AN ANALYSIS OF HOW *GEOGEBRA* CAN BE USED AS A VISUALISATION TOOL BY SELECTED TEACHERS TO DEVELOP CONCEPTUAL UNDERSTANDING OF THE PROPERTIES OF GEOMETRIC SHAPES IN GRADE 9 LEARNERS: A CASE STUDY IN NAMIBIA

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ABSTRACT

According to Rosken & Rolka (2006), learning mathematics through visualisations can be a powerful tool to explore mathematical problems and give meaning to mathematical concepts and relationships between them. "Visualisation can reduce the complexity of mathematical problems when dealing with a multitude of information" (p.458).

This case study focused on using *GeoGebra* as a visualisation tool to teach angle properties in Grade 9 geometry. This study set out to analyse how *GeoGebra* visualisations can be used by selected teachers to teach for conceptual understanding. The research is based on a constructivist view of learning and is oriented within an interpretive paradigm. The methodology used is a qualitative case study. The study was conducted in one school and involved 3 mathematics teachers who were purposefully selected because they showed willingness to use technology in their teaching. I used classroom observations and interviews to collect the data. The study identified a number of factors from the participants that related to using *GeoGebra* in teaching for conceptual understanding. These include the effective use of dynamic visuals to build on prior knowledge, using multiple representations through image generation and image transformation to make connections and using visuals to justify mathematics ideas. The results from this study indicated that *GeoGebra* can indeed be used effectively as a teaching tool to teach for conceptual understanding in mathematics.

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- Last but not least, deepest thanks go to all people who took part in making this thesis real.

DEDICATION

I dedicate this thesis to my parents: my father Festus Vatilifa Mwiikeni and my mother Selma Matheus. They instilled in me a culture of hard work. This contributed so much to my life today.

DECLARATION OF ORIGINALITY

I, Erasmus Mwiikeni, student number 15M7349, hereby declare that this thesis is my own work, composed in my own words. It has not been submitted in any form for another qualification or any assessment to another University or institution. Where I have drawn on the words or ideas of others, these have been fully acknowledged in accordance with Rhodes University, Education Department reference guide.

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Erasmus Mwiikeni 24 November 2016

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CHAPTER ONE

1. INTRODUCTION

This chapter introduces the research study of how *GeoGebra* can be used as a visualisation tool to develop conceptual understanding of angle properties in Grade 9 geometry. It begins with the research context background to rationale. This is followed by the research goals and questions. The research methodology is also highlighted and the significance of the study is revealed. The limitations of the study are also listed and the chapter concludes with a short outline of the structure of the thesis.

1.1 BACKGROUND OF THE STUDY

"Effective learning and teaching are closely linked to the use of teaching and learning materials (e.g. books, posters, charts or recycled waste materials, etc.) and ICTs (e.g. computers, audio and visual media) in the classroom" (Namibia. Ministry of Basic Education Sport and Culture, 2009, p. 27). Bishop (2003) in his review of research on visualisation in mathematics concludes that there is value in emphasising visual representations in all aspects of a mathematics classroom. He explains that mathematics is a subject that is concerned with the study of patterns, representations and sets of connected ideas. Many of these representations appear to be visual, having roots in visual sensed experiences.

As a Junior Secondary teacher of mathematics for over 14 years, I have observed learners struggling to understand geometric terminologies and concepts, in particular in the Junior Secondary (Grades 8 to 10) level where I teach. Generally, I noticed that teachers do not really have concrete and suitable teaching aids for teaching the concept of angle properties, particularly angles formed in parallel lines. Concepts such as corresponding angles, co-interior angles, alternate angles and vertically opposite angles formed within parallel lines are difficult to teach without using proper visuals. Mostly visuals used by teachers today are static images plus some rules from the textbook. In my view, using the textbooks visuals only does not support conceptual understanding sufficiently. Being passionate about mathematics and technology, I tried to use *GeoGebra* as a visualisation tool in my teaching. I started using it just to draw clear and accurate diagrams for worksheets or test papers. At first I did not use it for lesson presentation because I was not comfortable using it in front of my learners. To learn more about using *GeoGebra* in the classroom, I continued to watch videos on the internet of teachers using *GeoGebra* in their lessons. The first day I tried it in my classroom; I noticed that there was a big difference in terms of learners' participation and motivation. Even the way I presented my lesson was far more enjoyable than my usual ones. I was able to explain mathematical concepts more clearly with the use of dynamic images and diagrams.

Using *GeoGebra* dynamic images, teachers can deliver lessons that promote conceptual understanding. Teaching for conceptual understanding requires the teacher to make connections such as connecting mathematics to real world, use of multiple representations and building on prior knowledge, all of which can be done using *GeoGebra*.

GeoGebra is free licence software that integrates geometry and algebra. It enables one to make powerful connections between mathematical concepts through the use of dynamic visuals (M. Hohenwarter & Fuchs, 2005). Using *GeoGebra* in mathematics classrooms can be a way of providing opportunities for mathematical investigations, encouraging discussions and group work.

There is little work being done in Namibia in investigating the power of using technology as a visualisation tool in the mathematics classroom. I thus set out to analyse the potential role of *GeoGebra* visualisations when teaching angle properties in Grade 9 geometry. This study seeks to observe and analyse participants' perceptions and experiences when using *GeoGebra* as a visualisation tool for teaching for conceptual understanding.

1.2 RESEARCH GOALS AND QUESTIONS

The proposed case study aims firstly to investigate the role of *GeoGebra* as a visualisation tool by observing selected teachers teaching Grade 9 learners using *GeoGebra*. Secondly, this study analyses how these teachers use *GeoGebra* visualisations to enhance conceptual understanding of geometric angle properties. The study is framed by the following research questions:
What are selected teachers' perceptions and experiences of:

1. The role that *GeoGebra* visualisations can play in developing conceptual understanding in the teaching of properties of shapes in Grade 9 geometry?
2. How *GeoGebra* can be used as a teaching tool, in their view and experience, to enhance learners' conceptual understanding of angle properties in Grade 9 geometry?

1.3 RESEARCH METHODOLOGY

This qualitative study is located in the interpretive paradigm. According to Cohen, Manion, & Morrison (2007), interpretive research is characterized by a concern for the individual. Human actions are inseparable from meaning and experiences. Bertram & Christiansen, (2014) explains that:

> Within the interpretive paradigm, researchers do not aim to predict what people will do, but rather to describe and understand how people make sense of their worlds, and how they make meaning of their particular actions. The purpose is to develop a greater understanding of how people make sense of contexts in which they live and work. (p.26)

In this study, I specifically wished to understand and make sense of how teachers used *GeoGebra* in their teaching and what their perceptions were of that experience. I wanted to research what meaning the participating teachers attached to *GeoGebra* as a visualisation tool. I also wanted to know how the teachers decided whether *GeoGebra* was an effective tool to develop conceptual teaching or not.

A case study methodology was adopted to generate the data for this research study. "Case studies investigate and report the real-life, complex dynamic and unfolding interactions of events, human relationships and other factors in a unique instance" (Cohen, Manion, & Morrison, 2011, p.289). The case in my study consisted of three mathematics teachers using *GeoGebra* as a visualisation tool to teach angle properties in Grade 9 geometry. The unit of analysis was the perceptions and experiences of the participating teachers using *GeoGebra* visualisations to enhance conceptual understanding.

The data was collected using observations and semi-structured interviews. I collected data by video recording 10 lessons from 3 participants (two participants each teaching 4 lessons and one participant teaching 2 lessons). The recorded videos were transcribed and analyzed to help me gain insight into teachers' practice. After the lesson observation, two participants were individually interviewed. During the interviews, participating teachers were asked to reflect and give their perceptions and experiences while they were using *GeoGebra* as a visualisation tool to teach for conceptual understanding. These interviews were also audio recorded and subsequently transcribed for analysis purposes.

1.4 SIGNIFICANCE OF THE STUDY

This study offers insights into how *GeoGebra* can be used effectively as a visualisation tool to develop conceptual understanding of angle properties in Grade 9 geometry. It also highlights the role of *GeoGebra* visualisations when teaching angle properties. It is hoped that teachers and subject advisers who read this study will gain insight into how *GeoGebra* can be used as a visualisation tool to enhance conceptual understanding. The data and experience of this study could be workshopped and shared with mathematics teachers to help them use *GeoGebra* visualisations in their lessons.

1.5 LIMITATIONS

A limitation of this research is that it was a single case study performed by one researcher over a short period of time. Ideally, more time should have been spent on the participating teachers' training and familiarisation with the software because the knowledge required to teach using technology goes beyond subject content knowledge only. The study took place in one school using only 3 teachers. The outcome could have been more comprehensive if more than three teachers used *GeoGebra* for a longer period of time and in different topics and in different grades. Thus, the findings in this study cannot be generalised in a broader context. Another possible limitation could be my position in the Education structure. Knowing that I am a school principal it is possible that this influenced the participants' responses during the interviews. The participants may have felt that they needed to please me in order to maintain their integrity.

1.6 STRUCTURE OF THE THESIS

Chapter 2

This chapter reviews pertinent literature and provides the theoretical and conceptual framework for the study. It begins by reviewing the concept of visualisation and technology in mathematics. This leads to a review of how visualisation contributes to the theory of conceptual understanding and lastly links it to the learning theory of constructivism.

Chapter 3

This chapter provides an account of how the research was designed and conducted. It describes the research method employed, the research techniques, data analysis tools and ethical issues.

Chapter 4

This chapter presents research results. The chapter begins by outlining the lessons presented with *GeoGebra*. The 10 lesson presentations (where *GeoGebra* was used) and the two interviews are then presented noting the dominant themes. The analytical tool that was used to analyse the data was adapted from the work of Kosslyn (1994) and Kilpatrick et al. (2001).

Chapter 5

This chapter concludes the study. It presents a summary of the dominant themes, draws conclusions from the findings and makes recommendations and suggestions for future research. The chapter ends with my personal reflections of my research journey.

CHAPTER TWO

2. LITERATURE REVIEW

2.1 INTRODUCTION

The purpose of this chapter is to provide the context and theory to the study. It begins with a discussion on definitions of the terms *visualise* and *visualisation* followed by the analyses of some types and roles of visualisation. I also discuss some of the challenges of visualisation. This leads into a discussion of visualisation processes and the use of information and communication technology (ICT) in Namibian Education in general. The integration of technology into the teaching and learning of geometry is also discussed - this culminates in a discussion on the software package *GeoGebra*, and its use in teaching and learning mathematics, particularly in the topic of Geometry. The chapter concludes with a discussion of constructivist theory and visualisation that underpins this study.

2.2 VISUALISATION

2.2.1 Some definitions of visualisation

Visualisation is a noun derived from the verb visualise. The definition of the verb (visualise) according to the Oxford dictionary is "to make something visible to the eye" while the Cambridge dictionary's definition for (Visualize) is "to form a picture of someone or something in order to imagine or remember them". The noun (visualisation), according to the Merriam Webster online Dictionary, is the "formation of mental visual images" or "the act or process of interpreting in visual terms or of putting into visible form".

The literature offers more elaborate and varied definitions. Zazkis, Dubinsky & Dauterman (1996) define visualisation as:

> an act in which an individual establishes a strong connection between an internal construct and something to which access is gained through senses. Such a connection can be made in either of two directions. An act of visualisation may consist of any mental construction of an object, or

> processes that an individual associates with objects or events perceived by her or him as external. Alternatively, an act of visualisation may consist of the construction, on some external medium such as paper, chalkboard or computer screen of an object or events that the individual identifies with objects or process (es) in his or her mind (p.441).

This definition is not far from that of Arcavi's (2003) definition who defines visualisation as:

> the ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understanding. (p.217).

Arcavi (2003) suggests that "visualisation is no longer related to the illustrative purposes only, but is also being recognised as a key component of reasoning (deeply engaging with the conceptual and not the merely perceptual), problem solving, and even proving" (p.235). He proposed that visualisation offers an opportunity of seeing the unseen, which he referred to as visual imagery. Visual imagery is the ability to form mental representations of objects and manipulating them in the mind. "Therefore as a biological and a socio-cultural being we are encouraged and should aspire to 'see' not only what comes 'within sight', but also what we are unable to see" (Arcavi, 2003, p. 215).

Turning to definitions of vizualisatiuon within the mathematics arena, Zimmermann & Cunningham (1991) describe visualisation as applied to mathematics as, "the process of producing or using geometrical or graphical representations of mathematical concepts, principles or problems, whether hand drawn or computer generated" (p.1).
In his writing on learning with visualisation Van de Walle (2010) referred to visualisation as:

> 'Geometry done with the mind's eye'. It involves being able to create mental images of shapes and then turn them around mentally, thinking about how they look from different perspectives, and predicting the results of various transformations. It includes mental coordination of two or three dimensions predicting the unfolding of a box (or net) or understanding a two-dimensional drawing or a three-dimensional shape (p.429).

Makina (2010) says that, visualisation incorporates mental processes that make use of visual imagery, visual memory, visual processing, visual relationships, visual attention and visual imagination. These visuals help learners to construct useful mental schemata. Piaget, cited in Woolfork (1987), defined a schema as the mental representation of an associated set of perceptions, ideas or actions and schemata are the basic building blocks of thinking. These blocks are very important in constructing knowledge.

2.3 TYPES OF VISUALISATIONS

The literature divides visualisation into two groups namely, physical or external visualisations and mental or internal visualisations. External visualisations are visuals that can be seen with our naked eyes, while internal visualisations are those that happen in the mind or imagination of an individual (Guzmán, 2002; Mesaros, 2012 & Presmeg, 1997). These two groups are further categorised into static and dynamic visualisations. According to Guzmán (2002), we visualise by means of our imagination and representative ability, with the help of the normal tools at hand, such as paper and pencil, chalk and blackboard and other traditional physical materials. (Guzmán, 2002) asserts that:

> Mathematical concepts, ideas, and methods, have a great richness of visual relationships that are intuitively representable in a variety of ways. The use of them is clearly very beneficial from the point of view of their presentation to others, their manipulation when solving problems and in doing research. (p.2)

Guzmán (2002) classifies visualisations into isomorphic, homomorphic, analogical and diagrammatic interpretations.

2.3.1 Isomorphic visualisation

The word "isomorphism" applies when two complex structures can be mapped onto each other, in such a way that for each part of one structure there is a corresponding part in the other structure. A simple isomorphic visualisation could be congruent figures as is illustrated in Figure 2.1.

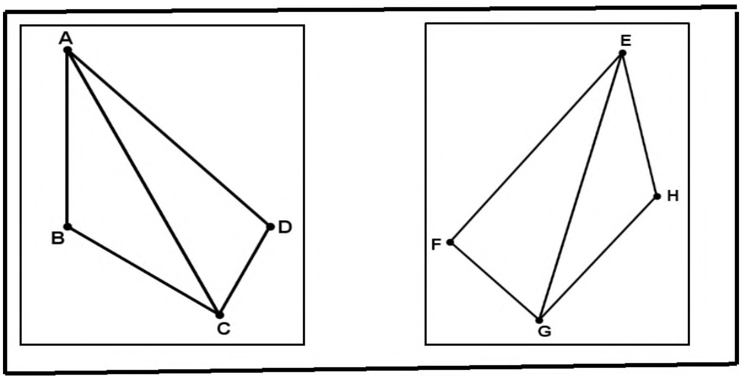


Figure 2.1: An example of an isomorphic visualisation

A maps to E,

B maps to H,

C maps to G, and

D maps to F.

A more complex isomorphic visualisation example is illustrated in Figure 2.2 where the condition of the isomorphisms are listed below.

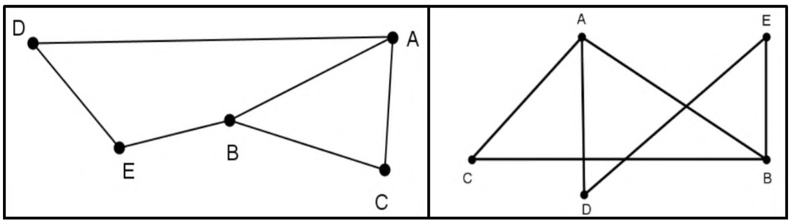


Figure 2.2: Isomorphic relationships between two diagrams

A maps to B, C, D

B maps to A, C, E

C maps to A, B

D maps to A, E

E maps to B, D

2.3.2 Homomorphic visualisation

In this visualisation, abstract mathematical objects or processes are depicted. The images have some of the elements that have mutual relationship between the abstract object to can guide our imagination in the mathematical processes of conjecturing, searching and proving. For example, the mapping illustrated in Figure 2.3 is a visualisation for the function $y = x^2$.

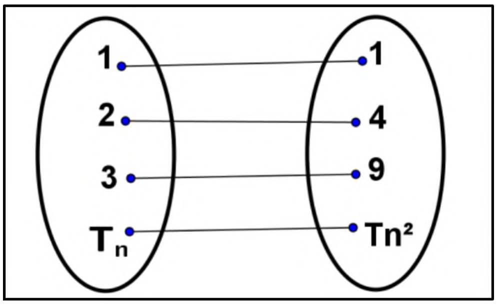


Figure 2.3: Illustration visualisation of $\boldsymbol{y = x^2}$

Learners should understand the concept of square numbers in order to make sense of Figure 2.3 . Guzmán (2002) observes that these visualisations are often very personalised and subjective as their functions are mainly to support and guide our imagination in the processes of conjecturing and proving.

2.3.3 Analogical visualisation

This visualisation makes use of analogous reasoning. Here the object or process is replaced by an analogy. According to Cheshire, Ball, & Lewis (2000), learning via analogy usually involves finding a set of systematic correspondences (a mapping) between a better-known source analog and a more novel target. Thinking of division as fitting things into a container is a visual analogy. Explaining multiplication in terms of directed movements on a number line combines a mathematics analysis and non-mathematics analysis analogy. Some analogies such as "to isolate terms is what you do to turn a complicated sentence into a simpler one" are propositional and literal; they involve a few surface similarities (Lee, 2007).

Some of the reasons why analogy is used in mathematics as a visual tool are to help learners to understand the meanings of new knowledge in an easier manner; to introduce a new formula without its proof and to motivate learners (Loc & Uyen, 2014). However, analogies used in textbooks are usually simple, verbal and structural, and are not necessarily used to enhance learners' learning activities. For example, an analogy made for Pythagoras's Theorem is a verbal expression that says, "In a right angle triangle the square of the hypotenuse is equal to the sum of the squares of the other two sides". What does this mean in developing conceptual understanding of the theorem? Even the written expression of the Pythagorean equation $c^2 = a^2 + b^2$ does not mean much to develop the conceptual knowledge to the mastery level. A static visual representation on the other hand, that shows a triangle with each side squared, may facilitate better conceptual understanding.

2.3.4 Diagrammatic visualisation

This visualisation includes the use of simple diagrams that assist our thinking processes. According to Guzmán (2002), in many cases such diagrams are similar to mnemonic rules. Teachers use mnemonics to link the information to be learned with familiar and already known information through the use of a visual picture or letter/word combinations. Wolgemuth, Cobb, & Alwell (2008) commented that the use of mnemonic strategies has been proven effective with learners at a wide range of ability levels (gifted, normally achieving, and those with mild and moderate disabilities) and at all grade levels. For example, we often find the mnemonic 'FUN' in textbooks to describe angles formed within parallel lines. The shape of the letters 'F', 'U' and 'N' assists mathematics teachers to explain to learners how they can identify corresponding angles, co-interior angles and alternates angle in parallel lines. Using dynamic software like *GeoGebra*, the teacher can better demonstrate the properties of angles formed in parallel lines as illustrated in Figure 2.4.

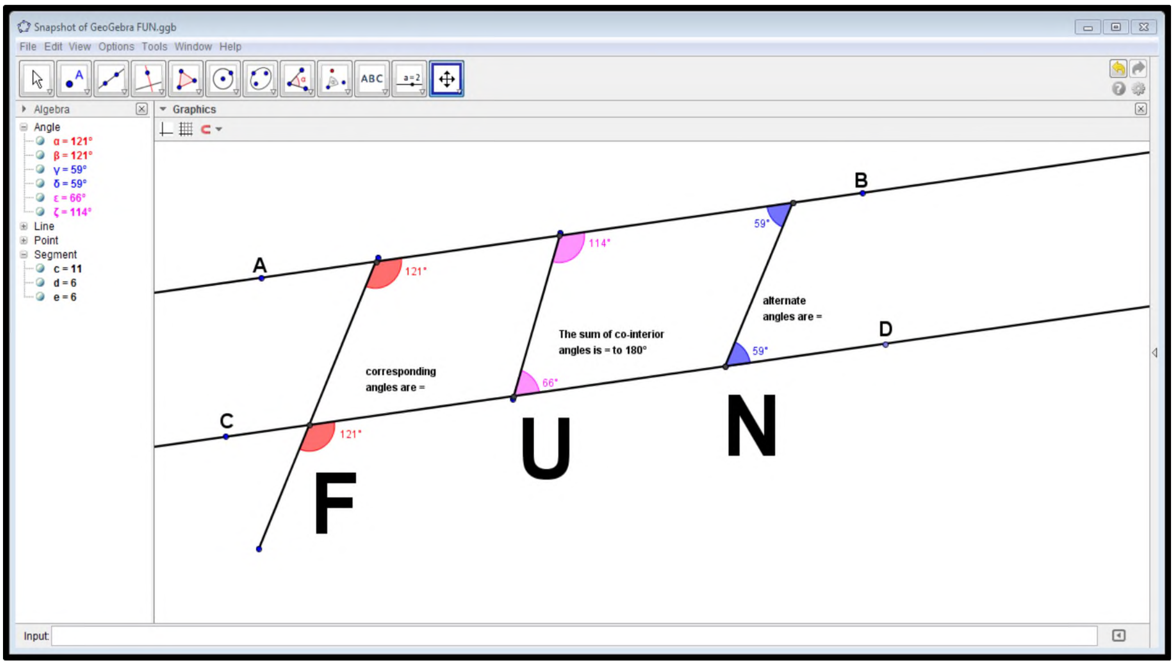


Figure 2.4: Mnemonic (F-U-N) of angles formed between parallel lines in *GeoGebra*

2.4 THE ROLE OF VISUALISATION IN TEACHING AND LEARNING MATHEMATICS

There is an abundance of evidence from the literature that visualisation plays an important role in learning and teaching of mathematics (Apostol & Blinn, 1993; Duval, 1999; Makina, 2010). The primary role of visualisation in teaching mathematics is to facilitate and support learners and teachers to transform a mathematical problem into a form or an image. This image enables the solver to better understand problems whose solutions would be inaccessible without using visualisation (Mesaros, 2012). This is in agreement with Duval (1999) who claimed that "representation and visualisation are the core of understanding mathematics" (p.3). This entails mathematical reasoning, solving mathematical problems and proving mathematical theorems. Similarly Makina (2010) affirms that visualisation is a very important cornerstone in "teaching for understanding" in mathematics because it helps the teacher to facilitate the lesson better as it creates a platform where learners are more engaged with visual images. According to Apostol & Blinn (1993) visual images make more sense than printed or spoken words only. People tend to forget words they hear or read, but images are retained for a long time because they have emotional as well as intellectual appeal. Thus Arcavi (2003) views visualisation as a powerful tool that plays three major roles in constructing mathematical knowledge: Firstly, visualisation can support and illustrate essential symbolic results and possibly provide proof in its own right. Secondly, visualisation can provide a way of resolving conflict between correct symbolic solutions and correct intuitions. Thirdly, visualisation can help learners to engage with concepts and meanings on a level that is not only symbolic and abstract but it help them to identify similarities and make connections and comparisons.

2.5 CHALLENGES OF VISUALISATION IN MATHEMATICS

Despite the positive recommendations from the literature, in my experience visualisation is not used strategically enough in the Namibian mathematics classroom. There are no general initiatives to ensure that the possibilities for learning and teaching school mathematics through the use of visualisations are realised. This may be due to the fact that using visualisation is not explicitly prescribed by the curricular documents; rather it is based on individual teachers' decisions as to whether or not to make visualisation a key part of their lesson presentations. Not incorporating visualisation as part of the official school mathematics curriculum leaves the ball entirely in the teachers' hands to adopt visualisation in their own teachings.

Teachers often use graphs and diagrams in order to illustrate mathematical concepts and thinking. It is noted however, that sometimes learners' attendance to the particularity of these visual aids are narrow and rather limited (Yerushalmy, 2005). Yerushalmy & Chazan (1990) grouped visualisation obstacles according to: (1) the particularity of diagrams, (2) the perception of standard diagrams as models as described by Hershkowitz (1989), and (3) the inability to 'see' a diagram in different ways. For example, learners find it difficult to move their attention from different parts of a diagram to the diagram as a whole. As a result, they struggle to embrace the diagram in the process of problem solving.

In the discussion of difficulties around visualisation, Arcavi (2003) noted that:

> What we see is not only determined by the amount of previous knowledge which directs our eyes, but in many cases it is also determined by the context within

> which the observation is made. In different contexts, the "same" visual objects may have a different meaning even for experts" (p.237).

Diagrams need to be appropriately "read" and cognitive processes are needed in order to understand and make sense of them (Eisenberg & Dreyfus, 1991). Even though this study advocates the benefit of using visualisation in mathematics, Eisenberg & Dreyfus (1989) indicated that many learners preferred algorithms to visualisation processes. They further revealed that thinking visually demands high cognitive skills from learners. Apart from reasoning from a conceptual understanding point of view, visualisation may not always have procedural routines to rely on, as in the case of algorithms where formal symbolic approaches and procedures are clearly set out.

Eisenberg & Dreyfus (1991) identified three reasons why learners are reluctant to visualise: "a *cognitive one* (visual is more difficult), a *sociological one* (visual is harder to teach) and a *cultural one* - related to beliefs about the nature of mathematics (visual is not mathematical)" (p. 30).

Cognitive difficulties refer to whether a visual is easier or more difficult to work with. Visual thinking requires higher cognitive demand than algorithmic thinking. Learners often experience difficulties extracting or translating information from visual representations due to the fact that visual representations require learners to utilize a variety of information, knowledge and skills. Moreover, learners who have been taught mathematics analytically find visual presentations more difficult and risky, so they prefer an analytic approach (Arcavi, 2003).

Sociological difficulties refer to the issue of teaching. School mathematics is sequential and algorithmic and many teachers believe that sequential analytic representations are more pedagogical and efficient (Arcavi, 2003). For example, a learner who likes to solve mathematical tasks by transferring questions into images may get marks for the tasks in the classroom because he/she can explain to the teacher the meaning of the images and how they came about. However, the same learner may fail the same task in the final examination because his/her solution strategies are not valued by the teacher (marker). In my experience, in many cases teachers expect learners to solve the task using standard algebraic methods. Many mathematics teachers prioritise mathematical rules rather than visualisations. They do not consider using images because they were not valued throughout their own schooling. They are not aware that in the early stages of development of mathematical theories, visualisation is a fundamental source of skill. To deny learners images or visuals in mathematics is like cutting them off from the historical roots of mathematics (Tall, 2009).

Cultural difficulties refer to beliefs and values held about what mathematics and doing mathematics means. It is about what is legitimate or acceptable and what is not. The idea that mathematics should be communicated in a non-visual framework is commonly shared in the mathematics community (Guzmán, 2002). Presmeg (1997) calls this attitude 'devaluation of visualisation', and it leaves little room for classroom practices to incorporate and value visualisation as an integral part of doing mathematics. If this 'devaluation' of visual thinking is passed to learners it can result in learners hesitating to use a visual approach (Eisenberg & Dreyfus, 1989) even when they become teachers.

2.6 VISUALISATION PROCESSES IN MATHEMATICS

Mathematics education researchers such as Hershkowitz (1989); Zimmermann & Cunningham (1991); Zazkis, Dubinsky & Dauterman (1996) and Makina (2010) categorized visualisation processes into physical and mental visualisation processes. With mental visualisation processes, the image is formed and processed mentally. With physical visualisation, the image is formed with pencil and paper, or with the aid of technology. Both images can be used for mathematical discovery and understanding. According to Kosslyn (1994), mental image processes are an important exercise because they help a person to, for example, see, prepare, or create, whether as a professional architect, engineer, or computer programmer, or simply to arrange furniture in a classroom. Kosslyn (1994) defined four classes of mental image processes:

- generating an image,
- inspection of an image,
- transforming an image,
- operating on an image or image use.

The creation of appropriate mental image processes is largely left to the learners, and from my experience, many of them fail to build appropriate representations that could help them acquire mathematical conceptual understanding. Of course, no one can see another person's mental imagery. That is why a model or a picture is needed to externalise mental image processes (Jonassen & Henning, 1999). In other words, a connection between mental and physical visualisation processes is important in any mathematics classroom. The use of images on a computer or video screen (Phillips & Pead, 1994) is one way of linking internal and external mathematical visualisations. However, they warn that it is not enough to make an animated film say, that depicts the mental images and just show this to your learners. Something more has to be done for learners to acquire images that are useful and more recognised. This has to do with the kind of activity that the teacher incorporates in her teaching. The teacher can use different screen images; however, this process requires computer skills such as drawing or painting skills. The combination of drawing tools and skills helps teachers to generate and visualise ideas and share them with learners. Software has been designed to convert mathematical mental representations into static or dynamic visual representations, resulting in better understanding.

According to Kosslyn (1994), mental representations (visual imagery) are important because they facilitate visualisation processes whereby images are generated, inspected, transformed or used for mathematical understanding. In his work, Kosslyn (1994) proposed four cognitive steps involved during visualisation processes. These are *image generation, image inspection, image transformation* and *image use:*

- **Image generation** occurs when a person produces a picture in his/her mind. In this process, the learner pictures himself/herself in an activity in which he/she is doing the moving of pictures or images in his/her imagination. The generation of an image is the most important of the four processes because there would be nothing to proceed with without the initial image. When teachers draw images, they operate from the constructed images and the nature and quality (clear with details or fuzzy and general) of the image will affect their concept or topic explanation.
- **Image inspection** involves examining an image in order to answer questions about it. It is therefore important for remembering shapes. Shapes during this process are recognized in terms of their properties. During image inspection, learners are estimating sizes,

creating, recognizing and naming shapes. This process allows learners to connect visual images and abstract conceptualizations by seeing, looking for and describing patterns as basic forms of mathematical thinking.

Image transformation is changing or operating upon an image, changing it into other related shapes. Proponents of dynamic software (Hohenwarter & Fuchs, 2005) would add to Kosslyn's model that it allows image transformation to be observed directly. Teachers can use dynamic software to change the image dynamically, for example by dragging a vertex of a square to form a kite, or holding and dragging a side of a square to form a rectangle or a parallelogram. Transformation such as sliding or turning is used to move and change geometric shapes in a way that is mathematically precise. It is however the responsibility of a teacher to make sure that learners experience different types of shapes, so that they do not form a narrow idea about any class of shapes. For example, learners might transform a rectangle or to a square or to a kite or to any quadrilateral figure. Clements, Battista, Sarama, & Swaminathan (1997) found that practical transformations in various situations such as turns, slopes, meeting, bends, directions, corners and opening could help learners understand angular relationships. Gradually learners develop general angle concepts by recognising common features of these situations. By practising visualisation processes, a teacher is able to explain why a shape belongs to a certain category. Computer-based shape manipulation and navigation of environment can help learners to experience visualisation processes (Clements & Sarama, 2004).

- **Image use** occurs when an image is employed in the service of some mental operation that includes comparing properties of images or answering questions about an image.

In relation to the above visualisation processes, Kosslyn (1994) suggests that these processes are hierarchical. A learner has to generate the image first, be able to inspect it, transform it and then be able to use it. He gives a case where a learner generated an image and failed to inspect or describe it in a way that would aid him/her to solve the problem. In such cases, there is a need to regenerate another image and take it through the processes above. This is because our human visualisation, even the apparently superficial phenomenon that we call ”vision” in its more physiological sense, is not a process that merely involves the optical processes of our eyes. It is much more complex, since it entails in a quite important form, the activity of our brain (Guzmán, 2002). This denotes that visualisation is not an immediate vision of the relationships, but rather an interpretation of what is presented for our contemplation. This consideration is one of the reasons why the introduction to visualisation, for example in teaching and learning of mathematics, is not an easy task.

2.7 TECHNOLOGY IN NAMIBIAN EDUCATION

Over the last few decades, computer technology has become very important in everyday life. Nowadays, computers are vital for business and the economy. Computer literacy is considered a very important skill in Namibian society, especially for young people who have grown up having access to computers and other technology devices at home. From my personal experience, computers are mostly used for text processing, information storage, basic calculations like using excel for ordering and budgeting, and to produce statistical data. However Information and Communication Technology (ICT) has been proven to be a useful tool in supporting and transforming teaching and learning (NCTM, 2000). In mathematics classrooms for example, ICT can help learners and teachers to perform calculations, analyze data and explore mathematical concepts, thus increasing the conceptual understanding in mathematics.

The Ministry of Education in Namibia has developed an ICT initiative in Education organization based on the ICT for education policy called TECH/NA! (Namibia:Ministry of Education Sport and Culture, 2005). TECHN/NA! is a comprehensive implementation strategy that the Namibian Ministry of Education developed based on its ICT for education policy. The main goals of TECH/NA! are “to equip educational institutions with hardware and software, connectivity, curriculum, content and technical support; educate school administrators, staff, teachers and learners in ICT literacy and ICT integration across the entire curriculum” (TECH/NA!, n.d. p.2). This initiative aims to turn ICT into a tool that provides new opportunities for teaching and learning. The policy places emphasis on the pedagogical use of ICT as an integrated tool in the teaching and learning processes at all levels in education. Looking at the increase of computer technologies in everyday life, several educational organizations started to develop technology-related software (Lawless & Pellegrino, 2007, p. 576). They are trying to foster the integration of new technology into teaching and learning. For example, the National Council of Teachers of

Mathematics (NCTM), which is the world's largest association of mathematics teachers, declared technology as one of their six principles for school mathematics (NCTM, 2000).

As per NCTM principles, "technology is essential in teaching and learning mathematics; it influences the mathematics that is taught and enhances learners' learning" (NCTM, 200 p. 11). Teachers and learners can benefit in different ways from technology integration into everyday teaching and learning. New learning and teaching opportunities are provided in technological environments, potentially engaging learners of different mathematical skills and levels of understanding in mathematical tasks and activities (Hollebrands, 2007). Visualisation of mathematical concepts and exploring mathematics in multimedia and technology-rich environments can foster learners' understanding in a new way. Technology is a useful tool for mathematics teachers in helping them to present mathematics less passively (Van Voorst 1999), not as a set of procedures, but to involve learners more actively in reasoning, exploring, solving problems, generating new information, and asking new questions. By integrating technological educational tools into everyday teaching practice, teachers can provide creative opportunities for supporting learners' learning and fostering the acquisition of mathematical knowledge and skills. Both weak and strong learners can be provided with activities that meet their special needs and help them to overcome their individual difficulties.

The other benefit of technology is that "learners can develop and demonstrate deeper understanding of mathematical concepts and are able to deal with more advanced mathematical contents than in 'traditional' teaching environments" (NCTM, 2000, p. 24). For example, Laborde (2001) distinguishes different types of software roles: Firstly, the software could facilitate "material aspects of the task while not changing it conceptually". Secondly, the software could be used as a 'visual amplifier' in order to facilitate observations, such as identifying properties of geometric figures and many more. Thirdly, the software allows learners to solve mathematical tasks in different ways. Finally, dynamic geometry software, for example, allows for the creation of new mathematical problems that could not be created in classrooms without technology. This is what Hohenwarter & Hohenwarter (2009) refers to as creative opportunities that support learners' learning and fostering the acquisition of mathematical knowledge and skills.

2.7.1 Integrating Technology in learning and teaching Geometry

Geometry is a substantial part of the mathematics curriculum that deals with many images and diagrams. It is the study of shapes and space (Gueven & Temel 2008). Teaching and learning geometry is not an easy process, thus a large number of learners fail to develop adequate understanding of geometry concepts, geometry reasoning and geometry problem-solving skills (Battisa, 1999; Idris, 2006). Without visualisation, learners cannot fully appreciate the natural world through geometry. Research carried out by Hodanbosi (2001) using Geometer's Sketchpad (GSP), a dynamic geometry software, found that learners in the GSP group had higher significant achievement scores on the Geometry Achievement Test than those in the traditional group. There are various types of commercial software packages available for teaching and learning mathematics in the open market such as *Geometer's Sketchpad, Derive, Cabri, Matlab, GeoGebra, Autograph* and others. Schools are encouraged to acquire such software in order to use them in the classroom. Cost however is usually the determining factor in acquiring new teaching and learning aids in schools, so some schools may end up not acquiring these resources.

Research on dynamic geometry software has shown that the software can offer an effective impact on mathematics education and has the potential to promote learner-centered education and active learning (Hodanbosi, 2001; Mohammad, 2004). Software can enhance learners' ability to visualise many mathematical concepts and in the process improve conceptual understanding. However, providing new technology to teachers does not guarantee its successful integration into mathematics teaching and learning (Mohammad, 2004). Some ineffective use of technology has been reported by the NCTM (2000). One reason frequently cited is that teachers are not trained in utilizing technology in the classroom context. This means that teachers need to be trained not only in the use of new software tools but also by introducing them to methods of how to effectively integrate technology into their teaching practices. This calls for programs such as continuous professional development in order to support teachers with this task.

Furthermore, teachers need to be prepared for the increasing complexity of technology and challenges in modern classrooms. The views of integrating technology in learning and teaching mathematics are worth heeding. This study focuses specifically on how *GeoGebra*, a free mathematics software, can be used as a visualisation tool to develop conceptual understanding of geometric shapes. With this in mind, the next discussion turns to *GeoGebra* as an affordable software in the field of mathematics education.

2.8 GEOGEBRA

GeoGebra is a freely available Dynamic Mathematics Software (DMS) package for teaching and learning mathematics from primary school through to university level. It is easy to use and includes basic features of Computer Algebra Systems (CAS) to bridge geometry, algebra and calculus (Hohenwarter & Fuchs, 2005). A study done by Royati, Ahmad, & Rohani (2010) on the effect of *GeoGebra* software on mathematics achievement has found that lessons incorporating *GeoGebra* are more effective in bringing about conceptual understanding than traditional instruction alone. However, in Namibia research on the effectiveness of integrating *GeoGebra* as a visualisation tool in teaching and learning mathematics is still limited. In the following discussion, some ideas are presented about *GeoGebra* and its applications.

2.8.1 *GeoGebra* and its application

GeoGebra is free, with no license issues associated with its usage. It can be freely downloaded from **www.*GeoGebra*.org** and once installed on a personal computer (PC), it is accessible at all times, because it does not need internet to access it. Teachers can use it in their classrooms for teaching at any time. *GeoGebra* has the feature of representing every object and every input in an algebra window. These objects and input have a simultaneous and corresponding object in the geometry window as illustrated in Figure 2.5. The drag mode of *GeoGebra* allows free movement between windows and automatically adjusts to any change in the algebraic representations. This provides good visualisation opportunities between geometry and algebra relationships and is far more accurate and precise than using paper and pencil (Laborde, 2001; Chrysanthou, 2008).

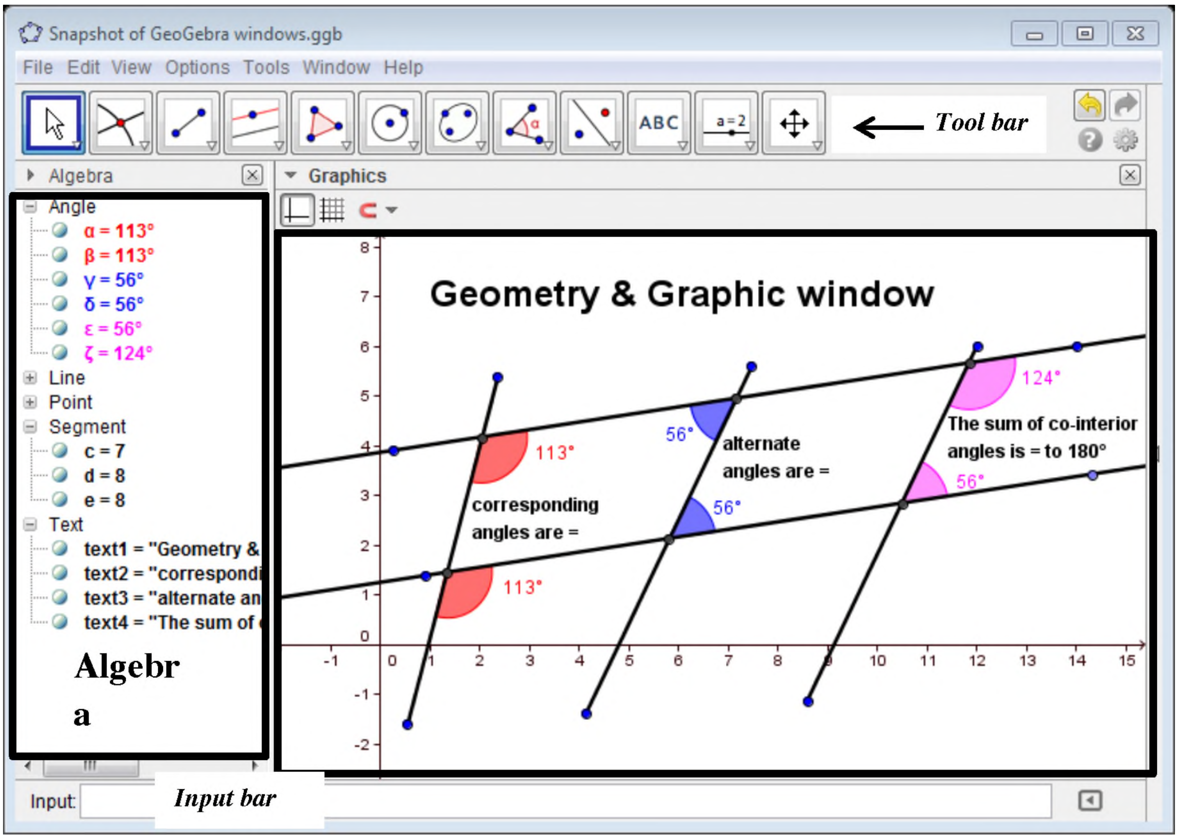


Figure 2.5: Snapshot of *GeoGebra* window

GeoGebra is a powerful tool for helping teachers to scaffold and accommodate the diverse needs of learners. Teachers can follow and assess individual learners' processes of thinking on a shared practice platform (Bruner, 1961; Preiner, 2008; Sangwin, 2007).

Learners can be engaged in deep mathematical modeling, problem exploration, and open-ended questioning. They can use *GeoGebra* for construction of mathematical objects. In the process they may make mistakes, but this is part of learning mathematics (Bu, Mumba, & Alghazo, 2011). The potential of engaging learners in building autonomy empowers them, and enables them to take responsibility for their own learning. It also increases their ability to take control of what they are learning (Doerr & Pratt, 2008).

From a task perspective, *GeoGebra* deeply enriches and enhances the learning environment with its multiple representations. Web-friendly features and customizable tools extend the scope of teaching and learning mathematics beyond the classroom walls (Lu, 2009; Bu, Mumba, & Alghazo, 2011).

Experiences from teachers workshops Hoenwater (2004) suggests that most teachers who are introduced to *GeoGebra* tend to use it as a tool to produce pictures only. As a first approach, they used it to create sketches and constructions for presentations, handouts, notes, or quizzes. This is another uniqueness of *GeoGebra* that even beginner teachers can use the software to produce handouts until such a time when they are comfortable to use it as a teaching and learning tool. According to Hohenwarter & Fuchs (2005) using the software for creating sketches of handouts and notes does not require technical support in the classroom and therefore, could be implemented by almost every teacher who is willing to enhance their everyday teaching of mathematics.

2.8.2 Why *GeoGebra* for teachers?

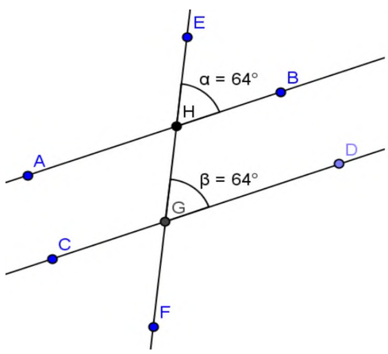
Teachers can use *GeoGebra* in many ways. They can use it for demonstration purposes, as a visualisation tool, as a discovering tool, as a tool for preparing teaching materials and as a construction tool (Hohenwarter & Fuchs, 2005). According to KoKol-Volji (2007), the construction of geometric objects is an important step in developing conceptual understanding, the step that is often lost when learners only use ready-made worksheets.

Bu & Schoen (2011) assert that *GeoGebra* is particularly well suited for teachers to represent diagrams in different ways on the screen and dynamically transform them. Entire diagrams and parts thereof can be moved around and manipulated in many ways. As a consequence, learners are able to gain rich experiences from the variety of forms of images. The dynamic nature of the software offers exciting opportunities for teaching and learning mathematics in schools. According to Hohenwarter & Hohenwarter (2009) and Stols (2009), if used effectively it helps learners and teachers to specifically make connections between **Geo**metry and al**Gebra.**

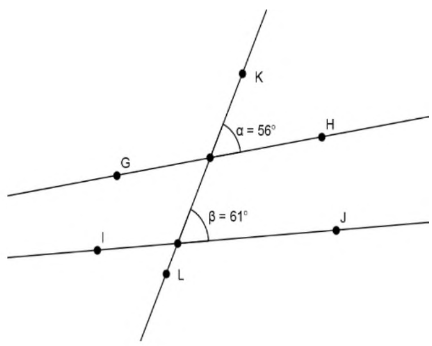
The ability to make mathematical connections is one of the key indicators of conceptual understanding.

2.8.3 The role of *GeoGebra* in Geometry

Using *GeoGebra*, teachers and learners can engage in a variety of exploratory activities such as drawing, constructing, testing, creating and manipulating any plane figure they desire to solve (Hohenwater & Jones, 2007). The software is designed to generate very accurate diagrams and images. For example, a teacher wishes to explain the behaviour of corresponding angles that are formed when a transversal crosses two lines. Corresponding angles are the ones at corresponding locations of the transversal line such as α and β as illustrated in Figure 2.6. In a textbook or on a chalkboard these representations are static. The dynamic nature of *GeoGebra* is such that the sizes of angles α and β can be manipulated by re-orientating lines AB and CD to different positions. In the process, the teacher can ask learners to immediately observe the behaviour of angles α and β. They then discover that once AB and CD are parallel lines, the corresponding angles α and β are equal. Conversely, they can also discover that GH and IJ are not parallel because the corresponding angles are not equal as shown in Figure 2.6**.**



Corresponding angles in parallel lines are equal



Corresponding angles in non-parallel lines are not equal

Figure 2.6: Corresponding angles in parallel lines and in non-parallel lines

As illustrated in Figure 2.6 above, *GeoGebra* allows the direct manipulation and reorientation of lines and points by its drag function. The movement produced by the drag function is a way to visualise the properties that define the figure when certain parameters of the lines are changed (Chiappini & Bottino, 1999). Learners can visualise, construct and manipulate mathematical concepts. The dynamic learning environment can enable learners to act mathematically, and to seek relationships between objects, that would not be as intuitive with static paper and pen representations, such as addressing *what-if* and *what-if-not* questions (Brown & Walter, 2005).

The notion that the use of *GeoGebra* encourages visualisation and enables teachers and learners to explore mathematical relationships and concepts in a dynamic manner aligns well with teaching for conceptual understanding. Researchers like Kokol-Volji (2007), find value in dynamic features of the software because one is able to generate more representatives of mathematical ideas than static representations. One of the conceptual understanding indicators is the ability to represent mathematical situations in different ways and knowing how different representations can be useful for different purposes. Kilpatrick, Swafford, & Findell (2001) assert, "It is important to see how the various representations connect with each other, how they are similar, and how they are different" (p.115).

2.9 CONCEPTUAL UNDERSTANDING

According to Barr, Graham, Hunter, Keown, & McGee (1997), a concept is a general idea, understanding or thought embodying a set of things that have one or more properties in common. Conceptual understanding can also be referred to as conceptual knowledge which Hiebert & Lefevre (1986), characterized as knowledge that is rich in relationship and can be thought of as a connected web of knowledge. Kilpatrick et al. (2001) describe "conceptual understanding as the comprehension of mathematical concepts, operations, and relations" (p.5). It involves the ability to integrate and connect mathematical ideas. These may be ideas about shapes and space, measures, patterns, functions, connections, proofs etc. Conceptual understanding enhances learners' confidence, which then provides a base from which they can move to further levels of understanding.

Learners with conceptual understanding know more than isolated facts and methods. They understand why mathematical ideas are important, and the contexts in which they can be used. Learners are able to organise their knowledge into a coherent whole, which enables them to learn new ideas by connecting those ideas to what they already know. According to Kilpatrick et al. (2001), one of the significant indicators of conceptual understanding is being able to represent mathematical situations in different ways and knowing how different representations can be useful for different purposes. These different representations include different visualisations such as diagrams, computer images and others. Knowledge that has been learned with understanding provides the basis for generating new knowledge and solving new and unfamiliar problems. Kilpatrick et al. (2001) believe that, "when learners have acquired conceptual understanding in an area of mathematics, they see the connections between concepts and procedures and can give arguments to explain why some facts are consequences of others" (p.119).

This study adapted five key conceptual understanding indicators from Kilpatrick et al. (2001). These are: (1) Connecting mathematics to prior knowledge, (2) Justifying and explaining mathematical ideas and solutions, (3) representing mathematical concepts in different ways, (4) Connecting ideas and concepts in mathematics and (5) connecting mathematics to real world. A brief description of the five indicators follows.

2.9.1 Connecting mathematics to prior knowledge

This indicator includes the ability of a teacher to make connections between the mathematics that learners are learning and what they already know. This approach creates an opportunity for learners to be actively building new knowledge from past experiences. This involves adapting acquired knowledge to new situations, and using it to solve new mathematical problems (Kilpatrick et al., 2001). Prior knowledge makes learning new knowledge easy because new information is connected to what is already known and forms something like a bridge between the old and the new information. It is easier to learn when we already have something on which to build our new information. Not only does it make it easier for us to learn, prior knowledge also makes it easier for us to remember what we have learned. When we remember what we have learned, we are able to use it, even outside classroom settings. Teachers can use simple graphic forms such as visual analogies, Venn diagrams or concept maps to engage learners' prior knowledge, to associate new content with topics learners already know (Gentner, 2010). For example, learning about angle properties, a teacher needs to connect primary grades contents such as identifying, comparing, sorting, and classifying shapes. Figure 2.7 shows the model of how *GeoGebra* can be used to construct a visual concept map of quadrilaterals. The teacher can use this concept map to demonstrate how other quadrilaterals relate to a square, which is likely to be a well-known quadrilateral from the primary phase. By using the drawing tool and dragging mode in *GeoGebra*, a teacher can explain more effectively the properties of each quadrilateral, and how they relate to other quadrilaterals. This can be done through visualisation processes such as image generation and image transformation as illustrated in Figure 2.7. The presence of the grid in the diagram serves as a construction guide that helps the teacher to create accurate shapes. It also assists the teacher in describing the relationship between the quadrilaterals.

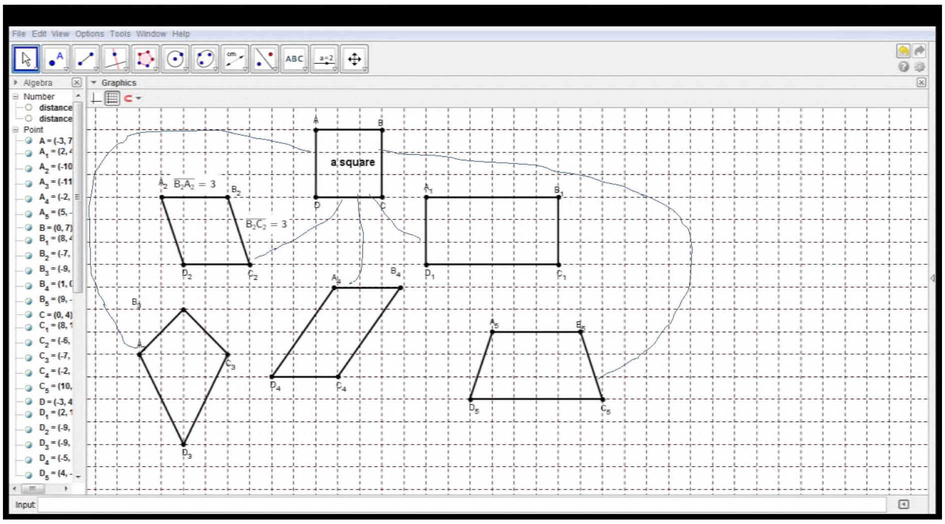


Figure 2.7: Snapshot of *GeoGebra* showing a concept map of quadrilaterals

Visually identifying similarities and differences among images and concepts can increase learners' thinking and understanding. The interaction between the new and previous knowledge entails a change in learners' cognitive structure (Ausubel, 1968). This is in agreement with a constructivist approach which advocates that learning is acquired when new knowledge is connected to previous relevant knowledge which already exists in learners' cognitive structure and is assimilated in it (Jonassen & Henning, 1999).

2.9.2 Justifying and explaining mathematical ideas and solutions

This conceptual understanding indicator refers to the ability to clearly explain and articulate mathematical concepts and ideas. Leaners are able to manipulate representations or compare concepts, and apply facts and definitions to justify solutions to mathematical problems (Kilpatrick et al., 2001).

Hanna (2000) suggests that dynamic software has the potential to encourage both exploration and proof, because it makes it easy to pose and test conjectures. This is particularly important in geometry where dynamic geometry software helps learners and teachers to perform geometric constructions with a high level of accuracy. The software expands the scope by making the ruler and compass constructions more dynamical (Gawlick, 2003). Thus, teachers and learners can easily test conjectures by exploring given properties of the constructions. In line with these claims, De Villiers (1995) asserted that explorers are able to investigate whether conjectures are true or false through continuous variations of geometric configurations. Angle properties such as the size of the exterior angle of a triangle is equal to the sum of its interior opposite angles, can be demonstrated very effectively using *GeoGebra* visualisations as illustrated in figure 2.8.

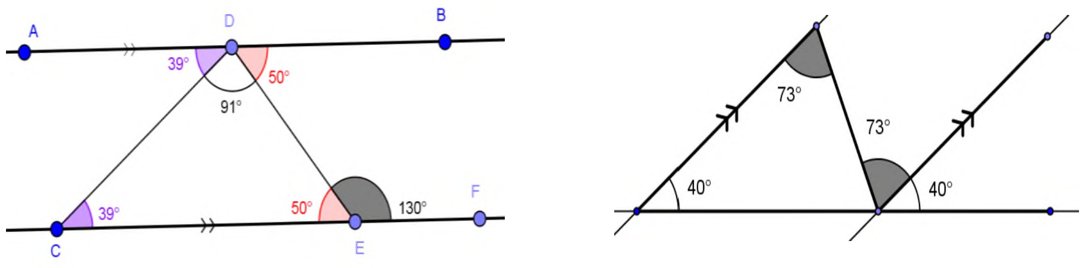


Figure 2.8: Exterior angle is equal to the sum of the two opposite interior angles in a triangle

Learners can observe the angles as the triangle is changed by dragging any of its vertices. The same construction, or a similar one, can be used to explain the proof that the sum of the three angles of a triangle is 180°.

2.9.3 Representing mathematics concepts in different ways – multiple representations

Using multiple representations in mathematics refers to teachers' abilities to show different representations of the same mathematical concepts. Good mathematics teachers use visuals, manipulatives and motion to enhance learners' understanding of mathematical concepts. National organizations for mathematics, such as the National Council for the Teaching of Mathematics (NCTM) and the Mathematical Association of America (MAA) have long advocated the use of multiple representations in teaching mathematics. The use of multiple representations is an important aspect of any teacher practice. Teachers can use multiple representations to explore and explain mathematical ideas (Leinhardt, Putnam, Stein, & Baxter, 1991). It is important that mathematics teaching and learning become more visual because there is not a single idea or concept that cannot be illustrated visually (Boaler, Chen, Williams & Cordero, 2016). *GeoGebra* provides multiple representations of mathematical objects, which potentially foster learners' understanding of mathematical concepts (Duval, 1999). Using multiple representations can assist learners in learning important geometric concepts, allowing them to easily manipulate models of geometric shapes and figures, which are supposed to be difficult to demonstrate with concrete materials or hand-drawn representations. The manipulation carried out in one window of *GeoGebra* is immediately updated in the other window and the impacts of actions taken upon one of the representations can be observed on the corresponding depiction. This feature of *GeoGebra* allows the teacher to demonstrate the conceptual links between the representations of a mathematical concept and eventually promotes learners' conceptual understanding.

2.9.4 Connecting ideas and concepts in mathematics

This is the ability to integrate related mathematical concepts and principles. It refers to teachers being able to discuss similarities and differences of representations and how they connect with each other (Kilpatrick et al., 2001). Learning is a process of drawing connections between what is already known or understood and new information. People make connections and draw conclusions based on a sense of what they already know and have experienced. Similarly, for learning to occur, facts, concepts and ideas must also be stored, connected to other facts, concepts, and ideas, and built upon (Jonassen & Henning, 1999). Knowing in advance what the big ideas are and how they relate to each other, help teachers to prepare lessons that make sense to learners. *GeoGebra* visualisations have the potential to create such opportunities where learners can make connections between the concepts both procedurally and conceptually.

2.9.5 Connecting mathematics to real world

This conceptual understanding indicator indicates the ability of teachers to connect and link mathematical knowledge to the outside world and see the practical relevance of this knowledge (Kilpatrick et al., 2001). For example in geometry, we learn to understand shapes of different objects, directions and locations in our world and the relationships between them. Connecting mathematics to the real world changes learning from a mere memorization act to being able to solve real world problems and communicating ideas with people around the world.

Geometric shapes can be used to represent objects in the world around us. Geometrical visuals can be composed into other shapes and structures, for example packaging boxes and tiles. One cannot speak of tiles without referring to the topic in mathematics called tessellation. Tessellations can be seen all around us, on the pavements, in our bathroom floor tiles and in hardwood floors, to name just a few. *GeoGebra* has functions that enable teachers to draw any geometric shape. When a teacher is able to draw (or sketch) any shape on the computer screen, then he/she can use such an image to demonstrate patterns such as basic tessellations. By so doing conceptual understanding is likely to be enhanced.

This study is framed by the theory of constructivism. Thus in the next discussion I first discuss constructivism as a learning theory, followed by a discussion of how constructivism and visualisation theories can be combined to enhance teaching for conceptual understanding.

2.10 THEORETICAL FRAMEWORK

2.10.1 Constructivism

Constructivism is a learning theory describing the process of knowledge construction in which individuals make meanings through the interactions with each other and with the environment they live in (Piaget, 1967). Constructivists believe that knowledge should not be just deposited into the learners' minds; instead, it should be constructed by the learners through active involvement in the learning process. Hausfather (1996) noted that constructivism is not a method but is a theory of knowledge and the learning should inform the practice but not a prescribed practice of knowledge. In other words learners are asked to deliberately take action to create meaning from what they are studying while teachers become facilitators and guides, rather than presenters of knowledge.

This study is informed by social constructivism (Papert, 1980). Constructivism places an overt emphasis on learners' creative performance, in this study expressed by the active exploration, construction, and modification of digital artefacts (Kafai & Resnick, 1996). Digital media and tools can be used by learners to construct meaningful objects as the tangible outputs of their meaning-making processes (Kynigos, 2007). Using *GeoGebra* as a visualisation tool can support visual thinking in the same manner because the software can facilitate the social interaction among learners and the teacher as they go through the visualisation process of image generation, image inspection, image transformation and image use.

Conceptual understanding and visualisation processes, which are the focus of this study, align well with social constructivism and active engagement learning environments. During the image generation process, the teacher generates an accurate image using *GeoGebra* to develop the mathematical basic competency to be acquired by learners. In this process, *GeoGebra* can facilitate the social platform whereby visualisation processes such as image inspection, image transformation and image use are observed and discussed by everyone in the class. However this activity must be carefully designed and developed by the teacher to engage learners in observing mathematical ideas and support them in learning mathematics (Hohenwarter, Jarvis, & Lavicza, 2009).

2.10.2 Constructivism and visualisation

Using *GeoGebra* as a visualisation tool in mathematics education aligns well with constructivism as a theory of learning. The theory claims that learners learn mathematics through active construction of their own knowledge, rather than receiving it as a finished product from the teacher or texts (Ernest, 1991). According to Vygotsky (1962), learners cannot be given knowledge, instead, they learn best when they discover things, build their own theories and try them out rather than simply consuming what they are told or instructed. Vygotsky strongly argues that, "direct teaching of concepts is impossible and fruitless. A teacher trying to do this accomplishes nothing but empty verbalism, a parrot-like repetition of words by the child, simulating a knowledge of the corresponding concept but actually covering up a vacuum" (Vygotsky, 1962, p.83). Using interactive software encourages leaners to interact with the mathematical concepts in ways that are exploratory. It encourages learners to construct knowledge by active engagement. This is central to my study as I intend to use *GeoGebra* interactively in such a way that learners explore the concept of angle properties using the visualisation processes and acquire the conceptual understanding of the said concept. It is envisaged that the use of *GeoGebra* in my study will enable teachers and learners to use visualisations in different ways on the computer screen. Learners will transform images to make connections and discoveries. Through activities that are consistent with social constructivism and active learning interaction, learners have the opportunity not only to learn mathematical skills and procedures, but also to explain, discuss and justify their own thinking and observations (Silver, 1996). This is supported by Hyles (1991), who argues that mathematics lessons can be enhanced by using computer technology that encourages social interaction and collaboration. The use of technology in the teaching and learning of mathematics offers an abundance of opportunities to make the classroom an interesting and inspiring space for learning. *GeoGebra*, if harnessed appropriately, is particularly well suited to facilitate a learning process that is interactive and activity-based (Hohenwarter et al., 2009). A key element of this study is the teachers' and learners' manipulation of *GeoGebra* images in combination with interacting with one another.

2.11 CONCLUSION

The purpose of this chapter was to provide a contextual and theoretical background to the study. It started with the definition and types of visualisations followed by the role of visualisation in teaching and learning mathematics. This was followed by the challenges of visualisation in mathematics. Visualisation processes were then discussed. The integration of technology in teaching and learning mathematics was discussed, with specific reference to *GeoGebra* as a mathematics visualisation tool. In this chapter, the theoretical issue was also explored in order to gain theoretical insight into how technology is related to visualisations in mathematics. In the light of this, the theory of constructivism and how it informed this study, was discussed.

CHAPTER THREE

3. METHODOLOGY

3.1 INTRODUCTION

Education research is a process of gaining an in-depth perspective of human experience (Merriam, 2009). The focus of this study was to analyze how *GeoGebra* could be used by mathematics teachers as a visualisation teaching tool to enhance conceptual understanding of angle properties in Grade 9 geometry. The structure of this chapter consists of articulating the research goals, describing the chosen research paradigm, the research methods used and the research design, discussing the techniques, the selection criteria of the participants and explaining the analytical framework used to analyze the data. The chapter then concludes with a discussion on validity and ethics used to enhance the quality of the research.

3.2 RESEARCH GOALS

The first goal of this case study was to analyse the role of *GeoGebra* as a visualisation tool by observing selected teachers teaching Grade 9 learners using *GeoGebra*. The second goal was to analyse how these teachers used *GeoGebra* visualisations to enhance conceptual understanding of geometric angle properties. This was done with the help of the following two specific research questions. What are selected teachers' perceptions and experiences of:

1. the role that *GeoGebra* visualisations can play in developing conceptual understanding in the teaching of properties of shapes in Grade 9 geometry?
2. how *GeoGebra* can be used as a teaching tool to enhance learners' conceptual understanding of angle properties in Grade 9 geometry?

3.3 RESEARCH PARADIGM/ORIENTATION

This qualitative study is located in the interpretive paradigm. According to Cohen, Manion, & Morrison (2007), interpretive research is characterized by a concern for the individual. Human actions are inseparable from meaning and experiences. Bertram & Christiansen, (2014) explains that:

> Within the interpretive paradigm, researchers do not aim to predict what people will do, but rather to describe and understand how people make sense of their worlds, and how they make meaning of their particular actions. The purpose is to develop a greater understanding of how people make sense of contexts in which they live and work. (p.26)

In this study, I specifically wished to understand and make sense of how teachers used *GeoGebra* in their teaching and what their perceptions were of that experience. I wanted to research what meaning the participating teachers attached to *GeoGebra* as a visualisation tool. I also wanted to know how the teachers understood whether *GeoGebra* was an effective tool to develop conceptual teaching.

3.4 CASE STUDY METHODOLOGY

A case study methodology was adopted to generate the data for this research study. “Case studies investigate and report the real-life, complex dynamic and unfolding interactions of events, human relationships and other factors in a unique instance” (Cohen, Manion, & Morrison, 2011, p.289). The case in my study consisted of three mathematics teachers using *GeoGebra* as a visualisation tool to teach angle properties in Grade 9 geometry. The unit of analysis was the perceptions and experiences of the participating teachers using *GeoGebra* visualisations to enhance conceptual understanding.

3.5 RESEARCH DESIGN

The research process was designed around five phases:

3.5.1 Phase 1 – installation of *GeoGebra* and training of participants

In this phase, I installed the *GeoGebra* software onto 30 laptops that are housed in the computer laboratory at the participating school. These laptops are meant to be used by learners during mathematics lessons. In this phase I also trained my two co-participants in *GeoGebra*. The training programme consisted of 3 workshops where I trained the two colleagues on the intricacies of how to use *GeoGebra*. Integral to the training programme was creating an awareness of conceptual understanding in mathematics and how *GeoGebra* could support the development of conceptual understanding. In addition to drawing on my *GeoGebra* skills and knowledge, I used *GeoGebra* tutorial videos on YouTube to supplement my input. At the end of the training programme, one further workshop was held to design the four lessons that formed the heart of this study. These lessons incorporated *GeoGebra* to teach angle properties as articulated in Phase 3. Initially each teacher was tasked to plan and design one lesson, which he/she would pilot in Phase 2.

3.5.2 Phase 2 – planning and piloting

During this phase, learners were introduced to *GeoGebra*. All three Grade 9 classes at the participating school were trained on how to use the *GeoGebra* software. This was a very challenging process as it was probably the first time that learners had worked with computers in this manner. It was the intention of the intervention in Phase 3 for learners to be fully aware and actively involved in using *GeoGebra*, and not, as is often the case, simply watching the teacher using software packages. The Pilot Curriculum Guide for Formal Basic Education under the Namibia. Ministry of Basic Education Sports and Culture [MBESC], (1996) emphasises that "Learners learn best when they are actively involved in the learning process through a high degree of participation, contribution and production" (p.26). It is thus important that the approach to any teaching encourages active involvement and participation of learners. There were a number of learners who were computer literate. I used them to assist in teaching those who were unfamiliar with using computers. I also used the same tutorial videos used during the teachers' training workshop in Phase 1. A further activity in this phase was the piloting of one lesson per teacher. The purpose of piloting was firstly to experience the practicalities of using *GeoGebra* as a visualisation tool to teach mathematics. Secondly, it was to test the appropriateness of my analytical tools. Thirdly, it was to see how comfortable the learners were to work with *GeoGebra* on the laptops. The piloted lessons were also used to experiment with optimising the position of the video camera in the classroom to cause as little disruption to the teacher and the learners as possible. The three piloted lessons were reflected upon by the participants. These reflections then contributed to the final planning and design of the 4 final lessons that the three participants (my two colleagues and I) taught in Phase 3.

3.5.3 Phase 3 – implementation and video recording

In this phase, the planned lessons using *GeoGebra* as a visualisation tool were taught. The lessons were about angle properties of geometric shapes. The two participant teachers and I were each supposed to teach four lessons. However, due to some reasons one participant only managed to teach lesson one and two while the other two participants taught all lessons as per the study plan. These were: **Lesson 1:** angles formed within parallel lines, **Lesson 2:** angles in a triangle, **Lesson 3:** angles in a quadrilateral, and **Lesson 4**: angles in a complex shape (a combination of lessons 1, 2 and 3). That added up to a total of 10 lessons, and they were all videotaped. The use of videos recordings enabled me, the researcher, to re-examine the data as many times as needed from different points of view during the data analysis. According to Cohen et al. (2011)

> the recording can be viewed several times; it is not a 'once-and-for-all' observation. Audio-visual data collection has the capacity for completeness of analysis, and comprehensiveness of material, reducing the dependence on prior interpretations by the researcher. Video recording also enables several playbacks to be conducted, to scrutinize the data carefully. (p.470)

3.5.4 Phase 4 – video transcribing

In this phase, I transcribed all the video recordings of the lessons. During the transcribing process, I was already able to recognise and identify some emerging themes as per my analytical tools, see tables 3.1 and 3.2. The data in the video recordings were transcribed exactly as per each video, including the non-verbal actions such as *pointing to…, demonstrating, and using the drawing tool to… use the image to…* and other actions and expressions. The issue here is that it is not enough to transcribe the spoken words only; other data are important as well for data analysis and findings. Cohen et al. (2011) noted that "if the transcript is of the video tape, then this enables the researcher to comment on all of the non-verbal communication that was taking place in addition to the features noted from the audiotape" (p.427). I numbered each line of the transcript in order to quickly identify particular utterances in a particular transcript.

3.5.5 Phase 5 – teachers' perceptions and experiences

In this phase, I conducted one-on-one semi-structured interviews with my two co-participant teachers to follow up on what emerged in lesson presentations. According to Nieuwenhuis (2011) a semi-structured interview "is commonly used in research projects to corroborate data emerging from another data source" (p.87). Interviewing is a useful method since it allows the researcher to ask probing questions and to discuss research participants' perceptions and understandings (Bertram & Christiansen, 2014). The focus of this interview was to tease out the teachers' own perceptions and experiences about using *GeoGebra* as a visualisation tool for enhancing conceptual understanding in the topic of angle properties.

3.6 TECHNIQUES/TOOLS

The techniques used to obtain the necessary qualitative and interpretative data in this case study were observation and semi-structured interviews.

3.6.1 Observations

"Observation methods are powerful for gaining insight into situations" (Cohen et al., 2011, p. 474). It allows the researcher to hear, see and begin to experience reality as participants do (Bertram & Christiansen, 2014; Nieuwenhuis, 2011). Cohen et al. (2011) note that observation data:

> Enables researchers to understand the context of programmes, to be open-ended and inductive, to see things that participants might not freely talk about in interview situations, to move beyond perception-based data (e.g. opinions in interviews) and to access personal knowledge (p.456).

This research employed video recordings for observation purposes. Video observation is a suitable tool for gathering verbal and visual interactions between teachers, learners and the software, which in this case is *GeoGebra*. Video recordings are permanent data which made it possible for me to analyse teachers' practice at any time. Two participants were observed teaching four geometry lessons each and one participant was observed teaching two lessons. The lessons were for Grade 9 learners covering the topic of angle properties and all participants used *GeoGebra* as a visualisation tool. On average each lesson lasted for 40 minutes. With these data I was able to analyse how teachers used *GeoGebra* to visualise concepts.

Cohen et al. (2011) pointed out that " the distinctive feature of observation as a research process is that it offers an investigator the opportunity to gather live data from naturally occurring social situations" (p.456). The observation in this study specifically focussed on teachers' actions with respect to using *GeoGebra* as a visualisation tool. In particular, the observation focus was on visualisation processes i.e. ***image generation, image inspection, image transformation*** and ***image use*** as proposed by Kosslyn (1994), see Table 3.1 under *analysis*.

3.6.2 Interviews

The semi-structured interviews proved to be useful tools to provide teachers with the opportunity to talk about their perceptions and experiences when using *GeoGebra* as a visualisation tool to enhance conceptual understanding. The interviews helped me to describe teachers' perceptions and experiences of the study. Bertram & Christiansen (2014) described a structured interview as a conversation where the researcher designed particular questions to be answered by the respondent because the researcher has in mind particular information that he or she wants from the participants, and further explained that:

> Interpretivist research uses the interview method extensively. In working towards its aim of exploring and describing people's perceptions and understandings that might be

> unique to them. Interviewing is a useful method since it allows the researcher to ask probing questions and discusses research participants' understandings with them. (p.82)

I only interviewed my two co-teachers i.e. Teacher B and Teacher C. As mentioned earlier the individual interview was semi-structured whereby the participants responded to a set of predetermined questions (see Appendix C). However, I was flexible and probed further for clarifications and elaboration where necessary. The interview sessions were recorded by means of a voice recorder. With a voice recorder I was able to concentrate on listening and responding to the participant, without being distracted by needing to write extensive notes.

3.7 SELECTION OF PARTICIPANTS

The research was conducted with three teachers (two colleagues and I). The two colleagues were purposefully selected. Cohen et al. (2011) emphasised that in "purposive sampling, a researcher hand-picks the participants to be included in the sample on the basis of their typicality or possession of the particular characteristics being sought" (p.156). Purposive sampling thus enabled me to select participants that were most suited for this research project, viz. mathematics teachers who are interested in using technology in their teaching. As Nieuwenhuis (2011) reminds us, "purposive sampling means that participants are selected because of some defining characteristics that makes them the holders of the data needed for the study" (p.79). The participants in this case study were all mathematics teachers who have advanced level computer skills. One of the selected teachers was a presenter in the Oshana regional e-learning conference in 2015. This is how I came to know his ability to use technology in the mathematics classroom. The other participant and I are teaching at the same school, both being passionate about teaching mathematics using technology. It was however, important for the participants to have a shared understanding of *GeoGebra* and visualisation as well as teaching for conceptual understanding – hence the training programme of Phase 1. For the purpose of this study the participants have been coded as follows: Teacher A (TA) refers to myself and the other two participants are called Teacher B (TB) and Teacher C (TC). I used codes in order to protect the participants in terms of research ethics, confidentiality and anonymity.

3.8 ANALYSIS

Analysis means "a close or systematic study or the separation of a whole into its parts, for the purpose of study" (Bertram & Christiansen, 2014, p.115). "At a practical level qualitative research rapidly amasses huge amount of data" (Cohen et al., 2011, p.539). According to Bertram & Christiansen (2014) this involves organising and sorting data into codes or categories and then looking for patterns or relationships between these categories. In qualitative research, data analysis involves the process of organising the data to develop understanding. This includes "making sense of data in terms of the participants' definitions of the situation, noting patterns, themes, categories and regularities" (Cohen et al., 2011, p. 537).

The two concepts that were central to my analysis are visualisation processes (Kosslyn, 1994) and the concept of conceptual understanding (Kilpatrick, Swafford, & Findell, 2001). These two concepts framed my analytical tool as illustrated in Tables 3.1 & 3.2 below.

The analytical tool A was used to analyse the role of *GeoGebra* visualisation during the lesson presentations. This was done by observing visualisation processes as listed in the first column of Table 3.1. The observable criteria for analysing the visualisation processes are listed in the second column.

Table 3.1: Analytical tool A - visualisation indicators and their codes

Visualisation **processes**	**The role of *GeoGebra*** visualisation**s** –visualisation **processes - external indicators** *Generate the image – inspect the image – transform the image – use the image*
Image generation **Code (IG)**	*Generate the image* The teacher generates an initial image in *GeoGebra* to develop the mathematical idea at hand. This image forms the basis from which the teacher will then manipulate certain elements and properties to either demonstrate or develop the mathematical idea further.
Image inspection **Code (II)**	*Inspect the image* The teacher uses *GeoGebra* images to scan, examine and scrutinise in order to distinguish similarities and differences between them. The teacher uses *GeoGebra* images to reinforce differences and similarities of various mathematical concepts. Similarities and differences are identified on the display and discussed by the whole class. These differences and similarities are also demonstrated dynamically by manipulating the image. Mathematics discovery and concept understanding is thus enhanced.
Image transformation **Code (IT)**	*Transform the image* The teacher uses *GeoGebra* to dynamically change and transform an image to demonstrate certain properties of angles in shapes. Evidence such as rotation, enlargement and translation of angles and shapes are observed on the computer screen.
Image use **Code (IU)**	*Use the image* The teacher uses *GeoGebra* images to emphasise and develop the appropriate properties of angles and shapes. For example, how does a teacher manipulate features of a rectangle to show that it is also a parallelogram? How does a teacher explain that the sum of angles in any triangle is equals to 180˚? etc.

Note: All ten observed lessons were subjected to this analytical tool

In order to analyse the development of conceptual understanding, I used the analytical tool B – see Table 3.2 below. This framework was used to analyse the data from the interviews. The interview protocols were based on Kilpatrick's proficiency strand of conceptual understanding (Kilpatrick et al., 2001; p.5).

Table 3.2: Analytical tool B - indicators of conceptual understanding and their codes

Conceptual understanding indicators or themes as defined by Kilpatrick et al. (2001, p. 5)	**Approaches to build conceptual understandings during teaching and learning. Description of indicators in relation to *GeoGebra* as a** visualisation **tool.**
Connecting ideas and concepts in mathematics **Code (CIC)**	The teacher uses *GeoGebra* to demonstrate connections between multiple concepts and establish relationships. The teacher uses *GeoGebra* to explore the relationship and makes conceptual links to other areas of mathematics such as algebra. The teacher encourages learners to explore connections between selected concepts that are conceptually rich. Using *GeoGebra* to emphasise the links or connections between different geometric concepts, ideas such as relationship of properties of shapes. The teacher provides accurate explanation of concepts through *GeoGebra* visualisations.
Connecting mathematics to real world **Code (CRW)**	The teacher use *GeoGebra* to connect mathematics to real world examples. Everyday shape is used and other properties are expressed. The teacher uses examples from learners' context that they can relate to.
Connecting mathematics to prior knowledge **Code (CPK)**	The teacher uses *GeoGebra* to construct dynamic Geometry diagrams that are familiar to learners and makes links to learners' prior knowledge. The teacher makes use of what the learners already know and draws from their past experiences.
Representing mathematical ideas and analysis in different ways **Code (RID)**	The teacher uses *GeoGebra* to represent mathematics in in different ways. The teacher uses diagrams and *GeoGebra* visualisation to illustrate geometric and algebraic properties. The teacher is able to drag around and change measurements, but maintaining the dependencies in construction. i.e generated shapes are dragged but they sustain the same properties.
Justifying and explaining mathematical ideas and solutions **Code - (JIS)**	The teacher uses *GeoGebra* visuals to explore dependencies, relationships and proofs of the central concepts and theorems.

3.9 VALIDITY

Validity is a continuous concept to which a researcher needs to constantly pay attention (Cohen et al., 2011). In interpretivist research, such as this study, validity refers to credible, confirmable or plausible work (Winter, 2000). Credibility in this research has been enhanced both during data collection and during the data analysis. According to Bertram & Christiansen (2014) "the researcher may use mechanical means to record the data" (p.188) to increase the credibility. For example in this study, I used technological devices to collect the data through observations and interviews tools. For observation, I video recorded the teachers as they used *GeoGebra* to teach mathematics. For the interviews, I used a voice recorder to capture teachers' experiences and perceptions when they used *GeoGebra* to enhance conceptual understanding. I believe that technology has contributed to the credibility because the transcripts are very accurate compared to notes jotted down during the lesson, observation or interviews. The raw data such as electronic recordings and transcripts are stored for crosschecking the data and confirming the final report of the study.

Another aspect of validity in the research is confirmability and plausibility. According to Bertram & Christiansen (2014) within an interpretive paradigm confirmability includes among others the articulation of how the data were collected and analysed. To reinforce the confirmability, the research process was divided into distinct phases. By working from one phase to another, I was able to manage well the coherence of data from the research goals, the data collection, the data analysis, findings and conclusions.

3.10 ETHICS

Ethics is an important part of the research process, particularly research that involves people and animals (Bertram & Christiansen, 2014). Ethics in this research is assured by employing the guiding ethical principles from Rhodes University, which are: respect and dignity, transparency and honest, accountability and responsibility as well as integrity and academic professionalism. These are now discussed in details.

3.10.1 Respect and dignity

I communicated the goal of my research to the participants in the study. I informed them that they had a right to withdraw from the research at any time. For the participants to remain anonymous I used pseudonyms. The school at which the research took place remained anonymous as well.

Since my research collected some data by means of video tape, these videos are only sharable between the participants and the supervisor. In the event of wishing to use these videos in a conference or other professional presentation, written consent will be obtained from the participants. Normal teaching time was not disturbed because research activities were scheduled after school hours and Saturdays.

3.10.2 Transparency and honesty

Permission to conduct the research at school was obtained from the Director of Education in Oshana Region, Namibia. This was done after the proposal was approved by Rhodes University High Degree Committee. I was then given permission by the principal of the school from where the co-participant teacher was sourced. I discussed the contribution of my study to the school with Head of Department of Mathematics and Science. I told her that that one of the teachers in her department was purposefully selected to participate in my study. Therefore, I needed permission from her as an immediate supervisor. The two teachers were informed about the purpose of the study and asked to volunteer to participate. They were then asked to complete a consent form that they agreed to be part of the research. Parents of learners taking part in the study were also informed well in advance about the study. A written brief about the research was sent to parents in their vernacular languages. In this document, they were also informed of their right to agree or disagree to their children participating in the research.

3.10.3 Accountability and responsibility

I have been accountable for the entire research process and ensured that the data are kept safe. Being a principal of the research school, I am aware that my authority might influence the participants in some way. To overcome this, a thorough outline of my study and its purpose was clearly spelled out to the participants from the beginning. Mutual trust between the participants and me was established before the research commenced. This was simple because I knew the participants and there has been a good collegial relationship amongst us. There has been a mutual passion and interest for technology amongst us. The initial workshops about *GeoGebra* and conceptual understanding, lesson planning, and discussion of the piloted lessons put the participants at ease. The essence that the co-participants are also mathematics teachers who are interested in technology, created a platform where both participants shared their experiences and learning freely without being threatened by issues of any position or authority.

3.10.4 Integrity and academic professionalism

Integrity has been upheld at all times in this research project. The co-participants and I conducted ourselves in a professional way and I do not think we have jeopardised the chances of future research at the same school. I made sure that the research findings and all the data are presented authentically without any distortion or manipulation to suit my assumptions and opinions. All raw data are kept safe and secure for anyone who might want to verify the research conclusions. Other academic work and ideas are appropriately acknowledged and referenced according to the Rhodes University guidelines for academic writing.

Objectivity is another form of the intellectual in research. I reported the research results as accurately I could and I interpreted the data rigorously and clinically with the use of my analytical tools. It is important to emphasise this as the analysis included analysing my own lessons. I made sure that I distanced myself from my own lessons. I only looked at my lessons through the lens of my analytical tool and did not let my subjective feelings interfere with this.

3.11 CONCLUSION

In this chapter, the research goals and research paradigm were discussed. The study was orientated in an interpretive paradigm. I used a case study method as my research approach. Observations and interviews were my dominant data gathering instruments. Video data was complemented by interviews to crosscheck and interrogate participants' personal perceptions and experiences. Sampling was purposive and the research site was one school in the Oshana region, Namibia. The chapter concluded with a discussion of the data analysis protocol, as well as interrogating issues relating to ethics and validity. In the next chapter, the data is presented and discussed.

CHAPTER FOUR

4. ANALYSIS AND DATA DISCUSSION

4.1 INTRODUCTION

The purpose of this chapter is to present and discuss the results of the study. The main purpose of the research was to analyse how the selected teachers could use *GeoGebra* visualisations to enhance conceptual understanding. The chapter begins with a general description of the 10 lessons (4 lessons taught by teacher A, 4 lessons taught by teacher B and 2 lessons taught by teacher C) that the participants taught. The chapter continues with a discussion of each teacher's lesson presentations noting the dominant themes about visualisation processes such as image generation, image inspection, image transformation and image use that were evident. Teaching for conceptual understanding themes such as connecting ideas and concepts in mathematics, connecting mathematics to the real world, connecting mathematics to prior knowledge, presenting mathematical ideas in different ways and justifying mathematical ideas and solutions are also discussed per teacher per lesson.

Data is presented as authentically as possible. I felt this to be important as it adds a richness and validity to the data presentation. Thus, where possible the actual works and words of the participants have been used as supporting evidence. Appropriate coding was used as follows:

Teacher A video transcript lesson 1, line 10	(TAV1, 10)
Teacher B video transcript lesson 3, line 15	(TBV3, 15)
Teacher C video transcript lesson 2, line 25	(TCV2, 25)
Teacher A interview, line 30	(TAI, 30)
Teacher B interview, line 14	(TBI, 14)

In the process of the data analysis (video footage from the recorded lessons and interviews) various themes dominated the data. Although these themes are discussed individually, it is imperative to mention that they complement each other. The chapter concludes by weaving together the data from the individual lesson observations and the interviews.

4.2 DESCRIPTION OF THE LESSONS IN GENERAL

All 10 lessons were on Grade 9 geometry. The learning and teaching contents revolved around angle properties in parallel lines, in triangles and in quadrilaterals. Teacher A and B taught lessons 1 to 4 each and teacher C only taught lessons 1 and 2 as indicated below.

Lesson 1: This lesson was taught by TA, TB and TC. The learning objectives of this lesson were for the learners to be able to identify pairs of equal angles in parallel lines and be able to find the missing angles when one or two angles are given in parallel lines. (See lesson plan Appendix D1).

Lesson 2: This lesson was taught by TA, TB and TC. The learning objectives of this lesson were for the learners to be able to understand that the exterior angle of a triangle is equal to the sum of the two interior opposite angles and to use angle properties of a triangle to find the missing angle in a triangle. (See lesson plan Appendix D2).

Lesson 3: This lesson was taught by TA and TB. The learning objectives of this lesson were for the learners to be able to determine the sum of interior angles in quadrilaterals and to use angle properties of quadrilaterals to solve problems. (See lesson plan Appendix D3).

Lesson 4: This lesson was taught by TA and TB. The learning objectives of this lesson were for the learners to be able to identify and use angle properties to calculate unknown angles in complex shapes i.e. a combination of parallel lines, triangles and quadrilaterals. (See lesson plan Appendix D4).

4.3 DESCRIPTION OF LESSONS WHERE *GEOGEBRA* WAS USED

The objective of the analysis below was to explore and identify how the participating teachers presented their lessons using *GeoGebra* visualisations to enhance conceptual understanding. The descriptions are in narrative form and describe what happened in each of the lessons per teacher. The lesson descriptions are arranged in such a way that I first described lesson one for each teacher, followed by the second, third and then finally the fourth lesson. This structure aimed to identify the common and different practices applied by the participating teachers when using *GeoGebra* as a visualisation tool.

4.3.1 Lesson 1 Teacher A: Angles in parallel lines

Teacher A began the lesson by welcoming learners to the mathematics class. He told learners that the lesson would be about angles formed in parallel lines. Learners in Grade 9 had already been introduced to parallel lines in Grade 8 but they had not learnt about angles formed in parallel lines yet. To check the prior knowledge of learners, the teacher asked learners if they could still remember what parallel lines looked like. One learner responded, *"Lines that are facing in one direction and they never meet" (TAV1, 4).* The teacher used *GeoGebra* and drew a pair of parallel lines as illustrated in Figure 4.1.

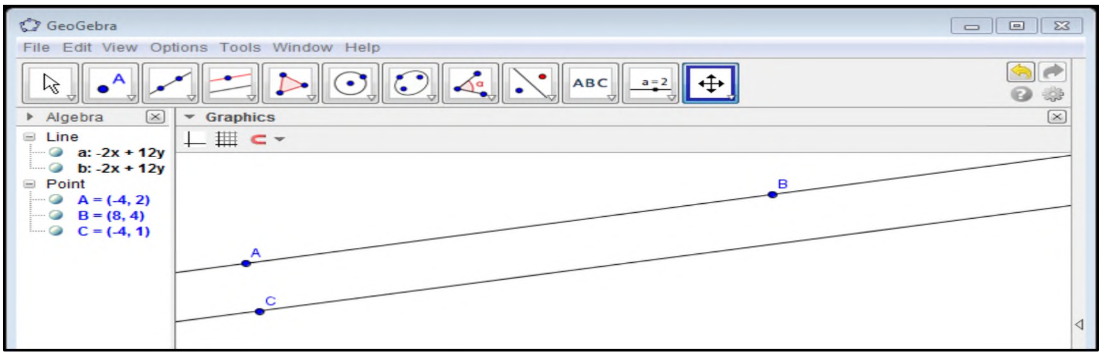


Figure 4.1: Parallel lines presented with GeoGebra

The teacher used the image to demonstrate and explain parallel lines properties, *"…I can scroll like this you see, you see, you see. The whole day you can pull these lines, they will never meet.*

They go parallel like that. So that is why we call them parallel lines…" (TAV1, 6). The teacher used *GeoGebra* to justify that parallel lines do not meet by scrolling the parallel lines across the screen. It was clearly observed that the two lines do not meet and the distance between the two lines remained the same.

To continue with the lesson presentation, Teacher A referred to parallel lines in real life examples such as railways, edges of a tarred road, elevators in shopping centres and ladders.

Later on, Teacher A added a transversal line to the parallel lines in order to form angles. In the process, the teacher emphasised two things, the direction of measuring angles, i.e. clockwise, and that angles always lie between three points. He asked learners to observe the images of different angles on the white board as he constructed them. He used a measuring tool in *GeoGebra to* measure the sizes of the angles and through dynamic manipulations, he was able to demonstrate and justify the direction of measuring angles. He started with the corresponding angles, followed by alternate angles and then co-interior angles. Each time he generated a pair of angles he was able to use in a dynamic way, the image that appeared, to support his explanations and to develop further the conceptual understanding through visuals. In addition, the teacher was also able to use the *GeoGebra* text tool to add labels to the images on the screen as illustrated in Figure 4.2.

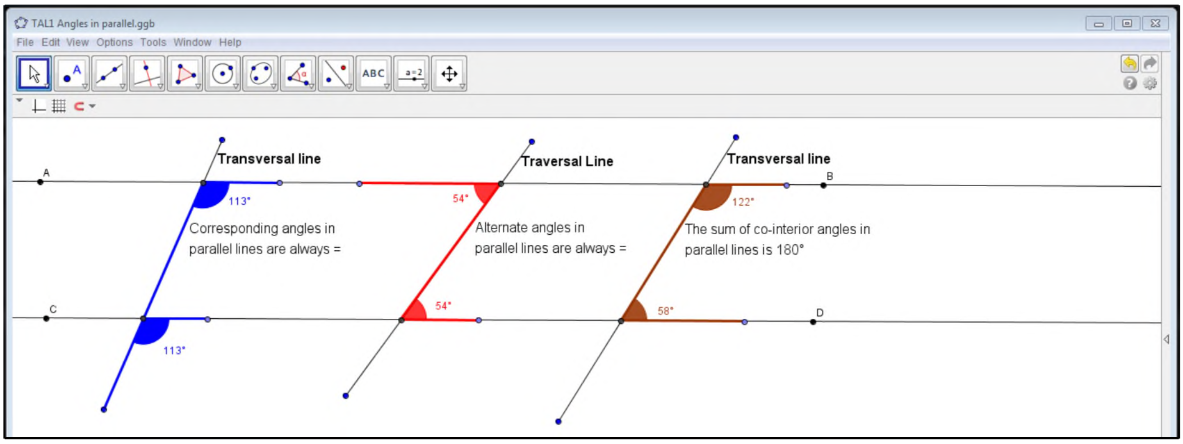


Figure 4.2: Teacher A's presentation of angles formed in parallel lines

In the lesson, Teacher A coloured all pairs of angles differently. This made it easy for him to describe them as well as more visible for learners to identify. Teacher A was able to change the sizes of angles dynamically by moving the transversal lines to the left and right. This image transformation helped the teacher and learners to observe and justify the properties of angles formed in the parallel lines. The teacher concluded the lesson by illustrating and referring the angles to the shapes of letters (F, Z, and U) as illustrated in Figure 4.2. The "F" shape could help leaners to identify the corresponding angles, "Z" shape can be used to identify the alternate angles and "U" shape for co-interior angles in parallel lines.

The following section illustrates how Teacher B presented the same lesson of angle properties formed in parallel lines. She also used *GeoGebra* visualisations to present the said lesson.

4.3.2 Lesson 1 Teacher B: Angles in parallel lines

Teacher B started the lesson by greeting learners. She told them that they were going to have an interesting lesson about angles between parallel lines. She asked learners to define the term *parallel line*. In response one learner responded, *"The lines that face the same direction" (TBV1, 2)*. Teacher B completed the learner's answer by adding that, *"…the space between them is the same and those lines never meet… " (TBV1, 2)*. Teacher B told learners that hand- drawn images are not very accurate. *"…now to make this very accurate we are going to use what we call GeoGebra… " (TBV1, 11)*. She removed off the grids, x- axis and y-axis. She drew parallel lines on the screen using *GeoGebra*. She then used the generated image to develop the concept of parallel lines. She explained, pointing to the parallel lines, *"…they are going in the same direction and the distance between them is always the same. I am trying to show you that you can move it as far to the right or as far to the left and they will never meet. Can you see that"? (TBV1, 11)*. The teacher used *GeoGebra* to develop the image further by adding transversal lines in order to form angles. This was followed by measuring the size of angles using the *GeoGebra* measuring tool. The aspect of how an angle can be formed also arose and the teacher emphasised that angles are always formed between three points. She also pointed out that angles are measured in a clockwise direction. Teacher B added another transversal line to the parallel lines to form more angles. She started with corresponding angles. By referring to the image, she explained how one could identify corresponding angles as shown in Figure 4.3.

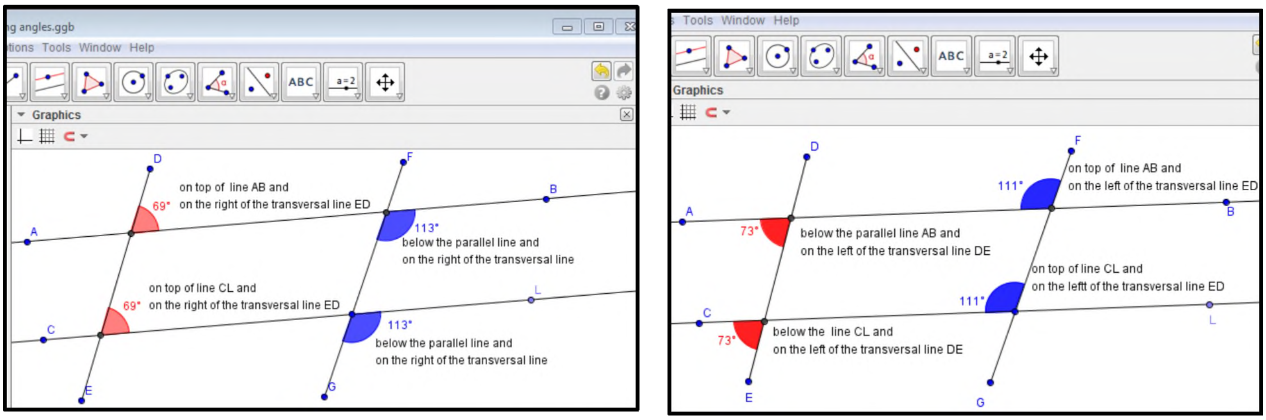


Figure 4.3: Teacher B's presentation of corresponding angles in parallel lines

She demonstrated that corresponding angles are formed on top or on the bottom of each parallel line, *"... wherever you have to identify corresponding angles please make sure that those angles are on the bottom or on top of parallel lines and can either be on the left or on the right side of the transversal line. Is that clear?"(TBV1, 37).*

Teacher B then drew another transversal lines and formed new angles. This time she targeted alternate angles. She asked learners how to identify alternate angles. There was silence in the class until one learner responded, *"We check if we are able to see letter Z" (TBV1, 42).* Teacher B continued to mark alternate angles and coloured them red. She used the image to describe and justify that alternate angles are not on the same side of the transversal line, *"... if one is on the left then the other one is on the right of the transversal line.... They are both between the parallel lines (TBV1, 47)*. Teacher B then added another transversal line to form co-interior angles. She asked learners to comment on the co-interior angles. One learner responded, *"When you add them together they will give you 180^0" (TBV1, 52).* Teacher B reminded the learners that co-interior angles can be recognised by assigning the letter "U" to them. She further used the image to show that co-interior angles are also between the parallel lines, *"... the co-interior angles are on one side of the transversal line" (TBV1, 69)*. She rounded off the lesson by emphasising the patterns with which learners could identify alternate angles, corresponding angles and co-interior angles in parallel lines. Teacher B then gave learners an activity about angles formed in parallel lines. The activity was to be done in pairs.

The following section illustrates how Teacher C presented the same lesson taught by Teacher A and Teacher B. He also presented the lesson using *GeoGebra* as a visualisation tool.

4.3.3 Lesson 1 Teacher C: Angles in parallel lines

Teacher C started the lesson by greeting learners. He told them that they were going to learn more about angles. He specifically mentioned that the lesson would focus on angles within parallel lines. He then opened *GeoGebra* and drew one line on the screen. To test the prior knowledge of learners, he asked them if one line could be called a parallel line. Learners replied together that it was not possible because it was only one line. Teacher C then drew another line and asked learners to tell whether they now had parallel lines on the screen. From the observation, it was clear that learners knew what parallel lines were. Teacher C then used *GeoGebra* to demonstrate and describe the properties of parallel lines. Properties such as lines never meet, and distance between them stays the same, were discussed. *"Ok, so for those who don't know parallel lines they will never meet even you continue them here, and the distance between the two lines must be the same. If you got two squares even you extent it like aa let me just move one, if we move this side we go, we go, we go the distance remain" (TCV1, 15).*

19.	*TC:*	*But if we are having the line which is not a parallel, let me just have this one (moving one line) we say that this one is not parallel if we are to move on you find that the distance here is increasing you see that?*
20.	*Ls:*	*Yes*
21.	*TC:*	*But if we move this side, the distance awuh, they even meet. Can you see that? So they are not parallel.*
22.	*Ls:*	*Yes*

Teacher C then added a transversal line to the parallel lines in order to form angles. He then measured the angles. At first, the corresponding angles did not have the same size because the lines were not really parallel. Teacher C and the learners observed that corresponding angles were not equal. Teacher C told the learners that if corresponding angles are not equal, it means that the two lines are not parallel. *"... Oh probably our lines they are not parallel. Look at this let me make them parallel; let me make them parallel because if we are getting the different values then those lines are not parallel. Thank you. Line AB is parallel to line CD are we there? (TCV1, 33).*

Teacher C continued to identify other pairs of corresponding angles on the image, and in conclusion, he referred to the letter ‘F’ shape as a symbol to locate or identify pairs of corresponding angles in parallel lines, demonstrating this by moving the transversal line right and left; in so doing, learners observed that the corresponding angles remained equal.

The next type of angle presented was alternate angles. With *GeoGebra* Teacher C drew a new pair of parallel lines and a transversal line to form up angles. For the alternate angles, he told learners to observe angles formed in the shape of the letter ‘Z’. Teacher C changed the angles dynamically and it was observed that the angle values changed simultaneously. *“Even we put it there, the other one become even bigger 109 the other one also become 109. That is the Z shape that we are having here. Let me move this part again, let me move this part. Look at that it is still the same .Are we together? (TCV1, 57).*

Co-interior angles in parallel lines was then presented. Teacher C told learners that to identify co-interior angles in the parallel lines; they must look at angles that are formed in a ‘C’ shape. After identifying the pair of co-interior angles in parallel lines, he asked learners to add the two angles together. At this stage, the lesson was fun. The following conversation tells the fun part of teaching and learning. ... *” Now that they are parallel, I just want us to explore here add 112 and 68. What do you get? Can I just have someone to give that answer? Just add 112 plus 68. (TCV1, 61)*

62. Ls: 180
63. TC: You get what?
64. Ls: 180
65. TC: 180 degrees. Are you sure?
66. Ls: Yes
67. TC: Ai, let me challenge you here, let me challenge you here. I will move this one. I will change the angles now. Ehe, now I have 79 and 101. Can you add them together again?
68. Ls: 180
69. TC: You get 180 again?
70. Ls: Yes
71. TC: You must be joking. Let me move it again. Are you sure let me move it again, I will make this one very big. Ok, 136 and that 44 add them together.
72. Ls: 180

73. TC: What? Ok, we can now conclude and say the angles, those two angles that we say are co-interior their total is 180 degrees. Are we together in that?

74. Ls: Yes

Teacher C then tested learners to find the missing co-interior angle if only one co-interior angle is given. He did this several times, by changing the image and coming up with the new angles. To make a distinction between parallel lines and non-parallel lines he changed the lines to non-parallel lines. He asked learners to add the pair of co-interior angles together. The answer was always 180^0.

To conclude the lesson Teacher C opened the activity that he had saved in Microsoft Word. Learners were asked to do it verbally in class. He also asked them to give reasons for their answers, such as corresponding angles, alternate angles or co-interior angles. The lesson was then adjourned with homework given to learners.

The following section illustrates how Teacher A presented lesson 2, about angles in triangles, using *GeoGebra* as a visualisation tool.

4.3.4 Lesson 2 Teacher A: Angles in triangles

The teacher welcomed learners to the class. He told them that they were going to learn about the sum of angles in triangles. They were also going to learn about the relationship between the exterior angle and the two opposite interior angles in a triangle. He further told learners that they were going to prove that the sum of angles in any triangle is equal to 180^0. Teacher A started the lesson presentation by drawing a triangle using *GeoGebra*. He then used *GeoGebra* to measure the angles inside the triangle. He reminded learners about the clockwise direction when measuring angles. To proceed with the lesson presentation, Teacher A moved the angles dynamically and asked learners to add all the interior angles in the triangle. Using *GeoGebra*, Teacher A transformed the initial triangle into different triangles. The first tringle was named triangle X, next was a scalene, then a right-angled triangle, followed by an equilateral triangle, and the last was an isosceles triangle. Teacher A used *GeoGebra to* measure both the angles and

the sides of triangles. Learners were able to observe the features of different types of triangles on the projector. On a separate whiteboard, he created a table, which he completed as he presented each triangle feature (or property) -
See Table 4.1.

Type of triangle	**Angles**			
	Angle sizes A	Angle sizes B	Angle sizes C	Sum of angles
1st triangle X	69	78	33	180
2nd triangle scalene	58	72	50	180
3rd triangle Right-angled	49	90	41	180
4th triangle Equilateral	60	60	60	180
5th triangle Isosceles	70	55	55	180

Table 4.1: Table used by Teacher A to record angle sizes of different triangles

In the process of transforming the one triangle, the teacher found it challenging to come up with an accurate equilateral triangle, but later he managed. The value for each angle per triangle was then entered in the table. After completing the table, the teacher asked learners to find the sum of angles for each triangle. It transpired that the sum of interior angles for each triangle was 180^0 as shown in Figure 4.1 above.

> *"Now all those you see that we take the different types of triangle, but then in each triangle when you add the sum of interior angles you will get 180. So now we know or we can prove that the sum of any, any the sum of interior angles in any triangle is equal to 180^0. Because all the triangles we have mentioned them here. Triangle X you just take any triangle, scalene triangle no angle is equal, right angled triangle one angle is 90^0, equilateral they are both 60, isosceles two are equal that is two base angles are equal and then one is different from the other two. This is enough to prove to us that the sum if interior angles in a triangle is equal to 180^0" (TAV2, 88).*

To present the second part of the lesson, Teacher A drew a triangle with an exterior angle to demonstrate the concept of exterior angles and the two opposite interior angles in a triangle as illustrated in Figure 4.4.

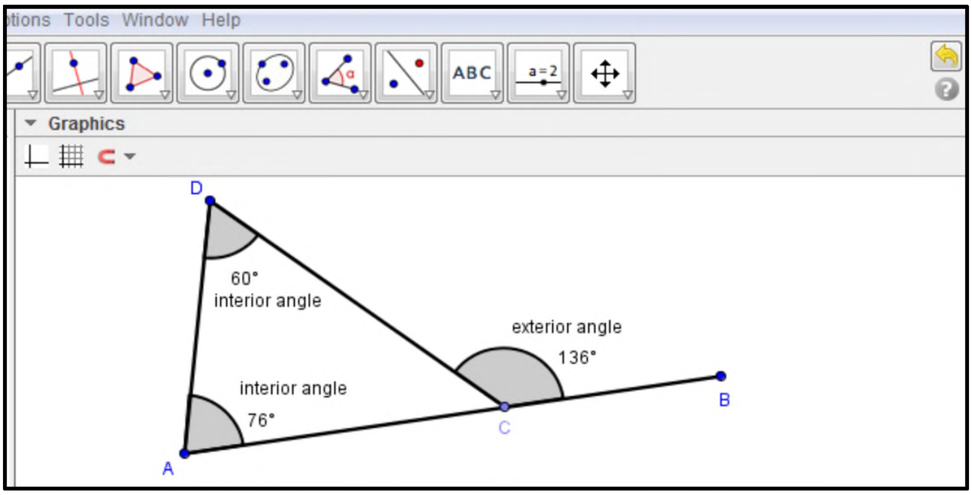


Figure 4.4: Opposite interior angles and the exterior angle of a triangle

Figure 4.4 was used in conjunction with table 4.2 to record the measurements of the two apposite interior angles and the exterior angle. Learners were required to transform the initial image in order to form different angles from the initial ones. They then had to fill in the table as illustrated in Table 4.2.

	Interior angles sizes		Total	Exterior angle sizes
	A	B	Sum (A+B)	Angle C
1st shape	41	90	**131**	**131**
2nd shape	30	96	**126**	**126**
3rd shape	21	118	**139**	**139**
4th shape	25	59	**84**	**84**
5th shape	35	86	**121**	**121**

Table 4.2: Table displaying the relationship between two opposite interior angles and the exterior angle of a triangle

The task of learners was first to record in a table the readings of angles A, B and the exterior angle C each time they transformed to a different triangle. Thereafter, Teacher A instructed learners to add the two interior angles together and complete the column of totals. It turned out that the sum of the two opposite interior angles is equal to the exterior angle. The result in table 4.2 confirmed the assertion that the sum of the two opposite interior angles of a triangle is equal to the exterior angle. Teacher A then concluded the lesson by emphasising the relationship between the two opposite interior angles and the exterior angle of a triangle.

The following section describes the presentation of lesson 2 by Teacher B. In this lesson, Teacher B again used *GeoGebra* visualisations.

4.3.5 Lesson 2 Teacher B: Angles in triangles

Teacher B greeted the learners and welcomed them to the lesson. She told the learners that they were going to learn about angles in triangles. She introduced the lesson by asking learners to define a triangle. One learner defined the triangle as a shape of three equal sides. Another learner responded that a triangle is a shape with three angles.

Teacher B then gave a definition of a triangle. *"It is a shape of three angles as long as there are three sides and three angles that is a triangle… " (TBV2, 12).* Teacher B then used *GeoGebra* to draw a triangle on the white board, and asked learners to name the shape. To develop the concept further she asked learners to mention different types of triangles that they knew. Learners managed to mention isosceles triangle, equilateral triangle, scalene triangle and right-angled triangle. Teacher B used *GeoGebra* to measure the angles of all triangles. As she was measuring the angles, she reminded learners that measuring of angles is in a clockwise direction. Apart from the four types of triangles mentioned by learners, Teacher B added a triangle that she called triangle X.

As she was measuring angles using *GeoGebra,* she asked learners to add all angles in a triangle. *"… We are saying that in a triangle, any triangle that we have the sum of three angles that are inside or the sum of interior angles are equal to 180^0… " (TBV2, 50).* She repeatedly changed the initial triangle into different triangles and measured the angles. She confirmed with learners that in all triangles the sum of interior angles was 180^0. One thing that she struggled with was to create an accurate equilateral triangle. However, she managed after several tries.

Teacher B then moved to the concept of exterior angles of a triangle. She used *GeoGebra* and drew a triangle with an external angle. She measured the exterior angle and the two opposite interior angles. She told the learners to observe the relationship between the exterior and the two opposite interior angles. She did not talk too much about the relationship because she wanted learners to discover the relationship for themselves. In conclusion, Teacher B gave a task to learners about the exterior angle and the two opposite interior angles. The task was already planned in *GeoGebra* and was saved on the laptops.

The following section illustrates how Teacher C presented lesson 2, using *GeoGebra* as a visualisation tool.

4.3.6 Lesson 2 Teacher C: Angles in triangles

After exchanging greetings with learners, Teacher C started the lesson by telling learners that the lesson of the day would be about interior angles in a triangle. He also told them that at the end of the lesson they were expected to know the relationship between the exterior angle and the two opposite interior angles of a triangle.

Teacher C opened *GeoGebra* and drew a triangle. He used *GeoGebra* to measure the interior angles. He asked learners to find the sum of the three interior angles. The answer that the learners got was 180^0. Teacher C kept on changing the triangle by dragging any point on the triangle. Every time he changed to a different triangle, he asked learners to find the sum of angles in such a triangle. The answer was 180^0 every time. He then summarised, that the sum of interior angles of any triangle is 180^0. *"Ok, let's have a conclusion here. The total, the total of the interior angles of the triangle is equal to 180^0. You get what I am saying"? (TCV2, 19)*

22.	*L:*	*180 -59 -90*
23.	*TC:*	*(Repeating after the learner) 180-59-90 oo, do we think she is right?*
24.	*Ls:*	*Yes*
25.	*TC:*	*While she is still standing, can we calculate this? Let us find out aa 180 – 90 – 59. I want you to get the answer on that one. Amos!*

26. L: *It is equal to 31*
27. TC: *Is equals to 31 degrees. Ok, now I want us to find it out whether it is true. Whether is true, whether is true. (he uses GeoGebra to show the answer) 31, so Amos is correct.*
28. Ls: *Yes*

Teacher C then tested learners by hiding the value of one angle then changing the image and asked learners to find the missing angle. During this time, he was also confirming learners' answers with *GeoGebra. "...Let's hide the number, the number I want to hide, come on; ok. Now we have those two but I want us to look at this (he changed the initial triangle into a right angled-triangle) we were talking about the sum of angles in a triangle is equal to 180 degrees. Now if one angle is missing. Which strategy can we use probably so that we get that angle"? (TCV2, 21).*

After checking learners' understanding, Teacher C emphasised the method of finding the missing interior angle if the two angles are given. *"So is very simple say that if you want to find another angle from the three angles you just take 180 minus the two angles that you are given. Ok, thank you very much you may sit. Are we together"? (TCV2, 29).* Teacher C proceeded to draw a triangle with one exterior angle. He asked learners to find the relationship between the exterior angle and the two opposite interior angles in a triangle. *"...Now what is the relationship? That is what I want us to explore first. I won't tell you but you find it on your own. Ok, I want you to add for me this two opposite, because these are the two opposite angles right. Can you add 72 + 52? (TCV2, 51).* He kept on changing the image to get the different measurement of angles. Every time he changed the image he asked learners to find the relationship between the exterior angle and the two opposite angles.

Teacher C then confirmed learners' answers with *GeoGebra*. The following extract shows how Teacher C explained the relationship between the exterior angle and the two opposite interior angles. *"...Right in conclusion we can say the exterior angle is equal to the sum of the opposite angles. Did you get what I said" (TCV2, 65).*

66. Ls: *Yes*
67. TC: *The sum of the two opposite angles is equal to the exterior angle. Are we together on that one?*
68. Ls: *Yes*

Teacher C continued to test learners by hiding the value of one angle and asking learners to find it. He again used *GeoGebra* to confirm learners' answers. He concluded the lesson and gave learners the homework.
The following section illustrates how Teacher A presented lesson 3. He presented the lesson using *GeoGebra* as a visualisation tool.

4.3.7 Lesson 3 Teacher A: angles in quadrilaterals

Teacher A started the lesson by greeting learners. He told them that they were going to learn about quadrilaterals. He told the learners that by the end of the lesson they were expected to determine the sum of interior angles in quadrilaterals and that they should be able to use angle properties of quadrilaterals to solve problems that are related to quadrilaterals. Teacher A drew a free-hand sketch of a rectangle on a different board. He asked learners to mention the name of the shape he drew. The leaners responded that it was a rectangle. He explained the properties of a rectangle. After the explanation Teacher A generated another quadrilateral (four-sided figure) using *GeoGebra*.

19. TA: ...I am going to generate some images in GeoGebra and from there I will interact with you. I will ask you questions and then I hope you will be able to answer. eee, I am going to put up a shape like this. Eee shape like that (as he constructs the quadrilateral) maybe I can pull this part here and... Ok. Can you all see the shape on the white board?

20. Ls: Yes

21. TA: Ok, I said this is a family of quadrilaterals. That means a quadrilateral is any figure or all figures with four sides and four angles. And like we said in the past that this program can measure the sizes of angles, that is what I am going to today. If you can remember we talked about the direction when we measure the angles. We measure in what direction again? In what direction again Maria?

When Teacher A was measuring the angles in a quadrilateral using *GeoGebra,* he emphasised the direction of measuring angles in a clockwise direction. When Teacher A had finished with the drawing and measurements, he asked learners to add all angles in the diagram. Because some angles were not whole numbers, the total was 359^0. Teacher A then changed all the points of the quadrilateral to whole number co-ordinates. After changing to whole number co-ordinates, angles now added to 360^0. Teacher A continued the lesson by asking learners to mention the types of quadrilaterals they knew. They (learners) started with trapezium, parallelogram, kite, square and rectangle. Teacher A drew a table on a board where he recorded the names of quadrilaterals see table 4.3.

	angle size A	angle sizes B	angle size C	angle size D	Sum of angle sizes
Trapezium	90	90	143	37	360
Parallelogram	124	56	124	56	360
Kite	113	105	37	105	360
Rhombus	53	127	53	127	360
square	90	90	90	90	360
Rectangle	90	90	90	90	360

Table 4.3: Table used by teacher A to record the names and angle sizes of different quadrilaterals

Teacher A then asked learners to open their laptops and complete the table together. *"…Now I want you to open your laptops, but then some of you, I will call you to use my laptop here…" (TAV3, 53).* Teacher A instructed learners to change the initial image into the required image.

For example, transform the trapezium into a kite. Learners worked in pairs or in groups of three per laptop.

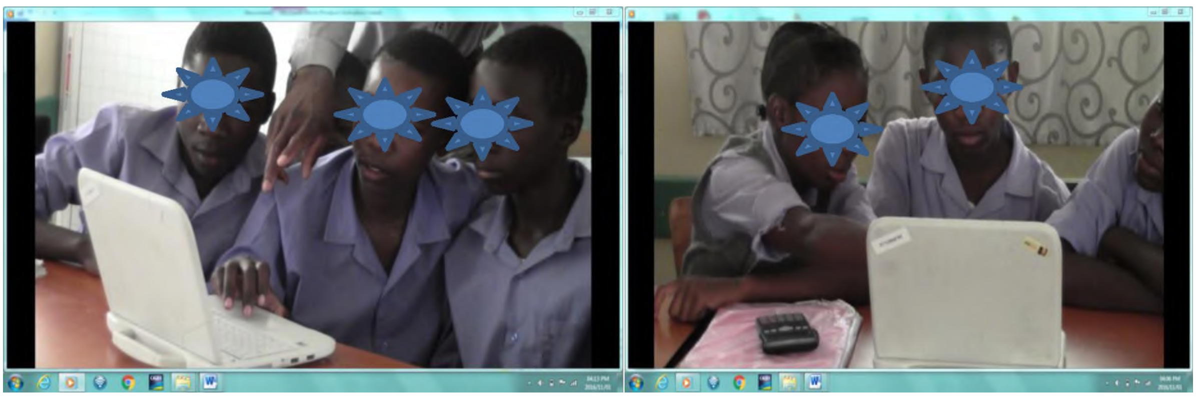


Figure 4.5: Snapshot of learners engaging with GeoGebra to complete the given task

The construction and the discussion of images were done according to Table 4.3. Teacher A moved from group to group, assisting learners who were struggling to operate *GeoGebra*.

During this time, learners were helping each other to transform the figures from one quadrilateral to another as shown in Figure 4.5. Learners were also able to discuss the properties of different quadrilaterals based on the *GeoGebra* visualisations.

Once a group had finished, their work was projected on the white board for the whole class to see and comment. *"...Any pair once you constructed parallelogram you just show us that you have constructed it. I will take your laptop and then I will put it here and then everybody can see, that yes, your parallelogram is correct" (TAV3, 71).* One group finished first and their work was ready to be displayed on the screen. *"...I have one group here which is done and I am going to project their work on the white board. Let's see. (the teacher connected the learners' laptop on the projector) Can you see?" (TAV3, 72)*

Teacher A then used the image that had been constructed by learners - see Figure 4.6 - to explain the properties of parallelograms, *"...so we have two sides which are parallel and the opposite angles, you can see here (pointing to the opposite angles angle B and D as well as angle A and C) this angle is equal to this angle and this angle is equal to this angle. That is one property of the parallelogram..." TAV3, 76).*

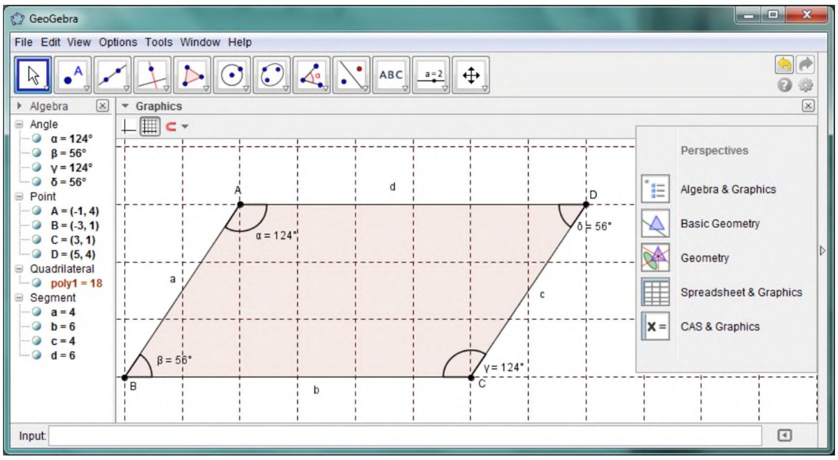


Figure 4.6: Screen shot of the parallelogram constructed by one group of learners

Teacher A used *GeoGebra* to measure the sides of the parallelogram on the screen. After several explanations and discussions, Teacher A took back the learners' laptop. The next quadrilateral to be constructed was a kite. It took some minutes for learners to transform the parallelogram into a kite. One group indicated that they were done but this group did not get the image correct, instead they had constructed a rhombus - see Figure 4.7.

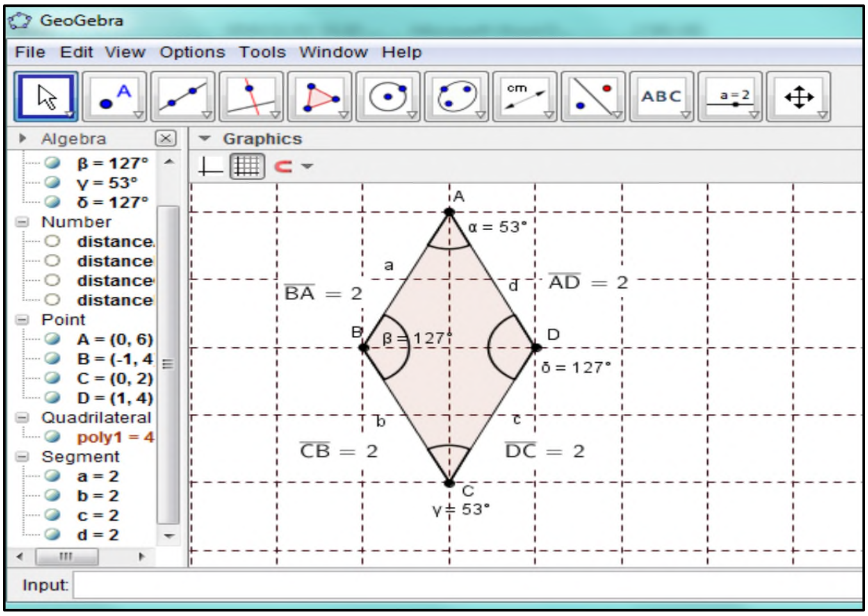


Figure 4.7: Screen shot of the quadrilateral constructed by one group of learners

Their work was projected on the screen, for the whole class to see and to discuss. When other learners saw the image on the screen, they started to comment immediately and discussion about the image proceeded.

93. Ls: It is not a kite
94. TA: If it is not a kite, Selma what is it?
95. Ls: It looks like a rhombus

After a little debate, the quadrilateral was identified as a rhombus. The properties were justified by using the *GeoGebra* measuring tool, which accurately measured and displayed the size of angles and the length of sides. *"Yes it looks like a rhombus. So it means this one is not a kite but* *it is a rhombus neeh. Ok, because rhombus all sides are equal and opposite angles are also equal…" (TAV3, 96).*

The measurements were then entered in the table. After that, one group was ready and their work was displayed on the screen. This group constructed an accurate kite. Teacher A used the image to describe the properties of a kite. *"You can see that DC is equal to 6 and also BC is equal to 6 and AB is 2 and this one also is 2. Ok, thank you that is an accurate diagram" TAV3, 108).* The information on the kite was then entered in a table. The next quadrilateral to be formed was a square. Learners were required to change the kite into a square. This was easy and almost all groups got it right in a short time. Teacher A used the images in the *GeoGebra* screen to explain the properties of a square. *"…This is a square, this is 3 units and 3 units, 3 units and 3 units, and the angles are all what?" (TAV3, 124).*

The last quadrilateral to be formed was a rectangle. This time Teacher A called one learner to the front to use the teacher's computer to transform the square into a rectangle. The learner transformed the square into a rectangle in a short time. The angle sizes of the rectangle were also entered in the table. This was followed by the exercise of transforming the initial quadrilateral into a new quadrilateral as per the teacher's instruction. This time individual learners were called to the front to use the teacher's computer, which was connected to the projector. Learners took it in turns to transform the images as per the instruction of the teacher.

Towards the conclusion of the lesson, Teacher A emphasised the advantages of using *GeoGebra* to construct, to measure angles and to measure sides. Teacher A together with learners then completed the table. They added all angles per figure and entered all the totals in the table. The conclusion was that the sum of interior angles of any quadrilateral (four-sided figure) is equal to 360^0. Learners were given homework to complete the task first without using *GeoGebra* and then later they (learners) could justify or check their answers using *GeoGebra*. That was the end of lesson 3 for teacher A.

The following section illustrates how Teacher B presented lesson 3 using *GeoGebra* as a visualisation tool.

4.3.8 Lesson 3 Teacher B: Angles in quadrilaterals

Teacher B started the lesson by greeting the learners and welcoming them. She told the learners that the lesson would be focusing on quadrilaterals. She asked learners the meaning of the word quadrilateral. One learner replied, *"Quadrilateral means four" (TBV3, 8).* The learner further added that quadrilateral means four sides. Teacher B then repeated that it means four sides. She then asked learners to give the names of quadrilaterals that they know. Learners mentioned the following quadrilaterals: square, rectangle, parallelogram, trapezium, rhombus and kite. Teacher B drew a rectangle on the board. To check the prior knowledge of learners, she asked them to give the name of the image. Learners responded that it was a rectangle. Learners went on to describe the properties of a rectangle such as parallel sides are equal and each interior angle is equal to 90^0.

After a lengthy discussion about the rectangle and its properties, teacher B opened *GeoGebra* and drew an accurate rectangle. *"…Fine now what we are going to do we are going to use GeoGebra to help us to have a clear and visible image instead of drawing them like that, because you cannot be sure that those lines are parallel. But otherwise if we use GeoGebra it will be easier for us…" (TBV3, 41).* Teacher B then drew a four-sided figure using *GeoGebra*. This time it was an irregular quadrilateral, which she called x-quadrilateral. She drew a table on the board where she recorded the names and size of angles of the quadrilaterals as illustrated in Table 4.4.

	angle size A	**angle size B**	**angle size C**	**angle size D**	**Sum of angle sizes**
x-quadrilateral	48	122	52	138	360
Trapezium	63	117	127	53	360
Square	90	90	90	90	360
Rhombus	67	113	67	113	360
Rectangle	90	90	90	90	360
Kite	105	106	105	44	360

Table 4.4: Table used by Teacher B for names and angle sizes of quadrilaterals

"…Ok, we are going to measure the angles of the quadrilateral that we have…" (TBV3, 41). Teacher B then used *GeoGebra* to measure all the angles, while measuring she emphasised that an angle is formed by three letters, reading them in a clockwise direction.

She clearly demonstrated this, using *GeoGebra* visuals. After she had measured all interior angles in the x-quadrilateral, she transferred the information into the table. She then asked learners to add the four angles together. *" Fine, now that we have all the angles inside the quadrilateral, can you please add those angles, to find out what the total we are going to get" (TBV3, 51).* The total was 360^0. *"The total we are having is 360^0. One can say if you add all the angles in a quadrilateral the total will be 360^0. Am right" (TBV3, 55)? "Let us prove it, we only having one quadrilateral which we do not even know the name. Now we have to try with quadrilateral that we know their names, right" (TBV3, 57).* Teacher B first added the name trapezium in the table; she then changed the x-quadrilateral into a trapezium, in the process of transforming the initial figure (x-quadrilateral) into a trapezium. She then asked learners to tell her the properties of a trapezium. She constructed the trapezium according to the properties. *"Ok, we will say that we are having a trapezium because we are having (pointing to the two parallel sides) this line and this line are parallel because they are facing in the same direction and the distance between them is the same and if you extend them they will never meet, right (TBV3, 67).*

This time the sizes of the angles were already present since this was only the initial image (x-quadrilateral) transformed into a trapezium. The angle sizes adjusted dynamically and automatically as the image was changed. Teacher B entered the angle sizes in Table 4.4. She then asked learners to add all the angles together. The total was 360^0. The next quadrilateral was a square. Teacher B started by transforming the trapezium into a square. In the process of transforming the trapezium into a square, the teacher used *GeoGebra* to measure the sides of the figure in order to show clearly that all sides of a square are equal.

81. TB Somebody said square (she added square on the table). So let's change that to become a square. The properties of a square?

82. Ls Both sides are equal.

83. TB Ok, we will try to draw all sides equal, now, the side we have to go too... angles, yes and then we get the distance and then we measure the distance of the sides that we have. From C to B (the distance did not show up) what happen?

84. Ls Nothing

85. TB *Ok, (moving the label outside the image) let me just take this one out like this. Ok, we are going to get the distance start from A to B and then from B to C from C to D and then fro D to A. ok, fine we can see that distance is 4cm, 10cm, 5cm and 15cm. we want all the ides to be equal so we can just change the size, we are going to select and then we change the size so that it can look like a square, looking like a square we can bring this one then we bring this one also up to here. The sides now are they equal.*

86. Ls *Yes*

87. TB *All the sides are equal?*

88. Ls *Yes*

After entering the angle measurements in the table, Teacher B asked learners to add all angles together. The total was 360^0. *"We are getting 360^0, so we are still supporting the idea of all the angles inside quadrilaterals is 360^0 Because we are able to prove it with three quadrilaterals" (TBV3, 101).* The next quadrilateral was a rhombus. Teacher B then transformed the square into a rhombus. In the process of transforming the square into a rhombus, properties such as all sides are equal and opposite angles are equal were observed and discussed. The total of all interior angles in the rhombus was also 360^0.

The next quadrilateral was a rectangle. While transforming the rhombus into a rectangle, the properties of the rectangle were also observed and discussed. The total of interior angles was also 360^0. The next quadrilateral was the kite. Like other quadrilaterals during the transformation process, the properties of a kite, such as adjacent sides are equal, were observed and discussed. However, for the kite, the sum of interior angles was 361, not 360^0. Teacher B then explained to learners the possible reason for not getting a total of 360^0. *"Maybe the reason why GeoGebra could not give us 360, we did not suit the properties of the kite. For example, we were having the two angles that were not equal. But then we said, to draw a proper kite you need to have one pair of opposite angles which are equal" (TBV3, 205).* She corrected the pair of equal angles and entered the information in the table.

Finally, Teacher B transformed the kite into a parallelogram. The properties of a parallelogram were also observed by the whole class and discussed. The teacher's explanation and discussion was supported by the image on the screen. The sum of interior angles was also 360^0. Teacher B then concluded that the sum of interior angles in any quadrilateral is 360^0.

229. TB *(TB filling in the table) Is 360 degrees. Ok because we are able to get 360 there, we tried the unknown quadrilateral, the trapezium, the square, the rhombus, rectangle, the kite and the parallelogram. Then we can say the sum of all the interior angles in any quadrilateral is?*

230. Ls *360*

231. TB *Is*

232. Ls *360*

The following section illustrates how Teacher A presented the lesson on angles in complex shapes using *GeoGebra* as a visualisation tool.

4.3.9 Lesson 4 Teacher A: Angles in a complex shape

Teacher A greeted the learners as usual and welcomed them to the lesson. He told them that they were going to continue working with *GeoGebra*. He further told them that they were going to learn about angle properties formed in different shapes. Teacher A opened the image, as shown in Figure 4.8, with *GeoGebra* and projected it on the white board. He asked learners to mention the name of the shape. Learners could not mention the name of the image as it was unfamiliar to them. He then asked learners to identify the names of geometric shapes that can be seen in the image on the screen. Learners responded quickly to this question.

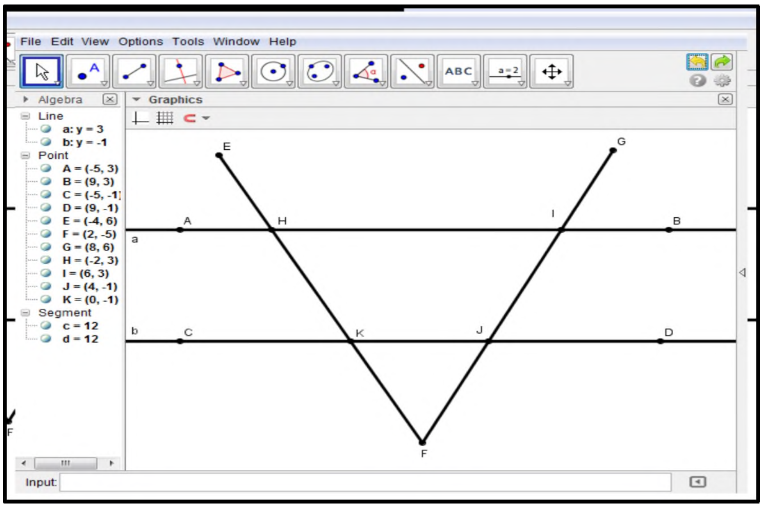


Figure 4.8: Screen shot of a complex geometrical shape used by Teacher A

The first geometric shape identified was the triangle, followed by the trapezium. Teacher A then added some points on the image at the intersections in order to form angles. *"...Ok, may be it is difficult, let me add some points so that maybe you can tell me exactly which one...?" (TAV4, 13)*. Learners were able to identify shapes by letters.

Teacher A introduced to learners the lesson for the day, which would be about angles in complex shapes. *"...complex shape is the shape where you have different types of Geometric shapes. In this case we have a triangle; we have a trapezium and another big triangle..." (TAV4, 29)*. The complex shape projected on the screen contained a pair of parallel lines. He used the parallel lines to remind learners about the angles formed in parallel lines, which they had done in lesson 1 using *GeoGebra*. Teacher A then told learners that at the end of the lesson they were expected to be able to identify and use angle properties to calculate missing or unknown angles in the complex shapes. He then filled in the angle sizes using *GeoGebra*.

While using *GeoGebra* to fill in the angle sizes, he was also discussing angle properties with learners. After displaying the sizes of some angles, Teacher A asked learners to use angle properties to find the size of some angles that he had purposely omitted. Learners were able to use the angle properties to form equations and solve them to find the omitted angles. Teacher A helped learners to use angle properties to find the missing angles. He asked questions such as *"...How do we get that size? Is there any clue...? (TAV4, 58)*. In the process, he told learners that one could fill in other angles even if they were not required. This would help them to find the required angles. Learners were using angle properties to find the missing angles while Teacher A was using *GeoGebra* for learners to check their answers, as illustrated by the following conversation (TAV4, 60 - 69):

60. TA: I hear 112, I am not going to calculate anything, I am just going to show and to see whether it is true, we can still check it. BIG
61. Ls: 68
62. TA: I also hear 68, I am about to .... Once I just point there the angle will come. Ok, let's check. So it is.....
63. Ls: 68
64. TA: Thank you, so 68 is the one which is correct. This one (referring to 112) is wrong. Maybe I don't know the one that got 68 what did you do? (Pointing to a learner) you got 68, how do you get 68?
65. Ls: I first get the size of KJF
66. TA: OK. How does that help you to get the other one? Ok let me put that one

KJF (TA used GeoGebra to get the angle size) KJF o, sorry, I must first get the angle, clockwise, KJF. That is also 68. And then from there…/

67. Ls: *Then from there you get a U-shape*

68. TA: *Use a what? Opposite ok, because this one is also 68. And then there is F-here*

69. Ls: *Yes*

After the class activity, some angles were discussed and added to the diagram as shown in Figure 4.9.

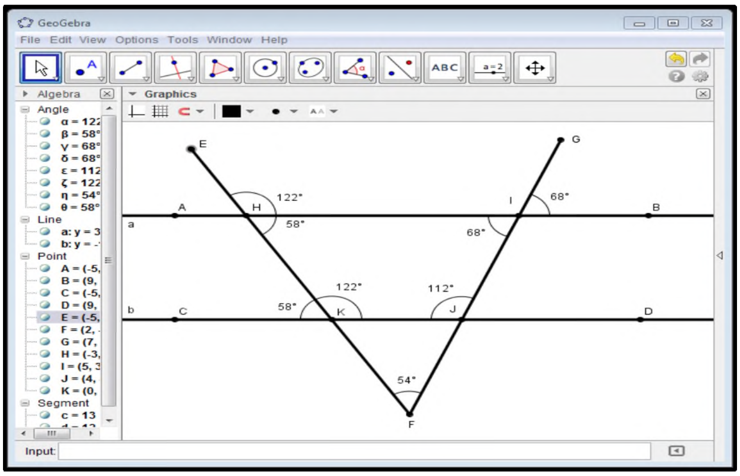


Figure 4.9: Screen shot of a complex geometrical shape showing measurements of some angles

During this time, Teacher A used one image dynamically to make the conceptual links between the algebra and geometry. *GeoGebra* visualisations made it easy for the teacher to provide accurate demonstrations about angle properties even in the complex shape. In this lesson, Teacher A discussed angles in parallel lines, angles in triangles, and angles in quadrilaterals, as illustrated in Figure 4.9.

After some minutes, Teacher A distributed the worksheets to learners to work in pairs or in groups of three per laptop. He told the learners to switch on their laptops. Then he instructed them to draw the complex shape that was projected on the white board. This was the most challenging moment in lesson 4 for Teacher A. Most of the groups could not draw the shape without support from the teacher. Teacher A concluded the lesson by justifying learners' answers using *GeoGebra*.

The next lesson description illustrates how Teacher B presented angles in complex shapes using *GeoGebra* as a visualisation tool.

4.3.10 Lesson 4 Teacher B: Angles in a complex shape

Teacher B started the lesson by greeting learners and welcoming them. She told the learners that they would continue with angle properties. She specified that the lesson was a continuation of angles, and would be a combination of the three lessons they had done with *GeoGebra* visualisations. Teacher B opened a complex shape with *GeoGebra* and asked learners to mention the name of the shape projected on the screen. The following extract tells how learners were struggling to give the name of the shape.

7. *TB* *Fine, aa... on the board we are having a shape like that. Ok, tell me what is the name of the shape? Tell me what is the name of the shape? Anybody please can you rise up your hand? And tell me what do you think is the name of that shape? Marius!*
8. *L* *I think it is a triangle*
9. *TB* *You think it is a triangle? Or you think there is a triangle? But it is one shape, right it is one big shape. Give it a name, give it a name. Kaetanus, what do you think it is? (After a while) Innocent? What is the name of that shape give it a name please. I think Malakia wants to try. The name?*
10. *L* *Triangle*
11. *TB* *Triangle? We are already said triangle has three sides. Are you only seeing three sides or three angles there? (After a while)*
12. *L* *No*
13. *TB* *Ok, I understand we cannot mention the name of the shape because it is too complicated. There are a lot shapes that are joined together to form up a shape like that. So we will call it a* ***complex shape****. It is a complex shape because it is complicated it consists of a lot of shapes, ok. Now which shapes can you see there? You have to tell me the name of that shape by using the letters Martha*

Teacher B continued to ask learners to identify and name the different geometrical shapes within the complex shape. Furthermore, she asked learners to identify pairs of angles that are co-interior angles, corresponding angles and alternate angles. Learners were able to identify these angles by first identifying shapes such as 'U" shape for co-interior angles, 'F' shape for corresponding angles and 'Z' shape for alternate angles.

From my observation, it was a very lively interaction between the teacher and learners because learners were able to identify pairs of angles from the different directions and positions. Teacher B then used *GeoGebra* to measure the identified pairs of angles. She first measured alternate angles, followed by corresponding angles and then co-interior angles. In the process of measuring these angles, she emphasised the clockwise direction when using *GeoGebra* to measure any angle. Teacher B facilitated the class activity by asking learners to identify pairs of angles that are equal to each other. She then confirmed learners' answers by measuring with *GeoGebra*. Another interesting thing she did was to colour similar pairs of angles with the same colour.

117.	*TB*	*But we are saying the one that are the same should have the same colour. 61 and 61 and I have to change this one so that it can also be the same colour. Ok, now we have this one we make it blue dark blue and then come to this one we make it the same colour as well. The one at the bottom same colour. Why do we make same colour again?*
118.	*Ls*	*Because they are of the same size.*
119.	*TB*	*Because they are of the same size. Why are you saying that they are of the same size?*
120.	*Ls*	*Because they are opposite*

Teacher B continued the lesson by asking learners to identify angle properties such as the exterior angle and the sum of the opposite interior angles of a triangle. Shapes of quadrilaterals such as trapeziums and triangles were also identified from a complex shape and discussed. The lesson was concluded with a task given to learners to do in pairs.

4.4 DISCUSSION OF THE DOMINANT THEMES AND FINDINGS FROM THE LESSON PRESENTATIONS AND INTERVIEWS

In this section, I discuss the data from the lesson observations and interviews. The data are discussed against the analytical framework, see Section 3.8 Tables 3.1 and 3.2. The themes that frame my analysis and discussion are:

- Image generation;
- Image inspection;
- Image transformation and image use;
- Connecting ideas and concepts in mathematics;
- Connecting mathematics to the real world;
- Connecting mathematics to prior knowledge;
- Representing mathematical ideas and analysis in different ways and
- Justifying and explaining mathematical ideas and solutions.

4.4.1 Image generation

All three teachers used *GeoGebra* to generate images during their lesson presentations, and used *GeoGebra* visuals to develop the mathematical idea at hand. All participants were able to manipulate dynamically the initial image to support their explanations. This is in agreement with Arcavi's (2003) definition of visualisation:

> The ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about, developing previously unknown ideas, and advancing understanding. (p.217)

In lesson 1, both teachers drew accurate parallel lines. They were able to scroll the image across the screen to visualise that the parallel lines do not meet. In lesson 2, both teachers used *GeoGebra* to generate different triangles accurately, which they used to explain the properties of angles in triangles. Teacher A and B used *GeoGebra* to generate an accurate quadrilateral, which they used to explain the angle properties in quadrilaterals - see Figures 4.2 and 4.3. Teachers also used diagrammatic visualisation such as mnemonic rules to simplify the concept of angles formed in parallel lines. According to Guzmán (2002), in many cases teachers use mnemonics rules to link the information to be learned with familiar and already known information through the use of a visual picture or letter/word combination. Mnemonics such as angles in parallel is the word F-U-N. This is found in most mathematics books but in a static form. However, by using *GeoGebra* teachers in this study were able to demonstrate the same mnemonic dynamically. According to the teachers, using mnemonics was a better way to advance learners' understanding about the properties of angles formed in parallel lines. This is in agreement with Duval (1999) who claimed that "representation and visualisation are the core of understanding mathematics" (p.3). Similarly Makina (2010) affirms that visualisation is a very important cornerstone in "teaching for understanding" in mathematics because it helps the teacher to facilitate the lesson, as it creates a platform where learners are more engaged with visual images.

4.4.2 Image inspection

According to Apostol & Blinn (1993), visual images make more sense than printed or spoken words only. People tend to forget words they hear or read, but images are retained for a long time because they have emotional as well as intellectual appeal.

In each lesson, images were generated by using *GeoGebra*. The generated image was then inspected in order to distinguish between angle similarities and differences. In lesson 1 for example, *GeoGebra* visualisations was used to display angles formed in parallel lines. *GeoGebra* visualisations created a platform where teachers and learners could study the image properties. This is supported by Hyles (1991), who argues that mathematics lessons can be enhanced by using computer technology, which encourages social interaction and collaboration. During their lessons, the teachers and learners inspected the position of corresponding angles, alternate angles and co-interior angles in parallel lines. With *GeoGebra*, teachers were able to colour pairs of angles with the same colour. This made it easy for the differences and similarities to be seen and discussed.

In lesson 2, Teachers A and B created tables to record the angle sizes of different triangles. This facilitated the inspection of interior angles in different triangles. Teachers and learners were able to inspect the features of different triangles, but the concept of 180^0 as the sum of interior angles in a triangle was observed in all triangles and justified by *GeoGebra* visualisations, as illustrated in Figure 4.10.

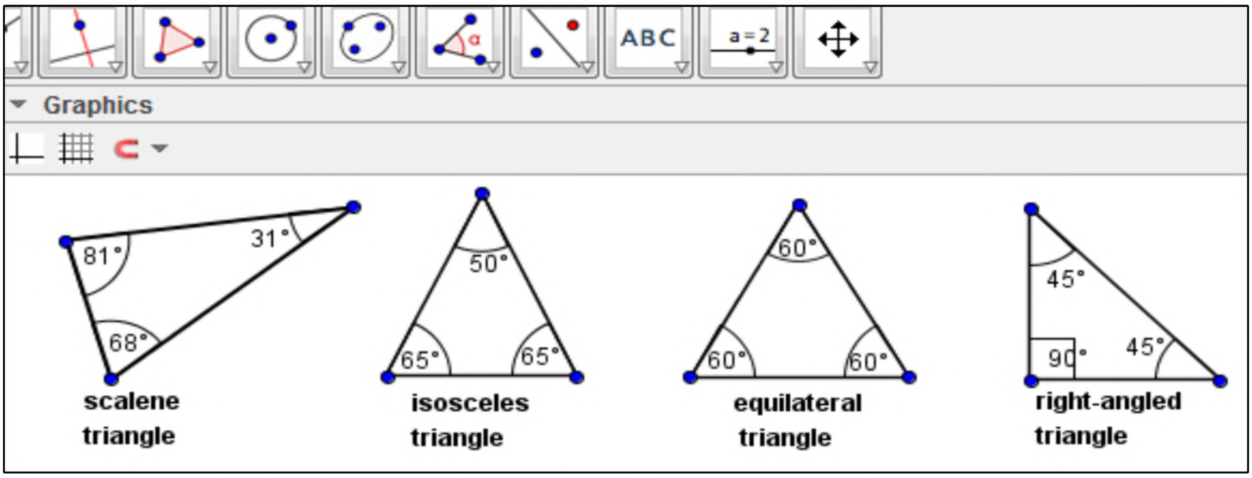


Figure 4.10: The sum of interior angles in triangles

During lesson 2, the relationship between exterior and interior angles was explored, as illustrated in Figure 4.11.

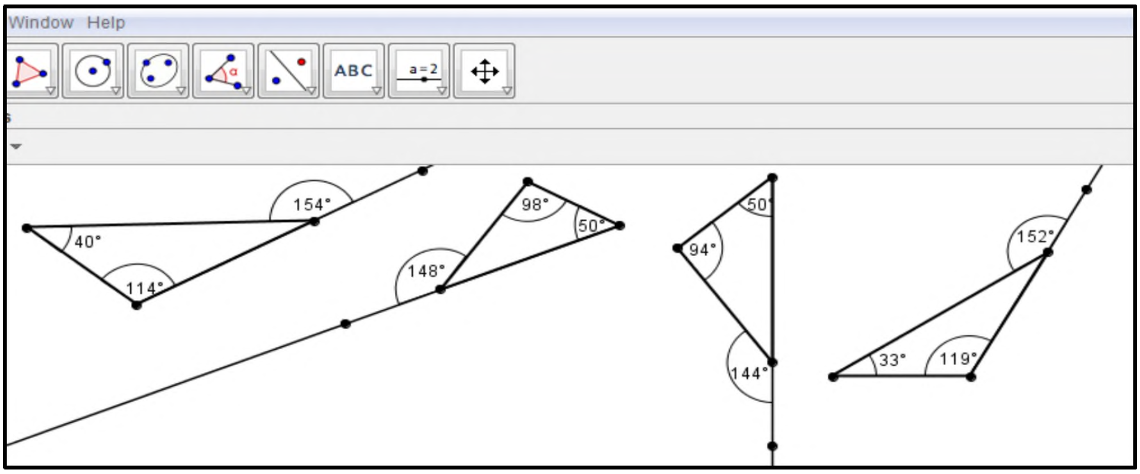


Figure 4.11: The relationship between exterior angles and the two opposite interior angles

In lesson 3, different quadrilaterals were also inspected as they were changed dynamically from one quadrilateral to the other. Teachers A and B used similar methods to teach the sum of interior angles in quadrilaterals as illustrated in Figure 4.12.

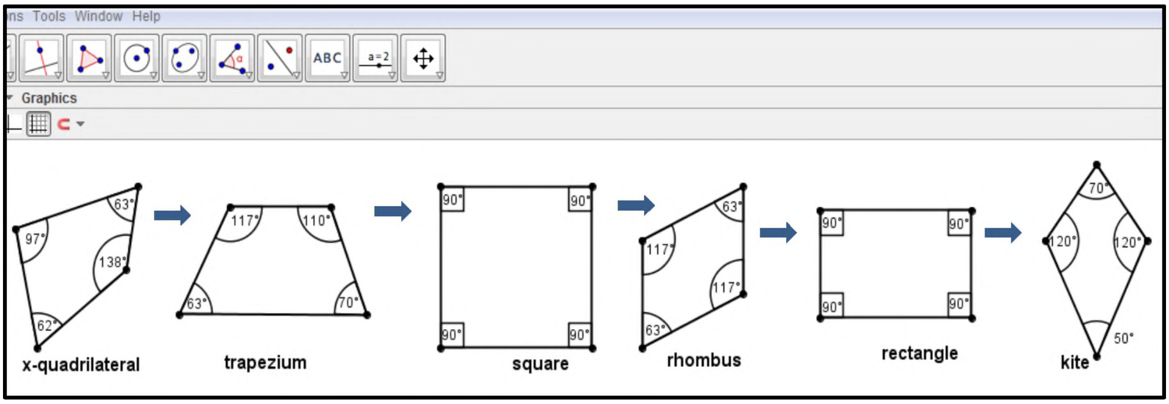
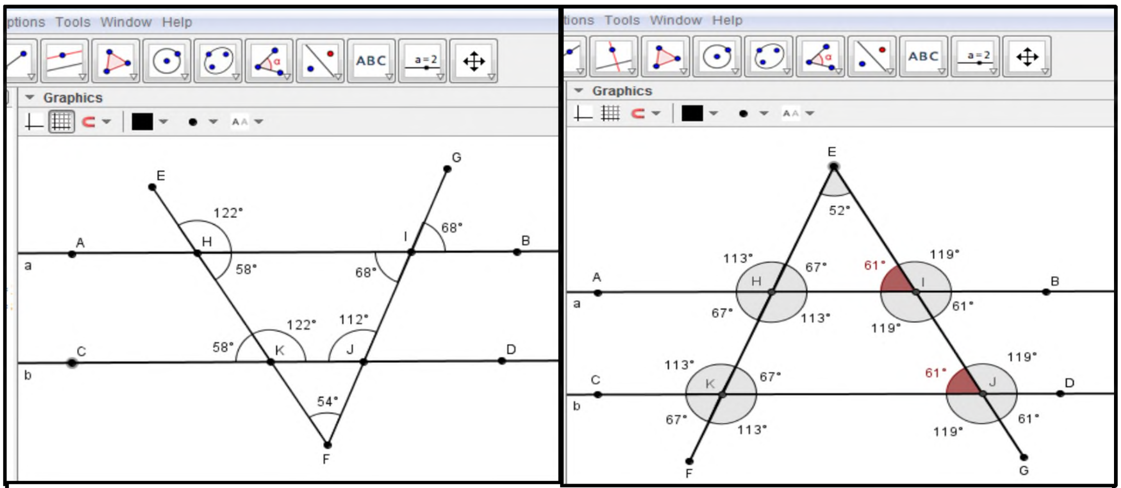


Figure 4.12: How GeoGebra was used to transform quadrilaterals

Figure 4.12 shows how the shape of quadrilaterals was transformed. Although some of the proportions of the quadrilaterals changed, the sum of the interior angles remained constant at 360^0.

In lesson 4, *GeoGebra* visualisations played a bigger role during image inspection. All teachers started by asking learners to give the name of the image on the screen. They further asked learners to identify the different geometrical shapes in the complex shape - see Figure 4.13. Learners had to examine the image and they identified shapes like trapeziums, triangles, and parallel lines. Angles formed in parallel lines were also identified.



complex shape by Teacher A complex shape by Teacher B

Figure 4.13: Complex shapes used by Teachers A and B

GeoGebra visualisations in Figure 4.13 created an environment where Teachers A and B each inspected the image together with the learners. This helped both teachers to reinforce various concepts of angle properties. In an attempt to enhance conceptual understanding, both teachers were able to support their explanations by dynamically manipulating the image on the screen.

4.4.3 Image transformation and image use

Image transformation happened concurrently with image use. In lesson 1, both teachers transformed parallel lines dynamically into new shapes. The teachers then used new shapes to demonstrate and justify the angle properties formed in parallel lines. In lesson 2, the initial triangle was transformed through (rotation, re-sized or translated) into different triangles. This demonstrated that irrespective of the size or type of any triangle, the sum of interior angles is always equal to 180^0 as shown in Figure 4.10. In lesson 3, *GeoGebra* was used to transform the initial quadrilateral into different quadrilaterals. The teachers manipulated the initial quadrilateral to demonstrate that it can be transformed into any type of quadrilateral. The key concept was to observe that the sum of interior angles in any quadrilateral was 360^0 as illustrated in Figure 4.12.

In lesson 4, transformations were used to re-size the image. Both Teachers A and B re-sized the image, and then hid some angles. They both used this as a class activity to create the discussion platform and to test learners' conceptual understanding. During this time, they used *GeoGebra* to justify learners' answers in each case.

4.4.4 Connecting ideas and concepts in mathematics

In the interview, teachers indicated that *GeoGebra* visualisation helped them to generate accurate images. They then used these images to make connections between the related mathematical topics. They drew multiple images through transforming the initial image from one shape to another. By using these multiple representations, the teachers were able to explain the concepts in depth, based on the dynamic evidence produced by *GeoGebra*. Teachers could actually make the link between geometry and algebra using *GeoGebra* visualisations. This is in agreement with Hohenwarter & Hohenwarter (2009) and Stols (2009), who stated that if *GeoGebra* is used effectively, it can help learners and teachers to specifically make connections between **Geo**metry and al**Gebra.** From the lesson observations, it was evident that teachers were able to switch smoothly from geometry to algebra, for example to find the missing angle or to form an equation and solve it. *"...Let's hide the number, the number I want to hide, come on; ok. Now we have those two but I want us to look at this (he changed the initial triangle into a right angled-triangle) we were talking about the sum of angles in a triangle is equal to 180 degrees. Now if one angle is missing. Which strategy can we use probably so that we get that angle"? (TCV2, 21).* The ability to make such mathematical connections is one of the key indicators of conceptual understanding suggested by Kilpatrick et al. (2001). *GeoGebra* is just one of the software packages available for teachers to demonstrate the conceptual links between mathematical concepts and the eventual teaching for conceptual understanding.

4.4.5 Connecting mathematics to the real world

The teachers felt that *GeoGebra* visualisations turned their lessons into a more real world context. Teacher B believed that *GeoGebra* helped her to connect parallel lines to a real world setting by using *GeoGebra* visualisations to demonstrate the properties of parallel lines. *"...you* *show them that you can extent it more and it will still have the same distance between, you know that would bring the real experience to the learners to put it into reality with other shapes or other thing like the escalators or the, the roads that we are having the parallel lines it doesn't matter how far you go it will be still parallel" (TBI, 20)* while Teacher C believed that the manipulation of points on the diagram changed the learning content to reality. *"...the manipulation of aa of points on the, on the item let say on GeoGebra is kind of visualizing and actually is not really an imagination in the lesson but is like the real world in the class... It is actually get into the mind of the learners. And I think they won't forget it (TCI, 22).*

4.4.6 Connecting mathematics to prior knowledge

Prior knowledge makes learning new knowledge easy because new information is connected to what is already known. This connection forms a bridge between old and new information (Kilpatrick et al., 2001). From the lesson observation, the teachers used *GeoGebra* to build a bridge between what is already known by learners to new concepts. The first thing the teachers did after they generated an image was to ask learners to say something about that image. The teachers were using instructions like *"...How do we call that line? The line that I just add, how do we call it"? (TAV1, 15). "... are those lines parallel"? (TBV1, 3). "... is that a parallel line"? (TCV1, 5).* This is an indication that the teachers were using the image as a stepping-stone to move from the known to the unknown. This worked very well with dynamic visualisations because in many cases it was the same initial image that was transformed into a new shape for the new concept to be learned.

4.4.7 Representing mathematical ideas and analysis in different ways

Bu & Schoen (2011) assert that *GeoGebra* is particularly well suited for teachers to represent diagrams in different ways on the screen and to dynamically transform them. Entire diagrams and parts thereof can be moved around and manipulated in many ways. As a consequence, learners are able to gain rich experiences from a variety of forms of images. There is evidence from the lesson observations that *GeoGebra* was used to present angle properties in different ways.

The common practice for both Teachers A and B was to transform the initial image into different shapes. However, the images used to demonstrate corresponding angles were different for each teacher - see for example Figures 4.2 and 4.3 for Teachers A and B respectively. This is an indication that *GeoGebra* has the potential to help teachers to represent the same mathematical idea in different ways. According to Kilpatrick et al. (2001), one of the significant indicators of conceptual understanding is being able to represent mathematical situations in different ways and knowing how different representations can be useful for different purposes. This is in agreement with Teacher B who used *GeoGebra* visualisations to connect ideas, images and concepts during her lesson presentation. She dynamically demonstrated the connection between all quadrilaterals to teach the concept of interior angles in quadrilaterals. *"...you can really see that this helped the learners to..to not just to memorize it but also to understand because they are able to visualise it. So they kept in their mind, you know they kept it because they were able to see it when it was done" (TBI, 26).*

4.4.8 Justifying and explaining mathematical ideas and solutions

One cannot talk about justifying an idea or a solution in the absence of accuracy, and proving and testing of conjectures in geometry. According to Hanna (2000), dynamic software has the potential to encourage both exploration and proof, because it makes it easy to pose and test conjectures. Dynamic geometry software helps learners and teachers to construct geometric shapes with a high level of accuracy. *GeoGebra* is one of the dynamic software packages used by teachers in this study to explore angle properties in parallel lines, in triangles and in quadrilaterals. Accuracy is one of the dominant themes from the interviews. *GeoGebra* helps teachers to construct accurate shapes. *"...Not every teacher or not every mathematics teacher is good at drawing and when you draw the shapes are not accurate... " (TBI, 16).* Another benefit noted by Teacher B was confirming the accuracy of the diagram, *"...because GeoGebra can measure angles as well as the sides' length of the figure..." (TBI, 30).* Teacher B noted that lessons with *GeoGebra* are rich in terms of image accuracy and dynamics. Using *GeoGebra* Teacher B was able to generate accurate diagrams on the screen for everybody in the class to view. *GeoGebra* visualisations influenced Teacher B's lessons in a positive way because she could use accurate diagrams. In line with these responses, accurate diagrams can help the teacher to justify and prove different concepts clearer than using an inaccurate free-hand sketch.

4.5 CONCLUSION

In this chapter, I have presented and discussed the data collected from the lesson observations and interviews. The data emanating from the lesson observations was based on visualisation processes. The data emanating from the interviews was based on the experiences and perceptions of the participating teachers. The themes used to categorise and discuss the data steadily emerged from the lesson observations and interviews.

Both teachers found that *GeoGebra* was very helpful in terms of generating accurate and dynamic visuals. In addition, teachers found that *GeoGebra* promoted active participation among learners and provided an active way to manipulate visual mathematical objects, facilitating the understanding of concepts. Dragging the mouse around made the teaching environment much more powerful than traditional paper and pencil learning. Teachers had an opportunity to dynamically represent geometric concepts that could not be easily illustrated without the use of *GeoGebra*.

CHAPTER FIVE

5. SUMMARY OF FINDINGS AND CONCLUSION

5.1 INTRODUCTION

This study aimed to investigate the role of *GeoGebra* visualisation in relation to developing learners' conceptual understanding when teaching angle properties in Grade 9. In this chapter, I summarise the key themes and research findings emerging from the study. The chapter also discusses some recommendations and suggestions for further research. Finally, I present my personal reflections of the research journey.

5.2 SUMMARY OF EMERGING THEMES AND FINDINGS

During the data analysis process, numerous themes and common threads emerged with respect to the use of *GeoGebra* in the classroom. These are image generation, image inspection, image transformation and image use, connecting mathematics to the real world, connecting mathematics to prior knowledge, representing mathematical ideas and analysis in different ways, and justifying and explaining mathematical ideas and solutions. Although each theme is summarised individually, it is acknowledged that they overlap and are intertwined.

5.2.1 Image generation

Using *GeoGebra* visualisation, teachers were able to produce accurate images and display them on the screen. They used these images to present the learning contents dynamically without necessarily employing strict definitions of theories and/or actual calculations. During the image generation process, *GeoGebra* images were generated systematically by the teachers throughout the lesson presentations.

5.2.2 Image inspection

When it was time to inspect the image on the screen, teachers and learners used the opportunity to discuss the similarities and differences of angles in a particular image. Since *GeoGebra* images were very clearly and accurately constructed, it was easy for the teachers and learners to inspect such images. There were clear links between the images on the screen and the discussion of the images by the learners and teachers.

5.2.3 Image transformation and image use

Throughout the lesson observations, the concept of angle properties was clearly demonstrated through *GeoGebra* image transformations. In all lessons, teachers demonstrated not only the geometrical relationship between images and angles, but the algebraic relationships as well. The values of angle sizes as they changed from one size to another were vividly observed. The transformation of images was always accompanied by teacher-learner discussions. The transformations were very helpful as they attracted the attention of all learners in the class and made them active participants of the topic under discussion.

5.2.4 Connecting ideas and concepts in mathematics

Teachers used *GeoGebra* visualisations to connect the mathematical concepts across mathematical domains. The findings of this study show that through displaying the images on the screen teachers were able to carefully explain the connections between geometry (the plain image), algebraic expressions (equations, formulae, theories and mathematical role) and mnemonics (special words used to help a person to remember something) such as angles in parallel lines forming the word F-U-N.

5.2.5 Connecting mathematics to the real world

Although there was not much evidence that the teachers connected all the *GeoGebra* images with the real world, mention was made of railway tracks and roads in relation to parallel lines. It was further noted that shapes are omnipresent in the real world.

5.2.6 Connecting mathematics to prior knowledge

Throughout the lesson presentations, teachers used *GeoGebra* to build bridges between what is already known by learners and the new concept to be learned.

5.2.7 Representing mathematical ideas and analysis in different ways

There is evidence from the lesson presentations that teachers were able to represent mathematics ideas through different *GeoGebra* images. The whole diagram or parts of it were moved around and manipulated in many ways. Angles were viewed from different directions. It was also interesting to notice that both teachers presented angles in parallel lines differently.

5.2.8 Justifying and explaining mathematical ideas and solutions

Using *GeoGebra* to justify and explain mathematical ideas facilitated much discussion in the lessons observed. The data gathered from the observation indicated that justification and multiple representations appeared to be tightly linked together. Through *GeoGebra* visualisations, teachers created an authentic context for justification. In the lessons observed, teachers created a space to explore and examine the truth of learners' answers.

5.3 SIGNIFICANCE OF THE STUDY

This study offers insights on how *GeoGebra* can be used effectively as a visualisation tool to develop conceptual understanding of the angle properties in Grade 9 geometry. It also highlights the role of *GeoGebra* visualisations when teaching angle properties. The study has certainly expanded my own understanding of *GeoGebra* contributions to the teaching of angle properties. It is hoped that teachers and subject advisers who read this study will gain insight into how *GeoGebra* can be used as a visualisation tool to enhance conceptual understanding. The data and experience of this study could be workshopped with mathematics teachers to help them practice *GeoGebra* visualisations in their lessons.

5.4 RECOMMENDATIONS

Although the findings of this study cannot be generalized, the results can contribute to those who wish to use visualisations for conceptual understanding in teaching angle properties in geometry. Based on the results of this study, I would like to make the following recommendations: The subject advisors for mathematics should be encouraged to provide workshops for teachers on how to use *GeoGebra* as a visualisation tool with special reference to aspects that support conceptual understanding such as:

- visualisation processes
- connecting mathematical ideas and concepts in mathematics
- connecting mathematics to the real world
- connecting mathematics to prior knowledge
- representing mathematical ideas and analysis in different ways
- justification and explaining mathematical ideas and solutions

Furthermore, the teacher-training institutions should incorporate the use of visuals from technology such as *GeoGebra* software to teach conceptual understanding in their training. From my own experience, Namibian schools in general are well resourced with computer facilities. All schools in Namibia should capitalise on the free distribution of *GeoGebra* and use it as a visualisation tool in the teaching and learning of mathematics.

5.5 LIMITATIONS

One limitation of the study is the small sample size. The findings can thus not be generalised. The study occurred only in one school and involved only three teachers. In addition, the study was conducted over a short period of tome focusing on a particular area of mathematics. Ideally, more time should have been spent on training the teachers and exposing them to the full potential of *GeoGebra*. The outcomes could have been more comprehensive if more than three teachers used *GeoGebra* for a longer period of time and in different topics and in different grades. Another possible limitation could be my own position in the Education structure. Knowing that I am a school principal could have influenced the participants' responses during the interviews. The participants may have felt that they needed to ingratiate themselves with me.

5.6 SUGGESTIONS FOR FURTHER RESEARCH

The results in this study show that *GeoGebra* can be effectively used as a visualisation tool in the teaching of angle properties in Grade 9 geometry. Suggestions for future research could include:

- expanding the mathematical domain to include other areas of mathematics
- researching the importance of *GeoGebra* in learning
- researching the impact of *GeoGebra* on mathematics performance

5.7 PERSONAL REFLECTION

My research journey taught me many lessons. This was the first time I engaged in 'proper' academic research. This not only exposed me to a wide spectrum of literature, but also improved my academic writing skills. Coming to the end of this M.Ed. study gives me a clear picture of the kind of researcher and person I am. My ambition is to fully integrate technology in mathematics education starting with my own teaching. During this study I gained deep insights into the practices of my participants (and my own), with regard to using technology in mathematics to teach for conceptual understanding. I am looking forward to implementing these skills in my own teaching as a mathematics teacher. I am also ready to share them with other mathematics teachers on platforms such as workshops and regional e-learning conferences, as well as in the National Mathematics Congresses.

5.8 CONCLUSION

Technology such as the *GeoGebra* software can be used as a powerful mathematical tool for teaching. It has the capacity and potential to be used as a visualisation tool that can aid and promote teaching for conceptual understanding. The software creates a dynamic environment where teachers and learners are engaged with the subject content and context. It encourages social interaction and is an interesting alternative to textbooks in terms of its images and visual representations. Visualisations in textbooks often focus only on the details of the images, but not on the visualisation process. Thus, the use of *GeoGebra* software has the potential to contribute to contemporary teaching strategies and projects such as VIZNAMZA, which seek to promote the effective use of visualisation processes in the mathematics classroom.

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APPENDICES

Appendix A: Letter from the Regional Director

REPUBLIC OF NAMIBIA

OSHANA REGIONAL COUNCIL
DIRECTORATE OF EDUCATION, ARTS AND CULTURE
Aspiring to Excellence in Education for All

Private Bag 5518
Oshakati, NAMIBIA

Tel: 065-230057
Fax: 065 – 230035
E-mail: otrc_physical_science@yahoo.co.uk
Enquiries: Maria Udjombala
Ref 12/2/1

3 March 2016

Mr Erasmus Mwiikeni
Ondjora CS
Ompundja Circuit
081 22406486

Dear Mr Mwiikeni

RE: **REQUEST FOR PERMISSION TO CONDUCT EDUCATIONAL RESEARCH AT ONDJORA COMBINED SCHOOL**

Your correspondence dated 26 January 2016 regarding the above mentioned subject has a reference.

Kindly be informed that permission to conduct research study at [redacted] Circuit, Oshana Region is herewith granted.

However, please take note that the research activities should not interfere with the normal programmes of the schools and the participation should be on a voluntary basis.

We wish you the best of luck with your research and hoping that your findings will be shared with other stakeholders in the Region and beyond.

Yours Sincerely

DIRECTORATE OF EDUCATION
OFFICE OF THE DIRECTOR OF EDUCATION
04 MAR 2016
PRIVATE BAG 5518
OSHAKATI
OSHANA REGIONAL COUNCIL
REPUBLIC OF NAMIBIA

04.03.16

MR IMMANUEL S. AIPANDA
ACTING REGIONAL DIRECTOR

CC: Inspector of Education, Ompundja Circuit

Appendix B: Teachers' consent forms

Teacher consent form

"An analysis of how GeoGebra can be used as a visualisation tool by selected teachers to develop conceptual understanding of the properties of geometric shapes in Grade 9 learners at Ondjora Combined School in Oshana Region".

I, the undersigned, confirm that (please tick box as appropriate):

1.	I have read and understood the aims, objectives and details of this research project.	✓
2.	I understand that my teaching, my interviews and discussions with Mr Mwiikeni will be video and audio-recorded. I recognize that these will be interesting and thought-provoking. If, however, I feel uncomfortable in any way during the interview sessions, I have the right to decline to answer any question or to end the interview.	✓
3.	My participation in this project is entirely voluntary. I understand that I will not be paid for my participation.	✓
4.	I may withdraw and discontinue participation at any time without penalty and prejudice.	✓
5.	I understand that Mr Mwiikeni will not identify me by name in any reports using information obtained from the videos and interviews. My confidentiality as a participant in this study will remain secure.	✓
6.	The technical processes of the audio and video recordings have been explained to me.	✓
7.	The use of the data by Mr Mwiikeni in his research, potential publications, sharing and archiving has been explained to me.	✓
8.	Notwithstanding the above ethical considerations I have no objections if Mr Mwiikeni wishes to use my name.	✓
9.	I have read and understood the explanation provided to me. I have had all my questions answered to my satisfaction, and I voluntarily agree to participate in this study.	✓
10.	I have been given a copy of this consent form.	✓

[redacted] ____________ 30/04/2016

Name of Participant — Signature — Date

ERASMUS MWIIKENI ____________ 30/04/2016

Name of Researcher — Signature — Date

Teacher consent form

"An analysis of how GeoGebra can be used as a visualisation tool by selected teachers to develop conceptual understanding of the properties of geometric shapes in Grade 9 learners at Ondjora Combined School in Oshana Region".

I, the undersigned, confirm that (please tick box as appropriate):

1.	I have read and understood the aims, objectives and details of this research project.	✓
2.	I understand that my teaching, my interviews and discussions with Mr Mwiikeni will be video and audio-recorded. I recognize that these will be interesting and thought-provoking. If, however, I feel uncomfortable in any way during the interview sessions, I have the right to decline to answer any question or to end the interview.	✓
3.	My participation in this project is entirely voluntary. I understand that I will not be paid for my participation.	✓
4.	I may withdraw and discontinue participation at any time without penalty and prejudice.	✓
5.	I understand that Mr Mwiikeni will not identify me by name in any reports using information obtained from the videos and interviews. My confidentiality as a participant in this study will remain secure.	✓
6.	The technical processes of the audio and video recordings have been explained to me.	✓
7.	The use of the data by Mr Mwiikeni in his research, potential publications, sharing and archiving has been explained to me.	✓
8.	Notwithstanding the above ethical considerations I have no objections if Mr Mwiikeni wishes to use my name.	✓
9.	I have read and understood the explanation provided to me. I have had all my questions answered to my satisfaction, and I voluntarily agree to participate in this study.	✓
10.	I have been given a copy of this consent form.	✓

[redacted] 30/04/2016

Name of Participant Signature Date

ERASMUS MWIIKENI 30/04/2016

Name of Researcher Signature Date

Appendix C: Interview questions

Semi-structured interview questions for the two participant teachers

1. In what ways (if any) did the use of *GeoGebra* facilitate your teaching?
2. Briefly share your experience of using *GeoGebra* in the four mathematics lessons you taught.
3. In your view, do you think it is important that mathematics teachers know about *GeoGebra*?
4. How does *GeoGebra* help you to incorporate real-world setting and applications in your mathematics lessons?
5. How do you use *GeoGebra* to teach for conceptual understanding? What did you do with *GeoGebra* to develop learner's understanding of concepts?
6. Can you share with me how you have used *GeoGebra* to teach concepts that are related and connected?
7. Would you share with me the benefit of using *GeoGebra* as a visualisation tool in your mathematics class?
8. How did you use *GeoGebra* as a visualisation tool?
9. How does *GeoGebra* influence your lessons presentations?
10. What challenges did you experience when you were using *GeoGebra*?
11. Does a lesson with the use of *GeoGebra* differ from the one without *GeoGebra*? If yes, how does it differ? What changed?
12. Did *GeoGebra* contribute in achieving your lesson learning objectives?
13. Would you share some advantages and disadvantages you noticed regarding the use of *GeoGebra* in mathematics?
14. Finally, do you have any comment about using *GeoGebra* as a visualisation tool in mathematics?

Appendix D1: Lesson plan for lesson One

Date:

GRADE: 9

Subject: Mathematics **Teacher:**

Topic: *Angles in parallel lines* **Duration:** *40 Min*

Lesson Contents: *corresponding, alternate and co-interior angles in parallel lines.*

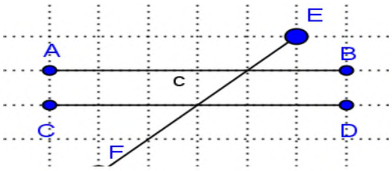
Teaching Resources: Chalkboard, *Projector, Laptops loaded with GeoGebra, Worksheet and Colour pencils*

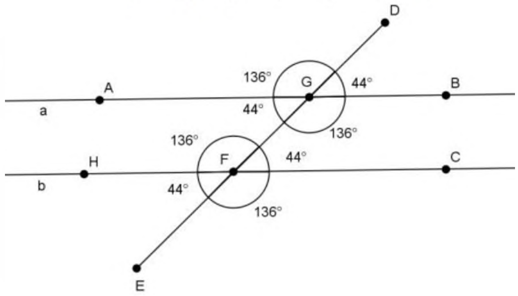
Lesson objective: By the end of the lesson, learners should be able to:

- *Identify pairs of equal angles in parallel lines.*
- *Find the missing angle formed within parallel lines.*

Lesson Presentation

Stage	Teacher activity	Learner activity	Remarks about the process of visualisation and the role of *GeoGebra* visualisation observed
Prior knowledge	Draw the pair of parallel lines on the chalk-board. Ask learners if they know anything about the lines on the chalkboard. Ask them any real life example where such representations can be found. Examples like roads, elevators, ladders and railways are expected.	Learners are expected to respond and reveal their pre-knowledge.	
Introduction	Introduce the topic by connecting the new concept with learners' prior knowledge. Example: Two lines are parallel if they are always the same distance apart and in the same plane.  Real life objects are like railway, ladders, elevators and others. Use *GeoGebra* draw line AB parallel to line CD. Explain by using *GeoGebra* example: same distance between line AB and CD and never meet.	Learners are expected to follow and observe visuals as the teacher demonstrates using *GeoGebra.*	

<table>
<tr>
<td>Content presentation</td>
<td>When a straight line (transversal) crosses parallel lines it crosses at the same angles. Use GeoGebra to show this.




Drag point A or D or B to change the size of angles, ask learners to observe and notice angles which are always equal. Identify and show learners angles, which are, corresponding, alternate and co-interior angles. By using GeoGebra, prove that the sum of co-interior angles is 180^0. Use check tool to uncheck some angles and ask learners to give the value of the missing angle.</td>
<td>Learners observe and take notes. They are also expected to respond to the questions posed by the teacher.</td>
<td></td>
</tr>
<tr>
<td>Lesson practice and assessment</td>
<td>Hand the work sheet to learners in pairs to complete by using GeoGebra as per instructions given in the worksheet.</td>
<td>Learners in their pairs will use GeoGebra to complete the work sheet.
Task 1. Learners will identify all equal pairs of angles formed within parallel lines and colour them with the same colour.
Task 2. Learners will find the size of angles marked with letters (see the work sheet) and later use GeoGebra to confirm their answers.</td>
<td></td>
</tr>
<tr>
<td>Conclusion</td>
<td>When a straight line crosses parallel lines, it crosses them at the same angles. Use GeoGebra to explain this. Emphasize alternate angles, corresponding angles and the co-interior angles, plus the sum of co-interior angles.</td>
<td>Learners observe, listen and take notes</td>
<td></td>
</tr>
</table>

Homework:

Lesson reflection

Worksheet for Lesson One

We have just learned that in parallel lines certain pairs of angles are equal.

Task 1: Open worksheet for lesson one task 1 in *GeoGebra.* Click and drag points A, B or E to change the angles.

Notice the pairs of angles that are always equal and color them with the some colour in figure 2. (4)

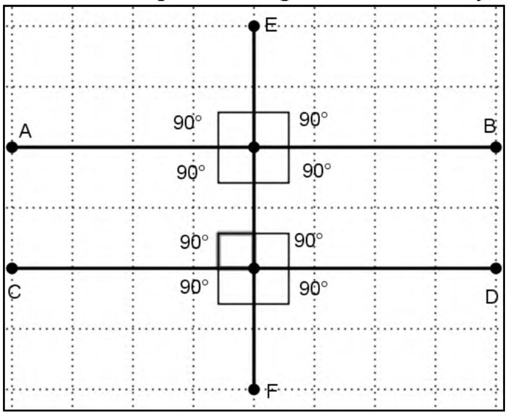


Figure 1 step 1

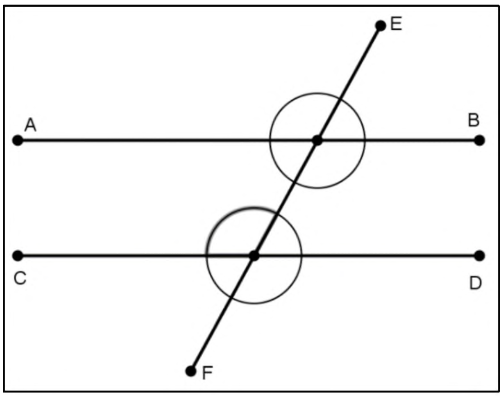
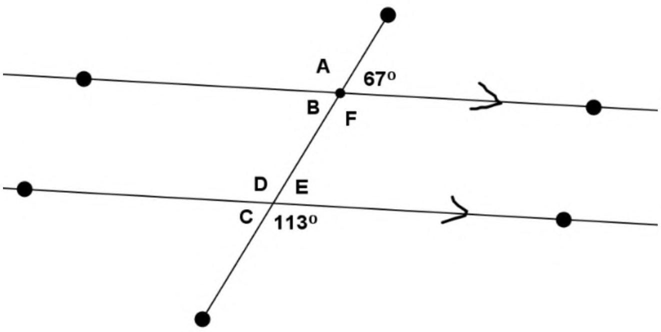


Figure 1 step 2

Task 2: (a) Using the patterns in task 1 find the size of angles marked with letters in the figure below.

(b) Open worksheet for lesson one task 2 in *GeoGebra* to confirm your answers.



A =

B =

C =

D =

E =

Appendix D2: Lesson plan for lesson Two

Date:

GRADE: 9

Subject: Mathematics

Teacher:

Topic: *Angles in triangles*

Duration: *40 Min*

Teaching Resources: *White board, Projector, Laptops loaded with GeoGebra, Calculators and a Worksheet*

Lesson Content: *Interior and exterior angles of a triangle.*

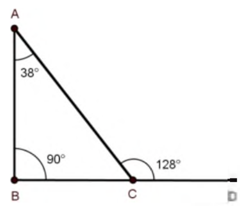
Lesson objective: By the end of the lesson, learners should be able to:

- *Understand that the exterior angle of a triangle is equal to the sum of the two interior opposite angles.*
- *Use angle properties to find the missing angle in a triangle.*

Lesson Presentation

Stage	Teacher activity	Learner activity	Remarks about the process of visualisation and the role of *GeoGebra* visualisation observed
Prior knowledge	Use GeoGebra to draw a right-angled triangle. Use the drawing to ask learners to reveal their pre-knowledge about triangles (three sided plane figures).Ask the following questions: What is the name of the shape on the screen? How many angles and how many sides does it have?	Learners are expected to respond and reveal their pre-knowledge about triangles	
Introduction	A triangle is a closed figure with three sides and three angles. We learnt that the sum of interior angles in a triangle is 180°. Use GeoGebra to show that the sum of interior angles in any triangle is 180°.	Learners are expected to follow and observe visuals as the teacher demonstrate the sum of angles in a triangle using GeoGebra.	

<table>
<tr>
<td>Content presentation</td>
<td>
Use GeoGebra to draw a tringle. By dragging any side or vertex of triangle, you make a different triangle. Draw the following table on a whiteboard and complete it together with learners.
<table>
<tr><th rowspan="2"></th><th rowspan="2">Type of triangle</th><th colspan="4">Size of angle in C^0</th></tr>
<tr><td>Angle A</td><td>Angle B</td><td>Angle C</td><td>Total</td></tr>
<tr><td>1st Triangle</td><td>X – triangle</td><td>34</td><td>104</td><td>42</td><td></td></tr>
<tr><td>2nd Triangle</td><td>Scalene triangle</td><td></td><td></td><td></td><td></td></tr>
<tr><td>3rd Triangle</td><td>Right-angled triangle</td><td></td><td></td><td></td><td></td></tr>
<tr><td>4th Triangle</td><td>Equilateral triangle</td><td></td><td></td><td></td><td></td></tr>
<tr><td>5th Triangle</td><td>Isosceles triangle</td><td></td><td></td><td></td><td></td></tr>
</table>
Complete the column for totals.
Ask learners to describe their observation and conclusion.
</td>
<td>Learners observe and participate by responding to what they are observing. They are also expected to ask questions or give contributions.</td>
<td></td>
</tr>
<tr>
<td>Lesson practice and assessment</td>
<td>
Draw a triangle on the white board extent one side to form exterior angle. Label clearly the interior angle and exterior angle. Tell the learners that they are expected to discover the relationship between interior angle A, B and exterior angle C.

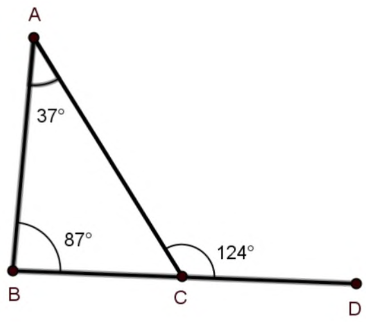


Hand the worksheet to learners in pairs. Let learners open the document titled 'worksheet for lesson two, task with GeoGebra'. A figure like the one on the whiteboard will be opened. Assist those that are unable to open the required file. Learners in pairs will complete the table in the work-sheet.
</td>
<td>Learners in pairs will open the diagram with GeoGebra and work according to the instructions in the worksheet.</td>
<td></td>
</tr>
<tr>
<td>Conclusion</td>
<td>The sum of angles in a triangle equals 180°
Exterior angle of a triangle is equal to the sum of the two opposite interior angles. Use GeoGebra to prove this.</td>
<td>Learners observe and take notes</td>
<td></td>
</tr>
</table>

Worksheet for Lesson Two

Task 1. Using the properties of Angles in triangles

Open the triangle below with *GeoGebra.*

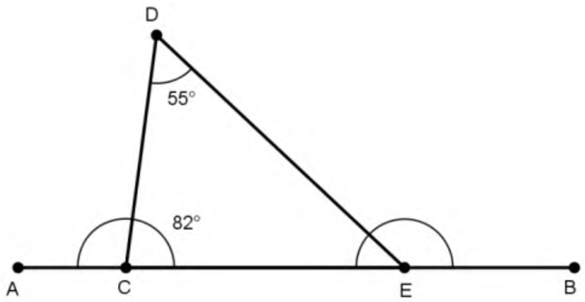


Drag any vertex on the shape to change the size of interior angles and the exterior angle. Fill in the table each time you make different angles. The first has been done for you.

	Interior angle A	Interior angle B	Sum of A+B	Exterior angle C
1st shape	41	90		131
2nd shape				
3rd shape				
4th shape				
5th shape				

(a) Complete the column for Sum (A+B)
(b) What is your conclusion about the table above?

Task 2. Using angle properties in triangles, give the size of the missing angles



angle DEC

angle BED

angle DCA

Open the work sheet for lesson two, task 2 with *GeoGebra* to confirm your answers.

Appendix D3: Lesson plan for lesson Three

Date:

GRADE: 9

Subject: Mathematics **Teacher:**

Topic: *Angles in quadrilaterals* **Duration:** *40 Min*

Lesson Contents: *Interior angles of a quadrilateral equal 360°*

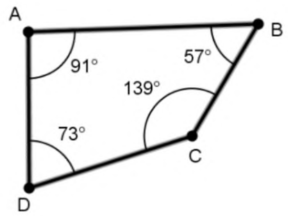
Teaching Resources: *Chalkboard, Projector, Laptops loaded with GeoGebra, Calculators and a Worksheet*

Lesson objective: By the end of the lesson, learners should be able to:

- *Determine the sum of interior angles in quadrilaterals;*
- *Use angle properties of quadrilaterals to solve problems.*

Lesson Presentation

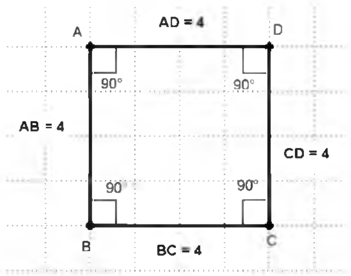
Stage	Teacher activity	Learner activity	Remarks about the process of visualisation and the role of *GeoGebra* visualisation observed
Prior knowledge	Draw a rectangle on a chalkboard. Use the drawing to ask learners to reveal their pre-knowledge about quadrilaterals (four-sided figures). Ask learners the following questions; *What is the name of the figure on the screen?* *How many sides and angles does the figure have?*	Learners are expected to respond and reveal their pre-knowledge.	
Introduction	A rectangle is an example of quadrilaterals. A quadrilateral is a closed figure that has four sides and four angles. Introduce the lesson about the sum of interior angles in quadrilaterals.	Learners are expected to follow and observe visuals as the teacher demonstrates the sum of angles in a quadrilateral (rectangle) using *GeoGebra*.	

Content presentation	Use GeoGebra to draw an irregular quadrilateral and show all interior angles. Ask learners to add all the interior angles together. The sum of all interior angles should be 360^0. 	Learners observe and respond to the questions posed to them by the teacher and take notes.	
Lesson practice and assessment	Hand the work sheet to learners in pairs to complete. Move around the class; assist learners who could not use *GeoGebra*.	Learners in pairs will use *GeoGebra* to complete the work sheet, as per worksheet instructions.	
Conclusion	The sum of angles in any quadrilateral equals 360°	Learners observe and take notes	

Homework:

Lesson reflection:

Task 1. Find the sum of interior angles in quadrilaterals. Draw the following quadrilateral with *GeoGebra*

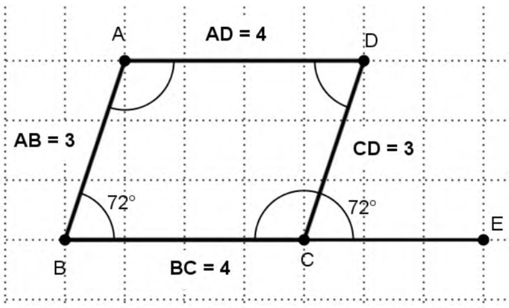


Drag the side or vertex of the given quadrilateral to make a different quadrilateral according to the table. Fill in the size of each angle in the table each time you make the required quadrilateral.

	Name of quadrilateral	Size of angles in C^0				
		Angle A	Angle B	Angle C	Angle D	Total
1^{st} quadrilateral	square	90	90	90	90	
2^{nd} quadrilateral	rectangle					
3^{rd} quadrilateral	parallelogram					
4^{th} quadrilateral	rhombus					
5^{th}quadrilateral	kite					
6^{th} quadrilateral	trapezium					

(a) Complete the column for totals
(b) What is your conclusion about the table?

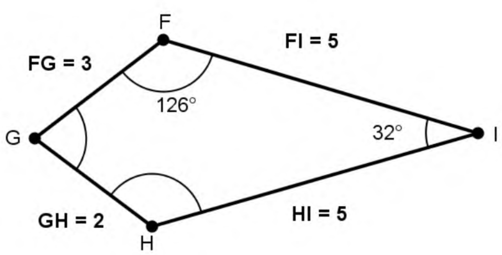
Task 2: By using the angle properties in quadrilaterals
(a) Find the value of the missing angles and then use *GeoGebra* to confirm your answers



$(i) angle$ ADC

$(ii) angle$ DCB

$(ii) angle$ BAD



$(iii) angle$ HGF

$(iV) angle$ IHG

Appendix D4: Lesson plan for lesson Four

LESSON PLAN FOR LESSON FOUR Date:

GRADE: 9

Subject: Mathematics **Teacher:**

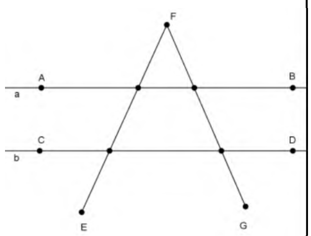
Duration: *40 Min*.

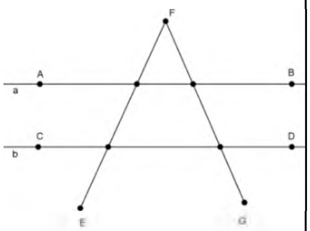
Lesson Contents: *Angles in the complex shapes. (A complex shape in this lesson is a combination of parallel lines, triangles and quadrilaterals).*
Teaching Resources: *Projector, Laptops loaded with GeoGebra, Calculators and Worksheet*

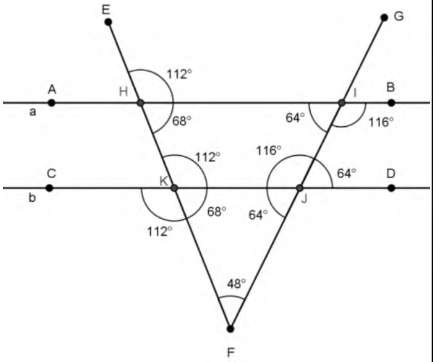
Lesson objective: By the end of the lesson, learners should be able to:

- *Identify and use angle properties to calculate unknown angles in complex shapes (combination of parallel lines, triangles and quadrilaterals).*

Lesson Presentation

Stage	Teacher activity	Learners activity	Remarks about the process of visualisation and the role of *GeoGebra* visualisation observed
Prior knowledge	To check the pre-knowledge of learners, open the complex shape with *GeoGebra*, ask learners the following questions.  *What is the name of the shape on the screen? How many Geometrical shapes can you identify in this shape? Can you name them?*	Learners are expected to respond and reveal their pre-knowledge.	

<table>
<tr><td>Introduction</td><td>Since the shape is made up of different geometrical shapes, we will call it a complex shape. The shape we see on the screen is a combination of parallel lines, triangles and quadrilaterals in one shape (complex shape).


Use GeoGebra visuals to show different Geometrical shapes in the complex shape.</td><td>Learners are expected to follow and observe visuals as the teacher changes the shape dynamically using GeoGebra.</td><td></td></tr>
<tr><td>Content presentation</td><td>Open the complex shape with GeoGebra.
Drag A or B or E or F or G to change the sizes of angles.
Discuss with learners the changes that occur in terms of the shape and the size of angles. Use the following points to lead the presentation and discussion;


- Sum of interior angles in a triangle.
- Sum of interior angles in a quadrilateral.
- Relationship between interior angles and the opposite exterior angle.
- Sum of angles at a point
-Identify pairs of angles that are; alternates, corresponding, vertical opposite and co-interior.</td><td>Learners expected to observe, participate through responding to the questions posed by the teacher and take notes</td><td></td></tr>
<tr><td>Lesson practice</td><td>Hand the work sheet to learners. Learners in pairs will complete the work sheet this time no need to use GeoGebra.</td><td>Learners in pairs will complete the worksheet as per instructions</td><td></td></tr>
<tr><td>Conclusion</td><td>Angle properties of a shape remain the same even in complex shapes.</td><td>Learners observe and take notes</td><td></td></tr>
</table>

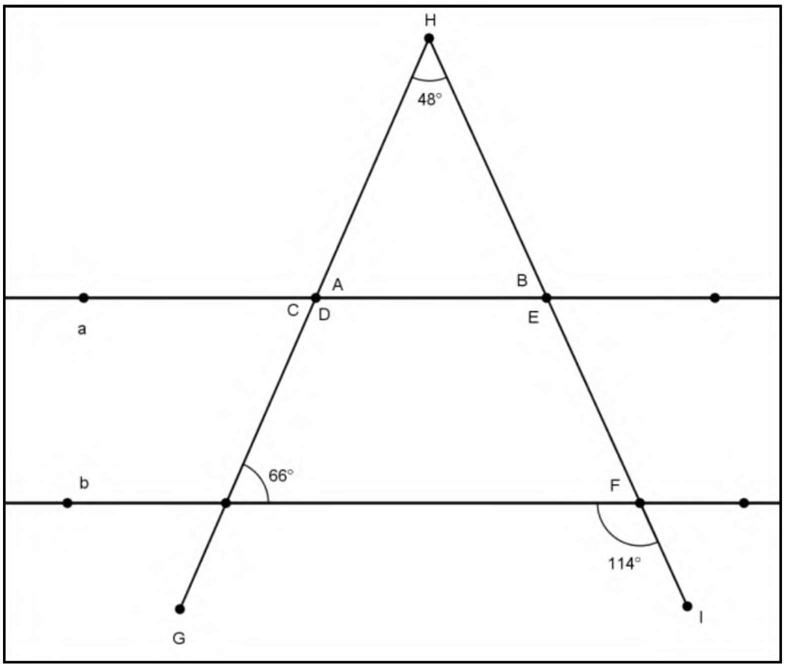
Homework:

Lesson reflection:

Worksheet for Lesson Four

Task 1.

Study the complex shape below and answer the questions that follow.



(a) Identify three Geometrical shapes in the complex shape above.

(b) Which pairs of angles are;
   i. Vertical opposite
   ii. Alternate angles
   iii. Two pairs of co-interior angles
   iv. Corresponding angles

(c) What is the sum of angles D+E+F+66? Give the reason for your answer.

(d) Find the size of the following angles. Give reasons for your answers:
   i. Angle A
   ii. Angle F
   iii. Angle C

(e) What is the sum of angle E+F? Give the reasons for your answer.

(f) What is the value of angle B + E? Give the reason for your answer.

(g) What is the sum of angle A+B+48? Give a reason for your answer.