# AN INVESTIGATION INTO THE EXTENT AND NATURE OF THE UNDERSTANDING FIRST YEAR COLLEGE OF EDUCATION STUDENTS HAVE OF ASPECTS OF ARITHMETIC AND ELEMENTARY NUMBER THEORY

VINCENT GEORGE OLIPHANT

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# DECLARATION

The author wishes to state that the whole thesis, unless specifically indicated to the contrary in the text, is his own original work.

V. G. OLIPHANT
January 1996
Port Elizabeth

# DEDICATION

This thesis is dedicated to my father
the late William Seth Oliphant (snr.)

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# ABSTRACT

First Year College of Education students who have done and/or passed mathematics at matric level, often lack adequate understanding of basic mathematical concepts and principles. This is due to the fact that formal tests and examinations often fail to assess understanding at anything but a basic level. It is against this background that this study uses alternative and more direct means of assessing the level and nature of the understanding such students have of aspects of basic arithmetic and number theory.

More specifically, the goals of the study are :

1. To determine the students' levels of understanding of the following number concepts :
   Rational numbers ; Irrational numbers ;
   Real numbers and Imaginary numbers.
2. To determine whether the students understand the rules governing operations with negative numbers and with zero as principles rather than conventions.
3. To determine whether the students understand the rule governing the order of operations as a matter of convention rather than as a matter of principle.

A survey of the literature concerning the nature of understanding as well as the nature of assessment is given.

The students' understanding in the above areas was assessed by means of a written test followed by interviews. A sample of 50 students participated in the study while a sub-sample of 6 were interviewed.

Some of the significant findings of the study were :

1. The students largely failed to draw clear distinctions between Real and Rational numbers as well as between Irrational and Imaginary numbers.

2. Very few of the students could explain the rationale behind the rules governing the operations with negative numbers and zero.

3. Only half of the students had any knowledge of the rule governing the order of operations. Only one student demonstrated an understanding of the rule as a convention.

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# CHAPTER 1

# INTRODUCTION

## 1.1 THE PLACE OF UNDERSTANDING IN THE MATHEMATICS CURRICULUM

" Whatever else an educated man is, he is one who has some understanding of something. He is not just a person with a know-how or knack. " ( Hirst and Peters , 1970 : 19 )

" What, then, do we look for in estimating whether a person is more or less educated ? We judge him by his understanding and his capacity for discrimination. "
( Barrow, 1981 : 38 )

The above views exemplify the necessary link between education and understanding. More precisely, they emphasise that the development of understanding is a necessary condition for the education of the learner. The school, which has as its overriding aim the education of the pupil, must, therefore, largely direct it's endeavour towards the development of understanding. This is especially true in the case of mathematics, which has the most fundamentally logical structure of all school subjects. It is therefore not surprising that statements of aims in mathematics syllabi invariably emphasise understanding. Where the references are not direct, they are implied by means of terms like " logical argument " and " application in problem situations " . Similarly, it is hard to imagine a mathematics teacher who would not claim that the main purpose of tests and examinations is to determine pupils' understanding of the subject-matter concerned.

All this, however, leaves us with an important question which is all too often regarded as trivial, namely : What is understanding ? That the meaning of " understanding " is not as self-evident as is often supposed is clear from the following discussion.

## 1.2 THE NATURE OF THE PROBLEM

In an unpublished essay regarding mathematics education entitled : A Learning Hierarchy For Three Standard Ten Topics, is claimed that

> Too many South African schools sacrifice true understanding for the sake of 'good ' results in the final standard ten examination.
> ( Stupard, 1984 : 1 )

This statement encapsules a serious shortcoming of current classroom practice and examination methods, namely, a lack of congruence between true understanding and success in formal examinations. It further implies that the major aim of classroom learning and instruction, as stated in the syllabus, is generally not satisfied. Referring to the practice in many South African schools of simply 'defining' concepts and then to proceed to the teaching of the algorithms, Stupard says

> The pupils 'taught' by such a teacher will probably be able to get the right answer, but will not understand any of the underlying concepts.
> ( Stupard, 1984 : 8 )

That the prevalence of algorithm-oriented teaching is not a peculiarly South African problem is confirmed by researchers world-wide. According to Dickson, Brown and Gibson,

> There has been and still is much emphasis placed on children being skilled in the standard written procedures of computation regardless of whether or

> not they understand the basis of such techniques.
> ( Dickson, Brown and Gibson, 1983 : 252 )

The FOURTH NATIONAL ASSESSMENT of EDUCATION PROGRESS MATHEMATICS ASSESSMENT showed that

> ... the emphasis on computational skills that generally characterizes precollege mathematics instruction has left many students with serious gaps in their knowledge of basic underlying concepts.
> ( Silver et al, in Mathematics Teacher, Dec. 1988 : 727 )

Thus sight is lost of the true value of algorithms as effective and economical tools. Instead they are largely being approached as if they constitute the essence of mathematics. Once again this results in the dubious kind of 'success ' referred to earlier. This problem is confirmed by Rees and Barr who point out that

> ... (students) can fool us by their apparent ability to solve many tasks correctly. Yet if and when their understanding is probed, we realise that they have not the foggiest understanding of why they are doing what they are doing.
> ( Rees and Barr, 1984 : 1 )

A very useful analysis of the above situation is given by the late Richard Skemp ( 1976 ) who distinguishes between two kinds of understanding, namely Relational (true) understanding which implies that the student knows the rules and the concepts underpinning them, and Instrumental understanding, in which the pupil knows what to do but does not know why the rule works.

Reflecting on the possible causes of the the failure of mathematics educators to popularise mathematics, Skemp says

> ... ( that ) there can be small doubt that the widespread failure to teach relational mathematics - a failure to be found in primary, secondary and further education and in 'modern ' as well as

> 'traditional ' courses - can be identified as a major cause.
>
> ( Skemp, 1976 : 25 )

The extent of this problem is further emphasised in the essay RATIONAL NUMBER CONCEPTS in which Behr and others refer to the results of studies such as the NAEP ( National Assessment of Education Progress ) and the SMSG ( the School Mathematics Study Group ). They point out that

> ... the generally poor performance may be a direct result of this curricular emphasis on procedures rather than the careful development of important functional understandings. "
>
> ( Lesh and Landau, 1983 : 91 )

Against the background of this brief survey, it would be reasonable to assume that :

1. There exists a disturbing gulf between the goal of developing true understanding on the one hand and classroom and examination practice on the other.

2. This problem plagues mathematics teaching and learning on a world-wide scale.

3. Although tentative causes have been variously postulated eg. overburdened syllabi, the development of concrete solutions to the problem is still largely in an exploratory stage

4. In view of the enormity of the problem, a lot of further research into this area is needed.

## 1.3 THE PURPOSE OF THE STUDY

The purpose of the study is, generally, to determine the extent and nature of the understanding that First Year College students who have done mathematics up to matric level, have of aspects of arithmetic and elementary number theory.

More specifically, the goals are :

1. To determine the level of the students' understanding of the following number concepts :
   Rational numbers; Irrational numbers;
   Real numbers; Imaginary numbers.
2. To determine whether students understand the rules governing operations with negative numbers and with zero as principles rather than conventions.
3. To determine whether students understand the rule governing the order of operations as a matter of convention rather than as a matter of principle.

## 1.4 THE POSSIBLE VALUE OF THE STUDY

The most immediate value would be to Dower College of Education, where the practical research has been conducted. The mathematical background of First Year students at the college varies from those who have done mathematics up to std. 7 level, to those who have done the subject up to Matric level. This last group represents, as far as mathematics is concerned, the academically more successful section of the First Year intake. It is therefore of interest to the college to determine, not only how well they have been able to write examinations, but how deep or superficial their understanding is. This will have implications for their further preparation as mathematics teachers.

Secondly, the students come from a wide variety of schools. The results could therefore be of general value to schools, tertiary institutions and other agencies concerned with mathematics education.

At a more fundamental level, the study would constitute a contribution to the ongoing debate about the appropriateness and efficacy of current methods of examining as well as serving to highlight the vexed question as to what understanding mathematics essentially entails.

## 1.5 THE LAYOUT OF THE STUDY

The next chapter (Chapter2) will present a review of the relevant literature.

Chapter 3 will discuss methods, techniques and instruments employed in this study.

Chapter 4 will report the results of this study

Chapter 5 will summarise the study and offer proposals and recommendations based on the results.

The next chapter will now present a review of the literature on the following key questions :

1. What does it mean to understand mathematics ?
2. How can understanding be assessed ?

# CHAPTER 2

# THE LITERATURE REVIEW

## 2.1 THE MEANING OF UNDERSTANDING

Skemp expressed the view that :

> To understand something means to assimilate it into an appropriate schema. This explains the subjective nature of understanding and also makes clear that this is not usually an all or nothing state.
> ( Skemp, 1971 : 46 )

recently, Hoffer gave a very similar analysis when, in attempting to define the concept of understanding, he said :

> ...understanding is not absolute ... different types of things need to be understood ... several levels of understanding exist ... understanding is also a function of time.
> ( Hoffer, 1993 : 316 )

As a starting point, the above description given by Skemp will be used as a basis for analysing the concept of understanding. This description contains three key elements , namely :

2.1.1 Assimilation into an appropriate schema.

2.1.2 That understanding is subjective and

2.1.3 That understanding is usually partial.

Each of these aspects will now be discussed in more detail.

### 2.1.1 Assimilation into an appropriate schema

The term 'schema' is described as a mental structure. Understanding a particular aspect of mathematics therefore implies that this aspect does not exist in isolation but that it fits into a mental structure pre-existing in the mind of the learner.

This idea of meaningful linkage is akin to that contained in the INFORMATION PROCESSING model of understanding as described in Resnick and Ford (1981) . This model describes a students' knowledge structure, and therefore his or her understanding, in terms of semantic networks. Each item in such a network is seen as

" ... not just connected to each other ... " but

" ... related to each other in definable and therefore meaningful ways ... " ( Resnick and Ford, 1981 : 202-203 )

It is important to note that the emphasis here is on knowledge being meaningfully connected. This contrasts with the earlier, and now largely discredited ASSOCIATIONIST model of Edward L. Thorndike. Although the ASSOCIATIONIST model is superficially similar to the INFORMATION PROCESSING model, it posits that the items in a knowledge structure are merely linked by means of bonds of association. It therefore excludes the idea of meaning. In terms of Thorndike's theory, there is no essential difference between meaningful and rote learning. The laws of readiness, exercise and effect are deemed to be adequate in accounting for all forms of learning. Resnick and Ford (1981 : 17) point out that William Brownwell critised Thorndike on two counts, namely, that his theory gives no account of

1. Qualitative differences in the computations of adults and children and
2. Meaning.

The importance of meaning in any account of understanding becomes clear in the following section on the subjectivity of understanding.

2.1.2 The subjective nature of understanding

The idea of the subjective nature of understanding is based on the quite reasonable assumption that the schema's of different people can differ in any number of ways. This should be seen against the background of Skemp's view that these schema's consist of primary concepts " ... which are derived from sensory and motor experience of the outside world " ( Skemp, 1971 : 25 ) and secondary concepts " ... which are abstracted from other concepts. "

( Skemp, 1971 : 25 )

The C.S.M.S. ( CONCEPTS IN SECONDARY MATHEMATICS AND SCIENCE) study conducted in Britain during the late 70's, uncovered the existence of a large variety of unusual and idiosyncratic views and methods among students. With reference to the results of this study, Booth (1981), says

> ... many children are not using the 'proper' mathematical methods taught them at school, but rather are relying on naive, intuitive strategies
> ( Booth, 1981 : 30 )

This view has subsequently been supported by so many studies that it has become part of the prevailing othordoxy of mathematics education.

The idea of subjectivity, which seem somewhat paradoxical in the context of an exact subject like mathematics, is further enlightened by Confrey, quoted in Karplus (1980), as :

> Concepts are valuable to education in that they exist at a private ( individual ) level as well as at a public ( consensus ) level
> ( Karplus, 1980 : 406 )

She further describes these aspects as

> ...comprehension ( individually making sense of the concept) and justification ( subjecting those reasons to public scrutiny )
> ( Karplus, 1980 : 406 )

The above-mentioned views of Confrey, as well as of Booth and others connected with the C.S.M.S. project, are based on clinical interviews which indicated that

> ...there are ... examples in which various interpretations of concepts exist which can only be labelled as alternative conceptions. In these the student provides a consistent and feasible alternative way of understanding the concept.
> ( Karplus, 1980 : 401 )

A mathematical concept, then, has two dimenions, namely, one based on an analysis of the subject-matter, and one based on how the learner actually experiences and perceives the subject-matter; or, in Confrey's terms, as expressed above, it exists at a public as well as at a private level. It stands to reason that, for the mathematics educator, the emphasis falls on the latter aspect. This is especially true in view of all the research findings referred to above. It is against this background that Larcombe says :

> There is no mathematical concept except that which exists in the mind of the person which has it and the concept he or she has will have been built up and is being built up continuously as new perceptions and new discriminations are made.
> ( Larcombe, 1985 : 35 )

Apart from the idea of subjectivity, Larcombe's view also implies that a concept can never be said to be completely formed in the learner's mind. This is, in fact, the third aspect of understanding lifted from the Skemp quote at the beginning of this section, namely, the essentially indeterminate nature of understanding.

2.1.3 The partial nature of understanding

According to Resnick and Ford, understanding involves
" ... grasping ... the interrelationships among concepts."
( Resnick and Ford, 1981 : 105 )

The number of possible interrelationships between any particular concept and the rest of mathematics is clearly infinite. The idea of absolute understanding of a particular concept is therefore extremely problematic. Taxonomic views of understanding are, in fact, based on the assumption that it is not an all-or-nothing state. In terms of Bloom's (1956) taxonomy of cognitive objectives, understanding can be viewed in terms of a hierarchy of levels ranging from the lowest, termed knowledge , to the highest, termed evaluation . These two extreme categories, as discussed by Frederick H. Bell comprise the following : Knowledge simply involves " ... recall of specific mathematical material in a form similar to the form in which the material was presented ... " whereas evaluation is " ... making judgments about the value of ideas, creations and methods ... "

(Bell, F H, 1978 : 169 - 171)

Significantly, several attempts have been made at adapting Bloom's taxonomy to mathematics. These adaptations use various numbers of categories. As Bell, Costello and Kucheman [1983] point out : " Wood (1968) produces five categories, Hollands (1972) has nine and Avital and Shettleworth (1968) just three. " (1983).

Avital and Shettleworth's taxonomy can be characterised in the following way :

> The first and lowest level is recall or recognition of material exactly as it was taught ... The second level is algorithmic thinking ... Here the student is expected to apply well-defined and familiar procedures but not precisely to the particular situation he has previously met ... The third level is called open search by Avital and Shettleworth. By this they mean the rearrangement of ideas and methods to produce new results or to solve new problems. It is to be emphasised that 'new' here means new to the student.
> ( Wynne-Wilson, 1978 : 183 )

Lesh, Landau and Hamilton also discuss understanding in the Information Processing vein of semantic networks. They, however, extend this idea by distiguishing between within-concept networks and between-concept systems. As an example of what is meant by the former, they use an analysis of the concept of rational number as used in the RATIONAL NUMBER PROJECT. (ch 4; p91). Here rational numbers are desribed as consisting of at least six different sub-constructs, namely : a part-whole comparison; a decimal; a ratio; an indicated division; an operator and a measure of continuous or discreet quantities.

The idea, then, is that an understanding of the concept of rational number, within the context of within-concept networks, requires a comprehension of each separate subconstruct and also how they interrelate.

Between-concept systems are effectively summed up by Lesh, Landau and Hamilton when they say that

> ... part of the meaning of individual ideas derives from relationships with other ideas in the system or from properties of the system as a whole. "
> (Lesh, Landau and Hamilton 1983 : 269)

By the very nature of mathematics as a deductive system, the between-concept component is obvious. It is, in other words, true that any mathematical concept is semantically linked to a number of other concepts in a deductive system. In the case of rational numbers Lesh, Landau and Hamilton (1983) state that

> ... between-concepts systems link rational number ideas with other concepts such as measurement, whole number division and intuitive geometry concepts related to areas and number lines. " (p269)

The information processing model of Greeno (1978) as discussed in Resnick and Ford (1981) adds yet another dimension to the concept of understanding. Greeno postulates three criteria for evaluating the degree of a person's understanding. These are given as :

> (1) internal integration of the representation ;
> (2) the degree of connectedness of the information to other things the person knows about; and
> (3) the correspondence of the representation with the material that is to be understood.
> ( Resnick and Ford, 1981 : 206 )

Correspondence refers to the extent to which one's view of an aspect of mathematics agrees with the way experts see it. It therefore implies the existence of an objective standard against which anybody's understanding can be measured. When a teacher, for instance, marks a test, he is using the correspondence criterion, with the "objective standard " being embodied in his memorandum.

Integration concerns the extent to which a person sees the different aspects of a mathematical system as being related instead of seeing it as consisting of separate pieces of information. In this regard, Resnick and Ford (1981) cite the example of multiplication and division which should be seen as inverse operations and not as two entirely separate entities. It is also interesting to note the similarity between the criterion of integration and Skemp's (1971) characterisation of understanding as assimilation into an "appropriate schema".

Greeno's third criterion, namely connectedness, has to do with a person's ability to apply his knowledge in problem situations. As Resnick and Ford (1981 : 207) frame it :

> A knowledge structure meeting the criterion of connectedness would allow a person to answer interpretive questions and apply the structure in new situations because of its links to other knowledge.

Connectedness, in other words, deals with problem-solving and is, therefore, concerned with the upper end of the taxonomic scale.

All the different aspects of understanding discussed above, although not leading us to a simple and conclusive answer, nevertheless help to define the picture. Before looking at the even more vexed question of how to determine understanding in terms of the above models, it would be helpful to briefly survey existing views concerning the related domain of memory .

## 2.2 MEMORY

Two types of memory, namely short-term (or working) memory and long-term (or semantic) memory are usually distinguished. Although Bell, Costello and Kucheman (1983) point out that this distinction is an oversimplification, they nevertheless feel that the model presents a useful framework within which to discuss the phenomenon of memory.

For the purpose of this study, it is the relationship between understanding and long-term memory that is important. In their discussion of the INFORMATION PROCESSING model of memory, Resnick and Ford [1981] make the following important points :

The information stored in long-term memory is basically organised and structured. Although the model has certain similarities with the early ASSOCIATIONIST views of Thorndike it differs fundamentally in that the items in memory are not simply bonded together by frequent occurrence together; they are, in fact, bonded by means of meaningful relationships. Most importantly, the human mind does not simply passively record information from outside, but can actively construct knowledge.

This implied relationship between understanding and long-term memory is supported by Bell, Costello and Kucheman [1983] When they say that

> ... information is better remembered if it is meaningful ...and ... non-meaningful material is more difficult to retrieve from long-term storage because fewer associations have been formed to act as retrieval cues.
> ( Bell, Costello and Kucheman, 1983 : 23 )

In view of their emphasis on meaning, it is reasonable to assume that the authors are using the term 'associations ' in the sense of 'meaningful relationships'.

They proceed to cite several research results which confirm that " ... it is 'depth' and 'elaborateness' of the processing which determines how well material is subsequently recalled. " [p24]. A further crucial point contained in the discussion is that the mind's processing capacity is dependent on its level of development. They then spell out the serious implications this has for teaching practice in the following way :

> This means, in contrast with a learning theory like Gagne's that to teach a complex concept or skill , it is necessary to ensure not only that the child has acquired all the prerequisite concepts or skills, but that the number of these that have to be integrated at any one time is kept to a certain minimum, depending on the level of development of the child's processing capacity.
> ( Bell, Costello and Kucheman, 1983 : 30 )

A failure to satisfy this important criterion may be a part of the explanation for pupils' lack of understanding as discussed earlier.

Finally, in summary, the different aspects of understanding examined in the foregoing discussion include the following :

1. Skemp's contribuition with regard to :

1.1 The distinction between instrumental and relational understanding.

1.2 The idea of a 'schema' as well as the view that understanding is essentially subjective and partial.

2. The Information Processing view of meaningful linkage of knowledge as opposed to the earlier view of associationist bonds.

3. Confrey's distinction between the private and public levels at which concepts exist.

4. The taxonomic views of understanding formulated by Bloom and others.

5. The ideas of within-concept networks and between-concept systems as formulated by Lesh and others.

6. Greeno's three criteria for evaluating a person's degree of understanding, namely :

6.1 Internal integration

6.2 Correspondence and

6.3 Connectedness.

7. The relationship between understanding and memory.

The question that needs to be faced in practice, however, is : How can understanding be determined ?

## 2.3 THE ASSESSMENT OF UNDERSTANDING :

" ... strategies for assessing knowledge structures are still rather ad hoc. " ( Resnick and Ford, 1981 : 207 )

" The roles, functions and effects of contemporary modes of assessment are neither clear, nor well understood. "
( Niss , 1993 : 5 )

The above views illustrate the fact that, although much progress has recently been made in terms of developing new modes of assessment, mathematics educators have not reached definitive positions in this regard.

Against this background, the purpose of this section is to :

1. Discuss the difficulties and limitations that inevitably attend assessment.
2. Consider the approaches and methodologies which are nevertheless available, especially with regard to the present research project.

Before attending to the above however, it is necessary to briefly discuss the meaning and purposes of assessment in mathematics.

### 2.3.1 THE MEANING OF ASSESSMENT

A view enjoying currency among innovative mathematics educators is that assessment should be integrated with the pupil's or student's entire experience of mathematics. In this vein, Webb defines assessment as

> ... the comprehensive accounting of an individual's or group's functioning within mathematics or in the application of mathematics.
> ( Webb, 1992 : 662 )

This represents the ideal of continuous assessment within the classroom context, where the assessor is also the teacher of the class and where optimum opportunity for integration of assessment with instruction therefore exists.

In the case of research of a more limited nature and focussing on the assessment of students' understanding of specific aspects of mathematics it would, however, be useful to adopt a slightly narrower view of assessment. In this regard the definition given by Bodin could be more useful. He defines assessment as

> ... to organize ( or to look at ) a situation in such a way that it enables us to gather information which, after processing, can reveal something that is reliable about personal knowledge ( or about the collective knowledge of a group ).
> ( Bodin, 1993 : 116 )

What, however, are the purposes of assessment ?

2.3.2 <u>THE PURPOSES OF ASSESSMENT</u>

Swann ( 1993 : 195-196 ) refers to the oft-made distinction between Formative, Summative and Evaluative assessment. Formative assessment refers to " ... the achievements of a pupil, so that appropriate follow-up may be designed ..." Summative assessment " ... records overall achievement in a systematic manner. "
Evaluative assessment " ... involves evaluating and reporting on the work of a teacher, school, textbook scheme or any other discreet part of the educational service. "
Webb ( 1992 : 663 ) distinguishes between the purposes of assessment as determining what students know; expressing what

is valued about students' knowledge; informing decision-makers and investigating the effectiveness of the education system as such.

These two descriptions of the various purposes are largely equivalent. All these purposes are valid, although the emphasis will differ from one case of assessment to another. In the present case, the main purpose is diagnostic assessment, which Swann subsumes under formative assessment and which he describes as "... where learning difficulties and misconceptions are identified." ( Swann, 1993 : 195 )

Whatever the purpose(s) of any particular case of assessment may be, however, there remain fundamental difficulties that need to be taken into account.

### 2.3.3 DIFFICULTIES AND LIMITATIONS OF ASSESSMENT

#### 2.3.3.1 THE INFERENTIAL NATURE OF ASSESSMENT

Hiebert and Carpenter point out that :

> ... theories and models of understanding are based on internal operations of the mind which cannot be observed directly
> ( Hiebert and Carpenter, 1992 : 65 )

This implies that understanding must be inferred from processes and products which are observable. Although more modern methods like interviews, talking through and written explanations have succeeded in making mental processes more explicit, they have not solved the problem entirely, as even oral responses to probing questions inevitably leave aspects of the processes concerned uncovered.

### 2.3.3.2 CONSTRUCT IRRELEVANT LEVELS OF DIFFICULTY

Messick, as quoted in Silver and Lane, formulates one of the fundamental difficulties of assessment when he says :

> An assessment instrument is an imperfect measure of a construct because it either underrepresents the construct domain ... or in addition to measuring the construct domain it also measures something that is irrelevant to the construct
> ( Silver and Lane, 1989 : 62 )

An assessment instrument could, therefore, be either too narrow or too wide or could even contain a mixture of these deficiencies.

### 2.3.3.3 THE INSTRUCTIONAL CONTEXT WITHIN WHICH ASSESSMENT OCCURS

In the tradional classroom, in which the transmission mode of instruction pre-dominates, assessment is done by means of conventional modes which are easier to grade and to replicate. This has led to a concentration on the testing of lower-order understanding and thinking skills.
In the innovative classroom, instruction is based on a constructivist philosophy. Because of the attendant difficulties of assessing in a similar mode, assessment practices have tended to lag behind in this regard and have remained rooted largely in behaviourist philosophy. This has led to what Galbraith (1993 : 79) describes as a conflict of paradigms.

When trying to assess "true" understanding, whether within the traditional or the innovative classroom, it is clear however that the researcher should move beyond traditional standard types of items.

2.3.3.4 THE EFFECT OF TIME AND THE LACK OF RELIABILITY

With reference to the fact that assessment is fundamentally unreliable, Bodin states that

> When one gives the same test to the same student one never knows what one will obtain
> ( Bodin, 1993 : 131 )

This is because new learning could have taken place, certain aspects might have been forgotten and the assessment situation can never be totally replicated.

2.3.3.5 OTHER LIMITATIONS

Other limitations of assessment which Bodin mentions are :

1 Changing the position of an item in a test can radically affect the outcome.

2 The form and formulation of the question almost inevitably introduces bias and influences the perceptions of the student and, ultimately, the information gathered.

Despite these shortcomings, assessment remains an essential part of the curriculum. Researchers and teachers are therefore compelled to continue the search for ever more effective ways of accomplishing this task.

2.3.4 APPROACHES AND METHODOLOGIES OF ASSESSMENT

In designing an assessment project, one needs to take into account the different components or dimensions which constitute assessment.

Niss ( 1993 : 12-14 ) lists some of the important components of an assessment mode as being :

The subject of assessment
the objects of assessment
the items of asessment
the occasions of assessment
the procedures and circumstances of assessment
the judging and recording in assessment and
the reporting of assessment outcomes

Webb ( 1992 : 668 ) names the four general components of assessment as :

the assessment situation
the response to the situation
the analysis of the response and
the interpretation of the results

The following could serve as a diagramatic comparison of the two descriptions :

| | |
|---|---|
| Subject<br>Objects<br>Items | Situation |
| Occasions<br>Procedures and circumstances | Response |
| Judging and Recording | → Analysis |
| Reporting | → Interpretation |

Webb further points out that the four components he mentions are interdependent and that formulation of the one influences the other. He then states that :

> A general principle of assessment is that the situation, response, analysis, and interpretation, as well as the mathematical knowledge being assessed, the characteristics of the individual or group who are to respond and the purpose of the assessment, must be in alignment.
>
> ( Webb, 1992 : 668 )

Apart from the above considerations concerning the general structure of the assessment process, it is also necessary to look more closely at the instruments of assessment that have been used in the past and that are being used today.

## 2.3.5 THE INSTRUMENTS OF ASSESSMENT

Traditionally, teachers have sought to assess pupils' understanding by means of written tests mostly consisting of standard types of problems. The main drawback of this approach is succinctly summed up by Skemp when he says :

> From the marks he makes on paper, it is very hard to make valid inference about the mental processes by which a pupil has been led to make them ; hence the ... difficulty of sound examining methods
> ( Skemp, 1976 : 24 )

The weakness pinpointed by Skemp is aggravated by the fact that the 'problems' with which pupils are confronted are usually standard types. The understanding tested in such a situation is of the instrumental type and can therefore not be considered as understanding in any true sense of the word. As Larcombe puts it :

> It is easy to do well, according to the criteria so often used for judging mathematics learning, provided you are an efficient rule learner and recaller.
> (Larcombe, 1985 : 37)

Faced with such a difficulty, it seems, at first, reasonable to suggested that one should move away from standard problems to more original types of problems. Such an approach, however, presents a number of difficulties, not the least of which are the following :

Firstly, genuine problem solving represents the top end of the taxonomic scale. A concentration on such an approach would, therefore, result in top-heavy testing. As such, it would be an inefficient discriminator along the whole spectrum of partial understandings. In this regard, Resnick and Ford, in a critique of the Piagetian tradition, pose the question :

> Does failure to perform a particular version of a problem reflect a lack of competence with regard to the presumed underlying logical structures ?
> ( Resnick and Ford, 1981 : 175 )

To this one could answer : not necessarily. After all, in problem solving, the effect of context is very important. Rees and Barr ( 1984) have, for instance, found that placing a problem in a particular context can either increase or decrease its level of difficulty depending on whether or not the context is familiar and meaningful to the learner. The question posed by Resnick and Ford represents but one side of the coin. The other side can be framed in the following way : Does the learner's ability to solve a particular problem imply that he understands the underlying structure of the algorithms used in the solution of the problem ? Again one would have to say : not necessarily, although, of course, an analytical understanding of the algorithms being used would increase the likelihood of success in solving the problems. What, however, can modern approaches offer in the way of offsetting the disadvantages associated with traditional written tests ?

Modern methods of assessing understanding are often characterised by a verbal ( oral ) dimension accompanying the written ( manual) dimension. The forerunner of this newer tradition is the clinical interview pioneered by Jean Piaget in the mid 1920's. According to Opper

> Piaget felt he needed a method that would allow

> the child to verbalise freely and thus provide the researcher with the opportunity of inferring the covert intellectual processes
> ( Opper, 1977 : 91 )

She further describes Piaget's method as involving a verbally presented problem relating to a concrete situation. The task of the interviewer is then to ask a series of questions concerning the learner's approach to the problem. The interviewer uses the responses as a basis for drawing inferences about the mental processes underpinning the learner's approach to the problem. Apart form the child's verbalisations, inferences are also drawn from direct observation of the child's manupilations of the experimental objects.

Piaget's method, and variations thereof, have since been used in many research projects, both large and small scale. It is important to note, however, that the relationship between the interview and written components differ from one research project to the next.

Hart (1981 : 1 ) reports that, in the case of the C.S.M.S. project the written test component was set in a problem solving format. In conjuction with this, the interview component had two main aims, namely to :

(1) assess the suitability of the written test items and
(2) to establish the methods used and errors made by the children, i.e. to inform and interpret the written test results.

Of the nature of the interviews she says :

> They lasted about an hour during which the child was asked to talk his way through the problems explaining what he was doing at each stage
> ( Hart, 1980 : 3 )

In the Assessment of Performance Unit ( A.P.U ) Primary Survey, interviews were also used in tandem with written tests. An important difference between the two, however, is that, in the case of the A.P.U., the practical tests ( interviews ) and the written tests were independent of each other. By this is meant that the pupils were not interviewed on the same problems that were presented to them in the written test. The results of the two tests could, however, be correlated in order to ascertain their degree of association.

In the case of the research of Rees and Barr (1984) students were presented with problems which they had to solve in the language laboratory. They were asked to talk through a problem, explaining their steps as they went along. These recordings were then used as a basis for constructing interviews in those cases where important leads needed to be followed up.

There are also the three criteria for assessing a person's knowledge structure as postulated by Greeno (Resnick and Ford, 1981 : 207 ).
These three criteria ( defined in Chapter 1 ) are integration, correspondence and connectedness. Different ways of assessing each of these aspects of understanding have been

developed.

According to Resnick and Ford (1981 : 207) an integrated knowledge structure is one in which a lot of related concepts are associated with central organising concepts, also known as nodal concepts. Based on the assumption that access time depends on the number of connections that need to be traversed, integration is assessed by means of timed free-recall patterns. In this respect, Resnick and Ford refer to the research of Larkin which showed that

> ... the expert appeared to retrieve information in chunks ... The novice, on the other hand, appeared to retrieve information more or less randomly
>
> (Resnick and Ford, 1980 : 208)

The authors do point out however that 'chunked' patterns of recall could also be attained by means of cramming and that it, therefore, does not necessarily mean that the students' knowledge structure is integrated.

As far as correspondence is concerned, Resnick and Ford (1981) name word association, graph building and card-sorting tests as methods which have been developed by Shavelson. The crux of word association is that a person is asked to state as many words or concepts as possible relating to a given word from the subject-matter domain. This is then measured against an analysis of the domain concerned. The authors briefly discuss research that confirms the validity of word association as well as some which contradicts it. The validity of the method has thus not yet been firmly established.

The third criterion, namely connectedness, deals with the pupil's ability to apply mathematical knowledge. In this case, more mainstream methods like direct problem-solving and clinical interviews are used.

Up to this stage, we have been considering methods of assessing understanding in general terms. Researchers have attempted, however, to narrow the focus by analysing the subject-matter of mathematics into different components and devising modes of assessment for each separate component.

With reference to Gagne's theory of learning, Bell says :

> The direct objects of mathematics learning - facts, skills, concepts and principles are the four categories into which mathematical content can be separated.
> ( Bell, 1978 : 108 )

The value of such analyses is that they allow a more focussed approach to the problem of assessment of understanding. This is true despite the fact that terms such as concepts, principles, etc., are not always used in the same sense by different writers.

Reviewing his categories, Davis describes concepts as " ... probably the most basic kind of subject-matter in mathematics " ( Davis, 1978 : 13 )

The NCTM Curriculum and Evaluation Standards state that :

> Concepts are the substance of mathematical knowledge. Students can make sense of mathematics only if they understand its concepts and their meanings and interpretations.
> ( NCTM, 1989 : 223 )

It therefore seems reasonable to start with the question :

How does one determine whether a learner understands a concept?

To this Bell answers :

> A learner who has learnt the concept of triangle is able to classify sets of figures into subsets of triangles and non-triangles
> ( Bell, 1978, 108 )

Larcombe, in criticising the prevalence of learning material which presents only examples, says : " It is only against non-examples that examples can be judged. " ( Larcombe, 1985 : 37 )

The NCTM Curriculum and Evaluation Standards, Standard 8 : Mathematical Concepts, states that understanding a concept implies the ability to :

* label, verbalize and define concepts
* identify and generate examples and non-examples
* use models, diagrams and symbols to represent concepts
* translate from one mode of representation to another
* recognise the various meanings and interpretations of concepts
* identify properties of a given concept and recognise conditions that determine a given concept
* compare and contrast concepts

( NCTM, 1989 : 223 )

The above will now serve as a framework in terms of which the design of the present research will be discussed.

# CHAPTER 3

# RESEARCH METHODOLOGY

## 3.1 BACKGROUND

This study is, to a large extent, a replication of a study conducted by this researcher in 1988. Although the 1988 study was never completed, most of its relevant results are still available. It was, therefore, thought prudent to regard it, for the purposes of this project, as a pilot study. Where appropriate and by way of distinction, reference will be made to the present study ( 1995 ) and the pilot study ( 1988 ). All discussion, however, unless otherwise indicated, will be with reference to the present study.

## 3.2 THE NATURE OF THE STUDY

This study concerns the assessment of the understanding of a group of mathematics students in their first year at a college of education. Because understanding, as can be seen from the extensive analysis in the Literature Survey, is a very complex idea, any single study can only hope to shed light on a limited aspect thereof. This is true irrespective of which of the various research methods available are adopted.

In this instance, it was decided to use a survey method, largely because the researcher wanted to replicate a study which unforeseen circumstances prevented him from completing a few years previously.

As Cohen and Manion ( 1984 : 94 ) point out, a survey can be large-scale or small-scale, depending on its scope. Fraenkel and Wallen ( 1990 : 331 ), further, state that all surveys have three major charcteristics in common, namely :

1. Information about a group of people is collected in order to describe some characteristic of the population from which they are taken.
2. Information is mainly obtained by asking questions .
3. Instead of questioning the whole population, information is collected from a sample .

3.3 THE SAMPLE

The kind of sample taken in this case is a convenience sample, which Fraenkel and Wallen describe as :

> a group of individuals who ( conveniently ) are available for study.
>
> ( 1992 : 75 )

More specifically, the sample consists of 50 First Year mathematics students at Dower College of Education, in the historically " Coloured " area of Port Elizabeth. All the students had done mathematics up to Matric level, with 47 of them having passed the subject. The relevant characteristics of the sample are reflected in tables 1 A and 1 B. ( See APPENDIX II, p101 )

TABLE 1 A

The sample used in the pilot study was much bigger due to the fact that, at that time, more students with matric mathematics aspired to become teachers. Amongst other factors, the current bleak employment prospects in the teaching profession have since led to an erosion of these numbers. To be more precise, the present sample is about half that used in the pilot study.

A further comparison of the two samples reveals the following :

In both cases, the numbers of male and female students are fairly well-balanced.

The most significant difference reflected in the table is that the present study sample contains 11 students from the former Department of Education and Training, which used to cater for pupils classified as Black. The pilot study, on the other hand, consists entirely of students from the former Department of Education and Culture, which used to cater for the pupils classified as Coloured. This change is due to the abolition of the Apartheid system which has led to the official de-racialisation of education in South Africa.

TABLE 1 B

The symbol-distribution of the two studies with both being basically normal, although the present study distribution is more skewed to the left.

In both cases, the percentage of students who had actually passed mathemtics is high, namely 94% in the case of the present study and 89 % in the pilot study.

3.4 THE CONTEXT

The mathematics content of the First Year college syllabus is more or less at the level of std. 5 and 6. None of the content included in this study had, by the time of assessment, been explicitly taught to the students. What is being tested would, therefore, be what they know and understand as a result of their school learning.

3.5 INSTRUMENTATION

Fraenkel and Wallen ( 1990 : 89 ) describe instrumentation as " The whole process of collecting data. "

The instruments used in this case are a written test of understanding and an interview which serves as a follow-up to the written test.

3.5.1 WRITTEN TEST

In view of the fact that this instrument has been personally developed by the researcher, the rationale and process of construction need to be discussed in some detail :

The test consists of three sections, namely :

Section A : The Number System

Section B : Operations with negative numbers and with zero

Section C : The order of operations

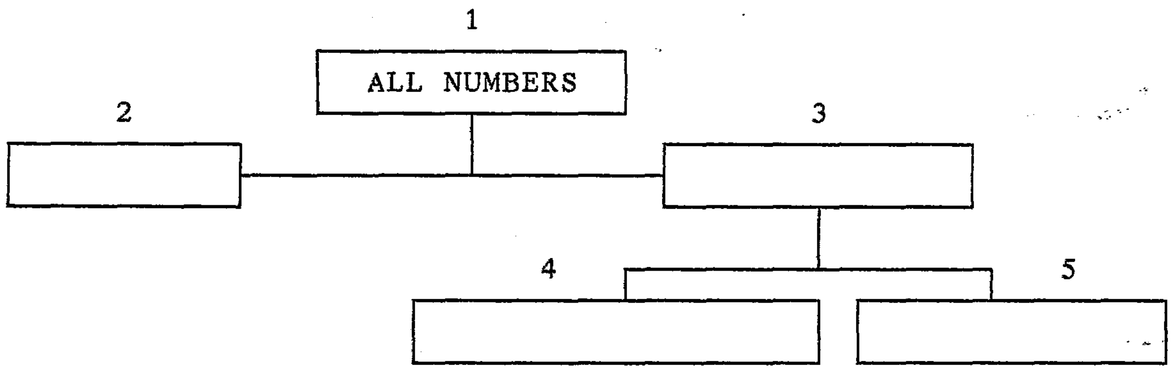
The rationale for the construction of the various items was as follows :

<u>SECTION A : THE NUMBER SYSTEM</u>

<u>ITEM 1</u> :

<u>OBJECTIVE</u> :

To see whether the students can illustrate the relationships between the Imaginary, Real, Rational and Irrational numbers by correctly placing the labels in the diagram given below :




( NB : The numbering of the frames is used for the purposes of the discussion below and does not appear in the actual test. )

<u>ITEM CONSTRUCTION</u>

Some of the immediate practical considerations were :

Firstly, in the case of frame 1, the term ALL NUMBERS was used instead of the term COMPLEX NUMBERS in order to avoid confusing students who might never have heard the latter term before.

Secondly, the sets of numbers were presented in an order which would not suggest the correct placements to the students. For this reason the order used was :

IRRATIONAL, IMAGINARY, RATIONAL, REAL

instead of a more expected order like :

REAL, RATIONAL, IRRATIONAL, IMAGINARY

In this item, the students had to show that they understand the complex numbers as partitioned into imaginary and real numbers and the real numbers as partitioned into rational and irrational numbers.

From this, it follows that the possible correct responses would be :

Frame 2 : Imaginary

Frame 3 : Real

Frame 4 : Rational ( or Irrational )

Frame 5 : Irrational ( or Rational )

Therefore, whilst frames 2 and 3 have only one possible correct answer each, frames 4 and 5 can have one of two possible correct answers : if frame 4 contains 'Rational', frame 5 must contain 'Irrational' and vice-versa.

Because of the objective nature of the item, it has a high degree of reliability. A serious limitation, however, is that a student who had learned the relationships between the sets off by heart, would be able to give a perfectly correct response. For this reason, this item had to be backed up by the other items in this section.

ITEM 2

OBJECTIVE :

To assess the pupils' ability to select examples of Real, Rational, Irrational and Imaginary numbers from a given list of numbers.

ITEM CONSTRUCTION

One of the ideas reflected in the Literature Review in Chapter 2, is that understanding a concept involves, among other things, the ability to identify and classify examples of the concept. It is this idea which forms the basis of this item.

The numbers were selected in such a way as to limit the degree of non-essential differences between them to a minimum. Thus numbers such as 2, -2, $\sqrt{-2}$ and $-\sqrt{2}$ were used instead of numbers like 2, -3, $\sqrt{-5}$ and $-\sqrt{7}$. In this way, the students' attention was focussed on the essential differences between the numbers. Because of the minimal presence of distractors, an inability to identify examples of a particular concept can reasonably be attributed to a lack of understanding of the concept concerned.

Like in the case of item 1, the student's grasp of the relationships between the various sets of numbers can be tested. In the case of item 2, however, these relationships can be assessed in a more specific manner. Incorrect responses would therefore also provide more concrete and specific leads to follow up during the interviews.

The format of this item was changed from what was used in the pilot study. In the pilot study, the students were asked to write down their selections in the indicated rows. Because the students wrote down the selected numbers in a variety of sequences, the analysis was a very difficult and tedious task.
In the present study, therefore, the numbers and set names were arranged in a two-dimensional grid so that the students simply had to make a tick in the appropriate block. This greatly enhanced the process of marking and analysis.

ITEM 3

OBJECTIVE

To assess the pupils' ability to give their own examples and non-examples of Imaginary, Real, Rational and Irrational numbers.

ITEM CONSTRUCTION

This item differs from item 2 in three respects :

Firstly, instead of selecting examples from a given list, students must now give their own examples.
Secondly, in addition to examples, this item also deals with non-examples. This, as has been discussed in the Literature Review in Chapter 2, is an essential aspect of conceptual understanding.

Thirdly, the more open-ended nature of the item reduces the likelihood of success based on guessing.

Item 3, further, provides a means of cross-checking any patterns that may emerge from the responses to item 2.

ITEM 4

OBJECTIVE

To assess the students' ability to define Imaginary, Real, Rational and Irrational numbers.

ITEM CONSTRUCTION

The item consists of a direct request to define the sets of numbers concerned.

SECTION B : OPERATIONS WITH NEGATIVE NUMBERS AND WITH ZERO

OBJECTIVE :

To assess the students' ability to explain a mathematical rule in terms of its underlying principles.

ITEM CONSTRUCTION :

This item consists of a direct request to explain how they would convince someone that the particular rule is valid.

SECTION C : THE ORDER OF OPERATIONS

OBJECTIVE :

To assess whether the students know and understand the rule governing the order of operations.

ITEM CONSTRUCTION :

The item consists of three parts

1. The first part consists of four simple operations to assess whether students can consistently apply the rule ( or a rule ) governing the order of operations. The items are arranged in order of increasing complexity.
   The second item, which only contains addition and subtraction should have been changed for the present study as it has very little diagnostic value. It could also have been left out as the other three items cover the required domain adequately.

2. In the second part they are asked to state the rule that they applied in the first part. The main purpose of this part of the item is to check for consistency of the student's conception of the rule.

3. The third part tests whether the student understands the rule as a convention.

3.5.2 INTERVIEWS

3.5.2.1 INTRODUCTION

Two of the most important considerations in planning a research interview are, the purpose of the interview and the type of interview that is required.

3.5.2.2 PURPOSE

Cohen and Manion ( 1984 : 292 ) distinguished three main purposes for the research interview :

Firstly, it may serve as the principal means of data gathering.

Secondly, it may serve as a means of testing hypotheses or suggesting new ones.

Thirdly, it may be used together with other methods in a research project.

In this case, the purpose of the interviews is of the third kind. More specifically, as Cohen and Manion put it, it is to

> ... follow up unexpected results ... to go deeper into the motivation of the repspondents and their reasons for responding as they do.
>
> ( 1984 : 293 )

3.5.2.3 <u>THE TYPE OF INTERVIEW</u>

The logical structure of the subject-matter should, on the face of it, make it possible not only to construct a fairly structured interview schedule, but to implement it as such. The purpose of the interview, as stated above, needs to be taken into account, however.

This purpose implies that the interviews would have to be based, in each particular case, on the student's responses to the written test. The fact that these responses differ from one student to the next, means that the line of questioning will have to differ accordingly.

The interviews will thus need to be semi-structured with the interview schedule mainly serving as a guide.

3.5.2.4 GENERAL GOALS

1. To follow up the written test with regard to the reasons for the given responses, unexpected responses, etc.
2. To elicit further responses in cases where the responses were either omitted, trivial or inconclusive.
3. To validate the provisional conclusions drawn from the analyses of the written test results.
4. To obtain a more qualitative and in-depth view of the students' understanding than can be gleaned from the written test alone.

3.5.2.5 OBJECTIVES

Section A

1. To see if the student understands how the different sets of numbers relate to one another.
2. To see to what extent students can justify their choices or omissions of examples and non-examples in Items 2 and 3
3. To probe for the underlying reasons for inconsistencies between the different items.
4. To probe students' understanding of certain terms used in the definitions e.g. existence / non-existence of numbers, numbers having no value / no exact value etc.

Section B

To follow up the students' replies in order to determine whether the rule is grasped as one which can be understood in terms of lower-order concepts and principles. More specifically, to approach the different classes of replies in the following manner :

1. CORRECT response : Probe from a different angle.
2. WRONG response : Use the wrong answer as starting point and follow through its consequences.
3. TRIVIAL response : Try to find out whether the student realises that his/her answer does not really amount to an explanation.
4. Response OMITTED : Try to elicit a response by probing from a different angle.

<u>Section C</u>

1. To see whether the students understand :

1.1 That multiplication can be done before division and vice-versa, depending on which occurs first, and that the same holds for addition and subtraction.

1.2 The rule as a convention.

2. To probe inconsistencies between rule application and rule definition.

3.5.2.6 <u>THE INTERVIEW SAMPLE</u>

6 of the 50 students were interviewed. It was very difficult to select an appropriate sample seeing that so many aspects of the written test results required follow-up. This difficulty was compounded by the great variation in the

responses. Against this background, any sample of this size was bound to suffer inadequacies.

One way of offsetting this could have been to interview more students. The interviews, however, had, due to time constraints, to be conducted at a very awkward time of the year. This was when the students had just completed their exams and many of them had left or were in the process of leaving. This restricted the number of students available for interviewing.

The sample was, nevertheless, as useful as most others would have been likely to be, and did, to a satisfactory extent, meet the requirements.

Also, in a study of this nature, the role of the interviews is less crucial than it would have been in the case of a problem-solving format.

Finally, the written test, given its nature, followed by the interviews, constitutes a design that is geared towards delivering results that are both valid and reliable. This much should become clear from the data analysis presented in the next chapter.

# CHAPTER 4

## STRATEGY FOR ANALYSIS

The scripts were marked using a colour-coding system in order to simplify the eventual process of analysis. The results of this written test are analysed in two phases :

Firstly, the results of the present study are analysed in terms of categories which emerged from the pilot study. This facilitates, inter alia, a comparison of the results of the two studies, which enables one to draw certain conclusions about the extent to which the two groups of students differ with respect to the level of their understanding of the concepts and procedures concerned.

Secondly, a more detailed analysis of the results of the present study is made in order to gain a clearer insight into the nature of the students' understanding of the concepts and procedures. In conjunction with this, an analysis of the follow-up interviews is done.

The analysis of each of the sections of the written test is now discussed in more detail. The test itself can be found in APPENDIX I. Tabulations of error frequencies are given in APPENDIX II. While the interview schedule is given in APPENDIX III.

## SECTION A : THE NUMBER SYSTEM

### Strategy for analysis

The following procedure is followed :

1. The nature and level of the students' understanding of each of the four sets of numbers, namely Real, Rational, Irrational and Imaginary, are analysed separately, taking each of the items in turn :

   Item 2

   Item 3

   Item 4

   Interviews

   In this way, the students' understanding of each particular number concept is analysed in four different ways. These analyses are then followed by a summary of the findings concerning the students' understanding of the particular number concept.

   The analysis of item 1 is done last because it concerns interrelationships between the four sets of numbers.

2. It was decided to start the analysis with the numbers which students are probably best acquainted with. A feasible sequence, in order of complexity, seemed to be :

   Real numbers

   Rational numbers

Irrational numbers

Imaginary numbers

It is in the above sequence that the analysis is now presented.

## REAL NUMBERS

### ITEM 2

Omissions : ( TABLE 2 A )

The numbers with the highest error frequencies are $-\sqrt{1/2}$ ; $-\sqrt{2}$ ; $\sqrt{2}$ ; $-1/2$ with percentage errors ranging from 98 % ($-\sqrt{1/2}$ ) to 92 % ($-1/2$ ).

This confirms the trend discovered in the pilot study when the same types of numbers had the highest error frequencies ranging from 79 % ($-\sqrt{2}$ ) to 58 % ($-1/2$ ).

This would seem to imply that a high percentage of students do not consider the following types of numbers as Real :

1. Surds ($-\sqrt{1/2}$ ; $-\sqrt{2}$ ; $\sqrt{2}$ )

2. Negative numbers ($-\sqrt{1/2}$ ; $-\sqrt{2}$ ; $-1/2$ ). This is confirmed by the fact that $-2$ was omitted as Real by a considerable number of students, namely 90 % in the present study and 45 % in the pilot study.

3. Fractions ($-\sqrt{1/2}$ ; $-1/2$ ). This is confirmed by the fact that 1/2 was omitted by 56 % of students in the present study and by 38 % of students in the pilot study.

Wrong Selections : ( TABLE 2 B )

Very few of the Imaginary numbers were selected as Real numbers. Two possible explanations for this come to mind : Firstly, this is in line with the tendency from the Omissions , as surds and/or negatives and/or fractions are not generally considered to be Real.

Secondly, more than for any other numbers, there are clear and straightforward visual cues for identifying Imaginary numbers. The students only need to look for a negative number under a root-sign to know that they are dealing with a number that is Imaginary and, therefore, not Real.

## ITEM 3

This item differs from item 2 in that, instead of simply being asked to select examples from a given list, the students were required to supply their own examples and non-examples.

Examples : ( TABLE 3 A )

In both the present study and the pilot study, the students had a great deal of success in providing their own examples. The percentage of students who had both examples correct was 82 % in the present study and 97 % in the pilot study.

The possible implication that the students know what Real numbers are, is however contradicted by the low success-rate achieved in item 2.

This apparent contradiction is cleared up by an examination of the types of example given. The following significant patterns emerge :

1. Of the total number of students, 98 % gave positive numbers as examples.
2. Further, of the total number, 86 % of the students gave positive whole numbers as examples.

From the above it would seem as if the students have a very limited idea of what Real numbers are. By far the majority of them seem to regard Real numbers as positive whole numbers, or, at best, positive numbers.

Non-examples ( TABLE 3 B )

In view of the students' restricted conception of Real numbers, as emerged from item 2 and the examples-part of item 3, one would expect a low success-rate in providing non-examples.

This is borne out in the case of both the present study ( 64 % of sudents got both wrong ) and, to a lesser extent, in the case of the pilot study ( 47 % of the students got both wrong ).

Types of non-examples given :

In view of the fact that mostly positive examples were given, one would expect a counter-balancing trend of mostly negative non-examples being given.

What emerges from an analysis of the types of non-examples given, is that 46 % of the students gave negative numbers as non-examples. Although this does not, on the face of it, confirm the expressed expectation, one needs to take into account that only 24 % of the students gave positive numbers as non-examples. This is due, mainly to the relatively high degree of omissions.

## ITEM 4

( TABLES 4A )

In the pilot study 29 % of the students managed to give correct definitions, while, in the present study only 8 % gave definitions which could in any way be considered as " correct ". These responses were of three types, and as should become clear in the following discussion, some of these could arguably be considered as trivial rather than as correct.

Type 1

1 student said : Real numbers are numbers of which the value can be found.

This is the most correct of the three types in that the value of any Real number, given in an unsimplified form, can be found by means of calculation eg. $\sqrt{25}$ or $\sqrt{6}$ or 3 ( 4 + 5 ).

Type 2

2 students said : Real numbers are numbers which exist.

Although, in a strict mathematical sense, Imaginary numbers also exist, this answer can be regarded as valid within the probable frame of reference of the student(s) concerned.

Type 3

1 student said : Real numbers are all numbers except Imaginary numbers. This is a factually correct statement although it may, arguably, be regarded as trivial.

Wrong Responses

In the pilot study the wrong responses stood at 51 % while, in the present study, they accounted for 84 % of the reponses.

Two clear types of wrong responses could be discerned :

Type 1

28 % of the students defined Real numbers as limited to whole numbers, positive numbers, natural numbers or counting numbers.

Type 2

30 % opted for a view of whole numbers as perfect squares or square numbers.

Apart from the 6 % of nonsensical answers ( those from which no clear statement could be discerned ) and the 6 % of omissions, the other reponses were so diverse that further classification was not viable.

## INTERVIEW RESPONSES

The following findings emerging from the written test, were confirmed by the interviews :

1. Real numbers are considered to be positive numbers, and more specifically, positive whole numbers.
2. There is a belief that Real numbers are numbers that are "real" in the everyday sense of the word. The implication is that negative numbers, fractions and surds are, somehow, less than "real".
3. The interviewees found it difficult to distinguish between Real and Rational numbers and, when probed, admitted that they did not really know the difference :

A typical response to the request to explain their understanding of Real numbers more fully and in terms of examples was :

> I would say it is a number which can stand on its own. Numbers like 2 or 3, not something with a square root or a negative or something else attached to it.

## RATIONAL NUMBERS

### ITEM 2

Omissions ( TABLE 5 A )

The numbers with the highest error frequency in both the pilot study and the present study are

-1/2 ; -2 ; 0 and 1/2

This would seem to indicate that the following types of numbers are not seen as Rational :

1. Negative numbers : -1/2 ; -2
2. Fractions : -1/2 ; 1/2
3. Zero : 0

An interesting and somewhat unexpected result is that, in both studies, 2 has an error frequency more than double that of $\sqrt{16}$.

A possible explanation for this is that many students connect the classification of numbers with a particular section of quadratic theory, namely " the nature of the roots of a quadratic equation ". This could be due to the fact that an equation has rational roots when the value of delta is equal to a perfect square. There is therefore a tendency to link the notion of Rational numbers with the idea of perfect squares.

<u>Wrong Selections</u> : ( <u>TABLE 5 B</u> )

The number with the highest error frequency in both studies is $\sqrt{2}$ . In the present study the % error of $\sqrt{2}$ is 24 % while, in the pilot study, it was 31,1 % . In both cases, this is more than thrice as high as the next highest error frequency. This could be due to the fact that, of all the numbers in the given set, $\sqrt{2}$ is the closest to " square root of a perfect square " in terms of the delta-association referred to earlier.

The lower error frequency of the other surds is possibly due to the fact that each of them have a negative sign associated with them.

ITEM 3

Examples : ( TABLE 6 A )

In both studies the students had a high level of success in providing their own examples ( 91,5 % and 82 % respectively for the pilot and present studies ). Like in the case of Real numbers, this is contradicted by the low level of success in item 2.

An investigation of the types of examples given, reveals the following :

1. 88 % of the students gave positive examples , with 56 % giving postive whole number examples.
2. 26 % of the students gave examples in the form of positive square roots. In all the cases the number under the root-sign is a perfect square.

Non-examples : ( TABLE 6 B )

In both studies the students had much less success in providing non-examples than they had with giving examples.

In the pilot study the comparative success rates for examples and non-examples were 92 % and 50 % respectively, while the comparative rates for the present study are 82 % and 16 %

respectively.

The most plausible explanation for this is that most of the numbers with which students generally work and which they are acquainted with are Rational numbers. They therefore have a relatively small pool from which to select possible non-examples.

As far as the <u>types</u> of non-examples given are concerned, the expected trend of negative non-examples ( as opposed to the trend of positive examples ) is not confirmed by a closer examination. There is, in fact, no significant difference between the numbers of postive and negative non-examples given .

<u>ITEM 4</u>

( <u>TABLE 7A</u> )

<u>Correct Responses</u>

The success-rate was 26 % in the pilot study and 8 % in the present study.

The 8 % of correct definitions were all given in terms of square roots, and not, as is normally the case, in terms of fractions. This, once again, could be due to an association with " the nature of the roots ".

<u>Wrong Responses</u>

Although there were four types of wrong response, only two types were significant :

Type 1

30 % of the students gave unsuccessful definitions either in terms of square roots or with reference to delta. This once again confirms the extent to which the classification of numbers is connected in the students' minds with the work on the nature of the roots of a quadratic equation.

Type 2

The only other significant group are the 28 % who defined Rational numbers as whole numbers or even numbers

## INTERVIEW RESPONSES

The following findings emerging from the written test, were confirmed by the interviews :

1. Rational numbers, like Real numbers, are considered to be positive numbers.
2. There is a sense of Rational numbers as numbers which need to be viewed in terms of square roots.
3. The difference between Real and Rational numbers is considered to be negligible, ie. the students think that nearly all Real numbers are Rational.
4. Rational numbers are seen as "rational" as in " numbers that make sense "

## IRRATIONAL NUMBERS

(a) Omissions : ( TABLE 8 A )

The percentage error for all the irrational numbers is fairly high in both the present study ( ave. 45 % ) and in the pilot study ( ave. 55 % ). In both studies there is no significant difference between the % errors for the three types of numbers concerned.

(b) Wrong Selections ( TABLE 8 B )

Like in the case of the Omissions-parts of the Real and Irrational numbers, the numbers with the highest error frequencies are the negative numbers. It is also interesting to note that, in the case of both studies, the sequence of numbers, arranged in order of decreasing error frequency, is virtually the same. The fact that -1/2 has the highest error frequency in both studies tallies with the earlier finding, that there is a tendency not to regard negative numbers and fractions as Rational.

ITEM 3

Examples : ( TABLE 9 A )

In the case of the pilot study, there is not a significant difference between the percentage of students who had both wrong ( 48 % ) and the percentage who had both correct ( 36 % ). In the present study, however, students had far more difficulty in giving examples, with only 8 % having both correct and 74 % having both wrong.

Non-examples : ( TABLE 9 B )

In the case of both studies the students had a high degree of success in giving non-examples. In the pilot study, 85 % had both non-examples correct, whilst, in the case of the present study, the success-rate was 68 % . This, again, is probably due to the fact that most of the numbers they are acquainted with are Rational numbers. The probability of a correct guess is, therefore, relatively high.

ITEM 4

( TABLE 10A )

Correct responses

The success-rate in the pilot study is 22 % , which is very close to the 23 % achieved in the present study. The correct definitions were all given in terms of square roots, e.g. " No clear square roots are obtained " or " the square root does not work out properly " .

Wrong responses

As could be expected from the low success-rates above, the percentage of wrong responses was high in the case of both studies, being 77 % in the pilot study and 74 % in the present study.

The most significant of the wrong responses is the definition given by 28 % of the students, of Irrational numbers as negative numbers. This is in line with the earlier finding that a considerable number of students do not see negative numbers as Rational.

INTERVIEW RESPONSES

The interviewees had extreme difficulty with the notion of Irrational numbers. What emerged was :

1. The only one who attempted some explanation, described Irrational numbers as negative numbers.
2. They failed to draw a distinction between Irrational and Imaginary numbers.

## IMAGINARY NUMBERS

### ITEM 2

Omissions ( TABLE 11 A )

The average %-error in the case of the pilot study is 34 % and in the case of the present study, it is 41 % . This is lower than is the case for the other three sets of numbers, namely Real, Rational and Imaginary numbers. This could be due to the fact that Imaginary numbers are relatively easy to recognise. There is, further, no significant difference between the error frequencies of the three types of numbers in either of the two studies.

It is interesting to note though that, in both studies, the number with the highest error frequency is $\sqrt{-16}$. This could be due to a possible confusion with $\sqrt{16}$, which is a Rational number.

Wrong Selections ( TABLE 11 B )

The most significant result emerging from both studies is that the highest error frequencies occurred in the case of the four negative numbers, namely $-\sqrt{1/2}$ , $-2$, $-\sqrt{2}$ and $-1/2$. This is in line with the tendency, as pointed out under the discussion of Real numbers, not to regard negative numbers as Real.

## ITEM 3

Examples ( TABLE 12 A )

In view of the fact that Imaginary numbers are relatively easy to recognise, the success-rates in both studies are surprisingly low. In the present study only 34 % of the students had both correct whereas, in the pilot study, 54 % had both examples correct.

Non-examples ( TABLE 12 B )

Against the background of the fact that most of the numbers the students usually work with are not Imaginary, it is reasonable to expect a fairly high success-rate in this regard. Although this higher success-rate was attained in the pilot study ( 86 % ), only 56 % of the students managed to get both correct in the present study.

## ITEM 4

( TABLE 13A )

Correct responses

The success-rate of 36 % in the pilot study, is very similar to the 38 % achieved in the pilot study.

It needs to be taken into account that, in both studies, the definition of Imaginary numbers as " ... numbers that do not exist ... " was taken as correct although it is not correct in a strict mathematical sense. This decision is considered to be justified in terms of the probable frame of reference of the students concerned. Imaginary numbers are, for instance, the only one of the four sets of numbers under consideration, with which students would not have performed calculations and which they would not have had the opportunity to represent graphically. Within their framework, which is limited to Real numbers, Imaginary numbers do indeed, therefore, not exist.

Wrong responses

The 56 % of wrong responses returned in the present study, is somewhat higher than the 40 % obtained in the pilot study. Like in the case of the Irrational numbers, the most significant group of wrong responses are the 26 % who defined the Imaginary numbers as negative numbers.

## INTERVIEW RESPONSES

Imaginary numbers were described as :

1. Negative numbers
2. Numbers that don't make sense
3. Numbers without a definite square root

## ITEM 1

In view of the students' relatively poor understanding of the four number sets, it would be reasonable to expect that their understanding of the inter-relationaships between the number sets would not be good. Although this expectation is borne out in the case of the present study ( 16 % right ), the students were more successful in the pilot study (44% right). As a closer examination of the responses did not reveal any clear pattern, it was decided to limit the analysis of these responses to a simple indication of the number of correct and wrong responses.

## SECTION B :

## OPERATIONS WITH NEGATIVE NUMBERS AND WITH ZERO

### STRATEGY FOR ANALYSIS

Like in the case of Section A, the results from the pilot study are used for purposes of comparison. The discussion will, however, focus on the results of the present study, of which a more detailed analysis will be presented.

In marking the items, categories derived from the pilot study were used for coding the students' responses. These categories are :

Correct :

In terms of Greeno's criterion of "Correspondence ", as discussed in Chapter 2 of this study, a response is taken to be correct if it corresponds to the conceptions of experts. " Experts " are here taken to be the researcher, his colleagues as well as various textbook writers.

Wrong :

A wrong response is one which is at odds with the views of experts.

Trivial :

A trivial response is one which amounts to a simple restatement of the rule in different terms.

Omission :

An omission is a case in which no response is given.

The responses of the students will now be discussed in terms of the above categories.

The results of this section, referred to in the discussion below, are reflected in TABLE 14.

ITEM 1

In this item, the following question was posed to the students :

How would you convince someone that the following statement is true ?

A NEGATIVE NUMBER TIMES A POSITIVE NUMBER IS EQUAL TO A NEGATIVE NUMBER.

The responses were as follows :

Correct reponses

In the case of the pilot study, 8 % of the responses were correct. This result is similar to that found in the present study in which 10 % of the responses were correct. These correct responses were of two types :

Type 1 :

2 Students explained the rule by means of a model based on borrowing / owing money. This way of interpreting the rule therefore demonstrates an understanding in terms of an

everyday application. An example is :

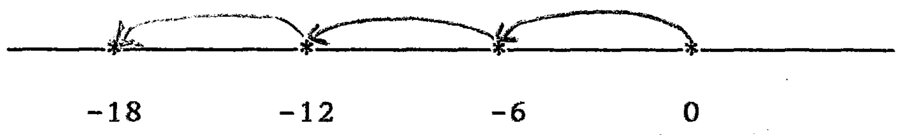
If you borrow R3 a month for 5 months, how much money would you owe ?

The answer is −R3 x 5 = −R15 , where debts are written as negative numbers.

Type 2 :

3 Students explained the rule by means of the number line.

An example is : $3 \times (-6)$




Both explanations amount to a concrete interpretation. The main difference is that the one is placed within an everyday context, while the other is placed within a more formal, de-contextualised mathematical setting.

There is, however, no apparent reason why one interpretation should be considered as representing a higher level of understanding than the other.

Wrong responses :

In the pilot study, 19 % of the responses were wrong. This is somewhat higher than in the case of the present study in which 8 % of the responses were wrong. The wrong responses given were all of different types. The lack of pattern and scope, leaves little room for further investigation.

Trivial responses :

The high percentage of trivial responses, 66 % in the pilot study and 82 % in the present study, would seem to imply that by far the majority of the students do not have a sense of the rule as being one that requires explanation. They seem to view the rule as a given. The only "explanation" in such a case is a simple restatement of the rule in different words. The following are examples of this :

Example 1 :

In multiplication, when you multiply a negative by a positive the answer is negative

or

Example 2 :

By showing an example, e.g. -2 x 3 = -6.

ITEM 2

The question posed in this case was :

How would you convince someone that the following statement is true ?

A NEGATIVE NUMBER ADDED TO A NEGATIVE NUMBER IS EQUAL TO A NEGATIVE NUMBER.

Correct Responses :

42 % of the students in the pilot study and 30 % in the present study gave correct responses. Two types of correct response were given :

Type 1

6 students gave a response based on a model of borrowing / owing money. They, in other words, demonstrated an understanding in terms of an everyday situation.

An example is :

If you borrowed R5 the first month and R5 the second month, you would owe R10 or  -R5 -R5 = -R10.

Type 2

9 of the students used the numberline to demonstrate an understanding of the rule eg.

(-2) + (-3) = -5

can be represented as

```
*---------*---------*---------*---------*---------*
-5        -4        -3        -2        -1        0
```

Wrong Responses

In both the pilot study (12 %) and the present study (4 %), the percentage of wrong responses was low.

The two wrong responses given are totally different and do not warrant further analysis.

Trivial Responses

The high percentage of trivial reponses would once again seem to imply that the rule is, to a large extent, being viewed as a convention which does not require an explanation beyond a simple restatement in different terms.

Examples of such responses are :

Example 1 :

It is a mathematical rule that negative plus negative equals negative

or

Example 2 :

$$(-2) + (-3) = -5$$

or

Example 3 :

When numbers having the same sign are involved in addition, the answer will always have that same sign.

Omissions

The fact that none of the students omitted a response could be an indication of the fact that none of them found the rule particularly puzzling. This could be due to the fact that addition is, generally, a simpler process than multiplication.

ITEM 3

The question posed in this case was :

How would you convince someone that the following statement is true ?

A NEGATIVE NUMBER TIMES A NEGATIVE NUMBER IS EQUAL TO A POSITIVE NUMBER.

Correct Responses :

Of the four items in Section B of the written test, item 4 yielded the lowest rate of success. The 0 % success of the pilot study is only just topped by the 2 % success achieved in the present study.

Wrong Responses :

The percentage of wrong responses in the pilot study ( 18 % ) is, once again, close to the percentage obtained in the present study ( 16 % ).

Although five types of wrong response were given, the only significant group is the four students who said that the negatives cancel each other.

Trivial responses

The percentage of trivial responses in both the pilot study (78 %) and the present study (76 %), is high.
This, again, would seem to imply that the students regard the rule as merely a convention.

Examples of such responses are :

Example 1 :

A negative times a negative is always positive.

or

Example 2 :

$$-3 \times -4 = 12$$

Omissions

Given the relatively abstract nature of this rule, the percentage of omissions in both the pilot study (4,7 %) and the present study (4 %) is low.

ITEM 4

The question posed in this case was :

How would you convince someone that the following statement is true ?

DIVISION BY NOUGHT IS NOT ADMISSIBLE

Correct Responses :

In keeping with the rest of this section, the percentage of correct responses was low, namely, 11 % in the pilot study and 4 % in the present study.

Both of the correct responses in the present study demonstrated an understanding of the rule by means of a sharing model. One student said :

If, for example, you have 10 sweets and you want to divide them amongst 0 children ( 10/0 ) you are not dividing anything.

The other one said :

If you have 16 sweets and you divide it by nought persons, you will not get an answer.

They, therefore both used a concrete representation set in the context of an everyday situation, rather than a more formal mathematical explanation.

Wrong Responses :

In the present study the percentage of wrong responses returned for this item ( 60 % ) is much more than the percentage of trivial responses returned ( 28 % ). This is in contrast to what was found in the case of the other items in this section.

Even in the case of the pilot study, the wrong responses (37 %), are not much less than the trivial responses (44 %).

The wrong responses were of seven different types, of which the three most significant types are briefly discussed below:

Type 1

14 of the students simply said that a number divided by nought is equal to nought.

Type 2

7 students said that you can't divide by 0 because 0 is nothing and to divide by nothing does not make sense.

Type 3

5 students gave the interesting response that division by 0 is not permissible because the calculator does not give an answer.

Trivial responses ( see Wrong responses above )

Omissions

The percentage of omissions in the present study ( 6 % ) is close to that in the pilot study ( 7 % ).

INTERVIEW RESPONSES

In this section, further probing did not reveal more than was already evident from the written test results.
In the cases where the written test responses were correct, the students could not provide alternative explanations.
Those students who had given wrong or trivial answers in the written test apparently also did not understand enough to utilise the hints that were given. Leads such as the following, for instance, did not help these students :

Could one perhaps explain the rule by means of the number line ?

or

What about a transaction involving money ?

## SECTION C : THE ORDER OF OPERATIONS

### Section C1 : The calculations : Items 1; 2; 3 and 4

Strategy for Analysis

The student responses were marked by focussing, not only on the answer, but, more specifically, on the procedure used to arrive at the answer. In this way, the reponses were classified into :
Correct, Wrong and Omissions.

Correct Responses :

As could be expected, given the nature of the items, the correct responses were all straightforward and of one type only ; there is, for instance only one way to follow the correct order in calculating 2 + 3 x 2.

Wrong Responses :

These were classified into different types in order to discern possible patterns in the ways in which students misunderstood the rules. A closer analysis of the wrong responses also provides possible leads to be followed up by means of the interviews.

The following is an analysis of the results of section C as reflected in TABLES 15, 16 and 17 :
( see Appendix II , p 111 - 112 )

ITEM 1 :

The question posed was :

CALCULATE THE ANSWER IN THE FOLLOWING CASE. SHOW YOUR STEPS :

2 + 3 x 2

Correct Responses

In view of the extreme simplicity of the item, the low success-rate in both studies, namely 54 % in the pilot study and 58 % in the present study, is surprising. It does seem to imply that a fair percentage of the students are not acquainted with the required convention. To gain a clearer insight into their thinking, it is, however, to the wrong reponses that we need to turn.

Wrong Responses

46 % of the students in the pilot study and 42 % in the present study gave wrong responses. An analysis of these reveals three types of wrong response, namely :

Type 1 :

17 of the students assumed that one should simply work from left to right, regardless of the kinds of operations involved, eg. :

2 + 3 x 2
= 5 x 2
= 10

Of the above, 6 found it necessary to facilitate the process by introducing their own brackets eg. :

2 + 3 x 2
= (2 + 3) x 2
= 5 x 2
= 10

Type 2

2 of the students demonstrated a belief that the order is optional eg.

| | | |
|---|---|---|
| 2 + 3 x 2 | or | 2 + 3 x 2 |
| = 2 x 6 | | = 5 x 2 |
| = 12 | | = 10 |

To investigate how strongly this view is held, it would be necessary to interview these students. The focus should be on their views concerning the fact that different orders produce different answers.

Type 3

2 of the wrong reponses were simply due to errors of calculation and are therefore non-diagnostic.

ITEM 2

The question posed was :

CALCULATE THE ANSWER IN THE FOLLOWING CASE. SHOW YOUR STEPS :

2 + 3 - 2

Correct Responses :

In view of the simplicity of the item, it is not surprising to find a high rate of succes in both studies ( 94 % ).

A weakness of this item, however, is that it does not discriminate between the correct convention and the " left to right " approach which emerged from item 1. This is due to the fact that it contains operations from only one of the " "pairs " of operations in the B O (DM) (AS) rule, namely, addition and subtraction. For better diagnostic value, one would have to turn to the other items in this section.

( As stated in Chapter 3, p41 of this study, this particular item could have been omitted without diminishing the study in any way )

Wrong Responses :

The wrong responses are all due to errors of calculation and are, therefore, non-diagnostic.

ITEM 3

The question posed in this case was :

CALCULATE THE ANSWER IN THE FOLLOWING CASE. SHOW YOUR STEPS.

$$2 + 3 \times 2 \div 2 - 1$$

Correct Responses :

In view of the relative complexity of the item ( it contains four different operations ) the lower rate of success, namely 27 % in the pilot study and 38 % in the present study, is in accordance with expectations.

Wrong Responses

The percentage of wrong responses are 60 % in the present study and 72 % in the pilot study.

The wrong responses are of five different types :

Type 1 :

More than half of the wrong reponses ( 19 out of 31 ) are of the type where the students simply worked from left to right. Like in the case of item 1, there are two sub-types :

12 students regarded the process as straightforward eg. :

$$
\begin{aligned}
& 2 + 3 \times 2 \div 2 - 1 \\
&= 5 \times 2 \div 2 - 1 \\
&= 10 \div 2 - 1 \\
&= 5 - 1 \\
&= 4
\end{aligned}
$$

5 students felt that it was necesseary to facilitate the process by first introducing brackets, eg. :

$$
\begin{aligned}
& 2 + 3 \times 2 \div 2 - 1 \\
&= (2 + 3) \times (2 \div 2) - 1 \\
&= 5 \times 1 - 1 \\
&= 5 - 1 \\
&= 4
\end{aligned}
$$

This interesting distinction needed to be followed up during the interviews.

Type 2 :

The second type of wrong response, given by 5 of the students, is one in which the student demonstrates a partial grasp of the rule by doing the calculations in a partially correct order, eg. :

$$\begin{aligned} & 2 + 3 \times 2 \div 2 - 1 \\ = \ & 2 + 3 \times 1 - 1 \\ = \ & 5 \times 0 \\ = \ & 0 \end{aligned}$$

Type 3 :

An interesting, and from the perspective of this researcher, unexpected, wrong reponse is the kind where the student treats everything following the division sign as a single term. 3 of the students simply transformed the entire expression into a fraction, with the " term " following the division sign acting as denominator. This amounts to effectively seeing the division sign as a line bracket, eg. :

$$\begin{aligned} & \frac{2 + 3 \times 2}{2 - 1} \\ = \ & \frac{2 + 6}{1} \\ = \ & 8 \end{aligned}$$

Type 4 :

The fourth type of wrong response echoes a type found in item 1 and item 3 where the student treats the order of operations as optional, using alternative sequences of steps and arriving at alternative answers. Although only one student gave such a response, it is worth noting because of the correspondence with the other items mentioned.

Type 5 :
3 of the wrong reponses were of the non-diagnostic type, where the wrong answer is due to an error of calculation.

ITEM 4

The question posed was :
CALCULATE THE ANSWER IN THE FOLLOWING CASE. SHOW YOUR STEPS.

$$(2 - 3) \times 12 \div 2 - 1$$

Correct Responses :
The percentage of 54 % correct responses in the present study is considerably higher than the 25 % achieved in the case of the pilot study.

Wrong Responses :
Only 44 % of the students in the present study got the item wrong, as opposed to 74 % in the case of the pilot study. The most significant type of wrong response in this case is that 6 students treated the division sign as a line bracket or as a sign that divides the expression into two distinct parts. Examples are :

Example 1 :

$$(2 - 3) \times 12 \div 4 - 1 + 3$$

$$= \frac{(2 - 3) \times 12}{4 - 1 + 3}$$

$$= \frac{-12}{6}$$

Example 2 :

$$(2 - 3) \times 12 \div 4 - 1 + 3$$

$$= -1 \times 12 \div 3 + 3$$

$$= -12 \div 6$$

$$= -2$$

Section C2 : Formulating the rule

This item required the students to state the rule in terms of which they had determined the order of operations.

Correct Responses

The 46 % success-rate achieved in the present study is very similar to the 49 % achieved in the pilot study.

In the case of the present study, however, a distinction is drawn between two kinds of correct answer :

Firstly, 20 of the students could state the "BODMAS" rule by simply writing down the operations in the required order eg. :

The operations are done in the following sequence :

Brackets

Of

Division

Multiplication

Addition

Subtraction

Secondly, 3 of the students demonstrated a deeper understanding of the rule by stating, in their own words, that division and multiplication on the one hand and addition and subtraction on the other, must be treated as an interchangable pair of operations. This means that multiplication can sometimes be done before division, depending on what occurs first in a given expression. The same goes for subtraction and addition eg. :

First the brackets

then of

then multiplication and division ( from left to right )

then addition and subtraction ( from left to right )

<u>Wrong Responses</u> :

In both studies, 50 % of the responses were wrong.

The wrong responses are of three types :

Type 1

9 students simply stated the order of operations wrongly.

Type 2

2 students said the order of operations is optional

Type 3

5 students said that, when doing a calculation containing more than one operation, one should first put in brackets.

Section C3 : Understanding the rule as a convention

The question posed in this case was :

COULD THE RULE HAVE BEEN DIFFERENT ?

WHY OR WHY NOT ?

Correct Responses :

Only "Yes" answers backed up with a plausible reason were taken as correct.

Because of the level of understanding required, it is not surprising to find that, in both studies, only 2 % of the responses were correct.

The correct response given in the present study is :

Yes, the rule could have been different. It could, for instance, have stated that we should all start with addition and subtraction. Our answers, however, would have been different.

Wrong responses :

"No" answers and "Yes" answers with wrong reasons were taken as wrong responses.

An interesting aspect, is that 20 of the 26 students who gave "No" responses, said the rule could not have been different because the rules of mathematics are fixed and cannot be different from what they are.

<u>INTERVIEW RESPONSE</u>

Only one of the students interviewed had applied the rule correctly in the written test. He, however, believed that division is always done before multiplication and addition always before subtraction as indicated by the acronym, BODMAS.

The other interviewees did not know the rule, apart from the "obvious" ones of working from left to right and "inserting one's own brackets".

When given the rule, two of them said that they could faintly recall something like that.

None of the interviewees seemed to understand that the rule is only a convention.

# CHAPTER 5

## 5.1 INTRODUCTION

This chapter summarises the findings of the study and makes recommendations based on the analysis of the data as reported in the previous chapter. A brief evaluation of the study is also given in which strengths are mentioned and limitations acknowledged. Suggestions for further research are also given.

## 5.2 SUMMARY OF FINDINGS AND RECOMMENDATIONS

### Section A

1. Generally, the students failed to draw clear distinctions between the four given sets of numbers, instead they tended to distinguish between two main categories of numbers, namely, numbers "which exist" and which, therefore "make sense" and numbers which "exist only in the imagination" and which, therefore, "do not make sense ".

The above seems to be due mainly to the fact that the students tend to interpret the names of the number sets in terms of the everyday meanings of the words instead of in terms of their mathematical meaning.

Real numbers are, for instance, seen as numbers which are "real", in the sense of " numbers which actually exist."

In contrast, Imaginary numbers are seen as numbers which exist only in the imagination.

In the same way, Rational numbers are regarded as "numbers

which make sense" whereas Irrational numbers "do not make sense".
This goes some way towards explaining why the students have such difficulty in distinguishing between Real and Rational numbers and also, on the other hand, between Irrational and Imaginary numbers. Thus we find that both Real and Rational numbers are largely regarded as positive numbers, and, more specifically, as positive whole numbers. On the other hand Imaginary and Irrational numbers are largely regarded as negative numbers, numbers without a square root or numbers that don't make sense.

It is plausible to assume that positive numbers, and more specifically, positive whole numbers, have meaning for students because they can attach concrete interpretations to such numbers. Moreover, these are numbers with which they would have worked from the beginning of their formal schooling careers and even beyond.

With regard to the other, more problematic numbers mentioned above, the following recommendations are made :

1. Pupils must be given the opportunity to use these numbers in meaningful situations set in different contexts, instead of just, as is often the case, being asked to work out lists of abstract, de-contextualised standard problems. Negative numbers should, for instance, be used

in the contexts of borrowing/owing money, temperature, direction, etc. in addition to their interpretation as points on the number line.

2. As many different representations as possible need to be used in the classroom. Fractions should, for instance, be represented by means of both discrete and continuous diagrams and also as points on the number line.

   It is especially with regard to their representation on the number line that fractions can be shown to be part of the Real numbers. The same is true for negative numbers.

   The different interpretations of Rational numbers, as analysed by Lesh, Landau and Hamilton ( 1983 ) and reported on p12 of this study, need to be developed in pupils.

3. Teachers should create more learning situations relevant to the classification of numbers. If pupils experience more such situations, they would acquire a better understanding of the meaning and relevance of the classification of numbers. They would also understand better why certain types of numbers belong to one set and not another.
   As has been pointed out in Chapter 4, the classification of the nature of the roots of a quadratic equation seems

to be the only such situation which most of these students have encountered at school.

Section B

1. The low percentage of correct responses ( 11 % on average) indicates that very few of the students know of a plausible way in which to explain why the rules work. An interesting aspect of the correct responses is, nevertheless, the fact that the successful students demonstrated different ways in which to understand the rules.

2. The high percentage of trivial responses has serious implications, not the least of which is the fact that so many students don't seem to have a sense of the rules as needing an explanation. This means that they regard fundamental principles of mathematics as being arbitrary, which in turn implies that they may regard mathematics as a loose and fractured collection of subject-matter. Such a subject perspective is a matter of concern, especially when exhibited by prospective teachers.

Recommendations in the above regards are :

Teachers should, at all times, make certain that they create learning situations in which pupils are given the opportunity to derive the rules for themselves. It would also be important to encourage pupils to share any alternative explanations they may be aware of. Pupils' understanding of basic, fundamental principles such as these should also be assessed at different stages of their school career. Having encountered these ideas at std. 6 level, they should be required to demonstrate an understanding of these rules in every standard up to std. 10.

Section C

1. If the non-diagnostic item 2 is not taken into consideration, many students ( 50 % on average ) do not know the rule governing the order of operations. This is quite alarming, especially in view of the fact that this rule is first done in std. 5 and then used throughout the pupils' and students' career, including in algebra.

2. The 50% of students who know the rule includes those who have a partial understanding of the rule as implying a fixed sequence of operations. They believe, for instance, that division should always be done before multiplication and that addition should always be done before subtraction. They, therefore, do not realise that multiplication can be done before division and vice-versa depending on which comes first, and that the same is true for addition and subtraction.

3. A matter for serious concern is that virtually none of the students could demonstrate an understanding of the rule as

a convention. They, for instance, did not realise that the rule could have been different on condition that it be applied consistently.

Recommendations :

The rule should not, as is often done, simply be given to the pupils to be learned by heart. The teacher should, rather, create learning situations in which the necessity for a universal rule becomes apparent to the pupil. Only after this is achieved should the universally agreed rule be given.

It is also apparent that the usual acronym, BODMAS, leads some pupils to believe that this is the one fixed sequence allowed. A slightly changed presentation like BO(DM)(AS) may be much more helpful in this regard.

General

Teachers should also become more aware of alternative modes of assessment, apart from giving standard types of "problems" which pupils can solve by means of algorithms which they may or may not properly understand. This study shows that the following types of items can lead to interesting and revealing results :

1. Selecting examples
2. Giving examples and non-examples
3. Defining concepts in your own words
4. Justifying or defending examples or non-examples
5. Explaining why a rule works

6. Distinguishing between rules which are principles and those which are merely conventions.

Also, in assessing such items, more than just correct or wrong answers are relevant, eg. :

Responses which are trivial may, for instance, yield different information from responses which are simply wrong. Different types of wrong answer need also to be considered. In the case of a particular item, there may be more than one response which is correct, in the sense of some responses demonstrating valid alternative understandings.

## 5.3 EVALUATION OF THE STUDY

### Generalisability

Survey studies are often large-scale with random sampling methods being used. As a result, one of the strong points of the survey method is, usually, generalisability.

This study, however, is a small-scale survey with a non-probability sampling method, namely convenience sampling, being used. This implies that the results are not statistically generalisable. The following, however, do offset this weakness to a certain extent :

Firstly, the study is, to a large extent, a replication of an earlier study conducted by this researcher.

Secondly, the results of these two studies are in very strong agreement, which enhances the validity of the results.

Further replication, in other settings, is, however also needed to further test the validity of the results.

## 5.4 SUGGESTIONS FOR FURTHER RESEARCH

As indicated above, this particular study would need to be replicated in other settings. In addition to this, other similar studies need to be conducted to investigate students' and pupils' understanding of other aspects of basic mathematics. In this regard, it is especially important to identify concepts and principles which are central to the whole mathematics curriculum and which pupils are likely to encounter right through their school careers.

Although this is important for all mathematics students and pupils, it is especially so for those who aspire towards becoming teachers. After all, if one is going to teach, your own understanding of the subject and your perspective on it need to be as sound as possible. A lot of research, therefore, needs to be focussed on the understanding and perspectives of teachers, both prospective and practising, so that remediation can also occur at that level.

As far as the research approach is concerned, survey methods, although currently less fashionable, and although inadequate in themselves, nevertheless remain important for uncovering trends which could be further investigated by other methods.

It is , therefore, important that other, more wholistic and participant methods, such as case studies and action research, be used to follow up the findings uncovered by means of surveys. This is all the more important in a subject like mathematics , where the focus is currently very much on the learning process in which the developing cognition of the learner is central.

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## APPENDIX 1

SURNAME .............................. CLASS ........

FIRST NAMES ..............................

### TEST OF UNDERSTANDING

Read the following carefully :

1. The purpose of the test is to find out to what extent you understand certain basic number concepts and procedures you encountered at school level. No marks will be awarded and the test will not have any bearing on your position at college. For the purposes of the research project however, your answers will be carefully analysed.

2. It is important that you answer each question as fully as possible even if you are not sure that your answer is correct.

3. Answer the questions in the space provided on the question paper. If this space is insufficient, carry on on the back of the page concerned.

4. Feel free to comment on the questions or on you own answer if the need should arise.

5. Please give your full co-operation. I am counting on it.

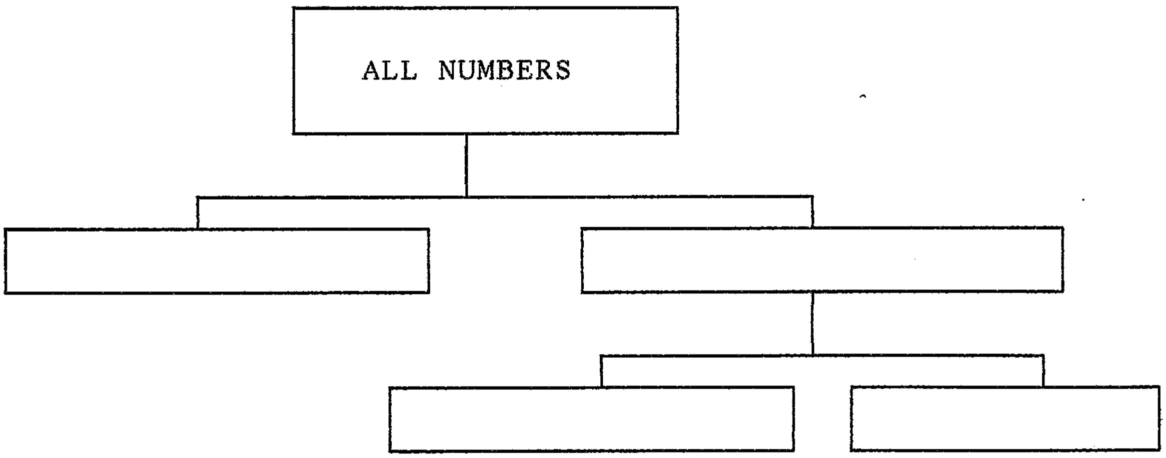
THANK YOU VERY MUCH

***

### SECTION A

### THE NUMBER SYSTEM

1. Place the terms IRRATIONAL, IMAGINARY, RATIONAL and REAL in the correct frames in the diagram below.



2. Complete the table below by selecting all the possible cases from the following set of numbers. Simply make a tick in the appropriate block.

| | IRRATIONAL | IMAGINARY | RATIONAL | REAL |
|---|---|---|---|---|
| 2 | | | | |
| -2 | | | | |
| 0 | | | | |
| $\sqrt{2}$ | | | | |
| $\sqrt{-2}$ | | | | |
| $-\sqrt{2}$ | | | | |
| 1/2 | | | | |
| -1/2 | | | | |
| $-\sqrt{1/2}$ | | | | |
| $\sqrt{-1/2}$ | | | | |
| $\sqrt{16}$ | | | | |
| $\sqrt{-16}$ | | | | |

3. Complete the following table without using the numbers given in (2) above :

| | GIVE TWO EXAMPLES | GIVE TWO NON-EXAMPLES |
|---|---|---|
| IRRATIONAL NUMBERS | | |
| IMAGINARY NUMBERS | | |
| RATIONAL NUMBERS | | |
| REAL NUMBERS | | |

4. Explain, in your own words, what each of the following terms mean to you :

4.1 IRRATIONAL NUMBERS

## 4.2 RATIONAL NUMBERS

## 4.3 IMAGINARY NUMBERS

## 4.4 REAL NUMBERS

## SECTION B

## OPERATIONS WITH NEGATIVE NUMBERS AND WITH ZERO

How would you convince someone that the following statements are TRUE ?
Use examples, as far as possible.

1. A negative number times a positive number is equal to a negative number.

2. A negative number added to a negative number is equal to a negative number.

3. A negative number times a negative number is equal to positive number.

4. Division by nought is not permissible

## SECTION C

## THE ORDER OF OPERATIONS

C1 Calculate the answer in each of the following cases. Show your steps.

1. 2 + 3 x 2

2. $2 + 3 - 2$

3. $2 + 3 \times 2 \div 2 - 1$

4. $( 2 - 3 ) \times 12 \div 4 - 1 + 3$

<u>C2</u> State the rule in terms of which you determined the order of operations.

<u>C3</u> Could this rule have been different ?
Why or why not ?

***

# APPENDIX II

## SAMPLE STATISTICS

### TABLE 1 A

| | SAMPLE SIZE | MALE | FEMALE | ex - DEC | ex - DET |
|---|---|---|---|---|---|
| PRESENT STUDY % | 50 | 23 | 27 | 39 | 11 |
| *PILOT STUDY %* | *106* | *56* | *50* | *106* | - |

### TABLE 1 B

### SYMBOL DISTRIBUTION

| | A | B | C | D | E | F | FF | G | H |
|---|---|---|---|---|---|---|---|---|---|
| PRESENT STUDY | 2 | 3 | 9 | 17 | 12 | 4 | - | 2 | 1 |
| *PILOT STUDY* | *3* | *3* | *15* | *18* | *31* | *21* | *3* | *7* | *2* |

## ITEM 2

## REAL NUMBERS

### TABLE 2 A : Omissions

| Numbers | $-\sqrt{\frac{1}{2}}$ | $-\sqrt{2}$ | $\sqrt{2}$ | $-\frac{1}{2}$ | $-2$ | $\frac{1}{2}$ | $\sqrt{16}$ | 0 | 2 | Ave% |
|---|---|---|---|---|---|---|---|---|---|---|
| PRESENT STUDY % | 98 | 96 | 92 | 92 | 90 | 56 | 46 | 40 | 26 | 71 |
| *PILOT STUDY %* | *78* | *79* | *68* | *58* | *45* | *38* | *48* | *28* | *23* | *52* |

### TABLE 2 B : Wrong Selections

| Numbers | $\sqrt{-16}$ | $\sqrt{-\frac{1}{2}}$ | $\sqrt{-2}$ | Ave % |
|---|---|---|---|---|
| PRESENT STUDY % | 10 | 4 | 0 | 5 |
| *PILOT STUDY %* | *8* | *6* | *5* | *6* |

## ITEM 3

## REAL NUMBERS

### TABLE 3 A : EXAMPLES

| | BOTH CORRECT | ONE CORRECT | BOTH WRONG | BOTH OMITTED |
|---|---|---|---|---|
| PRESENT STUDY % | 82 | 18 | 0 | 0 |
| *PILOT STUDY %* | *97* | *0* | *1* | *1* |

### TABLE 3B : NON-EXAMPLES

| | BOTH CORRECT | ONE CORRECT | BOTH WRONG | BOTH OMITTED |
|---|---|---|---|---|
| PRESENT STUDY % | 6 | 6 | 64 | 14 |
| *PILOT STUDY %* | *27* | *12* | *47* | *7* |

## ITEM 4

### TABLE 4 A : DEFINITIONS

| | "CORRECT" Numbers exist ; Value can be found; All except Imaginary | Whole; Positive Whole; Natural; Counting | Perfect square ; Square Number ; | Other | Nonsen-sical | Omission |
|---|---|---|---|---|---|---|
| f | 4 | 14 | 15 | 10 | 3 | 3 |
| % | 8 | 28 | 30 | 20 | 6 | 6 |

### *PILOT STUDY*
### *TABLE 4 B : DEFINITIONS*

| | *CORRECT* | *WRONG* | *TRIVIAL* | *OMISSION* |
|---|---|---|---|---|
| % | *29* | *59* | *8* | *4* |

## RATIONAL NUMBERS

### TABLE 5 A : Omissions

| Numbers | $-\frac{1}{2}$ | $-2$ | $0$ | $\frac{1}{2}$ | $2$ | $\sqrt{16}$ | Ave% |
|---|---|---|---|---|---|---|---|
| PRESENT STUDY % | 96 | 76 | 74 | 64 | 56 | 24 | 65 |
| *PILOT STUDY %* | *70* | *61* | *51* | *58* | *49* | *23* | *52* |

### TABLE 5 B : Wrong Selections

| Numbers | $\sqrt{2}$ | $-\sqrt{2}$ | $\sqrt{-16}$ | $-\sqrt{\frac{1}{2}}$ | $\sqrt{-\frac{1}{2}}$ | $\sqrt{-2}$ | |
|---|---|---|---|---|---|---|---|
| PRESENT STUDY % | 24 | 8 | 4 | 4 | 4 | 2 | 8 |
| *PILOT STUDY %* | *31* | *9* | *6* | *5* | *1* | *3* | *9* |

## ITEM 3

### RATIONAL NUMBERS

TABLE 6A : EXAMPLES

| | BOTH CORRECT | ONE CORRECT | BOTH WRONG | BOTH OMITTED |
|---|---|---|---|---|
| PRESENT STUDY % | 82 | 16 | 0 | 2 |
| *PILOT STUDY %* | *92* | *4* | *2* | *1* |

TABLE 6B : NON-EXAMPLES

| | BOTH CORRECT | ONE CORRECT | BOTH WRONG | BOTH OMITTED |
|---|---|---|---|---|
| PRESENT STUDY % | 16 | 12 | 58 | 14 |
| *PILOT STUDY %* | *50* | *14* | *31* | *2* |

## ITEM 4

TABLE 7 A : DEFINITIONS

| | In terms of square roots | | In terms of fractions | Direct reference to Delta | Whole ; Even; Counting | Real or Imaginary |
|---|---|---|---|---|---|---|
| | Correctly | Wrongly | | | | |
| f | 4 | 11 | 2 | 2 | 12 | 5 |
| % | 8 | 22 | 4 | 4 | 24 | 10 |

*PILOT STUDY*
*TABLE 7 B : DEFINITIONS*

| | *CORRECT* | *WRONG* | *TRIVIAL* | *OMISSION* |
|---|---|---|---|---|
| *%* | *26* | *70* | *0* | *4* |

## IRRATIONAL NUMBERS

### TABLE 8 A : Omissions

| Numbers | $\sqrt{2}$ | $-\sqrt{2}$ | $-\sqrt{\frac{1}{2}}$ | Ave % |
|---|---|---|---|---|
| PRESENT STUDY % | 48 | 46 | 40 | 45 |
| *PILOT STUDY %* | *54* | *59* | *53* | *55* |

### TABLE 8 B : Wrong Selections

| Numbers | $-\frac{1}{2}$ | $\sqrt{-2}$ | $\sqrt{-\frac{1}{2}}$ | $\sqrt{-16}$ | $-2$ | $\frac{1}{2}$ | 2 | 0 | $\sqrt{16}$ | Ave% |
|---|---|---|---|---|---|---|---|---|---|---|
| PRESENT STUDY % | 50 | 38 | 38 | 38 | 34 | 24 | 12 | 10 | 6 | 28 |
| *PILOT STUDY %* | *27* | *25* | *23* | *19* | *23* | *17* | *5* | *10* | *13* | *18* |

## ITEM 3

## IRRATIONAL NUMBERS

### TABLE 9A : EXAMPLES

| | BOTH CORRECT | ONE CORRECT | BOTH WRONG | BOTH OMITTED |
|---|---|---|---|---|
| PRESENT STUDY % | 8 | 14 | 74 | 0 |
| *PILOT STUDY %* | *36* | *12* | *48* | *1* |

### TABLE 9B : NON-EXAMPLES

| | BOTH CORRECT | ONE CORRECT | BOTH WRONG | BOTH OMITTED |
|---|---|---|---|---|
| PRESENT STUDY % | 68 | 10 | 6 | 12 |
| *PILOT STUDY %* | *85* | *8* | *3* | *2* |

## ITEM 4

### TABLE 10A : DEFINITIONS

| | "Correct" No clear square root or Don't work out | Negative numbers | Fractions | Undefined or Not on Number line | Not used often | Omis-sions and Trivial | Other |
|---|---|---|---|---|---|---|---|
| f | 11 | 14 | 3 | 4 | 2 | 4 | 14 |
| % | 22 | 28 | 6 | 8 | 4 | 8 | 28 |

### *PILOT STUDY*

### *TABLE 10 B : DEFINITIONS*

| | *CORRECT* | *WRONG* | *TRIVIAL* | *OMISSION* |
|---|---|---|---|---|
| *%* | *23* | *77* | *1* | *4* |

## IMAGINARY NUMBERS

### TABLE 11 A : Omissions

| Numbers | $\sqrt{-16}$ | $\sqrt{-\frac{1}{2}}$ | $\sqrt{-2}$ | |
|---|---|---|---|---|
| PRESENT STUDY % | 48 | 42 | 34 | 41 |
| *PILOT STUDY %* | *37* | *31* | *34* | *34* |

### TABLE 11 B : Wrong Selections

| Numbers | $-\sqrt{\frac{1}{2}}$ | -2 | $-\sqrt{2}$ | $-\frac{1}{2}$ | $\sqrt{2}$ | 0 | $\frac{1}{2}$ | 2 | $\sqrt{16}$ | Ave% |
|---|---|---|---|---|---|---|---|---|---|---|
| PRESENT STUDY % | 46 | 42 | 42 | 42 | 14 | 14 | 12 | 4 | 2 | 24 |
| *PILOT STUDY %* | *32* | *13* | *29* | *20* | *10* | *3* | *5* | *0* | *2* | *13* |

## ITEM 3

## IMAGINARY NUMBERS

### TABLE 12A : EXAMPLES

| | BOTH CORRECT | ONE CORRECT | BOTH WRONG | BOTH OMITTED |
|---|---|---|---|---|
| PRESENT STUDY % | 34 | 10 | 46 | 2 |
| *PILOT STUDY %* | *54* | *10* | *31* | *1* |

### TABLE 12B : NON-EXAMPLES

| | BOTH CORRECT | ONE CORRECT | BOTH WRONG | BOTH OMITTED |
|---|---|---|---|---|
| PRESENT STUDY % | 56 | 8 | 0 | 12 |
| *PILOT STUDY %* | *86* | *1* | *6* | *4* |

## ITEM 4

### TABLE 13 A : DEFINITIONS

| | Correct : Square root of a negative number | Numbers that don't exist | Negative numbers | Numbers without a square root | Other | Omissions |
|---|---|---|---|---|---|---|
| f | 11 | 7 | 13 | 6 | 9 | 3 |
| % | 22 | 14 | 26 | 12 | 18 | 6 |

### *PILOT STUDY*

### *TABLE 13 B : DEFINITIONS*

| *CORRECT* | *WRONG* | *TRIVIAL* | *OMISSION* |
|---|---|---|---|
| *38* | *44* | *14* | *4* |

## SECTION B

## OPERATIONS WITH NEGATIVE NUMBERS AND WITH ZERO

### TABLE 14

#### ITEM 1

| | CORRECT | WRONG | TRIVIAL | OMISSION |
|---|---|---|---|---|
| PRESENT STUDY % | 10 | 8 | 82 | 0 |
| *PILOT STUDY %* | *8* | *22* | *66* | *4* |

#### ITEM 2

| | CORRECT | WRONG | TRIVIAL | OMISSION |
|---|---|---|---|---|
| PRESENT STUDY % | 30 | 4 | 66 | 0 |
| *PILOT STUDY %* | *42* | *15* | *42* | *2* |

#### ITEM 3

| | CORRECT | WRONG | TRIVIAL | OMISSION |
|---|---|---|---|---|
| PRESENT STUDY % | 2 | 18 | 76 | 4 |
| *PILOT STUDY %* | *0* | *17* | *78* | *5* |

#### ITEM 4

| | CORRECT | WRONG | TRIVIAL | OMISSION |
|---|---|---|---|---|
| PRESENT STUDY % | 4 | 60 | 28 | 6 |
| *PILOT STUDY %* | *11* | *38* | *44* | *7* |

## SECTION C

## THE ORDER OF OPERATIONS

### C1 : The Calculations
### TABLE 15

#### ITEM 1

| | CORRECT | WRONG | OMISSION |
|---|---|---|---|
| PRESENT STUDY % | 58 | 42 | 0 |
| *PILOT STUDY %* | *54* | *46* | *0* |

#### ITEM 2

| | CORRECT | WRONG | OMISSION |
|---|---|---|---|
| PRESENT STUDY % | 94 | 6 | 0 |
| *PILOT STUDY %* | *94* | *5* | *1* |

#### ITEM 3

| | CORRECT | WRONG | OMISSION |
|---|---|---|---|
| PRESENT STUDY % | 38 | 60 | 2 |
| *PILOT STUDY %* | *27* | *72* | *1* |

#### ITEM 4

| | CORRECT | WRONG | OMISSION |
|---|---|---|---|
| PRESENT STUDY % | 54 | 44 | 2 |
| *PILOT STUDY %* | *25* | *74* | *1* |

## C2 : Formulating the Rule

TABLE 16

| | CORRECT | WRONG | OMISSION |
|---|---|---|---|
| PRESENT STUDY % | 46 | 50 | 4 |
| *PILOT STUDY %* | *49* | *50* | *1* |

## C3 : The Rule as a Convention

TABLE 17

| | CORRECT | WRONG | OMISSION |
|---|---|---|---|
| PRESENT STUDY % | 2 | 92 | 6 |
| *PILOT STUDY %* | *2* | *89* | *9* |

## APPENDIX III

## INTERVIEW SCHEDULE

### INTRODUCTORY QUESTIONS

1. I see that you attended school at ........ How did you find it there ?

2. How did you find mathematics as a subject at school ?

3. Would you say your mathematics teachers had a solid background in the subject ?

4. How would you describe their teaching style ?

### SECTION A

1. Why did you select / omit this number as an example of a Rational / Irrational / Real / Imaginary number ?

2. Is the number .... an example / non- example of a Rational / Irrational / Real / Imaginary number ?
   If it is, why was the similar number ..... omitted ?

3. If your definition of Rational / Irrational / Real / Imaginary numbers is ........., why did you select / omit the number ........ ?

4. What do you mean by the term ........... ?

SECTION B

1. Can you represent the rule diagrammatically /
   Can you show it on the number line ?

2. Can you explain it terms of an everyday situation ?
   How about using a financial transaction ?

3. I see that you have explained it correctly. Do you know an alternative explanation ?

SECTION C

1. Do you know why your answer has been marked wrong ?
   What do you think the correct answer is ?

2. Would it have made a difference if the order of ..... and .......... had been changed ?

3. Here is a further calculation ........ Explain how you would do it.

4. You define the rule as ............. , yet you did this item in a different way. Can you explain why ?

5. Have you heard of the BODMAS rule which governs the order of operations ?

6. Would it have created problems if, instead of the BODMAS rule, a totally different sequence had been universally agreed upon ?

7. In Section B we saw that the rules for operations with negative numbers and zero can be derived, and concretely illustrated. Can the same be done for the BODMAS rule ?