RHODES UNIVERSITY

DEPARTMENT OF MATHEMATICS

(L,M)–FUZZY TOPOLOGICAL SPACES

by

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A THESIS SUBMITTED IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE OF MASTER OF SCIENCE IN MATHEMATICS

JANUARY 1992

ACKNOWLEDGEMENTS

First of all I wish to thank Professor W. Kotzé for his extensive involement in my academic development at Rhodes University and for his advice concerning the choice of my topic.

Dr. T. Kubiak's tireless endeavours in directing and monitoring the presentation of this thesis deserves my deepest appreciation. I sincerely feel that without his kind assistance this thesis would not have been completed.

My appreciation is also directed to Dr. J. Chadwick who has been my supportive pillar of strength throughout my whole undergraduate and graduate Mathematics development.

Dr. V. Murali who was my Mathematics school teacher continued to boost my morale even at postgraduate level.

The friendly attitude of the Mathematics staff members at Rhodes University in general created a pleasant environment in which to work.

I am grateful to my sponsors FRD, DAAD, and Rhodes University for providing me with financial resources.

I must sincerely thank Mrs G. Harwood for her diligence and patience in typing this thesis.

This effort would not have been possible without the dedicated support of my father and mother, all the members of the family and my friends in general.

CONTENTS

ABSTRACT

The objective of this thesis is to develop certain aspects of the theory of (L,M)–fuzzy topological spaces, where L and M are complete lattices (with additional conditions when necessary). We obtain results which are to a large extent analogous to results given in a series of papers of Šostak (where L = M = [0,1]) but not necessarily with analogous proofs. Often, our generalizations require a variety of techniques from lattice theory e.g. from continuity or complete distributive lattices.

1991 Mathematics Subject

Classification : 54A40, 03E72, 04A72, 06D10, 06D30.

PREFACE

The concept of a fuzzy topology, introduced by Chang [Ch] in 1968, is similar to the concept of an ordinary topology, except that it consists of fuzzy subsets instead of sets. However if one knows the meaning of a "fuzzy set", then what should be called a fuzzy topology is quite natural. One has to think of the characteristic function of a topology rather than a subset of the power set, and then replace the two point lattice $\{0,1\}$ by a complete lattice L. Thus, a fuzzy topology on a set X might be understood as a map $\tau : \mathcal{P}(X) \longrightarrow L$ satisfying the following three properties :

(1) $\tau(\emptyset) = \tau(X) = 1$ (or possibly $\tau(\emptyset) = \tau(X) > 0$),

(2) $\tau(A \cap B) \geq \tau(A) \wedge \tau(B)$ for every $A, B \in \mathcal{P}(X)$,

(3) $\tau(\bigcup \mathcal{A}) \geq \bigwedge \{\tau(A) : A \in \mathcal{A}\}$ for every $\mathcal{A} \subset \mathcal{P}(X)$.

Once such a definition is introduced the way of defining morphisms between objects of the above types is easy. Namely, $f : (X,\tau) \longrightarrow (Y,\sigma)$ becomes a morphism if $f : X \longrightarrow Y$ is a map such that $\tau(f^{-1}(A)) \geq \sigma(A)$ for every $A \in \mathcal{P}(Y)$. These ideas essentially originate from Höhle [Hö]. One can go ahead and replace the power set $\mathcal{P}(X)$ by L^X, the collection of all maps from X to L. This idea was pursued in some (not easily accessible) papers by Hamburg (see e.g. [Ha]) who also considered filters defined in this spirit. Kubiak [K1] further defined the topology τ using two different complete lattices L and M. Thus
$\tau : L^X \longrightarrow M$ is called an (L,M)–fuzzy topology if it satisfies the following three conditions :

(1) $\tau(\underline{0}) = \tau(\underline{1}) = 1$,

(2) $\tau(u \wedge v) \geq \tau(u) \wedge \tau(v)$ for all $u, v \in L^X$,

(3) $\tau(\bigvee_{i \in J} u_i) \geq \bigwedge_{i \in J} \tau(u_i)$ for $\{u_i : i \in J\} \subset L^X$.

We shall adopt Kubiak's approach in this thesis. In fact, in many instances the extra structure of L^X does not play any role, as it is the case in the Chang–Goguen L–topological spaces, and which led Hutton [Hu] to consider point–free L–topologies where L just plays the role of L^X. In our case, it is possible to define $\tau : L \longrightarrow M$ satisfying (1) – (3). However this approach will not be pursued here. Šostak ([S1] – [S5]) has made a deep study of (I,I)–fuzzy topologies where I = [0,1]. The main objective of this thesis is to extend his work to the more general setting of (L,M)–fuzzy topologies. For obvious reasons, all our lattices will at least, be complete. We obtain results which are to a large extent analogous to results in Šostak's papers. However, some of his results do not naturally extend to complete lattices, in general. It is often necessary to impose further conditions on these lattices. As a result, our generalizations require a variety of techniques from lattice theory.

A brief summary or introduction is supplied at the beginning of each chapter. Nevertheless, we give details of contents of each chapter.

Chapter 1 is of an introductory nature, giving the foundations of what is to follow. In this chapter we concentrate on lattice theory. We define several distributivity axioms on lattices. These are finite, infinite (frame) and complete distributivity. We next consider the way below relation which can sometimes play the role of the strictly less than relation in a complete chain. We introduce continuity and its characterization which leads to a Corollary which states that complete distributivity implies continuity of a lattice. The upper, lower and Scott topologies on lattices are introduced. As already mentioned, all our lattices will be complete. We will assume that they have an order reversing involution. In actual fact, the latter is not necessary for some parts of the thesis. The reader will easily notice those places himself.

Chapter 2 is concerned with L–fuzzy sets and L–topological spaces. L–fuzzy sets were first introduced by Goguen [Go] and L–topological spaces just generalize Chang's topology from I to L. Although L–topological spaces are important on their own, in here they are used as a tool in developing properties of (L,M)–fuzzy spaces. In particular, we consider the so–called generalized Lowen functor $\omega_L : TOP \rightarrow TOP(L)$ and its left inverse $\iota_L : TOP(L) \rightarrow TOP$ with L being a continuous lattice carrying its Scott topology. They will later play a role in generating (L,L)–fuzzy spaces from L–topological ones. Also, we will prove some properties of the so called fuzzy inclusion which we shall need in what follows.

Chapter 3 is the longest chapter in this thesis. This is where the important notion of (L,M)–fuzzy topological spaces is introduced. An investigation of its level L–topologies, as well as the initial and final (L,M)–fuzzy topologies is made and examples are supplied in order to clarify these notions. In the construction of an initial (L,M)–fuzzy topology we need M to be completely distributive as confirmed in Proposition 6.7 in which we have filled the gap in Šostak's [S1] proof. All the (L,M)–fuzzy spaces together with continuous mappings between them form the category FTOP(L,M). The rest of the chapter is devoted to a discussion of embedding TOP and TOP(L) into FTOP(L,L). The functors $\Gamma : TOP(L) \rightarrow FTOP(L,L)$, $\psi : FTOP(L,L) \rightarrow TOP$ are introduced and ψ is found to be the left inverse of the composition functor $\Gamma \circ \omega_L$. A sufficient condition for Γ to be a functor is that L be a frame. We also note that $\Gamma \circ \omega_L$ (with L = I) is the same as the functor $\Phi : TOP \rightarrow FTOP(I,I)$ in [S1 – 4.12].

Chapter 4 is an attempt to explore possibilities of constructing new (L,M)–fuzzy topologies from old ones. Proposition 8.2 generalizes Šostak's method of constructing an (I,I)–fuzzy topology from a decreasing family of (I,I)–fuzzy topologies. To be able to generate an (L,M)–fuzzy topology from a decreasing family of (L,M)–fuzzy topologies, we assume that M must be a complete chain. In proving this result there is a step where we need to consider two cases, while Šostak has only one case since I is order dense.

This illustrates that extending his results does not happen naturally sometimes. In Proposition 8.5 we further show that each (L,M)–fuzzy topological space is generated by its level topologies. Šostak's [S2] analogue of this result is not correct since he puts a necessary and sufficient condition for it to hold. Proposition 8.8 expresses the supremum of (L,M)–fuzzy topologies in terms of the supremum of the levels of these (L,M)–fuzzy topologies. Similar conditions and strategy as in Proposition 8.2 are employed in proving this result. It is this expression which helps us to exploit the concept of a supremum of (L,M)–fuzzy topologies usefully. The remainder of the chapter concentrates on stratification of (L,M)–fuzzy topologies. We give an example of a stratified (L,M)–fuzzy topology. Proposition 9.3 shows the relationship of stratification of an (L,M)–fuzzy topology and the stratification of its level topologies. We further establish that the stratification is a functor from FTOP(L,M) to itself. Another, yet important result is that the product of the stratification of (L,M)–fuzzy topologies on a set is the same as the stratification of the product of these (L,M)–fuzzy topologies.

In Chapter 5 we deal with a certain theory of separation and compactness in FTOP(L,M). According to Šostak such a theory of separation and compactness in FTOP(I,I) is developed in [S3] and [S4] (but see introduction to Chapter 5). In this chapter, the spectrum of T_0, T_1 or T_2–separation of an (L,M)–fuzzy topological space at a certain level is introduced and also its degree of separation. We find a fuzzy analogue of the result which states that all points are closed in T_1–topological spaces. Also the spectrum of T_0–separation is contained in that of T_1–separation which is in turn contained in that of T_2–separation of an (L,M)–fuzzy space at the same level. The compactness spectrum of an L–fuzzy set at a particular level has a number of characterizations including a result which is similar to the Alexander subbase theorem (Proposition 11.9). We also obtain an analogue of the theorem which states that the continuous image of a compact set is compact.

To facilitate references, each chapter is divided into sections that are numbered sequentially throughout the thesis and all items within a given section are numbered sequentially throughout it.

At the end of each chapter a section of notes giving additional references will appear. The bibliography which follows, consisting only of those items to which direct reference is made, comprises only a portion of the background reading done.

CHAPTER 1

LATTICES

SUMMARY

This chapter is introductory in nature and considers definitions and results we need on lattices. Section 1 contains some general background material. In Section 2, the way below relation, complete distributivity and continuity are introduced. We then prove that complete distributivity implies continuity. We also introduce a number of topologies on lattices. The notion of quasi–complementation is considered (all our lattices will be endowed with quasi–complementation). We then prove that De Morgan laws hold in L.

1. A BACKGROUND IN LATTICES

We collect here some properties of lattices other than $I = [0,1]$. Basic information like definitions of lattices, totally ordered, directed sets will be assumed to be known.

1.1 DEFINITION

A complete lattice L is one in which every subset $A \subset L$ has a supremum ($\bigvee A$) and an infimum ($\bigwedge A$). We call a totally ordered complete lattice a complete chain.
We say a complete chain is order dense if $a < b$ implies $a < c < b$ for some $c \in L$.

NOTE : (a) All our lattices will at least have the completeness property.

(b) See [Ri] for a sufficient and necessary condition for an order dense chain to be isomorphic with I.

1.2 PROPOSITION

Let L be a lattice with 0 and 1. For L to be a complete lattice it is sufficient to assume the existence of suprema (or the existence of infima).

PROOF

Suppose all subsets of L have suprema. Let $X \subset L$, we must show that $\bigwedge X$ exists. Let $A = \bigcap_{x \in X} \{y \in L : y \leq x\}$, the set of lower bounds of X. If $X = \emptyset$, put $A = L$. We now show that $\bigvee A = \bigwedge X$.

If $x \in X$, then x is an upper bound of A, hence $\bigvee A \leq x$. Since $\bigvee A$ is a lower bound for each $x \in X$, we have $\bigvee A \in A$. Since for all $y \in A$, $y \leq \bigvee A$ and elements of A are lower bounds of X, $\bigvee A$ is the greatest lower bound of X.

Given $(L,\leq)$ one defines $(L, \leq^{op})$ with $a \leq^{op} b$ iff $b \leq a$ in $(L,\leq)$. We write L^{op} for $(L,\leq^{op})$. Suprema in $(L,\leq)$ become infima in L^{op} and vice versa. Therefore any formula in $(L,\leq)$ is valid in L^{op} after replacing $\bigwedge$ by $\bigvee$ and $\bigvee$ by $\bigwedge$. This new formula is valid for L^{op} and not in general in $(L,\leq)$. This is the duality principle.

1.3 DEFINITION

A lattice L is distributive if $\forall$ x, y, z $\in$ L

(a) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

(b) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

REMARK

(a) and (b) are equivalent .

1.4 DEFINITION

A complete lattice L is said to be infinitely distributive or a frame iff

$a \wedge \bigvee B = \bigvee_{b \in B} (a \wedge b)$ for each $a \in L$ and $B \subset L$.

1.5 REMARK

We have $a \wedge \bigvee B = \bigvee_{b \in B} (a \wedge b)$ for each $a \in L$ and $B \subset L$ iff

$(\bigvee A) \wedge (\bigvee B) = \bigvee \{a \wedge b : a \in A, b \in B\}$ for A, $B \subset L$.

PROOF

$\Rightarrow$ Let $a = \bigvee A$, thus

$$(\bigvee A) \wedge (\bigvee B) = \bigvee_{b \in B} (\bigvee A \wedge b)$$

$$= \bigvee_{b \in B} \bigvee_{a \in A} (a \wedge b)$$

$$= \bigvee \{a \wedge b : a \in A, b \in B\}$$

$\Leftarrow$ Suppose A has only one element a. Then we have $a \wedge \bigvee B = \bigvee \{a \wedge b : b \in B\}$.

We introduce some notation, which will remain fixed throughout.

NOTATION

(1) Let $\{T_s : s \in S\}$ be a nonempty collection of nonempty sets and write $\prod_{s \in S} T_s$ for the product of these sets. Its elements are those functions $f : S \longrightarrow \bigcup_{s \in S} T_s$ such that $f(s) \in T_s$ for all $s \in S$.

(2) T^S will denote the collection of all functions $\emptyset : S \longrightarrow T$.

2 COMPLETELY DISTRIBUTIVE AND CONTINUOUS LATTICES

In this section we shall discuss complete distributivity and continuity of lattices.

We now define complete meet distributivity and complete join distributivity. The two definitions are in fact equivalent, but we do not include a proof for that, because our lattices will always be endowed with a quasi–complementation, on account of which these two definitions are trivially equivalent.

2.1 DEFINITION

Let L be a complete lattice and $\{x_{st} : s \in S, t \in T_s\} \subset L$. We say that L is :

(a) Completely meet distributive (CMD) if

$$\bigwedge_{s \in S} \bigvee_{t \in T_s} x_{st} = \bigvee_{\varphi \in \prod_S T_s} \bigwedge_{s \in S} x_{s\,\varphi(s)}.$$

(b) Completely join distributive (CJD) if

$$\bigvee_{s \in S} \bigwedge_{t \in T_s} x_{st} = \bigwedge_{\varphi \in \prod_S T_s} \bigvee_{s \in S} x_{s\,\varphi(s)}.$$

A lattice which satisfies (a) and (b) is called completely distributive.

2.2 THEOREM

Every complete chain is completely distributive.

PROOF

We show that CMD holds in a complete chain L. If $\{x_{st} : s \in S, t \in T\} \subset L$, then for each $\varphi \in T^S$ and $s \in S$ we have :

$$\bigvee_{t \in T} x_{st} \geq x_{s\,\varphi(s)},$$

which implies

$$\bigwedge_{s \in S} \bigvee_{t \in T} x_{st} \geq \bigwedge_{s \in S} x_{s\,\varphi(s)}.$$

Thus

$$\bigwedge_{s \in S} x_{s\,\varphi(s)} \leq \bigwedge_{s \in S} \bigvee_{t \in T} x_{st} = y$$

for every $\varphi \in T^S$.

We show that y is the least upper bound of $\{ \bigwedge_{s \in S} x_{s\,\varphi(s)} : \varphi \in T^S\}$. Suppose $u \in L$ is another upper bound of this family. We show that $y \leq u$. Assume the converse

i.e. $y > u$. Since u is an upper bound of the family $\{ \bigwedge_{s \in S} x_{s\,\varphi(s)} : \varphi \in T^S\}$ we have,

(a) $u \geq \bigwedge_{s \in S} x_{s\,\varphi(s)}$ for each $\varphi \in T^S$,

also

$$u < \bigwedge_{s \in S} \bigvee_{t \in T} x_{st} = y,$$

hence

$$u < \bigvee_{t \in T} x_{st} \text{ for all } s \in S.$$

Thus for all $s \in S$, there exists $t \in T$ such that $u < x_{st}$, i.e. there is $\varphi_u : S \longrightarrow T$ such that for all $s \in S$, $u < x_{s\,\varphi_u(s)}$. Hence $u \leq \bigwedge_{s \in S} x_{s\,\varphi_u(s)}$. Hence $u = \bigwedge_{s \in S} x_{s\,\varphi_u(s)}$ by (a).

If $u \leq v < y$, then $u = v$ since by the same argument as above we get

$$v = \bigwedge_{s \in S} x_{s\,\varphi_u(s)}.$$

Since $u < x_{s\ \varphi_u(s)}$ for all $s \in S$ we must have $y \leq x_{s\ \varphi_u(s)}$ for all $s \in S$ (for otherwise, $u < x_{s\ \varphi_u(s)} < y$ implies $u = x_{s\ \varphi_u(s)}$). Thus

$y \leq \bigwedge_{s \in S} x_{s\ \varphi_u(s)} = u$ a contradiction to $y > u$.

The Theorem is proved.

The strict less–than relation on the unit interval has the following two important features :

(a) For each $a \in I$, $a = \bigvee \{b : b < a\}$, and

(b) For each $a < b$ in I there is $c \in I$ such that $a < c < b$ (ORDER DENSITY PROPERTY).

Our purpose is to introduce in each complete lattice L a new relation $<<$, stronger than the lattice ordering, and to distinguish a class of complete lattices in which the relation $<<$ has the above two features of the relation $<$.

2.3 DEFINITION

Let L be a complete lattice. We say x is way below y ($x << y$) iff for any directed $D \subset L$, $y \leq \bigvee D$ implies $x \leq d$ for some $d \in D$.

2.4 REMARK

Let L be a complete lattice. Then L satisfies the general associative law, i.e. for all $\{A_i : i \in J\} \subset \mathcal{P}(L)$ we have $\bigvee (\bigcup_{i \in J} A_i) = \bigvee_{i \in J} (\bigvee A_i)$.

We have the following useful characterization of the way below relation.

2.5 REMARK

For a complete lattice L and $x, y \in L$, the following are equivalent :

(1) $x << y$;

(2) If $A \subset L$ and $y \leq \bigvee A$, there is a finite subset $F \subset A$ such that $x \leq \bigvee F$.

PROOF

(1)$\Rightarrow$(2) Let $y \leq \bigvee A$. The set $D = \{\bigvee F : F \subset A \text{ is finite}\}$ is directed, and $\bigvee D = \bigvee A$ by the general associative law. Thus there is $\bigvee F$ with F a finite subset of A such that $x \leq \bigvee F$ by (1).

(2)$\Rightarrow$(1) If $y \leq \bigvee D$ with D directed, there is a finite $F \subset D$ with $x \leq \bigvee F$. Since D is directed there exists $d \in D$ such that $\bigvee F \leq d$, hence $x \leq d$.

The next proposition will be used in the sequel often without further explanation.

2.6 PROPOSITION

If L is a complete lattice, then we have these statements for all $a, x, y, z \in L$:

(1) $x << y$ implies $x \leq y$.

(2) $a \leq x << y \leq z$ implies $a << z$.

(3) $x << z$ and $y << z$ implies $x \vee y << z$.

(4) $0 << x$.

PROOF

(1) Putting $D = \{y\}$ we get the result.

(2) Let $z \leq \bigvee D$ for some directed set $D \subset L$. Since $x << y \leq z$ we have $x \leq y \leq z$. Thus $y \leq \bigvee D$, hence there exists $d \in D$ such that $x \leq d$. Thus $a \leq d$. Hence $a << z$.

(3) Assume $z \leq \bigvee D$ for a directed set $D \subset L$. Thus there exists $d_1, d_2 \in D$ such that $x \leq d_1$ and $y \leq d_2$. Since D is directed there exists $d \in D$ such that $d_1 \vee d_2 \leq d$. Thus we have $x \vee y \leq d_1 \vee d_2 \leq d$. Hence $x \vee y << z$.

(4) Obvious.

EXAMPLE

Let L be a complete chain. If $x \neq y$ in L, then $x < y$ iff $x << y$. One has $x << x$ iff either $\bigvee \{y \in L : y < x\} < x$ (we say x is isolated from below) or $x = 0$. Indeed, if $x < y \leq \bigvee D$, then clearly $x \leq d$ for some $d \in D$. Conversely, $x << y$ implies $x \leq y$, hence $x < y$ whenever $x \neq y$. For the second statement, if $x \neq 0$ is isolated from below and $x = \bigvee D$, then clearly $x \in D$.

Also, $0 << 0$. The converse is obvious. It follows that $x = \bigvee \{y \in L : y << x\}$ for all $x \in L$. We introduce some notation at this stage.

NOTATION

In a complete lattice L, for each $x \in L$ we write :

$$\downarrow x = \{a \in L : a \leq x\}$$

$$\uparrow x = \{a \in L : x \leq a\}$$

$$\twoheaduparrow x = \{a \in L : x << a\}, \text{ and}$$

$$\twoheaddownarrow x = \{a \in L : a << x\}.$$

2.7 DEFINITION

A complete lattice is called continuous if $x = \bigvee (\twoheaddownarrow x)$ for every $x \in L$.

The above property which always hold for the less than relation in the unit interval, is true for continuous lattices with the way below relation.

We have the following characterization of continuity of a lattice.

2.8 REMARK

A complete lattice L is continuous iff the following condition holds :
$x \not\leq y$ in L implies there exists an $a \in L$ such that $a \not\leq y$ and $a << x$.

PROOF

If L is continuous, then $x = \bigvee (\twoheaddownarrow x) \not\leq y$ implies that there is an $a << x$ with $a \not\leq y$.

Conversely, assume that L fails to be continuous. Since $\bigvee (\twoheaddownarrow x) \leq x$ always, we have

$x \not\leq \bigvee (\twoheaddownarrow x) = y.$

We have therefore found an $a << x$ such that $a \nleq y$ which contradicts the fact that $a << x$ implies $a \leq y$.

The next result provides an equational characterization of continuous lattices. This will show that completely distributive lattices are continuous.

2.9 THEOREM

For L a complete lattice, the following are equivalent :

(1) L is continuous.

(2) Let $\{x_{st} : s \in S,\ t \in T_s\} \subset L$ be such that $\{x_{st} : t \in T_s\}$ is a directed set for every $s \in S$. Then :

$$\bigwedge_{s \in S} \bigvee_{t \in T_s} x_{st} = \bigvee_{f \in \prod_{s \in S} T_s} \bigwedge_{s \in S} x_{sf(s)}.$$

PROOF

(1)$\Longrightarrow$(2) One easily sees that

$$y = \bigwedge_{s \in S} \bigvee_{t \in T_s} x_{st} \geq \bigvee_{f \in \prod_{s \in S} T_s} \bigwedge_{s \in S} x_{sf(s)} = z$$

We prove the reverse inequality by showing that $\Downarrow y \subset \downarrow z$. For if $a \in \Downarrow y$, then $a << \bigvee \{x_{st} : t \in T_S\}$ for every $s \in S$. Since $\{x_{st} : t \in T_s\}$ is directed, there is $t = f(s) \in T_s$ such that $a \leq x_{sf(s)}$ for all $s \in S$. This yields $a \leq \bigwedge \{x_{sf(s)} : s \in S\}$. Thus $a \in \downarrow z$.

(2)$\Longrightarrow$(1) Let $x \in L$ and S be the set of all directed subsets s of L with $\bigvee s \geq x$. For each $s \in S$ let $T_s = s$. Let $x_{st} = t$ for $s \in S$ and $t \in T_s$. Thus condition (2) holds. Suppose $f \in \prod_{s \in S} T_s$ and let

$a = \bigwedge \{x_{sf(s)} : s \in S\} = \bigwedge \{f(s) : s \in S\}$, since $x_{st} = t$.

Then $a << x$ since if $x \leq \bigvee D$ where D is directed, then $a \leq f(D) \in D \in S$ by definition of a. Now $x = y$, since $\{x_{st} : s \in S,\ t \in T_s\}$ is directed. Hence by (2), $x = z = \bigvee (\downarrow x)$ which is our result.

We have the following result as a consequence of 2.9.

2.10 COROLLARY

Every completely distributive lattice is continuous.

In a continuous lattice one has the so-called Interpolation Property (see below) which in some instances can play the role of the order density property in I. However, note that the element c with $a << c << b$ need not be distinct from both a and b.

2.11 PROPOSITION

Let L be a continuous lattice. If $a, b \in L$ and $a << b$, there exists $c \in L$ such that $a << c << b$. (INTERPOLATION PROPERTY).

PROOF

Since $a << b = \bigvee \{c : c << b\} = \bigvee \{\bigvee \downarrow c : c << b\} = \bigvee (\bigcup_{c << b} \downarrow c)$, there exist $d_1, \cdots, d_n \in \bigcup_{c << b} \downarrow c$ such that $a \leq d_1 \vee \cdots \vee d_n = d$ by 2.5. For every d_i there is $c_i << b$ with $d_i << c_i$. Put $c = c_1 \vee \cdots \vee c_n$. Then $d << c$. Since $c_i << b$ for each i, we have $c << b$. Thus $a \leq d << c$ and $c << b$, so that $a << c << b$.

2.12 PROPOSITION

Let L be a continuous lattice. For each directed set D of L, $y << \bigvee D$ iff $y << d$ for some $d \in D$.

PROOF

$\Rightarrow$ By 2.11 there exists $y_1, y_2 \in L$ such that $y << y_1 << y_2 << \bigvee D$. Thus $y_2 \leq \bigvee D$. Since $y_1 << y_2$ there exists $d \in D$ such that $y_1 \leq d$.

Thus $y << d$. For the converse, $y << d \leq \bigvee D$, hence $y << \bigvee D$.

2.13 DEFINITION

A subset X of a lattice L is said to be order generating iff $x = \bigwedge (\uparrow x \cap X)$ for all $x \in L$.

2.14 DEFINITION

An element p in a lattice L is called prime iff the relation $x \wedge y \leq p$ always implies $x \leq p$ or $y \leq p$. The set of prime elements is denoted by PRIME L.

2.15 DEFINITION

An element p in a lattice L is called coprime iff the relation $x \vee y \geq p$ always implies $x \geq p$. The set of coprime elements is denoted by COPRIME L.

2.16 DEFINITION

Let L be a complete lattice. We define :

(i) The upper topology SUP(L) to be the topology generated by the sets $L \backslash \downarrow x$ with $x \in L$.

(ii) The lower topology INF(L) to be the topology generated by the sets $L \backslash \uparrow x$ with $x \in L$.

(iii) We call $U \subset L$ Scott open, if :

(a) U is an upper set (i.e. $\uparrow x \subset U$ for each $x \in U$),

(b) $\bigvee D \in U$ implies $D \cap U \neq \emptyset$ for every directed set $D \subset L$.

The family of all Scott open sets form a topology which is called the Scott topology denoted by $\sigma(L)$.

2.17 PROPOSITION

Let L be a continuous lattice. Then $\{\twoheaduparrow x : x \in L\}$ is a basis for the Scott topology.

PROOF

We first show that $\twoheaduparrow a \in \sigma(L)$ for every $a \in L$. We have $\twoheaduparrow a = \{x : x >> a\}$. If $b \in \uparrow x$ then $a << x \leq b$, thus $b \in \twoheaduparrow a$ hence $\uparrow x \subset \twoheaduparrow a$. Also $\bigvee D \in \twoheaduparrow a$ implies $\exists\ c \in L$ such that $a << c << \bigvee D$ (by 2.11). There is $d \in D$ with $c \leq d$, whence $a << d$. Thus $D \cap \twoheaduparrow a \neq \emptyset$ for every directed set $D \subset L$. Now let $x \in U \in \sigma(L)$. Since $\twoheaddownarrow x$ is directed and $x = \bigvee \twoheaddownarrow x$ by property of $\sigma(L)$ there exists a such that $a \in \twoheaddownarrow x \cap U$. Hence $x \in \twoheaduparrow a \subset U$.

2.18 PROPOSITION

Let L be a continuous lattice. Then $\wedge, \vee : (L,\sigma(L)) \times (L,\sigma(L)) \longrightarrow (L,\sigma(L))$ are continuous.

PROOF

Suppose $a \vee b \in V \in \sigma(L)$. We shall show that there exist U_a and U_b, Scott open neighbourhoods of a and b respectively such that $U_a \vee U_b \subset V$.

Since $\{\twoheaduparrow x : x \in L\}$ is a basis of $\sigma(L)$, there exist $c \in L$ such that $a \vee b \in \twoheaduparrow c$. We have

$$c << a \vee b = {}'\bigvee \twoheaddownarrow a \vee {}'\bigvee \twoheaddownarrow b = {}'\bigvee (\twoheaddownarrow a \vee \twoheaddownarrow b).$$

Since $(\twoheaddownarrow a \vee \twoheaddownarrow b)$ is directed, there exists $x << a$ and $y << b$ such that $c << x \vee y$. We have $\twoheaduparrow x \vee \twoheaduparrow y \subset \uparrow (x \vee y) \subset \twoheaduparrow c$, where $U_a = \twoheaduparrow x$ and $U_b = \twoheaduparrow y$ are Scott open neighbourhoods of a and b respectively.

The proof of continuity of $\wedge$ is similar and therefore omitted.

We shall now define quasi–complementation. It is clear that this is quasi–complementation in the sense that $a \wedge a' = 0$ and $a \vee a' = 1$ do not hold in general. We then prove that De Morgan laws hold in a complete lattice L.

2.19 DEFINITION

A unary operation on a lattice $(L,\leq)$ is called quasi–complementation if for all $a, b \in L$ the following hold :

(1) $a'' = a$,

(2) $a \leq b \Longrightarrow b' \leq a'$.

Then $(L,\leq,')$ is called a lattice with quasi–complementation. When $L = [0,1]$ the quasi–complementation is understood always as $a' = 1-a$ for all $a \in [0,1]$.

2.20 PROPOSITION

If a complete lattice L has a quasi–complementation, then :

(1) $(\bigvee_r a_r)' = \bigwedge_r a_r'$

(2) $(\bigwedge_r a_r)' = \bigvee_r a_r'$

(3) $0' = 1$ and $1' = 0$

for all $\{a_r : r \in R\} \subset L$.

PROOF

(1) We have

$$\bigvee_r a_r \geq a_r \ \forall_r ,$$

$$\Rightarrow \quad (\bigvee_r a_r)' \leq a_r' \ \forall_r ,$$

$$\Rightarrow \quad (\bigvee_r a_r)' \leq \bigwedge_r a_r' \qquad \text{(a).}$$

We also have

$$\bigwedge_r a_r \leq a_r \ \forall_r ,$$

$$\Rightarrow \quad (\bigwedge_r a_r)' \geq a_r' \ \forall_r ,$$

$$\Rightarrow \quad (\bigwedge_r a_r)' \geq \bigvee_r a_r',$$

$$\Rightarrow \quad (\bigwedge_r a_r')' \geq \bigvee_r a_r'' = \bigvee_r a_r \text{ (Replacing } a_r \text{ by } a_r'),$$

$$\Rightarrow \quad \bigwedge_r a_r' \leq (\bigvee_r a_r)' \qquad \text{(b).}$$

Thus (a) and (b) prove (1).

(2) From (1) we have

$$(\bigvee_r a_r')' = \bigwedge_r a_r''$$

$$\Rightarrow \quad (\bigvee_r a_r')' = \bigwedge_r a_r$$

$$\Rightarrow \quad \bigvee_r a_r' = (\bigwedge_r a_r)' .$$

(3) Follows from (1) :

$$(\bigvee L)' = 1' = \bigwedge L' = 0.$$

NOTES

The source of the theory of distributive lattices is [B–D]. Information about order–reversing involution (which after [Ra] we call quasi–complementation) is standard in the literature about fuzzy sets, but a formal definition is in [Bi] or [Ra]. The remainder of the Chapter is obtained from [Co]. Note that 2.18 is in [Co] but half of the proof is different from that in [Co].

CHAPTER 2

L–FUZZY SETS AND L–TOPOLOGICAL SPACES

SUMMARY

In this chapter preliminary results and definitions on L–fuzzy sets and L–topological spaces are considered. In Section 3, L–fuzzy sets and properties of their images and preimages under maps are studied. We also develop results concerning the degree of inclusion of an L–fuzzy set by another. In Section 4, the category TOP(L) of L–topological spaces is defined and functors between TOP and TOP(L) are studied. These functors generalize the so called Lowen functors. In particular, with L a continuous lattice, the functor ω_L which replaces the topology of a topological space by the L–topology consisting of all Scott continuous functions is discussed. Among others, we establish that TOP is embedded as a full subcategory of TOP(L).

3 L–FUZZY SETS

3.1 DEFINITIONS AND NOTATION

In the sequel the two point lattice $\{0,1\}$ will be denoted by 2, the unit interval $[0,1]$ by I. If A is a subset of X, the characteristic function of A denoted by 1_A is defined by :

for any $x \in X$

$$1_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

Let L be a complete lattice. An L–fuzzy set A on a set X is a function $A : X \longrightarrow L$. In the sequel L–fuzzy sets will be denoted by symbols u, v, w etc.

If $c \in L$, then the constant L–fuzzy set with value c is denoted by $\underline{c}$.

Let L^X denote the family of all maps from a set X to L. The partial ordering on L^X is defined by :

Given $u, v \in L^X$,

$$u \leq v \text{ iff } u(x) \leq v(x) \text{ for all } x \in X.$$

For any family $\{u_i : i \in J\}$ we thus have :

$$(\bigvee_i u_i)(x) = \bigvee_i u_i(x) \quad \text{and}$$

$$(\bigwedge_i u_i)(x) = \bigwedge_i u_i(x)$$

for all $x \in X$.

If L has a quasi–complementation, then we define a quasi–complementation pointwise on L^X by :

$$u'(x) = (u(x))' \text{ for all } x \in X \text{ and } u \in L^X.$$

Hence de Morgan laws in L^X become,

$$(\bigvee_i u_i(x))' = \bigwedge_i u'_i(x)$$

$$(\bigwedge_i u_i(x))' = \bigvee_i u'_i(x)$$

for all $x \in X$ and $\{u_i : i \in J\} \subset L^X$.

Given $f : X \rightarrow Y$, we define the L–fuzzy set $f^{\leftarrow}(v)(v \in L^Y)$ in X by

$$f^{\leftarrow}(v)(x) = v(f(x)) \text{ for all } x \in X.$$

So $f^{\leftarrow}(v)$ is just vf, the composition of f and v. In what follows we will sometimes write vf for $f^{\leftarrow}(v)$. Conversely, if $u \in L^X$, then the L–fuzzy set $f^{\rightarrow}(u)$ in Y is defined by

$$f^{\rightarrow}(u)(y) = \begin{cases} \bigvee \{u(x) : x \in f^{-1}(y)\}, & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

for all $y \in Y$, where $f^{-1}(y) = \{x \in X : f(x) = y\}$.

We will sometimes write f(u) instead of $f^{\rightarrow}(u)$ for $u \in L^X$ in the sequel.

The following proposition can easily be verified.

3.2 PROPOSITION

If $f : X \longrightarrow Y$, $A \subset X$, $B \subset Y$, $\{U_i : i \in J\} \subset L^Y$, then the following properties hold :

1. $f^{\rightarrow}(1_A) = 1_{f(A)}$;

2. $f^{\leftarrow}(1_B) = 1_{f^{-1}(B)}$;

3. $f^{\leftarrow}(\bigvee_i u_i) = \bigvee_i f^{\leftarrow}(u_i)$;

4. $f^{\leftarrow}(\bigwedge_i u_i) = \bigwedge_i f^{\leftarrow}(u_i)$;

5. $f^{\leftarrow}(u') = (f^{\leftarrow}(u))'$, for any $u \in L^Y$;

6. $f^{\rightarrow}(v') \geq (f^{\rightarrow}(v))'$, for any $v \in L^X$;

7. $u \geq f^{\rightarrow}(f^{\leftarrow}(u))$, for any $u \in L^Y$;

8. $v \leq f^{\leftarrow}(f^{\rightarrow}(v))$, for any $v \in L^X$;

9. If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ and $u \in L^Z$, then $(g \circ f)^{\leftarrow}(u) = f^{\leftarrow}(g^{\leftarrow}(u))$, where $g \circ f$ is the composition of f and g.

From 1 and 2 it follows that $f^{\rightarrow}$ and $f^{\leftarrow}$ extend the usual image and preimage maps, respectively.

We now define the fuzzy inclusion. Here inclusion of an L–fuzzy set by another is allowed to vary from 0 to 1 in a lattice.

3.3 DEFINITION

Let L be a complete lattice with an order reversing involution. Fuzzy inclusion is a function $F : L^X \times L^X \rightarrow L$ defined by

$$F(u,v) = \bigwedge_{x \in X} (u' \vee v)(x).$$

The value F(u,v) shows the extent to which the L–fuzzy set u is contained in v.

We look at some of the properties of inclusion which will be useful to us.

3.4 PROPERTIES

1. If $A, B \subset X$, then $F(1_A, 1_B) = 1 \Leftrightarrow A \subset B$.

2. If $u, u_1, v, v_1 \in L^X$, then $v \leq v_1$ and $u_1 \leq u$ implies $F(u,v) \leq F(u_1,v_1)$.

3. Let L be a frame. If $\{u_i, v_j : i \in K, j \in J\} \subset L^X$, then

$$F(\bigvee_i u_i, \bigvee_j v_j) \geq \bigwedge_{i,j} F(u_i, v_j).$$

4. Let L be a frame. If $\{u, v_i : i \in J\} \subset L^X$, then

$$F(u, \bigwedge_i v_i) = \bigwedge_i F(u, v_i).$$

5. If $f : X \rightarrow Y$, then $F(uf, vf) \geq F(u,v)$ for all $u, v \in L^Y$.

6. If $u, v \in L^X$, then $F(u,v) = F(v', u')$.

PROOFS

(1) We have $F(1_A, 1_B) = \bigwedge_{x \in X} (1_{X\setminus A} \vee 1_B)(x) = 1$ iff $1_{X\setminus A} \vee 1_B = 1_X$ iff $A \subset B$.

(2) Since $u_1 \leq u$ implies $u' \leq u'_1$

and

$v \leq v_1$

we have

$$\forall x \in X, (u' \vee v)(x) \leq (u'_1 \vee v_1)(x).$$

Thus

$$\bigwedge_{x \in X} (u' \vee v)(x) \leq \bigwedge_{x \in X} (u'_1 \vee v_1)(x)$$

i.e.

$$F(u,v) \leq F(u_1, v_1).$$

(3) Let L be a frame. If $\{u_i, v_j : i \in K, j \in J\} \subset L^X$, then

$$F(\bigvee_i u_i, \bigvee_j v_j) = \bigwedge_{x \in X} (\bigwedge_i u_i'(x) \vee \bigvee_j v_j(x))$$

$$= \bigwedge_{x \in X} (\bigvee_j (\bigwedge_i u_i'(x) \vee v_j(x)))$$

(by associativity of supremum)

$$= \bigwedge_{x \in X} \bigvee_j \bigwedge_i (u_i'(x) \vee v_j(x))$$

(by dual frame law)

$$\geq \bigwedge_{x \in X} \bigwedge_i \bigwedge_j (u_i' \vee v_j)(x)$$

$$= \bigwedge_{i,j} \bigwedge_{x \in X} (u_i' \vee v_j)(x)$$

$$= \bigwedge_{i,j} f(u_i, v_j).$$

(4) Let L be a frame. If $\{u, v_i : i \in J\} \subset L^X$, then

$$F(u, \bigwedge_i v_i) = \bigwedge_{x \in X} (u' \vee (\bigwedge_i v_i))(x)$$

$$= \bigwedge_{x \in X} (u'(x) \vee \bigwedge_i v_i(x))$$

$$= \bigwedge_{x \in X} \bigwedge_i (u'(x) \vee v_i(x))$$

(by dual frame law)

$$= \bigwedge_i \bigwedge_{x \in X} (u' \vee v_i)(x)$$

$$= \bigwedge_i F(u, v_i).$$

(5) If $f : X \longrightarrow Y$, $u, v \in L^Y$, then

$$F(uf,vf) = \bigwedge_{x \in X} (u' f(x) \vee vf(x))$$

$$= \bigwedge_{f(x) \in f(X)} (u' f(x) \vee vf(x))$$

$$\geq \bigwedge_{y \in Y} (u'(y) \vee v(y))$$

$$= F(u,v).$$

(6) If $u, v \in L^X$, then

$$F(u,v) = \bigwedge_{x \in X} (u' \vee v)(x)$$

$$= \bigwedge_{x \in X} (v'' \vee u')(x)$$

$$= F(v',u').$$

4 L–TOPOLOGICAL SPACES.

We define L–topological spaces, interior of L–fuzzy sets, infima, suprema, stratification of L–topological spaces and some interesting properties of these notions.

4.1 DEFINITION

Let L be a complete lattice. An L–topology is a family $T \subset L^X$ which satisfies the following conditions :

(1) $\underline{0}, \underline{1} \in T$;

(2) If $u, v \in T$, then $u \wedge v \in T$;

(3) If $\{u_i : i \in J\} \subset T$, then $\bigvee_i u_i \in T$.

The pair (X,T) is an L–topological space.

4.2 DEFINITION

Let (X,T) be an L–topological space and $u \in L^X$. We define the interior of u by,

$$\mathrm{Int}_T\, u = \bigvee \{v \in T : v \leq u\}.$$

The following properties can easily be verified.

4.3 PROPERTIES

Let (X,T) be an L–topological space,

(1) $\mathrm{Int}_T u \leq u$;

(2) $\mathrm{Int}_T(u \wedge v) = \mathrm{Int}_T u \wedge \mathrm{Int}_T v$;

(3) $\mathrm{Int}_T 1 = 1$;

(4) $\mathrm{Int}_T(\mathrm{Int}_T u) = \mathrm{Int}_T\, u$;

(5) u is open iff $\mathrm{Int}_T u = u$;

4.4 DEFINITION

Let L be a complete lattice and $\mathcal{G} = \{S^i : i \in J\}$ be a family of L–topologies on X. The intersection of all members of $\mathcal{G}$ is called the infimum of the family $\mathcal{G}$. It will be denoted by $\bigwedge_{i \in J} S^i$.

4.5 PROPOSITION

Let $\{T^i : i \in J\}$ be a collection of L–topologies on X. Then T, defined by

$T = \bigwedge_{i \in J} T^i$, is an L–topology.

PROOF

(1) We have $\underline{0}, \underline{1} \in T$, since $\underline{0}, \underline{1} \in T^i$ for each $i \in J$.

(2) If $u, v \in T$, then $u, v \in T^i$ for each $i \in J$. Thus $u \wedge v \in \bigwedge_{i \in J} T^i = T$.

(3) If $\{u_j : j \in K\} \subset T$, then $\{u_j : j \in K\} \subset T^i$ for each $i \in J$.

Hence $\bigvee_{j \in K} u_j \in \bigwedge_{i \in J} T^i = T$.

4.6 DEFINITION

Let L be a complete lattice and let $\mathscr{T} = \{T^i : i \in J\}$ be a family of L–topologies on X. The intersection of all L–topologies containing $\bigcup_{i \in J} T^i$ is called the supremum of the family $\mathscr{T}$. We will denote it by $\bigvee_{i \in J} T^i$.

4.7 DEFINITION

Let (X,T) and (Y,S) be L–topological spaces. We say $f : (X,T) \to (Y,S)$ is continuous if for each $v \in S$, we have $vf \in T$. We will sometimes write f is (T,S)–continuous.

We supply the following characterization of continuity in addition to those listed in [P–L] and [Wa].

4.8 PROPOSITION

Let (X,T) and (Y,S) be L–topological spaces, $f : (X,T) \to (Y,S)$ be a mapping. Then the following two properties are equivalent :

1. f is (T,S)–continuous ;

2. $(\text{Int}_S\, v)f \leq \text{Int}_T(v\, f)$ for each $v \in L^Y$.

PROOF

(1) $\Longrightarrow$ (2)

Let $v \in L^Y$. We have :

$$\text{Int}_S v \leq v,$$

thus

$$(\text{Int}_S v)f \leq vf,$$

hence

$$(\text{Int}_S v)f \leq \text{Int}_T(vf),$$

since $(\text{Int}_S v)f$ is open.

(2) $\Longrightarrow$ (1)

Let $v \in S$, then

$$vf = (\text{Int}_S v)f \leq \text{Int}_T vf,$$

hence

$$vf = \text{Int}_T(vf),$$

thus

$$vf \in T.$$

4.9 PROPOSITION

Let (X,T^i) and (Y,S^i) be L–topologies for each $i \in J$. If f is (T^i,S^i)–continuous for each $i \in J$, then f is $(\bigwedge_{i \in J} T^i, \bigwedge_{i \in J} S^i)$–continuous.

PROOF

If $u \in \bigwedge_{i \in J} S^i$, then $u \in S^i$ for all $i \in J$. Thus $uf \in T^i$ for each $i \in J$, by (T^i,S^i)–continuity of f. Hence $uf \in \bigwedge_{i \in J} T^i$.

4.10 PROPOSITION

Let (X,T^i) and (Y,S^i) be L–topologies for each $i \in J$. If f is (T^i,S^i)–continuous for each $i \in J$, then f is $(\bigvee_{i \in J} T^i, \bigvee_{i \in J} S^i)$–continuous.

PROOF

If $S = \{u \in L^Y : uf \in \bigvee_{i \in J} T^i\}$, then S is the strongest L–topology on Y making f continuous from $(X, \bigvee_{i \in J} T^i)$ to Y. We have $S^i \subset S$, for all $i \in J$. Hence $\bigvee_{i \in J} S^i \subset S$, which is our result.

4.11 PROPOSITION

If $f : (X,T) \to (Y,S)$ and $g : (Y,S) \to (Z,W)$ are continuous functions, then $g \circ f : (X,T) \to (Z,W)$ is continuous.

PROOF

If $v \in W$, then $g^{\leftarrow}(v) \in S$ and hence by continuity of f, $f^{\leftarrow}(g^{\leftarrow}(v)) \in T$. But then $(g \circ f)^{\leftarrow}(v) = f^{\leftarrow}g^{\leftarrow}(v) \in T$.

4.12 DEFINITION

Let L be a complete lattice. An L–topology T on a set X is called stratified if it contains all the constants.

4.13 DEFINITION

Let L be a complete lattice. Let T be an L–topology on a set X. The weakest stratified L–topology T^c such that $T^c \geq T$ is called the stratification of the L–topology T. Thus the stratification T^c of T is $T \vee \{\underline{c} : c \in L\}$.

4.14 PROPOSITION

Let L be a complete lattice. If (X,T) and (Y,S) are L–topologies, then (T,S)–continuity of a map $f : X \longrightarrow Y$ implies (T^c,S^c)–continuity of f.

PROOF

Let the collection of constants in L^X and L^Y be C_X and C_Y respectively.
Then $f : (X,C_X) \longrightarrow (Y,C_Y)$ is continuous and also f is (T,S)–continuous. By 4.10 f is (T^c,S^c)–continuous.

NOTATION

(1) We denote by TOP(L) the category whose objects are L–topological spaces and morphisms continuous functions between them.

(2) As usual |TOP(L)| denotes all objects of TOP(L). The same convention applies to any other category.

REMINDER

A subset $U \subset L$ is Scott open if

(a) U is an upper set (i.e. $\uparrow x \subset U$ for each $x \in U$).

(b) $\bigvee D \in U$ implies $D \cap U \neq \emptyset$ for every directed set $D \subset L$.

The family of Scott open sets forms a topology which we call the Scott topology. We denote this by $\sigma(L)$. Also $\{\Uparrow x : x \in L\}$ is a basis for the Scott topology.

NOTATION

Let S_0 be a collection of L–fuzzy sets. The intersection of all L–topologies containing S_0 is denoted by $<<S_0>>$.

4.15 DEFINITION

Let L be a complete lattice. For each $(X,T) \in |TOP|$ define an L–fuzzy space $(X,\omega_L(T))$ with :

$$\omega_L(T) = <<C((X,T), (L,\sigma(L)))>>,$$

where $C((X,T), (L, \sigma(L)))$ is the collection of continuous functions $f : (X,T) \to (L,\sigma(L))$.

4.16 PROPOSITION

Let L be a complete lattice, (X,T) and (Y,S) be L–topological spaces, and $S = << S_0 >>$. Then $f : (X,T) \to (Y,S)$ is continuous iff $vf \in T$ for each $v \in S_0$.

PROOF

We will only show continuity of $f : (X,T) \longrightarrow (Y,S)$. We have that $S_1 = \{u \in L^Y : uf \in T\}$ is the strongest L–topology on Y making f continuous from (X,T) to Y. We thus have $S_0 \subset S_1$ and hence $S \subset S_1$ which completes the proof.

4.17 PROPOSITION

For a complete lattice L, if $f : (X,T) \longrightarrow (Y,S)$ is in TOP, then $f : (X, \omega_L(T)) \longrightarrow (Y, \omega_L(S))$ is in TOP(L).

PROOF

By 4.16 it is sufficient to prove that $vf \in \omega_L(T)$ for each $v \in C((Y,S), (L,\sigma(L)))$. We prove that $(vf)^{\leftarrow}(u) \in T$ for each $u \in \sigma(L)$. We have $v^{\leftarrow}(u) \in S$ by continuity of $v : (Y,S) \longrightarrow (L, \sigma(L))$ and hence $f^{\leftarrow}(v^{\leftarrow}(u)) \in T$ by continuity of $f : (X,T) \longrightarrow (Y,S)$.

4.18 COROLLARY

For each complete lattice L, $\omega_L : \text{TOP} \longrightarrow \text{TOP(L)}$ is a functor which assigns to each object (X,T) the object $(X, \omega_L(T))$ and which leaves morphisms unchanged.

4.19 DEFINITION

Let L be a complete lattice. For each $(Y,S) \in |\text{TOP(L)}|$ we define a topological space $(Y, \iota_L(S))$ by

$$\iota_L(S) = \bigvee \{u^{\leftarrow}(\sigma(L)) : u \in S\}.$$

NOTATION

If $v \in L^X$ and $a \in L$, then we denote $v^{\leftarrow}(\uparrow a)$ by $[a << v]$.

4.20 PROPOSITION

Let L be a complete lattice. If $f : (X,S) \longrightarrow (Y,S_1)$ is continuous, then $f : (X, \iota_L(S)) \longrightarrow (Y, \iota_L(S_1))$ is continuous.

PROOF

If u is a subbasic open set in $\iota_L(S_1)$, then $u = v^{\leftarrow}(w)$ for some $w \in \sigma(L)$ and some $v \in S_1$. $f^{\leftarrow}(u) = f^{\leftarrow}(v^{\leftarrow}(w)) = (vf)^{\leftarrow}(w)$ where $vf \in S$ and $w \in \sigma(L)$. Hence $f^{\leftarrow}(u) \in \iota_L(S)$.

4.21 COROLLARY

For each complete lattice L, ι_L : TOP(L) $\longrightarrow$ TOP is a functor which assigns to each object (Y,S) in TOP(L) the object $(Y,\iota_L(S))$ in TOP and leaves morphisms unchanged.

All along we have been using lattices which are complete. We now need the continuity property as well.

Kubiak [K2] gives a sketch of the proof of the next proposition but a detailed proof is supplied here.

With L continuous, 4.15 becomes simpler, and we have the following proposition.

4.22 PROPOSITION

If L is a continuous lattice, and (X,T) a topological space, then

$$\omega_L(T) = C((X,T),(L,\sigma(L))).$$

PROOF

By 2.18 $C((X,T), (L,\sigma(L)))$ is closed under finite infima and finite suprema. We now show that it is closed under arbitrary suprema. Let $\{f_j : j \in J\} \subset C((X,T), (L,\sigma(L)))$. In what follows, J_f denotes a finite subset of J. We know that

$$\bigvee_{j \in J} f_j(x) = \bigvee_{J_f \subset J} \bigvee_{j \in J_f} f_j(x) \quad \text{(by 2.4)} \qquad \text{(a).}$$

Next we show that $\bigvee_{j \in J_f} f_j(x)$ is directed. (b)

Suppose $\bigvee_{j \in J_{f_1}} f_j(x)$ and $\bigvee_{j \in J_{f_2}} f_j(x)$ are two of its elements.

Then $\bigvee_{j \in J_{f_1}} f_j(x) \vee \bigvee_{j \in J_{f_2}} f_j(x) \leq \bigvee_{j \in J_{f_1} \cup J_{f_2}} f_j(x)$ (in fact we have equality here)

which is still in the set since $J_{f_1} \cup J_{f_2}$ is finite.

We are now ready to prove our main step.

$$\begin{aligned}
(\bigvee_j f_j)^{\leftarrow}(\Uparrow a) &= \{x \in X : a \ll \bigvee_{j \in J} f_j(x)\}, \\
&= \{x \in X : a \ll \bigvee_{J_f \subset J} \bigvee_{j \in J_f} f_j(x)\} \text{ by (a),} \\
&= \bigcup_{J_f \subset J} \{x \in X : a \ll \bigvee_{j \in J_f} f_j(x)\} \text{ by (b) and 2.12,} \\
&= \bigcup_{J_f \subset J} (\bigvee_{j \in J_f} f_j)^{\leftarrow}(\Uparrow a).
\end{aligned}$$

4.23 PROPOSITION

Let L be continuous, $(X,T) \in |TOP|$ and $(Y,S) \in |TOP(L)|$. Then :

(1) $S \subset \omega_L \iota_L(S)$

(2) $T = \iota_L \omega_L(T)$

Note that (1) holds for any complete lattice L.

PROOF

(1) If $u \in S$, then $u : (Y,\iota_L(S)) \longrightarrow (L,\sigma(L))$ is continuous. Hence $u \in \omega_L \iota_L(S)$.

(2) We first show that $T \subset \iota_L \omega_L(T)$. If $u \in T$ then $1_u \in \omega_L(T)$, thus $u = 1_u^{\leftarrow}(\Uparrow a)$ where $a \neq 0$, hence $u \in \iota_L \omega_L(T)$. Next we show that $\iota_L \omega_L(T) \subset T$. Let $u \in \iota_L \omega_L(T)$ be subbasic open. Then $u \in v^{\leftarrow}(\sigma(L))$ for some $v \in \omega_L(T)$. Now $v : (X,T) \longrightarrow (L,\sigma(L))$ is continuous, thus $v^{\leftarrow}(\sigma(L)) \subset T$, hence $u \in T$.

4.24 COROLLARY

If L is continuous, then ω_L embeds TOP into TOP(L) as a full subcategory.

PROOF

Follows as a consequence of 4.17 and the fact that if f: $(X,\omega_L(T)) \longrightarrow (Y, \omega_L(S))$ is continuous, then so is $f : (X,T) \longrightarrow (Y,S)$ which results from 4.20 and 4.23(2).

NOTES

In Section 3 the definition of an L–fuzzy set is due to [Go]. Then 3.2 and the remainder of 3.1 generalize ideas in [Ch] from I to L and are part of the folklore. Also 3.3 and 3.4, extend inclusion in [S1] and its properties without proofs in [S3] respectively, from I to L. In Section 4, 4.1, 4.7 and 4.11 generalize results in [Ch] from I to L. Now 4.2 and 4.3 extend ideas which appear in [Lo] from I to L. Also 4.4, 4.5, 4.6, 4.9 and 4.10 belong to the fuzzy topological folklore and can be found, for example, in [Wa] and [W–L]. The proof of 4.16 is given by Höhle as confirmed in [K2]. We obtained 4.12, 4.13 and 4.14 from [R1]. We follow [K2] in the remainder of the Chapter. The proof of 4.22 is essentially new, but the result itself is an explicit corollary of a more general result in [Co, II – 4.17].

CHAPTER 3

(L,M)–FUZZY TOPOLOGICAL SPACES

INTRODUCTION AND SUMMARY

The chapter begins with some introductory material on (L,M)–fuzzy topological spaces. In Section 5, an (L,M)–fuzzy topological space, continuity, and level topologies are introduced and some examples are provided. The (L,M)–fuzzy topological spaces are quite general, because specifying the lattices L and M one arrives at several old notions, including ordinary topologies, L–topologies and some others. An a–level functor Δ_a which assigns to each (L,M)–fuzzy topological space an a–level L–topological space and leaves morphisms unchanged is introduced. In Section 6, the initial and final topologies are studied with an example of the initial topology supplied. In Section 7, we introduce a functor Γ which embeds the category of L–topological spaces into that of (L,L)–fuzzy topological spaces. We establish that $\Gamma \circ \omega_L$ is a functor which embeds the category of crisp topological spaces into L–fuzzy topological spaces.

5. (L,M)–FUZZY TOPOLOGICAL SPACES AND CONTINUOUS MAPPINGS

Each topology T on X can be described in terms of its characteristic function $1_T : 2^X \longrightarrow 2$. It satisfies the following conditions :

(i) $1_T(1_X) = 1_T(1_\phi) = 1$,

(ii) $1_T(1_A \wedge 1_B) \geq 1_T(1_A) \wedge 1_T(1_B)$, and

(iii) $1_T(\bigvee_i 1_{A_i}) \geq \bigwedge_i 1_T(1_{A_i})$,

for all A, B, $A_i \subset X$.

Conversely, if a function 1_T satisfies conditions (i)–(iii), then $1_T^{\leftarrow}(1)$ forms a topology on X.

Under the above interpretation, a map $f : (X,T) \longrightarrow (Y,S)$ is continuous iff

$$1_T(1_U f) \geq 1_S(1_U) \text{ for all } U \subset Y.$$

In the sequel, L and M will denote complete lattices unless specified to have additional properties.

The above observation has led Kubiak [K1] to a definition of a general fuzzy topology, where 2^X above is replaced by L^X, and 2 is replaced by M, with L and M complete lattices. Using this concept we can extend several notions and results from the case L = M = I considered by Šostak [S1], to this general lattice setting.

We now introduce Kubiak's definition of a general fuzzy topology and we will focus our attention on the properties of this fuzzy topology.

5.1 DEFINITION

Let L and M be complete lattices. A mapping $\tau : L^X \rightarrow M$ is called an (L,M)–fuzzy topology on X if it satisfies the following conditions :

(i) $\tau(\underline{0}) = \tau(\underline{1}) = 1$,

(ii) if $u, v \in L^X$, then $\tau(u \wedge v) \geq \tau(u) \wedge \tau(v)$,

(iii) if $\{u_i : i \in J\} \subset L^X$, then $\tau(\bigvee_i u_i) \geq \bigwedge_i \tau(u_i)$.

We call $\tau(u)$ the degree of openness of u for each $u \in L^X$

(In the sequel, we will denote (L,M)–fuzzy topologies by τ, δ, γ, σ etc.)

5.2 DEFINITION

The 4–tuple (X,L,M,τ) is called an (L,M)–fuzzy topological space.

We shall often write (X,τ) if no confusion will arise. We will call (L,M)–fuzzy topological spaces (L,M)–fuzzy spaces and (L,L)–fuzzy topological spaces will be called L–fuzzy spaces.

We give some examples of (L,M)–fuzzy topological spaces. The first example is the case where $M = 2^X$, the second one is for the case $L = M = I$ and the last one is for $M = 2$. The last two examples can easily be verified to be (L,M)–fuzzy spaces.

5.3 EXAMPLE

(1) Let L be a complete lattice, X be an arbitrary non–empty set and define

$$\tau : L^X \longrightarrow 2^X \quad \text{by}$$

$$\tau(u) = \begin{cases} u^{-1}\{0\} & \text{if } u \neq \underline{1} \\ X & \text{if } u = \underline{1}. \end{cases}$$

Then τ is an $(L,2^X)$–fuzzy topology.

PROOF

(a) We have $\tau(\underline{0}) = \tau(\underline{1}) = 1$ which is obvious.

(b) If $u, v \in L^X$, then

$$\begin{aligned} \tau(u \wedge v) &= (u \wedge v)^{-1}\{0\} \\ &= \{x \in X : u(x) \wedge v(x) = 0\} \\ &\supseteq \{x \in X : u(x) = 0\} \cap \{x \in X : v(x) = 0\} \\ &= u^{-1}\{0\} \cap v^{-1}\{0\} \\ &= \tau(u) \wedge \tau(v). \end{aligned}$$

(c) If $\{u_i : i \in J\} \subset L^X$, then

$$\begin{aligned} \tau(\bigvee_i u_i) &= (\bigvee_i u_i)^{-1}\{0\} \\ &= \bigwedge_i (u_i)^{-1}\{0\} \\ &= \bigwedge_i \tau(u_i). \end{aligned}$$

(2) Let X be a non–empty set and define $\tau : I^X \longrightarrow I$ by

$$\tau(u) = \begin{cases} \bigwedge_{x \in X} u(x) & \text{if } u \neq \underline{0} \\ 1 & \text{if } u = \underline{0} \end{cases}$$

We show that τ is indeed an I–fuzzy topology.

PROOF

(a) It is easy to see that $\tau(\underline{0}) = \tau(\underline{1}) = 1$.

(b) Let $u, v \in I^X$. If $u \wedge v = \underline{0}$, then there is nothing to prove. Assume that $u \wedge v \neq \underline{0}$. Then

$$\begin{aligned} \tau(u \wedge v) &= \bigwedge_{x \in X} (u \wedge v)(x) \\ &\geq \bigwedge_{x,\, y \in X} (u(x) \wedge v(y)) \\ &= \bigwedge_{x \in X} u(x) \wedge \bigwedge_{y \in X} v(y) \\ &= \tau(u) \wedge \tau(v). \end{aligned}$$

(c) If $\{u_i : i \in J\} \subset I^X$, we have for all $y \in X$:

$$\begin{aligned} (\bigvee_i u_i)(y) &= \bigvee_i u_i(y) \\ &\geq \bigvee_i \bigwedge_{x \in X} u_i(x) \\ &\geq \bigwedge_i \bigwedge_{x \in X} u_i(x) \\ &= \bigwedge_i \tau(u_i). \end{aligned}$$

Hence $\tau(\bigvee_i u_i) = \bigwedge_{y \in X} (\bigvee_i u_i)(y) \geq \bigwedge_i \tau(u_i)$.

If $\bigvee_i u_i = \underline{0}$, then there is nothing to prove.

(3) Let X be an arbitrary non–empty set, L and M be complete lattices. We define $\tau^0 : L^X \longrightarrow M$ and $\tau^1 : L^X \longrightarrow M$ by

$$\tau^0(u) = \begin{cases} 1 & \text{if } u = \underline{0} \text{ or } \underline{1} \\ 0 & \text{otherwise} \end{cases}$$

and

$$\tau^1(u) = 1 \text{ for all } u \in L^X.$$

(4) Let $L = \{0, a, b, 1\}$ with $a < b$ and $X = \{x,y\}$. Define sets $u, v \in L^X$ by $u(x) = a$, $u(y) = b$, $v(x) = b$, $v(y) = 1$. The mapping $\tau : L^X \longrightarrow 2$ is defined by

$$\tau(w) = \begin{cases} 1 & \text{if } w = \underline{0} \text{ or } \underline{1} \text{ or } v \text{ or } u \\ 0 & \text{otherwise} \end{cases}$$

REMARK

(1) In definition 5.1, letting L = I and M = 2 we get the characteristic function of Chang's topology [Ch].

(2) Letting L = M = 2 we get the characteristic function of a crisp topology as a subset of the power set.

(3) If L = M = I we obtain Šostak and Hamburg's fuzzy topology ([S1] and [Ha]).

(4) If L = 2 we get Höhle's [Hö] fuzzy topology.

(5) If M = 2 we get Goguen's [Go] fuzzy topology.

This is how we define continuity between (L,M)–fuzzy spaces.

5.4 DEFINITION

Let (X,τ) and (Y,σ) be (L,M)–fuzzy spaces. A mapping $f : X \longrightarrow Y$ is continuous iff $\tau(vf) \geq \sigma(v)$ for each $v \in L^Y$. We will occassionally say f is (τ,σ)–continuous.

The next proposition shows that continuity is closed under composition of functions.

5.5 PROPOSITION

Let (X,τ), (Y,σ), (Z,ρ) be (L,M)–fuzzy spaces and $f : X \longrightarrow Y$, $g : Y \longrightarrow Z$ continuous mappings. Then the composition $g \circ f : X \longrightarrow Z$ is also continuous.

PROOF

If $v \in L^Z$, then $\tau(v(gf)) = \tau((vg)f) \geq \sigma(vg)$ by continuity of f. We also have $\sigma(vg) \geq \rho(v)$ by continuity of g. Thus $\tau(v(gf)) \geq \rho(v)$.

The next corollary follows as a consequence of 5.5.

5.6 COROLLARY

We obtain a category whose objects are (L,M)–fuzzy spaces and morphisms are continuous functions between them.

NOTATION

The category in 5.6 will be denoted by FTOP(L,M).

NOTE

The category FTOP(L,M) is a subcategory of a more general category considered in [K1], which, in particular, contains the category FUZZ of Rodabaugh (see e.g.[R1]).

5.7 DEFINITION

Let $\tau : L^X \longrightarrow M$ be an (L,M)–fuzzy topology on X. For each $a \in M$, we let

$$\tau_a = \{u \in L^X : a \leq \tau(u)\}.$$

5.8 PROPOSITION

For each $a \in M$, τ_a is an L–topology on X. We call it an a–level L–topology of τ.

PROOF

(1) We have $\tau(\underline{0}) = \tau(\underline{1}) = 1 \geq a$ for each $a \in M$, hence $\underline{0}, \underline{1}, \in \tau_a$.

(2) If $u, v \in \tau_a$, then $\tau(u) \geq a$ and $\tau(v) \geq a$. Thus $\tau(u \wedge v) \geq \tau(u) \wedge \tau(v) \geq a$. Hence $u \wedge v \in \tau_a$.

(3) If $\{u_i : i \in J\} \subset \tau_a$, then $\tau(u_i) \geq a$ for each $i \in J$. Thus $\tau(\bigvee_i u_i) \geq \bigwedge_i \tau(u_i) \geq a$. Hence $\bigvee_i u_i \in \tau_a$.

We will sometimes write level topologies instead of a–level L–topologies if no confusion will arise. We now describe the level topologies for the examples of (L,M)–fuzzy topologies considered in 5.3. Examples (5.9) 1, 2, 3 and 4 are level topologies of (5.3) 1, 2, 3 and 4 respectively.

5.9 EXAMPLES

(1) If $A \in 2^X$, then $\tau_A : X \longrightarrow L$ becomes

$$\tau_A = \{\underline{1}\} \cup \{u \in L^X : u(A) = \{0\}\}.$$

(2) If $a \in I$, then

$$\tau_a = \{\underline{0}\} \cup \{u \in I^X : u^{-1}[a,1] = X\}.$$

(3) The 1–level topologies for τ^0 and τ^1 are the indiscrete and the discrete L–topologies respectively :

$$\tau_1^0 = \{\underline{0}, \underline{1}\}$$

and

$$\tau_1^1 = L^X.$$

(4) We have $\tau_1 = \{u, v, \underline{0}, \underline{1}\} \subset L^X$.

The converse of the next proposition holds (8.6) but we have to introduce a number of results before we can prove it. As a result of this, studying continuity between (L,M)–fuzzy spaces is the same as doing so between a–level L–topological spaces.

5.10 PROPOSITION

Let (X,τ) and (Y,σ) be (L,M)–fuzzy spaces, τ_a and σ_a their a–level L–topologies respectively. If f is (τ,σ)–continuous, then it is (τ_a,σ_a)–continuous for each $a \in M$.

PROOF

If $v \in \sigma_a$, then $\tau(vf) \geq \sigma(v) \geq a$. Hence $vf \in \tau_a$.

The next corollary follows as a result of 5.8 and 5.10.

5.11 COROLLARY

We obtain a functor Δ_a : FTOP(L,M) $\longrightarrow$ TOP(L) which assigns to each (L,M)–fuzzy topological space (X,τ) an L–topological space (X,τ_a) and leaves morphisms unchanged.

6 WEAK (L,M)–FUZZY TOPOLOGIES

In this section we define the initial (L,M)–fuzzy topology and hence the product of a family of (L,M)–fuzzy spaces. The subspace (L,M)–fuzzy topology is defined as a special case of the initial (L,M)–fuzzy topology. The final (L,M)–fuzzy topology is also introduced for a single function and hence defined for a family of continuous functions.

6.1 DEFINITION

Let τ_1 and τ_2 be (L,M)–fuzzy topologies on X. We say τ_1 is weaker than τ_2 or τ_2 is stronger than τ_1 if $\tau_1(u) \leq \tau_2(u)$ for each $u \in L^X$.

6.2 PROPOSITION

Let $\{\tau^i : i \in J\}$ be a collection of (L,M)–fuzzy topologies on X. The function $\tau : L^X \longrightarrow M$ defined by

$$\tau(u) = \bigwedge_{i \in J} \tau^i(u)$$

is an (L,M)–fuzzy topology on X.

PROOF

(1) We have $\tau(\underline{0}) = \bigwedge_{i \in J} \tau^i(\underline{0}) = 1$. Similarly $\tau(\underline{1}) = 1$.

(2) If $u, v \in L^X$, then

$$
\begin{aligned}
\tau(u \wedge v) &= \bigwedge_{i \in J} \tau^i(u \wedge v) \\
&\geq \bigwedge_{i \in J} (\tau^i(u) \wedge \tau^i(v)) \\
&\geq \bigwedge_{i,\, j \in J} (\tau^i(u) \wedge \tau^j(v)) \\
&= \bigwedge_{i \in J} \tau^i(u) \wedge \bigwedge_{j \in J} \tau^j(v) \\
&= \tau(u) \wedge \tau(v).
\end{aligned}
$$

(3) If $\{u_k : k \in K\} \subset L^X$, then

$$
\begin{aligned}
\tau\Big(\bigvee_{k \in K} u_k\Big) &= \bigwedge_{i \in J} \tau^i\Big(\bigvee_{k \in K} u_k\Big) \\
&\geq \bigwedge_{i \in J} \bigwedge_{k \in K} \tau^i(u_k) \\
&= \bigwedge_{k \in K} \bigwedge_{i \in J} \tau^i(u_k) \\
&= \bigwedge_{k \in K} \tau(u_k).
\end{aligned}
$$

We are now in a position to make the following definition of a supremum of (L,M)–fuzzy spaces.

6.3 DEFINITION

Let $\mathcal{S} = \{\tau^i : i \in J\}$ be a family of (L,M)–fuzzy topologies on X. The weakest (L,M)–fuzzy topology τ^0 on X such that $\tau^0 \geq \tau^i$ for all $i \in J$ is called the supremum of the family $\mathcal{S}$. We will denote it by $\bigvee_{i \in J} \tau^i$.

REMARK

On account of 6.2, $\bigwedge_{i \in J} \tau^i$ exists and thus by 1.2, $\bigvee_{i \in J} \tau^i$ exists in the complete lattice of all (L,M)–fuzzy topologies on X.

6.4 DEFINITION

Let X be a set, (Y,σ) an (L,M)–fuzzy space and $f : X \longrightarrow (Y,\sigma)$ a mapping. We define the initial topology for f, to be the weakest (L,M)–fuzzy topology on X making f continuous.

6.5 REMARK

The weakest (L,M)–fuzzy topology on X making f continuous exists.

PROOF

Let $\mathcal{F}$ = {all (L,M)–fuzzy topologies on X making f continuous}. Then $\tau : L^X \longrightarrow M$ defined by $\tau(u) = 1$ for all $u \in L^X$, belongs to $\mathcal{F}$. Thus $\mathcal{F} \neq \emptyset$. Hence the weakest (L,M)–fuzzy topology on X making f continuous is $\rho : L^X \longrightarrow M$ defined by

$$\rho(u) = \bigwedge_{\delta \in \mathcal{F}} \delta(u)$$

for each $u \in L^X$.

The description of τ in 6.6 is essentially the same as in Šostak (for L = M = I), but ours looks simpler and clearer, as his is defined in two steps.

6.6 DEFINITION

Let X be a set, (Y,σ) an (L,M)–fuzzy space and $f : X \longrightarrow (Y,\sigma)$ a mapping. Define $\tau : L^X \longrightarrow M$ by

$$\tau(u) = \bigvee \{\sigma(v) : v \in L^Y \text{ and } u = vf\}.$$

(We note that $\bigvee \emptyset = 0$).

Complete distributivity is going to come into play in the next results. The next proposition with L = M = I is in Šostak [S1]. However he has a gap in his proof, claiming that proving that τ of 6.6 satisfies the supremum axiom is similar to proving that it satisfies the infimum axiom. This is not so, as seen from our proof.

6.7 PROPOSITION

Let L be complete and M be completely distributive.

(1) The mapping τ of 6.6 is an (L,M)–fuzzy topology on X.

(2) τ is the initial (L,M)–fuzzy topology on X induced by (Y,σ) and f.

PROOF

(1) (a) We have $\tau(\underline{0}) = \tau(\underline{1}) = 1$, since $\sigma(\underline{1}) = \sigma(\underline{0}) = 1$ and $\underline{0} = \underline{0}f$, $\underline{1} = \underline{1}f$.

(b) We show that for $\{v_i : i \in J\} \subset L^X$.

$$\tau(\bigvee_{i \in J} v_i) \geq \bigwedge_{i \in J} \tau(v_i).$$

We have $\tau(\bigvee_{i \in J} v_i)$

$= \bigvee \{\sigma(u) : u \in L^Y \text{ and } \bigvee_{i \in J} v_i = uf\}$

$= \bigvee \{\sigma(\bigvee_i u_i) : \{u_i : i \in J\} \subset L^Y \text{ and } \bigvee_{i \in J} v_i = (\bigvee_{i \in J} u_i)f\}$

$\geq \bigvee \{\sigma(\bigvee_i u_i) : u_i \in L^Y \text{ and } v_i = u_i f \text{ for each } i \in J\}$

$\geq \bigvee \{\bigwedge_i \sigma(u_i) : u_i \in L^Y \text{ and } v_i = u_i f \text{ for each } i \in J\}.$

Now if we let $k_i = \{u \in L^Y : v_i = uf\}$, then we have

$\bigvee \{\bigwedge_i \sigma(u_i) : u_i \in L^Y \text{ and } v_i = u_i f \text{ for each } i \in J\}$

$= \bigvee_{\substack{\varphi : J \to L^Y \\ \varphi(i) \in k_i}} \bigwedge_{i \in J} \sigma(\varphi(i)),$

$= \bigvee_{\varphi \in \prod_{i \in J} k_i} \bigwedge_{i \in J} \sigma(\varphi(i)),$

$= \bigwedge_{i \in J} \bigvee_{u \in k_i} \sigma(u)$ (By complete distributivity of M),

$= \bigwedge_{i \in J} \bigvee \{\sigma(u) : u \in L^Y \text{ and } uf = v_i\},$

$= \bigwedge_{i \in J} \tau(v_i).$

(c) We show that for $v_1, v_2 \in L$,

$\tau(v_1 \wedge v_2) \geq \tau(v_1) \wedge \tau(v_2).$

We have,

$\tau(v_1 \wedge v_2)$

$$= \bigvee \{\sigma(u) : u \in L^Y \text{ and } v_1 \wedge v_2 = uf\},$$

$$= \bigvee \{\sigma(u_1 \wedge u_2) : u_1, u_2 \in L^Y \text{ and } (u_1 \wedge u_2)f = v_1 \wedge v_2\},$$

$$\geq \bigvee \{\sigma(u_1 \wedge u_2) : u_1, u_2 \in L^Y \text{ and } u_if = v_i, i = 1, 2\},$$

$$\geq \bigvee \{\sigma(u_1) \wedge \sigma(u_2) : u_1, u_2 \in L^Y \text{ and } v_i = u_if, i = 1,2\},$$

$$= \bigvee \{\sigma(u_1) : u_1 \in L^Y \text{ and } v_1 = u_1f\} \wedge \bigvee \{\sigma(u_2) : u_2 \in L^Y \text{ and } v_2 = u_2f\} \quad \text{(By complete distributivity of M)},$$

$$= \tau(v_1) \wedge \tau(v_2).$$

(2) Of course $f : (X,\tau) \longrightarrow (Y,\sigma)$ is continuous.

If $f : (X,\tau_1) \longrightarrow (Y,\sigma)$ is continuous, then $\tau_1(uf) \geq \sigma(u)$ for each $u \in L^Y$. We have, for each $u \in L^X$

$$\tau(u) = \bigvee \{\sigma(v) : v \in L^Y \text{ and } u = vf\},$$

$$\leq \bigvee \{\tau_1(vf) : v \in L^Y \text{ and } u = vf\},$$

$$= \tau_1(u).$$

Thus τ is weaker than τ_1.

6.8 EXAMPLE

We give an example of the initial topology $\sigma : L^{\mathbb{Z}} \longrightarrow 2$ (where $\mathbb{Z}$ is the set of integers) induced by $f : \mathbb{Z} \longrightarrow (X,\tau)$, where (X,τ) is the (L,2)–fuzzy topological space considered in 5.3(4). Define $f : \mathbb{Z} \longrightarrow (X,\tau)$ by

$$f(n) = \begin{cases} x & \text{if n is even} \\ y & \text{if n is odd} \end{cases}.$$

The initial topology $\sigma : L^{\mathbb{Z}} \longrightarrow 2$ is defined by

$$\sigma(w) = \bigvee \{\tau(w_1) : w_1 \in L^X \text{ and } w = w_1 f\}.$$

Observe that $v_1, u_1 \in L^{\mathbb{Z}}$ defined by

$$v_1(n) = \begin{cases} 1 & \text{if n is odd} \\ b & \text{if n is even} \end{cases}$$

$$u_1(n) = \begin{cases} a & \text{if n is even} \\ b & \text{if n is odd} \end{cases},$$

satisfy $v_1 = vf$ and $u_1 = uf$.

From the above observation we thus define $\sigma : L^{\mathbb{Z}} \longrightarrow 2$ by

$$\sigma(w) = \begin{cases} 1 & \text{if } w = u_1 \text{ or } v_1 \text{ or } \underline{0} \text{ or } \underline{1} \\ 0 & \text{otherwise} \end{cases}$$

Let M be a completely distributive lattice and $\{(Y_i,\sigma^i) : i \in J\}$ be a family of (L,M)–fuzzy spaces. Let X be a set and for each $i \in J$, let $\tau^i : L^X \longrightarrow M$ be the initial topology on X for $f_i : X \longrightarrow (Y_i,\sigma^i)$.

6.9 PROPOSITION

Let M be a completely distributive lattice. The mapping $\tau : L^X \longrightarrow M$ defined by $\tau = \bigvee_i \tau^i$, is the weakest (L,M)–fuzzy topology on X for which all the mappings $f_i : X \longrightarrow (Y_i,\sigma^i)$ are continuous.

NOTE

Šostak has two definitions (in two different papers [S1] and [S5]) for the topology on X making all these functions continuous and the one in [S1] is incorrect. He replaced $\bigvee \tau^i$ by $\bigwedge \tau^i$. It is easy to see that $\bigwedge \tau^i$ on X does not make all the mappings $f_i : X \longrightarrow (Y,\sigma_i)$ continuous.

PROOF

By 6.3, τ is indeed an (L,M)–fuzzy topology. Also τ makes all the mappings $f_i : (X,\tau) \longrightarrow (Y_i,\sigma^i)$ for each $i \in J$ continuous. For, given $v \in L^{Y_i}$, we get $\tau(vf) \geq \tau^i(vf) \geq \sigma^i(v)$ for each $i \in J$.

Suppose τ_1 is another (L,M)–fuzzy topology on X making all the mappings $f_i : (X,\tau_1) \longrightarrow (Y_i,\sigma^i)$ continuous. If $\tau \nleq \tau_1$, then $\exists_{i_0 \in J}$ such that $\tau^{i_0} \nleq \tau_1$, thus $f_{i_0} : (X,\tau_1) \longrightarrow (Y_{i_0},\sigma^{i_0})$ fails to be continuous, a contradiction.

We now define the product of a family of (L,M)–fuzzy spaces.

6.10 DEFINITION

Let M be a completely distributive lattice and $\{(X_i,\tau^i) : i \in J\}$ be a family of (L,M)–fuzzy spaces. A pair (X,τ) is called the product of these spaces if $X = \prod_i X_i$ is the set product and τ is the weakest (L,M)–fuzzy topology on X for which all projections $p_i : X \longrightarrow (X_i,\tau^i)$ are continuous.

6.11 PROPOSITION

Let (X,τ) be an (L,M)–fuzzy topological space and M be a completely distributive lattice. If $A \subset X$ and $id : A \longrightarrow X$ is the identity embedding, define

$$\tau_A : L^A \longrightarrow M$$

by
$$\tau_A(u) = \bigvee \{\tau(v) : v \in L^X \text{ and } u = v\,id\}.$$

Then τ_A is the weakest (L,M)–fuzzy topology on A making id continuous.

PROOF

Follows from 6.7.

6.12 DEFINITION

Let M be a completely distributive lattice. The (L,M)–fuzzy topology τ_A in 6.11 is called the subspace topology of A.

We now pay attention to the final topology, first for a single mapping and then for a family of mappings.

6.13 DEFINITION

Let Y be a set, and (X,τ) an (L,M)–fuzzy space and $f : (X,\tau) \longrightarrow Y$ a mapping. We define the final topology for f to be the strongest (L,M)–fuzzy topology σ on Y making f continuous.

6.14 PROPOSITION

Let L and M be complete lattices, τ an (L,M)–fuzzy topology, $f : (X,\tau) \longrightarrow Y$ be a mapping. If we define $\sigma(v) = \tau(vf)$ for each $v \in L^Y$, then σ is an (L,M)–fuzzy topology on Y.

PROOF

(1) We have $\sigma(\underline{0}) = \tau(\underline{0}f) = 1$. Similarly $\sigma(\underline{1}) = 1$.

(2) If $u, v \in L^Y$, then

$$\begin{aligned}\sigma(u \wedge v) &= \tau((u \wedge v)f) \\ &= \tau(uf \wedge vf) \\ &\geq \tau(uf) \wedge \tau(vf) \\ &= \sigma(u) \wedge \sigma(v).\end{aligned}$$

(3) If $\{u_i : i \in J\} \subset L^Y$, then

$$\begin{aligned}\sigma(\bigvee_{i \in J} u_i) &= \tau((\bigvee_{i \in J} u_i)f) \\ &= \tau(\bigvee_{i \in J} (u_i f)) \quad \text{(by 3.2(3))} \\ &\geq \bigwedge_{i \in J} \tau(u_i f) \\ &= \bigwedge_{i \in J} \sigma(u_i).\end{aligned}$$

6.15 PROPOSITION

The (L,M)–fuzzy topology σ in 6.14 is the strongest one making f continuous.

PROOF

Let σ_1 be another (L,M)–fuzzy topology on Y making f continuous. We show that $\sigma_1 \leq \sigma$. For each $v \in L^Y$ we have

$$\begin{aligned}\sigma(v) &= \tau(vf) \\ &\geq \sigma_1(v) \quad \text{by continuity of } f : (X,\tau) \to (Y,\sigma_1).\end{aligned}$$

6.16 PROPOSITION

Let $\{(X_i,\tau^i) : i \in J\}$ be a family of (L,M)–fuzzy spaces. Let Y be a set and for each $i \in J$, let $f_i : (X_i,\tau^i) \longrightarrow Y$ be a mapping. For each $i \in J$ let $\sigma^i : L^Y \longrightarrow M$ be the final topology on Y for f_i. The mapping $\sigma : L^Y \longrightarrow M$ defined by $\sigma(v) = \bigwedge_{i \in J} \sigma^i(v)$ for each $v \in L^Y$, is the strongest (L,M)–fuzzy topology on Y for which all the mappings $f_i : (X_i,\tau^i) \longrightarrow Y$ are continuous.

PROOF

That σ is an (L,M)–fuzzy topology is shown in 6.2. Also $f_i : (X_i,\tau^i) \longrightarrow (Y,\sigma)$ is indeed continuous for each $i \in J$ since $\sigma(u) \leq \sigma^i(u) \leq \tau^i(uf)$ for each $i \in J$ and $u \in L^Y$.

Suppose σ_1 is another (L,M)–fuzzy topology on Y making all the mappings $f_i : (X_i,\tau^i) \longrightarrow (Y,\sigma_1)$ continuous.
For $v \in L^Y$ we have :

$$\begin{aligned} \sigma(v) &= \bigwedge_{i \in J} \sigma^i(v), \\ &= \bigwedge_{i \in J} \tau^i(vf_i), \\ &\geq \sigma_1(v). \end{aligned}$$

7 EMBEDDING TOP AND TOP(L) INTO FTOP(L)

We now introduce some categories of fuzzy topological structures. We will first prove the following two properties of inclusion which we will need.

REMINDER

Let L be a complete lattice with an order reversing involution. The function $F : L^X \times L^X \longrightarrow L$ is defined by

$$F(u,v) = \bigwedge_{x \in X} (u' \vee v)(x).$$

7.1 PROPERTIES

Let (X,T) be an L–topological space.

(1) If L is a frame and $\{u_i : i \in J\} \subset L^X$, then

$$F(\bigvee_i u_i, \mathrm{Int}_T(\bigvee_i u_i)) \geq \bigwedge_i F(u_i, \mathrm{Int}_T u_i).$$

(2) If L is distributive and $u, v \in L^X$, then

$$F(u \wedge v, \mathrm{Int}_T(u \wedge v)) \geq F(u, \mathrm{Int}_T u) \wedge F(v, \mathrm{Int}_T v)$$

PROOF

(1) $F(\bigvee_i u_i, \mathrm{Int}_T(\bigvee_i u_i))$

$$= \bigwedge_{x \in X} ((\bigvee_i u_i)' \vee \mathrm{Int}_T(\bigvee_i u_i))(x)$$

$$\geq \bigwedge_{x \in X} ((\bigwedge_i u'_i) \vee (\bigvee_i \mathrm{Int}_T u_i))(x)$$

$$= \bigwedge_{x \in X} (\bigwedge_i (u'_i \vee \bigvee_j \mathrm{Int}_T u_j))(x) \quad \text{(by frame law for } L^{op})$$

$$\geq \bigwedge_{x \in X} \bigwedge_i (u'_i \vee \mathrm{Int}_T u_i)(x)$$

$$= \bigwedge_i \bigwedge_{x \in X} (u'_i \vee \mathrm{Int}_T u_i)(x)$$

$$= \bigwedge_i F(u_i, \mathrm{Int}_T u_i).$$

(2) $F(u \wedge v, \mathrm{Int}_T(u \wedge v))$

$$= \bigwedge_{x \in X} ((u \wedge v)' \vee (\mathrm{Int}_T u \wedge \mathrm{Int}_T v))(x)$$

$$= \bigwedge_{x \in X} ((u' \vee v') \vee (\mathrm{Int}_T u \wedge \mathrm{Int}_T v))(x)$$

$$\geq \bigwedge_{x \in X} ((u' \vee \mathrm{Int}_T u) \wedge (v' \vee \mathrm{Int}_T v))(x)$$

(by * below),

$$\geq \bigwedge_{x \in X} (u' \vee \mathrm{Int}_T u)(x) \wedge \bigwedge_{x \in X} (u' \vee \mathrm{Int}_T v)(x)$$

$$= F(u, \mathrm{Int}_T u) \wedge F(v, \mathrm{Int}_T v).$$

(*) $(u' \vee v') \vee (\mathrm{Int}_T u \wedge \mathrm{Int}_T v)$

$= [(u' \vee v') \vee \mathrm{Int}_T u] \wedge [(u' \vee v') \vee \mathrm{Int}_T v]$ (distributivity of L)

$= [(u' \vee \mathrm{Int}_T u) \vee v'] \wedge [u' \vee (v' \vee \mathrm{Int}_T v)]$ (commutativity of supremum)

$\geq (u' \vee \mathrm{Int}_T u) \wedge (v' \vee \mathrm{Int}_T v)$.

7.2 DEFINITION

Let L be a frame and $(X,T) \in |\mathrm{TOP}(L)|$. We define $\Gamma(T) : L^X \to L$ by

$$\Gamma(T)(u) = F(u, \mathrm{Int}_T u).$$

7.3 THEOREM

Let L be a frame. If $(X,T) \in |\mathrm{TOP}(L)|$, then $(X, \Gamma(T)) \in |\mathrm{FTOP}(L)|$.

PROOF

(1) The fact that $\Gamma(T)(\underline{0}) = \Gamma(T)(\underline{1}) = 1$ is obvious.

(2) If $u, v \in L^X$, then

$$\begin{aligned}\Gamma(T)(u \wedge v) &= F(u \wedge v, \mathrm{Int}_T(u \wedge v)) \\ &\geq F(u, \mathrm{Int}_T u) \wedge F(v, \mathrm{Int}_T v) \quad \text{(by 7.1(2))} \\ &= \Gamma(T)(u) \wedge \Gamma(T)(v).\end{aligned}$$

(3) If $u_i \in L^X$ for each $i \in J$, then we have :

$$\begin{aligned}\Gamma(T)(\bigvee_i u_i) &= F(\bigvee_{i \in J} u_i, \mathrm{Int}_T(\bigvee_i u_i)) \\ &\geq \bigwedge_i F(u_i, \mathrm{Int}_T u_i) \quad \text{(by 7.1(1))} \\ &= \bigwedge_i \Gamma(T)(u_i).\end{aligned}$$

7.4 THEOREM

Let L be a frame, and let $(X,T), (Y,S) \in |\mathrm{TOP}(L)|$. If $f : X \longrightarrow Y$ is (T,S)–continuous, then it is $(\Gamma(T), \Gamma(S))$–continuous.

PROOF

We have

$$\begin{aligned}\Gamma(T)(vf) &= F(vf, \mathrm{Int}_T(vf)) \\ &\geq F(vf, (\mathrm{Int}_S v)f) \qquad \text{(by 3.4(2) and 4.8)} \\ &\geq F(v, \mathrm{Int}_S v) \qquad \text{(by 3.4(5))} \\ &= \Gamma(S)(v).\end{aligned}$$

As a result of 7.3 and 7.4 we have the following corollary.

7.5 COROLLARY

Let L be a frame. We have that $\Gamma : \mathrm{TOP(L)} \longrightarrow \mathrm{FTOP(L)}$ is a functor, which assigns to every $(X,T) \in |\mathrm{TOP(L)}|$, the L–fuzzy space $(X,\Gamma(T))$ and which leaves morphisms unchanged.

7.6 and 7.7 appear in [K3] see also [K–K].

7.6 DEFINITION

Let (X,T) be a topological space and $u \in L^X$. A mapping

$$(\cdot)_* : L^X \longrightarrow L^X$$

is defined by

$$u_*(x) = \bigvee_{U_x \in T} \bigwedge_{y \in U_x} u(y)$$

for every $u \in L^X$ and $x \in X$, where U_x is a neighbourhood of x.

7.7 PROPOSITION

If (X,T) is a topological space and L a continuous lattice, then

$$u_* = \mathrm{Int}_{\omega_L(T)} u \quad \forall\, u \in L^X$$

and hence $(\cdot)_*$ is an interior operator on L^X.

The next proposition will be used in the proof of 7.9.

7.8 PROPOSITION

If (X,T) is a topological space, L a continuous lattice, and $A \subset X$, then

$$\mathrm{Int}_{\omega_L(T)}(1_A) = 1_{\mathrm{Int}_T A}.$$

PROOF

We have :

$$
\begin{aligned}
(\mathrm{Int}_{\omega_L(T)}(1_A))(x) = 1 \quad & \Leftrightarrow \bigvee_{U_x} \bigwedge_{y \in U_x} 1_A(y) = 1 \\
& \Leftrightarrow \exists_{U_x} \forall_{y \in U_x} 1_A(y) = 1 \\
& \Leftrightarrow \exists_{U_x} \; y \in U_x \Rightarrow y \in A \\
& \Leftrightarrow \exists_{U_x} \; U_x \subset A \\
& \Leftrightarrow x \in \mathrm{Int}_T A \\
& \Leftrightarrow 1_{\mathrm{Int}_T A}(x) = 1.
\end{aligned}
$$

Similarly one shows that

$$\mathrm{Int}_{\omega_L(T)}(1_A)(x) = 0 \quad \Leftrightarrow 1_{\mathrm{Int}_T A}(x) = 0.$$

$$\text{Hence } \mathrm{Int}_{\omega_L(T)}(1_A) \quad = 1_{\mathrm{Int}_T A}.$$

NOTE

We observe that, $\Gamma \circ \omega_L$ (with L = I) is the same as the functor

$$\Phi : \mathrm{TOP} \to \mathrm{FTOP(I)} \text{ in } [\mathrm{S1} - 4.12].$$

7.9 PROPOSITION

Let L be a continuous frame. The composition $\Gamma \circ \omega_L$ embeds TOP into FTOP(L) and is one–one on objects.

PROOF

Let $\Phi = \Gamma \circ \omega_L$. Define $\psi : \mathrm{FTOP(L)} \longrightarrow \mathrm{TOP}$ by

$$\psi(X,\tau) = (X, \psi(\tau)),$$

where

$$\psi(\tau) = \{A \subset X : \tau(1_A) = 1\},$$

and morphisms remain unchanged.

Now,

$$\begin{aligned} \psi\,\Phi(T) &= \{A : \Phi(T)(1_A) = 1\} \\ &= \{A : F(1_A, \mathrm{Int}_{\omega_L(T)}(1_A)) = 1\} \\ &= \{A : F(1_A, 1_{\mathrm{Int}_T A}) = 1\} \quad \text{(by 7.8)} \\ &= \{A : A \subset \mathrm{Int}_T A\} \quad \text{(by 3.4(1))} \\ &= T. \end{aligned}$$

Which shows that ψ is a left inverse of Φ.

Also, $\Phi : |\mathrm{TOP}| \longrightarrow |\mathrm{FTOP}|$ is an injective function since,

$$\begin{aligned} & \Phi(X,T) &&= \Phi(Y,S) \\ \Rightarrow\ & \psi\,\Phi(X,T) &&= \psi\,\Phi(Y,S) \\ \Rightarrow\ & (X,T) &&= (Y,S). \end{aligned}$$

NOTES

Except for 5.3, we obtain 5.1 – 5.5 from [K1]. Examples 5.3(1) and (2) were suggested by J. Chadwick. The rest of the examples are written by the author. Together with T. Kubiak we arrived at the remainder of the observations in this section and also 6.1 – 6.5. Apart from 6.8, the rest of the results in Section 6 generalize ideas in [S1]. Proof of 6.7 is due to T. Kubiak. Example 6.8 is supplied by the author. 7.1 – 7.5 extend results in [S1] from I to L. We obtain 7.6 and 7.7 from [K3] (see also [K–K]). T. Kubiak observed 7.8.

CHAPTER 4

NEW (L,M)–FUZZY TOPOLOGIES FROM OLD ONES

SUMMARY

In this chapter we investigate functors from FTOP(L,M) to itself. These functors only change the (L,M)–fuzzy topology on an object leaving the underlying set and morphism unchanged. Most results in Section 8 are used in proofs which are in Section 9. In Section 8, we give a method by which we generate an (L,M)–fuzzy topology from a family of decreasing (L,M)–fuzzy topologies. We further establish that an (L,M)–fuzzy topology is generated by its level topologies. Šostak [S2] puts a condition on level topologies for the above result to hold, but we did not see the necessity of this condition. We also find an expression of the supremum of (L,M)–fuzzy topologies in terms of the supremum of the a–level L–topologies. In Section 9 we concentrate on the functor from FTOP(L,M) to itself. In particular, if the functor $(\cdot)^c$ is restricted to TOP, then it reduces to the stratification functor for L–spaces.

8 GENERATION OF AN (L,M)–FUZZY TOPOLOGY BY A FAMILY OF (L,M)–FUZZY TOPOLOGIES.

At the beginning of this Chapter we will primarily be concerned with developing the machinery necessary to prove a number of results in the next section.

8.1 DEFINITION

A family $\{\tau^a : a \in M\backslash\{0\}\}$ of (L,M)–fuzzy topologies on X is said to be decreasing if for $b \leq a$ we have $\tau^a \leq \tau^b$.

The next result is very important. Šostak [S2] only gave the sketch of the proof (with M = L = I), but we have supplied a detailed one here. We shall assume that M is a complete chain.

8.2 THEOREM

Let M be a complete chain and $\{\tau^a : L^X \longrightarrow M|\ a \in M\backslash\{0\}\}$ a family of decreasing (L,M)–fuzzy topologies. Then the mapping defined by the equality,

$$\tau(u) = \bigvee_{a \neq 0} (\tau^a(u) \wedge a)$$

for each $u \in L^X$ is an (L,M)–fuzzy topology.

PROOF

(1) We have $\tau(\underline{0}) = \tau(\underline{1}) = \bigvee_{a \neq 0} a = 1$.

(2) Assume that $\tau(u \wedge v) < \tau(u) \wedge \tau(v)$ for some $u, v \in L^X$, then,

$$\tau(u \wedge v) < \tau(u) = \bigvee_{b \neq 0} (\tau^b(u) \wedge b)$$

$$\Longrightarrow \exists_{b_0 \neq 0} \bigvee_{a \neq 0} (\tau^a(u \wedge v) \wedge a) < \tau^{b_0}(u) \wedge b_0,$$

$$\Longrightarrow \exists_{b_0 \neq 0} \forall'_{a \neq 0}\ \tau^a(u \wedge v) \wedge a < \tau^{b_0}(u) \wedge b_0.$$

Similarly,

$$\exists_{b_1 \neq 0} \ \forall_{a \neq 0} \ \tau^a(u \wedge v) \wedge a < \tau^{b_1}(v) \wedge b_1.$$

Consequently, since $\{\tau^b : b \neq o\}$ is a decreasing family, we get

$$\forall_{a \neq 0} \ \tau^a(u \wedge v) \wedge a < \tau^{b_0 \wedge b_1}(u) \wedge b_0,$$

and

$$\forall_{a \neq 0} \ \tau^a(u \wedge v) \wedge a < \tau^{b_0 \wedge b_1}(v) \wedge b_1.$$

Hence

$$\tau^a(u \wedge v) \wedge a < \tau^{b_0 \wedge b_1}(u) \wedge \tau^{b_0 \wedge b_1}(v) \wedge b_0 \wedge b_1$$

for all $a \neq 0$. In particular with $c = b_0 \wedge b_1$ we obtain

$$\tau^c(u \wedge v) \wedge c < \tau^c(u) \wedge \tau^c(v) \wedge c,$$

which yields,

$$\tau^c(u \wedge v) < \tau^c(u) \wedge \tau^c(v),$$

contradicting the fact that τ^c is an (L,M)–fuzzy topology.

(3) Suppose we have a collection $\{u_i : i \in J\} \subset L^X$ such that

$$\tau(\bigvee_i u_i) < \bigwedge_i \tau(u_i).$$

CASE 1

There exists $c \in M$ such that

$$\tau(\bigvee_i u_i) < c < \bigwedge_i \tau(u_i),$$

$$\Longrightarrow \quad \exists_{c \neq 0} \ \forall_i \ \bigvee_{a \neq 0} (\tau^a(\bigvee_i u_i) \wedge a) < c < \bigvee_{b \neq 0} (\tau^b(u_i) \wedge b),$$

$$\Longrightarrow \quad \exists_{c \neq 0} \ \forall_i \ \exists_{b_0 \neq 0} \ \forall_{a \neq 0} \ \tau^a(\bigvee_i u_i) \wedge a < c < \tau^{b_0}(u_i) \wedge b_0,$$

$$\Longrightarrow \quad \exists_{c \neq 0} \ \forall_i \ \exists_{b_0 \neq 0} \ \tau^c(\bigvee_i u_i) \wedge c < c < \tau^{b_0}(u_i) \wedge b_0.$$

Observe that $c < \tau^{b_0}(u_i) \wedge b_0 \leq b_0$,

thus $$c < b_0 \Longrightarrow \tau^{b_0}(u_i) \leq \tau^c(u_i),$$

hence $$c < \tau^{b_0}(u_i) \wedge b_0 \leq \tau^c(u_i).$$

We thus have, $c \neq 0$ such that

$$\forall_i \; \tau^c(\bigvee_i u_i) \wedge c < c < \tau^c(u_i)$$

$$\Longrightarrow \quad \exists_{c \neq 0} \; \forall_i \; \tau^c(\bigvee_i u_i) < c < \tau^c(u_i),$$

$$\Longrightarrow \quad \exists_{c \neq 0} \; \tau^c(\bigvee_i u_i) < c \leq \bigwedge_i \tau^c(u_i),$$

a contradiction to our hypotheses.

CASE 2

Suppose there is no $c \in M$ satisfying $\tau(\bigvee_j u_j) < c < \bigwedge_j \tau(u_j)$. Let $j \in J$. We have

$$\tau(u_j) = \bigvee_{a \neq 0} (\tau^a(u_j) \wedge a) \geq \bigwedge_j \tau(u_j).$$

We claim that there exists $a_0 \in M$ such that

$$\tau^{a_0}(u_j) \wedge a_0 \geq \bigwedge_j \tau(u_j).$$

For if not then $\tau^a(u_j) \wedge a < \bigwedge_j \tau(u_j)$ for all $a \in M$. Since there is a gap between $\tau(\bigvee_j u_j)$ and $\bigwedge_j \tau(u_j)$ this forces $\tau^a(u_j) \wedge a \leq \tau(\bigvee_j u_j)$ for all $a \in M$.

Now $\tau(u_j) = \bigvee_{a \neq 0} (\tau^a(u_j) \wedge a) \leq \tau(\bigvee_j u_j)$, contradicting $\tau(\bigvee_j u_j) < \bigwedge_j \tau(u_j)$.

Hence, for each $j \in J$ we can find $a_j \in M$ with $\tau^{a_j}(u_j) \wedge a_j \geq \bigwedge_j \tau(u_j)$ and

hence $a_j \geq \bigwedge_j \tau(u_j)$ for all $j \in J$.

Since $\{\tau^a : a \in M\backslash\{0\}\}$ is decreasing, letting $\bigwedge_j \tau(u_j) = b$, we obtain

$$\tau^b(u_j) \geq \tau^{a_j}(u_j) \geq b \text{ for all } j \in J$$

and hence
$$\tau^b(u_j) \wedge b = b \text{ for all } j \in J.$$

It follows that $\tau^b(\bigvee_j u_j) \wedge b \geq \bigwedge_j (\tau^b(u_j) \wedge b) = b$

and hence, finally

$$\tau(\bigvee_j u_j) = \bigvee_{a \in M} (\tau^a(\bigvee_j u_j) \wedge a) \geq \tau^b(\bigvee_j u_j) \wedge b = b,$$

contradicting
$$\tau(\bigvee_j u_j) < \bigwedge_j \tau(u_j).$$

It may be of some interest to know whether Theorem 8.2 holds true for M a non–linearly ordered complete lattice, e.g. completely distributive or continuous.

We are now ready to define the generation of an (L,M)–fuzzy topology by a family of (L,M)–fuzzy topologies.

8.3 DEFINITION

Let M be a complete chain and $\{\tau^a : L^X \to M : a \in M\backslash\{0\}\}$ a decreasing family of (L,M)–fuzzy topologies. Then $\tau : L^X \to M$ defined by

$$\tau(u) = \bigvee_{a \neq 0} (\tau^a(u) \wedge a)$$

is said to be generated by the above family.

The proof of the next result is also different from Šostak's [S2] proof. He uses the order density property for M (M = L = I in [S2]), which we do not have.

8.4 PROPOSITION

Let M be a complete chain, $\{\tau^a : L^X \longrightarrow M : a \in M\backslash\{0\}\}$ and $\{\sigma^a : L^Y \longrightarrow M : a \in M\backslash\{0\}\}$ be decreasing families of (L,M)–fuzzy topologies on X and Y respectively, and τ,σ be the corresponding generated (L,M)–fuzzy topologies. If the mappings $f : (X,\tau^a) \longrightarrow (Y,\sigma^a)$ are continuous for each $a \in M\backslash\{0\}$, then the mapping $f : (X,\tau) \longrightarrow (Y,\sigma)$ is continuous.

PROOF

Continuity of $f : (X,\tau^a) \longrightarrow (Y,\sigma^a)$ implies that for all $v \in L^Y$ and $a \in M\backslash\{0\}$ we have

$$\tau^a(vf) \geq \sigma^a(v)$$

$$\Longrightarrow \qquad \tau^a(vf) \wedge a \geq \sigma^a(v) \wedge a$$

$$\Longrightarrow \qquad \bigvee_{a \in M\backslash\{0\}} (\tau^a(vf) \wedge a) \geq \bigvee_{a \in M\backslash\{0\}} (\sigma^a(v) \wedge a)$$

Thus $\qquad \tau(vf) \geq \sigma(v)$ for all $v \in L^Y$.

In the next result we establish that an (L,M)–fuzzy topology is generated by its level topologies. Contrary to Šostak's [S2] analogue of this result, we do not need any conditions on the level topologies.

8.5 PROPOSITION

If τ is an (L,M)–fuzzy topology, and $\{\tau_a : a \in M\backslash\{0\}\}$ is a family of its level L–topologies, then τ is the topology generated by this family.

PROOF

It is well known and obvious that for every $u \in M^X$ one has

$$u = \bigvee_{a \in M} (a \wedge 1_{u^{-1}(\uparrow a)}).$$

In particular, given an (L,M)–fuzzy topology $\tau \in M^{(L^X)}$ one gets,

$$\tau = \bigvee_{a \in M} (a \wedge 1_{\tau^{-1}(\uparrow a)}),$$

where $\tau^{-1}(\uparrow a) = \tau_a$ is just the a–level L–topology.

The next Corollary follows as a consequence of 8.4, 8.5, and 5.10.

8.6 COROLLARY

Let (X,τ) and (Y,σ) be (L,M)–fuzzy spaces, τ_a and σ_a be their a–level L–topologies respectively. Then f is (τ_a,σ_a)–continuous for each $a \in M$ iff it is (τ,σ)–continuous.

Šostak [S2] stated the following proposition without a proof but we have provided it here.

8.7 PROPOSITION

Let M be a complete chain, $\{\tau^a : a \in M\backslash\{0\}\}$ a decreasing family of (L,M)–fuzzy topologies and τ be generated by this family. Then, we have $\tau_a = \bigcap \{\tau^b_b : b < a\}$ for all $a \in M$ which are not isolated from below (i.e. $a = \bigvee \{b : b < a\}$).

PROOF

Let $u \in \bigcap \{\tau^b_b : b < a\}$. Then for all b such that $b < a$ we have :

$$\tau^b(u) \geq b$$

$$\Rightarrow \tau^b(u) \wedge b \geq b \wedge b = b$$

$$\Rightarrow \tau(u) \geq \bigvee_{b < a} (\tau^b(u) \wedge b) \geq \bigvee_{b < a} b = a$$

$$\Rightarrow \tau(u) \geq a$$

$$\Rightarrow u \in \tau_a$$

$$\Rightarrow \bigcap \{\tau^b_b : b < a\} \subset \tau_a.$$

For the reverse inclusion, if $u \in \tau_a$, then for $b < a$ we have,

$$b < a \leq \bigvee_{a \in M\setminus\{0\}} (\tau^a(u) \wedge a) = \tau(u)$$

$$\Rightarrow \exists_{c \in M} \; b < \tau^c(u) \wedge c$$

$$\Rightarrow b < c \text{ and } b < \tau^c(u)$$

$$\Rightarrow b < \tau^b(u) \text{ (since } b < \tau^c(u) \leq \tau^b(u))$$

Thus $$u \in \bigcap \{\tau^b_b : b < a\}.$$

In the next theorem suprema of families of (L,M)–fuzzy topologies are described in a useful manner.

The proof of the next theorem is different from Šostak's [S2] which we do not understand fully. He also uses the order density property of I, but we do not have it for our lattice M.

8.8 THEOREM

Let M be a complete chain and $\{\tau^k : k \in K\}$ be a family of (L,M)–fuzzy topologies on X. Let

$$\tau = \bigvee \{\tau^k : k \in K\}.$$

Then

$$\tau(u) = \bigvee \{a \in M : u \in \bigvee \{\tau^k_a : k \in K\}\}$$

for every $u \in L^X$.

PROOF

Let $\sigma(u) = \bigvee\{a \in M : u \in \bigvee\{\tau_a^k : k \in K\}\}$ for every $u \in L^X$. The assertion will be proved if it is shown that :

(1) $\tau^k \leq \sigma$ for every $k \in K$,

(2) $\sigma \leq \tau$,

(3) σ is an (L,M)–fuzzy topology on X.

Indeed, by (3) and (1) we have $\tau \leq \sigma$, so that $\tau = \sigma$ on account of (2).

To prove (1), let $u \in L^X$. We shall show that $\downarrow\tau^k(u) \subset \downarrow\sigma(u)\ \forall\ k \in K$. Let $k \in K$ and $a \in \downarrow\tau^k(u)$. Then $u \in \tau_a^k \leq \bigvee\{\tau_a^k : k \in K\}$, and we conclude that $a \leq \sigma(u)$. Therefore

$$\tau^k(u) = \bigvee\downarrow\tau^k(u) \leq \bigvee\downarrow\sigma(u) = \sigma(u).$$

Next we prove (2). Let $u \in L^X$. We show that if for all $a \in M$, $a < \sigma(u)$ implies $a < \tau(u)$, then $\sigma(u) \leq \tau(u)$. (For if the implication is not true, we have $\sigma(u) > \tau(u) = a$ which implies that $a = \tau(u) < \tau(u)$ by our assumption, which is absurd). Now if $a < \sigma(u)$, then there exists $b > a$ such that

$$u \in \bigvee_{k \in K} \tau_b^k \subset (\bigvee_{k \in K} \tau^k)_b = \tau_b.$$

Thus $a < b \leq \tau(u)$. Hence $\sigma(u) \leq \tau(u)$.

We now prove (3). It is clear that $\sigma(\underline{0}) = \sigma(\underline{1}) = 1$.

Let $u, v \in L^X$. Suppose that $\sigma(u \wedge v) < \sigma(u) \wedge \sigma(v)$.

Then $\sigma(u \wedge v) < \sigma(u)$, so that there exists $a \in M$ such that

$$\sigma(u \wedge v) < a \quad \text{and} \quad u \in \bigvee_{k \in K} \tau_a^k.$$

Similarly, there is $b \in M$ such that

$$\sigma(u \wedge v) < b \quad \text{and} \quad v \in \bigvee_{k \in K} \tau_b^k.$$

With $c = a \wedge b$ one gets

$$\sigma(u \wedge v) < c \quad \text{and} \quad u \wedge v \in \bigvee_{k \in K} \tau_c^k,$$

i.e. $\sigma(u \wedge v) < c$ and $\sigma(u \wedge v) \geq c$, a contradiction.

Finally to prove the supremum axiom, let $\{u_j : j \in J\} \subset L^X$. Let us assume that

$$\sigma(\bigvee_{j \in J} u_j) < \bigwedge_{j \in J} \sigma(u_j).$$

Again we consider two cases.

CASE 1

There is $c \in M$ such that

$$\sigma(\bigvee_{j \in J} u_j) < c < \bigwedge_{j \in J} \sigma(u_j).$$

The first inequality implies that

$$\bigvee_{j \in J} u_j \notin \bigvee_{k \in K} \tau_c^k,$$

which in turn implies that

$$u_{j_0} \notin \bigvee_{k \in K} \tau_c^k \text{ for some } j_0 \in J.$$

Therefore

$$(*) \qquad \sigma(u_{j_0}) \leq c.$$

Indeed, if

$$c < \sigma(u_{j_0}) = \bigvee \{b \in M : u_{j_0} \in \bigvee_{k \in K} \tau_b^k\},$$

then

$$u_{j_0} \in \bigvee_{k \in K} \tau^k_b \text{ for some } b > c,$$

i.e.
$$u_{j_0} \in \bigvee_{k \in K} \tau^k_b \subset \bigvee_{k \in K} \tau^k_c,$$

a contradiction.

Thus by (*) we get

$$\bigwedge_{j \in J} \sigma(u_j) \leq \sigma(u_{j_0}) \leq c, \text{ a contradiction.}$$

CASE 2

There is a gap between $\sigma(\bigvee_{j \in J} u_j)$ and $\bigwedge_{j \in J} \sigma(u_j)$,

i.e. there exists no $c \in M$ such that $\sigma(\bigvee_{j \in J} u_j) < c < \bigwedge_{j \in J} \sigma(u_j)$. Let us denote $\bigwedge_{j \in J} \sigma(u_j)$ by a.

Then $\sigma(\bigvee_{j \in J} u_j) < a$ means that $\bigvee_{j \in J} u_j \notin \bigvee_{k \in K} \tau^k_a$.

Thus there is $j_0 \in J$ such that

(*)
$$u_{j_0} \notin \bigvee_{k \in K} \tau^k_a.$$

Since a is isolated from below, we have by (*) that $\sigma(u_{j_0}) < a$.

Indeed, if $\sigma(u_{j_0}) \geq a$, i.e.

$$\bigvee \{b \in M : u_{j_0} \in \bigvee_{k \in K} \tau^k_b\} \geq a,$$

then (since a is isolated from below) there is $b \geq a$ such that

$$u_{j_0} \in \bigvee_{k \in K} \tau^k_b \subset \bigvee_{k \in K} \tau^k_a,$$

i.e.

$$u_{j_0} \in \bigvee_{k \in K} \tau^k_a,$$

a contradiction.

Thus

$$\bigwedge_{j \in J} \sigma(u_j) \leq \sigma(u_{j_0}) < a = \bigwedge_{j \in J} \sigma(u_j),$$

a contradiction.

The proof is complete.

We have the following corollary to Proposition 8.8.

8.9 COROLLARY

Let M be a complete chain and $\{\tau^k : k \in K\}$ be a family of (L,M)–fuzzy spaces. If $\tau^0 = \bigvee_{k \in K} \tau^k$, then $\tau^0_a = \bigcap_{a' < a} (\bigvee_k \tau^k_{a'})$ for all $a \in M$ which are not isolated from below. (i.e. $a = \bigvee \{a' : a' < a\}$).

PROOF

Let

$$u \in \bigcap_{a' < a} (\bigvee_k \tau^k_{a'}),$$

Then

$$\forall_{a' < a} \quad u \in \bigvee_k \tau^k_{a'}.$$

Thus

$$\begin{aligned}\tau^0(u) &= \bigvee \{b : u \in \bigvee_k \tau^k_b\}, \\ &\geq \bigvee \{a' : a' < a \text{ and } u \in \bigvee_k \tau^k_{a'}\}, \\ &= \bigvee \{a' : a' < a\}, \\ &= a.\end{aligned}$$

Hence $u \in \tau^0_a$.

For the reverse inclusion we do not need any condition on a.

If we let $u \in \tau^0_a$, then

$$\begin{aligned} &\forall_{a' < a} \quad a' < a \leq \tau^0(u) = \bigvee \{b : u \in \bigvee_k \tau^k_b\}, \\ \Rightarrow\ &\forall_{a' < a} \quad \exists_{c'}\ a' < c' \text{ and } \quad u \in \bigvee_k \tau^k_{c'}, \\ \Rightarrow\ &\forall_{a' < a} \quad u \in \bigvee_k \tau^k_{a'}. \\ &\text{Hence} \quad u \in \bigcap_{a' < a} (\bigvee_k \tau^k_{a'}).\end{aligned}$$

We note in passing the following proposition which will require that no member of M is isolated from below, in other words, that M is order dense.

8.10 PROPOSITION

Let M be an order dense chain, $\{\tau^i : i \in J\}$ and $\{\sigma^i : i \in J\}$ be families of (L,M)–fuzzy topologies on the sets X and Y respectively and let $\tau = \bigvee_i \tau^i$ and $\sigma = \bigvee_i \sigma^i$.

If a mapping $f : (X,\tau^i) \longrightarrow (Y,\sigma^i)$ is continuous for each $i \in J$, then the mapping $f : (X,\tau) \longrightarrow (Y,\sigma)$ is continuous.

PROOF

$$(\tau^i,\sigma^i)\text{–continuity of } f, \qquad \forall_{i \in J}$$

$$\Longrightarrow (\tau^i_a,\sigma^i_a)\text{–continuity of } f, \qquad \forall_{i \in J} \; \forall_{a \in M} \qquad \text{(by 5.10)},$$

$$\Longrightarrow \Big(\bigvee_i \tau^i_a, \bigvee_i \sigma^i_a\Big)\text{–continuity of } f, \qquad \forall_{a \in M} \qquad \text{(by 4.10)},$$

$$\Longrightarrow \Big(\bigcap_{a<b}\big(\bigvee_i \tau^i_a\big), \bigcap_{a<b}\big(\bigvee_i \sigma^i_a\big)\Big)\text{–continuity of } f, \quad \forall_{b \in M} \qquad \text{(by 4.9)},$$

$$\Longrightarrow (\tau,\sigma)\text{–continuity of } f \qquad \text{(by 8.6)}.$$

9 STRATIFICATION OF (L,M)–FUZZY TOPOLOGIES

9.1 DEFINITION

An (L,M)–fuzzy topology τ on a set X is called stratified if $\tau(\underline{c}) = 1$ for each constant L–fuzzy set $\underline{c} \in L^X$.

It is easy to see that all the examples of (L,M)–fuzzy topologies we have in 5.3 are not stratified. The following (L,M)–fuzzy topology is stratified.

EXAMPLE

Let X be a non–empty set, L and M be complete lattices. Define $\tau : L^X \longrightarrow M$ by

$$\tau(u) = \begin{cases} 1 & \text{if u is a constant L–fuzzy set} \\ 0 & \text{otherwise} \end{cases}$$

9.2 DEFINITION

Let τ be an (L,M)–fuzzy topology on a set X. The weakest stratified (L,M)–fuzzy topology τ^c such that $\tau^c \geq \tau$ is called the stratification of the (L,M)–fuzzy topology τ.

The following provides a relationship between τ^c and $(\tau_a)^c$ where τ is an (L,M)–fuzzy topology and $a \in M$.

9.3 PROPOSITION

Let M be a complete chain and (X,τ) be an (L,M)–fuzzy space. Then

$$\tau^c(u) = \bigvee_{a \neq 0} (1_{(\tau_a)^c}(u) \wedge a)$$

for all $u \in L^X$.

PROOF

Denote $\sigma = \bigvee \{1_{(\tau_a)^c} \wedge a : a \in M\backslash\{0\}\}$.

Since $\{(\tau_a)^c : a \in M\backslash\{0\}\}$ is decreasing, σ is an (L,M)–fuzzy topology by Theorem 8.2. Since $\tau \leq \tau^c$, we have $\tau_a \subset (\tau^c)_a$ for all $a \in M$. Clearly $(\tau^c)_a$ is a stratified topology, thus $(\tau_a)^c \subset (\tau^c)_a$. Consequently, by Proposition 8.5 we get

$$\begin{aligned}
\tau^c &= \bigvee_{a \in M} (1_{(\tau^c)_a} \wedge a) \\
&\geq \bigvee_{a \in M} (1_{(\tau_a)^c} \wedge a) \\
&= \sigma \\
&\geq \bigvee_{a \neq 0} (1_{\tau_a} \wedge a) = \tau.
\end{aligned}$$

Thus $\tau^c \geq \sigma \geq \tau$.

Finally note that σ is stratified. (For if $\underline{b} \in L^X$, then $\sigma(\underline{b}) = \bigvee\{1 \wedge a : a \in M\} = 1$). Therefore $\tau^c = \sigma$ by Definition 9.2.

The next proposition shows that the assignment $(X,\tau) \longrightarrow (X,\tau^c)$ is functorial.

9.4 PROPOSITION

Let M be a complete chain and $(X,\tau), (Y,\sigma) \in |\text{FTOP}(L,M)|$.
If $f : (X,\tau) \longrightarrow (Y,\sigma)$ is continuous, then $f : (X,\tau^c) \longrightarrow (Y,\sigma^c)$ is continuous.

PROOF

Given $f : X \longrightarrow Y$, we have for all $a \in M$,

(τ,σ)–continuity	$\Rightarrow (\tau_a,\sigma_a)$–continuity	(by 5.10)
	$\Rightarrow ((\tau_a)^c,(\sigma_a)^c)$–continuity	(by 4.14)
	$\Rightarrow (\tau^c,\sigma^c)$–continuity	(by 8.4)

9.5 COROLLARY

For M a complete chain, $(\cdot)^c : \text{FTOP}(L,M) \longrightarrow \text{FTOP}(L,M)$ is a functor such that :

$$(X,\tau)^c = (X,\tau^c),$$

and

$$(f : (X,\tau) \longrightarrow (Y,\sigma))^c = (f : (X,\tau^c) \longrightarrow (Y,\sigma^c))$$

i.e., $(\cdot)^c$ leaves mappings unchanged.

Recall that when dealing with weak (L,M)–fuzzy topologies M must be completely distributive (see 6.7). Denote by $f^{\leftarrow}(\sigma)$, the weak (L,M)–fuzzy topology on X induced by $f : X \longrightarrow (Y,\sigma)$. We note the following obvious fact :

9.6 REMARK

If σ is stratified, so is $f^{\leftarrow}(\sigma)$.

PROOF

Indeed, for all $b \in L$, $\underline{b}f$ is constant, hence

$$\begin{aligned} f^{\leftarrow}(\sigma)(\underline{b}) &= \bigvee \{\sigma(v) : v \in L^Y \text{ and } \underline{b} = vf\} \\ &\geq \sigma(\underline{b}) \\ &= 1 \end{aligned}$$

9.7 PROPOSITION

Let M be a complete chain, (Y,σ) be an (L,M)–fuzzy space and $f : X \longrightarrow Y$ be a mapping. If $\{\tau^i : i \in J\}$ is a family of (L,M)–topologies on X, then the following statements hold :

(1) $f^{\leftarrow}(\sigma^c) = f^{\leftarrow}(\sigma)^c$,

(2) $(\bigvee_{i \in J} \tau^i)^c = \bigvee_{i \in J} \tau^{i^c}$.

PROOF

(1) For each mapping f, we have

$$(f^{\leftarrow}(\sigma),\sigma)\text{–continuity} \Longrightarrow (f^{\leftarrow}(\sigma)^c,\sigma^c)\text{–continuity} \qquad \text{(by 9.4)}$$

hence

$$f^{\leftarrow}(\sigma^c) \leq f^{\leftarrow}(\sigma)^c.$$

Clearly

$$f^{\leftarrow}(\sigma) \leq f^{\leftarrow}(\sigma^c),$$

the latter being stratified by 9.6, hence

$$f^{\leftarrow}(\sigma)^c \leq f^{\leftarrow}(\sigma^c).$$

Then $f^{\leftarrow}(\sigma)^c = f^{\leftarrow}(\sigma^c)$.

(2) Clearly, $\bigvee_{i \in J} \tau^{i^c}$ is stratified. For, given $\underline{b} \in L^X$,

$$(\bigvee_{i \in J} \tau^{i^c})(\underline{b}) = \bigwedge \{\gamma(\underline{b}) : \tau^{i^c} \leq \gamma \text{ for all } i \in J\}$$

(γ's are (L,M)–fuzzy topologies).

Since $1 = \tau^{i^c}(\underline{b}) \leq \gamma(\underline{b})$, hence

$$(\bigvee_{i \in J} \tau^{i^c})(\underline{b}) = 1.$$

(**NOTE** : It suffices for $\bigvee_{i \in J} \tau^i$ to be stratified, that at least one τ^i is stratified).

Further,

$$(\bigvee_{i \in J} \tau^i)^c \geq \tau^{i^c} \text{ for all } i,$$

hence

$$(\bigvee_{i \in J} \tau^i)^c \geq \bigvee_{i \in J} \tau^{i^c} \geq \bigvee_{i \in J} \tau^i.$$

Since the (L,M)–topology $\bigvee_{i \in J} \tau^{i^c}$ is stratified, hence

$$(\bigvee_{i \in J} \tau^i)^c = \bigvee_{i \in J} \tau^{i^c}$$

by Definition 9.2.

9.8 PROPOSITION

Let M be a complete chain, $\{(X_i,\sigma^i) : i \in J\} \subset |FTOP(L,M)|$ and $f_i : X \rightarrow X_i$ for all $i \in J$. Then :

$$\Big(\bigvee_{i \in J} f_i^{\leftarrow}(\sigma^i)\Big)^c = \bigvee_{i \in J} f_i^{\leftarrow}(\sigma^{i^c})$$

PROOF

By 9.7.

9.9 COROLLARY

Let M be a complete chain and (X,τ), $(X_i,\tau^i) \in |FTOP(L,M)|$ $(i \in J)$. Let $A \subset X$. Then :

(1) $(\tau_A)^c = (\tau^c)_A$ (where τ_A is the subspace topology),

(2) $\Big(\prod_{i \in J} \tau^i\Big)^c = \prod_{i \in J} \tau^{i^c}$.

PROOF

(1) Let $e : A \rightarrow X$ be the identity embedding. Then,

$$\tau_A = e^{\leftarrow}(\tau).$$

Thus,

$$\begin{aligned}(\tau_A)^c &= e^{\leftarrow}(\tau)^c \\ &= e^{\leftarrow}(\tau^c) && \text{(by 9.7(1))} \\ &= (\tau^c)_A.\end{aligned}$$

(2) Let $p_i : \prod_{i \in J} X_i \longrightarrow X_i$ denote the i–th projection. We have,

$$
\begin{aligned}
(\prod_{i \in J} \tau^i)^c &= (\bigvee_{i \in J} p_i^{\leftarrow}(\tau^i))^c \\
&= \bigvee_{i \in J} p_i^{\leftarrow}(\tau^i)^c && \text{(by 9.8)} \\
&= \bigvee_{i \in J} p_i^{\leftarrow}(\tau^{i^c}) && \text{(by 9.7(1))} \\
&= \prod_{i \in J} \tau^{i^c}.
\end{aligned}
$$

Although this is not really important, we shall now show how the crucial equality $f^{\leftarrow}(\sigma^c) = f^{\leftarrow}(\sigma)^c$ can be obtained from 9.3, by avoiding the functoriality of $(\cdot)^c$. We need the following property of the stratification in TOP(L).

9.10 REMARK

If T is an L–topology on Y and

$$f : X \longrightarrow (Y,T), \text{ then } f^{\leftarrow}(T)^c = f^{\leftarrow}(T^c).$$

PROOF

We have,

$$
\begin{aligned}
f^{\leftarrow}(T^c) &= f^{\leftarrow}(T \vee \ell_Y) \quad (\text{where } \ell_Y = \{\underline{a} : \underline{a} \in L^Y\}) \\
&= f^{\leftarrow}(T) \vee f^{\leftarrow}(\ell_Y) \\
&= f^{\leftarrow}(T) \vee \ell_X \\
&= f^{\leftarrow}(T)^c.
\end{aligned}
$$

9.11 OBSERVATIONS

(a) We note that if $f : X \longrightarrow (Y,T)$ with $(Y,T) \in |\mathrm{TOP}(L)|$, then

$$f^{\leftarrow}(1_T) = 1_{f^{\leftarrow}(T)}.$$

Indeed,

$$\begin{aligned} f^{\leftarrow}(1_T)(u) &= \bigvee \{1_T(v) : u = vf\} \\ &= \begin{cases} 1 & \text{if } \exists\ v \in T : u = vf \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} 1 & \text{if } u \in f^{\leftarrow}(T) \\ 0 & \text{otherwise} \end{cases} \\ &= 1_{f^{\leftarrow}(T)}(u). \end{aligned}$$

(b) If (Y,σ) is an (L,M)–fuzzy topological space and $f : X \longrightarrow (Y,\sigma)$, then

$$f^{\leftarrow}(\sigma_a) \subset f^{\leftarrow}(\sigma)_a \quad \forall\ a \in M,$$

since

$$\begin{aligned} u \in f^{\leftarrow}(\sigma_a) &\Leftrightarrow \exists_{v \in L^X}\ u = vf \text{ and } v \in \sigma_a \\ &\Leftrightarrow \exists_{v \in L^X}\ u = vf \text{ and } \sigma(v) \geq a \\ &\Rightarrow a \leq \bigvee \{\sigma(v) : u = vf\} = f^{\leftarrow}(\sigma)(u) \\ &\Rightarrow u \in f^{\leftarrow}(\sigma)_a. \end{aligned}$$

We are now ready to prove this result. If M is a complete chain, (Y,σ) an (L,M)–fuzzy space and $f : X \longrightarrow (Y,\sigma)$ a mapping, then $f^{\leftarrow}(\sigma^c) = f^{\leftarrow}(\sigma)^c$.

ANOTHER PROOF OF 9.7(1)

We have,

$$f^{\leftarrow}(\sigma^c) \geq f^{\leftarrow}(\sigma),$$

hence

$$f^{\leftarrow}(\sigma^c) \geq f^{\leftarrow}(\sigma)^c.$$

For the converse we proceed as follows. Given $u \in L^X$,

$$f^{\leftarrow}(\sigma^c)(u) = \bigvee_{u = vf} \sigma^c(u)$$

$$= \bigvee_{u = vf} \bigvee_{a \in M} (1_{(\sigma_a)^c}(v) \wedge a) \qquad \text{(by 9.3)}$$

$$= \bigvee_{a \in M} (a \wedge \bigvee_{u = vf} 1_{(\sigma_a)^c})(v)$$

$$= \bigvee_{a \in M} (a \wedge f^{\leftarrow}(1_{(\sigma_a)^c}(u))$$

$$= \bigvee_{a \in M} (a \wedge 1_{f^{\leftarrow}((\sigma_a)^c)}(u)) \qquad \text{(by 9.11(a))}$$

$$= \bigvee_{a \in M} (a \wedge 1_{(f^{\leftarrow}(\sigma_a))^c}(u)) \qquad \text{(by 9.10)}$$

$$\leq \bigvee_{a \in M} (a \wedge 1_{((f^{\leftarrow}(\sigma))_a)^c}(u) \qquad \text{(by 9.11(b))}$$

$$= f^{\leftarrow}(\sigma)^c(u).$$

NOTES

In Section 8, everything apart from 8.5 and 8.6 extends Šostak's [S2] results from I to L and M. Note that 8.5 and 8.6 are T. Kubiak's observations. With both his and J. Chadwick's assistance the author has proved 8.2 and 8.8. Proofs of 8.7 and 8.9 are by the author. Everything in Section 9 except an example of 9.1 generalizes Šostak's [S2] work from I to L and M. All the proofs which are different from [S2] in this section are written by the author. Also the properties of the stratification functor extend those of [K4] where $M = 2$.

CHAPTER 5

SEPARATION AXIOMS AND COMPACTNESS IN (L,M)–FUZZY TOPOLOGICAL SPACES

INTRODUCTION

Šostak develops a theory of separation axioms and compactness for an I–fuzzy topological space at a particular level $a \in I$. At this level he actually considers the above notions for an a–level topology τ_a, which is an I–topology. In other words Šostak's study of the above notions concerns TOP(I) but not FTOP(I,I), in our opinion. Developing a theory of these notions for FTOP(L,M) would be an interesting problem for further research. It is our feeling that one would like to know what a Hausdorff, say, I–fuzzy space is supposed to be, without any reference to the a–level I–topologies. In other words, the (L,M)–fuzzy topological properties should generalize the L–topological ones and not be based on them (see [R2, 5.36(5)]). Šostak has however generalized the above notions, by letting their values vary from 0 to 1. Nevertheless we will discuss possible directions of extending Šostak's theory from (I,I) to (L,M).

10 SEPARATION AXIOMS IN (L,M)–FUZZY TOPOLOGICAL SPACES

We start with the degree of T_0–separation of (L,M)–fuzzy topological spaces. The next proposition motivates definition 10.2.

10.1 PROPOSITION

Let (X,τ) be an I–fuzzy space. The following are three possible definitions of the spectrum of T_0–separation of (X,τ) (denoted by $S_a^o(X)$), and they are all equivalent :

(1) DEFINITION

We have $b \in S_a^o(X)$ (where $b \in I$) iff for all $x, y \in X$, $x \neq y$, and for all $\epsilon > 0$ there exists $u \in \tau_a$ such that $u(x) \geq b$ and $u'(y) \geq b-\epsilon$ or $u(y) \geq b$ and $u'(x) \geq b-\epsilon$.

(2) DEFINITION

If $b \in I$, then $b \in S_a^o(X)$ iff for all $x, y \in X$, $x \neq y$ and for all $\epsilon > 0$ there exists $u \in \tau_a$ such that $u(x) \geq b-\epsilon$ and $u'(y) \geq b-\epsilon$ or $u(y) \geq b-\epsilon$ and $u'(x) \geq b-\epsilon$.

(3) DEFINITION

We say $b \in S_a^o(X)$ (where $b \in I$) iff for all $x, y \in X$ such that $x \neq y$ and for all $\epsilon > 0$ there exists $u \in \tau_a$ such that $u(x) > b-\epsilon$ and $u'(y) > b-\epsilon$ or $u(y) > b-\epsilon$ and $u'(x) > b-\epsilon$.

PROOF

It is easy to see that Definitions 1 and 2 are equivalent. We now show that definitions 2 and 3 are equivalent. Definition 3 $\Longrightarrow$ Definition 2 is obvious.

Definition 2 $\Longrightarrow$ Definition 3 : Let $x, y \in X$ be such that $x \neq y$ and let $\epsilon > 0$. Choose ϵ_1 such that $0 < \epsilon_1 < \epsilon$.

Now there exists $u \in \tau_a$ such that $u(x) \geq b-\epsilon_1 > b-\epsilon$ and $u'(y) \geq b-\epsilon_1 > b-\epsilon$ or $u(y) \geq b-\epsilon_1 > b-\epsilon$ and $u'(x) \geq b-\epsilon_1 > b-\epsilon$.

We know that in I, $>$ is the same as $\nleq$ or $>>$, hence extending Definition 2 to (L,M) we have the following two possibilities. We first consider the case of $\nleq$.

10.2 DEFINITION

If (X,τ) is an (L,M)–fuzzy space and $b \in L$ we say $b \in S_a^o(X)$ iff for all $x, y \in X$ such that $x \neq y$ and for all $b \nleq b_1$, there exists $u \in \tau_a$ such that $u(x) \geq b_1$ and $u'(y) \geq b_1$ or $u(y) \geq b_1$ and $u'(x) \geq b_1$.

NOTE : We could consider another possibility of extending Definition 2, *viz.* by replacing $\nleq$ by $>>$ in Definition 10.2, but this direction will not be pursued here. The symbol $S_a^o(X)$ will always mean that of Definition 10.2.

10.3 DEFINITION

The degree of T_0–separation of an (L,M)–fuzzy topological space (X,τ) at a level $a \in M\backslash\{0\}$ is the number $s_a^o(X) = \bigvee S_a^o(X)$.

In what follows all the propositions stated without proofs occur in Šostak [S3].

10.4 PROPOSITION

The spectrum of T_0–separation of an I–fuzzy topological space (X,τ) at a level $a \in I$ has the form

$$S_a^o(X) = [0, s_a^o(X)].$$

We have the following analogue for Proposition 10.4.

10.5 PROPOSITION

Let L be an order dense chain and (X,τ) an (L,M)–fuzzy space. Then the spectrum of T_0–separation of (X,τ) at a level $a \in M\backslash\{0\}$ has the form $S_a^o(X) = \downarrow s_a^o(X)$.

PROOF

To prove this assertion we must show the following two conditions :

(1) If $b \in S_a^o(X)$ and $0 \leq b_1 < b$, then $b_1 \in S_a^o(X)$.

(2) $s_a^o(X) \in S_a^o(X)$.

Since $b_2 < b_1$ implies $b_2 < b$, (1) follows.

To prove (2), we let $b = s_a^o(X)$. If $b_1 < b$, then by the order density property there exists $c \in L$ such that $b_1 < c < b$. Since $c \in \downarrow s_a^o(X) \setminus \{b\}$ we must have $c \in S_a^o(X)$. (For if $c \notin S_a^o(X)$, there exists no $b \in S_a^o(X)$ such that $c \leq b$, thus $\bigvee S_a^o(X) \leq c$ a contradiction).
Hence, for all $x, y \in X$ and $x \neq y$ there exists $u \in \tau_a$ such that $u(x) \geq b_1$ and $u'(y) \geq b_1$ or $u(y) \geq b_1$ and $u'(x) \geq b_1$.

10.6 PROPOSITION

If (Y,τ_Y) is a subspace of an (L,M)–fuzzy topological space (X,τ) (with M completely distributive), then $S_a^o(X) \subset S_a^o(Y)$ and consequently $s_a^o(X) \leq s_a^o(Y)$.

PROOF

If $id : (Y,\tau_Y) \longrightarrow (X,\tau)$ is the identity embedding, then recall that $\tau_Y : L^Y \longrightarrow M$ is defined by $\tau_Y(v) = \bigvee \{\tau(u) : (u)id = v \text{ and } u \in L^X\}$.

If $b \in S_a^o(X)$, then for all $x, y \in X$ such that $x \neq y$ and for all $b \nleq b_1$ there exists $u \in \tau_a$ such that $b_1 \leq u(x)$ and $b_1 \leq u'(y)$ or $b_1 \leq u(y)$ and $b_1 \leq u'(x)$. Let $v = (u)id$. Now for all $x, y \in Y$, $x \neq y$ and for all $b \nleq b_1$ we have $v \in (\tau_Y)_a$ and $v(x) \geq b_1$ and $v'(y) \geq b_1$ or $v(y) \geq b_1$ and $v'(x) \geq b_1$. Hence $b \in S_a^o(Y)$.

10.7 PROPOSITION

Let τ and τ' be two (L,M)–fuzzy topologies on X and $\tau \leq \tau'$. Then $S_a^o(X,\tau) \subset S_a^o(X,\tau')$ and hence $s_a^o(X,\tau) \leq s_a^o(X,\tau')$.

PROOF

Follows from the fact that $\tau_a \subset \tau'_a$ for each $a \in M\backslash\{0\}$.

We pay attention to the behaviour of the spectrum of T_0–separation in passage from an (L,M)–fuzzy space to its image under a continuous one–one map.

10.8 PROPOSITION

Let (X,τ) and (Y,σ) be (L,M)–fuzzy spaces. If there exists a one–one, continuous map $f : (X,\tau) \longrightarrow (Y,\sigma)$, then $S_a^o(Y) \subset S_a^o(X)$ and hence $s_a^o(Y) \leq s_a^o(X)$.

PROOF

Since $f : X \longrightarrow Y$ is one–one, $x_1 \neq x_2$ in X implies that $y_1 = f(x_1) \neq f(x_2) = y_2$ in Y.

If $b \in S_a^o(Y)$, then for all $b \nleq b_1$ there exists $u \in \sigma_a$ with $a \in M\backslash\{0\}$, such that $u(y_1) \geq b_1$ and $u'(y_2) \geq b_1$ or $u'(y_1) \geq b_1$ and $u(y_2) \geq b_1$. Letting $v = uf$ we get $v \in \tau_a$ by continuity of f (see 5.4). We also get $v(x_1) \geq b_1$ and $v'(x_2) \geq b_1$ or $v(x_2) \geq b_1$ and $v'(x_1) \geq b_1$. Hence $b \in S_a^o(X)$.

We observe the behaviour of the spectrum of T_0–separation of an I–fuzzy space in passage to its stratification as given by Šostak [S3].

10.9 PROPOSITION

Let (X^o,τ^o) be the stratification of an I–fuzzy space (X,τ). Then $S_a^o(X) \subset S_a^o(X^o)$ and $S_a^o(X) \cap (^1/_2,1] = S_a^o(X^o) \cap (^1/_2,1]$, where X^o is the stratification of X.

REMARK

The equality $S_a^o(X) = S_a^o(X^o)$ does not hold in general.

We now study the behaviour of the spectrum of T_0–separation of a family of I–fuzzy spaces in passage to their product space.

10.10 PROPOSITION

Let $\{(X_i,\tau^i) : i \in J\}$ be a family of I–fuzzy topological spaces and (X,τ) its product. Then $\bigcap_i S_a^o(X_i) \subset S_a^o(X)$ and we thus have $\bigwedge_i s_a^o(X_i) \leq s_a^o(X)$. We also have that

$$S_a^o(X) \cap (^1/_2,1] = \bigcap_i S_a^o(X_i) \cap (^1/_2,1]$$

and hence

$$s_a^o(X) \vee {}^1/_2 = (\bigwedge_i s_a^o(X_i)) \vee {}^1/_2.$$

10.11 PROPOSITION

If all I–fuzzy spaces $(\{X_i,\tau^i) : i \in J\}$ of Proposition 10.10 are non–empty and stratified, then

$$\bigcap_i S_a^o(X_i) = S_a^o(X)$$

and consequently

$$\bigwedge_i s_a^o(X_i) = s_a^o(X).$$

In extending Propositions 10.9 and 10.10 from (I,I) to (L,M), we must think of a reasonable way of finding a counterpart of $^1/_2$ in a complete lattice L. We have the following possibilities.

(a) Any $b \in L$ with $b = b'$
(if $L = I$, then $b = b'$ iff $b = {}^1/_2$).

(b) $b = \bigvee \{a \wedge a' : a \in L\}$
(if $L = I$, then $^1/_2 = \bigvee \{a \wedge (1-a) : a \in I\}$).

(c) $b = \bigwedge \{a \vee a' : a \in L\}$
(with $L = I$ we get $^1/_2 = \bigwedge \{a \vee (1-a) : a \in I\}$).

(d) $b = \bigvee \{a \in L : a \leq a'\}$
(if $L = I$, $a \leq a'$ iff $a \leq {}^1/_2$).

(e) $b = \bigwedge \{a \in L : a' \leq a\}$
(if $L = I$, $a' \leq a$ iff $^1/_2 \leq a$).

We now discuss possible notions of the T_1–separation axiom for (L,M)–fuzzy topological spaces. In what follows we just mimic everything for T_0–separation and eventually get these two possible definitions of the T_1–separation spectrum for (L,M)–fuzzy spaces.

10.12 DEFINITION

The spectrum of T_1–separation of an (L,M)–fuzzy topological space (X,τ) at a level $a \in M\backslash\{0\}$ is the set $S_a^{1^*}(X)$ consisting of all $b \in L$ such that for all $x, y \in X$, $x \neq y$ and $b_1 << b$ there exists $u \in \tau_a$ such that $u(x) \geq b_1$ and $u'(y) \geq b_1$.

10.13 DEFINITION

Replace $S_a^{1^*}(X)$ and $b_1 << b$ in 10.12 by $S_a^1(X)$ and $b \nleq b_1$ respectively.

10.14 DEFINITION

The degree of T_1–separation of an (L,M)–fuzzy topological space (X,τ) at a level $a \in M\backslash\{0\}$ is the number $s_a^1(X) = \bigvee S_a^1(X)$ (for 10.13) and $s_a^{1^*}(X) = \bigvee S_a^{1^*}(X)$ (for 10.12).

We know that in topology all points are closed in T_1–topological spaces. We now introduce the necessary theory which will enable us to establish a fuzzy analogue for that statement. We have the next two possible definitions of the spectrum of closedness of a subset W of X.

10.15 DEFINITION

Let (X,τ) be an (L,M)–fuzzy topological space. The spectrum of closedness $c\ell_a^*(W,X)$ at a level $a \in M\setminus\{0\}$ of a subset W of X, consists of all $b \in L$ such that for all $b_1 << b$ there exists $u \in \tau_a$ such that $u'(x) \geq b_1$ for every $x \in W$ and $u(x) \geq b_1$ for every $x \in X$ such that $x \notin W$.

Replacing $>>$ by $\not\leq$ in the previous definition gives us another definition of the spectrum of closedness of W at a level a. We will denote it by $c\ell_a(W,X)$.

10.16 DEFINITION

The spectrum of closedness of an (L,M)–fuzzy topological space (X,τ) is the intersection of the spectra of closedness of all its one–point subsets, i.e.

$$c\ell_a(X) = \bigcap_x c\ell_a(x,X)$$

or

$$c\ell_a^*(X) = \bigcap_x c\ell_a^*(x,X).$$

We make the following remark which summarizes 10.15 and 10.16.

10.17 REMARK

(1) We say $b \in c\ell_a^*(X)$ iff for each $x \in X$ and for all $b_1 << b$ there exists $u \in \tau_a$ such that $b_1 \leq u'(x)$ and $b_1 \leq u(y)$ for all $y \neq x$.

(2) We have $b \in c\ell_a(X)$ iff for each $x \in X$ and for all $b \not\leq b_1$ there exists $u \in \tau_a$ such that $b_1 \leq u'(x)$ and $b_1 \leq u(y)$ for all $y \neq x$.

We further establish our interesting result.

10.18 THEOREM

We have $S_a^1(X) = c\ell_a(X)$ for every (L,M)–fuzzy topological space X (similarly $S_a^{1*}(X) = c\ell_a^*(X)$).

PROOF

That $c\ell_a(X) \subset S_a^1(X)$ is obvious. Indeed, let $b \in c\ell_a(X)$. Fix $x, y \in X$ such that $x \neq y$ and for all $b \nleq b_1$ choose $u \in \tau_a$ such that $u'(x) \geq b_1$ and $u(y) \geq b_1$. Hence $b \in S_a^1(X)$. Thus $c\ell_a(X) \subset S_a^1(X)$.

For the reverse inclusion let $b \in S_a^1(X)$. Now for each $x \in X$, $y \neq x$ implies for all $b \nleq b_1$ there exists $u_y \in \tau_a$ such that $u_y'(x) \geq b_1$ and $u_y(y) \geq b_1$.

Letting $u = \bigvee_y u_y$, we obtain $\tau(\bigvee_y u_y) \geq \bigwedge_y \tau(u_y) \geq a$, hence $u \in \tau_a$.

Also $u'(x) = \bigwedge_y u_y'(x) \geq b_1$. Thus if $x \in X$, then for every $y \in X$ such that $y \neq x$ and for all $b \nleq b_1$, there exists $u \in \tau_a$ such that $u'(x) \geq b_1$ and $u(y) \geq b_1$.
Hence $b \in c\ell_a(X)$.

We have the following relationship between the T_0–separation and T_1–separation spectra of an (L,M)–fuzzy space (X,τ) at a level $a \in M$.

10.19 THEOREM

For every (L,M)–fuzzy space (X,τ) and $a \in M$, we have

$$S_a^1(X) \subset S_a^o(X)$$

and

$$S_a^{1*}(X) \subset S_a^{o*}(X)$$

PROOF

Obvious.

The T_2–separation (Hausdorffness) axiom is similar to the above separation axioms and therefore we will only state the next two definitions and some propositions.

10.20 DEFINITION

(1) The spectrum of Hausdorffness of an (L,M)–fuzzy topological space (X,τ) on a level $a \in M\backslash\{0\}$ is the set $H_a^*(X)$ of all $b \in L$ such that for all $x, y \in X$, $x \neq y$ and for all $c << b$ there exist $u, v \in \tau_a$ with $u(x) \geq c$, $v(y) \geq c$ and $F(u,v') \geq c$.

(2) $H_a(X)$ is defined in a similar manner, except that $>>$ is replaced by $\not\geq$.

Šostak has the fuzzy counterparts of the following results for topological spaces :

A space X is Hausdorff iff the diagonal in the product space $X \times X$ is closed.

If f and g are two continuous functions of a space X into a Hausdorff space, then $\{x : f(x) = g(x)\}$ is closed in X.

11 COMPACTNESS DEGREE OF L–FUZZY SETS

In this section L and M are complete lattices with order reversing involutions. We define the compactness degree of an L–fuzzy set at a level $a \in M\backslash\{0\}$. This compactness degree is allowed to vary from 0 to 1 in a lattice L. We have to introduce some notation.

NOTATION

For $\mathcal{U} \subset L^X$ let $\bigvee \mathcal{U} = \bigvee \{A : A \in \mathcal{U}\}$, $\bigwedge \mathcal{U} = \bigwedge \{A : A \in \mathcal{U}\}$. By $\mathcal{U}_0$ we denote an arbitrary finite subfamily of $\mathcal{U}$. In the sequel (X,τ) is an (L,M)–fuzzy space, $v \in L^X$ and $a \in M\backslash\{0\}$.

11.1 DEFINITION

The set $C_a(u)$ which consists of all $b \in L$ such that,

if $F(u, \bigvee \mathcal{U}) \geq b$ for some $\mathcal{U} \subset \tau_a$, then $\bigvee_{\mathcal{U}_0} F(u, \bigvee \mathcal{U}_0) \geq b$,

is called the compactness spectrum of an L–fuzzy set u at a level $a \in M\backslash\{0\}$.

Replacing L and M by I above we obtain Šostak's definition of the compactness spectrum of a fuzzy set at a particular level.

11.2 PROPOSITION

The compactness spectrum of an L–fuzzy set u at a level $a \in M\backslash\{0\}$ is always non–empty, in particular $0 \in C_a(u)$.

PROOF

Trivial.

11.3 DEFINITION

The number $d_a(u) = \bigwedge (L\backslash C_a(u))$ is the compactness degree of an L–fuzzy set u at a level $a \in M\backslash\{0\}$.

We investigate a number of characterizations of the compactness spectrum of an L–fuzzy set at a certain level.

11.4 DEFINITION

A subset $B \subset L^X$ is called a base for τ if for every $a \in M\backslash\{0\}$, each non–zero $u \in \tau_a$ can be expressed as a supremum $u = \bigvee_i u_i$ for some $u_i \in B \cap \tau_a$.

We cannot prove the next proposition for (L,M)–fuzzy topological spaces so we just write Šostak's [S3] result for (I,I).

11.5 PROPOSITION

Let $B \in I^X$ be a base for τ. Then $b \in C_a(u)$ iff for every $\mathscr{U} \subset \tau_a \cap B$,

$F(u, \bigvee \mathscr{U}) \geq b$ implies $\bigvee_{\mathscr{U}_o} F(u, \bigvee \mathscr{U}_o) \geq b$.

11.6 DEFINITION

We define $\tau^* : L^X \longrightarrow M$ by,

$$\tau^*(u) = \tau(u')$$

for each $u \in L^X$.

NOTATION

(1) If $a \in M$, then the family $\{u \in L^X : \tau^*(u) \geq a\}$ is denoted by τ^*_a.

(2) Let $\mathscr{F}$ be a family of subsets of L^X. We denote by $\mathscr{F}'$ the collection $\{u' : u \in \mathscr{F}\}$.

11.7 PROPOSITION

Let $u \in L^X$ and $\mathcal{F}$ be a family of subsets of L^X. We have $b \in C_a(u)$ iff $F(\bigwedge \mathcal{F}, u') \geq b$ implies

$$\bigvee_{\mathcal{F}_0} F(\bigwedge \mathcal{F}_0, u') \geq b \text{ for every } \mathcal{F}_0 \subset \tau_a^*.$$

PROOF

$\Rightarrow$ Let $b \in C_a(u)$, and $F(\bigwedge \mathcal{F}, u') \geq b$ with $\mathcal{F} \subset \tau_a^*$. Since

$F(\bigwedge \mathcal{F}, u') = F(u, \bigvee \mathcal{F}')$ (by 3.4(6)) we have

$$\bigvee_{\mathcal{F}'_0} F(u, \bigvee \mathcal{F}'_0) = \bigvee_{\mathcal{F}_0} F(\bigwedge \mathcal{F}_0, u') \geq b.$$

$\Leftarrow$ We have $\mathcal{F} \subset \tau_a^*$ and $F(\bigwedge \mathcal{F}, u') \geq b$ implies $\bigvee_{\mathcal{F}_0} F(\bigwedge \mathcal{F}_0, u') \geq b$. Now $\mathcal{F}' \subset \tau_a$, and since $F(\bigwedge \mathcal{F}, u') = F(u, \bigvee \mathcal{F}')$ and $\bigvee_{\mathcal{F}_0} F(\bigwedge \mathcal{F}_0, u') = \bigvee_{\mathcal{F}'_0} F(u, \bigvee \mathcal{F}'_0)$ we have $F(u, \bigvee \mathcal{F}') \geq b$ implies $\bigvee_{\mathcal{F}'_0} F(u, \bigvee \mathcal{F}'_0) \geq b$ which proves the proposition.

Šostak stated the next theorem without a proof. It is a fuzzy analogue of the theorem which states that the continuous image of a compact set is compact.

11.8 THEOREM

Let (X,τ) and (Y,σ) be I–fuzzy spaces and $f : X \longrightarrow Y$ a continuous mapping. Then $C_a(u) \subset C_a(f^{\rightarrow}(u))$ and hence $d_a(u) \leq d_a(f^{\rightarrow}(u))$ for every $u \in I^X$.

PROOF

It is sufficient to prove that $C_a(u) \subset C_a(f^{\rightarrow}(u))$.

Suppose $b \in C_a(u)$. Then we show that $b \in C_a(f^{\rightarrow}(u))$. We have :

$\mathscr{G} \subset \sigma_a$ and $F(f^{\rightarrow}(u), \bigvee \mathscr{G}) \geq b$ imply $f^{\leftarrow}(\mathscr{G}) = \{vf : v \in \mathscr{G}\} \subset \tau_a$ (by continuity of f).

Also by 3.4(5),

$$F(f^{\leftarrow}(f^{\rightarrow}(u)), f^{\leftarrow}(\bigvee \mathscr{G})) \geq F(f^{\rightarrow}(u), \bigvee \mathscr{G}) \geq b.$$

Thus, by 3.2(8) and 3.4(2),

$$F(u, f^{\leftarrow}(\bigvee \mathscr{G})) \geq F(f^{\leftarrow}(f^{\rightarrow}(u), f^{\leftarrow}(\bigvee \mathscr{G})) \geq b.$$

Hence,

$$\begin{aligned} b &\leq \bigvee_{\mathscr{G}_o} F(u, \bigvee f^{\leftarrow}(\mathscr{G}_o)) \quad \text{(since } b \in C_a(u)\text{)}, \\ &\leq \bigvee_{\mathscr{G}_o} F(f^{\rightarrow}(u), f^{\rightarrow}(\bigvee f^{\leftarrow}(\mathscr{G}_o))) \quad \text{(by (*) below)}, \\ &= \bigvee_{\mathscr{G}_o} F(f^{\rightarrow}(u), f^{\rightarrow}(f^{\leftarrow}(\bigvee \mathscr{G}_o))), \\ &\leq \bigvee_{\mathscr{G}_o} F(f^{\rightarrow}(u), \bigvee \mathscr{G}_o). \end{aligned}$$

(*) If $u, v \in I^X$ and $f : X \longrightarrow Y$ is a mapping, then $F(u,v) \leq F(f^{\rightarrow}(u), f^{\rightarrow}(v))$ (see [S1]).

This completes the proof.

NOTES

Everything in Section 10 extends [S4] from I to L and M. The author has proved 10.5, 10.6, 10.7 and 10.18. Section 11 extends observations in [S3] from I to L and M.

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