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Aspects of Renormalisation in some Quantum Field Theories

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Abstract

Renormalisation is an important aspect of Quantum Field Theory. It is used to create physically meaningful theories and some major developments took place in the 1970's and onwards. We consider Renormalisation in its application to the theories of ψ^4 , Quantum Electrodynamics, Quantum Chromodynamics and the Background Field Method. Feynman diagrams are used to illustrate many of the concepts.

KEYWORDS: Renormalisation, Functional, Feynman diagram, Quantum Electrodynamics (QED), Quantum Chromodynamics (QCD).

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Preface

This thesis contains four chapters which lead progressively to the subject of renormalisation which is mainly referred to in chapters 3 and 4.

Chapter one introduces much of the terminology which is used to describe such fundamental concepts as the path integral and the generating function. From this we define the free particle Greens functions and give the diagrammatic representation of the Feynman rules. ψ^4 theory is then introduced as an example of an interacting field and the 2 and 4 point functions are calculated. Emphasis is placed on the use of Feynman diagrams which summarise much of the mathematical notation. The end of the chapter deals with fermions and functional methods and the use of Grassmann algebra to describe anticommuting fields.

Chapter two deals with gauge field theories and begins by describing the Fadeev-Popov method and the motivation for its introduction. The Feynman rules for gauge theories are introduced and the ghost and gauge field propagators are calculated. Reference is also made to the coupling terms which arise from the cubic and quartic terms in the Lagrangian. Next the self energy operator and vertex function are considered followed by an introduction to the Ward Takahashi identities which describe a relationship between the vertex functions and the propagators and which are important in the renormalisation of QED. The renormalisation of QCD requires the use of the BRS transformation and the Slavnov-Taylor Identities and the end of the chapter contains two sections devoted to this.

Chapter three is concerned with the renormalisation of ψ^4, QED and QCD. Renormalisation is required so that the divergences occuring in the Feynman diagrams may be eliminated. The chapter begins by describing these divergences for ψ^4 theory and by describing the technique of dimensional regularisation which enables these divergences to be represented by poles in the dimension plane. It is then shown how these divergences may be removed by making suitable adjustments to the Lagrangian. An investigation is made into how the Greens functions change as the scale μ varies and this leads to a section on the Renormalisation Group Equation. Having completed our discussion on ψ^4 we turn our attention to QED. As before the graphs containing primitive divergences are examined but the renormalisability of QED is shown by means of the Ward Identities which were derived in Chapter 2. Following this section is a brief discussion of the BPHZ renormalisation scheme. This technique is illustrated by placing boxes around the primitive divergences. Finally we look at the one-loop structure of Yang Mills Theory (QCD). The approach here is similar to that of the previous discussion on QED. We examine the primitive divergences of the theory and use the Taylor Slavnov identities to prove renormalisability.

Chapter four deals with a more specialised area of Quantum Field Theory called the Background Field Method. The chapter begins by introducing standard quantities such as the functional $\tilde{Z}[J,A]$, $\tilde{\Gamma}[0,A]$. Unlike the conventional approach these quantities also depend on the background field A. Feynman rules are established and we check the mathematical expression for a particular

Feynman diagram. The chapter ends with a brief discussion on renormalisation.

Contents

Chapter 1

Introduction to the Path Integral Formulation

1.1 Path Integrals

Path integrals are used in the quantisation of classical systems. The path integral approach to Quantum Field Theory was developed by Feynman who continued the earlier work of Dirac in the 1950's. In the classical approach to quantisation one forms the classical action by integrating over the Lagrangian density $\tilde{\mathcal{L}}$. The Lagrange equations of motion may be derived by the Hamilton variational principle. One then sets up the Hamiltonian which is the starting point in the method of canonical quantisation.

The path integral approach to Quantum Mechanics is based on the notion of a time evolution operator (or propagator) $K(x_f t_f; x_i t_i)$ which relates the wavefunction $\Psi(x_i, t_i)$ at an initial time t_i to the wavefunction $\Psi(x_f, t_f)$ at a later time t_f. Following the approach of Frampton [10] we start by considering the one-dimensional Schroedinger equation for a particle of mass m moving in a potential $V(x)$. The equation governing the wave function (setting $\hbar = 1$) is

$$(-\frac{1}{2m}\frac{\partial^2}{\partial x^2} + V(x))\Psi(x,t) = i\frac{\partial}{\partial t}\Psi(x,t)$$

Let us suppose that the wavefunction at an inital time $t = t_i$ is given by $\Psi(x, t_i) = f(x)$. We would like to find the evolution operator K such that

$$\Psi(x_f, t_f) = \int dx K(x_f, t_f; x, t_i) f(x)$$

(This equation results from Huygens' principle) It is clear that if

$$K(x_f, t_f; x, t_i) = \delta(x_f - x)$$

then

$$\int dx K(x_f, t_f; x, t_i) f(x) = f(x_f) = \Psi(x_f, t_i)$$

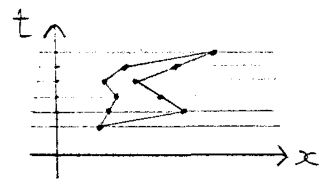
We now have to consider the time component. The trick is to express K as a path integral. We do this by dividing the time interval $(t_f - t_i)$ into $(n+1)$ parts such that

$$\begin{aligned} t_0 &= t_i \\ t_1 &= t_i + \epsilon \\ &\vdots \\ t_n &= t_i + n\epsilon \\ &\vdots \\ t_f &= t_i + (n+1)\epsilon \end{aligned}$$

where

$$\epsilon = \frac{t_f - t_i}{n+1}$$

We may represent typical paths from $(x_i t_i)$ to $(x_f t_f)$ as Markov chains.



The Lagrangian is

$$\begin{aligned} \mathcal{L} &= T - V \\ &= \frac{1}{2}\dot{x}^2 - V(x) \end{aligned}$$

We can therefore write the action as a discrete sum

$$\begin{aligned} S &= \int_{t_i}^{t_f} dt \mathcal{L} \\ &= \sum_{k=1}^{n+1} \epsilon\left(\frac{m\dot{x}_k^2}{2} - V(x_k)\right) \\ &= \sum_{k=1}^{n+1} \epsilon\left(\frac{m}{2\epsilon^2}(x_k - x_{k-1})^2 - V(x_k)\right) \end{aligned}$$

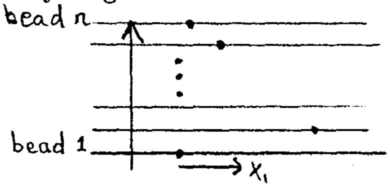
Note that $\dot{x}_k$ is the gradient of a particular segment and is given by

$$\frac{x_k - x_{k-1}}{t_k - t_{k-1}} = \frac{1}{\epsilon}(x_k - x_{k-1})$$

We now consider the following expression which is a sum over all possible paths.

$$\int_{-\infty}^{\infty} dx_1 \ldots dx_n e^{iS} \tag{1.1}$$

We may think of this integral in terms of the following illustration. Suppose we have n parallel infinitely long horizontal wires each of which contains a bead.



Each of the beads can move freely along the respective wire to which it is attached. If we associate x_1 with the position of bead 1, x_2 with bead 2 etc - then a particular path (or positioning of the beads) may be defined by an n-dimensional vector $(x_1, x_2, \ldots, x_n)$. A sum over all possible paths is therefore associated with the expression

$$\int_{-\infty}^{\infty} dx_1 \ldots dx_n$$

e^{iS} is simply a weighting factor dependent upon the particular path being considered. $S : R^n \to R$ is the action functional which associates a real number with each path. In the limit $\epsilon \to 0$ we find that the expression given by (1.1) vanishes - it is therefore necessary to introduce a normalisation factor; we <u>define</u>

$$K(x_f, t_f; x_i, t_i) = \begin{cases} \lim_{n\to\infty} (\frac{1}{N(\epsilon)})^{n+1} \int dx_1 \ldots dx_n e^{iS} & (t_f > t_i) \\ 0 & (t_f < t_i) \end{cases}$$

It can be shown that

$$N(\epsilon) = (\frac{2\pi i \epsilon}{m})^{\frac{1}{2}}$$

Using a shorthand notation we have

$$K(x_f, t_f; x_i, t_i) = \int \mathcal{D}x e^{iS} \qquad (t_f > t_i)$$

We now find the equation which K satisfies. We consider the time $t = t_f + \epsilon$ when the position is x_{n+2} Then

$$\begin{aligned} K(x_{n+2}, t_f + \epsilon; x_i, t_i) &= \int K(x_{n+2}, t_f + \epsilon; x_{n+1}, t_f) K(x_{n+1}, t_f; x_i, t_i) dx_{n+1} \\ &= \frac{1}{N(\epsilon)} \int dx_{n+1} e^{(i\epsilon[\frac{m}{2\epsilon^2}(x_{n+2}-x_{n+1})^2 - V(x_{n+2})])} K(x_{n+1}, t_f; x_i, t_i) \\ &= \frac{1}{N(\epsilon)} \int_{-\infty}^{\infty} d\eta e^{(\frac{im}{2\epsilon}\eta^2 - i\epsilon V(x_{n+2}))} K(x_{n+2} + \eta, t_f; x_i, t_i) \end{aligned}$$

where $x_{n+2} = x_{n+1} - \eta$. Expanding K as a Taylor series (setting $x_{n+2} = x$ and temporarily ignoring $t_f; x_i, t_i$) gives

$$K(x + \eta) = K(x) + \eta \frac{\partial K}{\partial x}(x) + \frac{1}{2}\eta^2 \frac{\partial^2 K}{\partial x^2}(x) + \ldots$$

Using the Gaussian integrals

$$\begin{aligned}
\int_{-\infty}^{\infty} d\eta e^{\frac{ia}{\epsilon}\eta^2} &= \sqrt{\frac{i\pi\epsilon}{a}} \\
\int_{-\infty}^{\infty} d\eta\eta e^{\frac{ia}{\epsilon}\eta^2} &= 0 \\
\int_{-\infty}^{\infty} d\eta\eta^2 e^{\frac{ia}{\epsilon}\eta^2} &= \frac{i\epsilon}{2a}\sqrt{\frac{i\pi\epsilon}{a}}
\end{aligned}$$

and the Taylor expansion

$$e^{-i\epsilon V(k)} = 1 - i\epsilon V(k) + O(\epsilon^2)$$

one finds

$$\begin{aligned}
K(x_{n+2}, t_f + \epsilon; x_i, t_i) &= (\frac{1 - i\epsilon V(x)}{N(\epsilon)})[\int_{-\infty}^{\infty} d\eta(K(x)e^{\frac{im\eta^2}{2\epsilon}}) \\
&+ \frac{\partial K}{\partial x}(x)\int_{-\infty}^{\infty} d\eta(\eta e^{\frac{im\eta^2}{2\epsilon}}) + \frac{\partial^2 K}{\partial x^2}(x)\int_{-\infty}^{\infty} d\eta(\eta^2 e^{\frac{im\eta^2}{2\epsilon}})] \\
&= \frac{1}{N(\epsilon)}(1 - i\epsilon V(x))\sqrt{\frac{2i\pi\epsilon}{m}}[K(x, t_f; x_i, t_i) \\
&+ \frac{i\epsilon}{2m}\frac{\partial^2 K}{\partial x^2}(x, t_f; x_i, t_i)] + O(\epsilon^2) \qquad (1.2)
\end{aligned}$$

We also know that

$$K(x, t_f + \epsilon; x_i, t_i) = K(x, t_f; x_i, t_i) + \epsilon\frac{\partial}{\partial t_i}K(x, t_f; x_i, t_i) + O(\epsilon^2) \qquad (1.3)$$

It follows by equating (1.3) and (1.2) that

$$(\frac{i\epsilon}{2m}\frac{\partial^2}{\partial x^2} - i\epsilon V(x))K(x, t_f; x_i, t_i) = \epsilon\frac{\partial}{\partial t_i}K(x, t_f; x_i, t_i)$$

or

$$(-\frac{1}{2m}\frac{\partial^2}{\partial x^2} + V(x))K(x, t_f; x_i, t_i) = i\frac{\partial}{\partial t_i}K(x, t_f; x_i, t_i)$$

Notice how our choice of $N(\epsilon)$ simplifies (1.2). We have therefore shown that K satisfies the Schrodinger equation. This result demonstrates that if $\Psi(x_i, t_i)$ satisfies the Schrodinger equation then so also does

$$\Psi(x_f, t_f) = \int dx K(x_f, t_f; x, t_i)\Psi(x, t_i)$$

The above discussion is a useful introduction to path integrals. One of the key results is equation (1.1) which demonstrates how the transition amplitude

$< x_i t_i | x_f t_f >$ may be expressed in terms of a path integral. In relativistic field theory the number of degrees of freedom is infinite. A field generalises the concept of a generalised coordinate. We consider not just several particles but a continuum of particles. If there is one coordinate for each point $\mathbf{x}$ of space we may write the field as $\psi(\mathbf{x}, t)$. In the case of a single degree of freedom (as discussed earlier) we have with $m = 1$

$$K(x_f, t_f; x_i, t_i) = \lim_{n\to\infty} \left(\frac{1}{2i\pi\epsilon}\right)^{\frac{n+1}{2}} \int dq_1 \ldots dq_n e^{iS} \tag{1.4}$$

$$\begin{aligned} S &= \sum_{k=1}^{n+1} \epsilon\left[\frac{1}{2\epsilon^2}(q_k - q_{k-1})^2 - V(q_k)\right] \\ &= \sum_{k=1}^{n+1} \epsilon\left[\frac{1}{2}\dot{q}_k^2 - V(q_k)\right] \end{aligned}$$

We now use the result that

$$\frac{1}{\sqrt{2i\pi\epsilon}} e^{\frac{1}{2}i\epsilon\dot{q}_k^2} = \int \frac{dp_k}{2\pi} e^{i\epsilon(p_k\dot{q}_k - \frac{1}{2}p_k^2)}$$

(which can be easily shown using a standard Gaussian integral) Substituting this into (1.4) gives

$$\begin{aligned} K &= \int \frac{\mathcal{D}p\mathcal{D}q}{2\pi} e^{i\int_{-\infty}^{\infty}(p\dot{q} - (\frac{1}{2}p^2 + V))dt} \qquad (1.5) \\ &= \lim_{n\to\infty} \int \prod_{i=1}^{n} \prod_{j=1}^{n+1} \frac{dp_j dq_i}{2\pi} e^{i\sum_{j=1}^{n+1}[p_j(q_j - q_{j-1}) - H(p_j, \frac{1}{2}(q_j + q_{j-1}))(t_j - t_{j-1})]} \end{aligned}$$

where $H(p, q) = \frac{1}{2}p^2 + V(q)$ is the Hamiltonian.
K may be expressed in terms of the Hamiltonian or alternatively in terms of the Lagrangian. Both formulae are equivalent. We have

$$\begin{aligned} K(x_f, t_f; x_i, t_i) &= \int \frac{\mathcal{D}q\mathcal{D}p}{2\pi} e^{i\int(p\dot{q} - H)dt} \\ &= \int \mathcal{D}q e^{i\int \mathcal{L}(q,\dot{q})dt} \\ &= \int \mathcal{D}q e^{iS} \qquad (1.6) \end{aligned}$$

The notion of the path integral is extended to field theory by allowing the number of variables to approach infinity. We consider the case of a single boson field. In this case the role of a coordinate is played by the field operator $\hat{\psi}(x)|\psi> = \psi(\mathbf{x})|\psi>$ where $\psi(\mathbf{x})$ is a function of the spatial coordinates. We now consider the transition amplitude $< \psi_2 t_2 | \psi_1 t_1 >$ from one field eigenstate at time t_1 to another eigenstate at t_2. Just as in the previous discussion for discrete quantum mechanics we may write this amplitude as a path integral.

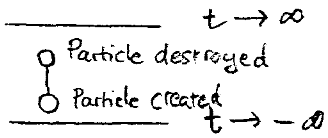
$$< \psi_2 t_2 | \psi_1 t_1 >= N \int \mathcal{D}\psi e^{i \int \mathcal{L} d^4 x}$$

$$(\psi_2(\mathbf{x}) \equiv \psi(\mathbf{x}, t_2) \qquad \psi_1(\mathbf{x}) \equiv \psi(\mathbf{x}, t_1))$$

Here we have made the substitution $\mathcal{D}q \rightarrow \mathcal{D}\psi$ and instead of dividing time up into segments, we divide space and time up, and Minkowski space is broken down into 4 dimensional cubes of volume δ^4 in each of which ψ is taken to be constant.

1.2 Coupling to External Sources

In quantum field theory it is useful to introduce a source field in the path integral. It has been stated (Guidry [13] 1991) that "all dynamical information about a system may be extracted by studying the response of the vacuum to an arbitrary external source." A source field represents the act of creating a particle. The act of destroying a particle may be represented by a sink. The boundary conditions of the problem may be represented in the following diagram.



As $t \rightarrow -\infty$ we assume that the system is in its ground state of lowest energy which we call the vacuum. The system evolves back into the vacuum as $t \rightarrow \infty$ via the creation, interaction and destruction of a particle, through the agency of a source. What we would like to investigate is the vacuum-to-vacuum transition amplitude in the presence of a source. We often write

$$< 0^+ | 0^- >_J \equiv < vacuum(t = +\infty | vacuum(t = -\infty) >_J$$

which measures the amplitude of the system to be in the ground state at $t = -\infty$ when it was known to be in the ground state at $t = -\infty$, while in the presence of an external source J that is turned off in the remote past and future. We incorporate the source term in the path integral by modifying the Lagrangian

$$\mathcal{L} \rightarrow \mathcal{L} + \psi(x) J(x)$$

We write

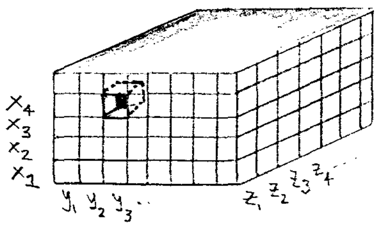
$$< \psi_2 t_2 | \psi_1 t_1 >_J \equiv N_0 \int \mathcal{D}\psi e^{i \int d^4 x (\mathcal{L}(\psi) + \psi(x) J(x))} \tag{1.7}$$

1.3 Generating Functionals

This section and the remainder of Chapter 1 follows closely the approach of Ramond [2] and Ryder [5]. The right hand side of (1.7) is a functional of J. We define the vacuum-to-vacuum transition amplitude in the presence of the source J as

$$\begin{aligned} Z[J] &= \int \mathcal{D}\psi e^{i\int d^4x[\mathcal{L}(\psi)+J(x)\psi(x)+\frac{i}{2}\epsilon\psi^2]} \\ &\propto < 0^+|0^- >_J \end{aligned} \tag{1.8}$$

Notice that the convergence factor $\frac{1}{2}i\epsilon\psi^2$ has been added to the Lagrangian. Without this factor the integrand in (1.8) is oscillatory and the path integral ill-defined. As mentioned earlier, we are to interpret (1.8) as a path integral over all field configurations. Further to the previous discussion we can think of the whole system as being enclosed in a (4-dimensional) box. We divide the volume of the box into N^4 small cubes each of volume δ^4. This idea is nicely illustrated in Guidry [13]



By imposing this lattice, the field ψ may be represented in each cube by a discrete value. In the diagram, we may label each cube by a 4-component object. For example, the cube highlighted may be labelled (x_4, y_3, z_1, t_1). The field value in this cube may be denoted by ψ_{4311} or in general ψ_{ijkl}. We may replace the four indices (i, j, k, l) by one index n and write

$$\begin{aligned} \mathcal{L}(\psi_{ijkl}, \partial_\mu\psi_{ijkl}) &= \mathcal{L}(\psi_n, \partial_\mu\psi_n) \\ &= \mathcal{L}_n \end{aligned}$$

Since each of i, j, k, l varies between 1 and N we see that n will vary between 1 and N^4. The action S is defined as being the integral of the Lagrangian.

$$S = \int \mathcal{L}\, d^4x$$

In the discrete case, in which we divide Minkowski space up into N^4 cubes we have

$$S \approx \sum_{n=1}^{N^4} \delta^4 \mathcal{L}_n$$

(Notice that d^4x has been replaced by δ^4 - the volume of each individual cube). The vacuum-to-vacuum amplitude $Z[J]$ is therefore given by

$$Z[J] = \lim_{N\to\infty} \int \prod_{n=1}^{N^4} d\psi_n e^{i\sum_{n=1}^{N^4}\delta^4(\mathcal{L}_n+\psi_n J_n+\frac{i}{2}\epsilon\psi_n^2)}$$

As N becomes very large the cubes will become very small, shrinking to points as $N \to \infty$. This is because the total volume of the box remains constant. We calculate $Z[J]$ in the case of the free particle field of mass m in which the Lagrangian $\mathcal{L}_0$ is given by

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu\psi\partial^\mu\psi - m^2\psi^2) \tag{1.9}$$

The vacuum-to-vacuum amplitude (taking the limit as $N \to \infty$) is

$$Z_0[J] = \int \mathcal{D}\psi e^{i\int(\frac{1}{2}[\partial_\mu\psi\partial^\mu\psi-(m^2-i\epsilon)\psi^2]+\psi J)d^4x} \tag{1.10}$$

This expression is most easily evaluated using a Fourier transformation to momentum space. The Fourier transforms we use are 4−dimensional

$$\tilde{F}(p) = \int_{-\infty}^{\infty}\frac{d^4x}{(2\pi)^4}e^{-ip.x}F(x)$$

$$F(x) = \int_{-\infty}^{\infty}\frac{d^4p}{(2\pi)^4}e^{ip.x}\tilde{F}(p) \tag{1.11}$$

We also consider a 4−dimensional Dirac delta function

$$\begin{aligned}\delta^4(x-x') &= \delta^3(\vec{x}-\vec{x}')\delta(x^0-x'^0)\\ &= \frac{1}{(2\pi)^4}\int_{-\infty}^{\infty} d^4p\; e^{i(x-x').p}\end{aligned}$$

Performing a Fourier transform on all the quantities under the d^4x in (1.10) gives

$$\begin{aligned}&i\int d^4x\int\frac{d^4p\, d^4p'}{(2\pi)^4}[\{\frac{1}{2}\partial_\mu e^{ip.x}\partial^\mu e^{ip'.x} - \frac{1}{2}(m^2-i\epsilon)e^{i(p+p').x}\}\\ &\qquad\times\tilde{\psi}(p)\tilde{\psi}(p') + e^{i(p+p').x}\tilde{J}(p)\tilde{\psi}(p')]\\ &= i\int d^4p\, d^4p'\frac{d^4x}{(2\pi)^4}e^{i(p+p').x}[\{-\frac{1}{2}p_\mu p'^{\,\mu} - \frac{1}{2}(m^2-i\epsilon)\}\tilde{\psi}(p)\tilde{\psi}(p') + \tilde{J}(p)\tilde{\psi}(p')]\end{aligned}$$

The integral

$$\int\frac{d^4x}{(2\pi)^4}e^{i(p+p').x} = \delta(p+p')$$

The dirac delta is then used to perform the integration with respect to p'. The result is

$$i\int d^4p[\frac{1}{2}(p^2-m^2+i\epsilon)\tilde{\psi}(p)\tilde{\psi}(-p)+\tilde{J}(p)\tilde{\psi}(-p)]$$

Next we make a transformation of functions -

$$\tilde{\psi}'(p)\equiv\tilde{\psi}(p)+\frac{\tilde{J}(p)}{(p^2-m^2+i\epsilon)} \tag{1.12}$$

so that the exponent becomes

$$i\int d^4p\,[\frac{1}{2}(p^2-m^2+i\epsilon)\tilde{\psi}'(p)\tilde{\psi}'(-p)-\frac{\tilde{J}(p)\tilde{J}(-p)}{2(p^2-m^2+i\epsilon)} \tag{1.13}$$

The first term in (1.13) gives the contribution to the vacuum amplitude with J switched off. The second term gives the effect of J only. We notice that in (1.12) ψ and ψ' differ only by a constant function. This means that

$$\mathcal{D}\psi=\mathcal{D}\psi'$$

Hence

$$Z_0[J]=e^{-\frac{i}{2}\int d^4p(\frac{|\tilde{J}(p)|^2}{p^2-m^2+i\epsilon})}\int\mathcal{D}\tilde{\psi}e^{i\int d^4x[\frac{1}{2}\partial_\mu\tilde{\psi}\partial^\mu\tilde{\psi}-\frac{1}{2}(m^2-i\epsilon)\psi^2\,]} \tag{1.14}$$

where we have used the property that $\tilde{J}(-p)=\tilde{J}^*(p)$. We may write (1.14) as

$$Z_0[J]=e^{-\frac{i}{2}\int d^4p(\frac{|\tilde{J}(p)|^2}{p^2-m^2+i\epsilon})}Z_0[0] \tag{1.15}$$

where

$$Z_0[0]=\int\mathcal{D}\psi' e^{i\int d^4x[\frac{1}{2}\partial_\mu\psi'\partial^\mu\psi'-\frac{1}{2}(m^2-i\epsilon)\psi'^{\,2}]}$$

We have therefore succeeded in factoring off the J dependence of $Z_0[J]$. Consider the J-dependent term in (1.14). We may revert back to spatial coordinate dependence by means of (1.11) to give

$$\begin{aligned}
&-\frac{i}{2}\int d^4p\;\tilde{J}(p)\frac{1}{p^2-m^2+i\epsilon}\tilde{J}(-p)\\
&=-\frac{i}{2}\int d^4p\frac{d^4xd^4y}{(2\pi)^4}e^{-ip.x}J(x)\frac{1}{p^2-m^2+i\epsilon}e^{ip.y}J(y)\\
&=-\frac{i}{2}\int d^4xd^4yJ(x)\{\frac{1}{(2\pi)^4}\int d^4p\frac{e^{-ip.(x-y)}}{p^2-m^2+i\epsilon}\}J(y)\\
&=-\frac{i}{2}\int J(x)\Delta_F(x-y)J(y)d^4xd^4y
\end{aligned} \tag{1.16}$$

where

$$\Delta_F(x-y)=\frac{1}{(2\pi)^4}\int d^4p(\frac{e^{-ip.(x-y)}}{p^2-m^2+i\epsilon}) \tag{1.17}$$

$\Delta_F(x-y)$ is called the Feynman propagator. The $i\epsilon$ term in (1.17) ensures that one gets the correct contour integration around the correct poles which occur at $k_0 = \pm(\mathbf{k}^2 + m^2)^{\frac{1}{2}}$. We may write (1.14) as (dropping the prime on ψ)

$$\begin{aligned} Z_0[J] &= e^{-\frac{i}{2}\int J(x)\Delta_F(x-y)J(y)d^4x d^4y} \\ &\times \int \mathcal{D}\psi e^{-\frac{i}{2}\int[\psi(\partial^\mu\partial_\mu + m^2 - i\epsilon)\psi]d^4x} \end{aligned} \tag{1.18}$$

We have used the fact that

$$\int \partial_\mu\psi\partial^\mu\psi \; d^4x = -\int \psi\partial^\mu\partial_\mu\psi d^4x \tag{1.19}$$

The surface term which arises will vanish if $\psi \to 0$ at infinity. We may rewrite (1.18) in terms of $Z_0[0]$ to give

$$Z_0[J] = [e^{-\frac{i}{2}\int J(x)\Delta_F(x-y)J(y)d^4x d^4y}]Z_0[0] \tag{1.20}$$

1.4 Free particle Green's functions

We can show that $Z_0[J]$ is the 'generating functional' for the free particle Green's functions which will shortly be defined. Introducing the shorthand notation

$$< J_1\Delta_{F12}J_2 >_{1,2} = \int J(x)\Delta_F(x-y)J(y) \; d^4x \; d^4y$$

we may expand (1.20) to give

$$\begin{aligned} Z_0[J] &= N\{1 - \frac{i}{2} < J_1\Delta_{F12}J_2 > \\ &+ \frac{1}{2!}(\frac{i}{2})^2\frac{1}{3}(< J_1\Delta_{F12}J_2 >< J_3\Delta_{F34}J_4 > \\ &+ < J_1\Delta_{F13}J_3 >< J_2\Delta_{F24}J_4 > \\ &+ < J_1\Delta_{F14}J_4 >< J_2\Delta_{F23}J_3 >) + \ldots\} \end{aligned} \tag{1.21}$$

In order to interpret (1.21) we recall the power series expansion of a functional.

$$F[y] = \sum_{n=0}^{\infty}\int dx_1 \ldots dx_n \frac{1}{n!}T_n(x_1, \ldots, x_n)y(x_1)\ldots y(x_n)$$

where

$$T_n(x_1, \ldots, x_n) = \frac{\delta}{\delta y(x_1)} \cdots \frac{\delta}{\delta y(x_n)} F[y]_{|y=0}$$

$F[y]$ is called the generating functional of the functions $T_n(x_1, \ldots, x_n)$. We shall relate this expansion to (1.21) after dealing with the question of normalisation. Since $Z_0[J]$ is the vacuum-to-vacuum expectation in the presence of the source J - it is sensible to set $Z_0[J = 0] = 1$. This means that we can write (1.20) as

$$Z_0[J] = e^{-\frac{i}{2}\int J(x)\Delta_F(x-y)J(y) \; d^4x \; d^4y} = e^{-\frac{i}{2}<J_1\Delta_{F12}J_2>_{1,2}} \tag{1.22}$$

$Z_0[J]$ defined in (1.22) is the generating functional for

$$\tau(x_1,\ldots,x_n) = \frac{1}{i^n}\frac{\delta^n Z_0[J]}{\delta J(x_1)\ldots\delta J(x_n)}_{|J=0}$$

The quantities $\tau(x_1,\ldots,x_n)$ are called the Greens functions or n -point functions of the theory. We shall now calculate the 2-point function

$$\tau(x,y) = -\frac{\delta^2 Z_0[J]}{\delta J(x)\delta J(y)}_{|J=0}$$

From (1.22) we have

$$\begin{aligned}
\frac{1}{i}\frac{\delta Z_0[J]}{\delta J(x)} &= \frac{1}{i}\frac{\delta}{\delta J(x)}e^{-\frac{i}{2}<J_1\Delta_{F12}J_2>_{1,2}} \\
&= (-\int \Delta_F(x-x_1)J(x_1)d^4x_1)e^{-\frac{i}{2}<J_1\Delta_{F12}J_2>_{1,2}}
\end{aligned}$$

$$\begin{aligned}
(\frac{1}{i}\frac{\delta}{\delta J(x)}\frac{1}{i}\frac{\delta}{\delta J(y)})Z_0[J] &= i\Delta_F(x-y)e^{-\frac{i}{2}<J_1\Delta_{F12}J_2>_{1,2}} \\
&\quad + \int \Delta_F(x-x_1)J(x_1)d^4x_1 \int \Delta_F(y-x_1)J(x_1)d^4x_1 \\
&\quad \times \ e^{-\frac{i}{2}<J_1\Delta_{F12}J_2>_{1,2}} \qquad (1.23)
\end{aligned}$$

Putting $J=0$ then yields

$$\frac{1}{i}\frac{\delta}{\delta J(x)}\frac{1}{i}\frac{\delta}{\delta J(y)}Z_0[J]_{|J=0} = i\Delta_F(x-y)$$

or

$$\tau(x,y) = i\Delta_F(x-y)$$

Concerning the physical significance of the 2-point function we should note that

$$\begin{aligned}
(\partial^\mu\partial_\mu + m^2)\Delta_F(x) &= \frac{1}{(2\pi)^4}\int d^4k\frac{(-k^2+m^2)e^{-ik.x}}{k^2-m^2+i\epsilon} \\
&= -\delta^4(x) \qquad (1.24)
\end{aligned}$$

Hence the name : Greens function! It turns out that the 3-point function is zero.

$$\begin{aligned}
\tau(x_1,x_2,x_3) &= \frac{1}{i}\frac{\delta}{\delta J(x_1)}\frac{1}{i}\frac{\delta}{\delta J(x_2)}\frac{1}{i}\frac{\delta}{\delta J(x_3)}Z_0[J]_{|J=0} \\
&= -i\Delta_F(x_2-x_3)\int \Delta_F(x_1-x)J(x)d^4x \ e^{-\frac{i}{2}<J_1\Delta_{F12}J_2>_{1,2}} \\
&\quad -i\Delta_F(x_2-x_1)\int \Delta_F(x_3-x)J(x)d^4x \ e^{-\frac{i}{2}<J_1\Delta_{F12}J_2>_{1,2}}
\end{aligned}$$

$$
\begin{aligned}
&-i\Delta_F(x_3-x_1)\int \Delta_F(x_2-x)J(x)d^4x\ e^{-\frac{i}{2}<J_1\Delta_{F12}J_2>_{1,2}} \\
&-\int \Delta(x_2-x)J(x)\ d^4x \int \Delta_F(x_3-x)J(x)d^4x \\
&\times\ \Delta_F(x_1-x)J(x)d^4x\ e^{-\frac{i}{2}<J_1\Delta_{F12}J_2>_{1,2}}
\end{aligned}
$$

(The differentiation is easy to see using (1.23)). Setting $J=0$ gives

$$\tau(x_1,x_2,x_3)=0$$

The 4 -point function is not zero.

$$
\begin{aligned}
\frac{1}{i}\frac{\delta}{\delta J(x_1)}\cdots\frac{1}{i}\frac{\delta}{\delta J(x_4)}Z_0[J] \quad = \quad & -\Delta_F(x_2-x_3)\Delta_F(x_1-x_4)e^{-\frac{i}{2}<J_1\Delta_{F12}J_2>} \\
& -\Delta_F(x_2-x_1)\Delta_F(x_3-x_4)e^{-\frac{i}{2}<J_1\Delta_{F12}J_2>} \\
& -\Delta_F(x_3-x_1)\Delta_F(x_2-x_4)e^{-\frac{i}{2}<J_1\Delta_{F12}J_2>} \\
& \ldots
\end{aligned}
$$

(The other terms in the 4-point function vanish when $J=0$). There is a standard diagrammatic representation for the n -point functions. We associate with $\Delta_F(x-y)$ a line connecting the two space-like points x and y.

$$\tau(x,y)=i\Delta_F(x-y)=x \text{———} y$$

For the 4 -point function $\tau(x_1,x_2,x_3,x_4)$ we have

$$\tau(x_1,x_2,x_3,x_4)=\begin{pmatrix} x_1\text{—}x_2 \\ x_3\text{—}x_4 \end{pmatrix}+\begin{pmatrix} x_1\text{—}x_3 \\ x_2\text{—}x_4 \end{pmatrix}+\begin{pmatrix} x_1\text{—}x_4 \\ x_2\text{—}x_3 \end{pmatrix} \tag{1.25}$$

It is always the case that when n is odd the n-point function will vanish when $J=0$.

1.5 Generating functionals for interacting fields

In equation (1.9) we dealt with the Lagrangian ($\mathcal{L}_0$) in the case of a free particle field. We now include an additional interaction term in the Lagrangian. A standard case is when the interaction term is proportional to ψ^4. This is known as ψ^4 theory and the Lagrangian we consider is

$$
\begin{aligned}
\mathcal{L} &= \mathcal{L}_0+\mathcal{L}_{int} \\
&= \frac{1}{2}\partial_\mu\psi\partial^\mu\psi-\frac{1}{2}m^2\psi^2-\frac{g}{4!}\psi^4
\end{aligned}
$$

We deal first with the case of a general interaction $\mathcal{L}_{int}$ and show how the Greens functions may be obtained. The normalised generating functional is

$$Z[J]=\frac{\int \mathcal{D}\psi e^{(iS+i\int J\psi d^4x)}}{\int \mathcal{D}\psi e^{iS}} \tag{1.26}$$

where

$$\begin{aligned} S &= \int \mathcal{L} d^4x. \\ &= \int (\mathcal{L}_0 + \mathcal{L}_{int}) d^4x \end{aligned}$$

Note that the denominator is just $Z[0]$. In the case when $\mathcal{L}_{int} = 0$ the numerator becomes equal to the right hand side of (1.20) and the denominator becomes $Z_0[0]$. The left hand side is (by definition) $Z_0[J]$. Hence we arrive at

$$Z_0[J] = \frac{(e^{-\frac{i}{2}\int J(x)\Delta_F(x-y)J(y)d^4x d^4y})\cancel{Z_0[0]}}{\cancel{Z_0[0]}}$$

which is the result in (1.22). (Note that we got round the problem of normalisation earlier by defining $Z_0[0] = 1$). Equation (1.22) was in a suitable form for functional differentiation with respect to J and we showed how the 2-point function and the 4-point function could be calculated. Our aim is to find the corresponding expression to (1.22) in the case of interacting fields. From (1.22) we have

$$\frac{1}{i}\frac{\delta}{\delta J(x)} Z_0[J] = -\int \Delta_F(x-y)J(y)d^4y e^{-\frac{i}{2}<J_1\Delta_{F12}J_2>_{1,2}}$$

We now let the operator $(\partial^\mu \partial_\mu + m^2)$ act on both sides of this equation to give

$$\begin{aligned} (\partial^\mu\partial_\mu + m^2)\frac{1}{i}\frac{\delta}{\delta J(x)} Z_0[J] &= -\int (\partial^\mu\partial_\mu + m^2)\Delta_F(x-y)J(y)d^4y \; e^{-\frac{i}{2}<J_1\Delta_{F12}J_2>_{1,2}} \\ &= -\int -\delta^4(x-y)J(y)d^4y \; e^{-\frac{i}{2}<J_1\Delta_{F12}J_2>_{1,2}} \\ &= J(x)e^{-\frac{i}{2}<J_1\Delta_{F12}J_2>_{1,2}} \\ &= J(x)Z_0[J] \end{aligned} \quad (1.27)$$

(using the result in (1.24) that $(\partial^\mu\partial_\mu + m^2)\Delta_F(x) = -\delta^4(x)$. Equation (1.27) is the differential equation which $Z_0[J]$ satisfies. From (1.26) we have

$$\frac{1}{i}\frac{\delta Z[J]}{\delta J(x)} = \frac{\int \mathcal{D}\psi (e^{iS + i\int J\psi d^4x})\psi(x)}{\int \mathcal{D}\psi e^{iS}} \quad (1.28)$$

We define the functional

$$\hat{Z}[\psi] = \frac{e^{iS}}{\int e^{iS}\mathcal{D}\psi}$$

Then from (1.26)

$$\begin{aligned} Z[J] &= \frac{\int \mathcal{D}\psi e^{iS} e^{i\int J(x)\psi(x)d^4x}}{\int \mathcal{D}\psi e^{iS}} \\ &= \int \mathcal{D}\psi \hat{Z}[\psi] e^{i\int J(x)\psi(x)d^4x} \end{aligned} \quad (1.29)$$

Note that

$$S = \int (\frac{1}{2}\partial_\mu\psi\partial^\mu\psi - \frac{1}{2}m^2\psi^2 + \mathcal{L}_{int})d^4x$$
$$= -\int (\frac{1}{2}\psi(\partial^\mu\partial_\mu + m^2)\psi - \mathcal{L}_{int})d^4x$$

Using this expression we obtain

$$i\frac{\delta\hat{Z}[\psi]}{\delta\psi(x)} = i\frac{\delta}{\delta\psi}\{e^{-i\int[\frac{1}{2}\psi(\partial^\mu\partial_\mu+m^2)\psi-\mathcal{L}_{int}]d^4x}\} \times [\int e^{iS}\mathcal{D}\psi]^{-1}$$
$$= (\partial^\mu\partial_\mu + m^2)\psi(x)\hat{Z}[\psi] - \frac{\partial\mathcal{L}_{int}}{\partial\psi}\hat{Z}[\psi]$$
$$= (\partial^\mu\partial_\mu + m^2)\psi(x)\hat{Z}[\psi] - \mathcal{L}'_{int}(\psi)\hat{Z}[\psi] \qquad (1.30)$$

Multiplying both sides of (1.30) by $e^{i\int J(x)\psi(x)d^4x}$ and then integrating over ψ gives

$$i\int \frac{\delta\hat{Z}[\psi]}{\delta\psi(x)}e^{i\int J(x)\psi(x)d^4x}\mathcal{D}\psi = (\partial^\mu\partial_\mu + m^2)\frac{1}{i}\frac{\delta Z[J]}{\delta J(x)} - \mathcal{L}'_{int}[\frac{1}{i}\frac{\delta}{\delta J}]Z[J] \qquad (1.31)$$

(we have used (1.28) and the argument of $\mathcal{L}'_{int}$ has been changed from ψ to $\frac{1}{i}\frac{\delta}{\delta J}$ since

$$\frac{1}{i}\frac{\delta}{\delta J(x)}e^{i\int J(x)\psi(x)d^4x} = \psi(x)e^{i\int J(x)\psi(x)d^4x} \quad)$$

The left hand side of (1.31) may be simplified

$$i\int \frac{\delta\hat{Z}[\psi]}{\delta\psi}e^{i\int J(x)\psi(x)d^4x}\mathcal{D}\psi = ie^{i\int J(x)\psi(x)d^4x}\hat{Z}[\psi]_{|\psi\to\infty}$$
$$+ \int J(x)\hat{Z}[\psi]e^{i\int J(x)\psi(x)d^4x}\mathcal{D}\psi$$
$$= J(x)Z[J] \qquad (1.32)$$

Combining (1.32) and (1.31) we have

$$(\partial^\mu\partial_\mu + m^2)\frac{1}{i}\frac{\delta Z[J]}{\delta J(x)} - \mathcal{L}'_{int}(\frac{1}{i}\frac{\delta}{\delta J(x)})Z[J] = J(x)Z[J] \qquad (1.33)$$

Our aim is to solve this equation for $Z[J]$. In the case of the free field $\mathcal{L}_{int} = 0$ and we reobtain (1.27). We can show that the solution to (1.33) is

$$Z[J] = Ne^{i\int \mathcal{L}_{int}(\frac{1}{i}\frac{\delta}{\delta J})d^4x}Z_0[J] \qquad (1.34)$$

where N is the normalising factor. The proof is as follows.

<u>Proof</u>

We start by proving the identity

$$e^{-i\int \mathcal{L}_{int}(\frac{1}{i}\frac{\delta}{\delta J(y)})d^4y}J(x)e^{i\int \mathcal{L}_{int}(\frac{1}{i}\frac{\delta}{\delta J(y)})d^4y} = J(x) - \mathcal{L}'_{int}(\frac{1}{i}\frac{\delta}{\delta J(x)}) \qquad (1.35)$$

Let us call

$$\theta_x = e^{-i\int \mathcal{L}_{int}(\frac{1}{i}\frac{\delta}{\delta J(y)})d^4y} J(x) e^{i\int \mathcal{L}_{int}(\frac{1}{i}\frac{\delta}{\delta J(y)})d^4y}$$

We evaluate this quantity by introducing a parameter λ by defining

$$\theta_x(\lambda) = e^{-i\lambda\int \mathcal{L}_{int}(\frac{1}{i}\frac{\delta}{\delta J(y)})d^4y} J(x) e^{i\lambda\int \mathcal{L}_{int}(\frac{1}{i}\frac{\delta}{\delta J(y)})d^4y} \tag{1.36}$$

We then create a first order ordinary differential equation for θ_x.

$$\begin{aligned}
\frac{d\theta_x(\lambda)}{d\lambda} &= -ie^{-i\lambda\int \mathcal{L}_{int}(\frac{1}{i}\frac{\delta}{\delta J(y)})d^4y}[\int \mathcal{L}_{int}(\frac{1}{i}\frac{\delta}{\delta J(y)})d^4y]J(x)e^{i\lambda\int \mathcal{L}(\frac{1}{i}\frac{\delta}{\delta J(y)})d^4y} \\
&\quad +[ie^{-i\lambda\int \mathcal{L}_{int}(\frac{1}{i}\frac{\delta}{\delta J(y)})d^4y}]J(x)[\int \mathcal{L}_{int}(\frac{1}{i}\frac{\delta}{\delta J(y)})d^4y]e^{i\lambda\int \mathcal{L}_{int}(\frac{1}{i}\frac{\delta}{\delta J(y)})d^4y} \\
&= -ie^{-i\lambda\int \mathcal{L}_{int}(\frac{1}{i}\frac{\delta}{\delta J(y)})d^4y}\int [\mathcal{L}_{int}(\frac{1}{i}\frac{\delta}{\delta J(y)}), J(x)]d^4y e^{i\lambda\int \mathcal{L}_{int}(\frac{1}{i}\frac{\delta}{\delta J(y)})d^4y}
\end{aligned}$$

To work out the commutator, we note the result

$$[\mathcal{L}_{int}(\frac{1}{i}\frac{\delta}{\delta J(y)}), J(x)] = -i\mathcal{L}'_{int}(\frac{1}{i}\frac{\delta}{\delta J(y)})\delta^4(x-y) \tag{1.37}$$

which is the functional analogue of the elementary result

$$[F(\frac{d}{dx}), x] = F'(\frac{d}{dx}) \tag{1.38}$$

We may verify (1.38) for a simple example. Suppose $F(x) = x^2$ then

$$\begin{aligned}
[F(\frac{d}{dx}), x]\psi &= (\frac{d}{dx})^2(x\psi) - x(\frac{d}{dx})^2\psi \\
&= \frac{d}{dx}(\psi + x\frac{d\psi}{dx}) - x\frac{d^2\psi}{dx^2} \\
&= \frac{d\psi}{dx} + \frac{d\psi}{dx} + x\frac{d^2\psi}{dx^2} - x\frac{d^2\psi}{dx^2} \\
&= 2\frac{d\psi}{dx} \\
&= F'(\frac{d}{dx})\psi
\end{aligned}$$

Equation (1.37) means that

$$\begin{aligned}
\frac{d\theta_x(\lambda)}{d\lambda} &= -ie^{-i\lambda\int \mathcal{L}_{int}(\frac{1}{i}\frac{\delta}{\delta J(y)})d^4y}[-i\mathcal{L}'_{int}(\frac{1}{i}\frac{\delta}{\delta J(x)})]e^{i\lambda\int \mathcal{L}_{int}(\frac{1}{i}\frac{\delta}{\delta J(y)})d^4y} \\
&= -\mathcal{L}'_{int}(\frac{1}{i}\frac{\delta}{\delta J(x)}) \qquad (1.39)
\end{aligned}$$

Equation (1.39) may be integrated to give

$$\theta_x(\lambda) = -\mathcal{L}'_{int}(\frac{1}{i}\frac{\delta}{\delta J(x)})\lambda + C$$

C may be evaluated by noting that $\theta_x(0) = J(x)$ from (1.36). We would like to calculate θ_x which is done by setting $\lambda = 1$.

$$\theta_x = -\mathcal{L}'_{int}(\frac{1}{i}\frac{\delta}{\delta J(x)}) + J(x)$$

We use this result to show that (1.34) is a solution of (1.33). Multiplying both sides of (1.34) by $J(x)$ gives

$$J(x)Z[J] = NJ(x)e^{i\int \mathcal{L}_{int}(\frac{1}{i}\frac{\delta}{\delta J(y)})d^4y}Z_0[J] \tag{1.40}$$

Using the identity in (1.35) we can write the right hand side of (1.40) as

$$Ne^{i\int \mathcal{L}_{int}(\frac{1}{i}\frac{\delta}{\delta J(y)})d^4y}[J(x) - \mathcal{L}'_{int}(\frac{1}{i}\frac{\delta}{\delta J(x)})]Z_0[J]$$

Using (1.27) for the term involving $J(x)Z_0[J]$ and interchanging the order of $e^{\mathcal{L}_{int}}$ and $\mathcal{L}'_{int}$ gives

$$\begin{aligned} J(x)Z[J] &= N[e^{i\int \mathcal{L}_{int}(\frac{1}{i}\frac{\delta}{\delta J(y)})d^4y}](\partial^\mu\partial_\mu + m^2)\frac{1}{i}\frac{\delta Z_0}{\delta J(x)} \\ &\quad -N\mathcal{L}'_{int}(\frac{1}{i}\frac{\delta}{\delta J(x)})[e^{i\int \mathcal{L}_{int}(\frac{1}{i}\frac{\delta}{\delta J(y)})d^4y}]Z_0[J] \\ &= (\partial^\mu\partial_\mu + m^2)\frac{1}{i}\frac{\delta Z[J]}{\delta J(x)} - \mathcal{L}'_{int}[\frac{1}{i}\frac{\delta}{\delta J(x)}]Z[J] \end{aligned} \tag{1.41}$$

We have shown that (1.34) satisfies the differential equation (1.33). Equation (1.41) is often rewritten as

$$(\partial^\mu\partial_\mu + m^2)(\frac{1}{i}\frac{1}{Z[J]}\frac{\delta Z[J]}{\delta J(x)}) = J(x) + \frac{1}{Z[J]}\mathcal{L}'_{int}[\frac{1}{i}\frac{\delta}{\delta J(x)}]Z[J] \tag{1.42}$$

Let us call

$$\psi_{cl}(x) = \frac{1}{i}\frac{1}{Z[J]}\frac{\delta Z[J]}{\delta J(x)}$$

($\psi_{cl}(x)$ is the field which satisfies the classical equations of motion. ie.

$$(\partial^\mu\partial_\mu + m^2)\psi_{cl}(x) = J(x)$$

in the case when $\mathcal{L}_{int} = 0$ see Chapter 2 -equation (2.42)). The last term in (1.42) represents a force since it involves the derivative of the potential function. Let us suppose that $\mathcal{L}_{int} = -\frac{g}{4!}\psi^4$ (ψ^4 theory). Then

$$-\frac{1}{Z[J]}\mathcal{L}'_{int}(\frac{1}{i}\frac{\delta}{\delta J(x)})Z[J] = \frac{g}{3!}(-i)^3\frac{1}{Z[J]}\frac{\delta^3}{\delta^3 J(x)}Z[J] \tag{1.43}$$

The right hand side of (1.43) may be evaluated in terms of $\psi_{cl}(x)$. We note the identity in which $\frac{1}{Z}\frac{\delta^3 Z}{\delta J^3}$ occurs :

$$\frac{\delta^2}{\delta J^2}\frac{1}{Z}\frac{\delta Z}{\delta J} = \frac{2}{Z^3}(\frac{\delta Z}{\delta J})^3 - \frac{3}{Z^2}(\frac{\delta^2 Z}{\delta J^2})(\frac{\delta Z}{\delta J}) + \frac{1}{Z}\frac{\delta^3 Z}{\delta J^3} \tag{1.44}$$

The left hand side and the first term on the right hand side may be easily interpreted in terms of $\psi_{cl}(x)$. The second term on the right hand side cannot. But then we may appeal to the identity

$$\frac{\delta}{\delta J}[(\frac{1}{Z}\frac{\delta Z}{\delta J})^2] = -\frac{2}{Z^3}(\frac{\delta Z}{\delta J})^3 + \frac{2}{Z^2}(\frac{\delta^2 Z}{\delta J^2})(\frac{\delta Z}{\delta J}) \tag{1.45}$$

(1.44) and (1.45) may be combined to eliminate the unwanted term $\frac{1}{Z^2}(\frac{\delta^2 Z}{\delta J^2})(\frac{\delta Z}{\delta J})$. Then the whole of the right hand side of (1.43) may be expressed in terms of ψ_{cl}. The result is

$$i\frac{\delta^2}{\delta J^2}(-\frac{i}{Z}\frac{\delta Z}{\delta J}) - \frac{3}{2}\frac{\delta}{\delta J}(-\frac{i}{Z}\frac{\delta Z}{\delta J})^2 - i(-\frac{i}{Z}\frac{\delta Z}{\delta J})^3 = \frac{1}{Z}\frac{\delta^3 Z}{\delta J^3}$$

Referring back to (1.42) we have

$$\begin{aligned}(\partial^\mu\partial_\mu + m^2)\psi_{cl}(x) &= J(x) - \frac{g}{3!}(-i)^3(i\frac{\delta^2}{\delta J^2}\psi_{cl}(x) - \frac{3}{2}\frac{\delta}{\delta J}\psi_{cl}^2(x) - i\psi_{cl}^3(x)) \\ &= J(x) - \frac{g}{3!}\psi_{cl}^3(x) + \frac{g}{3!}\frac{\delta^2}{\delta J^2}\psi_{cl}(x) + \frac{i}{4}\frac{\delta}{\delta J}\psi_{cl}^2(x)\end{aligned}$$

The first two terms on the right hand side give the classical equation of motion. The last two terms are purely quantum mechanical. We now examine the generating functional $Z[J]$ for ψ^4 theory. Using perturbation theory we show how the (disconnected) Green's functions may be obtained. The major difference between these functions and the free particle Green's functions (discussed earlier) is that in the former case the interaction term proportional to ψ^4 gives rise to vertex diagrams  . Such diagrams do not occur in the latter case in which $\mathcal{L}_{int} = 0$. The normalised generating functional $Z[J]$ for ψ^4 theory is -

$$Z[J] = \frac{e^{i\int \mathcal{L}_{int}(\frac{1}{i}\frac{\delta}{\delta J(z)})d^4z} e^{-\frac{i}{2}\int J(x)\Delta_F(x-y)J(y)d^4x d^4y}}{e^{i\int \mathcal{L}_{int}(\frac{1}{i}\frac{\delta}{\delta J(z)})d^4z} e^{-\frac{i}{2}\int J(x)\Delta_F(x-y)J(y)d^4x d^4y}}_{|J=0} \tag{1.46}$$

where $\mathcal{L}_{int} = -\frac{g}{4!}\psi^4$. Using perturbation theory we may expand the numerator as a power series in the coupling constant g -

$$[1 - \frac{ig}{4!}\int(\frac{1}{i}\frac{\delta}{\delta J(z)})^4 d^4z + O(g^2)]e^{-\frac{i}{2}\int J(x)\Delta_F(x-y)J(y)d^4x d^4y}$$

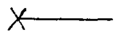
The term independent of g is simply the free particle generating functional $Z_0[J]$. It is a straightforward exercise to show that

$$\begin{aligned}&(\frac{1}{i}\frac{\delta}{\delta J(z)})^4 e^{-\frac{i}{2}\int J(x)\Delta_F(x-y)J(y)d^4x d^4y} = \\ &\{-3[\Delta_F(0)]^2 + 6i\Delta_F(0) \times [\int \Delta_F(z-x)J(x)d^4x]^2 \\ &+[\int \Delta_F(z-x)J(x)d^4x]^4\} \times e^{-\frac{i}{2}\int J(x)\Delta_F(x-y)J(y)d^4x d^4y}\end{aligned} \tag{1.47}$$

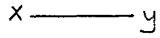
Using diagrammatic notation we may write (1.47) as

$$\left(\frac{1}{i}\frac{\delta}{\delta J(z)}\right)^4 e^{-\frac{i}{2}<J_1\Delta_{F12}J_2>_{1,2}} = \{-3\,\infty\ + 6i\ \ + \ \ \}e^{-\frac{i}{2}<J_1\Delta_{F12}J_2>_{1,2}} \quad (1.48)$$

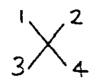
Notice that crosses have been introduced. This is due to the J terms. The standard convention within quantum field theory is to represent the expression $i(2\pi)^4 J(p)$ by a line with a cross attached as follows



The Feynman propagator $\Delta_F(x-y)$ is represented by a line



$\Delta_F(0) = \Delta_F(x-x)$ is represented by a closed loop. The interaction vertex is a consequence of the fact that $\mathcal{L}_{int}$ contains ψ^4. The coefficients 3, 6 and 1 in (1.48) follow from symmetry considerations and are known as symmetry factors. There are three possible ways of constructing the first diagram ∞ from the vertex $\times$.



We may join (1 to 4 and 2 to 3) or (1 to 2 and 3 to 4) or (1 to 3 and 2 to 4). Similarly there are six ways of constructing the diagram from $\times$. The double loop graph ∞ is known as a vacuum graph since it contains no external lines. Just as in 1.48 a factor of J is associated with each external line. (Hence there are 2 J's and hence 2 crosses associated with the graph and 4 J's and 4 crosses associated with the graph). The denominator of (1.46) is easily evaluated by setting $J = 0$ in (1.47). This gives

$$[e^{i\int \mathcal{L}_{int}} e^{-\frac{i}{2}<J_1\Delta_{F12}J_2>_{1,2}}]_{|J=0} = 1 - \frac{ig}{4!}\int(-3\,\infty\,)d^4z \qquad (1.49)$$

The complete generating functional to order g is found by combining (1.48), (1.49), and (1.46).

$$\begin{aligned} Z[J] &= \frac{[1 - \frac{ig}{4!}\int(-3\,\infty\ + 6i\ \ + \ \ )d^4z]e^{-\frac{i}{2}<J_1\Delta_{F12}J_2>_{1,2}}}{1 - \frac{ig}{4!}\int(-3\,\infty\,)d^4z} \\ &= [1 - \frac{\frac{ig}{4!}\int(6i\ \ + \ \ )d^4z}{1 - \frac{ig}{4!}\int(-3\,\infty\,)d^4z}]e^{-\frac{i}{2}<J_1\Delta_{F12}J_2>_{1,2}} \end{aligned}$$

The denominator may be expanded using the standard Taylor series

$$(1+x)^{-1} = 1 - x + \ldots$$

To order g we have

$$Z[J] = [1 - \frac{ig}{4!}\int(6i\ \ + \ \ )\ d^4z]\ e^{-\frac{i}{2}<J_1\Delta_{F12}J_2>_{1,2}} \qquad (1.50)$$

An important point to notice is that the vacuum diagram ∞ has disappeared in $Z[J]$. It turns out that this is true to all orders of perturbation theory.
2 point function The 2-point function is defined by

$$\tau(x_1,x_2) = -\frac{\delta^2 Z[J]}{\delta J(x_2)\delta J(x_1)}_{|J=0}$$

On differentiating the first term in (1.50) (namely $e^{-\frac{i}{2}<J_1\Delta_{F12}J_2>_{1,2}}$) we have

$$\begin{aligned}
\frac{1}{i}\frac{\delta}{\delta J(x_1)}e^{-\frac{i}{2}\int J(x)\Delta_F(x-y)J(y)d^4x d^4y} &= [(-\frac{i}{2}\int \Delta_F(x_1-y)J(y)d^4y) \\
&\quad \times e^{-\frac{i}{2}\int J(x)\Delta_F(x-y)J(y)d^4xd^4y}] \times \frac{2}{i} \\
\frac{1}{i}\frac{\delta}{\delta J(x_2)}\frac{1}{i}\frac{\delta}{\delta J(x_1)}(e^{-\frac{i}{2}\int J(x)\Delta_F(x-y)J(y)d^4xd^4y}) &= i\Delta_F(x_1-x_2)e^{-\frac{i}{2}\int J(x)\Delta_F(x-y)J(y)d^4xd^4y} \\
&\quad + \text{other terms}
\end{aligned}$$

The omitted terms will vanish when $J=0$ and we see that the first term in the 2-point function is $i\Delta_F(x_1-x_2)$ which is the free particle propagator. We now differentiate the second term in (1.50). We must evaluate

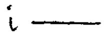
$$\frac{1}{i}\frac{\delta}{\delta J(x_2)}\frac{1}{i}\frac{\delta}{\delta J(x_1)}(\frac{g}{4}\Delta_F(0)\int d^4xd^4yd^4z\,\Delta_F(z-x)J(x)\Delta_F(z-y)J(y)e^{-\frac{i}{2}<J_1\Delta_{F12}J_2>_{1,2}})$$

The result (on setting $J=0$) is

$$-\frac{g}{2}\Delta_F(0)\int d^4z\Delta_F(z-x_1)\Delta_F(z-x_2)$$

We then have

$$\tau(x_1,x_2) = i\Delta_F(x_1-x_2) - \frac{g}{2}\Delta_F(0)\int d^4z\Delta_F(z-x_1)\Delta_F(z-x_2) + O(g^2) \quad (1.51)$$

$\tau(x_1,x_2) = i$  $-\frac{g}{2}$  $+ \;O(g^2)$

The effect of the correction term in (1.51) is to change the value of the physical mass away from m. Using (1.17) it is straightforward to show that

$$-\frac{1}{2}g\Delta_F(0)\int \Delta_F(z-x_1)\Delta_F(z-x_2)d^4z = -\frac{g}{2}\frac{\Delta_F(0)}{(2\pi)^4}\int(\frac{e^{-ip.(x_1-x_2)}}{(p^2-m^2+i\epsilon)^2})d^4p$$

The 2 -point function is therefore given by

$$\tau(x_1,x_2) = \frac{i}{(2\pi)^4}\int\frac{e^{-ip.(x_1-x_2)}}{p^2-m^2+i\epsilon}[1+\frac{\frac{1}{2}ig\Delta_F(0)}{p^2-m^2+i\epsilon}]d^4p$$

The term in square brackets may be written as

$$[1 - \frac{\frac{1}{2}ig\Delta_F(0)}{p^2 - m^2 + i\epsilon}]^{-1}$$

This gives

$$\tau(x_1, x_2) = \frac{i}{(2\pi)^4} \int (\frac{e^{-ip.(x_1 - x_2)}}{p^2 - m^2 - \frac{1}{2}ig\Delta_F(0) + i\epsilon}) d^4p$$

The Fourier transform of $\tau(x_1, x_2)$ will now possess a pole at

$$p^2 = m^2 + \frac{1}{2}ig\Delta_F(0) \equiv m^2 + \delta m^2 = m_r^2$$

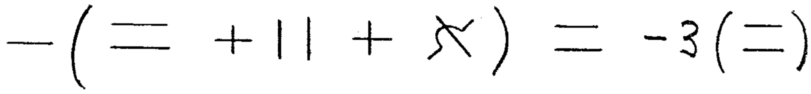
where

$$\delta m^2 = \frac{1}{2}ig\Delta_F(0).$$

m_r is called the physical or renormalised mass.
<u>4 point function</u>
The 4-point function is given by

$$\tau(x_1, x_2, x_3, x_4) = \frac{\delta^4 Z[J]}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)}_{|J=0}$$

The first term in the expansion of τ (to order g^0) is given by (1.25) which we may write as



We can now calculate the next term. We must evaluate

$$\frac{g}{4} \frac{\delta^4}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)} [\Delta_F(0) \int d^4z d^4x d^4y$$
$$\Delta_F(z - x)J(x)\Delta_F(z - y)J(y)e^{-\frac{i}{2}<J_1\Delta_{F12}J_2>_{1,2}}]_{|J=0} \qquad (1.52)$$

It is straightforward to show that (1.52) is equal to

$$\begin{aligned} -\tfrac{ig}{2}\Delta_F(0) \int d^4z[&\Delta_F(z - x_1)\Delta_F(z - x_2)\Delta_F(x_3 - x_4) \\ &+\Delta_F(z - x_1)\Delta_F(z - x_3)\Delta_F(x_2 - x_4) \\ &+\Delta_F(z - x_1)\Delta_F(z - x_4)\Delta_F(x_2 - x_3) \\ &+\Delta_F(z - x_2)\Delta_F(z - x_3)\Delta_F(x_1 - x_4) \\ &+\Delta_F(z - x_2)\Delta_F(z - x_4)\Delta_F(x_1 - x_3) \\ &+\Delta_F(z - x_3)\Delta_F(z - x_4)\Delta_F(x_1 - x_2)] \end{aligned}$$

which we may write as $-3ig$ 

Finally we calculate the term involving the 4-point vertex in (1.50). This is the last term to give a contribution of order g to $Z[J]$. We must evaluate

$$\begin{aligned}
& -\frac{ig}{4!}\frac{\delta^4}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)} \\
& \times\{[\int \Delta_F(z-x)J(x)d^4x]^4 d^4z e^{-\frac{i}{2}\int J(x)\Delta_F(x-y)J(y)d^4x d^4y}\}_{|J=0} \\
= & -ig\int \Delta_F(x_1-z)\Delta_F(x_2-z)\Delta_F(x_3-z)\Delta_F(x_4-z)d^4z \\
= & -ig[\times] \qquad (1.53)
\end{aligned}$$

The cross in (1.53) should not be confused with the cross in (1.50). The cross in (1.50) is taken to mean

$$[\int \Delta_F(z-x)J(x)\ d^4x]^4$$

The complete 4-point function, to order g is then

$$\tau(x_1,x_2,x_3,x_4) = -3[=] - 3ig(\ ) - ig(\times)$$

We summarise the Feynman rules for ψ^4 theory. In coordinate space they are

line	x $\quad$ y	$\Delta_F(x-y)$
vertex	integration over z $\times$	$-ig$

We call diagrams such as ( ) disconnected since it describes two particles moving independently. Notice that in the expression

$$-ig\Delta_F(0)\int d^4z\ \Delta_F(z-x_1)\Delta_F(z-x_2)\Delta_F(x_3-x_4)$$

$\Delta_F(x_3-x_4)$ does not take part in the integration. It is "disconnected" from the rest of the expression.

1.6 Generating functional for connected diagrams

We now define a generating functional W which generates only connected Feynman diagrams. It is given by $W[J] = -iln Z[J]$. For example, the connected 4-point function is given by

$$\frac{\delta^4 W}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)}_{|J=0}$$

It can be shown that this expression will not contain disconnected diagrams such as ( )

1.7 Path Integral Formulation with Fermions

In this section we consider a "path" integral over Grassmann (anticommuting) fields rather than ordinary commuting fields. We start by considering the case of a single Grassmann variable θ. It satisfies

$$\{\theta, \theta\} = 0 \quad \text{or} \quad \theta^2 = 0 \tag{1.54}$$

({ , } is the anti-commutator) The differential operator $\frac{d}{d\theta}$ is defined by

$$\{\frac{d}{d\theta}, \theta\} = 1 \tag{1.55}$$

Due to the anticommuting nature of the variables there are two types of differentiation, left differentiation and right differentiation. The left derivative of the product $\theta_1\theta_2$ is

$$\frac{\partial^L}{\partial\theta_i}(\theta_1\theta_2) = \delta_{i1}\theta_2 - \delta_{i2}\theta_1$$

and the right derivative is

$$\frac{\partial^R}{\partial\theta_i}(\theta_1\theta_2) = \delta_{i2}\theta_1 - \delta_{i1}\theta_2$$

Both these derivative operators satisfy

$$\{\frac{\partial}{\partial\theta_i}, \theta_j\} = \delta_{ij} \quad \text{and} \quad \{\frac{\partial}{\partial\theta_i}, \frac{\partial}{\partial\theta_j}\} = 0 \tag{1.56}$$

Notice that (1.55) follows from (1.56). For any function $f(\theta)$ we have a power expansion

$$f(\theta) = a + b\theta$$

This follows from (1.54) since the quadratic term vanishes. Integration is defined as follows.

$$\int d\theta = 0 \quad \text{and} \quad \int d\theta\ \theta = 1$$

For n Grassmann variables $\theta_1, \ldots, \theta_n$ we have

$$\int d\theta_i = 0 \quad \text{and} \quad \int d\theta_i\ \theta_i = 1 \tag{1.57}$$

(no summation over i). When more than one variable is involved the integration is performed according to a nested procedure. For example,

$$\int d\theta_1 \int d\theta_2\theta_1\theta_2 = -\int d\theta_1 \int (d\theta_2\theta_2)\theta_1 = -1$$

An integral which arises frequently in path integral evaluations with fermions is

$$I_N(M) = \int d\theta_1 \ldots d\theta_N e^{-\theta^T M\theta}$$

M is an $N \times N$ antisymmetric matrix with ordinary valued elements m_{ij}. If the exponential is expanded by means of a power series we will obtain a finite number of non-zero terms. (This is because of the Grassmann nature of the variables.) We guess at the general result by looking at some special cases. For $N = 2$,

$$\begin{aligned}
\theta^T M \theta &= (\theta_1 \ \theta_2) \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} \\
&= m_{11}\theta_1^2 + m_{21}\theta_2\theta_1 + m_{12}\theta_1\theta_2 + m_{22}\theta_2^2 \\
&= 0 + (-m_{12})(-\theta_1\theta_2) + m_{12}\theta_1\theta_2 + 0 \\
&= 2m_{12}\theta_1\theta_2
\end{aligned}$$

Note that at the outset we could have set the terms m_{11} and m_{22} equal to zero since M is antisymmetric. The last line follows from the antisymmetry of M and the Grassmann nature of θ_i. Hence

$$\begin{aligned}
I_2(M) &= \int d\theta_1 d\theta_2 e^{-2m_{12}\theta_1\theta_2} \\
&= \int d\theta_1 d\theta_2 (1 - 2m_{12}\theta_1\theta_2) \\
&= 2m_{12}
\end{aligned}$$

(Using the Grassmann rules for integration and noting that the power expansion of the exponential terminates because of the Grassmann property). For an antisymmetric 2×2 matrix

$$m_{12} = \sqrt{detM}$$

This gives

$$I_2(M) = 2\sqrt{detM} \tag{1.58}$$

In the case of an odd value of N, say $N = 3$ we may show that

$$e^{-\theta^T M \theta} = 1 - 2(m_{12}\theta_1\theta_2 + m_{23}\theta_2\theta_3 + m_{31}\theta_3\theta_1)$$

So then

$$\begin{aligned}
I_3(M) &= \int d\theta_1 d\theta_2 d\theta_3 (1 - 2(m_{12}\theta_1\theta_2 + m_{23}\theta_3\theta_2 + m_{31}\theta_3\theta_1)) \\
&= 0
\end{aligned}$$

Before making an intelligent guess we go one step further: $N = 4$
It is straightforward to show that

$$\begin{aligned}
\theta^T M \theta &= 2(m_{12}\theta_1\theta_2 + m_{13}\theta_1\theta_3 + m_{14}\theta_1\theta_4 + m_{23}\theta_2\theta_3 \\
&\quad + m_{24}\theta_2\theta_4 + m_{34}\theta_3\theta_4)
\end{aligned}$$

In the exponential $e^{-\theta^T M \theta}$ we need only retain the term involving $\theta_1\theta_2\theta_3\theta_4$ since all other terms will vanish by (1.57) and (1.54). We find that

$$\begin{aligned}
e^{-\theta^T M \theta} &= \ldots + \frac{1}{2!}(\theta^T M \theta)^2 + \ldots \\
&= \ldots + \frac{4.2}{2!}(m_{12}m_{34} - m_{13}m_{24} + m_{14}m_{23})\theta_1\theta_2\theta_3\theta_4
\end{aligned}$$

Hence

$$I_4(M) = 4(m_{12}m_{34} - m_{13}m_{24} + m_{14}m_{23} = 4\sqrt{detM} \tag{1.59}$$

This generalises to give

$$I_N(M) = 2^{\frac{N}{2}}\sqrt{detM} \tag{1.60}$$

We see from (1.58) and (1.59) that this formula gives the correct result for $N = 2$ and $N = 4$ respectively. It is perhaps early to infer the factor $2^{\frac{N}{2}}$. If we had performed a similar calculation for $N = 6$ it would have been more convincing evidence that $2^{\frac{N}{2}}$ is correct.

(1.60) may be generalised to the infinite-dimensional case and its importance is seen in Chapter 2 when ghost fields are introduced.

When describing Fermi fields we need to make the transition to an infinite dimensional Grassmann algebra. We denote the generators by $\theta(x)$. They obey the relations

$$\{\theta(x), \theta(y)\} = 0$$

$$\frac{\partial^{L,R}\theta(x)}{\partial\theta(y)} = \delta(x-y)$$

$$\int dC(x) = 0 \qquad \int C(x)dC(x) = 1$$

We consider the following Lagrangian for a free Dirac field :

$$\mathcal{L} = \psi^*(i\not{\partial} - m)\psi$$

The normalised generating functional for free Dirac fields is

$$Z_0[\eta, \eta^*] = \frac{1}{N}\int \mathcal{D}\psi^*\mathcal{D}\psi e^{i\int[\psi^*(x)(i\not{\partial}-m)\psi(x)+\eta^*(x)\psi(x)+\psi^*(x)\eta(x)]d^4x} \tag{1.61}$$

N is the normalisation factor and is given by

$$N = \int \mathcal{D}\psi^*\mathcal{D}\psi e^{i\int\psi^*(x)(i\not{\partial}-m)\psi(x)d^4x}$$

$\eta(x)$ and $\eta^*(x)$ are the sources for the fields $\psi^*(x)$ and $\psi(x)$ respectively. By differentiating (1.61) functionally with respect to the sources we can calculate the Greens functions. Firstly we simplify the appearance of (1.61) by introducing the notation

$$S^{-1} = i\not{\partial} - m$$

Then

$$Z_0[\eta, \eta^*] = \frac{1}{N}\int \mathcal{D}\psi^*\mathcal{D}\psi e^{i\int(\psi^*S^{-1}\psi+\eta^*\psi+\psi^*\eta)d^4x} \tag{1.62}$$

By putting

$$Q(\psi, \psi^*) = \psi^*S^{-1}\psi + \eta^*\psi + \psi^*\eta \tag{1.63}$$

we would like to find the value of ψ which minimises Q. This approach is similar to the saddle point evaluation of the path integral. Integrals of the form

$$I \equiv \int dx\; e^{-a(x)}$$

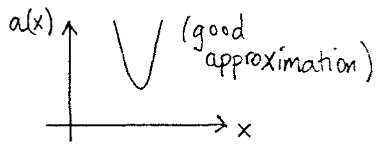
where $a(x)$ is a function of x can be approximated by expanding $a(x)$ around x_0 where $a(x)$ is stationary.

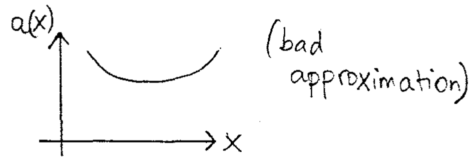
$$a(x) \simeq a(x_0) + \frac{1}{2}(x-x_0)^2 a''(x_0) + \ldots$$

Then

$$I \simeq e^{-a(x_0)} \int dx e^{-\frac{1}{2}(x-x_0)^2 a''(x_0)}$$

If $a''(x_0) > 0$ (ie. the function is concave up) the integral becomes a Gaussian and is easily evaluated. The approximation will be good if $a(x)$ climbs away rapidly from the minimum value at x_0





We apply this technique to approximate the integral in (1.62). Instead of minimising the function $a(x)$ we work with the function $Q(\psi, \psi^*)$. To find the value of ψ which minimises Q we must calculate the partial derivatives $\frac{\partial Q}{\partial \psi}$ and $\frac{\partial Q}{\partial \psi^*}$ and set them equal to zero.

$$\frac{\partial Q}{\partial \psi} = \psi^* S^{-1} + \eta^* \quad \text{and} \quad \frac{\partial Q}{\partial \psi} = 0 \quad \Rightarrow \quad \psi_m^* = -\eta^* S$$

$$\frac{\partial Q}{\partial \psi^*} = S^{-1}\psi + \eta \quad \text{and} \quad \frac{\partial Q}{\partial \psi^*} = 0 \quad \Rightarrow \quad \psi_m = -S\eta$$

We have assumed that S^{-1} has an inverse. The minimum value of Q is

$$Q_m = Q(\psi_m, \psi_m^*) = -\eta^* S \eta$$

We then have

$$Q = Q_m + (\psi^* - \psi_m^*) S^{-1} (\psi - \psi_m)$$

and

$$\begin{aligned} Z_0 &= \frac{1}{N} \int \mathcal{D}\psi^* \mathcal{D}\psi e^{i\int [Q_m + (\psi^* - \psi_m^*) S^{-1} (\psi - \psi_m)] d^4x} \\ &= \frac{1}{N} e^{[-i\int \eta^*(x) S \eta(y) d^4x d^4y]} \int \mathcal{D}\psi^* \mathcal{D}\psi e^{i\int [(\psi^* - \psi_m^*) S^{-1} (\psi - \psi_m)] d^4x} \end{aligned}$$

Notice that $e^{i\int Q_m d^4x}$ has been taken outside the path integral since it does not depend on ψ or ψ^*. If we make the change of variables

$$\psi'^* = \psi^* - \psi_m^* \quad \text{and} \quad \psi' = \psi - \psi_m$$

then we find

$$\mathcal{D}\psi'^* = \mathcal{D}\psi^* \quad \text{and} \quad \mathcal{D}\psi' = \mathcal{D}\psi$$

This gives

$$Z_0 = \frac{1}{N} e^{[-i\int \eta^*(x) S \eta(y) d^4x d^4y]} \int \mathcal{D}\psi^{*\prime} \mathcal{D}\psi' e^{i\int (\psi^{*\prime} S^{-1} \psi') d^4x}$$

Dropping the primes it is clear that $N = e^{i\int(\psi^* S^{-1}\psi)d^4x}$ will cancel with the path integral term on the numerator. This means that

$$Z_0[\eta, \eta^*] = e^{[-i\int \eta^*(x) S(x-y)\eta(y) d^4x d^4y]}$$

We now show that S exists. It is given by

$$S(x) = (i\,\partial\!\!\!/ + m)\Delta_F(x)$$

where $\Delta_F(x)$ is the Feynman propagator. Note that

$$\begin{aligned} S^{-1}S &= (i\,\partial\!\!\!/ - m)(i\,\partial\!\!\!/ + m)\Delta_F(x) \\ &= (-\partial^\mu \partial_\mu - m^2)\Delta_F(x) \\ &= \delta^4(x) \end{aligned}$$

The free particle propagator for the Dirac field is defined by

$$\tau(x,y) = -\frac{\delta^2 Z_0[\eta,\eta^*]}{\delta\eta(x)\delta\eta^*(y)}_{|\eta=\eta^*=0}$$

This is equal to

$$-\frac{\delta}{\delta\eta(x)}\frac{\delta}{\delta\eta^*(y)}\{-i\int \eta^*(x)S(x-y)\eta(y)d^4x d^4y\}_{|\eta=\eta^*=0}$$

This last line is just $iS(x-y)$. It is convenient to summarise our formulae for the free propagators of scalar and spinor fields. The free particle Lagrangian for scalar fields is given by

$$\mathcal{L}_0 = \frac{1}{2}\partial_\mu\psi\partial^\mu\psi - \frac{1}{2}m^2\psi^2 = -\frac{1}{2}\psi(\partial^\mu\partial_\mu + m^2)\psi$$

and the 2-point function was found to be

$$\tau(x,y) = i\Delta_F(x-y)$$

For spinor fields the Lagrangian is

$$\begin{aligned} \mathcal{L}_0 &= i\psi^*\gamma^\mu\partial_\mu\psi - m\psi^*\psi \\ &= \psi^*(i\,\partial\!\!\!/_\mu - m)\psi \\ &= \psi^* S^{-1}\psi \end{aligned}$$

and the 2-point function is i times the propagator

$$\tau(x, y) = iS(x - y)$$

In each case the propagator is the inverse of the operator appearing in the quadratic term of the Lagrangian.

Chapter 2

Gauge Field Theories

2.1 Quantisation of Gauge Field Theory

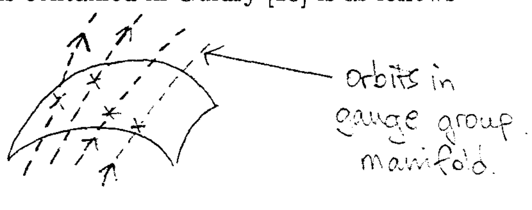
We begin by considering the generating functional

$$Z = \int \mathcal{D}A_\mu e^{iS[A_\mu]} \tag{2.1}$$

where the action $S[A_\mu]$ is invariant under a gauge transformation. We write

$$S[A_\mu] = S[A'_\mu]$$

where A'_μ is related to A_μ by a gauge transformation. Notice that in (2.1) the integration is taken over *all* fields, including those field configurations which are related to A_μ by a gauge transformation. This gives an infinite contribution to Z and leads to an overcounting of the physical configurations.
The set of all fields $A'_\mu(x)$ (for a given space-time point x) which are related to $A_\mu(x)$ by a gauge transformation constitute an *orbit.* Our aim is to devise an integration procedure which selects a unique representative from each orbit. The restriction which we introduce to achieve this is called gauge fixing. A useful illustration which is contained in Guidry [13] is as follows -



The gauge fixing constraint is of the form $f_i(A_\mu) = 0$ where $i = 1,2,3$. When performing the integration in (2.1) we would like to count each orbit once only. This is achieved by taking the point at which the orbit crosses the surface defined by the gauge fixing condition (2.1).
We now introduce the Fadeev-Popov procedure which applies the above ideas in the Feynman Path Integral.

2.2 The Fadeev-Popov method

A good description of this idea is found in Lee [17] and the following discussion follows this approach. As mentioned above the path integral (2.1) is to be performed over distinct orbits of $A_\mu(x)$. This idea was implemented using gauge fixing. We can define the gauge condition by the equation

$$F^a[A^b_\mu] = 0 \tag{2.2}$$

(a and b are internal indices.) Since the divergence due to "overcounting" in (2.1) is due to an integration over group space we recall some facts about integration over group representations. Given a symmetry group G and two elements $g, g' \in G$ we define the Hurwitz measure as an integration measure which is invariant in the sense that

$$dg' = d(g'g)$$

The infinitesimal form of the general gauge transformation associated with

$$U = e^{i\omega^a(x)M^a}$$

is

$$U(\omega) = 1 + i\omega^a M^a + O(\omega^2) \tag{2.3}$$

Notice that the gauge transformation is determined by the gauge parameters $\omega^a(x)$. We may then choose the group measure to be

$$dg = \prod_a d\omega^a = d\,\omega \qquad \text{where } \; g \simeq 1$$

We also have

$$A'^a_\mu \equiv (A_\omega)^a_\mu = A^a_\mu + f^{abc}A^b_\mu\omega^c + \partial_\mu\omega^a \tag{2.4}$$

where f^{abc} are the structure constants of the group G. Consider now the quantity

$$\Delta^{-1}[A] = \int \mathcal{D}\,\omega \;\; \delta(F[A_\omega]) = \int \mathcal{D}g\; \delta(F[A_g]) \tag{2.5}$$

where

$$\mathcal{D}\,\omega \;= \prod_x d\,\omega\;(x)$$

$$\mathcal{D}g = \prod_x dg(x)$$

and

$$\delta(F[A_\omega]) = \prod_{b,x} \delta(F^b[(A^a_\mu(x))_\omega] \tag{2.6}$$

is a product of Dirac delta functions one at each space time point. In this formula A is short for A^a_μ.

Notice that the measure $\mathcal{D}g$ in (2.5) has an infinite number of degrees of freedom, one at each space time point. If we let $g_i = g(x_i)$ then we may write (2.5) as

$$\int \prod_i dg_i \delta(F[A_{g_i}])$$

Consider a space time point $x = x_p$. We may write (2.5) as

$$\int \prod_{i,i \neq p} dg_i \delta(F[A_{g_i}]))dg_p \delta(F[A_{g_p}])$$

Examining the g_p integration we see that we will only obtain a contribution to the overall integral when g_p assumes the value which makes $F(A_{g_p}) = 0$. At each space time point x_q there is a unique element of the group (say g_{x_q}) which contributes to the integration.
We now observe that $\Delta^{-1}[A]$ is gauge invariant -

$$\begin{aligned} \Delta^{-1}[A_g] &= \int \mathcal{D}g' \ \delta(F[A_{g'g}]) \\ &= \int \mathcal{D}(g'g) \ \delta(F[A_{g'g}]) \\ &= \int \mathcal{D}(g'') \ \delta(F[A_{g''}]) \\ &= \Delta^{-1}[A] \end{aligned}$$

Notice that $g'' = g'g$ and $\mathcal{D}g'' = \mathcal{D}g'$. Since $\Delta[A]$ is defined by $\Delta^{-1}[A]\Delta[A] = 1$ we therefore have

$$\Delta[A] \int \mathcal{D} \ \omega \ \delta(F[A_\omega]) = 1 \tag{2.7}$$

We may insert this into the expression for the path integral giving -

$$\int \mathcal{D}A_\mu e^{iS} = \int \mathcal{D}A_\mu \Delta[A_\mu] \int \mathcal{D} \ \omega \ \delta[F((A_\mu)_\omega])e^{iS}$$

Performing a gauge transformation $A_\mu \to (A_\mu)_{\omega^{-1}}$ gives

$$\int \mathcal{D}A_\mu e^{iS} = \int \mathcal{D}A_\mu \Delta[A_\mu] \int \mathcal{D} \ \omega \ \delta(F[A])e^{iS} \tag{2.8}$$

Under the transformation the action and $\Delta[A]$ remain invariant. But now nothing in the expression depends on ω and the integral over gauge transformations $\mathcal{D} \ \omega$ is a multiplicative factor associated with the "overcounting". If we therefore *redefine* Z as

$$Z = \int \mathcal{D}A_\mu \Delta[A_\mu] \ \delta(F[A_\mu])e^{iS[A]} \tag{2.9}$$

the difficulties in (2.1) will be removed.
We must now find an expression for $\Delta[A_\mu]$

$$F^a[A_\omega] = F^a[A] + \frac{\partial F^a}{\partial A^b_\mu} \delta A^b_\mu + \ldots$$

$$= F^a[A] + \frac{\partial F^a}{\partial A^b_\mu}(D_\mu \omega)^b$$

for infinitesimal transformations ω close to zero. D_μ is the covariant derivative

$$D^{bd}_\mu = \delta^{bd}\partial_\mu + f^{bdc}A^c_\mu$$

Choose A such that $F^a[A] = 0$ then (for infinitesimal ω)

$$F^a[A_\omega] = \frac{\partial F^a}{\partial A^b_\mu}(D_\mu \omega)^b \tag{2.10}$$

and

$$\Delta^{-1}[A] = \int \mathcal{D}\omega^a \ \delta F^a[A_\omega] \tag{2.11}$$

It can be shown that this expression equals $(det\ M)^{-1}$ where

$$M_{ab}(x,y) = \frac{\delta}{\delta\omega^b(y)}(F^a[(A_\mu(x))_g])|_{g=identity} \tag{2.12}$$

Now one can apply (2.10) since ω is now infinitesimal. Hence using the more compact notation we have

$$\Delta(A) = det|\frac{\delta F}{\delta\omega}|_{F=0}$$

We should note that the right hand side of (2.12) is

$$\begin{aligned} &= \frac{\delta}{\delta\omega^b(y)}[\frac{\delta F^a}{\delta A^c_\mu(x)}D^{cd}_\mu \omega^d(x)]_{\omega=0} \\ &= \frac{\delta F^a}{\delta A^c_\mu(x)}[-\delta^{cb}\partial_\mu + f^{cbd}A^d_\mu]\delta^4(x-y) \end{aligned}$$

2.3 Feynman Rules for Gauge Theories

A clear treatment of this is contained in Ryder [5]. We begin with the generating functional

$$Z = \int \mathcal{D}A_\mu \delta(F[A]) det|\frac{\delta F}{\delta\omega}|e^{iS[A]} \tag{2.13}$$

Note that we could have chosen the more general gauge condition

$$F^a[A] - c^a(x) = 0 \tag{2.14}$$

where $c^a(x)$ is an arbitrary space-time function. Since $c^a(x)$ is independent of A the determinant $\Delta[A]$ is unaffected. We may then write (2.13) as

$$Z = \int \mathcal{D}A_\mu \Delta[A]\delta(F[A] - c)e^{iS[A]} \tag{2.15}$$

Since Z in (2.15) is independent of $c^a(x)$ we may include a weighting factor of

$$e^{-\frac{i}{2\alpha}\int c^{a2}(x)d^4x}$$

in (2.15). This just results in a change in the normalisation of Z and gives

$$Z = N\int \mathcal{D}A_\mu \Delta[A] e^{i\int d^4x(\mathcal{L}_{YM}-\frac{1}{2\alpha}F[A]^2)} \tag{2.16}$$

Here we have written $S[A]$ as the integral of the Yang-Mills Lagrangian ($\int \mathcal{L}_{YM}$) where $\mathcal{L}_{YM} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$. Dropping the N we may take the above equation as our starting point for the definition of the generating functional. The next step is to change the determinant in (2.13) into an exponential by introducing the Fadeev- Popov ghost fields. Letting $\Delta[A] = det(iM)$ and using the fact that

$$det(iM) = \int \mathcal{D}\eta\mathcal{D}\eta^* e^{-i\int d^4x(\eta_a^* M_{ac}\eta_c)} \tag{2.17}$$

(This last result arises from the properties of integration on a Grassmann algebra, the η and η^* being Grassmann fields. They are called Fadeev-Popov ghosts.) we get

$$Z = N\int \mathcal{D}A_\mu\mathcal{D}\eta\mathcal{D}\eta^* e^{i\int d^4x(\mathcal{L}_{YM}-\frac{1}{2\alpha}(F[A])^2-\eta^{a*}M_{ab}\eta^b)}$$

We may write this as

$$Z = N\int \mathcal{D}A_\mu\mathcal{D}\eta\mathcal{D}\eta^* e^{iS_{eff}[A,\eta^*,\eta]} \tag{2.18}$$

where the effective action is given by

$$S_{eff}[A,\eta^*,\eta] = \int d^4x[\mathcal{L}_{YM} - \frac{1}{2\alpha}(F[A])^2 - \eta_a^* M_{ab}\eta_b] \tag{2.19}$$

We are now in a position to write down the Feynman rules. We begin by computing M_{ab} in the Lorentz gauge which is given by $F^a = \partial^\mu A_\mu^a = 0$. In the abelian case we obtain

$$M_{ab}(x,y) = \partial_\mu\partial^\mu\delta^4(x-y)\delta_{ab} \tag{2.20}$$

In Quantum Electrodynamics (QED) we are dealing with the group $U(1)$ which is an abelian group. The rule for the gauge transformation of the potential becomes

$$\delta A_\mu = \partial_\mu\omega$$

If $F = \partial^\mu A_\mu$ we obtain $M = \frac{\delta F}{\delta\omega} = \partial_\mu\partial^\mu\delta^4(x-y)$ From (2.12) (in the non-abelian case) we have

$$M_{ab}(x,y) = \partial^\mu(\partial_\mu\delta^{ab} - f^{abc}A_\mu^c)\delta^4(x-y) \tag{2.21}$$

The ghost part of the generating functional is

$$\int \mathcal{D}\eta^* \mathcal{D}\eta e^{-i\int d^4x(\eta^{a*}\frac{\delta F^a}{\delta\omega^b}\eta^b)} \tag{2.22}$$

The exponent is (reintroducing g)

$$-i\int d^4x\ \eta^{a*}(x)\partial^\mu(\partial_\mu\delta^{ab} - gf^{abc}A^c_\mu(x))\eta^b(x) \tag{2.23}$$

This integral may be written as

$$-i\int d^4x\ \eta^{a*}(\partial^\mu\partial_\mu)\delta^{ab}\eta^b + igf^{abc}\int d^4x\ (\partial^\mu A^c_\mu)\eta^{a*}\eta^b + \eta^{a*}(\partial^\mu\eta^b)A^c_\mu$$

The ghost propagator may be defined as the inverse of the operator appearing in the quadratic term. By switching to momentum space and taking the inverse operator gives -

$$\frac{i}{k^2}\delta^{ab} \tag{2.24}$$

This may be represented diagramatically by

$$a \cdots\cdots k \cdots\cdots b$$

The (Lorentz) gauge field propagator is found in the same way but in this case we must examine the term in (2.19) which is quadratic in the gauge field. We find this term by considering the gauge fixing term (computed in the Lorentz gauge) and writing out $\mathcal{L}_{YM}$ explicitly in terms of the gauge field

$$\mathcal{L}_{YM} = \frac{-1}{4}F^a_{\mu\nu}F^{a\mu\nu} = \frac{-1}{4}[\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu]$$
$$\times[\partial^\mu A^{\nu a} - \partial^\nu A^{\mu a} + gf^{amn}A^{\mu m}A^{\nu n}]$$

This may be expanded to give -

$$\frac{-1}{4}[\text{quadratic term} + 2gf^{abc}A^b_\mu A^c_\nu(\partial^\mu A^{\nu a} - \partial^\nu A^{\mu a}) + g^2 f^{abc}f^{amn}A^b_\mu A^c_\nu A^{\mu m}A^{\nu n}] \tag{2.25}$$

The quadratic term is $2(\partial_\mu A^a_\nu)(\partial^\mu A^{\nu a}) - 2(\partial_\mu A^a_\nu)(\partial^\nu A^{\mu a})$. By combining this with the gauge fixing term we may write the quadratic part of S_{eff} as -

$$\int d^4x\ \left(\frac{-1}{2}(\partial_\mu A^a_\nu)(\partial^\mu A^{\nu a}) + \frac{1}{2}(\partial_\mu A^a_\nu)(\partial^\nu A^{\mu a}) - \frac{1}{2\alpha}(\partial_\mu A^{\mu a})(\partial_\nu A^{\nu a})\right)$$

Integrating by parts gives

$$\frac{1}{2}\int d^4x\ A^{\mu a}[g_{\mu\nu}\partial_\rho\partial^\rho - (1-\frac{1}{\alpha})\partial_\mu\partial_\nu]A^{\nu a}$$

By switching to momentum space via the Fourier transform

$$A^{\mu a}(x) = \int \frac{d^4k}{(2\pi)^4}e^{-ik.x}\tilde{A}^{\mu a}(k) \tag{2.26}$$

gives -

$$\frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\tilde{A}^{\mu a}(k)[-k^2 g_{\mu\nu} + (1-\frac{1}{\alpha})k_\mu k_\nu]\tilde{A}^{\mu a}(-k) \tag{2.27}$$

The propagator is calculated by finding the inverse of the operator in square brackets. Let us assume that the required operator is of the form

$$X(k)g_{\mu\nu} + Y(k)k_\mu k_\nu$$

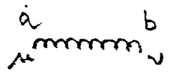
Then we require

$$g_{\mu\rho} = (X(k)g_{\mu\nu} + Y(k)k_\mu k_\nu)(-k^2 g_{\nu\rho} + (1-\frac{1}{\alpha})k_\nu k_\rho)$$

By solving for $X(k)$ and $Y(k)$ we find that

$$X(k) = -\frac{1}{k^2} \quad \text{and} \quad Y(k) = -\frac{\alpha - 1}{k^4} \tag{2.28}$$

The inverse operator and hence the propagator is given by



$$= -\frac{i}{k^2}[g_{\mu\nu} + (\alpha - 1)\frac{k_\mu k_\nu}{k^2}]\delta^{ab} \tag{2.29}$$

One of the interesting features about the Yang Mills Lagrangian is that the gauge field couples to itself. Referring back to (2.25) we see that $\mathcal{L}_{\text{eff}}$ contains terms which are cubic and quartic in A. The cubic term is -

$$-\frac{1}{4}[2gf^{abc}A^b_\mu A^c_\nu(\partial^\mu A^{\nu a} - \partial^\nu A^{\mu a})]$$

which equals

$$-gf^{abc}A^b_\mu A^c_\nu \partial^\mu A^{\nu a}$$

(found by relabelling the indices and noting the antisymmetry of f^{abc}). We therefore consider the integral

$$-gi\int d^4x \; f^{abc}A^b_\mu(x)A^c_\nu(x)\partial^\mu A^{\nu a}(x) \tag{2.30}$$

Switching to momentum space via (2.26) gives

$$-gf^{abc}\int \frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\frac{d^4r}{(2\pi)^4}p^\mu g^{\lambda\nu}(2\pi)^4\delta(p+q+r)\tilde{A}^b_\mu(q)\tilde{A}^c_\nu(r)\tilde{A}^a_\lambda(p)$$

($\delta(p+q+r)$ comes from the term $\int d^4x e^{-i(p+q+r)\cdot x}$.) Note that there are 3! ways of interchanging the terms of A. For example $(\lambda, a, p) \leftrightarrow (\mu, b, q)$ would mean that

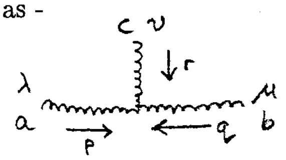
$$-gf^{abc}p^\mu g^{\lambda\nu}\tilde{A}^b_\mu(q)\tilde{A}^c_\nu(r)\tilde{A}^a_\lambda(p)$$
$$= -gf^{bac}q^\lambda g^{\mu\nu}\tilde{A}^a_\lambda(p)\tilde{A}^c_\nu(r)\tilde{A}^b_\mu(q)$$

$$= gf^{abc}q^{\lambda}g^{\mu\nu}\tilde{A}^a_{\lambda}(p)\tilde{A}^c_{\nu}(r)\tilde{A}^b_{\mu}(q)$$

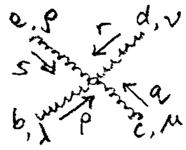
(using the antisymmetry of f^{abc}.) We find that (lowering the indices on the metric tensor)

$$gf^{abc}p_{\mu}g_{\lambda\nu} = \frac{1}{3!}gf^{abc}[(r-q)_{\lambda}g_{\mu\nu} + (q-p)_{\nu}g_{\lambda\mu} + (p-r)_{\mu}g_{\nu\lambda}] \tag{2.31}$$

We represent this coupling as -



(with $p+q+r=0$.) The quartic term gives rise to the coupling



which represents the mathematical expression

$$-ig^2[f_{abc}f_{ade}(g_{\lambda\nu}g_{\mu\rho} - g_{\mu\nu}g_{\lambda\rho}) + f_{adc}f_{abe}(g_{\nu\lambda}g_{\mu\rho} - g_{\mu\lambda}g_{\nu\rho})$$
$$+f_{abd}f_{ace}(g_{\lambda\mu}g_{\nu\rho} - g_{\nu\mu}g_{\lambda\rho})] \tag{2.32}$$

We may also compute the gauge field propagator in the axial gauge which is defined by the condition

$$t^{\mu}A^a_{\mu} = 0 \qquad t^{\mu}t_{\mu} = -1 \tag{2.33}$$

(t is a space-like vector.) We define the gauge-fixing term by

$$F^a = t^{\mu}A^a_{\mu} \tag{2.34}$$

then

$$\begin{aligned} \delta F^a &= t^{\mu}\delta A^a_{\mu} \\ &= f^{abc}\omega^b t^{\mu}A^c_{\mu} + t^{\mu}\partial_{\mu}\omega^a \\ &= t^{\mu}\partial_{\mu}\omega^a \end{aligned} \tag{2.35}$$

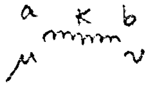
Hence

$$\frac{\delta F^a}{\delta\omega^b} = \delta^{ab}t^{\mu}\partial_{\mu} \tag{2.36}$$

We extract the term quadratic in A_{μ} by considering

$$\int d^4x\,[-\frac{1}{4}(\partial_{\mu}A^a_{\nu} - \partial_{\nu}A^a_{\mu})(\partial^{\mu}A^{\nu a} - \partial^{\nu}A^{\mu a}) - \frac{1}{2\alpha}(t^{\mu}A^a_{\mu})^2]$$

By proceeding as before we find (in the limit as $\alpha \to 0$)



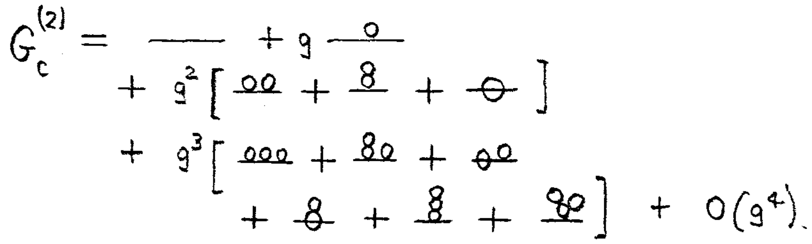
$$= -\frac{i}{k^2}[g^{\mu\nu} + \frac{t^2}{(k.t)^2}k^{\mu}k^{\nu} - \frac{k^{\mu}t^{\nu} + t^{\mu}k^{\nu}}{k.t}]\delta^{ab} \tag{2.37}$$

2.4 The self energy operator and the vertex function

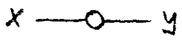
We recall from Chapter 1 that the connected Green's functions were defined by

$$G_c^{(n)}(x_1 \ldots x_n) = \frac{1}{i^{n-1}} \frac{\delta^n W[J]}{\delta J(x_1) \ldots \delta J(x_n)}_{|J=0} \tag{2.38}$$

where $Z = e^{iW}$. The 2-point function may be expanded to *all* orders (ignoring factors of i and numerical factors) as -



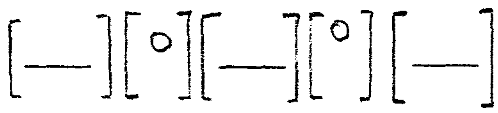
We wish to find a way of summing the above graphs. We call the sum the *complete* or *dressed* propagator and denote it by -



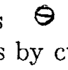
$$= G_c^{(2)}(x, y) \tag{2.39}$$

Note that the first graph

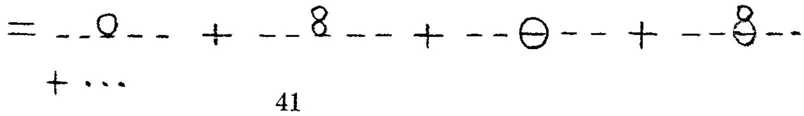
may be decomposed into a product



where the first and last terms are external propagators. These external propagators may be removed by mutiplying by inverse propagators. This multiplication produces a *truncated* graph which we can denote by

The second graph in g^2 becomes  and the third graph becomes  . Note that the first graph of order g^2 may be broken into two parts by cutting one internal line. We call this graph *1-particle reducible* . This property is not shared by the second or third graphs which are called *1-particle irreducible* (1PI) graphs. We define the *proper self-energy* as the sum of (1PI) graphs. We denote this as follows -

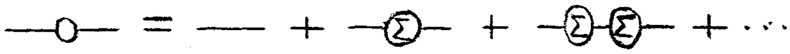
$$= \frac{1}{i}\Sigma(p)$$



We may write G_c^2 in momentum space as follows - (note that $G_0(p) = \frac{i}{p^2-m^2}$ is the bare propagator.)

$$
\begin{aligned}
G_c^2(p) &= G_0(p) + G_0(p)\frac{\Sigma(p)}{i}G_0(p) + G_0(p)\frac{\Sigma(p)}{i}G_0(p)\frac{\Sigma(p)}{i}G_0(p) + \dots \\
&= G_0(1 + \frac{\Sigma}{i}G_0 + \frac{\Sigma}{i}G_0\frac{\Sigma}{i}G_0 + \dots) \\
&= G_0(1 - \frac{\Sigma}{i}G_0)^{-1} \\
&= [G_0^{-1}(p) - \frac{1}{i}\Sigma(p)]^{-1} \\
&= \frac{i}{p^2 - m^2 - \Sigma(p)}
\end{aligned}
$$

This expansion in terms of $\Sigma(p)$ may be represented graphically as



This expansion is also used in the theory of QED when calculating the photon and electron propagators.
If we define the physical mass m_{phys} by

$$m_{\text{phys}}^2 = m^2 + \Sigma(p) \tag{2.40}$$

then we may write

$$G_c^2(p) = \frac{i}{p^2 - m_{\text{phys}}^2} \tag{2.41}$$

Note that $G_c^{(2)^{-1}}(p)$ is an example of a *vertex function.* We define $\Gamma^{(2)}(p)$ by demanding that

$$\Gamma^{(2)}(p)G_c^{(2)}(p) = i$$

$$\Leftrightarrow \Gamma^{(2)}(p) = i[G_c^{(2)}(p)]^{-1} = p^2 - m^2 - \Sigma(p)$$

We may derive an identity relating the connected 3-point function and the irreducible (1PI) 3 point vertex. Our starting point is to construct the Legendre transform of $Z[J]$ defined in Chapter 1. Let $\psi_{cl}(x)$ be the functional of J defined through

$$\psi_{cl}(x) = \frac{\delta W[J]}{\delta J(x)} \tag{2.42}$$

We denote the propagator by $G(x,y)$ and the inverse propagator by $\Gamma(x,y)$. The functional $\Gamma[\psi_{cl}]$ is by definition

$$\Gamma[\psi_{cl}] = W[J] - \int d^4x \; J(x)\psi_{cl}(x) \tag{2.43}$$

and this gives

$$\frac{\delta\Gamma[\psi_{cl}]}{\delta\psi_{cl}(x)} = -J(x) \tag{2.44}$$

As expressed in Itzykson [11] we have

$$\begin{aligned}
\delta^4(x-y) &= \frac{\delta\psi_{cl}(x)}{\delta\psi_{cl}(y)} \\
&= \frac{\delta}{\delta\psi_{cl}(y)}\frac{\delta W[J]}{\delta J(x)} \\
&= \int d^4z\, \frac{\delta J(z)}{\delta\psi_{cl}(y)}\frac{\delta^2 W[J]}{\delta J(z)\delta J(x)} \\
&= -\int d^4z\, \frac{\delta^2\Gamma[\psi_{cl}]}{\delta\psi_{cl}(z)\delta\psi_{cl}(y)}\frac{\delta^2 W[J]}{\delta J(z)\delta J(x)} \\
&= -\int d^4z\, \Gamma(z,y)G(x,z)
\end{aligned} \tag{2.45}$$

Changing the dummy indices we may write the relationship as

$$-\delta^4(x-z') = \int d^4z\, \frac{\delta^2 W}{\delta J(x)\delta J(z)}\frac{\delta^2\Gamma}{\delta\psi_{cl}(z)\delta\psi_{cl}(z')}$$

which implies that

$$\Gamma(z,y) = \frac{\delta^2\Gamma[\psi_{cl}]}{\delta\psi_{cl}(z)\delta\psi_{cl}(y)}$$

Differentiating with respect to J'' using

$$\begin{aligned}
\frac{\delta}{\delta J(x'')} &= \int d^4z''\, \frac{\delta\psi_{cl}(z'')}{\delta J(x'')}\frac{\delta}{\delta\psi_{cl}(z'')} \\
&= -\int d^4z'' G(x'',z'')\frac{\delta}{\delta\psi_{cl}(z'')}
\end{aligned}$$

gives

$$\begin{aligned}
&\int d^4z \frac{\delta^3 W}{\delta J(x)\delta J(x'')\delta J(z)}\frac{\delta^2\Gamma}{\delta\psi_{cl}(z)\delta\psi_{cl}(z')} \\
&-\int d^4z\, d^4z'' \frac{\delta^2 W}{\delta J(x)\delta J(z)}G(x'',z'')\frac{\delta^3\Gamma}{\delta\psi_{cl}(z)\delta\psi_{cl}(z')\delta\psi_{cl}(z'')} = 0
\end{aligned}$$

Hence

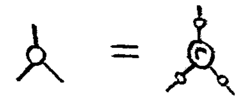
$$\begin{aligned}
&\int d^4z \frac{\delta^3 W}{\delta J(x)\delta J(x'')\delta J(z)}\Gamma(z,z') \\
&+\int d^4z\, d^4z' G(x,z)G(x'',z'')\frac{\delta^3\Gamma}{\delta\psi_{cl}(z)\delta\psi_{cl}(z')\delta\psi_{cl}(z'')} = 0
\end{aligned}$$

If we now multiply by $G(x',z')$ and integrate over z' using (2.45) we obtain

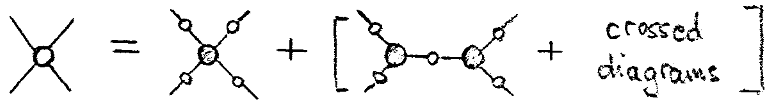
$$\frac{\delta^3 W}{\delta J(x)\delta J(x'')\delta J(x')} = -\int d^4z\, d^4z'\, d^4z''\, G(x,z)G(x',z')G(x'',z'')$$

$$\times \frac{\delta^3\Gamma}{\delta\psi_{cl}(z)\delta\psi_{cl}(z')\delta\psi_{cl}(z'')} \tag{2.46}$$

The effect of propagators in the right hand side is to add external legs to $\delta^3\Gamma\ldots$ which is the truncated 3-point function. We may represent this graphically as



We obtain a similar identity for the 4-point function. Its graphical representation is -



2.5 Ward Takahashi identities in QED

A relationship between 1PI vertex functions and propagators follows from the gauge invariance of QED. This relationship is necessary when proving the renormalisation of QED. We start with the generating functional Z for a system of photons and electrons given by -

$$Z = N\int \mathcal{D}A_\mu \mathcal{D}\psi^* \mathcal{D}\psi e^{i\int \mathcal{L}_{eff} d^4x} \tag{2.47}$$

$$\begin{aligned}\mathcal{L}_{eff} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\psi^*\gamma^\mu(\partial_\mu + ieA_\mu)\psi \\ &-\frac{1}{2\alpha}(\partial^\mu A_\mu)^2 + J^\mu A_\mu + \eta^*\psi + \psi^*\eta \end{aligned} \tag{2.48}$$

($F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$)
Note that the photon is described by the vector potential A_μ and that the Grassmann field describes the electron. The effective Lagrangian contains the free field part, the free field electron part, a gauge fixing term for the Lorentz gauge and source terms for A_μ, ψ and ψ^* (resp. J, η^*, η). We perform a local infinitesimal gauge transformation associated with the group $U(1)$ -

$$\begin{aligned}\psi(x) &\to \psi(x) - ie\Lambda(x)\psi(x) \\ \psi^*(x) &\to \psi^*(x) + ie\Lambda(x)\psi^*(x) \\ A_\mu(x) &\to A_\mu(x) - \partial_\mu\Lambda(x)\end{aligned} \tag{2.49}$$

Under this transformation the first three terms of (2.48) are invariant and

$$\mathcal{L}_{\text{eff}} \to \mathcal{L}'_{\text{eff}} = \mathcal{L}_{\text{eff}} - \frac{1}{\alpha}(\partial^\mu A_\mu)\partial_\rho\partial^\rho\Lambda - ie\Lambda(\eta^*\psi - \psi^*\eta) + (J^\mu\partial_\mu\Lambda)$$

Z transforms as

$$Z \to Z' = e^{i\int d^4x[-\frac{1}{\alpha}(\partial^\mu A_\mu)\partial^\rho\partial_\rho\Lambda) + J^\mu\partial_\mu\Lambda - ie\Lambda(\eta^*\psi - \psi^*\eta)]}Z$$

We may expand the exponential (ignoring terms of order $\Lambda^2, \Lambda^3 \ldots$) to give

$$Z' = \{1 + i \int d^4x \, [-\frac{1}{\alpha}\partial_\rho\partial^\rho(\partial^\mu A_\mu) - \partial^\mu J_\mu - ie(\eta^*\psi - \psi^*\eta)]\Lambda(x)\}Z$$

(the derivative operator has been removed from Λ by integrating by parts.) Since Λ is an arbitrary function $\delta Z = Z' - Z = 0$ (invariance of Z) implies that

$$[-\frac{1}{\alpha}\partial_\rho\partial^\rho(\partial^\mu A_\mu) - \partial^\mu J_\mu - ie(\eta^*\psi - \psi^*\eta)]Z = 0 \tag{2.50}$$

Making the substitutions

$$\psi \to \frac{1}{i}\frac{\delta}{\delta\eta^*} \qquad \psi^* \to \frac{1}{i}\frac{\delta}{\delta\eta} \qquad A_\mu \to \frac{1}{i}\frac{\delta}{\delta J_\mu} \tag{2.51}$$

we obtain the functional differential equation

$$[\frac{i}{\alpha}\partial_\rho\partial^\rho\partial^\mu\frac{\delta}{\delta J^\mu} - \partial^\mu J_\mu - e(\eta^*\frac{\delta}{\delta\eta^*} - \eta\frac{\delta}{\delta\eta})]Z[\eta, \eta^*, J] = 0$$

The corresponding equation for

$$W[\eta, \eta^*, J_\mu] = -i \ln Z[\eta, \eta^*, J_\mu]$$

is

$$[-\frac{\partial_\rho\partial^\rho}{\alpha}\partial^\mu\frac{\delta}{\delta J^\mu} - \partial^\mu J_\mu W^{-1} - ie(\eta^*\frac{\delta}{\delta\eta^*} - \eta\frac{\delta}{\delta\eta})]W[\eta, \eta^*, J^\mu] = 0 \tag{2.52}$$

The effective action which is given by

$$\Gamma(\psi, \psi^*, A_\mu) = W[\eta, \eta^*, J_\mu] - \int d^4x \, (\eta^*\psi + \psi^*\eta + J^\mu A_\mu)$$

implies that

$$\frac{\delta\Gamma}{\delta A_\mu(x)} = -J^\mu(x) \qquad \frac{\delta W}{\delta J_\mu(x)} = A^\mu(x) \qquad \frac{\delta\Gamma}{\delta\psi(x)} = -\eta^*(x)$$

$$\frac{\delta W}{\delta\eta^*(x)} = \psi(x) \qquad \frac{\delta\Gamma}{\delta\psi^*(x)} = -\eta(x) \qquad \frac{\delta W}{\delta\eta(x)} = \psi^*(x)$$

Equation (2.52) then becomes

$$-\frac{\partial_\rho\partial^\rho}{\alpha}\partial^\mu A_\mu(x) + \partial_\mu\frac{\delta\Gamma}{\delta A_\mu(x)} + ie\psi\frac{\delta\Gamma}{\delta\psi(x)} - ie\psi^*\frac{\delta\Gamma}{\delta\psi^*(x)} = 0 \tag{2.53}$$

Repeated differentiation of (2.53) at $\psi^* = \psi = A_\mu = 0$ generates relations between the 1PI diagrams. Letting the operator $\frac{\delta}{\delta A_\nu(y)}$ act on equation (2.53) gives

$$-\frac{1}{\alpha}(\partial_\rho\partial^\rho)_x\partial^\nu\delta(x-y) + \partial_\mu\Gamma^{\mu\nu}(x-y) = 0 \tag{2.54}$$

where

$$\Gamma^{\mu\nu}(x-y) = \frac{\delta^2\Gamma}{\delta A_\mu(x)\delta A_\nu(y)}$$

By switching to momentum space via the Fourier transform (2.26) we obtain

$$-\frac{1}{\alpha}k^2k^\nu - k_\mu\Gamma^{\mu\nu}(k) = 0 \tag{2.55}$$

$\Gamma^{\mu\nu}(k)$ is the Fourier transform of the inverse photon propagator. From (2.27) the inverse propagator for the free theory is

$$\Gamma_0^{\mu\nu}(k) = -g^{\mu\nu}k^2 + k^\mu k^\nu(1-\frac{1}{\alpha})$$

We recall that in $g\psi^4$ theory

$$\Gamma^{(2)} = p^2 - m^2 - \Sigma(p)$$

where $\Sigma(p)$ was the self-energy.
The analogous expression for the inverse photon propagator is -

$$\Gamma^{\mu\nu}(k) = -g^{\mu\nu}k^2 + k^\mu k^\nu(1-\frac{1}{\alpha}) - i\hbar\Pi^{\mu\nu}(k) \tag{2.56}$$

where $\Gamma^{\mu\nu}$ is a self-interaction term. By combining equations (2.55) and (2.56) we obtain the condition

$$k_\mu\Pi^{\mu\nu}(k) = 0 \tag{2.57}$$

This is a necessary condition for the photon to be massless and implies that

$$\Pi^{\mu\nu}(k) = (-g^{\mu\nu}k^2 + k^\mu k^\nu)\Pi(k) \tag{2.58}$$

By substituting this expression into (2.56) we see that the inverse photon propagator does not contain terms like $m^2g^{\mu\nu}$ which would give rise to mass. The identity in (2.52) has a second application. By letting the operator

$$\frac{\delta^2}{\delta\psi^*(x_1)\delta\psi(y_1)}$$

act on (2.52) at $A_\mu = \psi^* = \psi = 0$ we obtain

$$-\partial_x^\mu\frac{\delta^3\Gamma[0]}{\delta\psi^*(x_1)\delta\psi(y_1)\delta A^\mu(x)} = -ie\delta(x-x_1)\frac{\delta^2\Gamma[0]}{\delta\psi^*(x_1)\delta\psi(y_1)}$$

$$+ie\delta(x-y_1)\frac{\delta^2\Gamma[0]}{\delta\psi^*(x_1)\delta\psi(y_1)} \tag{2.59}$$

The left hand side is the derivative of the (1PI) electron-photon vertex whereas the terms on the right are the inverses of exact propagators. The expression becomes more transparent by performing a Fourier transform to momentum space. If the momenta conjugate to x, y_1 and x_1 are q, p and $p' = p + q$ we obtain the *Ward Takahashi* identity

$$q^\mu \Gamma_\mu(p, q, p+q) = S_F'^{-1}(p+q) - S_F'^{-1}(p) \tag{2.60}$$

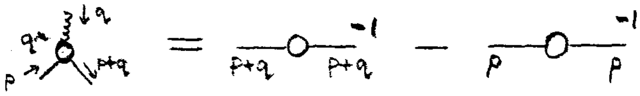
where we denote the inverse propagator

$$\frac{\delta^2 \Gamma}{\delta \psi^* \delta \psi}$$

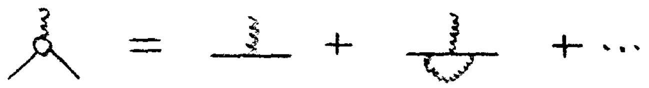
by $S_F'^{-1}$. In the limit as $q_\mu \to 0$ we have

$$\frac{\partial S_F'^{-1}}{\partial p^\mu} = \Gamma_\mu(p, 0, p) \tag{2.61}$$

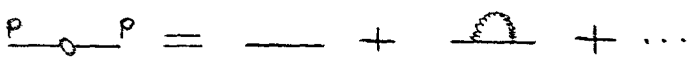
It may be expressed pictorially as



The expansion of $\Gamma_\mu(p, q, p+q)$ to the two lowest orders is



The expansion of $S_F'(p)$ is



Following Berestetskii [6] it is useful at this stage to make some comments about vertex parts. We start by examining graphs corresponding to three-ended (one photon and two electron) sections.

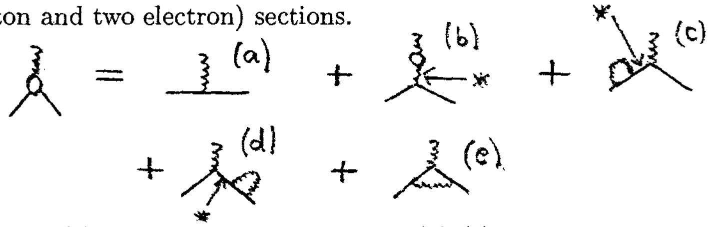


Diagram (a) is to first order. Diagrams (b)-(e) are third order correction terms but they do not all provide essentially new information. By performing cuts across the photon and electron lines (marked with *'s) in (b), (c) and (d) we obtain the simple vertex (a) and a second order self-energy part. We describe (b), (c), (d) as being 1P-reducible. They are correction terms of a particular type and are obtained by replacing the photon and electron lines in (a) by the first order correction terms in the expansion of the complete photon and electron propagators. Diagram (e) is 1PI and is an example of a vertex function.
Vertex parts may be *reducible* or *irreducible*. The *irreducible* ones are those

which do not contain self-energy corrections. eg. the third and fifth order diagrams

The fifth-order diagrams

are examples of reducible vertices because they contain the self-energy parts

To lowest order S'_F is the bare propagator S_F.

$$S_F^{-1}(p) = \gamma_\mu p^\mu - m \quad \text{and} \quad \frac{\partial S_F^{-1}(p)}{\partial p^\mu} = \gamma^\mu$$

We may calculate $\Gamma_\mu(p,0,p)$ to lowest order. Our starting point is

$$\frac{\delta^3\Gamma}{\delta\psi^*(x_1)\delta\psi(y_1)\delta A^\mu(x)} = -\int d^4u_1\ d^4v_1\ d^4u\ [S_F^{-1}(u_1-x_1)S_F^{-1}(v_1-y_1)]$$

$$\times\{-D_{\mu\nu}^{-1}(u-x)(-i)\frac{\delta^3 Z[0]}{\delta\eta(u_1)\delta\eta^*(v_1)\delta J^\mu(u)}\} \tag{2.62}$$

S_F is the bare electron propagator and $D_{\mu\nu}$ is the photon propagator function. It can be shown to lowest order that

$$\frac{\delta^3 Z[0]}{\delta\eta(u_1)\delta\eta^*(v_1)\delta J^\mu(u)} = ie\int d^4z\ S_F(u_1-z)S_F(v_1-z)D_{\mu\nu}(u-z)\gamma^\nu$$

By substituting this into (2.62) and performing a Fourier transform gives, to lowest order

$$\Gamma_\mu(p,q,p+q) = \gamma_\mu$$

This therefore verifies (2.61) to lowest order. We may verify the identity (2.61) to the next order. We shall proceed by rewriting the Ward identity in a different form. To see what is happening we require the following result. Differentiating the identity

$$S_F(p)S_F^{-1}(p) = 1$$

with respect to p^μ gives

$$\frac{\partial S_F(p)}{\partial p^\mu}S_F^{-1}(p) + S_F(p)\frac{\partial S_F^{-1}(p)}{\partial p^\mu} = 0$$

or

$$\frac{\partial S_F(p)}{\partial p^\mu} = -S_F(p)\frac{\partial S_F^{-1}(p)}{\partial p^\mu}S_F(p)$$

$$= -S_F(p)\gamma_\mu S_F(p) \tag{2.63}$$

($S_F^{-1}(p) = \gamma_\mu p^\mu - m$.) The term $-S_F(p)\gamma_\mu S_F(p)$ is associated with the vertex having zero momentum in the photon line. The overall effect of differentiating the propagator with respect to p_μ is to insert a photon line carrying zero momentum into each internal electron line. We now return to the task of verifying (2.61) to the next order. We may write out the complete electron propagator iS'_F in terms of the bare propagator iS_F as -

$$\begin{aligned} iS'_F &= iS_F + iS_F\frac{\Sigma}{i}iS_F + iS_F\frac{\Sigma}{i}S_F\frac{\Sigma}{i}iS_F + \ldots \\ &= iS_F(1 + \frac{\Sigma}{i}iS'_F) \end{aligned} \tag{2.64}$$

Σ is the electron self-energy which we may denote graphically by

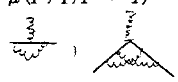
$\Sigma =$ 

It follows from (2.64) that $S_F^{-1\prime} = S_F^{-1} - \Sigma$ and hence

$$\begin{aligned} \frac{\partial S_F^{\prime -1}}{\partial p^\mu} &= \frac{\partial S_F^{-1}}{\partial p^\mu} - \frac{\partial \Sigma}{\partial p^\mu} \\ &= \gamma_\mu - \frac{\partial \Sigma}{\partial p^\mu} \end{aligned}$$

If we expand the vertex function as

$$\Gamma_\mu(p, q, p+q) = \gamma_\mu + \Lambda_\mu(p, q, p+q)$$

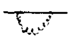
where Λ_μ represents the contributions from 
The Ward identity then implies that

$$\Lambda_\mu(p, 0, p) = -\frac{\partial \Sigma}{\partial p^\mu} \tag{2.65}$$

To verify this to lowest order we have to show that

$\Lambda_\mu(p, 0, p) =$  $= -\frac{\partial}{\partial p^\mu}($  $)$.

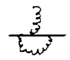
(This result is in logical agreement with the earlier remark about the insertion of a photon line.)
The Feynman rules tell us that

$\frac{\Sigma}{i} =$ 

$$\begin{aligned} &= (-ie)^2 \int \frac{d^4k}{(2\pi)^4} \frac{-ig_{\kappa\lambda}}{k^2} \gamma^\kappa \frac{i}{\gamma.(p-k) - m} \gamma^\lambda \\ &= -e^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2} \gamma^\lambda S_F(p-k) \gamma_\lambda \end{aligned}$$

Note that the photon propagator is taken in the gauge with $\alpha = 1$. Using (2.63) we have

$$\begin{aligned}
\frac{\partial \Sigma}{\partial p^\mu} &= -ie^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2} \gamma^\lambda \frac{\partial}{\partial p^\mu} S_F(p-k)\gamma_\lambda \\
&= ie^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2} \gamma^\lambda S_F(p-k)\gamma_\mu S_F(p-k)\gamma_\lambda
\end{aligned}$$

We now calculate 

$$-ie\Lambda_\mu(p,q,p+q) = (-ie)^3 \int \frac{d^4k}{(2\pi)^4} \frac{-ig_{\kappa\lambda}}{k^2} \gamma^\kappa \frac{i}{\gamma.(p-k)-m} \gamma_\mu \times \frac{i}{\gamma.(p-k+q)-m} \gamma^\lambda$$

By letting $q = 0$ we see that (2.65) is satisfied.
We now derive the more general form of the Ward identities for non-abelian gauge fields which are known as the Taylor-Slavnov identities. We begin by introducing the Becchi-Rouet-Stora transformation. Under this transformation the effective Lagrangian

$$\mathcal{L}_{\text{eff}} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} + \text{ gauge fixing term} + \text{ghost term}$$

is invariant.

2.6 Becchi-Rouet-Stora Transformation

Our starting point is the expansion for Z which is given by

$$Z = N \int \mathcal{D}A_\mu \mathcal{D}\eta \mathcal{D}\eta^* \; e^{i\int \mathcal{L}_{eff} d^4x}$$

where (in the Lorentz gauge)

$$\mathcal{L}_{\text{eff}} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} - \frac{1}{2\alpha}(\partial^\mu A^a_\mu)^2 + \text{ Fadeev Popov Ghost Terms}(\mathcal{L}_{FPG})$$

The Fadeev-Popov ghost terms may be written as

$$\begin{aligned}
\mathcal{L}_{FPG} &= -\eta^{*a}(\delta^{ab}\partial_\rho\partial^\rho - gf^{abc}\partial^\mu A^c_\mu - gf^{abc}A^c_\mu\partial^\mu)\eta^b \\
&= -\eta^{*a}\partial_\rho\partial^\rho\eta^a + gf^{abc}\eta^{*a}(\partial^\mu A^c_\mu + A^c_\mu\partial^\mu)\eta^b \\
|\text{ integrating by parts}| &= \partial^\mu\eta^{*a}\partial_\mu\eta^a - gf^{abc}(\partial^\mu\eta^{*a})A^c_\mu\eta^b + \text{total derivative} \\
&= \partial^\mu\eta^{*a}(\partial_\mu\eta^a + gf^{abc}A^b_\mu\eta^c) \\
&= \partial^\mu\eta^{*a}D_\mu\eta^a \\
&= -\eta^{a*}\partial^\mu D_\mu\eta^a + (\text{total derivative})
\end{aligned}$$

The total derivative terms may be ignored since they only contribute to surface terms of the action. We look for transformations which leave $\mathcal{L}_{\text{eff}}$ and the measure $\mathcal{D}A_\mu \mathcal{D}\eta^* \mathcal{D}\eta$ invariant. The transformations are global. For the A-fields they are of the form

$$\begin{aligned}
\delta A^a_\mu &= \frac{1}{g}\partial_\mu \Lambda^a + f^{abc} A^b_\mu \Lambda^c \\
&= \frac{1}{g}(D_\mu \Lambda)^a
\end{aligned}$$

Now we suppose that $\Lambda^a = -\eta^a \lambda$ where λ is a Grassmann constant.
The famous transformation discovered by Becchi, Rouet and Stora is defined by the following 3 equations

$$\begin{aligned}
\delta A^a_\mu &= -\frac{1}{g}(D_\mu \eta^a)\lambda \\
\delta \eta^a &= -\frac{1}{2} f^{abc}\eta^b \eta^c \lambda \\
\delta \eta^{*a} &= -\frac{1}{\alpha g}(\partial^\mu A^a_\mu)\lambda
\end{aligned} \tag{2.66}$$

We now show that $\mathcal{L}_{\text{eff}}$ is invariant under this transformation.

$$\begin{aligned}
\delta \mathcal{L}_{\text{eff}} &= -\frac{1}{4}\delta(F^a_{\mu\nu}F^{\mu\nu a}) + \delta\mathcal{L}_{GF} + \delta\mathcal{L}_{FPG} \\
&= \delta\mathcal{L}_{GF} + \delta\mathcal{L}_{FPG}
\end{aligned}$$

($\delta(F^a_{\mu\nu}F^{\mu\nu a}) = 0$ for any $\Lambda^a = -\eta^a\lambda$).

$$\begin{aligned}
\mathcal{L}_{GF} &= -\frac{1}{2\alpha}(\partial^\mu A^a_\mu)(\partial^\nu \delta A^a_\nu) - \frac{1}{2\alpha}(\partial^\mu \delta A^a_\mu)(\partial^\nu A^a_\nu) \\
&= -\frac{1}{\alpha}(\partial^\mu A^a_\mu)(-\frac{1}{g}\partial^\nu(D_\nu \eta^a)\lambda) \\
&= \frac{1}{\alpha}(\partial^\mu A^a_\mu)\frac{1}{g}(\partial.D\eta^a)\lambda \quad (\text{ where } \partial.D = \partial^\nu D_\nu)
\end{aligned} \tag{2.67}$$

$$\delta\mathcal{L}_{FPG} = -\delta\eta^{a*}[(\partial.D)\eta]^a - \eta^{a*}\partial_\mu(\delta D_\mu \eta^a)$$

By letting

$$\delta\eta^{*a} = -\frac{1}{\alpha g}(\partial^\mu A^a_\mu)\lambda$$

we find that

$$\begin{aligned}
-(\delta\eta^*)[(\partial.D)\eta]^a &= \frac{1}{\alpha g}(\partial^\mu A^a_\mu)\lambda(\partial.D\eta)^a \\
&= -\frac{1}{\alpha g}(\partial^\mu A^a_\mu)(\partial.D\eta^a)\lambda
\end{aligned} \tag{2.68}$$

(Using the fact that $\eta^a\lambda = -\lambda\eta^a$.) (2.67) and (2.68) cancel leaving

$$\delta\mathcal{L}_{\text{eff}} = -\eta^{*a}\partial^\mu(\delta D_\mu \eta^a)$$

For the overall variation to vanish we require that $\delta[D_\mu\eta]^a$ vanish. We have

$$\begin{aligned}
\delta[D_\mu\eta]^a &= \delta[\partial_\mu\eta^a + gf^{abc}A_\mu^b\eta^c] \\
&= \partial_\mu(\delta\eta^a) + gf^{abc}(\delta A_\mu^b)\eta^c + gf^{abc}A_\mu^b(\delta\eta^c) \\
&= \partial_\mu(-\frac{1}{2}f^{abc}\eta^b\eta^c\lambda) + gf^{abc}(-\frac{1}{g}\partial_\mu\eta^b - f^{bmn}A_\mu^m\eta^n)\lambda\eta^c \\
&\quad +gf^{abc}A_\mu^b(-\frac{1}{2}f^{cmn}\eta^m\eta^n\lambda)
\end{aligned}$$

Note that

$$\begin{aligned}
\partial_\mu(-\frac{1}{2}f^{abc}\eta^b\eta^c\lambda) &= -\frac{1}{2}f^{abc}\partial_\mu(\eta^b\eta^c)\lambda \\
&= -\frac{1}{2}f^{abc}[(\partial_\mu\eta^b)\eta^c + \eta^b(\partial_\mu\eta^c)]\lambda \\
&= -\frac{1}{2}f^{abc}[(\partial_\mu\eta^b)\eta^c - (\partial_\mu\eta^c)\eta^b]\lambda \quad \text{(using Grassmann property)} \\
&= -f^{abc}(\partial_\mu\eta^b)\eta^c \qquad (2.69)
\end{aligned}$$

(The last line follows by relabelling the indices and using the antisymmetry of f^{abc}.) Combining (2.69) with the last equation means that

$$\delta(D_\mu\eta^a) = -f^{abc}(\partial_\mu\eta^b)(\eta^c\lambda + \lambda\eta^c)$$

$$+gf^{abc}f^{bmn}A_\mu^m\eta^n\eta^c\lambda - \frac{1}{2}gf^{abc}f^{cmn}A_\mu^b\eta^m\eta^n\lambda$$

The first term vanishes since $\eta^c\lambda = -\lambda\eta^c$ (λ and η^c are Grassmann quantities.) In the last term the structure constants f obey the Jacobi identity

$$f^{abc}f^{cmn} = -f^{amc}f^{cnb} - f^{anc}f^{cbm}$$

This means that

$$\delta(D_\mu\eta^a) = gf^{abc}f^{bmn}A_\mu^m\eta^n\eta^c\lambda$$

$$+\frac{1}{2}gf^{amc}f^{cnb}A_\mu^b\eta^m\eta^n\lambda + \frac{1}{2}gf^{anc}f^{cbm}A_\mu^b\eta^m\eta^n\lambda \qquad (2.70)$$

The last term

$$\begin{aligned}
\frac{1}{2}gf^{anc}f^{cbm}A_\mu^b\eta^m\eta^n\lambda &= \frac{1}{2}gf^{amc}f^{cbn}A_\mu^b\eta^n\eta^m\lambda \\
&= \frac{1}{2}gf^{amc}(-f^{cnb})A_\mu^b(-\eta^m\eta^n)\lambda \\
&= \frac{1}{2}gf^{amc}f^{cnb}A_\mu^b\eta^m\eta^n\lambda
\end{aligned}$$

The last term is therefore equal to the second term in (2.70). The first term

$$gf^{abc}f^{bmn}A_\mu^m\eta^n\eta^c\lambda = gf^{apn}f^{pbm}A_\mu^b\eta^m\eta^n\lambda$$

(relabelling the indices.) We therefore have

$$\begin{aligned}\delta(D_\mu\eta^a) &= g(f^{apn}f^{pbm}+f^{amp}f^{pnb})A^b_\mu\eta^m\eta^n\lambda \\ &= g(-f^{apm}f^{pbn}+f^{amp}f^{pnb})A^b_\mu\eta^m\eta^n\lambda \\ &= 0 \end{aligned} \tag{2.71}$$

This shows that $\mathcal{L}_{\text{eff}}$ is invariant under the BRS transformation. Notice that

$$\delta^2(A^a_\mu) = -\frac{1}{g}(\delta(D_\mu\eta^a)\lambda) = 0$$

We say that the variation in A^a_μ in nilpotent.

2.7 Slavnov-Taylor Identities

We establish the Slavnov-Taylor identities by applying the BRS transformation to the following generalised generating functional which is dependent on five sources. We start with

$$Z[J,x,y;I_\mu,I] = \int \mathcal{D}\eta^*\mathcal{D}\eta\mathcal{D}A_\mu e^{iS_{tot}} \tag{2.72}$$

where

$$\begin{aligned} S_{tot} = \int d^4x\ [&\mathcal{L}_{eff} + J^a_\mu A^{a\mu} + \eta^a x^a + \eta^{a*}y^a \\ &+ I^{\mu a}(\frac{1}{g}D_\mu\eta)^a - \frac{1}{2}I^a f^{abc}\eta^b\eta^c] \end{aligned} \tag{2.73}$$

Of these sources x, y and I_μ are anticommuting. In order to construct identities that have ghosts on the external legs we need to introduce sources coupled to the ghosts in all possible ways.
Under the BRS transformation the only change in (2.72) comes from the second, third and fourth terms in (2.73). By the way the BRS transformation was constructed we know that $\mathcal{L}_{eff}$ is invariant. The result in (2.71) demonstrates that the coefficient of $I^{\mu a}$ is invariant.
To prove that the coefficient of I^a is invariant we have (using (2.66))

$$\begin{aligned} \delta(f^{abc}\eta^b\eta^c) &= f^{abc}[(\delta\eta^b)\eta^c + \eta^b(\delta\eta^c)] \\ &= -\frac{1}{2}f^{abc}(f^{bmn}\eta^m\eta^n\lambda\eta^c + \eta^b f^{cmn}\eta^m\eta^n\lambda) \\ |\text{Relabelling and using }\ \lambda\eta^c = -\eta^c\lambda\text{gives }| \quad &= \frac{1}{2}(f^{acb}f^{cmn}\eta^m\eta^n\eta^b + f^{abc}f^{cmn}\eta^m\eta^n\eta^b)\lambda \\ &= 0 \end{aligned} \tag{2.74}$$

The last line results from the antisymmetry of the structure constant f^{abc}. We may now observe that the change in η^a is nilpotent. ie. $\delta^2(\eta^a) = 0$.

To prove that the measure is invariant we must show that the Jacobian of the transformation defined by

$$A \to A + \delta A \qquad \eta \to \eta + \delta\eta \qquad \eta^* \to \eta^* + \delta\eta^*$$

is unity. The Jacobian is

$$J = \partial\left(\frac{A^a_\mu(x) + \delta A^a_\mu(x), \eta^a(x) + \delta\eta^a(x), \eta^{a*}(x) + \delta\eta^{a*}(x)}{A^b_\nu(y), \eta^b(y), \eta^{*b}(y)}\right)$$

The only non-zero elements of the determinant are

$$\begin{aligned}
\frac{\delta[A^a_\mu(x) + \delta A^a_\mu(x)]}{\delta A^b_\nu(y)} &= \delta^\nu_\mu \delta^4(x-y)\delta^{ab} + \frac{\delta}{\delta A^b_\nu(y)}[-\frac{1}{g}\partial_\mu \eta^a - f^{afc}A^f_\mu(x)\eta^c]\lambda \\
&= \delta^\nu_\mu \delta^4(x-y)\delta^{ab} - \delta^\nu_\mu \delta^4(x-y)\delta^b_f f^{afc}\eta^c\lambda \\
&= \delta^\nu_\mu \delta^4(x-y)(\delta^{ab} - f^{abc}\eta^c\lambda) \\
\frac{\delta[\eta^a(x) + \delta\eta^a(x)]}{\delta\eta^b(y)} &= \delta^4(x-y)[\delta^{ab} - \frac{1}{2}\frac{\delta}{\delta\eta^b}(f^{amn}\eta^m\eta^n)\lambda] \\
&= \delta^4(x-y)[\delta^{ab} - \frac{1}{2}(f^{amb}\eta^m - f^{abn}\eta^n)\lambda] \\
&= \delta^4(x-y)(\delta^{ab} + f^{abc}\eta^c\lambda)
\end{aligned}$$

(Note that the right derivative $\frac{\partial}{\partial\eta^b}(\eta^m\eta^n) = \delta_{nb}\eta^m - \delta_{bm}\eta^n$.)

$$\frac{\delta[A^a_\mu(x) + \delta A^a_\mu(x)]}{\delta\eta^b(y)} = \delta^4(x-y)f^{abc}A^b_\mu\lambda$$

To find the Jacobian we examine the determinant of the matrix (represented schematically) by

$$det\begin{bmatrix} \frac{\delta A'}{\delta A} & \frac{\delta A'}{\delta\eta} & 0 \\ 0 & \frac{\delta\eta'}{\delta\eta} & 0 \\ 0 & 0 & \frac{\delta\eta'^*}{\delta\eta^*} \end{bmatrix}$$

By working to first order (since $\lambda^2 = 0$) we find that the Jacobian is equal to unity. It follows from this last result and (2.74) that if the Z functional (2.72) is unchanged under the BRS transformation then we must have $Z = Z'$ ($Z' = Z + \delta Z$) where

$$Z' = Z\int \mathcal{D}A_\mu \mathcal{D}\eta\mathcal{D}\eta^* e^{i\int d^4x\ (J^a_\mu \delta A^{\mu a} + x^a\delta\eta^a + y^a\delta\eta^{a*})}$$

$$= Z\int \mathcal{D}A_\mu \mathcal{D}\eta\mathcal{D}\eta^*[1 + \int d^4x\ (J^a_\mu \delta A^{\mu a} + x^a\delta\eta^a + y^a\delta\eta^{*a})] \tag{2.75}$$

$Z' = Z$ implies that

$$Z\int \mathcal{D}A_\mu \mathcal{D}\eta\mathcal{D}\eta^* \int d^4x\ (J^a_\mu \delta A^{\mu a} + x^a\delta\eta^a + y^a\delta\eta^{*a}) = 0 \tag{2.76}$$

From (2.66) and (2.73) we can write (2.76) as

$$\lambda\int d^4x\{J^a_\mu(x)\frac{\delta Z}{\delta I^a_\mu(x)}+x^a(x)\frac{\delta Z}{\delta I^a(x)}-\frac{1}{\alpha g}y^a(x)[\partial_\mu\frac{\delta Z}{\delta J^a_\mu(x)}]\}=0$$

Since this contains first-order derivatives only (resulting from the introduction of the sources I_μ and I for the non-linear terms δA and $\delta\eta$) we may write it as

$$\int d^4x\ [J^a\frac{\delta W}{\delta I^a_\mu}+x^a\frac{\delta W}{\delta I^a}-\frac{1}{\alpha g}y^a(\partial_\mu\frac{\delta W}{\delta J^a_\mu})] \tag{2.77}$$

We now convert (2.77) into a condition on the generating functional Γ which is defined via the usual Legendre transformation

$$\Gamma[A_\mu,\eta,\eta^*;I_\mu,I]=W[J_\mu,x,y;I_\mu,I]-\int d^4x\ (J^a_\mu A^a+x^a\eta^a+y^a\eta^{*a})$$

Then

$$J^a_\mu=-\frac{\delta\Gamma}{\delta A^a_\mu},\qquad x^a=-\frac{\delta\Gamma}{\delta\eta^a},\qquad y^a=-\frac{\delta\Gamma}{\delta\eta^{*a}}$$

In addition,

$$\frac{\delta W}{\delta J^a_\mu}=A^{\mu a},\qquad \frac{\delta W}{\delta I^a_\mu}=\frac{\delta\Gamma}{\delta I^a_\mu},\qquad \frac{\delta W}{\delta I^a}=\frac{\delta\Gamma}{\delta I^a} \tag{2.78}$$

(2.77) therefore becomes

$$\int d^4x\ [\frac{\delta\Gamma}{\delta A^a_\mu}\frac{\delta\Gamma}{\delta I^{\mu a}}+\frac{\delta\Gamma}{\delta\eta^a}\frac{\delta\Gamma}{\delta I^a}-\frac{1}{\alpha g}(\partial^\mu A^a_\mu)\frac{\delta\Gamma}{\delta\eta^{*a}}]=0 \tag{2.79}$$

A simpler form of this equation may be obtained. In the expression for the generating functional Z the only terms involving η^* and I^a_μ are given by

$$Z=\int \mathcal{D}A_\mu\mathcal{D}\eta\mathcal{D}\eta^* e^{i\int d^4x\ [-\eta^{*a}(\partial.D\eta)^a+\eta^{*a}y^a+I^a_\mu(D^\mu\eta)^a+\ldots]}$$

This gives

$$\begin{aligned}\frac{\delta Z}{\delta\eta^{*a}} &= i[y^a-(\partial.D\eta)^a]Z\\ &= iy^aZ-g\partial_\mu\frac{\delta Z}{\delta I^a_\mu}\end{aligned}$$

This implies that

$$\frac{\delta W}{\delta\eta^{*a}}=y^a-g\partial_\mu(\frac{\delta W}{\delta I^a_\mu})$$

But

$$\frac{\delta W}{\delta\eta^{*a}}=0\ \text{ and }\ y^a=-\frac{\delta\Gamma}{\delta\eta^{*a}}$$

Therefore

$$\frac{\delta\Gamma}{\delta\eta^{*a}} = -g\partial_\mu(\frac{\delta\Gamma}{\delta I^a_\mu})$$

and (2.79) becomes (integrating by parts)

$$\int d^4x\ [\frac{\delta\Gamma}{\delta I^{a\mu}}(\frac{\delta\Gamma}{\delta A^a_\mu} - \frac{1}{\alpha}\partial^\mu(\partial^\nu A^a_\nu)) + \frac{\delta\Gamma}{\delta I^a}\frac{\delta\Gamma}{\delta\eta^a}] = 0$$

Defining

$$\Gamma = \Gamma' - \frac{1}{2\alpha}\int d^4x\ (\partial^\nu A^a_\nu)^2$$

yields

$$\int d^4x\ [\frac{\delta\Gamma'}{\delta I^{a\mu}}\frac{\delta\Gamma'}{\delta A^a_\mu} + \frac{\delta\Gamma'}{\delta I^a}\frac{\delta\Gamma'}{\delta\eta^a}] = 0$$

This expresses the Taylor-Slavnov identity in a form which will be used in Chapter 3 to show the renormalisability of Yang-Mills fields.

Chapter 3

Renormalisation

3.1 Divergences in Feynman diagrams

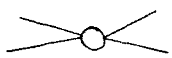
This chapter follows the approaches of Ramond [2] and Ryder [5]. In Chapter 1 ϕ^4 theory was introduced and it was discovered that the first order contribution to the propagation was given by the 'tadpole' graph

(3.1)

which, in momentum space corresponds to

$$g\int\frac{d^4q}{(2\pi)^4}\frac{1}{q^2-m^2}$$

(More formally, the 'tadpole' graph is the lowest order self-energy graph.) This integral is divergent as $q \to \infty$. This is because the measure has 4 powers of q and the denominator has only 2 powers of q. The integral therefore diverges quadratically at large q and is an example of what is known as ultra–violet divergence (i.e. it diverges as momentum goes to large values). Another divergence occurs in the $O(g^2)$ contribution to $G^{(4)}$ which is given by



(3.2)

In momentum space it corresponds to the integral

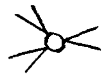
$$g^2\int\frac{d^4q_1}{(2\pi)^4}\frac{d^4q_2}{(2\pi)^4}\frac{\delta(q_1+q_2-p_1-p_2)}{(q_1^2-m^2)(q_2^2-m^2)}$$

$$=g^2\int\frac{d^4q}{(2\pi)^8}\frac{1}{(q^2-m^2)[(p_1+p_2-q)^2-m^2]}$$

This integral is logarithmically divergent since it behaves as

$$\sim\int\frac{d^4q}{q^4}$$

A useful example to study is the following graph



which behaves as

$$\sim \int \frac{d^4q}{q^6}$$

This integral is convergent. But if space–time had 6 dimensions this would have a logarithmic divergence. The simple examples above illustrates the idea of "power counting". We need to find some method of computing the degree of divergence of a particular Feynman diagram. It is clear that the propagator contributes a factor of q^2 to the denominator and that each integration contributes a factor of q^4 to the numerator, together with the delta function which gives overall momentum conservation. The numbers of loops (L) is equal to the number of independent $\int d^4k$ integrations. If a graph has n vertices, E external lines, I internal lines, L loops, and if we assume that the dimensionality of space–time is d, then the superficial degree of divergence is given by:

$$D = dL - 2I \tag{3.3}$$

The first term on the right–hand side counts the number of momentum integrations, the second counts the number of propagator factors $1/q^2$. The number of loops is given by

$$L = I - n + 1 \tag{3.4}$$

where $n - 1$ is the number of momentum conservations. We make use of a topological relation: each vertex (in ϕ^4 theory) has 4 lines attached – some of these being external lines, the remaining lines being internal lines. Each internal line is connected to two vertices and it is shared by two vertices. Hence,

$$4n = E + 2I \tag{3.5}$$

Combining (3.3), (3.4), and (3.5) gives

$$D = d - (\frac{d}{2} - 1)E + n(d - 4) \tag{3.6}$$

The more general expression, if the graph has n_l vertices, each with l lines attached is -

$$D = d - (\frac{d}{2} - 1)E + \frac{n_l}{2}[(d-2)(l-2) - 4] \tag{3.7}$$

If we replace n_l by n and l by 4 we obtain (3.6). If $d = 4$ then (3.6) gives

$$D = 4 - E \tag{3.8}$$

This gives the correct result of $D = 2$ for the "tadpole" graph (which has 2 external legs) and $D = 0$ for the 4 point function (which has 4 external legs).

If the coefficient of n in (3.6) is greater than 0 then D will increase with n. It means that the overall degree of divergence of any graph becomes as large as you like by expanding to a high enough order in g. Such a theory is called unrenormalisable. In ψ^4 theory we should note that D is only a function of the external legs and not of the order of g in perturbation theory. So we find that there are only a finite number of divergent graphs. It is useful to examine the more general case for ψ^l theory in which each vertice has l lines attached. The more general expression for the superficial degree of divergence becomes

$$D = d - (\frac{d}{2} - 1)E + \frac{n}{2}[(d-2)(l-2) - 4] \tag{3.9}$$

When $d = 4$ we have

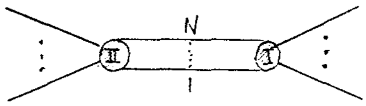
$$D = 4 - E + n(l-4) \tag{3.10}$$

We clearly obtain (3.8) when $l = 4$. In ϕ^6 theory we have

$$D = 4 - E + 2n \tag{3.11}$$

This theory is therefore unrenormalisable because D increases with n. See Callan [18]

We should note an important point at this stage which is that for a given graph, negative D does not necessarily imply convergence. This is why D is called *superficial.* Consider the following diagram



The diagram is "n-particle reducible" because it can be disconnected by cutting at least n internal lines. The overall superficial degree of divergence, D is given by

$$D = D_1 + D_2 + 4(n-1) - 2n$$

where D_1 and D_2 are the superficial degrees of divergence of the subdiagrams 1 and 2 respectively. This is because the diagram is connected together by n internal lines and $(n-1)$ loops. In the case of $n = 1$

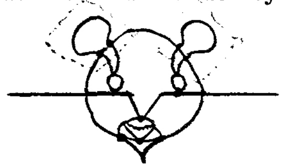
$$D = D_1 + D_2 - 2 \tag{3.12}$$

To illustrate this we may consider the case of two primitive logarithmically divergent diagrams connected together by a single internal line.

We find that

$$D = 0 + 0 - 2$$

where we have used (3.12) setting $D_1 = D_2 = 0$. Although D is negative the graph is clearly divergent due to the two divergent loop integrations. (We may note in passing that this graph is a contribution of order of order g^4 to the 6-point function G^6.) This happens with all Feynman diagrams which contain hidden 2 or 4-point functions with one loop or more. The diagrams will diverge despite the fact that D is negative. As g increases the diagrams become more complex and may contain a number of hidden divergences. We consider the "face" diagram which is a contribution of order g^{16} to the 2-point function. The hidden divergences have been surrounded by boxes.



"Chessa"

The important point about the above discussion is that in ψ^4 theory there are two basic divergent diagrams which are given by (3.1) and (3.2). They are called *primitive divergences.* The fact that there is a finite number of these primitive divergences is crucial if we are to be able to remove these divergences by making a redefinition of the Lagrangian. Since dimensional considerations are central in dimensional regularisation it is useful to present some results about the quantities with which we are dealing. Working in d dimensions the action is given by

$$S = \int d^d x \mathcal{L}$$

Because S is dimensionless and the measure has dimension L^d (where L is the length) the dimension of the Lagrangian is L^{-d}. We write $\mathcal{L} \sim L^{-d}$. We now consider the kinetic term $\partial_\mu \psi \partial^\mu \psi$. It is easily seen that $\psi \sim L^{1-\frac{d}{2}}$ because then $\partial_\mu \psi \sim L^{-\frac{d}{2}}$ and $\partial_\mu \psi \partial^\mu \psi \sim L^{-d}$. The interaction term is given by $g\psi^r$. It is seen that $g \sim L^{-d-r+\frac{rd}{2}}$. This is because $\psi^r \sim L^{r-\frac{rd}{2}}$.

3.2 Dimensional Regularisation of Feynman Integrals

Before studying the technique developed by t'Hooft and Veltman [15] we derive some formulae which are used in the evaluation of Feynman diagrams. The integrals which we typically have to deal with are of the form

$$I_d = \int d^d l F(l)$$

l_μ is a $d-$dimensional vector. We introduce polar coordinates

$$L, \psi, \theta_1, \ldots, \theta_{d-2}$$

where $L^2 = l_\mu l_\mu$ so that

$$\begin{aligned} d^d l &= L^{d-1}\, dL\, d\psi\, \sin\theta_1 d\theta_1\, \sin^2\theta_2 d\theta_2 \ldots \sin^{d-2}\theta_{d-2} d\theta_{d-2} \\ &= L^{d-1}\, dL\, d\psi \prod_{k=1}^{d-2} \sin^k \theta_k d\theta_k \\ &(0 \le L < \infty, \qquad 0 \le \psi \le 2\pi, \qquad 0 \le \theta_k \le \pi) \end{aligned}$$

Then

$$I_d = 2\pi \prod_{k=1}^{d-2} \int_0^\pi \sin^k \theta_k d\theta_k \int_0^\infty dL\, L^{d-1} F(L)$$

The angular integrations can be performed using the well-known formula from analysis

$$\int_0^{\frac{\pi}{2}} (\sin\ t)^{2x-1} (cos\ t)^{2y-1} dt = \frac{1}{2} \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

Putting $y = \frac{1}{2}$ gives

$$\int_0^\pi \sin^k(t) dt = \sqrt{\pi} \{ \frac{\Gamma(\frac{k+1}{2})}{\Gamma(\frac{k+2}{2})} \}$$

Hence

$$\begin{aligned} I_d &= \pi^{d/2} \frac{\Gamma(1)}{\Gamma(\frac{3}{2})} \frac{\Gamma(\frac{3}{2})}{\Gamma(2)} \cdots \frac{\Gamma(\frac{d}{2}-1)}{\Gamma(\frac{d-1}{2})} \frac{\Gamma(\frac{d-1}{2})}{\Gamma(\frac{d}{2})} \\ &\quad \times \int_0^\infty 2L\, dL\, L^{d-2} f(L^2) \\ &= \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} \int_0^\infty dx\, x^{\frac{d}{2}-1} f(x) \end{aligned}$$

where $F(L) = f(L^2) = f(x)$ with $x = L^2$. Typically $f(x)$ has the form

$$f(x) = \frac{1}{(x+a^2)^A} \qquad (A = 1, 2, 3 \ldots)$$

We therefore have

$$\int \frac{d^d l}{(a^2+l^2)^A} = \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} \int_0^\infty \frac{dx\, x^{\frac{d}{2}-1}}{(x+a^2)^A} \tag{3.13}$$

The integral on the right hand side of (3.13) may be rewritten (by using a simple substitution) as

$$a^{d-2A} \int dx\ (x)^{\frac{d}{2}-1} (1+x)^{-A}$$

This last expression is similar to the beta function which is given by

$$B(\frac{d}{2}, A - \frac{d}{2}) = \frac{\Gamma(\frac{d}{2})\Gamma(A-\frac{d}{2})}{\Gamma(A)} = \int_0^\infty dy\ y^{\frac{d}{2}-1} (1+y)^{-A}$$

(Valid for $\Re(A - \frac{d}{2}) > 0$ and $\Re(\frac{d}{2}) > 0$). By applying this to our expression for the Feynman integral (3.13) yields

$$\int \frac{d^d l}{(a^2 + l^2)^A} = \pi^{\frac{d}{2}} (a^2)^{\frac{d}{2} - A} \frac{\Gamma(A - \frac{d}{2})}{\Gamma(A)} \tag{3.14}$$

We assume that this expression is valid for non-integer d by analytic continuation. By letting $l = l' + p$ and relabelling $b^2 = a^2 + p^2$ we obtain

$$\int \frac{d^d l}{(l^2 + 2p.l + b^2)^A} = \pi^{\frac{d}{2}} \frac{\Gamma(A - \frac{d}{2})}{\Gamma(A)} [\frac{1}{(b^2 - p^2)^{A - \frac{d}{2}}}] \tag{3.15}$$

By differentiating with respect to p_μ we obtain the formula

$$\begin{aligned} \int d^d l \frac{l_\mu l_\nu}{(l^2 + 2p.l + a^2)^A} &= \{\frac{\pi^{\frac{d}{2}}}{\Gamma(A)(a^2 - p^2)^{A - \frac{d}{2}}}\} \\ &\times [\Gamma(A - \frac{d}{2}) p_\mu p_\nu + \frac{1}{2} \delta_{\mu\nu} \Gamma(A - 1 - \frac{d}{2})(a^2 - p^2)] \end{aligned} \tag{3.16}$$

This formula has been derived in Euclidean space. The equivalent formula, working in Minkowski space is found (technically) by replacing a^2 by $-a^2$ and $\delta_{\mu\nu}$ by $g_{\mu\nu}$. This is because

$$\int \frac{d^d l}{(l^2 - a^2)^A} = (-1)^A i \left[\int \frac{d^d l}{(l^2 + a^2)^A} \right]_{\text{Euclidean}}$$

All future calculations will be done in Minkowski space and a list of integrals (in Minkowski space) is given in the appendix. The above result demonstrates how they may be switched to Euclidean space. The formula in (3.16) requires some mathematical justification. The integral on the left hand side of equation (3.16) only converges if $d < 2A$. We show using the following technique by t'Hooft and Veltman that this type of integral may be analytically continued into one which is convergent for any d. In this section the Feynman diagrams will be evaluated. We start by introducing the a basic theorem on analytic continuation. This theorem will then be applied to Euler's $\Gamma-$ function. (see Leibbrandt [14])

Theorem(Knopp)

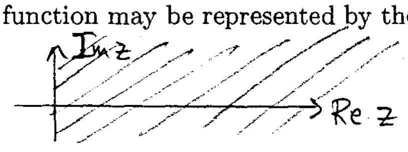
Let an analytic function $g_1(z)$ be defined in a region D_1 and let D_2 be another region which has a certain subregion R_1, but only this one, in common with D_1. Then if a function $g_2(z)$ exists which is analytic in D_2 and coincides with $g_1(z)$ in R, there can only be one such function. We call $g_1(z)$ and $g_2(z)$ analytic continuations of each other.

ie. $g_2(z)$ is unique if $R = D_1 \cap D_2 \neq \oslash$ The difference between the Euler and Weierstrass representations of the Γ-function serve as a useful illustration. For

$\Re(z) > 0$ Euler's Γ function may be written as

$$\Gamma(z) = \int_0^\infty dt\ t^{z-1} e^{-t} \tag{3.17}$$

The domain of this function may be represented by the shaded region.



In this region the function is analytic. The integral in (3.17) diverges when $\Re(z) < 0$. (ie it is in the left hand plane) because as $t \to 0$ the integral is not well defined. In order to discuss points lying in the divergent regions we find an analytic continuation of $\Gamma(z)$ which is valid in that region. Such a continuation is given by

$$\Gamma(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{\alpha^{n+z}}{(z+n)} + \int_\alpha^\infty dt\ e^{-t} t^{z-1}$$

When $\alpha = 1$ we obtain the Weierstrass representation which is analytic in the entire z-plane, except at the points $z = 0, -1, -2, -3 \ldots$.
The Weierstrass representation is unique (from the theorem) since its domain of definition clearly overlaps the domain of the Euler representation. In quantum field theory we are often faced with integral expressions of the form

$$I(d,k) = \int d^d l\ F(l,k)$$

where d is complex and F is a function associated with the form of the Feynman diagram it represents. This expression is similar to the Euler representation of the Γ function. Our aim is to find the equivalent Weierstrass representation.

The procedure to be followed is to

1. establish a finite domain of convergence in the ω-plane for which the integral of $F(l,k)$ converges. (typically it will lie to the left of the line $\Re(d) = 4$.)
2. derive a new function F' identical to F inside the domain of convergence of F but analytic in an enlarged domain containing the point $d = 4$.
3. take the limit on F' as $d \to 4$

We therefore illustrate the above procedure by considering the first order contribution to the self energy diagram  . Firstly we split the k-dimensional space into a 4 dimensional (physical) space and a $(k-4)$-dimensional subspace.

$$d^d l \to d^4 l\ d^{d-4} l$$

Introducing polar coordinates we may write the measure as

$$d^d l = d^4 l \; d\Omega_{d-4} \; L^{d-5} \; dL$$

We then have

$$I = \underline{\;\Omega\;} = \int d^4 l \int d\Omega_{d-4} \int dL \frac{L^{d-5}}{L^2 + l^2 - m^2}$$

Integrating over the angles one finds

$$I = \frac{2\pi^{\frac{d}{2}-2}}{\Gamma(\frac{d}{2}-2)} \int d^4 l \int_0^\infty dL \frac{L^{d-5}}{L^2 + l^2 - m^2} \tag{3.18}$$

This expression is still ultra-violet divergent for $d \geq 2$ since the original integral

$$\int \frac{d^{\frac{d}{2}} l}{l^2 - m^2}$$

contains this divergence and the above manipulations have not eliminated this. Note that the integral in (3.18) contains an infrared divergence for $d \leq 4$. These observations mean that (3.18) as it stands has no convergent domain. t'Hooft and Veltman [15] demonstrate how the infrared divergence may be removed by partial integration (throwing away surface terms). Firstly we observe that

$$L^{d-6} = \frac{2}{d-4} \frac{d}{dL^2} (L^2)^{\frac{d-4}{2}}$$

Then

$$\begin{aligned} \int_0^\infty dL \frac{L^{d-5}}{(L^2 + l^2 - m^2)} &= \frac{1}{2} \int_0^\infty \frac{d(L^2) L^{d-6}}{(L^2 + l^2 - m^2)} \\ &= \int_0^\infty \frac{dL^2}{d-4} \left(\frac{1}{L^2 + l^2 - m^2}\right) \frac{d}{dL^2} (L^2)^{\frac{d-4}{2}} \end{aligned}$$

($dL^2 = 2L \; dL$ - using the exterior derivative.) The last expression is equal to

$$\frac{1}{d-4} \left[\frac{(L^2)^{\frac{d-4}{2}}}{L^2 + l^2 - m^2} \right]_0^\infty - \frac{1}{d-4} \int_0^\infty dL^2 (L^2)^{\frac{d-4}{2}} \frac{d}{dL^2} \left\{ \frac{1}{L^2 + l^2 - m^2} \right\} \tag{3.19}$$

We discard the surface term which is zero in the region of convergence. Infrared divergence in the last term now occurs for $d \leq 2$. We may repeat this procedure.

$$(L^2)^{\frac{d-4}{2}} = \frac{2}{d-2} \frac{d}{dL^2} (L^2)^{(\frac{d-2}{2})}$$

So in (3.19) integrating and discarding the surface term gives

$$\int_0^\infty \frac{dL \; L^{d-5}}{L^2 + l^2 - m^2} = \frac{2}{(d-4)(d-2)} \int_0^\infty dL^2 (L^2)^{\frac{d-2}{2}} \left(\frac{d}{dL^2}\right)^2 \left\{ \frac{1}{L^2 + l^2 - m^2} \right\} \tag{3.20}$$

Infrared divergence now occurs for $d \leq 0$. The "tadpole" integral is therefore convergent for $0 < d < 2$ and we have

$$I = \frac{\pi^{\frac{d-4}{2}}}{\Gamma(\frac{d}{2})} \int d^4 l \int_0^\infty dL^2 (L^2)^{(\frac{d-2}{2})} (\frac{d}{dL^2})^2 \left\{ \frac{1}{L^2 + l^2 - m^2} \right\} \tag{3.21}$$

using the fact that

$$\Gamma(\frac{d}{2}) = (\frac{d}{2} - 1)(\frac{d}{2} - 2)\Gamma(\frac{d}{2} - 2)$$

(L is the length of l in the $(d-4)$-dimensional subspace.) If integration by parts is performed p times the "tadpole" integral will be convergent for

$$4 - 2p < d < 2$$

By taking p sufficiently large the domain of convergence may be extended to arbitrarily small values of d. The degree of convergence regarding the ultraviolet behaviour is $2 - d$ and it is $d - 4 + 2p$ for the infrared behaviour. Instead of starting with the tadpole diagram this technique could have been applied to more general diagrams. The final result would have been

$$\frac{\pi^{\frac{d-4}{2}}}{\Gamma(\frac{1}{2}(d-4) + p)} \int d^4 l \int_0^\infty dL^2 (L^2)^{\frac{d}{2}-3+p} (-\frac{\partial}{\partial L^2})^p f(l, L^2)$$

We now return to (3.21). This expression must be analytically continued to $d = 4$. Carrying out the differentiation given

$$I = \frac{2\pi^{(\frac{d-4}{2})}}{\Gamma(\frac{d}{2})} \int d^4 l \int_0^\infty dL^2 \frac{(L^2)^{\frac{d}{2}-1}}{(L^2 + l^2 - m^2)^3} \tag{3.22}$$

We now insert into this expression

$$\frac{1}{5}(\frac{\partial L}{\partial L} + \frac{\partial l_\mu}{\partial l_\mu}) = 1 \tag{3.23}$$

to give (integrating by parts)

$$I = -\frac{2\pi^{(\frac{d-4}{2})}}{5\Gamma(\frac{d}{2})} \int d^4 l \int_0^\infty dL^2 \left\{ l_\mu \frac{\partial}{\partial l_\mu} + \frac{\partial L}{\partial L} \right\} \frac{(L^2)^{\frac{d-2}{2}}}{(L^2 + l^2 - m^2)^3} \tag{3.24}$$

Note that (3.22) is true for the case of the "tadpole" graph. In passing we should note that the more general form of (3.22) is (see t'Hooft and Veltman [15])

$$I = \int d^\kappa p \frac{p_a^{\lambda_1} p_b^{\lambda_2} \ldots p_c^{\lambda_j}}{[(p + k_1)^2 - m_1^2]^{\alpha_1} [(p + k_2)^2 - m_2^2]^{\alpha_2} \ldots [(p + k_l)^2 - m_l^2]^{\alpha_l}} \tag{3.25}$$

Notice that (3.22) is of the form (3.25) when $\kappa = 5$. The integration over p_5 in (3.25) is nothing but the integration over L in (3.22) . The more general form of (3.23) which is also equal to unity is

$$\frac{1}{\kappa}\sum_{1}^{\kappa}\frac{\partial p_i}{\partial p_i}$$

By carrying out the differentiation in (3.24) and using (3.22) gives

$$I = \frac{3m^2}{(d-2)}\frac{4\pi^{\frac{d-4}{2}}}{\Gamma(\frac{d}{2})}\int d^4l \int_0^\infty dL^2 \frac{(L^2)^{\frac{d-2}{2}}}{(L^2+l^2-m^2)^4} \tag{3.26}$$

This expression has a pole at $d = 2$. Ultraviolet divergence occurs when $d \geq 4$. By inserting (3.23) in (3.26) and following the steps as before gives

$$I = \frac{4.2.3.4\, m^4\, \pi^{(\frac{d-4}{2})}}{(d-2)(d-4)\Gamma(\frac{d}{2})}\int d^4l \int_0^\infty \frac{dL^2(L^2)^{\frac{d-2}{2}}}{(L^2+l^2-m^2)^5}$$

This now displays ultraviolet divergence for $d \geq 6$. The integral will therefore converge if $d = 4$. The price we pay for making the integral converge is the pole term $\frac{1}{d-4}$ which gives rise to a singularity. The technique of dimensional regularisation discovered by t'Hooft and Veltman justifies the use of the formulas in Appendix A which were derived in a naive fashion earlier.

3.3 Evaluation of Feynman Integrals

We now apply the technique of dimensional regularisation to the evaluation of Feynman Integrals in ψ^4 theory. We must first generalise the Lagrangian defined by

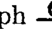
$$\mathcal{L} = \frac{1}{2}\partial_\mu\psi\partial^\mu\psi - \frac{m^2}{2}\psi^2 - \frac{g}{4!}\psi^4$$

from 4 dimensions to d dimensions. We introduce μ as an arbitrary mass parameter (whose units = 1 over length). From the end of section (3.1) we recall that ψ was of dimension $1 - \frac{d}{2}$. This means that $\psi^4 \sim 4 - 2d$. Since $\mathcal{L}$ has dimension $-d$ we must multiply g by a factor of μ^{4-d}. This is necessary if we wish to keep g dimensionless in d dimensions. Then $\mu^{4-d} g\, \psi^4 \sim -d$ as required. The Feynman rules need to be adjusted when working in d dimensions.

1. All 4-vectors become d-vectors. (summated indices in calculations run from 1 to d)

2. Integrals over 4-dimensional space-time become integrals in d-dimensional space.

3. The vertex strength $-ig$ becomes $-ig\mu^{4-d}$.

We now use these rules to calculate the correction of order g to the 'tadpole' graph . (We shall, as usual, work in Minkowski space). Omitting propagators of external lines we have

$$
\begin{aligned}
&= \frac{1}{2} g\mu^{4-d} \int \frac{d^d p}{(2\pi)^d} \left(\frac{1}{p^2 - m^2}\right) \\
&= -\frac{ig}{32\pi^2} m^2 \left(\frac{4\pi\mu^2}{m^2}\right)^{2-\frac{d}{2}} \Gamma\left(1 - \frac{d}{2}\right)
\end{aligned}
$$

-using $(A1)$ in the appendix which is the Minkowski analogue of (3.15). Central to the theme of dimensional regularisation is the ability to express the divergence of the integral as a pole. This is achieved by expanding around $d = 4$. To this end, note that,

$$
\left(\frac{4\pi\mu^2}{m^2}\right)^{2-\frac{d}{2}} = 1 + \left(2 - \frac{d}{2}\right) \ln\left(\frac{4\pi\mu^2}{m^2}\right) + \ldots \tag{3.27}
$$

and that

$$
\Gamma\left(1 - \frac{d}{2}\right) = -\frac{2}{4-d} - 1 + \gamma + \ldots \tag{3.28}
$$

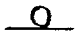
Equation (3.28) is obtained from the standard expansion of the Γ function -

$$
\Gamma(-n + \epsilon) = \frac{(-1)^n}{n!}\left[\frac{1}{\epsilon} + \psi(n+1) + \frac{1}{2}\epsilon\left[\frac{\pi^2}{3} + \psi_1^2(n+1) - \psi'(n+1)\right] + O(\epsilon^2)\right]
$$

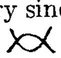
where

$$
\psi(n+1) = 1 + \frac{1}{2} + \ldots \frac{1}{n} - \gamma \tag{3.29}
$$

(γ is the Euler-Mascheroni constant). Equation (3.27) comes from the standard Taylor series expansion. We therefore have



$$
\begin{aligned}
&= \frac{-igm^2}{32\pi^2}\left[-\frac{2}{4-d} - 1 + \gamma + O(4-d)\right]\left[1 + \frac{4-d}{2}\ln\left(\frac{4\pi\mu^2}{m^2}\right)\right] \\
&= \frac{igm^2}{16\pi^2(4-d)} + \frac{igm^2}{32\pi^2}\left[1 - \gamma + \ln\left(\frac{m^2}{4\pi\mu^2}\right)\right] + O(4-d) \\
&= \frac{igm^2}{16\pi^2}\left\{\frac{1}{4-d} + \text{finite part}\right\}
\end{aligned}
$$

The divergence in  therefore takes the form of a single pole. Note that the finite part is arbitrary since it depends on the arbitrary mass. We now consider the 4-point function  to order g^2. It is given by

$$
\frac{1}{2} g^2 (\mu^2)^{4-d} \int \frac{d^d l}{(2\pi)^d} \frac{1}{(l^2 - m^2)} \frac{1}{(l-q)^2 - m^2} \tag{3.30}
$$

This integral is evaluated using the following general formula -

$$
\begin{aligned}
\frac{1}{D_1^{a_1} D_2^{a_2} \ldots D_k^{a_k}} = & \frac{\Gamma(a_1 + a_2 + \ldots a_k)}{\Gamma(a_1)\Gamma(a_2)\ldots\Gamma(a_k)} \int_0^1 \ldots \int_0^1 dx_1 \ldots dx_k \frac{\delta(1 - x_1 - \ldots x_k)}{(D_1 x_1 + \ldots + D_k x_k)^{a_1 + \ldots a_k}} \\
& \times \{x_1^{a_1 - 1} \ldots x_k^{a_k - 1}\}
\end{aligned}
$$

Applying this formula to the denominator of the integral in (3.30) gives

$$\begin{aligned}\frac{1}{(l^2-m^2)((l-q)^2-m^2)} &= \frac{\Gamma(2)}{\Gamma(1)\Gamma(1)}\int_0^1\int_0^1 dx_1 dx_2 \frac{\delta(1-x_1-x_2)x_1^0 x_2^0}{[(l^2-m^2)x_1+((l-q)^2-m^2)x_2]^2} \\ &= \int_0^1 \frac{dx_1}{[(l^2-m^2)x_1+((l-q)^2-m^2)(1-x_1)]^2} \\ &= \int_0^1 \frac{dx_1}{[l^2-m^2-2l.q(1-x_1)+q^2(1-x_1)]^2}\end{aligned}$$

By changing variables to

$$l' = l - q(1-x_1)$$

we see that the denominator can be written as

$$[l'^2 - m^2 + q^2 x_1(1-x_1)]^2$$

Under this transformation $d^d l' = d^d l$ and hence (3.30) becomes

$$\frac{1}{2}g^2(\mu^2)^{4-d}\int_0^1 dx \int \frac{d^d l}{(2\pi)^d}\frac{1}{[l^2-m^2+q^2x(1-x)]^2}$$

(We have replaced the dummy indices l' and x_1 by l and x respectively.) We now use the formulas in the appendix to perform the l integration to give

$$\begin{aligned}&\frac{i}{2}g^2(\mu^2)^{4-d}\int_0^1 \frac{dx\ \pi^{\frac{d}{2}}\Gamma(2-\frac{d}{2})}{(2\pi)^d(m^2-q^2x(1-x))^{2-\frac{d}{2}}} \\ =\ & \frac{ig^2}{32\pi^2}(\mu^2)^{2-\frac{d}{2}}\Gamma(2-\frac{d}{2})\int_0^1 dx[\frac{4\pi\mu^2}{m^2-q^2x(1-x)}]^{2-\frac{d}{2}}\end{aligned}$$

In the limit $d \to 4$,

$$\Gamma(2-\frac{d}{2}) = \frac{2}{4-d} - \gamma + O(4-d)$$

so we have

$$\begin{aligned}&\frac{ig^2\mu^{4-d}}{32\pi^2}(\frac{2}{4-d}-\gamma+O(4-d))\{1-\frac{4-d}{2}\int_0^1 dx\ \ln[\frac{m^2-q^2x(1-x)}{4\pi\mu^2}]\} \\ =\ & \frac{ig^2\mu^{4-d}}{16\pi^2(4-d)} - \frac{ig^2\mu^{4-d}}{32\pi^2}\{\gamma+\int_0^1 dx\ \ln[\frac{m^2-q^2x(1-x)}{4\pi\mu^2}]\}\end{aligned}$$

Setting $q^2 = s$ and letting

$$F(s,m,\mu) = \int_0^1 dx\ ln[\frac{m^2-sx(1-x)}{4\pi\mu^2}]$$

gives

$$\frac{ig^2\mu^{4-d}}{16\pi^2(4-d)} - \frac{ig^2\mu^{4-d}}{32\pi^2}[\gamma+F(s,m,\mu)] = \frac{ig^2\mu^{4-d}}{16\pi^2(4-d)} + \text{finite part}$$

We have now evaluated the 2 and 4-point functions to lowest order. This means that

$$\sum = -\frac{gm^2}{16\pi^2(4-d)} + \text{finite part} \tag{3.31}$$

where $\sum$ is the self-energy function defined in Chapter 2.
Since

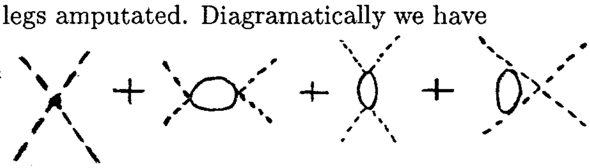
$$\Gamma^{(2)}(p) = p^2 - m^2 - \sum(p)$$

we have

$$\Gamma^{(2)}(p) = p^2 - m^2\left(1 - \frac{g}{16\pi^2(4-d)}\right)$$

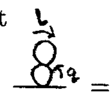
We may also apply the above calculation to $\Gamma^{(4)}(p)$ which is similar to $G^{(4)}(p)$ but with the external legs amputated. Diagramatically we have

$$\Gamma^{(4)}(p) =$$



$$\begin{aligned} &= -ig\mu^{4-d} + \frac{3ig^2\mu^{4-d}}{16\pi^2(4-d)} - \frac{ig^2\mu^{4-d}}{32\pi^2} \times \\ &\quad \{3\gamma + F(s,m,\mu) + F(t,m,\mu) + F(u,m,\mu)\} \end{aligned}$$

where $s = (p_1+p_2)^2$, $t = (p_1+p_3)^2$ and $u = (p_1+p_4)^2$. We now calculate the second order correction to $G^{(2)}$. Using the same technique as before we arrive at



$$\begin{aligned} &= \frac{i}{4}g^2(\mu^2)^{4-d}\int\frac{d^dl}{(2\pi)^d}\frac{1}{l^2-m^2}\int\frac{d^dq}{(2\pi)^d}\frac{1}{(q^2-m^2)^2} \\ &= \frac{ig^2m^2}{1024\pi^4}\left\{\frac{4}{(4-d)^2} + \frac{2}{4-d}[2ln(\frac{4\pi\mu^2}{m^2}) + \psi(2) + \psi(1)]\right. \\ &\quad +2(ln\frac{4\pi\mu^2}{m^2})^2 + 2ln(\frac{4\pi\mu^2}{m^2})[\psi(2)+\psi(1)] + \frac{1}{2}([\psi(2)+\psi(1)]^2 \qquad (3.32) \\ &\quad \left. +\frac{\pi^2}{3} - \frac{1}{2}\psi'(2) - \frac{1}{2}\psi'(1)) + O(\frac{d}{2}-2)\right\} \qquad (3.33) \end{aligned}$$

where ψ is defined by (3.27) and ψ' is defined by

$$\psi'(n+1) = \frac{\pi^2}{6} - \sum_{k=1}^{n}\frac{1}{k^2}$$

There are other techniques for evaluating Feynman diagrams. Brown [8] frequently employs the method of exponential parametrisation. This method is also referred to in Leibbrandt [14].

3.4 Renormalisation of ψ^4 theory

In the previous sections we have shown how the divergences arise in Feynman diagrams. The primitive divergences were found in the 2 and 4-point Greens functions. These exhibit quadratic and logarithmic divergences respectively. We were able to represent these divergences as poles in the complex plane using dimensional regularisation - a technique developed by t'Hooft and Veltman. We now consider the process of renormalisation which is associated with the removal of these divergences. We begin with the vertex functions $\Gamma^{(2)}$ and $\Gamma^{(4)}$. Our aim is to make $\Gamma^{(2)}(p)$ finite to the (1-loop) approximation. Let us consider

$$\Gamma^{(2)}(p) = p^2 - m_1^2$$

where m_1 is a *finite* parameter and is taken to represent the *physical* mass. We assume that the original mass m is infinite. The relationship between m and m_1 is given by

$$\begin{aligned} m^2 &= m_1^2 + \frac{m^2 g}{16\pi^2\epsilon} \\ &= m_1^2(1 + \frac{g}{16\pi^2\epsilon}) \end{aligned} \tag{3.34}$$

where we have replaced m by m_1 in the first order correction. (In this equation we have set $\epsilon = 4 - d$ and all future calculations shall use this notation.) To order g this does not create any error. The physical mass m_1 is also called the renormalised mass and is given by

$$m_1^2 = -\Gamma^{(2)}(0)$$

A similar approach may be taken to the vertex function $\Gamma^{(4)}$. It was shown that

$$\Gamma^{(4)}(p_i) = -ig\mu^\epsilon(1 - \frac{3g}{16\pi^2\epsilon}) + \text{finite part}$$

where the finite part is given by

$$-\frac{ig^2\mu^\epsilon}{32\pi^2}[3\gamma + F(s,m,\mu) + F(t,m,\mu) + F(u,m,\mu)]$$

We may write this as

$$i\Gamma^{(4)}(p_i) = g\mu^\epsilon - \frac{g^2\mu^\epsilon}{32\pi^2}[\frac{6}{\epsilon} - 3\gamma - F(s,m,\mu) - F(t,m,\mu) - F(u,m,\mu)] \tag{3.35}$$

Once again we can define a new parameter g_1 which is taken to be finite. We let

$$g_1 = g\mu^\epsilon - \frac{g^2\mu^\epsilon}{32\pi^2}[\frac{6}{\epsilon} - 3\gamma - 3F(0,m,\mu)] \tag{3.36}$$

We find that

$$g = g_1\mu^{-\epsilon} + \frac{3g_1^2\mu^{-2\epsilon}}{32\pi^2}[\frac{2}{\epsilon} - \gamma - F(0,m_1,\mu)] \tag{3.37}$$

satisfies (3.36) if we ignore terms of higher order than g^2. We check this by substituting the expresssion for g in (3.37) into the right hand side of (3.36). This gives

$$[g_1\mu^{-\epsilon} + \frac{3g_1^2\mu^{-2\epsilon}}{32\pi^2}(\frac{2}{\epsilon} + \ldots)]\mu^{\epsilon} - [g_1\mu^{-\epsilon} - \frac{3g_1^2\mu^{-2\epsilon}}{32\pi^2}(\frac{2}{\epsilon} + \ldots)]^2 \times \frac{\mu^{\epsilon}}{32\pi^2}[\frac{6}{\epsilon} - \ldots]$$

$$= g_1 - \frac{6g_1^2\mu^{-\epsilon}}{32\pi^2\epsilon} + \frac{6g_1^2\mu^{-\epsilon}}{32\pi^2\epsilon} = g_1$$

Terms of order g_1^3 etc. have been ignored. We therefore see that replacing g by the expression in (3.37) eliminates the divergence in (3.35). We obtain

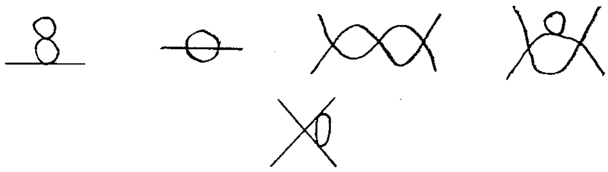
$$\begin{aligned} i\Gamma^{(4)}(p_i) &= g_1 + \frac{g_1^2\mu^{-\epsilon}}{32\pi^2}[F(s,m_1,\mu) + F(t,m_1,\mu) \\ &\qquad + F(u,m_1,\mu) - 3F(0,m_1,\mu)] \end{aligned}$$

and we see that

$$i\Gamma^{(4)}(0) = g_1$$

since $s = t = u = 0$ when $p_1 = p_2 = p_3 = p_4 = 0$. Note that there are other ways of defining the renormalised mass and coupling constant to make the theory finite.

The above discussion shows how the coupling constant and mass may be renormalised to one loop. When dealing with 2-loop diagrams we are faced with divergences arriving from the diagrams



The first two diagrams are contributions to $\Gamma^{(2)}$ of order g^2 and the remainder are contributions to $\Gamma^{(4)}$ of order g^3. It can be shown using dimensional regularisation that the divergence in  takes the form of a single pole and a double pole. We find that

$$= ig^2m^2[\frac{K}{\epsilon^2} + \frac{L}{\epsilon} + \text{finite part}] \tag{3.38}$$

It turns out that the double pole $\frac{K}{\epsilon^2}$ is cancelled by a new diagram. This diagram is induced by using the expression in (3.34) for m^2 in the Lagrangian. We may denote it by

The single pole in (3.38) remains. This can be removed by adding a term of the form

$$-\frac{m_1^2 g^2 L}{\epsilon}$$

to the right hand side of (3.34). In $\Gamma^{(4)}$ there are divergences which arise from the 2-loop graphs. Some of these disappear as a result of the new diagrams which are induced by the renormalised mass m_1. The other divergences can be removed by making a suitable redefinition of the renormalised coupling constant g_1. So it is possible to make $\Gamma^{(4)}$ finite to two loops. We find, however, that $\Gamma^{(2)}$ is still divergent. This is due to a pole term of the form $\frac{Kg^2p^2}{\epsilon}$ which comes from the diagram

(We do not need to know the precise expression for K.) The divergence multiplying p^2 is removed by making an overall "wavefunction" renormalisation. We define the renormalised 2-point function by

$$\Gamma_r^{(2)} = Z_\psi(g_1, m_1, \mu)\ \Gamma^{(2)}(p, m_1, \mu) \tag{3.39}$$

$\Gamma_r^{(2)}$ is now finite and Z_ψ is infinite. We may expand Z_ψ as a series in the renormalised coupling constant g_1.

$$\begin{aligned} Z_\psi &= 1 + g_1 Z_1 + g_1^2 Z_2 \ldots \\ &= 1 + g_1^2 Z_2 + \ldots \end{aligned}$$

There is no term in the expansion of order g_1 since we were able to make $\Gamma^{(2)}$ finite to one loop without introducing the multiplicative factor Z_ψ. In analogy with equation (3.36) we consider a renormalisation condition which reduces the degree of arbitrariness in (3.39). The following set of conditions, depending on the arbitrary mass scale μ may be used.

$$\Gamma_r^{(2)}(p^2)_{|p^2=0} = -m^2 \qquad \frac{d}{dp^2}\Gamma^{(2)}_{r_{|p^2=0}} = 1$$

$$\Gamma_r^{(4)}(0,0,0,0) = g_1$$

There is a great deal of choice in writing these conditions (see Itzkson [11]). We could have had

$$\Gamma_r^{(2)}(p^2)_{|p^2=\mu^2} = \mu^2 - m^2 \qquad \frac{\partial}{\partial p^2}\Gamma_r^{(2)}(p^2)_{|p^2=\mu^2} = 1$$

$$\Gamma_r^{(4)} = -g \ \text{ at } \left\{ \begin{array}{l} p_1^2 = p_2^2 = p_3^2 = p_4^2 = \mu^2 \\ s = t = u = \frac{4\mu^2}{3} \end{array} \right.$$

We recall that the expression for m^2 in (3.34) was enough to make $\Gamma^{(2)}$ finite to one loop. It was not sufficient, however, to make $\Gamma^{(2)}$ finite to two loops. Expressed in another way, we say that m_1 is infinite in the 2-loop approximation to $\Gamma^{(2)}$. The renormalised vertex function $\Gamma_r^{(2)}$ gives a <u>finite</u> value for the renormalised mass m_r. We have

$$m_1^2 = Z_\psi^{-1} m_r^2$$

$$\frac{m^2}{1+\frac{g}{16\pi^2\epsilon}} = Z_\psi^{-1} m_r^2 \qquad \text{(from (3.34))}$$

$$m^2 = (1+\frac{g}{16\pi^2\epsilon}) Z_\psi^{-1} m_r^2$$

We should note here that m^2 contains all the infinities and is defined in terms of the infinite multiplicative factor Z_ψ^{-1}. m^2 therefore diverges as $\epsilon \to 0$. We should note in passing that m^2 is called the bare mass. More will be said about this when we look at counter-terms. But the renormalised mass m_r remains finite and is to be identified with the physical mass of the theory. We now turn to $\Gamma^{(4)}$. $\Gamma^{(4)}$ and $\Gamma_r^{(4)}$ are linked together by the equation

$$\Gamma_r^{(4)} = Z_\psi^2 \Gamma^{(4)}(p, m_1, \mu)$$

Setting $p_i = 0$ we obtain a relationship

$$g_r = Z_\psi^2 g_1$$

A similar relationship holds between g and g_r. If the Lagrangian is written in terms of the <u>bare</u> quantities m, g and ψ then we obtain the Greens functions of the previous section. These functions contain pole terms due to the divergences. If we now replace m and g by their infinite expansions in terms of the renormalised quantities m_r and g_r the divergences cancel - making the Greens functions finite. This information is contained in the equality.

$$\Gamma^{(n)}(p_i, g, m) = Z_\psi^{-\frac{n}{2}} \Gamma_r^{(n)}(p_i, g_r, m_r, \mu) \tag{3.40}$$

We briefly summarise what we have done. Owing to the divergences which arise in the Feynman diagrams we were forced to make an order-by-order redefinition of the parameters of the theory. This procedure may be thought of as renormalising the parameters from their bare to physical values. In this way we were able to make the Greens functions finite.

An alternative approach is to regard the parameters m and g in the original Lagrangian as being the physical parameters of the theory. The fact that we obtain divergences in the Greens functions means that additional counter-terms must be added to the Lagrangian.

3.5 Counter-terms

We recall from the previous section that the "tadpole" graph was given by -

$$= \frac{igm^2}{16\pi^2\epsilon} + \text{finite terms}$$

When we considered mass renormalisation we set

$$m^2 = m_1^2(1+\frac{g}{16\pi^2\epsilon}) \tag{3.41}$$

(m_1 was the physical or renormalised mass in the 1-loop approximation) This is equivalent to changing our original Lagrangian (defined in terms of the physical parameters m_1 and g_1) from

$$\frac{1}{2}(\partial_\mu\psi)^2 - \frac{m_1^2}{2}\psi^2 - \frac{g_1\mu^\epsilon}{4!}\psi^4$$

to

$$\frac{1}{2}(\partial_\mu\psi)^2 - \frac{m_1^2}{2}(1+\frac{g}{16\pi^2\epsilon})\psi^2 - \frac{g_1\mu^\epsilon}{4!}\psi^4$$

We may represent this change as

$$\mathcal{L} \to \mathcal{L} + \delta\mathcal{L}_1$$

where

$$\delta\mathcal{L}_1 = -\frac{gm_1^2}{32\pi^2\epsilon}\psi^2 = \frac{1}{2}(-\frac{gm_1^2}{16\pi^2\epsilon})\psi^2$$

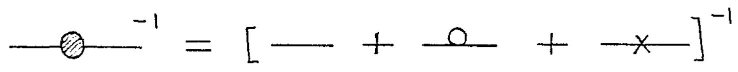
This extra term gives rise to an additional Feynman rule

$$= \frac{-igm_1^2}{16\pi^2\epsilon}$$

This term can be considered as an extra interaction term. Let us now assume that the original Lagrangian is defined in terms of m (and *not* m_1.) The complete inverse propagator is given by

$$\Gamma^{(2)}(p) = i[G^{(2)}(p)]^{-1}$$

or



$$\begin{aligned}[G^{(2)}(p)]^{-1} &= [\frac{i}{p^2-m^2-\sum(p)}]^{-1} \\ &= \frac{1}{i}[p^2-m^2-(-\frac{gm^2}{16\pi^2\epsilon}+\text{finite part})]\end{aligned}$$

(using the expression for $\sum(p)$ in (3.31).) We now suppose that at the start of the calculation we had replaced

$$m^2 \text{ by } m^2(1+\frac{g}{16\pi\epsilon})$$

(cf. (3.41)) Then we find

$$\Gamma^{(2)}(p) = p^2 - m^2 - (\frac{-gm^2}{16\pi^2\epsilon} + \text{finite part}) - \frac{m^2 g}{16\pi^2\epsilon}$$

The last two terms cancel which is as we would expect since the combination

is, by definition, finite.
So, ignoring the finite term

$$\Gamma^{(2)}(p) = p^2 - m^2$$

Here m is finite and represents the renormalised mass and is equal to $-\Gamma^{(2)}(0)$ in an appropriate order of perturbation theory. So we see that adding the divergent counter-term $\delta\mathcal{L}_1$ to the Lagrangian gives rise to the interaction . This interaction term cancels the divergence in and makes $\Gamma^{(2)}(p)$ finite (to order g). It is important to note that the counter term's dependence on the field is the same as that of a term already appearing in $\mathcal{L}$ (in this case the mass term). We explain why $\delta\mathcal{L}_1$ gives rise to a term which is considered as an interaction. Suppose we consider the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\psi)(\partial^\mu\psi) - \frac{1}{2}m^2\psi^2 \tag{3.42}$$

regarding ψ as a massless field with an interaction given by $-\frac{1}{2}m^2\psi^2$. Notice that we may write

$$\mathcal{L} = -\frac{1}{2}\psi\partial_\mu\partial^\mu\psi - \frac{1}{2}m^2\psi^2$$

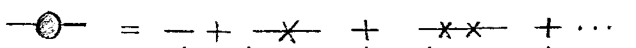
Switching the operator $\partial_\mu\partial^\mu$ to momentum space in the first term and taking the inverse gives

$$= \frac{i}{p^2}$$

The interaction term given to the vertex diagram

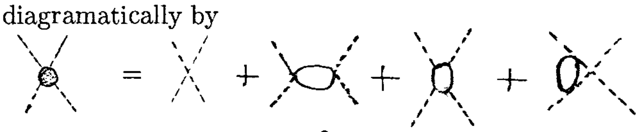
$$= -im^2$$

(The vertex has 2 external lines since we are dealing with ψ^2.) The complete propagator is given by



$$\begin{aligned} &= \frac{i}{p^2} + \frac{i}{p^2}(-im^2)\frac{i}{p^2} + \frac{i}{p^2}(-im^2)\frac{i}{p^2}(-im^2)\frac{i}{p^2} + \ldots \\ &= \frac{i}{p^2 - m^2} \end{aligned}$$

This is the usual expression for the propagator when we think of $\mathcal{L}$ in (3.42) as describing a *massive* field. $\Gamma^{(4)}$ can be treated in a similar way. $\Gamma^{(4)}$ is represented diagramatically by



$$= -ig\mu^\epsilon\left(1 - \frac{3g}{16\pi^2\epsilon}\right) + \text{finite terms} \tag{3.43}$$

This diverges as $\epsilon \to 0$. To remove this divergence we add the counter-term $\delta\mathcal{L}_2$ to the Lagrangian where

$$\delta\mathcal{L}_2 = -\frac{1}{4!}\frac{3g^2\mu^\epsilon}{16\pi^2\epsilon}\psi^4$$

This gives rise to an additional Feynman rule denoted by

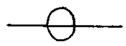
$$= \frac{-3ig^2\mu^\epsilon}{16\pi^4\epsilon}$$

This graph cancels the divergences in (3.43) rendering $\Gamma^{(4)}$ finite.

$$= -ig\mu^\epsilon + \text{finite terms}$$

We should note that these new rules will generate new diagrams whenever their counterparts among the rules operated. For example, the diagrams

are necessary when considering the renormalisation of the inverse propagator to order g^2. These diagrams contain double poles (1 pole from the crossed and dotted vertices and 1 pole from the loop integration) as well as single poles. It was previously stated that $\Gamma^{(2)}$ contained a divergence due to the term

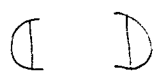
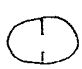


This divergence was due to a pole term of the form $\frac{Kg^2p^2}{\epsilon}$ the removal of which necessitated introducing the multiplicative factor Z_ψ. An equivalent approach is to introduce a counter-term

$$\delta\mathcal{L}_3 = -\frac{A}{2}(\partial_\mu\psi)^2$$

to the Lagrangian with $1 + A = Z_\psi$. Counter-terms of this type will produce terms of the form Ap^2. We only need to choose A carefully so that it cancels the term $\frac{Kg^2p^2}{\epsilon}$. The above discussion outlines how the insertion of counter-terms in the Lagrangian will cancel the unwanted divergences. We now ask whether this same technique may be applied to diagrams of any order in g. This leads to the question of renormalisability and what is known as "overlapping divergences". Consider the general two loop diagram

All external lines have been omitted. There are three one-loop diagrams contained in the two-loop diagram. They are

 and 

Owing to the divergences due to the various loop integrations we have counter-term diagrams associated with them -

These diagrams contain double poles and single poles. The single pole from the vertex multiplied by the finite part of the loop integrations leads to terms of the form

$$\frac{\ln\, p^2}{\epsilon}$$

This term can not be cancelled since it does not correspond to any term in $\mathcal{L}$. The theory will only be renormalisable if these divergences cancel with similar divergences from the two loop diagram. t'Hooft and Veltman [15] have shown that these divergences do indeed cancel - a crucial result in the proof of renormalisability. To make the theory finite we shall only require counter-terms which are of the same form as those appearing in the original Lagrangian. The Lagrangian which gives finite answers (often called the "bare Lagrangian") is given by

$$\mathcal{L}_B = \mathcal{L} + \mathcal{L}_{CT}$$

where $\mathcal{L}$ is the original Lagrangian given by -

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \psi)^2 - \frac{1}{2}m^2\psi^2 - \frac{g\mu^\epsilon}{4!}\psi^4$$

and

$$\mathcal{L}_{CT} = \frac{1}{2}A(\partial_\mu \psi)^2 - \frac{1}{2}m^2(\delta m^2)\psi^2 - \frac{Bg\mu^\epsilon}{4!}\psi^4$$

We therefore have

$$\mathcal{L}_B = (\frac{1+A}{2})(\partial_\mu \psi)^2 - \frac{(1+\delta m^2)}{2}m^2\psi^2 - (1+B)\frac{g\mu^\epsilon}{4!}\psi^4$$

Introducing the bare quantities ψ_B, m_B and g_B the Lagrangian may be rewritten as

$$\mathcal{L}_B = \frac{1}{2}(\partial_\mu \psi_B)^2 - \frac{m_B^2\psi_B^2}{2} - \frac{g_B\psi_B^4}{4!}\ .$$

giving the equation

$$\psi_B = (1+A)^{\frac{1}{2}}\psi = Z_\psi^{\frac{1}{2}}\psi \text{ where } Z_\psi = 1 + A$$

$$m_B = \sqrt{\frac{1+\delta m^2}{1+A}}m = Z_m m \text{ where } Z_m = \sqrt{\frac{1+\delta m^2}{1+A}}$$

$$g_B = (\frac{1+B}{(1+A)^2})\mu^\epsilon g = Z_g \mu^\epsilon g \text{ where } Z_g = \frac{1+B}{(1+A)^2}$$

The "bare" quantities are defined in terms of (infinite) multiplicative factors.

3.6 The Renormalisation Group

We now investigate how the Greens functions change as the scale μ varies. This question may be answered by recalling equation (3.40) which shows how the renormalised and unrenormalised Greens functions are linked. It is important to notice that the bare parameters are independent of μ. This means

$$\mu\frac{\partial}{\partial\mu}\Gamma^{(n)} = 0$$

where $\Gamma^{(n)}$ is unrenormalised and is written in terms of the bare parameters. From (3.40) we therefore have

$$\mu\frac{d}{d\mu}[Z_\psi^{-\frac{n}{2}}\Gamma_r^{(n)}(p_i, g_r, m_r, \mu)] = 0$$

Performing the differentiation (using the chain rule) and multiplying by $Z_\psi^{\frac{n}{2}}$ gives -

$$[\mu\frac{\partial}{\partial\mu} + \mu\frac{\partial g_r}{\partial\mu}\frac{\partial}{\partial g_r} + \mu\frac{\partial m_r}{\partial\mu}\frac{\partial}{\partial m_r} - \frac{n}{2}\mu\frac{\partial \ln Z_\psi}{\partial\mu}]\Gamma_r^{(n)} = 0 \qquad (3.44)$$

We note that $\Gamma_r^{(n)}$ depends implicitly on μ via g_r and m_r. Defining the coefficients

$$\beta(g) = \mu\frac{\partial g}{\partial\mu}$$

$$\gamma(g) = \frac{1}{2}\mu\frac{\partial \ln Z_\psi}{\partial\mu}$$

and

$$\gamma_m(g) = \frac{1}{2}\mu\frac{\partial \ln m^2}{\partial\mu}$$

equation (3.44) becomes

$$[\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} - n\gamma(g) + m\gamma_m(g)\frac{\partial}{\partial m}]\Gamma^{(n)} = 0 \qquad (3.45)$$

This is called the *Renormalisation Group Equation*. When the regularisation parameter μ varies the changes in the renormalised quantities g_r and m_r are such that the unrenormalised $\Gamma^{(n)}$ (which as stated above, does not depend on μ) does not change. We now perform a change of scale given by

$$p \to tp$$

(t is dimensionless.) $\Gamma^{(n)}$ has an engineering dimension of D_{eng} given by

$$D_{\text{eng}} = 4 - n + \frac{\epsilon}{2}(n-2)$$

Since $\Gamma^{(n)}(tp, g, m, \mu)$ is homogeneous of degree D_{eng} in p, m and μ we have

$$(t\frac{\partial}{\partial t} + m\frac{\partial}{\partial m} + \mu\frac{\partial}{\partial\mu})\Gamma^{(n)}(tp, g, m, \mu) = D_{\text{eng}}\,\Gamma^{(n)}(tp, g, m, \mu) \qquad (3.46)$$

Combining (3.46) with the renormalisation group equation given by (3.45) we eliminate $\mu\frac{\partial\Gamma}{\partial\mu}$ to obtain

$$\begin{aligned}[-t\frac{\partial}{\partial t}+\beta\frac{\partial}{\partial g}-n\gamma(g)+m(\gamma_m(g)-1)\frac{\partial}{\partial m}+D\mathrm{eng}]\\ \times\Gamma^{(n)}(tp,g,m,\mu)=0 \qquad (3.47)\end{aligned}$$

This equation shows how $\Gamma^{(n)}$ behaves as the momentum is scaled. This equation may be solved by introducing the *running coupling constant* which will be defined as the solution to the differential equation.

$$t\frac{\partial g(t)}{\partial t}=\beta(g)$$

This equation is derived as follows. We assume the existence of functions $g(t)$, $m(t)$ and $f(t)$ such that

$$\Gamma^{(n)}(tp,m,g,\mu)=f(t)\Gamma^{(n)}(p,m(t),g(t),\mu)$$

Differentiating this with respect to t gives

$$\begin{aligned}\frac{\partial}{\partial t}\Gamma^{(n)}(tp,m,g,\mu) &= \frac{df}{dt}\Gamma^{(n)}(p,m(t),g(t),\mu)\\ &\quad +f(t)(\frac{\partial m}{\partial t}\frac{\partial\Gamma^{(n)}}{\partial m}+\frac{\partial g}{\partial t}\frac{\partial\Gamma^{(n)}}{\partial g})\end{aligned}$$

or, multiplying both sides by t

$$\begin{aligned}t\frac{\partial}{\partial t}\Gamma^{(n)}(tp,m,g,\mu) &= (t\frac{df}{dt}+f(t)t\frac{\partial m}{\partial t}\frac{\partial}{\partial m}+f(t)t\frac{\partial g}{\partial t}\frac{\partial}{\partial g})\\ &\quad \times\Gamma^{(n)}(p,m(t),g(t),\mu)\\ &= (t\frac{df}{dt}+tf(t)\frac{\partial m}{\partial t}\frac{\partial}{\partial m}+tf(t)\frac{\partial g}{\partial t}\frac{\partial}{\partial g})\\ &\quad \times\frac{1}{f(t)}\Gamma^{(n)}(tp,m,g,\mu)\end{aligned}$$

Hence

$$(-t\frac{\partial}{\partial t}+\frac{t}{f}\frac{df}{dt}+t\frac{\partial m}{\partial t}\frac{\partial}{\partial m}+t\frac{\partial g}{\partial t}\frac{\partial}{\partial g})\Gamma^{(n)}(tp,m,g,\mu)=0 \qquad (3.48)$$

Comparing (3.48) with (3.47) we obtain the following differential equations.

$$t\frac{\partial g(t)}{\partial t}=\beta(g) \qquad t\frac{\partial m}{\partial t}=m[\gamma_m(g)-1]$$

$\frac{t}{f}\frac{df}{dt}=D_{eng}-n\gamma(g)$ with the initial conditions $g(1)=g$ $f(1)=1$ $m(1)=m$

The last equation is separable and can be integrated to give

$$f(t)=t^{D_{eng}}e^{-\int_1^t\frac{n\gamma(g(t))}{t}dt}$$

In the limit as $\epsilon \to 0$, $D_{eng} = 4 - n$ and we have

$$\Gamma^{(n)}(tp, m, g, \mu) = t^{4-n} e^{-\int_1^t \frac{n\gamma(g(t))}{t} dt} \Gamma^{(n)}(p, m(t), g(t), \mu) \tag{3.49}$$

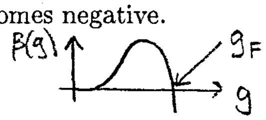
This equation demonstrates how the mass and coupling constant scale under a scaling in the external momenta. We should note that an "anomalous dimension" is now present in the exponential. Equation (3.49) represents a solution to the renormalisation group equation. We now investigate solutions to the differential equation

$$t\frac{\partial g(t)}{\partial t} = \beta(g)$$

given that we know the behaviour of $\beta(g)$. We examine the following cases (each with the assumption $\beta(0) = 0$).

1. $\beta(g) > 0$ for all values of g. Then g will always increase -showing upward or downward concavity depending on the sign of $\beta'(g)$.

2. $\beta(g)$ begins positive for small g then reaches a local maximum, decreases and finally becomes negative.



$\beta(g_F) = 0$ $g = 0$ and $g = g_F$ are called fixed points. We examine the behaviour of g near g_F by expanding β around g_F.

$$\beta(g) = \beta(g_F) + \beta'(g_F)(g - g_F) + \frac{\beta''(g_F)(g - g_F)^2}{2!}$$

To first order we have

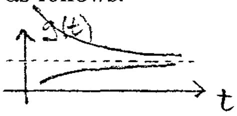
$$\beta(g) = \beta'(g_F)(g - g_F)$$

We therefore have

$$t\frac{\partial g(t)}{\partial t} = \beta'(g_F)(g - g_F)$$

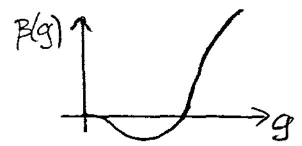
Let us choose the starting value of g to equal g_s. If $g_s < g_F$ then $g_s - g_F < 0$ and $\beta'(g_F)(g_s - g_F) > 0$. (since $\beta'(g_F) < 0$ from the diagram) This means that for $t > 0$, $\frac{\partial g(t)}{\partial t} > 0$ and implies that g starts off increasing. As $g \to g_F^-$, $\beta'(g_F)(g - g_F) \to 0^+$ and as $t \to \infty$ $\frac{\partial g(t)}{\partial t} \to 0^+$.

Arguing in this way we see that $g \to g_F^-$ as $t \to \infty$. If on the other hand $g_s > g_f$ a similar argument yields $g \to g_f^+$ as $t \to \infty$. We can sketch these two situations as follows.

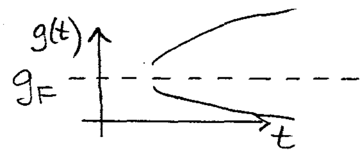


g_F is called an ultraviolet stable fixed point because $g(t) \to g_0$ as $t \to \infty$. This happens both from above and below depending on where our starting point is.

3. $\beta(g)$ behaves as follows -



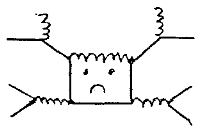
If $g_s < g_f$ then $\beta'(g_F)(g_s - g_F) < 0$. (since $\beta'(g_F) > 0$ from the diagram.) This implies that for $t > 0$ $\frac{\partial g(t)}{\partial t} < 0$ and means that $g(t)$ starts off decreasing. (ie. it moves <u>away</u> from g_F.) $g(t)$ continues decreasing and asymptotically approaches zero in the limit as $t \to \infty$. If, however, $g_s > g_F$ then using a similar argument g will be driven away from g_F. We can sketch these situations.



We say that $g = g_F$ is an infrared stable fixed point and $g = 0$ is an ultraviolet stable fixed point.

3.7 Divergences in Quantum Electrodynamics (QED)

We now turn to the dimensional regularisation of QED (see Berestetskii [6]). In this section the graphs containing primitive divergences shall be examined and the renormalisability of QED will be shown by means of the Ward Identities derived in Chapter 2. We begin by considering a diagram with n vertices, N_e external lines and N_γ external photon lines. If E_i are the number of internal electron lines then we have $2n = N_e + 2E_i$. This is because there are two electron lines attached to each vertex - some of which will be internal and hence count twice and the rest external counting once. One photon line emerges from each vertex. There will be N_γ vertices at which the photon line is external. There are $n - N_\gamma$ remaining vertices and $\frac{1}{2}(n - N_\gamma)$ internal photon lines since each photon line is attached to two vertices. For example, in the following diagram - the "alien"



$n = 8$ $N_\gamma = 2$ and the number of internal photon lines is $\frac{1}{2}(8-2) = 3$. $N_e = 6$ and the number of internal electron lines is $8 - \frac{1}{2}(6) = 5$. Since each internal photon line has a k^2 term in the denominator we associate a factor of -2 with it. Similarly we associate with each internal electron line a factor of -1. This means that the total power of momentum in the denominator is given by

$$2(\text{No. of internal photon lines}) + 1(\text{No. of internal electron lines})$$

$$\begin{aligned} &= 2(\tfrac{1}{2}(n - N_\gamma)) + n - \tfrac{1}{2}N_e \\ &= 2n - \tfrac{1}{2}N_e - N_\gamma \end{aligned}$$

We now compute the power factor in the numerator due to the loop integrations. The number of integrations (or loops L) over d^4k is equal to the number of internal lines minus the number $(n-1)$ of additional conditions. Multiplying by 4 we obtain the number of integrations over <u>all</u> the 4-momentum components. So

$$\begin{aligned} L \ &= 4[(n - \tfrac{1}{2}N_e) + \tfrac{1}{2}(n - N_\gamma) - (n-1)] \\ &= 2(n - N_e - N_\gamma + 2) \end{aligned}$$

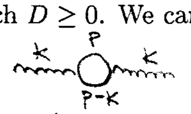
The superficial degree of divergence D is equal to the difference between the total power of the integration measure and the power of the momenta in the denominator. This gives

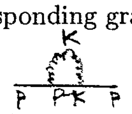
$$\begin{aligned} D \ &= 2(n - N_e - N_\gamma + 2) - (2n - \tfrac{1}{2}N_e - N_\gamma) \\ &= 4 - \tfrac{3}{2}N_e - N_\gamma \end{aligned} \tag{3.50}$$

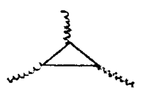
This expression is true in the case of $d = 4$. The more general formula working in d-dimensions is

$$D = d + n(\frac{d}{2} - 2) - (\frac{d-1}{2})N_e - (\frac{d-2}{2})N_\gamma$$

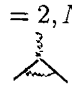
We notice that in (3.50), D is independent of the number of vertices - a condition which is crucial if we are to prove renormalisability of the theory. Just as in ψ^4 theory, the condition $D < 0$ is not sufficient to garuantee convergence - a diagram may contain internal sections (primitive divergences) for which $D > 0$. If $D \geq 0$ for the whole diagram, the integral always diverges. Since N_e and N_γ are positive integers there is a limited number of pairs of values of N_e and N_γ for which $D \geq 0$. We can draw the corresponding graphs as follows.

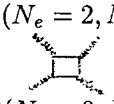






(a) $D = 2(N_\gamma = 2, N_e = 0)$ (b) $D = 1(N_e = 2, N_\gamma = 0)$ (c) $D = 1(N_\gamma = 3, N_e = 0)$





(d) $D = 0(N_\gamma = 1, N_e = 2)$ (e) $D = 0(N_e = 0, N_\gamma = 4)$

The above graphs are all primitively divergent in QED.

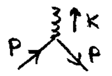
Graph(a) is part of the photon self-energy.

Graph (b) is the part of the electron self-energy.

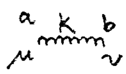
Graphs (c) and (d) are vertex graphs.

Graph (e) may be thought of as a scattering of light by light (see Aitchison [1]). We recall the Feynman rules for the matter propagator, the gauge-field-matter coupling and the gauge-field propagator given by

$$= \frac{i}{\gamma.p - m} = \frac{i}{\not p - m}$$



$$= -ie\gamma_\mu$$



$$= -\frac{i}{k^2}[g_{\mu\nu}] \text{ in the Feynman gauge } \alpha = 1$$

Since QED is based on the group $U(1)$ we lose the group matrix terms. Notice that we have replaced g by e. With these rules we can write down the expression for the electron self-energy in (b). We have

$$\frac{\sum(p)}{i} = \int \frac{d^4k}{(2\pi)^4}(-ie\gamma^\mu)(\frac{i}{\gamma.(p-k) - m})(-\frac{ig_{\mu\nu}}{k^2})(-ie\gamma^\nu)$$

$$= (-ie)^2 \int \frac{d^4k}{(2\pi)^4}\gamma^\mu(\frac{i}{\not p - \not k - m})(-\frac{ig_{\mu\nu}}{k^2})\gamma^\nu$$

Notice that the superficial degree of divergence is $4 - 3 = 1$ which agrees with equation (3.50) setting $N_e = 2$ and $N_\gamma = 0$. This four-dimensional integral is ultraviolet divergent at large k. It is also infrared divergent. The integral associated with the photon self-energy depicted in graph (a) is

$$i\Pi^{\mu\nu}(k) = -(-ie)^2 \int \frac{d^4p}{(2\pi)^4} Tr(\gamma^\mu \frac{i}{\not p - m}\gamma^\nu \frac{i}{\not p - \not k - m}) \tag{3.51}$$

is seen to be quadratically divergent. This result is in keeping with (3.50) setting $N_\gamma = 2$ and $N_e = 0$. $\Pi^{\mu\nu}(k)$ is known as the polarisation tensor. We may now write down the expression for the photon propagator $D'_{\mu\nu}$ which is, to one loop

$$iD'_{\mu\nu}(k) = \frac{-ig_{\mu\nu}}{k^2} + (\frac{-ig_{\mu\alpha}}{k^2})i\Pi^{\alpha\beta}(k)(\frac{-ig_{\beta\nu}}{k^2})$$

We may regard the photon-self-energy graph as the amplitude of the transition of a boson into itself via a decay into an electron-positron pair. In graphs (c), (d) and (e) the divergences are logarithmic. Diagram (d) is an example of a vertex graph and is given by

$$-ie\Lambda_\mu(p,q,p+q) = (-ie)^3 \int \frac{d^4k}{(2\pi)^4}\frac{-ig_{\rho\sigma}}{(k+p)^2}\gamma^\rho \frac{i}{\not k - \not q - m}\gamma_\mu \frac{i}{\not k - m}\gamma^\sigma$$

The integral is logarithmic divergent as $k \to \infty$. The only three primitively divergent diagrams in QED are given by graphs (a), (b) and (d). It follows from *Furry's Theorem* that graph (c) may be ignored. The graph is in fact, cancelled by a similar graph with the electron arrows reversed. The remaining graph (e) - although superficially divergent - happens to be convergent. We now show how to apply dimensional regularisation to the primitively divergent integrals. The Lagrangian which describes the interaction of photons and electrons is given by

$$\begin{aligned}\mathcal{L} \;&= i\psi^*\gamma^\mu\partial_\mu\psi - m\psi^*\psi - eA^\mu\psi^*\gamma_\mu\psi \\ &-\tfrac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - \tfrac{1}{2}(\partial_\mu A^\mu)^2\end{aligned}$$

We must generalise this expression to d-dimensions. If $\mathcal{L}$ is to have mass dimension d this means that A_μ must have dimensions $\frac{d-2}{2}$. This gives the correct dimension to all the terms excluding the third. This problem is corrected by multiplying e by $\mu^{2-\frac{d}{2}}$ where μ is the mass parameter of dimensional regularisation. We therefore deal with the Lagrangian given by

$$\begin{aligned}\mathcal{L} \;&= i\psi^*\gamma^\mu\partial_\mu\psi - m\psi^*\psi - e\mu^{2-\frac{d}{2}}A^\mu\psi^*\gamma_\mu\psi \\ &-\tfrac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - \tfrac{1}{2}(\partial_\mu A^\mu)^2\end{aligned}$$

We check the dimension of the third term.

$$\begin{aligned}e\mu^{2-\frac{d}{2}}A^\mu\psi^*\gamma_\mu\psi \;&\sim 2 - \tfrac{d}{2} + \tfrac{d}{2} - 1 + \tfrac{d}{2} - \tfrac{1}{2} + \tfrac{d}{2} - \tfrac{1}{2} \\ &= d\end{aligned}$$

Dimensional regularisation shall now be applied to the electron self-energy. Working in a space-time of complex dimensionality d we have

i  $= \sum(p) = -ie^2\mu^{4-d}\int \frac{d^dk}{(2\pi)^d}\gamma_\mu\frac{1}{\not p - \not k - m}\gamma_\nu\frac{g^{\mu\nu}}{k^2}$

$$= -ie^2\mu^{4-d}\int \frac{d^dk}{(2\pi)^d}\frac{\gamma_\mu(\not p - \not k + m)\gamma^\mu}{[(p-k)^2 - m^2]k^2}$$

Combining the propagator by means of the Feynman formula

$$\frac{1}{ab} = \int_0^1 \frac{dz}{[az + b(1-z)]^2}$$

yields

$$\sum(p) = -i\mu^{4-d}e^2\int_0^1 dz\int \frac{d^dk}{(2\pi)^d}\frac{\gamma_\mu(\not p - \not k + m)\gamma^\mu}{[(p-k)^2z - m^2z + k^2(1-z)]^2}$$

Feynman's formula works well in the case of one-loop diagrams. Care must be taken when using it in the case of multiple-loop diagrams - as pointed out by t'Hooft and Veltman (1972). It turns out that ultraviolet divergences find their way into parametric integrals. An example of this occurs in the diagram .
We now perform a shift of the integration variable given by

$$k' = k - pz$$

to give

$$\sum(p) = -i\mu^{4-d}e^2\int_0^1 dz\int\frac{d^dk'}{(2\pi)^d}\frac{\gamma_\mu(\not{p}-\not{p}z-\not{k}'+m)\gamma^\mu}{[k'^2-m^2z+p^2z(1-z)]^2}$$

We now have the integrand in a form in which the denominator depends only on k'^2. We use the following result which is the analog of symmetric integration in 4-space.

$$\int\frac{d^dk}{(2\pi)^d}\frac{k_\mu}{(k^2-H_0)^\alpha}=0 \tag{3.52}$$

(H_0 is a constant.) This means that the integral over the linear term in k' is zero. We therefore have

$$\sum(p) = -i\mu^{4-d}e^2\int_0^1 dz\ \ \gamma_\mu(\not{p}-\not{p}z+m)\gamma^\mu$$

$$\times\int\frac{d^dk'}{(2\pi)^d}\frac{1}{[k'^2-m^2z+p^2z(1-z)]^2}$$

From equation $(A1)$ in the appendix we have we have

$$\sum(p) = \mu^{4-d}e^2\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}}\int_0^1 dz\ \gamma_\mu[\not{p}(1-z)+m]\gamma^\mu$$

$$\times[m^2z-p^2z(1-z)]^{\frac{d}{2}-2}$$

Using the following properties of the gamma matrices

$$\begin{aligned}\gamma_\mu\gamma^\mu &= d\\ \gamma_\mu\gamma_\nu\gamma^\mu &= (2-d)\gamma_\nu\\ \text{or }\gamma_\mu\ \not{p}\gamma^\mu &= (2-d)\ \not{p}\end{aligned}$$

we have

$$\begin{aligned}\gamma_\mu[\not{p}(1-z)+m]\gamma^\mu &= (2-d)\ \not{p}(1-z)+md\\ &= (\epsilon-2)\ \not{p}(1-z)+m(4-\epsilon)\\ &= -[2\ \not{p}(1-z)-4m-\epsilon(\not{p}(1-z)-m)]\end{aligned}$$

This means

$$\sum(p) = \frac{-e^2}{16\pi^2}\Gamma(\frac{\epsilon}{2})\int_0^1 dz\{2\ \not{p}(1-z)-4m-\epsilon[\not{p}(1-z)-m]\}$$

$$\times\left(\frac{m^2z-p^2z(1-z)}{4\pi\mu^2}\right)^{-\frac{\epsilon}{2}}$$

Letting

$$\Gamma(\frac{\epsilon}{2})=\frac{2}{\epsilon}-\gamma+O(\epsilon)$$

from equation (3.28) and

$$\left(\frac{m^2z - p^2z(1-z)}{4\pi\mu^2}\right)^{-\frac{\epsilon}{2}} = 1 - \frac{\epsilon}{2} ln\left[\frac{m^2z - p^2z(1-z)}{4\pi\mu^2}\right]$$

(using a standard Taylor expansion.) Retaining only the pole and the finite term gives

$$\begin{aligned}
\sum(p) &= \frac{-e^2}{16\pi^2}\frac{2}{\epsilon}\int_0^1 dz\ (2\ \not{p}(1-z) - 4m) \\
&+\frac{e^2}{16\pi^2}\gamma\int_0^1 dz(2\ \not{p}(1-z) - 4m) + \frac{2e^2}{16\pi^2}\int_0^1 dz\ (\not{p}(1-z) - m) \\
&+\frac{e^2}{16\pi^2}\int_0^1 dz\ (2\ \not{p}(1-z) - 4m)\ ln[\frac{m^2z - p^2z(1-z)}{4\pi\mu^2}] \\
&= \frac{e^2}{8\pi^2\epsilon}(-\ \not{p} + 4m) + \frac{e^2}{16\pi^2}\{\not{p}(1+\gamma) - 2m(1+2\gamma) \\
&+2\int_0^1 dz\ [\not{p}(1-z) - 2m]\ ln(\frac{m^2z - p^2z(1-z)}{4\pi\mu^2})\} \\
&= \frac{e^2}{8\pi^2\epsilon}(-\ \not{p} + 4m) + \text{finite part}
\end{aligned}$$

We now calculate the vacuum polarisation graph. Generalising (3.51) to d dimensions we have

$$\begin{aligned}
\Pi_{\mu\nu}(k) &= i\mu^{4-d}e^2\int\frac{d^dp}{(2\pi)^d}Tr[\gamma_\mu\frac{1}{\not{p} - m}\gamma_\nu\frac{1}{\not{p} - \not{k} - m}] \\
&= ie^2\mu^{4-d}\int\frac{d^dp}{(2\pi)^d}\frac{Tr[\gamma_\mu(\not{p} + m)\gamma_\nu(\not{p} - \not{k} + m)]}{(p^2 - m^2)[(p-k)^2 - m^2]}
\end{aligned}$$

Introducing a Feynman parameter z and shifting the integration variable by putting $p' = p - kz$ gives

$$\Pi_{\mu\nu} = ie^2\mu^{4-d}\int_0^1 dz\int\frac{d^dp'}{(2\pi)^d}\frac{Tr[\gamma_\mu(\not{p}' + \not{k}z + m)\gamma_\nu(\not{p}' - \not{k}(1-z) + m)]}{[p'^2 - m^2 + k^2z(1-z)]^2}$$

The numerator of this expression may be simplified by means of the trace relations (Akyeampong and Delbourgo 1973)

$$Tr(\gamma_\mu\gamma_\nu) = f(d)g_{\mu\nu}$$

and

$$Tr(\gamma_\mu\gamma_\rho\gamma_\nu\gamma_\sigma) = f(d)(g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\rho\nu} - g_{\mu\nu}g_{\rho\sigma})$$

where $f(d) = 2^{\frac{d}{2}}$ We also have the relation that

$$Tr(\text{odd no. of } \gamma \text{ matrices}) = 0$$

and (3.52). This last identity means that we may write the numerator as

$$
\begin{aligned}
& [p^{\kappa\prime}p^{\prime\lambda} - k^{\kappa}k^{\lambda}z(1-z)]Tr(\gamma_\mu\gamma_\kappa\gamma_\nu\gamma_\lambda) + m^2 Tr(\gamma_\mu\gamma_\nu) \\
= \; & [p^{\kappa\prime}p^{\lambda\prime} - k^{\kappa}k^{\lambda}z(1-z)]f(d)(g_{\mu\kappa}g_{\nu\lambda} - g_{\mu\nu}g_{\kappa\lambda} + g_{\mu\lambda}g_{\kappa\nu}) \\
& + m^2 f(d) g_{\mu\nu} (\text{from the trace relations}) \\
= \; & f(d)\{2p'_\mu p'_\nu - 2z(1-z)(k_\mu k_\nu - k^2 g_{\mu\nu}) \\
& - g_{\mu\nu}[p'^2 - m^2 + k^2 z(1-z)]
\end{aligned}
$$

Hence, dropping the prime on p

$$
\Pi_{\mu\nu}(k) = ie^2\mu^{4-d} f(d) \int_0^1 dz \int \frac{d^d p}{(2\pi)^d}\{\frac{2p_\mu p_\nu}{[p^2 - m^2 + k^2 z(1-z)]^2}
$$

$$
-\frac{2z(1-z)[k_\mu k_\nu - g_{\mu\nu}k^2]}{[p^2 - m^2 + k^2 z(1-z)]^2} - \frac{g_{\mu\nu}}{[p^2 - m^2 + k^2 z(1-z)]}\}
$$

From the equation $(A2)$ in the appendix we observe that the first integral is

$$
\frac{(-1)^{-\frac{d}{2}} i\pi^{\frac{d}{2}} 2}{(2\pi)^d \Gamma(2)} \left\{ \frac{1}{2} g_{\mu\nu} \frac{\Gamma(1-\frac{d}{2})}{[k^2 z(1-z) - m^2]^{1-\frac{d}{2}}} \right\}
$$

The third integral is just equal to the negative of this and cancels. The middle term integrates to give

$$
\frac{(-1)^{-\frac{d}{2}} i\pi^{\frac{d}{2}}}{(2\pi)^d} \frac{\Gamma(2-\frac{d}{2})}{\Gamma(2)} \frac{-2z(1-z)[k_\mu k_\nu - g_{\mu\nu}k^2]}{(k^2 z(1-z) - m^2)^{2-\frac{d}{2}}}
$$

This gives

$$
\Pi_{\mu\nu}(k) = \frac{e^2}{2\pi^2}(k_\mu k_\nu - g_{\mu\nu}k^2)\{\frac{1}{3\epsilon} - \frac{\gamma}{6}
$$

$$
- \int_0^1 dz \; z(1-z) ln[\frac{m^2 - k^2 z(1-z)}{4\pi\mu^2}] + O(\epsilon)\}
$$

where we have used

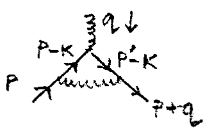
$$
\Gamma(\frac{\epsilon}{2}) = \frac{2}{\epsilon} - \gamma + O(\epsilon)
$$

and

$$
a^{-\epsilon} = 1 - \epsilon \; lna
$$

The latter result is from the standard Taylor expansion. We now evaluate the one-loop vertex correction -

$$
\Lambda_\mu(p, q, p') \quad =
$$



We have changed the notation from the way it was defined previously by letting $p' = p + q$. The Feynman rules give

$$
\begin{aligned}
(-ie\mu^{2-\frac{d}{2}})\Lambda_\mu(p,q,p') &= (-ie\mu^{2-\frac{d}{2}})^3 \int \frac{d^dk}{(2\pi)^d}\gamma_\nu(\frac{i}{\not p' - \not k - m}) \\
&\quad \times\gamma_\mu(\frac{i}{\not p - \not k - m})\gamma_\rho(\frac{-ig^{\nu\rho}}{k^2})
\end{aligned}
$$

$$
= -(e\mu^{2-\frac{d}{2}})^3 \int \frac{d^dk}{(2\pi)^d}\frac{\gamma_\nu(\not p' - \not k + m)\gamma_\mu(\not p - \not k + m)\gamma^\nu}{k^2[(p-k)^2 - m^2][(p'-k)^2 - m^2]}
$$

Owing to the 3 quadratic factors in the denominator we introduce two Feynman parameters. The delta function means that the y-integration goes from 0 to $1-x$.

$$
\Lambda_\mu(p,q,p') = \frac{-2ie^2\mu^{4-d}}{(2\pi)^d}\int_0^1 dx \int_0^{1-x} dy \int d^dk
$$

$$
\times\frac{\gamma_\nu(\not p' - \not k + m)\gamma_\mu(\not p - \not k + m)\gamma^\nu}{[k^2 - m^2(x+y) - 2k(px + p'y) + p^2x + p'^2y]^3}
$$

We now shift the integration variable by setting

$$
k' = k - px - p'y
$$

This changes the denominator to

$$
[k'^2 - m^2(x+y) + p^2x(1-x) + p'^2y(1-y) - 2p.p'xy]^3
$$

Dropping the prime on the k gives

$$
\Lambda_\mu(p,q,p') = \frac{-2ie^2\mu^{4-d}}{(2\pi)^d}\int_0^1 dx \int_0^{1-x} dy \int d^dk
$$

$$
\times\frac{\gamma_\nu[\not p'(1-y) - \not p x - \not k + m]\gamma_\mu[\not p(1-x) - \not p' y - \not k + m]\gamma^\nu}{[k^2 - m^2(x+y) + p^2x(1-x) + p'^2y(1-y) - 2p.p'xy]^3}
$$

This integral is divergent in the term quadratic in k. (As $d \to 4$ the integral behaves as $\int \frac{d^4k}{k^4}$ which is logarithmic divergent.) The rest of the integral is convergent. So we may write

$$
\Lambda_\mu = \Lambda_\mu^{(1)} + \Lambda_\mu^{(2)}
$$

where $\Lambda_\mu^{(1)}$ consists of the part quadratic in k and $\Lambda_\mu^{(2)}$ contains the finite part. The divergent part may be evaluated by means of the equation $(A2)$ in the Appendix to give

$$
\begin{aligned}
\Lambda_\mu^{(1)}(p,q,p') &= \frac{e^2}{2}\mu^{4-d}\frac{1}{(4\pi)^{\frac{d}{2}}}\Gamma(2-\frac{d}{2})\int_0^1 dx \int_0^{1-x} dy \\
&\quad \times\frac{\gamma_\nu\gamma_\rho\gamma_\mu\gamma^\rho\gamma^\nu}{[m^2(x+y) - p^2x(1-x) - p'^2y(1-y) + 2p.p'xy]^{2-\frac{d}{2}}}
\end{aligned}
$$

We have the following Dirac matrix identities in d-dimensions.

$$\gamma_\nu \gamma_\rho \gamma_\mu \gamma_\sigma \gamma^\nu = (2-d)\gamma_\rho \gamma_\mu \gamma_\sigma + 2(\gamma_\mu \gamma_\sigma \gamma_\rho - \gamma_\rho \gamma_\sigma \gamma_\mu)$$

$$\gamma_\nu \gamma_\rho \gamma_\mu \gamma^\rho \gamma^\nu = \gamma_\nu [(2-d)\gamma_\mu]\gamma^\nu = (2-d)^2 \gamma_\mu$$

Letting $\epsilon = 4 - d$ gives

$$\Gamma(2-\frac{d}{2}) = \Gamma(\frac{\epsilon}{2}) = \frac{2}{\epsilon} - \gamma + O(\epsilon)$$

$$(2-d)^2\gamma_\mu = (\epsilon-2)^2\gamma_\mu = (4-2\epsilon)\gamma_\mu + O(\epsilon^2)$$

$$\Lambda_\mu^{(1)}(p,q,p') = \frac{e^2}{2}\mu^{4-d}(\frac{1}{4\pi})^{\frac{d}{2}}(\frac{2}{\epsilon}-\gamma)(4-2\epsilon)\gamma_\mu$$

$$\times \int_0^1 dx \int_0^{1-x} dy[1 - \frac{\epsilon}{2} ln(m^2(x+y) - p^2x(1-x) - p'^2y(1-y) + 2p.p'xy)]$$

As $d \to 4$ we extract the divergent pole term which is

$$\frac{e^2}{2}\frac{1}{16\pi^2}\frac{2}{\epsilon}4\gamma_\mu \int_0^1 dx \int_0^{1-x} dy = \frac{e^2}{8\pi^2\epsilon}\gamma_\mu$$

We therefore have

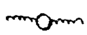
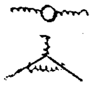
$$\Lambda_\mu^{(1)}(p,q,p') = \frac{e^2}{8\pi^2\epsilon}\gamma_\mu + \text{finite terms}$$

$\Lambda_\mu^{(2)}$ is convergent and so we may let $d = 4$. The integral over k is evaluated by means of equation $(A1)$ in the Appendix giving

$$\Lambda_\mu^{(2)}(p,q,p') = \frac{e^2}{16\pi^2}\int_0^1 dx \int_0^{1-x} dy$$

$$\times \frac{\gamma_\nu(\not{p}'(1-y) - \not{p}x + m)\gamma_\mu(\not{p}(1-x) - \not{p}'y + m)\gamma^\nu}{m^2(x+y) - p^2x(1-x) - p'^2y(1-y) + 2p.p'xy}$$

We have now calculated the primitively divergent graphs in QED. It is convenient to represent our results as follows.

		Superficial Deg. of Div.	Actual Deg. of Div.	Explicit expression
$\sum(p)$		1	*Logarithmic*	$\frac{e^2}{8\pi^2\epsilon}(-\not{p} + 4m)$ + finite part
$\Pi_{\mu\nu}(k)$		2	*Logarithmic*	$\frac{e^2}{6\pi^2\epsilon}(k_\mu k_\nu - g_{\mu\nu}k^2)$ + finite part
$\Lambda_\mu^{(1)}(p,q,p')$		0	*Logarithmic*	$\frac{e^2}{8\pi^2\epsilon}\gamma_\mu$ + finite part

3.8 1-Loop renormalisation of QED

We consider the counter-terms which, when added to the Lagrangian cancel the divergences. We start by considering the electron inverse propagator, $\Gamma^{(2)}(p)$

$$
\begin{aligned}
\Gamma^{(2)}(p) &= S_F(p)^{-1} - \sum(p) \\
&= \not{p} - m - \frac{e^2}{8\pi^2\epsilon}(-\not{p} + 4m) \quad \text{(We neglect the finite part.)} \\
&= \not{p}(1 + \frac{e^2}{8\pi^2\epsilon}) - m(1 + \frac{e^2}{2\pi^2\epsilon}) \qquad (3.53)
\end{aligned}
$$

Here we must add the two counter-terms to the Lagrangian. The first divergence in $\not{p}$ is similar to the divergence $\frac{kp^2}{\epsilon}$ occuring in the diagram  in ψ^4 theory. We recall that in order to cancel this term we needed to renormalise the wave-function. A similar technique is therefore applied to the above divergence. The second divergence in (3.53) is cancelled by a counter-term giving a contribution to the electron mass. To the Lagrangian $\mathcal{L}_1$ defined by

$$\mathcal{L}_1 = i\psi^* \not{\partial}\psi - m\psi^*\psi$$

we add

$$(\mathcal{L}_1)_{CT} = iB\psi^* \not{\partial}\psi - A\psi^*\psi$$

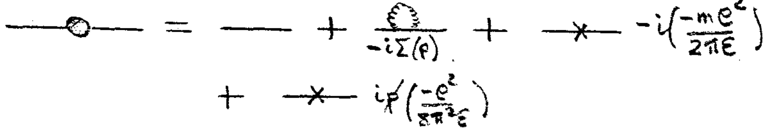
(B is chosen in such a way as to cancel the divergence in $\not{p}$; A cancels the divergence in m) We define the bare Lagrangian as

$$(\mathcal{L}_1)_B = \mathcal{L}_1 + (\mathcal{L}_1)_{CT} = i(1+B)\psi^* \not{\partial}\psi - (m+A)\psi^*\psi \qquad (3.54)$$

Clearly

$$B = \frac{-e^2}{8\pi^2\epsilon} \quad \text{and} \quad A = \frac{-me^2}{2\pi^2\epsilon}$$

To one loop, the electron propagator is finite. We have



The addition of the counter-terms $iB\psi^* \not{\partial}\psi$ is equivalent to a wave-function renormalisation. We define the bare wave-function by

$$\psi_B = \sqrt{Z_2}\psi$$

where $Z_2 = 1 + B$ and the bare Lagrangian by

$$(\mathcal{L}_1)_B = i\psi_B^* \not{\partial}\psi_B - m_B\psi_B^*\psi_B$$

We have from (3.54)

$$m_B\psi_B^*\psi_B = m_B Z_2\psi^*\psi = (m+A)\psi^*\psi$$

ie.

$$
\begin{aligned}
m_B &= \frac{m+A}{Z_2} \\
&= m\left(1-\frac{e^2}{2\pi^2\epsilon}\right)\left(1+\frac{e^2}{8\pi^2\epsilon}\right) \quad \text{(expanding } Z_2^{-1} \text{ to first order)} \\
&= m\left(1-\frac{3e^2}{8\pi^2\epsilon}\right) \quad \text{(ignoring terms of order } \tfrac{1}{\epsilon^2} \text{)} \\
&= m+\delta m
\end{aligned}
$$

m is called the physical or renormalised mass. We make a brief comment here about the Dirac equation. The last equation shows how the electron mass may be renormalised to first order. We could have based our perturbation theory on the equation

$$(i\,\partial\!\!\!/ - m + e\,A\!\!\!/ - \delta m)\psi = 0$$

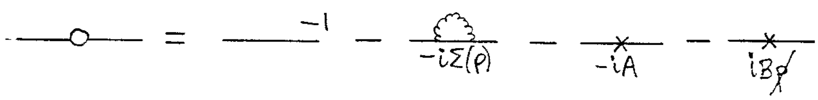
rather than on

$$(i\,\partial\!\!\!/ - m_B + e\,A\!\!\!/)\psi = 0$$

$\delta m = m_B - m$ is to be regarded as an "additional interaction". The Dirac equation for the physical free particle is now

$$(i\,\partial\!\!\!/ - m + e\,A\!\!\!/)\psi = 0$$

So we are beginning to see definite similarities between the renormalisation of QED and ψ^4 theory. The divergences which arise in the Feynman diagrams are cancelled by adding appropriate counter-terms to the Lagrangian. The expansion for the two-point vertex is



$$
\begin{aligned}
\Gamma^{(2)}(p) \quad &= i[S'_F(p)]^{-1} \\
&= p\!\!\!/ - m - (\textstyle\sum(p) + A - B\,p\!\!\!/) \\
&= p\!\!\!/ - m + \text{finite term}
\end{aligned}
$$

(since by definition $\sum(p) + A - B\ p\!\!\!/$ is finite to 1-loop by construction.) We now deal with the case of the photon propagator. Analogous to the expansion of G_c^2 in Chapter 2 we have

$$D'_{\mu\nu}(k) = D_{\mu\nu}(k) - D_{\mu\alpha}(k)\Pi^{\alpha\beta}(k)D_{\beta\nu} + \ldots$$

Working in the Feynman gauge ($\alpha \to 1$) means that

$$D_{\mu\nu}(k) = -\frac{g_{\mu\nu}}{k^2}$$

This gives

$$D'_{\mu\nu}(k) = \frac{-g_{\mu\nu}}{k^2} - \frac{g_{\mu\alpha}}{k^2}[\frac{e^2}{6\pi^2}(k^\alpha k^\beta - g^{\alpha\beta}k^2)(\frac{1}{\epsilon} + \frac{k^2}{10m^2})]\frac{g_{\beta\nu}}{k^2} + \ldots$$

The $\frac{k^2}{10m^2}$ is a finite term in the calculation of $\Pi^{\alpha\beta}(k)$. (This was not, however, shown in the earlier calculation.) Rearranging the expression yields

$$D'_{\mu\nu}(k) = \frac{-g_{\mu\nu}}{k^2}(1 - \frac{e^2}{6\pi^2\epsilon} - \frac{e^2}{60\pi^2}\frac{k^2}{m^2}) - \frac{e^2}{6\pi^2\epsilon}\frac{1}{k^2}\frac{k_\mu k_\nu}{k^2} + \ldots \tag{3.55}$$

We now add correction terms to the Lagrangian to cancel these divergences. The part of the Lagrangian which gives rise to the Feynman propagator

$$D_{\mu\nu}(k) = \frac{-g_{\mu\nu}}{k^2}$$

is

$$\mathcal{L}_2 = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(\partial_\mu A^\mu)^2 = \frac{1}{2}A^\mu g_{\mu\nu}\partial_\rho\partial^\rho A^\nu$$

(note that $\alpha = 1$) This is easily seen by switching the operator $g_{\mu\nu}\partial_\rho\partial^\rho$ to momentum space and taking the inverse. So by adding a suitable counter-term given by

$$-\frac{C}{4}F_{\mu\nu}F^{\mu\nu}$$

we may cancel the divergences

$$\frac{g_{\mu\nu}e^2}{k^2 6\pi^2\epsilon} \quad \text{and} \quad \frac{-e^2}{6\pi^2\epsilon}\frac{1}{k^2}\frac{k_\mu k_\nu}{k^2}$$

This is achieved by setting

$$C = \frac{-e^2}{6\pi^2\epsilon}$$

We define the required counter-term to cancel these divergences by

$$(\mathcal{L}_2)_{CT} = -\frac{C}{4}F_{\mu\nu}F^{\mu\nu}$$

The bare Lagrangian is

$$\begin{aligned}(\mathcal{L}_2)_B &= -(\frac{1+C}{4})F_{\mu\nu}F^{\mu\nu} + \text{gauge fixing terms} \\ &= -\frac{Z_3}{4}F_{\mu\nu}F^{\mu\nu} + \text{gauge fixing terms}\end{aligned}$$

with

$$Z_3 = 1 - \frac{e^2}{6\pi^2\epsilon} \tag{3.56}$$

We now show that the photon mass remains zero after renormalisation. This differs from the case of the electron-self-energy in which a bare mass (different from the physical mass) had to be introduced. The vacuum polarisation tensor may be written in the form

$$\Pi^{\alpha\beta}(k) = (k^\alpha k^\beta - g^{\alpha\beta}k^2)\Pi(k^2) \tag{3.57}$$

We note that

$$\begin{aligned} k_\alpha \Pi^{\alpha\beta} &= (k_\alpha k^\alpha k^\beta - k_\alpha g^{\alpha\beta}k^2)\Pi(k^2) \\ &= (k^2 k^\beta - k^\beta k^2)\Pi(k^2) \\ &= 0 \end{aligned}$$

Using (3.57) we may write (3.55), to one loop, as

$$D'_{\mu\nu} = D_{\mu\nu} - D_{\mu\alpha}(k^\alpha k^\beta - g^{\alpha\beta}k^2)\Pi(k^2)D'_{\beta\nu}$$

where

$$\Pi(k^2) = \frac{e^2}{6\pi^2}\left(\frac{1}{\epsilon} + \frac{k^2}{10m^2}\right)$$

Letting

$$D_{\mu\nu} = -\frac{g_{\mu\nu}}{k^2}$$

gives

$$\begin{aligned} D'_{\mu\nu} &= -\frac{g_{\mu\nu}}{k^2} - \frac{g_{\mu\alpha}}{k^2}(k^\alpha k^\beta - g^{\alpha\beta}k^2)\Pi(k^2)D'_{\beta\nu} \\ &= -\frac{g_{\mu\nu}}{k^2} + \frac{g_{\mu\nu}}{k^2}\Pi(k^2) - \frac{k_\mu k_\nu}{k^4}\Pi(k^2) \\ &= \frac{1}{k^2}(-g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\Pi(k^2)) + \frac{g_{\mu\nu}}{k^2}\Pi(k^2) \\ &= \frac{1}{k^2(1+\Pi(k^2))}(-g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\Pi(k^2)) \end{aligned}$$

(using the standard binomial expansion $(1+x)^n = 1 + nx + \ldots$ and neglecting terms of 2nd order ie. $\Pi^2(k^2)$). $\Pi(k^2)$ contains divergences. We may write

$$\Pi(k^2) = \frac{e^2}{6\pi^2\epsilon} + \Pi_f(k^2)$$

where

$$\Pi_f(k^2) = \frac{e^2k^2}{60\pi^2m^2} \to 0 \ \text{ as } \ k^2 \to 0$$

$\Pi_f(k^2)$ contains no divergences. The dressed propagator may be written as

$$\begin{aligned} D'_{\mu\nu} &= \frac{-g_{\mu\nu}}{k^2[1+\Pi(k^2)]} + \text{gauge terms} \\ &= \frac{-g_{\mu\nu}}{k^2[1+\frac{e^2}{6\pi^2\epsilon} + \Pi_f(k^2)]} + \text{gauge terms} \\ &= \frac{-Z_3 g_{\mu\nu}}{k^2[1+\Pi_f(k^2)]} + \text{gauge terms} \end{aligned} \tag{3.58}$$

In this last expression the divergence has been absorbed by the multiplicative constant Z_3 given by (3.57). The constant Z_3 relates the bare and renormalised A_μ-fields by the equation

$$A_B^\mu = Z_3^{\frac{1}{2}} A^\mu$$

$D'_{\mu\nu}$ - the propagator obtained from the bare field A_B^μ - is related to the renormalised complete propagator $\tilde{D_{\mu\nu}}$ by the equation

$$D'_{\mu\nu} = Z_3 \tilde{D'_{\mu\nu}} \tag{3.59}$$

Comparing (3.59) and (3.58) gives

$$\tilde{D'_{\mu\nu}} = \frac{-g_{\mu\nu}}{k^2[1+\Pi_f(k^2)]} + \text{gauge terms}$$

This means that the photon mass remains zero after renormalisation. The process of renormalisation removes the divergences in $D'_{\mu\nu}$. It does not, however, remove the finite parts and in the above expression for $D'_{\mu\nu}$ there is a finite correction of order k^2. viz.

$$D'_{\mu\nu} = \frac{-g_{\mu\nu}}{k^2}(1 - \frac{e^2}{60\pi^2}\frac{k^2}{m^2} + O(k^4))$$

The correction term means that Coulombs law requires to be modified. The potential separating two charges e a distance r apart is now

$$\frac{e^2}{4\pi r} + \frac{e^2}{60\pi^2 m^2}\delta^3(r)$$

We finally turn our attention to the vertex diagram and show how the divergence may be eliminated. By adding a counter-term to the Lagrangian given by

$$(\mathcal{L}_3)_{CT} = -De\mu^{2-\frac{d}{2}}\psi^* \not{A}\psi$$

We modify $\Lambda_\mu^{(1)}$ from

$$\frac{e^2}{8\pi^2\epsilon}\gamma_\mu + \text{finite part}$$

to

$$(D + \frac{e^2}{8\pi^2\epsilon})\gamma_\mu + \text{finite part}$$

Choosing $D = \frac{-e^2}{8\pi^2\epsilon}$ will therefore remove the divergence. The bare Lagrangian

$$\begin{aligned}(\mathcal{L}_3)_B &= -(1+D)e\mu^{\frac{\epsilon}{2}}A^\mu\psi^*\gamma_\mu\psi \\ &= -Z_1 e\mu^{\frac{\epsilon}{2}}A^\mu\psi^*\gamma_\mu\psi\end{aligned}$$

with $Z_1 = 1 - \frac{e^2}{8\pi^2\epsilon}$. To summarise our results we have

$$Z_1 = Z_2 = 1 - \frac{e^2}{8\pi^2\epsilon}$$

$$Z_3 = 1 - \frac{e^2}{6\pi^2\epsilon} \qquad A = \frac{-me^2}{2\pi^2\epsilon}$$

and we may write the total bare Lagrangian as

$$\begin{aligned}\mathcal{L}_B \quad &= iZ_2\psi^*\gamma^\mu\partial_\mu\psi - (m+A)\psi^*\psi - Z_1 e\mu^{\frac{\epsilon}{2}} A^\mu\psi^*\gamma_\mu\psi \\ &-\tfrac{Z_3}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + \text{gauge terms}\end{aligned}$$

Using this Lagrangian we obtain, (to one loop) finite calculations for the self energy and the vertex. Alternatively we could write the Lagrangian in terms of the bare quantities to give

$$\mathcal{L} = i\psi_B^*\gamma^\mu\psi_B - m_B\psi_B^*\psi_B - e_B A_B^\mu\psi_B^*\gamma_\mu\psi_B - \frac{1}{4}(\partial_\mu A_{B\nu} - \partial_\nu A_{B\mu})^2$$

$$e_B = e\mu^{\frac{\epsilon}{2}}\frac{Z_1}{Z_2 Z_3^{\frac{1}{2}}} = e\mu^{\frac{\epsilon}{2}} Z_3^{-\frac{1}{2}}$$

The bare quantities contain all the pole terms which are needed to cancel the divergences to one loop. We now investigate whether it is possible to cancel the divergences to all orders in perturbation theory. ie. whether QED is renormalisable. We begin by recalling some results which were derived in the previous chapter. The compete electron propagator and vertex functions were given by

$$S_F'(p)^{-1} = S_F(p)^{-1} - \sum(p) \tag{3.60}$$

and

$$\Gamma_\mu(p,q,p+q) = \gamma_\mu + \Lambda_\mu(p,q,p+q)$$

We also have the Ward Identity

$$-\frac{\partial\sum(p)}{\partial p^\mu} = \Lambda_\mu(p,0,p)$$

In the same way as (3.60) was derived we have the relationship

$$D'(k)^{-1} = D(k)^{-1} - \Pi(k) \tag{3.61}$$

where

$$\begin{aligned}D_{\mu\nu}(k) \quad &= g_{\mu\nu}D(k) \\ D'_{\mu\nu}(k) \quad &= g_{\mu\nu}D'(k) \\ \Pi_{\mu\nu}(k) \quad &= -g_{\mu\nu}\Pi(k)\end{aligned}$$

Analogous to (3.56) we have

$$\begin{aligned}D' \quad &= D + D\Pi D + D\Pi D\Pi D + \ldots \\ &= \tfrac{D}{1-\Pi D}\end{aligned} \tag{3.62}$$

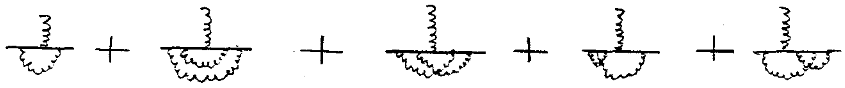
or

$$(D')^{-1} = D^{-1} - \Pi$$

This verifies (3.62). $\sum(p)$ is the proper self-energy given by the sum of the graphs

$\Pi(k)$ represents the proper vacuum polarisation diagrams which are given by

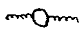
Λ is the 1PI contribution to Γ_μ and is given by

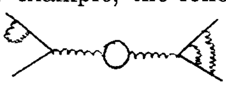


To show renormalisability we must absorb the divergences in the propagators and vertex function. This is done by introducing multiplicative constants Z_1, Z_2 and Z_3. We can define *finite* propagators and vertex functions.

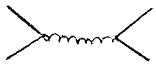
$$\tilde{S}_F = \frac{1}{Z_2} S'_F, \quad \tilde{D}_F = \frac{1}{Z_3} D'_F, \quad \tilde{\Gamma}_\mu = Z_1 \Gamma_\mu$$

We now turn our attention to the divergences on Feynman diagrams. In the previous chapter the term 1PI (1-particle irreducible) was introduced to describe graphs which could not be cut into two parts by cutting one internal line. These graphs were called proper (1PI) Feynman diagrams. We now introduce a subclass of diagrams called *irreducible.*

From the above expansion we see that the lowest order contributions to Σ, Π and Λ are given by  ,  and  . The first two are examples of self-energy graphs and the third is an example of a vertex graph. A *skeleton graph* is a graph for which no subgraphs containing self-energy parts or vertex parts can be formed. For example, the following Feynman diagram is not a skeleton graph



Its skeleton is obtained by replacing the self-energy part by a line and every vertex part by a "bare" vertex. Hence the skeleton is

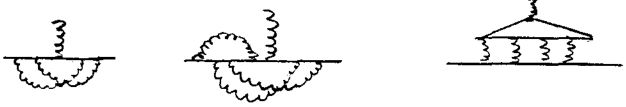


A graph which is its own skeleton is called irreducible. Skeleton graphs occur in ϕ^4 theory. Skeleton graphs for $\Gamma^{(n)}$ are graphs for which no subgraphs with positive degree of divergence (ie. non-trivial insertions of $\Gamma^{(2)}$ and $\Gamma^{(4)}$) can be found. The following is a skeleton graph for $\Gamma^{(6)}$:-

Clearly

is not a skeleton graph.

We firstly consider the divergences occuring in irreducible graphs. In the self-energy graphs the only irreducible graphs are  and  . In the vertex diagrams there are an infinite number of irreducible graphs. For example,



The superficial degree of divergence is given by

$$D = 4 - \frac{3}{2}N_e - N_\gamma$$

In each of the above graphs $N_e = 2$ and $N_\gamma = 1$ giving $D = 0$ for all of them. (ie. they are logarithmically divergent) The lowest order contribution to Λ_μ was the graph  and it was shown that

$$\Lambda_\mu = \frac{e^2}{8\pi^2\epsilon}\gamma_\mu + \text{finite part}$$

If we consider all the irreducible graphs in the expansion of Λ_μ (not just the lowest order) we obtain an expansion of the form

$$\Lambda_\mu = L\gamma_\mu + \Lambda_\mu^{(f)} \tag{3.63}$$

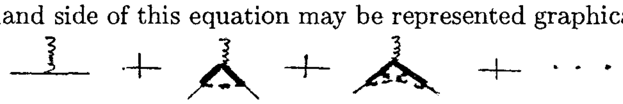
where L is an infinite multiplicative constant and $\Lambda_\mu^{(f)}$ is finite. Since Λ_μ is of this form all the divergences in irreducible vertex graphs may be isolated. We now return to the reducible graphs. To obtain a reducible graph it suffices to write down its skeleton and then to replace the propagators (lines) and bare vertices by the relevant self-energy and vertex parts. We first consider a vertex part V and its skeleton V_s. We have

$$\Lambda_\mu(p,p';S_F,D_F,\gamma,e) = \Lambda_{\mu s}(p,p';S'_F,D'_F,\Gamma,e)$$

from which it follows that

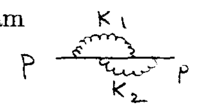
$$\Gamma_\mu(p,p') = \gamma_\mu + \Lambda_{\mu s}(p,p';S'_F,D'_F,\Gamma,e) \tag{3.64}$$

The right hand side of this equation may be represented graphically as



We have taken the lowest order terms in the skeleton expansion and replaced the internal lines by thick lines and the vertices by black dots. (ie. they have been replaced by the propagators S'_F and D'_F and the approximate vertex operator γ by Γ.

The self-energy parts are more awkward to handle due to the problem of overlapping divergences (which we recall occurred in ϕ^4 theory). Overlapping divergences occur in the diagram



It is difficult to separate out the divergences. It turns out that the problem is solved by appealing to the Ward-Takahashi identity.
A similar situation arises in the case of the photon self-energy diagrams. (eg. ). Just as the Ward identity implied that

$$\Lambda_\mu = -\frac{\partial \Sigma}{\partial p^\mu}$$

(in the case of the electron self-energy) we have the operator Δ_μ (analogous to Λ_μ) defined by

$$\Delta_\mu(k) = -\frac{\partial \Pi(k)}{\partial k^\mu}$$

Differentiating $\Pi(k)$ with respect to k^μ results in the insertion of a photon line in the internal electron lines.
Eg.

We denote the expression analogous to Γ_μ by W_μ. W_μ is defined by

$$W_\mu(k) = -2k_\mu + \Delta_\mu(k) \tag{3.65}$$

We now recall the equation for the complete photon propagator given by

$$D'(k)^{-1} = D(k)^{-1} - \Pi(k)$$

with

$$D = -\frac{1}{k^2} \quad \frac{\partial D^{-1}}{\partial k^\mu} = -2k^\mu$$

and

$$\begin{aligned}
\frac{\partial (D')^{-1}}{\partial k^\mu} &= -2k^\mu - \frac{\partial \Pi}{\partial k^\mu} \\
&= -2k^\mu + \Delta_\mu(k) \\
&= W_\mu
\end{aligned}$$

Analogous to (3.64) we have

$$\Delta_\mu(k; S_F, D_F, k, e) = \Delta_{\mu s}(k; S'_F, D'_F, W, e)$$

and hence (3.65) becomes

$$W_\mu(k) = -2k_\mu + \Delta_{\mu s}(k; S'_F, D'_F, W, e)$$

By performing the necessary subtractions we may define convergent functions

$$\begin{aligned}
\tilde{\Lambda}_\mu(p, p') &= \Lambda_{\mu s}(p, p') - \Lambda_{\mu s}(p_0, p_0)_{\not p_0 = m} \\
\tilde{\Delta}_\mu(k) &= \Delta_{\mu s}(k^2) - \Delta_{\mu s}(\mu^2)
\end{aligned} \tag{3.66}$$

where p_0 is the momentum of an on-shell electron, so that $p_0^2 \equiv m^2$ and μ is an invariant photon mass. Since

$$\Lambda_{\mu s}(p_0, p_0)_{|\not p_0 = m} = L\gamma_\mu \tag{3.67}$$

(L is an infinite constant) we may cancel the divergence occuring in (3.63). We define finite propagators and vertex functions (denoted by $\tilde{}$) by the following equations.

$$\begin{aligned}
\tilde{\Gamma}_\mu(p, p') &= \gamma_\mu + \tilde{\Lambda}_{\mu s}(p, p'; \tilde{S}_F, \tilde{D}_F, \tilde{\Gamma}, e_{ren}) && (3.68)\\
\tilde{S}_F(p)^{-1} - \tilde{S}_F(p_0)^{-1} &= (p - p_0)^\mu \tilde{\Gamma}_\mu(p, p_0) &&\\
\tilde{W}_\mu(k) &= -2k_\mu + \tilde{\Delta}_{\mu s}(k; \tilde{S}_F, \tilde{D}_F, \tilde{W}, e_{ren}) && (3.69)\\
\frac{\partial \tilde{D}_F(k)^{-1}}{\partial k^\mu} &= \tilde{W}_\mu(k) && (3.70)
\end{aligned}$$

The electron and photon propagator are normalised so that

$$\tilde{S}_F(p_0)^{-1} = \not p_0 - m$$

$$k^2 \tilde{D}_F(k^2)_{|k^2 = \mu^2} = 1$$

We now show the fundamental relations

$$\tilde{\Gamma}_\mu = Z_1 \Gamma_\mu \quad \tilde{S}_F = \frac{1}{Z_2} S'_F \quad \tilde{D}_F = \frac{1}{Z_3} D'_F \tag{3.71}$$

providing that the renormalised charge e_r is given by

$$e_r = \sqrt{Z_3} e \tag{3.72}$$

We first examine the expression for Λ_μ. Each irreducible vertex part of degree $2n$ has n photon lines, $2n$ electron lines and $2n+1$ corners. Introducing the transformations (3.71) and (3.72)

$$\begin{array}{ll}
S'_F \to \frac{1}{Z_2} S'_F = \tilde{S}_F & D'_F \to \frac{1}{Z_3} D'_F = \tilde{D}_F \\
\Gamma'_\mu \to Z_1 \Gamma'_\mu = \tilde{\Gamma}_\mu & e^2 \to Z_3 e^2 = e_r^2
\end{array} \tag{3.73}$$

we find $\Lambda_{\mu s} \to Z_1 \Lambda_{\mu s}$. (The Ward Identity means that $Z_1 = Z_2$) or,

$$\Lambda_{\mu s}[\frac{1}{Z_2} S'_F, \frac{1}{Z_3} D'_F, Z_1 \Gamma_\nu, Z_3 e^2] = Z_1 \Lambda_{\mu s}[S'_F, D'_F, \Gamma_\nu, e^2] \tag{3.74}$$

The left hand side of this equation is finite; Z_1 cancels the divergences in $\Lambda_{\mu s}[S'_F, D'_F, \Gamma_\nu, e^2]$. From (3.68), (3.66), (3.67) we have

$$\begin{aligned}
\tilde{\Gamma}_\mu &= \gamma_\mu + \tilde{\Lambda}_{\mu s}\\
&= \gamma_\mu + \Lambda_{\mu s} - L\gamma_\mu\\
&= (1 - L)(\gamma_\mu + \frac{1}{1 - L} \Lambda_{\mu s})\\
&= Z_1 \{\gamma_\mu + \frac{1}{Z_1} \Lambda_{\mu s}[\tilde{S}_F, \tilde{D}_F, \tilde{\Gamma}, e_r^2]\}\\
&= Z_1 \{\gamma_\mu + \Lambda_{\mu s}[S'_F, D'_F, \Gamma_\nu, e^2]\} \quad \text{using(3.74)}\\
&= Z_1 \Gamma_\mu && (3.75)
\end{aligned}$$

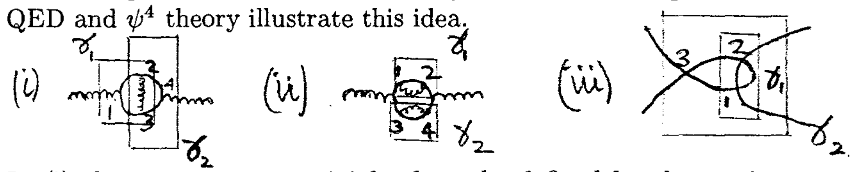
(Note that we have set $Z_1 = 1 - L$.) The above calculation shows that (3.68) holds when the renormalisation is performed according to (3.73). A similar argument can be used to show that (3.69) holds when W_μ is renormalised by

$$\tilde{W}_\mu = Z_3 W_\mu$$

The above discussion shows that renormalisation may be performed by means of a subtraction procedure (3.66). This procedure is equivalent to introducing a multiplicative constant (as shown in (3.75)). A more extensive treatment of the renormalisation of QED may be found may be found in Bjorken [7].

At this stage it is appropriate to introduce a renormalisation scheme developed and perfected by Bogoliubov, Parasiuk, Hepp and Zimmermann. To understand this method we introduce some appropriate terminology (see Chang [9] and Callan [18]).

(a) **Subgraph** A subgraph is defined by a set of vertices and all the lines connecting these vertices. It may be hidden inside a Feynman diagram and corresponds to a renormalisation part. The following examples from QED and ψ^4 theory illustrate this idea.



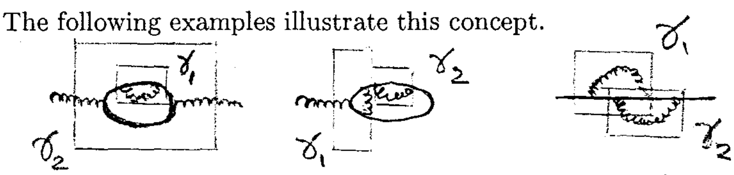
In (i) there are two non-trivial subgraphs defined by the vertices $\gamma_1 \equiv (1,2,3)$ and $\gamma_2 \equiv (2,3,4)$. (ii) contains two non-trivial subgraphs $\gamma_1 \equiv (1,2)$ and $\gamma_2 \equiv (3,4)$. (iii) (which is an example from ψ^4 theory) contains the subgraphs $\gamma_1 \equiv (1,2)$ and $\gamma_2 = (1,2,3)$.

(b) **Disjoint subgraphs** Two diagrams which have no common vertices are said to be disjoint. This relation is denoted by $\gamma_1 \cap \gamma_2 = \oslash$. Graph (ii), for example, contains two disjoint subgraphs γ_1 and γ_2.

Non-Overlapping graphs If any of the following conditions are satisfied two graphs γ_1 and γ_2 are said to be non-overlapping.

1. $\gamma_1 \subset \gamma_2$
2. $\gamma_2 \subset \gamma_1$
3. $\gamma_1 \cap \gamma_2 = \oslash$

The following examples illustrate this concept.



γ_1 and γ_2 in the first two graphs are non-overlapping. The third graph is overlapping.

(d) **Unrenormalised and renormalised amplitudes** We consider the unrenormalised amplitude M_Γ for a diagram Γ.

$$M_\Gamma = \int \prod_i \frac{d^4 k_i}{(2\pi)^4} I_\Gamma(\{p\},\{k\})$$

($\prod_i \frac{d^4 k_i}{(2\pi)^4}$ is the factor associated with internal integrations.) $\{p\}$ is the external momenta and $\{k\}$ is the internal momentum variable. By applying the Feynman rules we obtain M_Γ from Γ. We denote the renormalised amplitude by

$$\tilde{M}_\Gamma = \int \prod_i \frac{d^4 k_i}{(2\pi)^4} \tilde{I}_\Gamma(\{p\},\{k\})$$

$\tilde{I}_\Gamma$ is the renormalised integrand.

Our aim in this discussion is to construct $\tilde{I}_\Gamma$ from I_Γ. We shall do this for three classes of diagrams.

Class 1 - Diagrams without divergent subgraphs Superficial degrees of divergence of Γ are denoted by $D(\Gamma)$. If a diagram Γ contains no divergent subdiagrams and is not a renormalisation part, that is having a negative degree of divergence, we define

$$\tilde{I}_\Gamma = I_\Gamma$$

If Γ is a renormalisation part we have to introduce a Taylor operator t^Γ. We define

$$\tilde{I}_\Gamma = R_\Gamma I_\Gamma \equiv (1 - t^\Gamma) I_\Gamma$$

The operation $(1-t^\Gamma)$ on I_Γ (which we shall assume is a function $f(p_1, \ldots p_n)$ where $p_1 \ldots p_n$ are external momenta) is defined by subtracting from I_Γ the first $D(\Gamma)$ derivative terms in a Taylor expansion. t^Γ acting on $f(p_1, \ldots p_n)$ is therefore the first d derivative terms in the Taylor expansion of $f(\{p\})$. We have

$$\begin{aligned} t^\Gamma f(p_1, \ldots, p_n) &\equiv f(p_1^0, \ldots, p_n^0) + \ldots \\ &\quad + \tfrac{1}{d!} \sum (p - p^0)_i \ldots (p - p^0)_j \frac{\partial^d f(\{p\})}{\partial p_i \ldots \partial p_j} \end{aligned} \tag{3.76}$$

where $d = D(\Gamma)$. The point $\{p^0\}$ about which the Taylor expansion is made is called the renormalisation point. In equation (3.76), t^Γ is a shorthand for $t^{D(\Gamma)}$. This construction ensures that $\tilde{M}_\Gamma$ is finite.

Class 2 - Diagrams with no overlapping divergences In this case we have

$$\tilde{I}_\Gamma = \prod_i (1 - t^{\gamma_i}) I_\Gamma \tag{3.77}$$

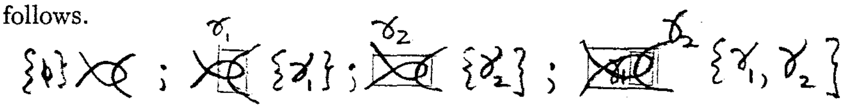
where $\{\gamma_i\}$ $(i = 1, \ldots, n)$ describes all the renormalisation parts. Equation (3.77) may be expanded out to give

$$\tilde{I}_\Gamma = \sum_U \prod_{\gamma \in U} (-t^\gamma) I_\Gamma \tag{3.78}$$

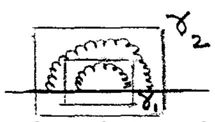
U is the set of possible divergent subgraphs. We have

$$U = \oslash; \gamma_1; \ldots; \gamma_n; \{\gamma_1, \gamma_2\}; \ldots; \{\gamma_1, \gamma_2, \ldots, \gamma_n\}$$

Each element in U represents a particular laying down of boxes. Diagram (iii) in part (a) serves as a useful example. Boxes may be layed down as follows.



Each element of U gives rise to a forest. We call $\{\gamma_1, \ldots, \gamma_m\}$ a forest. It is important to notice that a forest may be empty and it is legal to draw a box around the entire diagram provided that the diagram is a renormalisation part. A more formal definition of a forest will be given shortly. Another useful example to consider is the following self-energy diagram which has 2 non-overlapping renormalisation parts.



There are two divergent subgraphs γ_1 and γ_2. The renormalised amplitude is from (3.77) given by

$$\begin{aligned}\tilde{I}_\Gamma &= (1 - t^{\gamma_1})(1 - t^{\gamma_2}) I_\Gamma \\ &= I_\Gamma + (-t^{\gamma_1}) I_\Gamma + (-t^{\gamma_2}) I_\Gamma \\ &\quad + (-t^{\gamma^1})(-t^{\gamma_2}) I_\Gamma\end{aligned}$$

The forests are given by

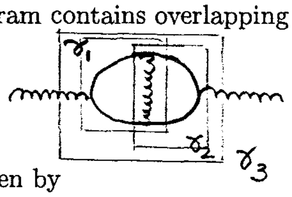
$$U = \oslash; \gamma_1; \gamma_2; \{\gamma_1, \gamma_2\}$$

and we see that this result is in agreement with (3.78)

Class 3 - Overlapping diagrams We now turn our attention to the case of diagrams containing overlapping subgraphs. A set of forests U belonging to a diagram Γ is defined as a family of superficially divergent subgraphs such that

1. Elements of the forest U are renormalisation parts.
2. Elements of U are non-overlapping.
3. The empty set $\oslash$ is a forest.

The following diagram contains overlapping subdiagrams γ_1 and γ_2.



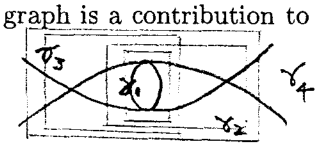
The forests are given by

$$U = \oslash; \gamma_1; \gamma_2; \gamma_3; \{\gamma_1, \gamma_3\}; \{\gamma_2, \gamma_3\}$$

$\{\gamma_1, \gamma_2\}$ and $\{\gamma_1, \gamma_2, \gamma_3\}$ are not forests since γ_1 and γ_2 are non-overlapping. The renormalised integrand $\tilde{I}_\Gamma$ is given by

$$\begin{aligned}
\tilde{I}_\Gamma &= \textstyle\sum_{\text{all } U} \prod_{\gamma \in U} (-t^\gamma) I_\Gamma \\
&= I_\Gamma + (-t^{\gamma_1}) I_\Gamma + (-t^{\gamma_2}) I_\Gamma + (-t^{\gamma_3}) I_\Gamma \\
&\quad + (-t^{\gamma_3})(-t^{\gamma_1}) I_\Gamma + (-t^{\gamma_3})(-t^{\gamma_2}) I_\Gamma \\
&= (1 - t^{\gamma_3})(1 - t^{\gamma_1} - t^{\gamma_2}) I_\Gamma \\
&\neq (1 - t^{\gamma_3})(1 - t^{\gamma_1})(1 - t^{\gamma_2}) I_\Gamma
\end{aligned}$$

Equation (3.78) may be used in calculating $\tilde{I}_\Gamma$ when overlapping subgraphs are present in Γ. Equation (3.77), however, may not be used and, as the above example shows, it gives a different result to that which is obtained when applying equation (3.78). We provide a further example from ψ^4 theory. The following graph is a contribution to the four point function of order g^4.



We note that γ_2 and γ_3 are overlapping - hence $\{\gamma_2, \gamma_3\}$ can not be included in the set of forests. The possible forests are

$$\begin{aligned}
U = \oslash; &\gamma_1; \gamma_2; \gamma_3; \gamma_4; \{\gamma_1, \gamma_2\}; \{\gamma_1, \gamma_3\}; \\
&\{\gamma_2, \gamma_4\}; \{\gamma_3, \gamma_4\}; \{\gamma_1, \gamma_4\}; \{\gamma_1, \gamma_2, \gamma_4\}; \\
&\{\gamma_1, \gamma_3, \gamma_4\}
\end{aligned}$$

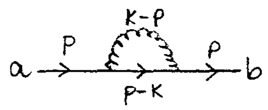
Applying equation (3.78) gives

$$\begin{aligned}
\tilde{I}_\Gamma &= [1 - t^{\gamma_1} - t^{\gamma_2} - t^{\gamma_3} - t^{\gamma_4} + (-t^{\gamma_1})(-t^{\gamma_2}) \\
&\quad + (-t^{\gamma_1})(-t^{\gamma_3}) + (-t^{\gamma_2})(-t^{\gamma_4}) + (-t^{\gamma_1})(-t^{\gamma_4}) \\
&\quad + (-t^{\gamma_3})(-t^{\gamma_4}) + (-t^{\gamma_1})(-t^{\gamma_2})(-t^{\gamma_4}) + (-t^{\gamma_1})(-t^{\gamma_3})(-t^{\gamma_4})] I_\Gamma \\
&= (1 - t^{\gamma_4})(1 - t^{\gamma_2} - t^{\gamma_3})(1 - t^{\gamma_1}) I_\Gamma
\end{aligned}$$

We do not obtain two factors $(1 - t^{\gamma_2})(1 - t^{\gamma_3})$ as we might expect from (3.77) because of the overlapping nature of the subdiagrams γ_2 and γ_3.

3.9 One-loop structure of Yang Mills theory

In this section we look at non-abelian gauge theory based on the group $SU(3)$. (We recall that QED was based on the abelian group $U(1)$.) This theory is known as QCD or Quantum Chromodynamics. We shall calculate the expressions for the self-energy, the vacuum polarisation and the vertex diagrams in QCD. At each stage we show how the renormalisation is performed by selecting appropriate counter-terms. We start with the quark self-energy diagram



The Feynman rules (in the Feynman gauge - see Chapter 2) give

$$-i\sum_{ab}(p) = -g^2\mu^{4-d}\int\frac{d^dk}{(2\pi)^d}\gamma_\mu\frac{1}{\not p - \not k - m}\gamma_\nu\frac{g^{\mu\nu}}{k^2}(T^c)_{ad}(T^c)_{db}$$

By recalling the integral expression for the electron self-energy in QED we see that

$$\begin{aligned}\sum_{ab}(p) &= (T^cT^c)_{ab}\sum^{QED}(p)\\ &= (T^cT^c)_{ab}\frac{g^2}{8\pi^2\epsilon}(-\not p + 4m)\end{aligned}$$

The only factor that remains to be calculated is $(T^cT^c)_{ab}$. $T^c = \frac{\lambda^c}{2}$ where λ^c $(c = 1, 2, \ldots 8)$ are the Gell-Mann matrices. These matrices are the generators of the group $SU(3)$ and are given in the Appendix. It is easy to verify that

$$\begin{aligned}T^cT^c &= \tfrac{1}{4}(\lambda_1^2 + \lambda_2^2 + \ldots\lambda_8^2)\\ &= \tfrac{4}{3}\mathbf{1} \quad (\text{ where } \mathbf{1} \text{ is the } 3\times 3 \text{ unit matrix })\end{aligned}$$

Hence

$$(T^cT^c)_{ab} = \frac{4}{3}(\mathbf{1})_{ab} = \frac{4}{3}\delta_{ab}$$

(In general, $(T^cT^c)^{ab} = C_2(F)\delta^{ab}$.) We therefore have

$$\sum^{ab}(p) = \frac{g^2}{6\pi^2\epsilon}(-\not p + 4m)\delta^{ab}$$

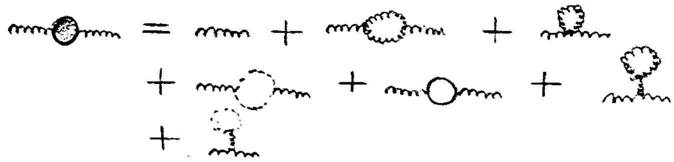
Since the divergence occurs in the term involving $\not p$ its removal is achieved by renormalising the wavefunction. We define the bare wavefunction (in analogy with the QED case) by

$$\psi_B = \sqrt{Z_2}\psi$$

where

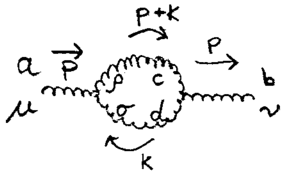
$$Z_2 = 1 - \frac{g^2}{6\pi^2\epsilon}$$

We shall now consider the gluon propagator. (This is the analogue of the photon propagator in QED.) The gluon propagator up to one loop is given by



This expression differs from the case of the photon propagator in that we have an interactions like  and 

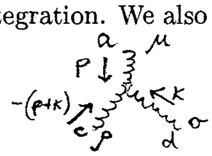
These interactions result from the non-abelian nature of the theory as may be recalled from Chapter 2. We calculate the gluon loop contribution to the vacuum polarisation in QCD as follows.

$$i\Pi^{ab}_{\mu\nu}(1) \quad =$$ 

The Feynman rules tell us to associate a factor of

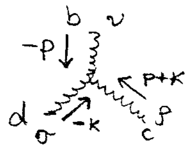
$$\int \frac{d^dk}{(2\pi)^d} \frac{1}{(p+k)^2k^2}$$

with the internal loop integration. We also need to analyse the couplings



The expression for this is (from Chapter 2)

$$\begin{aligned}
& -gf^{acd}[(k_\mu - (-(p+k))_\mu)g_{\rho\sigma} + (p-k)_\rho g_{\mu\sigma} \\
& +(-(p+k)-p)_\sigma g_{\mu\rho}] \\
= \quad & -gf^{acd}[(2k+p)_\mu g_{\rho\sigma} + (p-k)_\rho g_{\mu\sigma} - (k+2p)_\sigma g_{\mu\rho}]
\end{aligned}$$

The other coupling 

is given by

$$-gf^{bdc}[(2k+p)_\nu g_{\rho\sigma} - (2p+k)_\sigma g_{\rho\nu} + (p-k)_\rho g_{\sigma\nu}]$$

Combining these results we therefore have

$$i\Pi^{ab}_{\mu\nu}(1) = -\frac{1}{2}g^2\mu^{4-d}f^{acd}f^{bdc}\int\frac{d^dk}{(2\pi)^d}\frac{E_{\mu\nu}}{(p+k)^2k^2} \tag{3.79}$$

where

$$\begin{aligned} E_{\mu\nu}(k,p) &= [(2k+p)_\mu g_{\sigma\rho} + (p-k)_\rho g_{\mu\sigma} - (k+2p)_\sigma g_{\mu\rho}] \\ &\quad \times[(2k+p)_\nu g_{\rho\sigma} - (2p+k)_\sigma g_{\rho\nu} + (p-k)_\rho g_{\sigma\nu} \end{aligned}$$

The factor of $\frac{1}{2}$ in (3.79) comes from the symmetry of the diagram. (There are two ways of attaching the labels to the internal lines.)

$$\begin{aligned} E_{\mu\nu}(k,p) &= (2k_\mu + p_\mu)(2k_\nu + p_\nu)d - (2k_\mu + p_\mu)(2p_\nu + k_\nu) \\ &\quad +(2k_\mu + p_\mu)(p_\nu - k_\nu) - (k_\mu + 2p_\mu)(2k_\nu + p_\nu) \\ &\quad +(k_\sigma + 2p_\sigma)(2p_\sigma + k_\sigma)g_{\mu\nu} - (k_\nu + 2p_\nu)(p_\mu - k_\mu) \\ &\quad +(p_\mu - k_\mu)(2k_\nu + p_\nu) - (p_\nu - l_\nu)(2p_\mu + l_\mu) \\ &\quad +(p_\sigma - k_\sigma)(p_\sigma - k_\sigma)g_{\mu\nu} \\ &= k_\mu k_\nu(4d-6) + (p_\mu k_\nu + k_\mu p_\nu)(2d-3) + p_\mu p_\nu(d-6) \\ &\quad +g_{\mu\nu}[(2p+k)^2 + (p-k)^2](\text{using} g_{\mu\mu} = d) \end{aligned}$$

The integral

$$\int \frac{d^dk}{(2\pi)^d}\frac{E_{\mu\nu}(k,p)}{(p+k)^2k^2} \tag{3.80}$$

is both infrared and ultraviolet divergent at $d = 4$. We deal with this by using the standard technique of dimensional regularisation. Introducing Feynman parameters in (3.80) gives

$$\begin{aligned} &\frac{\Gamma(2)}{\Gamma(1)\Gamma(1)}\int_0^1 dx\int_0^1 dy\int\frac{d^dk}{(2\pi)^d}\frac{\delta(1-x-y)E_{\mu\nu}(k,p)}{[k^2x + (k+p)^2y]^2} \\ &= \int_0^1 dx\int\frac{d^dk}{(2\pi)^d}\frac{E_{\mu\nu}(k,p)}{[k^2x + (k^2 + 2k.p + p^2)(1-x)]^2} \end{aligned}$$

(Performing integration with respect to y causes the Dirac delta to disappear.) The denominator may be transformed as follows.

$$k^2 + (1-x)p^2 + 2(1-x)kp = (k + (1-x)p)^2 + p^2x(1-x)$$

Making the substitution

$$k' = k + (1-x)p$$

gives (dropping the prime on the k)

$$\int_0^1 dx\int\frac{d^dk}{(2\pi)^d}\frac{E_{\mu\nu}(k + p(x-1),p)}{[k^2 + p^2(1-x)x]^2}$$

Symmetry allows us to reduce the numerator to

$$E_{\mu\nu}(k - px, p)$$

to give

$$i\Pi^{ab}_{\mu\nu}(1) = -\frac{1}{2}g^2\mu^{4-d}f^{acd}f^{bdc}\int_0^1 dx\int\frac{d^dk}{(2\pi)^d}\frac{E_{\mu\nu}(k-px,p)}{[k^2+p^2x(1-x)]^2} \tag{3.81}$$

$E_{\mu\nu}(k-px,p)$ may be expanded as follows -

$$\begin{aligned} E_{\mu\nu}(k-px,p) &= (4d-6)(k_\mu - xp_\mu)(k_\nu - xp_\nu) \\ &\quad +(2d-3)[(k_\mu - xp_\mu)p_\nu + (k_\nu - xp_\nu)p_\mu] \\ &\quad +(d-6)p_\mu p_\nu + [(p-k+px)^2 + (k-px+2p)^2]g_{\mu\nu} \end{aligned}$$

Expanding this expression, dropping all terms linear in k (since they do not contribute to the integral) gives

$$\begin{aligned} E_{\mu\nu} &= (4d-6)k_\mu k_\nu + [(4d-6)x(x-1)+d-6]p_\mu p_\nu \\ &\quad +[2k^2 + p^2(2x(x-1)+5)]g_{\mu\nu} \end{aligned}$$

Substituting this expression in (3.81) gives

$$i\Pi^{ab}_{\mu\nu}(1) = \frac{1}{2}g^2\mu^{4-d}f^{acd}f^{bcd}\int_0^1 dx\int\frac{d^dk}{(2\pi)^d}$$

$$\times\frac{(4d-6)k_\mu k_\nu + [(4d-6)x(x-1)+d-6]p_\mu p_\nu + [2k^2+p^2(2x(x-1)+5)]g_{\mu\nu}}{[k^2+p^2x(1-x)]^2}$$

Using the formulas in the Appendix gives

$$\begin{aligned} \Pi^{ab}_{\mu\nu}(1) &= \frac{1}{2}g^2\mu^{4-d}\frac{f^{acd}f^{bcd}}{(2\pi)^d}\int_0^1\{(4d-6)\frac{1}{2}\frac{\pi^{\frac{d}{2}}}{\Gamma(2)}g_{\mu\nu}\frac{\Gamma(1-\frac{d}{2})}{(p^2x(1-x))^{1-\frac{d}{2}}} \\ &\quad +[(4d-6)x(x-1)+d-6]p_\mu p_\nu\pi^{\frac{d}{2}}\frac{\Gamma(2-\frac{d}{2})}{\Gamma(2)}\frac{1}{(p^2x(1-x))^{2-\frac{d}{2}}} \\ &\quad +[p^2(2x(x-1)+5)]g_{\mu\nu}\pi^{\frac{d}{2}}\frac{\Gamma(2-\frac{d}{2})}{\Gamma(2)}\frac{1}{(p^2x(1-x))^{2-\frac{d}{2}}} \\ &\quad +2g_{\mu\nu}\frac{1}{2}\frac{\pi^{\frac{d}{2}}}{\Gamma(2)}d\frac{\Gamma(1-\frac{d}{2})}{(p^2x(1-x))^{1-\frac{d}{2}}}\} \end{aligned}$$

or

$$\Pi^{ab}_{\mu\nu}(1) = \frac{g^2\mu^{4-d}}{2(4\pi)^{\frac{d}{2}}}f^{acd}f^{bcd}\int_0^1 dx\{\frac{(3d-3)g_{\mu\nu}\Gamma(1-\frac{d}{2})}{(p^2x(1-x))^{1-\frac{d}{2}}}$$

$$+\frac{\Gamma(2-\frac{d}{2})}{[p^2x(1-x)]^{2-\frac{d}{2}}}[g_{\mu\nu}p^2(5-2x(1-x)) + p_\mu p_\nu(d-6-(4d-6)x(1-x))]\}$$

This expression may be simplified by making the following algebraic manipulations. We write

$$(p^2x(1-x))^{1-\frac{d}{2}} = [p^2x(1-x)][p^2x(1-x)]^{-\frac{d}{2}}$$

$$\Gamma(2-\frac{d}{2}) = \Gamma(\frac{\epsilon}{2}) = \frac{2}{\epsilon} + \psi(1) \quad (\epsilon = 4-d)$$

$$[\frac{p^2x(1-x)}{4\pi\mu^2}]^{2-\frac{d}{2}} = [\frac{p^2x(1-x)}{4\pi\mu^2}]^{\frac{\epsilon}{2}} = 1 - \frac{\epsilon}{2}\ln(\frac{p^2x(1-x)}{4\pi\mu^2})$$

This gives

$$\Pi^{ab}_{\mu\nu}(1) = \frac{g^2 f^{acd} f^{bcd}}{32\pi^2}\int_0^1 dx[p^2x(1-x)(-3\epsilon+9)(-1)(\frac{2}{\epsilon}+\psi(2))$$

$$\times g_{\mu\nu}(1-\frac{\epsilon}{2}ln(\frac{p^2x(1-x)}{4\pi\mu^2}))] + \frac{g^2 f^{acd} f^{bcd}}{32\pi^2}\int_0^1 dx(\frac{2}{\epsilon}+\psi(1))(1-\frac{\epsilon}{2}ln(\frac{p^2x(1-x)}{4\pi\mu^2}))$$

$$\times[g_{\mu\nu}p^2(5-2x(1-x)) + p_\mu p_\nu(\frac{\epsilon}{2}(8x(1-x)-2) - (10x(1-x)+2))]$$

We now rearrange the order of the terms and write $\psi(2) = 1-\gamma$ to give

$$\Pi^{ab}_{\mu\nu}(1) = \frac{g^2 f^{acd} f^{bcd}}{32\pi^2}\left\{-(\frac{2}{\epsilon}+1-\gamma)g_{\mu\nu}(-3\epsilon+9)p^2\int_0^1 x(1-x)(1-\frac{\epsilon}{2}ln(\frac{p^2x(1-x)}{4\pi\mu^2}))dx\right\}$$

$$+\frac{g^2 f^{acd} f^{bcd}}{32\pi^2}\{(\frac{2}{\epsilon}-\gamma)\int_0^1 dx(1-\frac{\epsilon}{2}ln(\frac{p^2x(1-x)}{4\pi\mu^2}))$$

$$\times[g_{\mu\nu}p^2(5-2x(1-x)) + p_\mu p_\nu(\frac{\epsilon}{2}(8x(1-x)-2) - (10x(1-x)+2))]\}$$

Expanding and retaining only the terms in $\frac{1}{\epsilon}$ and ϵ^0 gives

$$\Pi^{ab}_{\mu\nu}(1) = \frac{g^2 f^{acd} f^{bcd}}{32\pi^2}\{-\frac{2}{\epsilon}g_{\mu\nu}p^2\frac{3}{2} + g_{\mu\nu}p^2 - (1-\gamma)\frac{9}{6}p^2 g_{\mu\nu}$$

$$+9g_{\mu\nu}p^2\int_0^1 dx[x(1-x)ln(\frac{p^2x(1-x)}{4\pi\mu^2})] + \frac{2}{\epsilon}g_{\mu\nu}p^2\frac{14}{3} - \frac{11}{3}\frac{2}{\epsilon}p_\mu p_\nu$$

$$-\int_0^1 dx\ ln(\frac{p^2x(1-x)}{4\pi\mu^2})[g_{\mu\nu}p^2(5-2x(1-x)) - p_\mu p_\nu(10x(1-x)+2)]$$

$$-\frac{14}{3}\gamma g_{\mu\nu}p^2 + \frac{11}{3}\gamma p_\mu p_\nu - \frac{2}{3}p_\mu p_\nu\}$$

$$= \frac{g^2 f^{acd} f^{bcd}}{32\pi^2}\{\frac{2}{\epsilon}(\frac{19}{6}g_{\mu\nu}p^2 - \frac{11}{3}p_\mu p_\nu) + g_{\mu\nu}p^2(-\frac{19}{6}\gamma - \frac{1}{2})$$

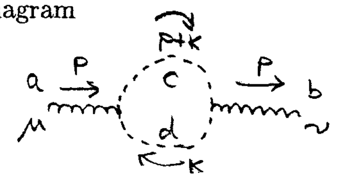
$$-p_\mu p_\nu(\frac{2}{3}-\frac{11}{3}\gamma) + \int_0^1 dx\ ln(\frac{p^2x(1-x)}{4\pi\mu^2})[p_\mu p_\nu(10x(1-x)+2) - g_{\mu\nu}p^2(5-11x(1-x))]\}$$

The pole term is therefore given by

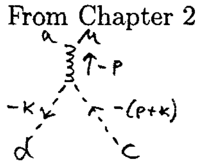
$$\frac{-g^2}{16\pi^2\epsilon}f^{acd}f^{bcd}[\frac{11}{3}p_\mu p_\nu - \frac{19}{6}g_{\mu\nu}p^2]$$

We now calculate the ghost contribution to the vacuum polarisation in QCD. This contribution is given by the following diagram

$$i\Pi^{ab}_{\mu\nu}(2) \quad = $$

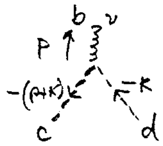


From Chapter 2 the Feynman rules tell us that



$$= -gf^{dca}(-k_\mu) \quad = gf^{dca}k_\mu$$
$$= gf^{cad}k_\mu$$

(using anti-sym. of the structure constants.)



$$= -gf^{cdb}(-(p+k)_\mu) \quad = gf^{cdb}(p+k)_\mu$$
$$= gf^{dbc}(p+k)_\mu$$

Combining this information with the integral which is associated with the internal loop momentum gives

$$i\Pi^{ab}_{\mu\nu}(2) = g^2 f^{cad} f^{dbc} \mu^{4-d} \int \frac{d^dk}{(2\pi)^d} \frac{(p+k)_\mu k_\nu}{(p+k)^2 k^2}$$

To evaluate this integral we introduce a Feynman parameter to give

$$i\Pi^{ab}_{\mu\nu}(2) = -g^2 f^{acd} f^{bcd} \mu^{4-d} \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{(k-px)_\nu (k+p(1-x))_\mu}{[k^2+p^2x(1-x)]^2} \quad (3.82)$$

(Note that $f^{cad} = -f^{acd}$. We now expand the numerator of (3.82) (ignoring terms which are linear since they disappear on integration.) Our aim, as usual, is to apply the standard formulas derived earlier dealing with dimensional regularisation. We have

$$i\Pi^{ab}_{\mu\nu}(2) = -g^2 \mu^{4-d} f^{acd} f^{bcd} \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{k_\mu k_\nu - x(1-x)p_\mu p_\nu}{[k^2+p^2x(1-x)]^2}$$

Applying the formulas in the Appendix gives

$$\begin{aligned}
\Pi^{ab}_{\mu\nu}(2) &= \frac{-g^2\mu^{4-d} f^{acd} f^{bcd}}{(2\pi)^d} \int_0^1 dx \{ \frac{1}{2} \pi^{\frac{d}{2}} g_{\mu\nu} \frac{\Gamma(1-\frac{d}{2})(p^2x(1-x))}{(+p^2x(1-x))^{2-\frac{d}{2}}} \\
&\quad -x(1-x)p_\mu p_\nu \frac{\pi^{\frac{d}{2}}\Gamma(2-\frac{d}{2})}{(p^2x(1-x))^{2-\frac{d}{2}}} \} \\
&= \frac{-g^2\mu^{4-d} f^{acd} f^{bcd}}{(4\pi)^{\frac{d}{2}}} \int_0^1 dx \{ \frac{1}{2} \frac{g_{\mu\nu}\Gamma(1-\frac{d}{2})}{(p^2x(1-x))^{1-\frac{d}{2}}} \\
&\quad - \frac{x(1-x)p_\mu p_\nu \Gamma(2-\frac{d}{2})}{(p^2x(1-x))^{2-\frac{2}{d}}} \}
\end{aligned}$$

We now expand the quantities μ^{4-d}, $(p^2x(1-x))^{1-\frac{d}{2}}$ and $\Gamma(2-\frac{d}{2})$ using the standard results.

$$\begin{aligned}\Gamma(-n+\epsilon) &= \tfrac{(-1)^n}{n!}[\tfrac{1}{\epsilon}+\psi(n+1)+O(\epsilon)]\\ \psi(n+1) &= 1+\tfrac{1}{2}+\ldots+\tfrac{1}{n}-\gamma\\ a^\epsilon &= 1+\epsilon\ \ln a+\ldots\end{aligned}$$

This gives

$$\Pi^{ab}_{\mu\nu}(2) = \frac{-g^2 f^{acd}f^{bcd}}{32\pi^2}\{-g_{\mu\nu}p^2(\frac{2}{\epsilon}+1-\gamma)\int_0^1 dx\ x(1-x)(1-\frac{\epsilon}{2}ln(\frac{p^2x(1-x)}{4\pi\mu^2}))$$

$$-2p_\mu p_\nu(\frac{2}{\epsilon}-\gamma)\int_0^1 dx\ x(1-x)(1-\frac{\epsilon}{2}ln(\frac{p^2x(1-x)}{4\pi\mu^2}))\}$$

$$= \frac{g^2 f^{acd}f^{bcd}}{32\pi^2}\{\frac{2}{\epsilon}(\frac{g_{\mu\nu}p^2}{6}+\frac{p_\mu p_\nu}{3})+\frac{1}{6}(1-\gamma)g_{\mu\nu}p^2$$

$$-g_{\mu\nu}p^2\int_0^1 dx(x(1-x)ln(\frac{p^2x(1-x)}{4\pi\mu^2}))-\frac{1}{3}\gamma p_\mu p_\nu$$

$$-2p_\mu p_\nu\int_0^1 dx(x(1-x)ln(\frac{p^2x(1-x)}{4\pi\mu^2}))+O(\epsilon)\}$$

$$= \frac{g^2 f^{acd}f^{bcd}}{32\pi^2}\{\frac{2}{\epsilon}(\frac{g_{\mu\nu}p^2}{6}+\frac{p_\mu p_\nu}{3})-\frac{1}{6}(\gamma-1)g_{\mu\nu}p^2$$

$$-\frac{1}{3}\gamma p_\mu p_\nu-\int_0^1 ln(\frac{p^2x(1-x)}{4\pi\mu^2})[x(1-x)g_{\mu\nu}p^2+2x(1-x)p_\mu p_\nu]+O(\epsilon)\}$$

The pole term is therefore given by

$$\frac{g^2}{16\pi^2\epsilon}f^{acd}f^{bcd}[\frac{1}{3}p_\mu p_\nu+\frac{1}{6}g_{\mu\nu}p^2] \tag{3.83}$$

Adding $\Pi^{ab}_{\mu\nu}(1)$ and $\Pi^{ab}_{\mu\nu}(2)$ which are the gluon and ghost contributions to the total pure Yang-Mills contribution to the vacuum polarisation in QCD. This gives

$$\Pi^{ab}_{\mu\nu}(1+2) = \frac{g^2 f^{acd}f^{bcd}}{32\pi^2}\{\frac{2}{\epsilon}(\frac{10}{3}(g_{\mu\nu}p^2-p_\mu p_\nu))-\frac{10}{3}\gamma(g_{\mu\nu}p^2-p_\mu p_\nu)$$

$$-\frac{1}{3}g_{\mu\nu}p^2-\frac{2}{3}p_\mu p_\nu+\int_0^1 dx\ ln(\frac{p^2x(1-x)}{4\pi\mu^2})[p_\mu p_\nu(8x(1-x)+2)-g_{\mu\nu}p^2(5-10x(1-x))]\} \tag{3.84}$$

The pole term is given by

$$\frac{g^2}{8\pi^2\epsilon}f^{acd}f^{bcd}\frac{5}{3}(g_{\mu\nu}p^2-p_\mu p_\nu)$$

We now concentrate on the integral term which may be rearranged into the following form.

$$-ln(\frac{p^2}{4\pi\mu^2})(g_{\mu\nu}p^2 - p_\mu p_\nu)\frac{10}{3} + 2p_\mu p_\nu \int_0^1 (4x(1-x)+1)\; ln(x(1-x))\; dx$$
$$-5g_{\mu\nu}p^2 \int_0^1 (1-2x(1-x))\; ln(x(1-x))\; dx$$

Elementary methods of integration yield the following results -

$$\int_0^1 (4x(1-x)+1)\; ln(x(1-x))dx = -\frac{28}{9}$$

$$\int_0^1 (1-2x(1-x))\; ln(x(1-x))dx = -\frac{13}{9}$$

Hence the integral in (3.85) reads

$$-ln(\frac{p^2}{4\pi\mu^2})(g_{\mu\nu}p^2 - p_\mu p_\nu)\frac{10}{3} - 2p_\mu p_\nu\frac{28}{9} + \frac{65}{9}g_{\mu\nu}p^2$$

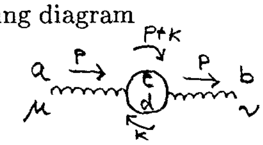
and

$$\begin{aligned}\Pi^{ab}_{\mu\nu}(1+2) &= \frac{g^2 f^{acd} f^{bcd}}{32\pi^2}\{\frac{2}{\epsilon}\frac{10}{3} - \frac{10}{3}\gamma + \frac{62}{9} - ln(\frac{p^2}{4\pi\mu^2})\frac{10}{3}\} \\ &\quad \times(p^2 g_{\mu\nu} - p_\mu p_\nu)\end{aligned}$$

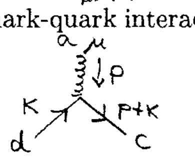
We should note that $\Pi^{ab}_{\mu\nu}(1+2)$ is proportional to the projection operator

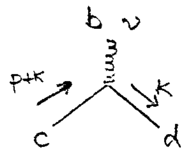
$$(p^2 g_{\mu\nu} - p_\mu p_\nu)$$

The term $ln(\frac{p^2}{4\pi\mu^2})$ gives rise to an infrared divergence when $p \to 0$.
We now turn to the quark contribution to the vacuum polarisation in QCD. This contribution is represented by the following diagram

$$i\Pi^{ab}_{\mu\nu}(3) =$$ 

To calculate $i\Pi^{ab}_{\mu\nu}(3)$ we shall require the expressions for the following gluon-quark-quark interactions.





The expressions can be found from Chapter 2. We find that

$$i\Pi^{ab}_{\mu\nu}(3) = -g^2\mu^{4-d}\int \frac{d^dk}{(2\pi)^d}(T^a)_{dc}(T^b)_{cd}$$

$$\times Tr[\gamma_\mu \frac{1}{\not p + \not k - m} \gamma_\nu \frac{1}{\not k - m}]$$
$$= iTr(T^a T^b)\Pi_{\mu\nu}(QED)$$

Here we have used the fact that $(T^a)_{dc}(T^b)_{cd} = Tr(T^a T^b)$. $\Pi_{\mu\nu}(QED)$ is the vacuum polarisation for QED which was calculated earlier in this chapter. It is therefore unnecessary to go through this calculation again. We therefore have

$$\Pi^{ab}_{\mu\nu}(3) = Tr(T^a T^b)\frac{g^2}{6\pi^2\epsilon}(p_\mu p_\nu - g_{\mu\nu}p^2) \tag{3.85}$$

As defined earlier we know that the matrices T^a $(a = 1 \ldots 8)$ are proportional to the well known Gell-Mann generators of SU(3). A standard result concerning the trace operator is that

$$Tr(\lambda^a \lambda^b) = 2\delta^{ab}$$

where $T^a = \frac{\lambda^a}{2}$. This result may be substituted in (3.85) if " only quarks belonging to one representation of SU(3) contributed to the vacuum polarisation." In general we have

$$Tr(T^a T^b) = \frac{n_F}{2}\delta^{ab}$$

n_F is a number. It refers to the number of *quark flavours*. In (3.83) it can be shown that

$$f^{acd}f^{bcd} = 3\delta^{ab} \tag{3.86}$$

The structure constants are for the group $SU(3)$ and are given by

$$\begin{array}{l} f_{123} = 1 \\ f_{147} = -f_{156} = f_{246} = f_{257} = f_{345} = -f_{367} = \frac{1}{2} \\ f_{458} = f_{678} = \frac{\sqrt{3}}{2} \quad \text{All other quantities are zero.} \end{array}$$

More generally, for an arbitrary group we have

$$\begin{aligned} f^{acd}f^{bcd} &= \delta^{ab}C_2(G) \\ C_2(G) &= N \text{ for } SU(N) \\ &= 3 \text{ for } SU(3) \end{aligned} \tag{3.87}$$

In the case of $SU(3)$ the sum of the gluon, ghost and quark loops give

$$\Pi^{ab}_{\mu\nu}(1+2+3) = \frac{g^2}{8\pi^2\epsilon}(g_{\mu\nu}p^2 - p_{\mu\nu})(5 - \frac{2n_F}{3})\delta^{ab}$$

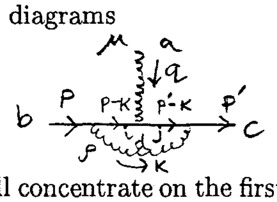
Since the divergence is proportional to the term

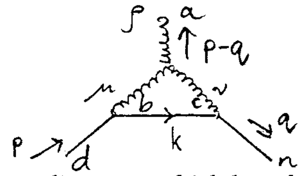
$$(g_{\mu\nu}p^2 - p_\mu p_\nu)\delta^{ab}$$

this suggests (in analogy with QED) setting

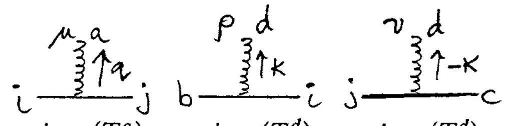
$$Z_3 = 1 + \frac{g^2}{8\pi^2\epsilon}(5 - \frac{2n_F}{3})$$

We now proceed to the case of the quark-gluon vertex function. This involves the two diagrams





We shall concentrate on the first of these diagrams which has three internal lines (two quark and one gluon) and involves three couplings given by



$$-ig\gamma_\mu(T^a)_{ji} \quad -ig\gamma_\rho(T^d)_{ib} \quad -ig\gamma_\nu(T^d)_{cj}$$

The Feynman rules give

$$-ig\mu^{2-\frac{d}{2}}(\Lambda_\mu^a)_{cd}(p,q,p')(1) = (-ig\mu^{2-\frac{d}{2}})^3 \int \frac{d^dk}{(2\pi)^d}\gamma_\nu(T^d)_{cj}(\frac{i}{\not{p}'-\not{k}-m})$$

$$\times\gamma_\mu(T^a)_{ji}(\frac{i}{\not{p}-\not{k}-m})(T^d)_{id}\gamma_\rho(\frac{-ig^{\nu\rho}}{k^2})$$

This expression is very similar to the corresponding diagram in QED and differs only by some gauge matrix terms. We find that

$$\Lambda_\mu^a(p,q,p')(1) = (T^dT^aT^d)\Lambda_\mu(p,q,p')(QED)$$

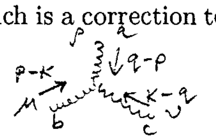
It can be shown that

$$\begin{aligned} T^dT^aT^d &= -\frac{1}{2}f^{adc}f^{dcb}T^b + C_2(F)T^a \\ &= [-\frac{1}{2}C_2(G) + C_2(F)]T^a \end{aligned}$$

This gives

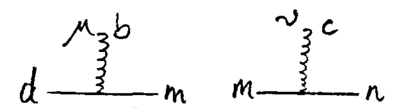
$$\Lambda_\mu^a(1) = \frac{g^2}{8\pi^2\epsilon}(-\frac{C_2(G)}{2} + C_2(F))\gamma_\mu T^a$$

The second diagram which is a correction to the quark-gluon vertex involves the 3-gluon vertex



$$\begin{aligned} = \quad & -gf^{abc}[((p-k)-(q-p))_\nu g_{\mu\rho} + ((k-q)-(p-k))_\rho g_{\mu\nu} \\ & +((q-p)-(k-q))_\mu g_{\nu\rho}] \end{aligned}$$

There are also internal lines. (Two gluon lines carrying momentum $k-p$ and $q-k$; one quark line carrying momentum k.) Finally there are the couplings



$-ig\gamma_\mu(T^b)_{dm} \qquad -ig\gamma_\nu(T^c)_{mn}$

Combining all this information means that the vertex contribution is given by

$$\begin{aligned}
-ig\mu^{2-\frac{d}{2}}\Lambda^a_\rho(2) &= (-ig)^2(-g)(\mu^{4-d})^{\frac{3}{2}}\int\frac{d^dk}{(2\pi)^d}\gamma_\mu(T^b)_{dm}\frac{-i}{(k-p)^2}f_{abc} \\
&\times[(p-k-q+p)_\nu g_{\mu\rho}+(k-q-p+k)_\rho g_{\mu\nu} \\
&+(q-k-p+q)_\mu g_{\nu\rho}](\frac{-i}{(q-k)^2})(\frac{i}{\not k-m})\gamma_\nu(T^c)_{mn}
\end{aligned}$$

or,

$$\Lambda^a_\rho(2)=\frac{g^2\mu^{4-d}}{(2\pi)^d}f^{abc}T^bT^cI_\rho \tag{3.88}$$

where

$$I_\rho=\int d^dk\frac{\gamma_\mu(\not k+m)\gamma_\nu[(2p-q-k)_\nu g_{\mu\rho}+(2k-p-q)_\rho g_{\mu\nu}+(2q-p-k)_\mu g_{\nu\rho}]}{(k^2-m^2)(k-p)^2(q-k)^2}$$

Our aim is to extract the divergent part of this integral. The technique is standard. We begin by making the substitution

$$k'=k-px-qy$$

obtaining an expression for I_ρ in terms of k'. (Note that $dk'=dk$.) We ignore terms which are linear in k' since they will integrate to zero. We may also neglect terms which do not involve k'. On carrying out the integration a finite result will be obtained in the limit $d\to 4$. The only important terms (which give rise to a divergence) are those which are quadratic in k'. Working through these steps it is found that the divergent term is

$$I_\rho=2\int dx\int dy\int d^dk\{\frac{(4-2d)k_\rho\not k-2\gamma_\rho k^2}{[k^2-m^2(1-x-y)+p^2x+q^2y-(px+qy)^2]^3}\}$$

Applying the standard formulae in the appendix yields the pole part

$$I_\rho=-\frac{6i\pi^2}{\epsilon}\gamma_\rho \qquad (\epsilon=4-d)$$

The group theoretic factors in (3.88) are easily calculated using (3.87) giving

$$f^{abc}T^bT^c=\frac{i}{2}C_2(G)T^a$$

The final result is

$$\Lambda^a_\mu(2)=\frac{g^2}{8\pi^2\epsilon}[\frac{3C_2(G)}{2}]\gamma_\mu T^a$$

Adding $\Lambda_\mu^a(1)$ and $\Lambda_\mu^a(2)$ yields the total vertex contribution

$$\begin{aligned}\Lambda_\mu^a &= \frac{g^2}{8\pi^2\epsilon}[C_2(G) + C_2(F)]\gamma_\mu T^a \\ &= \frac{g^2}{8\pi^2\epsilon}(\frac{13}{3})\gamma_\mu T^a \quad \text{(in the case of the group } SU(3))\end{aligned}$$

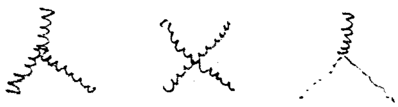
The divergence is eliminated by setting the renormalisation constant

$$Z_1 = 1 - \frac{g}{8\pi^2\epsilon}(\frac{13}{3})$$

(see analogous case in QED)

3.10 Renormalisation of Pure Gauge Theories

In this section a summarised account of the renormalisation of pure Yang-Mills theories will be presented. It is based on one of the original papers by Lee [17]. We begin with a preliminary result on power counting. We shall consider Feynman diagrams which do not contain quark (fermionic) lines. (As stated in the paper by Lee, a more general theory including matter fields, such as scalar and spinor fields may be treated with no difficulty.) This is equivalent to a theory in which no spinor fields are present. The basic building blocks are the couplings



These diagrams are, respectively, a 3-vector vertice, a 4-vector vertice and a vector-ghost-ghost interaction. We shall use the following notation to write down an equation for the superficial degree of divergence (D) of a diagram based on the above building blocks. Let

$$\begin{aligned} n_3 &= \text{number of 3-vector vertices} \\ n_4 &= \text{number of 4-vector vertices} \\ n_g &= \text{number of vector-ghost-ghost vertices} \\ V_e &= \text{number of external vector lines} \\ V_i &= \text{number of internal vector lines} \\ G_i &= \text{number of internal ghost lines} \end{aligned}$$

We find that

$$D = 4L - 2V_i - 2G_i + n_3 + n_g \tag{3.89}$$

for $d = 4$. The number of loops (L) is given by

$$\begin{aligned} L &= V_i + G_i - n_3 - n_4 - n_g + 1 \\ &= V_i - n_3 - n_4 + 1 \quad (\text{since } n_g = G_i) \end{aligned} \tag{3.90}$$

We also have the result that

$$4n_4 + 3n_3 + 3n_g = 2V_i + 2G_i + V_e \tag{3.91}$$

By combining (3.89), (3.90) and (3.91) we find that

$$D = 4 - V_e$$

Since $D < 0$ is associated with convergent diagrams we see that $D > 0$ for only a finite number of graphs. This demonstrates that the number of primitively divergent graphs is finite. So we would expect the Yang-Mills theory to be renormalisable. (It would certainly *not* be renormalisable if D increased with V_e.) We now turn to this task.
The basic idea which is central to renormalisation is the ability to scale the fields and coupling constant so that the generating function Γ is finite by introducing suitable multiplicative constants. Using the notation of equation (3.56) we have to show that the scaling can be accomplished according to the following equations.

$$\begin{array}{lll} A_{a\mu} = Z^{\frac{1}{2}}(\epsilon)A^r_{a\mu} & \eta_a = \tilde{Z}^{\frac{1}{2}}(\epsilon)\eta^r_a & \tilde{\eta}^*_a = \tilde{Z}^{\frac{1}{2}}(\epsilon)\eta^{*r}_a \\ I_{a\mu} = \tilde{Z}^{\frac{1}{2}}(\epsilon)I^r_{a\mu} & I_a = Z^{\frac{1}{2}}(\epsilon)I^r_a & g = \frac{X(\epsilon)}{\tilde{Z}(\epsilon)Z^{\frac{1}{2}}(\epsilon)}g^r \end{array}$$

$Z(\epsilon)$, $\tilde{Z}(\epsilon)$ and $X(\epsilon)$ must be chosen appropriately so that

$$\tilde{\Gamma}^r[A^r_{a\mu}, \eta^r_a, \eta^{*r}_a, I^r_{a\mu}, I^r_a; g^r] = \Gamma[A_{a\mu}, \eta_a, \eta^*_a, I_{a\mu}, I_a; g]$$

is a finite functional of its arguments as $\epsilon = 4 - d \to 0$. This requirement is directly linked with the Taylor-Slavnov identity (see Chapter 2)

$$\int dx\; (\frac{\delta\Gamma'}{\delta I^{a\mu}}\frac{\delta\Gamma'}{\delta A^a_\mu} + \frac{\delta\Gamma'}{\delta I^a}\frac{\delta\Gamma'}{\delta\eta^a}) = 0 \tag{3.92}$$

Following the notation introduced by Zinn-Justin we define

$$\Gamma_1 * \Gamma_2 = \int dx\; (\frac{\delta\Gamma_1}{\delta I^{a\mu}}\frac{\delta\Gamma_2}{\delta A^a_\mu} + \frac{\delta\Gamma_1}{\delta I^a}\frac{\delta\Gamma_2}{\delta\eta^a})$$

The Taylor-Slavnov identity may be written symbolically as

$$\Gamma * \Gamma = 0 \tag{3.93}$$

We expand Γ in a loop-wise fashion (= expansion in powers of $\hbar$)

$$\Gamma = \sum_{n=0}^{\infty}\Gamma_n \tag{3.94}$$

We expand the renormalisation constants in the same way

$$\begin{array}{ll} Z(\epsilon) & = 1 + \sum_{n=1}^{\infty} Z_n(\epsilon) \\ X(\epsilon) & = 1 + \sum_{n=1}^{\infty} X_n(\epsilon) \end{array}$$

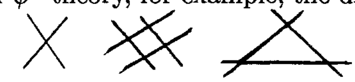
Substituting (3.94) into (3.92) gives (up to order n)

$$\sum_{p+q=n}\Gamma_p * \Gamma_q = 0 \tag{3.95}$$

Due to the divergences occurring in Γ an (infinite) number of counter-terms are added to the Lagrangian to make the theory finite. Earlier in this chapter in considering ψ^4 theory and QED we showed how divergences occurring in one-loop diagrams could be eliminated by adding appropriate counter-terms to the Lagrangian. It was stated that similar divergences occurring in higher-loop diagrams could be eliminated by adding further counter-terms to the Lagrangian. This procedure may be summarised using the following notation. At the outset, with no counter-terms the action is just the renormalised action. We put

$$\begin{aligned} S_0 &= S^r \\ S_1 &= S^r + (S_1)_{CT} \\ S_2 &= S^r + (S_2)_{CT} \\ &\vdots \\ S_\infty &= S^B \end{aligned} \tag{3.96}$$

Now $\Gamma_0(S_0)$ contains no divergences since the zero-loop diagrams are all convergent. In ψ^4 theory, for example, the diagrams



contain no loops and are finite. This means that $\Gamma_0(S_0)$ is finite. $\Gamma_1(S_0)$ is, however, divergent. (eg. consider the one-loop diagram ) and may be split up into finite and divergent parts as follows.

$$\Gamma_1(S_0) = \Gamma_1^f(S_0) + \Gamma_1^{\text{div}}(S_0) \tag{3.97}$$

To zeroth order we have (using (3.95))

$$\Gamma_0(S_0) * \Gamma_0(S_0) = 0 \tag{3.98}$$

To first order, we have (using (3.97))

$$\begin{aligned} \Gamma_0(S_0) * \Gamma_1^f(S_0) + \Gamma_1^f(S_0) * \Gamma_0(S_0) = 0 \\ \Gamma_0(S_0) * \Gamma_1^{\text{div}}(S_0) + \Gamma_1^{\text{div}}(S_0) * S_0 = 0 \end{aligned} \tag{3.99}$$

We now find a counter-term $(S_1)_{CT}$ such that $\Gamma_1(S_1)$ is finite. (This counter-term will eliminate the divergences in one-loop diagrams.) We also demand that $(S_1)_{CT}$ be chosen such that

$$S_1 * S_1 = 0 \tag{3.100}$$

We would like $\Gamma_1(S_0) + (S_1)_{CT}$ to be finite. This means setting

$$(S_1)_{CT} = -\Gamma_1^{\text{div}} \quad (\text{from}(3.97))$$

The new action S_1 is (from (3.96)) given by

$$\begin{aligned} S_1 &= S^r + (S_1)_{CT} \\ &= S_0 + (S_1)_{CT} \\ &= S_0 - \Gamma_1^{\text{div}} \end{aligned} \tag{3.101}$$

Then the renormalised generating functional is given by

$$\begin{aligned}\Gamma_1(S_1) &= \Gamma_1(S_0) + (S_1)_{CT} \\ &= \Gamma_1(S_0) + (S_1 - S_0)\end{aligned} \tag{3.102}$$

We show (3.100) -

$$\begin{aligned}S_1 * S_1 &= (S_0 - \Gamma_1^{\text{div}}) * (S_0 - \Gamma_1^{\text{div}}) \\ &= S_0 * S_0 - S_0 * \Gamma_1^{\text{div}} - \Gamma_1^{\text{div}} * S_0 + \Gamma_1^{\text{div}} * \Gamma_1^{\text{div}} \\ &= \Gamma_1^{\text{div}} * \Gamma_1^{\text{div}}\end{aligned}$$

(using (3.98) and (3.99).) We have used a result here which was not proved- namely $\Gamma_0(S_0) = S_0$. We have also assumed the distributivity property

$$A * (B + C) = (A * B) + (A * C)$$

Since $\Gamma_1^{\text{div}} * \Gamma_1^{\text{div}}$ is of order $\hbar^2$ we see that (3.100) is satisfied to one loop. (3.100) can be satisfied to two loops by adding an appropriate term to (3.101). We have shown how divergences may be eliminated in zero loop and one-loop functionals. The renormalisation procedure respects the Taylor-Slavnov identities.

We now assume that the theory may be renormalised to the $(n-1)$th loop approximation. (ie. we assume that $\Gamma_{(n-1)}$ is finite) This means that we have determined the renormalisation constants up to this order. Using induction our task is to show that the divergences in the n-loop approximation may be removed by determining the renormalisation constants to order n.

Working to order n (3.56) gives

$$\Gamma_{(n)}*\Gamma_{(0)}+\Gamma_{(0)}*\Gamma_{(n)} = -\Gamma_{(n-1)}*\Gamma_{(1)}-\Gamma_{(1)}*\Gamma_{(n-1)}-\Gamma_{(n-2)}*\Gamma_{(2)}-\Gamma_{(2)}*\Gamma_{(n-2)}-\ldots$$

From the induction hypothesis we know that the right hand side of this equation is finite. This is because it only involves quantities with less than n loops. The left hand side is divergent. The divergent part of Γ_n, say $\Gamma_n^{div}(S_{n-1})$ satisfies

$$\Gamma_n^{div}(S_{n-1}) * S_0 + S_0 * \Gamma_n^{div}(S_{n-1}) = 0$$

(Here we have used the result that $\Gamma_0 = S_0$.) Using the definition of the $*$ yields

$$[(\frac{\delta S_0}{\delta I_{a\mu}}\frac{\delta}{\delta A_a^\mu} + \frac{\delta S_0}{\delta I_a}\frac{\delta}{\delta \eta_a}) + (\frac{\delta S_0}{\delta A_a^\mu}\frac{\delta}{I_{a\mu}} + \frac{\delta S_0}{\delta \eta_a}\frac{\delta}{\delta I_a})]\Gamma_n^{div} = 0$$

We may write this as

$$G\Gamma_n^{div} = 0 \tag{3.103}$$

with $G = G_0 + G_1$ where

$$G_0 = \frac{\delta S_0}{\delta I_{a\mu}}\frac{\delta}{\delta A_a^\mu} + \frac{\delta S_0}{\delta I_a}\frac{\delta}{\delta \eta_a}$$

and

$$G_1 = \frac{\delta S_0}{\delta A_a^\mu}\frac{\delta}{\delta I_{a\mu}} + \frac{\delta S_0}{\delta \eta_a}\frac{\delta}{\delta I_a}$$

An important observation which enables us to solve (3.103) is that

$$G^2 = 0$$

Since $G = G_0 + G_1$ this means showing that

$$G_0^2 + \{G_0, G_1\} + G_1^2 = 0$$

From (2.73) we may write down expressions for $\frac{\delta S_0}{\delta I_{a\mu}}$ and $\frac{\delta S_0}{\delta I_a}$ and by direct calculation we find that $G_0^2 = 0$. It may also be shown that

$$\{G_0, G_1\} = -G_1^2$$

Hence

$$G^2 = 0$$

The fact that G is nilpotent means that, in general, $G(F)$ for arbitrary $F = F(A, \eta, \eta^*, I_{a\mu}, I_a)$ is a solution of (3.103).

Chapter 4

The Background Field Method

4.1 Introduction

In this chapter the background field method is introduced. This discussion closely follows the approach of Abbott [16]. The background field method is also discussed in Rivers [3] and Weinberg [12]. The idea of the background field method is to write the gauge field appearing in the classical action as $A + Q$ where A is the background field and Q is the quantum field which is the variable of integration in the functional integral. Normally the classical Lagrangian is constructed to be gauge invariant but on quantisation the explicit gauge invariance is lost in the Feynman rules because of the necessity to add gauge fixing and Fadeev-Popov ghost terms. This was discussed in Chapter 2. In the background field method gauge invariance is maintained at the quantum level.
We start by introducing the generating functional $Z[J]$ in the conventional functional approach to field theory. We recall equation (2.17). Including a source field J we may write this equation as

$$Z = \int \mathcal{D}Q det|\frac{\delta F^a}{\delta\omega^b}|e^{i\int(\mathcal{L}(Q)-\frac{1}{2\alpha}F[Q]^2+J^a_\mu Q^a_\mu)d^4x} \tag{4.1}$$

where

$$\mathcal{L}(Q) = -\frac{1}{4}(\partial_\mu Q^a_\nu - \partial_\nu Q^a_\mu + gf^{abc}Q^b_\mu Q^c_\nu)^2$$

F^a is the usual gauge fixing term and $\frac{\delta F^a}{\delta\omega^b}$ is the derivative of the gauge-fixing term under an infinitesimal gauge transformation -

$$\delta Q^a_\mu = -f^{abc}\omega^b Q^c_\mu + \frac{1}{g}\partial_\mu\omega^a$$

As discussed in Chapter 1 the disconnected Greens functions of the theory are evaluated by taking functional derivatives of $Z[J]$ with respect to J. The connected Greens functions are generated by

$$W[J] = -i ln Z[J] \tag{4.2}$$

The effective action is defined by making the Legendre transformation

$$\Gamma[\bar{Q}] = W[J] - \int d^4x \; J^a_\mu \bar{Q}^a_\mu \tag{4.3}$$

where

$$\bar{Q}^a_\mu = \frac{\delta W}{\delta J^a_\mu}$$

The derivatives of the effective action with respect to $\bar{Q}$ are the one-particle irreducible (1PI) Greens functions of the theory. Having now reviewed the conventional functional approach to field theory we now define the quantities analogous to Z, W and Γ in the background field method. These quantities shall be denoted by $\tilde{Z}$, $\tilde{W}$ and $\tilde{\Gamma}$. They are defined exactly like the conventional generating functions. The only difference is that the field Q in the classical Lagrangian of (4.1) is not written as Q but is decomposed into $A + Q$ where A is the background field. We define

$$\tilde{Z}[J, A] = \int \mathcal{D}Q det|\frac{\delta F^a}{\delta\omega^b}|e^{i\int d^4x[\mathcal{L}(A+Q) - \frac{1}{2\alpha}(F^a)^2 + J^a_\mu Q^a_\mu]} \tag{4.4}$$

The field Q is quantised while the background field A is left unquantised. $\frac{\delta F^a}{\delta \omega^b}$ is the derivative of the gauge-fixing term under the infinitesimal gauge transformation

$$\delta Q^a_\mu = -f^{abc}\omega^b(A^c_\mu + Q^c_\mu) + \frac{1}{g}\partial_\mu\omega^a$$

Notice that the background field is not coupled to the source.
We now generalise (4.2) and (4.3) and define

$$\tilde{W}[J, A] = -i ln \tilde{Z}[J, A]$$

and the background field effective action

$$\tilde{\Gamma}[\tilde{Q}, A] = \tilde{W}[J, A] - \int d^4x \; J^a_\mu \tilde{Q}^a_\mu \tag{4.5}$$

where

$$\tilde{Q}^a_\mu = \frac{\delta \tilde{W}}{\delta J^a_\mu}$$

At this stage it is useful to summarise our notation

$$\begin{aligned} Q^a_\mu &= \text{the quantum field, the variable of integration in (4.4)} \\ A^a_\mu &= \text{the background field} \\ \bar{Q}^a_\mu = \tfrac{\delta W}{\delta J^a_\mu} &= \text{the ``classical'' field in the conventional approach} \\ \tilde{Q}^a_\mu = \tfrac{\delta \tilde{W}}{\delta J^a_\mu} &= \text{the ``classical'' field using the background field method} \end{aligned}$$

We chose the background field gauge condition to be

$$F^a = \partial_\mu Q^a_\mu + g f^{abc} A^b_\mu Q^c_\mu$$

This is the covariant derivative with respect to A^b_μ acting on Q^a_μ. Now consider the infinitesimal transformations

$$\begin{aligned}
\delta A^a_\mu &= -f^{abc}\omega^b A^c_\mu + \frac{1}{g}\partial_\mu \omega^a \\
\delta J^a_\mu &= -f^{abc}\omega^b J^c_\mu \\
\delta Q^a_\mu &= -f^{abc}\omega^b Q^c_\mu
\end{aligned} \tag{4.6}$$

We now show that $\tilde{Z}[J,A]$ in (4.4) is invariant under these transformations.

4.2 Invariance of $\tilde{Z}[J,A]$

Firstly we show that $J^a_\mu Q^a_\mu$ is invariant.

$$\begin{aligned}
\delta(J^a_\mu Q^a_\mu) &= (\delta J^a_\mu) Q^a_\mu + J^a_\mu(\delta Q^a_\mu) \\
&= -f^{abc}\omega^b J^c_\mu Q^a_\mu + J^a_\mu(-f^{abc}\omega^b Q^c_\mu) \\
&= -f^{abc}\omega^b J^c_\mu Q^a_\mu - f^{cba}\omega^b J^c_\mu Q^a_\mu \\
&= 0
\end{aligned}$$

(Here we have used the antisymmetry of the structure constant. ie. $f^{abc} = -f^{cba}$). We now examine the gauge-fixing term

$$F^a = \partial_\mu Q^a_\mu + g f^{abc} A^b_\mu Q^c_\mu$$

$$\begin{aligned}
\delta F^a &= \partial_\mu(\delta Q^a_\mu) + g f^{abc}(\delta A^b_\mu) Q^c_\mu + g f^{abc} A^b_\mu(\delta Q^c_\mu) \\
&= \partial_\mu(-f^{abc}\omega^b Q^c_\mu) + g f^{abc}(-f^{bde}\omega^d A^e_\mu + \frac{1}{g}\partial_\mu \omega^b) Q^c_\mu \\
&\quad + g f^{abc} A^b_\mu(-f^{cde}\omega^d Q^e_\mu) \\
&= -f^{abc}(\partial_\mu \omega^b) Q^c_\mu - f^{abc}\omega^b(\partial_\mu Q^c_\mu) - g f^{abc} f^{bde}\omega^d A^e_\mu Q^c_\mu \\
&\quad + f^{abc}(\partial_\mu \omega^b) Q^c_\mu - g f^{abc} A^b_\mu f^{cde}\omega^d Q^e_\mu
\end{aligned}$$

We may rewrite the last term by interchanging the dummy indices. Setting

$$b \to e \qquad e \to c \qquad c \to b$$

gives

$$\delta F^a = -f^{abc}\omega^b(\partial_\mu Q^c_\mu) + g\omega^d A^e_\mu Q^c_\mu[-f^{aeb} f^{bdc} - f^{abc} f^{bde}]$$

From the Jacobi identity we have

$$f^{acb} f^{bde} = -f^{adb} f^{bec} - f^{aeb} f^{bcd}$$

or

$$-f^{abc}f^{bde} = -f^{adb}f^{bec} + f^{aeb}f^{bdc}$$

Hence

$$\delta F^a = -f^{abc}\omega^b(\partial_\mu Q^c_\mu) + g\omega^d A^e_\mu Q^c_\mu(-f^{adb}f^{bec})$$

Rearranging the dummy indices in the last term gives

$$\begin{aligned}
\delta F^a &= -f^{abc}\omega^b(\partial_\mu Q^c_\mu) + g\omega^b A^e_\mu Q^d_\mu(-f^{abc}f^{ced}) \\
&= -f^{abc}\omega^b(\partial_\mu Q^c_\mu + g f^{ced} A^e_\mu Q^d_\mu) \\
&= -f^{abc}\omega^b F^c
\end{aligned}$$

Then

$$\begin{aligned}
\delta(F^a F^a) &= 2F^a(\delta F^a) \\
&= 2F^a(-f^{abc}\omega^b F^c) \\
&= -2\omega^b f^{abc} F^a F^c \\
&= 0
\end{aligned}$$

The last line follows because we are summing an antisymmetric quantity f^{abc} over the symmetric quantity $F^a F^c$. Note that

$$\begin{aligned}
\delta(A^a_\mu + Q^a_\mu) &= \delta A^a_\mu + \delta Q^a_\mu \\
&= -f^{abc}\omega^b A^c_\mu + \frac{1}{g}\partial_\mu \omega^a - f^{abc}\omega^b Q^c_\mu \\
&= -f^{abc}\omega^b(A^c_\mu + Q^c_\mu) + \frac{1}{g}\partial_\mu \omega^a
\end{aligned}$$

This shows that $\delta(A^a_\mu + Q^a_\mu)$ is a gauge transformation and implies that $\mathcal{L}(A+Q)$ is invariant under the transformation in (4.6). It can also be shown that the Fadeev-Popov determinant

$$det\left|\frac{\delta F^a}{\delta \omega^b}\right|$$

is invariant. Combining all this information implies that the generating functional $\tilde{Z}[J, A]$ is invariant.
It then follows that the background field effective action $\tilde{\Gamma}[\tilde{Q}, A]$ is invariant under

$$\begin{aligned}
\delta A^a_\mu &= -f^{abc}\omega^b A^c_\mu + \frac{1}{g}\partial_\mu \omega^a \\
\delta \tilde{Q}^a_\mu &= -f^{abc}\omega^b \tilde{Q}^c_\mu
\end{aligned} \tag{4.7}$$

On setting $\tilde{Q} = 0$ it is clear that $\tilde{\Gamma}[0, A]$ is an explicitly gauge invariant functional of A since (4.7) becomes an ordinary gauge transformation of the background field. It will be shown that $\tilde{\Gamma}[0, A] = \Gamma[\bar{Q}]$ with $\bar{Q} = A$. This means that $\tilde{\Gamma}[0, A]$ may be used to generate the proper, one-particle-irreducible diagrams in the same way as $\Gamma[\bar{Q}]$ in the conventional approach.

4.3 Relation between $\tilde{\Gamma}[0, A]$ and $\Gamma[\bar{Q}]$

We now demonstrate the relation between $\tilde{\Gamma}[0, A]$ and $\Gamma[\bar{Q}]$. Using the translational invariance of the measure $\mathcal{D}Q$, (ie. $\mathcal{D}(A+Q) = \mathcal{D}Q$) $\tilde{Z}[J, A]$ in (4.4) may be rewritten as

$$\begin{aligned} \tilde{Z}[J, A] &= \int \mathcal{D}Q det|\frac{\delta F^a}{\delta \omega^b}| e^{i \int d^4x [\mathcal{L}(A+Q) - \frac{1}{2\alpha}(F^a)^2 + J^a_\mu (Q^a_\mu + A^a_\mu) - J^a_\mu A^a_\mu]} \\ &= Z[J] e^{-i \int d^4x (J^a_\mu A^a_\mu)} \end{aligned} \quad (4.8)$$

$Z[J]$ is the conventional generating functional evaluated with the gauge-fixing term

$$F^a = \partial_\mu Q^a_\mu - \partial_\mu A^a_\mu + g f^{abc} A^b_\mu Q^c_\mu$$

Note that $Z[J]$ is a functional of A as well as J through the dependence of F^a on A. (4.8) may be rewritten in terms of the conventional $W[J]$ to give

$$\tilde{W}[J, A] = W[J] - \int d^4x J^a_\mu A^a_\mu \quad (4.9)$$

Differentiating (4.9) with respect to J and recalling the definitions of $\bar{Q}$ and $\tilde{Q}$ we find that

$$\tilde{Q}^a_\mu = \bar{Q}^a_\mu - A^a_\mu$$

We are now able to deduce the relationship between Γ and $\tilde{\Gamma}$.

$$\begin{aligned} \tilde{\Gamma}[\tilde{Q}, A] &= \tilde{W}[J, A] - \int d^4x (J^a_\mu \tilde{Q}^a_\mu) \\ &= (W[J] - \int d^4x J^a_\mu A^a_\mu) - \int d^4x (J^a_\mu (\bar{Q}^a_\mu - A^a_\mu)) \\ &= W[J] - \int d^4x J^a_\mu \bar{Q}^a_\mu \\ &= \Gamma[\bar{Q}] \\ &= \Gamma[\tilde{Q}^a_\mu + A^a_\mu] \end{aligned}$$

Setting $\tilde{Q} = 0$ we have the identity

$$\tilde{\Gamma}[0, A] = \Gamma[A]$$

As mentioned earlier the effective action $\tilde{\Gamma}[0, A]$ is gauge invariant. From (4.5) we have

$$\tilde{\Gamma}[0, A] = \tilde{W}[J, A]$$

The condition

$$\tilde{Q} = 0$$

implies that

$$\bar{Q} = A$$

Hence A and J are related through the dependence of $\bar{Q}$ on J. Since $W[J]$ depends on A through the gauge-fixing term we must use the chain rule when differentiating the $W[J]$ with respect to J. We therefore have (using the condition that $\bar{Q} = A$)

$$\frac{\delta W}{\delta J^a_\mu} + \int d^4y [\frac{\delta W}{\delta A^b_\nu(y)} \frac{\delta A^b_\nu(y)}{\delta J^a_\mu}] = A^a_\mu$$

4.4 Feynman Rules

We now establish the Feynman rules to compute all 1PI diagrams with the background field A appearing only on external lines. The Q fields occur inside loops. The condition $\tilde{Q} = 0$ implies that no external lines will contain Q field propagators. Similarly no A field propagators occur inside loops (since the functional integral is only over Q). In order to derive the Feynman rules we must express the determinant factor

$$det \left| \frac{\delta F^a}{\delta \omega^b} \right|$$

appearing in (4.4) as an integral over anticommuting scalar ghost fields. This approach was used in Chapter 2 where we had the result

$$det(iM) = \int \mathcal{D}\eta \mathcal{D}\eta^* e^{-i\int d^4\ x \eta^*_a M_{ac} \eta_c}$$

where

$$M_{ab} = \frac{\delta F^a}{\delta \omega^b}$$

We calculate $\frac{\delta F^a}{\delta \omega^b}$ as follows -

$$\begin{aligned} \delta F^a &= \partial_\mu(\delta Q^a_\mu) + g f^{afc} A^f_\mu (\delta Q^c_\mu) \\ &= \partial_\mu(-f^{aec}\omega^e(A^c_\mu + Q^c_\mu) + \frac{1}{g}\partial_\mu \omega^a) + g f^{afc} A^f_\mu(-f^{cde}(A^e_\mu + Q^e_\mu)\omega^d + \frac{1}{g}\partial_\mu \omega^c) \end{aligned}$$

Then

$$\begin{aligned} \frac{\delta F^a}{\delta \omega^b} &= \partial_\mu(-f^{aec}\delta^{eb}(A^c_\mu + Q^c_\mu)) + \frac{1}{g}\partial_\mu \partial^\mu \delta^{ab} \\ &\quad - g f^{afc} f^{cde} A^f_\mu (A^e_\mu + Q^e_\mu)\delta^{db} + f^{afc} A^f_\mu \partial_\mu \delta^{cb} \\ &= \partial_\mu(-f^{abc}(A^c_\mu + Q^c_\mu)) + \frac{1}{g}\partial_\mu \partial^\mu \delta^{ab} \\ &\quad - g f^{afc} f^{cbe} A^f_\mu (A^e_\mu + Q^e_\mu) + f^{afb} A^f_\mu \partial_\mu \end{aligned}$$

We then have

$$det \left| \frac{\delta F^a}{\delta \omega^b} \right| = \int \mathcal{D}\theta \mathcal{D}\theta^\dagger e^{-\int d^4x\ \theta^\dagger_a (\frac{\delta F^a}{\delta \omega^b}) \theta_b}$$

The ghost Lagrangian ($\mathcal{L}_{ghost}$) is given by

$$-\theta_a^\dagger\left(\frac{\delta F^a}{\delta\omega^b}\right)\theta_b = -\theta_a^\dagger\left[\frac{1}{g}(\partial_\mu\partial^\mu\delta^{ab} + g\partial_\mu(f^{acb}(A^c_\mu + Q^c_\mu))\right.$$

$$\left.+gf^{afb}A^f_\mu\partial_\mu + g^2 f^{afc}f^{ceb}A^f_\mu(A^e_\mu + Q^e_\mu))\right]\theta_b$$

We can use integration by parts on the term

$$g\partial_\mu(f^{acb}(A^c_\mu + Q^c_\mu))$$

Discarding the integrated term we will obtain

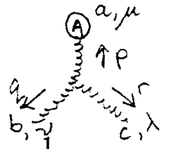
$$-g\partial^{\leftarrow}_\mu(f^{acb}(A^c_\mu + Q^c_\mu))$$

where $\partial^{\leftarrow}_\mu$ is assumed to act on $\theta_a^\dagger$. We therefore have

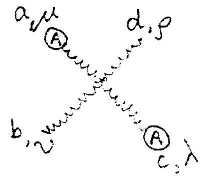
$$\mathcal{L}_{ghost} = -\theta_a^\dagger\left[\frac{1}{g}(\partial_\mu\partial^\mu\delta^{ab} - g\partial^{\leftarrow}_\mu f^{acb}(A^c_\mu + Q^c_\mu) + gf^{acb}A^c_\mu\partial^{\rightarrow}_\mu\right.$$

$$\left.+g^2 f^{acx}f^{xdb}A^c_\mu(A^d_\mu + Q^d_\mu))\right]\theta_b$$

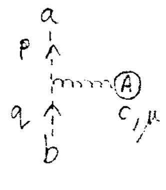
The Feynman rules in the background field gauge are as follows



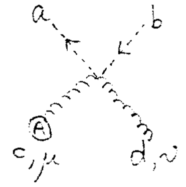
$$gf_{abc}[g_{\mu\lambda}(p - r - \frac{1}{\alpha}q)_\nu + g_{\nu\lambda}(r-q)_\mu + g_{\mu\nu}(q - p + \frac{1}{\alpha}r)_\lambda]$$



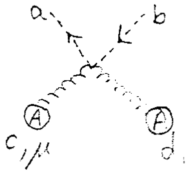
$$= ig^2[f_{abx}f_{xcd}(g_{\mu\lambda}g_{\nu\rho} - g_{\mu\rho}g_{\nu\lambda} + \frac{1}{\alpha}g_{\mu\nu}g_{\lambda\rho})$$
$$+f_{adx}f_{xbc}(g_{\mu\nu}g_{\lambda\rho} - g_{\mu\lambda}g_{\nu\rho} - \frac{1}{\alpha}g_{\mu\rho}g_{\nu\lambda})$$
$$+f_{acx}f_{xbd}(g_{\mu\nu}g_{\lambda\rho} - g_{\mu\rho}g_{\nu\lambda})]$$



$$= -gf_{acb}(p+q)_\mu$$



$$= -ig^2 f_{acx}f_{xdb}g_{\mu\nu}$$

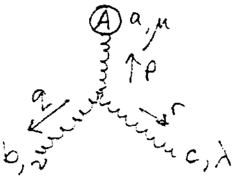


$$= -ig^2 g_{\mu\nu}(f_{acx}f_{xdb} + f_{adx}f_{xcb}$$

Wavy lines are quantum gauge field propagators; wavy lines ending with an A are external background fields and dashed lines are ghost fields.

4.5 Verification of Diagram

As an example, we check the mathematical expression for the diagram



(The other diagrams may be verified using a similar approach.) We begin by extracting the relevant terms in the Lagrangian. The gauge-fixing term is given by

$$
\begin{aligned}
-\frac{1}{2\alpha}(F^a)^2 &= -\frac{1}{2\alpha}[(\partial_\mu Q^a_\mu + gf^{abc}A^b_\mu Q^c_\mu)(\partial_\nu Q^a_\nu + gf^{ade}A^d_\nu Q^e_\nu)] \\
&= -\frac{1}{2\alpha}[(\partial_\mu Q^a_\mu)(\partial_\nu Q^a_\nu) + 2(\partial_\mu Q^a_\mu)(gf^{ade}A^d_\nu Q^e_\nu) + g^2 f^{abc}f^{ade}A^b_\mu A^d_\nu Q^c_\mu Q^e_\nu]
\end{aligned}
$$

The term which will give a contribution to the diagram is

$$
-\frac{1}{\alpha}(\partial_\mu Q^a_\mu)gf^{ade}A^d_\nu Q^e_\nu \tag{4.10}
$$

The cubic term in the Yang-Mills Lagrangian is

$$
\begin{aligned}
&-gf^{abc}(A^b_\mu + Q^b_\mu)(A^c_\nu + Q^c_\nu)\partial^\mu(A^{\nu a} + Q^{\nu a}) \\
&= -gf^{abc}(A^b_\mu A^c_\nu + A^b_\mu Q^c_\nu + Q^b_\mu Q^c_\nu + Q^b_\mu Q^c_\nu)(\partial^\mu A^{\nu a} + \partial^\mu Q^{\nu a})
\end{aligned}
$$

The relevant terms in this expansion are

$$
-gf^{abc}(A^b_\mu Q^c_\nu(\partial^\mu Q^{\nu a}) + Q^b_\mu(\partial^\mu Q^{\nu a})A^c_\nu + Q^b_\mu Q^c_\nu(\partial^\mu A^{\nu a}) \tag{4.11}
$$

Combining (4.10) and (4.11) (and reverting to standard summation convention) we perform a Fourier transform on the expression

$$
\begin{aligned}
& i\int d^4x(-\frac{1}{\alpha}(\partial_\mu Q^{\mu a}(x))gf^{abc}A^{\nu b}(x)Q^{\lambda c}(x)g_{\lambda\nu} \\
& -gf^{abc}[A^{\mu b}(x)Q^{\nu c}(x)(\partial_\mu Q^{\lambda a}(x))g_{\lambda\nu} + Q^{\mu b}(x)(\partial_\mu Q^{\nu a}(x))A^{\lambda c}(x)g_{\lambda\nu} \\
& +Q^{\mu b}(x)Q^{\lambda c}(x)(\partial_\mu A^{\nu a}(x))g_{\lambda\nu}])
\end{aligned}
$$

to give

$$
i\int d^4x \int \frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\frac{d^4r}{(2\pi)^4} g e^{-i(p+q+r).x}(-i)
$$

$$\times\{\frac{1}{\alpha}(-r_\mu)g_{\lambda\nu}f^{abc}\tilde{Q}^{\mu a}(r)\tilde{A}^{\nu b}(q)\tilde{Q}^{\lambda c}(p)$$
$$(-r_\mu)g_{\lambda\nu}f^{abc}\tilde{A}^{\mu b}(q)\tilde{Q}^{\nu c}(p)\tilde{Q}^{\lambda a}(r)$$
$$(-r_\mu)g_{\lambda\nu}f^{abc}\tilde{A}^{\lambda c}(p)\tilde{Q}^{\mu b}(q)\tilde{Q}^{\nu a}(r)$$
$$(-r_\mu)g_{\lambda\nu}f^{abc}\tilde{A}^{\nu a}(r)\tilde{Q}^{\mu b}(q)\tilde{Q}^{\lambda c}(q)\}$$

This may be rewritten as follows (switching around the dummy indices and using the antisymmetry of f^{abc})

$$\int d^4x \int \frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\frac{d^4r}{(2\pi)^4}ge^{-i(p+q+r).x}$$
$$\times\{(-\frac{1}{\alpha}r_\mu)g_{\lambda\nu} - r_\nu g_{\mu\lambda} + r_\lambda g_{\nu\mu} + q_\mu g_{\lambda\nu}\}f^{abc}\tilde{Q}^{\mu a}(r)\tilde{A}^{\nu b}(q)\tilde{Q}^{\lambda c}(p)$$

To use notation which is consistent with Abbott(1981) we swap the dummy indices according to the following scheme -

$$\begin{array}{lll} r\to q & q\to p & b\to a \\ \mu\to\nu & a\to b & \\ \nu\to\mu & p\to r & \end{array}$$

We then obtain

$$\int d^4x \int \frac{d^4r}{(2\pi)^4}\frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}ge^{-i(r+p+q).x}$$
$$\times\{(-\frac{1}{\alpha}q_\nu)g_{\lambda\mu} - q_\mu g_{\nu\lambda} + q_\lambda g_{\mu\nu} + p_\nu g_{\lambda\mu}\}(-f^{abc})\tilde{Q}^{\nu b}(q)\tilde{A}^{\mu a}(p)\tilde{Q}^{\lambda c}(r) \quad (4.12)$$

This expression is antisymmetric under

$$\nu\leftrightarrow\lambda \quad b\leftrightarrow c \quad q\leftrightarrow r$$

(since $f^{bac} = -f^{cab}$). We may then write (4.12) as

$$-\frac{1}{2}\int d^4x \int \frac{d^4r}{(2\pi)^4}\frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}e^{-i(r+p+q).x}$$
$$\times\{[g_{\mu\lambda}(p-r-\frac{1}{\alpha}q)_\nu + g_{\nu\lambda}(r-q)_\mu + g_{\mu\nu}(q-p+\frac{1}{\alpha}r)_\lambda]gf^{abc}\}$$
$$\times\tilde{Q}^{\nu b}(q)\tilde{A}^{\mu a}(p)\tilde{Q}^{\lambda c}(r)$$

The renormalisation of the quantum field Q poses difficulty since it involves the calculation of quantum field Greens functions. But in the background field method the ghost and quantum gauge fields appear only inside loops and it is unnecessary to renormalise them. Let us suppose, hypothetically, that the fields θ and Q were renormalised by writing

$$\theta_0 = \sqrt{Z}_\theta\theta \qquad Q_0 = \sqrt{Z}_Q Q$$

Each internal gauge propagator carries a factor of $\sqrt{Z}_Q$ at each end due to field renormalisation at each vertex and a factor of Z_Q^{-1} from propagator renormalisation. The two factors of $\sqrt{Z}_Q$ then cancel exactly with Z_Q^{-1}. In a similar fashion the two factors of $\sqrt{Z}_\theta$ at the end of each ghost line cancel with the factor Z_θ^{-1}. In view of these observations we see that it is redundant to renormalise the ghost and quantum gauge fields. It is necessary, however, to renormalise the coupling constant, the background field and the gauge fixing parameter. We therefore introduce the multiplicative constants Z_g, Z_A and Z_α and define the bare quantities g_0, A_0 and α_0 by

$$g_0 = Z_g g \quad A_0 = \sqrt{Z}_A A \quad \alpha_0 = Z_\alpha \alpha \tag{4.13}$$

Since gauge invariance is retained in the background field method we see that there is a relationship between the renormalisation factors Z_A and Z_g. The infinities in the gauge invariant effective action $\tilde{\Gamma}[0, A]$ must take a gauge invariant form in the background gauge. This means that the unrenormalised field tensor

$$F^a_{0,\mu\nu} = \partial_\mu A^a_{0,\nu} - \partial_\nu A^a_{0,\mu} + g_0 f^{abc} A^b_{0,\mu} A^c_{0,\nu} \tag{4.14}$$

and the renormalised field tensor

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$$

must be related by

$$F_{0,\mu\nu} = \sqrt{Z}_A F_{\mu\nu} \tag{4.15}$$

Substituting the expressions in (4.13) for A_0 and g_0 into (4.14) gives

$$(F^a_{\mu\nu})_0 = \sqrt{Z}_A [\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g Z_g \sqrt{Z}_A f^{abc} A^b_\mu A^c_\nu]$$

(4.15) then implies that

$$Z_g \sqrt{Z}_A = 1 \tag{4.16}$$

This is the relation between the charge and background field renormalisations in the background field gauge. The advantage here is that Z_A may be determined by calculating loop corrections to the A-field propagator only. In the conventional approach the gauge propagator, ghost propagator and gauge-ghost-ghost vertex all had to be computed. Here, only the gauge propagator is required. This is one of the great advantages of the background field method.
We can use the above theory to calculate the β function for pure Yang-Mills theory up to the two-loop level. The calculation is simplified because the β function can be determined by computing Z_A. From the equation

$$g_0 = Z_g g$$

we have that

$$\mu \frac{\partial}{\partial \mu} g_0 = 0 = Z_g \mu (\frac{\partial}{\partial \mu} g) + g \mu \frac{\partial}{\partial \mu} Z_g$$

and hence

$$\beta = \mu\frac{\partial}{\partial\mu}g \;=\; -\frac{1}{Z_g}g\mu(\frac{\partial}{\partial\mu}Z_g)$$
$$= -g\mu\frac{\partial}{\partial\mu}(ln\ Z_g)$$

But from (4.16) this becomes

$$\beta \;=\; \frac{1}{2}g\mu\frac{\partial}{\partial\mu}(ln\ Z_A)$$
$$= g\gamma_A$$

where $\gamma_A = \frac{1}{2}\mu\frac{\partial}{\partial\mu}ln\ Z_A$. Our task is now to compute Z_A. Abbott shows how this is done using dimensional regularisation and the minimal subtraction scheme in which Z_A is written as a series of poles. But the main advantage using the background field method (as mentioned above) is that to calculate Z_A we only require knowledge of the gauge propagator. At the 1-loop level we must consider the diagrams

and

CONCLUSION

In conclusion, we see the importance of renormalisation in quantum field theory. Different field theories have been considered in this thesis ($g\psi^4$, QED,QCD) and we see how renormalisation is used to make these theories more physically meaningful. The technique of renormalisation has been presented in some detail and we should note the significant contributions made by t'Hooft and Veltman in the 1970's. Apart from dealing with renormalisation this thesis aims to familiarise the reader with the mathematical language in which Quantum Field Theory is expressed. Concepts such as the path integral and generating functional form a central part of the theory. We should not, therefore, forget the pioneering work of Feynman in the 1950s who should surely be remembered as one of the great founders of Quantum Field Theory.

Appendix A

Integral Formulas in Minkowski Space

$$\int \frac{d^d l}{(l^2 + 2p.l - m^2)^A} = i(-1)^{-\frac{d}{2}} \pi^{\frac{d}{2}} \frac{\Gamma(A - \frac{d}{2})}{\Gamma(A)} \frac{1}{(-p^2 - m^2)^{A-\frac{d}{2}}} \tag{A.1}$$

$$\begin{aligned} \int \frac{d^d l l_\mu l_\nu}{(l^2 + 2p.l - m^2)^A} &= (-1)^{-\frac{d}{2}} \frac{i\pi^{\frac{d}{2}}}{\Gamma(A)} \frac{1}{(-p^2 - m^2)^{A-\frac{d}{2}}} \\ &\times \left\{ p_\mu p_\nu \Gamma(A - \frac{d}{2}) + \frac{1}{2} g_{\mu\nu} (-p^2 - m^2) \Gamma(A - 1 - \frac{d}{2}) \right\} \end{aligned} \tag{A.2}$$

Appendix B

The Gell-Mann Matrices

$$
\begin{aligned}
\lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \lambda_2 &= \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
\lambda_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \lambda_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \\
\lambda_5 &= \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} & \lambda_6 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\
\lambda_7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} & \lambda_8 &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}
\end{aligned}
\tag{B.1}
$$

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