisations "do not have meaning independent of the observer," (Bråting & Pejlare, 2008, p. 2). In other words, misconceptions can occur when employing either formal strategies or visualisation. This is referred to by Rivera (2011, p. 4) as "rational and sensuous modes of mathematical knowing." Integrating the two approaches could yield positive results in terms of enhancing conceptual understanding.

The formal and the visual strategies are not antagonistic approaches but rather complementary approaches. Teachers are expected to maximise learning opportunities for learners using both strategies. Visualisation can serve as a tool to provide learners with an opportunity to comprehend fractions because learners are disadvantaged when teachers cling to the formal strategies. Yilmaz and Argun (2018, p. 2) assert that "appropriate visuals help students engage actively in abstraction." Hence, the need to complement traditional teaching methods with visualisation processes in the pedagogy of fractions.

In this context, understanding a mathematical concept is associated with learners' ability to (1) form a concept image of a given concept; (2) to see the unseen elements of a mathematical concept; and (3) to interchangeably explain mathematical procedures using formal or visual strategies. As stated by Foreman and Bennett (1995, p. 11) "sensory experiences help students develop mental images that aid in understanding and recalling relationships and information." Although sensory experiences are associated with intuitive knowledge, these experiences develop a holistic approach to learning among learners. Sobanski (2002, p. 11) describes this type of learning as learning that occurs by first "envisioning the whole picture." While abstract learning is linear and sequential, visual learning is non-linear and context bound, progressing from whole to part.

2.2.3 The significance of visualisation in the 21st century mathematics

Teachers are continuously expected to align their teaching to the pressing needs of current life. Van de Walle et al. (2013) point out content and pedagogical knowledge as the two most important aspects required by mathematics teachers to teach effectively in today's world. Van de Walle et al. (2013, p. 1) further acknowledge the teacher's critical role in the 21st century teaching of mathematics.

> Ultimately, it is you, the teacher, who will shape mathematics for the students you teach. Your beliefs about what it means to know and do mathematics and about how students make sense of mathematics will affect how you approach instruction.

Since the teacher is expected to play a crucial role in preparing learners for the 21st century learning of mathematics, the pertinent question to consider is what the role of visualisation in contemporary mathematics is and whether using visualisation can be instrumental in achieving the existing goals of modern society. Visualisation serves as an important constructivist reasoning tool which can enhance learners' understanding of abstract mathematical concepts. Moreover, Makina (2010, p. 3) describes visualisation as "an important aspect of mathematical understanding, insight and reasoning, which, in turn, enhances the learner's critical thinking." Through visualisation, learners construct mental images of different mathematics concepts which aid their conceptual understanding. Yilmaz and Argun (2018, p. 2) infer that "having rich representations that include a large number of related pictures of concepts increases success in mathematics." Learners draw onto these "pictures of concepts" when they encounter new problems in mathematics.

In order to determine the role of visualisation as a pedagogical tool in 21st century mathematics, it is important to consider its effectiveness and applicability in different mathematics concepts. Rahim et al. (2011, p. 497) affirm that "visual reasoning is not restricted to geometry or spatially represented mathematics only. All branches of mathematics contain processes and properties that provide visual patterns and visually structured reasoning." In terms of its effectiveness, Boaler et al. (2016, p. 5) reiterate that

> if we ask the best teachers about the importance of visual representations they will usually share the rich knowledge they hold, of the deep understanding that is enabled – both from teachers introducing mathematical ideas visually, and students using visuals to think and make sense of mathematics.

Essentially, visualisation augments both content and pedagogical content knowledge and this is a fundamental aspect in Namibia's Vision 2030. Namibians are urged to respond with "innovation and commitment to new challenges" (National Planning Commission, p. 11) in order to achieve the goals of Vision 2030. In other words, modern teachers are expected to master content and effective teaching strategies. Van de Walle et al. (2013, p.1) caution that "as you prepare to help students learn mathematics, it is important to have some perspective on the forces that effect change in the classroom." A change in the learning environment is a crucial starting point in this endeavour. Warner and Kaur (2017, p. 194) assert that learning environments have the potential "to create a new type of graduate which the 21st Century demands." Teachers are therefore expected to possess skills that enable them to transform their classroom settings and provide opportunities that can enhance learners' creative and innovative skills. As a teaching strategy and a reasoning tool, visualisation is an appropriate consideration in this process.

A visual mathematics environment further supports equity, which is one of the NCTM principles for school mathematics (NCTM, 2000). The equity principle calls for equal learning opportunities for all learners "regardless of personal characteristics, backgrounds or physical challenges," (NCTM, 2000, p. 12). Visual mathematics environments, in terms of textbook content, teaching strategies, activities and so on, are said to be liberating for both teachers and learners. While enhancing deep engagement with the content, visualisation also alleviates status differences in classrooms, empowers and liberates learners (Boaler, et al., 2016, p. 10) by helping them embrace different teaching and learning strategies.

Contemporary mathematics dictates that we reflect on the traditional teaching approach and determine its effectiveness because "the history of mathematics shows visualization to have been cut back and even avoided to a certain extent" (Kadunz & Yerushalmy, 2015, p. 43). Moreover, Makina (2010, p. 1) explains that traditional teaching methods place more emphasis on teaching learners "what to think" rather than "how to think" In addition, the Australian

Association of Mathematics Teachers (AAMT) (2009, p. 2) interrogates questions such as "should the kind of 'procedural' calculus that has been the pinnacle of achievement in school mathematics in the 20th century remain so in the 21^{st} century?" and "Does the emphasis on algebraic skills serve students' and the society's needs?" should be interrogated in our quest to make mathematics relevant to the 21^{st} century society. It is vital to bear in mind that a lot has changed in terms of technology. There seems to be a close relationship between technological advancement and the relevance of school mathematics. As elaborated in the AAMT "technologies effectively 'submerge' the popular perception of what constitutes mathematics." With the emergence of technology in the educational arena, teachers are encouraged to be creative and innovative. In fact, technology should promote the use of visualisation in teaching instead of suppressing it because the 21st century requires mathematics of a higher order for citizens to be able to understand, work with and create mathematical models that are accessible and powerful in the context of current and emerging technologies. As a result, the important mathematics in schooling should be about this sort of mathematics.

In other words, the mathematical content and teaching methods that were embraced in the 20^{th} century may be discriminatory in the sense that it may not benefit some learners. By effecting change on learners' mathematical thinking and reasoning, the learning of mathematics is maximised (Van de Walle et al., 2013). This change (in thinking and reasoning) can be attained through the incorporation of visualisation processes in the pedagogy of mathematics. In order to develop innovative approaches to teach mathematics, visualisation can be used as a reasoning tool to improve learners' conceptual understanding, and subsequently improve their performance. The difficulties that learners encounter in learning mathematics require a rigorous, conceptually-based, learner-friendly and effective approach.

In addition to the equity principle discussed earlier, curriculum or mathematics content is one of the six principles and standards for school mathematics articulated in the NCTM (2000). A curriculum is described as something "more than a collection of activities: it must be coherent, focused on important mathematics, and well-articulated across the grades" (p. 2). Important mathematics refers to mathematics concepts that should be mastered by learners for them to acquire a broader, integrated understanding of the subject. The NCTM further explains that "mathematical ideas can be considered important if they help develop other ideas, link one idea to another, or serve to illustrate the discipline of mathematics as a human endeavour" (2000, p. 2).

In light of the above, it is important to consider whether visualisation has a place in what is termed ‘important mathematics’ or not. All too often, important mathematics is associated with learners’ abstract, computational skills. According to Yelland (2014, p. 4), computational skills are not enough for non-routine (novel) problems. “To solve novel problems, students need to have a broad mathematical knowledge base, to employ thinking skills and to have a positive attitude towards mathematics.” This should be the focus of the 21st century mathematics. Unfortunately, learners continue to think about and do mathematics in the same traditional manner due to the persistence of traditional teaching methods in mathematics classrooms (Ausiku, 2008). This trend is also attributed to the fact that teachers focus on teaching learners what to think and not how to think (Makina, 2010). In order to develop a broad ‘mathematical knowledge base’ and inculcate a positive attitude towards mathematics, innovative approaches encompassing visualisation should be promulgated.

Twenty-first century mathematics requires teachers to de-emphasise traditional teaching methods in order to reconceptualise mathematics (Coffland & Xie, 2015). Switching from one mode of teaching to another is not enough because the curriculum structure has an impact on the way teachers teach. Coffland and Xie (2015, p. 5) assert that one of the challenges (separations) emanating from the abstract mathematics structure is the fact that mathematics remains disconnected from reality. As stated by Healy and Hoyles (1996), learners are not motivated to engage in visual thinking when mathematics is presented in symbolic form. Moreover, learners are encouraged to learn when they see the relevance of the mathematical content that they are taught. Therefore, it is important to remember that change in 21st century mathematics cannot be achieved if the classroom practices and the mathematics curriculum remains the same.

2.3 CONCEPTUAL UNDERSTANDING

Although conceptual understanding is the goal of all the teaching and learning activities that occur in the classroom, it is seldomly achieved because it requires more effort from both the teachers and the learners (CCR, 2014). It is a complex process that is initiated by the teacher and requires the active participation of the learner. In the past, fractions were taught in a more traditional teacher-centred way, learners were expected to reproduce facts and in most instances, this resulted in little or no understanding. To date, this practice has continued in many mathematics classrooms and has resulted in the poor mastery of mathematical concepts. According the Center for Curriculum Redesign (CCR, 2014, p. 1) "Understanding mathematics is essential for full participation in society. Yet mathematics is learned mostly by rote, with little understanding or possibility for transfer." In an effort to improve learners' conceptual understanding, teachers use different strategies. However, a good starting point in ensuring that learners' conceptual understanding is enhanced, is by defining the term 'conceptual understanding'. It is also important to explain this concept in relation to fractions.

2.3.1 Definition of conceptual understanding in relation to fractions

Students' difficulties with fractions often stem from a lack of conceptual understanding. Many students view fractions as meaningless symbols or view the numerator and denominator as separate numbers rather than as a unified whole (Fazio & Siegler, 2011, p. 7)

Kilpatrick et al. (2001, p. 118), define conceptual understanding as an "integrated and functional grasp of mathematical ideas." The key words in this definition are 'integrated' and 'functional'. 'Integrated' refers to the 'connectedness' of the different mathematical concepts while 'functional' refers to the 'applicability' of the knowledge acquired. Similarly, Bossé and Bahr (2008, p. 3) describe conceptual knowledge "as a well-organized mental package of highly-connected concepts." This type of knowledge enables learners to develop a profound understanding of fundamental mathematics concepts (Ma, 1999, p. 108). However, it is not always achieved as teachers continue to rely on the traditional teaching methods. Bossé and Bahr (2008, p. 3) assert that "historically, traditional mathematics instruction has been characterized by an extreme commitment to the rote memorization of procedures with little concern for the associated concepts that underlie them." Molina (2014, p. 1) observes that "while there are promising changes occurring in mathematics instruction, we still need to help both teachers and students develop a more conceptual understanding of mathematics." Teaching for conceptual understanding does not happen spontaneously. It requires a paradigm shift in terms of pedagogy.

Conceptual understanding is also defined as the "knowledge of concepts" (Bisson et al., 2016, p. 142). By inference, the understanding of concepts can facilitate the attainment of conceptual understanding. Does this suggest that the understanding of fractions as a concept, translates into the conceptual understanding of fractions? The answer to this question is ‘yes’ because there is a direct relationship between the understanding of concepts and conceptual understanding. Kilpatrick et al. (2001) concur that the knowledge of mathematical concepts can improve learners’ conceptual understanding although Bisson et al. (2016, p. 142) argue that it is difficult to measure the learners’ understanding of concepts with “acceptable validity and reliability.” Some of the observable characteristics suggested by Molina (2014) of learners who have mastered a mathematical concept, include learners’ ability to provide comprehensive steps in solving mathematics problems, explain why the chosen strategies are appropriate and establish connections with related concepts.

The role of conceptual understanding in the acquisition of mathematical proficiency is a crucial aspect to consider. Although the five strands of mathematical proficiency proposed by Kilpatrick et al. (2001) are all important and intertwined, conceptual understanding has proved to be the cornerstone of mathematical proficiency. Ben-Zeev and Star (2001, p. 23) further point out that “too often symbolic procedures are learned by rote and suffer from an impoverished conceptual base”. This view is supported by Kilpatrick et al. (2001, p. 122), who claim that although “procedural fluency and conceptual understanding are often seen as competing for attention in school mathematics, putting skill against understanding creates a false dichotomy.” In addition, Rittle-Johnson and Alibali (1999, p. 176) assert that “children with greater conceptual understanding tend to have greater procedural skill.” Due to its fundamental role in the development of mathematical proficiency, conceptual understanding can enhance learners’ proficiency in the other four strands. As stated by Molina (2014, p. 3):

> Mathematics instruction must first ensure that students’ conceptual understanding is deeply embedded. When students have truly mastered a concept, they should be able to show all the detailed steps in a process, explain why those steps occur, and connect the process to related concepts.

With regard to fractions as a mathematical concept, this means that developing learners’ conceptual understanding of fractions can ultimately improve their fraction proficiency. It is important to consider what the conceptual understanding of fractions entails. Fazio and Siegler (2011, p. 6) provide a comprehensive definition of what it means to understand fractions conceptually

> We define conceptual knowledge of fractions as knowledge of what fractions mean, for example their magnitudes and relations to physical quantities, an understanding of why arithmetic procedures with fractions are mathematically justified and why they yield the answers they do.

Fazio and Siegler's definition includes fundamental aspects which are critical in developing learners' conceptual understanding of fractions. The poor understanding of fraction concepts such as fraction magnitude and operations with fractions are some of the factors that have contributed to challenges that learners encounter in understanding fractions. Drawing on Kilpatrick et al.'s (2002) explanation, Laswadi et al. (2016, p. 68) maintain that learners with conceptual understanding "are able to use several representations and communicate their ideas" and they "are able to choose a representation that is suitable for a specific situation." Therefore, visualisation is a key aspect in addressing the challenges related to the conceptual understanding of fractions.

2.3.2 The relationship between visualisation and conceptual understanding

The terms conceptual understanding and visualisation are interrelated in the sense that one encompasses the other. Conceptual understanding is actually a goal that can be achieved when teachers engage in various teaching activities. One of the activities that can augment learners' conceptual understanding of fractions is visualisation. Drawing on Lesh and Doerr's (2003) translation model of representations, Suh (2007, p. 164) concurs that "students make more meaningful connections when representing a mathematical idea in multiple modes: manipulatives, pictures, real-life contexts, verbal symbols and written symbols." It can be argued that the first three modes are all visual. Therefore, visualisation is an important tool in developing learners' conceptual understanding.

As argued above, visualisation has deep, long-lasting effects on learners' conceptual understanding. By definition, conceptual development refers to the build-up of networks among related concepts which learners can refer to in order to understand new concepts or solve mathematical problems. Boaler et al. (2016, p. 1) explain that "when students learn through visual approaches, mathematics changes for them, and they are given access to deep and new understandings." The networks connecting mathematics concepts are usually stored in a visual form, making it easier for learners to access stored knowledge. Moreover, the

National Council of Teachers of Mathematics (NCTM, 2000) lists representation as one of the Standards for School Mathematics from pre-kindergarten through to Grade 12.

> Representations should be treated as essential elements in supporting students' understanding of mathematical concepts and relationships; in communicating mathematical approaches, arguments, and understandings to one's self and to others; in recognizing connections among related mathematical concepts; and in applying mathematics to realistic problem situations through modelling (NCTM, 2000, p. 67)

Essentially, visualisation is embedded in the conceptual understanding of mathematics. As learners engage in visualisation, they form different mental images of mathematics concepts which can be perceived as a collection of pictures or diagrams of interrelated mathematics concepts. Thus, visualisation promotes the retention of mathematical knowledge. In fact, we usually think about and do mathematics visually. Boaler et al. (2016, p. 6) explain that "children may go through hundreds of hours of calculating only ever seeing numbers and symbols but mathematicians rarely if ever, solve a problem without visual representations." According to the NCTM (2000, p. 67), representation:

> … applies to processes and products that are observable externally as well as to those that occur internally in the minds of people doing mathematics. All these meanings of representation are important to consider in school mathematics.

The attainment of conceptual understanding may be hindered by the overuse of "abstraction and numbers" and as a result, learners could find mathematics content "inaccessible and uninteresting" (Boaler et al., 2016, p. 6). Visualisation can facilitate the conceptual understanding of content that is presented in an abstract form. Pelttari (2016, p. 106) states that visualisation can be used "to engage in the world depicted through the text." Since fractions are difficult to teach and learn, the incorporation of visuals is an alternative worth considering. In order to attain deep conceptual understanding of mathematics concepts, the NCTM recommends the use of multiple representations in teaching and learning. While there are success stories about number sense development among learners, the development of fraction sense remains a challenge. As a result, some learners continue to misapply their number knowledge when dealing with fractions (Dyson et al., 2020). An important question to consider in dealing with the challenges that learners encounter when learning fractions, is how teachers can overcome these challenges. Fennell and Karp (2016) suggest that the development of fraction sense can improve learners' proficiency in fractions.

2.4 FRACTION SENSE

2.4.1 Definition of fraction sense

Developing learners' fraction sense has proved to be a daunting task for teachers. Although adequate research (Griffin & Callingham, 2006; Kennedy & Tipps, 2000; Lock & Gurganus, 2004; Tsao & Lin, 2011 & Van de Walle et al., 2013) has been conducted in developing learners' number sense, the same cannot be said about fraction sense. Barnett (2016, p. 12) points out that "working with fractions is usually the first time in their schooling when students give up on making sense of a concept and resort to simply following the procedure demonstrated by the teacher." Van de Walle et al. (2013) asserts that the poor understanding of fractions is one of the factors contributing to learners' misapplication of their whole number knowledge when doing operations with fractions. While number sense refers to a good intuition about numbers and their relationships (Van de Walle et al., 2013), fraction sense refers to "a collection of fundamental perceptions and conceptions about fractions consistently associated with building conceptual understanding of fractions" (Way, 2011, p. 154). As explained earlier, the emphasis is on the conceptual understanding of fractions.

Woodward (1998) provides a more comprehensive definition of fraction sense which includes the characteristics of a learner who has acquired good fraction sense. Woodward defines fractions sense as:

> … an individual's ability to understand the meaning of fractions; to reason qualitatively about the absolute and relative size of fractions; and to make logical judgments about the reasonableness of calculations with fractions based on one's understanding of fractional numbers and the effect of operations on those numbers. (p. iii)

Teachers often grapple with the concepts mentioned in Woodward's definition of fraction sense. Learners' conceptual understanding of fractions rests on their ability to conceptually understand the meaning of fractions, to reason about the size of fractions and make sense of operations with fractions. These are considered as the foundational knowledge of fractions (Bruce et al., 2013; Dyson et al., 2020 & Van de Walle et al., 2013). The challenges that learners encounter in fractions are attributed to poor fraction sense, which in turn affects their understanding of other related mathematics concepts. Bruce et al. (2013, p. 6) assert that "the implications are broad (touching on, for example, a wide range of career fields), but they are also deep, effecting foundational understandings that help or hinder the learning of other areas of mathematics." It is therefore essential to consider the other mathematics concepts that depend on learners' mastery of fractions when teaching about fractions.

Thus, developing fraction sense in the foundation phase is crucial. Borenson (2015, p. 1) suggests that teachers "need to acquire the content knowledge and the instructional strategies," to enable them to teach fractions effectively and help learners make sense of fraction concepts. However, Huang et al.'s (2009) study on preservice teachers' content knowledge of fractions reveals that preservice teachers often have a poor conceptual knowledge of fractions. They attributed this to their weak understanding of the meaning of fractions.

In order to meaningfully comprehend the concept of fractions, learners are expected to understand the different constructs through which fractions can be interpreted (Bruce et al., 2013; Clarke, 2011 & Van de Walle et al., 2013). Clarke (2011) identifies part-whole, measures, operators, quotient and ratio as the five fraction constructs which ought to be presented to learners. Table 2.1 below presents the five fraction constructs:

Table 2.1: The five fraction constructs adapted from Richardson (2019, p. 21-22)

No	Fraction construct	Definition
1.	Part-whole	This is a way of representing part of a whole set of objects. It involves partitioning of a shape/number of discrete objects into equal parts, (unitising) or determining how many parts would be in a whole set based on a part of the set (re-unitising) (Richardson, 2019, p. 21)
2.	Measurement	Involves identifying a length and then using that length as a measurement piece to determine the length of an object (Van de Walle et al., 2013, p. 311). It is an appropriate construct to represent the magnitude of fractions
3.	Division	Understanding fractions as a result of division or sharing (Richardson, 2019, p. 22)
4.	Operator	Involves the multiplicative aspect of fractions. For example $\frac{2}{3}$ as finding two thirds *of* a quantity (Richardson, 2019, p. 22)
5.	Ratio	Involves the comparison of part-part or part-whole, for example, the fraction $\frac{3}{4}$ could mean the ratio of those wearing jackets (part) to those not wearing jackets (another part) or it could mean those wearing jackets (part) to the total number of learners in class (whole) (Van de Walle et al., 2013).

Bruce et al. (2013) attribute learners' poor understanding of fractions to traditional fraction instruction which is predominantly part-whole oriented. This has deprived learners of opportunities to see and extend their understanding of fractions. Zhang (2018, p. 5) further emphasises that "only through teaching based on understanding of these different sub-constructs of fractions, is it possible to help students truly understand the nature of fractions and master diverse complicated operations of fractions." Bruce et al. assert that this approach can deepen learners' conceptual understanding of fractions (2013, p. 9), which can prepare them "for a more seamless and coherent transition to operations with fractions." However, teachers often focus on the part-whole construct which has proved to be less effective when used in isolation.

Another crucial aspect of fraction sense as pointed out by Woodward (1998, p. 1), is the "ability to reason qualitatively about the absolute and relative size of fraction." Fraction size is a critical aspect in improving the conceptual understanding of fractions. As stated by Barnett (2016, p. 18),

> … understanding the magnitude of a fraction helps determine the reasonableness of an answer when operating with fractions, thus deterring students from employing flawed procedures, such as adding unlike denominators together when adding fractions.

Fractions can only make sense to learners if they have a deeper understanding of fraction size because it is a prerequisite for understanding other fraction concepts. In fact, understanding fraction size can be equated to the emphasis placed on understanding the magnitude of natural numbers in whole number concepts.

Qualitative knowledge or qualitative reasoning is described by Grouws (2006, p. 321) as "intuitive knowledge or knowledge that belongs to the individual, is constructed from real experience and provides for considerable flexibility of thought." Although mathematicians hold different views on the role of intuitive knowledge in mathematics, Bergsten (2004, p. 2) maintains that intuitive knowledge is "a reliable source of true knowledge" and "a mental strategy to search the essence of phenomena, beyond logical argumentation." Intuitive knowledge is foundational to the teaching and learning of mathematics as it helps learners understand formal mathematics. There is an inextricable interplay between intuitive and formal mathematics that is often evoked by the use of visuals. For instance, the reason why teachers often use visual representations to bring learners' prior (intuitive) knowledge to the fore, is to facilitate the process of "uncovering students' pre-existing knowledge in order to make connections between school-taught, formal knowledge and students' informal intuitions," (Ben-Zeev & Star, 2001, p. 6). The exclusion of intuitive knowledge in the teaching and learning of mathematics results in what Ben-Zeev and Star (2001, p. 10) refer to as an "overly procedural focus." As a result, learners begin to see mathematics as a subject that is detached from reality. Therefore, it is important for teachers to find common ground, preferably through visualisation, to accommodate both formal and intuitive mathematics knowledge.

Understanding fraction size through visualisation enables learners to reason about the effect of increasing or decreasing the numerator or denominator on the fraction (Grouws, 2006). In this study, I argue that fraction size or the meaning of fractions is a vital starting point in enhancing learners' conceptual understanding of fractions as it affects learners' ability to determine the size of a fraction (locate a fraction on the number line), compare and order fractions and do operations with them. Asku (1997, p. 373) elaborates that "a common type of error in teaching fractions is to have students begin computations before they have an adequate background to profit from such operations. Students must understand the meanings of fractions before performing operations with them." Most of the fraction misconceptions pointed out by Cramer and Whitney (2010) are associated with learners' poor conceptual understanding of fractions. Van de Walle et al. (2013) underscore the importance of manipulatives or fraction models in helping learners understand fractions meaningfully. "In fact, what appears to be critical in learning is that the use of physical models leads to the use of mental models and this builds students' fraction understanding" (p. 312). Therefore, the use of different fraction models should be encouraged among teachers as this "broadens and deepens" learners' understanding of fractions.

Apart from learners' individual ability to understand the meaning of fractions and to reason qualitatively about the absolute and relative size of fractions, learners' ability to make "logical judgements about the reasonableness of calculations with fractions based on one's understanding of fractional numbers and the effect of operations on those numbers" (Woodward, 1998, p. iii) is another vital element of fraction sense. In order to reason logically about operations involving fractions and the effect of these operations on the fractions, learners require fraction content that is more engaging and less abstract. According to Cramer and Whitney (2010, p. 341), one of the myths related to fractions is that "students mistakenly use the operation rules for whole numbers to compute with fractions for example." This can be attributed to the fact that learners' understanding of fraction size or the meaning of fractions is not well developed.

Although developing learners' fraction proficiency is not an easy task, Fennel and Karp (2016, p. 1) suggest that fraction sense can be developed

> … through instructional opportunities involving fraction equivalence and magnitude, comparing and ordering fractions, using fraction benchmarks, and computational estimation. Such foundations are then extended to operations involving fractions and decimals and applications involving proportional reasoning.

Unfortunately, this is not as easy as it sounds because the teachers who are entrusted with the responsibility of developing this kind of knowledge do not possess the required knowledge and skills themselves to carry out this important task effectively. Borenson (2015) attributes the challenges experienced by learners, in terms of learning fractions, to the observation that many teachers do not have adequate fraction knowledge. He suggests that "the acquisition of fraction sense cannot happen if teachers themselves do not have this conceptual understanding and the means to communicate it" (p. 1). Teachers are expected to have adequate content and instructional strategies in the context of teaching fractions. Fennell and Karp (2016, p. 2) recommend that teachers should consider "the progression of fraction instruction" or fraction trajectory as this could help them track the challenges learners encounter in learning fractions. They further suggest that fraction instruction ought to begin from

> … early work with conceptual understandings involving partitioning of regions to the magnitude-related topics of fraction equivalence and comparing and ordering fractions involving varied representations, including the number line.

This trajectory is fundamental to the understanding of operations with fractions (addition, subtraction, multiplication and division). Once a foundational base is established, learners can reason and make logical judgements about the procedures involved in fraction computation and the outcome thereof. According to Barnett (2016, p. 17), proficiency in fractions depends on two critical factors, that is, learners' "ability to understand what a fraction is and how different fractions relate to each other." This can be achieved by helping teachers revisit their instructional strategies. The incorporation of different fraction models provides learners with different ways of 'seeing' a fraction.

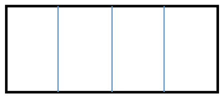
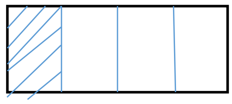
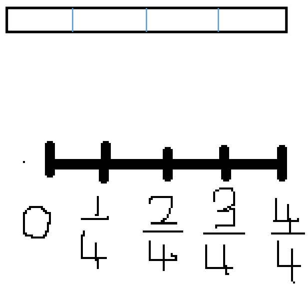
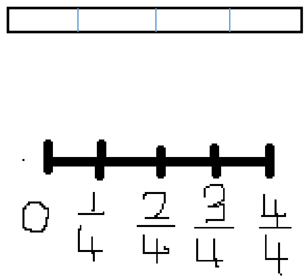
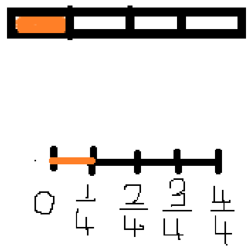
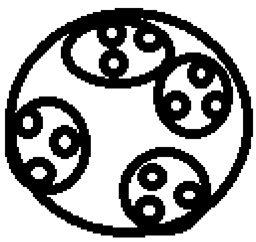
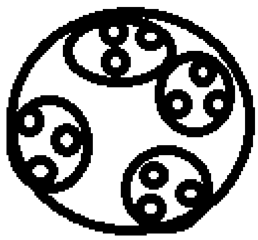
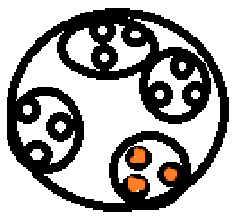
2.4.2 Fraction models

The complexity of this branch of mathematics (fractions) has steered mathematics educators into developing strategies that can enhance learners' conceptual understanding of fractions, and visualisation is one of them. Therefore, it is assumed that the appropriate use of fraction models such as area, length and set models can reduce the misconceptions that learners have about fractions. It is important that leaners develop representational competencies, that is, knowledge and skills that enable them to use visual representations to reason about and solve fractions. Bezuk and Cramer (1989, p. 157) specifically advocate for quantitative thinking as a suitable way to develop learners' fraction sense. They maintain that:

> To think quantitatively about fractions, students should know something about the relative size of fractions. They should be able to order fractions with the same denominators or same numerators as well as to judge if a fraction is greater than or less than 1/2. They should know the equivalents of 1/2 and other familiar fractions. The acquisition of a quantitative understanding of fractions is based on students' experiences with physical models and on instruction that emphasizes meaning rather than procedures.

Consequently, Van de Walle et al. (2013) argues that there are three models through which fractions can be represented to enhance quantitative thinking, namely the area, length and set models. The area model is derived from representing the whole of a fraction by any shape (such as a circle) and then slicing the shape into the fractional parts. The length model is a linear representation, usually a number line, indicating the fractions. However, despite its potential benefits in enhancing learners' conceptual understanding of fractions, this is rarely used by teachers (Cramer & Whitney, 2010). The length model enhances learners' understanding of fractions because it draws learners closer to the conceptualisation of fractions as numbers that expand the number system. Its connection to real-world contexts such as measurement is an added advantage. The set model refers to the representation of fractions using the set as a whole. It is recommended due to its practical application and the fact that it serves as the foundation for fractions and division. The sharing activities often used by teachers when dealing with fractions and division stem from this model. Table 2.2 below provides detailed definitions and illustrations of the three fraction models.

Table 2.2: Fraction models adapted from Cramer and Whitney (2010, p. 342)

No	**Fraction model**	**What defines the whole**	**What defines the part**	**What defines the fraction**
1.	Area model	The area of the defined region, e.g. a rectangle 	Equal parts, e.g. fourths 	The shaded part as it relates to the whole, e.g. $\frac{1}{4}$ 
2.	Number line model	Unit of measurement or length 	Length or measurement 	The location of a point in relation to zero and other 
3.	Set model	Representation of fractions using the set as a whole 	Equal number of objects 	The count of objects in a subset as it relates to the defined whole 

Teaching fractions through the three visual models presents different learning opportunities for learners. Murdock-Stewart's study (2005) reveals that the use of different models in presenting fractions has a positive impact on learners' performance because learners then 'see' fractions from different perspectives and appreciate the multi-dimensionality of fractions. Also, the use of different fraction models promotes differentiated instruction to cater for learners' diverse learning needs and experiences. This is a recommended approach for teachers to ensure that learning among learners with different abilities is enhanced. Mildenhall (2013) supports the use of a variety of models to enhance learners' conceptual understanding of fractions but cautions that these models must be selected carefully.

The area model is commonly used by teachers because it is well aligned with the definition of a fraction as part of a whole. Hence, teachers find it easier to demonstrate the 'part' and the 'whole' using the area model. Despite its suitability for the part-whole construct of fractions, the overuse of this model could limit learners' view of fractions and presents fewer opportunities for the conceptual understanding of fractions. "This means that students often do not get to explore fractions with a variety of models and/or do not have sufficient time to connect the visuals to the related concepts," (Cramer and Whitney, 2010, p. 342). The use of different models presents learners with alternatives to understand fractions as some learners find the area model more sensible, while others prefer the length or set models. Since one of the misconceptions learners have about fractions is that they see the numerator and the denominator as "two separate, unrelated whole numbers" (Jigyel & Afamasaga-Fuata'i, 2007, p. 8), the area model can be used to alleviate such misconceptions among learners because this model visibly presents the part (numerator) and the whole (denominator).

In addition, the number line is an important teaching tool (Dyson, et al., 2020; Ervin, 2017; Fennel & Karp, 2016). However, it has been observed that teachers seem to cling to the area model, without according learners opportunities to explore the other models (Jigyel & Afamasaga-Fuata'i, 2007). Research (Van de Walle et al., 2013; Cramer & Whitney, 2010; Hull, 2005) has shown that the length model is a vital teaching model which can be used to address the challenges that learners encounter in terms of seeing a fraction as an iterative unit. This property of fractions is not evident in the area model because learners do not usually perceive the parts of a whole as repeated parts. Jigyel and Afamasaga-Fuata'i's (2007) study reveals that there is a "tendency by some students to perceive the double count of shaded parts and total parts as two unrelated quantities," (p. 9). The length model addresses this misconception as it helps learners "understand a fraction as a number (rather than a number over another number) and helps develop other fraction concepts," (Van de Walle et al., 2013, p. 314). The linear model essentially supports the understanding of fractions as numbers that expand the number system. Dyson et al. (2020, p. 245) assert that the number line model "provides an underlying structure for learning a range of fraction concepts and skills," and as such it should form part of any fraction intervention programme.

While the length model primarily promotes measurement (Ervin, 2017) which is one of the five constructs of fractions, the set model is more aligned with the operator (and division) as a fraction construct. According to Hull (2005, p. 23), the set model "requires students to understand that a group of objects is considered the whole and the individual objects would be subsets or parts of the whole." The area and length models usually focus on one as a whole, while the set model takes a different facet, where the whole is a group of objects and learners are expected to work out the fraction represented by a subset of objects simply by comparing how the smaller set relates to the total number of objects, or by division, for instance, $\frac{1}{5}$ of 20. A challenge pointed out by Van de Walle et al. (2013), is the tendency among learners to "focus on the size of the set rather than the number of equal subsets in the whole." The set model is connected to the use of fractions in everyday life where learners are expected to understand the meaning of a half, a third, a quarter, two-thirds, etcetera. in different contexts.

If appropriately used, the fraction models can develop the five constructs of fractions outlined in Table 2.1, namely the part-whole, measurement, division, operator and ratio. According to Clarke et al. (2007, p. 24), the fact that learners "cannot synthesise the many different interpretations (sub-constructs) of fractions, and are not familiar with a variety of representations (models)" is the main reason why fractions are difficult to teach and learn. Hence, Hull (2005, p. 9) recommends that "to challenge and extend student understanding, multiple contexts and representations should be used to develop flexible interpretations and consolidate understanding of fractions." Due to the complex nature of fractions, the use of models should be encouraged because this gives learners an opportunity to comprehend aspects of fractions which might otherwise be symbolically incomprehensible. Ervin (2017, p. 3) explains that

> … models can be used to help clarify ideas that may be confusing when presented only in symbolic form. Also, models can provide students with opportunities to view problems in different ways and from different perspectives and some models may lend themselves more easily to particular situations than others.

The synchronised use of different models is encouraged because it gives learners opportunities to view fractions from different perspectives, thus enhancing their conceptual understanding. Kara and Incikabi (2018, p. 463) suggest that

> … taking advantage of different representations in the teaching of a mathematical concept and making transitions between different forms of representations are critical in terms of a complete internalization of mathematics.

Moreover, Nicolaou and Pitta- Pantazi (2011)'s theoretical model for understanding fractions considers representations as one of the key features in attaining learners' conceptual understanding. They postulate that learners' ability to "shift from one kind of representation to another is especially important for fractions' understanding," (Nicolaou & Pitta-Pantazi, 2011, p. 5). Therefore, a shift in the pedagogy of fractions should focus on discouraging the dependency of learners (and teachers) on one model (the area model) in order to allow learners to meaningfully understand fractions.

2.4.3 Operations with fractions

Fraction computation is one of the problematic areas in terms of conceptual understanding. As alluded to earlier, understanding fraction size or the meaning of fractions is a prerequisite to operating with fractions (Cramer et al., 2008), a position taken by Bruce, et al. (2013, p. 17) who claim that:

> Without the requisite conceptual understanding such as the importance of equivalence, estimation, unit fractions, and part-whole relationships, students struggle to complete calculations with fractions.

Often the procedures used to do operations with fractions and the results (answers) obtained do not make sense to learners, as their reasoning is informed by their whole number knowledge.

The incorporation of fraction models in the teaching of addition and subtraction has proved to be useful in helping learners make sense of the processes involved. Ervin (2018, p. 259) asserts that “modelling is an important step in the learning process before computational algorithms are examined.” In addition, Cramer et al. (2008) made the following observation in their study about learners who worked with visual models:

> … work with fraction circles helped them judge reasonableness, because they were able to visualize the relative sizes of the fractions and combine them by mentally adding on or taking away. Students were able to visualize almost any fraction, regardless of whether they had actually seen that fraction with the fraction circle model. (p. 493)

Kara and Incikabi (2018) also attribute the challenges learners encounter in fraction operations to the poor understanding of the concept of a fraction. The misconception that learners have regarding the numerator and the denominator came out strongly in Kara and Incikabi’s (2018) study on “sixth grade students’ skills of using multiple representations.” They observed that learners continued to see the numerator and the denominator as separate entities rather than a single value when adding or subtracting fractions. Such errors are attributed to cognitive factors rooted in the “complexity of the notion of fractions and in instructional approaches employed when teaching fractions,” (Kara & Incikabi, 2018, p. 467). Therefore, more emphasis should be placed on how teachers teach because according to Charalambous et al., 2010, effective fraction instruction is founded on 1) learners’ prior knowledge, 2) their conceptual understanding, 3) the comprehensive understanding of different aspects of the notion of fractions (delay the introduction of formal symbols and algorithms) and 4) promoting all fraction constructs.

Operations with fractions require learners to think and reason in a different way. For instance, learners are expected to apply multiplicative reasoning when adding or subtracting fractions with different denominators (or different wholes). Philipp (2000, p. 2) elaborates that “understanding the relationships among fractional quantities requires multiplicative reasoning, the cornerstone to proportional reasoning.” Moreover, this type of reasoning is necessary for learners to be able to convert fractions to their simplest forms and to compare and order fractions (Dyson et al., 2020). Additionally, Fielding (2012, p. 28) asserts that multiplicative thinkers are often proficient, correct and efficient in working with fractions.

2.4.3.1 Addition and subtraction

Since the mental processes involved in the addition and subtraction of fractions are complementary, it makes sense to discuss them concurrently. The challenges that learners encounter are mainly attributed to the fact that fraction instruction does not focus on promoting foundational understandings of fractions. According to Fennel and Karp (2016), the progression of fraction instruction is key in the development of foundational understandings. They further suggest that work with fractions extends from early work, with conceptual understandings involving partitioning of regions to the magnitude-related topics of fraction equivalence and comparing and ordering fractions involving varied representations, including the number line (Fennel & Karp, 2016, p. 2)

Since the addition and subtraction of fractions requires learners to have foundational knowledge of fractions, it is imperative to design a scheme of work that allows for the mastery of the foundational concepts prior to the introduction of addition and subtraction of fractions. Bruce et al. (2013, p. 53) identify the following as challenges associated with the addition and subtraction of fractions: 1) difficulties understanding and representing fraction relationships; 2) confusion about the roles of numerators and denominators and the relationships between them; 3) the use of a gap thinking approach; and 4) lack of attention to equivalence and equi-partitioning. For learners to be able to add and subtract fractions effectively, they must be assisted to abandon their whole number bias (Dyson et al., 2020, p. 244) since this often interferes with their learning of fraction addition and subtraction. Bruce et al. (2013) recommend the use of appropriate representations or models which can enable learners to see the numerator and the denominator of any given fraction as a single value rather than as two independent values.

Consequently, this leads to a poor understanding of the roles of the numerator and the denominator, which negatively affects learners' proficiency in the addition and subtraction of fractions. Duzenli-Gokalp and Sharma (2010) assert that the addition and subtraction of fractions with unlike denominators is a common challenge among learners. This can be attributed to Bruce et al.'s (2013, p. 18) first two challenges alluded to earlier. Learners usually find the addition and subtraction of fractions with the same denominators easy – not because they understand the procedures, but because they are able to memorise and apply rules such as *when adding or subtracting fractions with the same denominators, add the numerators only and retain the denominator*. Charalambous et al. (2010, p. 35) suggest that the instruction of fraction addition and subtraction should commence "with fractions with similar denominators to build students' confidence before moving to fractions with dissimilar denominators." However, the rules are applied by learners without understanding, for instance, $\frac{2}{4}+\frac{1}{4}=\frac{3}{4}$. As a result, they are challenged when presented with fractions that have different denominators.

Due to their poor background in fractions, learners often resort to their whole number knowledge when adding or subtracting fractions with different denominators, or instance, $\frac{2}{4}+\frac{1}{2}=\frac{3}{6}$. Instead of finding the Least Common Denominator (LCD) of the two fractions, learners tend to add the numerators (2 + 1) and the denominators (4 + 2) as shown in the example above, which results in an incorrect answer. Confusion and uncertainty is expected among learners who do not have a deep understanding of basic concepts of fractions since this knowledge is required to learn more advanced concepts.

Traditionally, learners are presented with rules for addition and subtraction of fractions and this approach has proved to be less successful as it does not enhance learners' conceptual understanding of the procedures. In terms of progression, Van de Walle et al. (2013, p. 339) suggest the use of contextual examples and invented strategies as a starting point in the teaching of addition and subtraction because this develops learners' conceptual understanding of fractions and fraction equivalence which are stepping stones to understanding standard algorithms. Cramer et al. (2008, p. 494) further assert that "students need to experience acting out addition and subtraction concretely with an appropriate model before operating with symbol." In their attempt to develop learners' conceptual understanding of the addition and subtraction of fractions, teachers should strive to establish connections between "concrete actions and symbols". Figure 2.1 below shows how fractions with the same and different denominators can be added using the area and number line models.

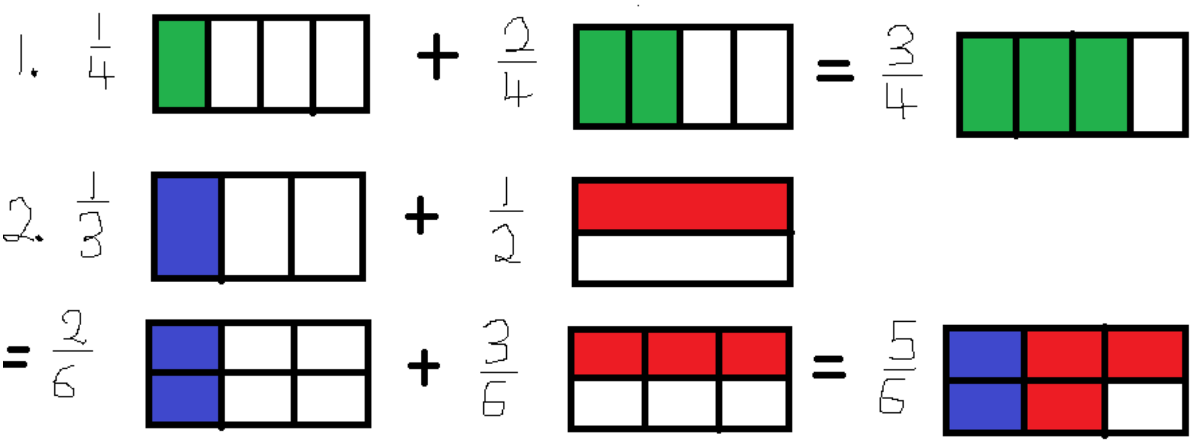
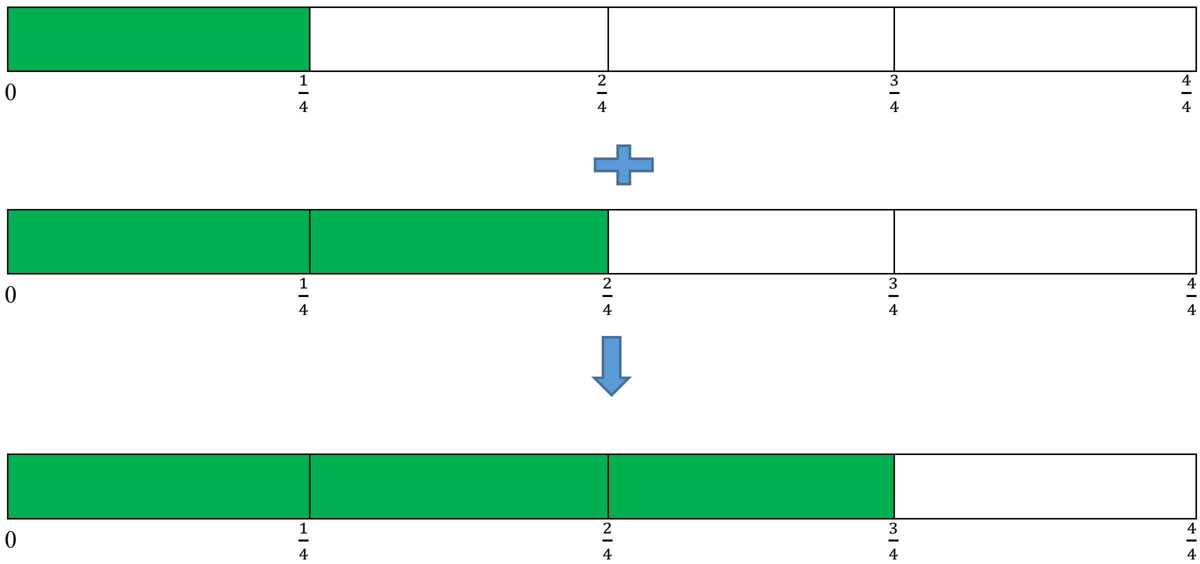
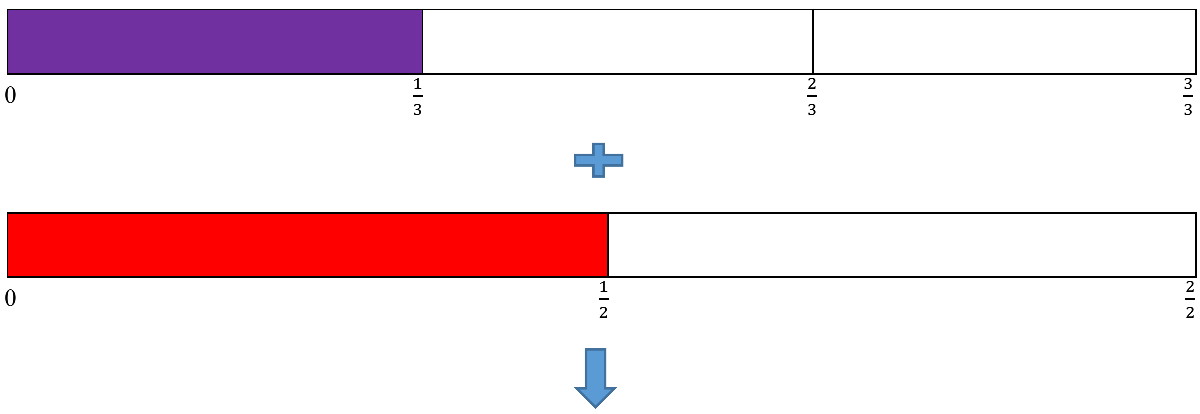


Figure 2.1: Addition of same-denominator and different-denominator fractions using the area model

$\frac{1}{4} + \frac{2}{4}$ using the number line model



$\frac{1}{3} + \frac{1}{2}$ using the number line



Repartition each *third* of the first diagram into halves and each *half* of the second diagram into thirds

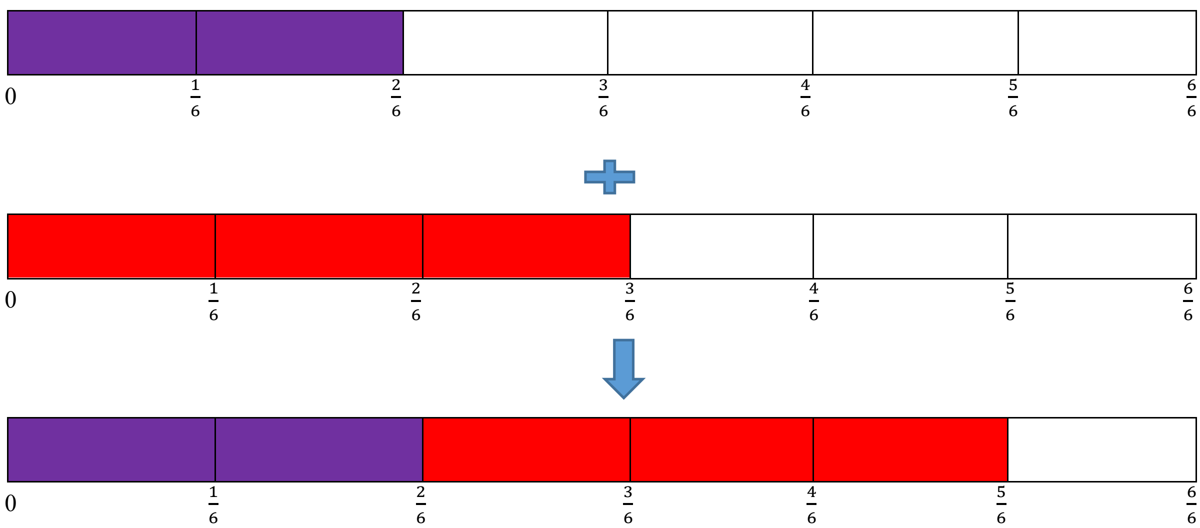


Figure 2.2: Addition of same-denominator and different-denominator fractions using the number line (or length) models

However, it is important to note that this should be preceded by the mastery of the basic fraction concepts such as fraction size, equivalence, ordering and comparing fractions because learners can only make sense of these models in addition and subtraction if they understand the magnitude of fractions and the relative size of fractions.

Substantial evidence exists to show that the mere application of rules in fraction addition and subtraction does not enhance learners' conceptual understanding of the procedures (Cramer & Whitney, 2010; Cramer et al., 2008; Fennel & Karp, 2016 & Van de Walle et al., 2013). This is an indication that the rules that are often overemphasised in the addition and subtraction of fractions do not make sense to the learners. Hence, the tendency among learners to resort to whole number rules when adding or subtracting fractions. De Castro (2008, p. 102) observes that one of the challenges in terms of fraction pedagogy is that:

> … teachers often take an adult-centered rather than a child-centered approach, emphasising a fully formed adult conception of rational numbers, not taking into consideration their schema and informal knowledge of fractions, thus denying children a spontaneous means of learning fractions.

Therefore, it is important for teachers to help learners see (visually) the 'parts' and the 'wholes' of the fractions that are being added or subtracted, in order for the rules and procedures to make sense to them.

2.4.3.2 Multiplication and division

Multiplication and division of fractions have proved to be problematic concepts (Bruce et al., 2014). In discussing the challenges related to these operations, it is imperative to consider the "conceptual underpinnings of multiplication and division with fractions" (Bruce, et al., 2014, p. 8). The multiplication of fractions is a difficult operation as it differs from the familiar procedures of multiplying whole numbers, which learners are used to. Learners have been taught to believe that to multiply means to 'make more,' or 'to increase' by a given factor. Feil (2010, p. 86) observes that "children tend to believe that in multiplication the product is always bigger than the factors and when multiplied, the number gets bigger." This is not the case with the multiplication of fractions because, unlike whole numbers, the product is usually less than the fractions that are being multiplied. Hence, the use of models could help learners to figure out "that the multiplication of fractions results in a smaller product and helps to build fractional number sense, number sense related to fractions as opposed to whole numbers," (Ervin, 2018, p. 261). According to Khairunnisak et al. (2012), misplaced priorities result in some teachers placing more emphasis on strategies or algorithms that help learners arrive at the answer quickly rather than those that focus on developing learners' conceptual understanding.

There are two multiplication strategies that are prevalent in mathematics classrooms: 1) multiply numerator by numerator, denominator by denominator and then simplify the product; or 2) search for the common factors and cancel before you multiply (Chen et al. 2013). The first strategy is straightforward and there is very little mental activity from learners who know how to multiply. The second strategy requires learners to find the common factors of the numerators and the denominators before they multiply. This can be challenging for learners who have not mastered multiplication and division. Neither of these strategies enhance learners' conceptual understanding because these rules are often applied without understanding.

The use of models can facilitate learners' understanding of how to multiply a fraction by a whole number, a fraction by a fraction, a whole number by a fraction or mixed numbers, because this process involves the visualisation of the whole procedure. In other words, learners can follow what is happening at each stage of the multiplication process. Van de Walle et al. (2013, p. 345) suggests that "different models must be used and aligned with contexts so that students get a comprehensive understanding of multiplication of fractions." Just like whole numbers, when using the area model to multiply two fractions, the "shared space" is considered (Bruce et al., 2014). The "shared space" is the space where the representations of the two models overlap. This is in line with de Castro's (2008, p. 105) fraction multiplication cognitive model which is based on the following sub-goals:

1) identify the multiplicand and draw a pictorial representation of it using a rectangle with vertical divisions; 2) identify the multiplier and draw a rectangle representation of the same size with horizontal divisions; 3) superimpose the two representations; and 4) represent the product using the double shaded regions as the numerator and the total number of regions made

on the superimposed model as the denominator. An example of de Castro's cognitive model is represented in Figure 2.2

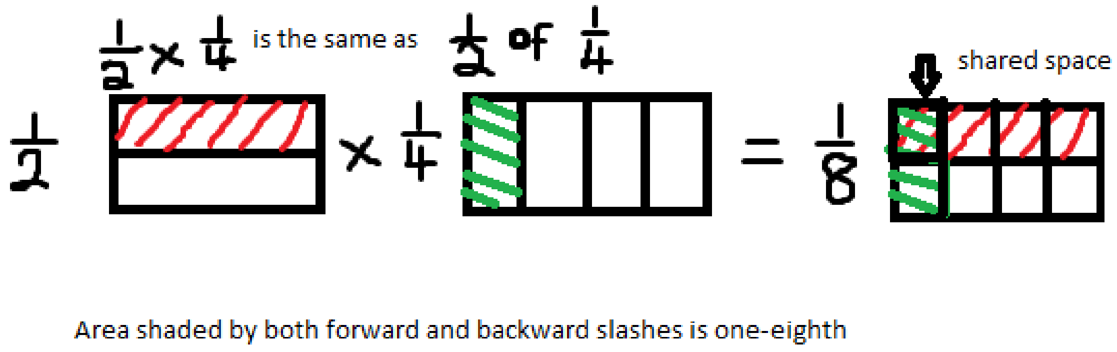


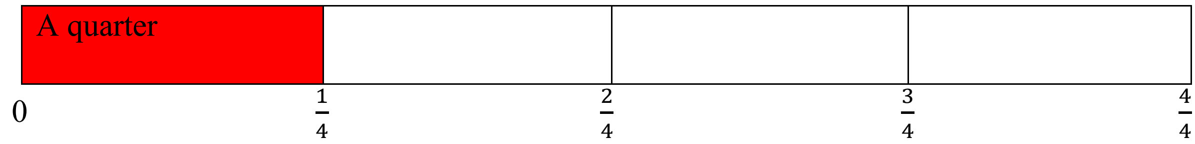
Figure 2.3: Multiplying fractions using the area model

The fraction sub-construct of an operator helps learners understand fraction multiplication visually. For instance, if learners learn to interpret $\frac{1}{2}$ x $\frac{1}{4}$ as a 'half of a quarter,' they will understand why the product of two proper fractions is always smaller than the two fractions being multiplied (Fazio & Siegler, 2011). Since a quarter is already smaller than a half, a half of it ($\frac{1}{4}$) cannot be bigger than the multiplicand or the multiplier.

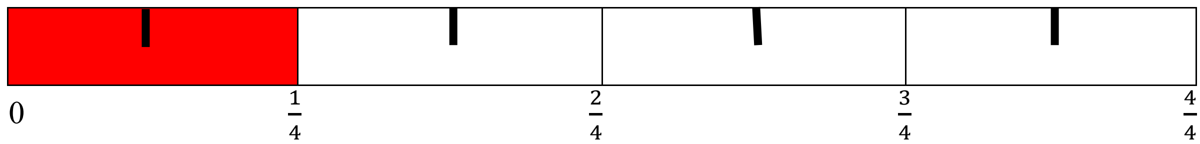
The length or number line model is also recommended for fraction operations because it promotes learners' conceptual understanding. "These models aid students in making connections to problems that are linear in context," (Ervin, 2017, p. 268). In the example above ($\frac{1}{2}$ x $\frac{1}{4}$), it is easier for learners to work out half of a quarter on a number line. Ervin (2017, p. 269) further asserts that:

> … the number line lends itself nicely to measuring and illustrates that a fraction is a number itself while at the same time showing students its relative size compared to other numbers and sometimes help students see multiplication.

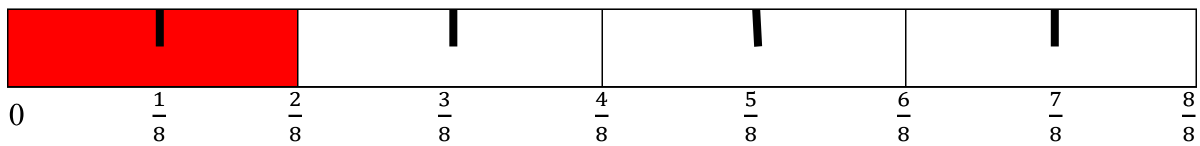
The number line model not only improves learners' understanding of multiplication but it deepens their understanding of fractions in general (Kolar et al., 2018; Van de Walle et al., 2013). According to Kolar et al. (2018, p. 76) "representing fractions on a number line improves the pupils' ability to bridge numerical and spatial properties and facilitates a deeper knowledge of magnitude concepts.” It is through the number line model that learners can begin to perceive fractions as numbers with “magnitudes that can be represented on a number line,” (Rodrigues et al., 2016, p. 135). In order to solve the same problem using a number line model, the procedure is much simpler. However, it requires learners to think about and locate fractions visually on a number line. See Figure 2.4 below. *Visually* think about a half of a quarter. Present the same example using the number line model.



What is a half of this quarter (the shaded part)? In order to get a half of the shaded part, all the four parts should be partitioned into halves as illustrated below:



After partitioning the number line, notice that the whole is no longer divided into four parts but eight parts.



Therefore, half of a quarter is equal to one eighth.

Figure 2.4: Linear representation of a $\frac{1}{2}$ x $\frac{1}{4}$

Multiplying a whole number by a fraction (4 x $\frac{1}{2}$) or a fraction by a whole number ($\frac{1}{2}$ x 4) requires learners to apply different visual strategies. Four times a half should be thought of as *four halves* which gives a product of two. On the other hand, a half times four should be perceived as a half of four which also gives a product of two.

$4 \text{ x} \frac{1}{2} = \frac{4}{1} \text{ x } \frac{1}{2} = \frac{4}{2} = 2$

Visually, this can be represented as:

4 x $\frac{1}{2}$ which should be understood as 4 times $\frac{1}{2}$ or 4 halves ($\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}$)

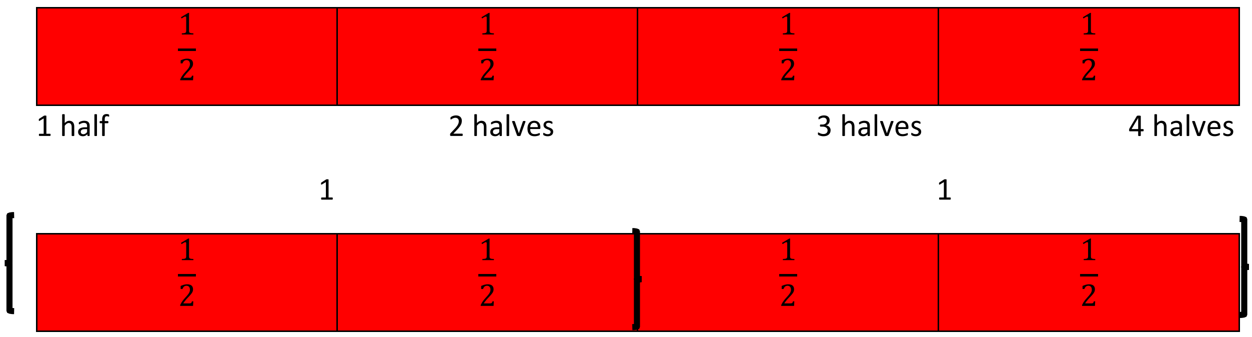


Figure 2.5: Linear representation of 4 x $\frac{1}{2}$

Van de Walle et al. (2013, p. 345) asserts that fluency in fraction multiplication is attained when "a student cannot only do the algorithm but also model problems, estimate and solve situations that involve multiplications." However, this is rarely achieved as fraction instruction often focuses on traditional teaching methods. Fazio and Siegler (2011, p. 7) identify "teachers' conceptual knowledge of fractions along with the knowledge of learners' common errors" as essential for improving learners' knowledge of fractions. Therefore, teachers should be equipped with adequate conceptual and procedural knowledge of fractions in order for their teaching to have a positive impact on the learners.

The invert-and-multiply is a common strategy that is used to teach fraction division. Teachers rely on this algorithm, a procedure that is poorly understood by both teachers and learners (Lamberg & Wiest, 2015). As stated by Sharp and Adams (2002, p. 336) "for many students, using the invert-and-multiply algorithm is an activity completely isolated from concepts and meaning." As a result, learners often make mistakes because they carry out these procedures without understanding the significance of the steps involved in the calculations. McNamara and Shaughnessy (2011) attribute the fraction-related errors that learners make to partial understandings, a view that is supported by Bruce et al. (2014, p. 8) who concur that "student misunderstandings of the meaning behind algorithmic shortcuts with fractions, can lead to later problems in other areas of mathematics, such as algebra." In most instances, the emphasis in mathematics lessons is on getting the right answer rather than developing learners' deep understanding of concepts.

Another algorithm for fraction division suggested by Van de Walle et al. (2013, p. 353) is the common-denominator algorithm which "relies on the measurement or repeated subtraction concept of division." This requires learners to find a common denominator for the two fractions involved in the division problem and divide the numerators only, for example, $\frac{3}{4} \div \frac{1}{2} = \frac{3}{4} \div \frac{2}{4} = 3 \div 2 = \frac{3}{2} = 1\frac{1}{2}$. Although this algorithm produces the right answer, it does not help learners understand the division of fractions conceptually. Therefore, rephrasing the problem to "how many halves are in three quarters?" could help learners relate it to whole number division and see the half as a referent unit (Lamberg & Wiest, 2015, p. 32). A visual approach to the division of fractions is recommended because it enhances learners' understanding of the process. The use of models can help learners see why the division of fractions can produce bigger numbers as answers, since learners expect the quotient in a division problem to always be smaller (Van de Walle et al., 2013). This misconception stems from their experience with whole number division. Feil (2010, p. 86) states that:

> … as for division, children tend to believe that the dividend is always bigger than the divisor and the quotient, and when you divide, the result gets smaller. When the number relation in a problem conflicts with these beliefs, children tend to change the numbers' relationships so that they will conform to these beliefs.

These misconceptions are promulgated by the dominant "measurement division, also called repeated subtraction, in which the group size is known and the number of groups is the unknown," (Kribs-Zaleta, 2006, p. 371). The second type of division problem which learners often struggle with is "partitive division or fair sharing, in which the number of groups is known but the size of each group is unknown or fair-sharing fraction division problems."

Due to the complexity of fraction division, the area, length or set models can be used to demonstrate fraction division by adopting de Castro's (2008, p. 106-107) fraction division cognitive model which is based on the following sub-goals:

> 1) identify the dividend and draw a pictorial representation of it using a number line or a rectangle, 2) identify the divisor and mark the picture drawn according to the size of the divisor, 3) count the number of marked groups and consider this as a whole number part of the quotient and 4) convert the remainder of the picture (if there is any) to a

fraction and this will be the fractional part of the quotient. See Figures 2.6 and 2.7 below:

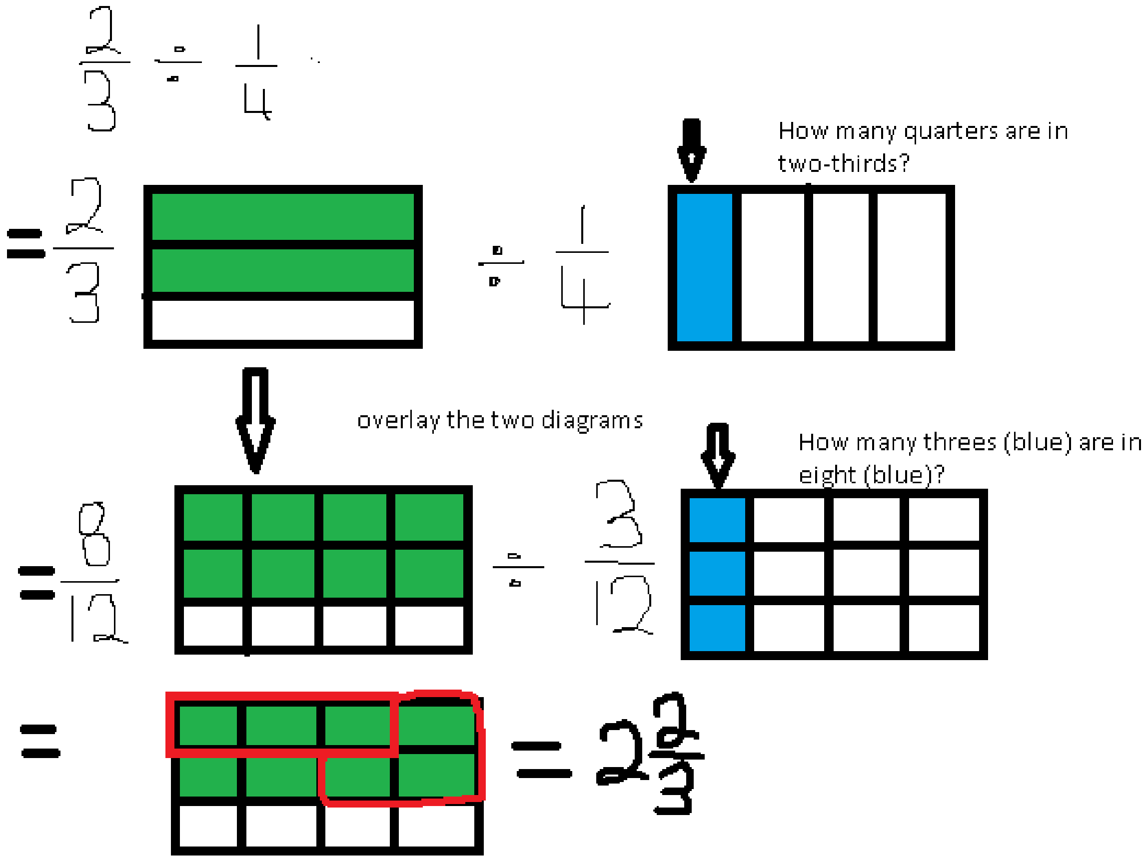


Figure 2.6: Dividing fractions using the area model

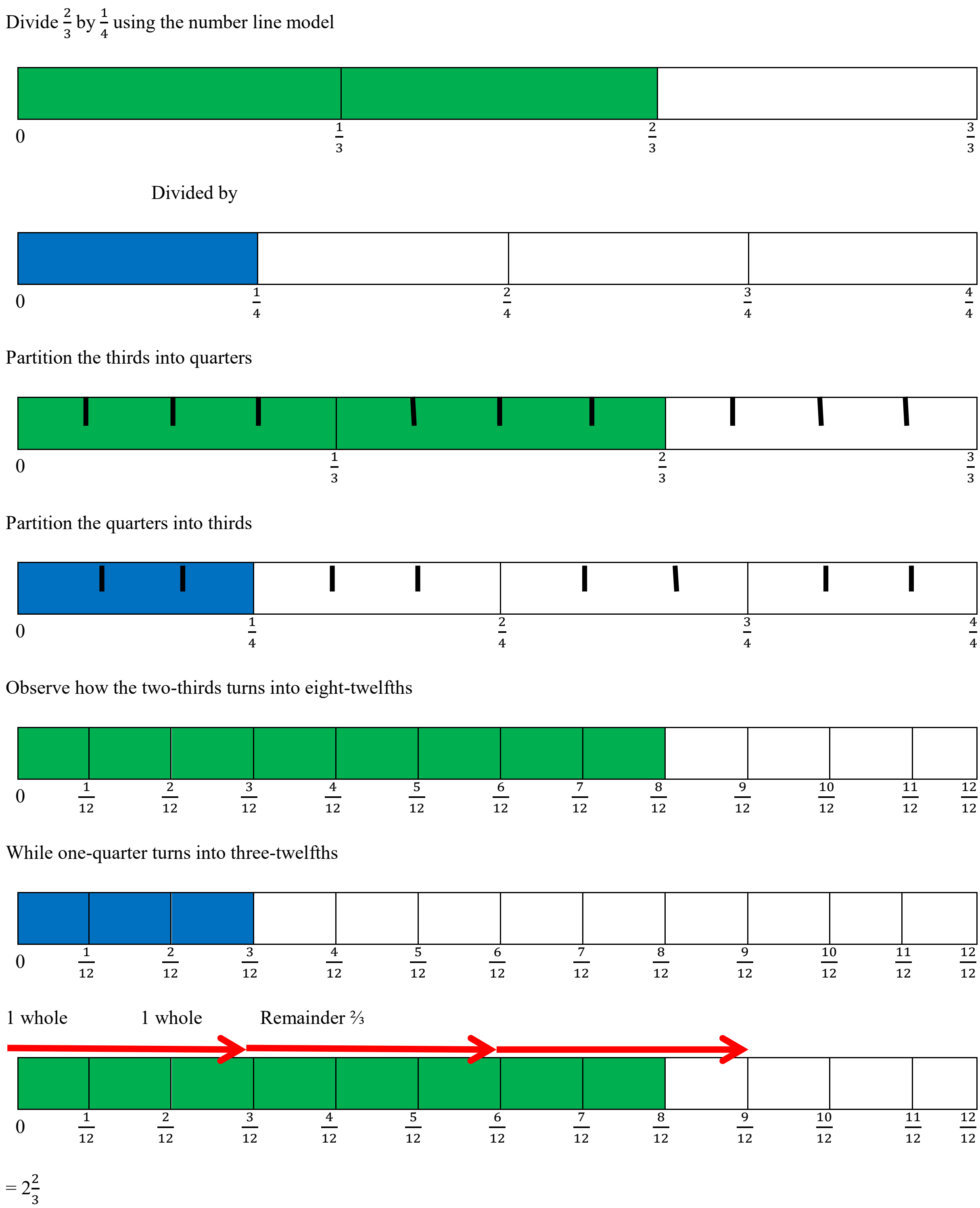


Figure 2.7: Dividing fractions using the number line model

Teachers are expected to play a crucial role in helping learners develop a conceptual understanding of fractions and reduce learners' misconceptions by revisiting their teaching strategies. Sadly, the teachers who are expected to possess deeper and robust conceptual knowledge of fractions often rely on procedural knowledge. According to Philipp (2000, p. 2), fractions is a topic that "most teachers understand only instrumentally." Lamberg and Wiest (2015, p. 32) describe the division of fractions as "problematic because both teachers and students have difficulty understanding the method conceptually." This is because many "teachers learn mathematics superficially and thus do not fully understand the underlying concepts," and as a result, "they cannot help students learn mathematics meaningfully" (p. 31). Hence, this develops into a vicious cycle of generations of teachers and learners who do not fully understand the concept of fractions. Hence, some teachers find the use

2.5 ENHANCING PEDAGOGICAL CONTENT KNOWLEDGE OF FRACTIONS THROUGH VISUALISATION

The definition of Pedagogical Content Knowledge (PCK) in relation to fractions is a vital aspect to consider in redefining fraction pedagogy. PCK refers to knowledge about content and pedagogy. It is a body of knowledge that Shulman (1986, p. 6) refers to as "indistinguishable" due to the fact that the two types of knowledge are intertwined and interdependent. Shulman (1986, p. 9) defines PCK as the "subject matter knowledge for teaching." Shulman further elaborates that this knowledge includes:

> … the most useful forms of representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations in a word, the ways of representing and formulating the subject that make it comprehensible to others. (1986, p. 8)

Founded on the notion above, Mishra and Koehler (2006, p. 1027) describe PCK as "knowledge of teaching strategies that incorporate appropriate conceptual representations in order to address learner difficulties and misconceptions and foster meaningful understanding," (2006, p. 1027). In this context, visualisation forms part of PCK since it is an approach that can improve learners' understanding of difficult mathematics concepts easily, if well incorporated. Moreover, the previous section (2.4) on fraction sense is very critical of fraction pedagogy. For teachers to be able to teach fractions effectively, they should possess the fundamental perceptions and conceptions of fractions (2.4.1). A good fraction sense can enable teachers to identify and address learners' difficulties pertaining to fractions.

Unfortunately, it is often assumed that teachers who have undergone teacher training are proficient in the subjects that they are certified to teach. On the contrary, Shulman (1986) considers this as a wrong assumption and suggests that the transition from an expert student to a novice teacher be taken into account in every teaching-learning situation. The need for a solid content knowledge base of fractions among mathematics teachers cannot be overemphasised. Ubah and Bansila (2018, p. 2) concur that “there is consensus in the teacher education literature that a strong knowledge of the subject taught is a core component of teacher competence.” However, in terms of fractions, studies (Huang et al., 2009; Kolar et al., 2018 & Kribs-Zaleta, 2006) have shown that preservice teachers lack the relevant content knowledge required to effectively teach fractions. These are the teachers who are later recruited in schools and expected to carry out this important yet very challenging task of developing learners’ fraction sense. Fielding (2011, p. 2) explains that

> … if student teachers, once qualified, are going to be able to teach the mathematics curriculum confidently, it is important that problematic areas such as fractions and the related aspects of mathematics are addressed so that teachers do not perpetuate misconceptions or impart any of their own anxieties to their pupils.

Since fractions are generally regarded as problematic, it is important to understand the PCK that can be used to address the challenges encountered in the teaching and learning of fractions. Therefore, teachers are required to have a balanced and comprehensive knowledge of content and pedagogy. Fazio and Siegler (2011, p. 7) asserts that

> Teachers with a firm conceptual knowledge of fractions, along with knowledge of students’ common errors and misconceptions, are essential for improving students’ learning about fractions.

The challenges associated with fractions are multifaceted. However, the main challenge is teachers’ poor understanding of fractions because this affects both content and pedagogy Kribs-Zaleta, 2006, p. 371) affirms that

> … complexities involved in working with rational numbers often challenge student and teacher alike, with the result that students are typically presented with the traditional algorithm for division of fractions before having an opportunity to construct meaning for it themselves.

Although teachers' good content knowledge does not always translate into effective pedagogy (Nambira, 2016), their poor content knowledge can definitely lead to poor pedagogical methods. Wu (2017, p. 12) emphasises the importance of teachers' content knowledge by clarifying that "teachers' content knowledge cannot afford to be polluted by any kind of mathematics that has no mathematical integrity." Consequently, it is difficult to understand how anyone can expect teachers to effectively teach mathematics content which they do not understand. In fact, Danişman and Tanişli (2017, p. 17) argue that "content knowledge is indispensable for effective teaching." The mastery of content thus precedes the pedagogy of content.

PCK and CK are essential components of teaching which can be used to define the quality of a teacher's knowledge. In their attempt to distinguish between knowledge quality and knowledge type, Star and Stylianides (2013, p. 6) define knowledge quality as "the way that something is known— essentially how well it is understood. Knowledge can be known at a deep level, at a superficial level, and anything in between the two extremes," while knowledge type refers to "what is known." Star and Stylianides further elaborate that deep-level thinking entails "understanding, flexibility, evaluation and critical judgement," (2013, p. 6) while superficial knowledge is associated with rote memorisation, repetition and dogmatism (Glaser, 1991). Since the appropriate use of visualisation enhances conceptual understanding and high levels of knowledge retention and retrieval, it could serve as an effective tool to attain knowledge quality.

Teachers are expected to possess a "confident level of subject knowledge" in order to teach mathematics effectively (Fielding, 2011, p. 1). Since fractions is a difficult concept in mathematics, it must be well mastered by teachers for them to confidently teach this concept and other mathematics concepts. Content knowledge refers to the mastery of concepts in mathematics (Setyaningrum et al., 2018, p. 1). Therefore, in terms of fractions, teachers are expected to possess important fraction concepts or fraction sense. As stated by Fennel and Karp (2016, p. 1) "fraction sense is developed through instructional opportunities involving fraction equivalence and magnitude, comparing and ordering fractions, using fraction benchmarks, and computational estimation." In other words, the absence of any of these concepts affects one's holistic understanding of fractions. This kind of knowledge is vital for both teachers and learners. It is a yardstick by which one's knowledge of fractions can be measured. For instance, fraction equivalence and magnitude, comparing and ordering fractions, using benchmarks and computational estimation are all prerequisites for the addition and subtraction of fractions. This is the kind of knowledge that should be enhanced by teachers through what Shulman (1986, p. 8) describes as “the most useful forms of representation of those ideas or the most powerful analogies, illustrations, examples, explanations, and demonstrations.” Therefore, it is not enough for teachers to have knowledge about equivalence but not magnitude. In fact, fraction magnitude is key to the conceptual understanding of fractions since this is a property that is shared by all real numbers. “Hence an important conceptual idea of fractions is understanding and differentiating between the magnitudes of fractions.” Teachers should therefore help learners understand fraction magnitude, just like whole numbers which can be sequentially placed on a number line.

Conversely, it is important to note that having conceptual knowledge of fractions is not enough because teachers are expected to possess relevant pedagogical methods to successfully develop fractional knowledge in their learners. Danişman and Tanişli (2017, p.1) concur that “qualified teachers should have very good pedagogical knowledge, as well as knowledge of mathematics.” Due to their abstract nature, teachers should strive to make fractions comprehensible by employing appropriate pedagogical methods. As stated by Fennema and Franke (1992: 153) “if teachers do not know how to translate those abstractions into a form that enables learners to relate the mathematics to what they already know, they will not learn with understanding”. Knowledge of learners is a key component of fraction pedagogy which all mathematics teachers are expected to have. Setyaningrum et al. (2018, p. 1) describe the knowledge of learners as “teachers’ knowledge of mathematical concepts that are difficult for students, common misconceptions that are frequently experienced by students, possible sources of students’ errors, and methods to eliminate the difficulties and misconceptions.” It is imperative for teachers to visually align their pedagogy to the type of learners they have, their level, their prior knowledge and the difficulties that they experience in learning fractions.

Generally, teacher characteristics have a major impact on pedagogy. Teachers produce different results because of their different characteristics in terms of teaching. According to Koopman et al. (2017, p. 16), three interrelated factors, namely teacher background characteristics, teacher knowledge and conceptions, and instructional characteristics have an impact on learner fraction proficiency development. These factors collectively contribute to the success or failure of a lesson as teachers choose different teaching strategies. Examples of issues associated with teacher knowledge, conceptions and instructional characteristics as pointed out by Charalambous (2010, p. 10) are a lack of consideration for prior knowledge, where emphasis is placed on rote memorisation rather than conceptual understanding; and the early introduction of formal symbols and algorithms where emphasis is placed on one fraction construct, as factors that may hinder effective fraction instruction. These challenges are related to teacher knowledge, conceptions and instructional characteristics while teacher training, qualifications and experience are examples of teacher background characteristics.

Despite the relevance of the diverse pedagogical views, expectations and challenges discussed earlier, the incorporation of visualisation in fraction instruction is a crucial consideration in each of the different teacher characteristics. According to Rahim et al. (2011, p. 497), "visually based pedagogy opens mathematics to students who are otherwise excluded." The best teachers whose pedagogy is visually-oriented, often take pride in the "rich knowledge they hold, of the deep understanding that is enabled – both from teachers introducing mathematical ideas visually, and students using visuals to think and make sense of mathematics," (Boaler, et al., 2016, p. 16). It is further observed that:

> … a child's difficulty in working with fractions may be related to teachers moving too quickly towards procedures so students can *do* fractions. Instead of rushing to procedures, students need the time and freedom to conceptualize, organize and assimilate the notion of a fraction into their already developed informal framework of fractions (Barnett, 2016, p. 4)

Visualisation is a powerful pedagogical tool because it helps learners to develop a broader and deeper understanding of difficult mathematics concepts. Learners are taught to see and view mathematics from a deeper conceptual perspective (Rahim et al., 2011). Therefore, in determining the effectiveness of a fraction lesson, it is important to find out whether individual teacher knowledge and instructional practices are inclined towards visual or symbolic modes.

2.6 THE NAMIBIAN CONTEXT

Despite the educational reform that occurred after independence in 1990, the Namibian education system is still characterised by remnants of the old colonial education system. The educational reform was based on four goals, namely access, equity, quality and democracy (Ministry of Education and Culture [MEC], 1993). However, these goals can only be achieved if the required infrastructure, teaching and learning materials, qualified teachers and teacher-training institutions are in place. This is a challenge that the Namibian government has been battling with since independence. It is important to acknowledge the fundamental role of a teacher in any teaching-learning situation. Kasanda (2004, p. 2) reiterates the importance of prioritising teacher education in African countries that were characterised by discriminatory apartheid laws such as Namibia. Many Namibians were denied the opportunity to do mathematics because the colonialists believed that Blacks (especially females) did not have the intellectual capacity to do mathematics (Naukushu, 2016). As such, Namibia inherited an education system that was characterised by a range of inequalities after independence. The new government was therefore faced with a daunting task of redressing the imbalances in order to ensure quality education for all Namibians, irrespective of their race, ethnic origin and socio-economic status (MEC, 1993).

From a pedagogical perspective, a shift from the “ineffective and frustrating” teacher-centred approach to learner-centred approach was deemed necessary (MEC, 1993, p. 12). Awe and Kasanda (2016) observe that the colonial Bantu educational system promoted rote memorisation of facts. Hence, success was measured by how well learners could reproduce facts. The MEC (1993, p. 4) therefore called for educational reform by advocating for quality education for all, which should not only focus on increasing the number of learners in schools but rather on “replacing the philosophy and practices of education suitable for educating elites with a new philosophy and practices appropriate for providing education for all our citizens.” This meant addressing the existing discrepancies in education, including pedagogical matters. The reform curriculum was based on the constructivist theory, which is founded on the notion that learners are active participants in the process of knowledge construction (von Glasersfeld, 1995). The learner-centred approach requires learners to actively construct their own knowledge rather than remain passive recipients of knowledge. Moreover, teachers are expected to do more than just talk and fill the chalkboards with a lot of notes or summaries.

Enhancing learners’ understanding of concepts should be the goal of every mathematics teacher. However, this is rarely achieved, mainly due to the poor teaching methods which continue to prevail in mathematics classrooms. Despite various attempts to develop learners’ conceptual understanding of mathematics concepts, Fuma (2018, p. 1) observes that learners still “perceive mathematics as an abstract subject that only emphasises on learning computing skills and memorisation of facts. It focuses on getting correct answers and labels those who remember mathematical facts as the brightest." In light of these persistent challenges experienced in terms of improving learners' interest and conceptual understanding of mathematics concepts, it is vital to critically consider the role of visualisation in the teaching and learning of mathematics. Chikiwa and Schafer (2019, p. 1) posit that "visualisation is generally accepted and considered as helpful in mathematics education because of its diverse pedagogic, cognitive and epistemic purposes." As explained earlier, visualisation is an important reasoning tool in the 21st century since it facilitates the development of a deep understanding of mathematics concepts through the creation of mental models (see Sections 2.2.2 and 2.2.3)

Generally, teachers determine the success or failure of a lesson. Therefore, it is important to have qualified teachers who can teach mathematics content effectively in all schools. As stated in the development brief for education, culture and training (MEC, 1993, p. 5) "when our assessment is that most learners have not grasped important concepts or adequately mastered skills, we must consider that primarily a failure of our teaching." Since access and equity are among the four educational goals, Hailombe (2011, p. 2) further observes that "equal access to qualified teachers and quality teaching has been a source of contention in the global debate over opportunity of quality education." Since the Namibian government made education and teacher training priorities after independence, a significant reduction in the number of unqualified and underqualified teachers has been observed.

Although teacher qualification has always taken centre stage in the educational context, it is important to consider the impact of teacher qualification on learner performance. Nambira (2016) suggests that teachers' competence in lesson presentation is a crucial aspect to consider besides teacher qualification. Notably, possessing the right qualification is not the only predictor of learners' performance in mathematics. Teacher qualification should be accompanied by other relevant competencies in order to improve learners' performance in mathematics. Nambira further suggests that:

> … to be able to assess high-level learning and critical thinking, it is essentially important for teachers to have a combination of skills that are above the level of content knowledge, pedagogical knowledge and specialised content knowledge required for the grade. Teachers' multifaceted teaching competencies, impact on the ability to develop teaching materials, improve the type and quality of learners' work. (p. 37)

The assumption that teachers who have undergone training in teacher-training institutions teach better and improve learners' understanding of the subject matter, is not always true. In spite of the improvement in teacher qualifications, the teaching of mathematics, especially the concept of fractions has remained problematic. Fuma (2018, p. 3) acknowledges the challenges associated with the pedagogy of mathematics by asserting that

> … the mathematical pedagogical process is incomprehensible to students, they do not understand its importance, and they despise it. It is further perceived that mathematics is designed to benefit a particular group of individuals, relatively based on intelligence or gender.

In the past, mathematics used to be a compulsory subject from Grade one to ten, after which learners were allowed to choose fields of study which did not have to include mathematics, especially if they did not obtain good grades in Grade ten. Following the realisation that mathematics was an essential subject for all learners up to Grade twelve, another reform took place in Namibia in 2012, which made mathematics compulsory (MEC, 2012). This decision resulted in a number of challenges for both teachers and learners, because all learners (including those who had failed) were compelled to do senior secondary level mathematics. Although the Namibian curriculum does not explicitly prescribe how the fraction content should be taught and learned, the incorporation of visuals is implicitly supported in the curriculum. While the syllabus (2015) encourages the use of visualisation in a few concepts, the subject policy guide (2008) recommends the use of a variety of prescribed textbooks as teaching and learning aids. The senior primary syllabus (Grade 4-7) mathematics syllabus recommends that

> … the teacher must be able to sense the needs of the learners, the nature of the learning to be done, and how to shape learning experiences accordingly. Teaching strategies must therefore be varied but flexible within well-structured sequences of lessons (Ministry of Education [ME], 2015, p. 3)

The aforementioned provision in the curriculum accords teachers the opportunity to use a variety of teaching strategies based on the needs of their learners. Twenty-first century mathematics teachers are expected to be creative, innovative and resourceful in order to have a positive impact on learners' perceptions of mathematics. Fuma (2018, p. 3) points out that "it is vital to introduce students to concepts at primary level using techniques that enable them to reflect on it throughout their school careers." This view is supported by Chikiwa and Schafer (2019, p. 1) who concur that

> … the teachers' primary task is thus to find and use teaching approaches that promote conceptual understanding of mathematical concepts, ideas and relationships. This has implications for every teacher in terms of how and what he/she teaches.

The incorporation of visualisation processes is often perceived as an appropriate alternative strategy to teach difficult mathematics concepts. Teachers intuitively resort to the use of visuals or concrete teaching aids when the content being presented is too abstract for learners to understand, because it is not appropriate to explain an abstract concept with purely abstract, symbolic or numerical examples. The use of concrete objects, pictures, diagrams and graphs comes naturally in response to difficult concepts that are presented in a purely abstract, verbal, symbolic or numerical form. The concept of fractions is one such example because it is difficult to teach and learn (Bruce et al., 2013). Although fractions are numbers, they are presented in a very unique and abstract manner, making it difficult for learners to comprehend.

In the Namibian mathematics curriculum, fractions are gradually introduced to learners from Grade one. Fractions are visually represented in a sequential manner, beginning with a full glass and a half glass in Grade one; a whole, a half and quarters in Grade two; and thirds in Grade three (National Institute for Education Development [NIED], 2015, p. 52). While the junior primary curriculum explicitly promotes the use of visuals, the senior primary fraction content is presented in a very abstract manner without a single illustration (NIED, 2015, p. 6, 8). Since the syllabus is a guiding document that is used by teachers every day, the way the content is presented in the syllabus can easily be misinterpreted by the implementers of the curriculum. Therefore, the material developers should be mindful about how the content is presented because the addition and omission of certain aspects can be misconstrued.

The sequence of fractions in the junior primary curriculum from Grade one to three is recommended on the premise that the fractions a half, a quarter and a third provide a good starting point in fraction instruction, since these are the most basic fractions and they are easy to represent visually. These fractions serve as a prerequisite for the understanding of other fractions. Moreover, Van de Walle et al. (2013) consider the fractions zero, a quarter, a half and one as benchmarks for understanding fraction magnitude and the ordering and comparing of fractions.

2.7 GAP IN RESEARCH

Considering the trends in the literature sources reviewed in this chapter, it is evident that the concept of fractions has been widely researched but not necessarily in relation to visualisation. Conceptual understanding of fractions and effective teaching strategies have been at the centre of many studies (Bruce et al. 2014; Charalambous et al., 2010; Dyson et al., 2020; Ervin, 2017; Fazio & Siegler, 2011; Fennel & Karp, 2016; Presmeg, 2006; Richardson, 2019 & Van de Walle et al., 2013). Moreover, studies conducted in this field bring to the fore the difficulties associated with the teaching and learning of fractions (Pattimukay et al., 2018; Richardson, 2019; Rodrigues et al., 2016; Siegler et al., 2018; Ubah & Bansila, 2018 & Zembat, 2015). Hence, the acknowledgement by many authors that fractions are difficult to teach and learn (Bruce et al., 2013; Huang et al., 2009; Idris & Narayanan, 2011; Jygel & Afamasaga-Fuata'I, 2007; Lambert and Wiest, 2014; Richardson, 2019; Siegler et al., 2018; Ubah & Bansila, 2018 & Van de Walle et al., 2013). This study, with its orientation in pedagogy, is inspired by Presmeg's (2014, p. 151) research questions based on "visualization as an epistemological learning tool in mathematics." Presmeg (2014) suggests that this field of visualisation needs further research and she guides prospective researchers in this field by generating a set of thirteen questions.

In identifying a topic for this study, specific consideration of question eight of Presmeg's thirteen questions was made. Presmeg's (2014, p. 151) eighth question which reads: "How can teachers help learners to make connections between idiosyncratic visual imagery and inscriptions and conventional mathematical processes and notations?" was used to frame this study. In my attempt to answer this question, I based my study on a specific mathematics concept called fractions, because this is one of the most difficult concepts in mathematics to teach. Therefore, visualisation as a teaching strategy, in conjunction with the two theories – namely constructivism and the DCT – were considered as crucial didactical instruments in bridging the gap between the traditional and modern approaches in the pedagogy of fractions.

Although visualisation has been used in other related studies (Barnett, 2016; Bruce et al., 2014; Bruce et al., 2013; Chen et al., 2013; Dyson et al., 2020; Fennel & Karp, 2016; Kara & Incikabi, 2018), it has mainly been conceived of as a teaching aid rather than a teaching strategy. I therefore argue that visualisation should be portrayed as a way of thinking about and doing fractions rather than as a mere tool that can be used to aid the understanding of abstract fraction concepts which can be shelved or discarded once it has served its purpose. As observed by Boaler et al. (2016, p. 7), the use of visuals in mathematics classrooms is often perceived as "a prelude to the development of abstract ideas, rather than a tool for seeing and extending mathematical ideas and strengthening important brain networks." When visualisation is presented as a way of conceptualising fractions (just like symbols and numbers), both learners and teachers begin to value and adopt it as an alternative and natural way to learn and teach fractions (Boaler et al., 2016). The DCT, a theory underpinning the use of visualisation, is another critical aspect that distinguishes this study from the other studies.

The DCT advocates for the recognition of both verbal and nonverbal codes in teaching and learning (Clark & Paivio, 1991). Since visualisation is embedded in the nonverbal code, this study aims to establish the extent to which participants of the Rundu Campus Fraction Project (RCFP) deliberately incorporate visualisation processes in their teaching and whether these processes are constructively integrated into the pedagogy of fractions. The RCFP was established to expose teachers to effective fraction instruction strategies encompassing both verbal and nonverbal codes. This intervention answers the 'how' part of Presmeg's (2014) eighth question because teachers can only be expected to incorporate visualisation processes in their fraction lessons once they have acquired skills on how to do so. However, since the nonverbal (visual) code is generally not well understood, emphasis (during the RCFP training sessions) was placed on the nonverbal code in order to help teachers understand its role in the teaching and learning of fractions. Thus, the use of visualisation is assessed to determine its impact on the pedagogy of selected RCFP participants.

It is important to acknowledge that the pedagogy of fractions in schools and tertiary institutions has been predominantly verbal and rule-oriented (Ahmed et al., 2004; Bruce et al., 2013 & Cramer et al., 2008). Since the existing literature (Bruce, et al., 2013; Cramer & Whitney, 2010 & Van de Walle et al., 2013) suggests that the difficulties associated with the teaching and learning of fractions are associated with the nature (mainly symbolic) of fractions, it is imperative to consider the incorporation of the nonverbal component of the DCT without disregarding the verbal code. Thus, although I advocated for and introduced the incorporation of visualisation strategies in teaching fractions, I was cautious about creating a compelling attitude in terms the participants' pedagogical choices. The aim of this project was primarily to provide alternative methods of teaching fractions not to totally discard their existing practices. Although maintaining a completely neutral position in interventions of this nature is a challenge, my findings are indicative of my position in the RCFP. For instance, in Table 4.2 (p. 109), the participants were at liberty to indicate the frequency at which they use visuals. Despite their participation in the RCFP, most of the participants indicated that they do not use visuals often. Hence, even in their lessons, the participants were not compelled to use the verbal or the nonverbal codes. Consequently, the use of the DCT as a theoretical framework in this context and the intervention project on the use of visualisation, sets this study apart from the other similar studies that have been conducted in this field.

2.8 THEORETICAL FRAMEWORKS

This study is informed by two theories: the social constructivist theory and the Dual Coding Theory (DCT). Although social constructivism is associated with active learning in social settings, it can also refer to teaching approaches that evoke active learning modes. Constructivism in learning can only occur if constructivist teaching approaches are used to present the content. In other words, we cannot continue to use behaviourist approaches in teaching and expect to achieve constructivist results in learning (von Glasersfeld, 1995). Therefore, it is important for teachers to incorporate approaches that can enhance constructivist ideologies from both the teaching and learning perspectives. Our teaching strategies should be aligned to the expected constructivist outcomes in learning.

The social constructivist theory is complemented by the DCT, which advocates for the use of both verbal (mathematical symbols) and nonverbal (visuals) cues in teaching to enhance learners' conceptual understanding. Mayer & Anderson (1991, p. 485) posit that "this theory predicts that learners will remember and transfer material better if they encode the material both visually and verbally because they have two separate ways of finding the information in memory." The over dependency of teachers on one mode of presenting subject content to the learners can be detrimental to learning, as some learners are visually inclined while others are inclined towards the verbal mode. In the following sections, I discuss the social constructivist theory and the Dual Coding Theory in relation to visualisation.

2.8.1 Social Constructivist theory in relation to visualisation

Vygotsky's' social constructivist theory is founded on the premise that meaningful learning occurs when learners actively construct their own knowledge in social settings. Dewey, Piaget and Vygotsky are major contributors to the development of constructivist ideologies (Jia, 2010). Knowledge is actively constructed in the mind of the learner where the ultimate understanding of an object is determined by the learners' individual "experiences and backgrounds," (Jia, 2010, p. 198). In this context, the term 'learners' refers to both teachers and learners. The mathematics teachers who are members of the RCFP and participants of this study are considered learners because of their role in the community of learning (RCFP). They are perceived as learners because they aspire to learn new strategies to teach fractions effectively and refine their existing strategies. Learners stand to benefit from this undertaking because it is anticipated that once teachers acquire constructivist strategies, their pedagogy will change and learners will in turn be expected to learn fractions using social constructivist methods. Kapur (2018, p. 2) postulates that

> Social constructivism and educational constructivism which includes theories of learning and pedagogy have had the maximum impact on instruction and curriculum design because they seem to be the most advantageous to incorporate into the current educational strategies.

Considering the nature and design of this study, social constructivism was opted for as a suitable theory informing this research. Kapur (2018, pp. 3-4) provides a comprehensive description of social constructivism by acknowledging the different facets that encompass social constructivism. She acknowledges the fact that this process involves individual constructions of knowledge by learners, which is connected to their prior knowledge; the important role that experience plays in the construction of knowledge; the fact that learning is social and it is linked to "languages, cultures and other social norms and values which have a direct impact upon learning"; the holistic nature of the knowledge construction process as it involves "social interaction, attitudes, emotions, values and actions" of an individual; and lastly calls for an inclusive and equitable learning environment where learners' different needs are taken into account. Ahmed et al. (2004, p. 318) further assert that "fundamental concepts of mathematics have to be drawn from human experiences and existence." Thus, the rich experiences that learners bring to the learning environment should be maximally utilised.

Semiotic mediation, a concept derived from the Vygotsky's social cultural theory, is an important concept to consider in this context because it makes provision for the visualisation activities embedded in this study. "Semiotic refers to the use of language, and other ways to convey cultural practices, such as diagrams, pictures and other action visuals and mediation means that these semiotics are exchanged between and among people," (Van de Walle et al., 2013, p. 20). Considering the challenges associated with the teaching and learning of fractions, such as the unusual way in which fractions are written, different fraction constructs, the misapplication of whole number knowledge on fractions and lack of emphasis on conceptual understanding in the teaching of fractions, semiotic mediation is a concept that validates the activities that the participants in this study are involved in as members of the RCFP.

The different visual materials developed by participants of the RCFP are used to enhance learners' understanding of fractions. As alluded to by Mariotti (2009, p. 427), "an artifact can be exploited by the teacher as a tool of semiotic mediation to develop genuine mathematical signs, that are detached from the use of the artefact, but that nevertheless maintain with it a deep semiotic link." Hence, teachers' understanding of mathematics in general and fractions in particular "as well as of the content of the curriculum needs to be understood in terms of a co-construction of knowledge through jointly conducted activities that are mediated by artefacts of various kinds, of which dialogue is the most powerful," (Wells, 2007, p. 244). It is envisaged that as teachers interact and work together to develop materials in the RCFP, this will affect their perceptions about fractions and how they can best be taught.

During the material development process in this study, teachers are expected to share ideas and learn from one another as they develop visuals together, reflect on their experiences and modify the visuals to enhance their teaching. This process helps teachers to acquire skills for material development and conceptual understanding of the fractions. Wu (1999, p. 1) asserts that "skills and understanding are completely intertwined." This basically means that as teachers are engaged in the development of visuals, there is an improvement in their fluency and conceptual understanding of fractions.

The social constructivist theory supports the notion of collaboration among teachers and learners. As teachers work together, they share ideas and learn from one another. Since teachers have different teaching strategies, collaboration is required to help teachers “arrive at a shared understanding of truth in a specific field” (Har, 2005, p. 3). The shared understanding of truth pertaining to the best practices regarding fraction pedagogy is crucial in this study. In other words, an improvement can be attained if teachers have a common understanding of the role of both the verbal and nonverbal codes in the pedagogy of fractions. These sentiments are echoed by Kapur (2018, p. 1) who concurs that effective learning occurs through socialisation because “learning in seclusion does not always prove to be advantageous to the individual.” Thus, although learning occurs individually, Vygotsky’s theory of Social Constructivism (Akpan, et al., 2020) argues that the content to be learned is more accessible when learning occurs collaboratively in social settings where learners receives support from their peers or more knowledgeable others. Akpan et al. (2020, p. 50) further explain that “while the constructivist sees knowledge as what students construct by themselves based on the experiences they gather from their environment, the social constructivist sees knowledge as what students do in collaboration with other students, teachers and peers.” Consequently, although both the Cognitive Constructivist Theory (CCT) and the Social Constructivist Theory (SCT) involve the construction of knowledge, the settings under which this occurs may differ and this has different implications for teaching.

Therefore, amidst the different challenges that teachers encounter in teaching fractions, meaningful socialisation should be encouraged to accord mathematics teachers the opportunity to meet regularly and share their successes, experiences, challenges and ideas pertaining to the best practices in the pedagogy of fractions. Mariotti (2009, p. 427) concurs that “starting from assuming the centrality of semiotic activities, collective mathematical discussion plays a crucial role.” This is an active meaning-making process which helps teachers become conscious of the impact of their pedagogy on the learners.

The two concepts – social constructivism and visualisation – address the challenges that learners experience in learning mathematics. While social constructivism is a theoretical viewpoint, visualisation is a technique that can be employed to operationalise this theory. Social constructivism differs from the “primarily psychological, in-the-head conceptions of learning,” (Dudley-Marling, 2012, p. 1). Although visualisation is a more psychological, mind-based activity, the social aspect of constructivism brings to the fore the different ways in which teachers perceive best practices in terms of fraction pedagogy. Kim (2001, p. 2) cautions that "to understand and apply models of instruction that are rooted in the perspectives of social constructivists, it is important to know the premises that underlie them." For instance, from a social constructivist perspective, knowledge is perceived as a human product that is socially and culturally constructed. Meaning is created through interactions with each other and with the environment we live in (Kim, 2001, p. 3).

The combination of social constructivism and visualisation provides an interface between the individual and the social aspect of fraction pedagogy. Klerkx, et al. (2014, p. 4) explain that "the main intent of information visualisation is to represent an abstract information space in a dynamic way, so as to facilitate human interaction for exploration and understanding." In line with this ideology, Bruce et al. (2013, p. 6) affirm "the importance of thoughtful selection of learning tasks and representations, as well as the long-term benefits of building a strong conceptual understanding of fractions." This is aligned with one of Gredler's (1997) four perspectives on the framework of social constructivism: namely, idea-based social constructivism. Kim (2001, p. 4) assert that this perspective "sets education's priority on important concepts in the various disciplines which expand learner vision and become important foundations for learners' thinking and on construction of social meaning." Fractions is one of the most important concepts in mathematics which if not well mastered could lead to serious repercussions in the learning of other mathematics concepts (Bruce et al., 2013), hence, the need to consider this concept in the category of idea-based social constructivism.

From a learning point of view, the knowledge construction process in mathematics classrooms can be enhanced through visualisation as learners are able to see 'the parts' and how these parts relate to one another and to the whole as opposed to rote memorisation of rules. Social constructivist learning environments and materials help learners to think about and reflect on their prior knowledge of different concepts in mathematics. Gordon (2009, p. 3) explains "how constructivism is sometimes misconstrued and constructivist teaching misused." This is mainly due to the fact that, often teachers do not understand what this theory entails. Gordon (2009, p. 3) further suggests that an important question to consider in this regard is: "what constitutes a successful constructivist learning environment?" A conducive constructivist learning environment calls for teachers to assume an active role in creating opportunities for learners to construct their own knowledge through interactions with their peers and the content.

Since teachers play a key role in the implementation of the Learner Centred Approach (LCA), they ought to be aware of the rationale and the principles underlying this approach in order for them to successfully implement it. Different interpretations of the Learner Centred Education (LCE) policy has been identified as a major obstacle in the implementation of LCE and this has led to inconsistencies in the way it is implemented.

> Consequently, it is not surprising that teaching in the classrooms, and teacher education in the colleges and university and through in-service programmes and professional development are not consistent either. Part of the problem has been identified as a lack of clarity about what the underlying principles and theory of learner-centred education are. (NIED, 2003, p. 2)

Although the focus of LCE is on the learner, teachers must be equipped with the necessary knowledge and skills to create what Gordon (2009, p. 3) refers to as a successful constructivist learning environment. The development of visual fraction materials is considered to be seminal in the creation of a LCE friendly environment. Since the difficulties experienced by learners stem from the way in which fractions are presented, a visual approach to fraction instruction can improve learners' foundational understandings of fractions (see Section 2.4.3.1). Teachers are cautioned from interpreting LCE as group work, a common misconception in the Namibian context (Adejoke, 2007). The successful implementation of LCE is determined by a number of factors such as the learning environment, the quality of tasks and the active participation of learners. The teacher's role in all of these aspects is quite immense. Therefore, this study, with its location in the RCFP, is informed by carefully selected theories which are immersed in the Namibian education context. The DCT is the second theory on which this study is founded.

2.8.2 The Dual Coding Theory

The Dual Coding Theory (DCT) proposed by scholars in the discipline of educational psychology is founded on the notion that there are two interconnected systems through which cognition may be attained. "Cognition according to DCT involves the activity of two distinct subsystems, a verbal system specialized for dealing directly with language and a nonverbal (imagery) system specialized for dealing with nonlinguistic objects and events" (Paivio, 2006, p. 3). The concurrent use of the verbal and the nonverbal code is known to produce good results as opposed to the use of either of them separately. Suh and Moyer-Packenham (2007, p. 209) assert that "when learners are presented with both visual and verbal codes, which are functionally independent, this has additive effects on their recall." Due to the duality of the DCT, it provides numerous options to address the existing challenges in mathematics. A paradigm shift in the pedagogy of fractions dictates an equal consideration of both the verbal and the nonverbal codes in teaching.

The verbal and the nonverbal codes are intertwined and complement each other. Therefore, there is a need to develop and nurture both codes during classroom instruction in order to produce the desired results. In mathematics, the challenge has always been the predisposition of teachers' instruction towards what Paivio (2006, p. 3) refers to as the "more abstract, common theories of cognition" or verbal code, hence, the emphasis of this study on the nonverbal or visual code. Stokes (2002, p. 2) explains that "as more visual elements are incorporated to achieve an optimal balance between verbal and visual cues in education, interdependence between the two modes of thought will be fostered." Moreover, the reformed approaches in the Namibian education system suggest that the integrated use of the verbal and nonverbal codes can enhance the quality of teaching and learning in mathematics classrooms. The confluence of the two codes is accentuated by Avgerinou and Petterson (2011, p. 9) who assert that:

> … memory for pictures is superior to memory for words. This is called the pictorial superiority effect. Memory for a picture-word combination is superior to memory for words alone or pictures alone.

In other words, teachers are encouraged to use both the verbal and the nonverbal codes. Paivio (2006) underscore the deep, robust understanding that is enabled by the incorporation of visuals in education. Similarly, the LCE policy adopted by the Namibian government supports the use of different teaching methods to cater for learners' different learning needs. As stated by Awe and Kasanda (2016, p. 31), the LCA "strives to be individualistic, flexible, competency-based, varied in methodology and not always constrained by time or place." Consequently, one of the teaching implications for the LCA is the use of a variety of teaching aids, including visuals, to provide more learning opportunities for learners. The use of the verbal or symbolic code was more prevalent in the Bantu education system and this proved to be problematic, hence the need to incorporate the nonverbal or visual code in teaching and learning. The DCT and the LCA are based on the constructivist principles, in the sense that both present teachers with more opportunities to present mathematics content in ways that can enhance learners' understanding of abstract mathematics concepts (Cutting, 2019; de Castro, 2008; Paivio, 2006) Thus, the LCA and the DCT share more similarities than differences.

Based on the theoretical underpinnings of the Dual Coding Theory (DCT) and the Learner Centre Approach (LCA), it is can be assumed that the DCT is appropriate for this study because both (the DCT and LCA) are vital in enhancing learners' conceptual understanding. The concurrent use of the DCT and the LCA can help to mitigate the inherited disparities in achievement between the elite and the disadvantaged learners. Moreover, mathematics is perceived as a difficult subject mainly due to the predominantly symbolic form in which it is presented in textbooks and classrooms. In fact, the symbolic form of mathematics can be a source of anxiety for many learners. Understandably, one of the challenges associated with the poor understanding of fractions is the unusual way in which fractions are written (Van de Walle et al., 2013). Since ordinary numbers in mathematics are already a source of uneasiness among learners, the unique fraction notation aggravates the situation. In view of these challenges, a theory that places emphasis on the verbal and nonverbal codes was deemed necessary for this study.

To remain relevant and practical, the two codes should be constantly developed through learners' exposure to a series of instructional materials comprising both the verbal and the nonverbal codes. This determines the extent to which learners can confidently use the two codes. As stated by Paivio (2006, p. 3) "cognition is this variable pattern of the interplay of the two systems according to the degree to which they have developed." Since it is the visual component of the DCT that is often neglected, it is important to consider incorporating it in the pedagogy of fractions. Avgerinou and Petterson (2011, p. 4) maintain that a visual language exists and it can be learned. Moreover, visual literacy is described as learners' ability to a) read, decode and interpret visual statements, b) to write, encode and create visual statements and c) to think visually. Therefore, providing teachers with a platform to develop their own visual teaching aids is an appropriate strategy for professional development. This gives teachers the opportunity to improve their own visual skills and those of the learners since they are provided with a platform to collaboratively learn and locate the place of visualisation in the pedagogy of fractions.

The relevance of the DCT in this study lies in its applicability to teaching and learning since it provides an "interconnected system of coding information," (Suh & Moyer-Packenham, 2007, p. 209). Instead of perceiving the two coding systems as being antagonistic in terms of their functions, the DCT attempts to attach equal value to both the verbal and the nonverbal codes in order to maximise learning outcomes in education. Since the difficulties associated with the learning of mathematics are often linked to the abstract nature of mathematics, it is imperative to consider incorporating visual strategies in the pedagogy of mathematics. Moreover, the DCT accords learners who are inclined towards the visual code, an equal opportunity to learn.

> The assertion is that a learner's preference should be taken into account, and if the teacher matches the form of instruction to the learner's preferred modality, then the learner will retain more information or will learn at an increased rate. (Cuevas & Dawson 2018, p. 41)

Aligning teaching strategies to the DCT can enhance learning in a meaningful way as learners can process knowledge through the two streams, that is the verbal and the visual codes. Rieber (1994, p. 1) affirms that "it is easier to recall information from visual processing codes than verbal codes because visual information is accessed using synchronous processing, rather than sequential processing." In other words, the use of visuals enables learners to make connections between related mathematical concepts (and unrelated ones). In using the visual code, Rieber (1994, p. 1) further cautions that "there are times when pictures can aid learning, times when pictures do not aid learning but do no harm, and times when pictures do not aid learning and are distracting". This suggests that if not properly handled, the use of visuals could result in more confusion than learning. In order to ensure that no learner is left behind in the mathematics classroom, the use of both codes is encouraged because this combination makes provision for both learning styles.

The combination of the two theories (constructivism and DCT) was inspired by a strong relationship that exists between them and the notion that they complement each other. It is almost incomprehensible to talk about the enhancement of the theory of cognitive or social constructivism in mathematics classrooms without linking it to the two modes of thinking, that is, the verbal and the visual mode. In other words, the use of verbal or nonverbal cues are instrumental in the knowledge (meaning) construction process. The absolutist view of mathematics has been disputed by a number of researchers (Davis & Hersh, 1980; Ernest, 1991;

Freudenthal, 1973) due to the fact that this view has undermined the social aspect of mathematics in human affairs (Barnes & Venters, 2008, p. 3). Thus, the infallibility of mathematics has been challenged due to the realisation that mathematics is "a product of human inventiveness and human activity," (Barnes & Venters, 2008). This is the interface of the visual aspect of the DCT and the social constructivist theory since they are both products of human creation aimed at enhancing learners' conceptual understanding.

2. 9 ANALYTICAL FRAMEWORK

The analytical framework is comprised of seven key indicators drawn from the literature review concepts discussed in this chapter. Since the overarching goal of this study is to determine the opportunities afforded by the incorporation of visualisation processes in the teaching and learning of fractions, it is important to ensure that the framework is founded on principles that can enhance visual, quality, fraction instruction. Consequently, the analytical framework was informed by research-based, quality classroom practices that distinguish a nonverbal (visual) fraction lesson from a verbal symbolic one.

To a large extent, the selection of these concepts was informed by the two theories (constructivism and DCT) discussed earlier. The seven key concepts which were used to develop the analytical framework for this study are: (1) Quality Fraction Knowledge (QFK), (2) Fraction Models (FM), (3) Visuality of Lessons (VL), (4) Verbal and Nonverbal Codes (VNVC), (5) Quantitative Thinking (QT), (6) Pedagogical Content Knowledge (PCK) and (7) Effective Questioning Strategies (EQS). In order to address the complexities associated with the pedagogy of fractions, the seven elements were deemed necessary. The seven key concepts are not completely independent, thus, concepts such as Fraction Models and Visuality of Lessons (VL) may overlap since the visuality of lessons is determined by the presence of fraction models.

1) **Quality Fraction Knowledge (QFK)**: There is enough evidence to show that both teachers and learners experience difficulties in terms of conceptualising fractions (Barnett, 2016; Duzenli-Gokalp & Sharma, 2010; Fennell & Karp, 2016 & Kara & Incikabi, 2018). Therefore, QFK should be enhanced through different teaching strategies to mitigate the challenges encountered by learners. Siegler et al.'s (2010, p. 1) fifth recommendation on the professional development programme for teachers indicates that QFK encompasses a deep understanding

of fractions and computational procedures, an ability to use pictorial and concrete representations of fractions and fraction operations, assess learners' understandings and misunderstandings of fractions and the ability to teach key concepts of fractions including fraction size, equivalent fractions, comparing and ordering fractions and fraction computation. These are the indicators of QFK for teachers, embedded in the constructivist theory and the DCT.

2) **Fraction Models (FM):** Considering the nature of this study, visualisation is an indispensable, fundamental element of the analytical framework. Therefore, the inclusion of different fraction models in the analytical framework was a critical consideration. Moreover, the use of different fraction models was used as a criterion to select participants using a questionnaire that was administered to participants prior to the commencement of the collection of empirical data. Siegler et al. (2010) recommends the use of area, length and set models to improve learners' understanding of different fraction concepts and their computational procedures, and to use estimation to predict or judge the reasonableness of answers involving operations with fractions. The use of the number line as a model to enhance learners' understanding of fractions as numbers and to locate and compare fractions is strongly encouraged (Siegler, 2010; Wu, 2011 & Van de Walle et al., 2013). Visualisation is a major component of this study and it is through the use of different fraction models that one can determine its impact on the pedagogy of fractions.

3) **Visuality of Lessons (VL):** Similar to the fraction models discussed above, the visuality of lessons is another important concept worth considering in the analytical framework. As defined by Presmeg (1986, p. 298), "a person's mathematical visuality is the extent to which that person prefers to use visual methods when attempting mathematical problems which may be solved by both visual and nonvisual methods." This aspect was deemed necessary, because for this specific study, it is important to establish the participants' mathematical visuality and how this affects their pedagogy. In addition, Vale and Barbosa (2018) suggest that depending on the extent to which learners (or teachers) prefer to use visuals in solving mathematical problems, they can be regarded as 1) visual or geometric 2) non-visual, analytical or verbal or 3) mixed, integrated or harmonic. Teachers whose pedagogy is inclined towards the use of visuals are known to model fractional amounts with more than one manipulative, engage learners in drawing activities, pose tasks with a visual component, use gestures to explain and introduce

fraction concepts using visuals (Boaler, 2015). Essentially, these are the indicators of mathematical visuality.

4) **Verbal and Nonverbal Codes (VCNC):** The other important concepts which form part of the analytical framework are the two main components of the DCT, are the verbal and nonverbal codes. The components of the DCT were included in the analytical framework in order to assess the extent to which the participants use the verbal and the nonverbal codes in teaching fractions. The DCT encourages the use of both codes. Hence, it is vital to assess, *when*, *how* and *why* the two codes are used in fraction instruction. Indicators of this element include teachers' ability to use both codes interchangeably, the incorporation of visual representations and symbolic notation in assessment activities, and teaching skills that support and build on learners' intuitive strategies. This is a crucial aspect because the prevalence of one code (at the expense of the other) can either enhance or hinder the conceptual understanding of fractions.

5) **Quantitative Thinking (QT):** As discussed earlier, quantitative thinking refers to the conception of the relative size of fractions. This is a critical element in the conceptual understanding of fractions, hence its inclusion in the analytical framework. In fact, learners' understanding of fraction size determines their understanding of fractions in general because fraction size hinges on all the other fraction concepts. Indicators for this element include learners' ability to compare and order same-denominator and different-denominator fractions, represent fractions using different fraction models, estimate fraction size using the benchmarks 'zero', 'a half' and 'one' and be able to estimate the answers to addition, subtraction, multiplication and division fraction problems. These indicators can be used to assess the strategies that teachers employ to enhance learners' understanding of fraction size.

6) **Pedagogical Content Knowledge (PCK):** Since this study intends to interrogate the redefining of the pedagogy of fractions through visualisation, it is imperative to include the concept of PCK in the analytical framework. According to Shulman (1986, p. 9), PCK refers to the ways of "representing and formulating the subject matter that makes it comprehensible to others." The difference in learners' performance is primarily determined by how the fraction content is presented to them. Consequently, the teacher's role in making fractions easy or difficult for learners cannot be overemphasised. The indicators for the PCK of fractions are based on Boaler's (2015) framework on the Mathematical Mindset Community and Bezuk and

Cramer's (1989) recommendations which include the following: using more than one manipulative, acknowledging learners' prior knowledge, addressing misconceptions, inviting curiosity when posing questions, creating an environment where learners freely ask and pose questions, encouraging learners to use and share different ideas, methods and perspectives, and learning to value and model fractions. These are very important considerations in the planning and execution of a fraction lessons.

7) **Effective Questioning Strategies (EQS)**: is the last component of the analytical framework. Although EQS is part of the PCK, it is important to have it as a separate component due its important role in aiding learners' conceptual understanding of fractions. The way fraction problems are designed and presented to learners can either enhance or impede learners' understanding of fractions. McCarthy et al., (2016, p. 80) concur that "the use of good questioning, by teachers, may mean the difference between constraining thinking and encouraging new ideas, and between recalling trivial facts and constructing meaning." Based on Boaler's (2015) recommendations, open tasks that encourage multiple methods, pathways and representations should be encouraged. EQS refers to the type of questioning that is aimed at developing learning and higher order thinking, promoting imagination, speculation and creative thinking, and pitched a suitable challenging level (Gast, 2009, p. 1). Therefore, this requires the formulation of questions with a visual component that allows all learners to contribute to the learning, have room for extension and make opportunities for learners to authentically share their thinking with their peers.

2.10 CONCLUSION

The reviewed sources of literature in this section all point towards the need for a paradigm shift in the way that fractions are taught (and learned). What came out strongly from the different sources is the observation and experience that fractions are difficult to teach and difficult to-learn. Although there are different suggestions on how to tackle this challenge, there is enough literature to support the use of visualisation as a powerful reasoning tool in the pedagogy of fractions. Despite the diverse views on the importance of visualisation in the pedagogy of mathematics, several authors (Boaler et al., 2016; Rosken & Rolka, 2006; Stokes, 2002 & Zimmermann & Cunningham, 1991) maintain that visualisation is not grade or phase-bound. Visualisation is a relevant strategy that can be used from preschool to tertiary education. The difficulties experienced in the teaching and learning of fractions have persisted because visualisation as an alternative to the abstract, rule-oriented strategies, has received little attention (Boaler, et al., 2016). This is often attributed to the perception that visualisation is regarded as a low level strategy which is appropriate for children in the foundation grades and concerns have also been raised regarding the transferability of knowledge (Rau & Mathews, 2017). However, Stokes (2002, p. 10) argues that "the connection of visual and verbal information is evident throughout history." It is interesting to note how visual human thinking is, yet this concept is less prevalent in most of the traditional classroom settings where the focus is on words and numbers.

Understanding the relationship between visualisation and conceptual understanding is key in comprehending how interconnected these two concepts are. Richardson (2019, p. 1) explains that "mathematics is understood if its mental representation is part of a network of representations." Therefore, the storage, construction and recall of mathematical knowledge can be perceived as a visual process. Richardson (2019, p. 2) further asserts that the building blocks for the construction of knowledge "originate in a person's experiences and the mental images derived from previous experiences." Hence, the absence of visualisation impedes learners' ability to create mental images of mathematical concepts and make connections between the new knowledge and the existing schema. This leads to the rote memorisation of facts and rules. Hence, visualisation has been the focus of educational research because it enhances human cognition. As stated by Hegarty (2004, p. 2) "educational research emphasizes the need to expose children to powerful external visualizations of data." External visualisations such as diagrams, graphs, pictures, concrete objects and animations are important because they enhance learners' ability to create mental representations of mathematics concepts.

Fraction sense and PCK were also identified as key concepts in enhancing learners' conceptual understanding of fractions. It is evident from the literature sources reviewed in this chapter that teachers can only be expected to teach fractions effectively if their fraction sense is well developed and they fully understand the enabling and constraining factors not only in teaching but also in learning fractions. Visualisation remains an integral component of fraction sense and PCK. Hence, the use of different fraction models is recommended for the development of learners' fraction sense. Besides the area and set models, "understanding fractions as magnitudes that can be represented on a number line provides an underlying structure for learning a range of fraction concepts and skills," (Rodrigues et al., 2016, p. 135). Consequently, through the use of the number line learners begin to understand fractions as numbers, thereby enhancing their fraction sense. Thus, the exposure of learners to a variety of fraction models (including the number line model) is a prerequisite to their conceptual understanding of fractions.

Despite the numerous benefits of the DCT, its success in the pedagogy of fractions depends on the ability of learners to establish links between the visuals and the standard mathematical procedures that they are eventually expected to use in solving mathematics problems. This is mainly determined by the mathematical culture or language that the teacher portrays in the classroom. Arcavi (2003, p. 38) argues that "when visualisation acts upon conceptually rich images, the cognitive demand is certainly high." However, when this link is missing, the use of visuals as opposed to symbols may be perceived by both teachers and learners as procedurally unsafe. Therefore, the cognitive difficulties associated with the use of visuals can be reduced by ensuring that the verbal and the nonverbal codes are used in tandem. The next chapter presents the methodology of this study.

CHAPTER THREE
METHODOLOGY

3.1 INTRODUCTION

In this chapter, I present the research methodology for this mixed-methods study which was designed to determine the role of visualisation in redefining the pedagogy of fractions. Basically, this chapter provides the research roadmap for this study, that is, the methodological choices made, the justifications for those choices, the selection of research instruments, the implementation of the research instruments, validity and reliability issues and ethical considerations. The Dual Coding Theory (DCT) and the Constructivist Theory (CT) framed this study. These two theories were instrumental in determining the type of visualisation processes that participants used in their teaching. Moreover, this theoretical framework provided a lens through which the data collected was analysed. Consequently, I was able to address pertinent questions related to the use of visualisation processes observed in the pedagogy of the participants. This chapter therefore describes the research process in its entirety by pointing out its uniqueness in the methodological process and its theoretical underpinnings.

3.2 ORIENTATION OF THE STUDY

A pragmatic paradigm was adopted for this study on the basis that it is a suitable philosophical justification for mixed-methods research (Maarouf, 2019). This paradgim embraces the use of different research methods (Kaushik & Walsh, 2019 & Maarouf, 2019). Considering the fact that this study relied on both qualitative and quantitative methods, a paradigm that encompassed the two methods was deemed necessary. Ndlovu (2021, p. 167) describes pragmatism as “a compromise between positivism and interpretivism by acknowledging that we can learn from both worldviews about reality” Moreover, a pragmatic paradigm portrays the view that a “mono-paradigmatic approach to research is limited and limiting” (Ndlovu, 2021, p. 171). The pragmatic paradigm was aligned to the two theories informing this study (Dual Coding Theory and constructivism) in the sense that it promotes the notion of a non-singular reality.

Teachers have different reasons for teaching the way they do, for the inclusion or exclusion of visualisation and for adopting certain teaching strategies. Therefore, current practices in mathematics classrooms should be defined through the participants’ interpretations and justifications for their practices in terms of their ontological and epistemological dispositions. Since this study is concerned with the implementation of visualisation processes in the teaching of fractions, it is important to understand the position of the participants from the interpretivist and the positivist point of view. Hence, the need for the pragmatic paradigm.

Thus, the discrepancies observed in teacher practice and learner performance on the concept of fractions can be attributed to the different ways in which teachers understand, interpret and present fraction concepts in their classrooms. This is a crucial starting point for this study because, by interrogating the pedagogy of the participants, it contributes immensely to the general belief that fractions are difficult to teach and learn. Therefore, identifying the underlying factors for the prevailing situation in mathematics classrooms is key in determining the primary cause of the existing challenges faced by both teachers and learners.

3.3 RESEARCH QUESTIONS

The research questions for this study comprised of one overarching research question and four sub questions as presented below.

Overarching question: How does the incorporation of visualisation processes in mathematics lessons as a result of teachers' participation in the RCFP, enhance the teaching of fractions, if at all?

Sub questions:

1. What type of visualisation processes do senior primary school teachers incorporate in their mathematics lessons?
2. How do senior primary school teachers incorporate visualisation processes in their mathematics lessons?
3. What significance do senior primary school teachers attach to the incorporation of visualisation processes in mathematics lessons?
4. What are the enabling and constraining factors in teaching fractions in an explicitly visual way at the senior primary phase?

3.4 METHODOLOGY

3.4.1 Mixed methods

Due to the nature of this study, a mixed-methods approach was deemed appropriate. The use of both qualitative and quantitative approaches provided opportunities for the collection of adequate and complementary sets of data. As stated by Alexander et al. (2008, p. 122) complementarity in mixed methods research is not necessarily used "to gauge a concrete number or a more accurate picture of a singular reality, but to reveal the different dimensions and enrich understanding of a multi-faceted, complex nature of the social world." Hodis and Hancock (2016, p. 301) assert that the use of mixed methods in educational research "opens the door to multiple and exciting opportunities to enhance knowledge." In addition, this design was useful in ascertaining the consistency participants' views about visualisation and practices.

For this study, the two approaches were sequential, that is, quantitative first, followed by qualitative (Bowen et al., 2017). This sequence was influenced by the research questions. Drawing on Leech and Onwuegbuzie's (2009) categories of mixed methods research, this can be classified as a partially mixed, sequential design lending itself towards a dominant qualitative data status. While the analysis of data from the lesson observations was exclusively qualitative, data from the semi-structured questionnaire and the interviews were quantitatively and qualitatively analysed. Punch and Oancea (2014, p. 5) posit that

> depending on the questions being asked, it is possible to transpose data to a different format in order to enable different types of analysis-for example numerical data may be interpreted qualitatively, or textual data may be converted into numbers to be analysed quantitatively.

Creswell and Plano Clark (2011) further suggest that although the use of mixed-methods is time consuming, it can help to address broader questions and clarifies other aspects of the research that may be attended to through the use of one method. In other words, this study was designed in such a way that the quantitative approach served as a prerequisite to the qualitative approach. However, the quantitative approach also complemented the qualitative approach throughout the study in terms of data analysis. Creswell and Plano Clark (2011, p. 58) further suggest that when using mixed methods, "researchers also need to be familiar with the timing, weighting, and mixing decisions that are made in each of the different mixed methods designs."

Hence, these considerations were taken into account in the integration of these the two approaches.

The combined effect of the two methods was more beneficial than the use of either method (used separately). Therefore, the use of the two methods advanced a complete and synergetic approach to this study.

3.4.2 Quantitative methods

The quantitative approach was instrumental in identifying participants for this study. A questionnaire comprising of questions that was informed by the theoretical framework and the research questions was administered to determine the teaching profiles of the members of the RCFP, from which my participants were selected. Twenty-one participants from the RCFP served as respondents to this questionnaire. Eventually, only ten participants were selected as participants of this study, based on their responses to the questionnaire. However, this number was reduced to eight after two of the participants withdrew from the study.

The questionnaire comprised of questions that were designed to identify participants with specific pedagogical profiles from the Rundu Campus Fraction Project (RCFP). Apart from the incorporation of verbal and nonverbal codes, other criteria were used in the selection of the participants and these included; their responses to questions regarding the types of visuals that the participants used and how often they used them, the significance attached to the use of visualisation, their views on the incorporation of visualisation in teaching fractions and the impact of the RCFP on their pedagogy.

Quantitative methods were also used to analyse qualitative data through the use of frequency tables, that is, the quantization of qualitative data. This was particularly instrumental in determining how often the participants used visualisation processes and the type of fraction models that they preferred to use in their teaching.

3.4.3 Qualitative methods

Qualitative data collection methods were also used to complement the data obtained from the questionnaire. Thus, besides the questionnaire, I also employed interviews and lesson observations as data collection methods. The interviews were used to determine the teaching profiles of the participants and their general knowledge regarding visualisation, the concept of fractions and the pedagogy of fractions. Moreover, I also intended to determine the impact of the RCFP on the pedagogy of the participants through the data collected from the three research methods, that is, the questionnaire, the interviews and the lesson observations. Hence, the triangulation of the three sets of data was instrumental in answering my research questions.

The interview schedule was made up of pre- and post-observation interview questions. The pre-observation interviews were general in nature, focusing on the importance of fractions, the importance of visualisation in teaching and learning fractions, factors contributing to learners' poor performance in the concept of fractions, their experience in the RCFP and how it had impacted their teaching, pedagogical considerations in teaching fractions, fraction models and the participants' perceptions about quality fraction knowledge. These factors were carefully selected because of their important role in answering my research questions. Although different aspects were identified, visualisation and fraction instruction were at the centre of them all.

Observations were identified as a very important research technique because of the nature of this study. Since the goal was to determine the incorporation of visualisation processes in the participants' fraction lessons, the inclusion of lesson observations as a research instrument was key. Thus, although the participants' responses to the questionnaire and the interviews were vital in addressing my research questions, the lesson observations played a pivotal role in this respect because what they did practically in their classrooms helped to affirm or disprove their responses to the questionnaire and the interviews.

3.4.4 Research design: A case study

In order to undertake an in-depth understanding of a phenomenon, a case study design was employed. An instrumental case study was adopted because by definition, an instrumental case study involves the selection of "a small group of subjects in order to examine a certain pattern of behaviour" (Zainal, 2007, p. 4). In addition, Clarke and Davies (2014, p. 3) explain that a "case study design is best applied when research addresses descriptive or explanatory questions and aims to produce a first-hand understanding of people and events." Although this was a mixed-methods study, it leaned more towards qualitative because the main goal was to determine how the participants incorporated visualisation processes in the teaching of fractions. Hence, this could be better established by deeply engaging with the participants through observations and interviews.

Since this study sought to understand the views of a specific group of teachers on the incorporation of visualisation processes in teaching fractions, a case study method was deemed necessary. This was an instrumental case of eight teachers from the RCFP who taught at different schools. The eight participants were purposefully selected (through a questionnaire) based on their teaching profiles. Participants with specific characteristics in terms of their teaching were considered in the design of this study. The selection of participants from the RCFP was informed by the two theories underpinning this study, that is, the Dual Coding Theory (DCT) and the Constructivist Theory (CT). Hence, the questionnaire was designed with specific emphasis on the teaching preferences of the participants.

The unit of analysis in this study was the incorporation of visualisation in fraction lessons by the eight participants. Roller (2019, p. 1) defines the unit of analysis as "the portion of content that will be the basis for decisions made during the development of codes." Mohajan (2018) further suggests that this may include individuals or groups of individuals. Thus, although the other units (sets of data) were important in the analysis of data, the focus was more on the incorporation of visualisation processes in the classroom by the eight participants. Moreover, this was instrumental in answering my research questions.

According to Mayer and Anderson (1991), the DCT predicts that the use of the verbal and nonverbal codes in teaching presents learners with greater opportunities of understanding the subject matter as opposed to the exclusive use of either the verbal or the nonverbal code. In light of this, the questionnaire included questions that presented participants with options to indicate best practices with regard to the pedagogy of fractions. The basic unifying criteria for all the selected participants was their participation in the RCFP, the phase they taught (senior primary) and the use of both verbal and nonverbal codes. However, additional criteria such as the use of fraction models, the value attached to the use of visualisation processes and the frequency at which they used visualisation processes were also considered (see Table 3.1)

3.5 RESEARCH SITES

The research sites used in this study included four public schools and the University of Namibia (UNAM), Rundu Campus. The identification and selection of the participants and the research sites was done concurrently since I had to follow my participants at their respective schools. Consequently, the following schools were identified as my research sites:

School RSP is an urban school, situated about one km east of Rundu town. This is a big, popular senior primary school, accommodating more than one thousand learners from different social economic backgrounds. On average, there were about sixty learners per class before the COVID 19 regulations were enforced, which compelled schools to split or alternate classes or grades to ensure that social distancing was maintained. Despite its location, the school is under resourced in terms of infrastructure, teaching and learning materials. The school lacks a proper administration block, a library, science laboratory and a computer laboratory. As a result, the school has converted two classrooms into a staffroom and the principal's office.

School SC is a big school situated in one of the informal settlements in Rundu which is about six km west of Rundu town. The school is one of the most overpopulated schools with about one thousand three hundred learners. The average number of learners per class at the time of the research was about forty learners per class and this was after the enforcement of the COVID 19 regulations. This means that under normal circumstances, the school accommodated more than seventy or eighty learners per class. Most of the learners admitted at this school are from poor socio-economic backgrounds. In terms of resources, this school can be described as an under-resourced school. The school lacks a proper administration block, a library, science laboratory and a computer laboratory. Hence, the school has converted two classrooms into a staffroom and the principal's office.

School NCS is another big school, situated in an informal settlement, which is about five km east of Rundu. This is one of the most densely populated schools in Rundu and it accommodates about one thousand four hundred learners, translating into an average of seventy learners per class. The school can be described as an under-resourced school with most of the learners hailing from poor family backgrounds. The school lacks a proper administration block, a library, science laboratory and a computer laboratory. As a result, the school converted two classrooms into a staffroom and a principal's office.

School SCS is a big, overpopulated school, situated in an informal settlement which is about six km east of Rundu. This under-resourced school accommodates about one thousand two hundred learners and an average of sixty learners per class. Most of the learners admitted at this school hail from poor family backgrounds. This school has a small administration block. However, it lacks a functional science laboratory, computer laboratory and a library.

Besides the identified schools, UNAM, Rundu Campus was also used as a research site mainly for the induction workshop and the development of materials. Under my supervision, the participants used to meet on a regular basis for a period of seven months from September 2019 and the first quarter of 2020 to develop visual materials and share their experiences on the use of visualisation processes.

3.6 POPULATION

Although the target population of this study are all teachers who teach fractions, the specific target population for this particular study were the participants of the RCFP. This is a group of teachers that is striving to improve the way fractions are taught. They meet twice a semester to discuss the difficulties associated with the pedagogy of fractions and to develop fraction visuals that can enhance the teaching of fractions. Thus, the RCFP was identified as a suitable empirical field for this study because its focus on fractions.

3.7 SAMPLING

Purposive sampling was used to select ten participants from the RCFP, using a questionnaire. However, the sample was later reduced to eight after two of the participants withdrew from the study. Demographically, the final sample was made up of four males and four females who taught at the senior primary phase. In terms of qualifications, three of the participants were in possession of a Bachelor of Education (Honours) degree (BEd) from the University of Namibia, while the rest were in possession of the Basic Education Teachers' Diploma (BETD).

Purposive sampling as defined by Etikan et al. (2016, p. 2) is "the deliberate choice of a participant due to the qualities the participant possesses." This sampling method was employed because it was appropriate for the identification and selection of information-rich cases.

3.7.1 Profiles of participants

As alluded to earlier, the profile of the participants was a major consideration in their selection. Hence, I considered the participants' position on the pedagogical aspects mentioned earlier. Although discrepancies were later observed between what the participants said and what was actually observed in the lesson observations, the initial findings from the questionnaire helped to classify the participants in terms of their teaching orientation. Moreover, since this study was informed by the DCT and the CT, the focus was on identifying participants who used both verbal and nonverbal codes. This section provides the profiles of the eight participants.

Teacher D was a young, male, BETD graduate who taught Grade seven mathematics at one of the densely populated schools in the vicinity of Rundu town. His personal position on the use of visualisation processes was that it is very important and appropriate for all phases. In addition, Teacher D indicated that although he did not use visuals very often, its incorporation in teaching fractions was vital, particularly when teaching the four basic operations. He singled out the area model as his preferred model of teaching fractions.

Teacher K was another young, male, BEd graduate who taught Grade six at the same school as Teacher D. Teacher K suggested that the incorporation of visualisation in teaching mathematics is very important and should be promoted at all levels. Despite his support for visualisation in the pedagogy of mathematics, he stated that he did not use it often because it was time-consuming. Furthermore, he pointed out that he preferred to use the area and set models in teaching fractions. Teacher K also suggested that visualisation was appropriate for teaching equivalent fractions and the four basic operations.

Teacher M1 was a BETD graduate in his late forties, who taught Grade seven at one of the biggest and most popular primary schools in Rundu. His views on the use of visualisation was that it was very important but more appropriate for the junior primary phase. Furthermore, he stated that visualisation was suitable for teaching all fraction concepts. He also indicated that he used visualisation processes frequently in his lessons. Moreover, Teacher M1 opted for the use of all three fraction models, that is area, length and set models in teaching fractions

Teacher S1 was a young, female, BETD graduate who taught Grade seven at a big school on the outskirts of Rundu. She perceived the use of visualisation as vital in enhancing learners' understanding of fractions, particularly unit fractions and equivalent fractions. However, she indicated that she did not incorporate it in her lessons frequently. According to Teacher S1, the incorporation of visualisation processes was more appropriate for the junior and senior primary phase. She further indicated her preference for the length model in teaching fractions.

Teacher H was a young, female, BETD graduate who taught Grade four and six at one of the big, densely-populated schools in Rundu. She valued the use of visualisation, specifically for introducing mathematics concepts, because according to her, it enhances learners' understanding. Although Teacher H supported the use of visualisation in all fraction concepts, she preferred to use one fraction model, that is, the area model. The junior primary phase was identified as the most appropriate phase for visualisation.

Teacher M2 was a young, female, BEd graduate who taught Grade seven at another big and overpopulated school on the outskirts of Rundu. She regarded the use of visualisation in teaching mathematics as important because it increases learners' retention of knowledge. However, she stated that she did not use visualisation often and that the nature of the topic dictated the incorporation or omission of visualisation processes in her lessons. She recommended the use of visualisation in all fraction concepts. Furthermore, she stated that the use of all three fraction models, that is area, length and set models is vital. She further pointed out that visualisation is more suitable for the senior primary phase.

Teacher I was a BETD graduate, in her early forties, who taught Grade six at a big school on the outskirts of Rundu. She valued the use of visualisation because according to her, it enables learners to retrieve information easily. Moreover, Teacher I also suggested that visualisation should be used in teaching all fraction concepts at all levels. She further indicated that the area model was more suitable for teaching fractions.

Teacher S2 was a young, male, BEd graduate, who taught Grade four at the same school as Teacher M1. He regarded visualisation as a very important concept that enhances learners' understanding and increases their retention levels. However, Teacher S2 provided contradicting versions pertaining to the incorporation of visualisation. On one hand, he indicated that visualisation should be used to teach all fraction concepts, while on the other hand, he stated that the incorporation of visualisation was dependent on the topic. While acknowledging the use of visualisation across all phases, he singled out the length model as an appropriate model for teaching fractions.

In summary, the selected participants shared some common characteristics. For instance, they all appreciated the incorporation of visualisation in the pedagogy of fractions. However, the participants had different views with regard to the use of models, the appropriate levels or phases for using visualisation and the fraction concepts for which visualisation was suitable. Despite the participants' divergent views on the use of visualisation processes, they were selected on the basis that they all concurred that the use of verbal and nonverbal codes was imperative.

3.8 IMPLEMENTATION OF THE STUDY

The research design was divided into three phases. Phase 1 encompassed an induction workshop for the research participants and the administering a questionnaire to all the RCFP participants. Phase two focused on a session on fraction models, while Phase three concentrated on lesson planning and material development sessions focusing on three fraction concepts: namely fraction size and equivalent fractions, addition and subtraction of fractions and multiplication and division of fractions. This design was useful because it helped to introduce participants to the core concepts of this study. The section below summarises the three phases.

3.8.1 Phase 1

3.8.1.1 Induction workshop

The aim of this workshop was mainly to familiarise the participants with this study by explaining its significance and what it entailed, and taking them through the different stages of the research. Since one of the most important items on the agenda of this workshop was material development, an invitation was extended to all the RCFP members, including those who were not participants of this study. Apart from introducing the participants to this study, I also planned to introduce the idea of visualisation kits to all the members of the RCFP.

A visualisation kit, in this context, refers to a collection of self-made visual teaching aids for fractions that the members of the RCFP could use in their lessons. Hence, this workshop was beneficial to all the members of the RCFP since it presented them with opportunities to develop materials for their own use, regardless of their participation in this study. During this workshop, the participants received materials such as posters, marker pens and rulers to create fraction walls and models of fraction size. The focus of this workshop was on helping the participants develop effective teaching strategies for the concept of a fraction and fraction size. These are foundational concepts in the teaching trajectory of fractions.

3.8.1.2 Administering a questionnaire to select participants

A semi-structured questionnaire with open-ended questions was administered to all the participants at the end of the induction workshop. Siniscalco and Auriat (2005, p. 4) posit that information obtained through questionnaires includes "facts, activities, level of knowledge, opinions, expectations and aspirations, membership of various groups, and attitudes and perceptions." Since my intention was to identify and select participants with specific traits, I found the use of a questionnaire in the initial stages of this study very helpful.

The questionnaire consisted of ten questions, which were all instrumental in determining the eligibility of the respondents for this study. The questions focused on the following aspects: 1) the phases that the respondents taught; 2) the incorporation of visualisation in teaching fractions; 3) the types and sources of visuals used by the respondents; 4) the use of verbal and nonverbal codes in teaching mathematics; 5) the value attached to the incorporation of visuals in the pedagogy of mathematics; 6) preferred fraction models; 7) the impact of the RCFP on their pedagogy; and 8) the appropriate phases for using visualisation. The questions were made up of closed and open-ended, because in some instances, the respondents were required to justify their choices. As stated by Parfitt (2005, p. 87), "the content of the questionnaire needs to be firmly rooted in the research questions or hypothesis under investigation." Therefore, this was an important consideration in the compilation of these questions.

The purpose of the questionnaire was clearly explained to the thirty participants and they were given a week to complete it. After one week, I visited their respective schools to collect the questionnaires. Moreover, I also visited the schools of some potential candidates who were absent from the induction workshop on that specific day and asked them to participate in the same exercise.

Based on their responses to the questionnaire, ten respondents were selected. The selected participants met the minimum qualifying criteria for this study. The three criteria that were instrumental in the selection of the participants were: 1) their participation in the RCFP, 2) their interest in the use of both verbal and nonverbal codes in teaching fraction concepts and 3) their teaching phase – since they had to be senior primary mathematics teachers to be eligible for this study. Initially, ten participants were selected although two of them withdrew from the study prior to its commencement. The table below presents the teaching profiles of the eight participants:

Table 3.1: Profiles of the selected participants

Respondents	Phase taught	Verbal/ nonverbal/both	How often they use visuals	Fraction models	Impact of RCFP on pedagogy	Appropriate phase for visualisation
Teacher D	Senior primary	Both	Not so often	Area	Good	All
Teacher K	Senior primary	Both	Not so often, time-consuming	Area Set	Good	All
Teacher M1	Senior primary	Both	Most of the time	Area Length Set	Good	Junior primary
Teacher S1	Senior primary	Both	Sometimes	Length	Good	Junior and senior primary
Teacher H	Senior primary	Both	Mostly when introducing the topic	Area	Good	Junior primary
Teacher M2	Senior primary	Both	Not very often. It depends on the topic	Area Length Set	Good	Senior primary
Teacher I	Senior primary	Both	Sometimes	Area	Good	All
Teacher S2	Senior primary	Both	Depends on the topic	length	Good	All

The additional criteria as indicated in column four to seven of Table 3.1 provided more insight on the participants' position regarding the use of visualisation. This study was conducted based on the assumption that the verbal code is more dominant in traditional mathematics classrooms than the visual code. Thus, the focus was more on the participants' views on the use of the nonverbal code (visualisation) than the verbal code. After identifying the participants, the remaining sessions targeted only the participants involved in this study rather than the entire RCFP group.

3.8.2 Phase 2: A session on fraction models

Several authors concur that the use of different fraction models, namely area, number line and length models can develop learners' fraction sense (Cramer & Whitney, 2010; Van de Walle, et al., 2013; Dyson, et al., 2016). Furthermore, they also suggest that the incorporation of the three fraction models in the pedagogy of fractions presents learners with different opportunities to understand fractions. It was against this background that I decided to have a session on this topic to acquaint the participants with the different ways of representing fractions and how this can improve the pedagogy of fractions.

During this session, I discovered that some of the participants possessed adequate content knowledge on this topic while others struggled with both content and pedagogical content knowledge. By the end of this session, the participants began to show a keen interest in the use of fraction models. Those who demonstrated a better understanding of the fraction models were given opportunities to demonstrate their teaching strategies to the rest of the participants. Generally, the participants found the area and length models much easier to understand than the set model. The set model differs from the other two models in the sense that a group of objects rather than a whole shape or length is used to represent a fraction. Thus, some participants with little exposure to fraction models found the set model confusing. However, most of them felt enlightened after this session.

3.8.3 Phase 3: Planning of lessons, creating materials, implementing and observing lessons

Having acquired skills in developing visualisation materials in the previous phase, the research participants were expected to plan their own lessons on selected fraction concepts and prepare relevant visualisation materials. This exercise was done in preparation for the topics that they were expected to teach. The topics included fraction size, equivalent fractions, addition and subtraction of fractions and the multiplication and division of fractions. These were identified as foundational concepts in the pedagogy of fractions. Therefore, they must be effectively taught. An observation schedule depicting the use of the three fraction models was used to assess the participants' use of visualisation in general.

3.8.3.1 Fraction size and equivalent fractions

In order to help learners acquire a deeper understanding of the concept of fractions, it is important that the process begins by exposing learners to representations of fractions across different constructs (Rau & Mathews, 2017; Van de Walle et al., 2013). These constructs enhance learners' understanding of fraction size. In addition, equivalency is another concept that forms part of the foundational knowledge that learners need in order to master fractions. According to Cooper et al. (2012, p. 7), equivalence "refers to the fact that many fractions can be used to name the same quantity." Bruce et al. (2013, p. 14) further suggest that "comparing fraction values requires a strong sense of equal partitions as well as equivalence." Equivalent fractions are an important starting point in teaching fractions as they lay the foundation for a better understanding of operations with fractions. The materials developed during this phase were banked in the participants' visualisation toolkits.

3.8.3.2 Addition and subtraction of fractions

The persistence of traditional teaching methods has been observed in mathematics classrooms (Ausiku, 2008). The dominance of rules in teaching fraction addition and subtraction has proved to be a challenge (Bruce, et al., 2013). Therefore, fraction models present an alternative approach to teaching fraction computation. My intention during this phase was to establish how the participants taught the addition and subtraction of fractions using the different fraction models. The use of visuals to teach fraction computation was considered because it helps learners to comprehend the rationale behind the need to find common denominators when adding or subtracting fractions with different denominators.

3.8.3.3 Multiplication and division of fractions

One of the misconceptions among learners is the expectation that, just like whole numbers, the multiplication of fractions should produce bigger numbers. The same applies to division with fractions as learners expect smaller numbers (as answers) when dividing any two given – which is not usually the case. This is a classic example of the wrong application of whole number knowledge by learners attributed to "very poor preparation in rational numbers and operations involving fractions and decimals" (Siegler et al., 2010, p. 6). Division with fractions produces bigger numbers and learners often grapple with this fact because of their experiences with whole numbers. Therefore, the focus during this phase was on observing participants' attempts to help learners understand the logic behind these unexpected results as far as the multiplication and division of fractions is concerned.

3.9 DATA COLLECTION TECHNIQUES

Three research instruments were used for data collection, that is, a questionnaire, an interview schedule and an observation schedule. These instruments were useful in comprehensively providing relevant data at the different stages of the study. The questionnaire was used to collect data which was essential in the selection of participants, while the interviews and the lesson observations were used to determine the extent to which visualisation processes were used in teaching different fractions concepts.

3.9.1 Questionnaire

Although the questionnaire was mainly used to identify participants, it was designed in such a way that it provided relevant data for the different stages of this study. In other words, the questionnaire covered a scope of questions that were useful in not only identifying participants, but in addressing the research questions as well (appendix one). These questions were structured in such a way that they provided data pertaining to the teaching profiles of the respondents, that is, their views on visualisation and its role in teaching mathematics.

In order to obtain rich data, the questionnaire was administered to the participants of the RCFP because they were all familiar with the concept of visualisation and its practical application in the context of fractions. The respondents were given a week to complete the questionnaire, after which I collected their individual responses. The response rate was very high because I was able to recover all the completed questionnaires.

Based on their responses to the questionnaire, ten respondents were selected. The selected participants met the minimum qualifying criteria for this study. The three criteria that were instrumental in the selection of participants were: 1) their participation in the RCFP, 2) the use of both verbal and nonverbal codes in teaching fraction concepts and 3) the phase they taught. In this case, I targeted senior primary teachers because this is the phase were the use of visuals diminishes and the use of rules becomes prevalent.

3.9.2 Interviews and interview schedules

The interviews were instrumental in ascertaining the participants' views on the incorporation (or omission) of visualisation processes in the pedagogy of fractions. Although some of the questions were addressed in the questionnaire, through the interviews, I was able to elicit the participants' "perceptions, thoughts, feelings and impressions in their own words," (Dilshad & Latif, 2013, p. 2) about visualisation and the impact of the RCFP on their teaching. This was clearly very important in answering the research (sub) questions and ultimately the overarching research question. In other words, the interviews provided opportunities for the participants to justify their positions on the use of visualisation and for the researcher to probe and seek further clarification.

The interview schedule was made up of semi-structured questions, primarily because I set out to obtain comprehensive sets of data that would establish the position of the participants on the incorporation of visualisation processes to teach fractions. The design of the interview schedule was such that it contained two sets of interviews, that is, the general pre-observation interviews and the stimulated recall interviews (appendix two). Thus, I was able to probe participants regarding their behaviour and decision-making processes based on specific video-recorded events. According to Nguyen et al. (2013, p. 2), “the technique of stimulated recall gives participants a chance to view themselves in action as a means to help them recall their thoughts of events as they occurred.” I found this technique useful for my study because participants may have diverse or conflicting views on the use of visualisation in their mathematics classrooms, thus, the interpretations of their actions could be regarded as the basis for the success (or failure) of visualisation as a teaching approach. The stimulated recall interviews were conducted by replaying the videos that were captured during lesson observations and this made it easier for the participants to explain or clarify specific events emanating from their lessons.

The interview schedule for the pre-observation interviews was designed in such a way that it encompassed questions about fraction content, performance of learners on the concept of fractions, best practices in teaching fractions, the impact of the RCFP on the participants' pedagogy, the use of different fraction models and the participants' views on the use of visualisation in teaching mathematics, particularly fractions. All the eight participants were exposed to the pre-observation and post-observation stimulated recall interviews. However, as alluded to earlier, the unit of analysis for this study was the twenty-five lesson observations, triangulated against the data from the interviews and the questionnaire.

3.9.3 Observation and observation schedules

Observation is a common and very important technique of data collection in qualitative studies as it allows us “to discover complex interactions in natural social settings” (Marshall, 2006, p. 99). It is defined as “the systematic noting and recording of events, behaviours, and artefacts (objects) in the social setting chosen for study,” (Marshall, 2006, p. 98). For this study, observations were used to ascertain change in participants' pedagogy of fractions as a result of participating in the RCFP. During the observations, I assumed the role of a non-participant observer. Non-participant observation implies that the observer does not play an active role in the activity being observed. In other words, the observer gives a detached and unbiased view in order to maintain objectivity and neutrality about the participants and the settings.

The observations focused on the participants' selection and use of visualisation materials to teach fraction concepts. Thus, the aim of the observations was twofold:

1) To determine the impact of the knowledge acquired from the RCFP on the pedagogy of fractions
2) To establish whether the participants' choice of visualisation materials enhanced the teaching and learning process (or not).

Although the initial plan was to observe each of the participants five times, this was not realised due to the COVID 19 pandemic. Out of the eight participants, three were observed in early March to mid-March before the first lockdown in Namibia due to the COVID-19 pandemic, while the remaining five were observed after the lockdown under the World Health Organisation (WHO) and the Ministry of Health and Social Services (MOHSS) regulations. Thus, the first three participants were observed under normal school and classroom settings. Evidence from the video recordings indicates that the school and classroom settings before COVID-19 were totally different from the new COVID-19 classroom settings as teachers taught learners in overcrowded classrooms and both teachers and learners were not compelled to wear masks, practise social distancing, wash hands regularly or use sanitisers.

Despite the challenges presented by COVID 19, I was able to observe twenty-five lessons from all the eight participants. The data collected from the lesson observations (videos and semi-structured observation notes) was adequate and it covered all the focus areas (fraction concepts) targeted in this study. An observation schedule was developed, depicting seven critical aspects, namely Quality Fraction Knowledge (QFK), Fraction Models (FM), Visuality of Lessons (VL), Verbal and Nonverbal Codes (VNVC), Quantitative Thinking (QT), Pedagogical Content Knowledge (PCK) and Effective Questioning Strategies (EQS). Essentially, the aim of these lesson observation was to establish the type of visuals used by the participants, how these visuals were incorporated in their lessons and the role of these visuals in enhancing teaching and learning. Thus, the lessons were video-recorded and transcribed for easy analysis.

3.10 DATA ANALYSIS

3.10.1 Analytical framework

The first step in the data analysis process was the development of an analytical framework. As stated by Coral and Bokelmann (2017, p. 1), an analytical framework helps to "organize research and provide a general list of areas or variables that will be used in any type of analysis." The analytical framework was instrumental in the analysis of the data obtained from the three research instruments. Besides, the research questions, I also took into account the theories and key concepts underpinning this study. Despite its focus on the concept of visualisation, the analytical framework was theory-driven, taking into account the DCT, CCT and SCT. Consequently, these components provided a lens through which the different sets of data were analysed.

I analysed the data from the lesson observations by using an analytical framework which was based on seven key concepts. These seven research-based concepts were carefully selected and were in conformity with the two theories underpinning this study, that is, the social constructivist theory and the Dual Coding Theory. Moreover, the concept of visualisation, quality fraction instruction, research questions and literature were major considerations in the development of this framework. This framework also draws on Bezuk and Cramer (1989) and Siegler et al.'s (2010) recommendations on effective fraction pedagogy while some of the pedagogical aspects are aligned with Dual Coding Theory (DCT) and Boaler's (2015) framework on the Mathematical Mindset Community and recommendations for task/lesson design. Data from the observations were analysed by summarising the overall performance of participants on each of the seven concepts, using the indicators for each concept as portrayed in the analytical framework below:

Table 3.2: Observation schedule and the beginnings of my analytical framework

<table>
<tr><th>No.</th><th>Pedagogical aspects</th><th>Definition</th><th>Description</th></tr>
<tr><td>1.</td><td>Quality Fraction Knowledge (QFK)</td><td>Achieved when teachers use visuals to enhance ‘deep and new understandings’ of fractions</td><td>• Selects relevant visuals that are aligned to the lesson objective(s)
• Employs at least two fraction models
Able to interchangeably provide explanations of selected fraction concepts using both the visual and the verbal codes</td></tr>
<tr><td></td><td colspan="3">Indicators based on Siegler et al.’s (2010) recommendations.

QFK1: Demonstrate the depth of understanding of fractions and computational procedures involving fractions.

QFK2: se varied pictorial and concrete representations of fractions and fraction operations.

QFK3: Demonstrate competence regarding the assessment of learners’ understandings and misunderstandings of fractions.

QFK4: Ability to teach different fraction concepts including fraction size, equivalent fractions, comparison and ordering of fractions and fraction computation</td></tr>
<tr><td>2.</td><td>Fraction Models (FM)</td><td>• Different ways of representing fractions, that is: area, length and set models</td><td>• Represents unit fractions using the area, length and set models
• Encourages the use of area, length and set models in fraction computation</td></tr>
<tr><td></td><td colspan="3">Indicators based on recommendations from Siegler et al. (2010).

FM1: Use area models, number lines, and other visual representations to improve learners’ understanding of formal computational procedures

FM2: Use area, length and set models to improve students’ understanding of fraction equivalence, unit and non-unit fractions

FM3: Provide opportunities for students to locate and compare fractions on number lines

FM4: Use measurement activities and number lines to help students understand that fractions are numbers, with all the properties that numbers share

FM5: Provide opportunities for students to use estimation to predict or judge the reasonableness of answers to problems involving computation with fractions</td></tr>
<tr><td>3.</td><td>Visuality of Lessons (VL)</td><td>• A person’s mathematical visuality is the extent to which that person prefers to use visual methods when attempting mathematical problems which may be solved by both visual and nonvisual methods</td><td>The use of more than one manipulative enhances students' understanding and promotes abstraction of the concept from irrelevant perceptual features of a manipulative, such as colour, size, or shape</td></tr>
</table>

	Indicators based on Boaler's (2015) framework for Mathematical Mindset Community VL1: Model fractional amounts with more than one manipulative and naming unit and non-unit fractions VL2: Ask learners to draw their ideas VL3: Pose tasks with a visual component VL4: Encourage learners to draw for their peers when they explain VL5: Learners should gesture when they explain VL6: Use visuals to introduce fraction concepts		
4.	Verbal and Nonverbal Codes (VNVC)	• Refers to the use of both visual and non-visual	• A fair integration of both visual and verbal codes in the presentation of fraction lessons and assessment activities
	Indicators based on the Dual Coding Theory VNVC1: Support and build on students' intuitive strategies and these are used as a basis for the development of written forms of mathematics that accord with students' verbally-based and pictorially-based strategies VNV2: Interchangeably use the verbal and nonverbal codes to enhance learners' understanding VNVC3: Assessment tasks are comprised of both symbolic notation and visual representations		
5.	Quantitative Thinking (QT)	• Refers to the conception of relative size of fractions	• In addition to the verbal representations, the teacher uses visualisation processes to enhance learners' understanding of: • Unit fractions • Comparing and ordering fractions • Equivalent fractions • Addition and subtraction of fractions • Multiplication and division of fractions
	Indicators based on Bezuk and Cramer's (1989) more specific suggestions for experiences with fractions in the intermediate grades. QT1: Engage learners in fraction activities with denominators less than twelve. For example, if a blue circular piece is called 1/3, then students can be asked to find the value of other circular pieces. QT2: Fraction concepts can also be extended to new physical models (number lines or Cuisenaire rods) and to a new interpretation QT3: Activities for generating equivalent fractions should be introduced with manipulatives, then with diagrams. In particular, equivalent forms of such common fractions as 1/2, 1/3, 1/4, and 3/4 should be stressed, with the greatest emphasis placed on 1/2. QT4: Comparisons of fractions with like denominators (2/7 and 3/7) and like numerators (2/4 and 2/8) should be developed with manipulatives. Children should be able to verbalise a rule for ordering fractions with like numerators that does not rely on changing them to equivalent fractions with like denominators		

	QT5: Ordering pairs of fractions by comparing them to 1/2 or 1 should be included. For example, 3/ 10 is less than 2/3 because 3/10 is less than 1/2 and 2/3 is greater than ½ QT6: The initial goal of instruction for addition and subtraction of fractions should be to model basic operations with manipulatives and diagrams. Instruction should emphasise estimation and judging the reasonableness of answers QT7: Children should demonstrate an understanding of multiplication and division by modelling a problem with manipulatives or by naming the problem for the manipulative model. Students should be able to create story problems for a multiplication or division sentence or write a multiplication or division sentence for a story problem.		
6.	Pedagogical Content Knowledge (PCK) and Content Knowledge (CK)	• PCK refers to the ways of representing and formulating the subject matter that makes it comprehensible to others while CK is the type of knowledge explicitly associated with knowledge of the subject matter (Shulman, 1986, p. 9)	• Tests learners' prior knowledge using visual representations • Enhances learners' understanding of fraction concepts by using both verbal and nonverbal codes • Constantly checks learners' understanding of unit fractions using different fraction models • Learners' misconceptions of fraction conceptions are addressed by using both the visual and verbal codes
	Indicators based on Bezuk and Cramer's (1989) recommendations and Boaler's (2015) framework on Mathematical Mindset Community PCK1: Modelling fractional amounts with more than one manipulative and naming unit and non-unit fractions; (*b*) generating equivalent fractions; (*c*) performing concept-of-unit activities; *(d)* ordering fractions PCK2: Address common misconceptions regarding computational procedures with fractions. PCK3: Present real-world contexts with plausible numbers for problems that involve computing with fractions. PCK4: Create opportunities for learners to extend their work and investigate PCK5: Invite curiosity when posing tasks PCK6: Present content in such a way that learners see fractions as an unexplored puzzle PCK7: Create an environment where learners freely ask and pose questions PCK8 : Evoke the "I have never thought of it like that before," statements PCK9: Invite learners to see fractions differently PCK10: Encourage learners to use and share different ideas, methods and perspectives PCK11: Value and model creativity		
7.	Effective Questioning Strategies (EQS)	• Refers to the type of questioning that is aimed at developing learning & higher order thinking, promoting imagination, speculation, creative thinking and to pitch a suitable challenge level (Gast, 2009, p. 1)	• Poses questions or assessment activities that encourage creative thought and imaginative or innovative thinking
	Indicators based on Boaler's (2015) recommendations for task/lesson design		

	EQS1: Open the task to encourage multiple methods, pathways and representations EQS2: Pose a problem before teaching the method EQS3: Design a task that allows all learners to contribute to the learning and have room for extension EQS4: Make opportunities for students to authentically share their thinking with peers EQS5: Add a visual component EQS6: Add the requirement to convince and reason

The first criterion (QFK), in this context refers to the effective use of visuals to enhance “deep and new understandings” (Boaler et al., 2016, p. 1) of fractions. QFK refers to knowledge that can change the way learners perceive fractions. This component was therefore used to determine the quality and relevance of the visuals used by participants. The second aspect (FM) refers to the different ways of representing fractions, that is the area, length and set models. The three FMs provide different opportunities for enhancing the conceptual understanding of fractions. As a component of the analytical framework, this aspect was used to establish the variability of models used by the participants. The visuality of lessons (VL), criterion 3, refers to how frequently visual representations are used in fraction lessons. Presmeg (1986, p. 42) defines mathematical visuality as the extent to which that person prefers to use visual methods when attempting mathematical problems which may be solved by both visual and nonvisual methods.” This was a crucial aspect in determining the extent to which the RCFP had impacted the pedagogy of the participants.

The fourth criterion, VNVC, was drawn from the DCT which advocates for the use of both verbal and nonverbal codes, that is, words or symbols and visuals. This aspect was deliberately included in the analytical framework to establish the teaching orientation of the participants in terms of the two codes. QT, criterion five, refers to the conception of the relative size of unit fractions. This is one of the aspects that learners grapple with as they often misapply whole number knowledge to fractions. The incorporation of visualisation processes can enhance learners’ ability to understand fraction size and equivalent fractions, and order and compare fractions (Bezuk & Cramer, 1989). Consequently, it was deemed necessary to establish whether the participants addressed this concept in their classrooms or not.

Criterion six, PCK and CK, formed part of the analytical framework because the pedagogical and content knowledge of the participants were among the key aspects of this study. The choices that teachers make in terms of content delivery are mainly determined by their PCK and CK. Shulman (1986, p. 9) defines PCK as "the ways of representing and formulating the subject matter that makes it comprehensible to others" while CK is the type of knowledge explicitly associated with knowledge of the subject matter. This aspect of the analytical framework was deemed necessary because it was vital in determining the participants' pedagogical choices in terms of the verbal or non-verbal codes. However, the focus was more on the incorporation of visualisation in teaching fractions and this was deliberately done to address the research questions. Essentially, all the research questions were based on the pedagogy of the participants, hence, this was an indispensable component of the analytical framework for this study.

Lastly, the criterion EQS was included, since effective questioning strategies promote the creation of mental images in the process of finding a solution to a problem, which was deemed necessary to reflect on the type of questions that participants presented to learners. As alluded to by McCarthy, et al (2016, p. 80), "the use of good questioning, by teachers, may mean the difference between constraining thinking and encouraging new ideas, and between recalling trivial facts and constructing meaning." In a study of this nature, it was imperative to pay attention to the types of questions posed by the participants in their lessons because that formed part of the pedagogy of the teachers. Thus, the type of questions posed by the teachers may advance the verbal or nonverbal code.

3.11 DATA ANALYSIS PROCESS

Since this study was based on empirical data, I took some time to deeply engage with the raw data as a whole and placed it into manageable categories (open coding) which was followed by establishing links between the categories (axial coding). The data was analysed quantitatively and qualitatively. As stated by Goundar (2012, p. 29-30), "empirical evidence (the record of one's direct observations or experiences) can be analyzed quantitatively or qualitatively." Moreover, selective coding was used to identify the "core category" (Kawulich, 2004, p. 99). Strauss and Corbin (1990) define the core category as the key theme underpinning the subcategories, which in this case, is visualisation. In analysing the data, the constructivist theory and the Dual Coding were instrumental. These are the two theories underpinning this study, from which the seven pedagogical concepts used in the analytical framework were derived. According to Kawulich (2004, p. 99), "theory driven coding begins with the researcher's theory of what occurs and the formulation of the indicators of evidence that would support the theory." Thus, data were coded according to predetermined pedagogical concepts from the analytical framework and key questions from the interviews and the questionnaire.

For this study, data from the questionnaire, interviews and lesson observations were analysed across each participant. This was necessitated by the need to establish synergies or tensions across a single participant because according to Kawulich (2009, p. 99), this allows one to make "comparisons for similarities and differences between incidents, events or other phenomena." Presenting data in this manner, made provision for an in depth understanding of each participant's position on the use of visualisation by analysing the sets of data obtained at different stages of the study. Quantitative data was analysed numerically in table form.

I began the data analysis process by employing Bruscia's (2005) six stages of preliminary processing for each case, involving: 1) making a sense of the whole, 2) making case notes, 3) culling the raw data, 4) segmenting the data, 5) formatting the data and 6) collaborating (p. 182-183).

3.11.1 Making sense of the whole

This refers to the process of going through all sets of data for each of the participants and presenting it in such a way that it helps the reader to get a clear picture of each set of data. This was achieved by reading through and scrutinising the data emanating from the questionnaire, the observations and interviews. This entails that for each of the eight participants, I had to read all the transcribed data and responses to the questionnaire, watch all the videotapes and listen to all the audiotapes. Bruscia (2005, p. 182) asserts that "whatever the data and format, the main idea is to get an overall picture of what gives each particular set of data its coherence, meaning, character or distinctiveness." Through this process, I was able to establish the differences and similarities among the different sets of data. Understanding data in its entirety is a crucial aspect of the data analysis process as it gives the researcher a sense of direction.

3.11.2 Making case notes

Taking note of striking or interesting features of different sets of data helps the researcher to "develop preliminary ideas about what themes or irregularities may be inherent in the data" (Bruscia, 2005, p. 182). This process began during the data collection process and continued through the data analysis process. Generally, case notes not only provide a record of unique details of the data but rather they also set the context under which the data collected may be analysed.

3.11.3 Culling the raw data

Although large amounts of data were collected during the data collection process, not all of it was useful. Hence, the need to filter it in order to discard redundant or irrelevant data. This process involved reviewing, sorting and condensing data to useful and manageable quantities that contributed immensely to my participants' stories. Redundant or irrelevant data refers to additional data that was not useful in providing insight or addressing the research questions. This type of data was prevalent in all sets of data, particularly in cases were questions were misunderstood or incorrectly answered.

3.11.4 Segmenting data

The segmentation of data refers to the categorisation or codification of data "into the most appropriate or meaningful units" (Bruscia, 2005, p. 182). For this study, the predetermined codes from the three research instruments and based on the seven aspects, were used to segment the data. The codes QFK, FM, VL, VNVC, QT, PCK and EQS, were all linked to visualisation and its role in the teaching and learning of fractions. This made the compartmentalisation and analysis of data less complex.

3.11.5 Formatting the data

The format refers to the way in which data was presented or summarised after it was segmented into the above mentioned categories. As alluded to earlier, a case-oriented approach analysis was opted for and for such cases, Bruscia (2005, p. 183) recommends the use of "a synopsis or condensation of the case." The synopsis should be a true reflection of the raw data. Therefore, it should "include actual words and phrases from the raw data. At the same time, the purpose of a synopsis is to summarise and reduce the raw data to its essential components." This was basically how I analysed my data and it was during this stage that I began to understand my data comprehensively because writing the synopsis required a deeper engagement with the segmented data.

3.11.6 Collaborating

It is important to verify the accuracy of each synopsis with the respective participants and that is what this stage entails. Therefore, upon the completion of the synopses, I went back to each of my participants and asked them to verify the accuracy of the synopsis, after which I then corrected, added or revised the data accordingly (Bruscia, 2005). This six-stage process was instrumental in authenticating the data.

3.12 VALIDITY AND RELIABILITY

Validity is an important aspect to consider in research as it validates the entire research process and its findings. While validity refers to the extent to which an instrument measures what it intends to measure, reliability is concerned with the consistency, dependability, repeatability and stability of the instrument in question (Bannigan & Watson, 2009). Thus, triangulation was achieved by the use of multiple data collection strategies. Since this study was informed by the Dual Coding Theory, it was deemed appropriate to ensure that the theoretical frameworks underpinning this study were incorporated in the design of all the research tools and this was attained by incorporating key concepts in each of these instruments.

Since this study was predominantly qualitative, it was practically impossible to completely eliminate administrative errors. Bannigan and Watson (2009, p. 3239) assert that "the premise for conducting reliability tests is that there will always be a degree of random error in the administration of measurement scales." Nonetheless, in conformity with the validity and reliability requirements, I implemented verification strategies necessary for attaining rigour and reducing variations and inconsistencies in the administration of the research instruments. In qualitative studies, this process is woven into the research process.

This was achieved by putting self-correcting measures in place to detect inconsistences, by moving back and forth through all the sections of this dissertation in order to ensure that all sections were aligned to the objectives of this study. As stated by Morse et al. (2002, p. 17) "a good qualitative researcher moves back and forth between design and implementation to ensure congruence among question formulation, literature, recruitment, data collection strategies, and analysis." Consequently, I was able to reduce the validity and reliability threats to my study as this process gave me a sense of direction as to when to "continue, stop or modify the research process in order to achieve reliability and validity and ensure rigor." Morse et al. further elaborates that verification strategies in this process include "ensuring methodological coherence, sampling sufficiency, developing a dynamic relationship between sampling, data collection and analysis, thinking theoretically, and theory development." The content, structure, scope and magnitude of my study served as the guiding principles in the attainment of validity and reliability.

Issues of validity and reliability of the study were partly addressed by the incorporation of literature and theories in the design of the research instruments. Bannigan and Watson (2009, p. 3238) suggest that "in any research study the theoretical basis of the study can serve as a sound foundation on which to build data collection and data analysis methods." This was a major consideration in the design of all the three research instruments and the analytical framework used in this study. Thus, the Dual Coding Theory and constructivism were instrumental in creating the questionnaire, interview schedule and the lesson observation schedule.

Some of the reliability issues worth mentioning in this study include the scope of participants covered in this study and the impact of COVID 19 on teaching, learning and the classroom settings. The questionnaire was administered to a limited number of respondents who belonged to the RCFP, which in itself is a limitation. However, since the RCFP was the target population for this study, the questionnaire could not be administered to teachers who were not members of the RCFP. COVID 19 was another reliability threat to this study because all the instruments were designed to be used under normal classroom settings. Nonetheless, the COVID 19 outbreak led to a number of changes in teacher and learner behaviour and the general classroom setup and routines. Despite my efforts to reduce the inconsistencies in the design of this case study, the findings cannot be generalised, considering its nature, scope and the inevitable administrative errors that are imminent in any kind of study.

3.13 ETHICAL CONSIDERATIONS

Ethical issues are very important considerations prior to the commencement of any research that involves human beings as participants (Flemming & Zegwaard, 2018). This is the beginning of standard, acceptable procedures in research. All ethical procedures were observed and clearly explained as per Rhodes University Ethical Standards Committee (RUESC) - Human Ethics (HE) sub-committee requirements. Subsequently, my ethics application was approved (see Appendix Ten) and a clearance certificate was issued to mark the beginning of my empirical study. Despite the surge in COVID 19, I decided to continue with this study because measures were put in place for all to operate under the "new normal." Considering the fact that this pandemic was unprecedented and no one knew when it was going to end, I deemed it necessary to carry on with my studies under these circumstances. Moreover, although there were several interruptions in the school activities, this was regarded as part of the "new normal" and we all had to either sink and swim.

3.13.1 Permission

Upon the receipt of the approval letter from the Human Ethics sub-committee, I wrote a letter to the Kavango East regional Director of Education seeking permission to conduct research in five (later reduced to four) selected schools. Fortunately, permission was granted (see Appendix Nine) without delay and I used this letter as a reference in seeking for permission from the school principals of the identified schools. All the principals responded positively and allowed me to conduct research at their schools.

3.13.2 Informed consent

After obtaining permissions from the regional director and the principals, I then sought consent from the actual participants. This is one of the most important ethical considerations in research because participants are directly involved in the study and the researcher should ensure that the rights, dignity and safety of participants are protected. Essentially, this defines the nature of the agreement I entered into with my research participants (Miller and Bell, 2002). Before seeking consent from the participants, I clearly explained the purpose of this study, how it was related to the RCFP and what their roles would be.

Information pertaining to data collection, handling, reporting and the potential risks involved was also shared with the participants (Flemming and Zegward, 2018). In addition, I also encouraged voluntary participation and explained that they could withdraw from the study at any stage of the research. I went through the consent form and explained the different sections of this form step by step. Thereafter, participants were given a week to complete the questionnaire, after which I collected them from their respective schools. However, those who completed this exercise during the RCFP meeting were allowed to submit their questionnaires immediately.

3.14 CHALLENGES ENCOUNTERED

Firstly, the university delay in the approval of my ethics proposal was a major setback in terms of the commencement of the data collection process, since all the planned activities came to a halt. This had some implications on my data collection schedule because the concept of fractions is only taught in the first trimester of the school calendar. Therefore, by the time I received the approval letter in August 2019, all my participants indicated that they had already taught this concept as per their scheme of work. As a result, I had to wait for the first trimester of 2020 to start the data collection process. Sadly, the whole of 2019 was wasted.

Secondly, my role as an insider researcher was another limitation to this study since I was involved in the design and facilitation of activities in the RCFP which were predominantly visual. This was an important consideration because despite my attempts to maintain a neutral position in the use of the verbal and nonverbal codes, my preconceptions and biases in the use of these two codes might have had a direct or indirect impact on the behaviour or responses of the participants during the research. Rose (1985, p. 77) asserts that one cannot claim to be neutral in qualitative research. "There is no neutrality. There is only greater or less awareness of one's biases. And if you do not appreciate the force of what you're leaving out, you are not fully in command of what you're doing." Therefore, although the verbal and nonverbal codes were used concurrently in some instances, the focus was mainly on the incorporation of visual strategies in the teaching of fractions, thus, this could have influenced the participants' pedagogical choices.

According to Dwyer and Buckle (2009), the insider role accords the researcher opportunities to get to engage with the participants and in the process, earn their trust. Since the RCFP had been in existence prior to the data collection process, a good working relationship had already been established between the researcher and the participants. Therefore, to some extent, the participants were free to express their opinions on the concept of visualisation despite my insider role. This was evident in their responses to the questionnaire, the interview questions and the teaching strategies that they used in their lessons. In other words, despite my role in this study, the participants did not feel compelled to do things my way. For instance, in lessons where visualisation strategies were not used, the participants openly explained why they could not use them. The DCT was another mitigating factor because it advocates for the use of both verbal and non-verbal codes. Hence, the participants did not feel 'unqualified or underrated' for choosing to use either the verbal or the nonverbal code.

Thirdly, conducting classroom observations under the novel COVID-19 pandemic can be described as challenging, risky, scary and unpredictable. Consequently, the second group of participants that was observed under the 'new normal' had to adapt to a lot of changes, which included reducing the number of learners per class to maintain social distancing, the compulsory wearing of masks, regular handwashing and applying sanitisers. The reduction of learners per class resulted in the adoption of new school routines by most public schools to accommodate all the learners by alternating school days or weeks. In other words, the schools placed learners in groups, and these groups attended school on different days or weeks depending on the arrangement of the school. This affected the schedule of my observations and I was compelled to adjust my timeline.

Thirdly, the ‘new normal’ slightly affected the design of my research because adjustments were inevitable. For instance, in my document analysis, I had planned to analyse the prevalence of visuals in the learners’ exercise books, however, this was one of the risky activities to engage in at the time. Hence, even teachers were advised to handle learners’ books with care because of the high risk of contracting COVID 19. Another related challenge was the impact of regulations such as social distancing on pedagogical matters, since participants could not move close to their learners due to the fear of COVID 19.

Despite the challenges experienced, the quality of the data collected was not compromised because the focus was on the incorporation of visualisation processes by the teachers (or participants) and not the learners. Learners’ books were replaced by other equally important documents such as textbooks which also depicted the visuals that learners were exposed to, since the participants relied on the textbooks for assessment activities such as homework or topic tasks.

3.15 CONCLUSION

In this chapter, I presented the practical aspect of my research journey in a nutshell. I explained the methodological choices that I made in terms of the research methods, orientation of the study, sampling methods, research instruments, the data analysis process, validity and reliability, ethical considerations and the challenges encountered. Most importantly, I also tried to justify the choices I made by demonstrating how these choices fit into the bigger picture and how the decisions I made were mainly informed by the theories underpinning this study.

It is important to be mindful that this study was conducted under very difficult circumstances due to the COVID 19 pandemic. However, despite the predicaments encountered in my research journey, I tried to ensure that standard research procedures were not compromised. For instance, in cases where I could not meet all my participants as a group, I resorted to meeting them individually or in pairs to ensure that they were assisted and kept abreast with the developments and progress of the research.

Moreover, the development of the RCFP manual helped to fill some of the gaps in the research design brought about by the novel COVID 19 pandemic. The RCFP manual was designed in such a way that the users found it interesting, useful and easy to follow, due to the incorporation of both the traditional, symbolic strategies and the visual strategies. Therefore, even in my absence, the participants were able to navigate through this manual and find alternative strategies for teaching fractions. Although the RCFP manual was developed before the outbreak of COVID 19, it was useful in addressing some of the challenges that emanated from COVID 19.

This chapter shows how I consciously navigated through the ‘new normal’ by modifying a few aspects of my empirical research, while trying to keep the changes minimal by ensuring that that I did not divert from the set goals. Although the COVID situation was beyond my control, I tried to minimise its impact on the outcome of my study. The data analysis and findings are discussed in the next chapter.

CHAPTER FOUR
DATA ANALYSIS

4.1 INTRODUCTION

In this chapter, I provide an analysis of the data collected from the questionnaire and the lesson observations, using a directed approach. The directed approach is described as a process that begins by using existing theory or prior research to identify key concepts or variables as initial coding categories. “The goal of a directed approach to content analysis is to validate or extend conceptually a theoretical framework or theory” (Hsieh & Shannon, 2005, p. 1281). For this study, a predetermined theoretical framework, based on pedagogical aspects, visualisation and conceptual understanding of fractions was used. These concepts were extracted from different literature sources and theories based on their relevance to this study. Seven key concepts, namely Quality fraction Knowledge (QFK), Fraction Models (FM), Visuality of fraction lessons (VL), Verbal and Nonverbal Codes (VNVC), Quantitative Thinking (QT), Pedagogical Content Knowledge (PCK) and Effective Questioning Skills (EQS) were identified as relevant components for this study. Moreover, these concepts were also instrumental in answering my research questions.

4.2 SUMMARY OF FINDINGS FROM THE QUESTIONNAIRE

Table 4.1 below provides a summary of the findings from the questionnaire. It clearly shows that despite the discrepancies in the participants’ views pertaining to the other aspects of the analytical framework, they all held the view that the use of both the verbal and nonverbal codes (VNVC) was vital and that their participation in the RCFP had a positive impact on their teaching. Attendance during the RCFP meetings was an important criterion in the selection of participants. Thus, only candidates who had attended at least five meetings were eligible for this study. This was mainly because I intended to select participants who had been exposed to the activities of the RCFP – since this was a major consideration in this study.

Table 4.1: Summary of findings from the questionnaire

Respondents	Gender	Level	Verbal/non verbal/both	Fraction concepts that require visuals	Fraction models used	Impact of RCFP on pedagogy	Appropriate phase for visualisation
Teacher D	Male	Senior primary	Both	Four basic operations	Area	Good	All
Teacher K	Male	Senior primary	Both	Equivalent fractions and the four basic operations	Area Set	Good	All
Teacher M1	Male	Senior primary	Both	All	Area Length Set	Good	Junior primary
Teacher S1	Female	Senior primary	Both	Unit fractions and equivalent fractions	Length	Good	Junior and senior primary
Teacher H	Female	Senior primary	Both	All	Area	Good	Junior primary
Teacher M2	Female	Senior primary	Both	All	Area Length Set	Good	Senior primary
Teacher I	Female	Senior primary	Both	All	Area	Good	All
Teacher S2	Male	Senior primary	Both	All	length	Good	All

Quality Fraction Knowledge (QFK) is achieved when teachers use both visuals and numerical notation to enhance deep and new understandings of fractions (Siegler, 2010). This is important because learners often come to mathematics classrooms with flawed views of the concept of fractions. Out of the thirty questionnaires initially administered, only twenty-one were returned and out of the twenty-one respondents, fifteen (71%) indicated that they would use visuals to complement the symbolic mode. The remaining (29%) opted for either the verbal or the nonverbal code. From the questionnaire, questions 2, 5, 6 and 7 were deliberately included to determine the respondents' stance on the concept of QFK. indicated that they would use visuals to complement symbolic notation (numbers) particularly for struggling learners, while

The use of visuals was an important consideration in the selection of participants because of the visual aspect of this study. Question 2 was based on determining the frequency at which the respondents used visuals in their lessons. In response to this question, all eight out of the twenty-one respondents (38%) indicated that they used visuals 'sometimes', depending on the need to use them. Participants used phrases such as *not so often, I don't often use them, it* *depends on the topic, most of the time, sometimes, mostly and not very often,* to describe the frequency at which they used visuals. Based on the participants' responses, it was evident that their use of visuals was circumstantial as they associated the use of visuals to the nature of topic (or content) and learners' level of competency. Among the remaining respondents (32%), some indicated that they do not use visuals at all while others (40%) did not fully understand the question as they referred to specific teaching aids such as concrete teaching aids, counters, abacus, fraction wall, etcetera.

Another important indicator of the visuality of lessons (VL) was the significance attached to the incorporation of visualisation in fraction lessons (question 5). The importance of visualisation in the pedagogy of fractions is at the core of this study. In answering this question, respondents posited that visuals provide opportunities for learners to participate, remember, visualise (see), clarify misconceptions and gain a deeper understanding of fractions. Visuals are presumed to arouse learners' interest and increase their participation in mathematics. For instance, one of the respondents explained that visuals *motivate most learners to participate. Learners learn best by observing or touching and through games.* The retention of knowledge is a fundamental consideration in the mastery of mathematics concepts as alluded to by some of the respondents.

Besides the visuality of lessons, it was necessary to find out whether respondents saw the need to use visualisation processes in the different sub concepts of fractions including fraction size, equivalent fractions and fraction computation. Thus, question 6 was set to gauge the participants' views on the use of visualisation as a relevant concept for fractions in general (Zhang, 2018; Barnett, 2016; Van de Walle et al., 2013; Bruce, 2013). Twelve of the respondents (57%) indicated that visualisation should be an integral component of all the sub concepts of fractions. While acknowledging the need to incorporate visuals in the pedagogy of fractions, some pointed out that they do not use them regularly, especially when teaching the four basic operations.

The use of different fraction models (FM) to enhance the pedagogy of fractions as recommended by several authors (Ervin, 2018; Dyson et al. 2016; Van de Walle et al., 2013; Bruce et al., 2013; Wu, 2011, Cramer & Whitney, 2010; Siegler et al., 2010 & Cramer et al., 2008) is a vital component of the analytical framework. The findings from the questionnaire indicate that the area model was predominantly opted for as the preferred model for teaching fractions and the reason for this choice was based on the view that area models are easy to visualise, draw and explain. Two of the respondents selected the length model. Although they explained how the length model enhances learners' understanding, none of them alluded to its important role in helping learners see fractions as numbers that expand the number system.

Although literature suggests that the use of all three models is vital in the pedagogy of fractions (Van de Walle et al., 2013; Bruce et al., 2013; Wu, 2011, Cramer & Whitney, 2010; Siegler et al., 2010 & Cramer et al., 2008), only two of the respondents (9%) backed this view. The respondents attributed the importance of using a variety of models to the learning opportunities they present, based on the visual nature of the models.

Only one respondent (5%) opted for the combination of the set model and the area model as their preferred models. According to Hull (2005, p. 23), the set model "requires students to understand that a group of objects is considered the whole and the individual objects would be subsets or parts of the whole." The respondent explained that the two models (area and set) are easy to use and understand. The participants' responses failed to elaborate how the use of preferred models could enhance learners' understanding of fractions. Their responses appeared to lend themselves towards the importance of visualisation rather than the importance of using the selected models.

Based on these findings, ten participants were quantitatively selected for this study, of whom two withdrew. The underlying criteria for all the selected participants was based on two aspects, that is, the importance attached to the use of verbal and nonverbal codes (VNVC) and their participation in the RCFP. Based on the theoretical framework of this study, it was necessary to include participants who understood the importance of the DCT and implemented it in their classrooms. Although the focus of this study was on the incorporation of visualisation processes in the pedagogy of fractions, this did not any way imply that the symbolic or verbal code would be discarded. On the contrary, my intention was to determine how the participants incorporated visual strategies into the predominantly verbal fraction pedagogy in order to enhance learners' understanding of fractions. This study was located in the RCFP because all the participants had been exposed to the use of both verbal and visual strategies for teaching fractions. Consequently, this study provided an opportunity to determine the participants' use of visualisation strategies in their fraction lessons, considering their exposure to these strategies.

Additional criteria were considered in the final selection of the ten participants as indicated in Table 4.2 below. These include, the frequency at which the participants used visuals, whether they used self-made or ready-made visuals and the value attached to visualisation. Questions pertaining to these criteria were included in the questionnaire (see appendix one, p. 227). In terms of the frequency of the use of visuals, the participants selected ranged from those who indicated that they did not use visuals often to those who used them very often. This selection allowed to observe and assess the use of visuals across this spectrum. The use of self-made or ready-made visuals was another important criterion since I needed to establish the involvement of the participants in the development of these visuals. As a result, only those who indicated that they would use the self-made only or both self-made and ready-made visuals were considered for this study. The value attached to the use of visualisation in teaching mathematics was an important criterion in the selection of the participants because visualisation was a key concept for this study.

Table 4.2: Additional criteria in the selection of participants

No.	**Name of participant**	**Frequency pertaining to the use of visuals**	**Self-made or ready-made visuals**	**Value attached to visualisation**	**Evidence**
1	Teacher D	Not so often,	Self-made	High	*A great way for learners to see information*
2	Teacher K	Not so often, time-consuming	Self-made	High	*Give a clear picture of the content taught*
3	Teacher M1	Most of the time	Self-made	High	*Learners learn best by observing or touching and through games*
4	Teacher S1	Sometimes	Both	High	*Learners may gain a deeper understanding of fractions*
5	Teacher H	Mostly when introducing the topic	Both	High	*Enhances learners' understanding…*
6	Teacher M2	Not very often. It depends on the topic	Self-made	High	*Learners will not forget what they have seen…*
7	Teacher I	Sometimes	Self-made	High	*Learners will be able to remember it easily when they see…*
8	Teacher S2	Depends on the topic	Both	High	*Enhance learners' understanding and recall the content taught*

Table 4.2 presents data pertaining to the use of visuals (frequency), type of visuals and the value attached to visualisation. Out of the eight selected participants, only one participant (participant 3) indicated that he used visualisation most of the time while the rest indicated that they used it inconsistently. However, all the selected participants seemed to agree that the use of visualisation is important in the pedagogy of fractions as portrayed in their justification of

the importance of visualisation. They all seemed to concur that visualisation enhances learners' conceptual understanding mainly because of its long-lasting effect on their memory.

Despite the relevance of the data collected through questionnaire, the findings are not conclusive. For more insight into the pedagogy of the participants, additional data was collected through lesson observations.

4.3 SUMMARY OF FINDINGS FROM LESSON OBSERVATIONS

Data from the observed lessons was analysed using the components of the analytical framework. Each lesson was analysed based on the key components portrayed by the participants and how these affected the whole lesson. Therefore, for each lesson, a brief overview is provided and this is followed by the summary of the lesson. The key components of the analytical framework featuring in the lessons are discussed in the first part (the overview) while the lesson summary presents supporting evidence of the identified components.

From a pedagogical perspective, the use of both standard algorithms and visualisation processes was observed because for each participant, at least one of the observed lessons involved the use of visualisation and in some cases all the lessons included visual methods. However, for most of the participants the challenge lay in their inability to select and prepare relevant visuals for enhancing learners' understanding of fraction size and fraction computation. This was evident in their lack of concern for equal partitioning in their diagrams and the fact that they deliberately avoided the use of visuals as an alternative strategy for adding and subtracting fractions with different denominators. Visual strategies were only observed in the addition and subtraction of fractions with the same denominators while rules were enforced for the addition and subtraction of fractions with different denominators. Moreover, most of the participants relied on chalkboard visuals which were drawn during the lesson rather than ready-made visual materials. Although chalkboard visuals are acceptable and allow learners to experience the process involved in drawing fraction visuals, the overuse of these diagrams can deprive learners of opportunities to see quality, accurate and relevant fraction diagrams. In one exceptional case, one of the participants used concrete teaching aids (oranges) to demonstrate fraction size and learners found this lesson very interesting.

In terms of preference, both teachers and learners preferred the conventional, rule-oriented symbolic notation as opposed to visualisation. Although the participants tried to balance the two, most of their lessons were predominantly symbolic. Visualisation was perceived as a ‘support strategy’ to enhance learners’ understanding of the abstract fraction concepts. In some instances, visualisation took centre stage and this mainly occurred in Grade four which is the first grade at the senior primary phase. Typically, the topics taught in this grade also lent themselves towards visualisation. For instance, Teacher H was compelled to incorporate visualisation in her lesson on the topic ‘fraction of a shape’. Thus, the use of visualisation in this lesson was inevitable. The following table presents a summary of the twenty-five observed lessons, focusing on the lessons taught, fraction models used and the grades taught.

Table 4.3: Fraction concepts presented by the participants

No.	Participant	Topic	Fraction Models	Grade
1	Teacher D	Multiplying fractions	Area and number line	7
		Dividing fractions	Area	7
		Homework corrections	None	7
2	Teacher K	Multiplying fractions	Set	6
		Fractional parts and quantities	Set	6
		Comparing and ordering fractions	None	6
3	Teacher M1	Fraction size and comparing fractions	Area and number line	7
		- Reading fractions - Adding fractions with the same denominators	Number line	7
		Adding and subtracting fractions with different denominators	None	7
		Adding and subtracting fractions with different denominators	None	7
4	Teacher S1	Adding and subtracting fractions with different denominators	Area-partly	7
		Adding and subtracting mixed numbers	None	7
		Adding and subtracting mixed numbers	None	7
5	Teacher H	Fraction of a shape	Area	4
		Comparing fractions	Area	4
		Comparing fractions with different denominators	None	6
		Ordering fractions	Area	4
6	Teacher M2	Multiplication of fractions	None	6
		Fraction part and word problems	None	6
		Addition and subtraction of fractions	Area and number line	6

7	Teacher I	Fractional part of a quantity	None	6
		Comparing fractions	Area and number line	6
8	Teacher S2	Comparing and ordering fractions	Area	4
		Fraction representation and notation	Area	4
		Calculations and word problems	Area and set model	4

As indicated in Table 4.3, the participants were given the prerogative to present topics as dictated by their syllabi. However, some of the participants taught similar topics. Therefore, in analysing these lessons I clustered them based on the topics presented, for easy identification of the differences and similarities in the participants' approaches. Table 4.4 below shows how the topics were clustered.

Table 4.4: Clusters of lessons taught

Participant	Fraction concept taught	Fraction Models used	Grade
Topic 1	**Multiplication**		
Teacher D	Multiplying fractions	Area and number line	7
Teacher K	Multiplying fractions	Set	6
Teacher M2	Multiplying fractions	None	6
Topic 2	**Division**		
Teacher D	Dividing fractions	Area	7
Topic 3	**Fractional parts**		
Teacher K	Fractional parts and quantities	Set	6
Teacher H	Fraction of a shape	Area	4
Teacher M2	Fraction part and word problems	None	6
Teacher I	Fractional part of a quantity	None	6
Topic 4	**Comparison**		
Teacher K	Comparing and ordering fractions	None	6
Teacher M1	Fraction size and comparing fractions	Area and number line	7
Teacher H	Comparing fractions	Area	4
Teacher H	Comparing fractions with different denominators	None	6
Teacher H	Ordering fractions	Area	4
Teacher I	Comparing fractions	Area and number line	7
Teacher S2	Comparing and ordering fractions	Area	4
Topic 5	**Addition**		
Teacher M1	Reading fractions	Number line	7
Teacher M2	Addition and subtraction of fractions	Area and number line	6
Teacher M1	Adding and subtracting fractions with different denominators	None	7

Teacher S1	Adding and subtracting fractions with different denominators	Area-partly	7
Teacher S1	Adding and subtracting mixed numbers	None	7
Topic 6	**Fraction representation**		
Teacher S2	Fraction representation and notation	Area	4
Topic 7	**Word problems and assessment**		
Teacher S2	Calculations and word problems	Area and set model	4
Teacher D	Assessment activities (homework)	None	7

4.3.1. Topic 1: Multiplication of fractions

Multiplication of fractions is one of the concepts that learners grapple with because of the nature of fractions. The transition from whole number multiplication to fraction multiplication can be very confusing if not carefully handled. This is mainly attributed to the different definitions attached to the concept of multiplication in these two contexts, that is, whole number multiplication and fraction multiplication. While the multiplication of whole numbers gives rise to bigger numbers as products, the multiplication of fractions produces smaller numbers as products.

Since learners' first encounter with multiplication involves whole numbers, they find the multiplication of fractions quite challenging (Feil, 2010). It is therefore vital to choose teaching strategies carefully to minimise potential misconceptions. In this context, three of the eight participants presented a topic on the multiplication of fractions. In the section below, I present the findings from the three participants focusing on the strategies used to teach multiplication of fractions.

Teacher D: Multiplying fractions (Grade 7)

Generally, all components, particularly QFK, FM, VL, VNVC and PCK of the analytical framework, were attained in Teacher D's lesson, as he switched from the area model to the number line model to enhance learners' understanding. The incorporation of the area model and the number line model to demonstrate the multiplication of fractions was an indicator of QFK. Although all the fraction models are important, Dyson et al. (2016, p. 135) argue that "understanding fractions as magnitudes that can be represented on a number line provides an underlying structure for learning a range of fraction concepts and skills." Moreover, it was interesting to observe how Teacher D managed to move learners from a point where they had no idea what a number line was, to a point where they were able to effectively use it to solve multiplication sums. This was definitely an indicator of good PCK. As stated by Star and

Stylianides (2013), the level at which a teacher understands the content and the pedagogy of a concept can determine the effectiveness of their lesson.

The use of different fraction models (FMs) presents learners with different opportunities to learn and to understand fractions meaningfully (Van de Walle et al., 2013). Although Teacher D did not use all the three models, at least he used two – the area model and the number line model. This means that a teacher's shortcomings in the use of one model (for example the area model) can be compensated for by another model. This was observed in Teacher D's lesson, because initially, he used the area model which some learners found challenging. However, when he switched to the number line model, learners found it quite interesting and easy to use. Presmeg (1986, p. 298), defines a teacher's mathematical visuality as "the extent to which a person prefers to use visual methods when attempting mathematical problems which may be solved by both visual and nonvisual methods." This in turn affects the visuality of lessons (VL) which refers to the presence of visuals in a mathematics lesson. Therefore, this lesson can be regarded as a visual lesson because of the deliberate attempt on the part of the teacher to incorporate visuals.

Teacher D introduced his lesson on multiplying fractions by relating the concept of multiplication to repeated addition. Although this helped learners to contextualise the "new" concept, Teacher D did not advance this relationship in his lesson development as he continued to remind learners about the multiplication rules for fractions.

Teacher D: Ok what happens in multiplication? What rule?

Learner LT: We multiply them the way they are?

Teacher D: How? Use this example $\frac{2}{3} x \frac{4}{5}$

Multiplication of fractions is one of the problematic areas in fractions since learners grapple with the idea of a smaller product obtained after multiplying two fractions. Based on their whole number knowledge, to multiply means to make more. Consequently, the product (answer) obtained after multiplying two fractions rarely makes sense to learners and as such the use of visuals could enhance learners' understanding of the product of two or more fractions.

According to Van de Walle et al. (2013, p. 345), fluency in fraction multiplication is attained when "a student cannot only do the algorithm but also model problems, estimate and, solve situations that involve multiplications." Although Teacher D used diagrams to illustrate the process of multiplying fractions, emphasis was placed on the how to draw and shade the diagrams rather than on the underlying concepts. Below, Figure 4.1 shows Teacher D's diagrams on the chalkboard.

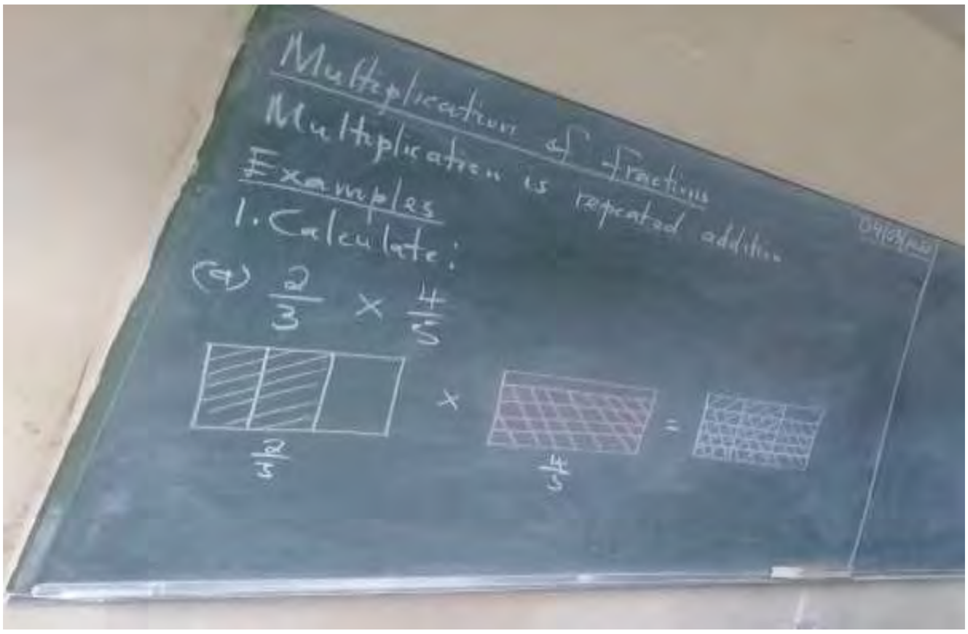


Figure 4.1: Teacher D's illustration of fraction multiplication using the area model

Figure 4.1 shows how Teacher D used a strategy involved the overlapping of the two fraction representations to find the shared space (de Castro, 2008). The use of diagrams is supposed to help learners understand the difference between the multiplication of fractions and whole number multiplication. This is important because the multiplication of fractions is not additive and learners find this difficult to comprehend. In other words, unlike whole numbers, the multiplication of two proper fractions produces an answer that is smaller than the two fractions involved in the computation. For example, when multiplying $\frac{1}{2}$ by $\frac{1}{4}$, the product is $\frac{1}{8}$ which is smaller than the multiplier and the multiplicand.

Thus, emphasis should be placed on the meaning of the multiplication sign between the two fractions, that is, 'of'. In this context, the multiplication sign should be read as 'of' to help learners make sense of the visualisation process. When a half times a quarter is read as 'a half of a quarter,' the (smaller) product obtained can be put into context and as a result, this can help learners understand the difference between whole number multiplication and fraction multiplication. In this example, learners should already be able to predict that the answer will be smaller because having a visual image of a quarter (in form of an area or length model) and halving that quarter helps learners to contextualise the problem. Instead of introducing the visual method as an alternative and equally important approach to the traditional, rule-orientated method, Teacher D presented it as a way of verifying the answers obtained using standard algorithms. This created an impression among the learners that the traditional method was the main method while the visual method was less mathematical, less significant and optional.

Although some learners did not understand the visual methods, they were fascinated by the fact that the visual method could produce correct answers. When Teacher D introduced the number line for learners to work out 4 x $\frac{3}{4}$, Grade seven learners had no idea what a number line was since they failed to identify it. Surprisingly, the concept of a number line seemed new to the learners as some referred to it as a line scape, a straight line, a bar graph, a linear scale and a space line. After several failed attempts to name it, the teacher had to tell them what it is called. The unfamiliarity with number lines shows just how important it is to establish common representations that go across learners' school experience. Teacher D partitioned the number line into quarters (from one quarter to twelve quarters). He demonstrated how to multiply a whole number by a fraction using arrows as illustrated below.

Figure 4.2: An example of multiplying fractions using the number line (Teacher D)

This figure was an illustration of fraction multiplication using the number line model with the arrows showing the number of times that a fraction is multiplied. It shows how a fraction can be multiplied by a whole number (4 x $\frac{3}{4}$). Figure 4.3 provides details of what Teacher D was trying to demonstrate in the picture above.

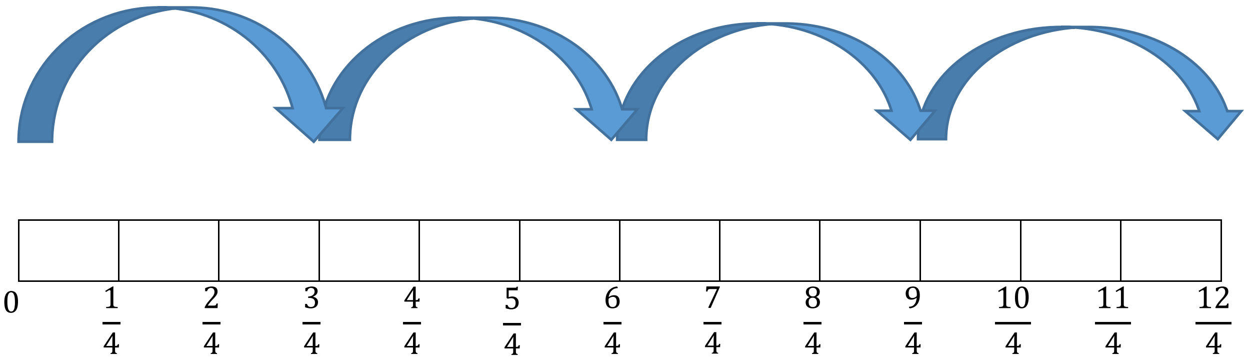


Figure 4.3: A representation of 4 x $\frac{3}{4}$ on a number line (Teacher D)

Teacher D showed learners how the arrows represented multiplication: moving the arrow once meant $\frac{3}{4}$ x 1, moving it twice meant $\frac{3}{4}$ x 2 and so on. In spite of the learners' initial inability to identify a number line they found its use a lot easier than the area model that was used earlier. Thus, learners effortlessly figured out the answer to 4 x $\frac{3}{4}$ after their exposure to the number line model.

Teacher K: Multiplication of fractions (Grade 6)

This lesson was characterised by the dominance of rules and standard algorithms. However, in spite of Teacher K's emphasis on multiplication rules, QFK indicators such as QFK1, QFK3 and QFK4 were prevalent in his lesson delivery. He demonstrated an adequate understanding of the content and procedures required to effectively teach this topic. QFK2 is associated with the use of different fraction models and this was not observed in this lesson. However, it was interesting to observe how Teacher K used the set model to address various misconceptions related to the representation and multiplication of fractions. It was therefore the teacher's CK and PCK that activated QFK in this lesson. Fazio and Siegler (2011) assert that teachers' solid conceptual understanding of fractions and their knowledge of learners' common misconceptions are important in improving the quality and pedagogy of fractions lessons. Other components of the analytical framework that were prevalent in this lesson include VNVC, QT and EQS.

Teacher K introduced the new topic on multiplication of fractions, by illustrating how multiplication in general is the same as repeated addition (applicable when the multiple is an integer). For instance, Teacher K explained how 4 x $\frac{3}{5}$ can be expressed as, $\frac{3}{5}+\frac{3}{5}+\frac{3}{5}+\frac{3}{5}$.

This four multiplied by three out of five, this simply means that this three out of five, we are going to add it how many times?

By representing 4 x $\frac{3}{5}$ as $\frac{3}{5} + \frac{3}{5} + \frac{3}{5} + \frac{3}{5} = \frac{12}{5} = 2\frac{2}{5}$, Teacher K attempted to draw on learners' prior knowledge of fraction addition to help them understand the new topic on multiplication of fractions. Furthermore, Teacher K tried to link the two basic operations by asking learners relevant questions to put the problems into context. For instance, the teacher asked the learners how many times three fifths should be added to get the same answer as 4 x $\frac{3}{5}$.

So, if you are seeing four multiplied by three, this means that this three out of five is going to be added how many times?

Furthermore, Teacher K provided a detailed explanation of the set model. He used visualisation to demonstrate how one quarter could represent five objects by using the following diagram:

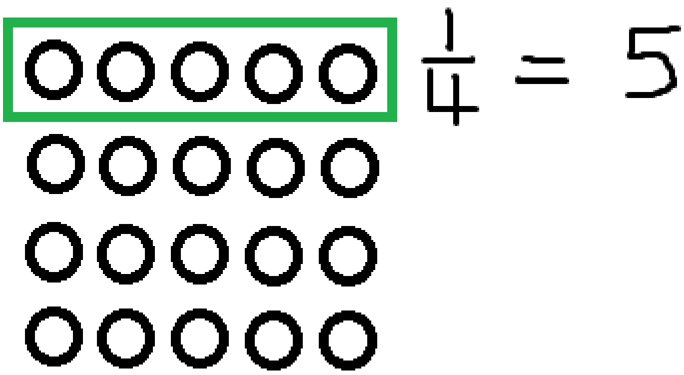


Figure 4.4: A representation of a $\frac{1}{4}$ using the set model (Teacher K)

The same diagram (Figure 4.4) was used to explain the concept of a whole in terms of the set model, as Teacher K demonstrated how the four rows can be added to get a whole as illustrated below:

$\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{4}{4}$

$5 + 5 + 5 + 5 = 20$

Moreover, Teacher K drew learners' attention to the fact that the denominator determines the number of times that a unit fraction should be added to get a whole. In his attempt to improve learners' conceptual understanding of fractions, Teacher K designed his questions in such a way that the questions addressed different aspects of fraction problems. In some cases, learners were asked to find the fraction while in other cases the fraction was given and learners were asked to find the number represented by the fraction. For instance, in the following example, learners were expected to find the total number of objects in a set if half of the set was equal to five.

Calculate the total number of beads if:

a) *Half of the beads is = 5*

In the second example, learners were asked to find the total number of beads if one third of the beads represented four beads.

One out of three of the beads is equal to four, calculate the total number of beads

This confused some learners because they assumed that they were being asked to find the missing fraction to make a whole and as such they were quick to state that the answer was two thirds. Other answers given included three out of three, two out of six, sixteen, twelve and eight. Hence, Teacher K had to remind the learners what they were expected to do.

I'm asking for the total number of beads…

Teacher K reiterated that they were required to find the total number of beads, not a fraction. In order to enhance learners' understanding of the process, Teacher K asked learners questions that helped them to understand the fractions that made a whole and what each of those fractions represented.

Effective Questioning Strategies (EQS) undoubtedly came out as one of strengths of this lesson as Teacher K strived to attain QFK by not only providing the right answers but also enhancing learners' conceptual understanding of the process involved in solving multiplication problems. Substantially, the weaknesses of this lesson can be attributed to two of the components that form part of the analytical framework, that is Fraction Models (FM) and Visuality of Lessons (VL). These are the two crucial components on which the visualisation of fraction lessons in this context, are founded. According Siegler et al. (2010), the concept of visualisation is attained when learners are allowed to experience fractions across different FMs, namely area, length and set models. Similarly, the VL was disregarded as Teacher K preferred to use more non-visual methods rather than visual methods. As stated by Presmeg (1986), one's mathematical visuality is the extent to which one prefers to use visual methods when solving mathematical problems which may be solved through both visual and non-visual methods.

Teacher M2: Multiplication of fractions (Grade 6)

The overuse of rules characterised this lesson and no reference was made to any visuals. This was a perfect example of a purely traditional, rule-based lesson in which multiplication rules were religiously adhered to. Although Teacher M2 included word problems that were visual, she missed opportunities to capitalise on these problems and include visuals in her teaching. For instance, the third example which read; *Rosa, Erica and James milked three cows. They got* $\frac{5}{6}$ *of a bucket full each. How many bucket fulls did they get all together?*

Instead of giving learners time to visualise what was actually happening in this question and address this question visually, Teacher H immediately resorted to standard algorithms and indicated that this problem should be worked out as 3 x $\frac{5}{6}$. Although some learners understood why this problem was represented in this way, there were definitely a number of them who found this confusing. Therefore, visualisation of the cows and the five-sixth bucketfuls should have been the point of departure. Moreover, learners were not accorded the opportunity to come up with their own strategies to solve this problem. Fraction language also proved to be a challenge because five-sixths of a bucket was read as five out of six of a bucket. Correct fraction language ought to be encouraged because the way a teacher reads fractions can affect the way learners understand a given problem.

Although learners followed the multiplication rules correctly, this did not guarantee their understanding of the process and what it entailed. It is one thing to apply the rules and quite another to understand what these rules imply. The focus was more on getting the right answer regardless of the learners' level of understanding. Teacher M2 can be commended for good subject matter knowledge and the inclusion of word problems in her examples. However, her pedagogical approach posed a threat to the attainment of all the components of the analytical framework. Teacher M2's lesson is depicted below.

Basically, this lesson was characterised by multiplication rules that are applicable to the multiplication of a fraction by a whole number. The rules were stipulated as follows: 1) convert the whole number to a fraction, 2) multiply numerator by numerator and denominator by denominator, 3) simplify your answer. Teacher M2 worked out the examples to demonstrate how these rules can be applied.

$4 \text{ x } \frac{3}{5} = \frac{4}{1} \text{ x } \frac{3}{5} = \frac{12}{5} = 2\frac{2}{5}$

$\frac{1}{2} \text{ x } 9 = \frac{1}{2} \text{ x } \frac{9}{1} = \frac{9}{2} = 4\frac{1}{2}$

Simplifying fractions proved to be a challenge to learners because they were struggling to divide the product of the numerators by the product of the denominators and get the correct remainder.

Despite the discrepancies in the pedagogical methods employed by the teachers on this topic, the use of visualisation was prevalent in two of the three lessons presented. Moreover, QFK was demonstrated by the two participants who flexibly used the verbal and the nonverbal code to enhance learners' understanding. The lack of visuals in Teacher M2's lesson, coupled with inappropriate fraction language, hindered learners' conceptual understanding of fraction multiplication because learners were exposed to one flawed approach. Thus, learners were compelled to adopt the teacher's strategy amidst a number of uncertainties.

4.3.2 Topic 2: Division of fractions

Just like multiplication, division of fractions is another concept that perplexes learners because of the transition between whole number division and fraction division. Contrary to whole number division, where learners get used to the idea that the division of any two whole numbers produces a smaller quotient, the division of proper fractions produces bigger quotients . Therefore, careful planning is necessary for a smooth transition between the two concepts. This can be aided by the use of FMs which in turn help learners to visualise the process. This concept was only taught by one teacher (Teacher D) who employed both symbolic and visual methods.

Teacher D: Dividing fractions (Grade 7)

Teacher D's lesson on fraction division promoted QFK, VL, VNVC, PCK and EQS. The strength of this lesson lay in Teacher D's ability to use effective questioning strategies (EQS). Setting appropriate questions is a vital component of fraction pedagogy because of the nature of fractions. For instance, in terms of context, there is a difference between $6 \div \frac{1}{2}$ and rephrasing it to *how many halves are in six*? In addition, the contextualisation of division sums encourages the use of visual methods (Van de Walle et al., 2013). Therefore, contextualising division sums enhances learners' understanding of the process involved in working out fraction problems and making sense of the answers obtained.

Although Teacher D relied on one fraction model (the area model), aspects of QFK and VL were also observed as Teacher D made an attempt to use the verbal and the nonverbal code (VNVC) to enhance learners' conceptual understanding of this topic. Teacher D's efforts can be attributed to his ability to make appropriate pedagogical choices that could benefit his learners. A summary of Teacher D's lesson on dividing fractions is presented below.

Teacher D spent some time on revising the previous lesson. His focus was on ensuring that learners could remember the addition and multiplication rules pertaining to fractions. Surprisingly, no reference was made to the visualisation processes used in the previous lessons. There was no consistence in the use of visualisation processes from lesson one to lesson two, specifically during the recap of the previous lesson, as Teacher D completely forgot about visualisation and focused on the rules instead.

The new topic on the division of fractions was introduced by focusing on the concept of reciprocal, a concept that seemed foreign to Grade seven learners. Seemingly, this was done to prepare learners for the invert-and-multiply strategy, a strategy that is poorly understood by both teachers and learners (Lamberg & Wiest, 2015). Learners grappled with this strategy, partly because the division of fractions (as a topic) was not covered in the previous grades. Although the other basic operations were tackled in the previous grades, the division of fractions is only taught in Grade seven as per curriculum outline. Hence, this topic proved to be a challenge for both teachers and learners.

The first division example ($\frac{1}{2} \div \frac{1}{2}$) was explained using the division rules alluded to earlier, that is, to replace the division sign by multiplication and invert the second fraction. This example was solved as follows: $\frac{1}{2} \div \frac{1}{2} = \frac{1}{2} \text{ x } \frac{2}{1} = \frac{2}{2} = 1$. Learners found the use of the symbolic method quite challenging since this was the first time they engaged in such calculations. After this example, Teacher D presented the second example $\frac{1}{2} \div \frac{1}{4}$ and explained it step by step using diagrams. He used the area model, as shown in Figures 4.5 and 4.6, to demonstrate the division process.

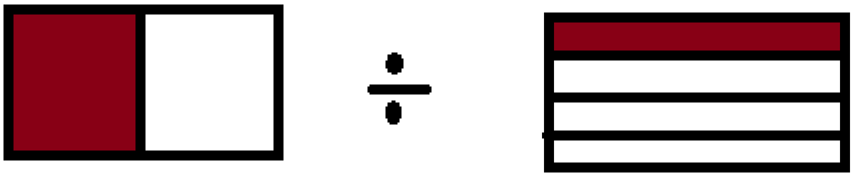


Figure 4.5: A representation of $\frac{1}{2} \div \frac{1}{4}$ (Teacher D)

Figure 4.6 below shows how Teacher D partitioned the first diagram (horizontally) into four parts and the second diagram was partitioned (vertically) into two parts.

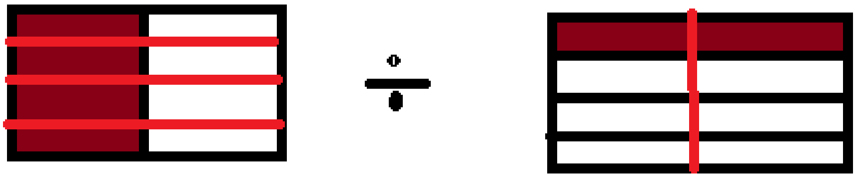


Figure 4.6: Using the area model to convert $\frac{1}{2} \div \frac{1}{4}$ to same-denominator fractions $(\frac{4}{8} \div \frac{2}{8})$

Without any explanation, Teacher D simply told the learners that the number of shaded parts in the first diagram represented the numerator while the shaded parts in the second diagram represented the denominator. This deprived learners of the opportunity to experience division in the real sense and merely followed the teacher's instructions. The same practice was also observed in the second example as Teacher D demonstrated the visual method of solving

$\frac{2}{3} \div \frac{2}{5}$.

Teacher D: Now, how many parts are shaded there? How many parts? So the first diagram will give us the numerator, the second diagram will give us the denominator. Ok I repeat, the first diagram will give us the answer as a numerator, the second diagram will give us the answer as a denominator. Are we clear?

Although the answers generated from the diagrammatic representations were correct, the learners did not understand why or how the teacher ended up getting the right answers. The teacher attempted to verify the answers by resorting to division rules which produced the same answers, however this did not help the learners to understand the visual method. Despite the fact that the teacher incorporated visuals in this lesson, the rationale behind the use of visuals did not come out clearly. This, according to McNamara and Shaughnessy (2011), can lead to partial understandings.

Although the interchangeable use of rule-oriented procedures and diagrams was explicitly demonstrated at this stage, neither method enhanced Quality Fraction Knowledge (QFK) as the deep and new understandings were not attained (Siegler, 2010). Nonetheless, the last example ($6 \div \frac{1}{2}$) was well illustrated visually since the teacher started by drawing the six wholes as shown in figure 4.7 below:

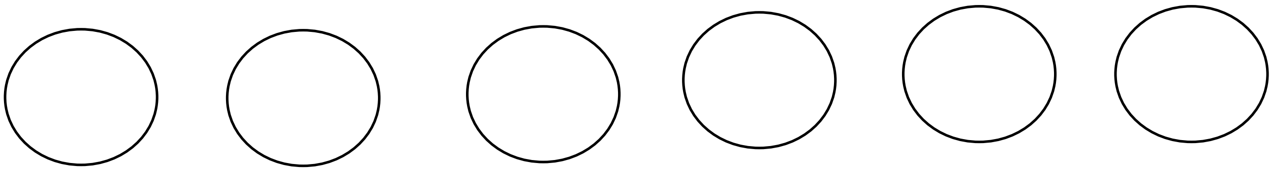


Figure 4.7: A representation of 6 wholes (Teacher D)

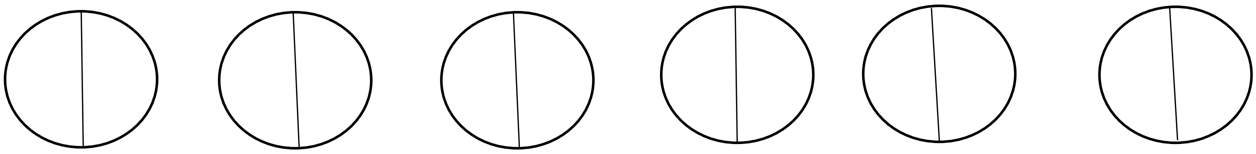


Figure 4.8: A representation of $6 \div \frac{1}{2}$ (Teacher D)

After drawing the six wholes, the teacher illustrated the halving process (Figure 4.8) where each whole was divided in half. Moreover, Teacher D actively engaged learners in the process by asking them relevant questions such as:

How many halves are in six? How many are they? So, you need to count. How many halves are in six?

Thus, the answer obtained made sense to the learners because they could see the six wholes and how they were split into half to generate twelve halves.

The notion that the use of VNVC yields positive results, depends on the teacher's ability to use both codes effectively. In this case, for example, Teacher D used both codes but learners' conceptual understanding of the division procedures was not enhanced. Therefore, the effectiveness of a lesson is not solely determined by the presence or absence of visuals. Other factors such as content knowledge (CK) and pedagogical content knowledge (PCK) are major contributors to the success of a lesson as portrayed here. However, a drastic change was observed in the last part of this lesson and this can be attributed to the appropriate integration of VNVC, PCK and EQS. Furthermore, effective questioning strategies (EQS) are essential in teaching fractions because the answers obtained – particularly the in multiplication and division of fractions – would only make sense if questions are contextualised.

4.3.3 Topic 3: Fractional parts and quantities

The essence of the concept of fractions is entrenched in this topic. Understanding the fractional parts of a quantity is a requisite to understanding fractions in general because a fraction is defined as a part of a whole. It is on this topic that the visual nature of fractions can be brought to the fore because learners ought to see the part and the whole to be able to understand fractions comprehensively (Van de Walle et al., 2013). The use of different fraction models helps learners to determine the size of any given fraction and estimate answers when presented with fraction sums.

In this study, four teachers namely Teacher K, Teacher H, Teacher M2 and Teacher I presented lessons on this topic and the extent to which understanding was enhanced varied from one teacher to the other. Out of the four teachers who presented lessons on this topic, Teacher K and Teacher H used visuals while Teacher M2 and Teacher I relied on traditional teaching methods or standard algorithms. The discrepancies in terms of pedagogy were noticeable, as the lessons in which visuals were used presented learners with better chances of understanding the concept. as opposed to the lessons in which visuals were not used.

Teacher K: Fractional parts and quantities (Grade 6)

Generally, Teacher K achieved all the components of the analytical framework except for the use of different Fraction Models (FM). This lesson was dominantly visual with aspects of non-visual methods (standard algorithms) appropriately embedded in the visual strategies. Teacher K's attempts to enhance learners' QFK were observed as he promoted learners' comprehensive understanding of fractional parts and quantities using diagrams.

The success of this lesson can also be attributed to the robust EQS employed by the teacher. The use of relevant word problems was observed from the introduction of the lesson through the lesson delivery, reinforcement to the conclusion. Another positive aspect of Teacher K's lesson was his appropriate level of PCK. It was mainly due to his PCK that he was able to achieve most of the components of the analytical framework. Shulman (1986, p. 9) summarises PCK as ways of "representing and formulating the subject matter that makes it comprehensible to others." Another vital feature of this lesson was its focus on QT. Teacher K advanced this important aspect through the use of the set model as he focused on helping learners understand the relative size of fractions while solving fraction problems visually.

This lesson was deliberately introduced with a word problem as a way of contextualising the topic. Teacher K presented the following problem:

Calculate the fraction of absent learners if 5 out of 35 learners are absent

After presenting this problem to the learners, the teacher asked them to represent the number of learners absent as a fraction of the total. Learners correctly stated that the fraction would be five out of thirty-five. At this point, Teacher K took a visual approach to this problem by using a diagram to explain the procedures involved in solving this problem. In Figure 4.9, the first column of five blocks was used to represent the number of learners who were absent while the whole diagram represented the total number of learners in the class.

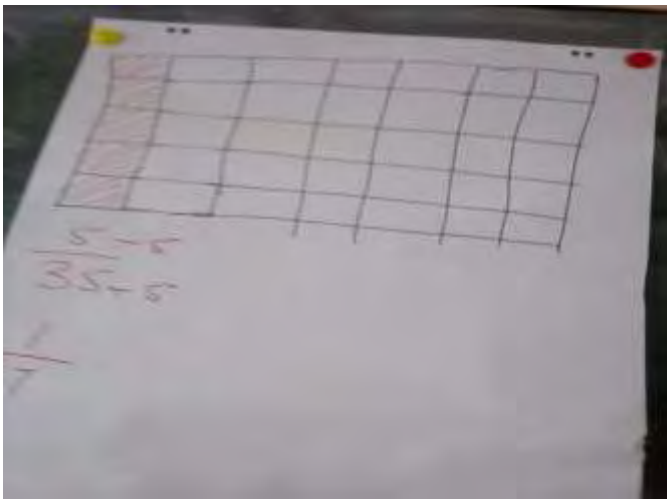


Figure 4.9: A representation of $\frac{5}{35}$ (Teacher K)

After representing the shaded blocks as $\frac{5 \div 5}{35 \div 5} = \frac{1}{7}$, Teacher K drew learners' attention to the fact that the shaded column (in relation to the whole) is also equivalent to one seventh. This was well demonstrated using the diagram as learners could see the part (one column) in relation to the whole (the seven columns). Hence, the answer obtained earlier of $\frac{1}{7}$ made sense to the learners.

In the second example, learners were asked to find three fifths of twenty marbles ($\frac{3}{5}$ of 20). Again, Teacher K employed the visual method to demonstrate the whole (twenty) and the part

($\frac{3}{5}$) as illustrated in Figure 4.10 below. With the assistance of the teacher, learners were able to figure out that each fifth represented four blocks which resulted in twelve, as a representation of three fifths.

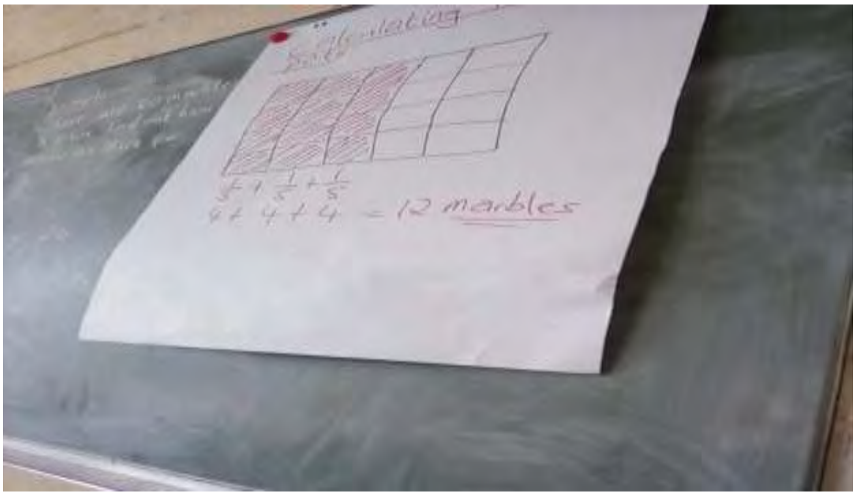


Figure 4.10: A representation of $\frac{3}{5}$ of 20 (Teacher K)

In both examples (Figures 4.9 and 4.10) the set model was employed since the focus was on helping learners realise that a unit fraction can represent a set of objects, for instance, in the first example one seventh represented five learners while one fifth represented four marbles in the second example. Furthermore, Teacher K explained the difference between the first and the second examples by stating that in the first example, learners were expected to derive a fraction from the given numbers while the second example required learners to find the exact number represented by a given fraction if the whole is given.

One of the shortcomings of this lesson was the fact that the use of a variety of FMs was not attained because Teacher K relied on one model. Although the set model was effectively used and enhanced learners understanding, Van de Walle et al. (2013, p. 312) encourages the use of different models because by so doing, learners are presented with alternative ways to learn fractions.

Teacher H: Fraction of a shape (Grade 4)

One of the key components that featured prominently in this lesson was its visuality (VL). According to Presmeg (1986, p. 298), "a person's mathematical visuality is the extent to which that person prefers to use visual methods when attempting mathematical problems which may be solved by both visual and nonvisual methods." Teacher H's lesson was inclined towards the visual mode and this was partly due to the nature of the topic and the grade level. In other words, the topic somehow dictated the use of visuals. For instance, the homework from the previous lesson was based on identifying the total number of parts into which the shape was divided (the denominator) and writing down the name of one fractional part.

The new lesson focused on identifying the name of one fractional part of a given diagram and to write down the fraction represented by the shaded part. Equally, this new topic required the use of diagrams because of its nature. Hence, Teacher H was compelled to incorporate visuals in her lesson in order to effectively present this lesson. Fraction language emerged as a challenge for Teacher H because for this specific lesson, a good command of fraction language was necessary since it involved naming fractional parts such as halves, thirds, quarters, fifths, etcetera. It seemed inept for the teacher to correctly label the fractional parts of a shape that was divided into three parts as thirds, but then to read one third as 'one over three' and two thirds as 'two over three'.

Despite the visual nature of this lesson, Teacher H relied on one fraction model. She used the area model in all her examples and this deprived her learners of opportunities to experience fractions across the other models. In terms of the VNVC, Teacher H used the visual code as a premise to gradually introduce the symbolic code. The other analytical framework component that was dominant in Teacher H's lesson was EQS and this can be attributed to the way she integrated questions in her teaching and the visual nature of her questions. The other aspect that featured the least in this lesson were QTs, because Teacher H did not pay attention to fraction size in drawing her diagrams.

Teacher M2: Fraction part and word problems (Grade 6)

Although this topic was more appropriate for incorporating visualisation processes, Teacher M2 clung to the symbolic mode. Relevant examples were presented but the lack of visuals rendered them abstract. For instance, the examples *find 5 litres of 20 litres* or express *20kg as a fraction of 50kg* did not make sense to the learners due to the lack of visual representations. According to Foreman and Bennett (1995 p. 11) "sensory experiences help students develop mental images that aid in understanding and recalling relationships and information." In this lesson, Teacher M2 did not make any attempt to help learners understand the relationships between the given quantities and see the whole picture. Consequently, some learners saw the given quantities as bits and pieces and only focused on applying the rules. Clearly, Teacher M2's emphasis on the rules overshadowed the importance of enhancing learners' conceptual understanding.

Teacher M2 introduced the new topic with examples to demonstrate how a fraction can be derived from two quantities. The abovementioned examples of *find 5 litres of 20 litres* and *express 20kg as a fraction of 50kg* were presented to the learners. As they collectively worked out the exercises, Teacher M2 stressed the importance of expressing one number as a fraction of the other and simplifying the two fractions by using the Highest Common Factor (HCF) method. However, the teacher did not use the term HCF to refer to this number. Instead, she referred to it as the *"biggest number that can go into the two numbers"*. The third problem, $\frac{3}{4}$ *of 8* was read by the teacher as "*three out of four of eight"*, instead of 'three quarters (or three fourths) of eight'. Fraction language plays an important role in putting problems into context and understanding them. In working out this problem, Teacher M2 placed emphasis on the meaning of 'of'' and the multiplication rules. At no point did she try to help learners see the eight as a whole. On the contrary, she was quick to convert the eight to a fraction for learners to apply the multiplication rules taught earlier.

This approach advances the perception that mathematics is an abstract subject that only emphasises learning computing skills and memorising facts (Fuma, 2018, p. 1). The synopsis below points to the fact that all the components of the analytical framework were compromised by Teacher M2's approach to this lesson.

Teacher I: Fractional part of a quantity and word problems (Grade 6)

Even though this lesson lends itself towards the use of both codes, Teacher I leaned more on the verbal code. Apart from the use of a simple visual representation in the introduction which was misplaced and did not have any impact on learners' understanding of the lesson, the entire lesson presentation was mostly verbal (symbolic). Hence, challenges associated with the omission of the visual code emerged despite Teacher I's attempt to enforce the application of rules.

Teacher I wrote down the lesson objectives of this lesson and clearly explained the anticipated outcomes of this lesson. This pointed towards a good start to the lesson, however Teacher I did not make an effort to establish learners' prior knowledge on this topic. As a result, she only discovered that her learners could not multiply or simplify fractions during her lesson presentation. According to Charalambous et al. (2010), learners' prior knowledge is one of the key indicators of effective fraction instruction. Drawing on the key components of the analytical framework, the only component that featured in this lesson was EQS, as Teacher I posed relevant questions before teaching the procedures (EQS 2 and 3). Despite Teacher I's good content knowledge, her approach to this topic did not support most of the components forming part of the analytical framework.

4.3.4 Topic 4: Fraction size, Comparing and ordering fractions

The three concepts, namely fraction size, comparing and ordering fractions are very helpful in developing learners' fraction sense because these concepts determine the extent to which learners can comprehensively understand fractions. Basically, these concepts are regarded as the foundational knowledge of fractions and can help alleviate most of the challenges that learners encounter in learning fractions. Woodward (1998, p. iii), learners who have attained good fraction sense should be able to "understand the meaning of fractions; to reason qualitatively about the absolute and relative size of fractions." Furthermore, Borenson (2015, p. 1) suggests that this can be achieved if teachers have acquired the appropriate content knowledge and the instructional strategies. Moreover, the variations observed in the following lessons point to the discrepancies in the participants' CK and PCK. Five of the participants involved in this study presented lessons on these concepts as summarised below.

Teacher K: Comparing and ordering fractions (Grade 6)

This lesson was characterised by several flaws which can be ascribed to inadequate content and pedagogical knowledge on the part of the teacher. Consequently, this negatively affected all seven components of the analytical framework. In fact, Danişman and Tanişli (2018) assert that CK and PCK are indispensable elements in the effective teaching of mathematics. The foundational nature of PCK in relation to the other components of the analytical framework is a vital aspect to consider in the pedagogy of fractions because it has a direct impact on the success or failure of a lesson. For this specific lesson, poor PCK resulted in the presentation of wrong content and the use of ineffective teaching strategies. Thus, the QFK, FM, VL, VNVC, QT and EQS were all compromised.

In his introduction, Teacher K focused on the relationship signs and rules pertaining to the ordering of fractions. On a positive note, he related this topic to the comparing and ordering of whole numbers which learners were familiar with. However, Teacher K wrongly explained equivalent fractions as fractions that have the same denominator and this was repeated several times.

Teacher K: When we talk about equivalent fractions, we are talking about whereby your fractions are having the same denominator. Are we together?

It is incorrect to assume that all fractions that possess the same denominator are equal and pass these kinds of misconceptions to learners. For instance, when asked to order the fractions $\frac{1}{2}$, $\frac{3}{4}$, $\frac{2}{3}$, and $\frac{2}{6}$, it is a common practice to convert all the fractions to the same denominator by finding the equivalence of each fraction, but that does not mean that these fractions are all equivalent. Misconceptions about very important concepts such as equivalent fractions could be one of the contributing factors to learners' poor understanding of fractions. The integration of visualisation processes before the introduction of the standard algorithms can help alleviate such misconceptions among teachers and learners.

Teacher K presented learners with two examples involving the comparison of fractions, both of which were solved by using traditional methods. The first example involved the comparison between $\frac{3}{7}$ and $\frac{5}{14}$ which was solved by converting placing both fractions to the same denominator

$\frac{3}{7} \square \frac{5}{14} = \frac{3 \, x \, 2}{7 \, x \, 2} \square \frac{5 \, x \, 1}{14 \, x \, 1} = \frac{6}{14} > \frac{5}{14}$. Teacher K further reminded learners that the original fractions should be used in the final answer, that is, $\frac{3}{7} > \frac{5}{14}$ and not $\frac{6}{14} > \frac{5}{14}$. The second example $\frac{7}{21} \square \frac{21}{63}$ was also solved by using the same strategy.

$$\frac{7x\,3}{21 \, x \, 3} \square \frac{21 \, x \, 1}{63 \, x \, 1}$$

$$\frac{21}{63} = \frac{21}{63}$$

$$\frac{7}{21} = \frac{21}{63}$$

Although the comparison of fractions seemed easy, learners found the ordering of fractions more challenging. This was primarily because learners struggled to find the LCD of more than two fractions which can be attributed to the fact that they had not mastered the two basic operations: multiplication and division. In finding the LCD, Teacher K relied on listing the multiples of all the denominators in a given sequence, which is a cumbersome process for learners at this level (Grade six). When the teacher presented learners with mixed numbers to be ordered, he was quick to tell learners to convert the mixed numbers to improper fractions instead of asking them to estimate the sizes of these fractions ($2\frac{1}{2}$, $1\frac{1}{2}$, $2\frac{1}{3}$, $2\frac{1}{6}$). Although learners found this method easier, it did not enhance their understanding of fraction size because learners can follow the procedures without paying attention to the size of the fractions.

The shortcomings of this lesson can be attributed to Teacher K's focus on traditional teaching methods and the total omission of the visual component in a lesson that required learners' mastery of fraction size. Moreover, a lesson of this nature should have been based on enhancing learners' QT, through visualisation processes. Just like the size of whole numbers, understanding fraction size is a critical component that is directly related to learners' ability to compare and order fractions. Hence, the repercussions observed in this lesson.

Teacher M1: Size of fractions and comparing fractions (Grade 7)

All the components of the analytical framework were prevalent in this lesson because Teacher M1 tried to ensure the attainment of QFK by focusing on enhancing learners' conceptual understanding of fractions through the incorporation of visualisation processes. As stated by Rosken and Rolka (2006, p. 458), "to understand a formal mathematical concept requires of the learner to generate a concept image for it." Evidently, the carefully selected visuals enabled learners to have a deeper understanding of fraction size and as a result they were able to compare fractions easily.

Although Teacher M1 did not use the set model in this lesson, he successfully used the area and length models. Van de Walle et al. (2013, p. 312) accentuates that "what appears to be critical in learning is that the use of physical models leads to the use of mental models and this builds students' fraction understanding." One of the most important components for this specific lesson was the enhancement of learners' QT, because understanding fraction size is a prerequisite to understanding fractions in general. Asku (1997, p. 373) argues that "students must understand the meanings of fractions before performing operations with them." The VL, PCK and EQS were also observed, mainly because of the effective teaching strategies that Teacher M1 used.

Teacher H: Comparing fractions (Grade 4)

This lesson was sequentially and logically presented from the introduction to the conclusion. The teacher's PCK was instrumental in the pedagogical methods that were used to present this lesson. Moreover, Teacher H placed emphasis on QT, and as a result learners were able to easily compare fractions. According to Bezuk and Cramer (1989, p. 157), "the acquisition of a quantitative understanding of fractions is based on students' experiences with physical models and on instruction that emphasizes meaning rather than procedures." Teacher H used a similar approach by drawing two circular diagrams to demonstrate how fractions with the same denominators could be compared.

Although Teacher H performed fairly well on the other components of the analytical framework, her drawback was the lack of different FMs in this lesson because she relied on only one model (area). As indicated earlier, exposing them to multiple representations of fractions enhances learners' understanding of fractions.

Teacher H: Comparing fractions with different denominators (Grade 6)

Generally, this lesson brought to light Teacher H's shortcomings regarding CK, because based on the corrections to the homework, she struggled to identify the factors of the numbers that learners were expected to simplify. This is one of the most basic concepts in mathematics that every mathematics teacher is expected to know. Thus, Teacher H's inability to identify the factors of numbers was a serious drawback. Although she relied on her notes, she still seemed uncertain about what she had written down. Lack of confidence in the way she presented this part of the lesson was also evident.

A similar trend was observed when Teacher H presented her new lesson on comparing fractions. The difference between Teacher H's lesson presentation on comparing fractions in the Grade four and Grade six classes confirms Boaler et al.'s (2016) argument regarding the view that visual mathematics is meant for younger or struggling learners and serves as a prelude to more important mathematics (2016, p. 7). This explains why Teacher H saw the need to use visuals in her Grade four class and omitted them in her Grade six class. The success attained in the Grade four class can be attributed to the incorporation of visuals. However, since the teacher did not use any visuals to present the same topic in Grade six and relied on the application of rules, the outcome was different.

The total omission of the visual code, coupled with Teacher H's inadequate content knowledge affected the acquisition of the all the components of the analytical framework. Hence, despite Teacher H's emphasis on the verbal code, QFK and QT was not achieved. These are very important concepts in comparing fractions because success can only be determined by the extent to which the pedagogical methods used can help learners determine or estimate fraction size.

Teacher H: Ordering fractions (Grade 4)

Teacher H's approach to this lesson was similar to the previous one on comparing fractions in the sense that it was predominantly verbal. Thus, the same challenges were observed as learners could not determine the size of the fractions in a sequence, for them to place them in ascending or descending order. Moreover, the teacher did not use any visuals to demonstrate fraction size, but instead, she simply introduced the rules that her learners were supposed to follow when ordering fractions with the same denominators or the same numerators. This did not enhance learners' understanding of fraction size. As a result, the misapplication of their whole number knowledge crept in as they attempted to order fractions. Hence, some learners perceived the same denominator fractions with bigger denominators as bigger than those with smaller denominators. In other words, the rules provided by the teacher did not make sense to the learners. Therefore, in the absence of concrete or visual representations, the rules are not convincing enough for learners to abandon their whole number knowledge.

Teacher H brought in a few visuals towards the end of this lesson when she realised that the rules did not help learners to understand fraction size. This intervention complemented the rules and learners began to see the logic in the rules that were introduced earlier. Despite the symbolic nature of this lesson, the incorporation of a few visuals changed the trajectory of this lesson. Hence, QFK, VNVC, QT and PCK were partly achieved.

Teacher I: Comparing fractions (Grade 6)

In this lesson, Teacher I paid more attention to important concepts such as prior knowledge and visualisation. While trying to establish learners' prior knowledge, Teacher I engaged learners in the process by providing them with opportunities to demonstrate what they knew about the topic. Moreover, the use of concrete teaching aids and visuals also featured in this lesson. Visual teachers are known for their ability to make connections between the mathematics content and other aspects such as the syllabus, other subjects, prior knowledge, aspects beyond the syllabus and most importantly, the real world (Presmeg, 2014). Although Teacher I may not be a visual teacher, the fact that this was a repeated lesson compelled her to reflect on the previous lesson and rectify some of the pedagogical shortcomings identified in the first lesson. Hence, the incorporation of visuals.

Another positive aspect about this lesson was the fact that Teacher I allowed learners to make mistakes and correct their own mistakes. She was not too fast to correct or reprimand learners for their mistakes. Instead, she was patient and accorded learners time to figure out their own mistakes. This is one of the indicators of good PCK, particularly PCK 7. Generally, Teacher I demonstrated good PCK and this was revealed through her selection of effective strategies. Substantially, this approach led to the accomplishment of other components of the analytical framework such as QFK, FM, VL, VNVC and QT.

A fair integration of the VNVC featured prominently in this lesson as Teacher I interchangeably used the two codes to enhance learners' understanding. Stokes (2002, p. 2) explains that achieving an optimal balance in the use of visual and verbal cues fosters interdependence between the two modes. Additionally, Suh and Moyer-Packenham (2007, p. 209) assert that the use of both codes has additive effects on learners' ability to recall subject content.

In her attempt to promote learners' QT, Teacher I engaged learners in practical activities of partitioning of concrete materials (oranges) as shown in Figure 4.11 below:

Figure 4.11: A demonstration of fraction size using oranges (Teacher I)

Despite the use of concrete teaching aids which learners enjoyed very much, fraction language emerged as a challenge since Teacher I referred to a quarter ($\frac{1}{4}$) as 'one out of four' or 'one divided by four' and a half ($\frac{1}{2}$) as 'one out of two' or 'one divided by two'. Although both are correct, Van de Walle et al. (2013) cautions against the wrong impressions that this kind of fraction language could have on learners since it does not emphasise fraction size. While drawing on the examples used in Figure 4.11, Teacher I tried to help learners understand how to use denominators to compare fractions but she did not specify the conditions under which the stated rule applies. She gave the learners a general rule that says ‘the bigger the denominator the smaller the fraction’, without making it clear that this rule only works when the numerators are the same.

Teacher I: Every time you must sing…every time you must sing, neh? The bigger the denominator the smaller it is, even at home. The bigger the denominator, the smaller it is. You must sing. The smaller the denominator, the bigger the fraction

While assuring learners that the use of concrete objects and diagrams could help them understand fractions better, she extracted examples from the RCFP manual to demonstrate fraction size and compare fractions. During this exercise she attended to individual learners who had challenges understanding the fraction parts on the fraction wall. Learners had a problem figuring out that a half and two quarters are equal on the fraction wall but with the teacher’s help, they were eventually able to see the equivalence between the two fractions. Moreover, Teacher I presented learners with another pair of fractions ($\frac{2}{5}$ and $\frac{2}{7}$) to compare. She moved around to ensure that learners could read fraction size correctly from the fraction wall and assist those who were struggling to compare fraction sizes.

After this exercise, some learners still insisted that two fifths was less than two sevenths until the teacher referred them to the rules taught earlier.

Teacher I: The bigger the denominator, the smaller the fraction. The…smaller the denominator, the bigger…

Learners: the fraction

She further reminded learners about the cutting of oranges (Figure 4.11) to demonstrate how the number of parts into which the whole is divided determines the size of fraction pieces. In addition, Teacher I introduced another simpler method (Figure 4.12) that involved cross-multiplication. For instance, in comparing a half and two quarters ($\frac{1}{2}$ and $\frac{2}{4}$), learners were instructed to cross-multiply the two fractions as shown below. After cross-multiplying 1 x 4 and 2 x 2, learners are expected to compare the answer on the left hand side (LHS) and the one on the right hand side (RHS).

$$\frac{1}{2} = \frac{2}{4}$$

$$4 = 4$$

In this case both sides gave a product of four, hence the two fractions are equal. The teacher used the same strategy to compare two fifths and two sevenths ($\frac{2}{5}$ and $\frac{2}{7}$) which resulted in fourteen on the LHS and ten on the RHS.

$$\frac{2}{5} \quad \frac{2}{7}$$

$$14 > 10$$

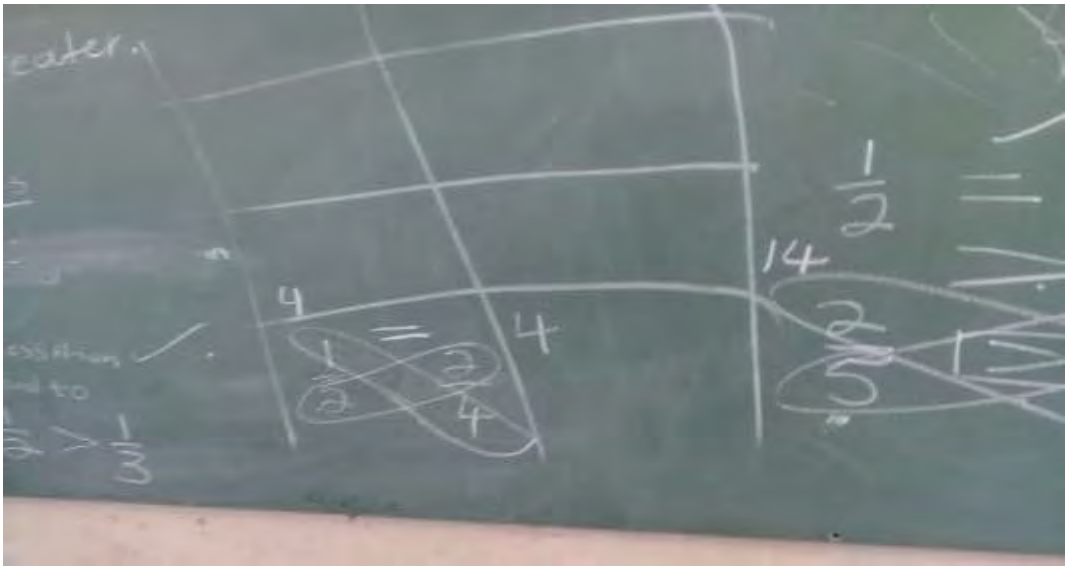


Figure 4.12: A demonstration of the cross-multiplication strategy used to compare fractions (Teacher I)

Furthermore, Teacher I used the fraction wall (from the RCFP manual) to demonstrate fraction size. Bezuk and Cramer (1989) assert that:

> … the acquisition of a quantitative understanding of fractions is based on students' experiences with physical models and on instruction that emphasizes meaning rather than procedures.

Quantitative thinking (QT) is a key concept in fraction pedagogy because it lays the foundation for understanding fraction size and operations with fractions. In this context, understanding the magnitude of fractions is as important as understanding the magnitude of numbers.

This lesson was characterised by visualisation, from the introduction to the conclusion as the teacher included more visualisation processes in her lesson presentation. The standard algorithms introduced towards the end of the lesson were effective but did not enhance learners' understanding of fraction size. Besides, the answers used to determine the size of fractions were whole numbers as demonstrated in the examples above. For instance, determining fraction size by cross-multiplying the two fractions $\frac{2}{5}$ and $\frac{2}{7}$ to get fourteen on the LHS and ten on the RHS does not say anything about the size of the two fractions being compared. Therefore, learners might easily get the correct answers and insert the correct relationship signs but they might not have a clue as to how the two fractions are related and why one is bigger than the other.

Teacher S2: Comparing and ordering fractions (Grade 4)

This lesson was predominantly visual because Teacher S2 focused on visualisation processes to compare and order fractions. The use of rules to compare and order fractions was also observed. However, Teacher S2's emphasis was mainly on the use of diagrams to illustrate fraction size. Based on the response of the learners on the examples presented, this can be described as a successful lesson in which all the components of the analytical framework, except the use of different FMs and QT were accomplished.

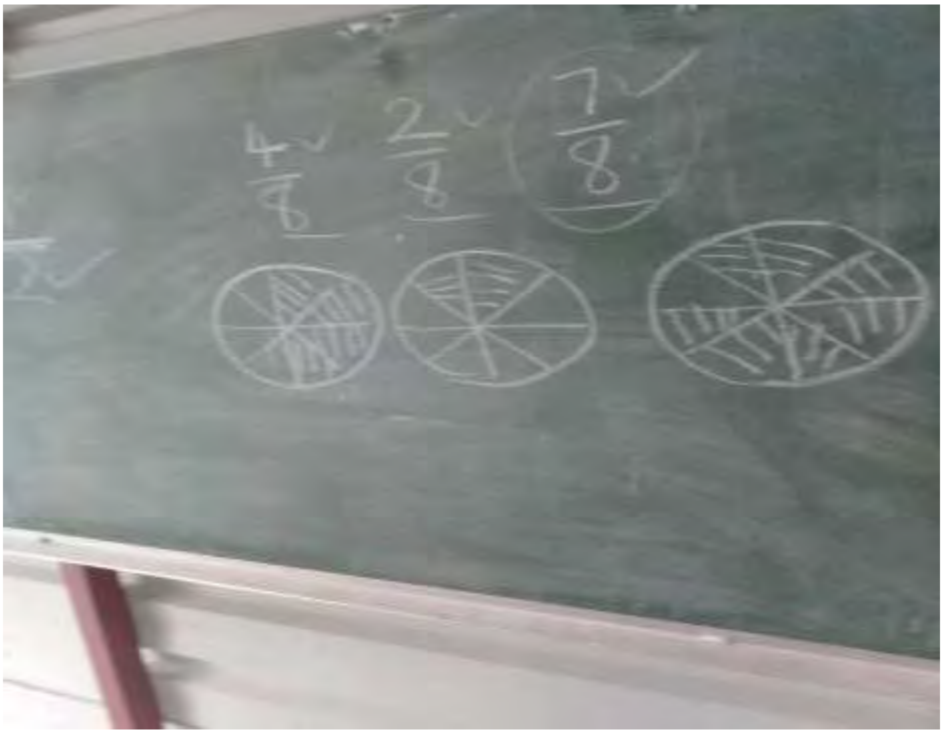


Figure 4.13: Ordering same denominator fractions using visual representations (Teacher S2)

It was strange to see how in his lesson presentation, Teacher S2 did not pay attention to fraction size (Figure 4.14) yet this did not hinder learners from comparing or ordering fractions correctly using both the visual and the verbal code. This is an indication that a weakness in one code can be compensated for by the strength in the other code. What transpired in Teacher S2's class confirms Mayer and Anderson's (1991, p. 485) view which is based on the notion that the use of both the visual and verbal code helps learners to remember and transfer content better because they have two separate ways of locating the information in memory.

Apart from the two shortcomings identified earlier (FM and QT), the other components of the analytical framework were fairly accomplished in this lesson, particularly VL. Although Teacher S2 relied on chalkboard diagrams, his lesson was more visual because he preferred to use visual methods throughout the lesson.

Generally, the incorporation of visualisation processes had a major impact on the success or failure of the lessons presented on this topic. In other words, the four lessons in which visuals were incorporated were more successful than the three in which visuals were omitted. This indicates that besides CK and PCK, visualisation plays a pivotal role in the pedagogy of fractions, particularly on the topics under discussion.

4.3.5 Topic 5: Addition and subtraction of fractions

The topics discussed earlier usually prepare learners for operations with fractions. Without this foundational knowledge, learners experience difficulties with addition and subtraction of fractions. The addition and subtraction of fractions are some of the most difficult fraction concepts and this is mainly because of the tradition, rule-based approach often introduced too early, that is before learners acquire the foundational knowledge required to cope with more complex concepts. To alleviate this challenge, Fennel and Karp (2016) assert that progression in terms of fraction instruction should take into account the use of fraction models to enhance learners' understanding. On the other hand, Van de Walle et al. (2013) recommend the use of contextual story problems and invented strategies to enhance learners' understanding of addition and subtraction of fractions.

One of challenges that learners encounter in trying to understand the addition and subtraction of fractions is the concept of numerator and denominator. Often, they struggle to understand why denominators have to change when adding fractions with different denominators and why the denominator remains the same when same denominator fractions are added. Furthermore, learners have difficulties comprehending the rule that says denominators cannot be added, because the pedagogy of fractions is based on the memorisation of such rules rather than the understanding of concepts. Therefore, standard algorithms should be carefully introduced by ensuring that learners' conceptual understanding precedes rules and formulas.

Teacher M1: Adding subtracting fractions with the same denominators (Grade 7)

Although this lesson started on bad note due to the learners' poor background in the addition and subtraction of fractions, Teacher M1 was able to turn it into a successful and interesting lesson. He founded his lesson on the learners' misconceptions identified during the introduction of his lesson. Based on the learners' responses, it was interesting to observe how this lesson changed from a difficult lesson into an easy and interesting lesson. This was mainly due to the teacher's good PCK as he was able to present the difficult content in such a way it became comprehensible to the learners. This, according to Shulman (1986) is the definition of PCK. The teacher's ability to select effective teaching strategies by using visual strategies, testing learners' prior knowledge, using both verbal and nonverbal codes and constantly checking learners' understanding of unit fractions using different fraction models, contributed to the success of this lesson. Subsequently, all the other components of the analytical framework were addressed through this approach.

Despite their level (Grade 7), Teacher M1's learners struggled to read simple fractions such as eight and a half ($8\frac{1}{2}$) which they read as 'eight and one-two'. The learners' inability to read $\frac{1}{2}$ as 'a half' can be attributed to the pattern observed in the naming of the other fractions such as $\frac{1}{4}, \frac{1}{5}, \frac{1}{6}$, and the fact that they had been exposed to inappropriate fraction language where the use of the terminologies 'over' and 'out of' were normalised. The acceptable way of reading fractions: 'one-half', 'one-third', 'one-fourth' (or 'a quarter'), 'one-fifth', etcetera, seemed new to the learners.

Teacher M1 introduced the topic on the addition of fractions by presenting learners with the following exercise: $\frac{1}{3} + \frac{1}{3} = \frac{}{}$. Before working it out, the teacher tested learners' ability to read fractions correctly.

Finding the answer to $\frac{1}{3} + \frac{1}{3}$ proved to be a challenge as learners provided a wide range of answers, including *one-six, two-six, half-six, one half, two out of three, two out of six, two third* (in the learners' terms). The interference of whole number knowledge in operations with fractions was observed in the range of answers that learners provided. For example, the answer *two-sixth* was obtained by adding the numerators (1 + 1 = 2) and the denominators (3 + 3 = 6). From the list of answers mentioned, it was clear that learners were not certain about the correct addition procedures to apply, and as a result, they came up with answers that were far-fetched such as *one-sixth, half-sixth and one-half.*

Consequently, Teacher M1 used real life examples to demonstrate why the answer to $\frac{1}{3} + \frac{1}{3}$ is two thirds ($\frac{2}{3}$). The use of a variety of examples also helped learners to understand that the two fractions have the same denominator and thus, they are addable.

….Ok this one is called a third and this is a third…yes!

If I ask you one book plus one book

One book plus one book…what is your answer?

One pen plus one pen is equals to?

Teacher M1 introduced the topic on addition of fractions in a logical and sequential manner. Initially, the teacher used real life examples such as *one book plus one or one pen plus one pen* which was then superseded by similar same-denominator fractions as a way of demonstrating to learners that we can only add or subtract fractions if they have the same denominators. Examples such as:

Now, one-third plus one-third ($\frac{1}{3} + \frac{1}{3}$)

One-quarter plus two-quarters ($\frac{1}{4} + \frac{2}{4}$)

Four-fifths minus three-fifths? ($\frac{4}{5} - \frac{3}{5}$)

Four-fifths minus three-fifths? ($\frac{4}{5} - \frac{3}{5}$)

Teacher M1 then explained that when adding or subtracting same-denominator fractions, learners should remember to keep the denominator unchanged and add or subtract only the numerators. It is worth noting that the process leading to the introduction of the rules helped learners to understand and apply the rules correctly. Figure 4.14 shows how Teacher M1 used a number line in a complementary exercise to add $\frac{2}{5} + \frac{2}{5}$. He illustrated how $\frac{2}{5} + \frac{2}{5}$ can be added using the number line step by step. Furthermore, a subtraction exercise ($\frac{4}{5} - \frac{3}{5}$) was presented and solved using a number line model. Together with the learners, the teacher counted three steps backwards from four fifths and landed on one fifths which learners easily identified as the answer.

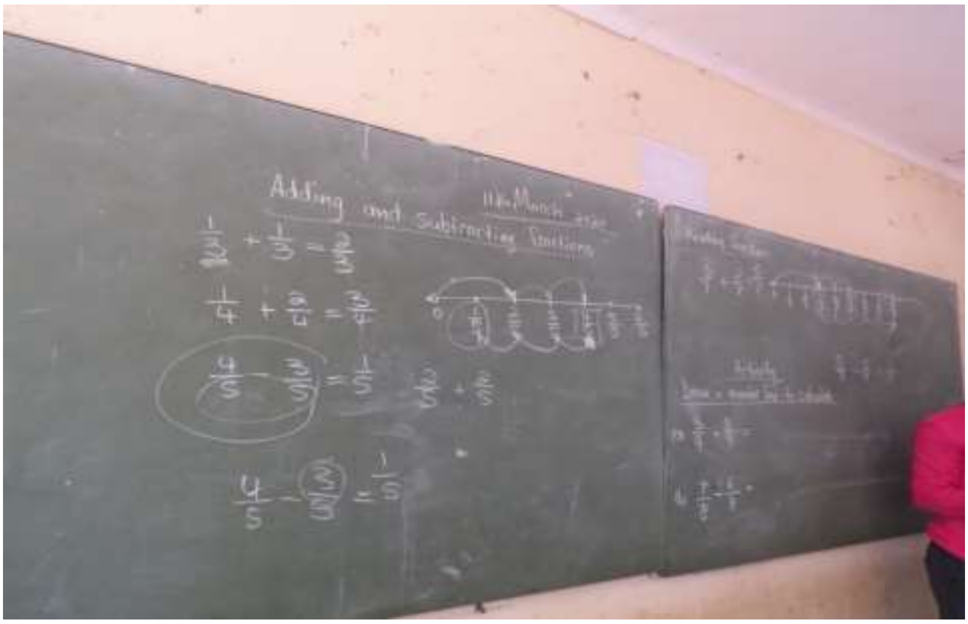


Figure 4.14: Teacher M1's strategy for adding and subtracting fractions on a number line

In addition, Teacher M1 drew another number line to work out three sevenths plus two sevenths ($\frac{3}{7} + \frac{2}{7}$) and as observed earlier, he actively involved his learners in the whole process, that is, from drawing the number line to adding the two fractions. Learners found the use of the number line to add and subtract fractions very interesting.

Teacher M1: Adding and subtracting fractions with different denominators (Grade 7)

This lesson was characterised by the use of visualisation processes in the first half of the lesson and the introduction of rules in the second half. Teacher M1 found the use of the number line (length) model easier because fractions with the same denominators were involved. However, the introduction of addition and subtraction of fractions with different denominators proved to be a challenge for the learners because the process was long and cumbersome. Firstly, it involved the identification of the LCD by listing the multiples of the denominators involved in the addition or subtraction problem. The listing of multiples was a challenge for some learners who saw this as an isolated task, not related to the task at hand (that is, the addition or subtraction of fractions with different denominators). Therefore, by the time the listing of multiples and the identification of common multiples and the LCD was completed, some learners could not link the process of listing multiples to the addition and subtraction of fractions. Thus, aspects of QFK, VL and QT were compromised because these were only observed in the first part of the lesson.

This lesson was a good demonstration of the role of FMs in enhancing learners' understanding of operations with fractions. The first part of the lesson was successful because Teacher M1 used the number line model to explain the addition and subtraction of fractions. However, the devastating impact of withdrawing the models was observed in the second half when Teacher M1 switched to standard algorithms. Ervin (2018) asserts that this confusion can be alleviated if the introduction of standard algorithms is preceded by the modelling of fractions. The teacher's PCK was compromised by the pedagogical strategies that he employed in the second half of this lesson. Moreover, Teacher M1 did not prepare questions that would encourage multiple strategies from the learners. Instead, he introduced one strategy that all learners were expected to follow. This had a direct impact on the last component of the analytical framework (EQS).

Teacher M1: Adding and subtracting fractions with different denominators (Grade 7)

This can be classified as a purely verbal lesson with an emphasis on the use of symbols which brought to the fore the role of visualisation in the teacher's PCK. As stated by Rahim et al. (2011, p. 497), "visually based pedagogy helps to make mathematics more inclusive as it paves way for learners to understand abstract mathematics concepts which would otherwise have been inaccessible to them." The overuse of the verbal code in this lesson did not yield any positive results. Boaler et al. (2016, p. 6) assert that learners usually find mathematics inaccessible and uninteresting because of the overuse of abstraction and numbers. This came out clearly in this lesson as Teacher M1 switched from one verbal strategy to another with no signs of improvement in the learners' conceptual understanding of fractions.

Despite Teacher M1's efforts to expose learners to multiple strategies to solve fraction problems, learners still struggled to switch from one type of problem to the other. Perhaps this required a different approach from the teacher, for instance dealing with one strategy at a time. Learners did not know when to find the LCD by listing the multiples of the two denominators, multiplying the two denominators, or using one of the two denominators as the LCD. This proved to be a challenge. Thus, Teacher M1's focus on the verbal code affected the quality, visuality, enhancement of quantitative thinking and the questioning strategies of this lesson.

Teacher S1: Adding and subtracting fractions with different denominators (Grade 7)

This lesson can be described as predominantly symbolic because the teacher emphasised the rules pertaining to the addition and subtraction of fractions more than the use of visualisation processes. Visualisation was merely brought in to demonstrate the concept of equivalence, while the actual addition and subtraction was dependent on the rules. Consequently, the components of the analytical framework such as QFK, FMs, VL, EQS were negatively affected because these components are closely intertwined. Zimmerman and Cunningham (1991) caution that leaning towards either the symbolic or the visual code could upset the balance between the two codes and this could produce undesirable pedagogical results.

Therefore, the incorporation of visuals should be a deliberate attempt on the part of the teacher to complement the symbolic code. Stokes (2002, p. 2) asserts that "as more visual elements are incorporated to achieve an optimal balance between verbal and visual cues in education, interdependence between the two modes of thought will be fostered." The incorporation of visualisation processes in mathematics alleviates the abstract nature of mathematics and enhances learners' conceptual understanding of fractions. Boaler et al. (2016, p. 7) accentuate that visual mathematics can "inspire students and teachers, to see mathematics differently, to see the creativity and beauty in mathematics and to understand mathematical ideas." Thus, the abstract nature in which this lesson was presented deprived learners of opportunities to see and experience mathematics differently.

The use of rules was more prevalent in the first part of the lesson as the teacher reminded learners about the procedures for the addition and subtraction of fractions. The rules involved the listing of multiples, identifying the common multiples (CMs) and the Lowest Common Multiples (LCM). Since learners could not remember the role of the LCM in the addition of fractions with different denominators, Teacher S1 took the learners through the procedures. She started by asking learners to list the multiples of four up to twenty-eight and the multiples of eight up to forty. After listing the multiples, the teacher explained that they should first identify all the Common Multiples (CM) before identifying the Lowest Common Multiple (LCM) as illustrated below:

4 = 4, 8, 12, 16, 20, 24, 28

8 = 8, 16, 24, 32, 40

Common multiples of 4 and 8

8, 16, 24, 32, 40

LCM of 4 and 8 = 8

The concept of multiples, CMs and LCMs seemed new to the learners because they had a challenge listing and identifying them, despite the teacher's efforts to simplify the process. Eventually, learners identified 8, 16 and 24 as CMs of 4 and 8 after which Teacher S1 asked them to choose the LCM from the CMs. Learners correctly identified 8 as the LCM. Thereafter, Teacher S1 tested learners' understanding by asking them to explain why the second fraction remained unchanged and this is a question that learners failed to answer.

The second part of this lesson focused on equivalent fractions. The example $\frac{1}{4} = \frac{2}{8}$ was used to explain this concept visually. The teacher also drew learners' attention to the definition of a fraction that is a part of a whole. As she drew a rectangle on the chalkboard, she reminded learners about the importance of ensuring that the whole should always be divided into equal parts.

Teacher S1: … then we are going to divide this loaf of bread in four parts and the four parts must be equal. When you are dividing…actually when you are dividing things, it should be always be equal isn't it? Not one is big and the other one is small.

Partitioning or dividing a whole into equal parts is a concept that should be emphasised because it is a prerequisite to meaningfully understanding fractions. Bruce et al. (2013) identified lack of equivalence and equi-partition as one of the challenges associated with learners' misconceptions of the addition and subtraction of fractions. To demonstrate that a quarter and two-eighths are equal, Teacher S1 drew two rectangles, one below the other (Figure 4.15). Step by step, Teacher S1 tried to help learners establish the relationship between a quarter and two-eighths. She started by partitioning both rectangles into fourths and then repartitioned each of the quarters in the second diagram to produce eighths. As learners could relate the quarters in

is equal to two-eighths.

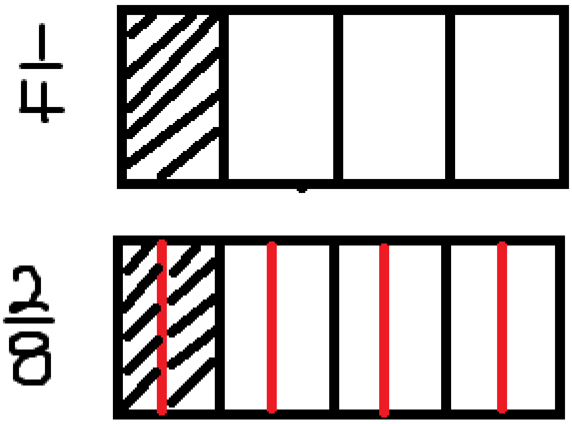


Figure 4.15: Teacher S1's illustration of the equivalent fractions $\frac{1}{4}$ and $\frac{2}{8}$

Although the focus was on the shaded part, Teacher S1 also drew learners' attention to the unshaded part and this helped learners to attain a holistic view of the shapes, that is, the part and the whole.

In addition, Teacher S1 also used the same diagrams to compare fractions using the example $\frac{2}{4}$ and $\frac{2}{8}$. Learners correctly identified $\frac{2}{4}$ as the bigger fraction and Teacher S1 probed them further to explain why they thought $\frac{2}{4}$ was bigger than $\frac{2}{8}$. One of the learners explained that two quarters was bigger because it was a half. Although initially the teacher was not in full agreement because she wanted learners to focus on the size of the parts, she later explained that despite the fact that both fractions had two as a numerator the size of the parts were not equal. She indicated that the two parts in two quarters were much bigger than the two parts in the two eighths. Moreover, she also explained that the two quarters represented half of the whole, while the two eighths represented a quarter. This opportunity could have been used to teach learners about how a half can be used as a benchmark when comparing fractions but the teacher missed that opportunity.

Despite spending time on the diagrams to teach learners about equivalence, Teacher S1 reverted to the traditional method of adding fractions where learners are expected to divide the LCM by the first denominators and multiply the numerators and the denominators by the quotient obtained from the division exercise. Before the bell rang to mark the end of the class, Teacher S1 reminded the learners about the addition and subtraction rules.

We can only add or subtract if the denominators are the same.

If they are not the same you look for the Lowest Common Multiple isn't it?

When the teacher asked learners what they were supposed to do next they correctly recited the addition rules.

Learner: We have to divide our old denominator with our new denominator

Teacher S1: Uhu…

Learner: Then we can multiply with our numerator to get our new numerator

This was presented as follows:

$$\frac{1x2}{4x2} - \frac{1x1}{8x1} = \frac{2}{8} - \frac{1}{8} = \frac{1}{8}$$

While using the diagrammatic representations to demonstrate how two different fractions can represent the same amount, Teacher S1 clung to the rule-based fraction computation strategies. In spite of this shortcoming, Teacher S1 taught in such a way that she allowed learners to come up with their own right or wrong answers. In cases where learners provided wrong answers, Teacher S1 gave learners ample time to figure out and rectify their mistakes.

Teacher S1: Adding and subtracting mixed numbers (Grade 7)

This lesson was based on the use of standard algorithms to either add or subtract mixed numbers. This proved to be a challenge because firstly, Teacher S1 enforced rules that learners did not understand. Secondly, learners did not understand the composition of a mixed number or what it entails and how to convert a mixed number to an improper fraction. Thirdly, visuals were not meaningfully incorporated.

The importance of incorporating visualisation processes when doing operations with fractions was apparent in this lesson because the use of rule-based standard algorithms proved to be a challenge. According to Cramer et al. (2008, p. 494), "students need to experience acting out addition and subtraction concretely with an appropriate model before operating with symbol." This develops learners' conceptual understanding and prepares them to handle standard algorithms. Moreover, operations with fractions should be preceded by a sound understanding of basic fraction concepts such as fraction size, equivalence, comparing and ordering fractions. This was evident in this lesson as learners (in Grade 7) struggled to do simple conversion of mixed numbers to improper fractions.

Another observation that came out clearly from Teacher S1's rule-based strategies was the alarming rate at which learners forgot the rules and procedures pertaining to the addition and subtraction of mixed numbers. The tendency of learners to forget procedures easily is in conformity with Fazio and Siegler (2011)'s findings.

Teacher S1's use of visuals to represent the answer obtained was not quality-driven because understanding the representation of the answer did not help learners to understand the addition process. Rieber (1994, p. 1) explains that "there are times when pictures can aid learning, times when pictures do not aid learning but do no harm, and times when pictures do not aid learning and are distracting". This was a perfect example of the scenario where teachers tend to use visuals that do not aid learning but do no harm.

Apart from a few elements of EQS observed as Teacher S1 encouraged learners to provide reasons for their answers, the other components of the analytical framework were not promoted in this lesson. This was mainly due to Teacher S1's adherence to standard algorithms despite clear indicators of learners' poor understanding of the rules.

Despite Teacher S1's emphasis on the rules, she encouraged learners to always give reasons for their answers. She asked them *"why"* questions to help learners get a deeper understanding of the concepts and to understand that just giving the answer is not enough.

Teacher S1: Why? There should be a reason why you are saying that four is the denominator.

Whatever answer that you give there should be a reason. Why are you saying that that answer is the one. Uhu…why are you saying that four will be the denominator?

Another observation was the fast rate at which learners forgot the rules in problems that involved multi-steps. For instance, the problem $1\frac{1}{2} + 2\frac{3}{4}$ required learners to first convert the mixed numbers to improper fractions before adding them. However, this proved to be a challenge because they had not mastered any of the steps involved. Learners struggled with the conversion of mixed numbers to improper fractions and after struggling to get the first step right, they had to be reminded about the addition procedures of fractions with different denominators, although these procedures were taught earlier. Hence, the teacher expressed her disappointment about the learners' forgetfulness.

Surprisingly, Teacher S1 only introduced visuals at the end of this exercise to show learners how the answer $4\frac{1}{4}$ obtained from $1\frac{1}{2} + 2\frac{3}{4}$ could be represented. The whole addition process was based on the correct application of rules. Using oranges as an example, the teacher drew four full circles and a quarter.

Teacher S1: So, you have four full oranges, uh…four full oranges…but then the fifth one is not full

At this juncture, Teacher S1 introduced the second strategy. She explained that the whole number and the fractions are added separately when using the second strategy. Using the same example ($1\frac{1}{4} + 2\frac{3}{4}$), Teacher S1 explained the difference between the first and the second method.

The second method…instead of you converting the mixed number to improper fraction, you separate the mixed from…ah! The whole numbers from the fraction

After working out the answer using the second method, again Teacher S1 used visuals towards the end of the exercise to show learners the wholes and the fractions. The purpose of the visuals used by the teacher was not clear as she only brought them in towards the end. Hence, the stage at which Teacher S1 introduced the use of visuals did not enhance learners' understanding of the process but the product, that is they were a mere representation of the answer.

Teacher M2: Addition and subtraction of fractions (repeated lesson) (Grade 6)

This lesson was unique in the sense that it was a repeated lesson. As observed in the introduction of Teacher M2's previous lessons, she consistently asked learners about the addition (subtraction) rules for fractions. However, learners seemed to forget these rules easily although the teacher repeated them a number of times. Thus, the teacher was compelled to reteach this lesson by incorporating visualisation processes. The integration of the verbal and the visual codes in this lesson was a deliberate attempt on the part of Teacher M2 to enhance learners' understanding of addition and subtraction procedures. Mayer and Anderson (1991) assert that using the verbal and the visual codes enhances learners' understanding of mathematics concepts because they are presented with two pathways to remember the information.

The incorporation of visuals resulted in an improvement in the learners' understanding of this topic. This was noticeable in the way that learners applied the rules. However, some learners found the visual strategy a bit confusing. This can be attributed to the fact that the use of visualisation process was new to them. As stated by Paivio (2006, p. 3) "cognition is this variable pattern of the interplay of the two systems according to the degree to which they have developed." Seemingly, visualisation was rarely used in Teacher M2's class as observed in her previous lessons. Therefore, the learners' visualisation had not been adequately developed to enable them to understand visual processes. Unfortunately, this also applied to the symbolic mode which the teacher had spent so much time on. Thus, this brought the complementary role of visualisation processes to the fore. Avgerinou and Petterson (2011, p. 9) point out that

> … memory for pictures is superior to memory for words. This is called the pictorial superiority effect. Memory for a picture-word combination is superior to memory for words alone or pictures alone.

Therefore, the use of one code is not recommended because it impedes learners' conceptual understanding of mathematical concepts. On the other hand, the use of both codes (verbal and visual) presents learners with ample opportunities to grasp and make sense of mathematics concepts.

In spite of learners' lack of exposure to visualisation, its introduction in this lesson yielded positive results because it aroused learners' interest in the lesson and most of them were eager to find out how the new visual strategy worked, as deduced from their questions. To some extent, the incorporation of visualisation processes improved the components of the analytical framework, particularly QFK, FM, VL and VNVC.

In summary, the nature of the lessons presented under this topic ranged between purely verbal and a combination of the verbal and visual codes. The verbal lessons turned out to be too abstract for the learners as they merely depended on the rules presented by the teacher with very little or no understanding of the underlying concepts. On the other hand, the combination of the two codes enhanced learners' understanding in some instances. The collection of lessons under this topic signifies the importance of visualisation as the withdrawal or total omission of visualisation processes proved to be a challenge in terms of enhancing learners' conceptual understanding.

4.3.6 Topic 6: Fraction representation and notation

This concept is basically the foundation of the concept of fractions. Knowing how to represent fractions visually and symbolically is a crucial aspect of the pedagogy of fractions. However, the whole process should focus on promoting learners' conceptual understanding of the notion of fractions rather than mere representations. This can be achieved by exposing learners to multiple representations of fractions or different fraction models. The NCTM (2000) asserts that "representations should be treated as essential elements in supporting students' understanding of mathematical concepts and relationships." Therefore, the use of visualisation in this integral concept of fractions cannot be overstated. The pedagogical choices that teachers make at this level can either make or break the learners. In this study, only one teacher (Teacher S2) presented a lesson on this topic although it was evident from the other observed topics that learners lacked knowledge on this pertinent concept.

Teacher S2: Fraction representation and notation (Grade 4)

In spite of the confusion among learners pertaining to the terms 'numerator' and 'denominator', this lesson was successfully taught because of the good pedagogical strategies employed by the teacher. A fairly balanced integration of the visual and verbal codes was observed in this lesson. The strength of this lesson lay in the visuality of the lesson (VL) because the use of visuals dominated the entire lesson. The other aspects of the analytical framework that were prevalent in this lesson are QFK, VL, VNVC, PCK and EQS.

One interesting aspect about Teacher S2's PCK was his ability to address learners' misconceptions pertaining to this topic. In reference to the misconception about the numerator and the denominator, for example, the use of both the visual and the verbal codes contributed to the misunderstanding that learners had. Therefore, misconceptions can occur even when both strategies are used. It is the meaning that learners attach to the symbolic or visual mode that actually determines the success or failure of a lesson (Bråting & Pejlare, 2008). Moreover, the challenges experienced by learners in terms of the numerator and denominator of a fraction can also be attributed to the misconception that learners perceive the numerator and the denominator as "two separate, unrelated whole numbers" (Jigyel & Afamasaga-Fuata'i, 2007, p. 8). This was portrayed in the learners' responses to questions related to the numerator and denominator because some learners insisted that if you take (shade) one out of a whole that is divided into four parts, you have three instead of a quarter. This was mainly because learners perceived the numerator and the denominator as two separate entities.

The term representation was defined as the diagrammatic way of representing fractions while notation was defined as the written form of fractions.

Teacher S2: To represent is…that is when a fraction is written into a diagram form

Teacher S2: You can see it with your eyes…Notation is the written part of the fraction or a number

Subsequently, the term 'fraction' was defined using a practical example. Teacher S2 brought to class a model of a pizza divided into four parts to explain the meaning of a fraction using the whole and the parts. He then drew the parts of the pizza on the chalkboard and asked learners to identify the halves and the quarters.

The term 'fraction' was defined as 'a part of a whole'. Before defining the terms 'numerator' and 'denominator', Teacher S2 asked learners to identify them using the example of a quarter ($\frac{1}{4}$). When asked to define the term 'numerator', some learners indicated that it means the small number because drawing on the example that they had been presented with, they developed the misconception that all fractions had smaller numerators. Thus, Teacher S2 gave them another example to demonstrate that some fractions can have bigger numerators. Defining the term numerator proved to be a challenge and this was partly due to the teacher's questioning skills. In his attempt to help learners understand the meaning of a numerator, Teacher S2 asked learners to indicate how many pieces they would have if they took one out of four. Despite the teacher's practical demonstrations, this question confused some learners because they did not understand whether the word 'have' implied the remaining pieces or the piece that they had taken out. Hence, some learners indicated that they would have three instead of one.

Moreover, learners also found the definition of the 'denominator' challenging because some interpreted it as the number of pieces that are left after taking out the numerator. Thus, Teacher S2 tried to help learners understand the concept of a denominator by demonstrating that both the part that is taken out ($\frac{1}{4}$) and the remaining part ($\frac{3}{4}$) can be expressed as fractions of the same whole (4) which is the denominator. He defined the denominator as 'the parts that one has in total'.

The components of the analytical framework that featured mostly in this lesson are the VNVC, VL and QFK. The importance of EQS came out clearly in this lesson because the manner in which the teacher phrased some of his questions caused confusion among the learners as they misinterpreted the questions. Another weak link in this lesson was the lack of a variety of FMs as Teacher S2 relied on one model, that is the area model.

4.3.7 Topic 7: Word problems

Word problems are not only an important topic in fractions but they also serve as a vital teaching strategy that cuts across all fraction concepts. This is a topic that should be integrated into all fraction concepts, particularly multiplication and division of fractions. Word problems or story problems help learners to put fraction sums into context since they barely make sense of the processes and outcomes involved in solving fraction problems. According to the NCTM (2000), there ought to be a paradigm shift in the pedagogy of mathematics which should focus more on conceptual understanding rather than procedural knowledge. Hence, the need for word problems which help learners to fully understand, make sense of the problem and select the appropriate computational strategies. In this study, only one participant (Teacher S2) presented this topic.

Teacher S2: Calculations and problem solving (Grade 4)

A fair integration of VNVC characterised this lesson. Unlike the previous lessons in which Teacher S2 relied on one fraction model (the area model) to represent fractions, the use of an additional model (the set model) was observed in this lesson. Based on the components of the analytical framework, the VNVC and FM is a strong combination that has a direct effect on the pedagogy of fractions. Evidently, the presence of these two components had a positive effect on the other components of the analytical framework such as QFK, VL, QT and PCK.

Although the set model is not commonly used by teachers, it helps learners to understand that a group of objects can represent a whole, while the individual objects form subsets (parts) of the whole (Hull, 2005). In addition, the set model enhances learners' understanding of equivalent fractions and the application of fractions in real-life contexts (PCK). For instance, learners' conceptual understanding of equivalent fractions depends on their understanding of the notion that a half does not only represent one out of two equal parts, but it could also mean a half of four, a half of ten, a half of hundred, etcetera (QT). Essentially, the set model helps learners to understand the application of fractions in everyday life.

Teacher S2 introduced the new topic on calculations and problem solving by directing the learners towards the lesson objective. He indicated that the focus would be on the two operations, that is addition and subtraction of fractions that have the same denominators. Consequently, he introduced the rules that learners were expected to apply when adding fractions, that is adding the numerators and retaining the denominator. However, learners did not understand this because they continued to add the denominators as well. This was followed by visual representations as shown in Figure 4.16, but the teacher did not clearly explain why the denominators could not be added.

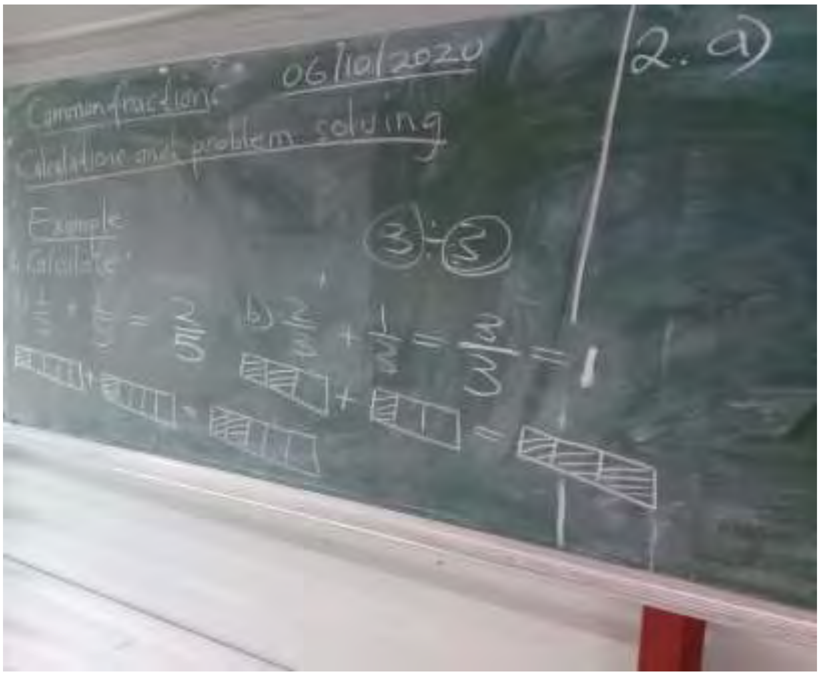


Figure 4.16: Teacher S2's strategy for adding same denominator fractions using the symbolic and visual mode

After solving the sums above, Teacher S2 wrote the following word problem and asked learners to solve it. *36 people live at a communal farm, one third of them are children. How many children live on the farm?* The teacher started by explaining what the learners were expected to do to find the answer, that is to find one third of thirty-six and to understand that 'of' in this context meant 'times' (or multiplication). Although the problem was supposed to be presented as one-third of thirty-six, Teacher S2 presented it as thirty-six of one-third. This resulted in the same answer because the product for both 36 of $\frac{1}{3}$ and $\frac{1}{3}$ of 36 is 12 (commutative property). However, the first notation is not a true representation of the problem. Thus, it conveyed a wrong message to the learners. This problem was solved in the following manner: 36 of $\frac{1}{3} = \frac{36}{1}$ x $\frac{1}{3} = \frac{36}{3} = 12$.

Teacher S2 also shared another method of solving this problem which involved the use of the set model. He did this by dividing the thirty-six across three sets or groups which resulted in each set having twelve children. This helped learners to understand why one third of thirty-six is twelve.

Despite the abstract nature of this topic, Teacher S2 was able to incorporate visualisation processes. By so doing, he accorded learners with different opportunities to solve addition, subtraction and multiplication fraction sums. Moreover, a balanced use of the symbolic and visual modes was observed in this lesson.

4.3.8 Topic 8: Assessment activity

This lesson was unique in the sense that it focused on the incorporation of visualisation in assessment strategies as opposed to teaching strategies. An assessment-based lesson was deliberately observed to determine the extent to which visualisation processes are incorporated in the design and execution of assessment activities. Teachers are expected to be consistent in their use of visualisation as a teaching strategy and an assessment strategy. Thus, observing how teachers incorporate visualisation in assessing their learners was deemed necessary.

Teacher D: Homework corrections (Grade 7)

This lesson was unusual because it focused on providing corrections to the previous homework. In spite of the nature of this lesson, there were no exceptions with regard to the observable indicators of the components of the analytical framework, because if well planned, lessons based on assessment can also include all the components of the analytical framework. However, what transpired in this lesson was an indication of poor integration of all the components: QFK, FMs, VL, VNVC, QT, PCK and EQS.

In analysing this lesson, emphasis was placed on EQS because the formulation of questions can determine the strategies that learners use to solve fraction problems. The shortcomings observed in this lesson can be attributed to two factors. Firstly, the homework questions did not include word problems. Secondly, Teacher D's silence on the overuse of the verbal code could have given learners the impression that this was the acceptable approach. Although Teacher D included the option to either use rules or visuals in working out the problems, the way in which the questions were presented, implicitly called for the use of the traditional, rule-oriented strategies. This lesson helped me realise that despite Teacher D's efforts to incorporate visuals in the previous lessons, he still advanced the verbal code in the assessment activities.

Since Teacher D had given learners an activity based on multiplying and dividing fractions the previous day, he used this lesson to assess learners' ability to answer questions based on the topics covered. The activity comprised of four exercises as shown below:

a) $\frac{3}{7}$ x $\frac{8}{9}$

b) $\frac{2}{3}$ x 8

c) $\frac{4}{5} \div 8$

d) $\frac{6}{15} \div \frac{6}{8}$

Teacher D's approach to this activity was such that he accorded learners the opportunity to work out the exercises on the chalkboard. He allowed learners to approve or disapprove the answers provided by their fellow learners rather than doing it himself. He only intervened to provide further clarification. Moreover, due to the learners' poor mastery of basic operations, especially multiplication and division, Teacher D had to divert from the topic on operations with fractions to basic operations which is a prerequisite to understanding operations with fractions. It took learners some time to understand why, for example, the correct answer for the problem $\frac{5}{15} \div \frac{5}{8}$ was $\frac{5}{15} \div \frac{5}{8} = \frac{5}{15}$ x $\frac{8}{5} = \frac{48 \div 3}{75 \div 3} = \frac{16}{25}$ and not $\frac{5}{15} \div \frac{5}{8} = \frac{5}{15}$ x $\frac{8}{5} = \frac{48}{75} = 1\frac{3}{75}$ as suggested by the learners.

The objective of this lesson rested on testing learners' understanding of multiplication and division of fractions using the visualisation processes or standard algorithms. Consequently, the option for learners to use visual strategies or standard algorithms was included in the instructions. However, apart from the first learner who attempted to use diagrams to solve the first exercise, the rest opted for the traditional, rule-oriented strategies. The inclination of both the teacher and learners to use standard algorithms as opposed to visualisation processes was observed in this lesson. This can be attributed to the cultural difficulties alluded to by Arcavi (2003). Cultural difficulties refer to the general perceptions about 'real' mathematics, which is often linked to the symbolic mode. Therefore, despite Teacher D's efforts to incorporate visuals in his lesson presentations, the use of visuals by both the teacher and the learners was short-lived because they preferred to use standard algorithms.

4.4 SUMMARY OF FINDINGS FROM THE INTERVIEWS

In order to obtain a comprehensive set of data for this study, an additional data collection instrument was employed, that is, semi-structure interviews. The semi-structured interviews were instrumental in the data collection process because they brought the participants' voices on the use of visualisation and its impact on their pedagogy to the fore. Through the interviews, participants were able to reflect and shed light on their own practices. Although I administered two sets of interviews, namely the pre-observation and post-observation interviews, I will only focus on the pre-observation interviews in this thesis because they were more comprehensive and addressed my research questions. The questions were designed with emphasis on key aspects of this study, such as the importance of fractions, the relevance of visualisation, pedagogical challenges and the impact of the RCFP on the pedagogy of the participants.

4.4.1 Importance of fractions

This question was deliberately asked to determine the participants' views on the importance of fractions as a mathematical concept and how this concept relates to other mathematics concepts. As stated by Bruce et al. (2013, p. 6), the pedagogy of fractions has broad and deep implications, "effecting foundational understandings that help or hinder the learning of other areas of mathematics." Therefore, through this question, I expected to establish the value attached to the concept of fractions.

Despite the participants providing various reasons pertaining to the importance of fractions, all eight participants indicated that fractions are an important concept in mathematics and in everyday life. Teacher D did not explicitly link fractions to other mathematics concepts but he made reference to its *"central role"* in learning mathematics.

Fractions is indeed important in mathematics. Fractions play a central role in mathematics learning. They require a deeper understanding of numbers than that typically gained from whole numbers

In his response, Teacher D also referred to the uniqueness of fractions in relation to whole numbers and how that renders them a more demanding concept in terms of teaching and learning. Generally, most of the participants focused on the application of fractions in everyday life. They explained how fractions are used in sharing activities and how that can help learners understand the meaning of fractions.

…fractions we are doing it…we are using it in everyday life. Ah…ah…let me say for example a loaf of bread, it's mos cut into slices. There you can just tell or even ask your kids that how many slices did you take…from the loaf of the bread

Although I expected participants to explain how the poor understanding of fractions affects other related mathematics concepts such as algebra, decimals, percentages, ratio and proportion (Fennel & Karp, 2016; Bruce et al. 2014), none of them alluded to that. This possibly points to the notion that mathematics concepts are taught in isolation which is a serious concern in the pedagogy of mathematics.

4.4.2 Visualisation

Since visualisation is a key concept in this study, it was necessary to establish the position of the participants on this concept. Therefore, questions were set to determine the relevance, general acceptance and implementation of visualisation. The participants' responses suggest that visualisation is generally accepted as a useful pedagogical approach.

4.4.2.1 Relevance of visualisation

All the eight participants concurred that visualisation is a vital aspect in the pedagogy of fractions. They all seemed to agree that learners learn best through visualisation. Teacher D pointed out that visualisation is an important component of a teacher's knowledge while Teacher K alluded to the importance of visualisation in the addition of fractions with different denominators as it helps learners to understand the addition rules.

Yeah, like I said…let's say you are adding up a half and a third. These two fractions have different denominators so, now you have to bring the concept of visuals whereby when you are going to make these fractions equivalent so that they have the same denominator and you add them up, learners need to know what are you really doing here?

Teacher K suggests that visualisation processes can enhance learners' understanding of one of the fraction concepts identified by the participants as a difficult concept. The difficulties associated with the addition of fractions that have different denominators were attributed to the cumbersome process involved, which requires the conversion of the original fractions to their equivalences. This step, in the addition and subtraction of different denominator fractions, is rarely comprehended by learners.

Fraction size is one of the fraction concepts that learners grapple with and this in turn affects their ability to compare and order fractions. In response to the question on the importance of visualisation, Teacher M2 suggested that visualisation can help learners understand fraction size through practical demonstrations. Consequently, this presents learners with opportunities to understand the rules related to the comparison of fractions.

Okay, ah…when you are using visuals, it makes them understand what you are teaching more or in the easiest way…whereby for example let's say I'm teaching ah…comparing fraction so when you tell them that rule that says the smaller the denominator the bigger the fraction…the bigger the denominator the smaller the fraction, learners really get confused but when you use visual things, they will understand it better.

Here, Teacher M2 alluded to the complementary role of fractions by explaining how the combined effect of the verbal and the nonverbal codes could enhance learners' understanding of the rules used in comparing fractions. Furthermore, the other participants suggested that visualisation enhances learners' comprehension and retention levels because through visualisation, learners are accorded the opportunity to see, touch and feel objects.

Ah…once you use visualisation, it's very difficult for a learner to forget. Learners always learn best when they see things. They will easily…easily remember things that they see ra…rather than remembering things that they only hear.

Despite the discrepancies observed in the practical use of visualisation in the participants' lessons, theoretically they all supported its use in the pedagogy of fractions. They all seemed to be aware of the long-lasting benefits of visualisation, although putting this into practice was a challenge.

4.4.2.2 The role of visualisation in enhancing learners' understanding of fraction size

The understanding of fraction size is a crucial aspect in understanding fractions comprehensively. It is as important as the understanding of the value of whole numbers. Thus, this aspect was deliberately included to determine whether participants understood the role of visualisation in enhancing learners' understanding of fraction size or not, since the symbolic notation of fractions has proved to be a challenge. All the participants seemed to concur that visualisation is a very important aspect in enhancing learners' understanding of fraction size. For instance, they justified their views by pointing out the role of visualisation in alleviating learners' misapplication of whole number knowledge.

Yes, the learners will know…like when you are talking about half. These are only two parts that are equal and then it have to be…halves have to be bigger than maybe quarters…uuh because sometimes learners do think, when you are talking about half, they will look at the number like one over two and the other one is one over four which is a quarter. Sometimes they do think quarter is bigger than the half but then when you give them example of an…an apple, for example which is cut into four pieces (Teacher M2)

Teacher M2 explained how learners tend to think that fractions that are made up of bigger numbers are bigger, while those with smaller numbers are smaller because of the influence of their whole number knowledge. Fraction size should be the starting point in the pedagogy of fractions because, just like whole numbers, it determines how well learners will be able to understand other fraction concepts. Barnett (2016, p. 18) asserts that:

> … understanding the magnitude of a fraction helps determine the reasonableness of an answer when operating with fractions, thus deterring students from employing flawed procedures, such as adding unlike denominators together when adding fractions.

The participants emphasised the role of visualisation in the pedagogy of fractions by, explaining how, from the onset, practical demonstrations, the visuality of lessons and the use of concrete teaching aids can enhance learners' understanding of fraction size. Therefore, the difficulties that learners encounter in understanding fractions in general can be attributed to the early introduction of the fraction notation.

4.4.2.3 Support from colleagues on the use of visualisation

A question on whether the participants were encouraged to incorporate visuals in their teaching by their colleagues was deliberately asked to determine the extent to which this concept is supported outside the RCFP. Teachers feel encouraged when their endeavours are supported by their colleagues. Hence, the relevance of this question.

In response to this question, all the participants indicated that they were supported by their colleagues and supervisors during their subject meetings. Teacher S2 explained how his supervisors encouraged the use of visuals in all subjects.

Yes…ah…like our principal and our HoD are encouraging us to use the visual...like not only in fractions…it's su…supposed to be any topic

While acknowledging that they are encouraged to incorporate visuals in their pedagogy, participants also indicated that this presents them with opportunities for collaboration with colleagues, since the concept of visualisation is fully supported.

4.4.3 Factors contributing to learners' poor performance in fractions

Through this question, I sought to establish the participants' views on factors that contribute to learners' poor performance on the concept of fractions. The underperformance of learners on the concept of fractions can be attributed to a number of factors which are mainly rooted in the complex nature of fractions and the pedagogical approaches employed by teachers

(Charalambous et al., 2010, p. 9). It is therefore important to consider these factors in a study of this nature.

The participants mentioned a number of factors which included finding the LCDs when adding fractions with different denominators, fraction notation, poor pedagogical strategies, lack of teaching materials, overcrowded classrooms, time allocation, poor foundation (prior knowledge), learners' negative attitude towards fractions, fraction terminology and fraction representation. These factors are divided into three categories as indicated in Table 4.5 below:

Table 4.5: Categories of factors that hinder the effective incorporation of visualisation

Nature of fractions	Instructional approaches	Administrative issues
Fraction notation	Finding the LCD	Lack of teaching materials
Fraction terminology	Poor pedagogical strategies	Overcrowded classrooms
	Poor foundation	Time allocation
	Learners' negative attitude towards fractions	
	Fraction representation	
	Lack of teaching materials	

As anticipated, most of the challenges raised by the participants fall into the second category, that is the instructional approaches. This shows how substantial a teacher's impact is on learners' performance. Teachers can often actually make or break the progression of learners in any mathematics concept. The pre-existing fear coupled with poor teaching strategies can hinder learners' understanding of fractions. Teacher K explained how the mere nature of fractions or the way fractions are written poses a threat to learners.

I think when they see…learners just generally when they see fractions…a number on top of another…they already fear

If not properly handled, this fear can develop into a negative attitude or disposition towards fraction concepts in general. Therefore, teachers should possess the necessary content and PCK to be able to mitigate against these challenges. Teacher M1 pointed out the importance of CK and PCK in the teaching and learning of fractions.

Some of the factors, just depend…sometimes on teachers…the way we introduce the topic or the way you know it, the background you are having, the content of the teacher on that topic can also contribute and even in most cases…you'll see, especially since I said teachers…some of them they say one over ten

Besides CK and PCK, Teacher M1 also referred to the role of the introduction in the teaching sequence. It is important to introduce fractions in such a way that learners' fears and misconceptions are alleviated by arousing their interest and building on their prior knowledge. Similarly, Teacher S1 referred to CK and PCK as a contributing factor to learners' poor performance. She acknowledged that the RCFP manual had compelled her to revisit her teaching methods.

Ah…I've learnt something today when I was teaching. I went through the book that you gave me (referring to the RCFP fraction manual) but then the method that I taught today…sometimes maybe okay…the teachers. Methods that we use also contributes to the failures of the learners. Ah…mmh…like the method that I used today…when we were adding and subtracting fractions with different denominators…I think I used a longer method…which is very difficult for learners…okay, it's only smart learners that will be able to catch up as to what I'm trying to teach… So, methodology of teachers…how to teach certain topics also leads to learners to not understanding of early…to understand what they are being taught.

It was very interesting and gratifying to witness how a product of the RCFP, the RCFP manual, was being used as a teaching resource by the participants. Most of them acknowledged that the content in the manual was useful because it enlightened and compelled them to reflect and improve their pedagogy.

The factors placed under the last category were beyond the participants' control because these were administrative matters that should be dealt with at a different level. However, the lack of teaching resources is both a pedagogical aspect and an administrative aspect because there are readily available teaching resources that teachers can draw on, while some may be inaccessible because of their commercial nature.

4.4.4 Rundu Campus Fraction Project (RCFP)

The Rundu Campus Fraction Project (RCFP) with its focus on visualisation processes, is an intervention programme that was primarily established to expose teachers (including the participants) to effective approaches in teaching fractions and to develop appropriate teaching aids. Consequently, the participants for this study were selected based on their membership to the RCFP. Thus, questions to determine the impact of the RCFP on the participants' pedagogy were included in the interview schedule. This section presents the participants' views on the impact and relevance of the RCFP.

4.4.4.1 The impact of the RCFP on participants' teaching approaches

All eight participants acknowledged that their participation in the RCFP had a great impact on their pedagogy. Below, Teacher D explained how his participation in RCFP activities had impacted his pedagogy:

Ah, it's a good one, so…ok. It...I would say it's…it's contributing to me positively, eh…because it came to remind me that, instead of using just the theoretical part of fractions, you can also introduce learners to…to visuals…using diagrams.

Generally, they all appreciated the visually-based strategies that they were exposed to in the RCFP and expressed their eagerness to implement these strategies in their classrooms. These strategies were described as beneficial to both teachers and learners because they enhance learners' understanding of complex concepts. Teacher M2 provided more insight on the importance of the visual approaches:

Oh, it helps! When you are using visuals it really has a good impact because learners understand in depth…like you are going in real life and they can see things rather than you teaching like according to the rule.

Moreover, two of the participants also appreciated the community of practice developed through the RCFP. They affirmed that their participation in the RCFP helped them to establish networks with colleagues that they could turn to for assistance. Teacher M1 described how his participation in the RCFP presented him with opportunities to share ideas on the pedagogy of fractions with other members.

Ah…it is helping. As a member, it is helping because we are sharing from…we learn from one another and what she does not know, I used to get it also from the colleagues at the campus. Yes, that's very important

In response to this question, two participants referred to the versatility of the RCFP manual. They indicated that they found the visual strategies on operations with fractions in the RCFP manual quite useful. Despite their different focus areas pertaining to the impact of the RCFP on their pedagogy, the participants had one thing in common, that is they all agreed that they benefited from the project and it had an impact on their pedagogy.

4.4.4.2 The pedagogy of the participants before joining the RCFP

One of the research questions that this study sought to address was the significance attached to the incorporation of visuals in the pedagogy of fractions. It was deemed necessary to include a question that would help divulge information on the pre-existing pedagogical practices of the participants. Their responses are presented in Table 4.6 below:

Table 4.6: Participants' use of visuals before joining the RCFP

Participants	Incorporated visuals	Incorporated a few visuals	Purely symbolic
Teacher D		x	
Teacher K			x
Teacher M1	x		
Teacher S1			x
Teacher M2	x		
Teacher S2	x		
Teacher I			x
Teacher S2		x	

Data from Table 4.6 shows that three participants indicated that they incorporated visuals in their pedagogy, two indicated that they incorporated a few visuals while three indicated that their pedagogy was purely symbolic. Teacher D, who indicated that he used a few visuals, pointed out that although the syllabus dictates that teachers should use diagrams, this was quite a challenge in some topics. Another participant, Teacher S1, explained that her pedagogy prior to joining the RCFP was purely symbolic but this changed after being exposed to the visual strategies in the RCFP.

I was more into symbolic but then since I joined the group, I noticed that more visual is important than symbolic

In addition, Teacher M1 who indicated that he used both symbolic and visual strategies, attributed his expertise to his long teaching experience.

4.4.4.3 The relevance of knowledge acquired from the RCFP

Another pertinent question for this study focused on finding out whether participants used the knowledge acquired from the RCFP or not. In response to this question, an overwhelming affirmation to confirm that they found the knowledge gained from the RCFP useful was received from all the eight participants. Teacher I stated that they found the knowledge acquired from the RCFP very useful.

Yes, yes, yes, indeed! Yeah, we are…we are using it. It's …it's…it's really helpful. It's very helpful…uuh

The excitement in the participants' responses was very explicit as they used words such as "*yes, exactly, definitely, yeah*" to ascertain the usefulness of the knowledge acquired from the RCFP in their pedagogy.

Since this study is informed by the Dual Coding Theory (DCT), it was necessary to gauge the participants' views on the use of one code (either symbolic or visual). The DCT advocates for the use of both codes because learners are presented with two separate pathways to retrieve information (Mayer & Anderson, 1991). In this question, I intentionally focused on one code to find out whether the participants would argue for the use both codes.

Generally, all the eight participants argued that ideally, teachers should use both codes because the use of one code could disadvantage some learners. Time emerged as a stumbling block for incorporating the visual code. Teacher K argued that the use of either the symbolic or verbal code saves time but does not enhance learners' understanding.

Yeah, like just using theory…that one okay, it saves time, like I can explain, explain but then not a lot of learners will understand

Disadvantages associated with the incorporation of visuals, included time and the delay in the introduction of the formal mathematical notation. According to Teacher S2, the incorporation of visuals enhances learners' conceptual understanding but it delays the development of the symbolic notation, which is dominant in the formal mathematical calculations.

…the disadvantage of like only using visual and no written…no notation item so learners will lack the notation skill.

Teacher M2 supported the use of the symbolic or verbal code on the basis that it paves the way for uniformity, because through the symbolic code, learners are exposed to conventional methods of doing mathematics, or standard algorithms.

You try to make sure that every learner should know or have the same way of answering things. So, when you are focusing on that, you just say this is how you are going to do it and that's it. So, it's like you are not giving them another choice…so then you…ok this is the right way…like you are just giving them one straight answer so then there's no need for them to think about other answers or other way but when they know that one rule, it helps them and then it also gives you time.

The participants justified the incorporation of visuals by focusing on the complementary role of the visuals. They explained how learning through seeing and touching deepens learners' understanding of mathematical concepts. The symbolic code was perceived as the common approach. However, they all held the view that this approach promoted rote memorisation of symbols among learners.

4.4.5 Pedagogical considerations

Questions under this section were designed with specific pedagogical aspects in mind, since the pedagogical choices that teachers make during their lesson planning may affect the teaching trajectory of fractions. This section is therefore important in the sense that it addresses issues that have a direct impact on the teaching and learning of fractions. For instance, the way a teacher introduces the concept of fractions to learners can either build or ruin their chances of meaningfully understanding fractions in advanced grades.

4.4.5.1 Introducing fractions

Since the introduction is an essential component of the learning trajectory, it was necessary to determine the importance that the participants attached to this component. The way fractions are introduced can determine how learners perceive fractions and the understanding that is developed thereof. I therefore included this question in the interview schedule to establish whether participants realised the key role that this component holds in the pedagogy of fractions through their choice of strategies. Their responses are summarised in Table 4.7 below:

Table 4.7: Strategies used by participants to introduce fractions

Participants	Strategies used to introduce fractions
Teacher D	Uses concrete materials such as chocolate and bread
Teacher K	Uses concrete materials such as apples
Teacher M1	Sharing activities and visuals
Teacher S1	Uses visuals (area model) to define fractions
Teacher M2	Uses shapes to help learners realise that fractions are derived from wholes
Teacher H	Starts by helping learners to make a distinction between whole numbers and fractions
Teacher I	Uses visuals to illustrate fractions
Teacher S2	Uses models

The use of visualisation was evident in the participants' responses. All participants, except Teacher H, alluded to the incorporation of visuals or concrete teaching materials in the introduction of fractions. This suggests that the participants were aware of the fundamental role of this component in setting the precedence for the comprehension of fractions. Although Teacher H did not make any reference to the incorporation of visualisation, she mentioned something very important by pointing out the importance of helping learners distinguish between whole numbers and fractions from the onset. In addition, Teacher M2 concurred that visuals should be used to help learners realise that fractions are derived from wholes.

4.4.5.2 Types of visuals used to enhance learners' understanding of fractions

The use of different types of visuals enhances learners' understanding of fractions because they are presented with more opportunities to learn fractions (Van de Walle et al., 2013; Cramer & Whitney, 2010). A question on the type of visuals used by the participants was deliberately set to determine the categories of visuals that they prefer to use in their classrooms. The categories of fraction models under which the visuals may be placed are area, length and set models. The type of visuals used by the participants are presented in Table 4.8 below.

Table 4.8: Visuals used by participants to enhance learners' understanding of fractions

Participant	Types of visuals	Category
Teacher D	Fraction wall	Length
Teacher K	Circles	Area
Teacher M1	Circles, rectangles	Area
Teacher S1	Diagrams,	Area
Teacher M2	Shapes and concrete materials, e.g. bread, apples, oranges	Area
Teacher H	Posters, concrete materials, e.g. apples	Area
Teacher I	Posters	Area
Teacher S2	Circles	Area

The findings reveal that most of the participants relied on the area model. This confirms Cramer and Whitney's (2010) concern about the overuse of the area model which deprives learners from learning fractions across a variety of models. Thus, learners acquire limited knowledge of fractions and this hinders their conceptual understanding. The overuse of the area model can be attributed to the participants' inadequate knowledge on the use of the length and set models as some admitted that despite their participation in the RCFP, they were not confident enough to use these models in their classrooms.

4.4.5.3 Preferred models

Visualisation is a broad concept and sometimes desired results are not obtained because teachers focus on one type of visual model instead of exposing learners to different types of visual models. According to Van de Walle et al. (2013), the meaningful understanding of visuals can be facilitated by the use of different fraction models because learners are presented with different opportunities to comprehend fractions. It is against this background that a question on the use of different models was posed. The participants' responses are presented in Table 4.9 below:

Table 4.9: Participants' preferred models

Participants	Preferred models			Justification
	Area	Length	Set	
Teacher D	x			- More convincing - Fear of introducing new models
Teacher K			x	- Easier to understand - Fear of introducing new models
Teacher M1	x	x		- Make the comparing of fractions easier
Teacher S1	x			- Easier to understand
Teacher M2	x			- Easier to understand
Teacher H	x			- Features often in exam papers
Teacher I	x	x		- Easier to understand
Teacher S2	x			- Easier to understand - Common in prescribed books for mathematics

As indicated in Table 4.9 above, the area model was preferred by all the participants except for Teacher K. Their main reason for opting for the area model was that it is easier for learners to understand. Although most of the participants associated the use of the area model to 'easy understanding', they found the use of the other models quite challenging, hence they were not confident enough to incorporate them in their lessons. For instance, Teacher D and Teacher K expressed their fear of introducing other models.

Teacher M1 explained how the use of the area model helps learners to understand fraction size and compare fractions. Meanwhile, Teacher H and Teacher S2 who both opted for the area model, pointed out that they are compelled to use the area model because this is the model that is dominant in examination papers and prescribed books. Since they wanted to prepare learners for examinations and perceived the area model as the commonly used model, these participants neglected the other models and focused on one model.

4.4.5.4 Preferred fraction constructs

There are five fraction constructs that are used to represent fractions: namely the part-whole, measurement, division, operator and ratio. Teaching fractions across these different constructs presents learners with more opportunities to understand fractions (Van de Walle et al., 2013).

The participants were asked to indicate the constructs that they most often use in their fraction lessons and their responses are reflected in Table 4.10 below:

Table 4.10: Participants' preferred fraction constructs

Participants	**Preferred models**					**Justification**
	Part-whole	**Measure-ment**	**Division**	**Operator**	**Ratio**	
Teacher D				x		Other constructs do not feature in the syllabus
Teacher K	x					No justification provided
Teacher M1	x	x				Part-whole is appropriate for introducing fractions while measurement is appropriate for operations with fractions
Teacher S1	x					Easier to understand
Teacher M2		x				Easier to count on a number line
Teacher H	x					Easier to understand
Teacher I	x					Common and easier to understand
Teacher S2			x			Learners can relate to it

Based on Table 4.10 above, five participants opted for the part-whole construct, two opted for measurement, one for division and another one for the operator. The most preferred construct was the part-whole and the two main reasons provided for choosing this construct were based on the notion that it is more common and easier to understand. One of the participants (Teacher M1) opted for both the part-whole construct and measurement and he explained that the part-whole construct is suitable for introducing fractions while measurement is suitable for doing operations with fractions. Ratio was not opted for by any of the participants although it is a very important construct that helps learners to compare part-part and part-whole relationships. The findings reveal Van de Walle et al. (2013)'s concern about the overuse of the part-whole construct. This hinders the meaningful understanding of fractions (Bruce et al., 2013; Clarke, 2011) because learners' understanding of fractions is enhanced when they are exposed to different fraction constructs.

4.4.5.5 The use of visualisation during different lesson components

A question on the use of visualisation during different lesson components was posed to determine whether participants understood the importance of visualisation in all the lesson components. The use of visuals in the introduction, lesson development, reinforcement and conclusion is imperative and teachers should be encouraged to use visualisation processes across different lesson phases.

Table 4.11: The use of visualisation in different lesson phases

Participants	Introduction	Lesson development	Reinforcement	Conclusion
Teacher D		x		
Teacher K	x			
Teacher M1	x			
Teacher S1	x	x	x	x
Teacher M2		x		
Teacher H	x			
Teacher I	x	x		
Teacher S2	x			

The data from Table 4.11 shows that visualisation is more prevalent in the introduction and the lesson development. The reasons provided by the participants for using visualisation in the introduction were based on building learners' conceptual understanding and arousing their interest. They suggested that it is vital to capture learners' attention and clarify concepts in the introduction before moving to other components of a lesson.

I only use the visualisation in introduction…just to capture the attention (Teacher S2)

Yeah, visualisation is very important to be used in the introduction because from there it's when they can know now how to apply the other components also (Teacher M1)

Meanwhile, some of the participants justified the use of visualisation in the lesson development by stating that that is where concepts are elaborated on in more detail. Two out of the four participants who indicated that they would rather use visualisation in the lesson development also supported its use in the introduction. Teacher D clearly stated that he would rather use the symbolic mode in the introduction and only introduce the use of visuals in the lesson development.

The lesson presentation. Yes, I think that is the part…because that is where the whole content of the lesson is learned. Yes, I can introduce it by using numbers only but then later, I need to go deep...where I need to present by using the figures also …

Surprisingly, only one participant (Teacher S1) indicated that she would use visualisation in all the lesson phases. The use of visualisation cuts across all lesson components from the introduction to the conclusion, but it takes a teacher with the appropriate PCK to acknowledge it (Fennema & Franke, 1992). As observed during the lesson observations, the participants did not incorporate visualisation in the last two components of their lessons, that is the reinforcement and the conclusion, which defeats the purpose of visualisation. Visualisation does not make sense to learners if it does not feature in the assessment activities because learners are more concerned about how useful the strategies introduced to them are, in solving mathematics problems.

4.4.5.6 The selection of visuals

Several factors – such as the topic, the lesson objective, the profile of the learners, the level of the learners – determine the selection of visuals. A question was therefore posed to determine what informs the participants' selection of visuals. In summary, most of the participants indicated that their choice of visuals is determined by the topic and the basic competency or lesson objective of a lesson as shown in Table 4.12 below.

Table 4.12: Determining factors in the selection of visuals

Participant	Major determining factors in selection of visual
Teacher D	Topic
Teacher K	Topic
Teacher M1	Learners' background
Teacher S1	Topic
Teacher M2	Topic
Teacher H	Lesson objective

Teacher I	Lesson objective
Teacher S2	Lesson objective and textbook

Although all the factors mentioned by the participants are crucial in the selection of visuals, the lesson objective is key because it is the driving force behind all the activities that teachers and learners engage in. The use of the lesson objective to select appropriate visuals can alleviate the incorporation of visuals that are too general or misplaced as observed in some of the participants' lessons. The selection of visuals should be properly aligned to the lesson objectives if they are to be regarded as useful. Furthermore, this can also prevent inconsistencies between the visual and the verbal codes since the two codes are supposed to speak to each other. Therefore, teachers should pay as much attention to the selection of visuals as they do to the selection of content for any specific lesson.

4.4.5.7 Strategies used to enhance learners' fraction sense

A question on the strategies used to enhance learners' fraction sense formed part of the interview schedule because this study intends to identify the visualisation strategies or processes used by the participants to teach fractions. In response to this question, participants made reference to the use of real-life examples such as bread, pizza and fruits to define fractions and demonstrate fraction size. For instance, Teacher M2 pointed out the importance of using concrete objects such as fruits to illustrate the sizes of fractions with different denominators and bring the concept of fractions closer to the learners. She further explained how the use of concrete objects can be used to reduce the misconception that fractions with bigger denominators are always bigger, irrespective of the size of the numerators.

Let me say small pieces but they are a lot and then a whole you divide it into pieces whereby they are less. Then learners really understand, ok this is what they mean by small and this is what they mean by big.

Teacher S1 on the other hand, suggested that the use of the number line is an appropriate strategy to enhance learners' understanding of fraction size, ordering and comparing of fractions.

…so the learners will understand that….uuh…from this to here, this is the fraction then the next one will be this…then we have a whole, after a whole this is the fraction that will be there

Cramer and Whitney (2010) assert that the use of the number line model helps learners understand that in essence, fractions are an expansion of the number system. Through its iterative property, the number line model enhances learners' understanding of fraction parts as repeated parts rather than as two unrelated parts.

On the contrary, Teacher S2 made reference to the curriculum of the grade that he taught (Grade 4) which stipulated that learners were only required to compare same denominator fractions. Unlike the other participants who proposed the incorporation of visual strategies, Teacher S2 alluded to the use of rules.

…you just tell them like when the fractions have the same denominator, you just look at the denominator so the numerator that has a big number is the one that is big

He further explained that since the focus was on same-denominator fractions, teachers were expected to teach learners to apply the rule that says when fractions have the same denominator, the fraction with a bigger numerator is always bigger. Although this rule-based strategy is common, it does not enhance learners' conceptual understanding of fractions.

4.4.5.8 Quality fraction instruction

A question on quality fraction instruction was relevant because it helped to determine the participants' understanding of this concept. Quality fraction instruction may be interpreted in different ways by different teachers, so I deemed it necessary to discover what this concept meant to my participants. The participants associated quality fraction instruction to the following factors: 1) the achievement of the basic competencies; 2) the ability to arouse learners' interest in your lesson; 3) the performance of learners; 4) learners' participation and their responses to questions; 5) the retention and applicability of knowledge; and 6) learners' ability to follow instructions or rules.

Basically, all the factors mentioned by the participants contribute to quality fraction instruction in one way or the other. For instance, while it is important to achieve the basic competencies, it is the strategies that the teacher employs to achieve those basic competencies, it is important to consider the strategies used by the teachers. The strategies used, determine whether the knowledge acquired will retained or not. As stated by Teacher S2, quality fraction instruction promotes the retention and applicability of knowledge.

So, it…it…it means that ah…the teacher will give that knowledge of fraction to learners and those learners they will never forget that information. So, they will use it in their entire lifespan

This view is supported by Fuma (2018, p. 3) who asserts that "it is vital to introduce students to concepts at primary level using techniques that enable them to reflect on it throughout their school careers." On the contrary, Teacher S1 associated quality fraction instruction to the learners' ability to follow instructions or rules religiously.

ok…instructions are very important. What is required from a learner? What should a learner do? What steps should a learner follow? In that the…instructions are very important. Let me just say instruction. What steps should a learner follow when they are either calc…calculating either adding or subtracting. They must just know the instructions…the steps. It will be very easier for them

This is a typical traditional, teacher-centred explanation of quality fraction instruction. However, this approach often does not enhance learners' understanding of fractions because some learners follow the rules or instructions without understanding. Moreover, these rules are easily forgotten and they cannot be applied in other contexts.

4.4.5.9 Participants' views on the incorporation of visualisation to teach fractions at the senior primary phase

This question was included in the interview schedule to gauge participants' views on the incorporation of visualisation processes at the senior primary phase. Although this question was addressed in the questionnaire, it was necessary to ascertain the participants' views in this regard since visualisation is a cross-curricular concept (Boaler et al., 2016). Generally, all the participants supported the use of visualisation in the senior primary phase and some even suggested that it should be used in all phases.

Oh, my view on that one is, it is…it's a lifelo…lifelong learning. You should just start from wherever you started and it go even up to the tertiary levels so that learners can get a better understanding for it (Teacher M1).

Teacher M2 justified her position on this aspect by stating that continuity in the use of visualisation provides opportunities for learners to understand mathematics concepts better, especially learners who have been taught by different teachers.

4.4.5.10 The sequence of rules versus visualisation

This question was specifically included to determine the participants' views on the sequence of visual and rule-based strategies. According to Charalambous et al. (2010), the early introduction of rules and standard algorithms can hinder effective fraction instruction. To answer this question, the participants were presented with the following choices:

a) introducing the rules first before explaining fraction concepts using visuals;
b) explaining concepts first using visuals before introducing the rules; and
c) introducing the rules and explaining fraction concepts using visuals concurrently.

Their responses are presented in Table 4.13 below:

Table 4.13: Participants' options on the sequence of rules versus visuals

Participant	Options
Teacher D	a)
Teacher K	b)
Teacher M1	b)
Teacher S1	a)
Teacher M2	a)
Teacher H	a)
Teacher I	a)
Teacher S2	a)

The data in Table 4.13 indicates that most of the participants opted for the first option which advocates for the use of rules first and visuals second. This entails that they introduce rules and standard algorithms before they bring in visuals, a view that is not supported by Barnett (2016, p. 4), who argues that:

> … a child's difficulty in working with fractions may be related to teachers moving too quickly towards procedures so students can *do* fractions. Instead of rushing to procedures, students need the time and freedom to conceptualise, organise and assimilate the notion of a fraction into their already developed informal framework of fractions.

Only Teacher K and Teacher M1 opted for option b), where instruction is expected to proceed from visual to standard algorithms. This option, according to Boaler et al. (2016, p. 16) enables learners to appreciate "the rich knowledge they hold, of the deep understanding that is enabled – both from teachers introducing mathematical ideas visually, and students using visuals to think and make sense of mathematics". Notably, although the simultaneous use of the verbal and nonverbal codes may help learners see how the two codes are connected, none of the participants opted for it.

4.5 SYNTHESIS OF MY ANALYSIS

Based on the analysis of the lessons presented by the participants, it was evident that the incorporation of visuals affected the lesson trajectories significantly. In cases where visuals were objective-driven and well integrated, quality fraction instruction was enhanced. However, the incidental use visuals did not yield any positive results in the teaching of fractions as some visuals were misplaced and confusing. Evidence gathered from the twenty-five lessons affirms the notion that the use of either the verbal or the nonverbal code is important in the teaching of mathematics, however, the combination of the two codes supersedes the impact of either of them (Mayer and Anderson, 1991). Notably, the misapplication of either the visual or the symbolic mode or both can have detrimental effects on the pedagogy of fractions

Undoubtedly, the incorporation of visualisation enhanced participants' fraction instruction but this was subject to a number of factors, such as the relevance of the visuals, teachers and learners' exposure to the use of visuals and most importantly, the teacher's PCK. The success of these lessons was not actually determined by the mere visuality of the lessons but rather by the teacher's ability to carefully choose the visuals and incorporate them effectively. Table 4.14 below is a summary of the use of visuals by the twenty-five participants. It shows that all the participants used visuals in at least one of their lessons.

Table 4.14: The overall use of visuals

No.	Participant	Incorporation of visuals	Percentage of use of visuals
1	Teacher D	Three out of four lessons ($\frac{3}{4}$)	75
2	Teacher K	Two out of three lessons ($\frac{2}{3}$)	67
3	Teacher M1	Two out of four lessons ($\frac{2}{4}$)	50
4	Teacher S1	One out of three lessons ($\frac{1}{3}$)	33
5	Teacher M2	One out of three lessons ($\frac{1}{3}$)	33
6	Teacher H	Three out of four lessons ($\frac{3}{4}$)	75
7	Teacher I	One out of two lessons ($\frac{1}{2}$)	50
8	Teacher S2	All three lessons ($\frac{3}{3}$)	100
			Avg: 60.4%

On average, 60% of the observed lessons were visual. Some of these visuals enhanced the participants' pedagogy while others did not. The findings also reveal that visualisation is a language that should be taught and learned. It is a language that should be carefully introduced to avoid confusion among learners. In this study, the participants used visuals in different ways, and thus they produced different results. The study further affirms that the mere incorporation of visualisation processes in the teaching of fraction concepts does not automatically translate into successful lessons. On the contrary, the incorporation of visuals must be goal-driven to have an impact on the pedagogy of teachers. As mentioned earlier, there are a number of factors that should be taken into consideration when using visualisation to complement the symbolic mode. This section provides a summary of this study by answering the research questions below:

4.5.1 Types of visuals used by senior primary school teachers in their mathematics lessons

Just like the symbolic mode, it is important to select the visualisation processes carefully because the random use of visuals can derail learners from the intended lesson objectives. There was a clear distinction in terms of impact between the lessons in which visuals were objective-driven and lessons in which participants used visuals incidentally. Based on the findings from the lesson observations, the participants incorporated different types of visuals. Table 4.4 and Table 4.8 provide evidence of the different types of visuals that were used by the participants. Although they incorporated a variety of visuals, some of the visuals were more dominant than others. For instance, in terms of fraction models, the area model was more prevalent than the length and set models.

Despite the participants' attempts to incorporate a variety of fraction models, they were challenged in terms of using the three models simultaneously. This is important because it accords learners with various contexts to experience fraction contexts. The use of one type of model – particularly the area model – was dominant in most of the lessons while the other models were seldomly used.

The following combinations of fraction models were also observed in the lessons of a few participants: 1) area and length models and 2) area and set models. However, the combination involving the length and set model was not observed in any of the lessons. This combination presents learners more opportunities to understand fractions. The infrequent use of the length model may be associated with Niemi's (1996, p. 353) claim which is based on the notion that the effective use of this model requires the "coordination of multiple units simultaneously and understanding that fractions are numbers representing relations between other numbers." Based on the challenges experienced by the participants, particularly the use of inappropriate fraction language, it can therefore be presumed that this is the understanding that seemed to be lacking. On the other hand, both the length and set models significantly enhanced the participants' pedagogy in terms of problem-solving exercises.

Generally, the models made a difference in the pedagogy of the participants, depending on how they were used. Some of the participants used the models in such a way that that they complemented the symbolic mode. This was the major distinguishing factor between the visuals that impacted the participants' lessons and those that did not. Some of the participants used visuals that were not properly integrated in their lessons. Moreover, in some of the lessons, the verbal code and the nonverbal code were not used in a coordinated manner and this rendered them useless.

Another interesting observation pertaining to the use of the fraction models when doing operations with fractions was the fact that the participants who used different fraction models in their lessons opted to use them selectively. In other words, they only used them to add, subtract, multiply or divide fractions that had the same denominators while none of them used them to do operations with different-denominator fractions. This can be attributed to the complex nature of the latter. Thus, learners were deprived of opportunities to experience the use of the different models across all types of fractions.

The participants' choices in terms of visuals can be attributed to the factors alluded to in the interviews as most of them indicated that their selection of visuals was informed by the topic rather than the lesson objective. This explains the discrepancies observed between some of the visuals used by the participants and the lesson objectives. Although the participants fully supported the use of visuals in the questionnaire and the interviews, their shortcomings were revealed in the actual implementation of the visual approach through the data from the lesson observations and the interviews. The interviews brought the participants' uncertainties and fears about the use of visualisation to light, because they were unable to distinctly justify their choices of visuals. The same trend was also observed in their teaching.

4.5.2 The incorporation of visuals in fraction lessons

Just like symbolic language, visual language is a language that has to be learned and practised in mathematics classrooms for it to have an impact on the pedagogy of fractions. As observed in the participants' lessons, the concept of fractions is delicate, regardless of the mode of teaching. This concept is difficult to teach and learn, not only in terms of the verbal code but the nonverbal code as well. Since fractions are complex in nature, the introduction of both codes should be cautiously done to avoid confusion among learners.

The incorporation of visuals by the participants in this study can be described as a secondary rather than a primary strategy. Although visuals were effectively used in some instances, the participants seemed to rely more on the verbal rather than the visual code. Drawing on their responses to the last question in the interview schedule, it is obvious that more emphasis was placed on the rules rather than the visuals. Moreover, the participants struggled to answer the question on the strategies used to enhance learners' understanding of fraction concepts.

The lesson samples show how participants who were competent in both the verbal and the visual codes were able to successfully incorporate visuals in their lessons by: 1) selecting the appropriate visuals, 2) using a variety of models, 3) correctly aligning the visuals to the lesson objectives and 4) infusing the visuals into the verbal code by interchangeably using the two codes. This was evident in some of the participants' lessons while others struggled to find a common ground between the verbal and the visual code. Another category of participants leaned more towards the verbal code and produced positive results. The success of these participants can be attributed to their good CK and PCK. This was evident in the manner in which they presented their lessons, by building on the learners' prior knowledge and asking relevant questions.

Lack of consistency in the use of visuals was also observed among the participants. This can be ascribed to the nature of the topic as the participants found some fraction concepts more visual-friendly than others. The findings from the questionnaires and interviews reveal that the addition and subtraction of fractions were identified as difficult concepts to teach visually by most of the participants. Furthermore, some participants were observed incorporating visuals in one or two lessons but none in the other lessons. This is an indication that the incorporation of visuals was influenced by a number of factors.

4.5.3 Significance attached to the use of visuals

The findings from the questionnaire and the interviews affirm that the participants valued the incorporation of visualisation in mathematics, particularly in fraction lessons. They perceived visualisation as an indispensable concept in the pedagogy of mathematics and explained how it facilitates the retention and retrieval of knowledge. The last column of Table 4.2 presents evidence regarding the value attached to visualisation. They justified their support for the incorporation of visualisation by using phrases such as: … *gives a clear picture of the content taught* (Teacher K), ... *learners may gain a deeper understanding of fractions* (Teacher S1), ... *learners will not forget what they have seen* (Teacher M2).

Similar sentiments were echoed by the participants in the interviews as they elaborated on the importance of visualisation in the pedagogy of fractions and alluded to its complementary role with standard algorithms. However, the strong views expressed in the questionnaire and interviews were not fully reflected in the lesson observations. Some of the participants were caught between the verbal and visual codes. This can be attributed to the deep-rooted traditional teaching strategies that they have acquired over the years. Thus, by inference, the value attached to the incorporation of visuals was partially demonstrated.

4.5.4 The enabling and constraining factors in teaching fractions in an explicitly visual way

Evidently, the desire to incorporate visuals in fraction lessons prevailed among the participants but they were held back by certain factors. The knowledge gained through their participation in the RCFP, the RCFP manual and the support from their colleagues at schools were identified as some of the enabling factors for the effective incorporation of visualisation. Their responses to the question about whether they were encouraged to use visuals by their colleagues or not (see Section 4.5.2.3) is enough proof that this concept was overwhelmingly supported beyond the RCFP. Therefore, they could have capitalised on the support that they received from the RCFP, the colleagues and supervisors at their schools. Moreover, the participants admitted that the RCFP had a positive impact on their pedagogy and that their experience had helped them to view the teaching and learning of fractions differently.

Besides the participants' tendency to cling to the symbolic, traditional approaches, several constraining factors were identified. Section 4.5.3 provides an overview of the constraining factors that prevent teachers from incorporating visuals in their fraction lessons. Several factors were identified by the participants, which were then placed under three categories, namely the nature of fractions, instructional approaches and administrative challenges. Additionally, Table 4.5 shows that most the identified challenges fell under instructional approaches which is the focus of this study. Therefore, there are several factors to consider in our attempt to help teachers adopt new teaching strategies.

4.6 CONCLUSION

Although there is enough evidence to suggest that the participation of the participants in the RCFP had a fairly visible impact on their teaching approaches, instances in which some of the participants reverted to their old practices were also observed. Based on the findings from the questionnaire and the interviews, the incorporation of visualisation processes was greatly appreciated by the participants and they strongly advocated for its incorporation in the teaching of fractions. However, the real impact of the RCFP on the pedagogy of the participants was drawn from the lesson observations.

The observed lessons brought to light the participants' attempts to incorporate visualisation processes in their lessons. Each of the eight participants used visuals in at least one of their lessons. However, the key distinguishing traits among the visual lessons was the extent to which the visuals impacted the participants' pedagogy. Based on the way the visuals were used, three categories of participants emerged. The first category of the participants used visuals to complement the symbolic code, the second group used visuals as a teaching strategy to enhance their pedagogy, while the third group used visuals without any clear intentions. The lack of knowledge and skills regarding the integration of visualisation was a major contributing factor to the differences observed in the use of visualisation processes among the participants. This is an indication that visualisation is a language that must be learned for it to be effectively incorporated in the pedagogy of different mathematics concepts.

The exposure of both teachers and learners to visualisation is an important consideration in the teaching of fractions because the meaningful understanding of fraction concepts can only be enhanced if teachers have adequate knowledge and experience in the use of visual language and can confidently merge it with symbolic language. Hence, there was less confusion between the two modes in fraction lessons where participants demonstrated adequate understanding of both visual and symbolic language because they were able to use the two modes interchangeably. Consequently, participants who possessed adequate symbolic and visual knowledge presented their lessons in a more comprehensive manner.

Despite the inconsistencies observed in some instances, the RCFP undoubtedly had an impact on the pedagogy of the participants. For some, this impact was subtle and insignificant while for others it was explicit and quite noticeable. Factors such as curriculum obligations, the participants' readiness to go beyond the curriculum requirements, their philosophical views on teaching and learning and learners' exposure to visualisation processes affected the extent to which the pedagogy of the participants was transformed. For instance, Teacher M2 explained the importance of ensuring uniformity in the way that content is presented. She indicated that learners must be taught and be prepared to answer questions in the same way (see Section 4.4.2.1). In addition, she held the view that visualisation could confuse learners because unlike the symbolic code, visuals can be interpreted differently by different individuals.

Another important factor that was cited by the participants was the view that the use of visuals was optional and this compelled some of them to adhere to teaching strategies that promoted the symbolic code. It was interesting to observe that the lessons of the participants who taught

Grade four (which is the transition grade for senior primary) were more visual than those of the other participants. This can be ascribed to the presence of visuals in the prescribed books and the syllabus. There was a tendency among the participants to value what was stipulated in the syllabus and prescribed books more than the research-based pedagogical approaches. Despite the diverse classroom experiences, most of the participants appreciated the knowledge acquired from the RCFP and the RCFP manual. They acknowledged the value of the pedagogical skills gained and how this had impacted their teaching.

CHAPTER 5
DISCUSSION, CONCLUSIONS AND RECOMMENDATIONS

5.1 INTRODUCTION

The Dual Coding Theory (DCT) with its emphasis on the use of VNVC was at the heart of this study. Thus, the research questions and processes were informed by the DCT. Despite the inconsistencies observed in the use of visuals, this study revealed that the RCFP had an impact on the pedagogy of the participants. This was evident in the participants' attempts to explain different fraction concepts using visualisation processes. This chapter provides a summary of the research findings by explaining how the research questions were addressed, and also proposes some recommendations.

5.2 A REFLECTION ON HOW THE RESEARCH QUESTIONS WERE ADDRESSED

5.2.1 What type of visualisation processes do senior primary school teachers incorporate in their mathematics lessons?

Three categories of fraction models, that is, area, length and set models were used by the participants. Of these three, the area model was the most prevalent, while the set model was the least prevalent. Examples of visuals observed in the participants' lessons included the fraction wall, geometrical shapes such as rectangles and circles, and concrete objects such as oranges and other diagrams. These models were presented in four different forms, that is, on posters, physically or in concrete form, and on chalkboards. However, the chalkboard visuals were most frequently used. In other words, most of the participants relied on the chalkboard for presenting both the visual and the verbal fraction content.

Some of the participants used posters to demonstrate how the set model can be used to solve word problems. Although no reference was made to the set model, the posters were effectively used to illustrate how a group of objects can be regarded as the whole, out of which smaller subsets or fractions may be derived (Hull, 2005). The use of posters helped learners to visualise the whole set and the smaller subsets that made up the whole. This was an important strategy in the sense that it presented learners with an alternative way of partitioning fractions, as opposed to what learners were familiar with: partitioning of shapes and number lines (Van de Walle et al., 2013). Therefore, the posters were effectively used to enhance learners' understanding of fractions in different contexts.

The use of concrete objects was observed in one of the participants' lessons, where oranges were used to demonstrate fraction size and equivalence. This was practically infused into the lesson and learners were actively involved in partitioning the oranges. In her attempt to illustrate how a whole orange relates to a half and a quarter and how the half and the quarter relate to each other, Teacher I presented learners with two oranges which they were expected to cut. She gave the learners clear instructions on how the oranges were supposed to be cut and asked them questions based on the parts of the oranges. According to Cramer et al. (2008), this is a good starting point in the pedagogy of fractions. Considering the fact that an orange is circular in nature, it is a recommended representation of the part-whole construct of fractions. Moreover, the circular model also illustrates the relative size of the part to the whole.

Another important visual model used by the participants was the fraction wall. The fraction wall was quite instrumental in the pedagogy of fractions. If effectively used, it can enhance learners' understanding of not only fraction size but comparing and ordering fractions, fraction equivalence and operations with fractions. Some of the participants extracted examples of fraction walls from the RCFP manual to teach all the above mentioned concepts. In most cases, it was not the mere use of the fraction wall that made a difference to the way in which the lessons were presented, but rather the types of questions that teachers posed. This is a very important consideration when using the fraction wall as a strategy to enhance learners' understanding of fractions because it helps to put the visuals into context.

Shapes such as rectangles and circles were also used to illustrate fraction size and operations with fractions. The participants conveniently used circles to add and subtract same-denominator fractions. Justifiably, circles presented learners with opportunities to understand the part-whole construct and this also made them suitable for demonstrating the addition and subtraction of fractions. Conversely, rectangles were used to represent addition, subtraction, multiplication and division problems involving both same-denominator and different-denominator fractions.

5.2.2 How do senior primary school teachers incorporate visualisation processes in their mathematics lessons?

As indicated in the previous section, the participants used a variety of fraction models. While some of the participants drew on their RCFP experience and prescribed books to choose appropriate fraction models, others relied on the RCFP manual which contains various examples of fraction models. Bruce et al. (2013, p. 32) assert that

> instructional decisions have a significant bearing upon students' ability to understand the concept of fraction, including the ability to represent fractions appropriately, compare the relative magnitude of two fractions, and complete calculations accurately.

Thus, the decisions that teachers make regarding the type of visuals to include or exclude in their fraction lessons are important considerations in the pedagogy of fractions. After selecting the visuals, it is important to consider how these visuals were incorporated in the fraction lessons and this is the focus of this section.

The participants incorporated visualisation processes in three ways. Based on the observed lessons, visuals were used: 1) to introduce fraction concepts, 2) to teach fractions and 3) as mere representations. Essentially, the usefulness of the visuals was determined by how they were integrated into the fraction lessons.

5.2.2.1 Using visuals to introduce fraction concepts

This refers to instances where visuals were used to introduce fraction concepts, but these visuals did not feature in the other stages of the lesson such as the lesson development, activities and conclusion. For instance, participants used circular and rectangular models to introduce the addition and subtraction of fractions but there was a lack of consistency in the use of these visuals to further enhance learners' understanding of fractions. They clearly represented fractions using the area, length and set models and explained what actually happens when adding or subtracting same-denominator or different-denominator fractions. Basically, they used these visuals as a way of introducing alternative methods of solving fraction problems, after which they resorted to the traditional, symbolic strategies of teaching fractions.

The manner in which fractions were used in these lessons pointed towards the inclination of teachers to perceive visualisation processes as a way of explaining difficult or abstract fraction concepts. Hence, it appeared that these teachers had no intention of using visuals from the beginning to the end of their lessons. They identified lesson stages, for example, the introduction, lesson development, activities or conclusion, which in their view, was suitable for incorporating visualisation processes and incorporated them only in those specific stages. Despite the presence of visualisation processes in this context, the visuals were mostly used in a single lesson component, that is, the introduction or the lesson development and they were inconsistently used to introduce fraction concepts. As a result, the impact of visuals in such lessons was difficult to determine because in some instances, their impact on fraction pedagogy was evident while in others, it was not so evident because of the short time span during which the visuals were used.

5.2.2.2 Using visuals to teach fractions

There was a clear distinction between the way in which the visuals were used in this context and the previous one. Essentially, differences were observed in the type, duration and purpose of visuals. In other words, the participants conscientiously identified a variety of fraction models and subsequently incorporated them into their lessons. Hence, the participants exhibited adequate knowledge and skills in the use of visuals to enhance the pedagogy of fractions. Consequently, the focus of the participants was not merely on the random use of visuals but rather on how the selected visuals would enhance learners' understanding of the different fraction concepts.

Although the visuals were mostly used in the first two components of the lessons (the introduction and the lesson development), the manner in which the visuals were used demonstrated a balance between the verbal and the nonverbal codes. The visuals in this context were used to complement the verbal code. Thus, this presented learners with two pathways to understand fractions (Mayer & Anderson, 1991). Despite the prevalence of visuals in the introduction and the lesson development, the incorporation of visuals was not fully attained because it was lacking in some of the lesson stages such as the activities and the conclusion. It appeared that the participants considered the use of visuals in the introduction and the lesson development to be more important than the other lesson stages. Thus, despite spending so much time on incorporating visuals in their teaching, these visuals barely featured in the assessment activities.

5.2.2.3 Misplaced visual representations

In this context, the incidental use of visuals was observed in some of the participants' lessons where they were randomly presented, but did not enhance the teaching of fractions. For example, there were instances where some participants taught about the addition of fractions and instead of using visuals to show the addition process visually, they just drew visual representations of the answers while the entire process was verbal. Similarly, some participants used wrong or misplaced visuals to introduce fraction concepts. Therefore, the use of visuals in these instances was incidental and not aligned with the lesson objectives.

5.2.3 What significance do senior primary school teachers attach to the incorporation of visualisation processes in mathematics lessons?

Evidence gathered from the questionnaire and the interviews suggests that that all the participants involved in this study supported the use of visuals in teaching mathematics. Besides, they also explained how the incorporation of visualisation processes can enhance the pedagogy of mathematics. Based on their responses regarding the impact of the RCFP on their teaching, it was evident that they all valued the incorporation of visualisation processes in the pedagogy of fractions, since they all indicated that the RCFP activities had impacted their pedagogy positively. In response to a direct question on the significance of visuals in teaching mathematics, the participants explained how visuals present learners with a holistic, lasting and explicit way of conceptualising mathematics content.

In addition, the participants often made reference to the relationship between seeing and remembering or understanding, hinting at the role of visuals in making the abstract fraction content accessible to all learners (Boaler et al., 2016). They all seemed to concur that learners' understanding is enhanced through the incorporation of visuals. Furthermore, the participants also pointed out that the use of visuals improves the retention and retrieval of content. All eight participants embraced the incorporation of visuals as a way of improving the pedagogy of fractions.

5.2.4 What are the enabling and constraining factors in teaching fractions in an explicitly visual way at the senior primary phase?

The manner in which this question was addressed provides a summary of what the participants intended to do, what they successfully did and the challenges that prevented them from realising their goals. As indicated in the previous section, all the participants supported the incorporation of visuals in their lessons, however, data from the observations suggested that visualisation processes were not used in all the lessons. The enabling factors include the participation of the participants in the RCFP, the RCFP manual, the presence of visuals in the mathematics syllabi and prescribed textbooks, and the support from their colleagues at their respective schools. The RCFP is a platform that promotes the incorporation of visualisation processes in the pedagogy of fractions through practical, hands-on activities.

The RCFP manual is another enabling factor for the incorporation of visualisation processes. The manual was developed based on selected fraction concepts which were considered to be instrumental in improving the pedagogy of fractions. All the participants were provided with a copy of the RCFP manual which was used as an additional teaching resource. Although visualisation is not mentioned explicitly in the mathematics syllabuses for Grades four to seven, it is embedded in the opening statement of the rationale for mathematics at the senior primary phase, which states that

> … mathematics involves observing, representing and investigating patterns and quantitative relationships in social and physical phenomena and between mathematical objects themselves. Through these processes new mathematical ideas and insights are generated.

The place of visualisation can be indirectly inferred in this statement as the terms used are interconnected with visualisation. For instance, the terminologies of representing, investigating patterns, quantitative relationships, physical phenomena and mathematical objects align well with the concept of visualisation. Therefore, the mathematics syllabus implicitly advances the notion of visualisation in the pedagogy of fractions. Additionally, the prescribed textbooks for learners contained some visuals to explain fraction concepts, particularly fraction size, equivalent fractions and fractional parts of a quantity.

In light of the above, the participants were presented with an enabling environment for the incorporation of visualisation processes. However, despite the existence of these enabling factors, some of the participants still relied more on the verbal code. This suggests that there were some constraining factors that deterred the participants from using visuals in all their lessons, especially in cases where the use of visuals was absolutely necessary.

The constraining factors identified include inadequate knowledge and skills in the use of visualisation processes, lack of resources, time, indecisiveness and the participants' inability to determine the confluence between visualisation and formal mathematics (see section 4.3.2 & 4.4.4.3). Apart from the RCFP manual, the visualisation fraction kit and the prescribed text books, the participants had no alternative sources of visuals, hence their dependency on self-drawn chalkboard visuals. This challenge was observed among all the participants. Also, some schools did not even have duplicating machines.

Time was also identified as a constraining factor because some of the participants perceived the incorporation of visualisation processes into their fraction lessons as a time-consuming exercise. In other words, the participants were more concerned about completing the syllabus content rather than spending time on the preparation and use of visuals which rarely featured in the exams. Although they acknowledged the role of visuals in enhancing learners' understanding of fractions, some of the participants still preferred to use the verbal code because it was perceived as the most acceptable and appropriate way to teach fractions.

Indecisiveness was also observed among some of the participants who were uncertain about how far they should go in using the nonverbal code (section 4.4.4.3). A sense of uneasiness was demonstrated by some of the participants, who, despite having experienced success stories in using the nonverbal code, still felt they were diverting from the conventional way of teaching fractions. This rendered the use of visuals less useful in achieving their teaching goals. Therefore, the inconsistencies observed in the use of visuals can be attributed to the participants' indecisiveness, their fears and lack of knowledge.

The participants' inability to find the middle ground for the two codes was a major constraint in the incorporation of visuals in the pedagogy of fractions (section 4.4.5.8). Despite their participation in the RCFP, they still perceived the use of visuals as a new and unconventional way of doing mathematics. Thus, merging the two codes proved to be a challenge as most of the observed lessons leaned towards the verbal code. This was a common trend in topics involving addition, subtraction, multiplication and division of fractions. Considering that some of the participants lacked both CK and PCK on certain fraction concepts, it was difficult to determine whether their reluctance to incorporate visualisation processes on these topics was due to the fear of diverting from the conventional teaching strategies, or their lack of knowledge.

Although there is no direct relationship between the knowledge that teachers possess and their ability to teach effectively, adequate CK is a prerequisite for the successful incorporation of visualisation in teaching fraction concepts. Therefore, for teachers to employ innovative teaching strategies, they are expected to have adequate CK of a given subject. It is from the comprehensive understanding of mathematics concepts that innovative pedagogical methods are derived. Similarly, participants who demonstrated a better understanding of fractions were able to represent and formulate "the subject matter in ways that made it comprehensible to others" (Shulman, 1986, p. 9) while those whose understanding was lacking relied more on the verbal code.

5.3 SUMMARY

In summary, the overarching research question was addressed through the sub questions discussed above. The participants incorporated visualisation processes in their fraction lessons and this had an impact on their pedagogy to some extent. However, the impact that these visualisation processes had on the teaching of fractions was determined by two main factors: the type of visuals selected and how these visuals were incorporated into the lessons. As discussed in Section 5.2, a variety of visuals such as the fraction wall, circular and rectangular diagrams and concrete materials were used by the participants.

Despite the overuse of area models, visuals falling under the length and set models were also employed. In lessons where at least two of these models were used, the pedagogy of fractions was enhanced as learners were presented with different ways of understanding fractions. However, the participants rarely incorporated two or more fraction models because most of them relied on one type of model – the area model. Thus, learners were not presented with enough opportunities to experience fractions across the length and set models as these were less frequently used. The reluctance of the participants to incorporate the length and set models can be attributed to their lack of knowledge and confidence in the use of these two models

There were discrepancies in terms of the impact that visualisation processes had on the teaching of fractions as some of the visuals were more impactful than others. Based on how the participants chose to incorporate visuals in their lessons, different results were obtained. While some used visuals to explain difficult fraction concepts and immediately switched to the verbal code, others used visuals to not only explain difficult concepts but to complement the verbal code, thus, the two codes were used concurrently up to a certain stage of their lessons. Those who used the visual and verbal codes concurrently did so mainly to introduce and explain fraction concepts and in some rare cases, visuals were also incorporated in the tasks. However, maintaining the use of the visual code from the introduction to the conclusion of the lesson proved to be a challenge.

The participants' fraction knowledge, their ability to use different fraction models and their ability to coherently use visuals in different contexts were major determining factors in the success or failure of their lessons. Therefore, although the participation of the selected participants in the RCFP empowered them to consider the incorporation of visualisation processes in their teaching, it was not enough to completely reshape the pedagogy of fractions. In other words, the RCFP had a positive impact on the pedagogy of fractions to some extent; however, the mere participation of the participants in the RCPF was not enough to warrant an overhaul of the teaching trajectory of fractions. Hence, despite the prevalence of visualisation processes in most of the fraction lessons observed, a manifestation of teaching practices entrenched in the verbal code still persisted in some lessons.

5.4 CONTRIBUTION TO NEW KNOWLEDGE

This study has proved that the Dual Coding Theory (DCT) can indeed be used to enhance the pedagogy of fractions, however, there are important factors to consider when using the two codes. These include, the context in which visuals are used and type and purpose of the visuals. The context herein refers to the use of the nonverbal code in relation to the verbal code and the teacher's content knowledge. These are important factors to consider in the application of the DCT because the teacher's content knowledge in both the verbal and the nonverbal codes determines their success in the use of the two codes. Besides, this study has shown that the teachers' competence in the verbal code has a direct impact on their ability to incorporate the nonverbal code into the pedagogy of fractions. Therefore, this implies that the two codes are intertwined and should not be considered in isolation.

This study also reveals that although the incorporation of visualisation processes can enhance the pedagogy of fractions, it is important to consider the circumstances under which they are used. Visuals should be carefully and consciously selected by aligning them with the lesson objectives, because visuals that are randomly picked can be an obstruction in the teaching and learning trajectory of fractions. Moreover, this study also points to the consistent use of visuals as an appropriate strategy to enhance the pedagogy of fractions.

Furthermore, this study also suggests that the meaningful incorporation of visualisation goes beyond the teaching phase of a lesson to all types of assessment activities. In other words, the incorporation of visualisation processes in teaching can only remain relevant if it features in class activities, homework, topic tasks, tests and examinations. Otherwise, it would be perceived as a strategy that is temporarily useful, resulting in its relevance in the pedagogy of fractions being questionable.

Generally, this study has demonstrated that the incorporation of visualisation can enhance the pedagogy of fractions if carefully integrated into the verbal code as a teaching strategy rather than an independent strategy.

5.5 RECOMMENDATIONS FOR TEACHING FRACTIONS

In light of the findings emanating from this study, I made the following recommendations. These recommendations are important considerations in the use of visualisation in general, and they provide suggestions on further research in this field.

I recommend that there should be a paradigm shift in the pedagogy of fractions where more emphasis should be placed on understanding rather than the rote memorisation of rules. Therefore, the completion of the syllabus should not be the primary focus in teaching. On the contrary, teachers should focus on ensuring that all the mathematics concepts are effectively taught and mastered by the learners. This can be achieved through the incorporation of visualisation processes in teaching because "a picture is worth a thousand words," (Rosken & Rolka, 2006, p. 457).

Furthermore, I recommend the use of objective-driven visuals rather than the random use of visuals. When using visuals, it is vital to consider the purpose and context in which they are used. The findings suggest that if appropriately incorporated, visuals have the potential to improve the pedagogy of fractions, however, the random use of visuals can result in unprecedented outcomes. Therefore, just like the verbal code, the visual code must be purpose driven. The random use of visuals can interfere with the teaching trajectory of fractions and it may deter the effective teaching and learning of fractions.

Based on the over dependency observed among the participants on one fraction model, the early introduction of visuals across all three fraction models is an important consideration in improving the pedagogy of fractions. I therefore recommend the incorporation of the area, length and set models in the design of visuals because this presents learners with different opportunities to understand fractions. Evidently, the confusion experienced by learners in some of the participants' lessons can be attributed to the inability of the participants to explain fraction concepts using different fraction models such as the length and set models, that are equally important in the pedagogy of fractions.

I further recommend that visualisation be used as a teaching strategy rather than as a mere teaching aid. Visuals have the potential to enhance the pedagogy of fractions provided that they are used to teach fractions rather than to merely explain or represent fraction concepts. This reduces their function to teaching aids that are only used when they are needed. Therefore, the verbal code remains dominant in contexts where the nonverbal code is not fully integrated into the teaching and learning of fractions.

In view of the above, I also recommend the full incorporation of visualisation processes in all the lesson stages. The value that teachers attach to the importance of visualisation determines how it is incorporated into their pedagogy. Depending on their perceptions about visualisation processes, their preparedness and exposure to visualisation processes, some teachers incorporate it fully while others incorporate it partially. Although the partial incorporation of visualisation processes could have an impact on the pedagogy of fractions, it is not enough to turn around the deeply entrenched verbally-based teaching practices that are prevalent in the pedagogy of fractions.

Since CK is instrumental in the incorporation of visualisation in the pedagogy of fractions, I recommend that the introduction of visualisation processes in the teaching of fractions be preceded with training sessions aimed at equipping teachers with adequate CK. This study reveals that there are potential risks involved in the use of visualisation processes among teachers who do not possess adequate CK. Therefore, it is important to establish the CK of teachers prior to the introduction of visualisation processes. Assuming that all teachers are at the same level in terms of their CK is a risky precedence.

5.6 RECOMMENDATIONS FOR FURTHER RESEARCH

Research into the incorporation of visual strategies in teacher training curricula is an important consideration for further research in light of the gaps identified in the pedagogy of fractions. Although there are several factors that contribute to the way teachers teach, the knowledge gained during their preservice training is one of the major contributing factors because it is during this period that preservice teachers are introduced to the content that they are expected to teach and the appropriate teaching methodologies (PCK) to present the content. It is therefore an important starting point in identifying gaps in the pedagogy of fractions.

Another important focus area for further research is the incorporation of visualisation in mathematics curricula. It is one thing to implicitly refer to visualisation and quite another to normalise its use by explicitly integrating it into the curriculum. The absence of visualisation in most mathematics curricula makes it insignificant to some extent because this affects the way content is presented, assessed and rated. Therefore, in order to reclaim its rightful position in the pedagogy of fractions, visualisation should be explicitly incorporated in the curriculum through teaching and assessment activities.

Furthermore, I recommend the incorporation of visualisation in all mathematics concepts since the focus on one mathematics concept (fractions) renders it irrelevant to the subject as a whole. Thus, for the relevance of visualisation to be impactful, it should be considered across all mathematics concepts.

Considering that mathematics content is predominantly verbally mediated, the nonverbal code should be gradually introduced into the mathematics curriculum to enhance learners' understanding of the verbal code, and not as an isolated strategy. Therefore, the selection and incorporation of visuals should be aligned with the objectives of the lesson which are usually presented using the verbal code.

5.7 LIMITATIONS OF THE STUDY

One of the major limitations to this study was the circumstances under which this study was conducted. This study was conducted during the COVID 19 pandemic and there were a number of restrictions on movement, gatherings, touching all kinds of objects and speaking without masks. These restrictions had a major impact on some of the planned activities and the design of this study. Generally, teaching is supposed to be a social activity that involves coming into close contact with the learners, marking their books, monitoring their progress during lessons and engaging learners in group activities. These activities were all regarded as risky during the data collection period, that is March – October 2020 because these were some of the common ways in which the disease was spread. Therefore, although we all tried to adapt to the 'new normal' and continue with the teaching and learning activities, this was done under completely new settings.

Despite their exposure to the RCFP, the lack of resources was a limitation in the full implementation of visual strategies in teaching because the visualisation fraction kit was not enough to cater for all the fraction concepts taught. As a result, some of the participants relied on chalkboard visuals while others resorted to the verbal code. It was therefore difficult to determine the visuality of some of the participants in instances where the use of visualisation processes was not supported by external factors such as the availability of resources. As stated by Presmeg (1986, p. 298), "a person's mathematical visuality is the extent to which that person prefers to use visual methods when attempting mathematical problems which may be solved by both visual and nonvisual methods." The lack of resources is a factor that could have been ruled out by ensuring that all the participants were provided with the necessary materials such as posters, marker pens, base ten blocks and one metre rulers. These materials, together with the fraction visualisation kit could have better enabled the researcher to determine the participants' inclination towards the visual mode.

A gap in fraction knowledge was also identified as a challenge because I worked with participants who were at different levels in terms of their fraction knowledge, which had a direct impact on their ability to understand and use visuals in their fraction lessons. Efforts were made to bridge this gap through the RCFP activities. However, this was not enough because the focus was more on the development of visual materials than on the content. Henceforth, I intend to balance the RCFP activities in terms of content and pedagogy to ensure that the members are equipped with adequate CK to address the pedagogical challenges related to fractions.

Time was also a limitation because it affected both the participants and the researcher. The abrupt change in the school settings due to the COVID 19 outbreak affected the teaching schedules and routines. As a result, teachers were compelled to adjust their schemes of work to be in conformity with the existing circumstances. In other words, teachers had to cover as much content as possible within a limited period of time. Subsequently, I also had to amend my research timeline to fit into the new realities on the ground. Unfortunately, there were no alternatives to these arrangements since we were all expected to follow the COVID 19 protocols from the Namibian Ministry of Health and Social Services (MHSS) and the Ministry of Education, Arts and Culture (MEAC).

In spite of these limitations, I tried to make the most out of the situation that I found myself in. Where humanly possible, I tried to minimise the effect of these limitations on my study by focusing on the enabling factors rather than the constraining factors which were beyond my control.

5.8 CONCLUSION

I embarked on this study to establish the impact of the RCFP activities on the pedagogy of fractions. The RCFP is a campus-based project that advocates for the incorporation of visualisation processes in the teaching of fractions. The participants were selected based on their ability to use both verbal and nonverbal codes and at least two fraction models. Considering the circumstances under which this study was conducted, there is enough evidence to suggest that the RCFP had an impact on the pedagogy of fractions.

Notwithstanding the type or quality of visuals, all the participants used visuals in one or more of the observed lessons. This is an indication that the participants' participation in the RCFP had an impact on their disposition towards the incorporation of visualisation processes in the pedagogy of fractions. In other words, visualisation processes were prevalent in all the twenty-five fraction lessons observed. Although the visuals did not have the same impact on the pedagogy of fractions, the visibility of visuals in all lessons was a good starting point in redefining the pedagogy of fractions.

From a pedagogical point of view, the impact of the visuals on the teaching of fractions was diverse as some visuals substantially improved the teaching of fractions; some partially improved the teaching of fractions; while others did not have any impact on the teaching of fractions. The discrepancies observed in the use of visualisation among the participants can be ascribed to the participants' exposure to visualisation prior to their participation in the RCFP and their perceptions about visualisation. Consequently, although all the participants indicated that visualisation is very important in the pedagogy of fractions, some of them had their own reservations about its relevance in advancing the goals and objectives of mathematics in traditional mathematics classrooms.

In instances where visualisation was effectively incorporated, its impact on the pedagogy of fractions was eminent as participants presented their learners with alternative ways to understand fractions. This study revealed that the effective use of visuals depends on the teacher's ability to help learners establish the link between the verbal and the nonverbal codes. Although literature (Avgerinou & Petterson, 2011; Suh & Moyer-Packenham, 2007; Paivio, 2006; Clark & Paivio, 1991) suggests that the verbal and the nonverbal codes are two separate pathways of understanding mathematics concepts such as fractions, the two pathways should be interconnected to help learners understand the concepts comprehensively. This is what actually distinguished the participants who successfully incorporated visualisation processes from those who did not. Therefore, the curriculum objectives should be considered when using the nonverbal code. In the final analysis, it is the teacher's ability to effectively use the two codes to enhance the pedagogy of fractions that really matters.

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APENDICES

APPENDIX ONE: QUESTIONNAIRE

This tool was administered to all RCFP members from which ten participants were selected for this study using the criteria stipulated in 3.8.1

This questionnaire is one of the research instruments for my PhD research entitled: *Analysing the role of visualisation in redefining the pedagogy of fractions in mathematics classrooms among primary school mathematics teachers.* Having participated in the Rundu Campus Fraction Project (RCFP), this questionnaire is set to establish your views on the incorporation of visualisation processes in your teaching. Moreover, this questionnaire is instrumental in identifying participants for this study. Your participation and honest responses to the following questions will be highly appreciated.

1. Indicate the phase that you teach:

    a) Junior Primary

    b) Upper Primary

    c) Secondary

2. How often do you use visuals in your lessons? Please elaborate.

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3. Do you create your own visuals or use ready-made ones?
    Please elaborate.

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4. Do you use any of the following to enhance learners' understanding of fractions?
   a) Mathematical symbols only

   b) Visuals only

   c) A combination of mathematical symbols and visuals in mathematics classrooms.

   Provide an explanation and example for your choice to question 3.

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5. What is the importance of visuals in teaching fractions?
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6. Which of the following fraction concepts do you think require the use of visuals?
   a) Unit fractions
   b) Equivalent fractions
   c) The four basic operations (addition, subtraction, multiplication and division)
   d) All of the above

   Provide an explanation and example for your choice to question 5

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7. Indicate the type(s) of models you prefer to use in fraction lessons.
   a) Area models
   b) Length models
   c) Set models
   d) All of the above

Please elaborate

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8. Briefly explain why you prefer to use the model (s) that you have indicated in question 4.

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9. Has your participation in the RCFP changed your perceptions on teaching fractions:
   a) Yes
   b) No
   c) Not sure
10. In your opinion, during which phase(s) is the use of visualisation appropriate?
    a) Junior primary
    b) Senior primary
    c) Junior secondary
    d) Senior secondary
    e) Tertiary
    f) All of the above

    Provide an explanation for your choice(s) to question 9.

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APPENDIX TWO: PRE-AND POST-OBSERVATION INTERVIEWS

The two sets of interviews were administered to the eight participants involved in this study

PRE-OBSERVATION INTERVIEWS

1. In your opinion, what is the most difficult fraction concept to teach?
2. Do you regard fractions in general as an important concept in mathematics? Motivate your answer.
3. Briefly discuss the relevance of visualisation in the pedagogy of fractions.
4. How would you rate the performance of your learners on the concept of fractions?
5. Discuss some of the factors contributing to learners' poor performance in fractions.
6. As a member of the Rundu Campus Fraction Project (RCFP), explain how you would rate the impact of the RCFP on your teaching.
7. Was your teaching before you joined the RCFP inclined towards the verbal or visual mode?
8. Is your current teaching informed by the knowledge and skills that you have gained from the RCFP?
9. What are the advantages/disadvantages associated with the use of one mode (either verbal or visual)?
10. Are you encouraged to use visuals by your fellow mathematics teachers?
11. Which of the following models do you often use to teach fractions?
    a) Area model
    b) Length model
    c) Set model
12. Why do you prefer the use of the model selected in question 11?
13. Fraction constructs such as part-whole, measurement, division, operator and ratio are different ways of representing fractions. Which of these constructs do you often use in your fraction lessons? Motivate your answer.
14. During which component of your lesson plan, for example introduction, development, reinforcement, or conclusion do you mostly use visualisation processes?
15. How do you introduce fractions to your learners?
16. What type of visuals do you use during your lessons to enhance learners' understanding of fractions?
17. Clearly explain how you use visuals to enhance learners' understanding.
18. Briefly explain what informs your selection and use of visuals.
19. Describe the role of visuals in enhancing learners' understanding of fraction size.
20. Describe the strategies that you use during your lessons to enhance learners' fraction sense?
21. What does quality instruction in terms of fractions mean to you?

22. What is your view on the use of visualisation to teach fractions at the senior primary phase?
23. What works best for you?
    a) Introducing rules first before explaining fraction concepts using visuals
    b) Explaining concepts first using visuals before introducing the rules
    c) Introducing the rules and explaining fraction concepts using visuals concurrently

POST-OBSERVATION STIMULATED RECALL INTERVIEWS

1. What informs your selection /preparation of visuals to teach the following fraction concepts:
   a) Equivalent fractions
   b) Addition and subtraction of fractions
   c) Multiplication of fractions
   d) Division of fractions
2. Which of the three fraction models, that is area, length and set did you find useful in teaching the following fraction concepts:
   a) Equivalent fractions
   b) Addition and subtraction of fractions
   c) Multiplication of fractions
   d) Division of fractions
3. Did the use of visuals in your lesson enhance learners' understanding of the concepts mentioned in no. 25?
4. In your opinion, which of the fraction concepts, if fully mastered can facilitate learners' understanding of other fraction concepts?
5. What are some of the misconceptions that learners have pertaining to:
   a) Equivalent fractions
   b) Addition and subtraction of fractions
   c) Multiplication of fractions
   d) Division of fractions
6. How can the misconceptions mentioned in no. 29 be addressed?
7. What is the role of visuals in teaching the concepts in no. 29 and how often do you use them?
8. Do you think the visuals you used in teaching the above mentioned concepts can enhance learners' understanding of fraction size? If so, how?
9. How did you support learners who struggled with these concepts?

APPENDIX THREE: LETTER OF CONSENT TO THE DIRECTOR OF EDUCATION

P.O. Box 2407

Rundu

26 October 2018

To The Director

Kavango East Educational Region

Rundu

Dear Sir/Madam

Re: Permission to conduct research at selected schools in Kavango East Region

My name is Charity Makwiliro Ausiku, a PhD student at Rhodes University. I intend to conduct a study in mathematics, focusing on fraction pedagogy at the senior primary phase. The title of my study is "An analysis into the role of visualisation in redefining fraction pedagogy in mathematics classrooms among senior primary school teachers at a few selected schools." Research has shown that the concept of fractions has proved to be problematic at the primary, secondary and tertiary levels. This prompted me to embark upon this study for my PhD as its findings could be used to mitigate the pedagogical challenges encountered by teachers in teaching fractions.

The overarching goal of this study is to determine how the incorporation of visualisation processes can enhance the teaching of fractions at the upper primary phase. Visualisation refers to the use of visuals or images to enhance the teaching and learning of different concepts.

The purpose of this letter is therefore to humbly request your highly esteemed office to grant me permission to conduct this study at selected schools in Kavango East region. This study will involve ten teachers (participants) from different schools in Kavango East region. I intend to uphold the acceptable research ethics such as respect and human dignity, transparency and honesty, integrity and academic professionalism during the data collection process. Issues of confidentiality and anonymity will be addressed by using pseudonyms rather than the real names of schools and teachers involved in this study. Moreover, data collected will be used exclusively for the purpose of this study and it will not be published without the consent of the participants. I undertake to make available to you my final thesis with its findings and recommendations

I look forward to hearing from you soon.

Yours Sincerely

……………………………………

Charity M. Ausiku

APPENDIX FOUR: LETTER OF CONSENT TO PRINCIPALS OF PARTICIPATING SCHOOLS

This letter was used to seek permission from the principals of the selected participants

P.O. Box 2407
Rundu

26 October 2018

To The Principal

Rundu

Dear Sir/Madam

Re: Permission to conduct research at your school

My name is Charity Makwiliro Ausiku, a PhD student at Rhodes University. I intend to conduct a study in mathematics, focusing on fraction pedagogy at the senior primary phase. The title of my study is "An analysis into the role of visualisation in redefining fraction pedagogy in mathematics classrooms among senior primary school teachers at a few selected schools." Research has shown that the concept of fractions has proved to be problematic at the primary, secondary and tertiary levels. This prompted me to embark upon this study for my PhD as its findings could be used to mitigate the pedagogical challenges encountered by teachers in teaching fractions.

The overarching goal of this study is to determine how the incorporation of visualisation processes can enhance the teaching of fractions at the upper primary phase. Visualisation refers to the use of visuals or images to enhance the teaching and learning of different concepts.

The purpose of this letter is therefore to humbly request your highly esteemed office to grant me permission to conduct this study at your school with Ms/Mr X (name withheld) as my participant. The regional Education Director has been informed about the intended study and permission has already been granted from his office for me to go ahead with this study. I intend to uphold all the acceptable research ethics such as respect and human dignity, transparency and honesty, integrity and academic professionalism during the data collection process. Issues of confidentiality and anonymity will be addressed by using pseudonyms rather than the real names of schools and teachers involved in this study. Moreover, data collected will be used exclusively for the purpose of this study and it will not be published without the consent of the participants. I undertake to make available to you my final thesis with its findings and recommendations

I look forward to hearing from you soon.

Yours Sincerely
………………………………
Charity M. Ausiku

APPENDIX FIVE: INVITATION LETTER TO PARTICIPANTS

This invitation was extended to all RCFP Members

P.O. Box 2407

Rundu

26 October 2018

Dear Sir/Madam

Re: Invitation to participate in Ms Charity Ausiku's PhD research study

My name is Charity Makwiliro Ausiku, a PhD student at Rhodes University. I intend to conduct a study in mathematics, focusing on fraction pedagogy at the senior primary phase. The title of my study is "An analysis into the role of visualisation in redefining fraction pedagogy in mathematics classrooms among senior primary school teachers at a few selected schools." Research has shown that the concept of fractions has proved to be problematic in both the teaching and learning processes at the primary, secondary and tertiary levels. This prompted me to embark upon this study for my PhD as its findings could be used to mitigate the pedagogical challenges encountered by teachers in teaching fractions.

The aim of this study is to determine how the incorporation of visualisation processes can enhance the teaching of fractions at the upper primary phase. Visualisation refers to the use of visuals or images to enhance the teaching and learning of different concepts.

The purpose of this letter is therefore to kindly request you to participate in this study. As a member of the RCFP, you will find the activities involved in this study quite familiar since you have been involved in the development and use of visual materials. It is against this backdrop that you have been selected as one of the participants in this study. However, I wish to reiterate that participation in this study is voluntary and you can withdraw from the study at any time. I intend to uphold all acceptable research ethics such as respect and human dignity, transparency and honesty, integrity and academic professionalism during the data collection process. Issues of confidentiality and anonymity will be addressed by using pseudonyms rather than the real names of schools and teachers involved in this study. Moreover, data collected will be used exclusively for the purpose of this study and it will not be published without the consent of the participants.

Please complete the consent form attached if you agree to participate in this study.
I look forward to hearing from you soon.

Yours Sincerely,

Charity M. Ausiku Signature:…………………….
Supervisor: Name…………………………. Signature:……………………

APPENDIX SIX: INFORMED CONSENT FOR PARTICIPANTS

This was completed by the selected participants only.

Research Title	An analysis into the role of visualisation in redefining fraction pedagogy in mathematics classrooms among senior primary school teachers at selected schools
Principal Researcher	Charity M. Ausiku

Information about the research	Pease tick if you agree
The aim of the research has been clearly explained to me	
I have made the decision to participate in this study voluntarily	
I understand that I can withdraw from the study at any time	
I agree with the data collection procedures explained to me by the researcher	
The benefits, challenges and potential risks involved in this study have been explained to me	
I accord the researcher to use the data collected from me for study purposes	
The researcher has not promised any rewards in monetary terms for my participation	
Details pertaining to this study were explained to me by the researcher	
Information regarding this study was clearly shared in English, a language which I fully understand	
I understand that the data obtained from me in any form will be used solely for the purpose of this study	

Voluntary consent

I, ………………………………………………………………………………….(name in full) hereby voluntarily give my consent to participate in Ms Charity Ausiku's research.

Signature-- Date ---------------------------------

Researcher's declaration

I, Charity Ausiku, hereby declare that I have shared all the relevant information regarding this study with the participant and I have truthfully addressed all the participant's questions, concerns and uncertainties.

Signature……………………………………. Date…………………………………………….

APPENDIX SEVEN: INFORMED CONSENT OF PARENTS

This was given to parents via their children for their consent since this study involved the use of video-recordings as a data collection method.

Dear parents

Your child's mathematics teacher will be involved in a research study intended to improve the teaching of fractions using different visualisation processes. Visualisation refers to the use of visual models to improve the teaching and learning of different concepts in mathematics, including fractions. Since learners struggle to understand fractions, exploring different teaching strategies could be beneficial for your child.

The purpose of this letter is to ask for your permission to allow your child to be present in the class where videos of their mathematics teacher will be recorded. I wish to video record five lessons next year where the focus will entirely be on the teacher. All efforts will thus be made not to record and include the faces of any child in the class. In the event of the face of a child being apparent, every effort will be made to blur the face on the video clip so as not to reveal the identity of that child. If this is not possible I would like to assure you that the visuals and videos where your child appears will only be used for the purpose of analysis of this study. These visuals will be handled with care and they will not be published without your consent.

Please feel free to discuss any misgivings that you have concerning the above, before signing the form below.

I, ---(father's name)

Date--------------------------------------

I, -- (mother's name)

Date-------------------------

□ give my unreserved consent for my child to be present in the class that you may video record for your research project.

□ have my reservations about allowing my child to be part of any class you may wish to video record.

APPENDIX EIGHT: ETHICAL GUIDELINES TO RESEARCH PARTICIPANTS

In order to ensure that the participants were acquainted with the framework under which we would be operating, I shared and explained these ethical guidelines with them during our RCFP meeting.

1. Participants must be informed that they are being asked to participate in a research study,

2. Participants must be provided an explanation of the purposes of the research and the expected duration of their participation

3. Participants must be given a description of the procedures to be followed and of any experimental procedures must be identified,

4. Participants must be given a description of any reasonably foreseeable risks or discomforts they may experience,

5. Participants must be given a description of any benefits to themselves or others that may reasonably be expected from the results of the study,

6. Appropriate alternative procedures or courses of treatment, if any, that might be advantageous to the subject of an experimental or quasi-experimental study must be disclosed

7. Participants must be given a statement describing the extent, if any, to which confidentiality of records identifying the subject/participant will be maintained

8. For research involving more than minimal risk, participants must be given an explanation about any treatments or compensation if injury occurs and, if so, what they consist of, or where further information may be obtained. (Note: A risk is considered "minimal" when the probability and magnitude of harm or discomfort anticipated in the proposed research are not greater, in and of themselves, than those ordinarily encountered in daily life or during the performance of routine physical or psychological examinations or tests).

9. Participants must be told whom to contact for answers to pertinent questions about the research and research subjects'/participants' rights, and whom to contact in the event of a research-related injury

10. Participants must be given a statement that participation is voluntary, refusal to participate will involve no penalty or loss of benefits to which the subject/participants is otherwise entitled, and the subject/participant may discontinue participation at any time without penalty or loss of benefits to which the subject/participant is otherwise entitled

APPENDIX NINE: PERMISSION TO CONDUCT RESEARCH IN THE KAVANGO REGION

REPUBLIC OF NAMIBIA
KAVANGO EAST REGIONAL COUNCIL

DIRECTORATE OF EDUCATION, ARTS AND CULTURE

OFFICE OF THE DIRECTOR

Tel (066) 218 9000 / 218 9201
Fax (066) 267 707

Private Bag 2134
RUNDU
Namibia

Enquiries: Maria I. Ribebe
Ref. No: 20/10/1

Date : 26 June 2019

TO: Ms. Charity M. Ausiku

SUBJECT: PERMISSION TO CONDUCT RESEARCH IN KAVANGO EAST REGION

Your letter on the above bears reference.

Kindly be informed that permission has been granted to Ms. Charity M. Ausiku to conduct research in Kavango East Region.

Please ensure that you first report your presence and roles to the Principals of the respective schools for authorization to enter schools.

The normal teaching and learning activities should <u>NOT</u> be disrupted in the process.

Yours sincerely,

Fanuel Kapapero
REGIONAL DIRECTOR
KAVANGO EAST REGIONAL COUNCIL

KAVANGO EAST REGIONAL COUNCIL
2019 -06- 26

26/6/2019
Date

CC. Inspectors of Education: Kavango East Region