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Sobriety of Crisp and Fuzzy Topological Spaces

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Abstract

The objective of this thesis is a survey of crisp and fuzzy sober topological spaces. We begin by examining sobriety of crisp topological spaces. We then extend this to the L-topological case and obtain analogous results and characterizations to those of the crisp case. We then briefly examine semi-sobriety of (L, M)-topological spaces.

KEYWORDS: Sober, Sobriety, Semi-sobriety, α-sober, Strongly sober, Ultrasober

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PREFACE

The purpose of this present work is a survey of Sobriety, starting with the crisp case and then extending it to the L-topological case and the (L, M)-topological case.

Chapter 1 is of an introductory nature, providing the reader with results concerning lattice theory, ideals and filters, fuzzy sets, topologies and neighbourhoods.

Chapter 2 concerns crisp sobriety. We begin with the initial definition of crisp sobriety of ordinary topological spaces and then characterize crisp sober spaces in three different ways. The first characterization is in terms of completely prime filters. The second characterisation is in terms of nets. The final characterization is in terms of irreducible closed sets.

Chapter 3 is about the properties of crisp sober spaces, it is divided up into three sections. The first section deals with the relationship of sobriety to the seperation axioms and we then go on to define a topology on $pt\mathcal{T}$. The second section deals with sober spaces and continuous maps. The third section is titled sobriety of partially ordered sets. In this section we define a partial order on a T_0 space, called the specialization ordering. We then proceed to examine topologies on partially ordered sets $(X, \leq)$ such that the specialization ordering on the space is $\leq$. In this connection we have a look at the Alexandrov topology and the upper interval topology. We then define the Scott topology and answer the question as to whether the Scott topology on a directed set is always sober.

Chapter 4 deals with fuzzy sobriety or L-sobriety. We define sobriety of an L-topological space in an analogous manner to sobriety of a crisp topological space. We then examine sobriety for the case $L = [0, 1]$. In this section we follow the work of Singh and Srivastava in [17] where we look at the definitions of α-sober spaces, strongly sober spaces and ultra-sober spaces. The remainder of the section deals with how these different concepts relate to one another.

Chapter 5 is titled properties of L-sober spaces and is analogous to Section 3.2 for the ordinary topological case.

Chapter 6 is titled fuzzy sobriety, semi-sobriety and the Hausdorff properties. In the first section we seek a counterpart to the characterization of sobriety of crisp topological spaces in terms of irreducibly closed sets. In doing this we discuss the L-topological equivalent, semi-sobriety due to Wesley Kotzé. We then go on to discuss the Hausdorff properties and how these relate to semi-sobriety and sobriety. The next section is a characterization of L-topological sober spaces, firstly in terms of completely prime filters and then in terms of nets. These characterizations are analogous to those in chapter 2. The last section of chapter 6 looks at how semi-sobriety is related to the concepts of Srivastava which we discussed in section 4.2. Sections 6.3 and 6.4 are based on as yet unpublished work by Wesley Kotzé.

In Chapter 7 we follow Kotzé's work in [13] this deals with the lifting of sobriety and semi-sobriety.

In Chapter 8 we again follow [13] but this time in connection with semi-sobriety in (L, M)-

topological spaces.

We are aware of at least two notable recent contributions about sober spaces which do not, in our opinion, fit this discussion. These are :
P. Taylor, *Sober spaces and continuations.* Theory and applications of Categories **10(12)**(2002), 248-300.
and
A. Pultr and S. E. Rodabaugh, *Examples for different sobrieties in fixed-basis topology.* Topological and Algebraic structures in fuzzy sets. Eds. S. E. Rodabaugh and E. P. Klement, Kluwer Academic Publishers (2003). ([15])

Chapter 1

Lattice Theory and Basic Concepts

1.1 Introductory Concepts

The following are well known. See e.g. [8].

1.1.1 Definition
Let A be a set. A *partial order* on A is a binary relation $\leq$ which is

1. reflexive : $a \leq a \quad \forall a \in A$
2. transitive : if $a \leq b$ and $b \leq c$ then $a \leq c$
3. antisymmetric : if $a \leq b$ and $b \leq a$ then $a = b$

A *poset* is a set equipped with a partial order.

1.1.2 Definition
Let A be a poset, S a subset of A. We say an element $a \in A$ is a *join (least-upper bound)* for S and write $a = \bigvee S$ if,

1. a is an upper bound for S i.e. $s \leq a \quad \forall s \in S$
2. if b satisfies $\forall s \in S$ $(s \leq b)$ then $a \leq b$

The antisymmetry axiom ensures that the join of S, if it exists, is unique. If S is a two-element set $\{s, t\}$ we write $s \vee t$ for $\bigvee\{s, t\}$ and $\bigvee \emptyset = 0$, should it exist; then 0 is clearly the least element of A.

1.1.3 Proposition
Let A be a poset in which every finite subset has a join and with least element 0, then the following are satisfied:

1. $a \vee a = a \quad \forall a \in A$
2. $a \vee b = b \vee a \quad \forall a, b \in A$
3. $a \vee (b \vee c) = (a \vee b) \vee c \quad \forall a, b, c \in A$
4. $a \vee 0 = a \quad \forall a \in A$

We say that $(A, \vee, 0)$ is a *commutative monoid* in which every element is idempotent.

1.1.4 Theorem

Let $(A, \vee, 0)$ be a commutative monoid in which every element is idempotent. Then there exists a unique partial order on A such that $a \vee b$ is the join of a and b and 0 is the least element. i.e. $a \leq b$ iff $a \vee b = b$

A set with the structure described in the theorem is called a *semilattice (join semilattice)*. The theorem shows that the notion of a join semilattice can be expressed equivalently either in terms of the order relation or in terms of the join operation.

1.1.5 Definition

Let A be a poset, S a subset of A. We say an element $a \in A$ is a *meet (greatest-lower bound)* for S and write $a = \bigwedge S$ if,

1. a is a lower bound for S i.e. $a \leq s \quad \forall s \in S$
2. if b satisfies $\forall s \in S \ (b \leq s)$ then $b \leq a$

If S is a two element set $\{s, t\}$, we write $s \wedge t$ for $\bigwedge\{s, t\}$ and $\bigwedge \emptyset = 1$, if it exists, will be the greatest element of A

1.1.6 Proposition

Let A be a poset in which every finite subset has a meet and with greatest element 1, then the following are satisfied,

1. $a \wedge a = a \quad \forall a \in A$
2. $a \wedge b = b \wedge a \quad \forall a, b \in A$
3. $a \wedge (b \wedge c) = (a \wedge b) \wedge c \quad \forall a, b, c \in A$
4. $a \wedge 1 = a \quad \forall a \in A$

$(A, \wedge, 1)$ is a commutative monoid in which every element is idempotent.

Dually, the analogue of Theorem 1.1.4 applies. If $(A, \wedge, 1)$ is a commutative monoid in which every element is idempotent, then there exits a unique partial order on A such that $a \wedge b$ is the meet of a and b and 1 is the greatest element i.e. $a \leq b$ iff $a \wedge b = a$. A set with this structure is again a semilattice, in this case a *meet semilattice*.

1.1.7 Definition

A *lattice* is a poset A in which every finite subset has both a join and a meet.

By Theorem 1.1.4 we see that a lattice is a poset A equipped with two binary operations $\vee$, $\wedge$ and two distinguished elements 0, 1 such that $(A, \vee, 0)$ and $(A, \wedge, 1)$ are semilattices and the partial orders induced on A by the semilattice structures are opposite each other.

1.1.8 Proposition

Suppose $(A, \vee, 0)$ and $(A, \wedge, 1)$ are semilattices. Then $(A, \vee, \wedge, 0, 1)$ is a lattice iff the *absorptive laws*,

1. $a \wedge (a \vee b) = a \quad \forall a, b \in A$
2. $a \vee (a \wedge b) = a \quad \forall a, b \in A$

are satisfied.

Proof. $a \vee b = b$ implies by (1) $a \wedge b = a \wedge (a \vee b) = a$ and by (2), $a \wedge b = a$ implies $a \vee b = b$. So the two partial orders on A agree.
The converse follows trivially.
□

1.1.9 Definition
A *distributive lattice*, is a lattice $(A, \vee, \wedge, 0, 1)$ which satisfies the additional identity, called the distributive law,

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \quad \forall a, b, c \in A$$

1.1.10 Lemma
If the distributive law holds in a lattice then so does its dual,

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \quad \forall a, b, c \in A$$

Proof. Using the absorptive laws of Proposition 1.1.8

$$\begin{aligned} (a \vee b) \wedge (a \vee c) &= ((a \vee b) \wedge a) \vee ((a \vee b) \wedge c) \\ &= a \vee ((a \wedge c) \vee (b \wedge c)) \\ &= (a \vee (a \wedge c)) \vee (b \wedge c) \\ &= a \vee (b \wedge c) \end{aligned}$$

□

1.1.11 Definition

1. A join-semilattice A is said to be *complete* if it has arbitrary joins and not just finite ones.

2. A meet-semilattice A is said to be *complete* if it has arbitrary meets and not just finite ones.

3. A lattice A is said to be *complete* if it has arbitrary joins and meets and not just finite ones.

1.1.12 Proposition
A poset is a complete join-semilattice iff it is a complete meet-semilattice.

Proof. Let A be a complete meet-semilattice, $S \subset A$. Consider the set T of all upper bounds for S, and let $a = \bigwedge T$. Since every $s \in S$ is a lower bound for T, we have $s \leq a$ and hence a is an upper bound for S. So a is the least element of T, i.e. $a = \bigvee S$.
□

1.1.13 Definition
A *frame* is a lattice which is closed under arbitrary suprema ($\bigvee$) and finite infima ($\wedge$) and satisfies the *frame distributive law*,

$$a \wedge (\bigvee b_i) = \bigvee (a \wedge b_i)$$

A *semi-frame* is a lattice which is closed under arbitrary suprema ($\bigvee$) and finite infima ($\wedge$).

1.1.14 Definition
An *order preserving map* is a map $f : (A, \leq) \longrightarrow (B, \leq)$ with the property that if $a \leq b$, $a, b \in A$ then $f(a) \leq f(b)$ in B.

1.1.15 Definition
A *lattice morphism* $f : A \longrightarrow B$ between two lattices is a map preserving the distinguished elements 1, 0 and the operations $\vee$ and $\wedge$.
A *frame morphism* between frames preserves arbitrary suprema and finite infima.

It should be noted that a lattice morphism is necessarily an order preserving map, but an order preserving map between lattices is not necessarily a lattice morphism.

1.1.16 Definition
An *order reversing involution* on a poset A is a map $' : A \longrightarrow A$ satisfying $a \leq b \Rightarrow b' \leq a'$ for all $a, b \in A$ and $a'' = a$ for all $a \in A$.

If A is a lattice with an order reversing involution it is easy to see that the *de Morgan laws* are satisfied, viz. $(a \vee b)' = a' \wedge b'$ and $(a \wedge b)' = a' \vee b'$, and $0' = 1$ and $1' = 0$; and in the case of a complete lattice $(\bigvee a_i)' = \bigwedge a_i'$ and $(\bigwedge a_i)' = \bigvee a_i'$.

1.1.17 Definition
A poset A is said to be *directed* if

1. $A \neq \emptyset$

2. every pair of elements of A has an upper bound in A.

We say A has *directed joins* if $\bigvee S$ exists for every subset $S \subseteq A$.

1.2 Ideals and Filters

1.2.1 Definition
An *ideal* $\mathcal{I}$ in a lattice, A, is a subset of A such that,

1. $0 \in \mathcal{I}$ and $a, b \in \mathcal{I}$ implies $a \vee b \in \mathcal{I}$

2. $\mathcal{I}$ is a lower set, i.e. $a \in \mathcal{I}$ and $b \leq a \Rightarrow b \in \mathcal{I}$

An ideal is *prime* if $1 \notin \mathcal{I}$ and $a \wedge b \in \mathcal{I} \Rightarrow a \in \mathcal{I}$ or $b \in \mathcal{I}$. If $a \in A$, then $\downarrow (a) = \{b \in A : b \leq a\}$ is an ideal of A, called the *principal ideal* generated by a.

1.2.2 Definition
A *filter* $\mathcal{F}$ in a lattice A is a subset of A such that

1. $1 \in \mathcal{F}$ and $a, b \in \mathcal{F} \Rightarrow a \wedge b \in \mathcal{F}$

2. $\mathcal{F}$ is an upperset i.e. $a \in \mathcal{F}$ and $b \geq a \Rightarrow b \in \mathcal{F}$

Clearly $\uparrow (a) = \{b \in A : b \geq a\}$ is a filter in A. A filter is *prime* if $0 \notin \mathcal{F}$ and $a \vee b \in \mathcal{F} \Rightarrow a \in \mathcal{F}$ or $b \in \mathcal{F}$ and *completely prime* if $\bigvee a_i \in \mathcal{F} \Rightarrow \exists i, a_i \in \mathcal{F}$.

1.2.3 Proposition

1. $\mathcal{F}$ is a prime filter in A iff $A \setminus \mathcal{F}$ is a prime ideal in A.

2. $\mathcal{F}$ is a completely prime filter in A iff $A \setminus \mathcal{F}$ is a principal(prime) ideal in A.

Proof.

1. trivial

2. $\mathcal{F}$ completely prime is equivalent to,

$$\forall i, a_i \in A \setminus \mathcal{F} \Rightarrow \bigvee a_i \in A \setminus \mathcal{F}$$

which (since $A \setminus \mathcal{F}$ is an ideal by part (a)) is equivalent to saying,

$$\begin{aligned} A \setminus \mathcal{F} &= \{b \in A : b \leq \bigvee a_i, a_i \in A \setminus \mathcal{F}\} \\ &= \downarrow (\bigvee\{a : a \in A \setminus \mathcal{F}\}) \end{aligned}$$

So $A \setminus \mathcal{F}$ is a principal prime ideal in A

□

1.2.4 Definition
An element $a \in A$ is *prime* in A iff $\downarrow (a)$ is a prime ideal. So a is prime iff $b \wedge c \leq a \Rightarrow b \leq a$ or $c \leq a$.

1.2.5 Definition
An element $a \in A$ is *irreducible* (*co-prime*) in A iff $b \vee c \geq a \Rightarrow b \geq a$ or $c \geq a$.

1.2.6 Proposition
If A has an order reversing involution then $a \in A$ is prime in A iff a' is irreducible in A'.

1.2.7 Theorem ([11], [12])

1. If $\mathcal{I}$ is a prime ideal of a lattice A, then $\mathcal{I} = \{a \in A : \varphi(a) \leq \alpha\}(= \varphi^{\leftarrow}[0, \alpha])$ where φ is a lattice morphism $\varphi : A \longrightarrow L$, where L is a lattice with 0 and 1 and $\alpha \neq 1$.

2. If $\mathcal{I}$ is a principal prime ideal of a frame A, then $\mathcal{I} = \varphi^{\leftarrow}[0, \alpha]$ where φ is a lattice morphism $\varphi : A \longrightarrow L$, where L is a lattice with 0 and 1 and $\alpha \neq 1$.

3. Conversely, if $\mathcal{I} = \varphi^{\leftarrow}[0, \alpha]$ with φ a lattice morphism $\varphi : A \longrightarrow L$, where L is a lattice with 0 and 1 and α is a prime element of L, $\alpha \neq 1$, then $\mathcal{I}$ is a prime ideal of A.

Proof.

1. Define $\varphi : A \longrightarrow L$ as :

$$\varphi(a) = \begin{cases} \alpha & \text{if } a \in \mathcal{I} \\ 1 & \text{if } a \notin \mathcal{I} \end{cases}$$

Then it can easily be checked that φ is a lattice morphism, i.e. if $a \in \mathcal{I}$ and $b \in \mathcal{I}$, then $a \vee b \in \mathcal{I}$ and so $\varphi(a) = \alpha = \varphi(b) = \varphi(a \vee b)$. Therefore $\varphi(a \vee b) = \varphi(a) \vee \varphi(b)$. If $a \in \mathcal{I}$ and $b \notin \mathcal{I}$, then $a \vee b \notin \mathcal{I}$ and so $\varphi(a) = \alpha$, $\varphi(b) = 1$ and $\varphi(a \vee b) = 1$. Therefore $\varphi(a) \vee \varphi(b) = \varphi(a \vee b)$. If $a \notin \mathcal{I}$ and $b \notin \mathcal{I}$, then $a \vee b \notin \mathcal{I}$ and so $\varphi(a) = \varphi(b) = 1$ and $\varphi(a \vee b) = 1$. Therefore $\varphi(a \vee b) = \varphi(a) \vee \varphi(b)$. If $a \in \mathcal{I}$ and $b \in \mathcal{I}$, then $a \wedge b \in \mathcal{I}$ and so $\varphi(a) = \alpha = \varphi(b) = \varphi(a \wedge b)$. Therefore $\varphi(a \wedge b) = \varphi(a) \wedge \varphi(b)$. If $a \in \mathcal{I}$ and $b \notin \mathcal{I}$ then $a \wedge b \in \mathcal{I}$ and so $\varphi(a) = \alpha, \varphi(b) = 1$ and $\varphi(a \wedge b) = \alpha$. Therefore $\varphi(a \wedge b) = \varphi(a) \wedge \varphi(b)$. If $a \notin \mathcal{I}$ and $b \notin \mathcal{I}$ then $a \wedge b \notin \mathcal{I}$ since $\mathcal{I}$ is a prime ideal, and so $\varphi(a) = \varphi(b) = \varphi(a \wedge b) = 1$. Therefore $\varphi(a \wedge b) = \varphi(a) \wedge \varphi(b)$.

2. If A is a frame we only have to add the following to the proof of (1):
   If $\bigvee a_i \in \mathcal{I}$, then $a_i \in \mathcal{I}$ for all i. So $\varphi(\bigvee a_i) = \alpha$ and $\varphi(a_i) = \alpha$. Hence $\varphi(\bigvee a_i) = \bigvee \varphi(a_i)$. If $\bigvee a_i \notin \mathcal{I}$ (so $\varphi(\bigvee a_i) = 1$): Suppose all $a_i \in \mathcal{I}$. Then $\bigvee a_i \in \mathcal{I}$ since $\mathcal{I}$ is a principal prime ideal. So there exists $a_j \notin \mathcal{I}$. Hence $\varphi(a_j) = 1$ and thus $\varphi(\bigvee a_i) = \bigvee \varphi(a_i)$.

3. Given $\mathcal{I} = \varphi^{\leftarrow}[0, \alpha]$, $\alpha \neq 1$, α prime. Then $1 \notin \mathcal{I}$. If $a \wedge b \in \mathcal{I}$, then $\varphi(a \wedge b) \leq \alpha$, hence $\varphi(a) \wedge \varphi(b) \leq \alpha$. Since α is prime, this means that $\varphi(a) \leq \alpha$ or $\varphi(b) \leq \alpha$. Thus $a \in \mathcal{I}$ or $b \in \mathcal{I}$. $\mathcal{I}$ is therefore a prime ideal.

□

1.2.8 Note

If $L = \{0, 1\}$, then $\alpha = 0$ and the φ in (1) and (2) above is unique and in (3), $\mathcal{I} = \varphi^{\leftarrow}\{0\}$.

1.3 Fuzzy sets and Zadeh's extension principle

If L is a complete lattice with 0 and 1 then $L^X = \{f : X \longrightarrow L\}$ becomes a lattice under the definition $(u \wedge v)(x) \equiv u(x) \wedge v(x)$ and $(u \vee v)(x) \equiv u(x) \vee v(x)$ for $u, v \in L^X$. $\underline{\mathbf{0}}$ and $\underline{\mathbf{1}}$ will indicate the functions identically 0 and 1 on X respectively. If L is distributive lattice or a frame so is L^X under these definitions. If L has an order reversing involution, so has L^X under $u'(x) \equiv u(x)'$. The members of $[0, 1]^X$ with these operations were called *fuzzy sets* by Zadeh([29]).

Given a function $f : X \longrightarrow Y$, Zadeh defined maps between L^X and L^Y (in both directions) as follows (of course Zadeh had $L = [0, 1]$ in mind) :

1.3.1 Definition (Zadeh's extension principle [29])

If f is a map from a set X into a set Y, then for $v \in L^Y$, $f^{\leftarrow}(v)$ is defined by $f^{\leftarrow}(v)(x) \equiv v(f(x))$, and for $u \in L^X$, $f^{\rightarrow}(u)$ is defined by,

$$f^{\rightarrow}(u)(y) = \begin{cases} \bigvee_{x \in f^{\leftarrow}\{y\}} u(x) & \text{if } f^{\leftarrow}(y) \neq \emptyset \\ 0 & \text{if } f^{\leftarrow}(y) = \emptyset \end{cases}$$

In the special case where $L = \{0, 1\}$, this corresponds with the classical notions of image and pre-image of "crisp" sets under a map.

The following summarizes the properties of $f^{\rightarrow}$ and $f^{\leftarrow}$. The proofs are routine and can be found in [1], [24], [25] and [26].

1.3.2 Theorem

Let X, Y, Z be sets, $f : X \longrightarrow Y$, $g : Y \longrightarrow Z$, $u, (u_j) \in L^X, v, (v_j) \in L^Y$, $w \in L^Z$.
Then

1. $(g \circ f)^{\rightarrow}(u) = g^{\rightarrow}(f^{\rightarrow}(u)$,

2. $(g \circ f)^{\leftarrow}(w) = f^{\leftarrow}(g^{\leftarrow}(w))$,

3. $f^{\leftarrow}(\sup_{j \in J} v_j) = \sup_{j \in J} f^{\leftarrow}(v_j)$,

4. $f^{\leftarrow}(\inf_{j\in J} v_j) = \inf_{j\in J} f^{\leftarrow}(v_j)$,

5. $f^{\leftarrow}(v') = (f^{\leftarrow}(v))'$,

6. $v_j \leq v_k \Rightarrow f^{\leftarrow}(v_j) \leq f^{\leftarrow}(v_k)$,

7. $f^{\rightarrow}(\sup_{j\in J} u_j) = \sup_{j\in J} f^{\rightarrow}(u_j)$,

8. $f^{\rightarrow}(\inf_{j\in J} u_j) \leq \inf_{j\in J} f^{\rightarrow}(u_j)$,

9. $f^{\rightarrow}(u') \geq f^{\rightarrow}(u)'$,

10. $u_j \leq u_k \Rightarrow f^{\rightarrow}(u_j) \leq f^{\rightarrow}(u_k)$,

11. $f^{\rightarrow}(f^{\leftarrow}(v)) \leq v$ with equality if f is surjective,

12. $f^{\leftarrow}(f^{\rightarrow}(u)) \geq u$ with equality if f is injective,

13. $f^{\rightarrow}(f^{\leftarrow}(v) \wedge u) = v \wedge f^{\rightarrow}(u)$.

1.3.3 Definition

A *fuzzy point* μ_x on X is a member of L^X with $\{x\}$ as support, i.e.

$$\mu_x(y) = \begin{cases} \alpha & \text{if } y = x \\ 0 & \text{otherwise} \end{cases}$$

for some $\alpha \in L \setminus \{0\}$. Where the value of α plays a role as in Section 6.2 we also denote such a point by (x, α).
If L has an order reversing involution, we say that a fuzzy point is *quasi-coincident* with $u \in L^X$ iff $\mu_x(x) > u(x)'$ (equivalently $u(x) > \mu_x(x)'$) (In the case $L = [0,1]$ this means that $\mu_x(x) + u(x) > 1$). We write μ_x *qco* u.

1.3.4 Proposition

1. μ_x *qco* $u \wedge v$ iff μ_x *qco* u and μ_x *qco* v

2. μ_x *qco* $\bigvee_{j\in J} u_j$ iff $\exists j \in J$ such that μ_x *qco* u_j

Proof. trivial
□

1.3.5 Definition

Let $\mu \in L^X$ we define the support of μ, suppμ, as follows,

$$\text{supp}\mu = \{x \in X : \mu(x) > 0\}$$

1.4 Topologies

C.L. Chang [1] was the first to define a "fuzzy topology" extending the "crisp" concept in the lattice $\{0,1\}^X$ to $[0,1]^X$.
More generally:

1.4.1 Definition
If L is a complete lattice with 0 and 1 then a subfamily $\mathcal{T}$ of L^X is an *L-topology* (fuzzy topology) on X if,

1. $\underline{\mathbf{0}}, \underline{\mathbf{1}} \in \mathcal{T}$

2. $u, v \in \mathcal{T} \Rightarrow u \wedge v \in \mathcal{T}$

3. $u_i \in \mathcal{T}, i \in I \Rightarrow \bigvee u_i \in \mathcal{T}$.

$(X, L, \mathcal{T})$ is called an *L-topological* (fuzzy topological) space and the members of $\mathcal{T}$ are the *L-open* (fuzzy open) sets.

In what follows, when considering L-topologies, we'll assume L to be a frame so that the frame distributive law of Definition 1.1.13 is also valid for L^X. Furthermore, if L has an order reversing involution then $\mathcal{T}'$ (the involutes of the members of $\mathcal{T}$) is a sublattice of L^X which is closed under arbitrary infima and finite suprema, the *L-closed* sets.

As in the crisp case we define,

1.4.2 Definition
Let $(X, L, \mathcal{T}_1)$ and $(Y, L, \mathcal{T}_2)$ be L-topological spaces and $f : X \longrightarrow Y$, then f is *L-continuous* (fuzzy continuous) iff $f^{\leftarrow}(v) \in \mathcal{T}_1$ for all $v \in \mathcal{T}_2$.

In view of Theorem 1.3.2 this means that f is L-continuous iff $f^{\leftarrow}$ is a frame morphism from $\mathcal{T}_2$ to $\mathcal{T}_1$.

A topology (and hence an L-topology) can also be viewed as a mapping. This probably goes back to [5].

1.4.1 Ordinary (crisp) topological case

If a topology $\mathcal{T}$ on a space X (i.e. a frame in $\mathcal{P}(X)$ with bottom $\emptyset$ and top X) is identified with a map,

$$\mathcal{T} : \mathcal{P}(X) \longrightarrow \{0,1\} \text{ where } \mathcal{T}(A) = 1 \text{ iff } A \text{ is open, then}$$

1. $\mathcal{T}(\emptyset) = \mathcal{T}(X) = 1$
2. $\mathcal{T}(U \cap V) \geq \mathcal{T}(U) \wedge \mathcal{T}(V)$ for all $U, V \in \mathcal{P}(X)$
3. $\mathcal{T}(\bigcup U_i) \geq \bigwedge \mathcal{T}(U_i)$ for all $U_i \in \mathcal{P}(X)$

$(X, \mathcal{T})$ is called a *topological space.*

In emulation of this we could also define,

1.4.2 L-topological case

(L is a frame with 0 and 1 and order reversing involution.)

If an L-topology $\mathcal{T}$ on X (i.e. a frame in L^X with $\underline{\mathbf{0}}$ and $\underline{\mathbf{1}}$) is identified with a map

$$\mathcal{T} : L^X \longrightarrow \{0,1\} \text{ where } \mathcal{T}(a) = 1 \text{ if } a \text{ is } L\text{-open, then also}$$

1. $\mathcal{T}(\underline{\mathbf{0}}) = \mathcal{T}(\underline{\mathbf{1}}) = 1$
2. $\mathcal{T}(u \wedge v) \geq \mathcal{T}(u) \wedge \mathcal{T}(v)$ for all $u, v \in L^X$
3. $\mathcal{T}(\bigvee u_i) \geq \bigwedge \mathcal{T}(u_i)$ for all $u_i \in L^X$

$(X, L, \mathcal{T})$ is called an L-*topological space*.

In what follows we identify the operator $\mathcal{T} : L^X \longrightarrow \{0,1\}$ with the frame $\mathcal{F} = \{u \in L^X : \mathcal{T}(u) = 1\}$ in L^X, and corresponding "closed sets" with the complements $u'(x) \equiv u(x)'$ where the second $'$ is the order reversing involution of L. If we write $\mathcal{T} \subset \mathcal{T}^*$ for two L-topologies on X, it is in the sense of the frame inclusion : $\mathcal{F} \subset \mathcal{F}^*$.

The advantage of these viewpoints is that it allows for the following generalization.

1.4.3 (L, M)-topological case

(L is a frame with 0 and 1 and order reversing involution and M is a frame with 0 and 1.)

An (L, M)-*topology* is a map $\mathcal{T} : L^X \longrightarrow M$ such that

1. $\mathcal{T}(\underline{\mathbf{0}}) = \mathcal{T}(\underline{\mathbf{1}}) = 1$
2. $\mathcal{T}(u \wedge v) \geq \mathcal{T}(u) \wedge \mathcal{T}(v)$ for all $u, v \in L^X$
3. $\mathcal{T}(\bigvee u_i) \geq \bigwedge \mathcal{T}(u_i)$ for all $u_i \in L^X$

$(X, L, M, \mathcal{T})$ is called an (L, M)-*topological space*.

This concept is due to Šostak (see e.g. [18] and [19] in case $L = M = [0,1]$) and Šostak and Kubiak [14] where these spaces are referred to as "(L, M)-fuzzy topologies" (cf. Chapters 3 and 4 of [4]). Mingsheng Ying [28] called the case $L = \{0,1\}$ and $M = [0,1]$ a "fuzzifying topology".

1.5 Neighbourhoods

1.5.1 Definition
In $(X, L, \mathcal{T})$ an α-*neighbourhood* ($\alpha \in L$, $\alpha \neq 1$) of a point $x \in X$ is a $u \in \mathcal{T}$ such that $u(x) > \alpha$

1.5.2 Definition
In $(X, L, \mathcal{T})$, L with an order reversing involution $'$ a *quasi-neighbourhood* of a fuzzy point μ_x is a $u \in \mathcal{T}$ such that μ_x *qco* u, i.e. $\mu_x(x) > u(x)'$.

1.5.3 Proposition
In $(X, L, \mathcal{T})$ (L with an order reversing involution) $u \in \mathcal{T}$ is an α neighbourhood of x iff u is a quasi-neighbourhood of $\mu_x = (x, \alpha')$.

Proof. Trivial.
□

Chapter 2

Crisp Sobriety

In this chapter we define sobriety of a crisp topological space in four different ways and show that the definitions are equivalent.

2.1 Initial definition and first characterization

2.1.1 Definition ([8])
Given a topological space $(X, \mathcal{T})$, $pt\mathcal{T}$ (or spectrum of $\mathcal{T}$) will denote the collection of all frame maps p from $\mathcal{T}$ into the frame $\{0,1\}$ (the topology on a singleton space).

2.1.2 Definition
Let

$$\Psi : X \longrightarrow pt\mathcal{T}$$

$$\Psi(x)(u) = \chi_u(x)$$

be the characteristic function of u, for $x \in X$ and $u \in \mathcal{T}$. Then $(X, \mathcal{T})$ is *sober* iff Ψ is a bijection.

2.1.3 Proposition ([8])
Ψ is injective iff $(X, \mathcal{T})$ is T_0.

Proof. If x, y are distinct points of X then we have $\Psi(x) = \Psi(y)$ iff x and y are contained in exactly the same open sets of X iff the T_0 axiom fails for the pair (x, y).
□

2.1.4 Corollary
$(X, \mathcal{T})$ sober $\Rightarrow$ $(X, \mathcal{T})$ T_0

2.1.5 Definition (see Definition 1.2.2)
Let $(X, \mathcal{T})$ be a topological space. A filter $\mathcal{F}$ of open sets is *completely prime* if for every family $(O_i)_{i \in I}$ with $\bigcup_{i \in I} O_i \in \mathcal{F}, \exists i \in I$ with $O_i \in \mathcal{F}$.

2.1.6 Theorem
A topological space $(X, \mathcal{T})$ is sober iff every completely prime filter $\mathcal{F}$ of open sets is the filter of open neighbourhoods of a unique $x \in X$.

Proof. Firstly observe that $\forall p \in pt\mathcal{T}$, $p^{\leftarrow}(1)$ is a completely prime filter in $\mathcal{T}$.
Filter :

1. $X \in p^{\leftarrow}(1)$
2.

$$\begin{aligned} u, v \in p^{\leftarrow}(1) &\Rightarrow p(u) = 1, p(v) = 1 \\ &\Rightarrow p(u \cap v) = p(u) \wedge p(v) = 1 \\ &\Rightarrow u \cap v \in p^{\leftarrow}(1) \end{aligned}$$

3.

$$\begin{aligned} u \in p^{\leftarrow}(1), u \subset v &\Rightarrow p(v) = 1 \text{ since } p(v) \geq p(u)\ (p(u) = p(u \cap v) = p(u) \wedge p(v) = 1) \\ &\Rightarrow v \in p^{\leftarrow}(1) \end{aligned}$$

Completely Prime :

$$\begin{aligned} \bigcup u_i \in p^{\leftarrow}(1) &\Rightarrow p(\bigcup u_i) = 1 \\ &\Rightarrow \bigvee p(u_i) = 1 \\ &\Rightarrow \exists u_i \text{ s.t. } p(u_i) = 1 \\ &\Rightarrow \exists u_i \text{ s.t. } u_i \in p^{\leftarrow}(1) \end{aligned}$$

We need to show that all completely prime filters in $\mathcal{T}$ are of the form $p^{\leftarrow}(1)$ for some $p \in pt\mathcal{T}$. Let $\mathcal{F}$ be a completely prime filter of open sets. Define,

$$p : \mathcal{T} \to \{0, 1\}$$

as

$$p(u) = \begin{cases} 1 & \text{if } u \in \mathcal{F} \\ 0 & \text{if } u \notin \mathcal{F} \end{cases}$$

Then p is a frame map as can be seen from,

1.

$$p(X) = 1, p(\emptyset) = 0$$

2.

$$p(u \cap v) = \begin{cases} 1 & \text{if } u \text{ and } v \in \mathcal{F} \\ 0 & u \text{ or } v \notin \mathcal{F} \end{cases}$$

$$= p(u) \wedge p(v)$$

3.

$$p(\bigcup u_i) = \begin{cases} 1 & \text{if } \bigcup u_i \in \mathcal{F} \\ 0 & \text{if } \bigcup u_i \notin \mathcal{F} \end{cases}$$

$$= \bigvee p(u_i)$$

because if $\bigcup u_i \in \mathcal{F}$, $\exists u_j \in \mathcal{F}$ and so $p(u_j) = 1$ hence $\bigvee p(u_i) = 1$. If $\bigcup u_i \notin \mathcal{F}$ then no $u_i \in \mathcal{F}$ and so $\bigvee p(u_i) = 0$. Thus $\mathcal{F} = p^{\leftarrow}(1)$ for a frame map $p \in pt\mathcal{T}$.

Now, if $(X, \mathcal{T})$ is sober then

$$\begin{aligned} p^{\leftarrow}(1) &= \{u \in \mathcal{T} : p(u) = 1\} \\ &= \{u \in \mathcal{T} : \chi_u(x) = 1 \text{ for some unique } x \in X\} \\ &= \{u \in \mathcal{T} : x \in u \text{ for some unique } x \in X\} \end{aligned}$$

Conversely, if $\mathcal{F}$ is a completely prime filter in $\mathcal{T}$ then

$$\mathcal{F} = \{u \in \mathcal{T} : x \in u \text{ for a unique } x \in X\}$$

Thus for any $p \in pt\mathcal{T}$

$$\begin{aligned} p^{\leftarrow}(1) &= \{u \in \mathcal{T} : x \in u \text{ for a unique } x \in X\} \\ &= \{u \in \mathcal{T} : \chi_u(x) = 1 \text{ for a unique } x \in X\} \end{aligned}$$

so Ψ is a bijection and $(X, \mathcal{T})$ is sober.
□

2.2 Two other characterizations of sobriety

Sobriety can also be defined in terms of nets. Before doing this we need to introduce some concepts concerning nets.

2.2.1 Definition ([23])
Given a topological space $(X, \mathcal{T})$. A net $(x_\lambda)_{\lambda\in\Lambda}$ in X is *observative* if , given $O \in \mathcal{T}$ and $x_\lambda \in O$ for some $\lambda \in \Lambda$, then the net is eventually in O.
Thus a net is observative iff it converges to each of its points (it is "self converging").
An observative net is said to *strongly converge* to x if it converges with respect to $\mathcal{T}$ and if it additionally satisfies that x is an element of every open set which eventually contains the net. In this case we write $x_\lambda \longrightarrow^* x$.
If $(x_\lambda)_{\lambda\in\Lambda}$ is a net on some set and $\lambda \in \Lambda$, we denote the λ-tail $\{x_j : j \geq \lambda\}$ of the net by $[x]_{\geq\lambda}$.

2.2.2 Example
In $\mathbb{R}$ with the ordinary topology the constant nets $(x_\lambda = r, \forall\lambda)$ are observative.
In $X = (-\infty, k]$ with the topology $\{X, \emptyset, (a, k] : a \in X\}$, $x_\lambda = \lambda \in X$ (with the usual $\leq$ on $\mathbb{R}$) is an observative net which converges strongly to k.

In the logical approach to topology, open sets correspond to observable properties of points and $x \in O$ means that x has the property O. Thus an observative net may be thought of as a stepwise computation where every property established at some stage will be satisfied eventually. In this setting, being the strong limit means having exactly all the properties established during the computation.

2.2.3 Definition ([23])
If $(X, \mathcal{T})$ is a topological space then $b - \mathcal{T}$, its *b-topology*, has as a subbase of open sets the family of all sets which are either $\mathcal{T}$-open or $\mathcal{T}$-closed.

2.2.4 Proposition ([23])
If $(x_\lambda)_{\lambda\in\Lambda}$ is an observative net in $(X, \mathcal{T})$ then $x_\lambda \longrightarrow^* x$ iff $x_\lambda \longrightarrow x$ with respect to the b-topology.

Proof. Suppose $x_\lambda \longrightarrow^* x$ and $x \in A$ for some $\mathcal{T}$-closed set A. If the net is not eventually contained in A, then it is frequently in the open set $X \setminus A$. The net is observative, hence $[x]_{\geq\lambda} \subseteq X \setminus A$ for some tail. But then $x \in X \setminus A$ by strong convergence, a contradiction. Hence strong convergence implies b-convergence. If on the other hand $x_\lambda \longrightarrow x$ with respect to $b - \mathcal{T}$, then strong convergence follows immediately: An open set O which eventually contains the net but does not contain x gives rise to a b-neighbourhood $X \setminus O$ of x which is

missed by the net.
□

For a net $(x_\lambda)_{\lambda\in\Lambda}$ in $(X, \mathcal{T})$ we construct the *derived filter* of open sets $\mathcal{F}_{(x_\lambda)}$ as follows,

$$\mathcal{F}_{(x_\lambda)} = \{O \in \mathcal{T} : \exists\lambda \in \Lambda, [x]_{\geq\lambda} \subseteq O\}$$

2.2.5 Proposition ([23])
A filter derived from an observative net is completely prime.

Proof. Let $(x_\lambda)_{\lambda\in\Lambda}$ be observative and $\bigcup_{j\in J} O_j \in \mathcal{F}_{(x_\lambda)}$. So $[x]_{\geq\lambda^*} \subseteq \bigcup O_j$. Then $\exists j_0 \in J$ such that $x_{\lambda^*} \in O_{j_0}$. Since the net is observative, this implies that some tail is contained in O_{j_0}. Thus by definition of $\mathcal{F}_{(x_\lambda)}$, O_{j_0} is a member of it.
□

2.2.6 Lemma ([23])
If $(x_\lambda)_{\lambda\in\Lambda}$ is an observative net then $x_\lambda \longrightarrow^* x$ iff $\mathcal{F}_{(x_\lambda)} = \mathcal{N}_x$.

Proof. $(x_\lambda)_{\lambda\in\Lambda}$ converges strongly to x iff it is true that $x \in O \in \mathcal{T}$ is equivalent to the existence of some $\lambda \in \Lambda$ with $[x]_{\geq\lambda} \subseteq O$.

$$x_\lambda \longrightarrow^* x \text{ iff } x \in O \Leftrightarrow \exists\lambda \in \Lambda, [x]_{\geq\lambda} \subseteq O \text{ iff } x \in O \leftrightarrow O \in \mathcal{F}_{x_\lambda}$$

□

2.2.7 Proposition ([23])
Suppose $\mathcal{F}$ is a filter of open subsets of the topological space $(X, \mathcal{T})$. Then $\mathcal{F}$ is completely prime iff $\forall O \in \mathcal{F}$, $\exists x \in O$ such that $\forall P \in \mathcal{T}$, $x \in P \Rightarrow P \in \mathcal{F}$

Proof. In case $\mathcal{F}$ has this property and $\bigcup_{j\in J} O_j \in \mathcal{F}$, pick $x \in \bigcup_{j\in J} O_j$ with $(x \in P \Rightarrow P \in \mathcal{F})$. Certainly we have $x \in O_{j_0}$ for some $j_0 \in J$. Hence $O_{j_0} \in \mathcal{F}$. Therefore, the filter is completely prime.
Conversely, suppose that $O \in \mathcal{F}$ does not have this property. This means that for each $x \in O$ there is $P_x \in \mathcal{T}$ with $x \in P_x$ and $P_x \notin \mathcal{F}$. Set $O_x \equiv P_x \cap O$. Then $O_x \notin \mathcal{F}$ for all $x \in O$ and $O = \bigcup_{x\in O} O_x \in \mathcal{F}$. This contradicts complete primality of $\mathcal{F}$.
□

Sünderhauf [23] constructs a net from a completely prime filter as follows.
For each $O \in \mathcal{F}$ choose $x_0 \in O$ with the property guaranteed by Proposition 2.2.7. Then $\Lambda_\mathcal{F} = \{(x_0, O) : O \in \mathcal{F}\}$ is a directed set with respect to $(x_0, O) \leq (x_S, S)$ iff $S \subset O$. Then $N : \Lambda_\mathcal{F} \longrightarrow X$, $N(x_0, O) \equiv x_0$ is called the *net derived from the completely prime filter* $\mathcal{F}$.

2.2.8 Lemma ([23])
A net derived from a completely prime filter is observative.

Proof. Let (x_0) be the net derived from the completely prime filter and let $x_0 \in O^*$ for $O^* \in \mathcal{T}$. Then $O^* \in \mathcal{F}$ by the choice of x_0 and hence $(x_0, O^*) \in \Lambda_\mathcal{F}$. If $(x_S, S) \geq (x_0, O^*)$ then $S \subset O^*$ and $x_S \in S \subset O^*$ and the net is eventually in O^*.
□

2.2.9 Proposition ([23])
Every completely prime filter $\mathcal{F}$ equals the derived filter of any of its derived nets.

Proof. If $O \in \mathcal{F}$ and $[x]_{\geq O} \subseteq O$ then $O \in \mathcal{F}_{(x_O)}$. Conversely, $O \in \mathcal{F}_{(x_O)}$ implies $[x]_{\geq P} \subseteq O$ for some $P \in \mathcal{F}$. Hence, $x_P \in O$, and this implies $O \in \mathcal{F}$ by the choice of x_P.
□

The preceeding results concerning nets provide us with the following theorem, which constitutes our second characterization of sobriety, now in terms of observative nets.

2.2.10 Theorem ([23])
A topological space $(X, \mathcal{T})$ is sober iff every observative net strongly converges to a unique point.

Proof. $(X, \mathcal{T})$ sober implies by Theorem 2.1.6 that every completely prime filter of open sets equals $\mathcal{N}_x$ for a unique $x \in X$. So given an observative net $(x_\lambda)_{\lambda \in \Lambda}$, then its derived filter is completely prime (by Proposition 2.2.5) and is therefore $\mathcal{N}_x$ for a unique x. Thus by Lemma 2.2.6, $x_\lambda \longrightarrow^* x$ for a unique x.
Conversely : Suppose every observative net in X strongly converges to a unique $x \in X$. Then by Lemma 2.2.6, $\mathcal{N}_x$ is the derived filter of such a net, and by Proposition 2.2.5 is completely prime. But by Proposition 2.2.9 every completely prime filter can be derived from such an observative net, and hence equals $\mathcal{N}_x$. So by Theorem 2.1.6, $(X, \mathcal{T})$ is sober.
□

The third characterization of sobriety is in terms of irreducible closed sets.

2.2.11 Definition ([8] see also Definition 1.2.5)
A subset F of a topological space $(X, \mathcal{T})$ is said to be irreducible closed if it is closed and it cannot be written as a union $F = F_1 \cup F_2$ where both F_1 and F_2 are proper closed subsets of F.

2.2.12 Theorem ([8])
A topological space is sober iff every non-empty irreducible closed set is the closure of a unique singleton.

Proof. For each $p \in pt\mathcal{T}$, $p^{\leftarrow}(1)$ is a completely prime filter in $\mathcal{T}$ (see proof of Theorem 2.1.6), i.e. $p^{\leftarrow}(0) = \downarrow \bigvee\{a \in p^{\leftarrow}(0)\}$ (p preserves arbitrary $\bigvee$) is a principal prime ideal in $\mathcal{T}$ (Proposition 1.2.3). So for each $p \in pt\mathcal{T}$, there exists a prime element $(\bigvee\{a \in p^{\leftarrow}(0)\})$ in $\mathcal{T}$ and hence (by Proposition 1.2.6) an irreducible closed set.
Conversely, given an irreducible closed set in X, then its complement in X is a prime element in $\mathcal{T}$ which generates a principal prime ideal in $\mathcal{T}$. Its complement in $\mathcal{T}$ is therefore a completely prime filter in $\mathcal{T}$. (Proposition 1.2.3). By the proof of Theorem 2.1.6, this is of the form $p^{\leftarrow}(1)$ for some $p \in pt\mathcal{T}$.
So there is a bijection between $pt\mathcal{T}$ and the irreducible closed sets in $(X, \mathcal{T})$ (i.e. a bijection between $pt\mathcal{T}$ and the prime elements in $\mathcal{T}$). Now if $(X, \mathcal{T})$ is sober by Definition 2.1.2, there is a bijection between X and the irreducible closed sets in X. Since for each $x \in X$, its closure $cl\{x\}$ is irreducible closed, sobriety implies that every irreducible closed set in X is the closure of a unique singleton.
Conversely suppose U is an irreducible closed set, therefore $U = cl\{x\}$ for a unique $x \in X$. Then $X \setminus U$ is a prime element in the lattice of open sets of X. Each prime element of $\mathcal{T}$ is associated bijectively with a point $p \in pt\mathcal{T}$, therefore we can associate the unique $x \in X$ that generates $X \setminus U$ with the $p \in pt\mathcal{T}$. Therefore ψ is a bijection.
□

Chapter 3

Properties of Sober Spaces

3.1 Results concerning crisp sobriety

3.1.1 Proposition ([8])
If $(X, \mathcal{T})$ is Hausdorff, then Ψ is a bijection.

Proof. If F is a closed subset of a T_2 space X containing two distinct points x and y, let U and V be disjoint open neighbourhoods of x and y. Then $F \setminus U$ and $F \setminus V$ are proper closed subsets of F whose union is $F \setminus (U \cap V) = F$ i.e. F is reducible. So the only irreducible closed sets of X are singletons. Therefore X is sober by Theorem 2.2.12.
□

This result (a Hausdorff space is sober) can also be deduced immediately from Theorem 2.1.6 or Theorem 2.2.10. Putting this together with Corollary 2.1.4, we get,

3.1.2 Corollary
$(X, \mathcal{T})$ Hausdorff $\Rightarrow$ $(X, \mathcal{T})$ sober $\Rightarrow$ $(X, \mathcal{T})$ T_0

3.1.3 Example
$\mathbb{R}$ with the ordinary topology, being Hausdorff, is sober whereas $\mathbb{R}$ with the cofinite topology, is T_1 but not sober since $\mathbb{R}$ itself is irreducibly closed but not the closure of a singleton. $X = \{-\infty, k]$ with the topology $\{X, \emptyset, \{(a, k] : a \in X\}\}$ is sober since every (irreducible) closed set $(-\infty, a]$ is the closure of $a \in X$.

3.1.4 Definition ([8])
Define,

$$\Phi : \mathcal{T} \longrightarrow \mathcal{P}(pt\mathcal{T})$$

$$\Phi(u) = \{p \in pt\mathcal{T} : p(u) = 1\}$$

3.1.5 Proposition ([8])
Φ is a frame homomorphism $\mathcal{T} \longrightarrow \mathcal{P}(pt\mathcal{T})$, in particular its image is a topology on $pt\mathcal{T}$.

Proof.

$$\Phi(X) = pt\mathcal{T} \text{ since } p(X) = 1 \text{ for all } p \in pt\mathcal{T}$$

$$\Phi(\emptyset) = \emptyset \text{ since } p(\emptyset) = 0 \text{ for all } p \in pt\mathcal{T}$$

$$\Phi(u) \cap \Phi(v) = \Phi(u \cap v) \text{ since } p(u \cap v) = p(u) \wedge p(v) \text{ for all } p \in pt\mathcal{T}$$

$$
\begin{aligned}
p \in \Phi(\bigcup u_i) &\Leftrightarrow p(\bigcup u_i) = 1 = \bigvee p(u_i) \\
&\Leftrightarrow \exists i, p(u_i) = 1 \\
&\Leftrightarrow \exists i, p \in \Phi(u_i) \\
&\Leftrightarrow p \in \bigcup \Phi(u_i)
\end{aligned}
$$

□

We can always consider $pt\mathcal{T}$ as a topological space, with topology given by the image of Φ.

3.1.6 Proposition ([8])
For any frame $\mathcal{T}$, the space $pt\mathcal{T}$ is sober.

Proof. Let F be an irreducibly closed subset of $pt\mathcal{T}$. Its complement in $pt\mathcal{T}$ is a prime open set (Proposition 1.2.6), which we can write as $\Phi(a)$ for some $a \in \mathcal{T}$; this a may not be uniquely determined, since we are not assuming $\mathcal{T}$ is a topology, but since Φ preserves joins there is a unique largest a with $\Phi(a) = pt\mathcal{T} \setminus F$, namely the join of all such a. It is now clear that this a is a prime element of $\mathcal{T}$, since if $b \cap c \subseteq a$ then $\Phi(b) \cap \Phi(c) \subseteq \Phi(a)$, whence either $\Phi(b) \subseteq \Phi(a)$ or $\Phi(c) \subseteq \Phi(a)$ and so $b \subseteq a$ or $c \subseteq a$. Moreover, it is easy to see that the point p of $\mathcal{T}$ defined by this prime element satisfies $\Psi(p) = pt\mathcal{T} \setminus \Phi(a) = F$. So Ψ is surjective. But if p, q are distinct points of $pt\mathcal{T}$, then there exists $a \in \mathcal{T}$ with $p(a) \neq q(a)$, so that the open set $\Phi(a)$ contains just one of p and q; thus $pt\mathcal{T}$ is a T_0 space. So Ψ is injective.
□

In view of this result $pt\mathcal{T}$ with the topology $\{\Phi(u) : u \in \mathcal{T}\}$ is called the *soberification* of $(X, \mathcal{T})$.

3.1.7 Note

1. $\{\Phi(u) : u \in \mathcal{T}\}$ is a topology on $pt\mathcal{T}$ under which Ψ becomes continuous and open since $\Psi^{\leftarrow}(\Phi(u)) = u$ and $\Psi^{\rightarrow}(u) = \Phi(u)$. Thus if $(X, \mathcal{T})$ is sober, Ψ is a homeomorphism. Of course if $(X, \mathcal{T})$ is sober it is its own soberification.

2. Note that $\Phi(u)$ as a subset of $pt\mathcal{T}$ can also be considered as a map from $pt\mathcal{T}$ into $\{0, 1\}$ where :
$$\Phi(u)(p) = \begin{cases} 1 & \text{if } p(u) = 1 \\ 0 & \text{if } p(u) = 0 \end{cases} = p(u).$$

3.2 Sober spaces and continuous maps

A map $f : (X, \mathcal{T}_1) \longrightarrow (Y, \mathcal{T}_2)$ is continuous iff $f^{\leftarrow}$ is a frame morphism from $\mathcal{T}_2$ to $\mathcal{T}_1$.
In [2] it was proved that:

3.2.1 Theorem
If $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ are two topological spaces with $(Y, \mathcal{T}_2)$ Hausdorff, and $\mu : \mathcal{T}_2 \longrightarrow \mathcal{T}_1$ is a frame morphism, then there exists a unique continuous function $f : X \longrightarrow Y$ such that $f^{\leftarrow} = \mu$.

This result can be considerably generalized:

3.2.2 Theorem
If $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ are two topological spaces with $(Y, \mathcal{T}_2)$ sober and $\mu : \mathcal{T}_2 \longrightarrow \mathcal{T}_1$ is a frame morphism, then there exists a unique continuous function $f : X \longrightarrow Y$ such that $f^{\leftarrow} = \mu$.

Sobriety of $(Y, \mathcal{T}_2)$ is also necessary in the following sense,

3.2.3 Theorem
If $(Y, \mathcal{T}_2)$ is T_0 (e.g. Ψ of Definition 2.1.2 is injective) and for every $(X, \mathcal{T}_1)$ and every frame morphism $\mu : \mathcal{T}_2 \longrightarrow \mathcal{T}_1$ there exists an $f : X \longrightarrow Y$ such that $f^{\leftarrow} = \mu$, then $(Y, \mathcal{T}_2)$ is sober.

On the other hand the T_0 requirement can be dropped in the case that f is unique :,

3.2.4 Theorem
If for every $(X, \mathcal{T}_1)$ and every frame morphism $\mu : \mathcal{T}_2 \longrightarrow \mathcal{T}_1$ there exists a unique $f : X \longrightarrow Y$ such that $f^{\leftarrow} = \mu$, then $(Y, \mathcal{T}_2)$ is sober.

These results are special cases of more general ones. See later in Chapter 5.

3.3 Sobriety of partially ordered sets

3.3.1 Definition ([8])
Let X be a T_0 space, we can define an order on X as follows $x \leq y$ iff x is in the closure of $\{y\}$ ($cl\{x\} \subseteq cl\{y\}$). If this relation holds we say x is a *specialization* of y, this order is called the *specialization ordering*.

The specialization ordering is reflexive, transitive and antisymmetric (antisymmetry is the T_0 axiom) so it is a partial ordering. This allows us to put a partial order on any T_0 space. Any continuous map between T_0 spaces is necessarily order preserving and the order is discrete iff X is a T_1 space.
Conversely, we want to ask the question, given a poset $(X, \leq)$, can we find a T_0 topology on X for which $\leq$ is the specialization ordering? The answer to this question is yes, as can be seen by the following results.

3.3.2 Definition ([8])
If $(X, \leq)$ is a poset the Alexandrov topology $\Upsilon(X, \leq)$, the collection of all upper sets of X, is a maximal T_0 topology on $(X, \leq)$ for which $\leq$ is the specialization ordering associated with this topology.

3.3.3 Definition ([8])
If $(X, \leq)$ is a poset the upper interval topology $\Phi(X, \leq)$, the smallest topology for which sets of the form $\downarrow(x)$ are closed, is the minimal topology on $(X, \leq)$ whose specialization ordering is $\leq$.

3.3.4 Proposition ([8])
Let $(X, \leq)$ be a poset, $\mathcal{T}$ a T_0 topology on X. Then $\mathcal{T}$ induces the ordering $\leq$ iff $\Phi(X, \leq) \subseteq \mathcal{T} \subseteq \Upsilon(X, \leq)$

Proof. Suppose $\mathcal{T}$ induces $\leq$. Then every $\mathcal{T}$-open set must be an upper set since if $cl\{y\} = \downarrow(y)$ meets an open set U then $y \in U$ and so $\mathcal{T} \subseteq \Upsilon(X, \leq)$. But since $\downarrow(y)$ must be $\mathcal{T}$ closed for every y we also have $\Phi(X, \leq) \subseteq \mathcal{T}$. Conversely if $\Phi(X, \leq) \subseteq \mathcal{T} \subseteq \Upsilon(X, \leq)$ then $\downarrow(y)$ is the smallest $\mathcal{T}$ closed set containing y and so $x \leq y$ iff $x \in cl\{y\}$ and $\mathcal{T}$ is T_0 since it contains the T_0 topology $\Phi(X, \leq)$.

□

3.3.5 Proposition ([8])
If $(X, \mathcal{T})$ is a sober space then the specialisation ordering on X has directed joins. Moreover if $U \in \mathcal{T}$ then U is not merely an upper set but also inaccessible by directed joins. ie. $S \subseteq X$ directed and $\bigvee S \in U$ imply $S \cap U \neq \emptyset$

Proof. Let S be a directed subset of X. Then the family of subsets $\{cl\{x\} : x \in S\}$ is also directed. Let T be the closure of its union. If $T = F_1 \cup F_2$ where F_1 and F_2 are closed, then for each $x \in S$ we have either $x \in F_1$ or $x \in F_2$ and by directedness we conclude either $S \subseteq F_1$ or $S \subseteq F_2$, since the F_i are lower sets. Hence $T = F_1$ or $T = F_2$ i.e. T is irreducible. So T is the closure of a unique point y, which is readily seen to be the join of S in $(X, \leq)$. Moreover if U is open and $y \in U$, then U meets $\bigcup\{cl\{x\} : x \in S\}$, which implies that $x \in U$ for some $x \in S$. So U is inaccessible by directed joins.
□

3.3.6 Definition ([8])
Let $(X, \leq)$ be a poset with directed joins then the set $\Sigma(X, \leq)$ of upper sets which are inaccessible by directed joins is a topology on X called the Scott topology.

The sets $X \setminus \downarrow (x)$ are inaccessible by directed joins and so $\Phi(X, \leq) \subseteq \mathcal{T} \subseteq \Sigma(X, \leq)$. For sober topologies we have improved the bounds of Proposition 3.3.4, a sober topology $\mathcal{T}$ on X induces a given partial order $\leq$ iff $\Phi(X, \leq) \subseteq \mathcal{T} \subseteq \Sigma(X, \leq)$. This leads to the question, is the Scott topology on a poset with directed joins always sober? Equivalently we can ask two seperate questions,

1. If $(X, \leq)$ has directed joins, is there a sober topology on X inducing $\leq$?
2. If $(X, \leq)$ is induced by some sober topology, is the Scott topology on X sober?

In general the answer to both of these questions is no. The following example is a counterexample to (1) and from it we can obtain a counterexample to (2).

3.3.7 Example ([7])
Consider the set

$$X = \mathbb{N} \times (\mathbb{N} \cup \{\infty\})$$

We define a partial order on X as follows,

$$(m, n) \leq (m', n') \text{ iff either } m = m' \text{ and } n \leq n' \ (\leq \infty) \text{ or } n' = \infty \text{ and } n \leq m'.$$

Evidently the elements (m, ∞) are all maximal in X, so if one of them is contained in a directed set $S \subset X$, it must be the greatest member of S. Remarking that if $m \neq m'$ the only upper bounds for (m, n) and (m', n') in X are elements of the form (m'', ∞), we see that the only directed subsets of X without greatest members are infinite and therefore cofinal subsets of $\{m\} \times \mathbb{N}$ for some m. Since any such subset has (m, ∞) as its only upper bound, we deduce that every directed subset of X has a least upper bound. On the other hand, if U is a nonempty Scott-open subset of X, U cannot be contained in $\mathbb{N} \times \{\infty\}$, since each point (m, ∞), is a directed join of elements not in this set. Now U contains some (m, n) with $n < \infty$, but since it is an upper set it must also contain (m', ∞) for all $m' \geq n$. Any two nonempty Scott-open subsets of X must therefore intersect, i.e. X is irreducible in the Scott topology, but it has no greatest member. Lastly, suppose $\mathcal{T}$ is any sober topology inducing the given partial order on X, since $\mathcal{T}$ is contained in the Scott topology, X must be irreducible (and closed) in it, and so no such topology can exist.

Determining the soberification of $(X, \Sigma(X, \leq))$ is now not difficult: if F is a proper Scott-closed subset of X, then it contains only finitely many points of the "top row" $\mathbb{N} \times \{\infty\}$, from which it follows that $\bigcup\{cl y : y \in F, y \not\leq x\}$ is closed for any maximal element x of F, and hence, if F is also irreducible, that is the closure of a singleton. Accordingly, the soberification of $(X, \Sigma(X, \leq))$ has underlying poset $(X^+, \leq)$ obtained by adding a single top element ∞ to $(X, \leq)$. Nevertheless, $\{\infty\}$ is open in the Scott topology on X^+, since ∞ is not expressible as a nontrivial directed join, so $(X^+, \Sigma(X, \leq))$ contains $(X, \Sigma(X, \leq))$ as a closed subspace and is therefore not sober, which provides our counterexample to (2).

Chapter 4

Fuzzy Sobriety or L-Sobriety

4.1 Sobriety of L-topological spaces

The following comes from [10] and [16].

4.1.1 Definition
Let $(X, L, \mathcal{T})$ be an L-topological space with L as in 1.4.4 (i.e. L is a frame with 0 and 1 and an order reversing involution). $Lpt\mathcal{T}$ is the set of all frame maps p from $\mathcal{T}$ into L.

4.1.2 Definition
Define,

$$\Psi : X \longrightarrow Lpt\mathcal{T}$$

$$\Psi(x)(u) = u(x) \quad \forall x \in X, \forall u \in \mathcal{T}$$

Ψ is an injection iff $x_1 \neq x_2$ implies there exists a $u \in \mathcal{T}$ such that $u(x_1) \neq u(x_2)$.
We call this property T_0.

We say $(X, L, \mathcal{T})$ is *sober* iff Ψ is a bijection.

4.1.3 Definition (see also Note 3.1.7(2))
Let $u \in \mathcal{T}$ then $\Phi(u)$ is a fuzzy subset of $Lpt\mathcal{T}$ defined as follows

$$\Phi(u) : Lpt\mathcal{T} \longrightarrow L$$

$$\Phi(u)(p) = p(u)$$

4.1.4 Proposition
$\{\Phi(u) : u \in \mathcal{T}\}$ is an L-topology on $Lpt\mathcal{T}$ under which Ψ becomes fuzzy continuous and fuzzy open.

Proof. The proof that $\{\Phi(u) : u \in \mathcal{T}\}$ is an L-topology follows that of Proposition 3.1.5

We can see that Ψ is fuzzy continuous because,

$$\begin{aligned}
\Psi^{\leftarrow}(\Phi(u))(x) &= \Phi(u)(\Psi(x)) \text{ (Zadeh's extension principle, Definition 1.3.1)} \\
&= \Psi(x)(u) \\
&= u(x) \text{ for all } x \in X \text{ and all } u \in \mathcal{T}.
\end{aligned}$$

Then $\Psi^{\leftarrow}(\Phi(u)) = u$ for all $u \in \mathcal{T}$.

Ψ is fuzzy open since

$$\begin{aligned} \Psi^{\rightarrow}(u)(p) &= \bigvee_{x \in \Psi^{\leftarrow}(p)} u(x) \text{ (Zadeh's extension priciple, Definition 1.3.1)} \\ &= p(u) \\ &= \Phi(u)(p) \text{ for all } p \in Lpt\mathcal{T} \end{aligned}$$

Thus $\Psi^{\rightarrow}(u) = \Phi(u)$
□

4.1.5 Corollary
If $(X, L, \mathcal{T})$ is sober then Ψ is a fuzzy homeomorphism.

We shall now provide examples of fuzzy L-topological spaces which are sober.

4.1.6 Example ([6])
The fuzzy real line $\mathbb{R}(I)$ and the fuzzy unit interval $[0, 1](I)$.

4.1.7 Example ([20] [21])
The fuzzy Sierpinski space of Srivastava is $(X, L, \mathcal{T})$ where $X = [0, 1]$, $L = [0, 1]$ and $\mathcal{T} = \{\underline{\mathbf{0}}, \underline{\mathbf{1}}, id\}$. $\mathcal{T}$ is clearly T_0, so $\Psi : X \longrightarrow Lpt\mathcal{T}$ is injective. To see the surjectivity of Ψ, pick any $p \in Lpt\mathcal{T}$. Then $p(\underline{\mathbf{0}}) = 0$, $p(\underline{\mathbf{1}}) = 1$ and $p(id) = \alpha$ for some $\alpha \in L$.
Also, by the definition of Ψ,

$$\Psi(\alpha)(\underline{\mathbf{0}}) = 0, \Psi(\alpha)(id) = \alpha, \text{ and } \Psi(\alpha)(\underline{\mathbf{1}}) = 1.$$

Thus $\Psi(\alpha) = p$. So Ψ is bijective, which implies that the fuzzy Sierpinski space is sober.

4.2 Fuzzy Sobriety for the case $L = [0, 1]$

Here we follow the work of Singh and Srivastava as contained in [17]

4.2.1 Definition
Let $(X, [0, 1], \mathcal{T})$ be a fuzzy topological space and $\alpha \in [0, 1)$, put $\iota_\alpha(\mathcal{T}) = \{u^{\leftarrow}(\alpha, 1] : u \in \mathcal{T}\}$. Then $\iota_\alpha(\mathcal{T})$ is a topology on X (usually called the *α-level topology* on X induced by $\mathcal{T}$). Let $\iota(\mathcal{T})$ be the topology on X generated by the family $\{u^{\leftarrow}(\alpha, 1] : u \in \mathcal{T}, \alpha \in [0, 1)\}$.

4.2.2 Definition
1. $(X, [0, 1], \mathcal{T})$ is *α-sober* (where $\alpha \in [0, 1)$), iff $(X, \iota_\alpha(\mathcal{T}))$ is sober.

2. $(X, [0, 1], \mathcal{T})$ is *strongly sober*, iff $(X, \iota_\alpha(\mathcal{T}))$ is sober, $\forall \alpha \in [0, 1)$.

3. $(X, [0, 1], \mathcal{T})$ is *ultrasober* iff $(X, \iota(\mathcal{T}))$ is sober.

4.2.3 Proposition
Every sober fuzzy topological space is ultrasober.

Proof. Let $(X, [0,1], \mathcal{T})$ be a sober fuzzy topological space. Then $\Psi_1 : X \longrightarrow Lpt\mathcal{T}$ (of Definition 4.1.2) is bijective. It follows that $(X, [0,1], \mathcal{T})$ is T_0, whereby $\Psi_2 : X \longrightarrow pt\iota(\mathcal{T})$ (of Definition 2.1.2) is injective. It remains to be shown that Ψ_2 is surjective also. This we show as follows.

For each $p \in pt\mathcal{T}$, let $p^* : \iota(\mathcal{T}) \longrightarrow \{0,1\}$ be defined, first on the subbasic open sets of $\iota(\mathcal{T})$, as

$$p^*(u^{\leftarrow}(\alpha,1]) = \begin{cases} 1 & \text{if } p(u) > \alpha \\ 0 & \text{if } p(u) \leq \alpha \end{cases}$$

where $u \in \mathcal{T}$ and $\alpha \in [0,1)$. To complete the definition of p^*, the image of an arbitrary $\iota(\mathcal{T})$-open set U of the form $U = \bigcup_{j\in J}(\bigcap_{k\in K_j}(u_{jk}^{\leftarrow}(\alpha_{jk},1])$, where J is an arbitrary index set, for each $j \in J$, K_j is a finite index set and for each $k \in K_j$, $u_{jk} \in \mathcal{T}$, and $\alpha_{jk} \in [0,1)$, is defined as $p^*(U) = \bigcup_{j\in J}(\bigcap_{k\in K_j} p^*(u_{jk}^{\leftarrow}(\alpha_{jk},1]))$. It is tedious, but straightforward otherwise, to verify that $p^* \in pt\iota(\mathcal{T})$. Now define a map $\eta : Lpt\mathcal{T} \longrightarrow pt\iota(\mathcal{T})$ as $\eta(p) = p^*$. We show that $(\eta \circ \Psi_1)(x) = \Psi_2(x)$ as follows. For $x \in X$, $(\eta \circ \Psi_1)(x) = (\Psi_1(x))^*$ and for $u \in \mathcal{T}$ and $\alpha \in [0,1)$,

$$\begin{aligned}(\Psi_1(x))^*(u^{\leftarrow}(\alpha,1]) &= \begin{cases} 1 & \text{if } (\Psi_1(x))(u^{\leftarrow}(\alpha,1]) > \alpha \quad \text{(i.e. } \mu(x) > \alpha) \\ 0 & \text{otherwise} \end{cases} \\ &= \Psi_2(x)(u^{\leftarrow}(\alpha,1])\end{aligned}$$

Since both $(\Psi_1(x))^*$ and $\Psi_1(x)$ are frame maps, we have $(\Psi_1(x))^*(U) = (\Psi_2(x))(U) \quad \forall U \in \iota(\mathcal{T})$. Hence $\eta \circ \Psi_1 = \Psi_2$. In view of this, the surjectivity of Ψ_2 will follow from the surjectivity of η, which we prove next. For each $q \in pt\iota(\mathcal{T})$, define $p_q : \mathcal{T} \longrightarrow [0,1]$ as

$$p_q(u) = \begin{cases} 1 & \text{if } q(u^{\leftarrow}(\alpha,1]) = 1 \quad \forall \alpha \in [0,1) \\ int\{\beta \in [0,1) : q(u^{\leftarrow}(\beta,1]) = 0\} & \text{otherwise} \end{cases}$$

We show that $p_q \in Lpt\mathcal{T}$.

Assertion 1. p_q preserves order.
Let for $u, v \in \mathcal{T}, u \leq v$. Then $u^{\leftarrow}(\alpha,1] \subseteq v^{\leftarrow}(\alpha,1] \quad \forall \alpha \in [0,1)$. Hence $q(u^{\leftarrow}(\alpha,1]) = 0$ if $q(v^{\leftarrow}(\alpha,1]) = 0$ (as q is a frame map). Thus, $\{\beta \in [0,1) : q(u^{\leftarrow}(\beta,1]) = 0\} \supseteq \{\beta \in [0,1) : q(v^{\leftarrow}(\beta,1]) = 0\}$, whereby $\inf\{\beta \in [0,1) : q(u^{\leftarrow}(\beta,1]) = 0\} \leq \inf\{\beta \in [0,1) : q(v^{\leftarrow}(\beta,1]) = 0\}$, i.e., $p_q(u) \leq p_q(v)$.

Assertion 2. p_q preserves finite infima, i.e., for $u, v \in \mathcal{T}, p_q(u \wedge v) = p_q(u) \wedge p_q(v)$.
Case(1): When $p_q(u \wedge v) = 1$.
We note that

$$\begin{aligned}p_q(u \wedge v) = 1 &\Leftrightarrow q((u \wedge v)^{\leftarrow}(\alpha,1]) = 1 \quad \forall \alpha \in [0,1) \\ &\Leftrightarrow q(u^{\leftarrow}(\alpha,1] \cap v^{\leftarrow}(\alpha,1]) = 1 \quad \forall \alpha \in [0,1) \\ &\Leftrightarrow q(u^{\leftarrow}(\alpha,1]) \wedge q(v^{\leftarrow}(\alpha,1]) = 1 \quad \forall \alpha \in [0,1) \\ &\Leftrightarrow q(u^{\leftarrow}(\alpha,1]) = 1 \text{ and } q(v^{\leftarrow}(\alpha,1]) = 1 \quad \forall \alpha \in [0,1) \\ &\Leftrightarrow q(u^{\leftarrow}(\alpha,1]) = 1 \quad \forall \alpha \in [0,1) \text{ and } q(v^{\leftarrow}(\alpha,1]) = 1 \quad \forall \alpha \in [0,1) \\ &\Leftrightarrow p_q(u) = 1 \text{ and } p_q(v) = 1 \\ &\Leftrightarrow p_q(u) \wedge p_q(v) = 1\end{aligned}$$

Case(2): when $p_q(u \wedge v) \neq 1$.
We note that

$$\begin{aligned}
p_q(u \wedge v) &= \inf\{\beta \in [0,1) : q((u \wedge v)^{\leftarrow}(\beta,1]) = 0\} \\
&= \inf\{\beta \in [0,1) : q(u^{\leftarrow}(\beta,1] \cap v^{\leftarrow}(\beta,1]) = 0\} \\
&= \inf\{\beta \in [0,1) : q(u^{\leftarrow}(\beta,1]) \wedge q(v^{\leftarrow}(\beta,1]) = 0\} \\
&= \inf\{\beta \in [0,1) : q(u^{\leftarrow}(\beta,1]) = 0 \text{ or } q(v^{\leftarrow}(\beta,1]) = 0\} \\
&= \inf\{\{\beta \in [0,1) : q(u^{\leftarrow}(\beta,1]) = 0\} \cup \{\beta \in [0,1) : q(v^{\leftarrow}(\beta,1]) = 0\}\} \\
&= \inf\{\inf\{\beta \in [0,1) : q(u^{\leftarrow}(\beta,1]) = 0\}, \inf\{\beta \in [0,1) : q(v^{\leftarrow}(\beta,1]) = 0\}\} \\
&= \inf\{p_q(u), p_q(v)\} \\
&= p_q(u) \wedge p_q(v)
\end{aligned}$$

Assertion 3. p_q preserves arbitrary suprema, i.e.,

$$\bigvee_{j \in J} p_q(u_j) = p_q(\bigvee_{j \in J} u_j) \quad \forall \{u_j : j \in J\} \subseteq \mathcal{T}$$

Case(1): When $\bigvee_{j \in J} p_q(u_j) = 1$.
We note that

$$\begin{aligned}
\bigvee_{j \in J} p_q(u_j) = 1 &\Leftrightarrow p_q(u_j) = 1 \text{ for some } j \\
&\Leftrightarrow \forall \alpha \in [0,1), q(u_j^{\leftarrow}(\alpha,1]) = 1 \text{ for some } j \\
&\Leftrightarrow \bigvee_{j \in J} q(u_j^{\leftarrow}(\alpha,1]) = 1 \quad \forall \alpha \in [0,1) \\
&\Leftrightarrow q(\bigcup_{j \in J} u_{j \in J}^{\leftarrow}(\alpha,1]) = 1 \quad \forall \alpha \in [0,1) \\
&\Leftrightarrow q((\bigvee_{j \in J} u_j)^{\leftarrow}[0,1)) = 1 \quad \forall \alpha \in [0,1) \\
&\Leftrightarrow p_q(\bigvee_{j \in J} u_j) = 1
\end{aligned}$$

Case(2): When $\bigvee_{j \in J} p_q(u_j) \neq 1$.
Let $\bigvee_{j \in J} p_q(u_j) = \alpha \quad \alpha \neq 1$. Then

(a) $p_q(u_j) \leq \alpha, \quad \forall j \in J$ and

(b) for $\varepsilon > 0, \quad \exists j = j_0$ with $p_q(u_{j_0}) > \alpha - \varepsilon$.

From (a), it follows that $\inf\{\beta \in [0,1) : q(u_j^{\leftarrow}(\beta,1]) = 0\} \leq \alpha, \quad \forall j \in J$,

$$\text{i.e., } q(u_j^{\leftarrow}(\alpha,1]) = 0 \quad \forall j \in J, \text{ i.e. } \bigvee_{j \in J} q(u_j^{\leftarrow}(\alpha,1]) = 0,$$

$$\text{i.e., } q(\bigcup_{j \in J} u_j^{\leftarrow}(\alpha,1]) = 0, \text{ i.e., } q((\bigvee_{j \in J} u_j)^{\leftarrow}(\alpha,1]) = 0. \qquad (1)$$

From (b) it follows that, for $\varepsilon > 0, \exists j = j_0$ with $q(u_{j_0}^{\leftarrow}(\alpha - \varepsilon, 1]) = 1$,

$$\text{i.e., for } \varepsilon > 0, \bigvee_{j \in J} q(u_j^{\leftarrow}(\alpha - \varepsilon, 1]) = 1, \text{ i.e., } q((\bigvee_{j \in J} u_j)^{\leftarrow}(\alpha - \varepsilon, 1]) = 1 \qquad (2)$$

Combining (1) and (2), we get $\inf\{\beta \in [0,1) : q((\bigvee_{j\in J} u_j)^{\leftarrow}(\beta, 1]) = 0\} = \alpha$, i.e. $p_q(\bigvee_{j\in J} u_j) = \alpha$. This proves the Assertion. Assertions (1), (2) and (3) together show that $p_q \in Lpt\mathcal{T}$.

We now claim that $\eta(p_q) = p_q^* = q$. To see this, it is easily seen that $p_q^*(u^{\leftarrow}(\alpha, 1]) = q(u^{\leftarrow}(\alpha, 1])$, for any subbasic open set $u^{\leftarrow}(\alpha, 1]$ of $\iota(\mathcal{T})$ and so, as p_q^* and q are frame maps, we get $p_q^*(U) = q(U)$ for any $\iota(\mathcal{T})$-open set U. Thus, η is surjective. Hence $(X, \iota(\mathcal{T}))$ turns out to be sober, i.e., $(X, [0,1], \mathcal{T})$ is ultrasober.

4.2.4 Proposition

Every strongly sober fuzzy topological space is ultrasober.

Proof. Let $(X, [0,1], \mathcal{T})$ be a strongly sober fuzzy topological space. Then $\Psi^{\alpha} : X \longrightarrow pt\iota_{\alpha}(\mathcal{T})$ is bijective, $\forall \alpha \in [0,1)$. Clearly, $(X, \iota(\mathcal{T}))$ is T_0 as $\iota_{\alpha}(\mathcal{T}) \subseteq \iota(\mathcal{T})$, whereby $\Psi : X \longrightarrow pt\iota(\mathcal{T})$ is injective. So, it remains to be proved that Ψ is surjective also. For $\alpha \in [0,1)$, define $\eta_{\alpha} : pt\iota(\mathcal{T})) \longrightarrow pt\iota_{\alpha}(\mathcal{T})$ as $\eta_{\alpha}(p) = p \circ i$, where $p \in pt\iota(\mathcal{T})$, and $i : \iota_{\alpha}(\mathcal{T}) \longrightarrow \iota(\mathcal{T})$ is the inclusion map. We show that $\eta_{\alpha} \circ \Psi = \Psi^{\alpha}$ as follows.
Note that for $x \in X$ and $u^{\leftarrow}(\alpha, 1] \in \iota_{\alpha}(\mathcal{T})$,

$$\begin{aligned} ((\eta_{\alpha} \circ \Psi)(x))(u^{\leftarrow}(\alpha, 1]) &= [\Psi(x)]_{\alpha}(u^{\leftarrow}(\alpha, 1]) \\ &= (\Psi(x) \circ i)(u^{\leftarrow}(\alpha, 1]) \\ &= \Psi(x)(u^{\leftarrow}(\alpha, 1]) \\ &= \Psi^{\alpha}(x)(u^{\leftarrow}(\alpha, 1]) \end{aligned}$$

Thus, $\eta_{\alpha} \circ \Psi = \Psi^{\alpha}$.
As Ψ^{α} is bijective and $\eta_{\alpha} \circ \Psi = \Psi^{\alpha}$, η_{α} is surjective. To prove the injectivity of η_{α}, suppose $p_1 \neq p_2$ for $p_1, p_2 \in pt\iota(\mathcal{T})$. Then $\exists U \in \iota(\mathcal{T})$ such that $p_1(U) \neq p_2(U)$. Let $U = \bigcup_{j\in J}(\bigcap_{k\in K_j} u_{jk}^{\leftarrow}(\alpha_{jk}, 1])$, where J is an arbitrary index set, for each $j \in J$, K_j is a finite index set and for each $k \in K_j$, $u_{jk} \in \mathcal{T}$ and $\alpha_{jk} \in [0,1)$. Then $p_1(U) = \bigvee_{j\in J}(\bigwedge_{k\in K_j} p_1(u_{jk}^{\leftarrow}(\alpha_{jk}, 1]))$ and $p_2(U) = \bigvee_{j\in J}(\bigwedge_{k\in K_j} p_2(u_{jk}^{\leftarrow}(\alpha_{jk}, 1]))$. As $p_1(U) \neq p_2(U)$, $\exists$ some $u_{jk}^{\leftarrow}(\alpha_{jk}, 1] \in \iota_{\alpha_{jk}}(\mathcal{T})$ such that $p_1(u_{jk}^{\leftarrow}(\alpha_{jk}, 1]) \neq p_2(u_{jk}^{\leftarrow}(\alpha_{jk}, 1])$.
Then $(p_1 \circ i)(u_{jk}^{\leftarrow}(\alpha_{jk}, 1]) \neq (p_2 \circ i)(u_{jk}^{\leftarrow}(\alpha_{jk}, 1])$, i.e., $\eta_{\alpha}(p_1) = \eta_{\alpha}(p_2)$, which shows that η_{α} is injective. Hence η_{α} is bijective and $(X, \iota_{\alpha}(\mathcal{T}))$ is sober, i.e., $(X, [0,1], \mathcal{T})$ is ultrasober.
□

We'll now provide counterexamples to show that the converses of the previous theorems are not necessarily true. The first example shows that sober $\not\Rightarrow$ α-sober, sober $\not\Rightarrow$ strongly sober, ultrasober $\not\Rightarrow$ α-sober and ultrasober $\not\Rightarrow$ strongly sober.

4.2.5 Example

Consider the fuzzy Sierpinski space $(X, [0,1], \mathcal{T})$ of Srivastava, where $X = [0,1]$, $\mathcal{T} = \{\underline{\mathbf{0}}, \underline{\mathbf{1}}, id\}$ with id the identity map on $[0,1]$, which is known to be sober (see Example 6.1.6 later). Using Proposition 4.2.3, $(X, [0,1], \mathcal{T})$ is ultrasober also. Now, it is clear that $\iota_{\alpha}(\mathcal{T}) = \{\emptyset, [0,1], (\alpha, 1]\}$. Evidently, $([0,1], \iota_{\alpha}(\mathcal{T}))$ is not T_0 and so $(X, [0,1], \mathcal{T})$ cannot be α-sober for any $\alpha \in [0,1)$, whence $(X, [0,1], \mathcal{T})$ is not strongly sober also.

The next example shows that α-sober $\not\Rightarrow$ sober, strongly sober $\not\Rightarrow$ sober and ultrasober $\not\Rightarrow$ sober.

4.2.6 Example

Consider the fuzzy topological space $(X, [0,1], \mathcal{T})$, where $X = \{a, b\}$ and $\mathcal{T} = \{\underline{\mathbf{0}}, \underline{\mathbf{1}}, (\alpha, 1)\}$, where $(a, 1)$ the fuzzy point with value 1 at a (see Definition 1.3.3). Then $\iota_\alpha(\mathcal{T}) = \{\emptyset, X, \{a\}\}$, $\forall \alpha \in [0,1)$. Evidently, $(X, \iota_\alpha(\mathcal{T}))$ is sober, $\forall \alpha \in [0,1)$ and so $(X, [0,1], \mathcal{T})$ is strongly sober too. As $\iota(\mathcal{T}) = \{\emptyset, X, \{a\}\} = \iota_\alpha(\mathcal{T})$, $(X, [0,1], \mathcal{T})$ is ultrasober also.
Note, however, that the map $\Psi : X \longrightarrow Lpt\mathcal{T}$ (Definition 4.1.2) cannot be onto. For example if we fix $\alpha \in (0,1)$, then the function $p : \mathcal{T} \longrightarrow [0,1]$, defined as $p(\underline{\mathbf{0}}) = 0$, $p((a,1)) = \alpha$ and $p(\underline{\mathbf{1}}) = 1$, is clearly such that $p \in Lpt\mathcal{T}$ and for no $x \in X$, $\Psi(x) = p$. Thus $(X, [0,1], \mathcal{T})$ cannot be sober.

The next example shows that α-sober $\not\Rightarrow$ strongly sober.

4.2.7 Example

Consider the fuzzy topological space $(X, [0,1], \mathcal{T})$, where $X = \{a, b, c\}$ and $\mathcal{T} = \{\underline{\mathbf{0}}, \underline{\mathbf{1}}, (0.7, 0.6, 0.1), (0.8, 0.3, 0.2), (0.8, 0.6, 0.2), (0.7, 0.3, 0.1)\}$. Then for $\alpha \in [0, 0.1)$, $\iota_\alpha(\mathcal{T}) = \{\emptyset, X\}$. Clearly, $(X, \iota_\alpha(\mathcal{T}))$ is not sober for $\alpha \in [0, 0.1)$. For $\alpha \in [0.3, 0.6)$, $\iota_\alpha(\mathcal{T}) = \{\emptyset, X, \{a, b\}, \{a\}\}$. The irreducible closed sets of $(X, \iota_\alpha(\mathcal{T}))$ are $\{c\}$, $\{b, c\}$ and X. Clearly, $\{c\} = \overline{c}$, $\{b, c\} = \overline{b}$ and $X = \overline{a}$. Thus, $(X, [0,1], \mathcal{T})$ is α-sober, $\forall \alpha \in [0.3, 0.6)$ but is not strongly sober.

The following concepts and results from Wutys [27] are needed to provide a counterexample showing that α-sober $\not\Rightarrow$ ultrasober.

4.2.8 Definition

If $\{T_\alpha : \alpha \in [0,1)\}$ is a family of topologies on X, $\alpha \in [0,1)$, $G \in T_\alpha$, and $x \in G$, then $\{G_\beta : \alpha < \beta < \sigma \leq 1\}$ is said to be an *(LT)-family* for (x, G, α) *relatively to the family* $\{T_\alpha : \alpha \in [0,1)\}$ if the following properties hold:

1. $\forall \beta \in (\alpha, \sigma), G_\beta \in T_\beta, x \in G_\beta \subset G$

2. $\forall \beta' \in (\alpha, \sigma), \forall \beta'' \in (\alpha, \sigma), \beta' < \beta'', G_{\beta''} \subset G_{\beta'}$.

If $\{T_\alpha : \alpha \in [0,1)\}$ is a family of topologies on X, we say that it has the (LT)-property if for each $\alpha \in [0,1)$, $G \in T_\alpha$, and $x \in G$, there exists an (LT)-family for (x, G, α) relatively to $\{T_\alpha : \alpha \in [0,1)\}$.
If $(X, [0,1], \mathcal{T})$ is a fuzzy topological space, the family $\{\iota_\alpha(\mathcal{T}) : \alpha \in [0,1)\}$ of its level topologies turns out to have the (LT)-property.

4.2.9 Proposition ([27])

If $\mathcal{F} = \{T_\alpha : \alpha \in [0,1)\}$ is a family of topologies on a set X, then there exist fuzzy topologies $\mathcal{T}$ on X having $\mathcal{F}$ as their level topologies, i.e., such that $\forall \alpha \in [0,1)$, $\iota_\alpha(\mathcal{T}) = T_\alpha$ iff $\mathcal{F}$ has the (LT)-property.

The following counterexample shows that α-sober $\not\Rightarrow$ ultrasober.

4.2.10 Example

Let $X = (-\infty, t] \subseteq \mathbb{R}$ be equipped with the topology $T = \{X, \emptyset\} \cup \{(a, t] : a \in X\}$. Clearly (X, T) is sober. Enlarge T to the topology T^* on X generated by T together with the cofinite topology on X. Then the closed subsets of (X, T^*) are $(-\infty, a]$ (where $a \in X$), all finite subsets, X, and $\emptyset$. Each $(-\infty, a]$, $a \in X$, is easily seen to be irreducible T^*-closed but is not the T^*-closure of any singleton.
Let us fix $k \in (0,1)$ and for each $\alpha \in [0,1)$, let the topology T_α on X be given by $T_\alpha = T^*$ for all $\alpha \in [0, k)$ and $T_\alpha = T$ for all $\alpha \in [k, 1)$. Then for each $\alpha \in [0, k)$, $G \in T_\alpha$, and $x \in G$, there exists a family $\{G_\beta : G_\beta = G, \forall \beta \in (\alpha, k)\}$ which is an (LT)-family relatively to $\{T_\alpha : \alpha \in [0,1)\}$. Again, for each $\alpha \in [k, 1)$, $G \in T_\alpha$ and $x \in G$, there exists a family

$\{G_\beta : G_\beta = G, \forall \beta \in (\alpha, 1)\}$ which is an (LT)-family relatively to $\{T_\alpha : \alpha \in [0,1)\}$.
Thus for each $\alpha \in [0,1)$, $G \in T_\alpha$, and $x \in G$, there exists an (LT)-family for (x, G, α) relatively to $\{T_\alpha : \alpha \in [0,1)\}$. Hence the family $\{T_\alpha : \alpha \in [0,1)\}$ has (LT)-property.
Thus, by Proposition 4.2.9, there exists a fuzzy topology $\mathcal{T}$ on X such that $\forall \alpha \in [0,1)$, $\iota_\alpha(\mathcal{T}) = T_\alpha$. This $(X, [0,1], \mathcal{T})$ is α-sober for $\alpha \in [k, 1)$ but not ultrasober (since $T^* \supset T \Rightarrow \iota(\mathcal{T}) = T^*$ and T^* is not sober).

Putting together the foregoing results and counterexamples, we now arrive at the following conclusion.

4.2.11 Theorem
The following implications exist among the various sobriety concepts in fuzzy topology. Moreover, no other implications exist between any of these concepts.

$$\begin{array}{ccc} \text{Strongly sober} & \Longrightarrow & \text{Ultrasober} \\ \Downarrow & & \Uparrow \\ \alpha\text{-sober} & & \text{Sober} \end{array}$$

Chapter 5

Properties of L-sober spaces

In section 3.2 we referred to the following results. Here again L is as in 4.1.

5.0.12 Theorem ([10])
$(X, L, \mathcal{T}_1)$ and $(Y, L, \mathcal{T}_2)$ are two L-topological spaces with $(Y, L, \mathcal{T}_2)$ sober. If $\mu : \mathcal{T}_2 \longrightarrow \mathcal{T}_1$ is a frame map, there exists a unique L-continuous function $f : X \longrightarrow Y$ such that $f^{\leftarrow} = \mu$

Proof. We have $\Psi_X : X \longrightarrow Lpt\mathcal{T}_1$ and $\Psi_Y : Y \longrightarrow Lpt\mathcal{T}_2$ with the latter a bijection. Define $G : Lpt\mathcal{T}_1 \longrightarrow Lpt\mathcal{T}_2$ as $G(p) = p \circ \mu$ for $p \in \mathcal{T}_1$ and therefore define $f : X \longrightarrow Y$ as $f = \Psi_Y^{\leftarrow} \circ G \circ \Psi_X$ we need to check that $f^{\leftarrow} = \mu$ i.e. if $\nu \in \mathcal{T}_2$ and $x \in X$ we need to show that $f^{\leftarrow}(\nu)(x) = \mu(\nu)(x)$.

$$\begin{aligned} f^{\leftarrow}(\nu)(x) &= \nu(f(x)) \\ &= \nu(\Psi_Y^{\leftarrow}(G(\Psi_X(x)))) \\ &= \nu(\Psi_Y^{\leftarrow}(\Psi_X(x) \circ \mu)) \end{aligned}$$

Now if $\Psi_Y^{\leftarrow}(q) = y$ ($q \in Lpt\mathcal{T}_2$), then $\Psi_Y(y) = q$ i.e. for every $\nu \in \mathcal{T}_2$, $q(\nu) = \Psi_Y(y)(\nu) = \nu(y)$ by definition of Ψ_Y. So for $y = \Psi_Y^{\leftarrow}(\Psi_X(x) \circ \mu)$ we have $\nu(y) = \Psi_X(\mu(\nu)) = \mu(\nu)(x)$. Hence $f^{\leftarrow}(\nu)(x) = \mu(\nu)(x)$ for every $\nu \in \mathcal{T}_2$ and every $x \in X$. Thus $f^{\leftarrow} = \mu$. Uniqueness of f: Suppose we also have $\hat{f} : X \longrightarrow Y$ such that $\hat{f}^{\leftarrow} = \mu$. We must show that $f = \hat{f}$, or that $\hat{f} = \Psi_Y^{\leftarrow} \circ G \circ \Psi_X$. Now $f(x) = \Psi_Y^{\leftarrow}(G(\Psi_X(x))) = \Psi_Y^{\leftarrow}(\Psi_X(x) \circ \mu)$. And again $\Psi_Y^{\leftarrow}(\Psi_X(x) \circ \mu) = y$ iff for every $\nu \in \mathcal{T}_2$, $\nu(y) = \mu(\nu)(x) = \hat{f}^{\leftarrow}(x) = \nu(\hat{f}(x))$. But this implies, since Ψ_Y is injective that $y = \hat{f}(x)$. Hence $f = \hat{f}$.
$\square$

5.0.13 Theorem ([10])
If $(Y, L, \mathcal{T}_2)$ is T_0 and for every $(X, L, \mathcal{T}_1)$ and every frame map $\mu : \mathcal{T}_2 \longrightarrow \mathcal{T}_1$ there exists $f : X \longrightarrow Y$ such that $f^{\leftarrow} = \mu$ then $(Y, L, \mathcal{T}_2)$ is sober.

Proof. Since $(Y, L, \mathcal{T}_2)$ is T_0, we have that Ψ_Y is an injection. We therefore only have to prove that Ψ_Y is surjective (i.e. for every $p \in Lpt\mathcal{T}_2$ there exists a $y \in Y$ such that for every $\nu \in \mathcal{T}_2$, $p(\nu) = \mu(y)$). Now take $X = \{x\}$ with $\mathcal{T}_1 = L^{\{x\}}$ and any $\mu : \mathcal{T}_2 \longrightarrow \mathcal{T}_1$. Then by hypothesis there exists an $f : X \longrightarrow Y$ such that $\varphi^{\leftarrow} = \mu$. Call $f(\{x\}) = f(x) = y$. Then for any $\nu \in \mathcal{T}_2$,

$$\begin{aligned} \mu(\nu)(x) &= f^{\leftarrow}(\nu)(x) \\ &= \nu(f(x)) \\ &= \nu(y) \end{aligned}$$

Identify $L^{\{x\}}$ with L. Then the frame maps $\mu : \mathcal{T}_2 \longrightarrow \mathcal{T}_1$ are identified with the $p \in Lpt\mathcal{T}_2$, and after that identification $\mu(\nu)(x) = p(\nu)$. We therefore have that for every $p \in Lpt\mathcal{T}_2$, there exists a $y \in Y$ such that for every $\nu \in \mathcal{T}_2$, $p(\nu) = \nu(y)$. Thus Ψ_Y is surjective. $\square$

We can drop the T_0 requirement in Theorem 5.0.13 in the case f unique.

5.0.14 Theorem ([10])
If for every $(X, L, \mathcal{T}_1)$ and every frame map $\mu : \mathcal{T}_2 \longrightarrow \mathcal{T}_1$ there exists a unique $f : X \longrightarrow Y$ such that $f^{\leftarrow} = \mu$ then $(Y, L, \mathcal{T}_2)$ is sober.

Proof. If $(Y, L, \mathcal{T}_2)$ is not T_0 we can shuffle the points of Y with the same values for all $\nu \in \mathcal{T}_2$ around without any effect on the frame of the latter, but violating the uniqueness of f. $\square$

Chapter 6

Fuzzy sobriety, Semi-Sobriety and the Hausdorff properties

6.1 The relationship between fuzzy sobriety and semi-sobriety

The following is contained in [12].

We are to seek a counterpart for Theorem 2.2.12 for fuzzy sobriety. Let $(X, L, \mathcal{T})$ be an L-topological space as in 4.1.
Choose $\alpha \in L$, prime and $\neq 1$. Then

1. for each $p \in Lpt\mathcal{T}$, there exists a prime element in $\mathcal{T}$, viz. $\bigvee\{u \in \mathcal{T} : p(u) \leq \alpha\}$ (by Theorem 1.2.7(3)), and conversely,

2. for each prime element $u \in \mathcal{T}$ (i.e. $\downarrow u$ is a principal prime ideal), there exists a $p \in Lpt\mathcal{T}$ (not necessarily unique. See the following example) such that $\downarrow u = \{v \in \mathcal{T} : p(v) \leq \alpha\} = \downarrow \bigvee\{v \in \mathcal{T} : p(v) \leq \alpha\}$ (by Theorem 1.2.7(2)). So the prime elements of $\mathcal{T}$ are $\bigvee\{u \in \mathcal{T} : p(u) \leq \alpha\}$ for $p \in Lpt\mathcal{T}$, α prime, $\alpha \neq 1$.

Now if $(X, L, \mathcal{T})$ is sober, then for each $p \in Lpt\mathcal{T}$, there exists a unique $x_1 \in X$ such that $p = \Psi(x_1)$ and $\{u \in \mathcal{T} : p(u) \leq \alpha\} = \downarrow \bigvee\{u \in \mathcal{T} : p(u) \leq \alpha\} = \downarrow \bigvee\{u \in \mathcal{T} : \Psi(x_1)(u) = u(x_1) \leq \alpha\}$.
So by (1) above, in the sober case, the prime element in $\mathcal{T}$ associated with p and α is $\bigvee\{u \in \mathcal{T} : u(x_1) \leq \alpha\}$ for some $x_1 \in X$. Furthermore by (2) above, in the sober case, for every prime element in $\mathcal{T}$, there exists a $p \in Lpt\mathcal{T}$, hence an $x_1 \in X$ such that the prime element is $\bigvee\{u \in \mathcal{T} : u(x_1) \leq \alpha\}$.

6.1.1 Note
Given a prime element $u \in \mathcal{T}$, the corresponding $p \in Lpt\mathcal{T}$ of (2) above, in the general L case is not necessarily unique as can be seen from the following example,

Let $X = \{x\}$, a singleton, $L = [0, 1]$, $\mathcal{T} = L^X \cong L, \alpha = \frac{1}{2}; u \in \mathcal{T}, u(x) = \frac{1}{2}$ (or $u = \frac{1}{2}$); $p_1 = id_L = id_{\mathcal{T}}$ and p_2 defined as

$$p_2(w) = \begin{cases} w & \text{if } w \in (\frac{1}{2}, 1] \\ \sqrt{\frac{w}{2}} & \text{if } w \in [0, \frac{1}{2}] \end{cases}$$

Then $p_1^{\leftarrow}[0,\alpha] = p_2^{\leftarrow}[0,\alpha] = \downarrow u$.

So if $(X, L, \mathcal{T})$ is sober, the corresponding $x_1 \in X$ is not necessarily unique. See Example 6.1.6 below. The uniqueness of x_1 must therefore not be confused with the injectivity of Ψ in the general L case.

6.1.2 Definition
A fuzzy topological space $(X, L, \mathcal{T})$ is *semi-sober* iff the only non-zero irreducible members of $\mathcal{T}'$ are the (β)-$\mathcal{T}$-closures of the singletons of X where $\beta \neq 0$ and irreducible in L.
A (β)-$\mathcal{T}$-closure of a singleton $x_1 \in X$ is the smallest member of $\mathcal{T}'$ with at least the value of β at x_1.

6.1.3 Note
If $L = \{0,1\}$, α must be 0 and the $p \in Lpt\mathcal{T}$ is unique (see Note 1.2.8) and so if $(X, \{0,1\}, \mathcal{T})$ is sober then $x_1 \in X$ is unique. This is exactly in agreement with Theorem 2.2.12, where sobriety of $(X, \{0,1\}, \mathcal{T})$ is "every irreducible closed set of X is the closure of a *unique* singleton in X".

6.1.4 Theorem
$(X, L, \mathcal{T})$ sober $\Rightarrow$ $(X, L, \mathcal{T})$ semi-sober.

Proof.

$(X, L, \mathcal{T})$ sober	$\Longrightarrow$	the only prime elements in $\mathcal{T}$ are the largest members of $\mathcal{T}$ with value at most α ($\alpha \in L$, α prime, $\alpha \neq 1$) at an $x_1 \in X$.
	$\Longleftrightarrow$	the only non-zero irreducible elements in $\mathcal{T}'$ are the least members of $\mathcal{T}'$ with at least the value β ($\beta \in L$, β irreducible, $\beta \neq 0$) at an $x_1 \in X$.
	$\Longleftrightarrow$	the only non-zero irreducible members of $\mathcal{T}'$ are the "β-closures of singletons of X".

□

On the other hand, if $(X, L, \mathcal{T})$ is semi-sober, i.e., the only prime elements of $\mathcal{T}$ are the largest members of $\mathcal{T}$ with value at most α ($\alpha \in L$, α prime, $\alpha \neq 1$) at an $x_1 \in X$, then $\bigvee\{u \in \mathcal{T} : p(u) \leq \alpha\} = \bigvee\{u \in \mathcal{T} : u(x_1) \leq \alpha\}$ or $\{u \in \mathcal{T} : p(u) \leq \alpha\} = \{u \in \mathcal{T} : u(x_1) \leq \alpha\} = \{u \in \mathcal{T} : \Psi(x_1)(u) \leq \alpha\}$. Thus $p(u) = k_1 \leq \alpha$ iff $\Psi(x_1)(u) = k_2 \leq \alpha$, which does not necessarily mean that $\Psi(x_1) = p$ (the case $\alpha = 1$ was excluded), unless $L = \{0,1\}$ in which case $p(u) = 0$ iff $\Psi(x_1)(u) = 0$, hence $p(u) = \Psi(x_1)(u)$, or Ψ is surjective.
But semi-sobriety of $(X, \{0,1\}, \mathcal{T})$ does not necessarily imply that Ψ is injective, e.g. If X is an indiscrete topological space with more than one element, it is not sober (being not T_0) but is semi-sober as its only non-empty irreducible closed set, viz. X, is the closure of every x in X.

6.1.5 Theorem
In the crisp case $(X, \{0,1\}, \mathcal{T})$, sobriety is equivalent to semi-sobriety plus T_0.

A further notable difference in the general L case is the non-uniqueness of the point $x_1 \in X$. This was pointed out in [21] through the first of the following examples :

6.1.6 Example
Let $X = [0,1]$, $L = [0,1]$ with $\mathcal{T}$ consisting of the following three functions from X into $[0,1]$: the function $\underline{\mathbf{0}}$ which is identically 0, the function $\underline{\mathbf{1}}$ which is identically 1 and *id*,

the identity map (the fuzzy Sierpinski space of Srivastava [21]). The non-zero irreducible members of $\mathcal{T}'$ are then $\underline{\mathbf{1}}$ and $\underline{\mathbf{1}} - id$. Now $\underline{\mathbf{1}} - id$ is the $\beta = 1$ closure of 0 whereas $\underline{\mathbf{1}}$ is the $\beta = 1$ closure of any $x \in [0, 1]$; or alternatively $\underline{\mathbf{1}} - id$ is the $\beta = 1 - x$ closure of any $x \in [0, 1)$ and $\underline{\mathbf{1}}$ is the $\beta = 1$ closure of $x = 1$.
Note that this example is also sober as was shown in Example 4.1.7.

6.1.7 Example
Let X and L be as in the example above with $\mathcal{T} = \{\underline{\mathbf{0}}, \underline{\mathbf{1}}, \Delta\}$
where

$$\Delta(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq \frac{1}{2} \\ -2(x-1) & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}$$

Then

$$v(x) = \Delta'(x) = \begin{cases} -2x & \text{if } 0 \leq x \leq \frac{1}{2} \\ 2x - 1 & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}$$

is the β-closure of any point in $[0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$ whereas $\underline{\mathbf{1}}$ is the $\beta = 1$ closure of $\frac{1}{2}$. Thus $(X, L, \mathcal{T})$ is semi-sober, but not sober since it is not T_0, e.g. $x = \frac{1}{4}$ and $\frac{3}{4}$ cannot be separated by a member of $\mathcal{T}$

6.1.8 Example ([13])
$X = (-\infty, k]$ on the real line with the topology $\{X, \emptyset, (a, k] : a \in X\}$ is sober and semi-sober but not T_1 (or T_2), whereas $\mathbb{R}$ with the cofinite topology is T_1 but not sober because $\mathbb{R}$ itself is irreducibly closed but not the closure of a singleton.

Thus in the general case, sobriety is definitely a stronger requirement than semi-sobriety.

6.2 Hausdorff Properties

Semi-sobriety seems to be useful in terms of the Hausdorff properties proposed for $(X, L, \mathcal{T})$.

6.2.1 Definition ([22])
$(X, L, \mathcal{T})$ is *fuzzy Hausdorff* iff for every pair of distinct "points" in L^X, (x_1, α), (x_2, β), $x_1 \neq x_2$, there exist u_A and u_B from $\mathcal{T}$ which are disjoint (i.e. $u_A \wedge u_B = \underline{\mathbf{0}}$) and such that (x_1, α) belongs to u_A and (x_2, β) belongs to u_B (i.e. $\alpha \leq u_A(x_1)$ and $\beta \leq u_B(x_2)$).

6.2.2 Definition ([9])
$(X, L, \mathcal{T})$ is *quasi fuzzy Hausdorff* iff for every pair of distinct points (x_1, α), (x_2, β), $x_1 \neq x_2$, there exist u_A and u_B from $\mathcal{T}$ which are disjoint , and such that $\alpha > u'_A(x_1)$ and $\beta > u'_B(x_2)$.

6.2.3 Corollary ([12])
If $(X, L, \mathcal{T})$ is fuzzy Hausdorff, then it is semi-sober.

Proof. If u_c is a closed set in $(X, L, \mathcal{T})$ which is not a β-$\mathcal{T}$-closure of a singleton, then supp u_c must contain two distinct points x_1 and x_2 say. (Otherwise u_c is the fuzzy point (x_1, β) with β reducible, in which case u_c is reducible.)
Consider the fuzzy points $(x_1, u_c(x_1))$ and $(x_2, u_c(x_2))$. Since $(X, L, \mathcal{T})$ is fuzzy Hausdorff there exists u_1, u_2 open and disjoint such that $u_1(x_1) \geq u_c(x_1)$ and $u_2(x_2) \geq u_c(x_2)$. Then $u'_1 \vee u'_2 = (u_1 \wedge u_2)' = 1$ and $u_c = [u_c \wedge u'_1] \vee [u_c \wedge u'_2]$ with the components containing $(x_2, u_c(x_2))$ and $(x_1, u_c(x_1))$ respectively. Thus u_c is reducible. So the only irreducible closed sets are the β-$\mathcal{T}$-closures (β irreducible) of the singletons. Hence $(X, L, \mathcal{T})$ is semi-sober.
□

6.2.4 Corollary ([12])
If $(X, L, \mathcal{T})$ is quasi fuzzy Hausdorff, then it is semi-sober.

Proof. If u_c is a closed set in $(X, L, \mathcal{T})$ which is not a β-$\mathcal{T}$-closure of a singleton, then supp u_c must contain two distinct points x_1 and x_2 say. (Otherwise u_c is the fuzzy point (x_1, β) with β reducible, in which case u_c is reducible.)
Consider the fuzzy points $(x_1, u_c(x_1))$ and $(x_2, u_c(x_2))$. Since $(X, L, \mathcal{T})$ is quasi fuzzy Hausdorff, there exist u_1, u_2 from $\mathcal{T}$ such that $u_c(x_1) > u_1'(x_1)$ and $u_c(x_2) > u_2'(x_2)$ and $u_1 \wedge u_2 = 0$. So $u_c = u_c \wedge (u_1' \vee u_2') = (u_c \wedge u_1') \vee (u_c \wedge u_2')$ and $u_c \wedge u_1'(x_2) = u_c(x_2)$ and $u_c \wedge u_2'(x_1) = u_c(x_1)$. Thus u_c is irreducible. So the only irreducible closed sets are the β-$\mathcal{T}$-closures (β irreducible) of the singletons. Hence $(X, L, \mathcal{T})$ is semi-sober.
□

Note that Example 6.1.7 above is semi-sober but not fuzzy Hausdorff (e.g. the points $(\frac{1}{4}, \frac{1}{2})$ and $(\frac{3}{4}, \frac{1}{2})$ cannot be seperated by disjoint open sets). Likewise it is not quasi fuzzy Hausdorff.

6.2.5 Theorem ([12])
Provided $\mathcal{T}$ contains the constant functions (level sets),

1. $(X, L, \mathcal{T})$ fuzzy Hausdorff $\Longrightarrow (X, L, \mathcal{T})$ is T_1 (singletons/points are closed).
2. $(X, L, \mathcal{T})$ quasi fuzzy Hausdorff $\Longrightarrow (X, L, \mathcal{T})$ is T_1. Here $L = [0, 1]$, the unit interval.

For the relationship between fuzzy Hausdorff and quasi fuzzy Hausdorff see [9].

Proof.

1. Consider the point (x_1, α). Then since $(X, L, \mathcal{T})$ is fuzzy Hausdorff, for every $x \neq x_1$, there exists an open set $u_x \in \mathcal{T}$ such that $(x, 1) \in u_x$ $(u_x(x) = 1)$ and $u_x(x_1) = 0$. Then
$$\bigvee\{u_x : x \neq x_1\} = \left\{ \begin{array}{ll} 1 & \text{if } x \neq x_1 \\ 0 & \text{if } x = x_1 \end{array} \right\} = (x_1, 1)'$$
an open set. So $(x_1, 1)$ is closed. Then $\alpha \wedge (x_1, 1) = (x_1, \alpha)$ is closed since $\underline{\alpha}$, the function which is identically α on X, is a member of $\mathcal{T}'$.
2. Consider the point (x_1, α) with $0 < \alpha < 1$. Then since $(X, L, \mathcal{T})$ is quasi fuzzy Hausdorff, for every $x \neq x_1$ and every ϵ, $0 < \epsilon < 1 - \alpha$, there exist $u_{x,\epsilon}$ and $v_{x,\epsilon}$, disjoint open sets such that $u'_{x,\epsilon}(x) < \epsilon$ and $v'_{x,\epsilon}(x_1) < 1 - \alpha - \epsilon$. So $u = \bigvee\{u_{x,\epsilon} : x \neq x_1, \epsilon\}$ is open and $u'(x) = \bigwedge u'_{x,\epsilon}(x) = 0$ for $x \neq x_1$, i.e. $u(x) = 1$ for $x \neq x_1$.
Also $v = \bigwedge\{v_{x,\epsilon} : x \neq x_1, \epsilon\}$ is disjoint from u, so $v(x) = 0$ for $x \neq x_1$ and $v(x_1) \geq \alpha$, so $u(x_1) = 0$. Thus $u' = (x_1, 1)$ is closed and so is $u' \wedge \underline{\alpha} = (x_1, \alpha)$ since $\underline{\alpha} \in \mathcal{T}'$ by assumption.

□

6.2.6 Summary
Some of the results above may be summarized as follows:

1. **The Case** $L = \{0, 1\}$

$$\begin{array}{ccccc} & & \text{semi-sober} & & \\ & & \uparrow \;\; \downarrow +T_0 & & \\ \text{Hausdorff} & \longrightarrow & \text{sober} & \longrightarrow & T_0 \\ & \searrow & & \nearrow & \\ & & T_1 & & \end{array}$$

2. **The General** $(X, L, \mathcal{T})$ **case**.

$$\begin{array}{ccccc} & & \text{sober} & & \\ & & \downarrow & \searrow & \\ \text{fuzzy Hausdorff} & \longrightarrow & \text{semi-sober} & \not\longrightarrow & T_0 \\ (\mathcal{T} \text{ with constants}) & \searrow & & \nearrow & \\ & & T_1 & & \end{array}$$

6.3 Characterizations of semi-sobriety for the case $L = [0, 1]$

The question arises as to whether a counterpart to Theorem 2.1.6 for fuzzy sobriety or semi-sobriety can be proved. Copying the arguments in the proof of Theorem 2.1.6 , we have the following:

6.3.1 Theorem
If $(X, L, \mathcal{T})$ is sober, then the only completely prime filters of L-open sets are the α-neighbourhoods (α prime, $\alpha \neq 1$) of an $x \in X$ where an α-nbd. of x is a $u \in \mathcal{T}$ with $u(x) > \alpha$ (Definition 1.5.1).

Proof. We firstly observe that for every $p \in Lpt\mathcal{T}$ and α prime, $\alpha < p(\underline{\mathbf{1}})$, $p^{\leftarrow}(\alpha, 1]$ is a completely prime filter in $\mathcal{T}$:

$$\begin{array}{rcl} (\text{Note: } p \in \Psi(X) & \Rightarrow & p = \Psi(x) \text{ for some } x \in X \\ & \Rightarrow & p(u) = \Psi(x)(u) = u(x) \text{ for some } x \in X \text{ and all } u \in \mathcal{T} \\ & \Rightarrow & p(\underline{\mathbf{1}}) = \underline{\mathbf{1}}(x) = 1) \end{array}$$

Filter:

1. $\underline{\mathbf{1}} \in p^{\leftarrow}(\alpha, 1]$ since $p(\underline{\mathbf{1}}) > \alpha$.

2.

$$\begin{array}{rcl} u, v \in p^{\leftarrow}(\alpha, 1] & \Rightarrow & p(u) > \alpha, \quad p(v) > \alpha \\ & \Rightarrow & p(u \wedge v) = p(u) \wedge p(v) > \alpha \quad \text{since } \alpha \text{ is prime} \\ p(u) \wedge p(v) \leq \alpha & \Rightarrow & p(u) \leq \alpha \quad \text{or } p(v) \leq \alpha \text{ if } \alpha \text{ is prime} \\ & \Rightarrow & u \wedge v \notin p^{\leftarrow}(\alpha, 1]. \end{array}$$

3.

$$\begin{array}{rcl} u \in p^{\leftarrow}(\alpha, 1], \quad u \leq v & \Rightarrow & p(v) \geq p(u) > \alpha \\ & \Rightarrow & v \in p^{\leftarrow}(\alpha, 1]. \end{array}$$

Completely prime:

$$\begin{array}{rcl} \bigvee u_i \in p^{\leftarrow}(\alpha, 1] & \Rightarrow & p(\bigvee u_i) > \alpha \\ & \Rightarrow & \bigvee p(u_i) > \alpha \\ & \Rightarrow & \exists u_i \text{ s.t. } p(u_i) > \alpha \\ & \Rightarrow & \exists u_i \text{ s.t. } u_i \in p^{\leftarrow}(\alpha, 1]. \end{array}$$

Conversely, every completely prime filter $\mathcal{F}$ of open sets is of the type $p^{\leftarrow}(\alpha, 1]$, α prime and $\alpha < p(\underline{\mathbf{1}})$:
Define for $\beta \neq 0$

$$p_\beta(u) = \begin{cases} \beta & \text{if } u \in \mathcal{F} \\ 0 & \text{if } u \notin \mathcal{F} \end{cases}$$

Then

$$\begin{aligned} p_\beta(u \wedge v) &= \begin{cases} \beta & \text{if } u \wedge v \in \mathcal{F}, \text{ i.e. if } u \text{ and } v \in \mathcal{F} \\ 0 & \text{if } u \wedge v \notin \mathcal{F}, \text{ i.e. } u \text{ or } v \notin \mathcal{F} \end{cases} \\ &= p_\beta(u) \wedge p_\beta(v) \end{aligned}$$

$$\begin{aligned} p_\beta(\bigvee u_i) &= \begin{cases} \beta & \text{if } \bigvee u_i \in \mathcal{F} \\ 0 & \text{if } \bigvee u_i \notin \mathcal{F} \end{cases} \\ &= \bigvee p_\beta(u_i) \text{ since } \mathcal{F} \text{ is completely prime.} \end{aligned}$$

(If $\bigvee u_i \in \mathcal{F}$, then $\exists u_j \in \mathcal{F}$ and so $p_\beta(u_j) = \beta$ and hence $\bigvee p_\beta(u_i) = \beta$; and if $\bigvee u_i \notin \mathcal{F}$ then no $u_i \in \mathcal{F}$ and so $\bigvee p_\beta(u_i) = 0$).
So $\mathcal{F} = p^{\leftarrow}(\alpha, 1]$ for $\alpha < p(\underline{\mathbf{1}})$ and α prime, for any completely prime filter $\mathcal{F}$ (Note that the p need not be unique). If $(X, L, \mathcal{T})$ is sober (so $p(\underline{\mathbf{1}}) = 1$ for all $p \in Lpt\mathcal{T}$) then every completely prime filter $\mathcal{F}$ of $\mathcal{T}$-open sets is of the type

$$\begin{aligned} p^{\leftarrow}(\alpha, 1] &= \{u \in \mathcal{T} : p(u) > \alpha \text{ for } \alpha \neq 1 \text{ and } \alpha \text{ prime.}\} \\ &= \{u \in \mathcal{T} : u(x) > \alpha \text{ for a unique } x \in X\}. \end{aligned}$$

the filter of open "α-nbds" of x.
□

6.3.2 Corollary

$(X, L, \mathcal{T})$ sober $\Rightarrow$ $(X, L, \mathcal{T})$ semi-sober.

Proof. By Proposition 1.2.3(2) we deduce that sobriety of $(X, L, \mathcal{T})$ implies that the only principal prime ideals in $\mathcal{T}$ are $\{u \in \mathcal{T} : u(x) \leq \alpha\}$ for α prime, $\alpha \neq 1$ and $x \in X$. This statement is equivalent to that of Theorem 6.1.4. In fact Theorem 6.3.1 could have been deduced from Theorem 6.1.4 using Proposition 1.2.3.
□

6.3.3 Corollary

$(X, [0,1], \mathcal{T})$ sober implies that the only completely prime filters of $[0,1]$-open sets are the quasi-neighbourhoods of a fuzzy point μ_x on X.

Proof. Follows from Proposition 1.5.3 and Theorem 6.3.1.
□

6.3.4 Corollary (First characterization of semi-sobriety)

$(X, [0,1], \mathcal{T})$ is semi-sober iff the only completely prime filters of $[0,1]$-open sets are the quasi-neighbourhoods of a fuzzy point μ_x on X

Proof. Follows from Definition 6.1.2 and Proposition 1.2.3
□

To find a counterpart for Theorem 2.2.10 at least for the case $L = [0, 1]$, proceed as follows in emulation of Section 2.2.

6.3.5 Definition
A *net* $(\mu_\lambda)_{\lambda\in\Lambda}$ in $(X, L, \mathcal{T})$ is a map (image of a map) from a directed set Λ to fuzzy points.
A net is *eventually in* $u \in \mathcal{T}$ iff $\exists\lambda^* \in \Lambda$ such that $\lambda \geq \lambda^* \Rightarrow \mu_x\ qco\ u$.
A net in $(X, L, \mathcal{T})$ *converges to* μ_y with respect to $\mathcal{T}$ iff it is eventually in every $u \in \mathcal{T}$ which is a quasi neighbourhood of μ_y.
A net (μ_λ) in $(X, L, \mathcal{T})$ is *observative* if, given $u \in \mathcal{T}$ and $\mu_\lambda\ qco\ u$ for some λ, then the net is eventually in u.
An observative net *converges strongly* to μ_x ($\mu_\lambda \longrightarrow^* \mu_x$) if it converges with respect to $\mathcal{T}$ and it additionally satisfies that μ_x is quasi-coincident with every $u \in \mathcal{T}$ which eventually contains the net.

6.3.6 Definition
Given a net $(\mu_\lambda)_{\lambda\in\Lambda}$ in $(X, L, \mathcal{T})$, its *derived filter* of L-open sets is

$$\mathcal{F}_{(\mu_\lambda)} \equiv \{u \in \mathcal{T} : \exists\lambda^* \in \Lambda, \mu_\lambda\ qco\ u \text{ for } \lambda \geq \lambda^*\}$$

6.3.7 Proposition
A filter derived from an observative net is completely prime

Proof. Let $(\mu_\lambda)_{\lambda\in\Lambda}$ be observative and $\bigvee_{j\in J} u_j \in \mathcal{F}_{(\mu_\lambda)}$. So $\mu_\lambda\ qco \bigvee u_j$ for $\lambda \geq \lambda^*$. Then $\mu_{\lambda^*}\ qco \bigvee u_j$ and hence by Proposition 1.3.4, $\exists j_0 \in J$ with $\mu_{\lambda^*}\ qco\ u_{j_0}$. Since the net is observative, this implies that some tail of (μ_λ) is *qco* with u_{j_0}. So by definition of $\mathcal{F}_{(\mu_\lambda)}$, $u_{j_0} \in \mathcal{F}_{(\mu_\lambda)}$.
□

6.3.8 Lemma
If (μ_λ) is an observative net, then $\mu_\lambda \longrightarrow^* \mu_x$ iff $\mathcal{F}_{(\mu_\lambda)} = Q\mathcal{N}_{\mu_x}$, the quasi-neighbourhoods of μ_x.

Proof.

$$(\mu_\lambda) \longrightarrow \mu_x \text{ iff for } u \in \mathcal{T},\ \mu_x\ qco\ u \Rightarrow \exists\lambda^* \in \Lambda,\ \mu_\lambda\ qco\ u \text{ for } \lambda \geq \lambda^*.$$

$$(\mu_\lambda) \longrightarrow^* \mu_x \text{ iff for } u \in \mathcal{T},\ \mu_x\ qco\ u \Leftrightarrow \exists\lambda^* \in \Lambda,\ \mu_\lambda\ qco\ u \text{ for } \lambda \geq \lambda^*.$$

□

6.3.9 Proposition
Suppose $\mathcal{F}$ is a filter of L-open ($L = [0, 1]$) subsets of $(X, [0, 1], \mathcal{T})$. Then $\mathcal{F}$ is completely prime iff $\forall u \in \mathcal{F}$, $\exists$ a fuzzy point μ_x which is quasi-coincident with u and such that $\forall v \in \mathcal{T}$, $\mu_x\ qco\ v \Rightarrow v \in \mathcal{F}$.

Proof.

1. Suppose $\mathcal{F}$ has this property and consider $u = \bigvee u_j \in \mathcal{F}$. Let μ_x be the point with the property. Then $\exists j$ such that μ_x *qco* u_j (Proposition 1.3.4) and $u_j \in \mathcal{T}$. So $u_j \in \mathcal{F}$. Thus $\mathcal{F}$ is completely prime.

2. Suppose the condition is not fulfilled. Thus $\exists u \in \mathcal{F}$ such that $\forall \mu_x$ which is quasi-coincident with u, we have $\exists v_x \in \mathcal{T}$ such that μ_x *qco* v_x and $v_x \notin \mathcal{F}$. Put $w_x = v_x \wedge u$. Then $w_x \notin \mathcal{F}$. However $w = \bigvee_{\mu_x \, qco \, u} w_x = u$:-
Clearly $w \leq u$. So suppose $\exists x^*$ such that $w(x^*) < u(x^*)$. Then choose $\varepsilon > 0$ so that $w(x^*) < u(x^*) - \varepsilon$ and consider the fuzzy point μ_{x^*} defined as $\mu_{x^*}(x^*) = 1 - u(x^*) + \varepsilon$. So μ_{x^*} *qco* u and thus $\exists v_{x^*} \in \mathcal{T}$ such that μ_{x^*} *qco* v_{x^*}, hence μ_{x^*} *qco* w_{x^*}, i.e. $w_{x^*}(x^*) > 1 - \mu_{x^*}(x^*) = u(x^*) - \varepsilon$. This is a contradiction.
We conclude that $\mathcal{F}$ is not completely prime.

□

Construction of a net from a completely prime filter $\mathcal{F}$:-

6.3.10 Definition
For each $u \in \mathcal{F}$ completely prime choose the μ_x with the property guaranteed by Proposition 6.3.9. Then $\Lambda_{\mathcal{F}} = \{(\mu_x, u) : u \in \mathcal{F}\}$ is directed with respect to $(\mu_x, u) \leq (\mu_y, w)$ iff $w \leq u$. Then $N_x : \Lambda_{\mathcal{F}} \longrightarrow X$, $N(\mu_x, u) = \mu_x$ is called the *net derived from the completely prime filter $\mathcal{F}$.*

6.3.11 Lemma
A net derived from a completely prime filter is observative.

Proof. Let (μ_x) be the net derived from the completely prime filter and let μ_x *qco* u^* for $u^* \in \mathcal{T}$. Then $u^* \in \mathcal{F}$ by the choice of μ_x and hence $(\mu_x, u^*) \in \Lambda_{\mathcal{F}}$. If $(\mu_y, w) \geq (\mu_x, u^*)$ then $w \leq u^*$ and μ_y *qco* u^* and the net is eventually in u^*.
□

6.3.12 Proposition
Every completely prime filter equals the derived filter of any of its derived nets.

Proof. If $\mu \in \mathcal{F}$ and μ_λ *qco* u, for $\lambda \geq \lambda^*$ then $u \in \mathcal{F}_{(\mu_\lambda)}$.
Conversely, $u \in \mathcal{F}_{(\mu_\lambda)}$ implies μ_λ *qco* u for $\lambda \geq \lambda^* = (\mu_x, w)$, $w \in \mathcal{F}$. Hence μ_{λ^*} *qco* u which implies, since $\mathcal{F}$ is completely prime, that $u \in \mathcal{F}$ (Proposition 6.3.9)
□

6.3.13 Theorem (Second characterization of semi-sobriety)
A $(X, [0,1], \mathcal{T})$ topological space is semi-sober iff every observative net strongly converges to a fuzzy point μ_x.

Proof. $(X, [0,1], \mathcal{T})$ semi-sober implies by Corollary 6.3.4 that every completely prime filter of $[0,1]$-open sets are the quasi-neighbourhoods of a fuzzy point μ_x on X. So given an observative net $(\mu_\lambda)_{\lambda \in \Lambda}$ then its derived filter is completely prime (by Proposition 6.3.7) and is therefore the quasi-neighbourhoods of a fuzzy point μ_x. Thus by Lemma 6.3.8 $(\mu_\lambda) \longrightarrow^* \mu_x$.
Conversely: Suppose every observative net in X converges strongly to a μ_x on X. Then by Lemma 6.3.8, the derived filter of such a net is the quasi-neighbourhoods of the μ_x and by Proposition 6.3.7 is completely prime. But by Proposition 6.3.12 every completely prime filter can be derived from such an observative net, and hence equals the quasi-neighbourhoods

of μ_x. So by Corollary 6.3.4, $(X, [0,1], \mathcal{T})$ is semi-sober.
□

6.4 The relationships of semi-sobriety towards the concepts of Srivastava as in Section 4.2

6.4.1 Theorem
If $(X, [0,1], \mathcal{T})$ is α-sober ($\alpha \in [0,1)$) and $\mathcal{T}$ contains the constant functions then it is semi-sober.

Proof. By definition of α-sobriety (Definition 4.2.2), the crisp topological space $(X, \iota_\alpha(\mathcal{T}))$ is sober, and thus by Theorem 2.1.6, the only completely prime filters of $\iota_\alpha(\mathcal{T})$-open sets are $\mathcal{N}_{x_\alpha}$ for some $x_\alpha \in X$.
Let $\mathcal{F}$ be a completely prime filter of $[0,1]$-open sets in $(X, [0,1], \mathcal{T})$. Then

$$\begin{aligned} \mathcal{F}_\alpha &= \{u^{\leftarrow}(\alpha, 1] : u \in \mathcal{F}\} \\ &= \{(u \vee \underline{\alpha})^{\leftarrow}(\alpha, 1] : u \vee \underline{\alpha} \in \mathcal{F}\} \end{aligned}$$

since $\underline{\alpha}(x) \equiv \alpha$ is in $\mathcal{T}$.
$\mathcal{F}_\alpha$ is a completely prime filter of $\iota_\alpha(\mathcal{T})$-open sets :-
Filter :

1. $X \in \mathcal{F}_\alpha$ since $\underline{\mathbf{1}} \in \mathcal{F}$

$$\begin{aligned} U, V \in \mathcal{F}_\alpha \quad \Rightarrow \quad U &= u^{\leftarrow}(\alpha, 1] \quad , V = v^{\leftarrow}(\alpha, 1] \quad , u, v \in \mathcal{F} \\ \Rightarrow \quad U \cap V &= u^{\leftarrow}(\alpha, 1] \cap v^{\leftarrow}(\alpha, 1] \\ &= (u \wedge v)^{\leftarrow}(\alpha, 1], \quad u \wedge v \in \mathcal{F} \\ &\in \mathcal{F}_\alpha \end{aligned}$$

2.

$$\begin{aligned} U \in \mathcal{F}_\alpha, U \subset V \quad &\Rightarrow \quad u^{\leftarrow}(\alpha, 1] \subset V \text{ where } u \in \mathcal{F} \text{ and } V \in \iota_\alpha(\mathcal{T}) \\ &\Rightarrow \quad (u \vee \underline{\alpha})^{\leftarrow}(\alpha, 1] \subset v^{\leftarrow}(\alpha, 1] \text{ for a } v \in \mathcal{T} \\ &\Rightarrow \quad u \vee \underline{\alpha} \leq v \quad , u \vee \underline{\alpha} \in \mathcal{F} \\ &\Rightarrow \quad v \in \mathcal{F} \\ &\Rightarrow \quad V \in \mathcal{F}_\alpha \end{aligned}$$

Completely prime :

Let $\bigcup U_j \in \mathcal{F}_\alpha$. So

$$\begin{aligned} \bigcup U_j &= \bigcup u_j^{\leftarrow}(\alpha, 1] \quad , u_j \in \mathcal{F} \\ &= (\bigvee u_j)^{\leftarrow}(\alpha, 1] \end{aligned}$$

Now $\bigvee u_j \in \mathcal{F}$ and since $\mathcal{F}$ is completely prime, $\exists j_0$ such that $u_{j_0} \in \mathcal{F}$. Then $u_{j_0}^{\leftarrow}(\alpha, 1] \in \mathcal{F}_\alpha$. So by Theorem 2.1.6, $\mathcal{F}_\alpha = \mathcal{N}_{x_\alpha}$ for some $x_\alpha \in X$. Thus $\forall u \in \mathcal{F}$, $u^{\leftarrow}(\alpha, 1] \in \mathcal{N}_{x_\alpha}$ or u is an α-neighbourhood of x_α. Thus by Proposition 1.5.3, u is a quasi-neighbourhood of the fuzzy point $(x_\alpha, 1-\alpha)$. So $\mathcal{F} = Q\mathcal{N}_{\mu_{x_\alpha}}$ and thus by Corollary 6.3.4, $(X, [0,1], \mathcal{T})$ is semi-sober.
□

6.4.2 Corollary
If $(X, [0,1], \mathcal{T})$ is strongly sober and $\mathcal{T}$ contains the constant functions, then it is semi-sober.

Proof. Follows from Theorems 6.4.1 and 4.2.11.
□

6.4.3 Theorem
If $(X, [0,1], \mathcal{T})$ is ultrasober and $\mathcal{T}$ contains the constant functions, then it is semi-sober

Proof. Follow the Proof of Theorem 6.4.1 so as to obtain that for each $\alpha \in [0,1)$, $\mathcal{F}_\alpha = \{u^{\leftarrow}(\alpha,1] : u \in \mathcal{F}\}$ is the neighbourhood filter $\mathcal{N}_{x_\alpha}$ of a point $x_\alpha \in X$. Hence $\forall \alpha \in [0,1)$, $\mathcal{F} = Q\mathcal{N}_{\mu_{x_\alpha}}$. Thus by Corollary 6.3.4, $(X, [0,1], \mathcal{T})$ is semi-sober.
□

Reverse implications from semi-sober to ultrasober or strongly sober or α-sober do not hold according to Section 4.2.

Chapter 7

Lifting of Sobriety

7.1 Lifting of sobriety and semi-sobriety

A property P of a topological space (ordinary or L-valued) is said to *lift* if $(X, \mathcal{T}^*)$ has property P whenever $(X, \mathcal{T})$ has property P for $\mathcal{T} \subset \mathcal{T}^*$. Clearly many seperation axioms lift eg. T_0, T_1 and T_2.
For ordinary (crisp) topological spaces $(X, \mathcal{T}) = (X, \{0,1\}, \mathcal{T})$ we have,

7.1.1 Theorem ([13])
If a T_0 topological space $(X, \mathcal{T})$ is sober and $\mathcal{T} \subset \mathcal{T}^*$, and if H is irreducibly closed in $(X, \mathcal{T}^*)$, then $H \subset cl_{\mathcal{T}}(x_1)$ for some $x_1 \in cl_{\mathcal{T}}(H)$.

Proof. Firstly, $cl_{\mathcal{T}}(H)$ is irreducibly closed in $(X, \mathcal{T})$; because if not, $cl_{\mathcal{T}}(H) = F_1 \cup F_2$, F_1 and F_2 closed in $(X, \mathcal{T})$ with $cl_{\mathcal{T}}(H)$ not contained in either F_1 or F_2.
Now $H = (F_1 \cap H) \cup (F_2 \cap H)$ with both $F_1 \cap H$ and $F_2 \cap H$ closed in $(X, \mathcal{T}^*)$. So $H \subseteq F_1 \cap H$ or $H \subseteq F_2 \cap H$, i.e. $H \subseteq F_1$ or $H \subseteq F_2$. But then $cl_{\mathcal{T}}(H) \subseteq F_1$ or $cl_{\mathcal{T}}(H) \subseteq F_2$. Contradiction.
Since $cl_{\mathcal{T}}(H)$ is irreducibly closed in $(X, \mathcal{T})$ and $(X, \mathcal{T})$ is sober, by Theorem 2.2.12, $cl_{\mathcal{T}}(H) = cl_{\mathcal{T}}\{x_1\}$ for some $x_1 \in cl_{\mathcal{T}}(H)$. Thus $H \subseteq cl_{\mathcal{T}}\{x_1\}$.
□

7.1.2 Corollary ([13])
Sobriety lifts for T_1 topological spaces.

7.1.3 Example ([13])
Here is a non-T_1 sober space where sobriety does not lift: Let $X = (-\infty, k] \subset \mathbb{R}$ with the sober topology,

$$\mathcal{T} = \{X, \emptyset, (a, k] : a \in X\}$$

Enlarge this to the topology $\mathcal{T}^*$ on X generated by $\mathcal{T}$ together with the cofinite topology on X. Then the closed sets are: $(-\infty, a], a \in X$, all finite sets and X and $\emptyset$. The sets $(-\infty, a]$ are again irreducible, but are not the closures of singletons since the space is now T_1.

7.1.4 Corollary
Referring to Section 4.2, we observe that if $\iota_\alpha(\mathcal{T})$ is T_1, then α-sober $\Rightarrow$ ultrasober.

7.1.5 Definition ([13])
An L-topological space $(X, L, \mathcal{T})$ is T_1 iff the map $\mathcal{T} : L^X \longrightarrow \{0,1\}$ assigns to the complement of a fuzzy point the value 1. (L with an order reversing involution)

We can extend Theorem 7.1.1 to the semi sober L-topological case so as to obtain,

7.1.6 Theorem ([13])
If $(X, L, \mathcal{T})$ is semi-sober and T_1 and $\mathcal{T} \subseteq \mathcal{T}^*$, then $(X, L, \mathcal{T}^*)$ is also semi-sober.

Proof. Here $\cup, \cap$ and $\subseteq$ in the proof of Theorem 7.1.1 should be read as $\vee, \wedge$ and $\leq$ respectively in L^X. Then $H \leq cl_{\mathcal{T}}\{x_1\}$, where $cl_{\mathcal{T}}\{x_1\}$ denotes the β-closure of x_1. Since $(X, L, \mathcal{T})$ is T_1, $cl_{\mathcal{T}}\{x_1\} = (x_1, \beta)$ (the fuzzy point $\mu_{x_1}(x_1) = \beta$ see Definition 1.3.3). Since H is irreducibly closed in $(X, L, \mathcal{T}^*)$, H is also a fuzzy point (x_1, β^*) where β^* is irreducible and $0 < \beta^* \leq \beta$; hence H is the β^*-closure of a singleton, and so $(X, L, \mathcal{T}^*)$ is semi-sober.
□

The converse of the preceding Theorem, if $(X, L, \mathcal{T})$ is semi-sober and if for any $\mathcal{T}^* \supseteq \mathcal{T}, (X, L, \mathcal{T}^*)$ is semi-sober, then $(X, L, \mathcal{T})$ is not necessarily T_1 as can be seen from the following two counterexamples, the first a "crisp" case and the second an L-topological one,

7.1.7 Example ([13])
Consider the two point Sierpinski space $(\{0,1\}, \mathcal{T})$ with $\mathcal{T} = \{\emptyset, \{0,1\}, \{1\}\}$ is sober (hence semi-sober), but all larger topologies on $\{0,1\}$, and there is only one such, namely the discrete topology, are sober (semi-sober). However $(\{0,1\}, \mathcal{T})$ is not T_1.

7.1.8 Example ([13])
Take $X = \{0,1\}$ and $\mathcal{T} = \{\mathbf{0}, \iota, \mathbf{1}\}$ where ι is the inclusion map of $\{0,1\}$ into $[0,1]$. Denote $\mu : X = \{0,1\} \longrightarrow [0,1]$, defined by $\mu(0) = \alpha$, $\mu(1) = \beta$ by $< \alpha, \beta >$. (So $\iota =< 0, 1 >$.)
The irreducibly non-zero L-closed sets are $\mathbf{1}$ and $\mathbf{1} - \iota =< 1, 0 >$ and $\mathbf{1} = cl_{\mathcal{T}}(1,1)$ while $\mathbf{1} - \iota = cl_{\mathcal{T}}(1,0)$. So $(X, L, \mathcal{T})$ is semi-sober.
Now let $\mathcal{T}^*$ be another L-topology on X with $\mathcal{T} \subseteq \mathcal{T}^*$ and let $\mu =< \alpha, \beta >$ be a non-zero irreducibly $\mathcal{T}^*$ L-closed set. Then as $< 0, 1 >$ is $\mathcal{T}$-closed, hence $\mathcal{T}^*$-closed,

$$< \alpha, \beta > \wedge < 1, 0 >=< \alpha, 0 > \text{ is } \mathcal{T}^*\text{-closed.}$$

If $\beta = 0$, then $< \alpha, \beta >=< \alpha, 0 >$ is clearly $cl_{\mathcal{T}^*}(0, \alpha)$.
If $\beta \neq 0$ and $\alpha = 0$, then $< \alpha, \beta >=< 0, \beta >$ is clearly $cl_{\mathcal{T}^*}(1, \beta)$.
If $\beta \neq 0$ and $\alpha \neq 0$, then also $< \alpha, \beta >= cl_{\mathcal{T}^*}(1, \beta)$, for if not we could have an $\mathcal{T}^*$-closed set $< \alpha', \beta >$ with $\alpha' < \alpha$, which would mean that $< \alpha, \beta >=< \alpha, 0 > \vee < \alpha', \beta >$, contradicting the $\mathcal{T}^*$-irreducibility of $< \alpha, \beta >$.
So again, $(X, L, \mathcal{T}^*)$ is semi-sober.
But $(X, L, \mathcal{T})$ is not T_1 since the fuzzy point $(1,1) = \iota =< 0, 1 >$ is not $\mathcal{T}$-closed.

7.2 Further Questions for the Crisp Case

The question as to whether we could find a converse to Theorem 7.1.6 was answered for sobriety (but not semi-sobriety) in the crisp case by R-E. Hoffman in [3] where he showed that the following two statements are equivalent,

1. $(X, \mathcal{T})$ is sober and T_D;

2. every space finer than $(X, \mathcal{T})$ is sober.

The T_D property is : every singleton is the intersection of an open and closed set, or equivalently, for any $A \in \mathcal{P}(X)$, $derA$ is closed.
It is easy to see that $T_1 \Rightarrow T_D \Rightarrow T_0$. This begs the question to come up with a T_D axiom

for L-topological spaces, weaker than the T_1 axiom, which will give an analogous result for semi-sobriety.

A further interesting question in this connection is, can we have two (crisp) topologies $\mathcal{T}$ and $\mathcal{T}^*$ on X, $\mathcal{T} \subseteq \mathcal{T}^*$ with $(X, \mathcal{T})$ sober, $|pt\mathcal{T}| = |pt\mathcal{T}^*|$ but $(X, \mathcal{T}^*)$ not sober? The following example answers the question in the affirmative :
Consider $X = \mathbb{Z} \cap (-\infty, k]$, $k \in \mathbb{Z}$ (i.e. all integers up to and including k) with the topology $\mathcal{T}$ consisting of X, $\emptyset$ and all $(a, k] \cap \mathbb{Z}$, $a \in X$ (simply $(a, k]$ in what follows).
$(X, \mathcal{T})$ is sober since the irreducibly closed sets are $(-\infty, a] \cap \mathbb{Z}$ (simply$(-\infty, a]$ in what follows), each of which is the closure of $\{a\}$ (see also Example 6.1.8). $pt\mathcal{T}$ consists of, for $a \in X$,

$$p_a(l, k] = \begin{cases} 1 & \text{if } l < a \\ 0 & \text{if } l \geq a \end{cases}$$

Note that $p_a(\emptyset) = p_a((k, k]) = 0$ and

$$p_k(l, k] = \begin{cases} 1 & \text{if } l < k \\ 0 & \text{if } l = a \end{cases}$$

So $|pt\mathcal{T}| = |X| = \aleph_0$. Furthermore $\Psi : X \longrightarrow pt\mathcal{T}$ defined as $\Psi(x)(u) = \chi_u(x)$ (Definition 2.1.2) gives

$$\begin{aligned} \Psi(a)(l, k] &= \begin{cases} 1 & \text{if } l < a \\ 0 & \text{if } l \geq a \end{cases} \\ &= p_a(l, k] \end{aligned}$$

Now define $\mathcal{T}^*$ on X as X; $\emptyset$; $(a, k]$, $a \in X$; $\{k\}$. So the closed sets are $\emptyset$, X, $(-\infty, a]$, $(-\infty, k)$. $(X, \mathcal{T}^*)$ is not sober since $(-\infty, k)$ is irreducibly closed but is not the closure of a singleton. Further $pt\mathcal{T}^*$ consists of the following :

1. for $a \in X$

$$p_a(l, k) = \begin{cases} 1 & \text{if } l < a \\ 0 & \text{if } l \geq a \end{cases}$$

$$p_a\{k\} = 0$$

2.

$$p'_k(l, k) = \begin{cases} 1 & \text{if } l < k \\ 0 & \text{if } l = a \end{cases}$$

$$p'_k\{k\} = 1$$

So $|pt\mathcal{T}^*| = \aleph_0 + 1 = \aleph_0$.

Chapter 8

Semi-sobriety in (L, M)-Topological spaces

Let $(X, L, M, \mathcal{T})$ be an (L, M)-topological space and put $\mathcal{T}_\alpha = \mathcal{T}^{\leftarrow}\{[\alpha, 1]\}$ where $[\alpha, 1] = \{\beta \in M : \beta \geq \alpha\}$. $\mathcal{T}_\alpha$ is a frame in L^X (an L-topology on X) for each $\alpha \in M$ with $\mathcal{T}_0 = L^X$ and $\alpha \geq \beta \Rightarrow \mathcal{T}_\alpha \subseteq \mathcal{T}_\beta$.

8.0.1 Definition
In a complete lattice L, x is *way below* y ($x << y$) iff for directed subsets D of L, the relation $y \leq \sup D$ implies the existence of a $d \in D$ such that $x \leq d$.
In L $x << y \Rightarrow x \leq y$.

8.0.2 Definition
A complete lattice L is a *continuous lattice* if for all $x \in L$, $x = \sup\{u \in L : u << x\}$.

L a frame $\Rightarrow$ L a continuous lattice $\Rightarrow$ L a semi-frame, but these implications are not reversible.

8.0.3 Proposition ([13])
If M is a continuous lattice, then $\mathcal{T}_\alpha = \bigcap_{\beta << \alpha} \mathcal{T}_\beta$.

Proof. Since $\alpha \geq \beta \Rightarrow \mathcal{T}_\alpha \subseteq \mathcal{T}_\beta$ it follows that $\mathcal{T}_\alpha \subseteq \bigcap_{\beta << \alpha} \mathcal{T}_\beta$. On the other hand, if $u \in \mathcal{T}_\beta$ for all $\beta << \alpha$, then since $\alpha = \sup\{\beta : \beta << \alpha\}$ we have

$$[\mathcal{T}(u) \geq \beta, \text{ for all } \beta << \alpha] \Rightarrow \mathcal{T}(u) \geq \alpha \Rightarrow u \in \mathcal{T}_\alpha$$

□

8.0.4 Definition ([13])
An (L, M)-topological space $(X, L, M, \mathcal{T})$ is T_1 iff the map $\mathcal{T} : L^X \longrightarrow M$ assigns to the complement of a fuzzy point on X the value 1 in M.

If $(X, L, M, \mathcal{T})$ is T_1, then all the complements of fuzzy points in L^X are in $\mathcal{T}_\alpha$. This means that for each $\alpha \in M$ we get a T_1 L-topological space $(X, L, \mathcal{T}_\alpha)$ after identification of the frame $\mathcal{T}_\alpha$ in L^X with $\mathcal{T}_\alpha : L^X \longrightarrow \{0, 1\}$ through $u \in \mathcal{T}_\alpha$ iff $\mathcal{T}_\alpha(u) = 1$.

8.0.5 Definition ([13])
A T_1 (L, M)-fuzzy topological space $(X, L, M, \mathcal{T})$ is *semi-sober of degree* $\geq m$ iff $(X, L, \mathcal{T}_m)$ is semi-sober.

Then in view of Theorem 7.1.6 all the $\mathcal{T}_\alpha$ between $\mathcal{T}_m$ and $\mathcal{T}_0 = L^X$ are semi-sober.

In the definition above we cannot specify $(X, L, M, \mathcal{T})$ to be semi-sober of degree m if m is the supremum of members of M for which $(X, L, \mathcal{T}_m)$ is semi-sober, since the infimum (intersection) of semi-sober topologies is not necessarily semi-sober, as can be seen through the following example :

8.0.6 Example ([13])
Consider $\mathbb{R}$ with the topology $\mathcal{T} = \{\mathbb{R}; \emptyset; (a, \infty) : a \in \mathbb{R}\}$. Then $(\mathbb{R}, \mathcal{T})$ is not semi-sober (nor sober) since $\mathbb{R}$ is irreducibly closed but not the closure of a singleton. Now define $\mathcal{T}_x$ on $\mathbb{R}$ as the topology which contains $\mathcal{T}$ as well as sets of the form $\{(-\infty, b) : b > x\}$. The non-empty closed sets are then $\mathbb{R}$, $\{[b, \infty) : b > x\}$, $\{[b, b'] : b > x\}$, $\{(-\infty, a] : a \in \mathbb{R}\}$, and $\{\{b\} : b > x\}$. The first three types are reducibly closed whereas the fourth is reducible for $a > x$, but in the case $a \leq x$ is both irreducibly closed and the closure of $\{a\}$. The fifth type is irreducible and the closure of $\{b\}$. So $\mathcal{T}_x$ is semi-sober (and sober). But $\mathcal{T} = \bigcap_{x \in \mathbb{R}} \mathcal{T}_x$.
On the other hand, given L-topologies $\{\mathcal{T}_\alpha : \alpha \in M\}$ on X with $\alpha \geq \beta \Rightarrow \mathcal{T}_\alpha \subseteq \mathcal{T}_\beta$ and $\mathcal{T}_0 = L^X$, then $\mathcal{T}^* : L^X \longrightarrow M$ defined by

$$\mathcal{T}^*(u) = \bigvee \{\alpha : u \in \mathcal{T}_\alpha\}$$

is an (L, M)-topology on X. Now consider $\mathcal{T}^*_\alpha = \mathcal{T}^{*\leftarrow}\{[\alpha, 1]\}$, $\alpha \in M$. Then as in Proposition 8.0.3, if M is a continuous lattice, then

$$\mathcal{T}^*_\alpha = \bigcap_{\beta << \alpha} \mathcal{T}^*_\beta$$

8.0.7 Proposition ([13])
If M is a continuous lattice and $\beta << \alpha$, then

$$\bigcap_{\gamma << \alpha} \mathcal{T}_\gamma \subseteq \mathcal{T}^*_\beta \text{ and } \bigcap_{\gamma << \alpha} \mathcal{T}^*_\gamma \subseteq \mathcal{T}_\beta.$$

Proof. Suppose there exists a u such that $u \in \mathcal{T}_\gamma$ for all $\gamma << \alpha$ but $u \notin \mathcal{T}^*_\beta$. Since $\sup\{\gamma : \gamma << \alpha\} = \alpha$ and $\beta << \alpha$, there exists a $\gamma_0 << \alpha$ such that $\gamma_0 \geq \beta$. Hence $\mathcal{T}_\beta \subseteq \mathcal{T}_{\gamma_0}$. Now $u \in \mathcal{T}_{\gamma_0}$ and so $\mathcal{T}^*(u) \geq \gamma_0$. Hence $u \in \mathcal{T}^*_{\gamma_0}$ and so $u \in \mathcal{T}^*_\beta$. Contradiction.
The other case follows similarly.
□

8.0.8 Corollary ([13])
If M is a continuous lattice, then $\mathcal{T}_\alpha = \mathcal{T}^*_\alpha$.

Proof.

$$\mathcal{T}^*_\alpha = \bigcap_{\beta << \alpha} \mathcal{T}^*_\beta = \bigcap_{\gamma << \alpha} \mathcal{T}_\gamma = \mathcal{T}_\alpha$$

□

So if all the $\mathcal{T}_\alpha$, $\alpha \in [0, m]$ are semi-sober (and the rest not), then $(X, L, M, \mathcal{T}^*)$ obtained is an (L, M)-topological space which is semi-sober of degree $\geq m$ $(= m)$.

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