# Keychains and Preferential Fuzzy Sets with Applications

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by

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**Abstract**

In this thesis, we study the preferentiality behaviour of choices under uncertainties using keychains, where a keychain is defined as an $(n+1)-$tuple of non-increasing real numbers in the unit interval, $I = [0, 1]$. We look at the representations of uncertainties or sets defined by vague properties using the idea of keychains, pins and pinned flags. We then apply the ideas of preferential fuzzy sets to voting patterns, economics and decision making. For voting patterns, we simulate mock trials to investigate the behaviours of choices of different individuals, the outcomes of such voting and make specific conclusions about voting strategies. It can be argued that preferentiality in voting can enhance the democratic processes in national elections. This thesis contains various representations of keychains such as binary digits, weight order, lattice and simplex representations.

Another useful aspect of keychains and preferential fuzzy sets is to study the outcomes of decision making linking it to the study of keychains and finite fuzzy sets. We envisage that this study will throw light on computational aspects of any countable situations.

**KEYWORDS**:

Fuzzy sets, Posets, Lattices, Equivalence Relation, Keychains, Voting Patterns, Simplices, Preference relation, Decision Making.

**A.M.S SUBJECT CLASSIFICATION**:

06A06; 03B52; 91B12; 91B86; 05C99; 06C99; 62C86; 06D99; 94D05; 94D05; 03E72; 20E99;

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## NOTATIONS

| | |
|---|---|
| $\mathbb{N}^0$ | The set of natural numbers with zero (0) included. |
| $I = [0, 1]$ | The unit interval. |
| $\mu_X$ | Fuzzy subset of set $X$. |
| $\mu^C$ | The complement of the fuzzy set $\mu$. |
| $\mu_\alpha$ | A weak $\alpha-$cut of $\mu$, the set is $\{x \in X : \mu(x) \geq \alpha\}$. |
| $\mu^\alpha$ | A strong $\alpha-$cut of $\mu$, the set is $\{x \in X : \mu(x) > \alpha\}$. |
| $core(\mu)$ | The core of $\mu$ denoted by $\{x \in X : \mu(x) = 1\}$. |
| $supp(\mu)$ | The support of $\mu$ denoted by $\{x \in X : \mu(x) > 0\}$. |
| $co - supp(\mu)$ | The core support of $\mu$ denoted by $\{x \in X : \mu(x) = 0\}$. |
| $\mu \vee \nu$ | Intersection of two fuzzy sets, $\mu$ and $\nu$. |
| $\mu \wedge \nu$ | union of two fuzzy sets, $\mu$ and $\nu$. |
| $(X, \preceq)$ | A partial ordered set of $X$ ordered by $\preceq$, (poset). |
| $(X, \preceq: \vee, \wedge)$ | A lattice of $X$ ordered by $\preceq$. |
| $1 = \lambda_0 > \cdots > \lambda_n$ | Is a totally ordered set or a chain, where $n \in \mathbb{N}^0$. |
| $1 = \lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_n$ | Is a finite chain called a keychain, where $n \in \mathbb{N}^0$. |
| $1\lambda_1\lambda_2 \cdots \lambda_n$ | Another way of representing a keychain, where $n$ refers to its length. |
| $(C, \ell)$ | Its a Pinned flag, where $C$ is a maximal chain of a given poset and $\ell$ a keychain of length $n$. |
| $(n_1, n_2, \cdots, n_t)$ | An order of pins for a keychain of length $n$, for $1 \leq t \leq n$ such that $n_1 + n_2 + \cdots + n_t = n$, and $n$ & $t \in \mathbb{N}$. |
| $n_1, n_2, \cdots, n_t$ | These are components of keychain of length $n$. |
| Weight order | Is a partial order on the set of all keychains of the same length |
| $V$, $E$ and $F^{(n)}$ | Vertex ($V$), Edge ($E$), and $n^{th}$ sided Face ($F^{(n)}$) of a simplex. |

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## PREFACE

Our primary purpose of this study is to explore the applications of the notion for keychains, a concept which was proposed by V. Murali and B.B. Makamba, [33], [35], [36], [37], [40], [41]. By keychains $\ell$, we imply a finite chain of the form

$$\ell : \ 1 = \lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_n \geq 0 = 1\lambda_1\lambda_2\cdots\lambda_n,$$

where $\lambda_n$ denotes a pin, and $n$ the length of the keychain. Any keychain of length $n$ has $(n+1)-$pins. The pinned flag is a pair $(C, \ell)$, where $C$ is the maximal chain of a fuzzy set $X$, where $\ell$ is a keychain.

We discuss how choices under uncertainties and in preference relations can be made simpler using this notion of keychains. We further show how the $(n+1)-$pins and the notion of pinned flags for keychains of length $n$, can be used in modeling the different possible choices and preferences that could be made. Simulating examples illustrating how keychain pins can be linked to any choice exercised are given. Below we give a breakout of the contents of this study.

**In Chapter 1**, we recall the well established basics of the theory of Fuzzy set with some applications as discussed by different researchers, viz [2], [3], [6], [5], [9], [13], [12], [14], [27], [29], [26]. Prominently, discussions on the definitions, characteristics and properties of some concepts of fuzzy sets that will be useful in the build-up of the project will be observed. These concepts include but not limited to, $\alpha-$cuts, intersections, unions and complements.

Furthermore, notions of Posets together with Lattices, with the assistance of [22], [23] and others, are analysed with their characteristics by means weight-diagrams. This chapter serves as an introduction to the whole study.

**In Chapters 2 & 3**, we explore the notion of totally ordered sets under the concept of keychains, where we consider a unit interval $I = [0,\ 1]$ and an $n-$chain,

$$\ell : \ 1 = \lambda_0 > \lambda_1 > \lambda_2 > \cdots > \lambda_n,$$

see [29] and [38] and others. We review the basic definitions pertaining the evolvement of keychains. This includes definitions of Weight order, Pins, Components, and Padidity of keychains. We proceed to show that keychains can be represented by binary digits, weight order diagrams and simplex diagrams. We discuss the vertices, edges, faces and cores that formed from our simplices and compare them with Euler's characteristic equation,[21].

It is in this chapter where we show the representation of Simplices/ Simplexes, [17], [18] by means of keychains.

**In Chapter 4**, we provide relationships between keychains and other fuzzy set concepts which include, Flags and Pinned Flags with the help in [38], [39], [40], [43]. We further unpack the definition of the notion of equivalence relation, [48] and use it to show how it could be applied in the notion of keychains. Later on we look at the enumerations of fuzzy sets using the notion of keychains where it's pins are of more importance.

**In Chapter 5**, we outline the keychain applications in voting patterns by Miklos, [31] which defines voting as a mechanism where the opinions from a set of votes are evaluated in order to select the alternatives that best represent the collective preferences. We explore Hosseingholizadey's work, [15] about voting using the keychains to determine how well keychains are exercised in a Fuzzy ballot paper, (FBP) model by Cagman and Aktas, [42], to make an informed decision.

This Fuzzy Ballot paper in conjunction with the ideas of keychains, we discover how well keychains are exercised in this fuzzy ballot paper by using pins of each keychain of specific length that will be generated. Simulating examples of casted votes from the FBP are done and presented to show how keychains can be used to come up with any most preferred situation or best possible choice among countless possibilities. Enumerations of casted votes by pins of the generated keychains are well displayed from the given simulation examples (Mock examples).

**In Chapter 6**, we seek to utilise the notion of keychains in the ideas of economists which among few include, [4], [19], [30], [45]. We show that keychains can be used to model how one can make choices of commodities, consumables or budget as the socio-economic world necessitates one to make choices, of which each choice can be uncertain.

In the end we bring in the notion of a choice rule, [11] where we show that membership values of different choices in a pool of alternative choices $x \in X$ can be clearly represented by keychains. We conclude this section by displaying examples that model three fuzzy linguistic words.

**In Chapter 7**, we introduce a procedure of utilising the notion of keychains in decision making as according to Bellman and Zadeh, [7]. We also use ideas from Zhou, [50]. Our aim is to exercise these ideas at any institution that deals with any service delivery.

Simulating examples are used to illustrate how keychains can be used to come with suitable and just decisions to the benefit of the decision makers.

**In Chapter 8**, we give summary of the work done in these studies and recommendations for future research which could be regarded by researchers.

THROUGHOUT THIS THESIS, ACKNOWLEDGEMENTS TO VARIOUS AUTHORS ARE GIVEN WHERE THEY ARE DUE.

# Chapter 1

# Fuzzy Sets

We start our studies by first looking at the basic ideas of representation of vagueness that led to the concept of Fuzzy set. These ideas are helpful in the representation of keychains. We discuss the operations on fuzzy sets and alpha-cuts. We show with few examples how vagueness is captured using the fuzzy set theory. We briefly raise the notions of Posets and Lattices.

It is common knowledge that sets can be informally described as collections of objects. The objects we refer to are said to be members or elements of the set. Zimmerman, [51] states that a set is an abstract object as it does not necessarily have to contain a physical collection of elements to be a set.

- In ordinary set theory the membership criteria of elements in a set must be well defined.
- Each single element in a collection can either belong to or not belong to a given set.

Now we consider a set $X$ and an element $x$. If $x$ is a member of $X$, we denote that by

$$x \in X,$$

if not, we denote it by

$$x \notin X.$$

But according to Kaufmann, [26] elements in or not in a set can be determined by means of a characteristic function, $\mu$. For a set $X$ and an element $x$, the characteristic function will be denoted by;

$$\mu_X(x) = \begin{cases} 0 & \text{if } x \text{ is not a member of } X \\ 1 & \text{if } x \text{ is a member of } X. \end{cases}$$

The above description of elements belonging to or not in a set unfairly discards elements that could differ slightly to those that might be regarded as belonging to some set. Those that belong to a set are given the value **1** and that do not a value **0**. In the coming sections we will entertain the real numbers in the interval $I = [0,\ 1]$ and link the values with the notion of membership idea in a set.

## 1.1 Fuzzy Sets Review

Chakraborty, R.C. [10], says, in the real world, there exist much fuzzy knowledge; knowledge that is vague, imprecise, uncertain, ambiguous, inexact or probabilistic in nature. He further says that human thinking and reasoning frequently involve fuzzy information, originating from inherently inexact human concepts and that humans can give satisfactory answers, which are probably true.

The goal of this section is to illustrate the use of fuzzy subsets in modeling vague propositions/arguments that need one to take reasonable decisions. The concept of fuzzy subset was first introduced by Professor L.A Zadeh in the year 1965. According to [16], Zadeh in his theory of fuzzy sets, proposed using a membership function (with a range covering the interval $[0, 1]$) operating on the domain of all possible values.

He proposed new operations for the calculus of logic and showed that fuzzy logic was a generalisation of classical and Boolean logic. He also proposed fuzzy numbers as a special case of fuzzy sets, as well as the corresponding rules for consistent mathematical operations (fuzzy arithmetic).

From [27], a fuzzy subset $A$ is defined as a set of ordered pairs as follows;

$$A = \{(x, \mu_A(x)) | x \in \mathbf{X}, \mu_{\mathbf{A}}(\mathbf{x}) \in [\mathbf{0}, \mathbf{1}]\}$$

where the function $\mu_A(x)$ is the membership function that associates to each element $x_i \in X$ a number $\mu_A(x_i)$ in the closed unit interval $[0, 1]$. This number $\mu_A(x_i)$ represents the degree of membership of $x_i$ in $\mathbf{A}$. This can also be viewed as a mapping;

$$A : X \to [0, 1].$$

This is unlike the crisp sets whose elements can either be in a set or not. The following figure shows the membership functions of both Crisp Sets, $\mathbf{C}$ and Fuzzy Sets, $F$.

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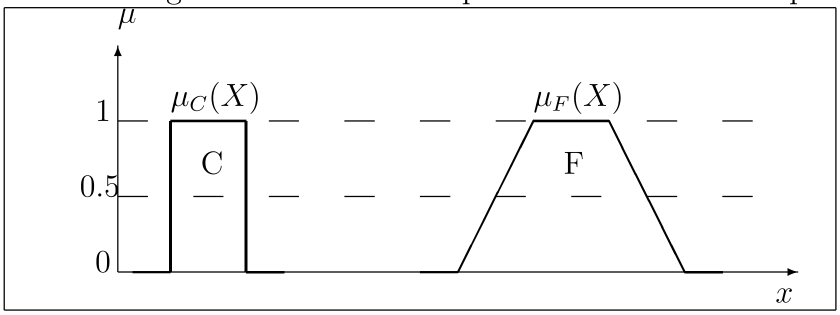
Figure 1.1: Membership functions of both Crisp Sets




The above figure clearly shows that the members of a crisp set are given a membership degree of "**1**" with those that do not belong are given a membership degree of "**0**".

From the definition of fuzzy sets we notice that sets allow elements to be partially members by assigning a value or a degree of membership to each element in the given fuzzy subset within an interval of $I = [0,\ 1]$. So one can conclude and say,

$$\text{Crisp Sets} \subseteq \text{Fuzzy Sets.}$$

### 1.1.1 Fuzzy subsets in Economics

In economics, the notion of fuzzy subsets brings a reasonable perspective when analysing some vague concepts. In [1], Adzic and Sedlak mention the fact that fuzzy set theory represents an appropriate mathematical apparatus for dealing with uncertainties, subjectivity and vagueness.

In its applications, they highlight that fuzzy sets have been advanced to treat imprecise and uncertain information, where fuzzy instruments can overcome the restrictions of classical quantitative methods when describing and defining problems of economic development, thus enabling a more exact approach to qualitative modeling.

These authors revealed that fuzzy sets play a big role in the process of decision making within macroeconomic planning in systems undergoing large changes.

### 1.1.2 Fuzzy subsets in Social Science

The fuzzy set theory on the other hand grades collections with uncertainty with values from 0 to 1. That gives elements a chance to have a degree of membership in the said fuzzy set. If the grading of a given element is 1, then that particular element belongs absolutely to the set. On the other hand, if the grading of an element is 0 then that element is absolutely not a member of that set. The values between 0 and 1 strictly indicate the relative degrees of membership of elements to the given fuzzy subset.

Now we look at examples of situations that could be represented by fuzzy subsets and bring a clear description of what a given proposition mean.

**Example 1.1.1.**

Here we look at the concepts of people that are little educated, highly educated and very highly educated.

The following graph is showing levels of education of different people, i.e. (i) Little Educated, (ii) Highly Educated and (iii) Very Highly Educated.

**On education level 1**

Little educated $\longrightarrow$ 0.8

Highly educated $\longrightarrow$ 0

**On education level 2**

Little educated $\longrightarrow$ 0.5

Highly educated $\longrightarrow$ 0.2

Very highly educated $\longrightarrow$ 0

**On education level 3**

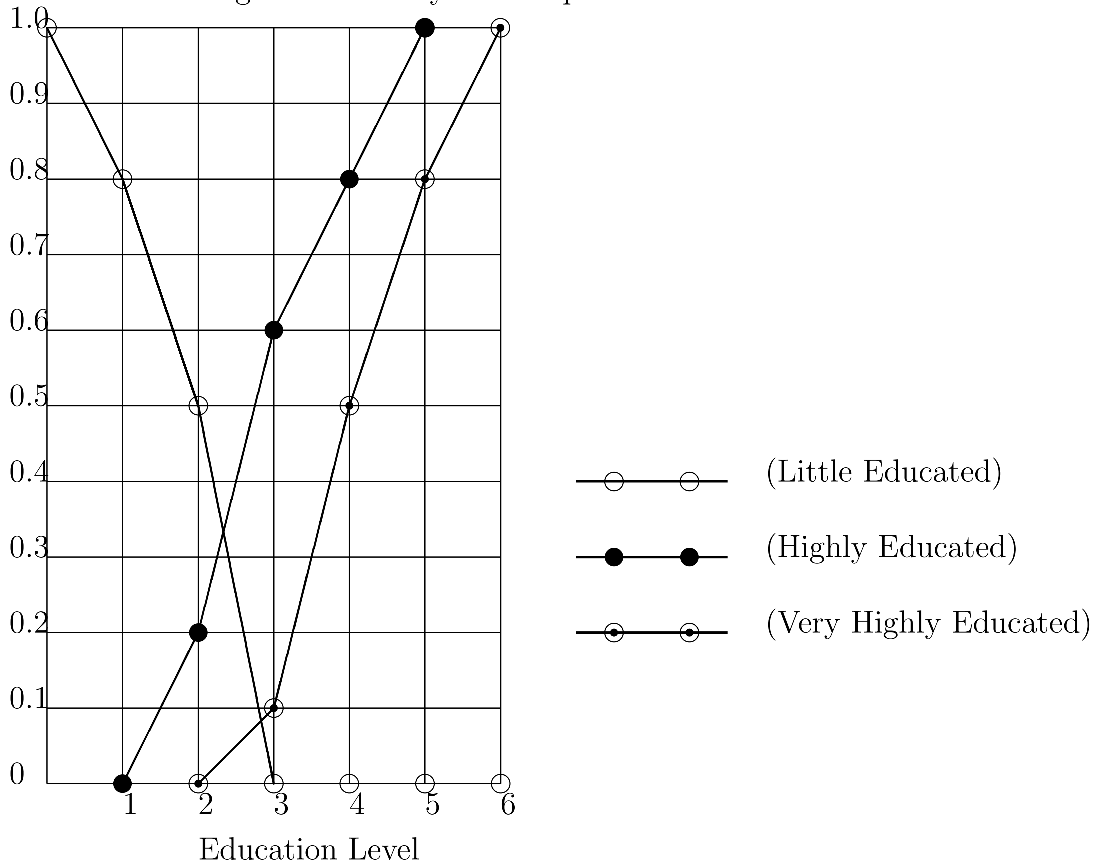
Little educated $\longrightarrow$ 0

Highly educated $\longrightarrow$ 0.6

Very highly educated $\longrightarrow$ 0.1

and so on.

Figure 1.2: Fuzzy Set Graph for Education Levels




## 1.2 Operations on fuzzy subsets

Suppose $\mu$ and $\nu$ are two fuzzy subsets of $U$.

### 1.2.1 The Inclusion and Equality

- **The inclusion** of $\mu$ and $\nu$, is denoted by $\mu \subseteq \nu$ and it means that $\mu(x) \leq \nu(x)$ for all $x \in X$ and at least for one $x \in U$, $\mu(x) < \nu(x)$.

- **The Equality** of $\mu$ and $\nu$, denoted by $\mu = \nu$, this means that $\mu$ is equal to $\nu$ if and only if $\mu(x) = \nu(x)$ for all $x \in X$.

### 1.2.2 The Union, Intersection and Complement

- **The Union** of $\mu$ and $\nu$, denoted by $\mu \cup \nu$ is defined as a maximum of the two individual membership functions. i.e.

$$\mu \cup \nu = Max(\mu, \nu) = \mu \vee \nu,$$

and for $x \in X$,

$$(\mu \vee \nu)(x) = Max(\mu(x), \nu(x)) = \mu(x) \vee \nu(x).$$

- **The intersection** of $\mu$ and $\nu$, denoted by $\mu \cap \nu$ is defined as the **minimum** value of the membership function of the two fuzzy subsets, i.e.

$$\mu \cap \nu = Min(\mu, \nu) = \mu \wedge \nu,$$

and for $x \in X$

$$(\mu \wedge \nu)(x) = Min(\mu(x), \nu(x)) = \mu(x) \wedge \nu(x).$$

R. R Yager, [49] proved the following theorem about union and intersection of two or more fuzzy sets.

**Theorem 1.2.1.**

Assume that $\mu$ and $\nu$ are two fuzzy subsets of $X$. If $\alpha(x) = (\mu \cup \nu)(x)$ and $\beta(x) = (\mu \cap \nu)(x)$ for $x \in X$, then:

**(a)** $\beta \subset \alpha$.

**(b)** $\mu \subset \alpha$ and $\nu \subset \alpha$.

**(c)** $\beta \subset \mu$ and $\beta \subset \nu$.

**Proof 1.2.2.**

For each $x \in X$,

$\beta(x) = Min(\mu(x), \nu(x))$ and $\alpha(x) = Max(\mu(x), \nu(x))$

hence

$\beta(x) \le \alpha(x)$ and thus $\beta \subset \alpha$.

Since

$\alpha(x) = Max(\mu(x), \nu(x))$, then $\alpha(x) \ge \mu(x)$

and

$\alpha(x) \ge \nu(x)$ for all $x \in X$,

hence

$\mu \subset \alpha$ and $\mu \subset \alpha$.

Since

$\beta(x) = Min(\mu(x), \nu(x))$ then $\beta(x) \le \mu(x)$

and

$\beta(x) \le \nu(x)$ thus $\beta \subset \mu$ and $\beta \subset \nu$.

This completes the proof.

- **The Complement**

The complement of $\mu$, denoted by $\mu^C$, is a fuzzy subset with membership function defined by

$$\mu^C(x) = 1 - \mu(x), \quad \text{for} \quad x \in X.$$

## 1.3 Fuzzy Subsets and Alpha-Cuts

Let us consider the set of all fuzzy subsets of $X$, denoted by $I^X$. The Greek letters $\mu$, $\nu$, etc denote the fuzzy subsets of $X$ and $0 \le \alpha \le 1$ is a real number.

### 1.3.1 Weak and Strong Alpha-Cuts

**Definition 1.3.1.**

The *weak $\alpha$-cut* of $\mu$ denoted by $\mu_\alpha$ is the crisp subset given by

$$\mu_\alpha = \{x \in X : \mu(x) \ge \alpha\}.$$

**Definition 1.3.2.**

The *strong* $\alpha$*-cut* of $\mu$ denoted by $\mu^\alpha$ is the crisp subset of $X$ given by

$$\mu^\alpha = \{x \in X : \mu(x) > \alpha\}.$$

**Note 1.3.3.**

If $\alpha = 1$ then $\mu^\alpha = \emptyset$;

If $\alpha = 0$ then $\mu_\alpha = X$ ;

Depending on the fuzzy set, $\mu^\alpha$ or $\mu_\alpha$ may be empty for certain $\alpha < 1$ and $\mu^\alpha$ or $\mu_\alpha$ may be $X$ for $\alpha > 0$.

**Proposition 1.3.4.**

If $\mu,\ \nu \in I^X$ then

1. $\mu = \nu$ if and only if $\mu^\alpha = \nu^\alpha$ for all $\alpha \in I$.

2. $\mu = \nu$ if and only if $\mu_\alpha = \nu_\alpha$ for all $0 < \alpha < 1$.

**Proof 1.3.5.**

Assume $\mu = \nu$. This means

$\mu(x) = \nu(x)$ for all $x \in X$.

Therefore for any $\alpha \in I$,

$\mu^\alpha = \{x : \mu(x) > \alpha\}$

and

$\nu^\alpha = \{x : \nu(x) > \alpha\}$ by definition.

So $x \in \mu^\alpha$ if and only if $x \in \nu^\alpha$.

This implies that $\mu^\alpha = \nu^\alpha$.

**Conversely**

$\mu^\alpha = \nu^\alpha$ for all $\alpha \in I$ for any two fuzzy sets,

$\mu : X \longrightarrow I$

$\nu : X \longrightarrow I.$

To show that $\mu(x) = \nu(x)$ for all $x \in X$,

let $\mu(x) = \alpha$ then

$x \notin \mu^\alpha \Leftrightarrow \alpha(x) \not> \alpha$ and

$x \in \nu^\alpha \Leftrightarrow \nu(x) < \alpha.$

Suppose not, then

$\mu(x_0) \neq \nu(x_0)$ for some $x_0 \in X$,

then there is an $\alpha$ such that

$\mu(x_0) < \alpha < \nu(x_0)$, for $x_0 \notin \mu^\alpha$ and $x_0 \in \nu^\alpha$.

This is a contradiction because $x_0 \in \nu^\alpha$ implies that $x_0 \in \mu^\alpha$.

### 1.3.2 The Core, Support and Co-Support of a Fuzzy Set

**Definition 1.3.6.**

The *core* of $\mu$ denoted by $core(\mu)$, is the crisp subset such that

$$core(\mu) = \{x \in X : \mu(x) = 1\}.$$

We note that the core is the subset of all elements that belong to the fuzzy subset absolutely.

**Definition 1.3.7.**

The *support* of $\mu$ denoted by $supp(\mu)$ is the crisp subset such that

$$supp(\mu) = \{x \in X : \mu(x) > 0\}.$$

If $supp(\mu) = X$ then every element belongs to $\mu$ with some degree $\mu(x) > 0$.
It is possible that $supp(\mu) = X$ without $\mu^\alpha = X$ or $\mu_\alpha = X$ for any $\alpha > 0$.

**Definition 1.3.8.**

The *co-support* of $\mu$ denoted by co$supp(\mu)$, is the crisp set such that

$$\text{co}supp(\mu) = \{x \in X : \mu(x) = 0\},$$

which consists of elements that do not belong to the given fuzzy subset absolutely. The characteristic function of the alpha cut of $\mu$ is denoted by $\chi_{\mu^\alpha}$.

**Proposition 1.3.9.**

Any fuzzy subset $\mu$ of $X$ can represented by its $\alpha$-cuts as follows:

$$\mu = \bigvee\{\alpha\chi_{\mu^\alpha} : 0 < \alpha < 1\}.$$

**Proof 1.3.10.**

For any fixed $x \in X$, we need to prove that

$\mu(x) = \bigvee_\alpha(\alpha\chi_{\mu^\alpha})(x)$ $\quad\quad *$

The R.H.S. of * is equal to $(\bigvee \alpha\chi_{\mu^\alpha})(x)$

If $x \in \mu^\alpha$ then $\alpha\chi_{\mu^\alpha}(x) = \alpha$, otherwise it is 0.

Suppose $x \in \mu^\alpha$ then $\mu(x) > \alpha$. Also if $\mu(x) > \alpha$ then $x \in \mu^\alpha$.

The above imply that $x \in \mu^\alpha$ if and only if $\mu(x) > \alpha$.

Also, $\mu(x) > \alpha$ implies that $\alpha\chi_{\mu^\alpha}(x) = \alpha$.

Therefore $\mu(x) > \alpha\chi_{\mu^\alpha}(x)$ for every $\alpha$ such that $\mu(x) > \alpha$.

Thus $\mu(x) > \bigvee \alpha\chi_{\mu^\alpha}$. The L.H.S of * > R.H.S of *.

**Conversely**

For fixed $x \in X$ and $\mu(x) = \beta$.

If $\mu(x) = \beta \leq \alpha < 1$ then $\alpha < 1$, $\alpha\chi_{\mu^\alpha}(x) = 0$.

Also, if $0 < \alpha < \beta = \mu(x)$ then $\alpha X_{\mu^\alpha}(x) = \alpha$.

Therefore $\bigvee \alpha X_{\mu^\alpha}(x) = \bigvee \alpha \leq \beta = \mu(x)$. We claim the equality holds.

Suppose not, then $\bigvee \alpha\chi_{\mu^\alpha}(x) < \beta = \mu(x)$.

If we choose $\varphi$ such that $\bigvee \alpha\chi_{\mu^\alpha} < \varphi < \beta = \mu(x)$

then $\varphi\chi_{\mu^\varphi}(x) = \varphi$

Thus $\bigvee \alpha\chi_{\mu^\alpha} < \varphi$ is a contradiction. Therefore $\bigvee \alpha\chi_{\mu^\alpha} = \mu$.

## 1.4 Partially Ordered Sets

From [22], a partial order on a set is an arrangement such that, for certain pairs of elements, one precedes the other. This basically means not all pairs of elements are required to be comparable in that the order. So sets into which a partial order is defined are called partially ordered sets, abbreviated as *POSETS.* In general, the *'partial order'* is denoted by $\leq$, and in its usual sense may be strict as in $<$ or non-strict as in $\leq$.

### 1.4.1 Partial Orders

Now we look below at the definitions of 'Strict' and 'Non-strict' partial orders as previously tackled in [29].

**Definition 1.4.1.**

Let $X$ be a non-empty set.

A *strict partial order* $R$ on X is a subset of $X \times X$ and will satisfy the following conditions:

1. Irreflexitivity:- for no $x \in X$ does $(x, x) \in R$ hold;

2. Antisymmetry:- if $(x, y) \in R$, then $(y, x) \notin R$;

3. Transitivity:- if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$.

**Definition 1.4.2.**

Let $X$ be a non-empty set.

A *non-strict partial order* $R$ on X is a subset of $X \times X$ satisfying the following conditions.

1. Reflexitivity:- for all $x \in X$ does $(x, x) \in R$;

2. Antisymmetry:- if $(x, y) \in R$, and $(y, x) \in R$ then $x = y$;

3. Transitivity:- if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$.

**Note 1.4.3.**

A set $X$ together with a strict partial order as defined above is denoted as $(X, <)$. This shows that a poset is a set $X$ carrying a partial order.

### 1.4.2 Properties of Posets

Let us consider a poset $X$,

- An element $x \in X$ is called a maximal element if there is no element $y \in X$ such that $x < y$. On the other hand, $x$ is minimal if no element y satisfies $y < x$.

- If in a poset $X$, every two elements are comparable, e.g $x_1 \geq x_2$ then the partial order is said to be a total order in $X$. A totally ordered set is also called a *chain*. The set of real numbers is one good example of a chain, because any two real numbers are comparable with respect to $\leq$ .

**Partially Ordered Sets in Diagrams**

Here we look at how to represent the ordering of elements of a poset $X$ that could be compared in a diagram form. The number of nodes in each diagram will represent the number of elements in the given poset. If nodes are not connected by a line, then such nodes are not comparable. Nodes on the same line, horizontally, vertically or otherwise can be compared by $\leq$ or $\geq$. The diagrams will be referred to as weight diagrams which could also represent preferences/arrangements and so on, by individuals.

Let $X$ be a poset on the relation $\Re$.

**(i) A one element poset**

If $X = \{a\}$ then $a \leq a$.

This shows that there is only one way of ordering one element in a set represented. We can show that as follows:


**(ii) A two element poset**

If $X = \{a,\ b\}$ then it is either

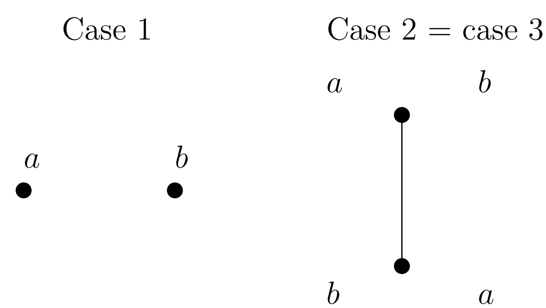
(1) $a \leq a$ and $b \leq b$, comparable to themselves but not to each other,

(2) $a \leq a$, $b \leq b$ and $a \leq b$, or

(3) $a \leq a$, $b \leq b$ and $b \leq a$.

Cases (2) and (3) above are referred to as *duals*.

Diagrammatically, these three possibilities can be shown as follows:




**(iii) A three element poset**

If $X = \{a,\ b,\ c\}$ then we will have nineteen possibilities which fall under the following 5 cases:

(1) $a \leq a$, $b \leq b$ and $c \leq c$, comparable to themselves but not to each other,

(2) $a \leq b \leq c$;   $a \leq c \leq b$;   $b \leq a \leq c$;  $b \leq c \leq a$;   $c \leq a \leq b$;   $c \leq b \leq a$.

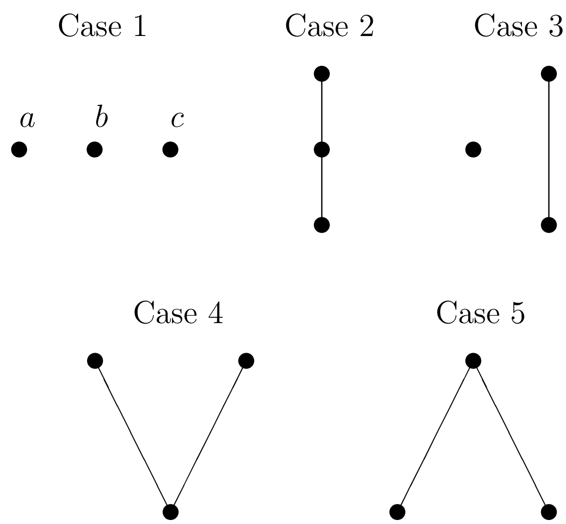
(3) $a \leq a$; $b \leq c$;   $a \leq a$; $c \leq b$;   $b \leq b$; $a \leq c$;   $b \leq b$; $c \leq a$;   $c \leq c$; $a \leq b$;

$c \leq c$; $b \leq a$.

(4) $a \leq b$; $a \leq c$;   $b \leq a$; $b \leq c$;   $c \leq a$; $c \leq b$.

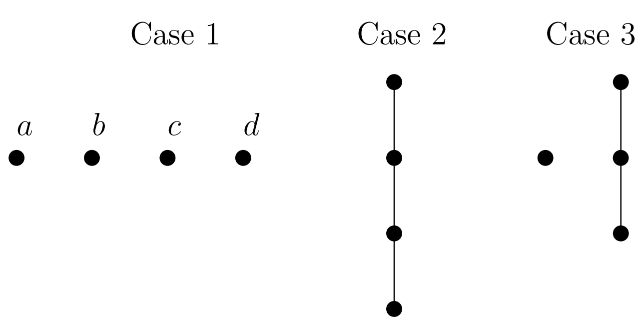
(5) $a \geq b$; $a \geq c$;   $b \geq a$; $b \geq c$;   $c \geq a$; $c \geq b$.

Below is a diagram representing the five cases.

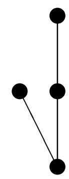



**(iv) A four element poset**

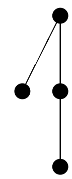
If $X = \{a,\ b,\ c,\ d\}$ then we will have *16 cases of non-isomorphic* posets which are represented in the following diagrams:



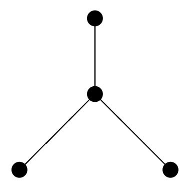
Case 4



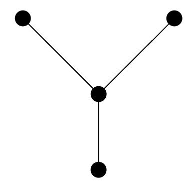
Case 5



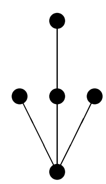
Case 6



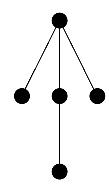
Case 7



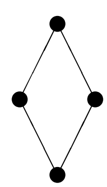
Case 8



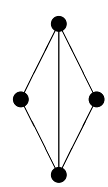
Case 9



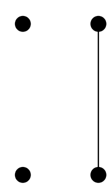
Case 10



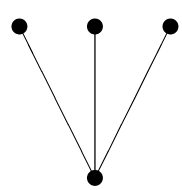
Case 11



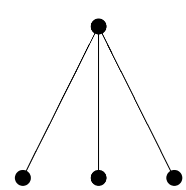
Case 12



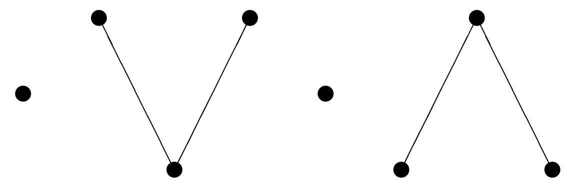
Case 13



Case 14



Case 15 Case 16



The above weight diagrams of the posets are representing all orderly possible arrangements of $n-$element posets, where $n = 1, 2, \cdots$ . These arrangements can mimic some real life situations. If we consider the cases 3, 4 and 5 under a poset with three elements then:

**Case 3** This case can be viewed as gathering of three members of an organization in any situation. We could regard any two members as related according to the organisations office positions and the third person being just an observer.

**Case 4** This can be viewed as a gathering of three members from two different organisations, like, two departments sharing a secretariat office.

**Case 5** This is the opposite of case 4 above and it can be viewed as a gathering of three people from two different organizations where one is the chairperson of both organizations.

## 1.5 Lattices

In this section we look at lattice definition and some few examples of lattice structures.

### 1.5.1 Lattice Definition

We cannot talk about lattices without mentioning posets which was discussed from the above sections. As stated by [3], [12], [14], and [53]. Lattices are partially ordered sets where any pair of elements have unique supremum and infimum also called a least upper bound (join) and a greatest lower bound (meet) respectively. The above can be clearly put according to the following table:

**Figure 1.5.1.** *Supremum and Infimum notations.*

| Name | Synonyms | Notation | Symbol |
|---|---|---|---|
| Supremum | Least upper bound, join | lub | $\vee$ |
| Infimum | Greatest lower bound, meet | glb | $\wedge$ |

Lattices can be defined either as a partially ordered set or as an algebraic structure. Now we look at the following definitions of Upper and Lower bounds together with a complete Lattice.

**Definition 1.5.2.** *Least Upper Bound*

Suppose $X$ is a poset, $u \in X$ is said to be an upper bound for the pair $(x, y)$ in $X$ if $x \leq u$ and $y \leq u$.

Thus the **least upper bound** for the pair $(x, y)$ is an element $u_0 \in X$ such that the element $u_0$ is an upper bound for the pair $(x, y)$ in $X$. Thus whenever $u$ is an upper bound for the pair $(x, y)$ in $X$ then $u_0 \leq u$.

The least upper bound is unique if it exists in $X$ and is denoted by

$$x \vee y.$$

that is,

If $x, y \in X$ where $X$ is a poset,

$$x \vee y = lub(x, y)$$

is a least upper bound.

**Definition 1.5.3.** *Greatest Lower Bound*

Let $X$ be a poset, an element $u \in X$ is said to be a lower bound for the pair $(x, y)$ in $X$ if $x \geq u$ and $y \geq u$.

The **greatest lower bound** for the pair $(x, y)$ is an element $u_0 \in X$ such that $u_0$ is a lower bound for $(x, y)$ in $X$. So whenever $u$ is a lower bound for $(x, y)$ in $X$ then $u_0 \geq u$ and is unique if it exists in $X$ and is denoted by

$$x \wedge y.$$

That is,

if $x, y \in X$ where $X$ is a poset,

$$x \wedge y = glb(x, y)$$

is a greatest lower bound.

**Definition 1.5.4.** *Complete Lattice*

A poset $(X, \leq)$, is said to be a *lattice* if for $x, y \in X$ there exist a greatest lower bound in the pair of $(x, y)$ and a least upper bound pair of $(x, y)$.

If every subset of poset $(X, \leq)$ has a greatest lower bound and a least upper bound then $(X, \leq)$ is said to be a *complete lattice*.

For a lattice $X = \{x_1, \cdots, x_n\}$ ordered by $\leq$, its least upper bound for all elements in $X$ will be denoted by

$$1 = x_1 \vee ... \vee x_n.$$

Its greatest lower bound of all its elements in $X$ will be denoted by

$$0 = x_1 \wedge ... \wedge x_n.$$

The digits 1 and 0 are referred to as the top and bottom elements of $X$.

The partial order $\leq$ is related to $\wedge$ and $\vee$ in $X$ in the following ways:

$$x_1 \leq x_2 \text{ if and only if } x_1 \wedge x_2 = x_1$$

and

$$x_1 \leq x_2 \text{ if and only if } x_1 \vee x_2 = x_2.$$

A lattice which satisfies at least one of the above mentioned properties is said to be a **conditionally complete** lattice, [23].

Other kinds of lattices include, Bounded Lattices, Complemented Lattices, Distributive Lattices, and Modular Lattices.

Below is an example of a lattice, $(X, \leq)$.

**Example 1.5.5.**

If $X = \{0,\ a,\ b.\ 1\}$ with the partial ordering, $\leq$, will $(X, \leq)$ be a lattice.

- The possible pairs from $X$ will be as follows:

  $\{(0,0),\ (0,a),\ (0,b),\ (0,1),\ (a,0),\ (a,a),\ (a,b),\ (a,1),\ (b,0),\ (b,a),\ (b,b),\ (b,1),\ (1,0),\ (1,a),\ (1,b),\ (1,1)\}$

  then their *lub* and *glb* will be as follows:

**Figure 1.5.6.** *lub and glb example from pairs above.*

| | $\vee$ (lub) | | | | | | $\wedge$ (glb) | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 0 | a | b | 1 | — | | 0 | a | b | 1 |
| 0 | 0 | a | b | 1 | — | 0 | 0 | 0 | 0 | 0 |
| a | a | a | b | 1 | — | a | 0 | a | a | 0 |
| b | b | b | b | 1 | — | b | 0 | a | b | b |
| 1 | 1 | 1 | 1 | 1 | — | 1 | 0 | a | b | 1 |

Since every pair of $X$ has both a join and a meet, so $(X, \vee, \wedge)$ is a lattice.

## 1.5.2 Lattice Algebraic Structures

Like posets from the previous sections, lattices can be represented geometrically by means of a graphs. The nodes that will be found on the graphs represent elements of any given set which could be comparable or not.

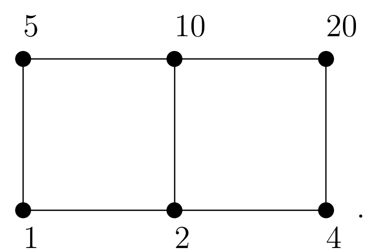
We now look at the following definitions of comparability as stated in [52]:

**Definition 1.5.7.** *Comparability*

If $a$ and $b$ are elements of a poset $(X, \preceq)$, then $a$ and $b$ are said to be *comparable* if either $a \preceq b$ or $b \preceq a$. If this does not hold, then $a$ and $b$ are said to be *incomparable.*

**Example 1.5.8.**

Let us consider a set $X = \{1, 2, 4, 5, 10, 20\}$, which is a set of all divisors of 20. By means of lattice diagram, this set can be represented as follows:




From the above diagram, any two nodes (picked either from left to right or from bottom to top) representing a pair of elements of set $X$, are comparable by the order divisibility.

**Some More Examples of Lattices**

Here we look at some posets that could be lattices or not in the form of diagrams/ graphs as according to our previous studies in [29].

Let us consider a poset $X = \{x_1, x_2, x_3, x_4\}$ and a lattice, $(X, \leq: \vee, \wedge)$.
The following four lattice and non-lattice diagrams could be depicted from the given poset, $X$.

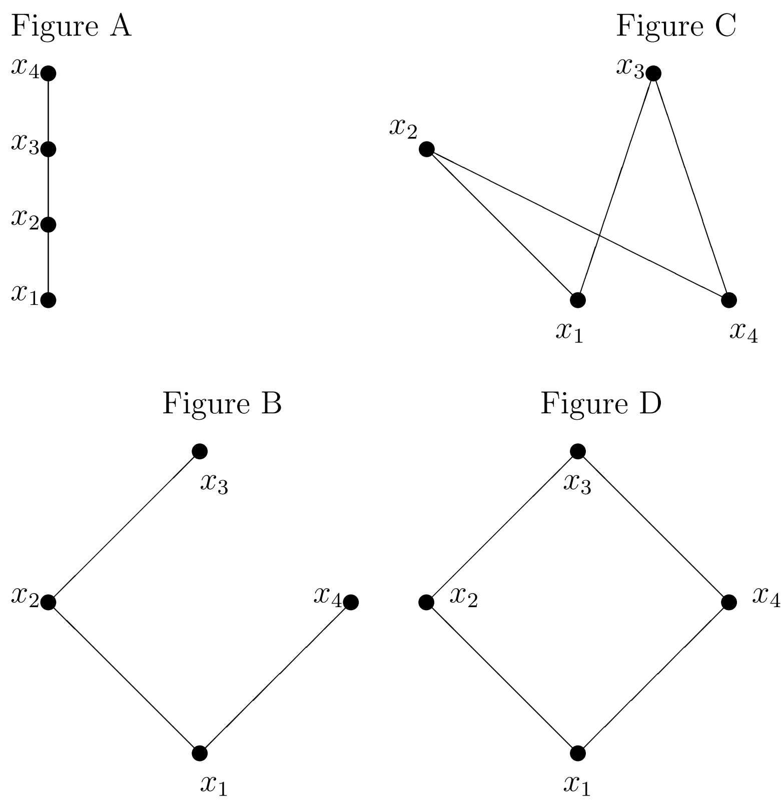



Figure 1.3: Lattice and Non-Lattice diagrams

From the above diagrams, we note the following:

**(i)** Figure A

* is not bounded,

* is a lattice as any pair elements of elements of $X$ possess a supremum and an infimum.

**(ii)** Figure B

* is not bounded,

* elements $x_2$ and $x_4$ do not have a least upper bound, hence, this is not a lattice.

**(iii)** Figure C

* is bounded (closed diagram),

* $x_1$ and $x_4$ have no *lub* and *glb*, thus it is not a lattice.

**(iv)** Figure D

* is bounded (closed diagram),

* in any pair of elements, we will get the Infimum and the Supremum, thus this is a lattice, and is associated with a two element Boolean Algebra.

**Definition 1.5.9.** *Total Order*

A poset $(X, \preceq)$ is called **totally ordered** if every two elements of $X$ are comparable, where $(\preceq)$ is called a *total order*.

**Definition 1.5.10.**

A totally ordered set is also called a chain, in [52].

# Chapter 2

# Keychains and its Representations

## 2.1 Background of Keychains

Keychains arise naturally from the study of finite fuzzy sets. A novel way of representing keychains as diagrams with nodes and edges, and code keychains using binary digits and decimal representations is shown with the help of, [37], [41] and [43].

**Definition 2.1.1.**

Let $I = [0, 1]$ be the unit interval. A chain of the form

$$1 = \lambda_0 \geq \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n \geq 0$$

where $\lambda_i \in I$ and for $i = 0, 1, 2, \cdots, n$, is called a *keychain* of length $n$. We can denote the keychain by a symbol, viz $\ell$, and we can also represent it as

$$\ell = 1\lambda_1\lambda_2 \cdots \lambda_n.$$

The numbers $1, \lambda_1, \lambda_2, \cdots, \lambda_n$ in a keychain are called *pins*. The pins are in a non-decreasing order. We have $2^{(n+1)} - 1$ possible keychains of length, $n$, where each has $n + 1$ pins. The pins of a keychain can either be distinct or not and the keychains with only pins 1 and 0 may be regarded as keychains of crisp sets.

**Remark 2.1.2.**

Naturally all keychains start with digit 1 as their first pin, that is, Pin $\lambda_0$ in any keychain is always equal to 1.

**Remark 2.1.3.**

There is no keychain such that $1 = \lambda_0 = \lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$.

**Remark 2.1.4.**

A keychain such that $1 = \lambda_0 > \lambda_1 > \lambda_2 > \lambda_3 > \cdots > \lambda_n > 0$, is said to be a chain.

## 2.2 Keychain Representation

### 2.2.1 Pins, Components and Padidity of Keychains

In this section we look at how keychains can be represented using pins.

Let us consider a keychain of length 3 where all the pins are distinct. Out of fifteen keychains of length 3, we will only have the following two keychains with distinct pins, and they will be represented as follows,

$$1 > \lambda_1 > \lambda_2 > \lambda_3 > 0$$

or

$$1 > \lambda_1 > \lambda_2 > \lambda_3 = 0.$$

Using shorthand, we can also represent them as follows;

$$1\lambda\beta\gamma$$

and

$$1\lambda\beta 0$$

respectively.

From the above representation, the differences in pins are clearly illustrated, as each pin represents a positive real number in the interval $I = [1, 0]$.

Now we look at another situation when we have a keychain with same pins as to how to represent them. We consider the following keychain of length 4

$$1 > \lambda_1 = \lambda_2 = \lambda_3 > \lambda_4 > 0 \ = \ 1\lambda\lambda\lambda\beta.$$

From the latter keychain, we see the consecutive occurrences of equality signs (pins which are the same). According to [40] this occurrence of an interlocking position of pins is called a **component**. The first pin, 1 in the first position in any keychain does not interlock with any other pins. This pin is not considered as a component of any chain. In keychains of length $n$ with $n+1$ pins, there could be $k$ distinct components, for $1 \leq k \leq n$ and we refer to these types of keychains as $k$-pads.

Below, we look at an example that will illustrate the component idea in keychains with pins in interlocked positions.

**Example 2.2.1.**

Let us consider the following keychains of the same length, 12.

$$\ell_1 = 111\lambda\beta\beta\beta\gamma\gamma0000$$

and

$$\ell_2 = 11\lambda\lambda\beta\gamma\gamma\gamma\gamma\gamma\zeta\zeta0,$$

$\ell_1$ is a 5-pad keychain whilst $\ell_2$ is a 6-pad one.

If one looks at the $k$-pad, the number, $k$ indicates the number of distinct types of pin (values) in a keychain. The number of the pins found in any component of a keychain is called the **padidity** of the component.

From the above example, $\ell_1$ and $\ell_2$ have the following padidities of its components;

$$\ell_1 : \ \ 2,\ 1,\ 3,\ 2,\ 4$$

$$\ell_2 : \ \ 1,\ 2,\ 1,\ 5,\ 2,\ 1$$

respectively.

### 2.2.2 Keychains and Binary Digits

In this section, we represent keychains as binary strings, to do so we let the equality sign (=) be denoted by 0 and the greater than sign (>) by 1, that is

$$= \sim\ 0$$

$$> \sim\ 1.$$

We know that the numbers 0 and 1 are referred to as binary digits. We know by definition that a keychain of length $n$, is denoted by

$$1 = \lambda_0 \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0,$$

or

$$1 \geq \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n \geq 0.$$

We will illustrate the binary string representations of keychains using some few keychains of length, 0, 1, 2, 3, $\cdots$ , $n$, respectively.
Recall, we have $2^{(n+1)} - 1$, keychains of length $n$.

**Table 2.2.2.** *Keychains of length zero (0).*

| Symbol representation of key chains | Binary string |
|---|---|
| - = $[1 \neq 0]$ | - |
| $1 = [1 > 0]$ | 1 |

**Table 2.2.3.** *Keychains of length one (1).*

| Symbol representation of key chains | Binary string |
|---|---|
| $11 = [1 = \lambda > 0]$ | 01 |
| $1\lambda = [1 > \lambda > 0]$ | 11 |
| $10 = [1 > \lambda = 0]$ | 10 |

**Table 2.2.4.** *Keychains of length two (2).*

| Symbol representation of key chains | Binary string |
|---|---|
| $111 = [1 = \lambda_1 = \lambda_2 > 0]$ | 001 |
| $11\lambda = [1 = \lambda_1 > \lambda_2 > 0]$ | 011 |
| $110 = [1 = \lambda_1 > \lambda_2 = 0]$ | 010 |
| $1\lambda\lambda = [1 > \lambda_1 = \lambda_2 > 0]$ | 101 |
| $1\lambda\beta = [1 > \lambda_1 > \lambda_2 > 0]$ | 111 |
| $1\lambda 0 = [1 > \lambda_1 > \lambda_2 = 0]$ | 110 |
| $100 = [1 > \lambda_1 = \lambda_2 = 0]$ | 100 |

**Table 2.2.5.** *Keychains of length three (3).*

| Symbol representation of key chains | Binary string |
|---|---|
| $1111 = [1 = \lambda_1 = \lambda_2 = \lambda_3 > 0]$ | 0001 |
| $111\lambda = [1 = \lambda_1 = \lambda_2 > \lambda_3 > 0]$ | 0011 |
| $1110 = [1 = \lambda_1 = \lambda_2 > \lambda_3 = 0]$ | 0010 |
| $11\lambda\lambda = [1 = \lambda_1 > \lambda_2 = \lambda_3 > 0]$ | 0101 |
| $11\lambda\beta = [1 = \lambda_1 > \lambda_2 > \lambda_3 > 0]$ | 0111 |
| $11\lambda 0 = [1 = \lambda_1 > \lambda_2 > \lambda_3 = 0]$ | 0110 |
| $1100 = [1 = \lambda_1 > \lambda_2 = \lambda_3 = 0]$ | 0100 |
| $1\lambda\lambda\lambda = [1 > \lambda_1 = \lambda_2 = \lambda_3 > 0]$ | 1001 |
| $1\lambda\lambda\beta = [1 > \lambda_1 = \lambda_2 > \lambda_3 > 0]$ | 1011 |
| $1\lambda\lambda 0 = [1 > \lambda_1 = \lambda_2 > \lambda_3 = 0]$ | 1010 |
| $1\lambda\beta\beta = [1 > \lambda_1 > \lambda_2 = \lambda_3 > 0]$ | 1101 |
| $1\lambda\beta\gamma = [1 > \lambda_1 > \lambda_2 > \lambda_3 > 0]$ | 1111 |
| $1\lambda\beta 0 = [1 > \lambda_1 > \lambda_2 > \lambda_3 = 0]$ | 1110 |
| $1\lambda 00 = [1 > \lambda_1 > \lambda_2 = \lambda_3 = 0]$ | 1100 |
| $1000 = [1 > \lambda_1 = \lambda_2 = \lambda_3 = 0]$ | 1000 |

The diagrams of keychains of binary digits are shown on the appendix on figures 8.1 and 8.2.

Observing the binary string representations of the keychains above we noticed that their numeric order changes the weight order of the keychains. We will discuss weight order in sections below.

### 2.2.3 Decimal Value Representations of Binary Digit Keychains

Here we look at how binary digit keychains are represented as decimal values or ordinary integer numbers. It is a general knowledge that any binary digit string can be converted by means of the following process:

- Let $n$ be the length of binary string, with $(n+1)$ binary digits.
- We let $B_n$ be a binary digit in position $n$ of a binary string, smallest position in each string is on far right whilst the highest on far left of the given string, and
- Let $2^{(n-1)}$ be the weight of the binary digit in position $n$ of the binary string.

To get the decimal equivalent value of a binary string, we add together the product of the position and weight of the various binary digits in the given binary string, as follows:

$$\sum_{r=0}^{n-1} B_{(r+1)} * 2^r = B_1 * 2^0 + B_2 * 2^1 + \cdots + B_n * 2^{(n-1)}.$$

The following tables represent the decimals of keychains as done above.

**Table 2.2.6.** *Decimal of Keychains of length zero (0).*

| keychains | Binary digits | Decimal representations |
|---|---|---|
| - | - | - |
| 1 | 1 | 1 |

**Table 2.2.7.** *Decimal of Keychains of length one (1).*

| keychains | Binary digits | Decimal representations |
| --- | --- | --- |
| 11 | 01 | 1 |
| $1\lambda$ | 11 | 3 |
| 10 | 10 | 2 |

**Table 2.2.8.** *Decimal of Keychains of length two (2).*

| keychains | Binary digits | Decimal representations |
| --- | --- | --- |
| 111 | 001 | 1 |
| $11\lambda$ | 011 | 3 |
| 110 | 010 | 2 |
| $1\lambda\lambda$ | 101 | 5 |
| $1\lambda\beta$ | 111 | 7 |
| $1\lambda0$ | 110 | 6 |
| 100 | 100 | 4 |

**Table 2.2.9.** *Decimal of Keychains of length three (3).*

| keychains | Binary digits | Decimal representations |
|---|---|---|
| 1111 | 0001 | 1 |
| $111\lambda$ | 0011 | 3 |
| 1110 | 0010 | 2 |
| $11\lambda\lambda$ | 0101 | 5 |
| $11\lambda\beta$ | 0111 | 7 |
| $11\lambda 0$ | 0110 | 6 |
| 1100 | 0100 | 4 |
| $1\lambda\lambda\lambda$ | 1001 | 9 |
| $1\lambda\lambda\beta$ | 1011 | 11 |
| $1\lambda\lambda 0$ | 1010 | 10 |
| $1\lambda\beta\beta$ | 1101 | 13 |
| $1\lambda\beta\gamma$ | 1111 | 15 |
| $1\lambda\beta 0$ | 1110 | 14 |
| $1\lambda 00$ | 1100 | 12 |
| 1000 | 1000 | 8 |

When we numerically arrange the above binary digit keychains of length 0, 1, 2, and 3, we also observe some interesting orders whenever we neglect the first pin. Now let us look at the following examples so as to observe how the said orders are formed:

### 2.2.4 Order of Keychains

We look at the definition of an *Order* of a keychain, as discussed in [29].

**Definition 2.2.10.**

Let $\ell$ be a keychain of length $n$.

By *order* of the keychain $\ell$, we mean a set of positive integers $(n_1, n_2, \cdots, n_t)$ for $1 \leq t \leq n$ such that $n_1 + n_2 + \cdots + n_t = n$.

**Note 2.2.11.**

- That the order of keychains is the arrangement of pins in the $(n+1)-$positions of a keychain of length $n$ when the first pin is omitted.
- The numbers $n_1,\ n_2,\ \cdots,\ n_t$ represent the number of pins in the interlocking positions called *components* and are the compositions of the integer $n$.
- An order $(n_1,\ n_2,\ \cdots,\ n_t)$ may contain one or more positive integers.

Now we consider the orders in the keychains of lengths. 0, 1, 2, 3 and 4.

**Table 2.2.12.** *Order in a 0-element set.*

| Keychains | Order |
|---|---|
| 1/0 | 0 |

**Table 2.2.13.** *Order in a 1-element set.*

| Keychains | | | Order |
|---|---|---|---|
| 11 | $1\lambda$ | 10 | 1 |

**Table 2.2.14.** *Orders in a 2-element set.*

| Keychains | | | | Order |
|---|---|---|---|---|
| $11\lambda$ | 110 | $1\lambda\beta$ | $1\lambda 0$ | 11 |
| | 111 | $1\lambda\lambda$ | 100 | 2 |

**Table 2.2.15.** *Orders in a 3-element set.*

| Keychains | | | | Order |
|---|---|---|---|---|
| $11\lambda\beta$ | $11\lambda 0$ | $1\lambda\beta\gamma$ | $1\lambda\beta 0$ | 111 |
| $11\lambda\lambda$ | 1100 | $1\lambda\beta\beta$ | $1\lambda 00$ | 12 |
| $111\lambda$ | 1110 | $1\lambda\lambda\beta$ | $1\lambda\lambda 0$ | 21 |
| | 1111 | $1\lambda\lambda\lambda$ | 1000 | 3 |

**Table 2.2.16.** *Orders in a 4-element set.*

| Keychains | | | | Order |
|---|---|---|---|---|
| $11\lambda\beta\gamma$ | $11\lambda\beta 0$ | $1\lambda\beta\gamma\zeta$ | $1\lambda\beta\gamma 0$ | 1111 |
| $11\lambda\beta\beta$ | $11\lambda 00$ | $1\lambda\beta\gamma\gamma$ | $1\lambda\beta 00$ | 112 |
| $11\lambda\lambda\beta$ | $11\lambda\lambda 0$ | $1\lambda\beta\beta\gamma$ | $1\lambda\beta\beta 0$ | 121 |
| $11\lambda\lambda\lambda$ | 11000 | $1\lambda\beta\beta\beta$ | $1\lambda 000$ | 13 |
| $111\lambda\lambda$ | 11100 | $1\lambda\lambda\beta\beta$ | $1\lambda\lambda 00$ | 22 |
| $111\lambda\beta$ | $111\lambda 0$ | $1\lambda\lambda\beta\gamma$ | $1\lambda\lambda\beta 0$ | 211 |
| $1111\lambda$ | 11110 | $1\lambda\lambda\lambda\beta$ | $1\lambda\lambda\lambda 0$ | 31 |
| | 11111 | $1\lambda\lambda\lambda\lambda$ | 10000 | 4 |

Looking at the above illustrated orders of keychains, we observe that there are certain orders or patterns into which the pins of keychains formed are ordered.

**Example 2.2.17.**

Let us consider a keychain of length 5; The possible orders of such keychains are:

(5), (41), (32), (311), (23), (221), (212), (2111), (14), (131), (122), (1211), (113), (1121), (1112) and (11111).

The components in the order can be distinct, the same or a combination of distinct and same, e.g (131), (221) and (1121).

Distinct keychains associated with any given $n$-element set may share the same order.

**Example 2.2.18.** .

Consider a keychain of length 3, where $11\lambda\beta$, $11\lambda 0$, $1\lambda\beta\gamma$ and $1\lambda\beta 0$ share the order, (111). There is no other keychain in this set other than these four with this order.

### 2.2.5 Weight Order and Diagrams

**Weight Order**

The keychain diagrams that are in [29] represent the keychain weight diagrams, as the diagrams clearly show keychains that could be compared. We could also see that Keychains are ordered by means of their particular *weight order*.

By *weight order* of a keychain of length $n$, we mean the arrangement and the comparison of all the pins of distinct Keychains. The distinct keychains of the same length have weights attached which are defined by pins in any given position. We provide the following definition of *weight order* for a Keychain.

**Definition 2.2.19.**

Consider the following two keychains of the same length,

$$1\lambda_1\lambda_2\cdots\lambda_n$$

and

$$1\lambda_1'\lambda_2'\cdots\lambda_n'.$$

We define a partial order $\leq$ on the set of all keychains of the same length by

$$1\lambda_1\lambda_2\cdots\lambda_n \;\leq\; 1\lambda_1'\lambda_2'\cdots\lambda_n'$$

if and only if $\lambda_i \;\leq\; \lambda_i'$ for all $i = 1, 2, \cdots, n$.

**Proposition 2.2.20.**

The weight order is a partial order on the set of all keychains of the same length.

**Proof 2.2.21.** It is clear that the three properties, reflexivity, anti-symmetry and transitivity all satisfied.

For the keychains of same length 11111 and $1111\lambda$, where $1 \geq \lambda \geq \beta$. By the weight order

$$11111 > 1111\lambda.$$

Since

$$1111\lambda \geq 111\lambda\beta$$

we can expect that

$$11111 \geq 111\lambda\beta$$

as these keychains are comparable.

However, the keychains $11\lambda\lambda\beta$ and 11100 are not comparable. It's the order of their pins that make us to come to that conclusion.

They could relatively have the same weight or not.

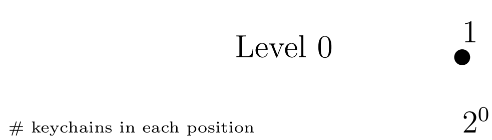
### Weight Diagrams

In our previous studies we looked at the weight diagrams of keychains of different lengths, 0, 1, 2, 3, 4 and 5.
We represented these weight diagrams as follows:

### (i) Keychains of length 0.

This is a keychain with one pin and that pin is 1.

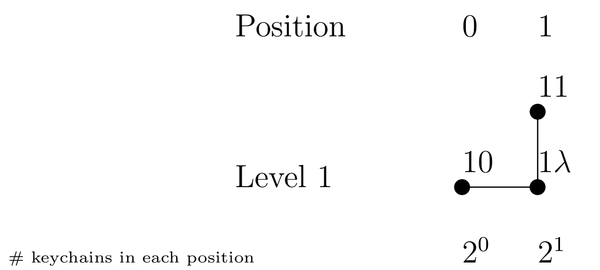
**Figure 2.2.22.** *Keychain diagram of length 0.*



**(ii) Keychains of length 1.**

The total number of possible keychains of length 1 is three and are as follows; $11, 1\lambda, 10$. Their weight diagram is illustrated below.

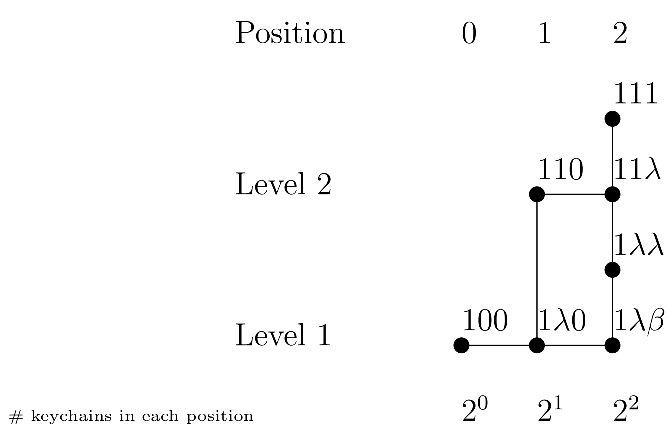
**Figure 2.2.23.** *Keychain diagram of length 1.*




**(iii) Keychains of length 2.**

There are seven distinct keychains of this length 2 which are:
$111, 11\lambda, 110, 1\lambda\lambda, 1\lambda\beta, 1\lambda 0, 100$. The weight diagram of the above keychains is depicted as follows:

**Figure 2.2.24.** *Keychain diagram of length 2.*



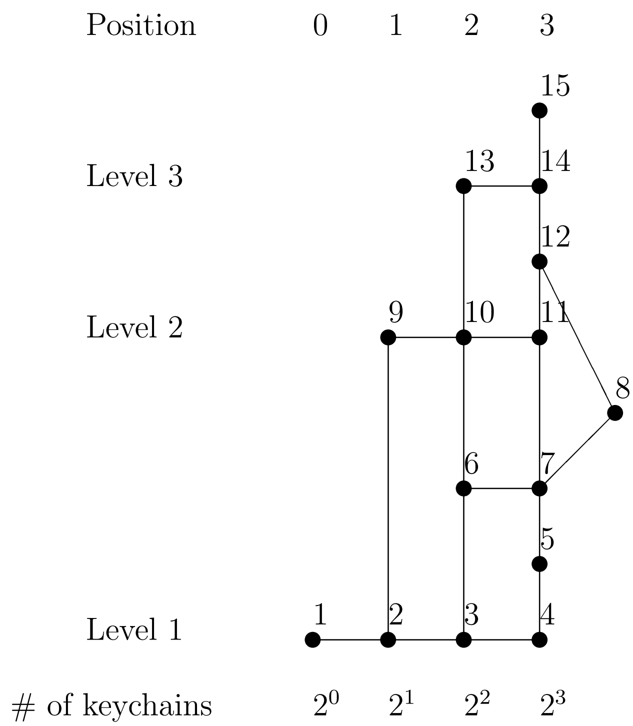
**(iv) Keychains of length 3.**

The 15 keychains of length 3 are

$1 = 1000$, $2 = 1\lambda 00$,$3 = 1\lambda\beta 0$, $4 = 1\lambda\beta\gamma$, $5 = 1\lambda\beta\beta$, $6 = 1\lambda\lambda 0$, $7 = 1\lambda\lambda\beta$, $8 = 1\lambda$ $\lambda\lambda$, $9 = 1100$, $10 = 11\lambda 0$, $11 = 11\lambda\beta$, $12 = 11\lambda\lambda$, $13 = 1110$, $14 = 111\lambda, 15 = 1111$,

The weight diagram of the above keychains is as follows;

**Figure 2.2.25.** *Keychain diagram of length 3.*



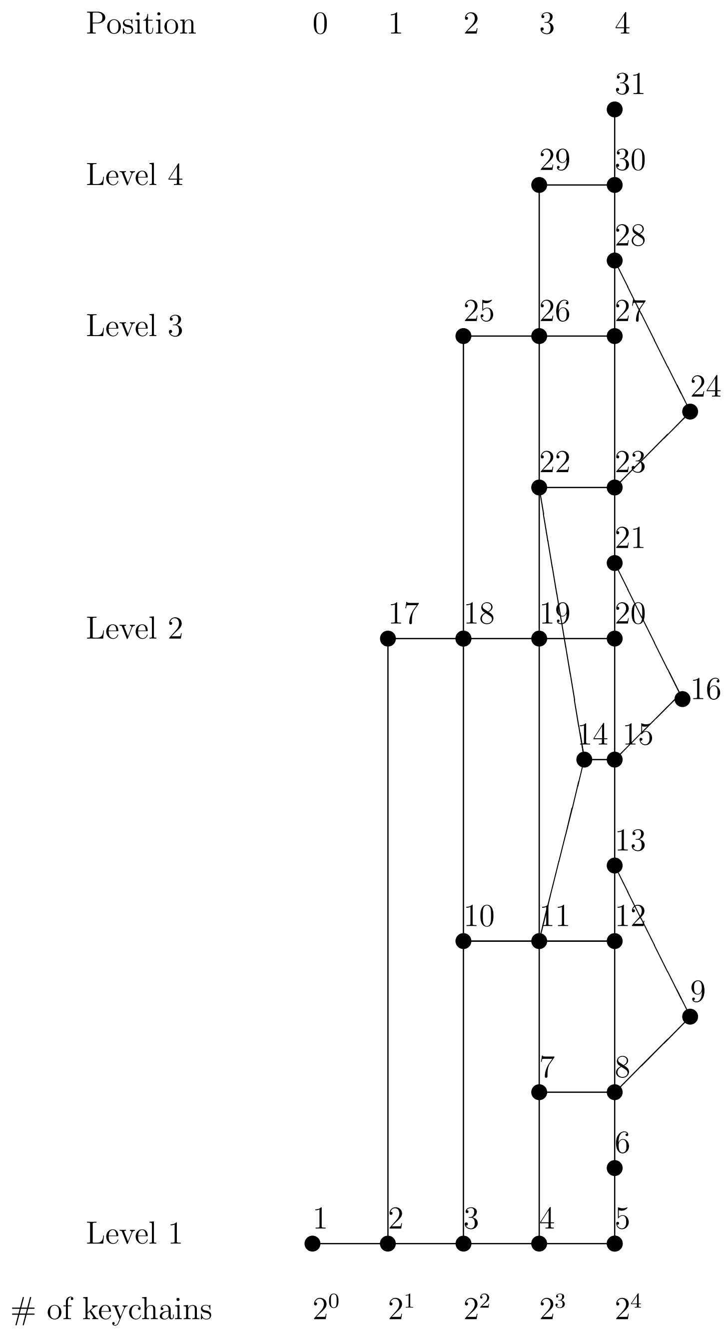

**(v) Keychains of length 4.**

The 31 keychains of length 4 are listed below:

1. $10000$, 2. $1\lambda 000$, 3. $1\lambda\beta 00$, 4. $1\lambda\beta\gamma 0$, 5. $1\lambda\beta\gamma\tau$, 6. $1\lambda\beta\gamma\gamma$, 7. $1\lambda\beta\beta 0$,
8. $1\lambda\beta\beta\gamma$, 9. $1\lambda\beta\beta\beta$, 10. $1\lambda\lambda 00$, 11. $1\lambda\lambda\beta 0$, 12. $1\lambda\lambda\beta\gamma$, 13. $1\lambda\lambda\beta\beta$, 14. $1\lambda\lambda\lambda 0$,
15. $1\lambda\lambda\lambda\beta$, 16. $1\lambda\lambda\lambda\lambda$, 17. $11000$, 18. $11\lambda 00$, 19. $11\lambda\beta 0$, 20. $11\lambda\beta\gamma$, 21. $11\lambda\beta\beta$,
22. $11\lambda\lambda 0$, 23. $11\lambda\lambda\beta$, 24. $11\lambda\lambda\lambda$, 25. $11100$, 26. $111\lambda 0$, 27. $111\lambda\beta$, 28. $111\lambda\lambda$,
29. $11110$, 30. $1111\lambda$, 31. $11111$,

The weight diagram of the set of keychains above is shown as follows,

**Figure 2.2.26.** *Keychain diagram of length 4.*




The nodes shown from these weight diagrams represent the keychains and are arranged in such a way that they form rows and columns.

The connection of nodes by lines is indicated according to weight order. As one moves from *bottom* to *top* the degree of membership of elements increases and also as one moves from *left* to *right* the degree of membership increases.

In all we can summarize this by saying that the line connecting the lower node to the upper node means that the lower node is of a lower degree than the upper one.

### 2.2.6 Lattice Representation of Keychains

In this section we illustrate the representation of keychain by lattices. We do so in line with Murali's discussions in [37].

#### Lattices and Keychains

Since keychains have an order, as discussed in the previous sections, then some two or more can be comparable or not. In [37], it is stated that, if

$$\ell_1 = (\lambda_0, \lambda_1, \cdots, \lambda_n)$$

and

$$\ell_2 = (\nu_0, \nu_1, \cdots, \nu_n)$$

then $\ell_1$ & $\ell_2$ will be ordered $\ell_1 \leq \ell_2$ whenever $\lambda_i \leq \nu_1$ for all $0 \leq i \leq n$.
The pins of keychains follow the ordering of real numbers in a unit interval. The bounded unit interval, $I = [0, 1]$, is a complete lattice under the usual ordering, thus the set of keychains $(\mathcal{K}, \leq)$ is also a bounded lattice with under the ordering $\leq$, where $\mathcal{K}$ denotes the set of all keychains. For a keychain of length $n$, the top and bottom elements are denoted by $(1, \underbrace{1, \cdots, 1}_{\text{n−times}})$ and $(1, \underbrace{0, \cdots, 0}_{\text{n−times}})$ respectively.

Since the set of keychains $(\mathcal{K}, \leq)$ is also a lattice, then the operations on keychains $\ell_1$ and $\ell_2$ can be considered, where

$$\ell_1 \vee \ell_2 = (\lambda_0 \vee \nu_0, \cdots, \lambda_n \vee \nu_n)$$

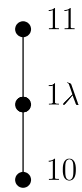
and

$$\ell_1 \wedge \ell_2 = (\lambda_0 \wedge \nu_0, \cdots, \lambda_n \wedge \nu_n).$$

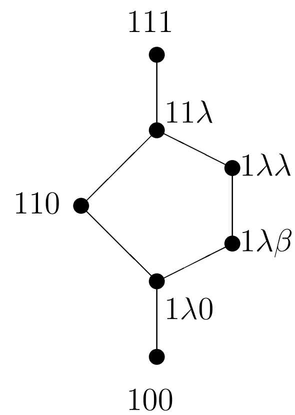
**Lattice diagrams of Keychains**

Here we look at some lattice diagram representations of keychains.

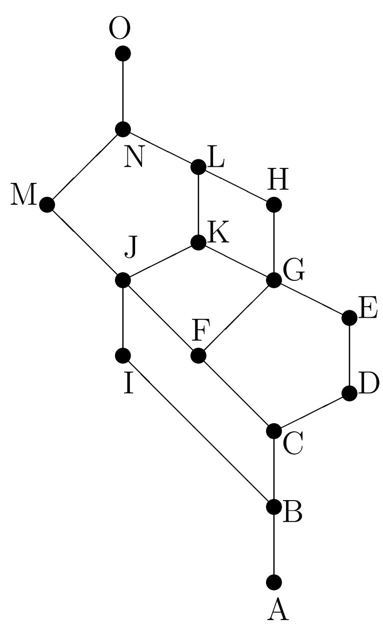
**(i)** Now we draw the lattice diagrams of keychains of lengths, 1, 2 and 3.




(a) Length 1




(b) Length 2




(c) Length 3

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**Indicators:**

$A = 1000$, $B = 1\lambda 00$, $C = 1\lambda\beta 0$, $D = 1\lambda\beta\gamma$, $E = 1\lambda\beta\beta$, $F = 1\lambda\lambda 0$, $G = 1\lambda\lambda\beta$, $H = 1\lambda\lambda\lambda$, $I = 1100$, $J = 11\lambda 0$, $K = 11\lambda\beta$, $L = 11\lambda\lambda$, $M = 1110$, $N = 111\lambda$,

$0 = 1111$

From the keychains of the lengths mentioned above that are associated with nodes of the lattice diagrams displayed, the numbers, $1, \lambda, \beta, \gamma$ & $0$, are called the pins of the keychains and $1 > \lambda > \beta > \gamma > 0$.

Murali, in [37] argues that the lattice diagram (c) of keychain of length 3 is made by doubling the graphs of keychains of length 2, then join the corresponding nodes.

Thus, the two lattices are drawn on the same axes with different positions, where the second one is shifted vertically down and horizontally towards the right by 2 units each. Thereafter, corresponding nodes are joined. In terms of functions, the lattice in (c) would be viewed as two functions $f_1$ and $f_2$ which are drawn on the same plane according to the following equation: $f_2(x) = f_1(x-2) - 2$.

**(ii)** Let us consider the keychain, $\ell = (1\underbrace{1\cdots1)}_{\text{n}-2\text{ times}}\lambda_1\lambda_2$ of length $n \geq 2$, with pins 1, where we will maneuver the pins in the last two positions with all possible numbers in the unit interval such that the two last pins are $0 \leq \lambda_1, \lambda_2 \leq 1$. The possible set of keychains (lattice) that could be generated from this keychain are as follows:

$\ell_1 = 1\underbrace{1\cdots1)}_{\text{n}-2\text{ times}}11$, Top element

$\ell_2 = 1\underbrace{1\cdots1)}_{\text{n}-2\text{ times}}1\lambda$

$\ell_3 = 1\underbrace{1\cdots1)}_{\text{n}-2\text{ times}}10$

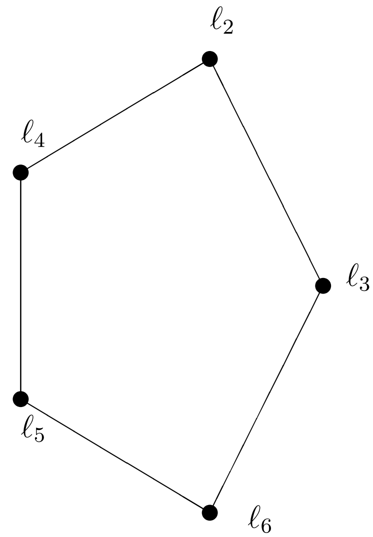
$\ell_4 = 1\underbrace{1\cdots1)}_{\text{n}-2\text{ times}}\lambda\lambda$

$\ell_5 = 1\underbrace{1\cdots1)}_{\text{n}-2\text{ times}}\lambda\beta$

$\ell_6 = 1\underbrace{1\cdots1)}_{\text{n}-2\text{ times}}\lambda0$

$\ell_7 = 1\underbrace{1\cdots1)}_{\text{n}-2\text{ times}}00$, Bottom element.

If we are neglecting the top and bottom element we get the sub-lattice $\{\ell_2, \ell_3, \ell_4, \ell_5, \ell_6\}$ and its diagram is as follows;




From the definitions of lattice discussed from section 1, the above diagram shows a lattice that is not distributive, so the sub-lattice $\{\ell_2, \ell_3, \ell_4, \ell_5, \ell_6\}$ is not modular.

# Chapter 3

# Keychains and Simplex diagrams

According to [17], a simplex is a geometrical concept, which is a generalisation of the notion of a triangle or a tetrahedron to arbitrary dimensions and its one of the tools of Mathematical optimization.

Here we discuss the properties of simplex diagrams then show the representations of keychains by simplices.

## 3.1 Properties of Simplex Diagrams

### 3.1.1 Vertices and Lines

According to [18], a **point/ vertex** denoted by **V**, is a location represented by a dot, geometrically. They further on and say that a point does not have any length, width, shape or size. They stress that a point has only a position.

We know that between any two points a **straight line** denoted by **L**, joining them can be drawn, but this is not always possible for spherical geometry. In the simplex diagrams, we note that in any two corresponding keychains denoting two corresponding vertices we will be able identify a line keychain that joins the said two vertices.

### 3.1.2 Faces and a Core

If three or more lines come together, a three or four or more sided faces can be formed. To differentiate these faces **F** as from one shape to the other by identifying them by the following symbols:

$$\begin{aligned} F^3 &= \text{Triangular face} \\ F^4 &= \text{Rectangular face} \\ &\vdots \\ F^n &= n^{th}-\text{sided face.} \end{aligned}$$

A keychain of order $n$ with $(n+1)-$distinct pins excluding pin 0 is a **Core.** In the following section, we discuss the Properties under the notions mentioned above.

## 3.2 Understanding Keychains in Simplex Diagram Properties

**Remark:** The keychains with only pins 1 or/ and 0 are regarded as keychains of a *crisp set.*

- **Vertices**

  If a keychain is a $1-$pad keychain of ones, $1^s$ or a $2-$pad keychain, that will be comprised by pins ones $1^s$ and zeros $0^s$, then such a keychain will represent the **vertex** in a simplex diagram formed.

  The number of vertices that a simplex diagram will poses depend on the length of a given keychain. That is, for a keychain of length $n$, we will have $(n+1)-$vertices. They are of the form;

$$\underbrace{11\cdots1}_{(n+1)-\text{times}}$$

$$\underbrace{11\cdots1}_{n-\text{times}}0$$

$$\underbrace{11\cdots1}_{(n-1)-\text{times}}\ 00$$

$$\vdots$$

$$1\underbrace{00\cdots0}_{n-\text{times}}.$$

Now, let us consider a keychain of length 3. The following will be the vertices of the simplex diagram that will be formed:

$$1111,\ 1110,\ 1100,\ 1000.$$

- **Lines**

  A 2−pad keychain made of only two distinct types of pins where there could be a 0− pin(s) to make it a 3−pad keychain represents a **lines (edges)** between any two vertices of the simplex diagram.

  Since each keychain has its own length, $n$, which denotes $(n+1)-$vertices, then for any two vertices chosen out of $(n+1)-$vertices, we will,

$$1-\text{vertex} \longrightarrow 0-\text{lines}$$

$$2-\text{vertices} \longrightarrow 1-\text{line}$$

$$3-\text{vertices} \longrightarrow 3-\text{lines}$$

$$4-\text{vertices} \longrightarrow 6-\text{lines}$$

$$5-\text{vertices} \longrightarrow 10-\text{lines}$$

$$\vdots$$

$$(n+1)-\text{vertices} \longrightarrow C(n,2)-\text{lines}.$$

**Remark**: The vertex of the keychain of length 0 is excluded as no definite/ absolute line can be produced from it.

Thus only keychains of lengths $1, 2, 3, \cdots, n$, that could produce $C(n, 2)$ lines. This is done by choosing 2 keychain vertices out of $n$ vertices, where $n \in \mathbb{N}$, and $n \geq 2$.

From the keychains of order 3, we will have the following keychains that represents the lines or edges.

$$111\lambda,\ 11\lambda\lambda,\ 11\lambda0, 1\lambda\lambda\lambda,\ 1\lambda\lambda0,\ 1\lambda00.$$

- **Triangular Faces** - $F^3$

  Are keychains that are made by only three types of distinct pins where zero pin(s) may exist or not at the end of each keychain.

  Thus, any three corresponding lines/ edges form a keychain representing a triangular face.

  When choosing any three corresponding keychain lines of length $n$, a triangular face will develop as follows:

$$\text{keychain length } 0 \longrightarrow 0 - \text{triangular faces}$$

$$\text{keychain length } 1 \longrightarrow 0 - \text{triangular faces}$$

$$\text{keychain length } 2 \longrightarrow 1 - \text{triangular face}$$

$$\text{keychain length } 3 \longrightarrow 4 - \text{triangular faces}$$

$$\text{keychain length } 4 \longrightarrow 10 - \text{triangular faces}$$

$$\vdots$$

$$\text{keychain length } (n+1) \longrightarrow C(n,3) - \text{triangular faces.}$$

  From above, $C(n,3)$ means choosing 3 keychain lines out of $C(n,2)$ keychain lines. The keychains of lengths $2,\ 3,\ \cdots,\ n$, are the only keychains that could produce triangular faces.

  From keychains of length 3, we will have the following keychains that represents the triangular faces generated, viz

$$11\lambda\beta,\ 1\lambda\lambda\beta,\ 1\lambda\beta\beta,\ 1\lambda\beta0.$$

- **Four-sided Faces** - $F^4$

  Any four corresponding keychains of length $n$ representing lines or edges will give rise to $C(n,4)$ to 4−sided faces out of $C(n,2)$ lines, and so on.

  For a keychain of length 5, we will have the 15 four-sided faces out of 15 lines, which are;

  $$111\lambda\beta\gamma,\ 11\lambda\lambda\beta\gamma,\ 11\lambda\beta\beta\gamma,\ 11\lambda\beta\gamma\gamma,\ 11\lambda\beta\gamma0,\ 1\lambda\lambda\lambda\beta\gamma,\ 1\lambda\lambda\beta\beta\gamma,\ 1\lambda\lambda\beta\gamma\gamma$$

  $$1\lambda\lambda\beta\gamma0,\ 1\lambda\beta\beta\beta\gamma,\ 1\lambda\beta\beta\gamma\gamma,\ 1\lambda\beta\beta\gamma0,\ 1\lambda\beta\gamma\gamma\gamma,\ 1\lambda\beta\gamma\gamma0,\ 1\lambda\beta\gamma00.$$

- **A Core**

  A keychains of length $n$ made up of $n-$unique pins excluding $0-$pin, is called the **core** of the simplex diagram. Thus any keychain of the following form,

  $$1\lambda_1\lambda_2\cdots\lambda n$$

  which is a keychain of distinct $(n+1)-$pins. This type of a keychain excludes the pin, 0 and any simplex diagram of keychains is comprised of one core.

  From keychains of length 3, we will have the following keychain represents the core generated.

  $$1\lambda\beta\gamma$$

## 3.3 Simplex Diagrams of Keychains

In this section we introduce the keychains in simplex, where keychains of lengths, 1, 2, 3, 4 and 5 will be considered. Recall, these keychains were also represented above by weight diagrams.

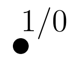



**Figure 3.3.0.** *Simplex Keychain of length 0.*

**Note 3.3.1.** The vertex could either be **1** or **0**, a crisp set representing an empty set.

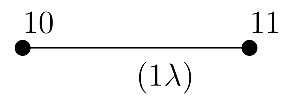



**Figure 3.3.2.** *Simplex Keychain of length 1.*

**Note 3.3.3.** The above diagram has two vertices with one line and the vertices are all representing a crisp set.

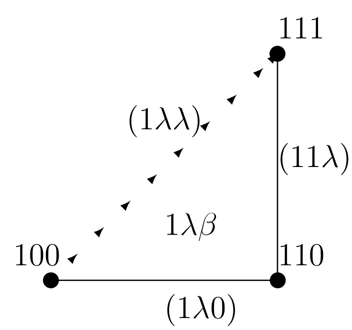



**Figure 3.3.4.** *Simplex Keychain of length 2.*

**Note 3.3.5.** The above diagram has three vertices, three lines and a core. The vertices indicate the crisp set.

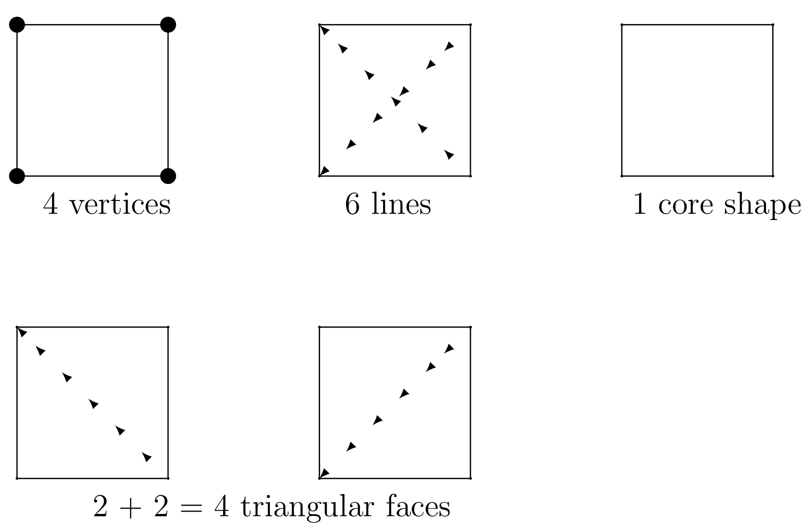



**Figure 3.3.6.** *Simplex Keychain of length 3.*

**Note 3.3.7.** The diagram above has four vertices, six lines, four triangular faces and a core.

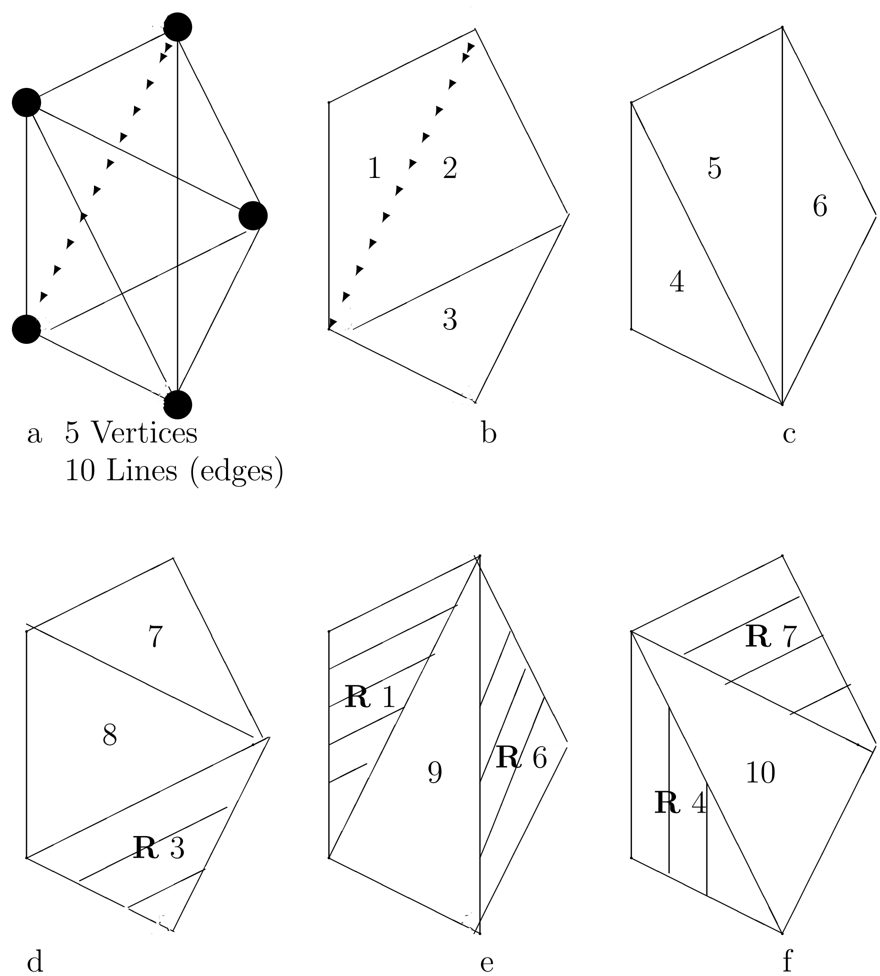



**Figure 3.3.8.** *Simplex Keychain of length 4.*

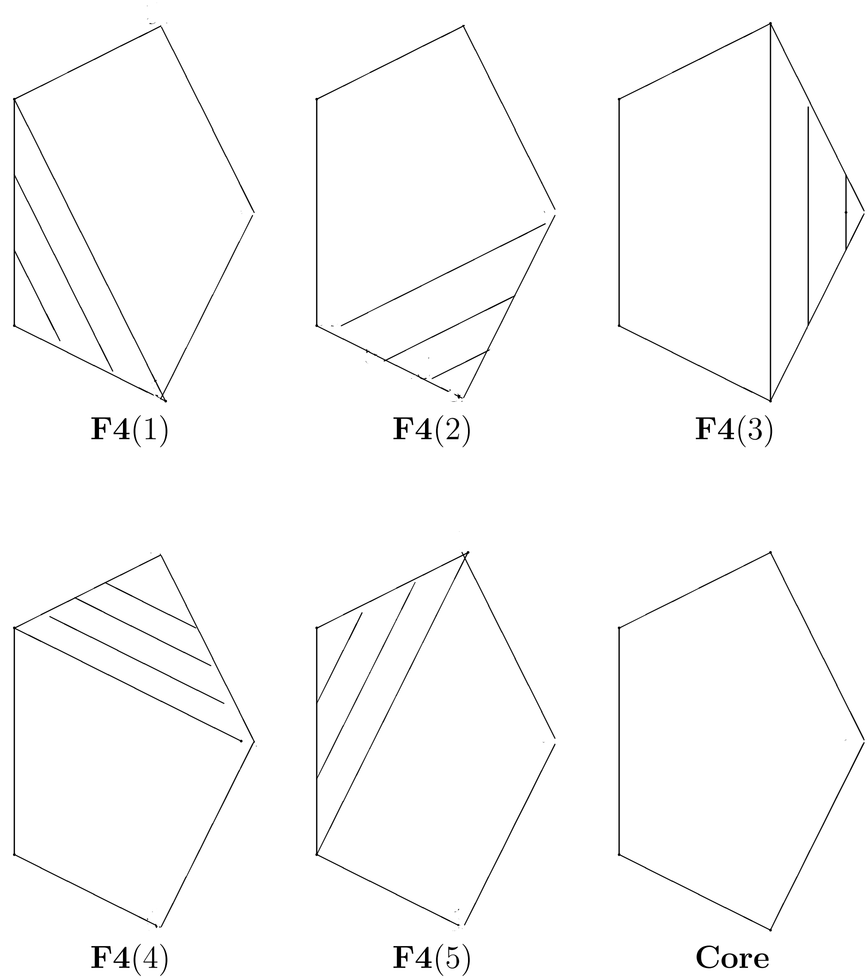



**Figure 3.3.9.** *Simplex Keychain of length 4, Continuation*

**Note 3.3.10.** The above two diagrams respectively, show that a keychain of length 4 has five vertices, ten lines, ten triangular faces, five rectangular faces and a core. From the triangular faces, the shaded faces **R3**, **R1**, **R6**, **R4** and **R7** denote, the repeats of faces 3, 1, 6, 4 and 7 respectively.

The simplex keychain diagrams above develop into polyhedrons/ polygons. Their vertices, edges (lines), faces, can be named using keychains.

The notion of keychains in this situation can be used to differentiate the different positions of the vertices, edges and faces of each polyhedron/polygon formed.

Now we look at briefly some properties of simplex diagrams of keychains of lengths 2 and 4 respectively:

**(i) Simplex diagram of Keychain, length 2.**

We note from its simplex diagram that we have three vertices, namely

$$111,\ 110,\ 100,$$

three edges or lines, namely,

$$11\lambda,\ 1\lambda\lambda,\ 1\lambda 0$$

and one triangular face or an inner called a core,

$$1\lambda\beta.$$

**(ii) Simplex diagram of Keychain, length 4.**

From its diagrams we have the following five vertices;

$$11111,\ 11110,\ 11100,\ 11000,\ 10000,$$

ten edges (lines),

$$1111\ \lambda,\ 111\lambda\lambda,\ 111\lambda 0,\ 11\lambda\lambda\lambda,\ 11\lambda\lambda 0,\ 11\lambda 00,\ 1\lambda\lambda\lambda\lambda,\ 1\lambda\lambda\lambda 0,\ 1\lambda\lambda 00,\ 1\lambda 000,$$

ten triangular faces,

$$111\ \lambda\beta,\ 11\lambda\lambda\beta,\ 11\lambda\beta\beta,\ 11\lambda\beta 0,\ 1\lambda\lambda\lambda\beta,\ 1\lambda\lambda\beta\beta,\ 1\lambda\lambda\beta 0,\ 1\lambda\beta\beta\beta,\ 1\lambda\beta\beta 0,\ 1\lambda\beta 00$$

five rectangular shapes,

$$11\lambda\beta\gamma,\ 1\lambda\lambda\beta\gamma,\ 1\lambda\beta\beta\gamma,\ 1\lambda\beta\gamma\gamma,\ 1\lambda\beta\gamma 0$$

and a 5-sided inner/ core,

$$1\lambda\beta\gamma\tau.$$

Below is the table showing vertices, lines and type of faces that develop from the simplex diagrams of keychains of different lengths.

**Table 3.3.11.** *Data Analysis of Simplex Diagrams of Keychains*

| No of elements | Keychain Lengths | Vertices/ Corners | Line Edges | Tria Faces | Rect Faces | Inner Inner | **Total Keychains** |
|---|---|---|---|---|---|---|---|
| 0 | 1 | 1 | | | | | 1 |
| 1 | 2 | 2 | 1 | | | | 3 |
| 2 | 3 | 3 | 3 | 1 | | | 7 |
| 3 | 4 | 4 | 6 | 4 | 1 | | 15 |
| 4 | 5 | 5 | 10 | 10 | 5 | 1 | 31 |

The results found on the data above match with those of [24], which are shown at the appendix for which can be used to validate/ verify what we got, see figures 8.3, 8.4 and 8.5 on the appendix.

We observed that the representation of keychains on simplex diagrams could be vertices, faces, edges, or cores. We further observed that any two keychains representing distinct vertices can develop to a new keychain representing an edge. Then a combination of three distinct edges will develop into a triangular face representing a distinct keychain.

## 3.4 Reversal of Lines to Points and Faces to Lines

In this section, we will look at how to get back the points that made a specific line and the lines that made a specific face.

**Note 3.4.1.**

The type of padded pins in any keychain are the ones that guide one to determine whether a given keychain represents a point, edge or face.

Let us consider the following two keychains:

$$111\lambda\lambda\lambda 0.$$

and

$$11\lambda\beta\beta 0$$

Our aim here is to:

(i) identify as to what is represented by these two keychains,

(ii) to find the lines and points that developed them.

### (a) Illustrations on keychain, $111\lambda\lambda\lambda 0$

This is a keychain of length 6 and it represents a line. It is formed by joining two keychains that represent vertices. We notice from this keychain that the last four pins of the keychain are $\lambda\lambda\lambda 0$. This means, for this keychain to be line, the pins $\lambda\lambda\lambda 0$ were either 1110 or 0000. Thus, the two points/ vertices that developed to this line keychain are as follows;

$$111\lambda\lambda\lambda 0 \longrightarrow \begin{cases} 1111110 \\ 1110000. \end{cases}$$

### (b) Illustrations on keychain, $11\lambda\beta\beta 0$

We notice that this keychain, :

(i) is of length 5,

(ii) has three unique pins excluding pin 0,

(iii) and is identified as the keychain representing a triangular face.

Any keychain represented by a triangular face is made by three lines keychains of which each line keychain is made by two keychains represented by vertices. The last four pins $\lambda$, $\beta$, $\beta$, and 0 of the keychain $11\lambda\beta\beta 0$ will be replaced by new last four pins to form the three line keychains that formed the triangular face $11\lambda\beta\beta 0$; Below, please see all the three (3) possible replacements of $\lambda\beta\beta0$

$$\lambda\beta\beta0 \longrightarrow \begin{cases} 1\lambda\lambda0 \\ \lambda\lambda\lambda0 \\ \lambda000 \end{cases}$$

Thus the three lines, $L_1$, $L_2$ and $L_3$, forming this triangular face are

$$L_1 \;=\; 111\lambda\lambda0,\; L_2 \;=\; 11\lambda\lambda\lambda0,\; L_3 \;=\; 11\lambda000.$$

These lines are in turn formed by the following vertices:

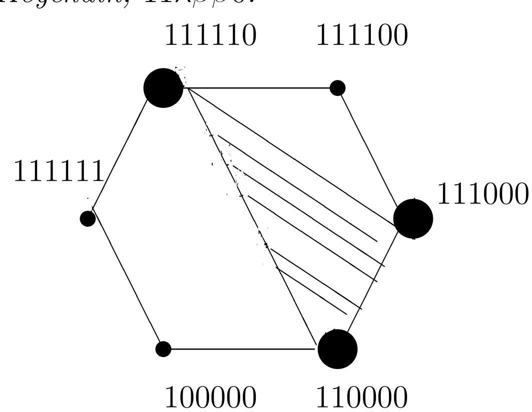
$$L_1 \;=\; 111\lambda\lambda0 \longrightarrow \begin{cases} 111110 \\ 111000 \end{cases}$$

$$L_2 \;=\; 11\lambda\lambda\lambda0 \longrightarrow \begin{cases} 111110 \\ 110000 \end{cases}$$

$$L_3 \;=\; 11\lambda000 \longrightarrow \begin{cases} 111000 \\ 110000. \end{cases}$$

We can show this three faced keychain, $11\lambda\beta\beta0$ as in the following diagram:

**Figure 3.4.2.** *Simplex diagram of Keychain,* $11\lambda\beta\beta0$.




From the above diagram, the shaded triangle, represents the keychain $11\lambda\beta\beta0$. The nodes from the shaded triangle represents the keychains, 110000, 111000 and 111110. The edges of the shaded triangle are, $11\lambda000$, $11\lambda\lambda\lambda0$ and $111\lambda\lambda0$.

## 3.5 Counting of Vertices, Lines and Faces in Keychain Simplex Diagrams

From the data analysis table of simplex diagrams of keychains shown above, we noticed that the number of vertices, edges, faces and cores develop into the form of a pascal triangle, where each row matches the number of vertices, lines (edges), faces and cores when we exclude all the one's on the upper left hand side of the Pascal triangle.

Recall, a Pascal triangle, which represents the coefficients in the powers of a binomial

$$(a+b)^n = {}_nC_0\ a^n \ +\ {}_nC_1\ a^{n-1}b \ +\ {}_nC_2\ a^{n-2}b^2 \ +\ \cdots \ +\ {}_nC_{n-1}\ ab^{n-1} \ +\ {}_nC_n\ b^n,$$

is given as follows:

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}$$

**Figure 3.5.1.** *Pascal Triangle*

From the simplex diagrams of keychains of lengths, 1, 2, 3, 4, 5, above, we get the following triangle that predicts the numbers of vertices, lines, Faces and Cores of any keychains of length $n+1$. One will note that, the differences between the Pascal triangle and the one below is missing upper left hand array of ones ($1^s$).

Each row from the triangle below represents keychain lengths, where from to bottom will be lengths, 1, 2, $\cdots$ , up to $n+1$. The entries in each row represents, the number of vertices, lines (edges), faces, which could be triangular, rectangular, pentagonal, hexagonal, and so on...

1

2 1

3 3 1

4 6 4 1

5 10 10 5 1

6 15 20 15 6 1

7 21 35 35 21 7 1

**Figure 3.5.2.** *Simplex Keychain diagram Triangle*

The above is able to predict the numbers of vertices, lines and faces of each simplex diagram of keychain of length $n+1$. We also note that these predictions come the same as $n-$simplex element diagrams as mentioned in, [17].

## 3.6 Keychain Simplex Diagrams and Euler Formula

In [20], Mathematics is defined as the science of structure, order, and relation that has evolved from elemental practices of counting, measuring, and describing the shapes of objects. They further say that Mathematics deals with logical reasoning and quantitative calculation.

From the development of the simplex diagrams of keychains, we have noted the resemblance of an Euler characteristic $\chi$, which is said to be a Topological characteristics. These characteristics include various classes of geometric figures based only on a relationship between the numbers of vertices (V), edges (E), and faces (F) of a geometric figure.

These mentioned geometric figures are also found in simplex diagrams of keychains. We also note a same behaviour when using a Euler Formula. [20], states that the Euler formula is, $\chi =$ *number of vertices* - *number of edges* + *number of faces.* Using shorthand, this formula can be written as follows

$$\chi = |V| - |E| + |F|,$$

where,

$V =$ set of vertices

$E =$ set of edges

$F =$ set of faces.

From [21], it is stated that the Euler Formula for any polyhedron that doesn't intersect itself, then

number of Vertices (corner points)

minus the number of Edges

plus the number of Faces

is always equals 2. Using the equation form, we write

$$\chi = |V| - |E| + |F| = 2.$$

It is also stated that $\chi$ is not always equal to two, it could be equal to zero, one or negative one.

At this point, we look at the link between ordinary simplices and keychain simplices.

| Keychain | Vertx | | Line | | F | | A | | C | | E | | S | | **Total** |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | V | - | E | + | $F^3$ | - | $F^4$ | + | $F^5$ | - | $F^6$ | + | $F^7$ | = | $\chi$ |
| 1 | 1 | - | 0 | + | 0 | - | 0 | + | 0 | - | 0 | + | 0 | = | 1 |
| 2 | 2 | - | 1 | + | 0 | - | 0 | + | 0 | - | 0 | + | 0 | = | 1 |
| 3 | 3 | - | 3 | + | 1 | - | 0 | + | 0 | - | 0 | + | 0 | = | 1 |
| 4 | 4 | - | 6 | + | 4 | - | 1 | + | 0 | - | 0 | + | 0 | = | 1 |
| 5 | 5 | - | 10 | + | 10 | - | 5 | + | 1 | - | 0 | + | 0 | = | 1 |
| 6 | 6 | - | 15 | + | 20 | - | 15 | + | 6 | - | 1 | + | 0 | = | 1 |
| 7 | 7 | - | 21 | + | 35 | - | 35 | + | 21 | - | 7 | + | 1 | = | 1 |
| $\vdots$ | $\cdots$ | | | | | | | | | | | | | | |

Table 3.1: Euler characteristics of keychain simplex diagrams

The above Euler characteristics of the simplex diagram of keychains was calculated using the equation;

$$|V| - |E| + |F^3| - |F^4| + |F^5| - |F^6| + \cdots + (-1)^n |F^{n-1}| = \chi^*,$$

where $n = 3,\ 4,\ 5,\ \cdots$ , and $F^n$ is a core. The characteristic number $\chi$, is always equal to one, i.e, $\chi\ =\ 1$ in this situation.

If we are excluding the core face, $F^n$, we will have the following characteristic table of keychains:

| KC | Vertx | | Line | | F | | A | | C | | E | | S | | **Tot** |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | V | - | E | + | $F^3$ | - | $F^4$ | + | $F^5$ | - | $F^6$ | + | $F^7$ | = | $\chi$ |
| 1 | 0 | - | 0 | + | 0 | - | 0 | + | 0 | - | 0 | + | 0 | = | 0 |
| 2 | 2 | - | 0 | + | 0 | - | 0 | + | 0 | - | 0 | + | 0 | = | 2 |
| 3 | 3 | - | 3 | + | 0 | - | 0 | + | 0 | - | 0 | + | 0 | = | 0 |
| 4 | 4 | - | 6 | + | 4 | - | 0 | + | 0 | - | 0 | + | 0 | = | 2 |
| 5 | 5 | - | 10 | + | 10 | - | 5 | + | 0 | - | 0 | + | 0 | = | 0 |
| 6 | 6 | - | 15 | + | 20 | - | 15 | + | 6 | - | 0 | + | 0 | = | 2 |
| 7 | 7 | - | 21 | + | 35 | - | 35 | + | 21 | - | 7 | + | 0 | = | 0 |
| $\vdots$ | $\ldots$ | | | | | | | | | | | | | | |

Table 3.2: Core-less Euler characteristics of keychain simplex diagrams

The above characteristics are from the equation;

$$|V| - |E| + |F^3| - |F^4| + |F^5| - |F^6| + \cdots + (-1)^n |F^{n-1}| = \chi^*,$$

where $n = 3,\ 4,\ 5,\ \cdots$ , and $F^{n-1}$ is the face before the core, $F^n$. Thus the number $\chi^* = 0$ or $\chi^* = 2$, when the keychain length is odd or even respectively.

# Chapter 4

# Keychains and Flags

## 4.1 Flags

Consider a non-empty finite set, $X = \{x_1, x_2, \cdots, x_n\}$.

**(i)** A fuzzy subset $\mu$ of $X$ is a mapping $X \longrightarrow [0, 1]$.

**(ii)** A fuzzy subset with membership value zero on all of $X$ is called an empty fuzzy subset of $X$.

**(iii)** If $X$ is empty, then a fuzzy subset of $X$ does not have a conventional meaning.

We always assume that $X$ is non-empty set. We also denote $X_0$ to be the empty subset of $X$. Further we assume $\mu : \emptyset \longrightarrow I$ takes value 1. Also any fuzzy subset $\mu$ from a non-empty set $X$ restricted to $X_0$ takes the value 1.

If $X_1$ denotes the one element set then there are three possible fuzzy sets. These three possible fuzzy subsets on $X_1$ arise from the keychains 11, $1\lambda$ and 10 where $1 > \lambda > 0$.

**(a)** The keychain 11 represents a fuzzy subset $\mu_1 : X_1 \to I$ with $\mu_1(x) = 1$ for $x \in X_1$.

**(b)** The keychain $1\lambda$ represents another fuzzy subset $\mu_2 : X_1 \to I$ with $\mu_2(x) = \lambda$ for $x \in X_1$.

**(c)** The keychain 10 represents yet another fuzzy subset $\mu_3 : X_1 \to I$ with $\mu_3(x) = 0$ for $x \in X_1$.

We can loosely say that for a one element set there are only three different fuzzy subsets corresponding to the three possibilities, namely; $x$ may belong to $X_1$ absolutely or may belong to $X_1$ relatively to a degree $0 < \lambda < 1$ or does not belong to $X_1$ absolutely.
Similarly for two element sets and so on.

Generally when $X = \{x_1, x_2, \cdots, x_n\}$, the chain under inclusion

$$X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_n = X$$

is a maximal chain of subsets of $X$, which is referred to as a *flag*.
The way flags and keychains are related to express fuzzy subset will be discussed below.

## 4.2 Pinned flags

Since fuzzy sets/ keychains are ordered, they could clearly show or model a clearer picture of all preferences of consumers in what ever choice situation. This preferentiality can be represented by *Pinned Flags*.

If we let $X$ be a non-empty set. [32] states that the pair $(C,\ell)$ is a pinned flag and is denoted by

$$X_0^1 \subset X_1^{\lambda_1} \subset \cdots \subset X_n^{\lambda_n},$$

where $C$ is a maximal chain on $X$ and $\ell$ a keychain of length $n$, that is

$$\ell : 1 = \lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_n.$$

He refers to it as a *pinned flag* on $X$.

According to [34] a fuzzy set, $\mu$ associated with the above pinned flag $(C,\ell)$ will be

$$\mu(x) = \begin{cases} 1 & if \quad x \in X_0 \\ \lambda_1 & if \quad x \in X_1 \setminus X_0 \\ \vdots & \quad\quad \vdots \\ \lambda_n & if \quad x \in X_n \setminus X_{n-1} \end{cases}$$

where the component $X_n$ represents the whole set of $X$.

**Note 4.2.1.**

From the above fuzzy subset $\mu$, $\mu(x)$ may or may not take the value zero. We recall that the support of a fuzzy subset $\mu$ is

$$Supp\,\mu = \{x \in X : \mu(x) > 0\}.$$

If $\lambda_n = 0$ then $Supp\,\mu$ is strictly contained in $X$ but if $\lambda_n$ is different from zero then every element belongs to $\mu$ with some degree $\mu(x) > 0$; so in the latter case $Supp\,\mu = X$. Therefore the first $\lambda_i$ not equal to zero determines the support.

## 4.3 Equivalence Relation in Keychains

In this section we look at equivalence of keychains as explained in [37]. We explained earlier as to what keychains are and what they are comprised of, that is the pins. Pins were said to be the set of real numbers in the unit interval, $I = [0\ ,\ 1]$. [37] states that the equivalence relation of a set represented by keychains is based on the ordering of pins, rather than the actual value of the pin as a real number.

**Definition 4.3.1.**

Let

$$\ell_1 = \lambda_n\lambda_{(n-1))}\cdots\lambda_1\lambda_0$$

and

$$\ell_2 = \nu_n\nu_{(n-1)}\cdots\nu_1\nu_0$$

be two keychains of length $n$. We say, $\ell_1$ is equivalent to $\ell_2$ denoted by

$$\ell_1 \cong \ell_2,$$

if and only if the following condition is met;

$\lambda_i > \lambda_j$ if and only if $\nu_i > \nu_j$, $\forall\ i,\ j = 0,\ 1,\ \cdots,\ n$ and $i \neq j$.

To illustrate the above definition, we consider a three element set,

$$X_3 = \{x_1, x_2, x_3\}$$

This set $X_3$, has length three, with fifteen possible keychains that could represent it. Keychains $\ell_1 = 111\lambda$ and $\ell_2 = 1\lambda\lambda 0$ could arise from this set. The two keychains $\ell_1$ and $\ell_2$ could be described as keychains of the fuzzy subsets $\mu$ and $\nu$ belonging in $I^{X_3}$ respectively as follows:

$$\mu(x) = \begin{cases} 1 & if \quad x = x_1, x_2, \\ \lambda & if \quad x = x_3. \end{cases}$$

$$\nu(x) = \begin{cases} \lambda & if \quad x = x_1, x_2, \\ 0 & if \quad x = x_3. \end{cases}$$

It is clear that the condition of definition above is not satisfied as $\mu(x_3) = \lambda$ and $\nu(x_3) = 0$. Therefore $\mu$ and $\nu$ are in-equivalent, so we say $\ell_1 \ncong \ell_2$.

If we could have the fuzzy sets,

$$\mu(x) = \begin{cases} 1 & if \quad x = x_1 \\ \lambda & if \quad x = x_2 \\ \beta & if \quad x = x_3 \end{cases}$$

$$\nu(x) = \begin{cases} 1 & if \quad x = x_1 \\ \lambda & if \quad x = x_2 \\ \zeta & if \quad x = x_3 \end{cases}$$

We could clearly see that there is no contradiction to the definition of equivalence relation. So the above fuzzy sets, $\mu$ and $\nu$ will have the keychains, $\ell_3 = 11\lambda\beta$ and $\ell_4 = 11\lambda\zeta$ respectively, where $1 > \lambda > \beta > \zeta > 0$. Thus $\ell_3 \cong \ell_4$.

## 4.4 Equivalence Representation of Fuzzy Subsets Using Keychains

By definition the equivalence relation $\sim$ on $I^X$ is as follows;

**Definition 4.4.1.**

Let $\mu$ and $\nu$ be any two fuzzy subsets on $X$. We say $\mu$ is *equivalent* to $\nu$ denoted by $\mu \sim \nu$ if and only if

(i) for all $x, y \in X$, $\mu(x) > \mu(y)$ if and only if $\nu(x) > \nu(y)$,

(ii) $\mu(x) = 1$ if and only if $\nu(x) = 1$,

(iii) $\mu(x) = 0$ if and only if $\nu(x) = 0$,

where the condition

$$\mu(x) \;=\; 0 \iff \nu(x) \;=\; 0,$$

implies that

$$Supp\, \mu = Supp\, \nu$$

and the replacement $>$ by $\geq$ in condition (i) will not affect the results as the two are equivalent.

The following is the proposition of equivalence as proposed by [41], it captures the notion of equivalence of two fuzzy subsets in terms of pinned-flags.

**Proposition 4.4.2.** Consider the fuzzy subsets $\mu$ and $\nu$ that correspond to the following pinned flags

$$(C_\mu, \ell_\mu) : X_0^1 \subset X_1^{\lambda_1} \subset \cdots \subset X_n^{\lambda_n}$$

and

$$(C_\nu, \ell_\nu) : Y_0^1 \subset Y_1^{\beta_1} \subset \cdots \subset Y_m^{\beta_m}.$$

We say $\mu \sim \nu$ on $X$ if and only if,

**(i)** $n = m$,

**(ii)** $X_i = Y_i$ for $i = 0, 1, \cdots, n$ provided the $\lambda_i'^s$ and $\beta_i'^s$ are distinct,

**(iii)** $\lambda_i > \lambda_j$ if and only if $\beta_i > \beta_j$ for $1 \leq i, j \leq n$ and $\lambda_k = 0$ if and only if $\beta_k = 0$ for some $k$ between 1 and $n$.

**Proof 4.4.3.**

Let $\mu \sim \nu$.

**(i)** Define a function $f : \mu(X) \to \nu(X)$ from the subset $\mu(X)$ of $I$ to the subset $\nu(X)$ of $I$ by $f(\mu(x)) = \nu(x)$ for $x \in X$.

It is easy to check that $f$ is firstly well defined and secondly bijective since $\mu \sim \nu$. Thus $|\mu(X)| = |\nu(X)|$. Therefore $m = n$.

**(ii)** We prove by induction on $n$.

- for $n = 0$, $X_0 = Y_0$ since each set is the empty set.

- We assume that $X_k = Y_k$ for $k \geq 0$.

- We show that $X_{k+1} = Y_{k+1}$. Let $g \in X_{k+1}$.

If $g \in X_k$, then $g \in Y_k$ which is contained in $Y_{k+1}$.

On the other hand, suppose $g \notin X_k$, then $\mu(g) = \lambda_{k+1}$. We claim that $g \in Y_{k+1}$.

Now suppose $g \notin Y_{k+1}$. Then $\nu(g) < \beta_{k+1}$.

Choose $x \in Y_{k+1}$ but $x \notin Y_k = X_k$. Then $\nu(g) < \nu(x)$ which implies $\mu(g) < \mu(x)$ by equivalence. Hence $\lambda_{k+1} = \mu(g) < \mu(x) = \alpha$ where $\lambda_k > \alpha > \lambda_{k+1}$.

This implies $x \in \mu^\alpha \subseteq X_k = Y_k$, a contradiction. Therefore $X_{k+1} \subseteq Y_{k+1}$. Similarly, we can show that $Y_{k+1} \subseteq X_{k+1}$. This completes the induction.

**(iii)** The property (iii) follows from (i) and (ii) and from the definition of equivalence pertaining to support.

Conversely

Suppose $\mu$ and $\nu$ are two fuzzy subsets as defined in the proposition with pinned flags satisfying (i), (ii) and (iii). Then from (iii) $\lambda_k = 0$ if and only if $\beta_k = 0$ for some $k$ between 1 and $n$ is true.

By (i) $n = m$. Therefore, $supp\, \mu = supp\, \nu$.

For $x, y \in supp\, \mu$, suppose $\mu(x) > \mu(y)$. Then $\mu(x) = \lambda_i$ and $\mu(y) = \lambda_j$ for some $i, j$ in 1 to $n$.

But from (iii) $\lambda_i > \lambda_j$ if and only if $\beta_i > \beta_j$ for $1 \leq i, j \leq n$.

Now, (ii) and (iii) together imply $\nu(x) > \nu(y)$.

From (iii) it is clear that $\mu(x) = 1$ if and only if $\nu(x) = 1$.

Thus $\mu \sim \nu$ as required.

## 4.5 Keychains enumerations of Fuzzy subset

Recall that keychains of length $n$ are denoted as follows,

$$1 = \lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_n \geq 0$$

where

$$\lambda_i \in [0, 1] \text{ for } i = 1, 2, \cdots, n.$$

From the orders of the $n$-element sets discussed earlier, we observed on the order tables of the few keychains of lengths 0, 1, 2, 3, that a set of four keychains of any given $n$-element set share an order.

We conjecture that any order, $(n_1, n_2, \cdots, n_t)$ will always be shared by four distinct keychains. There is only one exception. For an example the following keychains of length 3, i.e. 1111, $1\lambda\lambda\lambda$ and 1000 are the only keychains that have the order **(3)** from all the possible 15 keychains of length 3.

We similarly make a conjecture that the order $(n)$ will always be shared by $n$ distinct keychains.

The following table shows the generation of orders of keychains from 1-element set to an $n$-element set. It also shows the number of orders that will be formed.

$\downarrow$

$\downarrow$

$\downarrow$

$\downarrow$

$\downarrow$

$\downarrow$

(continuing on the next page)

$\downarrow$

$\downarrow$

$\downarrow$

$\downarrow$

$\downarrow$

Table 4.1: Orders of Keychains

| # elements | # orders | Collection of orders |
|---|---|---|
| 1 | 1 | 1 |
| 2 | 2 | 2,11 |
| 3 | 4 | 3,21,12,1111 |
| 4 | 8 | 4,31,22,211,13,121,112,1111 |
| ⋮ | ⋮ | ⋮ |
| $n$ | $2^{n-1}$ | $(n), (n-1,1), (n-2,2), (n-2,1,1)$,...,(11...1) |

From the above table we see that the orders of each set forms a set with elements of the form

$$s_1,\ s_2,\ s_3,\ \cdots,\ s_k$$

for $s_i$ is an order where $i = 1,\ 2,\ \cdots,\ k$ where $k$ is a positive integer. The value $k$ corresponds to the number of blocks of orders.

This table further suggests that, with $2^{n+1} - 1$ keychains of length $n$, there will always be $2^{n-1}$ orders into which the keychains are classified. Each order generates four different keychains of the same length with the exception of order $(n)$ where each will generate three distinct keychains.

**Proposition 4.5.1.**

There are $2^{n-1}$ orders associated with keychains of length $n+1$.

The following proof is a modified version of the proof given in [47].

**Proof 4.5.2.**

Consider an infinite set with an order $(x + x^2 + x^3 + \cdots)$ where $x$ represents a component of the order in a block and powers of components showing the number of components in a block such that

$(x + x^2 + x^3 + \cdots)$ is the first block.

$(x + x^2 + x^3 + \cdots)^2$ the second block.

$\vdots$

$(x + x^2 + x^3 + \cdots)^n$ the $n^{th}$ block.

$\vdots$

We want to show that a keychain of length $n + 1$ will have $2^{n-1}$ orders.

Let the sum of blocks be

$$(x + x^2 + x^3 + \cdots) + (x + x^2 + x^3 + \cdots)^2 + \cdots + (x + x^2 + x^3 + \cdots)^n + \cdots$$

This is an infinite geometric series, so

$$s_\infty = (x + x^2 + x^3 + \cdots) + (x + x^2 + x^3 + \cdots)^2 + \cdots + (x + x^2 + x^3 + \cdots)^n + \cdots$$

$$s_\infty = \Sigma_1^\infty x^i + (\Sigma_1^\infty x^i)^2 + \cdots + (\Sigma_1^\infty x^i)^{n-1} + (\Sigma_1^\infty x^i)^n + \cdots$$

We let $U = \Sigma_1^\infty x^i$,

$$= x + x^2 + x^3 + \cdots$$

$$= \frac{x}{1-x}$$

Now $s_\infty = U + U^2 + U^3 + \cdots + U^{n-1} + U^n + \cdots$

$$= \frac{U}{1-U}$$

$$= \frac{\frac{x}{1-x}}{1-\frac{x}{1-x}}$$

$$= \frac{x}{1-2x}$$

Thus, using the binomial expansion/series, we find

$$s_\infty = \frac{x}{1-2x} = x(1 + 2x + 2^2x^2 + \cdots + 2^{n-1}x^{n-1} + 2^nx^n + \cdots)$$

$$s_\infty = \frac{x}{1-2x} = x + 2x^2 + 2^2x^3 + \cdots + 2^{n-1}x^n + 2^nx^{n+1} + \cdots$$

Thus, the number $2^{n-1}$ which is the coefficient of $x^n$, shows the number of orders for a keychain of length $n + 1$. This completes the proof.

# Chapter 5

# Keychain Applications in Voting Patterns

## 5.1 Introduction

In [31], voting is defined as a mechanism where the opinions from a set of votes are evaluated in order to select the alternatives that best represent the collective preferences.

So, in this chapter we wish to put our focus in applications fuzzy sets and keychains in voting behaviours and conclusions. The use of a fuzzy ballot paper will be used to show how uncertainty can be reduced in voting situation, since voting is fuzzy in nature.

Hosseingholizadey, [15], in his introduction, he says "Collecting experts into a group for any organization costs a lot. The organization undertakes the costs to achieve some aims. For two reasons, in electing a manager or leader for such groups, manner of election is important. First, who **deserve** leadership, in group's members opinion? Second, election **must hinder from creating unnecessary favoritism**. In fuzzy election not only the candidates are confronted by an opinion poll with great care of candidate's background but also fuzzy election cause members to be unable to create favoritism."

He further on says, "Choosing one from a group of experts is always difficult. It will be more difficult if the person wants to be the leader of the group. In traditional elections members have to vote someone and don't vote another one, absolutely rejected or absolutely accepted. The group members, who have suffrage, have to vote in favor of only one candidate. Therefore, they vote someone whereas they do not entirely accept him or her and deprive others of their ballot whereas in group's opinion they are almost qualified too."

## 5.2 Ballot Paper and Fuzzy Set Theory

**What is a ballot paper?** - A ballot paper is defined as a slip of paper used to register a vote. Cagman and Aktas [42], describe a ballot paper as the most important tool of an election where its design can aid or inhibit clarity in the elections.

Let us consider the following Fuzzy ballot paper (FBP) as designed by [42],

Table 5.1: Ballot Paper

| | 1 | 2 | $\cdots$ | k |
|---|---|---|---|---|
| $x_1$ | ○ | ○ | $\cdots$ | ○ |
| $x_2$ | ○ | ○ | $\cdots$ | ○ |
| $\vdots$ | $\vdots$ | $\vdots$ | | $\vdots$ |
| $x_n$ | ○ | ○ | $\cdots$ | ○ |

### 5.2.1 Fuzzy sets and Elections

Matiki, in [46], states that a fuzzy subset has a kind of preference built in it. He further on states that, if one element, $x \in \mathbf{X}$ has a higher membership value than $y$

to the fuzzy subset $\mu$, then we can interpret that as, *'x is preferred more than y'*, and this kind of preference naturally leads to an election.

According to Matiki, [46], elections could be done by putting elements of set $X$ according to their preferentiality.

This arrangement he is referring to could be represented by keychains as in the following subsection:

### 5.2.2 Elections and Keychains

Let us consider the position of a chairperson to be contested by four (4) candidates, by a panel of $m-$voters who are eligible to do so, i.e to choose their preferential candidate by voting. The choices that could be made could bring in a lot of uncertainty if a voter is allowed to prefer any candidate(s) of the voter's choice, i.e a voter can choose, one, two, three or all as an absolute preferred candidate, the voter can also elect not to prefer anyone.

Out of $m-$voters, there will be $2^{n+1}-1$ ways of choosing one of the $n-$candidates, (i.e 31 ways from the four candidates by a single voter), where $1, \lambda, \beta, \gamma, \zeta$ and 0 are degrees of preferentiality as follows:

**(i)** $1 \Longrightarrow$ an absolute choice by the voter,

**(ii)** $0 < \lambda, \beta, \gamma, \zeta < 1 \quad \Longrightarrow$ partial choices of different degrees.

**(iii)** If $\lambda, \beta, \gamma, \zeta = 0 \quad \Longrightarrow$ not a choice.

The following could be a ballot paper to be used by voters, $V_s$ to make selections of a preferred candidate, $C_k$, where $s = 1, 2, 3, \ldots, n$, and $m$ denotes the number of voters, and $k = 1, 2, 3, \ldots, n,$, where $n$ denotes the number of candidates.

Table 5.2: Selection Paper/ Preferences

| Candidate | 1 | $\lambda$ | $\beta$ | $\gamma$ | $\zeta$ |
|---|---|---|---|---|---|
| $C_1$ | | | | | |
| $C_2$ | | | | | |
| $C_3$ | | | | | |
| $\vdots$ | | | | | |
| $C_m$ | | | | | |

**Rules of voting**

From the above selection paper, a voter from the panel is expected to fill the paper as follows:

* A mark or a cross under each preference can be made to indicate a preferred choice of degree for any candidate.

* All candidates can be voted/ selected by a voter but **ONLY** one preference degree can be allocated to each candidate, otherwise, the paper will be a spoilt paper, or **no vote** to all candidates listed in that particular ballot paper.

* A paper should at least have one candidate with a preferential value/degree, otherwise it will be treated as no vote.

### 5.2.3 Calculations of the Fuzzy sets using Keychains in voting

From the above subsection, we note that, $m-$ voters would be participating in this voting scenario. Now, what we will be looking at is how to get the overall choice of the $n-$ voters. From each ballot paper casted, we will look for the types of pins chosen for each individual candidate. These types will be collected from each casted paper then get the sum of each preference of the voter to the $n-$ candidates.

The following table could be used to collect the choices towards the candidates by the $n-$ voters:

$\downarrow$

$\downarrow$

$\downarrow$

$\downarrow$

$\downarrow$

$\downarrow$

$\downarrow$

$\downarrow$

$\downarrow$

(continuing on the next page)

$\downarrow$

$\downarrow$

$\downarrow$

$\downarrow$

*Counting Sheet*

| Voter's Choice | $C_1$ | $C_2$ | $C_3$ | $\cdots$ | $C_m$ |
|---|---|---|---|---|---|
| $\Sigma P_1$ | | | | | |
| $\Sigma P_\lambda$ | | | | | |
| $\Sigma P_\beta$ | | | | | |
| $\vdots$ | | | | | |
| $\Sigma P_0$ | | | | | |

$P_1$, $P_\lambda$, $\cdots$, $P_0$, represents the pins from the ballot paper where,

- $P_1$ = absolute choices,

- $P_\lambda$, $P_\beta$,$\cdots$ = partial choices of different degrees and,

- $P_0$ = no choice.

The sums, $\Sigma P_1$, $\Sigma P_\lambda$, $\Sigma P_\beta$, $\cdots$, $\Sigma P_0$ in each candidate, indicates the number with which candidate $C_n$ was preferred by the $m-$ voters under the specified degrees $1, \lambda, \cdots, 0$

How do we get an OVERALL choice? Does the candidate who gets the LARGEST "$\Sigma P_1$" becomes one? The following subsection will try to illustrate such:

### 5.2.4 Enumerations using Keychains

In this subsection we will try to come up with ballot papers that will produce uncertainty in voting. We will also try to show how to use keychains in it. We will look at the following example to illustrate the enumerations of uncertain situations using the idea of Keychains.

**Voting Examples**

**Example 5.2.1.**

Let us consider the simulation example of a situation of 4 candidates looking for a position of a Chairperson who will be chosen by 10 people. See below the ten (10) ballot papers by ten voters which mimic the choice of a chair from four (4) candidates.

Ballot paper 1

| Candidate | 1 | 3/4 | 2/4 | 1/4 | 0 |
|---|---|---|---|---|---|
| $C_1$ | X | | | | |
| $C_2$ | X | | | | |
| $C_3$ | X | | | | |
| $C_4$ | X | | | | |

Ballot paper 2

| Candidate | 1 | 3/4 | 2/4 | 1/4 | 0 |
|---|---|---|---|---|---|
| $C_1$ | X | | | | |
| $C_2$ | | X | | | |
| $C_3$ | | | | | |
| $C_4$ | | | X | | |

Ballot paper 3

| Candidate | 1 | 3/4 | 2/4 | 1/4 | 0 |
|---|---|---|---|---|---|
| $C_1$ | | | | | |
| $C_2$ | X | | | | |
| $C_3$ | | | | | |
| $C_4$ | | | | | |

Ballot paper 4

| Candidate | 1 | 3/4 | 2/4 | 1/4 | 0 |
|---|---|---|---|---|---|
| $C_1$ | X | | | | |
| $C_2$ | | | X | | |
| $C_3$ | X | | | | |
| $C_4$ | | | | | |

Ballot paper 5

| Candidate | 1 | 3/4 | 2/4 | 1/4 | 0 |
|---|---|---|---|---|---|
| $C_1$ | X | | | | |
| $C_2$ | | X | | | |
| $C_3$ | | | | | |
| $C_4$ | X | | | | |

Ballot paper 6

| Candidate | 1 | 3/4 | 2/4 | 1/4 | 0 |
|---|---|---|---|---|---|
| $C_1$ | X | | | | |
| $C_2$ | | | X | | |
| $C_3$ | | | | | |
| $C_4$ | | | | | |

Ballot paper 7

| Candidate | 1 | 3/4 | 2/4 | 1/4 | 0 |
|---|---|---|---|---|---|
| $C_1$ | | X | | | |
| $C_2$ | | X | | | |
| $C_3$ | | | | X | |
| $C_4$ | X | | | | |

Ballot paper 9

| Candidate | 1 | 3/4 | 2/4 | 1/4 | 0 |
|---|---|---|---|---|---|
| $C_1$ | | | | | |
| $C_2$ | | X | | | |
| $C_3$ | X | | | | |
| $C_4$ | X | | | | |

Ballot paper 8

| Candidate | 1 | 3/4 | 2/4 | 1/4 | 0 |
|---|---|---|---|---|---|
| $C_1$ | | | | | |
| $C_2$ | X | | | | |
| $C_3$ | | X | | | |
| $C_4$ | | | | | |

Ballot paper 10

| Candidate | 1 | 3/4 | 2/4 | 1/4 | 0 |
|---|---|---|---|---|---|
| $C_1$ | X | | | | |
| $C_2$ | | | | | |
| $C_3$ | | | X | | |
| $C_4$ | | | | | |

**(i) Representation of the Ballots by Keychains**

From the ten ballot papers above, we will use keychains to represent a candidate is ranked/ preferred by a voter in any given ballot paper.

Please note; the preferences of the for 4−candidates from the 10 ballots, will be represented by $2^{4+1} - 1 = 31$ keychains of length 5. Each pin on the keychain will represent the degree of choice of each candidate in a ballot paper.

This tells us that each voter $V_n$ has 31−ways of choosing his / her preferred candidate(s) out of the 4 candidates. The ballots given illustrates only seven ways exercised out of the thirty one ways, where five are unique, with one exercised by two voters and another one by three voters.

We now from ballot papers 1, 2, ..., to 10, generate the following keychains represented there:

**Ballot 1** 11111 - This means, **all** the four (4) have been absolutely chosen by a voter.

**Ballot 2** $11\lambda\beta\beta$ - All four were preferred with only one who was absolutely preferred with the other preferred with different degrees.

**Ballot 3** 11000 - Only one candidate preferred and is preferred absolutely.

**Ballot 4** 11100 - Two candidates are preferred absolutely, the rest not chosen/ preferred at all.

**Ballot 5** $11\lambda00$ - One absolutely preferred candidate, another preferred to some degree and two not preferred at all.

**Ballot 6** 11000 - Same as ballot 3.

**Ballot 7** $1\lambda\lambda\beta0$ - No candidate preferred absolutely, but three are preferred to some different degrees with one without preferred at all.

**Ballot 8** $11\lambda00$ - Same as ballot 5.

**Ballot 9** $111\lambda0$ - Two absolutely preferred candidates, another one preferred to some degree and one not preferred at all.

**Ballot 10** 11000 - Same as ballot 3.

We note or observe from the above casted ballots 1, 3 and 4, that they represent a crisp set.

**(ii) Keychain representations of all possible votes**

The following table describes all the possible keychains that could be generated from votes of four candidates, i.e how one voter can be preferred by the four candidates:

Keychains for ALL possible choices/ preferences by the Voters

| | | |
|---|---|---|
| 1 | 11111 | All four (4) candidates are preferred absolutely. |
| 2 | 1111 $\lambda$ | 3 candidates preferred absolutely, with one preferred to degree $\lambda$. |
| 3 | 11110 | 3 candidates preferred absolutely, with one not preferred at all. |
| 4 | $111\lambda\lambda$ | Out of four, two were absolutely preferred whilst the other two were reasonable preferred with equal lower degrees. |
| 5 | $111\lambda\beta$ | Out of four, two were absolutely preferred whilst the other two were reasonable preferred with different lower degrees |
| 6 | $111\lambda0$ | Out of four, two were absolutely preferred, another preferred with lower degree whilst one was not preferred at all. |
| 7 | 11100 | Only two candidates were preferred outright and absolutely |
| 8 | $11\lambda\lambda\lambda$ | Out of 4, one was absolutely preferred whilst the other three were reasonable preferred with equal lower degrees |
| 9 | $11\lambda\lambda\beta$ | Out of 4, 1 was absolutely preferred, 2 were reasonable preferred with equal lower degrees and 1 preferred at the least degree |
| 10 | $11\lambda\lambda0$ | Out of 4, 1 was absolutely preferred, 2 were reasonable preferred with equal lower degrees and one not chosen at all. |
| 11 | $11\lambda\beta\beta$ | Out of 4, 1 was absolutely preferred, abd 1 was reasonable preferred with lower degree and 2 with equal least degree. |
| 12 | $11\lambda\beta\gamma$ | All four candidates, one preferred absolutely and rest three were reasonable preferred but with different smaller degrees. |
| 13 | $11\lambda\beta0$ | Three candidates preferred, one absolutely and rest two were reasonable preferred but with different smaller degrees |
| 14 | $11\lambda00$ | Two candidates preferred, one absolutely and the other one at a reasonable lower degree. |
| 15 | 11000 | Only one candidate was preferred and it was absolutely. |
| 16 | $1\lambda\lambda\lambda\lambda$ | All candidates were preferred, but with equal reasonable lower degree. |
| 17 | $1\lambda\lambda\lambda\beta$ | All candidates are preferred, with 3 with equal reasonable lower degree and one with least degree. |

Keychains for ALL possible choices/ preferences by the Voters (continuation)

| | | |
|---|---|---|
| 18 | $1\lambda\lambda\lambda0$ | Only three candidates are preferred, but with equal reasonable lower degree |
| 19 | $1\lambda\lambda\beta\beta$ | All candidates preferred, with each pair having with equal different reasonable lower degrees |
| 20 | $1\lambda\lambda\beta\gamma$ | All candidates were preferred, where two had equal reasonable lower degree followed by two with two different lower degrees. |
| 21 | $1\lambda\lambda\beta0$ | Three candidates were preferred, where two had equal reasonable lower degree followed by one different lower degrees |
| 22 | $1\lambda\lambda00$ | Only two candidates preferred with equal reasonable low degree |
| 23 | $1\lambda\beta\beta\beta$ | All 4 candidates reasonable preferred but with different smaller degrees where three shared the least degree. |
| 24 | $1\lambda\beta\beta\gamma$ | All candidates reasonable preferred but with different smaller degrees where two are sharing a degree before the least one. |
| 25 | $1\lambda\beta\beta0$ | 3 candidates were reasonable preferred but with different smaller degrees where 2 share the least degree. Only one who was not preferred at all. |
| 26 | $1\lambda\beta\gamma\gamma$ | All candidates were reasonable preferred but with different smaller degrees where two are sharing the least degree. |
| 27 | $1\lambda\beta\gamma\tau$ | All candidates were reasonable preferred but with different smaller degrees. |
| 28 | $1\lambda\beta\gamma0$ | 3 out of 4 candidates are reasonable preferred but with different smaller degrees. |
| 29 | $1\lambda\beta00$ | 2 out of 4 candidates are reasonable preferred but with different smaller degrees. |
| 30 | $1\lambda000$ | out of 4, only one candidate preferred to a degree $\lambda$ |
| 31 | 10000 | no candidate preferred |

**(iii) Counting of Votes**

Our aim here is to find the most preferred candidate by the voters for a specific position.

**Note 5.2.2.**

For this fuzzy voting, if there are $m-$voters, a single candidate can be out of $n-$candidates, can be voted to whatever degree of choice in

$m-$times, $(m-1)-$times, $\cdots$, 3$-$times, 2$-$times, 1$-$time or 0$-$times.

This means, out of the preferences represented by the pins, $1, \lambda, \beta, \gamma$ *or* 0, a candidate $C_n$ can get $m, (m-1), (m-2), \cdots, 3, 2, 1,$ *or* 0 preferential choices out of $m-$voters.

**Recall:** $1 = 1$, $\lambda = 3/4$, $\beta = 2/4$, $\gamma = 1/4$ and $0 = 0$.

The preferences by $m-$voters for candidate $C_n$ could be displayed as follows:

Preferences of $m-$voters for a candidate

| 1 | $\lambda$ | $\beta$ | $\gamma$ | 0 |
|---|---|---|---|---|
| $m$ | $m$ | $m$ | $m$ | $m$ |
| $m-1$ | $m-1$ | $m-1$ | $m-1$ | $m-1$ |
| $m-2$ | $m-2$ | $m-2$ | $m-2$ | $m-2$ |
| $\vdots$ | $\vdots$ | $\ldots$ | $\vdots$ | $\vdots$ |
| 2 | 2 | 2 | 2 | 2 |
| 1 | 1 | 1 | 1 | 1 |
| 0 | 0 | 0 | 0 | 0 |

**(iv) Individual counting of votes using the pins of Keychains generated.**

Now we look at how each of the four candidates were voted from our example and the table below will illustrate that.

Counting of votes

| Pin | $C_1$ | $C_2$ | $C_3$ | $C_4$ |
|---|---|---|---|---|
| $\sum(1)$ | 6 | 3 | 3 | 4 |
| $\sum(\frac{3}{4})$ | 1 | 4 | 1 | 0 |
| $\sum(\frac{2}{4})$ | 0 | 2 | 1 | 1 |
| $\sum(\frac{1}{4})$ | 0 | 0 | 1 | 0 |
| $\sum(0)$ | 3 | 1 | 4 | 5 |

From the above table, we can pick out a lot information, viz

Seven voted $C_1$, of which six were absolute votes and one not.

Three voters could not prefer $C_1$.

$C_4$ was not preferred by five (5) voters, and so on.

This table therefore, helps in getting how were the candidates voted for.

### (v) Voting scores of candidates

The table below represents votes obtained by individual candidates.

Scores of candidates

| Candidate | 1 | $\lambda = 3/4$ | $\beta = 2/4$ | $\gamma = 1/4$ | 0 | Voting scores |
|---|---|---|---|---|---|---|
| $C_1$ | 6*1 | $1*\frac{3}{4}$ | $0*\frac{2}{4}$ | $0*\frac{1}{4}$ | 3*0 | **6.75** |
| $C_2$ | 3*1 | $4*\frac{3}{4}$ | $2*\frac{2}{4}$ | $0*\frac{1}{4}$ | 1*0 | **7** |
| $C_3$ | 3*1 | $1*\frac{3}{4}$ | $1*\frac{2}{4}$ | $1*\frac{1}{4}$ | 4*0 | **4.5** |
| $C_4$ | 4*1 | $0*\frac{3}{4}$ | $1*\frac{2}{4}$ | $0*\frac{1}{4}$ | 5*0 | **4,5** |

These tables represent the voting scores of each candidate, $C_n$.

We could pick a lot of benefits from the table made by the pins of the keychains generated from the ballots. Such benefits can include information about the aspects of absolute choices, some differing degrees of choices up to no choices made at all.

**Note 5.2.3.**

**(A)** Voters absolutely preferred candidates are as follows:

- $C_1 \longrightarrow 6$,
- $C_2 \longrightarrow 3$,
- $C_3 \longrightarrow 3$,
- $C_4 \longrightarrow 4$.

We order these preferences as follows: $\mathbf{P_1} : C_2 = C_3 < C_4 < C_1$.

This *order* shows that candidate $C_1$, is the candidate with most absolute choices (1 or 100%).

**(B)** Candidates preferred to degree $\lambda = \frac{3}{4}$ are as follows:

- $C_1 \longrightarrow 1$,
- $C_2 \longrightarrow 4$,
- $C_3 \longrightarrow 1$,
- $C_4 \longrightarrow 0$.

We order these preferences as follows: $\mathbf{P}_\lambda : C_4 < C_1 = C_3 < C_2$.

This indicates that candidate $C_2$ is the candidate with most choices of degree, $\dfrac{3}{4}$, 0.75 or 75%.

**(C)** Candidates preferred to degree $\beta = \frac{2}{4}$ are as follows:

- $C_1 \longrightarrow 0$,
- $C_2 \longrightarrow 2$,
- $C_3 \longrightarrow 1$,
- $C_4 \longrightarrow 1$.

We order these preferences as follows: $\mathbf{P}_\beta : C_1 < C_3 = C_4 < C_2$, thus candidate $C_2$ got most choices of degree, $\frac{1}{2}$, 0.5 or 50%.

**(D)** Candidates preferred to degree $\gamma = \frac{1}{4}$ are as follows:

- $C_1 \longrightarrow 0$,
- $C_2 \longrightarrow 0$,
- $C_3 \longrightarrow 1$,
- $C_4 \longrightarrow 0$.

We order these preferences as follows: $\mathbf{P}_\gamma : C_1 = C_2 = C_4 < C_3$. This indicates that candidate $C_3$ is the candidate with most choices of degree, $\frac{1}{4}$, 0.25 or 25%.

**(E)** Not preferred at all are as follows:

- $C_1 \longrightarrow 3$,
- $C_2 \longrightarrow 1$,
- $C_3 \longrightarrow 4$,
- $C_4 \longrightarrow 5$.

We order these preferences as follows: $\mathbf{P_0} : C_2 < C_1 < C_3 < C_4$. This order shows us that $C_4$ was not chosen by a bigger number (by half) of the voters.
The *candidate* with the highest voting score, will be the **most** preferred candidate from the above example. We could see that absolute choices by voters do not always guarantee an absolute winner to candidates.

**Example 5.2.4.**

Here we look at the same example given above by looking at a situation where counting of scores is the same, viz:

Scores of candidates

| Candidate | 1 | $\lambda = 3/4$ | $\beta = 2/4$ | $\gamma = 1/4$ | 0 | Voting scores |
|---|---|---|---|---|---|---|
| $C_1$ | 7*1 | $0*\frac{3}{4}$ | $0*\frac{2}{4}$ | $0*\frac{1}{4}$ | 3*0 | **7** |
| $C_2$ | 3*1 | $2*\frac{3}{4}$ | $5*\frac{2}{4}$ | $0*\frac{1}{4}$ | 0*0 | **7** |
| $C_3$ | 3*1 | $1*\frac{3}{4}$ | $1*\frac{2}{4}$ | $1*\frac{1}{4}$ | 4*0 | **4.5** |
| $C_4$ | 4*1 | $0*\frac{3}{4}$ | $1*\frac{2}{4}$ | $0*\frac{1}{4}$ | 5*0 | **4,5** |

From the above table, we observe a tie, as $C_1$ is equal to $C_2$ score. We also notice that, $C_1$ is being preferred by seven (7) voters compared to the ten (10) of $C_2$.

**(i) Breaking a Tie**

To break a tie, we make use of weighted scores of candidates, where we let,

$$m, (m-1), (m-2), \cdots, 3, 2, 1, \text{ or } 0$$

to be the number of times at which candidate $C_n$ is/ was chosen by the $m-$voters.

If

$$P_s = P_1, P_\lambda, P_\beta, \cdots, P_0$$

represents the degrees of choice by the $m-$voters, then

$$\frac{m, (m-1), \cdots, 1, \text{ or } 0}{m} \times P_s$$

is weighted score of candidate, $C_n$, under the choice of degree $P_s$. Thus the OVERALL score for each candidate $C_n$ will be

$$\frac{m, (m-1), \cdots, 1, \text{ or } 0}{m} \times P_1 + \cdots + \frac{m, (m-1), \cdots, 1, \text{ or } 0}{m} \times P_0$$

or

$$\Sigma P_1 \times 1 + \Sigma P_\lambda \times \lambda + \Sigma P_\beta \times \beta + \cdots + \Sigma P_0 \times 0.$$

For this specific example, we will have the following weighted scores of candidates;

Weighted scores of candidates

| Candidate | 1 | $\lambda = 3/4$ | $\beta = 2/4$ | $\gamma = 1/4$ | 0 | Voting scores |
|---|---|---|---|---|---|---|
| $C_1$ | $\frac{7}{10}*1$ | $\frac{0}{10}*\frac{3}{4}$ | $\frac{0}{10}*\frac{2}{4}$ | $\frac{0}{10}*\frac{1}{4}$ | $\frac{3}{10}*0$ | **0.7** |
| $C_2$ | $\frac{3}{10}*1$ | $\frac{2}{10}*\frac{3}{4}$ | $\frac{5}{10}*\frac{2}{4}$ | $\frac{0}{10}*\frac{1}{4}$ | $\frac{0}{10}*0$ | **0.7** |
| $C_3$ | $\frac{3}{10}*1$ | $\frac{1}{10}*\frac{3}{4}$ | $\frac{1}{10}*\frac{2}{4}$ | $\frac{1}{10}*\frac{1}{4}$ | $\frac{4}{10}*0$ | **0.45** |
| $C_4$ | $\frac{4}{10}*1$ | $\frac{0}{10}*\frac{3}{4}$ | $\frac{1}{10}*\frac{2}{4}$ | $\frac{0}{10}*\frac{1}{4}$ | $\frac{5}{10}*0$ | **0.45** |

**(ii) Choosing an overall choice**

By first noting the pin $P_1$ to check which candidate got the highest choices, it will be easy to determine the absolute choice/ a preferred candidate. If the highest choice is not determined from $P_1$, we proceed to the next, and so on and so on. In this situation/ example $\mathbf{C_1}$ is the absolute choice.

With these two displays represented by the above tables, it could be a little bit easier to make some decisions of some outcomes.

# Chapter 6

# Keychains in Economics

According to [19], Economics is a social science that mainly looks at the description and analysis of the production, distribution and consumption of goods and services.

From [29], it was stated that terms that are looked at in the theory of Economics are fuzzy, that is: are ambiguous and vague and to some extent are based on linguistic connectives. It was further stated that some mathematical tools such as calculus, linear algebra and matrices are of great use in bringing a precise explanation or understanding of Economics concepts.

According to A, Pfeilsticker [45], these mathematical tools do not contribute as much as expected to the better understanding of economics concepts, however he agrees that there are advantages of using the above mentioned mathematical tools.

In this chapter, we seek to display one of the mathematical tools, i.e the idea of keychains to unlock the fuzziness of some of the economics terms and concepts. We will further on look at the behavior of the consumers, when making decisions about their preferential choices in choosing a preferred object(s) from a given list of different products.

We will also try to link the idea of preferences and choices made by consumers in reaching some conclusions or decisions using the concept of keychains

## 6.1 Consumer Choices

P Barten and V Bohm, from [4] state that the main objective of consumer theory is to determine the impact on observable demands for commodities of alternative assumptions on the objectives, behavioral rules of the consumer and constraints on which one faces when making decisions.

When it comes to commodities, they could be divided into goods and services. Each commodity is completely specified by its physical characteristics, its location and date at which it is available or produced.

From our previous studies, [29], we discussed the commodities, and consumption bundles.

### 6.1.1 Commodities

Let us assume that there exist a finite number of commodities with finite specification of aspects mentioned above. Let us also consider a *commodity bundle*, a list of real numbers which indicates the quantity of each *commodity*, [4]. The commodity bundle is also referred as a vector and: can be represented as follows

$$x = (x_1, x_2, \cdots, x_l) \in R^l,$$

where

$x_1$ is the amount of the $1^{st}$ good
$x_l$ is the amount of the $l^{th}$ good.

These commodities can be ranked by means of keychains according to the preferences or choices of the consumer. If we are having $l$-commodities then a keychain of length $l+1$ may be formed as follows

$$1 = \lambda_0 \geq \lambda_1 \geq \lambda_2, \geq \cdots, \geq \lambda_l \geq 0.$$

A keychains $1\underbrace{11\cdots 1}_{l\,-\,times}$ means that all the $l-$goods are equally chosen absolutely.

Among the $l-$goods there is a possibility that under certain circumstances some goods are not ranked or chosen at all by the consumer. So in this case the pins representing the corresponding goods are zero. So the number of zero pins in each keychain indicates the number of goods which are out of the consumer's preferences, this means the consumer has no intention of choosing such good.

The following keychain will represent the above situation,

$$111 \quad \underbrace{00\cdots 0}_{(n-2)\text{ times}} \quad .$$

Otherwise if one is uncertain about her choice then the pins different from 1 and 0 may be used to represent the situation. As before we use symbols for numbers in unit interval for the degree of choice, for example,

$$1 \ \underbrace{\lambda\lambda\cdots\lambda}_{(n-1)\ times} \ \beta.$$

We can note from above that all the preferences of an individual can be ranked using the idea of keychains.

### 6.1.2 Consumption Bundles and Budget

If there are $l$ number of goods to choose from then there will be

$$2^{l+1} - 1$$

possible preferences of a single individual in choosing one of the $l$ goods.

Now, we let $X$ be a set of possible consumption bundles and assume that;

1. a consumption set is non-empty,
2. inputs in consumption sets are positive quantities,
3. outputs in consumption sets are negative quantities.

Let $x \in X$ and $X \subset R^l$.

According to [28], the value of a commodity bundle $P.x$ is a net outlay i.e. expenses - receipts. The value of a consumption of a consumer should not exceed his/ her initial wealth (income).
We let the initial wealth given by the price vector $P$ be $w$, so that

$$w = P.\omega.$$

Let the set of possible consumption bundles whose values do not exceed $w$ , of the consumer be called a budget set, $\beta$ , so

$$\beta(P.\omega) = \{x \in X | P.x \leq \omega\}.$$

For a consumer to make a decision to choose an item from a consumption set, a preference has to take place. If for any two consumption bundles i.e. $x$ and $y \in X$, it is likely that $x$ is to be least preferred as to $y$ or the vice versa. If there are bundles that are most preferred than the others then we let the most preferred ones as her demand.

So the demand is denoted as follows;

$$\varphi(P.\omega) = \{x \in \beta(P.\omega) | x^l \in \beta(P.\omega)\} \Longrightarrow x \geq x^l \text{ not } x^l \geq x.$$

## 6.2 Keychain Preference Relations and Choices

### 6.2.1 Preferences

Let $X$ be a consumption set of all commodity bundles.
According to [4], a consumer is assumed to make a choice and to have some preferences

among commodity bundles in the consumption set $X$. The consumer choices from a universal set are always absolute.

When one is looking at his or her choices from the commodity bundles, you may find out that chosen commodities have each a preferential ranking which is totally ordered on $X$.
These consumer preferences can be represented by a binary relation, $\succeq$, on $X$ called a preference relation.

### 6.2.2 Preference Relations

Let $X$ be a set of alternatives (e.g shares, stocks, etc.) and $\succeq$ be any relation on $X$. Now we consider any 2 commodity bundles $x, y \in X$, as stated in [29]. The statement

$$x \succeq y$$

is read as '$x$ is at least as good as $y$'.

The three basic axioms that are satisfied by " $\succeq$ " are

**Axiom 1** Reflexivity,

$\forall x \in X$, $x \succeq x$ i.e. any bundle is as good as itself.

**Axiom 2** Transitivity,

for any $x,\ y,\ z\ \in X,\ x \succeq y\ \ \&\ \ y \succeq z\ \Rightarrow\ x \succeq z$

**Axiom 3** Completeness,

for any two bundles $x,\ y\ \in X,\quad x \succeq y$ or $y \succeq x$.

We call the relation $\succeq$ a *preference relation.*

From the above explanation we see that any two preferences must be comparable. The comparability of preferences means that preferences can be ranked using the idea of keychains.

**(a)** A preference relation that satisfies all the above 3 axioms is a complete pre-ordering on $X$ known as preference order. From this preference order two relations can be immediately derived, one is a strict preference relation and the other is a relation of *indifference*.

- The strict preference relation ($\succ$).

  A commodity bundle $x$ is said to be strictly preferred to $y$ if

  $$x \succ y \iff x \succeq y \text{ but definitely not } y \succeq x.$$

- The indifference relation ($\sim$) is defined by

  $$x \sim y \iff x \succeq y \text{ and } y \succeq x$$

  where $x \sim y$ is read as $x$ is indifferent to $y$.

**(b)** Individual preferences are assumed to be rational. That is if it possesses the following two properties:

- Completeness i.e. for all $x,\ y\ \in X$ we have $x \succeq y$ or $x \succeq y$ (or both).
- Transitivity i.e. for all $x,\ y,\ z\ \in X, x \succeq y\ \&\ y \succeq z \implies x \succeq z$.

So from the assumption $\succeq$ is complete, it means that an individual has a well defined preference between any two possible alternatives in this case.

If the relation, $\succeq$ is rational, then the

1. Strict preference relation, ($\succ$) is both irreflexive and transitive i.e.

   $x \succ y$ never holds,

   $x \succ y\ \&\ y \succ z$ always holds.

2. The indifference relation, ($\sim$) will be,

   - reflexive, that is $x \sim x$ for all $x$,

   - transitivity, that is $x \sim y\ \&\ y \sim z$ then $x \sim z$,

   - complete, that is $x \sim y$ then $y \sim x$.

**(c)** A preference relation, $\succeq$ on $X$ is said to be monotone if $x \in X$ and $y \succeq x$ imply that $y > x$.

Preference relations are often described by means of utility functions which assign a value to each element in the set of alternatives indicating the degree of one's preferences.

Any preference relation can be represented by utility function only if it is rational. The results on the connections between utility functions and preferential relations can be found in [30].

### 6.2.3 Choice Rule

From [8], [44], the behavior of consumers is reflected in their preferences or choice of goods. Here we will look at how the idea of a choice structure arises.

Let us consider a consumer with a list of consumables/ commodities according to his/ her affordability. We refer to this list as a budget $\beta$.

From this budget $\beta$, a consumer can choice one or more commodities or none at all. When that is exercised, a consumer will have different alternatives.

If $X$ is the set of all choices, then

$$\beta \subseteq X.$$

We will now consider the chosen commodities in $\beta$ to be $C(\beta)$ called the choice rule. We observe that

$$C(\beta) \subseteq \beta \subseteq X.$$

De Wilde [11], states that a fuzzy subset is defined as

$$C(\beta) \subseteq \beta \Leftrightarrow \mu_{C(\beta)}(x) \leq \mu_\beta(x) \ \forall x \in X,$$

$\mu_\beta -$ indicates the membership value of a set of alternative choices $x$ in $X$. These membership values of different choices can be clearly represented by keychains. The keychains can determine the optimum or a final choice of a consumer.

## 6.3 Some Few Examples of Keychains in Economics

In this section we pick the examples mentioned in [29], that is, *unemployment, employable individuals and choosing a consumable.* The three mentioned concepts are uncertain or fuzzy.

**Example 6.3.1.** *Unemployment*
Here, we consider someone/ a person who cannot be employed. The following model is trying to explain/ show someone/ a person who is regarded as being unemployed. We will use the idea of fuzzy set theory to illustrate how an unemployed person can be represented.

We look at the normal working hours of an average worker for someone who is working for a government funded institution, normally, it's 8 hours per day, this makes one to work for almost 40 hrs per week.

Using the total time for a normal worker in a week we may have the following model which represents the fuzzy concept unemployment.
For this model $\mu(t)$, a fuzzy subset, represents a person who works $t$ hours per week and is defined as

$$\mu(t) = \begin{cases} 1 & \text{if} \quad t = 0 \\ \dfrac{40-t}{40} & \text{if} \quad 0 \leq t \leq 40 \\ 0 & \text{if} \quad t \geq 40. \end{cases}$$

From the above model, if $\mu(t) = 1$, means that the the person is totally unemployed since $t = 0$, whilst when $\mu(t) = 0$ represents a fully employed person, that is someone who is working for 40 hours or more per week. For $\leq t \leq 40$, we get the degree at which one is employable.

**Example 6.3.2.** *Employable*

Here we look at the fuzzy concept employable using the idea of keychains. We know that for anyone to qualify for a job, one has to meet certain qualifications. Sometimes the qualifications of candidates for a job differ from one person to another, or they may all have the same qualifications to do a certain job. In the case of candidates who are for a certain job but are having the same qualifications, then a look at their experience and competence will be an advantage.

The following is a keychain model that will explain the concept of employable candidates. Fully employable are going to be represented by a pin 1, whilst those that are not employable at all will be represented by pin 0. The pins whose values lie within the interval, $I = [0, 1]$ will indicate the degrees of employability of each candidate relative to the job. This will make a very good platform to compare candidates, so the idea of keychains will be of great help in making selections of a suitable candidates for a given vacancy.

Now let us look at a situation where three people who have applied for the same vacancy. Using the idea of the keychains the fuzzy concept " employable " will be represented by 15 keychains of length, 3. Each keychain formed will have four pins, $3 + 1$ and these keychains are as follows;

$1111$; $111\lambda$; $1110$; $11\lambda\lambda$; $11\lambda\beta$; $11\lambda 0$; $1\lambda\lambda\lambda$; $1\lambda\lambda\beta$; $1\lambda\beta\beta$; $1\lambda\beta\gamma$; $1100$; $1\lambda\lambda 0$; $1\lambda\beta 0$; $1\lambda 00$; $1000$.

Discussing the above keychains, we neglect the first pin in each keychain, then the three remaining pins will represent the degrees of being employable. From keychain 1000 we see that none of the three candidates is employable for the given vacancy.

The following table explains the possible outcomes that may be encountered in selecting the three candidates.

**Table 6.3.3.** *Employable vs Keychains*

| Keychain | Interpretation of the keychain |
|---|---|
| 1111 | This shows that all the three candidates are fully employable, they meet all the requirements for the job with their qualifications.<br>A toss of coin can be used to choose one candidate from the three candidates or experience, age, enthusiasm can have a big impact in choosing that one candidate. |
| $111\lambda$ | Two candidates are fully employable and one is reasonable employable. |
| 1110 | Only two are fully employable and one is not employable. |
| $11\lambda\lambda$ | Only one is fully employable and the other two are equally reasonable employable. |
| $11\lambda\beta$ | Only one is fully employable and the other two are reasonable employable but one is better than the other |
| $11\lambda0$ | Only one candidate is full employable with one of the other two reasonable employable whilst the third one is not employable at all. |
| 1100 | Only one candidate is full employable the others are not employable at all |
| $1\lambda\lambda\lambda$ | All the three candidates are equally reasonable employable |
| $1\lambda\lambda\beta$ | Two of the three candidates are equally better employable than the third one |
| $1\lambda\lambda0$ | Two of the three candidates are equally better employable whilst the third one is not employable at all. |
| $1\lambda\beta\beta$ | One of the three candidates is better employable than the other two who are reasonable equally employable. |

| Keychain | Interpret of the keychain (continuation) |
|---|---|
| $1\lambda\beta\gamma$ | All the three are reasonable employable but they differ in degrees of employability. |
| $1\lambda\beta 0$ | Two of the three candidates are reasonable employable but they differ in degrees of employability whilst the third one is not employable at all. |
| $1\lambda 00$ | One of the three candidates is reasonable employable whilst the others are not employable at all. |
| 1000 | All the three candidates are not employable at all. |

The general norm that is taken by most institutions when looking for candidates for a certain post is to advertise a position and short list 5 or fewer candidates for possible employment. If there are $n$ applicants then there will be $2^{n+1} - 1$ possible ways of employing suitably qualified candidates based on the degrees of suitability.

It is at the state of short listing that the idea of keychains can be fruitful, whereby the applicants can be ordered according to the possible keychains. Using this idea means that suitable candidates can be short listed for the given position. In the case of unavailability of one or more of the fully qualified candidates then the next set of candidates to be considered will be those with higher degree of membership in the keychain.

So in general, a total of $2^{n+1} - 1$ keychains of length $n$ will be formed where $n$ will denote the number of elements in the set on which a fuzzy concept is modelled. The pins that are found in the keychains are graded with respect to degree of membership to the fuzzy subset. Hence the method of keychains can have a very big impact in selecting or looking for a well suited candidate for a job if qualification is a priority or in any other context in which a fuzzy concept is modeled.

**Example 6.3.4.** *Choices*

If one is in a shop and want to buy coffee then he/ she is confronted with a host of different brands of coffee $\{x_1, x_2 \cdots x_n\}$. Thus one is led to make choices. Hence arises a preference relation.

Now if we consider two brands of coffee, say $x$ and $y$, the fuzzy preference relation will indicate a degree to which $x$ is preferred at least as much as $y$ or vice versa. This comparison can be done with associated numbers in the unit interval $I = [0, 1]$. This situation can be represented by the 7 keychains of length 3, which are 111, 11$\lambda$, 110, 1$\lambda\lambda$, 1$\lambda\beta$, 1$\lambda$0 and 100. These keychains can be interpreted as follows in relation to the choices and preferences that can be made by a consumer;

**Table 6.3.5.** *Choice of a Coffee Brand*

| Keychain | Interpretation |
|---|---|
| 111 | Both brands of coffee are equally preferred, and one can buy them both depending on one's budget otherwise the cheaper one would do. |
| 11$\lambda$ | One of the brands is preferred absolutely and the other is preferred to a lesser degree. It can be bought if one has sufficient money at disposal. |
| 110 | From the two brands available only one is fully preferred by the consumer. The other is not a choice at all. |
| 1$\lambda\lambda$ | The 2 brands available are both preferred relative to the same degree. |
| 1$\lambda\beta$ | The 2 brands are preferred one more than the other. |
| 1$\lambda$0 | One of the brand is preferred relative to some degree and the other is not preferred at all. |
| 100 | Not a single brand of coffee is preferred by the consumer. |

As one can see from the above example, the idea of keychains helps in modeling or formulating some decisions that can be taken by consumers when they are facing choices based on their preferences. This is done by interpreting the pins found in each keychain.

# Chapter 7

# Keychain Applications in Government Departments

## 7.1 Decisions-Making

From [7], Bellman and Zadeh states that decision-making in a fuzzy environment meant a decision process into which the goals and/or the constraints, but not necessarily the system under control, are fuzzy in nature. They further say that the decision-making in the real world takes place in an environment in which the goals, the constraints and the consequences of possible actions are not known precisely or not sharply defined. Below we look at a formalised definition of a decision as explained in [7].

**Definition 7.1.1.** *Decision:*
Assume that we are given a fuzzy goal $G$, and a fuzzy constraint $C$ in a space of alternatives $X$. Then, $G$ and $C$ combine to form a decision, $D$, which is a fuzzy set resulting from intersection of $G$ and of $C$.

In symbols, the above is shown as follows:

$$D = G \cap C$$

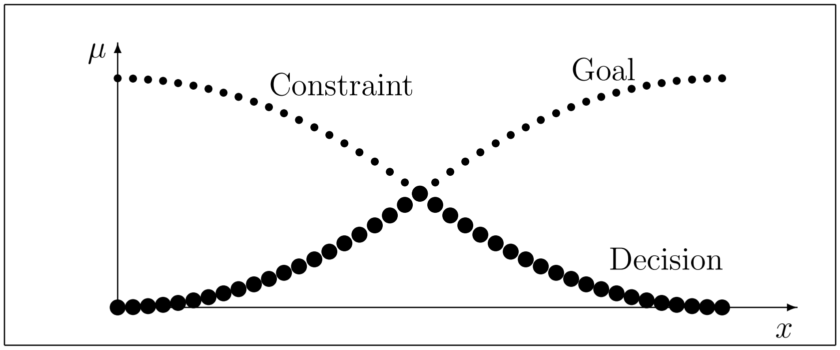



Figure 7.1: Relation between the Goal, the Constraint and the Decision.

and correspondingly

$$\mu_D \ = \ \mu_G \ \wedge \ \mu_C.$$

The graph below shows the relationship between $G$, $C$ and $D$.
In this definition it is further stated by [7], that if we have a set of $m-$goals,

$$G \ = \ \{G_1, \ G_2, \ \cdots , \ G_m\}$$

and a set of $n-$constraints,

$$C \ = \ \{C_1, \ C_2, \ \cdots , \ C_n\},$$

then the resultant decision is the intersection of the given goals, $G$ and the given constraints, $C$.
Thus the decision will be denoted by

$$D \ = \ G_1 \ \cap \ G_2 \ \cap \ \cdots \ G_n \ \cap \ C_1 \ \cap \ C_2 \ \cap \ \cdots \ C_n,$$

which corresponds to

$$\mu_D \ = \ \mu_{G_1} \ \cap \ \mu_{G_2} \ \cap \ \ \cdots \ \mu_{G_n} \ \cap \ \mu_{C_1} \ \cap \ \mu_{C_2} \ \cap \ \cdots \ \mu_{C_n}.$$

Right through this chapter we will try to deal with imprecision by employing the concepts and techniques of keychains so as to come up with more precise decisions.

## 7.2 Decisions and Keychains

In this section we will make use of definition mentioned in the above section in illustrating examples using Keychains.

### 7.2.1 Decision Making in Institutions

Most of the time, we make choices as individuals, and take time to make decisions due to many factors. The factors could be other uncertain situations, but if some factors are factored out before choices are made or final decisions, then it will be a quick thing to come up with wise preferences with well thought decisions.

We find out that in most government departments interviews for vacant posts are conducted almost every month. What is amazing is the time it takes for officials to come up with decisions on a preferred candidates from any particular interview.

Conducting interviews is a complex situation, where many factors are at play, but there seems to be the most high ones after the less high ones have been factored out. The main things in interview includes:

- Number of interviewers and interviewees,

- A list of questions to ask candidates,

- Scores by the interviewers,

- Consolidation of scores of each candidate.

The last point seems to be a milestone as senior officials take time to complete it.

### 7.2.2 Illustrating Examples to consolidate scores

In this subsection we look at simulating examples that looks at the applications of keychains to come up with interviewing decisions.

**Example 7.2.1.**

In this example, we wish to illustrate how keychains can be used when an interviewing team is selecting a candidate for a vacant post.

We will use a score sheet from one of the Eastern Cape's governmental departments, in South Africa.

Let

$$C = \{ c_1, c_2, c_3, c_4, c_5 \}$$

be finite set of shortlisted candidates for a certain governmental department. We could refer to these candidates as alternatives for the vacant post and they will be evaluated using a set of linguistic label set as shown on score sheet on the appendix. These linguistic labels will be used to evaluate each candidate on four questions, say

$$Q = \{q_1, q_2, q_3, q_4\}.$$

Below are the given linguistic labels are:

↓

↓

↓

↓

↓

↓

(continuing on the next page)

↓

↓

↓

↓

Table 7.1: Scoring key for individuals

| | | |
|---|---|---|
| 0 | Did Not/Could Not Answer the Question. | Did Not/Could Not Answer the Question |
| 1 | Unsatisfactory Performance | Lacks knowledge of relevant job content. Displays no technical understanding or competence |
| 2 | Performance Needs Improvement | Demonstrates little knowledge of relevant job content and may require close supervision in terms of applying technical knowledge and competence. |
| 3 | Competent/Satisfactory | Demonstrates knowledge of job content and demonstrates the ability to apply technical knowledge and competence under normal supervision |
| 4 | Performance Significantly Above Expectations | Shows above average knowledge of job content and technical knowledge and competence is displayed in terms of the job content and wider work environment, little guidance/supervision will be required. |
| 5 | Outstanding Performance | Demonstrates outstanding knowledge on a wide spectrum of job-related areas as well as broader work environment issues. Displays superior technical knowledge and competence and will be able to work without supervision/guidance. |

Using shorthand for linguistic labels with the help of [50] on the scoring keys on the table 7.1 above, we will have

$$L = \{ l_0, l_1, l_2, l_3, l_4, l_5\}$$

where

$$
\begin{aligned}
l_0 &= \text{Did not / could not answer the question,} \\
l_1 &= \text{Unsatisfactory Performance,} \\
l_2 &= \text{Performance Needs Improvement,} \\
l_3 &= \text{Competent/Satisfactory,} \\
l_4 &= \text{Performance Significantly Above Expectations,} \\
l_5 &= \text{Outstanding Performance}
\end{aligned}
$$

In the unit interval $I = [0, 1]$, we can represent these linguistic labels as,

$$L = \{ 0, 0.2, 0.4, 0.6, 0.8, 1\}.$$

We also consider a team of interviewers or panelists which we will refer to as decision makers,

$$D = \{d_1, \cdots, d_k\}, \quad \text{where } k = 1, 2, \cdots, n,$$

and $n$ is the number of decision makers.

If we had six panelists (Decision makers), let us assume that their decisions in matrix format using the interview scoring sheet are as follows,

| | $c_1$ | $c_2$ | $c_3$ | $c_4$ | $c_5$ |
|---|---|---|---|---|---|
| $q_1$ | 0.8 | 0.6 | 0.6 | 0.8 | 0.8 |
| $q_2$ | 0.6 | 0.8 | 0.8 | 0.4 | 0.6 |
| $q_3$ | 0.6 | 0.8 | 0.8 | 0.6 | 0.8 |
| $q_4$ | 0.6 | 0.4 | 0.4 | 0.8 | 0.4 |

**Table 7.2.2.** *Scoring Matrix Decision for $d_1$.*

| | $c_1$ | $c_2$ | $c_3$ | $c_4$ | $c_5$ |
|---|---|---|---|---|---|
| $q_1$ | 0.6 | 0.6 | 0.8 | 0.8 | 0.8 |
| $q_2$ | 0.8 | 0.6 | 0.6 | 0.6 | 0.8 |
| $q_3$ | 0.6 | 0.8 | 0.8 | 0.8 | 0.6 |
| $q_4$ | 0.8 | 0.8 | 0.6 | 0.6 | 0.4 |

**Table 7.2.3.** *Scoring Matrix Decision for* $d_2$.

| | $c_1$ | $c_2$ | $c_3$ | $c_4$ | $c_5$ |
|---|---|---|---|---|---|
| $q_1$ | 0.8 | 0.4 | 0.8 | 0.8 | 0.6 |
| $q_2$ | 0.8 | 0.6 | 0.8 | 0.6 | 0.8 |
| $q_3$ | 0.6 | 0.6 | 0.6 | 0.8 | 0.6 |
| $q_4$ | 0.4 | 0.4 | 0.6 | 0.6 | 0.8 |

**Table 7.2.4.** *Scoring Matrix Decision for* $d_3$.

| | $c_1$ | $c_2$ | $c_3$ | $c_4$ | $c_5$ |
|---|---|---|---|---|---|
| $q_1$ | 0.6 | 0.8 | 0.8 | 0.6 | 0.6 |
| $q_2$ | 0.8 | 0.8 | 0.6 | 0.8 | 0.6 |
| $q_3$ | 1.0 | 0.6 | 0.6 | 0.8 | 0.6 |
| $q_4$ | 0.8 | 0.6 | 0.4 | 0.6 | 0.6 |

**Table 7.2.5.** *Scoring Matrix Decision for* $d_4$.

| | $c_1$ | $c_2$ | $c_3$ | $c_4$ | $c_5$ |
|---|---|---|---|---|---|
| $q_1$ | 1 | 0.6 | 0.8 | 0.8 | 0.6 |
| $q_2$ | 0.6 | 0.6 | 0.6 | 0.8 | 0.6 |
| $q_3$ | 1 | 0.8 | 0.6 | 0.6 | 0.8 |
| $q_4$ | 0.8 | 0.4 | 0.6 | 0.4 | 0.6 |

**Table 7.2.6.** *Scoring Matrix Decision for* $d_5$

| | $c_1$ | $c_2$ | $c_3$ | $c_4$ | $c_5$ |
|---|---|---|---|---|---|
| $q_1$ | 0.4 | 0.6 | 0.6 | 0.8 | 0.6 |
| $q_2$ | 0.8 | 0.8 | 0.6 | 0.6 | 0.6 |
| $q_3$ | 0.6 | 0.6 | 0.8 | 0.6 | 0.6 |
| $q_4$ | 0.4 | 0.4 | 0.6 | 0.8 | 0.4 |

**Table 7.2.7.** *Scoring Matrix Decision for* $d_6$ .

From the six matrix decisions above, we can represent the preferences of each decision maker of any candidate by a keychain. The keychains representing any candidates preference by any decision maker, will be keychains of length **four**. Thus from the decisions in table 7.2.2, we will get the following keychains,

$$d_1 \longrightarrow \begin{cases} c_1 \rightarrow & 1\lambda\beta\beta\beta \\ c_2 \rightarrow & 1\lambda\lambda\beta\gamma \\ c_3 \rightarrow & 1\lambda\lambda\beta\gamma \\ c_4 \rightarrow & 1\lambda\lambda\beta\gamma \\ c_5 \rightarrow & 1\lambda\lambda\beta\gamma \end{cases}$$

From the decisions in table 7.2.3, we have the following keychains,

$$d_2 \longrightarrow \begin{cases} c_1 \rightarrow & 1\lambda\lambda\beta\beta \\ c_2 \rightarrow & 1\lambda\lambda\beta\beta \\ c_3 \rightarrow & 1\lambda\lambda\beta\beta \\ c_4 \rightarrow & 1\lambda\lambda\beta\beta \\ c_5 \rightarrow & 1\lambda\lambda\beta\gamma \end{cases}$$

From the decisions in table 7.2.4, we have the following keychains,

$$d_3 \longrightarrow \begin{cases} c_1 \rightarrow & 1\lambda\lambda\beta\gamma \\ c_2 \rightarrow & 1\lambda\lambda\beta\beta* \\ c_3 \rightarrow & 1\lambda\lambda\beta\beta \\ c_4 \rightarrow & 1\lambda\lambda\beta\beta \\ c_5 \rightarrow & 1\lambda\lambda\beta\beta \end{cases}$$

From the decisions in table 7.2.5, we have the following keychains,

$$d_4 \longrightarrow \begin{cases} c_1 \rightarrow & 11\lambda\lambda\beta \\ c_2 \rightarrow & 1\lambda\lambda\beta\beta \\ c_3 \rightarrow & 1\lambda\beta\beta\gamma \\ c_4 \rightarrow & 1\lambda\lambda\beta\beta \\ c_5 \rightarrow & 1\lambda\lambda\lambda\lambda* \end{cases}$$

From the decisions in table 7.2.6, we have the following keychains,

$$d_5 \longrightarrow \begin{cases} c_1 \rightarrow & 111\lambda\beta \\ c_2 \rightarrow & 1\lambda\beta\beta\gamma \\ c_3 \rightarrow & 1\lambda\beta\beta\beta \\ c_4 \rightarrow & 1\lambda\lambda\beta\gamma \\ c_5 \rightarrow & 1\lambda\beta\beta\gamma \end{cases}$$

From the decisions in table 7.2.7, we have the following keychains,

$$d_6 \longrightarrow \begin{cases} c_1 \rightarrow & 1\lambda\beta\gamma\gamma \\ c_2 \rightarrow & 1\lambda\beta\beta\gamma \\ c_3 \rightarrow & 1\lambda\beta\beta\beta \\ c_4 \rightarrow & 1\lambda\lambda\beta\beta \\ c_5 \rightarrow & 1\lambda\lambda\lambda\beta* \end{cases}$$

**Note 7.2.8.**

The generated keychains from decision makers $d_3$, $d_4$ and $d_6$ with asterisks, (*), indicates the equivalence not that the keychains are equal to those that are written the same as them.

Recall that keychains are based on the ordering of pins, rather than the actual value of the pin as a real number. Thus, from the six panelists, the top generated keychains are,

- $111\lambda\beta \times 1$,
- $11\lambda\lambda\beta \times 1$,
- $1\lambda\lambda\beta\beta \times 10$.

The above are top three keychains resembling top three preferences by the six decision makers, $D$.

One will notice that these top three keychains are illustrating that;

* candidate $c_1$ is the only one preferred according to keychain $111\lambda\beta$,
* again, candidate $c_1$ is the only one preferred according to keychain $11\lambda\lambda\beta$,
* candidates $c_1$ (1), $c_2$ (2), $c_3$ (2), $c_4$ (3), $c_5$ (2), are preferred under the preference $1\lambda\lambda\beta\beta$.

The decision matrices displayed together with the keychains reveal candidate $c_1$ as the most preferred choice for the vacant.

If candidate $c_1$ rejects to take post, the next candidate of choice could be candidate $c_4$ as the candidate was preferred three times under the preference $1\lambda\lambda\beta\beta$, which is one of the top three preferences from the decision-makers.

# Chapter 8

# Conclusion

This work, looked at the broad study of finite fuzzy subsets which was introduced by L.A Zadeh in 1965, [26] where he dealt with vagueness, uncertainty and impreciseness. This concept of fuzzy sets as clearly discussed in the above chapters with some definitions and applications, clearly shows that it is capable of making linguistic variables that were unclear or imprecise to be more and more understandable.

This was shown by focusing on the degrees of membership of different events.

The membership values of fuzzy subsets are the ones that led to the development of the notion of keychains, a concept which is the influence of studies done by Professors Murali and Makamba, [38]. This notion of keychains is the one that is being explored in this thesis so as to come up with its applications or to develop its applications to some existing or proposed life scenarios.

Since keychains represent totally ordered sets, a build-up to them by explaining posets and lattices at length, we think that it shows an understandable way of representing keychains graphically.

The ideas of keychain were applied to represent some existing concepts, like, how binary digits can be represented by keychains. It was further on illustrated as how feasible regions of solutions, that simplices, in the theory of optimization can be represented by keychains.

Since keychains of specified length $n$ represent any possibility out of all $2^{(n+1)} - 1$, possibilities at disposal, from the simplex diagrams, then any vertex $(V)$, edge $(E)$, or $n^{th}-$sided face represents a keychain which could be easily located from the simplex keychain diagrams.

The weight order and interpretations of pins of the keychains developed can be useful in analysing of what is happening at any given keychain (region) in the diagram.

A comparison of simplex keychain diagrams and the normal simplexes diagrams shows a very close link when it comes to their Euler's characteristic equation given by

$$\chi = |V| - |E| + |F|.$$

From the simplex keychain diagrams, we have the following formula that give close results to the Euler formula:

$$\chi^* = |V| - |E| + |F^3| - |F^4| + |F^5| - |F^6| + \cdots + (-1)^n |F^{n-1}|,$$

where $n = 3,\ 4,\ 5,\ \cdots$, and
$F^{n-1}$ is the face before the core, $F^n$,
The notation used has been taken from other authors from the study
of Euler formula, [17] and others.

We also showed how keychains can used to model how one makes choices, undergo preferentiality and decision-making.

These ideas of keychains discussed in this study can be developed into many applications other than the ones mentioned here.

# Appendix

1. Distribution of nodes in keychain diagram of binary digits
2. Keychain diagrams of binary digits
3. Higher-Dimensional Simplexes
4. Properties of Platonic Solids
5. Regular Polyhedron

**<u>Keychains and Binary Digits Diagrams:</u>** <u>Distribution of nodes</u>

* equal sign, (**=**) is represented by **0**.
* greater than sign, (**>**) is represented by **1**.

* the number of possible nodes/ points is (***2^n–1*** ), where ***n*** represents the number of elements in a subset.

| # elements (***n*** ) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | |
|---|---|---|---|---|---|---|---|---|---|---|
| # of nodes $(2^n - 1)$ | 0 | 3 | 7 | 15 | 31 | 63 | 127 | 255 | 511 | |

| | |
|---|---|
| For *n = 0* , no possibilities. | 0 |
| For *n = 1* , three (3) possibilities. | 01; **11**; 10 |
| For *n = 2* , seven (7) possibilities. | **001**; 011; **010**; 101; **111**; 110; **100** |
| For *n = 3* , fifteen (15) possibilities. | 0001; **0011**; 0010; **0101**; 0111; **0110**; 0100; **1001**; 1011; **1010**; 1101; **1111**; 1110; **1100**; 1000 |

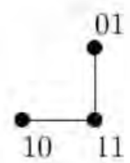
**Distribution of nodes/ points on the keychain diagrams of binary digits.**

| **Step** | **base** | **top** | **Possibly nodes** | **# of boxes** | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| # elements (***n*** ) | | | | | | | | | |
| 1 | 0 | | | 0 | | | | | |
| 2 | 2 | 1 | 3 | 0 | | | | | |
| 3 | 4 | 3 | 7 | 2 | | | | | |
| 4 | 8 | 7 | 15 | 6 | | | | | |
| 5 | 16 | 15 | 31 | 14 | | | | | |
| 6 | 32 | 31 | 63 | 30 | | | | | |
| 7 | 64 | 63 | 127 | 62 | | | | | |
| 8 | 128 | 127 | 255 | 126 | | | | | |
| 9 | 256 | 255 | 511 | 254 | | | | | |

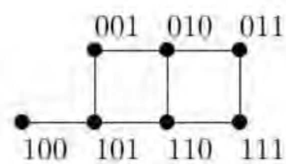
Figure 8.1: Distribution of nodes in keychain diagram of binary digits, from [29]

# Keychain Diagrams of Binary Digits

Keychain Diagram of length 1




Keychain Diagram of length 2.




Keychain Diagram of length 3.

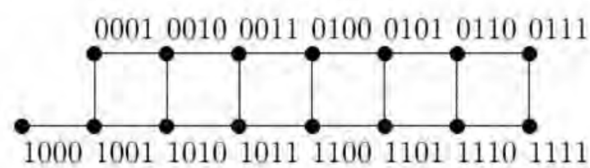
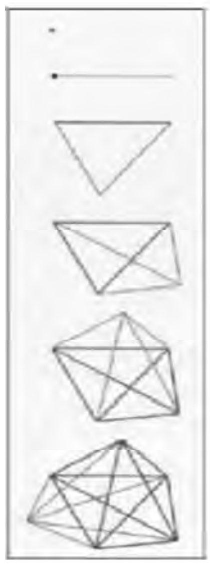


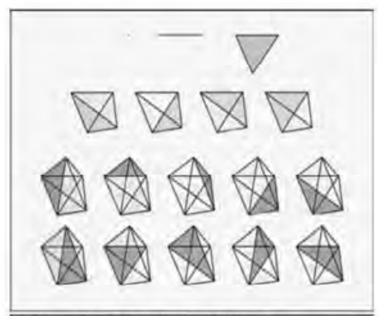

Figure 8.2: Keychain diagrams of binary digits, from [29]

## The First five (5) Dimensional Simplexes



| | Dimension of simplex | | | | | |
|---|---|---|---|---|---|---|
| | 0 | 1 | 2 | 3 | 4 | 5 |
| Number of vertices | 1 | 2 | 3 | 4 | 5 | 6 |
| Number of edges | 0 | 1 | 3 | 6 | 10 | 15 |

| | Dimension of simplex | | | | | |
|---|---|---|---|---|---|---|
| | 0 | 1 | 2 | 3 | 4 | 5 |
| Number of triangles | 0 | 0 | 1 | 4 | 10 | ? |



The triangular faces of the simplexes in the first five dimensions.

**From**: Higher-Dimensional Simplexes (brown.edu)

Figure 8.3: Higher-Dimensional Simplexes, from [24]

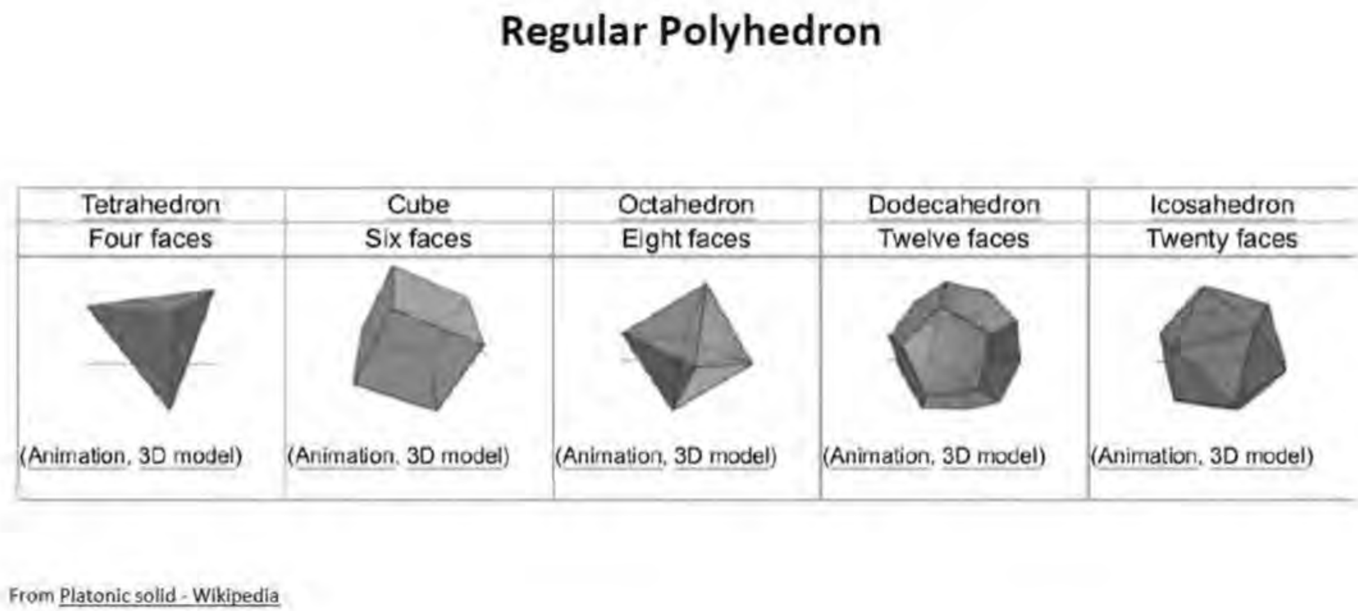



Figure 8.4: Properties of Platonic Solids, from [25]

## Properties of Platonic Solids

| Polyhedron | | Vertices | Edges | Faces | Schläfli symbol | Vertex configuration |
|---|---|---|---|---|---|---|
| tetrahedron |  | 4 | 6 | 4 | {3, 3} | 3.3.3 |
| cube |  | 8 | 12 | 6 | {4, 3} | 4.4.4 |
| octahedron |  | 6 | 12 | 8 | {3, 4} | 3.3.3.3 |
| dodecahedron |  | 20 | 30 | 12 | {5, 3} | 5.5.5 |
| icosahedron |  | 12 | 30 | 20 | {3, 5} | 3.3.3.3.3 |

From Platonic solid - Wikipedia

Figure 8.5: Regular Polyhedron, from [25]

# Bibliography


[1] Adzic, S. and Sedlak, O. *Economic modelling and Theory of Fuzzy Set*, Yugoslav Journal of Operations Research 8 (1998) Number 2, 331-341.

[2] Baise, P. *Cogitator A Parallel, Fuzzy, Database-Driven Expert System*, Thesis, Rhodes University, South Africa, 1994.

[3] Balbes, R. and Dwinger, P. *Distributive Lattices*, University of Missouri Press, Columbia, 1975.

[4] Barten, P. A. and Bohm, V, *Consumer theory: Handbook of mathematical economics*, KJ ARROW and MD Intriligator Editors, Amsterdam North-Holland, Vol 2, 1982.

[5] Baliamoune, M. N, *On the measurement of human well being: Fuzzy set theory and Sen"s capability approach,* Coggin College of Business, University of North Florida, 2003.

[6] Bell, J. and Moshe Machover, *A Course in Mathematical Logic*, Amsterdam, North-Holland, 1977.

[7] R. E. Bellman and L. A. Zadeh *Decision-Making on a Fuzzy Environment* Prepared under Grant No.NGL-05-003-016 by ELECTRONICS RESEARCH LABORATORY; University of California, Berkely, 94720.

[8] Bouyssou, D. *A note on the 'min in favour' choice procedure relations,* http://www.lamsade.dauphine.fr/bouyssou/minhagen.pdf.

[9] Britz, T. and Cameron, P. *Partially ordered sets*, 2001, http://www.designtheory.org/library/encyc/topics/posets.pdf.

[10] Chakraborty, R.C. *Fuzzy Set Theory*, 2010, http://www.myreaders.info/html/softcomputing.html.

[11] De Wilde, P. *Fuzzy Constraints, Choice and Utility*, IEEE International Conference, 2002, 1-6.

[12] Gawron J. M. *Partially ordered sets and lattices*, http://www-rohan.sdsu.edu/gawron/mathling/lectures/posets-slides.pdf.

[13] Gierz, G. *A Compendium of continuous lattices*, Berlin, Springer, 1980.

[14] Gratzer, G. *General Lattice Theory*, 2Ed, Basel; Boston; Berlin: Birkha¨user, 2003.

[15] R. Hosseingholizadeh. *Fuzzy Voting in Internal Elections of Educational and Party Organizations* World Academy of Science, Engineering and Technology Vol:4 2010-06-24;

[16] L, Zadeh. Fuzzy sets and systems: *https://en.wikipedia.org/wiki/Lotfi A. Zadeh Fuzzy sets and systems*

[17] Wikipedia. Simplex, *https://en.m.wikipedia.org/wiki/Simplex*

[18] Cuemath. Points and Lines, *https://www.cuemath.com/geometry/points-and-lines/*

[19] Merriam-Webster. Economics Definition and Meaning - *https://www.merriam-webster.com/dictionary/economics*

[20] Britannica. Mathematics Definition, History, and Importance - *https://www.britannica.com/science/mathematics*

[21] Maths is Fun. Eulers-Formula - *https://www.mathsisfun.com/geometry/eulers formula.html*

[22] Wikipedia. Partially Ordered Sets *https://en.wikipedia.org/wiki/Partially ordered set*

[23] Wikipedia. Complete Lattice - https://en.wikipedia.org/wiki/Complete lattice

[24] https://www.math.brown.edu/tbanchof/Beyond3d/chapter4/section06.html :text=The

[25] https://en.wikipedia.org/wiki/Platonicsolid :text=Kepler

[26] Kaufmann, A. *Introduction to the theory of fuzzy subsets*, New York: Academic Press, 1975.

[27] Klir, G. J. and Folger, T. A. *Fuzzy Sets, Uncertainty, and Information*, Prentice Hall New Jersey USA, 1988.

[28] Lubczonok, G. *Mathematical Economics lecture*, RU, 2003.

[29] Mahlasela, Z, *Finite Fuzzy Sets, Keychains and Their Applications,* Rhodes University, 2007.

[30] Mas-Colell A., Whinston Michael D., Green J. R. *Microeconomic theory*, Oxford University Press (New York), 1995.

[31] Miklos Nagy1, Maria Vargas-Vera2, and Enrico Motta1 *Introducing Fuzzy Trust for Managing Belief Conflict over Semantic Web Data* Knowledge Media Institute (KMi), The Open University Walton Hall, Milton Keynes MK7 6AA, United Kingdom,

[32] Murali, V, *Professorial Inaugural Lecture,* RU, 2002.

[33] Murali, V, *Lattice Theory and Fuzzy sets lecture,* RU, 2002.

[34] Murali, V, *Combinatorics lecture notes,* RU, 2003.

[35] Murali, V, *Fuzzy points of equivalent Fuzzy subsets,* J Information Sciences, 158, 2004, 277 - 288.

[36] Murali, V, *Young Diagrams of Equivalent Fuzzy Subsets,* RU, 2003.

[37] Murali, V *Lattice of Keychains,* J. Appl. Math & Computing, vol 20, 2006.

[38] Murali, V and Makamba, B. B. *Operations on Equivalent Fuzzy Subgroups,* Rhodes University, 2001.

[39] Murali, V and Makamba, B. B. *On an equivalence Fuzzy Subgroups I, J Fuzzy sets and Systems 123*, 2001, 259 - 264.

[40] Murali, V and Makamba, B. B. *On an equivalence Fuzzy Subgroups II, J Fuzzy Sets and Systems 136*, 2003, 93 - 104.

[41] Murali, V and Makamba, B. B. *Finite Fuzzy sets, International Journal of General Systems*, Vol 34, no. 1, 2005, 61 - 75.

[42] Naim Cagman and Haci Aktas, *Fuzzy Ballot Paper and Its Election System, Selçuk J. Appl. Math* Vol. 6. No.2. pp. 53-58, 2005.

[43] Ngcibi, S. L. *Case Studies of Equivalent Fuzzy Subgroups of Finite Abelian Groups*, Thesis, Rhodes University - South Africa, 2002.

[44] Nurmi, H. *From Group Choice Puzzles to Mechanism Design, Three Decades of Fuzzy Social Choice*, http://www-bisc.cs.berkeley.edu/BISCSE2005

[45] Pfeilsticker A. *The systems approach and fuzzy set theory bridging the gap between mathematical and language-oriented economists,* Fuzzy set and systems Journal, Vol 6-7 1981/82.

[46] Taliwanga Matiti, *Counting of Finite Subsets with Applications to Fuzzy Recognition and Selection Strategies*, Rhodes University 2012.

[47] Van Lint, J.H. and Wilson, R.M. *A Course in Combinatorics,* Cambridge University Press - Great Britain, 1993.

[48] Warren, R. H. *Equivalent fuzzy sets(Short communication),* J Fuzzy set and systems, Vol. 6-7 1981/82.

[49] Yager, R. R.and Dimatar, P. F. *Essentials of Fuzzy Modeling and Control*, Wiley-Interscience Publication USA 1994.

[50] Zhou Ligang, Huayou Chen *A generalization of the power aggregation operators for linguistic environment and its application in group decision making* School of Mathematical Sciences, Anhui University, Hefei, Anhui 230601, China, 2011.

[51] Zimmermann, H. J. *Fuzzy Set Theory and Its Applications,* Kluwer Academic Publishers - USA 1988.

[52] Geeks for Geeks. Partial orders and Lattices. *Mathematics — Partial Orders and Lattices - GeeksforGeeks*

[53] Zukisa Q Xhotyeni. A Study of Existence of Equilibrium in Mathematical Economics Rhodes University 2008.