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ON A CLASS OF PSEUDO-DIFFERENTIAL OPERATORS IN $\mathbb{R}^n$

by

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ABSTRACT

The class of pseudo-differential operators with symbols from $S^m_{\rho\delta}(\Omega \times \mathbb{R}^n)$ has been extensively studied. The main assumption which characterises this class of symbols is that $a(x,\xi) \in S^m_{\rho\delta}(\Omega \times \mathbb{R}^n)$ should have a polynomial growth in the ξ-variable only. The x-variable is controlled on compact subsets of Ω.

A polynomial growth in both the x and ξ variables on a $C^\infty(\mathbb{R}^{2n})$ function $a(x,\xi)$ gives rise to a different class of symbols and a correspondig class of operators. In this work, such symbols and the action of the operators on the functional spaces $S(\mathbb{R}^n)$, $S'(\mathbb{R}^n)$ and the Sobolev spaces $Q^s(\mathbb{R}^n)$ ($s \in \mathbb{R}^n$) are studied. A study of the calculus (i.e. transposes, adjoints and compositions) and the functional analysis of these operators is done with special attention to L^2-boundedness and compactness.

The class of hypoelliptic pseudo-differential operators in $\mathbb{R}^n$ is introduced as a subclass of those considered earlier. These operators possess the property that they allow a pseudo-inverse or parametrix.

In conclusion , the spectral theory of these operators is considered. Since a general spectral theory would be beyond the scope of this work, only some special cases of the pseudo-differential operators in $\mathbb{R}^n$ are considered. A few applications of this spectral theory are discussed.

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PREFACE

The pseudo-differential operators are just natural extansions of the linear partial differential operators and singular integral operators. The Fourier Transform is employed to convert the theory of linear partial differential operators into an algebraic theory of characteristic polynomials or symbols. These pseudo-differential operators are classified in accordance with the growth behaviour of their symbols.

An extensive study on the class $S^m_{\rho\delta}(\Omega \times \mathbb{R}^n)$ and some of its generalisations is done in e.g. Unterberger [27] , Beals[2] ,Hörmander [11],[12] ,Taylor[24], and Treves [25].Maslov and Fedorjuk [A] introduce a class of symbols that have a given polynomial growth with all derivatives. They express the opinion that a class of a symbol depends on the problem under consideration. Since the Hörmander and Maslov classes , we have not observed an appearance of any new important class of symbols.

Combining the concepts of $S^m_{\rho\delta}$ classes and the Maslov classes, a different class of symbols is born . We denote this class by $\Gamma^m_\rho(\mathbb{R}^{2n})$ and the corresponding class of pseudo-differential operators by $G^m_\rho(\mathbb{R}^n)$. We shall basically follow the book by Shubin [21] to develop this work.The proofs of most of the theorems in this book are brief and even omitted in certain cases. We attempt to complete these proofs,using the ideas of Feigin [8] , Shubin [22] ,Tulovskii and Shubin [26] and the knowledge of the $S^m_{\rho\delta}(\Omega \times \mathbb{R}^n)$-classes. The ideas of Catchpole [4] , Weder [28] and Wong [30] are used to develop a spectral theory for the unperturbed and the perturbed pseudo-differential operators in $\mathbb{R}^n$. A few applications are given with special reference to quantum mechnics. Furthermore ,a study of the spectrum of hypoelliptic pseudo-differential operators in $\mathbb{R}^n$ is done and some applications given.

In the first chapter , we study the Γ^m_ρ-classes and the Π^m_ρ-classes, introducing the language which is used throughout. Asymptotic expansions, properties of the Γ^m_ρ-classes and the effects of the $G^m_\rho(\mathbb{R}^n)$ operators on the functional spaces $S(\mathbb{R}^n)$ and $S'(\mathbb{R}^n)$ are considered and we prove that the Π^m_ρ-classes have asymptotics in the Γ^m_ρ-classes. This result is analogous to the result that the $S^m_{\rho\delta}(X \times Y \times \mathbb{R}^n)$-classes have asymptotics in $S^m_{\rho\delta}(\Omega \times \mathbb{R}^n)$-classes (see [24]). This result is used to define the left,right and the Weyl symbol and deduce their asymptotic expansions. These definitions are of vital importance for the development of our discussion.

The calculus of these operators is studied in the second chapter, and the asymptotic formulas for the transposes, adjoints and compositions of these operators are determined. This chapter is preliminary to chapter 4 , where we shall give the construction of a parametrix of a hypoelliptic pseudo-differential operator in $\mathbb{R}^n$.

In Chapter 3 we do a functional analytic study of these operators. We study in particular their L^2-boundedness and compactness. The notion of aW-symbols is used. Basically , two important results are proved; firstly it is proved that if $A \in G^0_\rho(\mathbb{R}^n)$ then A extends to a bounded operator on $L^2(\mathbb{R}^n)$; secondly if $A \in G^m_\rho(\mathbb{R}^n)$ and $m < 0$, then A extends to a compact operator on $L^2(\mathbb{R}^n)$.

The fourth chapter is devoted to the study of the hypoelliptic pseudo-differential operators in $\mathbb{R}^n$, namely the $HG^{m,m_0}_\rho(\mathbb{R}^n)$ operators. We prove a very important result that every $A \in HG^{m,m_0}_\rho(\mathbb{R}^n)$ has a parametrix $B \in HG^{-m_0,-m}_\rho(\mathbb{R}^n)$ such that $AB - I = R_1$ and $BA - I = R_2$ are integral operators with kernels from $S(\mathbb{R}^{2n})$. The construction of this parametrix is studied. Furthemore the elliptic pseudo-differential operators in $\mathbb{R}^n$ and the classical symbols are introduced. We prove that every operator with a classical elliptic symbol is hypoelliptic and then show that the construction of the parametrix of this operator yields an explicit formula. Finally in this chapter , the Sobolev spaces $Q^s(\mathbb{R}^n)$ $(s \in \mathbb{R}^n)$ and the effects of the $G^m_\rho(\mathbb{R}^n)$ operators on these spaces are considered.

Chapter 5 contains the spectral theory of the pseudo-differential operators in $\mathbb{R}^n$. We give some examples of pseudo-differential operators as discussed in Catchpole [4], Weder [28] and Wong [30]. This may be regarded as the spectral theory of constant coefficient differential operators. The spectrum of the hypoelliptic pseudo-differential operators is considered and we prove that these operators are essentially self-adjoint and have discrete spectrum only. In conclusion some applications of this spectral theory are given.

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NOTATION

$\mathbb{R}^n$: n-dimensional real Euclidean space.

$x = (x_1,\dots,x_n)$, $\xi = (\xi_1,\dots,\xi_n)$; variables and coordinates in in $\mathbb{R}^n$.

$\|x\| = (\sum_{i=1}^{n} x_i^2)^{1/2}$: norm in $\mathbb{R}^n$.

We shall use the following version of the Fourier Transform troughout the text;

We define the Fourier Transform (F) of u as,

$$(Fu)(\xi) = \hat{u}(\xi) = \int e^{-ix\xi}u(x)dx = F_{\xi\to x}\{u(x)\},$$

and the inverse Fourier Transform (F^{-1}) as;

$$(F^{-1}\hat{u})(x) = u(x) = (2\pi)^{-n}\int e^{ix\xi}\hat{u}(\xi)d\xi$$

$\langle z\rangle = (1 + \|z\|^2)^{1/2}$, for $z = (x,\xi)\in\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$ and $\|z\|^2 = \|x\|^2 + \|\xi\|^2$ in this case, also $\langle x\rangle = (1+\|x\|^2)^{1/2}$.

Z: set of integers.

Z_+: set of non-negative integers.

Z_+^n: set of n-tuples $\alpha = (\alpha_1,\dots,\alpha_n)$ with $\alpha_j \in Z_+$ for each $j = 1,2,3,\dots,n$. Z_+^n is also referred to as a set of multi-indices.

For $\alpha,\beta \in Z_+^n$, $\beta \leq \alpha$ means $\beta_j \leq \alpha_j$; $j = 1,\dots,n$.

$|\alpha| = \sum_{i=1}^{n} \alpha_i$; the length of $\alpha \in Z_+^n$.

$\alpha! = \alpha_1!\dots\alpha_n!$, $\binom{\alpha}{\beta} = \binom{\alpha_1}{\beta_1}\dots\binom{\alpha_n}{\beta_n} = \dfrac{\alpha!}{(\alpha-\beta)!\,\beta!}$ if $\alpha,\beta \in Z_+^n$ and $\beta \leq \alpha$.

$x^\alpha = (x_1^{\alpha_1})\dots(x_n^{\alpha_n})$ if $x\in\mathbb{R}^n$ and $\alpha \in Z_+^n$.

$\partial_x^\alpha = (\frac{\partial}{\partial_{x_1}})^{\alpha_1}\dots(\frac{\partial}{\partial_{x_n}})^{\alpha_n}$ also denoted by ∂^α.

D(T) : domain of the operator T.

$D^{\alpha}_{x} = (\frac{1}{i}\frac{\partial}{\partial x_1})^{\alpha_1} \dots\dots (\frac{1}{i}\frac{\partial}{\partial x_n})^{\alpha_n}$ also denoted by D^{α}, $i = \sqrt{-1}$.

$\langle D_{\xi}\rangle^{2k} = (1 + D^{2}_{\xi_1} + \dots + D^{2}_{\xi_n})^{k}$ for $\xi \in \mathbb{R}^n$, $k \in Z_+$

If $u \in C^{\infty}$ is a function of x then,

$u^{(\alpha)} = \partial^{\alpha}_{x} u$, $\partial_x u$ = grad $u = (\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n})$ and

$D_j u = \frac{1}{i}\frac{\partial u}{\partial x_j}$, $j = 1, \dots, n$.

Taylor Expansion

$$u(x) = \sum_{|\alpha| \le N-1} \frac{1}{\alpha!}(x-y)^{\alpha} \partial^{\alpha} u(y) + \sum_{|\alpha| = N} (N/\alpha!)(x-y)^{\alpha} \int_0^1 (1-t)^{N-1} \partial^{\alpha} u[y + t(x-y)]\, dt.$$

Leibniz Formula

$$\partial^{\alpha}(uv) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} (\partial^{\alpha-\beta} u)(\partial^{\beta} v) .$$

General Leibniz Formula

Let $P(\partial)$ be a partial differential operator with constant coefficients, u and v be C^{∞} functions. Then

$$P(\partial)(uv) = \sum_{\alpha} (\frac{1}{\alpha!}) \partial^{\alpha} u \, P^{(\alpha)}(\partial) v ,$$

where $P^{(\alpha)}(\xi) = \partial^{\alpha}_{\xi} P(\xi)$.

CHAPTER 1

THE OPERATORS AND SYMBOLS

1.1 Introduction to $G_\rho^m(\mathbb{R}^n)$ operators

We begin our discussion by defining the class of pseudo-differential operators in $\mathbb{R}^n$. This is the $G_\rho^m(\mathbb{R}^n)$ class of operators characterised by the symbols from $\Gamma_\rho^m(\mathbb{R}^{2n})$. First of all we recall the definition of the $S_{\rho\delta}^m(\Omega\times\mathbb{R}^n)$ classes. They will aid us in defining the $\Gamma_\rho^m(\mathbb{R}^{2n})$-classes.

Suppose m, ρ and δ are real numbers with $0 \leqslant \delta < \rho \leqslant 1$. Then we say $a(x,\xi) \in S_{\rho\delta}^m(\Omega\times\mathbb{R}^n)$ ($\Omega\subset\mathbb{R}^n$ open), if $a(x,\xi) \in C^\infty(\Omega\times\mathbb{R}^n)$ and for every compact set $K \subset \Omega$ and all multi-indices $\alpha, \beta \in Z_+^n$ there exists a positive constant $C_{\alpha\beta K}$ such that

$$(1) \qquad |D_x^\beta D_\xi^\alpha a(x,\xi)| \leqslant C_{\alpha\beta K} \langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|}$$

for all $x \in K$ and $\xi \in \mathbb{R}^n$. The corresponding pseudo-differential operator is expressed as:

$$(2) \qquad Au(x) = (2\pi)^{-n} \int\int e^{i(x-y)\xi}\, a(x,\xi)\, u(y)\,dy\,d\xi$$

ff for every $u \in S(\mathbb{R}^n)$, where $S(\mathbb{R}^n)$ is the Schwartz space (see p15).

The estimates in (1) mean that $a(x,\xi)$ has a polynomial growth on the ξ-variable only. The x-variable is confined to a compact subset of Ω in $\mathbb{R}^n$.

For the pseudo-differential operator $A \in G_\rho^m(\mathbb{R}^n)$ we assume that the symbol $a(x,\xi)$ of A should have a polynomial growth in both the variables x and ξ. Thus formally, we have the following definition.

Definition 1.1.1 $[\Gamma^m_\rho(\mathbb{R}^{2n})]$. Let $m \in \mathbb{R}$, $0<\rho\leq 1$.

We shall denote by $\Gamma^m_\rho(\mathbb{R}^{2n})$ a class of $C^\infty(\mathbb{R}^{2n})$ functions $a(z)$ such that for every multi-index α, there is a positive constant C_α such that the following estimates hold,

(1) $|D^\alpha_z a(z)| \leq C_\alpha \langle z\rangle^{m-\rho|\alpha|}$

for all $z\in\mathbb{R}^{2n}$, where $\langle z\rangle = (1+\|z\|^2)^{\frac{1}{2}}$.

We give an example to make our definition lucid.

Example 1.1.2 Let $P(z)$ be a polynomial of degree $\leq m$. In other words

$P(z) = \sum\limits_{|\beta|\leq m} a_\beta z^\beta$. Then for $\alpha\leq\beta$ we have

$\partial^\alpha_z P(z) = \sum\limits_{\substack{\alpha\leq\beta\\|\beta|\leq m}} \frac{\beta!}{(\beta-\alpha)!} a_\beta z^{\beta-\alpha}$. Therefore

$$|\partial_z{}^\alpha . P(z)| \leq \sum_{\substack{\alpha\leq\beta\\|\beta|\leq m}} \frac{\beta!}{(\beta-\alpha)!} |a_\beta|\, |z^{\beta-\alpha}|$$

Since $|z_i^{\alpha_i}| \leq \langle z\rangle^{\alpha_i}$, we have

$$\begin{aligned}|\partial^\alpha_z P(z)| &\leq \sum_{\substack{\alpha\leq\beta\\|\beta|\leq m}} \frac{\beta!}{(\beta-\alpha)!} |a_\beta| \langle z\rangle^{|\beta|-|\alpha|}\\ &\leq \langle z\rangle^{m-|\alpha|} \sum_{\substack{\alpha\leq\beta\\|\beta|\leq m}} \frac{\beta!}{(\beta-\alpha)!} |a_\beta|\\ &\leq C_\alpha \langle z\rangle^{m-|\alpha|} \quad \text{where } C_\alpha = \sum_{\substack{\alpha\leq\beta\\|\beta|\leq m}} \frac{\beta!}{(\beta-\alpha)!}|a_\beta| .\end{aligned}$$

The above estimates prove that $P(z)\in\Gamma^m_1(\mathbb{R}^{2n})$.

Before giving the second example we make the following observation, if $a(z)\in\Gamma^m_\rho(\mathbb{R}^{2n})$ and $a'(z) = a(z) + b(z)$, where $b(z)\in S(\mathbb{R}^{2n})$ then $a'(z)\in\Gamma^m_\rho(\mathbb{R}^{2n})$. Actually, $|\partial^\alpha a'(z)| \leq |\partial^\alpha a(z)| + |\partial^\alpha b(z)| \leq C_\alpha\langle z\rangle^{m-\rho|\alpha|} + M_\alpha\langle z\rangle^{m-\rho|\alpha|}$ by (1) and the definition of the Schwartz space (see p15). Thus the following conclusion is justfied; if $a\in C^\infty(\mathbb{R}^{2n})$ then $a\in\Gamma^m_\rho(\mathbb{R}^{2n})$ provided (1) is true for $\|z\|\geq R$, where R is any positive real number.

Example 1.1.3 Any positively homogeneous function $P(z) \in C^\infty(\mathbb{R}^{2n})$ of degree m for large z also belongs to $\Gamma_1^m(\mathbb{R}^{2n})$.

Recall: P(z) is positively homogeneous of degree m for large z if there is a real number $R > 0$ such that for any $\lambda > 0$ and $\|z\| \geq R$
$P(\lambda z) = \lambda^m P(z)$.

For example we can take $P(z) = \Phi(z) \| z \|^m$, where $\Phi(z) \in C^\infty(\mathbb{R}^{2n})$, $\Phi(z) = 1$ for $\|z\| \geq R$ and $\Phi(z) = 0$ for $\| z\| \leq R/2$.

We consider the estimates (1) for $\| z \| \geq R+1$ only.

We have

$$\begin{aligned} |P(z)| &= |P[\|z\|(z/\|z\|)]| \\ &= \|z\|^m \, |P(z/\|z\|)| \\ &\leq [\sup_{\|w\|=1} |P(w)|] \|z\|^m \\ &\leq C \langle z\rangle^m, \quad \text{where } C = \sup_{\|w\|=1} |P(w)|, \text{ since } \|z\| < \langle z\rangle . \end{aligned}$$

For the partial derivatives consider (for $t > 0$)

$$\begin{aligned} \partial/\partial z_k \, P(tz) &= \lim_{h\to 0} \frac{P(tz + he_k) - P(tz)}{h} \\ &= \lim_{h\to 0} \frac{P[t(z + \frac{h}{t}e_k)] - P(tz)}{h} \\ &= t^{m-1} \lim_{h\to 0} \frac{P(z + \frac{h}{t}e_k) - P(z)}{h/t} \\ &= t^{m-1} \frac{\partial P(z)}{\partial z_k}, \quad \text{where } e_k = (0,\ldots,0,\underset{\substack{\uparrow \\ \text{k-th position.}}}{1},0,\ldots 0) \in \mathbb{R}^n \end{aligned}$$

Therefore we conclude that $\partial_z^\alpha P(z)$ is positively homogeneous of degree $m- |\alpha|$ and thus

$$|\partial_z^\alpha P(z)| = |\,\|z\|^{m-|\alpha|}\,(\partial_z^\alpha P)\,(z/\|z\|)\,|$$

$$\leq (\sup_{\|w\|=1} |\partial_w^\alpha P(w)|)\,\|z\|^{m-|\alpha|}$$

$$\leq C_\alpha <z>^{m-|\alpha|} \quad , \text{ where } C_\alpha = \sup_{\|w\|=1} |\partial_w^\alpha P(w)| .$$

Some properties of these symbols are given in the Lemma below. These properties will be of vital importance in the development of our discussion.

1.2 PROPERTIES OF SYMBOLS IN $\Gamma_\rho^m(\mathbb{R}^{2n})$ CLASSES

Lemma 1.2.1 The $\Gamma_\rho^m(\mathbb{R}^{2n})$ classes have the following properties.

If $a(z) \in \Gamma_\rho^{m_1}(\mathbb{R}^{2n})$ and $b(z) \in \Gamma_\rho^{m_2}(\mathbb{R}^{2n})$ then

1. $a(z)\,b(z) \in \Gamma_\rho^{m_1+m_2}(\mathbb{R}^{2n})$.
2. $\partial_z^\alpha a(z) \in \Gamma_\rho^{m_1-\rho|\alpha|}(\mathbb{R}^{2n})$.
3. $a(z) + b(z) \in \Gamma_\rho^{\max(m_1,m_2)}(\mathbb{R}^{2n})$.
4. $\Gamma_\rho^m(\mathbb{R}^{2n}) \subset \Gamma_\rho^{m'}(\mathbb{R}^{2n})$ for $m' > m$.

Proof

1. From Leibnitz formula.

$$|\partial_z^\alpha (a(z)\,b(z))| = |\sum_{\gamma+\delta=\alpha} \frac{\alpha!}{\gamma!\,\delta!}\,\partial_z^\gamma a(z)\,\partial_z^\delta b(z)|$$

$$\leq \sum_{\gamma+\delta=\alpha} \frac{\alpha!}{\delta!\,\gamma!}\,|\partial_z^\gamma a(z)|\,|\partial_z^\delta b(z)|$$

$$\leq \sum_{\delta+\gamma=\alpha} \frac{\alpha!}{\gamma!\,\delta!}\,C_\gamma <z>^{m_1-\rho|\gamma|}\,C_\delta <z>^{m_2-\rho|\delta|}$$

$$= \sum_{\gamma+\delta=\alpha} \frac{\alpha!}{\gamma!\ \delta!} C_\gamma C_\delta <z>^{(m_1+m_2)-\rho|\gamma+\delta|}$$

$$\leq <z>^{m_1+m_2-\rho|\alpha|} \sum_{\gamma+\delta=\alpha} \frac{\alpha!}{\gamma!\ \delta!} C_\gamma C_\delta \quad ,$$

which proves 1.

2. Now $|\partial_z^\mu a(z)| \leq C_\mu <z>^{m_1-\rho|\mu|}$. Thus

$$|\partial_z^\alpha \partial_z^\mu a(z)| \leq C_\mu <z>^{m_1-\rho|\alpha| -\rho|\mu|}$$ which

shows clearly that $\partial_z^\alpha a(z) \in \Gamma_\rho^{m_1-\rho|\alpha|} (\mathbb{R}^{2n})$.

3. 3 & 4 is obvious.

The following proposition gives us the relationship between the $\Gamma_\rho^m (\mathbb{R}^{2n})$ and the functional space $S(\mathbb{R}^{2n})$.

Proposition 1.2.2

$\bigcap_m \Gamma_\rho^m (\mathbb{R}^{2n}) = S(\mathbb{R}^{2n}) = \bigcap_j \Gamma_\rho^{m_j} (\mathbb{R}^{2n})$, where $m_j \to -\infty$ as $j \to -\infty$.

Proof

The first equality follows from the definition of $\Gamma_\rho^m (\mathbb{R}^{2n})$ classes and the definition of the seminorm on the space $S(\mathbb{R}^{2n})$ which are given on page 15 or Treves [25] , page xix. The second equality is a trivial consequence of the first one and the property 4 of Γ_ρ^m symbols.

1.3 Asymptotics in $\Gamma_\rho^m (\mathbb{R}^{2n})$ classes

Firstly we shall define the asymptotics in Γ_ρ^m - classes.

Definition 1.3.1. Let $a_j(z) \in \Gamma_\rho^{m_j} (\mathbb{R}^{2n})$, $j = 1,2,3...$ Suppose also that $a(z) \in C^\infty(\mathbb{R}^{2n})$. We write

(1) --- $a(z) \sim \sum_{j=1} a_j(z)$

if there exist $\{m_j\}_{j=1}^{\infty}$ with $\lim_{j\to\infty} m_j = -\infty$ such that

$$(2) \text{ --- } a(z) - \sum_{j=1}^{r-1} a_j(z) \in \Gamma_\rho^{\overline{m}_r}(\mathbb{R}^{2n}) , \quad \text{if for any } r \geq 2,$$

where $\overline{m}_r = \max\{m_j ; j \geq r\}$ .The formal expansion (1) is called the asymptotic expansion of the symbol $a(z)$ in $\Gamma_\rho^m(\mathbb{R}^{2n})$ classes.

NOTE: One can replace the series in (1) by a new one for which

(2') $m_j > m_{j+1}$, (see Appendix A).

Now we shall show that given a formal asymptotic expansion in the Γ-classes,the exists a function from $\Gamma_\rho^m(\mathbb{R}^{2n})$ such that this asymptotic expansion defines the function at our disposal uniquely mod a function from $S(\mathbb{R}^{2n})$. Without loss of generality we restrict ourselves to the case (2').

From now on we shall omit the argument z except where confusion may arise. In other words by a ,we shall refer to $a(z)$,where $z \in \mathbb{R}^{2n}$.

Theorem 1.3.2. Let $a_j \in \Gamma_\rho^{m_j}(\mathbb{R}^{2n})$,$m_j > m_{j+1}$, $j = 1,2,3,\ldots$, $m_j \to -\infty$ as $j \to +\infty$ .Then there exists $a \in C^\infty(\mathbb{R}^{2n})$ such that

$$(1) \text{ --- } a \sim \sum_{j=1}^{\infty} a_j \quad ,$$

if $a' \in C^\infty(\mathbb{R}^{2n})$ is another function with property (1) then,

$$(2) \text{ --- } a - a' \in S(\mathbb{R}^{2n}) .$$

Proof Take $\psi(z) \in C^\infty(\mathbb{R}^{2n})$ defined by

$$(R) \text{ --- } \psi(z) = \begin{cases} 0 & \text{for } \|z\| \leq \tfrac{1}{2} \\ 1 & \text{for } \|z\| \geq 1 \end{cases} .$$

and $0 \leq \psi(z) \leq 1$ for all z .

Choose a sequence $\{t_j\} \in \mathbb{R}^+$ such that $t_j \to +\infty$ and $t_j > t_{j-1}$, for $j = 1,2,3,\ldots$

and let

$$(3)\quad a(z) = \sum_{j=1}^{\infty} \psi(t_j^{-1} z)\ a_j(z) .$$

The series (3) is well-defined since it is locally finite. In fact if $\| t_j^{-1} z \| < ½$ then $\psi(t_j^{-1} z) = 0$. So given a positive constant R, there exists $N > 0$ such that if $j > N$ then $\|t_j^{-1} z\| < ½$ for $\|z\| \leqslant R$. Therefore $\psi(t_j^{-1} z) = 0$ for $j > N$. Therefore in the ball $\|z\| \leq R$ the series (3) has at most N non zero terms.

Now $a(z) \in C^\infty(\mathbb{R}^{2n})$ since $\psi(z)$ and $a_j(z)$ are all from $C^\infty(\mathbb{R}^{2n})$.

We shall now focus our attention on the partial derivatives of a(z). By Leibnitz formula we have

$$|\partial_z^\alpha[\psi(t_j^{-1} z)\, a_j(z)]| \leqslant \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} |\partial_z^\beta \psi(t_j^{-1} z)\, \partial_z^{\alpha-\beta} a_j(z)| .$$

Observe that $\partial_z^\beta\ \psi(t_j^{-1} z) = 0$ for $\|t_j^{-1} z\| \geq 1$ and $\| t_j^{-1} z\| \leq ½$ if $|\beta| > 0$. We consider the case

a) $½ \leq \| t_j^{-1} z\| \leq 1$ or $\| z \| \leq t_j \leq 2 \| z \|$,

For t_j in this annulus we have,

$$|\partial_z^\beta\ \psi(t_j^{-1} z)| = t_j^{-|\beta|} |(\partial_z^\beta \psi)\ (t_j^{-1} z)|$$

$$\leq \sup_{z \in \mathbb{R}^{2n}} |(\partial_z^\beta \psi)\ (t_j^{-1} z)|\ t_j^{-|\beta|} .$$

Since $\|z\| < \langle z\rangle$ we have

$$(4)\quad |\partial_z^\beta\ \psi(t_j^{-1} z)| \leq t_j^{-|\beta|} C_\beta \langle z\rangle^{-|\beta|} \quad \text{where}$$

$$C_\beta = \sup_{z \in \mathbb{R}^{2n}} |(\partial_z^\beta \psi)\ (t_j^{-1} z)| = \sup_{z \in \mathbb{R}^{2n}} |\partial_z^\beta \psi(z)| .$$

To prove this theorem it is sufficient to show that

$$|\partial_z^\beta\ \psi(t_j^{-1} z)| \leq C_\beta \langle z\rangle^{-|\beta|} .$$

From Leibnitz formula and (4) together with the $\Gamma_\rho^m(\mathbb{R}^{2n})$ estimates we obtain

$$|\partial_z^\alpha\{\psi(t_j^{-1}z)\,a_j(z)\}| \leqslant \sum_{\beta\leqslant\alpha}\binom{\alpha}{\beta}t_j^{-|\beta|}C_\beta\langle z\rangle^{-|\beta|}C_{\alpha-\beta,j}\langle z\rangle^{m_j-\rho|\alpha-\beta|}$$

if $\|z\|<t_j<2\|z\|$.

Since $0<\rho\leqslant 1$, we have $\rho|\beta|\leqslant|\beta|$ and so

$\langle z\rangle^{-|\beta|}\langle z\rangle^{-\rho|\alpha|+\rho|\beta|}\leqslant\langle z\rangle^{-\rho|\alpha|}$ and therefore,

$$(5)\text{---}\quad |\partial_z^\alpha(\psi(t_j^{-1}z)a_j(z))| \leqslant \sum_{\beta\leqslant\alpha}\binom{\alpha}{\beta}t_j^{-|\beta|}C_\beta\, C_{\alpha-\beta,j}\langle z\rangle^{m_j-\rho|\alpha|} \leqslant \tilde{C}_{\alpha,j}\langle z\rangle^{m_j-\rho|\alpha|}$$

where $\tilde{C}_{\alpha,j}=\sum_{\beta\leqslant\alpha}\binom{\alpha}{\beta}t_j^{-|\beta|}C_\beta\, C_{\alpha-\beta,j}$.

We have

$\langle z\rangle^{m_j-\rho|\alpha|}=\langle z\rangle^{m_{j-1}-\rho|\alpha|}\langle z\rangle^{m_j-m_{j-1}}$. Since by the assumption $m_j<m_{j+1}$ one can choose t_j so large that for all z such that $t_j\leqslant 2\|z\|$ we have

$\langle z\rangle^{m_j-m_{j-1}}\leqslant 2^{-j}/\tilde{C}_{\alpha,j}$

Then for $\|z\|\leqslant t_j\leqslant 2\|z\|$, $|\alpha|\leqslant j$ we have

$$(6)\text{---}\quad |\partial_z^\alpha(\psi(t_j^{-1}z)\,a_j(z))| \leqslant 2^{-j}\langle z\rangle^{m_{j-1}-\rho|\alpha|}\ \text{for}$$

$\|z\|\leqslant t_j\leqslant 2\|z\|$ and any finite number of derivatives, say for all $|\alpha|\leqslant j$.

We now deal with those z which are not in the annulus;

(b) $\|t_j^{-1}z\|\leqslant \frac{1}{2}$ or $\|t_j^{-1}z\|\geqslant 1$.

If $\|t_j^{-1}z\|\leqslant\frac{1}{2}$ then $\psi(t_j^{-1}z)=0$ and (6) does hold.

If $\|t_j^{-1}z\|\geqslant 1$ which is equivalent to saying that $\|z\|\geqslant t_j$ then

$\partial_z^\alpha\psi(t_j^{-1}z)=0$ for $|\alpha|>0$ since $\psi(t_j^{-1}z)=1$ by (R).

Therefore the following estimates are valid:

$$
\begin{aligned}
|\partial_z^\alpha[\phi(t_j^{-1}z)\,a_j(z)]| &= |\phi(t_j^{-1}z)|\;|\partial_z^\alpha a_j(z)| \\
&= |\partial_z^\alpha a_j(z)| \\
&\leqslant \tilde{C}_{\alpha,j} <z>^{m_j-\rho|\alpha|} \\
&= \tilde{C}_{\alpha,j} <z>^{m_{j-1}-\rho|\alpha|} <z>^{m_j-m_{j-1}} \\
&\leqslant 2^{-j} <z>^{m_{j-1}-\rho|\alpha|} .
\end{aligned}
$$

Thus we have proved (6) for every $z \in \mathbb{R}^{2n}$.

Therefore
$a(z) \in \Gamma_\rho^m(\mathbb{R}^{2n})$ where $m = m_1$.

At this juncture, we shall show that (1) is an asymptotic series, that is

$$a - \sum_{j=1}^{r} a_j \in \Gamma_\rho^{m_{r+1}}(\mathbb{R}^{2n}).$$

Clearly,

$$a - \sum_{j=1}^{r} a_j = a(z) - \sum_{j=1}^{r} \phi(t_j^{-1}z)\,a_j(z) + \sum_{j=1}^{r} [\phi(t_j^{-1}z)-1]\,a_j(z).$$

Since $\phi(t_j^{-1}z) - 1 \in C_0^\infty(\mathbb{R}^{2n})$ then the second sum above is also in $C_0^\infty(\mathbb{R}^{2n})$.
We need to show that,

$$(6') \quad a(z) - \sum_{j=1}^{r} \phi(t_j^{-1}z)\,a_j(z) = \sum_{j=r+1}^{\infty} \phi(t_j^{-1}z)\,a_j(z) \in \Gamma_\rho^{m_{r+1}}(\mathbb{R}^{2n}).$$

Given α take $N>|\alpha|$ and split the sum $\sum_{j=r+1}^{\infty} \phi(t_j^{-1}z)\,a_j(z)$ into
$\sum_{j=r+1}^{N} \phi(t_j^{-1}z)\,a_j(z) + \sum_{j=N+1}^{\infty} (t_j^{-1}z)\,a_j(z)$. The estimates (6) yield;

$$(7)\text{---}\ |\partial_z^\alpha\{\sum_{j=N+1}^{\infty} \phi(t_j^{-1}z)\,a_j(z)\}| \leqslant \sum_{j=N+1}^{\infty} 2^{-j} <z>^{m_{j-1}-\rho|\alpha|} \leqslant 2^{-N} <z>^{m_N-\rho|\alpha|}$$

and thus the infinite sum is in $\Gamma_\rho^{m_{r+1}}(\mathbb{R}^{2n})$. Since $j \geqslant r+1$ then the finite sum; $\sum_{j=r+1}^{N} \phi(t_j^{-1}z)\,a_j(z) \in \Gamma_\rho^{m_{r+1}}(\mathbb{R}^{2n})$, by 3. of Lemma 1.2.1.

This proves that (3) is uniformly convergent with all derivatives.

Hence from (6')

$$a - \sum_{j=1}^{r-1} a_j \in \Gamma_\rho^{m_r}(\mathbb{R}^{2n}) \text{ as required.}$$

Finally we prove (2):

Observe that $a - a' = a - \sum_{j=1}^{k-1} a_j - \left(a' - \sum_{j=1}^{k-1} a_j \right)$

$$\in \Gamma_\rho^{m_k}(\mathbb{R}^{2n}) \quad \text{for } k > 1.$$

Letting $k = 1,2,3,\ldots$ we get (see Appendix B)

$$a - a' \in S(\mathbb{R}^{2n}) \text{ as required by Proposition 1.2.2.}$$

The theorem that follows give us even weaker conditions than condition (2) of Definition 1.3.1, under which the series (1) of Theorem 1.3.2 is valid.

Theorem 1.3.3 Let $a_j \in \Gamma_\rho^{m_j}(\mathbb{R}^{2n})$, $j = 1,2,3,\ldots$ $m_j \to -\infty$ as $j \to +\infty$. Suppose $a \in C^\infty(\mathbb{R}^{2n})$ and for every multi-index α there are constants μ_α, $C_\alpha > 0$ such that

(1) $$|\partial_z^\alpha a(z)| \leqslant C_\alpha \langle z \rangle^{\mu_\alpha}.$$

Suppose further that for some sequences $\{\ell_j\}$ and $\{C_j\}$ $j=1,2,3\ldots$ such that $\ell_j \to -\infty$ as $j \to +\infty$, the following estimates hold

(2) $$\left| a(z) - \sum_{j=1}^{r-1} a_j(z) \right| \leqslant C_r \langle z \rangle^{\ell_r}, \text{ then}$$

$$a(z) \sim \sum_{j=1}^{\infty} a_j(z) .$$

To prove this theorem we need two lemmas.

We formulate the first Lemma for $t \in [-1,1]$. The proof for any interval $[a,b]$ is the same.

Lemma 1.3.4 Let $f \in C^2(\mathbb{R})$ and $A_j = \sup_{t \in [-1,1]} |f^{(j)}(t)|$ for $j = 0,2$. Then $|f'(0)|^2 \leqslant A_0(A_0 + A_2)$.

Proof By the Mean-Value Theorem we have

$$(3) \quad ||f'(t)| - |f'(0)|| \leq |f'(t) - f'(0)| \leqslant A_2 |t| , \text{ for } |t| \leq 1 .$$

Now we shall show that

$$(4) \quad |f'(t)| \geqslant ½ |f'(0)| \text{ for } A_2 |t| \leqslant ½ |f'(0)| , |t| \leqslant 1 .$$

From (3) $||f'(t)| - |f'(0)|| \leqslant A_2 |t|$ or

$$|f'(0)| - A_2 |t| \leqslant |f'(t)| \leqslant |f'(0)| + A_2 |t| .$$

So if $A_2 |t| \leqslant ½ |f'(0)|$ we get

$½ |f'(0)| = |f'(0)| - ½ |f'(0)| \leqslant |f'(0)| - A_2 |t| \leqslant |f'(t)|$ which proves (4).

Let $\delta = \min \{|f'(0)|/2A_2 , 1\}$, then by (4),

$$(5) \quad |f'(t)| \geqslant ½ |f'(0)| \text{ for } t \in [-\delta, \delta].$$

Mean Value Theorem yields

$$(6) \quad |f(\delta) - f(-\delta)| = 2\delta |f'(\zeta)| \geqslant 2\delta |f'(0)|/2 , \zeta \in (-\delta, \delta) , \text{by } (5).$$

Also $|f(\delta) - f(-\delta)| \leqslant 2 A_0$. Therefore

$$\delta |f'(0)| \leqslant |f(\delta) - f(-\delta)| \leqslant 2A_0$$

i.e. $|f'(0)| \leqslant 2A_0/\delta$.

By definition of δ we have

$$|f'(0)| \leqslant 2A_0 \max \{2A_2/|f'(0)|, 1\} .$$

This inequality implies

$|f'(0)| \leq 4A_0A_2/|f'(0)|$ or $|f'(0)| \leq 2A_0$

which means $|f'(0)|^2 \leqslant 4 A_0A_2$ or $|f'(0)|^2 \leq 4A_0^2$

These two inequalities yield

$$(5) \quad |f'(0)|^2 \leqslant 4A_0 (A_0 + A_2) \text{ as required.}$$

Remark: Replacing $f(t)$ by $f_h(t) = f(t+h)$ in the Lemma above, we get $|f'(h)| \leqslant 4A_0(A_0 + A_2)$, for any h in $[-1,1]$.

Second Lemma:

Lemma 1.3.5. Let K_1 and K_2 be two compact set in $\mathbb{R}^n$ such that $K_1 \subset \operatorname{Int} K_2$ (Int = interior) and $\operatorname{dist}(K_1, \partial K_2) > 1$ (dist = distance and ∂K_2 = boundary of K_2). Then for any function $f \in C^2(\mathbb{R}^n)$ we have.

$$(6)\quad \Big(\sup_{z\in K_1} \sum_{|\alpha|=1} |D^\alpha f(z)| \Big)^2 \leq 4n^2 \sup_{z\in K_2} |f(z)| \Big(\sup_{z\in K_2} |f(z)| + \sup_{z\in K_2} \sum_{|\alpha|=2} |D^\alpha f(z)| \Big)$$

Proof: Put $\varphi_z(t) = f(z+te_i)$, $i = 1, \dots, n$ and $e_i = (0,\dots,0,\underset{\substack{\uparrow \\ \text{ith position}}}{1},0,\dots,0)$

Applying Lemma 1.3.4 we get, (we assume that φ_z is defined for $|t| < 1$)

$$|\varphi_z'(0)|^2 < 4 \sup_{|t|\leq 1} |\varphi_z(t)| \Big(\sup_{|t|\leq 1} |\varphi_z(t)| + \sup_{|t|\leq 1} |\varphi_z''(t)|\Big)$$

Observe that $\varphi_z'(t) = \dfrac{\partial t}{\partial z_i}(z+te_i)$ and $\varphi_z''(t) = \dfrac{\partial^2 f}{\partial z_i^2}(z+te_i)$.

From the assumption $\operatorname{dist}(K_1, \partial K_2) \geq 1$, we infer that

$\varphi_z(t)$ is C^2 for $|t| \leqslant 1$. Thus

$$(7)\quad \Big|\frac{\partial f}{\partial z_i}(z)\Big|^2 \leqslant 4 \sup_{|t|\leq 1} |f(z+te_i)| \Big(\sup_{|t|\leq 1} |f(z+te_i)| + \sup_{|t|\leq 1} \Big|\frac{\partial^2 f}{\partial z_i^2}(z+te_i)\Big| \Big)$$

and thus

$$(8)\quad \sum_{i=1}^{n} \Big|\frac{\partial f}{\partial z_i}(z)\Big|^2 \leq 4n \sup_{\substack{|t|\leq 1 \\ i=1,\dots,n}} |f(z+te_i)| \Big(\sup_{\substack{|t|\leq 1 \\ i=1,\dots,n}} |f(z+te_i)| + \sup_{\substack{|t|\leq 1 \\ i=1,\dots,n}} \Big|\frac{\partial^2 f}{\partial z_i^2}(z+te_i)\Big| \Big)$$

We can take sup over $z \in K_1$ and K_2 to get

$\sup_{z\in K_1} \big(\sup_{|t|\leq 1} |f(z+te_i)|\big) \leqslant \sup_{z\in K_2} |f(z)|$; $i = 1,\dots,n$ and

$\sup_{z\in K_1} \Big(\sup_{|t|\leq 1} \dfrac{\partial^2 f}{\partial z_i^2}(z+te_i)\Big|\Big) \leqslant \sup_{z\in K_2} \Big|\dfrac{\partial^2 f}{\partial z_i^2}(z)\Big|$; $i = 1,\dots,n$.

Hence,

$$(9) \quad \sum_{i=1}^{n} \left|\frac{\partial f}{\partial z_i}(z)\right|^2 \leqslant 4n \sup_{z\in K_2} |f(z)| \left\{ \sup_{z\in K_2} |f(z)| + \sum_{i,j=1}^{n} \sup_{z\in K_2} \left|\frac{\partial^2 f}{\partial z_i \partial z_j}(z)\right| \right\} .$$

Now using the inequality

$$\left(\sup_{z\in K_1} \sum_{i=1}^{n} \left|\frac{\partial f}{\partial z_i}(z)\right| \right)^2 \leqslant n \sup_{z\in K_1} \left(\sum_{i=1}^{n} \left|\frac{\partial f}{\partial z_i}(z)\right|^2 \right)$$

we get (6). This completes the proof.

Proof of Theorem 1.3.3 Since $\ell_j \to -\infty$ as $j \to +\infty$ by Theorem 1.3.2 there exists $b(z) \in C^\infty(\mathbb{R}^{2n})$ such that

$$b(z) \sim \sum_{j=1}^{\infty} a_j(z) .$$

Let

$$\begin{aligned} d(z) &= a(z) - b(z) \\ &= a(z) - \sum_{j=1}^{r} a_j(z) - \left(b(z) + \sum_{j=1}^{r} a_j(z)\right) . \end{aligned}$$

Hence $|d(z)| \leqslant \left|a(z) - \sum_{j=1}^{r} a_j(z)\right| + \left|b(z) - \sum_{j=1}^{r} a_j(z)\right| .$

Using the asymptotic form of $b(z)$ and (2) of the Theorem we have

$$|d(z)| \leqslant C_r \langle z\rangle^{\ell_r} , \text{ for } z \in \mathbb{R}^{2n} \text{ and any } r.$$

We need to show that the same estimates also hold for the derivatives of $d(z)$. Now put $K_1 = \{ z : n \leqslant \|z\| \leqslant n+1 \}$ and $K_2 = \{ z ; n-\varepsilon \leqslant \|z\| \leqslant n+1-\varepsilon \}$, $0<\varepsilon<1$ .. Applying Lemma 1.3.5. to K_1 and K_2 we obtain, (the condition $\operatorname{dist}(K_1, \partial K_2) > 1$ is not satisfied by the sets defined here. The condition is not, however, necessary for an inequality of the type of equation (6), but the constant would change)

$$\left(\sup_{z\in K_1} \sum_{|\alpha|=1} |\partial_z^\alpha d(z)|\right)^2 < 4n^2 \sup_{z\in K_2} |d(z)| \left(\sup_{z\in K_2} |d(z)| + \sup_{z\in K_2} \sum_{|\alpha|=2} |D_z^\alpha d(z)|\right)$$

Hence for any ℓ_r there is C_r such that

$$\Big(\sup_{z\in K_1}\sum_{|\alpha|=1}|\partial_z^\alpha d(z)|\Big)^2 \lesssim C_r \sup_{z\in K_2}\langle z\rangle^{\ell_r}\Big(C_r\sup_{z\in K_2}\langle z\rangle^{\ell_r}+\sup_{z\in K_2}\sum_{|\alpha|=2}C_\alpha\langle z\rangle^{\mu_\alpha}\Big)$$

If $\ell_r<0$ (which is not restrictive since $\ell_r\to-\infty$, as $r\to\infty$) then for some $\tilde{C}_r$

$$\begin{aligned}\Big(\sup_{z\in K_1}\sum_{|\alpha|=1}|\partial_z^\alpha d(z)|\Big)^2 &\le \tilde{C}_r \sup_{z\in K_2}\langle z\rangle^{\ell_r}\sup_{z\in K_2}\langle z\rangle^{\mu}\\ &= \tilde{C}_r\,[1+(n-\varepsilon)^2]^{\ell_r/2}\,[1+(n+1-\varepsilon)^2]^{\mu/2},\end{aligned}$$

where $\mu=\max_{|\alpha|=2}|\mu_\alpha|$.

Clearly $[1+(n+1-\varepsilon)^2]^{\mu/2}\le M[1+(n-\varepsilon)^2]^{\mu/2}$, where M is independent of n.

Thus we get

$$\tilde{C}_r[1+(n-\varepsilon)^2]^{\ell_r/2}\,[1+(n+1-\varepsilon)^2]^{\mu/2}\le\tilde{C}_r M\,[1+(n-\varepsilon)^2]^{\ell_r/2+\mu/2}.$$

Since $\ell_r\to-\infty$ as $r\to+\infty$ we have $\ell_r+\mu<0$ for sufficiently large r. We observe that for $\ell<0$ $\sup_{w\in K_2}\langle w\rangle^{\ell}=[1+(n-\varepsilon)^2]^{\ell/2}\le Q\,\langle z\rangle^{\ell}$ for all $z\in K_2$,

where Q is independent of n. Thus we get

$$\tilde{C}_r\,[1+(n-\varepsilon)^2]^{\ell_r/2+\mu/2}\le\tilde{C}_r M\,Q\,\langle z\rangle^{\ell_r+\mu}$$ and thus

$$\Big(\sup_{z\in K_1}\sum_{|\alpha|=1}|\partial_z^\alpha d(z)|\Big)^2\le\tilde{C}_r M\,Q\,\langle z\rangle^{\ell_r+\mu}\qquad\text{for all } z\in\mathbb{R}^{2n}.$$

This proves that $\partial_z^\alpha d(z)$ are rapidly decreasing at infinity for $|\alpha|=1$. Similarly we can prove that $\partial_z^\alpha d(z)$ decreases rapidly at infinity for any α. This completes the proof.

1.4 $G^m_\rho(\mathbb{R}^n)$ Pseudo-Differential Operators

We shall now define a class of pseudo-differential operators corresponding to the $\Gamma^m_\rho(\mathbb{R}^{2n})$ classes.

Definition 1.4.1 Let $a(x,\xi)\in\Gamma^m_\rho(\mathbb{R}^{2n})$. We define the corresponding pseudo-differential operator $A\in G^m_\rho(\mathbb{R}^n)$ as follows.

(a) $$Au(x)=(2\pi)^{-n}\iint e^{i(x-y)\xi}\,a(x,\xi)\,u(y)\,dy\,d\xi$$

where $u\in S(\mathbb{R}^n)$ or equivalently

(b) $Au(x) = (2\pi)^{-n} \int e^{ix\xi}\, a(x, \xi)\, \hat{u}(\xi) d\xi$

where $u \in S(\mathbb{R}^n)$ and $\hat{u}(\xi)$ is the Fourier Transform of u. The formula (a) must be treated as an iterated integral and not as a double integral (see Appendix C).

The functional spaces $S(\mathbb{R}^n)$ (the Schwartz space) and its dual $S'(\mathbb{R}^n)$ (the space of tempered distributions) are important.
We therefore need to know the action of the pseudo-differential operators on these spaces. It turns out that the operators from $G^m_\rho(\mathbb{R}^n)$ define continuous maps on $S(\mathbb{R}^n)$ and $S'(\mathbb{R}^n)$. These are the important theorems in this section.

Before we formulate these theorems, we briefly discuss the concepts of continuity and the principle of duality.

The space $S(\mathbb{R}^n)$ is equipped with semi-norms $\{q_{\alpha\beta}\}$, where ;
(1)... $q_{\alpha,\beta}(u) = \sup\{ <x>^{|\alpha|} |\partial^\beta u(x)| : x \in \mathbb{R}^n \} < \infty$; $\alpha, \beta \in Z^n_+$ and $u \in S(\mathbb{R}^n)$.

We say a linear map $f: S(\mathbb{R}^n) \to S(\mathbb{R}^n)$
is continuous if (as $n \to \infty$)

$q_{\alpha,\beta}(u_n) \to 0$ implies $q_{\alpha,\beta}[f(u_n)] \to 0$ for any sequence $u_n \to 0$ in $S(\mathbb{R}^n)$.

A dual map $A': S'(\mathbb{R}^n) \to S'(\mathbb{R}^n)$ we shall define as follows;
(2)...$A'f(u) = f(Au)$ for $f \in S'(\mathbb{R}^n)$ and $u \in S(\mathbb{R}^n)$.
We shall prove that $A \in G^m_\rho(\mathbb{R}^n)$ maps $S(\mathbb{R}^n)$ into itself in a continuous way, and thus A'f is a continuous functional on $S(\mathbb{R}^n)$.
We shall adopt a pointwise topology on $S'(\mathbb{R}^n)$ i.e. $f_n \to f$ in $S'(\mathbb{R}^n)$ if and only if $f_n(u) \to f(u)$ as $n \to \infty$ for every $u \in S(\mathbb{R}^n)$. Clearly the dual map $A': S'(\mathbb{R}^n) \to S'(\mathbb{R}^n)$ is continuous, if $f_n \to f$ in $S'(\mathbb{R}^n)$ then $A'f_n \to A'f$ i.e.

$A'f_n(u) = f_n(Au) \to f(Au) = A'f(u)$ for every $u \in S(\mathbb{R}^n)$ in view of (2).

Therefore we conclude that a continuous map on $S(\mathbb{R}^n)$ induces a sequentially continuous map on its dual space $S'(\mathbb{R}^n)$. This is the principle of duality restricted to $S(\mathbb{R}^n)$ and $S'(\mathbb{R}^n)$.

Theorem 1.4.2. The operator $A \in G^m_\rho(\mathbb{R}^n)$ defines a continuous operator $A : S(\mathbb{R}^n) \to S(\mathbb{R}^n)$.

On $S'(\mathbb{R}^n)$ we have a similar result;

Theorem 1.4.3. The operator $A \in G^m_\rho(R^n)$ defines a continuous map $A : S'(R^n) \to S'(\mathbb{R}^n)$.

We will prove these theorems for the $\Pi G^m_\rho(\mathbb{R}^n)$ operators because this class of pseudo-differential operators is more general than the $G^m_\rho(\mathbb{R}^n)$ class of operators.

1.5 $\Pi^m_\rho(\mathbb{R}^{3n})$ Classes.

We shall discuss a class of symbols $\Pi^m_\rho(\mathbb{R}^{3n})$ which is more general than the $\Gamma^m_\rho(\mathbb{R}^{2n})$.We shall prove that every member of the $\Pi^m_\rho(\mathbb{R}^{3n})$ has an asymptotic expansion in the $\Gamma^m_\rho(\mathbb{R}^{2n})$ class of symbols.

Definition 1.5.1 $[\Pi^m_\rho(\mathbb{R}^{3n})]$. We shall denote by $\Pi^m_\rho(\mathbb{R}^{3n})$ the set of all $C^\infty(\mathbb{R}^{3n})$ functions $a(x,y,\xi)$ such that for some $m \in \mathbb{R}$ and every multi-indices α, β and γ, there is a positive constant $C_{\alpha\beta\gamma}$ for which the following estimates hold:

$$|D^\alpha_\xi \; D^\beta_x \; D^\gamma_y \; a(x,y\,\xi)| \leqslant C_{\alpha\beta\gamma} <(x,y,\xi)>^{m-\rho|\alpha+\beta+\gamma|} <x-y>^{m'+\rho|\alpha+\beta+\gamma|}$$

for some suitable $m' \in \mathbb{R}$.

Example 1.5.1. If $a(x,y,\xi) = \sum_{|\alpha|\leqslant p} a_\alpha(x,y)\,\xi^\alpha$ where $a_\alpha(x,y)$ are polynomials in x and y variables of degree $\leqslant k$, then $a(x,y\,\xi) \in \Pi^m_1(\mathbb{R}^{3n})$ where $m = p+k$. To see this, observe that

$$\begin{aligned} |a(x,y,\xi)| &\leqslant \sum_{|\alpha|\leqslant p} |a_\alpha(x,y)| \langle\xi\rangle^{|\alpha|} \\ &\leqslant \langle\xi\rangle^{p} \sum_{|\alpha|\leqslant p} |a_\alpha(x,y)| \\ &\leqslant \langle\xi\rangle^{p}\, C_{k,p} \langle(x,y)\rangle^{k} \\ &\leqslant C'_{k,p} \langle(x,y,\xi)\rangle^{m} \end{aligned}$$

where $m = p+k, C_{k,p}$ and $C'_{k,p}$ are some appropriate positive constants. Similar estimates also hold for the partial derivatives of $a(x,y,\xi)$.

1.6 Properties of $\Pi_\rho^m(\mathbb{R}^{3n})$

Lemma 1.6.1

If $a \in \Pi_\rho^{m_1}(\mathbb{R}^{3n})$ and $b \in \Pi_\rho^{m_2}(\mathbb{R}^{3n})$ then

1. $ab \in \Pi_\rho^{m_1+m_2}(\mathbb{R}^{3n})$.
2. $\partial_\xi^\alpha \partial_x^\beta \partial_y^\gamma a \in \Pi_\rho^{m_1-\rho|\alpha+\beta+\gamma|}(\mathbb{R}^{3n})$
3. $a + b \in \Pi_\rho^{\max(m_1, m_2)}(\mathbb{R}^{3n})$
4. $a(x,x,\xi) \in \Gamma_\rho^m(\mathbb{R}^{2n})$.

The properties 1, 2 and 3 are analoguous to those of the Γ_ρ^m - classes in Lemma 1.2.1. Thus we shall give the proof of 4 only.

Proof of 4: From the definition of $\Pi_\rho^m(\mathbb{R}^{3n})$ with $\alpha = \beta = \gamma = 0$ we have

$$|a(x,x,\xi)| \leqslant C \langle(x,x,\xi)\rangle^{m} \text{ when } x = y.$$

Clearly $\langle(x,x,\xi)\rangle \leqslant 2\langle(x,\xi)\rangle$.

Thus

$$\langle(x,x,\xi)\rangle^{m} \leqslant 2^m \langle(x,\xi)\rangle^m \quad \text{for } m > 0$$

and $\langle(x,x,\xi)\rangle^{m} \leqslant \langle(x,\xi)\rangle^m$ for $m \leqslant 0$.

Hence we conclude that

$| a(x,x\ \xi) | \leqslant C <(x,\ \xi)>^{m}$ where C is some suitable positive constant.

For the partial derivatives we have

$|D^{\alpha}_{\xi}\ D^{\beta}_{x}\ a(x,x,\ \xi)| \leqslant C_{\alpha\beta}\ <(x,x,\ \xi)>^{m-\rho|\alpha+\beta|}$ and

we argue as above. This proves our assertion.

The pseudo-differential operator corresponding to the $\Pi^{m}_{\rho}(\mathbb{R}^{3n})$ symbol is defined just in the same way as for the $\Gamma^{m}_{\rho}(\mathbb{R}^{2n})$.

Definiiton 1.6.2 [$\Pi G^{m}_{\rho}(\mathbb{R}^{n})$] We say $A \in \Pi G^{m}_{\rho}(\mathbb{R}^{n})$ if and only if A has a symbol $a(x,y,\xi) \in \Pi^{m}_{\rho}(\mathbb{R}^{3n})$ so that

$$(1) \qquad Au(x) = (2\pi)^{-n} \int\int e^{i(x-y)\xi}\ a(x,y,\xi)\ u(y)\ dy\ d\xi$$

for every $u \in S(\mathbb{R}^{n})$, where (1) is an iterated integral.

Using the identities $<x-y>^{-N} <D_{\xi}>^{N} e^{i(x-y)\xi} = e^{i(x-y)\xi}$ and $<\xi>^{-M} <D_{y}>^{M} e^{i(x-y)\xi} = e^{i(x-y)\xi}$, where M and N are even positive integers.

Carrying out integration by parts on y and subsequently on ξ to get the new iterated integral

$$(2) \qquad Au(x) = (2\pi)^{-n} \int\int <x-y>^{-N}\ e^{i(x-y)\xi} < D_{\xi}>^{N} < D_{y}>^{M} \times [<\xi>^{-M} a(x,y,\xi)\ u(y)]\ dy\ d\xi\ .$$

For details see Appendix C.

Theorem 1.6.3 The operator $A \in \Pi G^{m}_{\rho}(\mathbb{R}^{n})$ defines a continuous operator $A: S(\mathbb{R}^{n}) \to S(\mathbb{R}^{n})$.

Proof: First we shall show that A transforms $S(\mathbb{R}^n)$ into itself. The form (2) of Definition 1.6.2. will be handy in this case.

Using Peetre's inequality (lemma 1.1 p6 of [27]) one has $\langle x-y\rangle^{-N} \leq 2^N\langle x\rangle^{-N}\langle y\rangle^{N}$, for any positive integer N. We claim that with sufficiently large M and N, (2) is absolutely integrable on the pair y,ξ.

Now,

$$|Au(x)| \leq (2\pi)^{-n}2^N \iint \langle x\rangle^{-N}\langle y\rangle^{N}\,|\langle D_\xi\rangle^N\langle D_y\rangle^M[\langle\xi\rangle^{-M}a(x,y,\xi)u(y)]|\,dy\,d\xi .$$

Applying Leibnitz Formula to the differential operators $\langle D_\xi\rangle^N$ and $\langle D_y\rangle^M$ we get;

$$(3)\ldots\ \langle D_\xi\rangle^N\langle D_y\rangle^M[\langle\xi\rangle^{-M}a(x,y,\xi)u(y)] = \sum_{\substack{|\alpha+\gamma|\leq N\\ |\beta+\rho|\leq M}} C_{\alpha\beta\gamma\rho}\,D_\xi^\gamma\langle\xi\rangle^{-M}\,D_\xi^\alpha D_y^\beta a(x,y,\xi)\,D_y^\rho u(y)$$

where $C_{\alpha\beta\gamma\rho}$ are some appropriate constants. Thus,

$$|Au(x)| \leq \langle x\rangle^{-N}2^N(2\pi)^{-n}\iint \sum_{\substack{|\alpha+\gamma|\leq N\\ |\beta+\rho|\leq M}} C_{\alpha\beta\gamma\rho}|D_\xi^\gamma\langle\xi\rangle^{-M}|\,|D_\xi^\alpha D_y^\beta a(x,y,\xi)|\,|D_y^\rho u(y)|\,dy\,d\xi .$$

By the $\pi_\rho^m(\mathbb{R}^{3n})$ estimates and Peetre's inequality,

$$\begin{aligned}|D_\xi^\alpha D_y^\beta a(x,y,\xi)| &\leq C_{\alpha\beta}\,\langle x,y,\xi\rangle^{m-\rho|\alpha+\beta|}\langle x-y\rangle^{m'-\rho|\alpha+\beta|}\\ &\leq C_{\alpha\beta}\,\langle x,y,\xi\rangle^{m}\langle x-y\rangle^{m'}\\ &\leq 2^{m'}C_{\alpha\beta}\langle x,y,\xi\rangle^{m}\langle x\rangle^{-m'}\langle y\rangle^{m'}\end{aligned}$$

for some positive constant $C_{\alpha\beta}$ and $m'>0$. Hence

$$|Au(x)| \leq 2^{m'}\langle x\rangle^{-m'-N}(2\pi)^{-n}\iint \sum_{\substack{|\alpha+\gamma|\leq N\\ |\beta+\rho|\leq M}} C_{\alpha\beta\gamma\rho}C_{\alpha\beta}\langle\xi\rangle^{-M-|\gamma|}\langle x,y,\xi\rangle^{m}\langle y\rangle^{N+m'}|D_y^\rho u(y)|\,dy\,d\xi .$$

Observe that

$$(4)\ \ldots\ \langle x,y,\xi\rangle^{m} \leq \begin{cases}\langle x\rangle^{m}\langle y\rangle^{m}\langle\xi\rangle^{m} & \text{if } m>0\\ 1 & \text{if } m\leq 0 .\end{cases}$$

For $m>0$ we get

$$|Au(x)| \leq 2^{m'}\langle x\rangle^{m-m'-N}(2\pi)^{-n}\iint \sum_{\substack{|\alpha+\gamma|\leq N\\ |\beta+\rho|\leq M}} C_{\alpha\beta\gamma\rho}C_{\alpha\beta}\langle\xi\rangle^{-M-|\gamma|+m}\langle y\rangle^{N+m'+m}|D_y^\rho u(y)|\,dy\,d\xi .$$

The case $m \leqslant 0$ yields a similar result. Thus,

$$(5)\ldots \quad |Au(x)| \leqslant C_{MN} \langle x \rangle^{m'+m-N},$$

where,

$$C_{MN} = (2\pi)^{-n} 2^{m'} \iint \sum_{\substack{|\alpha+\gamma| \leq N \\ |\beta+\rho| \leq M}} C_{\alpha\beta\gamma\rho} C_{\alpha\beta} \langle \xi \rangle^{-M-|\gamma|+m} \langle y \rangle^{N+m'+m} |D_y^{\rho} u(y)| \, dy d\xi .$$

Integration with respect to the ξ-variable yields a finite value provided that the exponent $-M-|\gamma|+m < -n$. This is indeed the case since M can be chosen arbitrarily large to satisfy this requirement. Clearly $C_{MN} < \infty$, and thus (2) is absolutely integrable as claimed.

The number $m'+m-N$ can be made negative by a suitable choice of the even positive integer N and thus making $Au(x)$ to decrease rapidly at infinity. This state of affairs is valid for the partial derivatives of $Au(x)$ as well. Therefore we conclude that $Au \in S(\mathbb{R}^n)$.

In the final analysis we prove that the map $A : S(\mathbb{R}^n) \to S(\mathbb{R}^n)$ is continuous. Using (5) we have,

$\langle x \rangle^k |Au(x)| \leqslant C_{MN} \langle x \rangle^{m'+m-N+k}$ for $k > 0$. Hence

Hence,

$$(6)\ldots \quad \langle x \rangle^k |Au(x)| \leqslant C_{MN},$$

because $\langle x \rangle^{m'+m-N+k} \leqslant 1$ for the choice $m'+m-N+k \leqslant 0$.

The estimates in (6) hold for partial derivatives of $Au(x)$ also.

To be explicit the relation (6) becomes,

$$\langle x \rangle^k |Au(x)| \leqslant (2\pi)^{-n} \iint 2^N \sum_{\substack{|\alpha+\gamma| \leq N \\ |\beta+\rho| \leq M}} C_{\alpha\beta\gamma\rho} C_{\alpha\beta} \langle \xi \rangle^{-M-|\gamma|+m} \langle y \rangle^{N+m'+m} |D_y^{\rho} u(y)| \, dy d\xi$$

$$\leqslant (2\pi)^{-n} 2^N \Big\{ \sum_{\substack{|\alpha+\gamma| \leq N \\ |\beta+\rho| \leq M}} C_{\alpha\beta\gamma\rho} C_{\alpha\beta} \sup \left[\langle y \rangle^{2N+m'+m} |D_y^{\rho} u(y)| \right] \Big\} \iint \langle \xi \rangle^{m-M} \langle y \rangle^{-N} \, dy d\xi .$$

The integral

$$\iint \langle\xi\rangle^{-M+m} \langle y\rangle^{-N} dy d\xi < \infty ,$$

for M-m and $N > n$.

Thus

$$(7) \ldots \langle x\rangle^{k} |Au(x)| \leqslant C \sum_{|\rho| \leqslant M} q_{2N+m+m',\rho}(u) .$$

The estimates (7) hold for any $k > 0$ and for the partial derivatives of $Au(x)$.

Now take a sequence $\{u_n\}$ in $S(\mathbb{R}^n)$ such that $u_n \to 0$ as $n \to \infty$. Then $q_{K,0}(Au_n) \leqslant C \sum_{|\rho| \leqslant M} q_{2N+m+m',\rho}(u_n) \to 0$ as $n \to \infty$ by (7). Similarly for $q_{k,\alpha}(Au_n)$. This proves the continuity of the operator A and the theorem.

By duality we obtain:

<u>Theorem 1.6.4.</u> The operator $A \in \pi G^{m}_{\rho}(\mathbb{R}^n)$ can be extended to a continuous operator $A : S'(\mathbb{R}^n) \to S'(\mathbb{R}^n)$.

<u>Remark</u>: We know that $S(\mathbb{R}^n) \subset S'(\mathbb{R}^n)$ if we identify $u \in S(\mathbb{R}^n)$ with the $v \to (u,v) = \int u(x)v(x)dx$. Suppose $A' : S'(R^n) \to S'(\mathbb{R}^n)$ be the dual map to the map $A : S(\mathbb{R}^n) \to S(\mathbb{R}^n)$ .Then,

$$A'u(v) = u(Av) = \int u(x)\,(Au)(x)\,dx$$

$$= (2\pi)^{-n} \int u(x)\Big[\iint e^{i(x-y)\xi}\, a(x,y,\xi)v(y)dyd\xi\Big]\,dx$$

$$= \int v(y)\Big[(2\pi)^{-n} \iint e^{i(x-y)\xi}\, a(x,y,\xi)u(x)dxd\xi\Big]\,dy.$$

The above change of the order of integration is possible since we are dealing with functions from $S(\mathbb{R}^n)$.

Therefore A' acts on $S(\mathbb{R}^n)$ as a transposed operator

$$ {}^tAu(y) = (2\pi)^{-n} \iint e^{i(y-x)\xi} a(x,y,-\xi)\, u(x)\,dx\,d\xi . $$

We shall define the transposed operator accordingly in Chapter 2. Clearly ${}^tA \in \pi G_\rho^m(\mathbb{R}^n)$ since $a(x,y,-\xi) \in \pi_\rho^m(\mathbb{R}^{3n})$. Therefore the map ${}^tA : S(\mathbb{R}^n) \to S(\mathbb{R}^n)$ is continuous in view of Theorem 1.6.3. Hence A has an extension $({}^tA)'$ which is a continuous map of $S' \to S'$.

Now we shall define a class of operators which is the intersection of all $\pi G_\rho^m(\mathbb{R}^n)$ classes. We denote this class of operators by $G^{-\infty}(\mathbb{R}^n)$. The operators from $G^{-\infty}(\mathbb{R}^n)$ are characterized by kernels from $S(\mathbb{R}^{2n})$ i.e. $A \in G^{-\infty}(\mathbb{R}^n)$ if and only if the kernel $K_A(x,y) = (2\pi)^{-n} \int e^{i(x-y)\xi}\, a(x,y,\xi)\,d\xi$ of A belong to $S(\mathbb{R}^{2n})$. The following result is useful:

<u>Theorem 1.6.5.</u> If $A \in G^{-\infty}(\mathbb{R}^n)$, then $A : S'(\mathbb{R}^n) \to S(\mathbb{R}^n)$ is continuous.

<u>Proof</u>: We shall use the results of Theorem 1.6.6 which we shall present later. The results are that the amplitude $a(x,y,\xi) \in \pi_\rho^m(\mathbb{R}^{3n})$ can be replaced by $b(x,\xi) \in S(\mathbb{R}^{2n})$. Hence consider

$$ Au(x) = (2\pi)^{-n} \iint e^{i(x-y)\xi}\, b(x,\xi)\, u(y)\,dy\,d\xi . $$

As we observed, the dual operator $A' : S'(\mathbb{R}^n) \to S'(\mathbb{R}^n)$ is an extension of the transposed operator ${}^tA$. Similarly, we can regard the dual operator of ${}^tA$ as being an extension of A. As in Theorem 1.6.4. denote this extension by $\hat{A} : S'(\mathbb{R}^n) \to S'(\mathbb{R}^n)$, $\hat{A}u = Au$ for $u \in S(\mathbb{R}^n)$. We need to show that $\hat{A}$ maps $S'(\mathbb{R}^n)$ into $S(\mathbb{R}^n)$ continuously.

This problem arises in the case of Hormander $S^m_{\rho\delta}(\mathbb{R}^{3n})$ classes and therefore we treat it briefly.

If $a(x, \xi) \in S(\mathbb{R}^{2n})$ then one can integrate on ξ to get

$K(x,y) = (2\pi)^{-n} \int e^{i(x-y)\xi} \, a(x,\xi) d\xi$

which belongs to $S(\mathbb{R}^{2n})$ on x and y.

Now one can extend A to $u \in S'(\mathbb{R}^n)$ by

$Au(x) = u_y \, (K(x,y))$

i.e. the tempered distribution u acts on y variable.

From Corollary [1] page 180 every $u \in S'(\mathbb{R}^n)$ has the form

$u = \partial^\alpha [\, (1+ ||x||^2)^{k/2} f(x)]$

for some α and k where f is a bounded continuous function on $\mathbb{R}^n$.

Thus,

$Au(x) = \int (1+ ||y||^2)^{k/2} f(y) \, \partial^\alpha_y \, K(x,y) dy$

and clearly $Au(x) \in S(\mathbb{R}^n)$.

For the complete proof of the continuity we refer to [25] paragraph 2.

The following theorem tells us that every symbol from the $\Pi^m_\rho(\mathbb{R}^{3n})$ classes has an asymptotic series in the $\Gamma^m_\rho(\mathbb{R}^{2n})$ classes. We have applied this result in the proof of Theorem 1.6.5. Furthermore, with the results of this theorem we shall define the $\mathcal{T}$-symbols and also deduce their asymptotic expansions. The latter are important in the next chapter.

Theorem 1.6.6. Let $A \in \pi G^m_\rho(\mathbb{R}^n)$ in other words

$$(0) \qquad A\,u(x) = (2\pi)^{-n} \iint e^{i(x-y)\xi}\; a(x,y\,\xi)\; u(y)\; dy\; d\xi$$

for $u \in C_0^\infty(\mathbb{R}^n)$ and $a(x,y,\xi) \in \pi^m_\rho(\mathbb{R}^{3n})$. Then for each $\mathcal{T} \in [0,1]$,

$$(1) \qquad Au(x) = (2\pi)^{-n} \iint e^{i(x-y)\xi}\; b_{\mathcal{T}}\,((1-\mathcal{T})x + \mathcal{T}y, \xi)\; u(y)\; dy\; d\xi$$

where

$$(2) \qquad b_{\mathcal{T}}[(1-\mathcal{T})x + \mathcal{T}y, \xi] \in \Gamma^m_\rho(\mathbb{R}^{3n}) \text{ and}$$

$$(3) \qquad b_{\mathcal{T}}(x,\xi) \sim \sum_{\beta\gamma} \frac{1}{\beta!\,\gamma!}\, \mathcal{T}^{|\beta|}\,(1-\mathcal{T})^{|\gamma|}\, \partial_\xi^{\beta+\gamma}\, (-D_x)^\beta\, D_y^\gamma\, a(x,y,\xi)\Big|_{y=x}$$

Before we prove this theorem, we give a definition.

Definition 1.6.7. The function $b_{\mathcal{T}}(x,\xi)$ in (3) is called the $\mathcal{T}$-symbol of the operator A. When $\mathcal{T}=0$, 1 and ½, we call the corresponding symbols

$b_0(x,\xi) = \sigma_{A,\ell}(x,\xi)$: the left symbol

$b_1(x,\xi) = \sigma_{A,r}(x,\xi)$: the right symbol

and

$b_{1/2}(x,\xi) = \sigma_{A,w}(x,\xi)$: the Weyl symbol of the operator A

(respectively). These symbols have the following asymptotic expansions (obtained from (3) by inserting $\mathcal{T} = 0, 1$ and ½)

$$\sigma_{A,\ell}(x,\xi) \sim \sum_\alpha \frac{1}{\alpha!}\, \partial_\xi^\alpha\, D_y^\alpha\, a(x,y\,\xi)\Big|_{y=x}$$

$$\sigma_{A,r}(y,\xi) \sim \sum_\alpha \frac{1}{\alpha!}\, \partial_\xi^\alpha\, (-D_x)^\alpha\, a(x,y,\xi)\Big|_{x=y}$$

$$\sigma_{A,w}(x,\xi) \sim \sum_{\beta\gamma} \frac{1}{\beta!\,\gamma!}\, (½)^{|\beta+\gamma|}\, \partial_\xi^{\beta+\gamma}\, (-D_x)^\beta\, D_y^\gamma\, a(x,y\,\xi)\Big|_{y=x} .$$

Remark 1: Since $a(x,y,\xi) \in \Pi_{\rho}^{m}(\mathbb{R}^{3n})$ we have

$$|\partial_{\xi}^{\beta+\gamma}(-D_x)^{\beta} D_y^{\gamma} a(x,y,\xi)| \leqslant C_{\beta\gamma} <(x,y,\xi)>^{m-2\rho|\beta+\gamma|} \times <x-y>^{m'+2\rho|\beta+\gamma|} .$$

Thus for x=y we get

$$|\partial_{\xi}^{\beta+\gamma}(-D_x)^{\beta} D_y^{\gamma} a(x,y,\xi)| \Big|_{y=x} \leqslant C_{\beta\gamma}<(x,x,\xi)>^{m-2\rho|\beta+\gamma|}$$

or equivalently

$$\partial_{\xi}^{\beta+\gamma}(-D_x)^{\beta} D_y^{\gamma} a(x,y,\xi) \Big|_{y=x} \in \Gamma_{\rho}^{m-2\rho|\beta+\gamma|}(\mathbb{R}^{2n}).$$

Since $\rho>0$ it means that as $|\beta+\gamma| \to +\infty$, $m-2\rho|\beta+\gamma| \to -\infty$. So the condition on the asymptotic expansion is satisfied (as required by Definition 1.3.1). Note that the above estimates give the required estimate (1) of Theorem 1.3.3 which we shall use to justify that the series (3) is an asymptotic expansion of b_{τ}.

Proof of Theorem 1.6.6

Take $0 \leqslant \tau \leqslant 1$ and put

$$(S) \quad \begin{cases} v = (1-\tau)\,x + \tau y \\ w = x-y \end{cases} \quad \text{then} \quad \begin{aligned} x &= v + \tau w \\ y &= v-(1-\tau)\,w . \end{aligned}$$

Then

$$(4) \qquad a(x,y,\xi) = a(v+\tau w,\ v-(1-\tau)w,\ \xi) .$$

Applying the Taylor Expansion to (4) on w at w=0 we obtain:

$$(5) \qquad a(x,y,\xi) = \sum_{|\varkappa|\leqslant N-1} {}^{1}/_{\varkappa!}\ \partial_w^{\varkappa}\ a(v+\tau w,\ v-(1-\tau)w,\ \xi)\Big|_{w=0} w^{\varkappa} + R_N ,$$

where $R_N = \sum_{|\varkappa|=N} \frac{N\, w^{\varkappa}}{\varkappa!} \int_0^1 (1-t)^{N-1}\, \partial^{\varkappa}_{w}\, a(v+t\,\tau w,\ v-t(1-\tau)w,\ \xi)\, dt$

is the integral form of the remainder term, see [5].

Furthermore, by the chain rule one has

$$(6)\qquad \partial^{\varkappa}_{w}\, a(v+\tau w,\ v-(1-\tau)w,\ \xi)\Big|_{w=0}$$

$$= \sum_{\beta+\gamma=\varkappa} \frac{\varkappa!\ (-1)^{|\gamma|}}{\beta!\ \gamma!}\, \tau^{|\beta|}\, (1-\tau)^{|\gamma|}\, (\partial^{\beta}_{x}\, \partial^{\gamma}_{y}) a(v,v,\xi)\ ,$$

and from (5) and (6) we obtain

$$(7')\qquad a(x,y,\xi) = \sum_{|\varkappa|\leq N-1} \frac{1}{\varkappa!} \sum_{\beta+\gamma=\varkappa} \frac{\varkappa!\ (-1)^{|\gamma|}}{\beta!\ \gamma!}\, \tau^{|\beta|}\, (1-\tau)^{|\gamma|}\, (x-y)^{\beta+\gamma}$$

$$\times\ (\partial^{\beta}_{x}\ \partial^{\gamma}_{y})\ a(v,v,\xi) + R'_N$$

where

$$R'_N = \sum_{|\varkappa|=N} \frac{N}{\varkappa!} \sum_{\beta+\gamma=\varkappa} \varkappa!\ C_{\beta\gamma\tau}\, (x-y)^{\beta+\gamma} \int_0^1 (1-t)^{N-1}\, (\partial^{\beta}_{x}\ \partial^{\gamma}_{y})$$

$$a(v+t\tau w, v-t(1-\tau)w,\ \xi)\ dt\quad ,$$

where $C_{\beta\gamma\tau}$ are suitable constants . Simplifying (7') we get

$$(7)\qquad a(x,y,\xi) = \sum_{|\beta+\gamma|\leq N-1} \frac{(-1)^{|\gamma|}}{\beta!\,\gamma!}\, \tau^{|\beta|}\, (1-\tau)^{|\gamma|} (x-y)^{\beta+\gamma} (\partial^{\beta}_{x}\ \partial^{\gamma}_{y})\ a(v,v,\xi\ )$$

$$+ \sum_{|\beta+\gamma|=N} N\, C_{\beta\gamma\tau} (x-y)^{\beta+\gamma} \int_0^1 (1-t)^{N-1}\, (\partial^{\beta}_{x}\ \partial^{\gamma}_{y})\ a(v+t\tau w,\ v-t(1-\tau)w,\ \xi)dt\ .$$

Remark 2 : An operator with amplitude

$(x-y)^{\beta+\gamma}(\partial_x^\beta\,\partial_y^\gamma)\,a(v,v,\,\xi)$ is equal to an operator with the amplitude

(8) $(-D_\xi)^{\beta+\gamma}(\partial_x^\beta\,\partial_y^\gamma)\,a(v,v,\xi)$ or $(-1)^{|\beta+\gamma|}(\partial^{\beta+\gamma}\,D_x^\beta\,D_y^\gamma)\,a(v,v,\,\xi)$.

The formulas (8) follow from the following calculations.

Let $(\partial_x^\beta\,\partial_y^\gamma)\,a(v,v,\,\xi)=\sigma(x,y,\,\xi)$. Then for $u\in C_0^\infty(\mathbb{R}^n)$

Integrating by parts we get

(9) $$(2\pi)^{-n}\int\!\!\int e^{i(x-y)\xi}\,(x-y)^\mu\,\sigma(x,y,\,\xi)\,u(y)dy\,d\xi$$
$$=(2\pi)^{-n}\int\!\!\int e^{i(x-y)\xi}\,(-D_\xi)^\mu\,\sigma(x,y,\,\xi)\,u(y)\,dy\,d\xi\,.$$

Note that the integrals in (9) are not double integrals but iterated ones.

The validity of (9) for our class of functions follows from the regularisation process as in the Appendix B.

Returning to the v-variable we get from (9) ($\mu=\beta+\gamma$),

(10) $$(2\pi)^{-n}\int\!\!\int e^{i(x-y)\xi}\,(x-y)^{\beta+\gamma}(\partial_x^\beta\,\partial_y^\gamma)a(v,v,\,\xi)\,u(y)\,dy\,d\xi=$$
$$(2\pi)^{-n}\int\!\!\int e^{i(x-y)\xi}\,(-D_\xi)^{\beta+\gamma}(\partial_x^\beta\,\partial_y^\gamma)a(v,v,\,\xi)\,u(y)\,dy\,d\xi\,.$$

$$= (-1)^{|\beta+\gamma|} \; (\partial_\xi^{\beta+\gamma} \; i^{-|\beta|}\partial_x^\beta \; i^{-|\gamma|}\partial_y^\gamma) \; a(x,y,\xi)|_{x=y}$$

$$= (-1)^{|\beta+\gamma|} \; (\partial_\xi^{\beta+\gamma} \; D_x^\beta \; D_y^\gamma) \; a(x,y,\xi)|_{x=y} \text{ as required.}$$

Employing the above results in (7) we get

$$\text{(7")} \qquad a(x,y,\xi) \sim \sum_{|\beta+\gamma|\leq N-1} \frac{(-1)^{|\gamma|}}{\beta!\gamma!}\tau^{|\beta|}(1-\tau)^{|\gamma|} \; (-1)^{|\beta+\gamma|} \; (\partial_\xi^{\beta+\gamma} D_x^\beta \; D_y^\gamma) \; a(v,v,\xi)$$

$$+ \sum_{|\beta+\gamma|=N} N \; C'_{\beta\gamma\tau}\int_0^1 (1-t)^{N-1}\,\partial_\xi^{\beta+\gamma}D_x^\beta \; D_y^\gamma \; a(v+t\tau w, \; v-t(1-\tau)w,\xi) \; dt \quad .$$

To establish formula (3) we need to show that the integral remainder in (7") belongs to $\Pi_\rho^{m-2\rho N}(\mathbb{R}^{3n})$. It is enough to prove that the terms

$$\text{(11)} \quad \int_0^1 (D_\xi^{\beta+\gamma}D_x^\beta \; D_y^\gamma) \; a(v+t\tau w, \; v-t(1-\tau)w,\xi) \; (1-t)^{N-1} \; dt$$

are in $\Pi_\rho^{m-2pN}(\mathbb{R}^{3n})$ for $|\beta+\gamma| = N$.

Observe that

$$\text{(12)} \quad \begin{cases} v = (1-\tau) \; (v+t\tau w) + \tau \; [v-t(1-\tau)w \;] \\ tw = (v+t\tau w) - [v-t(1-\tau)w \;]. \end{cases}$$

Then we have,

$$\text{(13)} \qquad C^{-1} \leq \frac{\|v+t\tau w\| + \|v-t(1-\tau)w\|}{\|v\| + \|tw\|} \leq C$$

where $C > 0$ is independent of $t \in [0,1]$.

Really the left inequality in (13) follows from (12) in the following way:

$$\|v\| \leq (1-\tau) \; \|v+t\tau w\| + \; \tau \; (\|v-t(1-\tau) \; w\|$$

and $\|tw\| \leq \|v+t\tau w\| + \|v-t(1-\tau)w\|$.

Therefore,

$$\|v\| + \|tw\| \leq (1+ \; 1-\tau \;) \; (\|v+t\tau w\| \;) + (1+ \; \tau \;) \; (\|v-t(1-\tau)w\| \;)$$

$$\leq C_1 \; [\|v+t\tau w\| \; + \|v-t(1-\tau)w\|]$$

where $1 \leq C_\tau = \max \; \{1+ \; 1-\tau \; , \; 1+ \; \tau\} = 2 \; = \; C_1$.

On dividing by $C_1 \, (\| v + \|tw\|)$ we get

$$C_1^{-1} \leq \frac{\| v + t\tau w \| + \| v - t\,(1-\tau) w \|}{\|v\| + \|tw\|} \quad .$$

Continuing we have:

$$\begin{aligned} \| v+t\tau w \| + \| v-t(1-\tau)w \| &\leq \|v\| + |\tau|\,\|tw\| + \|v\| + |1-\tau|\,\|tw\| \,. \\ &\leq 2\|v\| + (|\tau| + |1-\tau|)\,\|tw\| \\ &\leq C_2\,(\|v\| + \|tw\|) \end{aligned}$$

where $C_\tau = \max\,\{2,\ |\tau| + |1-\tau|\} = 2 = C_2$.

These prove that

$$2^{-1} \leq \frac{\| v + t\tau w \| + \| v - t(1-\tau) w \|}{\|v\| + \|tw\|} \leq 2 \,.$$

Thus any $C \geq 2$ satisfies (13).

Choose N large enough such that $m-2\rho N < 0$. Then by the definition of Π^m_ρ classes and formula (13) we have:

$$(14) \quad |\partial_\xi^{\beta+\gamma}\partial_x^{\beta}\,\partial_y^{\gamma}\ a(v+t\tau w,\ v-t(1-\tau)w,\xi)|$$
$$\leq \quad C_{\beta\gamma}\,(1+\|v+t\tau w\| + \|v-t(1-\tau)w\| + \|\xi\|\ )^{m-2\rho N}\,(1+\|tw\|)^{m'+2\rho N} \,.$$

From (13) we have

$$C^{-1}\,(\|v\| + \|tw\|) \leq \|v+t\tau w\| + \|v-t(1-\tau)\,w\|$$

and thus

$$C^{-1}\,(1+\|v\| + \|tw\| + \|\xi\|\ ) \leq 1 + \|v+t\tau w\| + \|v-t(1-\tau)w\| + \|\xi\| \,.$$

Since we have chosen $m-2\rho N<0$, we obtain;

$$(1+\|v+t\tau w\|+\|v-t(1-\tau)w\|+\|\xi\|)^{m-2\rho N}\leq C^{-1}(1+\|v\|+\|tw\|+\|\xi\|)^{m-2\rho N}.$$

Now (14) becomes

$$|\partial_\xi^{\beta+\gamma}\partial_x^\beta\partial_y^\gamma a(v+t\tau w,\ v-t(1-\tau)w,\ \xi)|$$

$$\leq C_{\beta\gamma}C^{-1}(1+\|v\|+\|tw\|+\|\xi\|)^{m-2\rho N}(1+\|tw\|)^{m'+2\rho N}$$

Since in the definition of Π_ρ^m classes, any sufficiently large exponent m' is "good", we may assume that $m'+\rho N\geq 0$ and $m'+m>0$. Under these assumptions we have

$$(15)\quad (1+\|v\|+\|tw\|+\|\xi\|)^{m-2\rho N}(1+\|tw\|)^{m'+2\rho N}$$

$$\leq (1+\|v\|+\|w\|+\|\xi\|)^{m-2\rho N}(1+\|w\|)^{m'+2\rho N}.$$

We shall justify (15) by using the methods of calculus. In fact we shall treat the left hand side (LHS) of (15) as a function of t ($t\in[0,1]$) and then differentiate to obtain the extrema :

Let $\|w\|=a>0$ and $\|v\|+\|\xi\|=b>0$. Then LHS becomes

$$g(t)=(1+ta+b)^{m-2\rho N}(1+ta)^{m'+2\rho N}$$

and
$$g'(t)=(m-2\rho N)\,a(1+ta+b)^{m-2\rho N-1}(1+ta)^{m'+2\rho N}$$
$$+\,a(m'+2\rho N)(1+ta+b)^{m-2\rho N}(1+ta)^{m'+2\rho N-1}.$$

Let $g'(t)=0$ for extrema, then

$$(m-2\rho N)(1+ta)+(m'+2\rho N)(1+ta+b)=0$$

thus $\quad ta(m+m')+(m'+m)+b(m'+2\rho N)=0.$

Since we can choose m' such that $m'+m > 0$ and $m'+2\rho N > 0$,we conclude that $g(t)$ has no local extrema on $[0,1]$ and $g'(t) > 0$ for $t \in [0,1]$.Thus the global maximum of $g(t)$ occurs when $t = 1$.This yields (15) .

Thus we have

$$(16) \quad |\partial_\xi^{\beta+\gamma} \partial_x^{\beta} \partial_y^{\gamma} a(v + t\tau w, v - t(1-\tau)w, \xi)$$
$$\leqslant C C_{\beta\gamma} (1 + \|v\| + \|w\| + \|\xi\|)^{m-2\rho N} (1 + \|w\|)^{m'+2\rho N} .$$

Therefore,

$$(17) \quad \left| \int_0^1 (\partial_\xi^{\beta+\gamma} \partial_x^{\beta} \partial_y^{\gamma}) a (v + t\tau w, v - t(1-\tau), \xi) \right.$$
$$\left. \times (1-t)^N \, dt \right|$$
$$\leqslant C C_{\beta\gamma} (1 + \|v\| + \|w\| + \|\xi\|)^{m-2\rho N} (1 + \|w\|)^{m'+2\rho N}$$
$$\times \int_0^1 (1-t)^{N-1} dt .$$

The estimates in (17) also hold for partial derivatives. The proof of this fact is analogous to that above. The inequality (17) implies that the integral remainder belongs to $\Pi_\rho^{m-2\rho N}(\mathbb{R}^{3n})$ and thus (3) is proved.

The inequality (17) and the remark implies that all the assumptions of Thoerem 1.3.3 are satisfied. Thus the asymptotic expansion (3) is true. This completes the proof.

1.7. CONCLUDING REMARKS

According to Theorem 1.6.6 every $a \in \pi_\rho^m(\mathbb{R}^{3n})$ can be replaced by a function with an asymptotic expansion given by formula (3).

Conversely, given a formal asymptotic expansion in Γ_ρ^m classes,

$$\sum_{\beta\gamma} \frac{1}{\beta!\,\gamma!} \tau^{|\beta|} (1-\tau)^{|\gamma|} \partial_\xi^{\beta+\gamma} (-D_x)^\beta D_y^\gamma \; a(x,y,\xi)\Big|_{y=x}$$

in (3) we can apply Thoerem 1.3.2 to construct a symbol

$$\tilde{b}_\tau(a,\xi) = \sum_{\beta\gamma} \psi\,[t_{(\alpha,\beta)}^{-1}(x,\xi)]\, \frac{1}{\beta!\,\gamma!} \tau^{|\beta|} (1-\tau)^{|\gamma|} \partial_\xi^{\beta+\gamma}$$

$$\times\; (-D_x)^\beta D_y^\gamma \, a(x,y,\xi)\Big|_{y=x}$$

by taking a $\{t_{(\alpha,\beta)}\}$sequence and the regularisation function $\psi(x,\xi)$ in (R) of Theorem 1.3.2. For any two different $\{t_{(\alpha,\beta)}\}$sequences, the corresponding amplitudes differ by a function from $S(\mathbb{R}^{2n})$. Furthermore for the corresponding pseudo-differential operators $A, \tilde{A} \in \pi G_\rho^m(\mathbb{R}^n)$ one has $A - \tilde{A} \in G^{-\infty}(\mathbb{R}^n)$ or equivalently $A-\tilde{A}$ has a kernel from $S(\mathbb{R}^{2n})$.

We therefore conclude that two pseudo-differential operators with the same asymptotic expansion differ by a smoothing operator or by an operator with a kernel from $S(\mathbb{R}^{2n})$.

These results define a one-to-one correspondence (mod smoothing operators) between the $\pi_\rho^m(\mathbb{R}^{3n})$classes and a set of asymptotic expansions in $\Gamma_\rho^m(\mathbb{R}^{2n})$ classes and thus a one-to-one correspondence between the $\pi G_\rho^m(\mathbb{R}^n)$operators and the $G_\rho^m(\mathbb{R}^n)$operators mod operators from $G^{-\infty}(\mathbb{R}^n)$.

This correspondence will be extended to an algebra of symbols (transpose, adjoints and compositions) and the algebra of the corresponding pseudo-differential operators in the next chapter.

We close this chapter with an example.

Example 1.7.1. Let $A\,u(x) = \sum_{|\alpha|\leq m} a_\alpha(x)\, D_x^\alpha u(x), \quad u \in S(\mathbb{R}^n)$

with the symbol $a(x,\xi) = \sum_{|\alpha|\leq m} a_\alpha(x)\xi^\alpha \in \Gamma_1^m(\mathbb{R}^{2n})$.

We shall determine the τ-symbol, $\sigma_{A,\ell}(x,\xi)$, $\sigma_{A,r}(x,\xi)$ and $\sigma_{A,w}(x,\xi)$ of A .

We apply (3) of Theorem 1.6.5 directly. We have

$$b_\tau(x,\xi) \sim \sum_\beta \frac{1}{\beta!}\tau^{|\beta|}\partial_\xi^\beta(-D_x)^\beta a(x,\xi) \quad .$$

Since differentiation with respect to the ξ-variable can be done up to $|\beta| = m$ it means that the asymptotic expansion above is in fact finite, i.e. the remainder term is zero.

Therefore

$$(1) \qquad b_\tau(x,\xi) = \sum_{\substack{\beta\leq\alpha \\ |\alpha|\leq m}} \frac{(-1)^{|\beta|}}{\beta!}\frac{\alpha!\,\tau^{|\beta|}}{(\alpha-\beta)!} D_x^\beta a_\alpha(x)\xi^{\alpha-\beta} \quad .$$

Letting $\tau = 0$ in (1) we are compelled to set $|\beta| = 0$ to get a non-zero term. Thus

$$b_0(x,\xi) = \sigma_{A,\ell}(x,\xi) = \sum_{|\alpha|\leq m} a_\alpha(x)\xi^\alpha$$

$$= a(x,\xi) \; .$$

The above observation suggests that if a symbol depends on the x-variable, a natural thing to do is to treat it as a left symbol.

When $\tau = 1$:

$$b_1(x, \xi) = \sigma_{A,r}(x, \xi) = \sum_{\substack{\beta \leqslant \alpha \\ |\alpha| \leqslant m}} \frac{(-1)^{|\beta|}\alpha!}{\beta!(\alpha-\beta)!} D_x^\beta a_\alpha(x)\, \xi^{\alpha-\beta}$$

and when $\tau = \frac{1}{2}$

$$b_{1/2}(x, \xi) = \sigma_{A,w}(x, \xi) = \sum_{\substack{\beta \leqslant \alpha \\ |\alpha| \leqslant m}} \frac{(-1)^{|\beta|}\alpha!}{\beta!(\alpha-\beta)!} 2^{-|\beta|} D_x^\beta a_\alpha(x)\, \xi^{\alpha-\beta} .$$

In particular for a constant coefficient differential operator, we see that $a_\alpha(x) = a_\alpha$ = constant. Therefore differentiation on the x-variable results in a zero contribution to the sum in (1) and thus we are led to choose $|\beta| = 0$ and hence

$$b_0(\xi) = \sigma_{A,\ell}(\xi) = \sum_{|\alpha| \leqslant m} a_\alpha \xi^\alpha .$$

This fact suggests that every symbol of a constant coefficient differential operator should be treated as a left symbol.

Example 1.7.2. Consider once again the differential operator $A = \sum_{|\alpha| \leqslant m} a_\alpha(x) D_x^\alpha$,where the coefficients $a_\alpha(x)$ are polynomials in the x-variable.The left symbol of A is $a(x,\xi) = \sum_{|\alpha| \leqslant m} a_\alpha(x)\xi^\alpha$ as suggested in the preceeding example.Now we shall develop a formula for determining the left symbol of differential operators similar to A and then extend this result to $\pi G_\rho^m(\mathbb{R}^n)$ operators.

Observe that,

$$\begin{aligned} e^{-ix\xi}(Ae^{i(\cdot)\xi})(x) &= e^{-ix\xi}\Big(\sum_{|\alpha| \leqslant m} a_\alpha(x) D_x^\alpha\Big)e^{ix\xi} \\ &= e^{-ix\xi}\Big(\sum_{|\alpha| \leqslant m} a_\alpha(x)\,\xi^\alpha\Big)e^{ix\xi} \\ &= \sum_{|\alpha| \leqslant m} a_\alpha(x)\xi^\alpha = a(x,\xi). \end{aligned}$$

Hence for differential operators, the left symbol can be determined by the formula; $\sigma_{A,1}(x,\xi) = e^{-ix\xi}(Ae^{i(.)\xi})(x)$.

We now extend the above results to the $\pi G_\rho^m(\mathbb{R}^n)$ operators. Let $A \in \pi G_\rho^m(\mathbb{R}^n)$, we shall show that the left symbol $\sigma_{A,1}(x,\xi)$ of A is given by, (mod $S(\mathbb{R}^{2n})$)

$$\sigma_{A,1}(x,\xi) = e^{-ix\xi}(Ae^{i(.)\xi})(x).$$

Initially we assume that $a(x,y\xi) = 0$ if $y \notin K$, K compact

$$Au(x) = (2\pi)^{-n} \iint e^{i(x-y)\xi} a(x,y,\xi)u(y)dyd\xi, \ u \in S(\mathbb{R}^n) \text{ and } a(x,y,\xi) \in \pi_\rho^m(\mathbb{R}^{3n}).$$

Thus $$e^{-ix\theta}(Ae^{i(.)\theta})(x) = (2\pi)^{-n} \iint e^{i(x-y)\xi} e^{-ix\theta} a(x,y,\xi) e^{iy\theta} dyd\xi$$

$$= (2\pi)^{-n} \iint e^{i(\xi-\theta)x} e^{-i(\xi-\theta)y} a(x,y,\xi)dyd\xi$$

put $\xi-\theta=\eta$ then $d\xi = d\eta$.

So, $$e^{-ix\theta}(Ae^{i(.)\theta})(x) = (2\pi)^{-n} \iint e^{i(x-y)\eta} a(x,y,\eta+\theta)dyd\eta.$$

Let $z = y-x$, then $dz = dy$.

Then, $$e^{-ix\theta}(Ae^{i(.)\theta})(x) = (2\pi)^{-n} \iint e^{-iz\eta} a(x,z+x,\eta+\theta)dzd\eta .$$

Expanding $a(x,z+x,\eta+\theta)$ about $\eta=0$, we obtain;

$$a(x,z+x,\eta+\theta) = \sum_{|\alpha| \leqslant N-1} \frac{1}{\alpha!} \partial_\theta^\alpha a(x,z+x,\theta)\, \eta^\alpha + R_N' \quad \text{, where}$$

$$R_N' = \sum_{|\alpha| = N} \frac{1}{\alpha!} N \eta^\alpha \int_0^1 \partial_\theta^\alpha a(x,z+x,\eta+t\theta)(1-t)^{N-1} dt.$$

Hence,
$$e^{-ix\theta}(Ae^{i(.)\theta})(x) = (2\pi)^{-n} \sum_{|\alpha| \leqslant N-1} \iint e^{-iz\eta}\left(\frac{\eta^\alpha}{\alpha!}\right) \partial_\theta^\alpha a(x,z+x,\theta)dzd\eta + R_N,$$

where,
$$(1)\ldots \quad R_N = (2\pi)^{-n} \iint e^{-iz\eta} R_N' dzd\eta .$$

Using the results of formula (9) of Theorem 1.6.5., we obtain;

$$(2\pi)^{-n} \iint e^{-iz\eta}\left(\frac{\eta^\alpha}{\alpha!}\right) \partial_\theta^\alpha a(x,z+x,\theta)dzd\eta = \frac{1}{\alpha!} D_z^\alpha \partial_\theta^\alpha a(x,z+x,\theta)\big|_{z=0} .$$

Therefore, $$e^{-ix\theta}(Ae^{i(.)\theta})(x) = \sum_{|\alpha| \leqslant N-1} \frac{1}{\alpha!} D_y^\alpha \partial_\theta^\alpha a(x,y,\theta)\big|_{x=y} + R_N ,$$

where R_N is given by formula (1).

Finaly, we have to show that $R_N \in \pi_\rho^{m-\rho N}(\mathbb{R}^{3n})$. We argue as in Theorem 1.6.5. to establish this result.

$$R_N = \sum_{|\alpha| = N} \frac{N}{\alpha!} \int (1-t)^{N-1} D_y^\alpha \partial_\theta^\alpha a(x,y,\theta+t\eta)|_{x=y} dt \ .$$

Now,

$$|D_y^\alpha \partial_\theta^\alpha a(x,y,\theta+t\eta)| < C_\alpha <x,x,\theta+t\eta>^{m-2\rho|\alpha|} <x,x,\theta+t\eta>^{\tilde{m}'-2\rho|\alpha|}$$

We then argue as in the expression (17) of Theorem 1.6.5., that

$$|D_y^\alpha \partial_\theta^\alpha a(x,x,\theta+t\eta)| < C_\alpha <x,x,\theta>^{m-2\rho|\alpha|} <x,x,\theta>^{m'+2\rho|\alpha|+m} \ .$$

Therefore,

$$e^{-ix\theta} (Ae^{i(\cdot)\theta})(x) = \sigma_{A,1}(x,\theta) \sim \sum_{|\alpha| \geq 0} \frac{1}{\alpha!} D_y^\alpha \partial_\theta^\alpha a(x,y,\theta)|_{x=y} \quad \text{as required.}$$

To extend this result to any $a(x,y,\xi) \epsilon \pi_\rho^m (\mathbb{R}^{3n})$ we use the regularisation as applied in the Appendix D.

CHAPTER 2

SYMBOLIC CALCULUS

2.1 INTRODUCTION

This section is devoted to the study of the algebra of symbols and the algebra of their corresponding pseudo-differential operators.We shall study transposes,adjoints and compositions of these operators.

A celebrated result is that the Weyl-symbol yields a clean formula as far as transposes and adjoints are concerned. For the composition of symbols,we shall use the left-symbol quite often because it furnishes us with a nice illustration of this concept.

We shall begin by establishing a relationship between the τ-symbols for different values of τ.This result will be of vital importance in our discussion here.

2.2 RELATIONSHIP BETWEEN THE τ-SYMBOLS FOR DIFFERENT VALUES OF τ.

Theorem 2.2.1. Let $0 \leq \tau,\ \tau_1 \leq 1$. Then the symbols $b_\tau(x,\xi)$ and $b_{\tau_1}(x,\xi)$ satisfy the following,

(a)--- $b_\tau(x,\xi) \sim \sum\limits_{\beta\gamma} \frac{(-1)^{|\beta|}}{\beta!\,\gamma!} \tau^{|\beta|} (1-\tau)^{|\gamma|} \tau_1^{|\gamma|} (1-\tau_1)^{|\beta|} \partial_\xi^{\beta+\gamma} D_x^{\beta+\gamma} b_{\tau_1}(x,\xi)$

and when $\beta+\gamma=\alpha$,then

(b)--- $b_\tau(x,\xi) \sim \sum\limits_{\alpha} C_\alpha \partial_\xi^\alpha D_x^\alpha b_{\tau_1}(x,\xi)$

where,

(c)--- $C_\alpha = \sum\limits_{\beta+\gamma=\alpha} \frac{(-1)^{|\beta|}}{\beta!\,\gamma!} [\tau(1-\tau_1)]^{|\beta|} [(1-\tau)\tau_1]^{|\gamma|}$.

Proof:

Letting $a(x,y,\xi) = b_{\tau_1}[(1-\tau_1)x + \tau_1 y, \xi]$ in Theorem 1.6.5. we get

$$b_\tau(x,\xi) \sim \sum_{\beta\,\gamma} \frac{1}{\beta!\gamma!} \tau^{|\beta|} (1-\tau)^{|\gamma|} \partial_\xi^{\beta+\gamma} \{(-D_x)^\beta D_y^\gamma b_{\tau_1}[(1-\tau_1)x + \tau_1 y, \xi]\}|_{y=x}$$

$$= \sum_{\beta\,\gamma} \frac{1}{\beta!\gamma!} \tau^{|\beta|} (1-\tau)^{|\gamma|} (1-\tau_1)^{|\beta|} \tau_1^{|\gamma|} \partial_\xi^{\beta+\gamma} (-1)^{|\beta|} D_x^{\beta+\gamma} b_{\tau_1}(x,\xi)$$

after applying the chain rule.On simplifying we obtain,

$$b_T(x,\xi) \sim \sum_{\beta\gamma} \frac{(-1)^{|\beta|}}{\beta!\,\gamma!} [(1-T_1)T]^{|\beta|} [(1-T)T_1]^{|\gamma|} \partial_\xi^{\beta+\gamma}[D_x^{\beta+\gamma} b_{T_1}(x,\xi)]$$

which proves (a).

When $\beta+\gamma=\alpha$,the formula (b) follows immidiately by letting

$$C_\alpha = \sum_{\beta+\gamma=\alpha} \frac{(-1)^{|\beta|}}{\beta!\,\gamma!} [(1-T_1)T]^{|\beta|} [(1-T)T_1]^{|\gamma|} .$$

Now we are going to improve the results of Theorem 2.2.1.

Theorem 2.2.2. Under the assumptions of Theorem 2.2.1 we have,

(a) --- $b_T(x,\xi) \sim \sum_\alpha \frac{1}{\alpha!}(T_1-T)^{|\alpha|}\partial_\xi^\alpha D_x^\alpha b_{T_1}(x,\xi)$.

Proof

We shall use the binomial formula;

(b) --- $(x+y)^\alpha = \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} x^\gamma y^\beta$, for $x,y \in \mathbb{R}^n$.

Now let $x=(1-T)T_1 e$ and $y=(1-T_1)Te$,where $e=(1,\ldots,1)\in\mathbb{R}^n$.

Applying (b) to Theorem 2.2.1 we get

$$(T_1-T)^{|\alpha|} = \sum_{\beta+\gamma=\alpha} \frac{\alpha!(-1)^{|\beta|}}{\beta!\ \gamma!} [(1-T)T_1]^{|\gamma|} [(1-T_1)T]^{|\beta|} .$$

We observe that $C_\alpha = \frac{1}{\alpha!}(T_1-T)^{|\alpha|}$ and thus (a) is established.

2.3. TRANSPOSE OPERATOR

Define $(u,v) = \int u(x)\,v(x)\,dx$.

Given a pseudo-differential operator $A\in G_\rho^m(\mathbb{R}^n)$ acting on the space $S(\mathbb{R}^n)$, we define the transpose operator ${}^tA$ of A as follows :

(1) --- $(Au,v) = (u,{}^tAv)$ for every $u,v\in S(\mathbb{R}^n)$.

Now

$$(Au,v) = (2\pi)^{-n}\iiint e^{i(x-y)\xi} b_T[(1-T)x+Ty,\xi]u(y)v(x)dyd\xi dx$$

Since the interpretation on y yields a function of from $S(\mathbb{R}^n)$ we have

$$(Au,v) = (2\pi)^{-n}\iiint e^{i(x-y)\xi}\, b_T[(1-T)x+Ty,\xi]\, u(y)v(x)dydxd\xi .$$

$$= (2\pi)^{-n}\iiint e^{i(x-y)\xi} b_{\tau}[(1-\tau)x + \tau y,\xi]\, v(x)u(y)dxdyd\xi$$

since $u,v \in S(\mathbb{R}^n)$ and b_τ has a polynomial growth in x and y. Finally, integration on x yields a function of ξ from $S(\mathbb{R}^n)$, thus,

$$(Au,v) = (2\pi)^{-n}\iiint e^{i(y-x)\xi}\, b_\tau\,[(1-\tau)x+\tau y,-\xi]\, u(x)v(y)\, dxd\xi dy = (u,{}^tAv)$$

Therefore,

$$\text{(2)---}\quad {}^tAv(y) = (2\pi)^{-n}\iint e^{i(y-x)\xi} b_{\tau'}[(1-\tau')y+\tau'x,-\xi]\, v(x)dxd\xi$$

where $\tau' = 1-\tau$.

Thus the symbol of ${}^tb_\tau(x,\xi) = b_{1-\tau}(x,-\xi)$. Consequently we get from Theorem 2.2.2 the following asymptotic formula for ${}^tb_\tau(x,\xi)$;

Theorem 2.3.1 If $A \in G^m_\rho(\mathbb{R}^n)$ then ${}^tA \in G^m_\rho(\mathbb{R}^n)$ and has a symbol ${}^tb_\tau(x,\xi)$ given by

$$\text{(a)---}\quad {}^tb_\tau(x,\xi) \sim \sum_\alpha \frac{1}{\alpha!}(1-2\tau)^{|\alpha|}\partial^\alpha_\xi D^\alpha_x b_\tau(x,-\xi)$$

If $\tau = 0,1$ and $1/2$ we get

$$\sigma_{{}^tA,l}(x,\xi) = {}^tb_0(x,\xi) \sim \sum_\alpha \frac{1}{\alpha!}\partial^\alpha_\xi D^\alpha_x b_0(x,-\xi)$$

$$\text{(b)---}\quad \sigma_{{}^tA,r}(x,\xi) = {}^tb_1(x,\xi) \sim \sum_\alpha \frac{1}{\alpha!}\partial^\alpha_\xi D^\alpha_x b_1(x,-\xi)$$

$$\sigma_{{}^tA,w}(x,\xi) = {}^tb_{1/2}(x,\xi) = b_{1/2}(x,-\xi)$$

(respectively).

Proof

That ${}^tA \in G^m(\mathbb{R}^n)$ follows from (2) since $b_{\tau'}(x,-\xi) \in \Gamma^m_\rho(\mathbb{R}^{2n})$.

To prove the rest of the theorem, we employ the results of Theorem 2.2.2.

We have,

$${}^tb_\tau(x,\xi) = b_{\tau'}(x,-\xi) \sim \sum_\alpha \frac{1}{\alpha!}(\tau'-\tau)^{|\alpha|}\partial^\alpha_\xi D^\alpha_x b_{1-\tau'}(x,-\xi).$$

Letting $\tau = 1-\tau'$ we get (a).

Formulas (b) follow form (a) with $\tau = 0,1$ and $1/2$ respectively.When $\tau = 1/2$,then $\tau' = 1/2$ and thus $\tau'-\tau = 0$.Therefore we get a nonzero contribution only when $|\alpha| = 0$ and this gives formula for $\sigma_{{}^tA,w}(x,\xi)$.

Remark: The formula for $\sigma_{{}^tA,w}(x,\xi)$ shows clearly that it is obtained directly from $\sigma_{A,w}(x.\xi)$ upon replacing ξ by $-\xi$.

2.4 ADJOINT OPERATOR

To define the adjoint operator A^* of A,we shall use the hermitian product

$$(3)\text{---}\quad (u,v) = \int u(x)\overline{v(x)}dx$$

For the pseudo-differential operator $A \in G^m_\rho(\mathbb{R}^n)$ on $S(\mathbb{R}^n)$, we define A^* as follows:

$$(4)\text{---}\quad (Au,v) = (u,A^*v) \quad \text{for } u,v \in S(\mathbb{R}^n) .$$

Applying (3) in (4) we get:

$$\begin{aligned}(Au,v) &= (2\pi)^{-n}\iiint e^{i(x-y)\xi}b_\tau[(1-\tau)x+\tau y,\xi]u(y)\,\overline{v(x)}\,dyd\xi dx\\ &= \int u(y)\Big[(2\pi)^{-n}\iint \overline{e^{i(y-x)\xi}\overline{b_\tau[(1-\tau)x+\tau y,\xi]}v(x)dxd\xi}\Big]dy\\ &= (u,A^*v).\end{aligned}$$

Therefore,

$$(5)\text{---}\quad A^*u(y) = (2\pi)^{-n}\iint e^{i(y-x)\xi}\overline{b_\tau[(1-\tau)x+\tau y,\xi]}u(x)dxd\xi .$$

In the same way as for the transpose operator,we get the following theorem:

Theorem 2.4.1 If $A \in G^m_\rho(\mathbb{R}^n)$ then $A^* \in G^m_\rho(\mathbb{R}^n)$ and the τ-symbol $b^*_\tau(x,\xi)$ of A^* is given by:

$$(a)\text{---}\quad b^*_\tau(x,\xi) = \overline{b_{1-\tau}(x,\xi)}$$

and therefore

$$(b)\text{---}\quad b^*_\tau(x,\xi) \sim \sum_\alpha \frac{1}{\alpha!}(1-2\tau)^{|\alpha|}\partial^\alpha_\xi D^\alpha_x\overline{b_\tau(x,\xi)}.$$

The proof of Theorem 2.4.1. can be obtained by direct application of Theorem 2.2.2. and the definition of the adjoint operator in the discussion above.

When $\tau = 1/2$, which corresponds to the Weyl-symbol, we observe that the factor $(1-2\tau) = 0$. Thus in the formula (b) above, we get a nonzero contribution to the sum when $|\alpha| = 0$. This observation yields the following corollary;

Corollary 2.4.2. If $A \in G^m(\mathbb{R}^n)$ then

$$\text{(a)---}\quad \sigma_{A^*,w}(x,\xi) = \overline{\sigma_{A,w}(x,\xi)}$$

where $\sigma_{A,w}$ and $\sigma_{A^*,w}$ are the Weyl-symbols of A and A* respectively.

Remark: From Corollary 2.4.2. we conclude that $A^* = A$ if and only if $\sigma_{A,w}(x,\xi)$ is real-valued.

Example 2.4.3. Consider once again the differential operator of Example 1.7.2. that is $Au(x) = \sum_{|\alpha|\leq m} a_\alpha(x) D_x^\alpha u(x)$ for $u \in S(\mathbb{R}^n)$ with a symbol $a(x,\xi) = \sum_{|\alpha|\leq m} a_\alpha(x)\xi^\alpha \in \Gamma_1^{m'}(\mathbb{R}^{2n})$ *). We shall determine $b_\tau^*(x,\xi)$ and $b_{1/2}^*(x,\xi)$.

By formula (b) of Theorem 2.4.1 we obtain

$$\begin{aligned} b_\tau^*(x,\xi) &= \sum_{\substack{\beta\leq\alpha\\ |\alpha|\leq m}} \frac{1}{\beta!}(1-2\tau)^{|\beta|} D_x^\beta\overline{a_\alpha(x)}\frac{\alpha!}{(\alpha-\beta)!}\xi^{\alpha-\beta} \\ &= \sum_{\substack{\beta\leq\alpha\\ |\alpha|\leq m}} \binom{\alpha}{\beta}.(1-2\tau)^{|\beta|} D_x^\beta\overline{a_\alpha(x)}\,\xi^{\alpha-\beta}\ . \end{aligned}$$

When $\tau = 1/2$, we see that a nonzero contribution to the sum is only when $|\alpha| = 0$. Therefore,

$$b_\tau^*(x,\xi) = \sigma_{A^*,w}(x,\xi) = \overline{\sum_{|\alpha|\leq m} a_\alpha(x)\xi^\alpha} = \overline{a(x,\xi)}.$$

---

*) See Example 1.5.1.

Furthermore, we observe that in the case of a differential operator, the asymptotic expansion of the transpose and adjoint are exact formulas; in other words their remainder terms turn out to be zero.

2.5 COMPOSITION OF PSEUDO-DIFFERENTIAL OPERATORS IN $\mathbb{R}^n$

Theorem 2.5.1. Suppose $A' \in G_\rho^{m_1}(\mathbb{R}^n)$ and $A'' \in G_\rho^{m_2}(\mathbb{R}^n)$, then $A' \circ A'' \in G_\rho^{m_1+m_2}(\mathbb{R}^n)$ and if $b'_{\tau_1}(x,\xi)$ is a τ_1-symbol of A' and $b''_{\tau_2}(x,\xi)$ a τ_2-symbol of A'', then the τ-symbol, $b_\tau(x,\xi)$ of $A' \circ A''$ has the following asymptotic expansion;

$$\text{(a)---}\quad b_\tau(x,\xi) \sim \sum_{\substack{\alpha\beta\gamma\delta \\ \alpha+\gamma=\beta+\delta}} C_{\alpha\beta\gamma\delta}\,(\partial_\xi^\alpha D_x^\beta b'_{\tau_1}(x,\xi))(\partial_\xi^\gamma D_x^\delta b''_{\tau_2}(x,\xi))$$

where the coefficients $C_{\alpha\beta\gamma\delta}$ depend on τ, τ_1 and τ_2 only, $C_{0000} = 1$. In particular

$$b_\tau(x,\xi) - b'_{\tau_1}(x,\xi)\,b''_{\tau_2}(x,\xi) \in \Gamma_\rho^{m_1+m_2-2\rho}(\mathbb{R}^n).$$

Proof

We shall prove the theorem for the special case when $\tau_1=0$ and $\tau_2=1$. This particular case simplifies our calculations considerably. In this case the pseudo-differential operators A' and A'' are given by;

$$\text{(b)---}\quad A''u(x) = (2\pi)^{-n} \iint e^{i(x-y)\xi}\, b''_1(y,\xi)\,u(y)\,dy\,d\xi$$

or equivalently,

$$\text{(c)---}\quad \widehat{A''u}(\xi) = \int e^{-iy\xi}\, b''_1(y,\xi)\,u(y)\,dy\,.$$

Similarly,

$$\text{(d)---}\quad A'v(x) = (2\pi)^{-n} \iint e^{i(x-y)\xi}\, b'_0(x,\xi)\,v(y)\,dy\,d\xi$$

$$= (2\pi)^{-n} \int e^{ix\xi}\, b'_0(x,\xi)\,\hat{v}(\xi)\,d\xi$$

where $u, v \in S(\mathbb{R}^n)$.

Applying the definition of composition of operators we get;

$$(e)\text{---}\ A'\circ A''u(x) = A'(A''u(x))$$

$$= (2\pi)^{-n}\int e^{ix\xi} b_0'(x,\xi)(\widehat{A''u}(\xi))d\xi$$

$$= (2\pi)^{-n}\int e^{ix\xi} b_0'(x,\xi)\Big(\int e^{-iy\xi} b_1''(y,\xi)u(y)dy\Big)d\xi$$

$$= (2\pi)^{-n}\iint e^{i(x-y)\xi} b_0'(x,\xi)b_1''(y,\xi)u(y)dyd\xi \quad .$$

Hence the composition operator $A'\circ A''$ has the symbol;

$$(f)\text{---}\ a(x,y,\xi)=b_0'(x,\xi)b_1''(y,\xi) \in \Gamma_\rho^{m_1+m_2}(\mathbb{R}^{3n}).$$

Thus $A'\circ A'' \in G_\rho^{m_1+m_2}(\mathbb{R}^n)$.

Using formula (3) of Theorem 1.6.5 we get

$$(g)\text{---}\ b_\tau(x,\xi) \sim \sum_{\beta\ \gamma} \frac{(-1)^{|\beta|}}{\beta!\ \gamma!}\tau^{|\beta|}(1-\tau)^{|\gamma|}\partial_\xi^{\beta+\gamma}\{D_x^\beta b_0'(x,\xi)D_y^\gamma b_1''(y,\xi)\}\big|_{y=x} \ .$$

Now we use the Leibnitz Formula to establish the required results:

$$\partial_\xi^{\beta+\gamma}\{D_x^\beta b_0'(x,\xi)D_y^\gamma b_1''(y,\xi)\} = \sum_{\delta+\mu=\beta+\gamma} \frac{(\beta+\gamma)!}{\delta!\ \mu!}\partial_\xi^\delta D_x^\beta b_0'(x,\xi)\partial_\xi^\mu D_y^\gamma b_1''(y,\xi).$$

Finally we have

$$(h)\text{---}\ b_\tau(x,\xi) \sim \sum_{\substack{\beta\gamma\delta\mu \\ \beta+\gamma=\delta+\mu}} \frac{(-1)^{|\beta|}}{\beta!\,\gamma!\,\delta!\,\mu!}(\beta+\gamma)!\,\tau^{|\beta|}(1-\tau)^{|\gamma|}(\partial_\xi^\delta D_x^\beta b_0'(x,\xi))(\partial_\xi^\mu D_y^\gamma b_1''(y,\xi))\big|_{y=x}.$$

This proves the theorem for $\tau_1 = 0$ and $\tau_2 = 1$. To obtain a general case of the theorem that is for any τ_1 and τ_2, we use Theorem 2.2.2. to express b_0 and b_1 in terms b_{τ_1} and b_{τ_2} and substitute into formula (h).
In particular if $\tau = 0$ we get the following theorem;

Theorem 2.5.2 Under the assumptions of Theorem 2.5.1 we get;

$$(i)\text{---}\ b_0(x,\xi) \sim \sum_{\delta+\mu=\gamma} \frac{1}{\delta!\,\mu!}\partial_\xi^\delta b_0'(x,\xi)(\partial_\xi^\mu D_y^\gamma b_1''(y,\xi))\big|_{y=x}$$

Proof:

Follows from the fact that when $\tau = 0$, we get a nonzero contribution to our series provided $|\beta| = 0$ in formula (h).

We shall derive the composition formula for the left-symbols independently because they will be of vital importance in the development of our discussion.

Theorem 2.5.3. Under the assumptions of Theorem 2.5.1. we get

(a)--- $b_0(x,\xi) \sim \sum \frac{1}{\alpha!} \partial_\xi^\alpha b_0'(x,\xi) D_x^\alpha b_0''(x,\xi)$.

The symbol $b_0(x,\xi)$ mod $S(\mathbb{R}^{2n})$ is called a composition of symbols b_0' and b_0''.

Proof:

From the definition of A' and A" we have

$$A'u(x) = (2\pi)^{-n} \iint e^{i(x-y)\xi} b_0'(x,\xi) u(y) dy d\xi$$

$$A''v(x) = (2\pi)^{-n} \iint e^{i(x-y)\xi} b_0''(x,\xi) v(y) dy d\xi \quad .$$

Using the identity $^t({}^tA'') = A''$ we express the operator A" in terms of its transpose symbol.

$$A''u(x) = (2\pi)^{-n} \iint e^{i(x-y)\xi} \, {}^tb_0''(y,-\xi) u(y) dy d\xi$$

where,

(b)--- ${}^tb_0''(y,-\xi) \sim \sum_\alpha \frac{1}{\alpha!} \partial_\xi^\alpha b_0''(y,\xi)$ (by Theorem 2.3.1.(b)).

Thus we have;

$$\widehat{A''u}(\xi) = \int e^{-iy\xi} \, {}^tb_0''(y,-\xi) u(y) dy$$

and

$$A'u(x) = (2\pi)^{-n} \int e^{ix\xi} b_0'(x,\xi) \hat{u}(\xi) d\xi \quad .$$

Therefore,

$$(A'\circ A'')u(x) = (2\pi)^{-n}\int e^{ix\xi}\, b_0'(x,\xi)\widehat{A''u}(\xi)d\xi$$

$$= (2\pi)^{-n}\int e^{ix\xi}\, b_0'(x,\xi)\Big(\int e^{-iy\xi}\, {}^t b_0''(y,-\xi)u(y)dy\Big)d\xi$$

$$= (2\pi)^{-n}\iint e^{i(x-y)\xi} b_0'(x,\xi)\, {}^t b_0''(y,-\xi)u(y)dyd\xi \quad .$$

Let $a(x,y,\xi) = b_0'(x,\xi)\,{}^t b_0''(y,-\xi)$, then we get (for $\tau = 0$) from Theorem 1.6.5. (2)

$$b_0(x,\xi) \sim \sum_\alpha \frac{1}{\alpha!}\,\partial_\xi^\alpha D_y^\alpha[b_0'(x,\xi)\,{}^t b_0''(y,-\xi)]\big|_{y=x}$$

$$= \sum_\alpha \frac{1}{\alpha!}\,\partial_\xi^\alpha[b_0'(x,\xi)D_x^\alpha\, {}^t b_0''(x,-\xi)] \ .$$

Substituting the asymptotic formula for ${}^t b_0''(x,-\xi)$, namely

$${}^t b_0''(x,-\xi) \sim \sum_\beta \frac{(-1)^{|\beta|}}{\beta!}\,\partial_\xi^\beta D_x^\beta b_0''(x,\xi)$$

we get,

$$b_0(x,\xi) \sim \sum_{\alpha\beta} \frac{(-1)^{|\beta|}}{\alpha!\beta!}\,\partial_\xi^\alpha\,(b_0'(x,\xi)\partial_\xi^\beta D_x^{\alpha+\beta}\, b_0''(x,\xi)).$$

Using Leibnitz Formula we obtain,

$$b_0(x,\xi) \sim \sum_{\alpha\beta} \frac{(-1)^{|\beta|}}{\alpha!\,\beta!} \sum_{\delta+\gamma=\alpha} \frac{\alpha!}{\delta!\gamma!}\,\partial_\xi^\delta b_0'(x,\xi)\partial_\xi^{\gamma+\beta} D_x^{\alpha+\beta}\, b_0''(x,\xi)$$

$$= \sum_{\beta\delta\gamma} \frac{(-1)^{|\beta|}}{\beta!\gamma!\delta!}\,\partial_\xi^\delta b_0'(x,\xi)\partial_\xi^{\gamma+\beta}\; D_x^{\gamma+\delta+\beta}\, b_0''(x,\xi) \ .$$

Let $\delta + \beta = \mu$, then

$$b_0(x,\xi) \sim \sum_{\mu\gamma} \frac{1}{\gamma!}\Big(\sum_{\delta+\beta=\mu} \frac{(-1)^{|\beta|}}{\beta!\delta!}\Big)\{\partial_\xi^\gamma b'_0(x,\xi)\,[\partial_\xi^\mu\, D_x^{\gamma+\mu} b_0''(x,\xi)]\} \quad .$$

If $e = (1,\dots,1) \in \mathbb{R}^n$ then

$$(e-e)^\mu = \sum_{\beta+\delta=\mu} \frac{\mu!}{\beta!\delta!}(-1)^{|\beta|}e^{\beta+\delta} \qquad (e^\beta = e^\delta = 1)$$

$$= \mu! \sum_{\beta+\delta=\mu} \frac{(-1)^{|\beta|}}{\beta!\delta!} .$$

Clearly $(e-e)^\mu = 0$ for $\mu \neq (0,0,\dots,0)$.

So we get a nonzero contribution to the last expression only if $\mu = (0,0,\dots,0)$ and this implies that $\delta = \beta = (0,0,\dots,0)$. Hence

$$b_0(x,\xi) \sim \sum_\gamma \frac{1}{\gamma!} \partial_\xi^\gamma b_0'(x,\xi) D_x^\gamma b_0''(x,\xi) \qquad \text{as required.}$$

Example 2.5.4. Let $a'(x,\xi) = \xi^2 + V(x) \in \Gamma_1^2(\mathbb{R}^2)$ and $a''(x,\xi) = \dfrac{1}{\xi^2 + V(x)} \in \Gamma_1^{-2}(\mathbb{R}^2)$. Suppose A' and A" are pseudo-differential operators with left symbols $a'(x,\xi)$ and $a''(x,\xi)$ respectively. We shall determine the left symbol of the composition operator $A = A' \circ A''$. Put $b_0'(x,\xi) = a'(x,\xi)$ and $b_0''(x,\xi) = a''(x,\xi)$. By formula (a) of Theorem 2.5.3 we get

$$b_0(x,\xi) \sim \sum_{|\alpha| \leq 2} \frac{1}{\alpha!} \partial_\xi^\alpha b_0'(x,\xi) D_x^\alpha b_0''(x,\xi) .$$

In this case differentiation on the ξ-variable can be done up to the second order only. Therefore the symbol of the composition operator becomes

$$b_0(x,\xi) \sim \sum_{j=0}^{2} \frac{1}{j!} \partial_\xi^j b_0'(x,\xi) D_x^j b_0''(x,\xi)$$

$$= 1 - \frac{2i\xi V'(x)}{[\xi^2 + V(x)]^2} + \frac{2\xi[2(V'(x))^2 - V''(x)[\xi^2 + V(x)]}{[\xi^2 + V(x)]^3} .$$

Since the subsequent terms in the expansion above improve at infinity upon differentiating, we conclude that $A' \circ A'' \sim I + K$, where I is an identity operator on $S(\mathbb{R}^n)$ and K is some integral operator with a smooth kernel. We shall dicuss these type of examples in the section on hypoelliptic operators.

CHAPTER 3

BOUNDEDNESS AND COMPACTNESS OF PSEUDO-DIFFERENTIAL OPERATOS ON $L^2(\mathbb{R}^n)$

3.1 INTRODUCTION

In this section we study boundedness and compactness of pseudo-differential operators in $\mathbb{R}^n$ in order to use it in the spectral analysis of Chapter 5.

Important results in this section are that every $A \in G_\rho^0(\mathbb{R}^n)$ extends to a continuous or bounded operator on $L^2(\mathbb{R}^n)$ and that $A \in G_\rho^m(\mathbb{R}^n)$ $(m<0)$ extends to a compact operator on $L^2(\mathbb{R}^n)$.

Consider a pseudo-differential operator $A \in G_\rho^m(\mathbb{R}^n)$ with a τ-symbol $b_\tau \in \Gamma_\rho^m(\mathbb{R}^{2n})$. In other words, for $u \in S(\mathbb{R}^n)$

$$(1)\text{---}\quad Au(x) = (2\pi)^{-n} \iint e^{i(x-y)\xi}\, b_\tau[(1-\tau)x + \tau y, \xi]u(y)dyd\xi \quad .$$

Assume further that $b_\tau(v,\xi) \in S(\mathbb{R}^{2n})$. Then we can integrate on the ξ-variable first in (1) and get

$$Au(x) = \int ((2\pi)^{-n}\int e^{i(x-y)\xi} b_\tau\,[(1-\tau)x + \tau y, \xi]d\xi)u(y)dy$$

$$= \int K_A(x,y)u(y)dy$$

where,

$$(2)\text{---}\quad K_A(x,y) = F^{-1}_{\xi\to(x-y)}\{b_\tau[(1-\tau)x + \tau y, \xi]\} \quad ,$$

where F^{-1} is the inverse Fourier transform on $S(\mathbb{R}^n)$ and the arrow $(\to)$ indicates that we pass from the ξ-variable to the $(x-y)$-variable. Clearly $K_A(x,y) \in S(\mathbb{R}^{2n})$.

It is apparent from the above calculations that if we have a pseudo-differential operator A with a τ-symbol $b_\tau \in S(\mathbb{R}^{2n})$ then it has a kernel

$K_A(x,y) \in S(\mathbb{R}^{2n})$ given by (2).

The question may arise as to how can we rediscover b_T if we a pseudo-differential operator A with a kernel $K_A(x,y) \in S(\mathbb{R}^{2n})$? We simply consider the Fourier Transform (F) applied to (2) with a change of variables $v = (1-T)x + Ty$ and $w = x-y$. This yields

$$(3)\text{---}\ b_T(v,\xi) = F_{w\to\xi}K_A(v+Tw, v-(1-T)w).$$

Thus we are led to conclude that every pseudo-differential operator A with kernel $K_A(x,y) \in S(\mathbb{R}^{2n})$ has a T-symbol $b_T \in S(\mathbb{R}^{3n})$ given by (3). The converse is also true.

In particular for the Weyl-symbol ($T = 1/2$) we have

$$(4)\text{---}\ \sigma_w(x,\xi) = F_{w\to\xi}K_A(x + w/2, x - w/2).$$

The above discussion is an introduction to the anti-Wick symbols.

3.2. ANTI-WICK SYMBOLS

We shall first present a scheme that will aid us in defining the anti-Wick symbols.

Let $\Phi_o(x) = \pi^{-n/4}e^{-x^2/2}$.

Clearly $\Phi_o(x) \in S(\mathbb{R}^n)$ and also,

$$\|\Phi_o\|^2_{L^2(\mathbb{R}^n)} = \int \pi^{-n/2}e^{-x^2}dx$$
$$= 1 , \qquad \text{since} \qquad \int_{\mathbb{R}} e^{-t^2}dt = \sqrt{\pi}$$

and

$$(F\Phi_o)(\xi) = \hat{\Phi}_o(\xi) = \pi^{-n/4}\int e^{-ix\xi - x^2/2}dx$$
$$= \pi^{-n/4}e^{-\xi^2/2}$$
$$= \Phi_o(\xi).$$

Let $P_o : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ be the operator with kernel $K(x,y) = \pi^{-n/2}e^{-(x^2+y^2)/2}$.

It is clear that $K(x,y) \in S(\mathbb{R}^{2n})$. We now show that P_o as defined above is the orthogonal projection onto the subspace of $L^2(\mathbb{R}^n)$ spanned by Φ_o .

$$(1)\text{---}\quad P_o u(x) = \pi^{-n/2}\int e^{-(x^2+y^2)/2}u(y)dy \qquad \text{for } u \in S(\mathbb{R}^n)$$

$$= \pi^{-n/4}\, e^{-x^2/2}\, \pi^{-n/4} \int e^{-y^2/2}u(y)dy$$

$$= C\Phi_o(x) \qquad \text{where } C = \pi^{-n/4}\int e^{-y^2/2}u(y)dy .$$

Next we show that P_o is orthogonal.

Now from (1) taking $u(x) = \Phi_o(x)$, we get

$$C = \pi^{-n/4} \int e^{-y^2/2}\,(\pi^{-n/4}\, e^{-y^2/2})\, dy$$

$$= \pi^{-n/2} \int e^{-y^2}\, dy$$

$$= 1$$

Thus $P_o\, \Phi_o(x) = \Phi_o(x)$. This proves that P_o is orthogonal since $P_o^2 = P_o$ and evidently $(P_o u,v) = (u,P_o v)$ on $S(\mathbb{R}^n)$ and this completes the proof. (Note that the kernel of P_o is real-valued).

Let us now determine the Weyl-symbol of our operator P_o with kernel $K(x,y) = \pi^{-n/2}e^{-(x^2+y^2)/2}$. We shall apply formula (4) of section 3.1 directly.

Observe further that $F_{v\to\xi}(e^{-\alpha v^2}) = (\pi/\alpha)^{n/2}e^{-\xi^2/4\alpha}$ $,\alpha > 0$.

Hence for $K(x,y)$ above we get,

$$(2)\text{---}\quad \sigma_W(x,\xi) = F_{v\to\xi}(\pi^{-n/2}e^{-\{(x+v/2)^2+(x-v/2)^2\}/2})$$

$$= F_{v\to\xi}(\pi^{-n/2}e^{-(x^2+v^2/4)})$$

$$= 2^n e^{-(x^2+\xi^2)}$$

or equivalently,

$$(3)\text{---}\quad P_o\, u(x) = 2^n(2\pi)^{-n}\iint e^{i(x-y)\xi}e^{-(x^2+\xi^2)}u(y)dyd\xi \quad .$$

Next consider the function

$$(4)\text{---}\quad \sigma_w^{z_o}(x,\xi) = \sigma_w[(x-x_o),(\xi-\xi_o)]$$
$$= 2^n\, e^{-[(x-x_o)^2+(\xi-\xi_o)^2]}$$

where $z_o = (x_o,\xi_o) \in \mathbb{R}^{2n}$.Let P_{z_o} be the operator with the Weyl-symbol $\sigma_w^{z_o}(x,\xi)$.Then we write

$$(5)\text{---}\quad P_{z_o}u(x) = \pi^{-n}\iint e^{i(x-y)\xi}e^{-[(x-x_o)^2+(\xi-\xi_o)^2]}u(y)dyd\xi \quad .$$

Furthermore,let us introduce the operators M_{ξ_o} and T_{x_o} defined as follows;

$$(6)\text{---}\quad \begin{cases} M_{\xi_o}u(x) = e^{ix\xi_o}u(x) & \\ T_{x_o}u(x) = u(x-x_o) & \text{for} \quad u \in S(\mathbb{R}^n). \end{cases}$$

Evidently M_{ξ_o} and T_{x_o} are unitary operators on $L^2(\mathbb{R}^n)$.We shall show that P_{z_o} is unitarily equivalent to P_o by using M_{ξ_o} and T_{x_o}.

Proposition 3.2.1 If $U_{z_o} = U_{(x_o,\xi_o)} = M_{\xi_o}T_{x_o}$ then

$$(1)\text{---}\quad P_{z_o} = U_{z_o}P_oU_{z_o}^{-1} \quad .$$

Proof:

Take $u \in S(\mathbb{R}^n)$.Then $U_{z_o}^{-1}u(w) = T_{x_o}^{-1}M_{\xi_o}^{-1}u(w)$

i.e.

$$U_{z_o}^{-1}u(w) = e^{-i(w+x_o)\xi_o}u(w+x_o)$$

so,

$$(P_0 U_{z_\circ}^{-1} u)(x) = \pi^{-n} \iint e^{i(x-w)\xi} e^{-(x^2+\xi^2)} e^{-i(w+x_\circ)\xi_\circ} u(w+x_\circ)\,dw\,d\xi$$

and thus,

$$(U_{z_\circ} P_0 U_{z_\circ}^{-1}) u(x) = \pi^{-n} \iint e^{ix\xi_\circ} e^{i(x-x_\circ-w)\xi} e^{-[(x-x_\circ)^2+\xi^2]} e^{-i(w+x_\circ)\xi_\circ} u(w+x_\circ)\,dw\,d\xi \;.$$

Put $x_\circ + w = y$, then $dw = dy$. Hence ,

$$\begin{aligned}(U_{z_\circ} P_0 U_{z_\circ}^{-1} u)(x) &= \pi^{-n} \iint e^{ix\xi_\circ} e^{i(x-y)\xi} e^{-\{(x-x_\circ)^2+\xi^2\}} e^{-iy\xi} u(y)\,dy\,d\xi, \\ &= \pi^{-n} \iint e^{i(x-y)(\xi+\xi_\circ)} e^{-\{(x-x_\circ)^2+\xi^2\}} u(y)\,dy\,d\xi \\ &= \pi^{-n} \iint e^{i(x-y)\xi} e^{-\{(x-x_\circ)^2+(\xi-\xi_\circ)^2\}} u(y)\,dy\,d\xi \\ &= P_{z_\circ} u(x)\end{aligned}$$

after making a substitution $\xi+\xi_\circ = \eta$ and $\xi_\circ = \eta_\circ$.

Since $P_{z_\circ}$ is unitarily equivalent to P_0 ,we conclude that $P_{z_\circ}$ is an orthogonal projection onto the subspace of $L^2(\mathbb{R}^n)$ spanned by $U_{z_\circ}\Phi_0$.

Now we are in a position to introduce the anti-Wick symbols.Take $a(x,\xi) \in \Gamma_\rho^m(\mathbb{R}^{2n})$ and define the operator;

$$\begin{aligned}(W)\text{---}\ Au(w) &= (2\pi)^{-n} \iint a(x,\xi)(P_{(x,\xi)}u)(w)\,dx\,d\xi \\ &= (2\pi)^{-n} \iint a(x,\xi)\Big((2\pi)^{-n} \iint e^{i(w-y)\eta} \sigma_W^{(x,\xi)}(y,\eta)u(y)\,dy\,d\eta\Big)dx\,d\xi \\ &= (2\pi)^{-n} \iint e^{i(w-y)\eta} \Big\{(2\pi)^{-n} \iint a(x,\xi)\sigma_W^{(x,\xi)}(y,\eta)\,dx\,d\xi\Big\} u(y)\,dy\,d\eta ,\end{aligned}$$

where $\sigma_W^{(x,\xi)}(y,\eta)$ is given by (4).

Observe that if $u \in S(\mathbb{R}^n)$,then $(P_{(x,\xi)}u)(x_\circ) \in S(\mathbb{R}^{2n})$ as a function of x and ξ .Thus (W) is well-defined.

Definition 3.2.2. The operator (W) is called an operator with an anti-Wick symbol $a(x,\xi)$.We shall write aW-symbol for anti-Wick symbol.

The relationship between the $\mathcal{T}$-symbol and the aW-symbol is contained in the theorem below.This theorem furnishes us with a formula for determining the Weyl-symbol of a pseudo-differential operator from its aW-symbol and also gives the asymptotic expansion of the $\mathcal{T}$-symbol in terms of the aW-symbol of this operator.

Theorem 3.2.3. Let A be an operator with an aW-symbol $a(z) = a(x,\xi) \in \Gamma^m_\rho(\mathbb{R}^{2n})$.Then $A \in G^m_\rho(\mathbb{R}^n)$ and the Weyl-symbol of A is given by

$$(1)\text{---}\quad b_w(z) = \pi^{-n} \int e^{-\|z-z'\|^2} a(z')dz'$$

$$(\text{recall}\quad \|z-z'\|^2 = \sum_{i=1}^{2n} |z_i - z_i'|^2 \quad .$$

Moreover,the $\mathcal{T}$-symbol $b_{\mathcal{T}}(z)$ of A has the following asymptotic expansion;

$$(2)\text{---}\quad b_{\mathcal{T}}(z) \sim \sum_\alpha C_\alpha \partial_z^\alpha a(z)$$

where C_α are constants dependent on $\mathcal{T}$ and α with $C_0 = 1$ and $C_\alpha = 0$ for $|\alpha|$ odd.In particular

$$(3)\text{---}\quad b_{\mathcal{T}}(z) - a(z) \in \Gamma^{m-2\rho}_\rho(\mathbb{R}^{2n}).$$

Proof:

Formula (1) follows directly from (W). Explicitly we have

$$\begin{aligned} b_w(y,\eta) &= (2\pi)^{-n} \iint a(x,\xi)\,\sigma_w^{(x,\xi)}(y,\eta)dxd\xi \\ &= (2\pi)^{-n} \iint a(x,\xi)2^n e^{-\{(y-x)^2+(\eta-\xi)^2\}}dxd\xi \\ &= \pi^{-n} \iint a(x,\xi).e^{-\|(y-x,\eta-\xi)\|^2}dxd\xi \\ &= \pi^{-n} \iint a(x,\xi)e^{-\|(y,\eta)-(x,\xi)\|^2}dxd\xi \ . \end{aligned}$$

Put $(x,\xi) = z'$. and $(y,\eta) = z$, then (1) follows immediately.

Next consinder the Taylor Expansion of $a(z')$ about z.

(4)--- $a(z') = \sum_{|\alpha|\leq N-1} \frac{1}{\alpha!}(\partial^{\alpha}_{z'}a)(z)(z'-z)^{\alpha} + R'_N(z',z)$

where,

(5)--- $R'_N(z',z) = \sum_{|\alpha|=N} C'_{\alpha}(z'-z)^{\alpha} \int_0^1 \partial^{\alpha}_{z'}a[z + t(z'-z)](1-t)^{N-1}dt$.

Substituting (4) into (1) we get

(6)--- $b_w(z) = \sum_{|\alpha|\leq N-1} C_{\alpha}\partial^{\alpha}_{z}a(z) + R_N$

where,

(7)--- $C_{\alpha} = (1/_{\alpha!})\pi^{-n}\int u^{\alpha}e^{-\|u\|^2}du$ and $u = z'-z$.

From (7) we get $C_0=1$ and when $|\alpha|$ is odd,then the integrand is an odd function and thus the integral vanishes over $\mathbb{R}^{2n}$.Hence $C_{\alpha}=0$ for $|\alpha|$ odd.

The remainder term takes the form,

(8)--- $R_N(z) = \pi^{-n}\int e^{-\|z-z'\|^2}R'_N(z',z)dz'$.

We shall show that $R_N(z)\in\Gamma^{m-\rho N}_{\rho}(\mathbb{R}^{2n})$.It is enough to estimate the terms

(9)--- $\partial^{\gamma}_{z}R_N(z) = \int_0^1 u^{\alpha}e^{-\|u\|^2}(\partial^{\beta}_{z}a)(z+tu)(1-t)^{N-1}du\,dt$,where

(9')--- $\beta = \alpha+\gamma$ and $|\beta| = N+|\gamma|$.

In fact we need to estimate;

(10)--- $I_{\alpha\beta}(z) = \int u^{\alpha}e^{-\|u\|^2}(\partial^{\beta}_{z}a)(z+tu)du$.

We shall estimate $I_{\alpha\beta}$ by splitting the integration over $\mathbb{R}^n$ as follows; $\mathbb{R}^n = \{\|u\|\leq\|z\|/2\}\cup\{\|u\|>\|z\|/2\}$ then,

$$I_{\alpha\beta}(z) = \int_{\|u\|\leq\|z\|/2} u^{\alpha}e^{-\|u\|^2}(\partial^{\beta}a)(z+tu)du + \int_{\|u\|\geq\|z\|/2} u^{\alpha}e^{-\|u\|^2}(\partial^{\beta}_{z}a)(z+tu)du.$$

We can choose N sufficiently large such that $m-(N+|\gamma|) = m-|\beta| \lesssim 0$.

We have,

$$|\partial_z^\beta a(z+tu)| \leq C_\beta <z+tu>^{m-\rho|\beta|}$$

$$\leq \begin{cases} 2^m C_\beta <z>^{m-\rho|\beta|} & \text{for } \|u\| \leq \|z\|/2 \\ 2^m C_\beta <z>^m <u>^m & \text{for } \|u\| \geq \|z\|/2 . \end{cases}$$

In fact , if $m-\rho|\beta| \leq 0$ then $<z+tu> \leq <z>$ and the inequality is true. If $m-\rho|\beta| > 0$ then $<z+tu> \leq <2z>$ for $\|u\| \leq \|z\|/2$ and thus $<z+tu> \leq 2<z>$. This proves the inequality for $\|u\| \leq \|z\|/2$ and every β .The inequality when $\|u\| \geq \|z\|/2$ follows from the inequality $<z+tu> \leq 2<z> <tu> \leq 2<z><u>$. Therefore ($|\alpha| = N$),

$$|I_{\alpha\beta}(z)| \leq 2^m C_\beta <z>^{m-|\beta|} \int_{\|u\| \leq \|z\|/2} <u>^N e^{-\|u\|^2} du + 2^m C_\beta <z>^m \int_{\|u\| \geq \|z\|/2} <u>^{N+m} e^{-\|u\|^2} du$$

$$\leq C_N 2^m C_\beta <z>^{m-|\beta|} + 2^m C_\beta <z>^m \int_{\|u\| \geq \|z\|/2} <u>^{N+m} e^{-\|u\|^2} du ,$$

where $C_N = \int_{\|u\| \geq \|z\|/2} <u>^N e^{-\|u\|^2} du < \int <u>^N e^{-\|u\|^2} du$.

The integral

$\int_{\|u\| \geq \|z\|/2} <u>^{N+m} e^{-\|u\|^2} du$,yields a function of the variable z from $S(\mathbb{R}^n)$.

Using polar coordinates we get

$$\int_{\|u\| \geq \|z\|/2} <u>^{N+m} e^{-\|u\|^2} du \leq V \int_{\nu \geq \|z\|/2} <\nu>^{N+m+n} e^{-\nu^2} d\nu ,$$

where V = volume of the unit sphere in $\mathbb{R}^n$.

Now let $\sigma = \nu - \|z\|/2$,then

$$\int_{\nu \geq \|z\|/2} <\nu>^{N+m+n} e^{-\nu^2} d\nu < \int_0^\infty <\sigma+\|z\|/2>^{N+m+n} e^{-\sigma^2-\|z\|^2/4} d\sigma$$

$$= e^{-\|z\|^2/4} 2^{N+m+n} <\|z\|/2>^{N+m+n} \int_0^\infty <\sigma>^{N+m+n} e^{-\sigma^2} d\sigma.$$

This proves that the integral defines a function of the order.

$C e^{-\|z\|^2/4} <\|z\|/2>^{N+m+n}$ which belongs to $S(\mathbb{R}^n)$ (C = some positive constant). We conclude that $\partial_z^\gamma R_N(z) \in \Gamma_\rho^{m-\rho(N+|\gamma|)}(\mathbb{R}^{2n})$.

Next we study some of the properties of the operator A in (W). We consider the special case when A is a positive operator and then deduce its L^2-norm estimates. First we recall the definition of a positive operator.

Definition 3.2.4. A symmetric operator T on a Hilbert Space H is said to be positive if $(x,Tx) \geqslant 0$ for all x in D(T). We denote this by $T \geqslant 0$. It is strictly positive if $(x,Tx) > 0$ for all $x \neq 0$, x from D(T). We shall denote this by $T > 0$ or $0 < T$. For any two operators A and B we shall write $A < B$ if $B-A > 0$. Similarly $A < B$ if $0 < B-A$.

Proposition 3.2.5. Let A be the operator of Definition 3.2.2. and $a(z) = a(x,\xi) \geq 0$. Then A is symmetric and $A \geq 0$ i.e. $(Au,u) \geq 0$ for all $u \in S(\mathbb{R}^n)$.

Proof:

From (1) of Theorem 3.2.3., we get that P_z is unitarily equivalent to P_o. Take $u \in S(\mathbb{R}^n)$, then

$$(P_o u,u) = \pi^{-n}\int \left(\int e^{-(x^2+y^2)/2} u(y)dy\right)\overline{u(x)}dx$$

$$= \pi^{-n} \left|\int e^{-x^2/2} u(x) dx\right|^2 \geqslant 0.$$

In other words P_o is a positive operator and thus we conclude that P_z is also a positive operator. Hence $a(x,\xi)P_{(x,\xi)} \geqslant 0$. This means that,

$$(Au,u) = \int (Au)(w)\overline{u(w)}dw$$

$$= \iint a(x,\xi)(P_{(x,\xi)}u)(w)\overline{u(w)}dxd\xi dw = \iint (P_{(x,\xi)}u)(w)\, \overline{u(w)}\, a(x,\xi)\, dwdxd\xi \geq 0$$

which shows that a is symmetric and non-negative.

We are now in a position to estimate the norm of A in $L^2(\mathbb{R}^n)$.
(recall $\|A\|_{L^2(\mathbb{R}^n)} = \sup\{\frac{\|Au\|}{\|u\|}\ ;\ u \neq 0 \text{ and } u \in S(\mathbb{R}^n)\}$ since $S(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$).

Proposition 3.2.6. Let the aW-symbol a(z) of A be real-valued and $\sup\{\ |a(z)|; z \in \mathbb{R}^{2n}\} = M$ .Then A is a bounded operator on $L^2(\mathbb{R}^n)$ and $\|A\|_{L^2(\mathbb{R}^n)} \leqslant \sup\{|a(z)|\ ; z \in \mathbb{R}^{2n}\}$.

Proof

From the proof of Proposition 3.2.5 A is symmetric.

Now $\sup\{|a(z)|\ ; z \in \mathbb{R}^{2n}\} = M$ implies that $M-a(z) \geqslant 0$ and $M+a(z) \geqslant 0$.
Thus by Proposition 3.2.5.,it means that $MI - A \geq 0$ and $MI + A \geq 0$. Therefore A is bounded on $L^2(\mathbb{R}^n)$ and

$$\|A\|_{L^2(\mathbb{R}^n)} \leqslant M$$

and thus

$$\|A\|_{L^2(\mathbb{R}^n)} \leqslant \sup\{|a(z)|\ ; z \in \mathbb{R}^{2n}\}$$

as required.

The following theorem is just the consequence of the latter.

Proposition 3.2.7. If a(z) in the above proposition is complex-valued then

$$\|A\|_{L^2(\mathbb{R}^n)} \leqslant 2\sup\{|a(z)|\ ; z \in \mathbb{R}^{2n}\}$$

Proof:

We know that $|Re(a(z))| \leqslant |a(z)|$ and $|Im(a(z))| \leq |a(z)|$.
Now let B and C be operators with aW-symbols $b(z) = Re(a(z))$ and $c(z) = Im(a(z))$, respectively.The operators B and C so defined satisfy the conditions of Proposition 3.2.6.Thus

$$\|B\|_{L^2(\mathbb{R}^n)} \leqslant \sup\{|b(z)|\,;z\in\mathbb{R}^{2n}\} \leqslant \sup\{|a(z)|\,;z\in\mathbb{R}^{2n}\}$$

and

$$\|C\|_{L^2(\mathbb{R}^n)} \leqslant \sup\{|c(z)|\,;z\in\mathbb{R}^{2n}\} \leqslant \sup\{|a(z)|\,;z\in\mathbb{R}^{2n}\}$$

Therefore

$$\|A\|_{L^2(\mathbb{R}^n)} \leqslant \|B\|_{L^2(\mathbb{R}^n)} + \|C\|_{L^2(\mathbb{R}^n)}$$

consequently

$$\|A\|_{L^2(\mathbb{R}^n)} \leqslant 2\sup\{|a(z)|\,;z\in\mathbb{R}^{2n}\} \qquad \text{as required.}$$

Now we shall determine the relationship between the T-symbol and the aW-symbol of a pseudo-differential operator.The following theorem tells us that every symbol $a(z)\in\Gamma_\rho^m(\mathbb{R}^{2n})$ is also an aW-symbol of some pseudo-differential operator from the corresponding class.

Theorem 3.2.8. Let $A'\in G_\rho^m(\mathbb{R}^n)$.Then there is an operator $A\in G_\rho^m(\mathbb{R}^n)$ with an aW-symbol $a(z)\in\Gamma_\rho^m(\mathbb{R}^{2n})$ such that $A-A'\in G^{-\infty}(\mathbb{R}^n)$.(Recall that $A-A'\in G^{-\infty}(\mathbb{R}^n)$ if and only if $A-A'$ has a kernel form $S(\mathbb{R}^n)$).

Proof:

Let A' have a Weyl-symbol $b'(z)\in\Gamma_\rho^m(\mathbb{R}^{2n})$.Consider an operator A_0 with aW-symbol $a_0(z) = b'(z)$.Then by Theorem 3.2.3. we have $A'-A_0\in G_\rho^{m-2\rho}(\mathbb{R}^n)$

Next,let A_1 be an operator with an aW-symbol $a_1(z)$ equal to the Weyl-symbol of $A'-A_0$.Then we have $A'-(A_0+A_1)\in G_\rho^{m-4\rho}(\mathbb{R}^n)$.Continuing by induction we get a sequence of operators $A_0,A_1,A_2,\dots,A_p$ with aW-symbols $a_0(z),a_1(z),a_2(z),\dots,a_p(z)$ (respectively) such that $A'-(A_0+A_1+A_2+,\dots,+A_p)\in G_\rho^{m-2\rho p}(\mathbb{R}^n)$.If $a_k(z)$ is the Weyl-symbol of the operator $A_0+A_1+A_2+,\dots,+A_k$,then (since $b'(z)-a_k(z)\in\Gamma_\rho^{m-2\rho k}(\mathbb{R}^{2n})$ and

$m-2\rho k \to -\infty$ as $k \to +\infty$) we can find an operator with an aW-symbol $a(z)$ such that (by Theorem 1.3.2.)

$$a(z) \sim \sum_{k=0}^{\infty} a_k(z) .$$

We know that A-A' has a symbol from $S(\mathbb{R}^{2n})$. Therefore A-A' is an integral operator with a smooth kernel and this completes the proof.

We are now going to present the main results of this section.

Theorem 3.2.9. Let $A \in G_\rho^0(\mathbb{R}^n)$. Then A extends to a continuous operator on $L^2(\mathbb{R}^n)$.

Proof:

By Theorem 3.2.8 we can find $\tilde{A} \in G_\rho^0(\mathbb{R}^n)$ such that $A-\tilde{A} \in G^{-\infty}(\mathbb{R}^n)$.

So $A = \tilde{A}+K$, where K is an integral operator with a kernel $k(x,y) \in S(\mathbb{R}^{2n})$. Since $\tilde{a}(z) \in \Gamma_\rho^0(\mathbb{R}^{2n})$ we then have $|\tilde{a}(z)| \leq C$, where C is some positive constant. Thus,

$\sup\{ |\tilde{a}(z)| ; z \in \mathbb{R}^{2n}\} \leq C$. By Proposition 3.2.6. we conclude that,

$$\|\tilde{A}\|_{L^2(\mathbb{R}^n)} \leq C ,$$

and therefore,

$$\|A\|_{L^2(\mathbb{R}^n)} \leq \|\tilde{A}\|_{L^2(\mathbb{R}^n)} + \|K\|_{L^2(\mathbb{R}^n)} .$$

The above inequality proves that A is bounded on $L^2(\mathbb{R}^n)$ since $\tilde{A}$ and K are bounded on $L^2(\mathbb{R}^n)$.

Finally we have

Theorem 3.2.10. If $A \in G_\rho^m(\mathbb{R}^n)$, $m < 0$, then A extends to a compact operator on $L^2(\mathbb{R}^n)$.

Proof:

Suppose $A \in G_\rho^m(\mathbb{R}^n)$, $m < 0$ with an aW-symbol $a(z) \in \Gamma_\rho^m(\mathbb{R}^{2n})$. Let $\chi(z) \in C_0^\infty(\mathbb{R}^{2n})$,

$0 \leq x(z) \leq 1$, ($x(z) = 1$ for $\|z\| < 1/2$ and $x(z)=0$ for $\|z\| > 1$) .

Put $a_L(z) = x(L^{-1}z)a(z)$ and A_L be an operator with aW-symbol $a_L(z)$.

Then

$\sup\{|a(z)-a_L(z)| ; z \in R^{2n}\} \to 0$ as $L \to \infty$

and

$|a(z)| \leq C<z>^m$ where C is some positive constant.Thus given $\varepsilon>0$ choose $<z> \geq M$ (M is some positive constant) such that $C<z>^m < \varepsilon$,(this is possible since $m < 0$). Since $|a_L| \leq |a|$ we have $|a_L(z)| < \varepsilon$ for $<z> > M$. Then for every $<z> \geq M$ we can choose L such that

$| a(z)-a_L(z)| < \varepsilon$

precisely

$$|a(z)-a_L(z)| = |a(z)- x(L^{-1}z)a(z)|$$

$$= |a(z)| \, |1-x(L^{-1}z)|$$

$$\leq |a(z)| \leq C<z>^m < \varepsilon \; .$$

Therefore,for sufficiently large L, $|a(z)-a_L(z)| < \varepsilon$, and $a(z) = a_L(z)$ for $|z| < L$. Since $a_L(z) \in C_o^\infty(\mathbb{R}^{2n})$ we conclude that A_L is a compact operator.From Proposition 3.2.6. $\|A-A_L\|_{L^2(\mathbb{R}^n)} \leq \sup\{|a(z)-a_L(z)| ; z \in \mathbb{R}^{2n}\}$.

Therefore $\|A_L - A\| \to 0$ as $L \to \infty$ and thus A is also a compact operator. We have used the fact that a limit (in the norm) of a sequence of compact operators is also a compact operator.

CHAPTER 4

HYPOELLIPTIC OPERATORS

4.1 INTRODUCTION

The class of hypoelliptic pseudo-differential operators in $\mathbb{R}^n$ is a subset of $G^m_\rho(\mathbb{R}^n)$ and a special property that every hypoelliptic pseudo-differential operator has a parametrix or pseudo-inverse. We shall study the construction of the parametrix of any such an operator. We shall employ the classical symbols to demonstrate the construction of this parametrix explicitly.

4.2 HYPOELLIPTIC SYMBOLS

Definition 4.2.1. We write $a(z) \in H\Gamma^{m,m_0}_\rho(\mathbb{R}^{2n})$ if

(1) --- $a(z) \in C^\infty(\mathbb{R}^{2n})$.

(2) --- there is a constant $R > 0$ such that for every multi-index α, there are constants C, C_1 and C_α all positive, such that

(H1) --- $C\,(\|z\|)^{m_0} \leq |a(z)| \leq C_1\|z\|^m$.

(H2) --- $|D^\alpha_z a(z)| \leq C_\alpha\, |a(z)|\|z\|^{-\rho|\alpha|}$ for $\|z\| \geq R$.

We shall denote by $HG^{m,m_0}_\rho(\mathbb{R}^n)$ the class of pseudo-differential operators with Υ-symbols belonging to $H\Gamma^{m,m_0}_\rho(\mathbb{R}^{2n})$.

Remark: H1 and H2 imply that $a(z) \in \Gamma^m_\rho(\mathbb{R}^{2n})$ and also $HG^{m,m_0}_\rho \subset G^m_\rho$.

Firstly, we give some properties of the $H\Gamma^{m,m_0}_\rho(\mathbb{R}^{2n})$ classes. These properties are contained in the Lemma below.

Lemma 4.2.2. The classes $H\Gamma^{m,m_0}_\rho(\mathbb{R}^{2n})$ have the following properties.

(a) (i) If $a(z) \in H\Gamma^{m,m_0}_\rho(\mathbb{R}^{2n})$ then $a^{-1}(z) = 1/_{a(z)} \in H\Gamma^{-m_0,-m}_\rho(\mathbb{R}^{2n})$

provided we multiply $a^{-1}(z)$ by $\varphi(z) \in C^\infty(\mathbb{R}^{2n})$ such that $\varphi(z) = \begin{cases} 0 & \text{if } \|z\| \leq \frac{1}{2} \\ 1 & \text{if } \|z\| \geq 1 \end{cases}$

(ii) If $a(z) \in H\Gamma_\rho^{m,m_0}(\mathbb{R}^{2n})$ and $a'(z) \in H\Gamma_\rho^{m',m_0'}(\mathbb{R}^{2n})$, then $a(z)a'(z) \in H\Gamma_\rho^{m+m',m_0+m_0'}(\mathbb{R}^{2n})$.

(iii) If $a(z) \in H\Gamma_\rho^{m,m_0}(\mathbb{R}^{2n})$ and $r(z) \in \Gamma_\rho^{m_1}(\mathbb{R}^{2n})$ where $m_1 < m_0$, then $a(z) + r(z) \in H\Gamma_\rho^{m,m_0}(\mathbb{R}^{2n})$.

(b) (α) If $A \in HG_\rho^{m,m_0}(\mathbb{R}^n)$ and $A' \in HG_\rho^{m',m_0'}(\mathbb{R}^n)$ then $A \circ A' \in HG_\rho^{m+m_0',m\ +m'}(\mathbb{R}^n)$.

(β) If $A \in HG_\rho^{m,m_0}$ then $A^*, {}^tA \in HG_\rho^{m,m_0}(\mathbb{R}^n)$.

(γ) If $A \in HG_\rho^{m,m_0}(\mathbb{R}^n)$ and $R \in G_\rho^{m_1}(\mathbb{R}^n)$ where $m_1 < m_0$, then $A+R \in HG_\rho^{m,m_0}(\mathbb{R}^n)$.

Proof: (a) (i). From H1 it is clear that the following is true

$$C_1^{-1}\|z\|^{-m} \leq |a^{-1}(z)| \leq C\,\|z\|^{-m_0} \quad \text{for } \|z\| \geq R.$$

It is enough to give the estimates for partial derivatives. After expressing $\partial_z^\alpha a^{-1}(z)$ in terms of the partial derivatives of $a(z)$ and using H2 we get the required estimates;

$$|\partial_z^\alpha a^{-1}(z)| \leq C_\alpha |a^{-1}(z)|\, \|z\|^{-\rho|\alpha|}.$$

The proof of (a) (ii) follows immediately upon the application of the Leibnitz formula.

(a) (iii). Write $a(z) + r(z) = a(z)\ (1 + \frac{r(x)}{a(x)})$.

From (i) and (ii) we have $\frac{r(z)}{a(z)} \in \Gamma_\rho^{m_1-m_0}(\mathbb{R}^{2n})$ where $m_1 - m_0 < 0$.

Therefore $1 + \frac{r(z)}{a(z)} \in \Gamma_\rho^0(\mathbb{R}^n)$.

Observe that our task is simplified since we only have to prove that $1 + \frac{r(z)}{a(z)} \in H\Gamma_\rho^{0,0}$ since by (ii) we shall be in a position to draw the conclusion that $a(z)(1 + {}^{r(z)}/_{a(z)}) \in H\Gamma_\rho^{m,m_0}(\mathbb{R}^{2n})$ and thus establishing the required results. Really, from (i) and the definition of the Γ^m_ρ-classes, there exist some positive constants C_1 and C_2 such that (2)--- $C_1 \|z\|^{m_1-m_0} \leq |r(z)/a(z)| \leq C_2 \|z\|^{m_1-m_0}$. Since $m_1-m_0<0$, we conclude that there are constants C' and C" (both positive) such that

$$C' \leq \left|1 + \frac{r(z)}{a(z)}\right| \leq C'' \quad \text{for} \quad \|z\| \geq R \quad (R>0) \quad .$$

Also

$$\left|\partial_z^\alpha\left(1 + \frac{r(z)}{a(z)}\right)\right| = \left|\partial_z^\alpha \left[\frac{r(z)}{a(z)}\right]\right|$$

applying Leibnitz formula we get

$$\begin{aligned}
|\partial_z^\alpha(1 + {}^{r(z)}/_{a(z)})| &= \Big|\sum_{\delta+\gamma=\alpha} \frac{\alpha!}{\delta!\,\gamma!}\, \partial_z^\delta r(z)\; \partial_z^\gamma a^{-1}(z)\Big| \\
&\leq \sum_{\delta+\gamma=\alpha} \frac{\alpha!}{\delta!\,\gamma!}\, |\partial_z^\delta r(z)|\; |\partial_z^\gamma a^{-1}(z)| \\
&\leq \sum_{\delta+\gamma=\alpha} \frac{\alpha!}{\delta!\,\gamma!}\, C_\delta \|z\|^{m_1-\rho|\delta|}\, C_\gamma \|z\|^{-m_0-\rho|\gamma|} \\
&= C_\alpha \|z\|^{m_1-m_0-\rho|\alpha|} \quad \text{, where } C_\alpha = \sum_{\delta+\gamma=\alpha} \frac{\alpha!}{\delta!\;\gamma!} C_\delta C_\gamma \\
&\leq C_\alpha \|z\|^{-\rho|\alpha|} \quad \text{since } m_1-m_0 \leq 0
\end{aligned}$$

Hence $1 + {}^{r(z)}/_{a(z)} \in H\Gamma_\rho^{0,0}(\mathbb{R}^{2n})$. This proves the required results.

The remaining properties follow from the relationship between symbols and their corresponding pseudo-differential operators.

Remark: Property (a) (iii) tells us that perturbation of a symbol by a lower class symbol does not alter its hypoelliptic class. Hence perturbation of a pseudo-differential operator, by another pseudo-differential operator from a lower class, does not change the hypoellipticity.

By Theorem 2.2.2. we expect that if $b_{\tau_1} \in H\Gamma_\rho^{m,m_0}(\mathbb{R}^n)$ then $b_\tau \in H\Gamma_\rho^{m,m_0}(\mathbb{R}^n)$ for any two different values of τ_1 and τ. This presumption is true.

Proposition 4.2.3. If for some $\tau_1 \in [0,1]$, $b_{\tau_1} \in H\Gamma_\rho^{m,m_0}(\mathbb{R}^{2n})$, then $b_\tau \in H\Gamma_\rho^{m,m_0}(\mathbb{R}^{2n})$ for all $\tau \in [0,1]$.

Proof: Since $H\Gamma_\rho^{m,m_0}(\mathbb{R}^{2n}) \subset \Gamma_\rho^m(\mathbb{R}^{2n})$, we have $b_{\tau_1}(z) \in \Gamma_\rho^m(\mathbb{R}^{2n})$ and thus by Theorem 2.2.2.

(1)---$b_\tau(x,\xi) \sim \sum_\alpha \frac{1}{\alpha!} (\tau_1-\tau)^{|\alpha|} \partial_\xi^\alpha D_x^\alpha b_{\tau_1}(x,\xi)$ and therefore

$b_\tau(x,\xi) - b_{\tau_1}(x,\xi) \in \Gamma_\rho^{m-2\rho}(\mathbb{R}^{2n})$.

Of course the leading term of the asymptotic expansion (1) is $b_{\tau_1} \in H\Gamma_\rho^{m,m_0}(\mathbb{R}^{2n})$. By property (a) (iii) of Lemma 4.2.2., we conclude that $b_\tau(x,\xi) \in H\Gamma_\rho^{m,m_0}(\mathbb{R}^{2n})$.

The action of $H\Gamma_\rho^{m,m_0}$-classes on the spaces $S(\mathbb{R}^n)$ and $S'(\mathbb{R}^n)$ is the same as for the Γ_ρ^m-classes since in fact $H\Gamma_\rho^{m,m_0}(\mathbb{R}^n) \subset \Gamma_\rho^m(\mathbb{R}^n)$. Hence we discuss some more interesting results about the action of the $H\Gamma_\rho^{m,m_0}$-classes on the function space $S'(\mathbb{R}^n)$. These results are contained in the Corollary below. We invoke the concept of a parametrix to prove these results. This concept will be discussed in the next section.

Corollary 4.2.4. If $A \in HG_\rho^{m,m_0}(\mathbb{R}^n)$, furthermore, $u \in S'(\mathbb{R}^n)$ and $Au \in S(\mathbb{R}^n)$, then $u \in S(\mathbb{R}^n)$.

Proof: We know that the map $A: S(R^n) \to S(\mathbb{R}^n)$ is continuous. Take the operator B which is also a parametrix of A, i.e. $AB = I + R_1$ and $BA = I + R_2$, where R_1 and R_2 are both from $G^{-\infty}(\mathbb{R}^n)$. Choose $u \in S'(\mathbb{R}^n)$ and let $Au = f \in S(\mathbb{R}^n)$. Then $B(Au) = Rf = u + R_2u$, hence $u = Bf - R_2u$. Since $B \in HG_\rho^{-m_0,-m}(\mathbb{R}^n)$, then $Bf \in S(\mathbb{R}^n)$ and $R_2u \in S(\mathbb{R}^n)$ as well (Theorem 1.6.5). Therefore we conclude that $u \in S(\mathbb{R}^n)$.

4.3 PARAMETRIX OF A HYPOELLIPTIC OPERATOR

Before the formulation of the theorem on parametrix of a hypoelliptic pseudo-differential operator, we recapitulate some basic facts about the pseudo-differential operators and their symbols.

Let $Au(x) = (2\pi)^{-n} \iint e^{i(x-y)\xi} a(x,\xi)\, u(y)\, dy\, d\xi$ where $a(x,\xi) \in \Gamma^{m}_{\rho}(\mathbb{R}^{2n})$ is a left symbol of A and $u \in S(\mathbb{R}^{n})$. Suppose further that

(a)---$a(x,\xi) \sim \sum_{j=1}^{\infty} a_i(x,\xi)$ is an asymptotic expansion of $a(x,\xi)$ in Γ^{m}_{ρ} classes, where $a_j(z) \in \Gamma^{m_j}_{\rho}(\mathbb{R}^{2n})$ such that $m_1 > m_2 > m_3 > \dots$ and $m_1 = m$. Then the asymptotic expansion (a) defines A mod $G^{-\infty}(\mathbb{R}^{n})$ operators.

To recover A from the asymptotic expansion, we use a smoothing function $\Psi(x,\xi)$ and the appropriate sequence $\{t_j\}$, such that $t_j \to +\infty$ as $j \to +\infty$, and set

(b)---$\tilde{a}(x,\xi) \sim \sum_{j=1}^{\infty} \Psi[t_j^{-1}(x,\xi)] a_j(x,\xi)$ converges in Γ^{m}_{ρ}-classes to an amplitude in $\Gamma^{m}_{\rho}(\mathbb{R}^{2n})$ such that

$$a(x,\xi) - \tilde{a}(x,\xi) \in \Gamma^{-\infty}(\mathbb{R}^{2n}) = S(\mathbb{R}^{2n}).$$

This correspondence extends to the algebra of operators and algebra of symbols as in Chapter 2.

Consider

(c)---$Bu(x) = (2\pi)^{-n} \iint e^{i(x-y)\xi} b(x,\xi)\, u(y)\, dy\, d\xi$.

If the symbol of the composition $A \circ B$ describes an identity operator or (after the composition formula)

(d)---$\sum_{\alpha} \frac{1}{\alpha!} \partial^{\alpha}_{\xi} a(x,\xi) D^{\alpha}_{x} b(x,\xi) \sim 1$

(we assume that $b(x,\xi)$ is a left symbol of the operator B in (c) and then apply the results of Theorem 2.5.2) then the composition $A \circ B$ of A and B is the identity operator plus a smoothing operator since by repeating the construction of the asymptotic (d) (as for (c)) we get an operator which differs from the identity operator by an operator from $G^{-\infty}(R^{n})$ or an operator with a symbol from $\Gamma^{-\infty}(\mathbb{R}^{2n}) = S(\mathbb{R}^{2n})$. The operator B satisfying (1) of Theorem 4.3.1 is called a parametrix of A.

This reduces the problem of finding B with property (d) to a pure algebraic problem of finding an asymptotic expansion for B which defines a symbol of B and has property (d).

Theorem 4.3.1. If $A \in HG_\rho^{m,m_o}(\mathbb{R}^n)$ then there exists $B \in HG_\rho^{-m_o,-m}(\mathbb{R}^n)$ such that

(1)---$AB = I + R_1$ and $BA = I + R_2$, where R_1 and R_2 are both from $G^{-\infty}(\mathbb{R}^n)$. Moreover, if $B' \in HG_\rho^{-m_o,-m}(\mathbb{R}^n)$ is another such an operator, i.e. $B'A - I$ and $AB' - I$ are both from $G^{-\infty}(\mathbb{R}^n)$, then $B' - B \in G^{-\infty}(\mathbb{R}^n)$.

Proof: Consider

(2)---$$Au(x) = (2\pi)^{-n} \iint e^{i(x-y)\xi} a(x,\xi)\, u(y)\, dy\, d\xi$$

where $a(x,\xi) \in H\Gamma_\rho^{m,m_o}(\mathbb{R}^{2n})$ and $u \in S(\mathbb{R}^n)$.

Then $a^{-1}(z) \in H\Gamma_\rho^{-m_o,-m}(\mathbb{R}^{2n})$ by proprty (a)(i) of Lemma 4.2.2.Let $b_o(x,\xi) = a^{-1}(x,\xi)$ for $\|(x,\xi)\| \geq R$,where $R \geq 0$.From Theorem 2.5.3 the symbol $b_o \circ a$ of the composition of the symbols b_o and a,we have

(3)---$$\begin{aligned}(b_o \circ a)(x,\xi) &\sim \sum_\alpha \frac{1}{\alpha!}\, \partial_\xi^\alpha b_o(x,\xi)\, D_x^\alpha a(x,\xi) \\ &= 1 + \sum_{|\alpha| \geq 1} \frac{1}{\alpha!}\, \partial_\xi^\alpha a^{-1}(x,\xi)\, D_x^\alpha a(x,\xi) \\ &= 1 + \sum_{|\alpha| \geq 1} \frac{1}{\alpha!} \frac{\partial_\xi^\alpha a^{-1}(x,\xi)}{a^{-1}(x,\xi)} \frac{D_x^\alpha a(x,\xi)}{a(x,\xi)}\end{aligned}$$

for all $\|(x,\xi)\| \geq R$.

Now take the operator C_o with amplitude

(4)---$$c_o(x,\xi) \sim \sum_{|\alpha| \geq 1} \frac{1}{\alpha!} \frac{\partial_\xi^\alpha a^{-1}(x,\xi)}{a^{-1}(x,\xi)} \frac{D_x^\alpha a(x,\xi)}{a(x,\xi)}$$

i.e. the symbol of C_o has an asymptotic expansion given by the series (4) above. Clearly $c_o(x,\xi) \in \Gamma_\rho^{-2\rho}(\mathbb{R}^{2n})$ by (H2) of Definition 4.2.1.

Explicitly, we observe that

$$a^{-1}(x,\xi) \in H\Gamma_\rho^{-m_0,-m}(\mathbb{R}^{2n}) \subset \Gamma_\rho^{-m_0}(\mathbb{R}^{2n}) .$$

Therefore $\partial_\xi^\alpha a^{-1}(x,\xi) \in \Gamma_\rho^{-m_0-\rho|\alpha|}(\mathbb{R}^{2n})$ by Lemma 4.2.2

thus $\dfrac{\partial_\xi^\alpha a^{-1}(x,\xi)}{a^{-1}(x,\xi)} \in \Gamma_\rho^{-\rho|\alpha|}(\mathbb{R}^{2n})$ by Lemma 1.2.1 ,1.

also, $a(x,\xi) \in H\Gamma_\rho^{m,m_0}(\mathbb{R}^{2n}) \subset \Gamma_\rho^{m}(\mathbb{R}^{2n})$.

Therefore $\dfrac{\partial_\xi^\alpha (x,\xi)}{a(x,\xi)} \in \Gamma_\rho^{-\rho|\alpha|}(\mathbb{R}^{2n})$.

Hence $\dfrac{\partial_\xi^\alpha a^{-1}(x,\xi)}{a^{-1}(x,\xi)} \cdot \dfrac{\partial_x^\alpha a(x,\xi)}{a(x,\xi)} \in \Gamma_\rho^{-2\rho|\alpha|}(\mathbb{R}^{2n})$ by Lemma 1.2.1,1.

Thus the leading term of $c_0(x,\xi)$ (the term corresponding to $|\alpha| = 1$) belongs to $\Gamma_\rho^{-2\rho}(\mathbb{R}^{2n})$. Since differentiation improves the class, we conclude that $c_0(x,\xi) \in \Gamma_\rho^{-2\rho}(\mathbb{R}^{2n})$. Consequently we can write

$B_0A = I + C_0$, were $C_0 \in G_\rho^{-2\rho}(\mathbb{R}^n)$. Thus by Theorem 3.2.10., C_0 is a compact operator. This fact allows us to express $(I + C_0)^{-1}$ in terms of a Neumann Series i.e. $(I + C_0)^{-1} = I - C_0 + C_0^2 ---- .$

We employ it to algebra of symbols,(see Appendix E).

(5)--- $1+r(x,\xi) \sim [1 + c_0(x,\xi)]^{-1} = \sum_{i=0}^{\infty} (-1)^i [c_0(x,\xi)]^i$ where

$[c_0(x,\xi)]^i = c_0(x,\xi) \circ c_0(x,\xi) \circ \ldots \circ c_0(x,\xi)$ = i-factors of composition of $c_0(x,\xi)$, ($\circ$ denotes the product of left symbols).

Next take an operator R with a symbol $1+r(x,\xi)$. From (5) it follows that $R \circ (I + C_0) = I + R_1$ and $(I + C_0) \circ R = I + R_2$ where R_1 and R_2 are both from $G^{-\infty}(\mathbb{R}^n)$. So $B_0 A = I + C_0$ and thus

$$R(B_0A) = R(I + C_0) = I + R_1 .$$

If we define the operator $B = RB_0$, then we get the required results. By a similar construction as above we can produce B_2 such that $AB_2 = I + R_2$, $R_2 \in G^{-\infty}(\mathbb{R}^n)$. Now $BAB_2 = B(I+R_2) = (I+R_1)B_2$.

So, $B - B_2 = R_1B_2 - BR_2 \in G^{-\infty}(\mathbb{R}^n)$. Thus is the right parametrix.

Let $BA = I + R_1$ and $AB = I + R_2$; $B'A = I + R_1'$ and $AB' = I + R_2'$.

Then $BAB' = B' + R_1B'$. Using $AB' = I + R_2'$ we get

$B(I + R_2') = B + BR_2' = B' + R_1B'$. Thus $B' - B = -R_1B' + BR_2' \in G^{-\infty}(\mathbb{R}^n)$.

This establishes the required result.

4.4 ELLIPTIC OPERATORS AND CLASSICAL SYMBOLS

We shall define a class of pseudo-differential operators in $\mathbb{R}^n$ which we call elliptic pseudo-differential operators in $\mathbb{R}^n$ and then show that this class is just a subset of the class of hypoelliptic pseudo-differential operators in $\mathbb{R}^{2n}$.

In particular, let $A = \sum_{|\alpha| \leq m} a_\alpha(x) D_x^\alpha$ be a differential operator on $S(\mathbb{R}^n)$. Suppose further that the functions $a_\alpha(x)$ are positively homogeneous of degree $|\alpha|$ in the x-variable. The symbol (left symbol in this instance) of A is then given by $b(x,\xi) = \sum_{|\alpha| \leq m} a_\alpha(x)\xi^\alpha$.

Now introduce $a_k(x,\xi) = \sum_{|\alpha| = k} a_\alpha(x)\xi^\alpha$. Then $b(x,\xi) = \sum_{k=0}^{m} a_k(x,\xi)$.

It is evident that the term with the highest order, i.e. $a_m(x,\xi)$ (also called the leading term of the symbol) decides the nature of the operator in view of Lemma 4.2.2. (a) (iii). We define ellipticity as follows:

Definition 4.4.1. A differential operator A having a symbol with the leading term $a_m(x,\xi)$ is called elliptic if

$$a_m(x,\xi) \neq 0 \quad \text{for} \quad (x,\xi) \neq 0 .$$

Next we shall define the classical symbol and then employ this notion to give a demonstration of the construction of a parametrix of a classical elliptic pseudo-differential operator in $\mathbb{R}^n$. They happen to yield an explicit formula in this regard.

Definition 4.4.2. Let $a(x,\xi) \in \Gamma_\rho^m(\mathbb{R}^{2n})$. We say $a(x,\xi)$ is a classical symbol if its asymptotic formula is given by the series

(1)---$a(x,\xi) \sim \sum_{j=0}^{\infty} a_{m-j}(x,\xi)$ where $a_{m-j}(x,\xi)$ are positively homogeneous of degree $m-j$ in (x,ξ), furthermore we say that $a(x,\xi)$ is a classical elliptic symbol if its leading term $a_m(x,\xi) \neq 0$ when $(x,\xi) \neq 0$.

Remark: In view of Example 1.1.3.,we infer that $a_{m-j}(x,\xi) \in \Gamma_1^{m-j}(\mathbb{R}^{2n})$. In the case of $S_{\rho\delta}^m$ symbols a homogeneity in ξ is required.

The following proposition tells us that every elliptic pseudo-differential operator is hypoelliptic.

Proposition 4.4.3. Every classical elliptic symbol $a(z) \in \Gamma_\rho^m(\mathbb{R}^{2n})$ belongs to $H\Gamma_\rho^{m,m}(\mathbb{R}^{2n})$.

Proof: We have $a(x,\xi) \sim a_m(x,\xi) + \sum_{j=1}^{\infty} a_{m-j}(x,\xi)$ and $a_m(x,\xi) \in \Gamma_1^m(\mathbb{R}^{2n})$ by definition.Furthermore,observe that

$a_m(z) = \|z\|^m a_m\left(\frac{z}{\|z\|} \right)$ since $a_m(z)$ is positively homogeneous of degree m.

Therefore the following estimates are valid (for $\|z\| \geq R > 0$) ,

$$C_1 \|z\|^m \leq |a_m(z)| \leq C_2 \|z\|^m ,$$

where $C_1 = \min_w \{ |a_m(w)| : \|w\| = 1 \}$ and $C_2 = \max_w \{ |a_m(z)| : \|w\| = 1 \}$.

Therefore $a_m(z) = a_m(x,\xi) \in H\Gamma_\rho^{m,m}(R^{2n})$ since $a_m(z)$ satisfies $H2^-$(see Appendix F).

Also we can show that $\sum_{j=1}^{\infty} a_{m-j}(x,\xi) \in \Gamma_1^{m-1}(\mathbb{R}^{2n})$ (Lemma 4.2.2. (a)(iii)).

Hence by (a)(iii) of Lemma 4.2.2. we conclude that ,

$$a(x,\xi) \sim a_m(x,\xi) + \sum_{j=1}^{\infty} a_{m-j}(x,\xi) \text{ belongs to } H\Gamma_\rho^{m,m}(\mathbb{R}^{2n}).$$

4.5.PARAMETRIX OF PSEUDO-DIFFERENTIAL OPERATORS IN $\mathbb{R}^n$ WITH CLASSICAL SYMBOLS.

Suppose $a(z)$ is a classical elliptic symbol with an asymptotic series

$$(1)\text{---}\ a(z) \sim \sum_{j=0}^{\infty} a_{2m-2j}(z)\ ,$$

where the $a_{2m-2j}(z)$'s are positively homogeneous functions of degree $2m-2j$ (all elliptic operators in $\mathbb{R}^n$ are of even order when $n \geq 2$) . By a parametrix of $a(z)$ we mean an amplitude of the form;

$$(2)\text{---}\ b(z) \sim \sum_{j=0}^{\infty} b_{-2m-2j}(z)\ ,$$

where the $b_{-2m-2j}(z)$ are positively homogeneous functions of degree $-2m-2j$,such that the composition of (1) and (2) equal to

$$(3)\text{---}\ \begin{cases} 1 \sim a(z)\circ b(z) \\ \text{or} \\ 1 \sim \sum\limits_{\alpha}^{\infty} \frac{1}{\alpha!}\ \partial_{\xi}^{\alpha}\, a(z)\ D_x^{\alpha}\, b(z), \end{cases}$$

here we treat $a(z)$ and $b(z)$ as left symbols.

Inserting(1) and (2) into (3) we get

$$(4)\text{---}1 \sim \sum_{|\alpha|,k,\ell=0} \frac{1}{\alpha!}\ \partial_{\xi}^{\alpha}\ a_{2m-2k}(z)\ D_x^{\alpha}\ b_{-2m-2\ell}(z)\ .$$

Observe that $\partial_{\xi}^{\alpha}\ a_{2m-2k}(z)$ is positively homogeneous of degree $2m-2k-|\alpha|$ and $D_x^{\alpha}\ b_{-2m-2\ell}(z)$ of degree $-2m-2\ell-|\alpha|$. Now grouping terms of the same degree we get

$$(5)\text{---}1 \sim \sum_{j=0}\ \sum_{k+\ell+|\alpha|=j} \frac{1}{\alpha!}\ \partial_{\xi}^{\alpha}\ a_{2m-2k}(z)\ D_x^{\alpha}\ b_{-2m-2\ell}(z)$$

with the observation that when $j=0$ then $k=\ell=|\alpha|=0$. Consequently we get the following triangular system for determining the parametrix

$$(6)\text{---}\quad\begin{cases} 1 = a_{2m}(z)\ b_{-2m}(z) \\ \text{(we shall leave out the argument "z" for clarity)} \\ 0 = a_{2m}\ b_{-2m-2j} + \sum\limits_{\substack{k+\ell+|\alpha|=j \\ \ell<j}} \frac{1}{\alpha!}\ \partial_\xi^\alpha\, a_{2m-2k}\ D_x^\alpha\ b_{-2m-2\ell}\ . \end{cases}$$

The system (b) defines a recurrence relation for finding the b_{-2m-2j}'s. Let us try to deduce a formula for the b_{-2m-2j}'s.

Now we have $b_{-2m}(z) = {}^1/a_{2m}(z) = a_{2m}^{-1}(z)$ $(j=0)$. When $j=1$, we have

$$0 = a_{2m}\ b_{-2m-2} + \sum_{k+|\alpha|=1} ({}^1/_{\alpha!})\, \partial_\xi^\alpha\, a_{2m-2k}\ D_x^\alpha\ b_{-2m}$$

$(\ell=0$ in this case)

$$= a_{2m}\ b_{-2m-2} + a_{2m-2}\ b_{-2m} + \sum_{s=1}^{n} \partial_{\xi_s} a_{2m}\ D_{x_s} b_{-2m}\ .$$

Therefore:

$$b_{-2m-2} = -(a_{2m-2}\ b_{-2m})\ a_{2m}^{-1} - \Big(\sum_{s=1}^{n} \partial_{\xi_s} a_{2m}\ D_{x_s} b_{-2m}\Big) a_{2m}^{-1}$$

$$= -\ a_{2m-2}\ a_{2m}^{-2} - \Big(\frac{1}{i} \sum_{s=1}^{n} \partial_{\xi_s} a_{2m}\ \partial_{x_s} a_{2m}^{-1}\Big)\ a_{2m}^{-3})\ .$$

A similar calculation can be done for b_{-2m-4}, b_{-2m-6} etc .

We can write in general

(7)---$b_{-2m-2j}(z) = \sum\limits_{s=2}^{j+1} c_s(z) a_{2m}^{-s}(z)$ where the coefficents $c_s(z)$ are given in terms of the polynomials of a_{2m-2s} and their partial derivatives of degree less than or equal to $2j$. Note that

$\deg[c_s(z)\, a_{2m}^{-s}(z)] = \deg[c_s(z)] + \deg[a_{2m}^{-s}(z)]$, and thus

$$-2m-2j = \deg(c_s(z)) + (-2ms). \quad \text{Therefore}$$

(8)---$\deg[c_s(z)] = 2[m(s-1-j]$ where "deg" means "degree of".

Example 4.5.1. Let us determine the parametrix of the operator $A = -\frac{d^2}{dx^2} + x^2$. It is the quantum harmonic oscillator operator. The left symbol of this operator is;

$a(x,\xi) = \xi^2 + x^2$.

By formula (1) we can write ;

$a(x,\xi) = a_2(x,\xi) \in Hr_1^{2,2}(R^{2n})$. In other words $m = 1$ in this particular case.

Then we have

$$b_{-2} = \frac{1}{a_2}$$
$$= \frac{1}{\xi^2 + x^2}$$

$$b_{-4} = -a_0 a_2^{-2} + \frac{1}{i}(a_2\, \partial_x a_2^{-1}) a_2^{-3}$$
$$= \frac{i4x\xi}{(\xi^2 + x^2)^3}$$

The subsequent terms, i.e. b_{-6}, b_{-8}, b_{-10} etc. are expressible in the form;

$$\frac{p_k(x,\xi)}{(\xi^2 + x^2)^k},$$

where $p_k(x,\xi)$ are homogeneous polynomials in x and ξ.

For $\|(x,\xi)\| \geqslant R$, where R is some suitably large positive constant we can write the parametrix of $a(x,\xi) = \xi^2 + x^2$ as

$$b(x,\xi) \sim \frac{1}{\xi^2 + x^2} + \frac{i4x\xi}{(\xi^2 + x^2)^3} + \cdots .$$

For large values of R the terms $\frac{p_K(x,\xi)}{(\xi^2 + x^2)^k} \to 0$ as $k \to \infty$ and thus in this case the series on the right hand side of $b(x,\xi)$ above is the asymptotic formula of $b(x,\xi)$.

Finally, it may be useful to mention that the process of solving the system (6) may be stopped at any stage. If for example $m > 0$, then by taking $b_{-2m}(x,\xi)$ only, we obtain an operator B_0 with a symbol $b_{-2m}(x,\xi)$ which is an approximation of the parametrix of A. In other words

$A \circ B_0 = I + K_0$,

where K_0 is an operator with amplitude $b_{-2m-2}(x,\xi) \in \Gamma^{-2m-2}(\mathbb{R}^{2n})$ and thus K_0 is a compact operator by Theorem 3.2.10. So this first approximation of the parametrix reduces the problem of finding a pseudo-inverse of A (if it does exist), to Fredholm Theory (see Zabreyko [31]).

4.6. THE SOBOLEV SPACES

The Sobolev Spaces which we shall denote by $Q^s(\mathbb{R}^n)$ is a subspace of $S'(\mathbb{R}^n)$. It is in fact the domain of the pseudo-differential operator L_s. The operator L_s is characterized by the left symbol;

$b(x,\xi) = (1 + \|x\|^2 + \|\xi\|^2)^{s/2}$.

It is apperent that $L_s \in HG_1^{s,s}$ and thus has a parametrix. We shall denote this parametrix by L_{-s}. Clearly L_{-s} belongs to $HG_1^{-s,-s}(\mathbb{R}^n)$.

Some of the stunning results we are going to present in this sequel are the facts that ;

(a) every $A \in G_\rho^m(\mathbb{R}^n)$ defines a continuous operator $A : Q^s(\mathbb{R}^n) \to Q^{s-m}(\mathbb{R}^n)$

(b) $Q^s(\mathbb{R}^n)$ has a Hilbert Space structure on it.

The results of this section are preliminary to the proof of the fact that every hypoelliptic pseudo-differential operator has a discrete spectrum only. This result will be presented in the section on spectral theory.

Definition 4.6.1. We say $u \in S'(\mathbb{R}^n)$ belongs to $Q^s(\mathbb{R}^n)$ if $L_s u \in L^2(\mathbb{R}^n)$, where L_s is a hypoelliptic pseudo-differential operator with a left symbol $b(x,\xi) = (1+\|x\|^2+\|\xi\|^2)^{s/2}$ or $b(z) = \langle z\rangle^s$, $z = (x,\xi)$. Clearly $b(x,\xi) \in Hr_1^{s,s}(\mathbb{R}^{2n})$, and

$$(1)\text{---}L_s u(x) = (2\pi)^{-n} \iint e^{i(x-y)\xi}(1+\|x\|^2+\|\xi\|^2)^{s/2}\, u(y)\,dy\,d\xi \quad \text{for } u \in S(\mathbb{R}^n).$$

The proposition below furnishes us with knowledge of the effect of the $G_\rho^m(\mathbb{R}^n)$ operators on the space $Q^s(\mathbb{R}^n)$.

Proposition 4.6.2.

(a) If $A \in G_\rho^m(\mathbb{R}^n)$ then A defines an operator $A: Q^s(\mathbb{R}^n) \to Q^{s-m}(\mathbb{R}^n)$.

(b) If $A \in HG_\rho^{m,m_0}(\mathbb{R}^n)$, $u \in S'(\mathbb{R}^n)$ and $Au \in Q^s(\mathbb{R}^n)$ then $u \in Q^{s+m_0}(\mathbb{R}^n)$.

Proof:

(a) Consider $L_s \in HG_\rho^{s,s}(\mathbb{R}^n)$ and its parametrix $L_{-s} \in HG_\rho^{-s,-s}(\mathbb{R}^n)$. Then

$(0)\text{---}\ L_{-s} \circ L_s = I + R_s$ where $R_s \in G^{-\infty}(\mathbb{R}^n)$.

Take $u \in Q^s(\mathbb{R}^n)$, then $L_s u = u_\circ \in L^2(\mathbb{R}^n)$ by the definition of $Q^s(\mathbb{R}^n)$. Thus we can write

$$\begin{aligned} u &= L_{-s}u_\circ - R_s u \\ &= L_{-s}u_\circ + v \quad \text{where } v = -R_s u \in S(\mathbb{R}^n) \text{ by Theorem 1.6.5.} \end{aligned}$$

Hence

$$\begin{aligned} L_{s-m}Au &= L_{s-m}A(L_{-s}u_\circ + v) \\ &= L_{s-m}A\,L_{-s}u_\circ + L_{s-m}Av \end{aligned}$$

Observe that $L_{s-m}A\,L_{-s} \in G_\rho^\circ(\mathbb{R}^n)$ and $Av \in S(\mathbb{R}^n)$ and since

$L_{s-m}\,AL_{-s} \in G^{\circ}_{\rho}\,(\mathbb{R}^n)$, it extends to a continuous operator on $L^2\,(\mathbb{R}^n)$. Hence $L_{s-m}AL_{-s}\,u_\circ \in L^2(\mathbb{R}^n)$.

Furthermore, since $L_{s-m} \in HG_1^{s-m,s-m}(\mathbb{R}^n) \subset G_1^{s-m}(\mathbb{R}^n)$ we conclude that $L_{s-m}:S(\mathbb{R}^n)\to S(\mathbb{R}^n)$ is continuous and therefore $L_{s-m}Av \in S(\mathbb{R}^n)$. Therefore $L_{s-m}Au \in L^2(\mathbb{R}^n)$ which means that $Au \in Q^{s-m}(\mathbb{R}^n)$. This proves the required result.

(b) Let $A \in HG^{m,m_\circ}_{\rho}(\mathbb{R}^n)$ and $Au \in Q^s(\mathbb{R}^n)$. We want to prove that $u \in Q^{s+m_\circ}(\mathbb{R}^n)$. Take a parametrix B of A. Then $B \in HG_{\rho}^{-m_\circ,-m}(\mathbb{R}^n)$, and $BAu = u + R_su$, where $R_s \in G^{-\infty}(\mathbb{R}^n)$. Thus $u = RAu - R_su$. Since $R_s \in G^{-\infty}(\mathbb{R}^n)$ and $Au \in Q^s(\mathbb{R}^n)$ then from (a) above we have

$BAu \in Q^{s-(-m_\circ)}(\mathbb{R}^n)$ or $BAu \in Q^{s+m_\circ}(\mathbb{R}^n)$. We know that R_s transforms $S'(\mathbb{R}^n)$ into $S(\mathbb{R}^n)$. This proves that

$u = BAu - R_su \in Q^{s+m_\circ}(\mathbb{R}^n)$.

In the proposition above, we did not make mention of the continuity of the maps we have considered. We only give proofs of how these maps transform the space $Q^s(\mathbb{R}^n)$. This fact is not surprising because we have not adopted any topology on $Q^s(\mathbb{R}^n)$. Thus in what follows, we shall build a Hilbert space structure on $Q^s(\mathbb{R}^n)$ so as to help us in considering the continuity of the maps in the above Proposition.

Let us define the following inner product on $Q^s(\mathbb{R}^n)$.

(1)---$(u,v)_s = (L_su,L_su)_{L^2(\mathbb{R}^n)}$ for $u,v \in Q^s(\mathbb{R}^n)$.

It is easy to check that (1) satisfies all the properties of an inner product with the following exception:

Observe that $(u,v)_s = 0$ if and only if $(L_su,L_su) = 0$. But we do not know if L_s is one-to-one so that we are not in a position to conclude that $L_su = 0$ implies that $u = 0$. To emphasize our problem, consider the Laplace's operator ($-\Delta$-operator)with the symbol $\|\xi\|^2$.Then from $(\Delta u,\Delta u)_{L^2(\mathbb{R}^n)} = 0$,it follows that $\Delta u = 0$.But $\Delta u = 0$ only means that u is a harmonic function.

It is upon this observation that we are lead to define the following inner product on $Q^s(\mathbb{R}^n)$.

$$(2)\text{---}(u,v)_s = (L_s u, L_s v)_{L^2(\mathbb{R}^n)} + \sum_{|\alpha|+|\beta|\leq p} (Q_{\alpha\beta}R_s u, Q_{\alpha\beta}R_s u)_{L^2(\mathbb{R}^n)}$$

where R_s is given by (0),p is an even integer such that $p > s$,and $Q_{\alpha\beta} = x^\alpha D^\beta$ for multiindices α and β.

Now we shall show that in (2),

$$(u,u)_s = 0 \text{ if and only if } u = 0$$

Clearly $(u,u)_s = 0$ implies that $L_s u = 0$ and $Q_{\alpha\beta}R_s = 0$ $(|\alpha| + |\beta| \leq p)$, this implies that $R_s u =$ constant, but $R_s u \in S(\mathbb{R}^n)$. Therefore $R_s u = 0$. Also $u = L_{-s}L_s u - R_s u$. Therefore $u = 0$.

Next we need to show that the inner product $(,)_s$ defined in (2) yields a complete normed space with

$$\|u\|^2 = (u,u)_s.$$

<u>Proposition 4.6.3.</u> $Q^s(\mathbb{R}^n)$ is a Hilbert space under the scalar product (2).

<u>Proof</u>: If $\{u_n\}$ is a Cauchy sequence in $Q^s(\mathbb{R}^n)$ then $\{L_s u_n\}$ is also Cauchy in $L^2(\mathbb{R}^n)$ by definition of $Q^s(\mathbb{R}^n)$. Since $L^2(\mathbb{R}^n)$ is complete, we conclude that $\{L_s u_n\}$ has a limit in $L^2(\mathbb{R}^n)$. Let $\lim_{n\to\infty} L_s u_n = v$

Since $R_s : S'(\mathbb{R}^n) \to S(\mathbb{R}^n)$ is continuous and the topology on $Q^s(\mathbb{R}^n)$ is stronger than the topology induced on $Q^s(\mathbb{R}^n)$ by the topology on $S'(\mathbb{R}^n)$, and the topology of $S(\mathbb{R}^n)$ is stronger than the topology of $L^2(\mathbb{R}^n)$ it implies that R_s defines a continuous map taking $Q^s(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$.

We know that every linear bounded map between normed spaces transforms Cauchy sequences into Cauchy sequences. Therefore $\{R_s u_n\}$ is a Cauchy sequence in $L^2(\mathbb{R}^n)$ and thus converges. Let $\lim_{n\to\infty} R_s u_n = w$. Hence,

$$(3)--- \begin{cases} L_s u_n \to v \\ \\ R_s u_n \to w \end{cases} \qquad \text{in } L^2(\mathbb{R}^n).$$

We have $u_n = L_{-s} L_s u_n - R_s u_n$. Since L_{-s} is a bounded operator in $L^2(\mathbb{R}^n)$ we can write $u = L_{-s} v - w$.

Now we only have to show that $u_n \to u$ in $Q^s(\mathbb{R}^n)$ or equivalently $\|u_n - u\|_s \to 0$ as $n\to\infty$. We have

$$\begin{aligned} u_n - u &= u_n - L_{-s} v + w \\ &= L_{-s} L_s u_n - R_s u_n - L_{-s} v + w \\ &= L_{-s}(L_s u_n - v) - (R_s u_n - w) . \end{aligned}$$

Therefore

$$\begin{aligned} (4)---\|u_n - u\|_{..} &\le \|L_s L_{-s}(L_s u_n - u)\|_{L^2(\mathbb{R}^n)} \\ &+ \sum_{|\alpha|+|\beta|\le p} \|Q_{\alpha\beta} R_s [L_{-s}(L_s u_n - v)]\|_{L^2(\mathbb{R}^n)} \\ &+ \|L_s (R_s u_n - w)\|_{L^2(\mathbb{R}^n)} \\ &+ \sum_{|\alpha|+|\beta|\le p} \|Q_{\alpha\beta} R_s (R_s u_n - w)\|_{L^2(\mathbb{R}^n)}. \end{aligned}$$

Since $L_{-s}L_s \in G^{\circ}_{\rho}$, it is bounded as an operator on $L^2(\mathbb{R}^n)$. It follows from the above fact that $L_s[L_{-s}(L_s u_n - v)] \to 0$ in $L^2(\mathbb{R}^n)$ as $n\to\infty$. Also L_{-s} is compact in $L^2(\mathbb{R}^n)$, thus $L_{-s}(L_s u_n - v)\to 0$ in $L^2(\mathbb{R}^n)$ as $n\to\infty$. Therefore $Q_{\alpha\beta}R_s[L_{-s}(L_s u_n - v)] \to 0$ in $L^2(\mathbb{R}^n)$ as $n\to\infty$, since $Q_{\alpha\beta}\ R_s$ is bounded in $L^2(\mathbb{R}^n)$. By the same token we have $Q_{\alpha\beta}R_s(R_s u_n - w)\to 0$ in $L^2(\mathbb{R}^n)$.

Finally, we shall estimate $\|L_{-s}(R_s u_n - w)\|_{L^2(\mathbb{R}^n)}$. $R_s: S'(\mathbb{R}^n)\to S(\mathbb{R}^n)$ is continuous and $u_n \to v$ in $L^2(\mathbb{R}^n)$, it means that $R_s u_n \to w$ in $S(\mathbb{R}^n)$ and $L^2(\mathbb{R}^n)$. Therefore $\|L_{-s}(R_s u_n - w)\|_{L^2(\mathbb{R}^n)} \to 0$ as $n\to\infty$, since L_{-s} is bounded. This proves that $\|u_n - u\|_s \to 0$ as $n\to\infty$. Therefore $Q^s(\mathbb{R}^n)$ is a Hilbert space.

Now we are going to prove that the map of Proposition 4.6.2. is in fact continuous by using the results that $Q^s(\mathbb{R}^n)$ is a Hilbert space.

Proposition 4.6.4. If $A \in G^m_{\rho}(\mathbb{R}^n)$ then A extends to a bounded operator $A: Q^s(\mathbb{R}^n) \to Q^{s-m}(\mathbb{R}^n)$.

Proof: We need to estimate the norms $\|L_{s-m}Au\|_{L^2(\mathbb{R}^n)}$ and $\|Q_{\alpha\beta}R_{s-m}Au\|_{L^2(\mathbb{R}^n)}$ in terms of $\|L_s u\|_{L^2(\mathbb{R}^n)}$ and $\|Q_{\alpha\beta}R_s u\|_{L^2(\mathbb{R}^n)}$, where $L_{m-s}L_{s-m} = I + R_{s-m}$ and $u \in Q^s(\mathbb{R}^n)$.

Really, $u = L_{-s}L_s u - R_s u$ and therefore

(1)---$Au = (AL_{-s})L_s u - AR_s u$ and thus we have

(2)---$L_{s-m}Au = (L_{s-m}A\ L_{-s})(L_s u) - (L_{s-m}A\ R_s)u$ and

(3)---$Q_{\alpha\beta}R_{s-m}Au = Q_{\alpha\beta}R_{s-m}L_{-s}L_s u + Q_{\alpha\beta}R_{s-m}R_s u$.

We know that $L_{s-m}AL_{-s} \in G^{\circ}_{\rho}(\mathbb{R}^n)$ and therefore extends to a continuous operator on $L^2(\mathbb{R}^n)$. Also $L_{s-m}AR_s \in G^{-\infty}(\mathbb{R}^n)$. So,

$$\|L_{s-m}Au\|_{L^2(\mathbb{R}^n)} \le \|L_{s-m}AL_s\|_{L^2(\mathbb{R}^n)}\|L_s u\|_{L^2(\mathbb{R}^n)} + \|L_{s-m}AR_s L_{-s}\|_{L^2(\mathbb{R}^n)}\|L_s u\|_{L^2(\mathbb{R}^n)} .$$

Therefore $\|L_{s-m}Au\|_{L^2(\mathbb{R}^n)} \le C_1\|L_s u\|_{L^2(\mathbb{R}^n)}$ and

$$\|Q_{\alpha\beta}R_{s-m}Au\|_{L^2(\mathbb{R}^n)} \leq \|Q_{\alpha\beta}R_{s-m}L_{-s}\|_{L^2(\mathbb{R}^n)}\|L_s u\|_{L^2(\mathbb{R}^n)} + \|Q_{\alpha\beta}R_{s-m}R_sL_{-s}\|_{L^2(\mathbb{R}^n)}\|L_s u\|_{L^2(\mathbb{R}^n)}$$

$$\leq C_2\|L_s u\|_{L^2(\mathbb{R}^n)}$$

Where ,

$$C_1 = \|L_{s-m}AL_s\|_{L^2(\mathbb{R}^n)} + \|L_{s-m}AR_sL_{-s}\|_{L^2(\mathbb{R}^n)} \quad \text{and}$$

$$C_2 = \|Q_{\alpha\beta}R_{s-m}L_{-s}\|_{L^2(\mathbb{R}^n)} + \|Q_{\alpha\beta}R_{s-m}R_sL_{-s}\|_{L^2(\mathbb{R}^n)} \ .$$

It is of course clear that C_1 and C_2 are finite and thus the proposition is proved.

Finally we shall formulate a simple conclusion from Proposition 4.6.2 (b).

Corollary 4.6.5. $\bigcap_s Q^s(\mathbb{R}^n) = S(\mathbb{R}^n)$.

Proof Since $\bigcup_m G_1^m(\mathbb{R}^n)$ contains all the differential operators with polynomial coefficients, then Proposition 4.6.2 (b) yields the results immediately.

Corollory 4.6.6 $S(\mathbb{R}^n)$ is dense in $Q^s(\mathbb{R}^n)$ for all s.

Proof

By Proposition 4.6.4 the linear map L_s: $Q^s(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is continuous. From the definition of $Q^s(\mathbb{R}^n)$ the map L_s is onto.
Therefore by the Interior Mapping Theorem [7] p.55 L_s is an open map since L_s maps $S(\mathbb{R}^n)$ onto $S(\mathbb{R}^n)$. The preimage of $S(\mathbb{R}^n)$ under L_s contains by Proposition 4.6.2 (b) only the functions from $S(\mathbb{R}^n)$.
If closure $(L_s^{-1}(S(\mathbb{R}^n))) \neq Q^s(\mathbb{R}^n)$ then the complementary open set in $Q^s(\mathbb{R}^n)$ to this closure transforms onto an open subset of $L^2(\mathbb{R}^n)$ which does not contain any element from $S(\mathbb{R}^n)$ whicn contradicts the density of $S(\mathbb{R}^n)$ in $L^2(\mathbb{R}^n)$.

CHAPTER 5

THE SPECTRAL THEORY OF PSEUDO-DIFFERENTIAL OPERATORS IN $\mathbb{R}^n$

5.1. INTRODUCTION

In this section we study the spectral theory of the pseudo-differential operators in $\mathbb{R}^n$.We shall treat these operators as the unbounded operators on $L^2(\mathbb{R}^n)$ with their domain of definition being either $C_0^\infty(\mathbb{R}^n)$ or $S(\mathbb{R}^n)$.Both these domains are appropriate for our theory because they are both dense in $L^2(\mathbb{R}^n)$. The following books discuss a general theory of such operators, [7], [10], [13] , [18] , [19] and [29] jointly.

It is quite demanding to build a general spectral theory for the pseudo-differential operators we have discussed thus far.Hence for the sake of clarity,we shall deal with some extreme special cases.

For hypoelliptic pseudo- differential operators in $\mathbb{R}^n$, we prove that they are essentially self -adjoint and have a discrete spectrum only.

5.2. THE SPECTRUM OF SOME $G_\rho^m(\mathbb{R}^n)$ OPERATORS.

We shall determine the spectrum of a pseudo-differential operator $A \in G_\rho^m(\mathbb{R}^n)$ with a symbol $a(x,\xi) \in \Gamma_\rho^m(\mathbb{R}^{2n})$.

If $m = 0$,then by Theorem 3.2.9.,A extends to a bounded operator on $L^2(\mathbb{R}^n)$.The spectral theory of bounded operators is treated in Helmberg [10].

If $m<0$, then by Theorem 3.2.10., A extends to a compact operator on $L^2(\mathbb{R}^n)$. By Theorem 6.10.2. of [18] or Theorem 6.7. of [29], we conclude that A has discrete spectrum only. [10] also makes an extensive study of the spectrum of compact operators.

For $m>0$, we shall consider two cases. The first case will be on the unperturbed operator and the second case on the perturbed operator as discussed in [4] and [28]. For the unperturbed operator we make a crucial assumption that the symbol $a(x,\xi)$ of A be dependent upon the ξ-variable only. In this particular case we have,

$$(1)\text{---}\quad \begin{cases} a(x,\xi) = a(\xi) & \text{and the estimates,} \\ |a(\xi)| \leqslant C\langle\xi\rangle^m & \text{where } C \text{ is some positive constant.} \end{cases}$$

Then we have,

$$(2)\text{---}\quad Au(x) = (2\pi)^{-n}\int e^{ix\xi}\, a(\xi)\hat{u}(\xi)d\xi \quad \text{for every } u \text{ in } S(\mathbb{R}^n)$$

or equivalently,

$$(3)\text{---}\quad \widehat{Au}(\xi) = a(\xi)\hat{u}(\xi)\ .$$

Of course (3) means that A is unitarily equivalent to the multiplication operator by a function $a(\xi)$. Therefore, in view of Corollary 2 in [4], we conclude that A has a continuous spectrum equal to the closure of the range of $a(\xi)$. Because the discrete spectral set of A is empty, we infer that this spectrum of A is in fact the essential spectrum.

A typical example of an unperturbed pseudo-differential operator is the constant coefficient differential operator;

$$Au(x) = (2\pi)^{-n}\iint e^{i(x-y)\xi}\Big(\sum_{|\alpha|\leqslant m} C_\alpha \xi^\alpha\Big)u(y)dyd\xi$$

$Au(x) = \int e^{ix\xi} a(\xi)\hat{u}(\xi)d\xi$ for every $u \in S(\mathbb{R}^n)$, where,

$a(\xi) = \sum_{|\alpha| \leq m} C_\alpha \xi^\alpha$ and clearly,

$|a(\xi)| \leq \sum_{|\alpha| \leq m} |C_\alpha| \, \|\xi^\alpha\| \leq C\langle\xi\rangle^m$,where $C = \sum_{|\alpha| \leq m} |C_\alpha|$

The dependence of the spectrum of A on its symbol $a(\xi)$ is advantegeous in the sense that we are able to deduce the nature of A easily.If for example $a(\xi)$ is real-valued,we infer that its spectrum is contained in $\mathbb{R}^n$.

In particular,suppose $A = -\frac{d^2}{dx^2}$ on $S(\mathbb{R}^n)$.Then the operator A so defined has a symbol $a(\xi) = \|\xi\|^2$. So the spectrum is $\sigma(A) = \overline{\{\|\xi\|^2 ; \xi \in \mathbb{R}^n\}}$. Therefore $\sigma(A) = [0,\infty)$.We shall discuss more examples later.

Let us define the terms symmetric and formal adjoint as in Shubin [21] and Dundford & Schwartz [7] .Let $(u,v)_{L^2(\mathbb{R}^n)} = (u,v)$.

Definition 5.2.1. We say a pseudo-differential operator $A \in G^m_\rho(\mathbb{R}^n)$ is formally symmetric if $(Au,v) = (u,Av)$ for all $u,v \in C^\infty_0(\mathbb{R}^n)$.Furthermore, we say $A^+ \in G^m_\rho(\mathbb{R}^n)$ is a formal adjoint of $A \in G^m_\rho(\mathbb{R}^n)$ if $(Au,v) = (u,A^+v)$ for all $u,v \in C^\infty_0(\mathbb{R}^n)$.

5.3.SPECTRUM OF THE HYPOELLIPTIC PSEUDO-DIFFERENTIAL OPERATORS IN $\mathbb{R}^n$

We continue our discussion by studying the spectrum of the $HG^{m,m_0}_\rho(\mathbb{R}^n)$ operators ($m_0>0$).We shall prove that they are essentially self-adjoint and have a discrete spectrum only.We begin by giving some important definitions and results.

Let H be a Hilbert space.

Definition 5.3.1. An operator $T : H \to H$ is said to be essentially self-adjoint if the domain $D(T)$ of T is dense in H and $\overline{T} = T^*$, i.e. the closure of T is equal to T^*.

The relationship between an essentially self-adjoint operator and a symmetric operator is contained in the proposition below.

Proposition 5.3.2. An essentially self-adjoint operator is symmetric i.e. $(Tx,y) = (x,Ty)$ for all $x,y \in D(T)$.

Proof: Let $T : H \to H$ be an essentially self-adjoint operator on H. This means that $\overline{T}$ does exist and in actual fact $\overline{T} = T^*$. Now consider $x,y \in D(T)$. Then,

$$(Tx,y) = (\overline{T}x,y) = (x,T^*y) = (x,\overline{T}y) = (x,Ty),$$

as required.

The following is a characterization theorem for essentially self-adjoint operators. The proof of this theorem can be found in [21], pp 190-191.

Theorem 5.3.3. A symmetric operator $A : H \to H$ with a domain $D(A)$ dense in H is essentially self-adjoint if and only if;

(i) --- $\ker(A^*-iI) \subset D(\overline{A})$

and

(ii)--- $\ker(A^*+iI) \subset D(\overline{A})$.

Note that for a symmetric operator $A, A \subset A^*$ and therefore $\overline{A}$ is well-defined, since A^* is closed (see [7], Lemma 8. page 1226).

The following theorem tells us that a formally symmetric operator $A \in HG_{\rho}^{m,m_0}(\mathbb{R}^n)(m_0>0)$, treated as an unbounded operator on $L^2(\mathbb{R}^n)$, is essentally self-adjoint.

Theorem 5.3.4. Let $A \in HG_{\rho}^{m,m_\circ}(\mathbb{R}^n)$,where $m_\circ > 0$,be a formally symmetric operator.Consider an unbounded operator $A_\circ$ on $L^2(\mathbb{R}^n)$ with domain $D(A_\circ) = C_\circ^\infty(\mathbb{R}^n)$ such that $A_\circ u = Au$ for every $u \in C_\circ^\infty(\mathbb{R}^n)$.Then $A_\circ$ is essentially self-adjoint and its closure is equal to the restriction of A acting on $S'(\mathbb{R}^n)$ to the subspace;

(1)--- $D(\bar{A}_\circ) = \{u; u \in L^2(\mathbb{R}^n) \text{ and } Au \in L^2(\mathbb{R}^n)\}$.

Proof:

Since $HG_{\rho}^{m,m_\circ}(\mathbb{R}^n) \subset G_{\rho}^{m}(\mathbb{R}^n)$,then the map $A:S(\mathbb{R}^n) \to S(\mathbb{R}^n)$ is continuous. Denote by $\tilde{A}:S'(\mathbb{R}^n) \to S'(\mathbb{R}^n)$ the extension of A to $S'(\mathbb{R}^n)$ guaranteed by by Theorem 1.6.4. We have the following inclusions;

$C_\circ^\infty(\mathbb{R}^n) \subset S(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \subset S'(\mathbb{R}^n)$ (see [9],page118 or [23],page 162).

Clearly,$A_\circ \subset A \subset \tilde{A}$ (see Remark p.21).

Step 1: The equality $(A_\circ u, v) = (u, A_\circ v)$ is true for all $u, v \in C_\circ^\infty(\mathbb{R}^n)$ and therefore extends to $u, v \in S(\mathbb{R}^n)$ by continuity of A and the fact that $C_\circ^\infty(\mathbb{R}^n)$ is dense in $S(\mathbb{R}^n)$.

We shall prove that,

(2)--- $(\tilde{A}u, v) = (u, \tilde{A}v)$ if $u, v, \tilde{A}u$ and $\tilde{A}v$ are all from $L^2(\mathbb{R}^n)$.

If $\tilde{A}u, \tilde{A}v \in L^2(\mathbb{R}^n) = Q^\circ(\mathbb{R}^n)$ then $u, v \in Q^{m_\circ}(\mathbb{R}^n)$,since $A \in HG_{\rho}^{m,m_\circ}(\mathbb{R}^n)$,by Proposition 4.6.2 (b). By Corollary 4.6.6 $S(\mathbb{R}^n)$ is dense in $Q^s(\mathbb{R}^n)$ since it is dense in $L^2(\mathbb{R}^n)$ and the map $A : Q^m(\mathbb{R}^n) \to Q^\circ(\mathbb{R}^n)$ is continuous. Take sequences $\{u_n\}$ and $\{v_n\}$ in $S(\mathbb{R}^n)$ such that $u_n \to u$ and $v_n \to v$ in $Q^{m_\circ}(\mathbb{R}^n)$.Then $Au_n \to \tilde{A}u$ and $Av_n \to \tilde{A}v$ in $Q^\circ(\mathbb{R}^n)$ or $L^2(\mathbb{R}^n)$ and $(Au_n, v_n) \to (\tilde{A}u, v)$ also $(u_n, Av_n) = (Au_n, v_n) \to (u, \tilde{A}v)$ and this proves (2).

Step 2: We shall show that,

(3)--- $D(A_\circ^*) = \{u \in L^2(\mathbb{R}^n); \tilde{A}u \in L^2(\mathbb{R}^n)\} = D.$

In actual fact, since A_o is defined on $C_o^\infty(\mathbb{R}^n)$, which is a dense subset of $L^2(R^n)$, then the operator A_o^* is well-defined. From the equality $(A_o u,v) = (u,A_o u)$ for $u,v \in C_o^\infty(\mathbb{R}^n)$, it becomes clear that $A_o \subset A_o^*$ i.e. A_o^* is an extension of A_o. Since A_o^* is closed, then the closure of the graph of A_o, defines an operator i.e. $\overline{A}_o \subset A_o^*$ or $D(\overline{A}_o) \subset D(A_o^*)$.

From (2) we have; for every $u \in C_o^\infty(\mathbb{R}^n)$,

(4) --- $(\tilde{A}u,v) \doteq (u,\tilde{A}v) = (A_o u,v)$,

provided that v and $\tilde{A}v \in L^2(\mathbb{R}^n)$.

The operator A_o^* is characterized by the equality;

$(A_o u,v) = (u,A_o^* v)$, for $v \in D(A_o^*)$.

The identity $(u,\tilde{A}v) = (A_o u,v)$, $(u \in C_o^\infty(\mathbb{R}^n))$ shows that

(5) --- $v \in D(A_o^*)$ and $A_o^* v = \tilde{A}v$, provided that $v \in D$

or

(6) --- $D \subset D(A_o^*)$.

Step 3: To prove the reverse inclusion

(7) --- $D(A_o^*) \subset D$,

we proceed as follows :

Let $v \in D(A_o^*)$. By the definition of A_o^*, the functional $u \to (A_o u,v)$ extends to a continuous functional on $L^2(\mathbb{R}^n)$ and $(A_o u,v) = (u,A_o^* v)$, $A_o^* v \in L^2(\mathbb{R}^n)$. Consider a sequence $\{v_n\}$ in $C_o^\infty(\mathbb{R}^n)$ such that $v_n \to v$ in $L^2(\mathbb{R}^n)$. Then,

$(A_o u,v_n) = (u,A_o v_n) = \int u(x)\overline{A_o v_n(x)}\,dx = (\overline{\tilde{A}v_n})(u)$,

($\overline{}$ refers to complex conjugation).

Since $\tilde{A}$ is continuous on $S'(\mathbb{R}^n)$, it means,

$$(A_o u,v_n) = (\overline{\tilde{A}v_n})(u) \to (\overline{\tilde{A}v})(u) \text{ as } n\to\infty \quad \text{and } (\overline{\tilde{A}v})(u) = (u,A_o^* v)$$

for all $u \in C_o^\infty(\mathbb{R}^n)$.

Hence a tempered distribution $u \to (\overline{\tilde{A}v})(u)$ extends to a continuous functional on $L^2(\mathbb{R}^n)$.As we know,such a distribution represents a function from $L^2(\mathbb{R}^n)$.On using (2) we get ,

$\overline{(\tilde{A}v)}(u) = (u,\tilde{A}v) = (u,A_o^*v)$ for $u \in C_o^\infty(\mathbb{R}^n)$ and thus ,$A_o^*v = \tilde{A}v$ and this proves that $v \in D$ as required.

Consequently (6) and (7) yield (3).

Step 4: We use Theorem 5.3.3. to prove that A_o is essentially self-adjoint. Since $m_o > 0$,the operator $A_o - iI \in HG_p^{m,m_o}(\mathbb{R}^n)$ by Lemma 4.2.2 and from (3) and (5) ,it is clear that $\ker(A_o^* - iI) \subset \operatorname{Ker}(\tilde{A} - iI)$.Also $\tilde{A}u \in S(\mathbb{R}^n)$ implies that $u \in S(\mathbb{R}^n)$ by Theorem 1.6.3.We therefore conclude that $\ker(\tilde{A} - iI) \subset S(\mathbb{R}^n)$ and thus $\ker(A_o^* - iI) \subset D(A_o^*)$ since $S(\mathbb{R}^n) \subset D(A_o^*)$.Similarly we can show that $\ker(A_o^* + iI) \subset D$.

By (3),(5) and essential self-adjointness,wehave: if $v \in D$ then $\tilde{A}v = \bar{A}_o v = A_o^* v$.This completes the proof.

Next we prove that these formally self-adjoint hypoelliptic operatos have a discrete spectrum only.

Theorem 5.3.5. Let $A \in HG_p^{m,m_o}(\mathbb{R}^n)$,$m_o > 0$,be a formally symmetric operator. Then A has discrete spectrum $\{\lambda_j\}$ only,j = 1,2,3,...,such that $|\lambda_j| \to +\infty$ as $j \to +\infty$ .The eigenfunctions $\psi_j \in S(\mathbb{R}^n)$ form a complete orthonormal set in $L^2(\mathbb{R}^n)$ and the spectrum;

$\sigma(\bar{A}) = \sigma(A^*) = \{\lambda_j\}$;j = 1,2,3,... .

Proof:

Let A_o be the unbounded operator defined on $C_o^\infty(\mathbb{R}^n)$ as in Theorem 5.3.4. Then $\bar{A}_o = A_o^*$,in other words A_o is essentially self-adjoint, and A_o is the same as A acting on $S(\mathbb{R}^n)$ .Therefore the spectrum $\sigma(\bar{A})$ is contained in the real line,see [7], Lemma 1,p1191.

Take λ_o with $\operatorname{Im}\lambda_o \neq 0$.

Thus $\lambda_o \notin \sigma(\overline{A})$ and the operator $\hat{A} = A-\lambda_o I$ is invertible.Furthermore, $\hat{A}^{-1}$ is bounded on $L^2(\mathbb{R}^n)$.Consider now an operator $\hat{B}$,which is a parametrix of $\hat{A}$.Since $I \in G_1^0(\mathbb{R}^n)$ and $A \in HG_\rho^{m,m_o}(\mathbb{R}^n)$,$m_o > 0$,then $\hat{A} \in HG_\rho^{m,m_o}(\mathbb{R}^n)$.

We know that $\hat{B} \in HG_\rho^{-m_o,-m}(\mathbb{R}^n)$ and since $m_o > 0$,by Theorem 3.2.10., we conclude that $\hat{B}$ is a compact operator on $L^2(\mathbb{R}^n)$.Now,

(1)--- $\hat{A}\hat{B} = I - \hat{K}$ and $\hat{B}\hat{A} = I - \hat{R}$,where $\hat{K}$ and $\hat{R}$ are integral operators with kernels from $S(\mathbb{R}^{2n})$.

Form (1) we get;

(2)--- $\hat{A}^{-1} = \hat{B} + \hat{A}^{-1}\hat{K}$.

We need to show that $\hat{A}^{-1}\hat{K}$ is a compact operator.We know that the composition of a compact operator and a bounded operator is a compact operator,Lemma 8.3-2 of [13] .$\hat{K} \in G^{-\infty}(\mathbb{R}^n)$ and $\hat{A}^{-1}$ is bounded on $L^2(\mathbb{R}^n)$.

Therefore $\hat{A}^{-1}\hat{K}$ is compact.Hence $\hat{A}^{-1}$ is compact and self-adjoint.Theorem 8.3-1. of [13] ,tells us that $\hat{A}^{-1}$ has a discrete spectrum with the corresponding eigenfunctions forming a complete orthonormal set in $L^2(\mathbb{R}^n)$. Let the spectral set of $\hat{A}^{-1}$ be $\{\hat{\mu}_j\}$, where $j = 1,2,3,\ldots$ .Then

$\hat{A}^{-1}\hat{\phi}_j = \hat{\mu}_j\hat{\phi}_j$,where $\{\hat{\phi}_j\}$ are the corresponding eigenfunctions.

Hence $\hat{A}$ has a spectrum equal to $\{\hat{\lambda}_j\}$,where $\hat{\lambda}_j = \hat{\mu}_j^{-1}$,$j = 1,2,3,\ldots$ with the corresponding set of eigenfunctions $\{\hat{\phi}_j\}$.

Now we have $A = \hat{A}+\lambda_o I$.So A has a spectrum $\{\lambda_j\} = \{\hat{\lambda}_j + \lambda_o\}$ with the corresponding set of eigenfunctions $\{\phi_j\} = \{\hat{\phi}_j\}$,$j = 1,2,3,\ldots$. Cleaarly $|\lambda_j| \to +\infty$ as $j \to +\infty$.This arises as a result of the fact that $\hat{\mu}_j \to 0$ as $j \to +\infty$ and $\hat{\lambda}_j = \hat{\mu}_j^{-1}$.

Finally, we shall show that $\psi_j = \hat{\psi}_j \in S(\mathbb{R}^n)$ for $j = 1,2,3,\dots$. Observe that $A^k \psi_j = \lambda_j^k \psi_j$, for every positive integer k. Also $A^k \in HG_\rho^{km,km_o}(\mathbb{R}^n)$. By Proposition 4.6.4 we conclude that

$\psi_j \in Q^{km}(\mathbb{R}^n)$ for every integer k. We know that $Q^s(\mathbb{R}^n) \subset Q^{s'}(\mathbb{R}^n)$ for $s > s'$ and $\bigcap_{k=1}^{\infty} Q^{km}(\mathbb{R}^n) \supset \bigcap_{s>0} Q^s(\mathbb{R}^n)$. Furthermore $\bigcap_{k=1}^{\infty} Q^{km}(\mathbb{R}^n) = S(\mathbb{R}^n)$ by Appendix F.

Therefore $\psi_j \in S(\mathbb{R}^n)$ for $j = 1,2,3,\dots$.
This establishes the required proof.

So far we have discussed the spectrum of an unperturbed $A_o \in HG_\rho^{m,m_o}(\mathbb{R}^n)$, $m_o > 0$. The spectrum of the perturbed operator is dependent upon the class of the perturbing operator. But of interest to us are those perturbations which leaves the spectrum invariant.

It is under the above conditions that we consider only perturbations by the pseudo-differential operators from $G_\rho^{m'}(\mathbb{R}^n)$ with $m' < m$ in view of Lemma 4.2.2. These operators will leave the operator under consideration hypoelliptic. Therefore the spectrum of the new operator will be purely discrete by Theorem 5.3.5.

Consequently, by Lemma 4.2.2. and Theorem 5.3.5., we have ;
Corollary 5.3.6. Let $A_o \in HG_\rho^{m,m_o}(\mathbb{R}^n)$ and $B \in G_\rho^{m'}(\mathbb{R}^n)$, where $m_o > 0$ and $m_o > m'$. Furthermore suppose A_o and B are both formally symmetric. Then the operator $\hat{A} = A_o + B$ is essentially self-adjoint and has discrete spectrum $\{\lambda_j\}$ only, $j = 1,2,3,\dots$, such that $|\lambda_j| \to +\infty$ as $j \to +\infty$. The eigenfunctions $\{\psi_j\} \subset S(\mathbb{R}^n)$ form a complete orthonormal set in $L^2(\mathbb{R}^n)$.

5.4. PSEUDO-DIFFERENTIAL OPERATORS AND QUANTUM MECHANICS

The theory of pseudo-differential operators in $\mathbb{R}^n$ finds very wide application in geometrical optics and quantum mechanics. For example the various symbols; τ-symbols, Weyl-symbol etc., first appeared in attempts to find a natural arrangement of the operators of the classical quantum mechanics q and $-ih\frac{\partial}{\partial q}$ i.e. the position and the momentum operators (respectively).

The concept of asymptotic expansion on the other hand is related to the various methods (like the stationary phase method, just to mention one) developed in geometrical optics.

The spectral theory of pseudo-differential operators can be applied to deduce the spectrum of the Schrödinger operators. In this section we briefly discuss a few examples. We discuss them for illustration.

The $\Gamma^m_\rho(\mathbb{R}^{2n})$ classes are too restrictive for some applications, for example the well known operators of quantum mechanics such as $H = -(\frac{h^2}{8\pi^2 m})\Delta + V(x)$, where Δ = Laplacian operator and $V(x)$ is the potential energy operator, (Schecter [20]) do not belong to our classes.

5.4.1 Free Particle: The Hamiltonian for a single free particle is $H_o = -(\frac{h^2}{8\pi^2 m})\Delta$, (see Schecter [20]). The symbol of this operator is $a_o(\xi) = (\frac{h^2}{8\pi^2 m})\xi^2$ Clearly $a_o(\xi)$ is real-valued. As a pseudo-differential operator H_o takes the form

$$H_o\, u(x) = (2\pi)^{-n} \int e^{ix\xi} \frac{h^2}{8\pi^2 m} \xi^2 \hat{u}(\xi)\, d\xi,$$

$$D(H_o) = C_o^\infty(\mathbb{R}^n).$$

Now H_o is self-adjoint since $a_o(\xi)$ is real-valued ([28] Lemma 1.3). Therefore $\sigma(H_o)$ = closure $\{a_o(\xi); \xi \epsilon \mathbb{R}^n\} = [0, \infty)$.
By Corollary 2 of [4], we conclude that the essential spectrum $\sigma_e(H_o) = \sigma(H_o)$ and from the Remark below we conclude that $\sigma_d(H_o)$ consists of negative eigenvalues of finite multiplicity. Physically, it means that the energy states corresponding to these eigenvalues are bound states of this free particle and these bound states are discrete.

Remark

From [29] p.202, the essential spectrum $\sigma_e(T)$ of a self-adjoint operator T is the set of those points of $\sigma(T)$ that are either accumulation points of $\sigma(T)$ are isolated eigenvalues of infinite multiplicity. The discrete spectrum $\sigma_d(T) = \sigma(T) - \sigma_e(T)$ is the set of these eigenvalues of finite multiplicity. The spectrum of T said to be purely discrete if $\sigma_e(T)$ is empty.

5.4.2. Consider the relativistic spin-zero hamiltonian of a particle of mass m as presented in Weder [28] .The free-field hamiltonian expressed in the language of pseudo-differential operators takes the form;

$$A_o u(x) = (2\pi)^{-1} \iint e^{i(x-y)\xi} \sqrt{\xi^2 + m^2}\, u(y)\,dy\,d\xi$$

$$= \int e^{ix\xi} \sqrt{\xi^2 + m^2}\, \hat{u}(\xi)\,d\xi \;,$$

where $D(A_o) = C_o^\infty(\mathbb{R})$.The symbol of A_o is $a_o(\xi) = \sqrt{\xi^2 + m^2}$ and is real-valued.Therefore A_o is self -adjoint on $C_o^\infty(\mathbb{R})$.Hence the spectrum is given by ,

$$\begin{aligned} \sigma(A_o) &= \text{closure } \{a_o(\xi); \xi \in \mathbb{R}\} \\ &= \text{closure } \{\sqrt{\xi^2 + m^2}; \xi \in \mathbb{R}\} \\ &= [m, \infty). \end{aligned}$$

We therefore make the inference that the relativistic spin-zero particle has bound states as well.These bound states correspond to the eigenvalues which are less than m.

Remark: In the preceeding examples we have assumed that all the physical constants are equal to 1 i.e. $h = c = 1$.We shall adopt this condition throughout our discussion.

5.4.3. We shall now discuss the quantum harmonic oscillator operator, (see [13] ,[14] and [17] for this problem). The operator takes the form

$$A_o u(x) = - \frac{d^2u(x)}{dx^2} + x^2u(x),$$

where $u \in D(A_o) = S(\mathbb{R})$.

The symbol of this operator is $a_o(x,\xi) = \xi^2 + x^2$.Clearly $a_o(x,\xi) \in H\Gamma_1^{2,2}(\mathbb{R}^2)$.

In actual fact,

$$\begin{aligned} |a_o(x,\xi)| &= |\xi^2 + x^2| \\ &\leq |\xi|^2 + |x|^2 \\ &\leq \langle(x,\xi)\rangle^2 . \end{aligned}$$

Furthermore,we can choose a positive constant C such that the following inequalities are true;

$$C \langle(x,\xi)\rangle^2 \leq |a_o(x,\xi)| \leq \langle(x,\xi)\rangle^2$$

thus proving the assertion that $a_o(x,\xi) \in H\Gamma_1^{2,2}(\mathbb{R}^2)$.

Since $m_o=2$,we conclude that A_o is essentially self-adjoint and has discrete spectrum.The spectral set is contained in the real line;the eigenvalues are, $\lambda_n = n + \frac{1}{2}$.

Suppose we perturb A_o by some function $V(x)$. Physically, it means that we subject the oscillator to some external force with gradient $V(x)$.The new operator takes the form;

$$A = A_o + \lambda V(x),$$

where λ is some non-zero constant.

For every function $u \in S(\mathbb{R})$ we have;

$$Au(x) = -\frac{d^2u(x)}{dx^2} + x^2u(x) + \lambda V(x)u(x).$$

In fact we shall consider only those potentials $V(x)$ with estimates which fall into our scheme.

Let $V(x) \in \Gamma_\rho^m(\mathbb{R}^{2n})$, where m is a real number. Then, of course, for these perturbations we need $m < 2$. With this choice of m we get that the symbol of A viz. $a(x,\xi) = \xi^2 + x^2 + \lambda V(x)$ belongs to $H\Gamma_1^{2,2}(\mathbb{R}^2)$ .Therefore A is essentially self-adjoint and thus has a discrete spctrum only.Lathouwers [14],discusses a particular case when $V(x) = x^{-2}$. A discrete spectrum is obtained.

5.4.4. It is possible to represent the operators of the above examples by a general pseudo-differential operator.This is an operator A with symbol $a(x,\xi) = \xi^k + \lambda m(x,\xi)$,where k is some positive integer and λ is some constant.Then for $u \in D(A) = S(\mathbb{R}^n)$ we have,

$$\begin{aligned} Au(x) &= (2\pi)^{-n} \iint e^{i(x-y)\xi} [\xi^k + m(x,\xi)]u(y)dyd\xi \\ &= (2\pi)^{-n} \iint e^{i(x-y)\xi}\xi^k u(y)dyd\xi + (2\pi)^{-n} \iint e^{i(x-y)\xi} m(x,\xi)dyd\xi \\ &= \frac{d^k u(x)}{dx^k} + (2\pi)^{-n} \iint e^{i(x-y)\xi} m(x,\xi)\, dyd\xi . \end{aligned}$$

Clearly the pseudo-differential operator A as defined above is also an integro-differential operator.This is an important observation because it shows that pseudo-differential operators can unite integral operators and differential operators.

Hopefully the theory of pseudo-differential operators will be an important tool for determining the spectrum of integro-differential operators such as those encountered in neutron transport theory like the one presented in Mika [15] .These are operators of the form;

$$\frac{\partial}{\partial t} - A ,$$

where A is a pseudo-differential operator.

Numerical analysis of these operators seems possible.

APPENDIX A

Let us initially assume that $m_j \neq m_k$ for $j \neq k$, $j,k, = 1,2,3,\ldots$. Given an interval $(i,i+1]$, there is a finite number of m_j's in this interval, since $m_j \longrightarrow -\infty$ as $j \longrightarrow +\infty$. We arrange m_j's in every $(i, i+1]$, $i \in Z$ in a decreasing order. The new sequence satisfies the assumptions of Theorem 1.3.2 and therefore can be transformed into an absolutely convergent series by an appropriate choice of $\{t_j\}$ and Ψ. Thus the original series is absolutely convergent with all derivatives and satisfies the condition (2). If there are repetitions in the sequence $\{m_j\}$ we consider the decreasing sequence of distinct elements $\{\hat{m}_j\}$ of the $\{m_j\}$ sequence.

Let $\hat{m}_j = m_{j_1} = \ldots\ldots = m_{j_p}$. In this case we have $<z>^{m_j-\rho|\alpha|} = <z>^{\hat{m}_{j-1}-\rho|\alpha|} <z>^{\hat{m}_j-\hat{m}_{j-1}}$.

By an appropriate choice of t_j's one can make $<z>^{\hat{m}_j-\hat{m}_{j-1}}$ sufficiently small as required by (6).

It thus shows that Theorem 1.3.2 is true for any sequence $\{m_j\}$, $m_j \longrightarrow -\infty$ as $j \longrightarrow \infty$. In practice we encounter the case $m_j \neq m_k$, $j \neq k$, $j,k, = 1,2,3,\ldots$ only.

APPENDIX B

We need to show that $\bigcap_{m \in \mathbb{R}} \Gamma^m_\rho (\mathbb{R}^{2n}) = \bigcap_j \Gamma^{m_1}_\rho (\mathbb{R}^{2n})$ if $m_j \longrightarrow -\infty$ as $j \longrightarrow \infty$. The property 4 of Lemma 1.2.1 reduces this problem to a standard question of set theory.

APPENDIX C

To show that the iterated integral (1) is well–defined, we demonstrate that $\int e^{-iy\xi}\, a(x,y,\xi)u(y)dy$, $u \in S(\mathbb{R}^n)$ is a rapidly decreasing function of ξ.

Observe that if $a \in \Pi^m_\rho (\mathbb{R}^{3n})$ then $a(x,y.\xi)u(y) \in S(\mathbb{R}^n)$ as a function of y. Actually

$$<y>^k \,|\, \partial^\alpha_y \left[a(x,y,\xi)u(y)\right] | \leq \sum_{\gamma+\sigma=\alpha} \frac{\alpha!}{\gamma!\,\sigma!} |\partial^\gamma_y a(x,y,\xi)| \; |\partial^\sigma_y u(y)| \; <y>^k$$

$$\leq \sum_{\gamma+\sigma=\alpha} \frac{\alpha!}{\gamma!\,\sigma!} C_\gamma <x,y,\xi>^{m-\rho|\gamma|} <x-y>^{m'+\rho|\gamma|} \times <y>^k \,|\, \partial^\sigma_y u(y) \,| .$$

Since $\partial_y^{\sigma} u \in S(\mathbb{R}^n)$, we conclude that

$|\partial_y^{\alpha} u(y)| \leq C_N <y>^{-N}$ for any $N > 0$.

Observe that

$$<x,y,\xi>^{m-\rho|\lambda|} \leq \begin{cases} 1 & \text{if } m-\rho|\lambda| \leq 0 \\ (<x>\ <y>\ <\xi>)^{m-\rho|\lambda} & \text{if } m-\rho|\lambda| > 0 \end{cases}$$

Also

$$<x-y>^{m'+\rho|\lambda|} \leq \begin{cases} 1 & \text{if } m'+\rho|\lambda| \leq 0 \\ (<x><y>)^{m'+\rho|\lambda|} & \text{if } m'+\rho|\lambda| > 0 \end{cases}$$

In general we can write

(*) $<x,y,\xi>^{m-\rho|\gamma|}\ <x-y>^{m'+\rho|\gamma|} \leq <x>^p <y>^q <\xi>^r$ for all $j \leq \alpha$, where p, q and r are appropriate positive real numbers.

Finally, we have

$$\sum_{\gamma+\sigma=\alpha} \frac{\alpha!}{\gamma!\,\sigma!} C_\gamma <x,y,\xi>^{m-\rho|\gamma|}\ <x-y>^{m'+\rho|\gamma|}\ <y>^k\ |\partial_y^{\sigma} u(y)|$$

$$\leq C_N <x>^p <\xi>^r \sum_{\gamma+\sigma=\alpha} \frac{\alpha!}{\gamma!\,\sigma!} C_\gamma <y>^{q+k-N}$$

$$\leq \text{Const } <x>^p <\xi>^r \text{ for } N > q+k.$$

It follows that given x, ξ;

$\sup\limits_y <y>^k |\partial_y^{\sigma} (a(x,y,\xi)\, u(y)| < \infty$ as required of a function from $S(\mathbb{R}^n)$.

Since $a(x,y,\xi)\, u(y) \in S(\mathbb{R}^n)$ as a function of y, we are allowed to integrate by parts. Let N be an even positive integer. Then on using the identity $e^{-ix\xi} = <\xi>^{-N} <D_\xi>^N e^{-ix\xi}$ and integrating by parts we obtain

$$\int e^{-iy\xi}\, a(x,y,\xi)\, u(y)dy = \int e^{-iy\xi} <\xi>^{-N} <D_y>^N \left[a(x,y,\xi)u(y)\right] dy.$$

Since $<D_y>^N$ is a constant coefficient differential operator, by (3) of Theorem 1.6.3 and (*) we get,

$$|<D_y>^N \left[a(x,y,\xi)\, u(y)\right]| \leq C_N <x>^p <y>^{-M} <\xi>^{-N+r},$$

where M and N can be chosen arbitrarily large. This proves that, given x

$$\sup_\xi <\xi>^r \left|\int e^{-ix\xi}\, a(x,y,\xi)\, u(y)dy\right| < \infty$$

and therefore the integration on y yields a function of ξ from $S(\mathbb{R}^n)$. Thus we conclude that the iterated integral (1) of Definition 1.6.2 is well defined.

APPENDIX D

Let $\chi(x,y,\xi) \in C_0^\infty(\mathbb{R}^{3n})$, $\chi(0,0,0) = 1$. Consider $\sigma_{\epsilon,\omega}(x,y,\xi) = \chi(\epsilon x,\epsilon y,\omega\xi)\ \sigma(x,y,\xi)$. The formula (10) of Theorem 1.6.6 is valid for $\sigma_{\epsilon\omega}(x,y,\xi)$. We have

$$\iint e^{i(x-y)\xi}(x-y)^\mu\ \sigma_{\epsilon,\omega}(x,y,\xi)\ u(y)dyd\xi = \int e^{ix\xi}\left[\int e^{iy\xi}(x-y)^\mu\ \sigma_{\epsilon\omega}(x,y,\xi)u(y)dy\right]d\xi.$$

From the Lebesgue's Dominated Convergence Theorem applied to the dominant function $y \longrightarrow \sigma(x,y,\xi)\ u(y)$ from $S(\mathbb{R}^n)$ we are in a position to take $\epsilon \longrightarrow 0$ under the sign of the integral in brackets. We get

$$\int e^{ix\xi}\left[\int e^{iy\xi}\chi(0,0,\omega\xi)\ (x-y)^\mu\ \sigma(x,y,\xi)\ u(y)dy\right]d\xi$$

$$= \int e^{ix\xi}\ \chi(0,0,\omega\xi)\left[\int e^{iy\xi}(x-y)^\mu\ \sigma(x,y,\xi)\ u(y)dy\right]d\xi.$$

On using Appendix B and the fact that $\sigma(x,y,\xi) \in \Pi_p^m(\mathbb{R}^{3n})$ we conclude that the result of the integration of the term in brackets is a function of ξ from $S(\mathbb{R}^n)$. Thus we can take $\omega \longrightarrow 0$ under the sign of the integral. This justifies the LHS of (10) for $\sigma(x,y,\xi) \in \Pi_p^m(\mathbb{R}^{3n})$ as an iterated integral.

The same argument applies to RHS of (10). Thus (10) is valid for any $\sigma(x,y,\xi) \in \Pi_p^m(\mathbb{R}^{3n})$ if both sides of (10) are considered as repeated integrals.

APPENDIX E

Put $C(x,\xi) = \sum_{j=1}^{\infty}(-1)^j\ \varphi(t_j^{-1} <x,\xi>)\ C_0^{(j)}(x,\xi)$

$$\varphi(z) = \begin{cases} 0 \text{ if } \|z\| \leq \frac{1}{2} \\ 1 \text{ if } \|z\| \geq 1 \end{cases}$$

Let $t_{j+1} > t_j$, $j = 1,2,3,\ldots$. If $<x,\xi> \geq t_j$, then

$\varphi(t_j^{-1} <x,\xi>) = 1$. Since $t_j \geq t_k$, $k \leq j$, we have

$\varphi(t_k^{-1} <x,\xi>) = 1$ for $<x,,\xi) \geq t_j$. Thus the product of the symbols $1+C_0(x,\xi)$ and $\sum_{k=0}^{j} (-1)^k \varphi(t_k^{-1}<x,\xi>)\, C_0^{(k)}(x,\xi)$ yields a symbol $1 + r_j$, where $r_j \in \Gamma_p^{-2j\rho}(\mathbb{R}^{2n})$ since the calculations take place for $<x,\xi> \geq t_j$. So the infinite sum yields a symbol $1+r$, where $r \in \bigcap_{j=1}^{\infty} \Gamma_p^{-2j\rho}(\mathbb{R}^{2n}) = S(\mathbb{R}^{2n})$.

APPENDIX F

We need to show that $|\partial_z^\alpha a_m(z)| \leq C_\alpha |a_m(z)|\, \|z\|^{-\rho|\alpha|}$.

Both sides of the inequality above are homogeneous functions of degree $m-\rho|\alpha|$ (see Example 1.1.3). Thus it is sufficient to prove the result for z on the unit sphere. By definition $\|a_m(z)\| \geq C_1$, for $\|z\| = 1$.

Thus $\partial_z^\alpha a_m(z)$ is bounded on the unit sphere. Therefore there exists a positive constant C_α such that $|\partial_z^\alpha a_m(z)| \leq C_\alpha \|a_m(z)|\, \|z\|^{-\rho|\alpha|}$ for $\|z\| = 1$.

By homogeneity, the above inequality is true for all $z \in \mathbb{R}^{2n}$.

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