The initial grounding of rational numbers: An Investigation

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Abstract

This small scale exploratory research project investigated the grounding of rational number concepts in informal, everyday life situations.

A qualitative approach was taken to allow for the identification and then in depth investigation, of issues of importance for such a grounding of rational number understanding. The methodology followed could be seen as a combination of grounded theory and developmental research. And the data was generated through in-depth and clinical interviews structured around a number of grounded tasks related to rational numbers. The research comprised three cycles of interviews that were transcribed and then analysed in detail, interspersed with periods of reading and reflection. The pilot cycle involved a single grade three teacher, the second cycle involved 2 grade three teachers and the third cycle involved 2 grade three children.

The research identified a number of different perspectives that were all important for the development of a fundamental intuitive understanding that could be considered personally meaningful to the individual concerned and relevant to the development of rational number concepts. Firstly in order to motivate and engage the child on a personal level the grounding situation needed to be seen as personally significant by the child. Secondly, coordinating operations provided a means of developing a fundamental intuitive understanding, through coordination with affording structures of the situation that are relevant to rational numbers. Finally, goal directed actions that are deliberately structured to achieve explicit goals in a situation are important for the development of more explicit concepts and skills fundamental for rational number understanding. Different explicit structures give rise to different interpretations of rational numbers in grounding situations. In addition to these perspectives, it became evident that building and learning representations was important for developing a more particularly mathematical understanding, based on the fundamental understanding derived from the child's grounded experience.

The conclusion drawn in this research as a result of this complexity, is that to achieve a comprehensive and meaningful grounding, children's learning of rational numbers will not follow a simple linear trajectory. Rather this process forms a web of learning, threading coordinating operations for intuitive development, interpretations for explicit grounding and representations to develop more formal mathematical conceptions.

Acknowledgements

Without this environment and encouragement of of the staff at RUMEP, this research would never have happened. When I joined the unit, I was encouraged to explore the teaching and learning of fractions. A large number of fraction and then rational number learning activities were also made available to me to adapt and augment and I was given access to many recent articles on this topic. The staff at RUMEP, in particular John Stoker and Tom Penlington, were also always ready to discuss my developing thinking and offer advice and encouragement. This provided an ideal environment for me to take on and pursue this research project.

My thanks to my supervisor, Marc Schäfer for his enjoyment of the project and his useful advice. Also for encouraging me to write up the research in a manner fitting to the process, even though this resulted in a non-standard appearance to the thesis.

Finally, I must acknowledge the critical contribution to this project made by the participating teachers and children. It was their contributions in our interviews that forced me to critically examine my thinking at each stage of the project and in this way led to most of the development of my understanding.

Dedication

To my wife Lisa,

whose interest and encouragement kept this project on track

and to Taryn,

who provided endless distraction in the final two years.

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Chapter 1: Introduction

This chapter provides a brief introduction to the research project, the research process and some of the important concepts and terms used in the research. The form of the dissertation will also be discussed. The introduction begins with a statement of the goals of the research. These provided direction and coherence to the process. Following the statement of the research goals, I provide some detail about a number of concepts involved in the research. I then briefly outline my fundamental orientation to the research.

Initially, an important feature of the goals was present only as an informal intuition which was difficult to state explicitly. Through the course of the project, I was able to develop the insight and the conceptual tools to describe this feature in a more precise and formal manner. I have used the term 'personal meaning' to refer to this feature. In order to provide a concise statement of the research goals, I will use the term 'personal meaning' in my goal statement, even though it will only be carefully described later in the introduction.

1.1 Goals of the Research

The primary goal of this research was to carry out a small scale in-depth analysis of the learning processes involved in a person's:

1. Development of an informal and intuitive foundation for rational number understanding.
2. Initial steps towards the construction of a reflective and formal understanding of rational numbers, which is personally meaningful and based on the intuitive and informal foundation.

This small scale analysis culminated in the development of a conceptual model for a nonlinear web of learning, encompassing fundamental rational number understanding.

As a master's study, an implicit goal of the project was to develop a mastery of the literature and methodologies current in the research on learning mathematics, which are relevant to the topic of study. To do this, an element of reflexivity was necessary in the study and the thesis. In addition, the requirement that the literature and methodologies be relevant to the chosen topic, necessitated a selection that was guided by the personal meaning that I ascribed to the topic. This selection and engagement with the literature was not static, occurring only at the start of the research. Instead it was a dynamic process, changing with my own developing understanding of the topic. The reflexive study of this process, and the influence of personal meaning on the process, led naturally to the secondary goal of the project:

3. To develop an understanding of the function of personal meaning in my own learning of the processes involved in developing rational number understanding.

1.2 Background

My initial interest in the meaningful teaching and learning of rational numbers was stimulated by my work in teaching mathematics. As a teacher of mathematics at tertiary level, I was struck by the number of students who did not appear to have achieved mastery of rational number concepts. After this, I worked for a number of years at RUMEP (Rhodes University Mathematics Education Project) in professional development programmes for teachers of mathematics. Here too, many teachers appeared to experience a great deal of difficulty with rational number concepts.

It generally appeared to me as if these students approached rational numbers from a purely formal perspective, without reference to any possible meaning of these numbers. In responding to this, I found that many errors in their formal calculations could be counteracted by encouraging students to keep in mind meaningful interpretations of the numbers.

In its professional development programmes, RUMEP (2005) encourages a similar orientation to teaching mathematics. That is, to encourage the development of a meaningful conceptual understanding of mathematical objects, rather than a predominantly formal development of mathematical skills. In order to achieve this, the teaching approach advocated by RUMEP involves engaging children in the exploration of concrete patterns and the creation and justification of reasonable solutions to meaningful problems. As well as helping children to master these problems and explorations, the teacher facilitates the development of a more abstract understanding involving mathematical concepts, based on these investigations.

It was my intuition that this more 'grounded' and 'investigative' approach to teaching and learning could overcome many of the problems I had noticed with students' and teachers' rational number understanding. Understanding that these opinions were largely informal and not rigorously justified, I became motivated to embark on a research project to develop a more rigorous and complete understanding.

In this research project, I will at times refer to 'my approach' to the teaching of mathematics. By this, I will mean an approach similar to that described briefly above as advocated by RUMEP (2005), for teaching mathematics.

1.3 Fundamental Concepts in the research

A number of concepts, which are themselves the subject of active research and debate, are fundamental to this research. The research will involve specific interpretations, rather than a globally accepted understanding, of these concepts. It is thus important at the outset of this dissertation, to spell out the interpretation that I have taken of these concepts.

My interpretations of these fundamental concepts have developed in the course of this research from quite vague and imprecise intuitions, to more specific, detailed and precisely formulated opinions. In this section, I provide a compact discussion of the final forms of these interpretations, as held at the end of the project. In keeping with the secondary goal (3) of my research, more detail about these interpretations and their development from my earlier intuitions will be provided in the body of the dissertation.

1.3.1 Knowledge and understanding

In this work, I will use the term 'understanding' in a loose sense, to indicate learners' individual mastery of the knowledge, cognitive skills and practical tools that are associated with a concept, in a manner which is personally meaningful to them. Used in this sense, it is evident that subjectivity is inherent in understanding and that a people's understanding is continually developing and changing over time in response to their experience of life.

In contrast, in much mathematical and mathematical education research, 'mathematical knowledge' was traditionally interpreted to be objective, universal, immutable and timeless (Rowe, 2004). As a result, the questions of how such knowledge was constituted and developed were held to be immaterial to the character and validity of the knowledge. In fact, attempting to justify knowledge by appealing to its development was dubbed the 'genetic fallacy'.

The focus of this research is the development of mathematical understanding, rather than the justification of mathematical knowledge. In this dissertation, I will not attempt to precisely define 'understanding', or 'knowledge'. I will use both terms in a loose, informal sense, making the distinction between impersonal, objective and immutable 'knowledge' and personal, subjective and continually developing 'understanding'. Where precision is desired, I will use other terms which I have defined more precisely.

1.3.2 Cognition

When regarding the question of what constitutes knowledge, I will not follow the usage of the terms 'knowledge' and 'cognition' which is common in the information processing approach to cognition. This view has been termed 'encodingism' by Bickhard (1993). It is that knowledge consists of information, or data, which is symbolically encoded and stored in the brain. Cognition then involves the process of storing, retrieving and computing with data in order to arrive at conclusions and make decisions (Newell, 1990). To explain just how meaning may be ascribed to these symbols by the cognizing subject is a difficult problem for this approach, known as the symbol grounding problem (Harnard, 1990).

Due to the importance of personal meaning and intuitive understanding in this research, I chose instead to follow the view of cognition provided by the research programs of situated cognition (Lave and Wenger, 1991) and embodied cognition (Clark, 1997). These programs stress the importance of context for cognition and learning. Researchers in these programs see the function of cognition as enhancing the ability of the individual to act in, adapt to and control their environment.

Situated and embodied cognition are not mutually exclusive research programs. Rather, the different names are derived from a focus on different environmental factors. Situated cognition tends to focus largely on the social and cultural environment and less on the physical environment. Embodied cognition asserts the importance of goal directed physical action and emphasizes the interaction between the physical environment, and the individual's sensorimotor activity and cognition. These issues will all have relevance for my research and I will thus view an individual's environment from a holistic perspective, including the physical (natural and constructed), the social and the cultural environment. The importance of goal directed action (both physical and social) for cognition will also be accepted.

To carry out goal directed action successfully, individuals need to respond rapidly and effectively to their continually changing environment. Thus, many of the models of cognition proposed by researchers in embodied cognition involve rapid and automatic processes, without including means for reflection on representations. See for example, Brooks (1991) and Thelen et al. (2001). Others, such as Clark (1997) and Glenberg (1997) see a need for both automatic, non-reflective processes, and slower, reflective or symbolic processes. This view is particularly important for my research. For I am interested in the development of intuitive understanding of rational numbers, which could be seen as embodied by automatic processes; as well as in the development of a more reflective, symbolic understanding of rational numbers, which relates meaningfully to this intuitive basis.

Finally, any discussion of this issue would be incomplete without some mention of the work of Piaget. For an important goal of Piaget's genetic epistemology (Piaget, 1970), was to show the genesis of formal scientific understanding in the sensorimotor activity of the child. To do this, he pioneered the general interpretation of learning which is taken in this research — that of constructivism.

1.4 Orientation of the research — Ontology and epistemology

This research will build upon a realist ontology and a radical constructivist epistemology.

1.4.1 Ontology — realism

As an ontological realist, I will accept the existence of reality independent of my experience. This is taken to be both prior to and independent of a person's experience and conception of reality. A person's acts, however, both practical and communicative, do affect reality. Here reality is the object both of our knowledge and our acts. Following my earlier discussion of environment, in this research I will take a holistic view of reality as physical, social and cultural. Hence, I will use the term 'reality' only in this sense, in this dissertation.

Paul Ernest (1998) has added a number of additional theses to this basic standpoint, to develop the comprehensive ontological and epistemological approach, that he terms social constructivism. But the focus of my research is on the development of personally meaningful understanding by the individual as a result of practical physical and social action — not the social communicative process through which children negotiate a common understanding . Thus, the full social complexity of Ernest's (1998) social constructivism is not needed as a foundation of this work and so I have not adopted his position in this research.

The argument taken to reach this conclusion has the same form as the transcendental arguments followed by Roy Bhaskar (1989) in his development of the ontology of critical realism. He starts with elements of practice that are commonly accepted as basic to science and from these deduces properties of reality that must hold in order for these practices to be possible, or well founded. To provide a sufficient foundation for this research, the full complexity of a critical realist ontology is, however, not necessary. A sufficient base will be provided by two ontological assumptions. First, that a person is able to carry out instrumental interactions with physical reality and to participate in social interactions, that they perceive as significant. Secondly, that it is possible for a person to perceive regularities or structure in such interactions. Forming mathematical concepts on the basis of such perceptions of regularities or structure in significant interactions will be termed **grounding** the mathematical concept. This idea of 'grounding' mathematical understanding will be fundamental to this work.

My commitment to ontological realism is distinct from my views of how people may come to understand their experience of reality. This is an epistemological issue. To help maintain this distinction, I will not use the term 'reality' in an experiential sense, to refer to a person's experience of what is real. Instead, following von Glasersfeld (1995), I will make the experiential sense explicit by referring to this as a person's 'experienced world' or 'experience of reality'.

1.4.2 Constructivism

The most basic principles of constructivist learning (Wadsworth, 1989) are currently accepted by mathematics teachers from in a wide variety of different epistemological positions (Davis, 1990). These principles provide a broad metaphorical description of mental processes involved in learning and will provide a basic foundation for the descriptive process models developed in this research. As a rough metaphorical foundation, I will not attempt undue precision in formulating these principles. Rather, I will attempt to describe them broadly in a manner that could be interpreted acceptably by most positions:

- An individual's cognition is governed by cognitive structures of some form.
- These cognitive structures are not static and fixed, but change through maturation and in response to the individuals lived experience.

- The changes in an individuals cognitive structures are of at least three types:
  - An accumulation of perceived data which is acted upon by stable mental structures. This accumulation allows for effective applications of these structures to a broader range of experience.
  - The further development of mental structures to improve the efficacy of their action on the perceived data from experience.
  - A more fundamental adaptation of the mental structures over a period of time, resulting in a qualitative change in the individual's cognition and the corresponding development of different responses to experience.

1.4.3 Epistemology — radical constructivism

Radical constructivism is an epistemological standpoint that was formulated by Ernst von Glasersfeld, and discussed in detail in his book (von Glasersfeld, 1995). It involves more than the basic constructivist metaphor of the processes involved in learning. The following two basic principles define radical constructivism:

- Knowledge is not passively received, but built up by the cognizing subject.
- The function of cognition is adaptive and serves the organization of the experiential world, not the discovery of ontological reality.

(von Glasersfeld, 1995, p. 18)

Thus, radical constructivism takes the view that the domain of cognition is limited to the experiential world of the cognizing subject. Cognition does not describe or represent ontological reality, and it definitely does not yield certainty about reality. However, cognition is not arbitrary, because it is adaptive and so may be judged according to the adaptive viability of the corresponding organization of the experiential world. We may also be able to judge two alternate conceptions against each other, according to their adaptive viability. This possibility of judgement allows us some confidence in our conceptions of experience, even if we do not possess certainty.

Note that such a judgement of adaptive viability will of necessity be relative to the purpose of the adaptation. Different purposes of adaptation could well yield different judgements of viability. This has important implications for the learning process, for learners with different ideas of the purpose of a learning activity will quite probably learn something different from the activity.

It is important to understand that limiting the domain of cognition to a person's experiential world, does not imply that people do not accept the existence of objects. Radical constructivism makes the following points regarding this issue (von Glasersfeld, 1995):

- A critical function of the organizing of experience provided by cognition is to delineate experiences as being the result of interaction with 'objects' that are accepted as existing external to the person.
- However, cognition is not able to prove absolutely that such objects do in fact exist in ontological reality.
- Instead, the person ascribes reality to such an object (a reflective, or automatic, act of belief). A person's cognitive processes organize experience in order to achieve alignment with this ascription of reality. In effect, cognition enables a person to organize their experience to provide evidence justifying their belief. This evidence may be extremely convincing, as it is for (ascribed) macroscopic objects. But, certainty cannot be achieved.

Finally, radical constructivism holds that through its organization of the experiential world, cognition also effects the way in which experiences are perceived. Thus a change in cognition may lead to a reframing of the corpus of experienced data, as well as the individual's existing mental skills for dealing with this data. This type of change will be important for this work, both in terms of children's intuitive understanding of rational numbers and my own understanding of the learning process.

1.5 Personal meaning

The following two aspects of meaning will be important in this dissertation:

- The significance, or purpose of something. In particular, when referring to some act, this relates to the 'why' or the ultimate cause of the act.
- The sense, or signification of something. When dealing with an act, this relates to what is done, or the structure of the act.

The importance of purpose, or significance, follows from my epistemological view, that sees purpose as necessary for judging the viability of a concept. From this view, the sense of a concept corresponds to the manner in which the individual objectifies the concept, or the organization of the experiential world yielded by the concept.

'Personal meaning' will be taken to incorporate both the sense and significance individuals ascribe to something in relation to themselves. In my interpretation, I am using the term 'personal' to emphasize the relating of the sense and significance to the self. Because cognition is adaptive, the relation of the sense and significance of a concept to the self, will be constituted through the purposive acts of the person which are mediated by the concept.

Some fundamental ideas of activity theory (Engestrom, 1999), allowed me to develop a more detailed interpretation of personal meaning for the purposes of this research. This theory originated in the work of Leontev (1978), building on Vygotsky's (1986) socio-cultural approach to psychology. Activity theory distinguishes three different views of active processes carried out by people.

- An **activity** is an active process carried out to satisfy some motivating need or desire, be it physical, psychological or social. Generally this motivation imbues the activity with significance for the persons involved.
- An **action** is an active process directed towards a conscious practical purpose, or goal. Generally an action is not invested with personal significance, but it makes sense, in that whatever is done to constitute the action is aimed at achieving the desired goal.
- An **operation** is an active process in which a person uses a tool to achieve a desired end. Operations are often carried out by means of tools. Certain instrumental conditions need to be satisfied in order for the operation to be successfully carried out.

In this dissertation, I will generalize the concept of an operation somewhat, to include processes in which a person interacts with the environment in some way, in order to take advantage of opportunities it affords to achieve the desired end. For the interaction to be effective, the individual must coordinate their acting to a fine level of detail, through immediate and subtle responses to the pervading feedback provided by the senses. For example, the tactile and visual feedback experienced by a master carpenter allows a tool to be used almost as an extension of the body. Through this process of coordination the person develops an intuitive sense of the operation and the structure of the environment which makes this operation possible.

The terms 'activity', 'action' and 'operation' are the accepted technical terms in activity theory and so will be adopted in this dissertation. An unfortunate consequence of this is that the term 'operation' will be used in three different senses in this dissertation. Each sense being an accepted usage in a different field. The second sense of the term is that used by Piaget. He used 'operation' to refer to a mental act which had a certain basic form (Piaget and Inhelder, 1969). In contrast to the first sense where it is used to describe an embodied act coordinating with instrumental conditions offered by the environment. The third sense of the term is that of a mathematical operation, such as addition, subtraction, multiplication and division. Because of this ambiguity, it will be necessary for the reader to use the context to determine the desired sense.

The insight into purposive acts derived from activity theory may now be combined with my interpretation of the personal meaning of a concept as constituted through the purposive acts of the person mediated by the concept. The personal meaning of a concept may thus be interpreted as a multilayered idea, with different layers understood from different perspectives on purposive acts.

- Viewing the act as part of an activity, the mediating concept is ascribed with personal significance according to the personal motives for the activity.
- Viewing the act as an action or element of an action, the mediating concept is ascribed with personal sense according to the place and form of the act as contributing to the achievement of the goal of the activity.
- Viewing the act as an operation or combination of operations, the mediating concept is ascribed with personal sense according to the intuitive sense of the operations and the structure of the environment, developed by the individual through coordinating the operation.

Although the layers of personal meaning may be interpreted separately, the development of personal meaning occurs concurrently in all layers at each moment of experience.

1.6 The Research Process and the Form of the Dissertation

During the process of the research, my understanding of the learning of rational numbers developed in response to my work with teachers and children, as well as the reading of relevant literature.

In order to more accurately depict the constructive process, this dissertation will not merely present my final understanding as a finished whole. Instead, it will be structured to echo the cycles of reading, interaction, evaluation and reflection, through which my understanding developed. This will facilitate the communication of my reflexive study of the research process, and allow me to highlight the effects of the personal meaning that I held at different times in the research, on the research process at that point in time.

Three cycles of inquiry were followed in this project. Each cycle began with a period of reflection on the state of my understanding to identify the issues that I considered most pressing at that point. This was followed by a review of the literature relevant to the identified issues, in order to formulate a effective responses to incorporate into my understanding. A number of tasks were then designed which formed the basis for a sequence of clinical interviews in order to evaluate my extended understanding. A preliminary analysis of the interview data then formed the basis for the next cycle of inquiry. After the third inquiry cycle, a final analysis of the interview data was carried out and the conclusions of the research were formulated.

The second chapter of the dissertation describes the methodology followed in this project. Following this, each cycle of the project is discussed:

- Chapter 3 details my initial orientation to the project and the pilot cycle of research.
- Chapter 4 describes the literature review and subsequent conceptual development at the start of the second cycle. The structure of the interviews for this cycle and the selection of the tasks for the interviews is also discussed.
- Chapter 5 is concerned with the analysis of the interview data from this cycle.
- Chapter 6 discusses the preparation for the third cycle. This includes my theoretical investigations at this point and subsequent conceptual development. After this, the structure of these children interviews is detailed.
- Chapter 7 describes the analysis of the interview data from this, child focussed, cycle.
- Chapter 8 presents a discussion and synthesis of both the theoretical investigations and the interview analyses of all three cycles.
- Finally, a brief conclusion to the study is given in chapter 9.

Chapter 2: Methodology

2.1 Overall structure of the research process

The aim of this research was to develop an improved understanding of the initial development of personally meaningful rational number understanding. More specifically, to identify and investigate possible issues of importance, when engaging children in classroom activities, with the aim of providing a grounded and comprehensive foundation for rational number learning. At all stages of the research, the ultimate focus was on developing a framework for understanding this learning process.

To do this, a number of cycles of development were carried out. The first cycle involved working with a single teacher known to me in order to pilot the approach and provide an initial exploration of the basic ideas. The second cycle involved a deeper theoretical investigation, a follow up interview with the original teacher and more extensive discussions with two other teachers. The third cycle included a more focussed theoretical investigation, and was largely concerned with six clinical interviews with two grade three children.

Each cycle began with a number of issues to be explored and I carried out a review of the literature relevant to these issues. Reflection on this reading helped to crystallize my developing understanding of these issues, and I would then develop a number of examples to flesh out this understanding. These examples then formed a basis for the interviews in the cycle. In the interviews, I discussed the examples with participating teachers or children, in order to elicit their views and responses. These interviews were taped and transcribed and the transcriptions were then analysed in order to evaluate and modify my understanding, and to identify further issues to be explored.

2.2 Selection of participants

Unlike the learning of whole numbers and counting, there is little development of explicit rational number concepts and representations outside of school (Saxe, 1988). So investigating possible, or actual, school oriented activities for learning rational numbers, is well suited to the investigation of the initial grounded learning of this conceptual field. Also, some fractions, such as halves and the half family, are introduced in grades 1 and 2, but a more comprehensive introduction to fractions generally occurs in grade 3 (Department of Education, 2002). As a result, I decided to work with grade 3 teachers and children.

To pilot this research, I approached a teacher that I had worked with before in a RUMEP teacher professional development programme and so was familiar with my general orientation to meaningful learning. I did this because at that stage of the process, I was not able to explicitly identify and describe many of the issues that I intuitively believed were important, as well as my views on these issues. Because of our common experience, I hoped that my attempts to communicate these issues and opinions would resonate with her and that her responses would enable me to achieve a more explicit understanding. Furthermore, I hoped that because of her experience of following an approach similar to that advocated by RUMEP in teaching, that she would be able to identify other issues of importance that I had overlooked.

In contrast to this, the teachers and children approached for interviews in the second and third cycles of research were not known to me and were unfamiliar with the orientation that I adopted to mathematics teaching. This selection was carried out by first selecting and requesting the involvement of a suitable primary school. Important considerations for this selection were that the medium of instruction in the school was English, that it be reasonably accessible for research visits, and that the school teachers be willing to be involved in the research project. I was fortunate in that the principal of the first school approached was prepared for the school to be involved in the research and that both grade 3 teachers in the school expressed a keen interest in the research.

For the third cycle, I decided to interview only two children, one from the class of each of the teachers involved in the second cycle. The reason for this was that I expected to conduct a number of interviews with each child participant and this would generate a considerable volume of transcription data for analysis. In our initial discussion, I asked the teachers to identify a suitable child in each class to be involved in the research. Effort was made to identify two children that were of different genders and were reasonably verbal and so would be expected to respond well in an interview. Furthermore, children that could be seen as average performers in each class were selected. The parents of these children were approached and gave permission for their involvement in the research.

2.3 Research Paradigm

This research falls naturally within the interpretive paradigm, because of the importance of personal meaning in the study and of my interpretation of the concept of personal meaning. For research within the interpretive paradigm is oriented towards people's interpretations of their experience within a situation, as related to their intentional actions within the situation. (Cohen and Manion, 1994; Schwandt, 2000).

In this research, the children's learning of rational number concepts is seen as constituted by their interpretations of, and intentional actions within, the learning situation. Developing an understanding of this learning is thus fundamentally an interpretive process. In addition, the focus of the interviews and analysis is on the interpretations and intentional actions of the participants. With the learners, to achieve better understanding of the learning process. With the teachers, to explore their reactions to my developing understanding.

Finally, the ontological and epistemological positions I have assumed, are incompatible with many of the other paradigms currently common in social science research (Lincoln and Guba, 2000). The assertion that reality is not knowable in any absolute sense conflicts with the basic assumptions of positivism, in particular, that observations provide direct and positive access to reality. On the other hand, the assertion that reality exists independent of my experience of it, contradicts the anti-realist assumption of constructionism.

2.4 Methodology

Two methodological approaches important for this research were developmental research (Gravemeijer, 1994; Richey and Nelson, 1996) and grounded theory (Strauss, 1987; Strauss and Corbin, 1990).

2.4.1 Developmental Research

The methodology of developmental research systematizes the development, or evaluation of products (Richey and Nelson, 1996). Type I developmental research is concerned with the development or evaluation of specific products, and type II research focusses on the processes involved in such development or evaluation. The methodology proposes a cyclical process of development and research, or verification, of either the product, or the process concerned.

The aim of this research was the development of an intellectual, or cognitive product. Namely, my own understanding of the initial learning of rational numbers by children. Because the objective was not to develop a physical, or symbolic artifact as product, this was not standard developmental research. In the final stages of the research, I was able to formulate key aspects of my understanding as a web of learning. If this was taken as the aim from that start, the research could be classified simply as developmental. But the web of learning was not a product expected at the start of the project. Rather it emerged during the process of research. This provides further support for the judgement that the research process was not a straightforward application developmental research as described by Richey and Nelson (1996).

But the process followed was strongly influenced by developmental research. In particular, a cyclical process of development and research was indeed followed. For, in each cycle, a series of prototype activities were developed and these formed the basis for the research interviews and data analysis. In the process of this research, the activities were critically analysed and modified in the light of the teachers' and children's responses to in the interviews. An important difference though, is that the aim of the research was not the development of the set of activities as a product. Rather, these activities were developed in order to allow the critical analysis of my conception of principles important for the development of a grounded, fundamental rational number understanding. To do this, I attempted to design these activities in line with my conception at that time. Then in the interviews and subsequent analysis, I searched for responses that suggested weaknesses or omissions in this conception. The refinement in the activities was thus a secondary, rather than a primary effect of the research.

This process is more in line with developmental research as described by Gravemeijer (1994) as followed in the research program of Realistic Mathematical Education. He interprets developmental research as primarily theory oriented, as opposed to curriculum development, which is primarily product oriented. The theories in question are constituted by the developers views of the process of teaching and learning in arising from a sequence of instructional activities — forming a "local instruction theory" (Gravemeijer, 1994). Furthermore, to properly develop such a theory, he holds that it is necessary to consider the mental activity of students. This was adopted as an important underlying principle in this research. On the other hand, the theoretical development in this research was more general than such a local theory. It was concerned with the identification and investigation of general issues of importance in the grounded foundational learning of rational number concepts. And the final web of learning that emerged, categorized and related a number of important issues that need to be considered for a comprehensive grounding of rational number concepts. But it did not link these to a specific sequence of instructional activities, or describe possible trajectories of learning.

As is mentioned by Treffers (1993), the process of developmental research in the program of Realistic Mathematical Education, proceeds by means of constant comparisons. This constant comparative method was the early form of what has now become known as the methodology of grounded theory (Hueser,1996). This is a qualitative research technique that uses an inductive approach guided by a set of procedures, in order to explore a particular phenomenon in the lived experience of people (Strauss and Corbin, 1990). The theory which emerges from this method of untiring comparison between categories, is grounded in the data, which then supports the existence of the theory.

2.4.2 Grounded Theory

Glaser and Strauss (1967) first proposed the method as a way for undertaking social research. According to them, the unique goals and strategies of grounded theory were a valuable alternative to the prevailing empiricism of the time. They believed in a pragmatic approach where "the theory must fit the situation being researched, and work when put into use" (1967, p.3) and therefore saw the value of their method as being its ability to fit the different social problems being researched and providing a wider application than traditional empirical methods. Since Glaser and Strauss' initial writing, grounded theory has evolved, resulting in the emergence of various forms of the method. The different forms seem to agree on the basic tenets of the theory but differ on issues of implementation and practice. However, it would be fair to say that for all versions the concept of constant comparison remains important (Hueser,1996).

This methodology also had a strong influence on the research process. But there were also a number of important differences that distinguished it from grounded theory. The most fundamental difference being that the aim of the research was not to develop a theory about the particular lived experience of those being interviewed. Rather, the primary data was analysed in order to inform and critically analyse my own understanding of general issues important for children's initial meaningful learning about rational numbers. The differences in practice that arose, generally stemmed from this difference in orientation.

Grounded theory describes a set of procedures which should be followed to analyse the data. Strauss and Corbin (1990) argue that if one is willing to engage with the subtleties of meaning of the data (theoretically sensitive), it will be possible to develop a conceptually dense, and well integrated theory of the phenomenon being studied, that is well grounded in the data. Theoretical sensitivity is informed through reading literature, professional experience, personal experience and through following the analytic process rigorously and meticulously.

A key characteristic of a grounded theory is that it is not formulated on prior assumptions and then tested through data collection and analysis, but rather emerges from the data. The analytical approach followed in the grounded theory methodology is designed to facilitate the emergence of rigorous and well-constructed theories. But it should be noted that this approach is not expected to yield final and all-encompassing theories. In fact, an important analytical attitude encouraged in grounded theory, is that the developing, or developed, theory should always be seen as provisional and open to revision in order to accommodate conflicting data. As the aim of the research was to develop a fitting theoretical formulation, the emphases of this methodology on building a theory grounded in the data, and on adapting theories to better fit the data, was judged to particularly relevant to the research.

In traditional research methods the literature review is a fundamental aspect of the initial phase of the research. An extensive search of the existing literature pertaining to the topic is used to inform and guide the hypothesis and questions being asked. However, the grounded theory method utilises literature resources differently (Strauss and Corbin, 1990). The researcher does not know ahead of time what areas will become relevant, because the method dictates that the themes and issues be allowed to emerge as the data is collected. Therefore, doing a detailed literature review in grounded theory is difficult and it is recommended that the difficulty be overcome by becoming familiar with the broad area of study (Glaser and Strauss, 1967; Strauss and Corbin, 1990). This absence of an extensive literature review prior to data collection makes certain that literature is not given a position of privilege when compared to data (Dick, 2000; Strauss and Corbin, 1990). Rather, any available literature is treated as data and in some cases may be analysed together with any other data collection, thereby becoming an integral part of the data collection and analytical procedure (Dick, 2000). The process of interweaving the emergent theory with existing literature is valuable in enhancing the internal validity, generalisability and the theoretical level of the theory being created, particularly as the emerging theory is often based on a very limited sample (Pandit, 1996).

I began this project having developed some professional and personal sensitivity to the primary issue of research: issues in developing children's initial learning of rational numbers in a grounded fashion. Unfortunately, the common approach to introducing rational numbers in South Africa has a symbolic rather than a grounded focus, aimed at developing fraction notation to describe part-whole comparisons (Newstead and Murray, 1998). As a result, most participants could not be expected to have experienced this grounded approach in their learning or teaching of fractions. I thus decided that I would need to attempt to generate this experience for discussion. Developing activities to form the basis of this experience required the formulation of an initial approach to the research question prior to generating and analysing the data, and this was done by carrying out an initial literature review. Following the analysis of the interviews in the first two cycles, that literature was consulted to deepen and provide a more theoretical grounding of the developing theory. However, it was also necessary to use this theory as a basis for developing the activities on which the following cycle was based. Thus, the research goal of investigating issues of relevance to a changed practice of rational number instruction, rather than investigating people's experience of a common practice, necessitated an adaptation of the approach commonly followed in grounded theory.

Most applications of grounded theory can be classed as naturalistic inquiry. That is, the major aim of the research is to "reconstruct the perspectives of those being studied" (Lincoln and Guba (1985). And that such a reconstruction describes these perspectives from the point of view of the actors themselves (Erickson, 1986). However, this research is concerned more with developing a general perspective on the learning process, than with the actors specific experience of this process. In effect, the actors' specific experiences are used to critically analyse and refine the more general perspective.

Data analysis in grounded theory is based on the concept of coding. There are three different types of coding: open, axial and selective coding. These are defined as follows:

- Open coding involves examining the data to identify themes that are then coded into categories (Dey, 1999). Here, the researcher names and categorizes phenomena in the data, breaking it down into parts which can then be compared to other parts for similarities or differences (De Vos and van Zyl,1998). A theme is any specific issue that was expressed within the data. Themes, once labelled, are grouped together via a process of constant comparison, to form categories.

- Axial coding generally follows the process of open coding, although in reality they may occur interchangeably (Strauss and Corbin, 1990). This coding aims to interconnect the categories (Cresswell, 1998) and allows the formation of initial hypotheses about the possible relationships between categories and subcategories (Hueser,1996). This coding method requires the researcher to examine the categories in detail so that relationships and dimensions can be highlighted and explained. The outcome of axial coding is a detailed definition of the categories initially established by the open coding stage.
- Finally, in selective coding, core categories are identified and systematically related to the other categories, while constantly validating the relationships and filling in other categories that need to be refined or developed (De Vos and van Zyl, 1998). This process of relating the categories to the core category forms the basis of the grounded theory that emerges. Selective coding builds a narrative about the relationships between categories, resulting in a discursive set of theoretical propositions (Cresswell, 1998) which explains and integrates the categories and relationships that have emerged from the earlier analytical steps (Strauss and Corbin, 1990).

As the research progresses, new data is collected ana analysed in the light of the categories and relationships that have been identified in the coding process. This is termed "theoretical sampling" by Glaser and Strauss (1967). Further data collection and analysis stops when theoretical saturation has been reached. This occurs when the concepts in the data no longer reveal any new categories, properties or relationships.

2.4.3 Data analysis and grounded theory

For the reasons mentioned earlier, the interviews for each cycle of the research were structured by the form of the developing theory at that stage of the process. Furthermore, the data analysis was constrained by an orienting focus that was formulated at the start of the analysis for each cycle. At each stage a general orientation, focussing on themes that may be of interest to the general research question was taken. This allowed the emergence and inclusion of particular themes that were not clearly evident in earlier cycles, but that may have been of importance for the meaningful learning of rational numbers. In addition, specific focal categories were identified for investigation in each cycle. They were identified after an initial reading and 'rough' open coding of the data. This involved reading the data and identifying interpretations of blocks of the interview text that suggested interesting themes that may, or may not, have been relevant to the research question. These categories were chosen for four purposes:

- to facilitate the deepening of the developing theory,
- to fill in gaps and allow the generation of a more comprehensive theory,
- to better inter-relate the themes and categories in the emerging theory,
- to encourage a critical analysis of the developing theory by identifying shortcomings and conflicts.

The process of theoretical sampling was thus very important in this research. In fact, the initial reading and rough open coding of the data tended to be rather confusing because of the number and wide variety of themes and issues that were evident in the data. Formulating the focal questions brought focus and direction back to the research, by removing many interesting, but irrelevant, themes and by suggesting relationships and avenues of approach for those remaining.

The data analysis in each cycle could be seen as a combination of all three types of coding.

Open coding was carried out in all three cycles before a more focussed and detailed analysis based on the focal questions. This allowed the emergence of previously unrecognized themes even in the third cycle. An important instance of this was the identification of the important influence of developing representations in the open coding of the third cycle. The themes and interpretations that emerged from the process of open coding were also documented and used for a process of subjective verification, or viability testing. This will be discussed further in section (2.6). Finally, carrying out such open coding also ensured that the focal questions for further analysis were set bearing in mind the themes that had been identified in the data. Setting these questions thus involved a consideration of both the possibilities for further understanding offered by the themes identified, and the theoretical identification of issues related to the developing theory that needed to be investigated.

The focussed analysis could be seen as a combination of axial and selective coding. The general categories of 'interpretations' and 'operations' were identified and named in the initial theoretical analysis, and that following the pilot cycle, based on the my prior understanding and themes emerging from the open coding of the pilot cycle. The focussed analysis involved fleshing out these general categories and specific examples in each category, as well as exploring relationships between them. This aligns well with the form and function of axial coding. The data analysis also resulted in a number of reorganizations in the categories identified, as well as the manner in which they were interrelated. These changes largely occurred through attempts to systematize relationships between categories, and through the theoretical imperative to ensure that each category did indeed form a coherent unit of analysis. This analysis could be seen as occurring in the process of selective coding.

2.5 Interview methods

Data for this research was collected largely through interviews and surveys of the literature published on topic of relevance. Both the teacher and children interviews can be seen as clinical interviews.

Clinical interviews are extensively described by Ginsburg (1981 and 1997). This method was developed by Piaget, with the aim of investigating a child's "natural mental inclination" and underlying thought processes, while bearing in mind the larger mental context to each response. These interviews are structured in a flexible and open manner, to allow the exploration of the child's spontaneous thinking, through the ongoing interaction. In the course of this exploration, the interviewer actively builds hypotheses about the child's thinking and structures the interaction to test these hypotheses. Furthermore, this is generally not done by directly asking what the child was thinking, because the underlying thought processes are not expected to be directly accessible to the child. Rather the situation or problem being discussed is manipulated so that the responses of the child themselves provide the evidence for or against the hypotheses.

It is evident from the above discussion that an interview process of the same form will achieve the same function, whether the interviewee is a child or an adult. That is, clinical interviews would be suitable for use with both children and teachers. There are some important differences between standard clinical interviews and the approach taken in the interviews. Firstly, in clinical interviews, the interviewer seeks to probe and challenge the understanding of the participant. However, in the interviews with teachers, my aim was to probe and challenge my own understanding. To make it more possible for the views of the participants to challenge my own, my stance was to be more accepting and less challenging of the participants' responses. My goal was to understand what they were trying to communicate, in order to identify conflicts between my views and theirs. These conflicts were then used as a means to probe and challenge my own understanding. Identifying these conflicts became an important feature of the final methodology and means of doing this were developed throughout the research.

Secondly, in clinical interviews, the aim is to investigate the understanding of the participant, at that time. To carry out a detailed investigation, the interviewer may respond in ways that challenge the apparent thinking of the interviewee. But the aim of these challenges is to further probe the understanding of the participant. In my research, the precise understanding of teacher, or child, at that time is important only in terms of how it informs my understanding of the possible learning process. As a result, responses that teach, or that facilitate the development of the interviewee's understanding are quite acceptable. In fact, such responses may even be quite productive for this research, because the resulting interchange may yield valuable insight into the learning process. With the teacher interviews, I also spent some time discussing my ideas at that time about meaningful rational number learning in order to gauge their opinions about whether it may show some fit with and even possibly be useful in, their teaching of rational numbers.

The children's interviews are more similar to clinical interviews, but they do not merely attempt to probe the learners thinking. For my aim is also to teach and develop the children's understanding when the opportunity presents itself. I am also interested in 'modelling' the children's thinking as they learn. In this respect these interviews are quite similar to the "teaching experiments" described by Steffe (1991), a methods which extends the standard Piagetian clinical interview in a teaching setting. Even so, these are not precisely teaching experiments, as described by Steffe (1991), because the time period is much shorter and the number of interviews is much reduced.

2.6 Viability of interpretations

2.6.1 Interpretations

The starting point of the data analysis was the rough open coding in which I developed interpretations of subsets of the data. I then worked to incorporate these interpretations into my developing understanding of the issues. This incorporation was not a mere assimilation into my existing understanding, in fact, at times it resulted in radical restructuring of my understanding.

These interpretations carried the meaning that I attributed to the data. That is, an interpretation is a statement that **I** make, which specifies the elements, features or interactions in **my** perception of reality, to which I believe the subset of data refers.

2.6.2 What is the data?

The data for this study consists of an annotated transcription of the interviews. The transcription of the interviews was carried out by a third person, who was provided with a tape recording of the interviews, together with a copy of the 'situated questions' which were explored in the course of the interview. This person was an experienced copy/dictaphone typist, with little interest in mathematics or mathematics teaching, and little knowledge of the project. This 'lack of context' resulted in some difficulties in creating a verbatim transcription, especially in particularly dense, or quiet passages.

The transcription was reviewed by the researcher. With my deeper contextual knowledge of the project and of my interview style, I was able to correct a number of words which were misheard. I was also able to clarify many of the difficult passages. The result was a 'final' transcript, which still had a number of unclear passages and probably some misheard words.

2.6.3 A simple classification of interpretations

The interpretations of the data could be subdivided into two broad categories.

The first concerns references to specific things that people do or say. These things could be termed public, because they were open to observation by the people in the interaction. Such interpretations could be seen as a simple re-writing of the data in terms of the publicly observable features of the interaction, and as such could be loosely termed observations. I will use the term 'observation' in this sense in this work. Such observations will be relatively unproblematic, because it is expected that most people would agree with the interpretation if they were in the position of observer. It was decided that a sufficient test for viability of these interpretations would be agreement by the project supervisor that these interpretations could indeed be judged as observations in the sense described above.

The second category are interpretations of peoples meanings, intentions or understanding. These interpretations do not deal with directly public features of the interaction. Instead they involve inferences about internal states or mental processes of the people involved. In the remainder of this chapter, I will use 'interpretation' to refer to such inferential interpretations. That is, as a short form for 'interpretation of meaning, intention, understanding or thinking'. Judging such an interpretation will actually be judging an inference.

2.6.4 Viability, truth and validity

An important question for any research based on a realist ontology concerns the fit of the developed theory with reality. This is often framed as a consideration of the validity of the research (Eisenhart and Howe, 1992). Following my radical constructivist epistemology, I prefer to frame this in terms of the viability of the theory being developed. This explicitly acknowledges the lack of objective standards of 'truth'. That is, the question asked is: "Is this **a** viable, or fitting theory?" rather than "Is this **the** correct theory?"

Following Smith and Glass's (1987) eclecticism, once viability of the interpretations is established, the viability of the developed theory will follow from its logical viability. That is, from whether the argument put forward in this dissertation is consistent, showing that the theory does indeed follow from the interpretations.

To judge the viability of the interpretations of the data made in this research, the following question was formulated: "Is the given interpretation of the person's meaning, intention or understanding reasonable and fitting, given the data?" How to make this judgement is the key issue in judging the viability of this research.

2.6.5 Viability and purpose of interpretations.

Again, it follows from my perspective on epistemology, that 'viability', or 'fit' is not an absolute concept. It will always be viability, or fit for some given purpose.

When interpretations are made for the objective purpose of accurately representing the experience of another person, a judgement of viability may be criticized in the same fashion as the judgement of objective representations of reality. The only difference is that the other person may be asked to verify the judgement themselves. A valid response by the other could be very useful in making the judgement of viability. Unfortunately, there is no certainty that this response is not misleading, either because:

- the other person may deliberately and knowingly chooses to provide the researcher with a misleading response, for reasons of their own, or
- the other person may unknowingly provides a misleading response, because their own interpretation of their recalled experience is not an accurate representation of their experience at the time.

The second possibility is particularly important when there is a good chance that, since the time of the interview, the other person has changed in ways that would effect their interpretation of their recalled experience.

This criticism is less relevant to this research, because most interpretations were made for the purpose of providing insight into a specific issue and not at forming a general representation of the experience if those being interviewed. I will term these 'directed interpretations' because they are directed at specific issues related to meaningful rational number learning.

A systematic and formalized process of developing directed interpretations was not followed for the interviews of the pilot cycle. Reflecting on this process, I decided to develop a more systematized approach for the following cycles, that would allow the inclusion of a process of viability testing. The directed interpretations for these cycles were created as follows:

- To start, a rough open coding of the data was carried out. That is, I would read through the interviews, to identify and code data which was potentially of relevance to the research.
- Focal issues, or questions, would then be developed through a consideration of both emergent themes and themes judged importance for reflecting on the developing theory.
- For each specific issue, I would then identify and generate a report of the coded data related to that issue.
- Finally, I would work through each extract in the report, to see what inferences that I thought reasonable, could be made from this extract about the issue. Extracts that did not give rise to inferences judged as reasonable were removed from the report. Each inference made would then be a directed interpretation of the data.

The theory was then critically reviewed and further developed, on the basis of these directed interpretations. But before this was done, the directed interpretations were subjected to external review in order to reduce the effect of possible mis-readings of the data.

2.6.6 Judging the viability of directed interpretations.

A common way of evaluating the interpretations in qualitative research is to use member checks (Lincoln and Guba, 1985). This involves showing a draft report of the analysis of the data gathered from each participant, to the participant concerned for comment. Based on their responses, the report would then be edited to achieve better alignment with the participants own view of their experiences. It is accepted that the analysis would probably not achieve perfect alignment with participants self reports because the analysis would generally attempt to include participants tacit knowledge, of which they may not be reflectively aware, and also because the researchers would approach the analysis from a perspective different from that of the participants. But this emphasis on alignment with the participants views of their own experiences remains very important for naturalistic inquiries.

But this research did not have a naturalistic focus, as analysis was undertaken less to precisely reflect participant's own experience and more to refine and develop a more general theoretical approach. For this reason, it was expected that the feedback obtained from participants in the member checks would have a less relevance for the viability of the final development and so it was decided not to perform member checks. Instead, a process of consensual validation (Pitman and Maxwell, 1992) was followed. Consensual validation involves providing reports of the interpretations to other researchers for their critical response.

I began this process by developing the criteria that respondents were asked to use in judging interpretations. To be viable, I required that the interpretation be compatible and consistent with a reasonable reading of the extract, carried out from the perspective of inquiry about the identified issue. I did not require that the participant intended to communicate the interpretation made.

More specifically, the viability of a directed interpretation may be judged using the following three criteria:

- Relevance: The interpretation is relevant to the issue.
- Reasonable reading: The inference must follow from an informal and reasonable reading of the extract, when related to the identified issue, based on the common language usage of the participants.
- Consistency: If an alternative interpretation of the same extract can be made which is

inconsistent with the first, then the reading of the extract yielding the alternative must be judged as less reasonable than the reading yielding the first.

If an alternative interpretation was not available, then the criteria of viability are quite weak. For all that need to be satisfied are the judgements of relevance and being a reasonable reading. The judgement becomes more restrictive when an incompatible interpretation is available. In this case, the third criterion comes into play, requiring a choice between the two. Note that the criterion of being a reasonable reading is hermeneutic, and so the decision which is the better will be subjective.

In this research, I did not attempt to prescribe how such a judgement should be made. Instead, I will use these viability criteria to provide a simple external judgement of the acceptability of my directed interpretations. To do this, I asked two other qualitative researchers — the supervisor of the project and another, who had no links with the project — to judge the viability of the interpretations made. They were provided with a report for each issue, giving each extract and the accompanying interpretations and were asked to judge the acceptability of the interpretation according to the two criteria of relevance and reasonable reading. Also, they were asked to think of alternative interpretations of the data extract, which provided a reasonable reading of the extract, were relevant to the issue in question and were inconsistent with the first. Only the interpretations which do not have reasonable alternatives were included in the further analysis.

2.7 Overview of the research process

In the final section of this chapter, I will provide a brief overview of the research process. This will provide an overall picture of the data collection and analysis that will be fleshed out by the more detailed discussion in the following chapters.

The research process could best be viewed as consisting of three cycles of data collection and analysis. In the first two cycles, I worked with teachers and in the third cycle, with children. In each cycle, the primary data was generated from a number of interviews in which I worked with participants to explore a number of example situations that I considered relevant to the issues I wished to investigate in that cycle. The interview data was then analysed using a guiding framework that was formulated for that particular cycle. Literature was also consulted during the course of each cycle in order to obtain more insight into these issues.

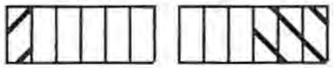
Table 2.1 details the interviews that were conducted in each cycle, as well as the participants in each interview. In the interests of confidentiality, pseudonyms have been used for all the participants in the interviews. The transcripts of these interviews, edited to change the names, have been included in the CD-rom supplied with this dissertation. These transcripts will be taken as the 'raw data' for this research.

Table 2.1: Map of the interviews carried out for each cycle.

	Pilot Cycle (Chapter 3)		
	Pilot interview 1:	Lungiswa	
	Pilot interview 2:	Lungiswa	
	Pilot interview 3:	Lungiswa	
	Teacher Cycle (Chapters 4 and 5)		
	Pilot interview 4:	Lungiswa	
	Teacher interview 1:	Nadine and Sarah	
	Teacher interview 2:	Nadine and Sarah	
	Teacher interview 3:	Nadine and Sarah	
	Teacher interview 4:	Nadine and Sarah	
	Teacher interview 5:	Nadine and Sarah	
	Teacher interview 6:	Nadine and Sarah	
	Teacher interview 7:	Nadine and Sarah	
Children Cycle (Chapters 6 and 7)			
Interview 1:	Ruth	Interview 1:	Jason
Interview 2:	Ruth	Interview 2:	Jason
Interview 3:	Ruth	Interview 3:	Jason
Interview 4:	Ruth	Interview 4:	Jason
Interview 5:	Ruth	Interview 5:	Jason
Interview 6:	Ruth	Interview 6:	Jason

In each cycle, much of the discussion was based on examples selected to draw out the issues of interest. The following tables show these examples. The situations shown in table 2.2 exemplify

Table 2.2: Examples of different rational number interpretations.

Fraction Interpretation	Example	Description
Part-Whole	Zingiswa cuts a loaf of bread into 6 equal pieces. • How much of the whole loaf is one piece? • Three pieces make up how much of the whole loaf? 	Here a piece of an object is compared with the whole object, of which it forms a part.
Quotient (Sharing)	Nombulelo shares 8 loaves of bread between 6 people. How much does each person get?	Fair sharing of a quantity consisting of a whole or a number of wholes. The size of the share is named by comparing it with a whole.
Quotient (Grouping)	Sipho has 34 oranges to sell. He sells them in bags of 6 oranges each. How many bags does he have to sell? How much of a bag is left over?	Subdivision of a quantity to form groups of a certain size. The remainder needs to be compared with the size of a group to determine how much of a group is left.
Measurement	Bulelani measures the length of the classroom wall using a piece of string. The wall is 5¾ lengths of his string. Zukiswa's string is a quarter of Bulelani's string. She measures the wall with her string. How many lengths of her string is the wall?	Here we look at measuring a quantity using different units of measurement. These units all different unit fractions of a given whole.
Ratio	John ran 50 metres in 10 seconds, Thamo ran 60 metres in 11 seconds. Who ran faster? **Pilot Cycle:** This class has 3 boys for every 4 girls. There are 16 girls in the class. How many boys? **Teacher Cycle:** Yesterday, Mrs Mpati used 3 cups of flour and 2 cups of sugar to make small cakes. Today she uses the same recipe to make small cakes. She uses 6 cups of flour today. How many cups of sugar does she use?	Here fractions allow us to compare the relative sizes of two generally unrelated quantities.
Operator	Some children are picking prickly pears. I will swop 2 chocolates for every 8 prickly pears they give me. They bring me 24 prickly pears. How many chocolates do I give them?	Here we see a fraction as a multiplicative operator, which changes an input quantity into an output quantity. (The change is always proportional.)

different rational number interpretations and were used in both the pilot and the teacher cycles. A more detailed discussion of the selection of these problems is provided in chapter 3 (The pilot cycle). The response of the teacher in the pilot cycle to the one problem given for the ratio interpretation led me to change this problem for the teacher cycle. Both of these problems are given in the table. They are labelled by the cycle in which they were used.

Table 2.3: Examples of different operations relevant to rational number understanding.

Dominant Operation	Example
Unitizing	Mthunzi buys and sells apples. He buys a box of 86 apples from a farmer. Then he makes up bags of 8 apples to sell. He sells each bag of oranges for R2.00. a. How many bags of apples can he make? How many apples are left over? b. Lulama says she will buy the apples that are left over. What is a fair price for her to pay Mthunzi?
Unitizing	Draw pictures for each of these situations: a. Lunga's mother gave him 3/4 of a chocolate for a treat. Show what he ate. b. Thabisa's mother has 3 chocolates. One plain, one with nuts and one raisins. She cut each chocolate into 4 equal pieces and gave one piece of each to Thabisa. Show what she ate. c. James has bought a packet of three chocolates. He cuts the packet of chocolates into 4 equal parts without opening it. He ate the chocolate from one of the parts. Show what he ate.
Equal subdivision	Lulama shares 5 slices of bread fairly between her 3 children. Martha shares 7 slices of bread fairly between her 4 children. Who gets more?
Equal subdivision	Zolani shares a slab of chocolate with Sibeko. Sibeko then shares his chocolate with 2 other friends. How much of the chocolate does Sibeko get?
Iteration / counting	Lungiswa bakes small apple pies. Each pie needs ½ an apple. How many apples does she need for 35 pies?
Iteration / counting	Vutani buys long pieces of wire to make wire cars to sell. Each car needs 1/3 of a piece of wire. How many pieces of wire does he need to make 20 cars?
Coordinated counting	John's dog eats 5 bags of food in 7 weeks. Kholeka's dog eats 3 bags of food in 4 weeks. Whose dog eats more?

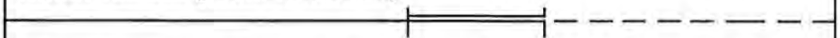
Coordinated counting	Mboleli walks to the sports field with his father. Every time that his father takes 2 steps, Mboleli has to take 5 steps. His father takes 34 steps to get to the field. How many steps does Mboleli take?
Scaling	Pitiwe works every day, painting the lines in the middle of the road. One day she paints three different types of lines – solid lines, dotted lines and double lines. Here is a picture of how much she has painted that day.  Pitiwe wants to compare how much she has painted of each line. a. How much of the solid line is the double line? b. How much of the whole distance is the double line?
Scaling	Nolitha is visiting a museum. From far away, she sees a picture of a pot of gold. It looks like this:  She likes the picture, so she goes close to it to see it better. Which of these pictures is the same picture from close? Picture 1  Picture 2  Picture 3 
Exchanging	Yolande has a bag of apples and Lulama has a box of chocolates. Yolanda wants some chocolates. Lulama says she will swop three chocolates for 5 apples. a. Yolanda gives Lulama 5 apples. How many chocolates should Lulama give her? b. Yolanda gives Lulama 10 apples. How many chocolates should Lulama give her? c. Lulama gives Yolanda 3 chocolates. How many apples should Yolanda give her? d. Lulama gives Yolanda 12 chocolates. How many apples should Yolanda give her? e. Yolanda gives Lulama 32 apples. How many chocolates should Lulama give her?
Unequal subdivision (taking reasoned unequal parts)	Maria is dishing up supper for her family. She cuts up a long sausage for herself, her husband and her son, who is 6 years old. Show how she should cut the sausage so that they all get enough to eat.

Table 2.3 shows the examples selected at the start of the teacher cycle, for the purpose of introducing different operations relevant to rational number understanding. The initial conceptualization of these operations, and the development of these examples is discussed in chapter 4 (Teacher cycle preparation).

Each interview of the children cycle involved the exploration of a different situation. These situations are described for each of the six interviews, in table 2.4.

Finally, table 2.5 details the issues selected as the focus for analysis in each cycle of the research.

Table 2.4: Situations explored in the six children interviews.

Interview 1	Sharing loaves of bread between people. Different questions involved different numbers of loaves and different numbers of people. The amounts obtained from different sharings were then compared.
Interview 2	Slicing loaves of bread into slices of equal thickness. Different questions involved cutting different numbers of slices, and thus slices of different thicknesses. The relationships between the numbers of slices and their thicknesses were then compared
Interview 3	Buying loaves of bread at a given price from a shop. Two different types of loaves were used, costing different amounts. Different questions involved buying for different amounts of money or buying different numbers of loaves.
Interview 4	Dividing an amount of bread dough between two tins of different sizes, for baking. Two questions were asked, one in which the dough was presented as a single 'sausage' and the other in which the dough was presented as a number of 'balls'.
Interview 5	The children were shown decorative shapes on bread before baking and asked to draw the corresponding shapes after baking. Alternatively, they were shown shapes after baking and asked to draw then before baking. Finally, they were asked to draw the shape of the bread after (or before) baking, when they were given the original (final) shape and the way the length or width had changed.
Interview 6	Buying sausages at different prices, for different amounts of money.

Table 2.5: Focal issues for analysis of the interviews in each cycle of the research.

Cycle	Focal Issues
Pilot cycle	1. Teachers views about the utility of the general approach I took to teaching mathematics in general and rational numbers in particular. 2. The teacher's personal understanding of the interpretations. 3. The teacher's views about the utility of the interpretations for introducing rational number concepts.
Teacher cycle	1. Issues of meaning, motivation and significance. When viewing the discussion problems from different perspectives. 2. Insights obtained about interpretations and operations, from the teachers understanding and analysis of the problems. 3. Emerging themes about teaching and learning rational numbers
Children cycle	1. Issues of meaning, motivation and significance. 2. Insights about interpretations obtained from the children's understanding and analysis of the problems. 3. Insights about operations obtained from the children's understanding and analysis of the problems. 4. What issues relevant to the development of more formal, conceptual understanding were apparent in the children's interviews? 5. Issues relevant to the learning of representations.

Chapter 3 – Pilot Cycle

3.1 Initial Engagement

As described in section 1.2, my initial interest in the meaningful teaching and learning of rational numbers was stimulated by my work in teaching mathematics to students and to mathematics teachers in professional development programmes. The problematical nature of rational number learning became evident in this work and in response, I developed a number of ideas and opinions on this issue. Understanding that these opinions were largely informal and not rigorously justified, I became motivated to embark on a research project to develop a more rigorous and complete understanding.

These initial ideas formed the initial conceptual background from which my understanding developed in the research. In this section, I will specify my starting point in more detail by briefly outlining these ideas.

I noticed that a sizeable proportion of students and teachers experienced difficulty when working with rational numbers. In general, they appeared to approach these numbers from a purely formal perspective, without reference to any possible meanings. In responding to this, I found that a number of these errors could be counteracted by encouraging them to interpret these numbers meaningfully.

Developing an understanding of rational numbers is an aspect of mathematical learning that starts early in a child's school career. A basic experiential foundation for this learning is laid in the first two years of schooling, with a more explicit development of fractions in grades three to six (Department of Education, 2002). Teachers shared that, in most cases, learning about rational numbers only occurred within the school environment. This generally resulted in children developing an abstract view of rational numbers, with little understanding of the relevance and use of these numbers in their home environments. Teachers were thus faced with the problem of how to introduce rational numbers within the classroom environment so that children would be able to effectively use what they had learned in the very different environment out of school.

This situation could be contrasted with the development of whole number understanding, where the foundational learning occurs in the child's experience of counting and quantity before school (Ginsburg, 1977, Gelman and Gallistel, 1978). As a result, children see the relevance of whole numbers in their out of school environment, and are willing and able to use their basic number skills to good effect in this environment. This is evidenced in the activities of largely unschooled Brazilian children who make their living as street vendors (Carraher, et al., 1985). In addition, the ability to effectively use these basic skills persists into adulthood, as shown by the studies of Lave (1988) on adults dieting and shopping activities. This is the case even when the adults are unable to perform many of the more abstract tasks involving whole numbers, that they learned later in school.

Both whole numbers and rational numbers form systems with a complex structure. It was generally accepted by most teachers that one of the aims of school mathematical education was to develop a deep and powerful understanding of these number systems. In addition, for this learning to be useful, or meaningful, it should include the relationships between these systems and the individual's environment, both in and out of school. To do this is a difficult task, as evidenced in the studies of Lave (1988) which showed the difficulties experienced by large numbers of adults in performing more complex and abstract number operations, as well as the reluctance of adults to use the more complex number procedures learned in school, to solve problems in other situations.

My underlying concern at the start of the project, was how to engage children in activities within the classroom environment, in order to provide an effective experiential foundation for rational number learning. This foundation would need to satisfy the following two conditions. It should be **comprehensive,** in that the important structures and processes involved in rational number thinking may be developed naturally from this base. Also, it should be **grounded,** in that the children should see it as stemming from their experience out of school and be willing to use the subsequent rational number learning in this environment.

3.2 Literature review

To begin the research, I consulted the literature in order to further clarify this thinking.

3.2.1 Foundations for Mathematical Learning

The assumption that learning is constructive in the broad sense of the term has a number of implications for this project. According to the constructivist principle, children construct their own new knowledge when engaging in an activity, by building on the basis of their pre-existing knowledge, in response to the physical and social influences in the activity (Ernest, 1990).

Contrasting everyday activities with more formal school activities, it has been shown that concepts which are developed naturally from patterns of everyday activity are more effectively learned than formal systems of rules (Nisbett et al., 1987). Conversely, children may demonstrate an understanding of number concepts in contexts which are relevant (Hughes, 1988) or meaningful to them (Gelman, 1972; Gelman and Gallistel, 1978), even though they do not demonstrate a corresponding understanding in formal contexts. This justified my choice of less formal activities in relevant, or meaningful contexts, for the initial development of rational number understanding.

Furthermore, children who have developed their understanding of a concept authentic engagement in meaningful contexts, develop the representational means to use these concepts in activities which are physically and socially similar to the learning activity (Brown et al.,1989). It is hypothesized that this is due to the similarity of the contextual cues in the actual and the learning activity. Thus, to provide an adequate grounding for general rational number knowledge, it will be important to develop a broad selection of activities, providing links to the most important contexts and activities outside the school environment, for which rational number thinking is useful.

For teaching purposes, it is possible to identify three important aspects of knowledge that learners construct for themselves when engaging in meaningful activity in everyday contexts.

- This knowledge is **situated**, in that it is constituted in the interaction between the situation and the learner's activity. As such, is not immediately available to the learner in a different situation (Lave, 1988; Lave and Wenger, 1991).
- It is often **intuitive**, in that it is perceptually cued and the learner finds it difficult to provide supporting evidence for the knowledge (Leinhardt, 1988).
- Finally, it is **informal**, in that it is not incorporated as part of a system which can be symbolically and formally represented and taught (De Corte et al., 1996).

For the purposes of this research, the term **'informal knowledge'** will be taken to represent a composite of these three meanings.

To investigate the initial development of learners informal knowledge of rational numbers, I approached the literature to find information about:

- different interpretations of rational number concepts in terms of everyday activities in meaningful situations, and
- different approaches currently being investigated for introducing rational number thinking.

3.2.2 Semantic analysis of the rational number domain

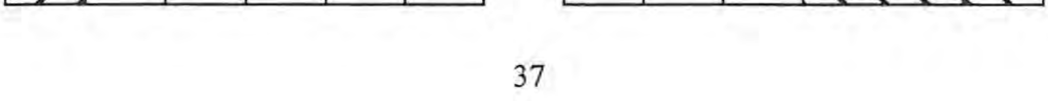
In order to identify the important structures and processes in rational number thinking, a semantic analysis of the rational number domain was carried out by Kieren (1988; 1993) and by Behr and associates (Behr et al., 1983; Behr et al., 1992; Behr et al., 1993).

These analyses could be viewed as a classification of the different logical and mathematical operations resulting in rational numbers, or conversely, as detailing different mathematical or logical interpretations of a rational number – the sub-constructs comprising the domain. Although not precisely a classification of different interpretations of rational numbers in terms of everyday activities, this analysis has formed the basis of a number of different approaches to improving rational number teaching. As such, it was chosen as a good starting position for this research. This mismatch did however, introduce a tension into the initial research and the resolution of this tension was expected to provide some of the impetus for the development of the project.

With minor disagreements, Kieren and Behr et al. concluded that the rational number domain is composed of five major sub-constructs, namely part-whole, quotient, measure, ratio and operator (Marshall, 1993). These different sub-constructs and the corresponding interpretations are briefly described below, with clarifying examples taken from those used in the interviews. The descriptions are based on the above references, as well as the writing of Lamon (1993, 1999 and 2001).

The **part-whole** sub-construct relates to the situation when a part of some given whole is distinguished. For example:

Zingiswa cuts a loaf of bread into 6 equal pieces.



- How much of the whole loaf is one piece?
- Three pieces make up how much of the whole loaf?

This interpretation formed the foundation for the classical introduction to fractions, and hence rational numbers (Kieren 1988). In most cases, the whole (which may be continuous or discrete) is partitioned into elements of equal size in such a way that the part is made up of a number of these elements. In general, the distinguished part must satisfy the condition that such a partition is possible. It follows from this that a rational number (between 0 and 1) may be used to relate the size of the part to the size of the whole. The fraction representing the rational number has the number of elements in the part as the numerator, and the number of elements in the whole as the denominator.

For the **measure** sub-construct, unit fractions are used as the units of measure when measuring some quantity. For example:

> Bulelani measures the length of the classroom wall using a piece of string. The wall is 5¾ lengths of his string. Zukiswa's string is a quarter of Bulelani's string. She measures the wall with her string. How many lengths of her string is the wall?

That is, a quantity to be measured and a comparison quantity (the whole, or measurement unit) are given and the aim is to describe the size of the quantity to be measured in terms of the comparison quantity. What yields rational numbers in this interpretation is that the comparison quantity need not equally subdivide the quantity to be measured. However, it must be possible to evenly partition the measurement whole into units which do evenly subdivide the quantity to be measured. These unit fractions of the measurement whole may then be used as different measurement units to accurately measure the quantity to be described. The rational number resulting from the measurement may then be represented by the product of the measurement and the unit fraction. Another important issue in this interpretation is the way the measurement (relative to the unit fraction) varies with the change in the size of the unit fraction.

The **quotient** sub-construct relates rational numbers to the mathematical operation of division. It is well known that the mathematical operation of division has two very different interpretations in terms of everyday activities (Neuman, 1999, Vergnaud, 1983 and Fischbein et al., 1985). These are partitive division, or sharing, and quotitive, or measurement division, which corresponds to forming groups of given size. Thus, for the purposes of this research, the quotient sub-construct was divided into two separate interpretations: **sharing** and **grouping**.

The **sharing** interpretation sees rational numbers as arising from the process of sharing a given quantity into a number of equal parts. Generally by sharing the quantity fairly between that number of people. A simple example is:

Nombulelo shares 8 loaves of bread between 6 people. How much does each person get?

This sub-construct is different from the part-whole sub-construct. For sharing always results in an even partition, which is then distributed to form the desired number of groups (often one to each person sharing). This distribution destroys the unity of the whole, so that it is not available for visual comparison with the parts. On the other hand, the part-whole interpretation always involves a primary partition of the whole into two pieces (the identified part and the remainder) and whole remains visible as a unit, allowing the comparison of the whole with the identified part. An even partition is an important intermediate step in the process of describing this relationship, but the actual partition is arbitrary (as long as it satisfies the commensurability condition that both the whole and the identified part are evenly subdivided by the partition).

The **grouping** interpretation involves dividing a quantity into groups of a given size. In contrast to sharing, the size of the groups is determined by the situation, while the number of groups is determined as a result of the grouping operation. For example:

Sipho has 34 oranges to sell. He sells them in bags of 6 oranges each. How many bags does he have to sell? How much of a bag is left over?

Grouping is operationally similar to measurement, although these interpretations are practically distinct. In measurement, the quantity being measured remains as a distinct entity and the measurement describes the magnitude of this entity by comparing it to a number of repeated measurement units. On the other hand, in grouping, the initial quantity is subdivided into a number of distinct entities of the given size and thus no longer exists as a complete entity. Because of this, it was decided to retain grouping as an additional interpretation, distinct from measurement.

The **ratio** sub-construct is used when comparing two quantities of different type. For example:

This class has 3 boys for every 4 girls. There are 16 girls in the class. How many boys?

This ratio is a multiplicative comparison of the values of the two quantities and may be expressed as a rational number. An important feature of this interpretation is that there is no distinguished reference quantity which is treated as a 'whole'. Both of the quantities being compared have the same status.

Lastly, the **operator** sub-construct deals with the conversion of some quantity into a different quantity by proportional means. The following example was chosen for this:

> Some children are picking prickly pears. I will swop 2 chocolates for every 8 prickly pears they give me. They bring me 24 prickly pears. How many chocolates do I give them?

Often these quantities are of different types, as is the case in contexts involving exchange, or shopping. Two types of operators which are very important for this interpretation are stretching and shrinking operators, of which doubling and halving are the simplest examples.

3.2.3 Rational number sub-constructs and approaches to teaching

Two important questions arise when using the research about sub-constructs as a framework for developing children's rational number understanding from informal knowledge. Firstly, whether children's informal knowledge encompasses critical concepts from each of these sub-constructs, or whether it is constrained within a single sub-construct. Secondly, if the understanding grows from an initial single sub-construct, whether any sub-construct could serve as a basis, or whether there is a specific, privileged sub-construct which will always form the foundation.

In the search for answers to these questions, different research studies have investigated learners' use of informal strategies in the solution of problems involving rational numbers. Studies such as Leinhardt (1988) and Mack (1990; 1993) focussed on learners introduction to fractions and rational numbers, while others such as Lamon (1993) focussed on the teaching of ratio and proportional thinking. In these studies, it was found that learners informal solutions to problems drawn from any of the five sub-constructs, were based on partitioning – and thus developed from the part-whole sub-construct. These partitioning strategies have different levels of complexity, from simple partitioning of whole objects to the formation of multiple levels of composite units. From this perspective, developing rational number understanding involves developing the ability to form and relate higher levels of composite units. This led to the position of Mack (2000), that

> students were able to build on their informal knowledge to grow in their understanding of a domain such as multiplication of fractions even though their initial conceptions were very different from the complex mathematical ideas they

were building towards. (pg. 309)

However, in a more recent longitudinal study by Lamon (2001), different groups of children followed different four year programmes, each based on a single sub-construct, to develop rational number understanding from informal knowledge. It was found that, although there were many differences between the learning in each approach:

> After four years, all five groups of students had developed a deeper understanding of rational numbers than the students in the control group had, as measured by the number of subconstructs that the students were using. The numbers of proportional reasoners in the five classes far exceeded the number in the control group, and even in computation, achievement was greater in all five groups than it was in the control group. (pp. 159–160)

This suggested that, contrary to the earlier research, no particular sub-construct is privileged as a starting point for rational number learning. In addition, some interesting features of a combined approach using both part/whole and measurement interpretations are mentioned. However, the use of an integrated approach using a combination of all the interpretations is left as an open question.

3.2.4 Additive and multiplicative thinking

Not all research into the initial learning or rational numbers is based on the semantic analysis given above. Another important perspective, based on the difference between additive and multiplicative thinking, was developed independently by Case (1985) and Resnick and Singer (1993). They suggested that current school practices foster children's slow development of multiplicative knowledge, resulting in an early preference for additive solutions to proportionality problems. To counter this, for rational number learning in elementary schooling, they advocate the development of learning activities based on alternative informal knowledge structures which favour children's development of multiplicative knowledge.

An approach to teaching fractions emphasizing multiplicative thinking through the operation of repeated subdivision, was developed by Moss and Case (1999). A number of additional features were incorporated into this approach. Following Case's Neo-Piagetian theory of cognition (Case 1985), they stressed the importance of qualitative comparison as the basis for children's numerical schemata and thus included this as an integral part of their approach. Also, they identified two factors which increased the information processing complexity of the task in many common approaches to introducing rational numbers. (Complexity in the sense of the number of independent dimensions which had to be mentally represented in the children's cognitive processing.) The first was the complexity of the standard fraction representation, which children were expected to master early in their learning of rational numbers. To overcome this difficulty, they designed a teaching programme which used percentages as the initial representation. The second was the use of physical grounding materials, which were measured and compared using area (a two dimensional property). To simplify this, they used materials which could be described by a length (one dimensional). They showed that fourth grade learners taught using this approach, were able to effectively solve rational number problems using multiplicative strategies.

The success of this approach suggested that the semantic analysis may not encompass all the important features of initial rational number learning. I thus decided to base the pilot interviews on the semantic analysis and the different interpretations it provided for rational numbers, but also to be deliberately open to the teacher's responses which could suggest other issues which of importance for initial rational number learning based on grounded experience.

3.3 Structuring the pilot interviews

The aims of the pilot interviews were twofold. The first aim was to obtain an improved insight into the research issues, through interaction with a teacher involved with developing fundamental rational number understanding. The second was to develop basic experience, and means of analysis for the empirical process of this research.

In order to make the communication process as simple as possible. I approached a teacher with whom I already had a working relationship, to reduce the need for building rapport. Previously, we had worked together with her grade 3 class, to trial concrete activities for introducing rational numbers. These activities involved sharing in a number of different contexts where length was the relevant property for comparison (including some motivated by the work of Moss and Case; 1999). Because of this, I expected the teacher to understand the rationale of the approach to teaching mathematics that I advocated, as well as to be committed to introducing rational numbers through meaningful and concrete activities. I hoped that this would reduce the possibility for mis-communication in the interviews.

Through the interviews and the subsequent analysis, I wished to gain insight into the teacher's personal opinions and understanding about early rational number learning, and then use this insight to adapt and improve my own personal understanding. Because of our very different personal backgrounds, to develop this insight I would need to interpret the teacher's words and actions in the light of my understanding of her personal world.

A sequence of three interviews was planned for the pilot study. In the attempt to deepen my understanding of her personal world and so enrich my insight into her opinions and understanding, the first interview was aimed at building a personal history of the teacher. This would include the personal background of the teacher, as well as her views on teaching in general and mathematics teaching in particular. The aim of the second interview was to begin the investigation of the different interpretations provided by the semantic analysis by introducing them to the teacher and exploring her personal response. In the third interview, the teacher would be asked to discuss her opinion of the general approach being advocated for introducing rational numbers, and of how these different interpretations could usefully contribute to such an approach. This interview would also provide the opportunity for the teacher to explore any misunderstandings arising in the second.

It should be noted that I expected to play an active role in the interviews. This would be particularly evident when introducing the different interpretations in the second interview, where the role almost of a teacher would be taken. In order to do this consistently, a standard approach was developed, to be followed in the second interview for each interpretation. To ensure that the interpretations were adequately grounded, this approach included the investigation of examples and a more abstract description of the interpretation.

One or two examples of each interpretation were developed when preparing for the interviews. These examples were formulated as simple word problems, which were set out on a sheet of paper. To introduce an interpretation, I would present the teacher with the prepared problems, and ask her to discuss what she would do to solve these problems in the given situation. I would then collaborate with her to a greater or lesser degree, in developing an approach which could yield the solution to the problem. After this, I would name the interpretation and offer a brief abstract description as an encapsulation of the solution approach developed. The teacher would then be asked if she thought this was a valid description of the solution process. This was done after the example in order to allow the teacher freedom to follow any approach that she deemed relevant when solving the problem. If the teacher did follow an approach fitting with the interpretation, this would attest to the suitability of the choice of example, and to the point of view that the particular interpretation processes would be used naturally in certain situations. On the other hand, if the teacher followed a different solution process, either, or both of these claims would be questioned. Finally, I would ask the teacher what she thought of the given interpretation and its relevance to rational number thinking.

In addition, I developed a table that showed the name and a brief description of each interpretation, together with the examples investigated in the interview. This is reproduced as table 2.2 in the methodology chapter. The table would be given to the teacher for reference at the conclusion of the second interview. To prepare for the third interview, the teacher would be asked to think about teaching activities which could be implemented to introduce rational numbers using each of these interpretations. The third interview was a free discussion based on the teachers response to this request.

3.4 Analysis of the interviews

The main aim of the interviews was to inform my thinking about initial rational number learning, and how the different interpretations could effectively contribute to this. Initially I followed an open approach to the analysis, carrying out open coding and identifying emergent themes as advocated in grounded theory. This analysis identified a wide range of themes, not all of relevance to the research issues. Because of this, I found it useful to provide more structure to the analysis by identifying three broad areas of discussion, which were more directly related to the research issues and then focussing my analysis on these areas. The areas identified were:

1. The utility of the general approach I took to teaching mathematics in general and rational numbers in particular. Analysed in terms of the teacher's general responses, her use of the approach in her teaching and her reports of children's responses.
2. The teacher's personal understanding of the interpretations. Displayed through her manner of solving the chosen word problems and her responses in our subsequent discussion.
3. The teacher's views about the utility of the interpretations for introducing rational number concepts. This would include her opinions about possible ways that she could use these interpretations in her teaching and possible ways that children could work with

situations involving these interpretations. It would also include her reports about classroom incidents in her experience that had relevance to these interpretations.

During the interviews and the process of analysis, it became evident that it would be difficult to separate the teacher's personal responses from those involving teaching and children's responses. For when solving each problem, most of her responses related what she was doing to what children would do when faced with a similar problem, and to ways that she could incorporate this in her teaching. This was especially the case when dealing with interpretations with which she was confident, such as sharing and part-whole. Because of this, I decided to analyse the second and third areas of interest together.

3.4.1 Teacher background

The aim of the first interview was to allow the teacher to tell her story. In particular, the story of who she was as a teacher and a teacher of mathematics. This included her personal history and her history as a teacher, as well as her aims and goals in teaching and what she thought was important for achieving these goals. This background information about the teacher was important in that it provided a personal context and allowed a richer interpretation of the teacher's responses. A deep analysis of the teacher's story in the first interview, was not carried out. Rather it was read as a contextualizing narrative, providing a personal perspective from which I attempted to view the teacher's responses in the remaining interviews. In this subsection, I briefly summarize this story.

Lungiswa had spent almost her whole life in the region of Grahamstown. Her father, mother and grandmother were teachers and she learned much about teaching from her father. As a young child, she saw herself as becoming a teacher and many of her favourite games were teaching games. She attended different schools in Grahamstown and on attaining her junior certificate, she attended the Healdtown teachers training college where she studied for her teacher's diploma, specializing in junior primary. After graduating, she returned to Grahamstown to teach grades three and four in a junior primary school. She has taught in this school since then.

As a teacher, Lungiswa saw her work as very important in contributing to the personal and intellectual growth of the children in her class. She saw this as helping these children develop to become skilled and responsible members of society, able to care for themselves and work to uplift their community. Uplifting her community was a very important motivating factor for her and she was actively involved in a number of different initiatives to improve the teaching and learning in her community. These include inviting children to her home over weekends, for extra learning activities; working with other teachers (both primary and secondary) to improve their teaching; inviting parents into her class to observe and participate in the children's learning activities; and organizing homework clubs in the community.

The mode of teaching in grades 3 and 4 is class teaching and so Lungiswa taught all subjects from the start of her teaching career. However, for a long time, she did not like teaching mathematics and "had many failures in maths". In fact, she went as far as asking the inspector if she could teach Afrikaans to other classes in return for another teacher teaching mathematics to her class. A few years ago, her principal convinced her to attend a professional development workshop for mathematics teachers. At this workshop, she was introduced to an approach to teaching mathematics in a meaningful way, by encouraging children to investigate and discuss patterns in concrete activities. Trying this approach and these activities in her classroom, she found that her children became interested and excited about mathematics. This positive feedback led to her becoming increasingly interested in teaching mathematics and taking a leading role in the teaching of mathematics in her school and community. She became a strong advocate of co-operative and investigative approaches to teaching, seeing them as developing a positive attitude, as well as personal responsibility and initiative in her learners. In addition, she stressed a number of times in her interviews that it was important to develop mathematical understanding by means of concrete activities which provide a meaningful grounding to this knowledge.

3.4.2 General approach

Because of her prior involvement with me, Lungiswa was familiar with the general approach that I followed to teach mathematics in a meaningful fashion. This involved encouraging children to investigate and discuss patterns in activities that made sense to them. As mentioned in the discussion of her background, she found that following this approach her children became interested and excited about mathematics.

She was committed to using concrete experiences to help her children develop a meaningful understanding of mathematics. In her own words (having confused the meaning of abstract and concrete):

703 LS: Ja. I ... I ... yes, I ... I think I ... I think I understand your question. But, to my belief ... I don't think ... it's possible ... for one to teach ... without an abstract.
704 BB: Okay.
705 LS: More especially ... with our learners. That is ... uh ... the foundation phase.
706 BB: Explain in more detail?
707 LS: Uh ... uh ... uh ... the abstract ... helps a lot ... so that ... a child has got ... uh ... uh ... an imagination ... uh ... further than now... we are dealing of ... of ... we are doing of an apple now.
708 BB: Okay.
709 LS: They see this apple ...
710 BB: Yes.
711 LS: (Hesitation). They divide this apple ...
712 BB: Ja.
713 LS: And they see the pieces from this apple.
714 BB: That's right. Okay.
715 LS: Now... say now ... if ... there was no apple.
716 BB: Mhmm.
717 LS: And I say, ... "You have an apple ... you share this apple ... fairly."
718 BB: Ja.
719 LS: (Hesitation). Definitely ... this ... this child would say, ... "We'll not ... uh ...we'll ... we'll not ... we'll not be able ... to cut ... this apple ... so as to get ... equal pieces."
720 BB: Mhmm.
721 LS: In her mind ... she'll think that ... the pieces ... are equal.
722 BB: Okay.
723 LS: In the first place ... they must see ... what equal means.
724 BB: Okay.
725 LS: What fairly means.
726 BB: Mhmm.
727 LS: So that's why ... I think that's why we must use abstract.

(Pilot Interview 1)

She had been following such an approach when teaching rational numbers before the research project. Most of the teaching activities that she had used, developed the mathematical concepts based on the concrete and meaningful activity of sharing.

643 LS: When I said, ... "Uh ... What fraction do you get here? She ... this child ... was ... um ... sharing ... an apple ... to a friend, ... ne? And then the friend got ... a half ... of a half .
644 BB: Yes. Okay.
645 LS: Yeah?
646 BB: Mhmm.
647 LS: She got a half of a half.
648 BB: Okay. So she started with her half ...
649 LS: Mmm.
650 BB: ... and shared that with her friend.
651 LS: Ja.
652 BB: Okay.
653 LS: And the friend ... got a half of a half.
654 BB: Okay.
655 LS: Then I said, ... "What fraction is this one? ..."
656 BB: Mhmm.
657 LS: "... from a whole apple ... from an whole apple."
658 BB: Yes.

659 LS: Ne?
660 BB: Okay.
661 LS: And now she said, ... "If it was an whole ... it was a whole apple ... she would get ... two halves."
662 BB: Yes.
663 LS: "And then ... when those two halves ... are shared ..."
664 BB: Mhmm..
665 LS: "... in the middle ..."
666 BB: Ja.
667 LS: "... you get four pieces.
668 BB: Okay.
669 LS: And then when you put those four pieces together they make a whole apple."
670 BB: Okay.
671 LS: "So if they are four pieces ... each piece is called ... a quarter ... a quarter ... a quarter. That's what they've got.'

(Pilot interview 1)

It was apparent that the focus of the mathematical development in much of her rational number teaching was on identifying and naming fractions. That is, even though the representation was developed through actions that the children perceived as significant, the representation was developed as an end in itself. It was not developed as a means to achieve an end which the children would see as significant outside of school.

As a consequence of this, the motivation and significance of this learning was derived from the activity of schooling, as opposed to other, non school based activities.

3.4.3 Rational number interpretations

To carry out the analysis, a category was created for each interpretation. I then worked through the interview transcript and any interchange relevant to an interpretation was coded in the corresponding category. The data coded in each category was extracted and read reflectively a number of times, in order to develop a coherent interpretation of the teachers response to that manner of thinking about rational numbers. This culminated in the writing of a short description of the researcher's interpretation of the teachers response, with supporting extracts from the interviews. Also in this reflective reading process, other items of interest which emerged within and between categories, were identified and coded.

In this subsection, I present the six descriptions developed through this interpretive analysis, accompanied by illustrative extracts from the interviews.

In the interviews, we examined the part-whole interpretation first, followed by the sharing interpretation. However, in the interviews, the teacher used sharing to make sense of a number of part-whole situations. For this reason, I will report on the sharing interpretation before te part-whole.

3.4.3.1 Sharing

The teacher seemed to find this interpretation easy and natural. She also stated that her learners found this interpretation straightforward.

The importance of fair sharing was evident in her responses. She saw this as important for rational number learning, but added that the children also considered it important in their social interactions. Children were often faced with social situations in which they needed to share some resource fairly. She stated that in such situations they often used the term 'square' to denote fairness:

634 LS: Come back with the term ...square ... (laughing).
635 BB: They're all square.
636 LS: Ja ... ja. Are you okay? Ja ... ja, miss. It's square ... it's square.
(Pilot interview 3)

Though it derives, in her opinion, from the four equal sides of a square, it has come to be used for equality in general:

644 LS: ... So if they are ... equalness ... that ... that they say is square ...
645 BB: Ja..
646 LS: If ... eh ... if all things now are equal ... even if they are six ... even if eight ...
647 BB: Yes. That's right.
648 LS: ... they said it's square.
(Pilot interview 3)

The teacher found it interesting that when sharing, the children act to share and then compare. They then adapt their subdivision in order to endure that the sharing is fair. Using this procedure, they are able to accurately share between three children.

529 LS: Ne? Okay ... if you give them this ... teacher ... share this ... for you ... you are three,
530 ne?
531 BB: Mhmm. Ja.
532 LS: ... and then ... usually what they do ... they will share it for two people ...
533 BB: Yes.
534 LS: ... and then they will say, "I'm going to give you a piece ... and this one's going to give you a piece ...
535 BB: Uh huh ...
536 LS: Ne? I don't know how they do it, but they do it ... they do that ... good ...
537 BB: Ja.
538 LS: ... because they all get ... equal pieces.

539 BB: So they ... so they wouldn't divide into ... they'd divide into two pieces first.
540 LS: Ehhh.
541 BB: And then each with a small piece would actually then ... break off a piece of theirs to make ...
543 LS: Yo! I ... I want ask the other one... I said. "Okay. If you are going to share now with this one they are both going to share with this one ...
544 BB: Mhmm.
545 LS: ... that one, is it ... is she not going to get more ... more share?"
546 BB: Yes. Ja.
547 LS: They said, "No we know that. I'm going to ..." They always refer to ... to the sharing pieces ... quarters, even if they're not quarters.
548 BB: Do they call them quarters?
549 LS: They call them quarters.
550 BB: (Laughing) ... Anything smaller than a half is a quarter.
551 LS: Everything that that is smaller than a half is a quarter with them.

(Pilot interview 3)

She found this flexible use of the word 'quarter' remarkable and enjoyed sharing it with the researcher.

This interpretation of rational numbers dominates for the teacher to such an extent that it may even be considered as an over-generalized conception. This is evident when she

- initially identified the part-whole interpretation as sharing,
- used sharing situations in part-whole discussions, and
- used sharing situations in the discussion of grouping.

3.4.3.2 Part–whole

The teacher was confident that the part-whole interpretation was easy to understand and was sure that "The kids will understand when they see it." In discussing her teaching, it was evident that she naturally used this interpretation when responding to children's queries. She described using folding of paper or string for introducing fractions, as well as asking her children to subdivide a drawing of a bar, in order to help them understand the naming of fractions:

842 LS: Ja. I said, "Now we've got this bar" ... I said, "This is a half ... this is a half, ne?" I said, "Now from this half ... make another half ..."
843 BB: Mhmm.
844 LS: "... from this half."
845 BB: Ja.
846 LS: So ... he did that And from the other one ... make another one. " I said, "How many parts do you see now? Can you count those parts?"
847 BB: Uh huh.
848 LS: They counted them, ne?
849 BB: Ja.
850 LS: Said, "They are four."
851 BB: Yes.
852 LS: I said, "Now ... you say they are four. Can you give a name to each one?"

853 BB: Mhmm.
854 LS: I knew he was going to be right because they know a quarter.
855 BB: (Laughing). Yes. Okay.
856 LS: Ja.
857 BB: Ja.
858 LS: He said, "It's a quarter ... it's a quarter ... it's a quarter ... it's a quarter. I want to take the quarter now."
859 BB: Mhmm.
860 LS: And the other one said, "It's a half of that half." I said, "Yooo! You are bright!"
861 BB: Okay.
862 LS: "You are very clever."
(Pilot interview 3)

On the other hand, after the part-whole and sharing interpretations were introduced in the second interview, she returned to the card with the part-whole interpretation , discussing it in the context of sharing:

229 LS Ja. You know what I like is this one.
230 BB Mhmm ...
231 LS When we are doing fractions ...
232 BB Ja.
233 LS ... so ... here they take ... uh ... say... This is a ... Cut this, in fact share it for one, two, three, four, five, six people.
234 BB Yes.
235 LS So now ... when they share this ...
236 BB Mhmm ...
237 LS ... they say, "One gets one sixth. There for!
(Pilot interview 2)

In this discussion, a transient state in a sharing situation was interpreted from a part-whole perspective. That is, the part-whole drawing was taken as a picture of an object to be shared, after it had been cut and before the pieces had been shared out. This is a transient state, created after the sharing has begun and destroyed by the time the sharing is complete. Although the teacher was happy to acknowledge the difference between sharing, which ended with no whole and all the pieces handed out, and part-whole, where the pieces were together forming the whole, this conflation persisted. For the interpretation of a transient state of a situation from a part-whole perspective also arose in the third interview. This was when discussing another realistic situation, that of creating a 'quarter' of bread:

468 LS: You know ... It's very easy for them ... because at break time at the shops ...
469 BB: Yes.
470 LS: They ... they buy what is called ... they ...they called quarter ...
471 BB: I've heard people talk about the quarter.
472 LS: Yes.
473 BB: Tell me about it? Ja.
474 LS: (Laughing). Can ...
475 BB: Ja. Ja.
476 LS: Can I show you?
477 BB: Go for it.

478 LS: Here's bread. At shop, ne?
479 BB: Mhmm.
480 LS: It's cut like this ... (showing how it is cut).
481 BB: Length wise?
482 LS: Length wise.
483 BB: That's interesting.
484 LS: Ja.
485 BB: Okay.
486 LS: Like this. So ... they buy this part ... (showing a part) ...
487 BB: Uh huh.
488 LS: ... and they take out ... the inner part ...
489 BB: The starch ... Yes.
490 LS: ... the inner part ...
491 BB: Mhmm.
492 LS: ... and put something ... chips, whatever ...
493 BB: Okay.
494 LS: ... fried chips ...
495 BB: Ohhkay.
496 LS: ... and they put that against the ...
497 BB: Woaa ... (laughing)
498 LS: (Laughing).... And they put that back in a pack ...
499 BB: Okay.
500 LS: ... and sell it to the children.

(Pilot interview 3)

In this case, the 'quarter' sold in the shops is displayed and sold as a separate entity, not reassembled as one piece of four making a whole. That is, the quarter is not part of a physical whole in the situation, the whole is a mental construction. Physically, the quarter exists as an isolated part, which could possibly be interpreted as a unit from the measurement perspective, or a grouping unit from the grouping perspective. However, in her discussion, she identifies the quarter by drawing a whole loaf of bread, showing how it is cut into four and identifying the quarter as one of these four parts. Again, this is a transient state in the creation of the final separated 'quarters'. But this transient state is convenient for showing the validity of the name 'quarter' for the piece of stuffed bread.

The teacher did identify one context for which the part-whole interpretation was totally appropriate. She talked about her children playing hopscotch, and in particular, how they described the situation when the player stepped on a line.

710 LS: ... if ... uh ... you find the foot here ...
711 BB: Sort of ... yes ... half of ..
712 LS: ... it's half of ...
713 BB: Oh! Do they actually use half? With sort of ...
714 LS: They use halves ... they use fractions.
715 BB: Oh.
716 LS: It's half ...

717 BB: Mhmm.
718 LS: ... so you are out.
719 BB: Okay.
720 LS: Your foot is half on the line.
721 BB: Yes.
722 LS: Ne? And sometimes ... it's only the ... the toe that is here.
723 BB: Yes.
724 LS: They'll say the three quarters of the foot ... is on the other circle ...
725 BB: Oh. So they actually ...
726 LS: This is a quarter ...
727 BB: Okay.
728 LS: ... That toe is a ... I ...I ...
729 BB: A quarter, yes.
730 LS: ... is a quarter.
731 BB: Ja.
732 LS: With the other break here.

(Pilot interview 3)

It was notable that, as in the discussion about sharing, the word 'quarter' was used very flexibly to describe any piece smaller than a half.

3.4.3.3 Grouping

The grouping interpretation was not familiar to the teacher. In the second interview it was evident that she did not find this interpretation straightforward and we worked together extensively to come to grips with it. Even then, she was not confident that she had mastered the approach and was still finding it confusing in the third interview.

The interpretation was introduced in the second interview in the context of selling bags of oranges.

310 LS: Sipho has 34 oranges to sell. He sells bags of oranges with 6 oranges each. How many bags does he have to sell? How much of a bag is left over?

(Pilot interview 2)

Even though it is directly asked in the problem, it took some thought before she agreed that the defining issue was the number of bags. In working to understand the situation, she was initially confused between grouping and sharing. By carefully following what would be done in the given situation, she was able to conclude that the relevant question is the number of bags. (Pilot interview 2: 312-325)

Finding the number of bags was then easy, but she then faced the problem of what to do with the remaining oranges.

327 LS: And ... it's where they can see now the ... the remaining oranges.
328 BB: How many oranges will remain?

329 LS: Four.
330 BB: Will they have four? So how many bags will there be?
331 LS: Five.
332 BB: So there are five bags and four remaining. Then what will they do? Now they know, they have four oranges left.
333 LS: Will they cut them? (Chuckling).
334 BB: (Chuckling). Well ... I'm not ... what was the ques ... so how many bags
335 LS: How many bags ... how many bags did they have to sell?
336 BB: How many full bags?
337 LS: No, they won't cut the oranges.
338 BB: Okay.
339 LS: ... because there's nothing says here "Half oranges". Half of the oranges.
340 BB: So, what is this, half of what? Or a fraction of what?
341 LS: Of ... um ... of the bag.
342 BB: So, it's a fraction of the bag. How do we get to work out the fractions
343 of the bag?
344 LS: I think ... (Thinking.)
345 BB: They've got ... they've got four oranges ... six in a bag. Now the question is
346 if they've got four oranges out of six, what fraction of a bag is that?
347 LS: Four out of six. It's ... it's two thirds.

(Pilot interview 2)

Again she initially thought of subdividing the oranges, rather than subdividing the bag. After some work, including some prompting from me, she was able to change her focus to finding a fraction of a bag. But she was then unable to proceed from this. She seemed unable to decide on which features of the situation were important for solving the problem. When I provided this focus, she was able to complete the solution. In the remainder of the interchange, I attempted to underline the differences between grouping and sharing and to ask about using grouping in teaching. However, a full engagement did not develop about either of these issues – the teacher seemed a little distracted, possibly still grappling with this interpretation.

In the third interview, there were two interchanges involving the grouping interpretation. In the first interchange, she provided playing with marbles as an example context for this interpretation. She started with a discussion of building up a total from a number of groups – asking each group in the class to bring 20 marbles, making a total of 120. But when discussing dividing these marbles into bags, she again showed confusion between sharing and grouping. Initially she responded as if the number of bags to fill had been specified in this problem (a sharing interpretation), rather than on the number of marbles in a bag. After some discussion, she changed her focus, and began to approach this as a grouping situation.

945 LS: How many marbles do we have? We've got ... uh ... one-twenty marbles. Okay. Now ... I want to put these marbles into ... four bags ... or three bags. Whatever, ne?
946 BB: It's interesting, if you're saying you're putting it into three bags ... that's a sharing

situation ...
948 LS: Uhhh.
949 BB: ... isn't it? Because you want one for this ... one for that ... I'm thinking of say, would you be able to say something ... I want to put them into bags so that there are ...
950 LS: Ten bags
951 BB: ... ten in a bag. How many bags will I ... will I need.
952 LS: One-twenty, ne.
953 BB: Okay. Now that's a grouping because I'm not saying share between ...
954 LS: Okay.
955 BB: ... twelve bags. I'm saying I want ten to a bag. That's do an interesting one. If I say, put them eight to a bag. How many bags would I get?

(Pilot interview 3)

Once she took this approach, she was able to find the expected solution (Pilot interview 3: 956–995). And at the conclusion of the interchange, she excitedly acknowledged that she had grasped this interpretation.

In the second interchange, I brought up packing items in bags to sell at a shop. She responded

1038 LS: But they will know that.
1039 BB: Mhmm.
1040 LS: Taking ... still from their homes.
1041 BB: Okay.
1042 LS: Their mothers ... are hawkers there.
1043 BB: Yes.
1044 LS: They buy ... apples and they sell it, ne?

(Pilot interview 3)

Once she had grasped the grouping interpretation, she seemed convinced that this packing context would be relevant and meaningful for her learners and could be used to build rational number understanding using this interpretation. It was interesting that this was the context used to introduce the interpretation and that she was hesitant about the context at that time. This hesitancy seemed to be linked with the difficulty that she was having in understanding the interpretation.

A further interesting point arose in connection with packing items for hawking. She saw this as a good context for the grouping interpretation. However, when I talked about counting the same number into each packet, she disagreed, saying that

1061 LS: You know what they do? They don't count.
1062 BB: Uh huh ...
1063 LS: Because for ... for potatoes or ... or apples ...
1064 BB: Ja.
1065 LS: ... they don't count them.
1066 BB: Mhmm.
1067 LS: Because they are not the same.
1068 BB: Not the same size?
1069 LS: Same size.

1070 BB: Yes.
1071 LS: Ja. They ... they put in a small one ... a big one ... a small one ...
1072 BB: A big one ... (Laughing). So if we said ... I'll take a big count to make so many apples, would it still make sense for the children or would they say, "Ag, that's just a school problem, I'm not really not that interested."
1073 LS: No.
1074 BB: Do you think it would still make sense
1075 LS: It would work.]
(Pilot interview 3)

That is, the approach taken by the children in real life would be much more flexible and approximate. Yet this flexible approach would provide a good grounding for a child's developing understanding of the more rigid structure provide by rational numbers.

3.4.3.4 Measurement

Measurement was a natural and straightforward process to the teacher, as was the idea of choosing appropriate units for measurement. She was also confident that her children would find measurement natural and useful. She stated that they were often faced with measurement situations when buying liquids, such as paraffin:

229 LS: ... something that ... uh ... they ... they always do at home ... they're always sent to buy a bottle of paraffin ...
230 BB: So they're used to buying paraffin and ...
231 LS: ... too much ...
232 BB: Okay.
233 LS: They're sent ... er, "Go and buy ... half a bottle of paraffin ... half a litre
234 BB: Yes ...
235 LS: ... go and buy a full ... a full bottle that is a litre ...
236 BB: Yes ...
237 LS: ... ne? ... So they know those ... measurements.
(Pilot interview 3)

There were a number of different units in which paraffin could be bought, so comparing different units and choosing appropriate units were commonly done by the children in this context. Another familiar context is buying Coca-Cola, and she shared a story of when a child had brought a teachers attention to the use of fractions in this context (Pilot interview 3: 386–410).

It was evident that the teacher had taught measurement in her class, especially with regard to measuring volume, building on a number of different real contexts in her children's experience. By using a number of different containers of different volumes that the children had brought to class, she had been able to help them to develop a good understanding of the relationships between the different measurements. Some of this work was directly relevant to the measurement interpretation of rational numbers. For example, investigating the quantity obtained by repeatedly pouring quarters of a litre into a bottle, and so learning to visually recognize a quarter, a half and three quarters of a bottle:

257 LS: So that I will get from them ... to get an empty bottle ...
258 BB: Uh huh ... and then empty bottles of different sizes.... an empty half bottle or an empty quarter bottle ...
259 LS: Mmm. In fact ... uh ... uh ... a whole bottle first ...
260 BB: Yes.
261 LS: ... ne? ... where they can see now ... how much does a quarter shows in this bottle?
262 BB: Mhmm.
263 LS: How much does a half shows ...
264 BB: Ja.
265 LS: If I put ...the ... the ... two quarters together ... and put them in this bottle ... what does it show?
266 BB: Okay.
267 LS: If I add another quarter ...
268 BB: Mmmm.
269 LS: ... what does it show now?
270 BB: Yes.
271 LS: Now how many quarters are here?
272 BB: ... you got, ja. And that's ...
273 LS: That's ... so definitely a child would say, ..."I first poured ... a quarter ..."
274 BB: Mhmm.
275 LS: "... and it shows this in this bottle ..."
276 BB: Yes.
277 LS: "... and then again I ... I poured out this quarter ... I had a half ..."
278 BB: Yes.
279 LS: "... I had poured the half into the same bottle ... "
280 BB: Uh huh ...
281 LS: "... and the mark is here. It shows like this, ne?"

(Pilot interview 3)

One point of interest is that the focus of her work with the children in measurement was on the product of measurement. That is, through what they did in her class, the children had developed basic skills in the process of measurement, with the aim of producing measurements of an acceptable accuracy. These skills included those of choosing appropriate units and comparing measurements with different units – fundamental to the measurement interpretation of rational numbers. But these skills were largely implicit, developed through practice, under her guidance and feedback. To be able to use this in developing rational number understanding, she would need to build on this foundation to draw out a more explicit understanding.

3.4.3.5 Ratio

The teacher appeared to find difficulty with making sense of the problem situation that was used to introduce this interpretation. In particular, she was not able to explicitly formulate the requirements of the problem.

502 LS: (Thinking). ... Okay. Now ... Are we looking for how they can ... get into the solution? The classroom? What?
504 BB: Well ... how would you say ...
505 LS: (Thinking).
506 BB: What's the important thing you're doing with this?
507 LS: If ... if this says the class has 3 boys for every 4 girls and there are sixteen girls in the class ...
508 BB: Mhmm ...
509 LS: ... I think that what ... uh ... they're supposed to do first is to get ... uh ... To take out the ... (chuckles). I don't know!
511 BB: Difficult, isn't it? Look at ...
512 LS: How about taking ...
513 BB: Mhmm ...
514 LS: ... uh ... (thinking) I uh ... don't get about this one.

(Pilot interview 2)

I responded to this by trying to bring out the process of comparing the numbers of boys and girls, by relating this to the comparison of water levels earlier in the interview. However, she rejected this analogy quite strongly, arguing that it would be much more difficult to imagine the problem situation in a way that made the required quantities evident (Pilot interview 2: 519–538). Eventually, I began to model the process of proportional comparison as one of aligning groups of four girls with groups of three boys. This model made sense to her and she was able to calculate the number of boys.

548 BB: This is confusing. It is confusing this. It's interesting because this is part of what I want to consider. I included this in order to see ... what I'm trying to get here ... what I'm trying to work with is that you work with a proportional comparison ... and you're comparing ... three boys ... four girls. So every time you see four girls ... how many boys do you see?
549 LS: Three ...
550 BB: So when you see eight girls ...
551 LS: Ohhhh!
552 BB: ... how many girls do you see?
553 LS: I see now! Uhuh! (Chuckling). Now I see.
554 BB: It's quite complicated, hey?
555 LS: If the ... if the ... there're sixteen girls ...
556 BB: Mhmm ...
557 LS: ... and then four ... four ... four girls ... there are three boys ...
558 BB: Mhmm ...
559 LS: So ... the boys ... they are twelve.

(Pilot interview 2)

It is interesting that once she had accepted the modelled process of aligning groups, she was able to make sense of the problem situation. This response was quite similar to what had occurred when working with the grouping interpretation. Another point of interest was that she calculated the number of boys mentally, by a direct multiplication, rather than by counting aligned groups. This interpretive jump was immediate, so much so that she found it difficult to describe how multiplication related to counting repeated groups.

559 LS: So ... the boys ... they are twelve.
560 BB: Tell me how you get that? (Chuckling together). Sounds good.
561 LS: You multiply three by four.
562 BB: Okay. Why by four? Sixteen is not a good number to choose. If ... let's
563 change the problem a little bit. ... so instead of sixteen shall we take twenty
564 girls.
565 LS: Mhmm. And then?
566 BB: How many boys will there be now?
567 LS: (Clicking tongue). I get three boys for every four girls ... (thinking) ...
568 fifteen.
569 BB: Fifteen. And how did you work that out?
570 LS: (Clicking tongue.) Um ... from this twenty ... from these twenty girls ...
571 BB: Mhmm ...
572 LS: ... you take out ... how many? ... You take out ... four girls ... four girls ...
573 four girls ... four girls ...
574 BB: Okay.
575 LS: ... and you take ... five boys.
576 BB: Yes. Okay.
577 LS: Mmm.
578 BB: Okay.
579 LS: So ... which one?
580 BB: Five groups ... five groups of four girls.
581 LS: Mmmm.
582 BB: For each group of girls ...
583 LS: For each group of girls ... you get ... three boys.
584 BB: Okay.
585 LS: So ... they are five groups ...
586 BB: Mhmm ...
587 LS: ... so these groups ... I mean this group ...
588 BB: Mhmmm ...
589 LS: ... they are three boys ... in this group, three boys ... in this group, three boys ...
591 BB: Okay. Yes.
592 LS: So the groups are five.
593 BB: Mhmm ...
594 LS: It's five times ... three.
595 BB: Yes.
596 LS: That it's fifteen.

(Pilot interview 2)

Here the production and application of an appropriate solution process was quick, appearing almost automatic, while articulating the relationship between the process and the situation was difficult and took concentrated effort. Finally, in this interchange, the teacher found the fact that rational numbers could be used to directly calculate proportional comparisons enlightening and interesting. However, she thought that even at secondary school level, children would only be able to cope with simple problems of this form (Pilot interview 2: 597–624).

The observation that the ratio interpretation concerns the comparison of two quantities, rather than a physical change of one quantity into another, arose in the discussion of the operator interpretation. To her, the fact that there was no active change was one of the aspects that had made the ratio problem difficult to understand. To help overcome this difficulty, she suggested that comparing two separate classes – one of boys and one of girls – could make the problem more tractable (Pilot interview 2: 746–770). The teacher was of the opinion that the focus on comparison, rather than change was what made this interpretation so difficult to understand.

Her reformulation of the problem situation could be interpreted as supporting this by re-introducing the idea of change, although in a weaker and more abstract form. In this case it would be a change in the actions of the person solving the problem – a change in the focus of counting (who is being counted). However, this change of focus was also a necessary feature of the initial formulation, although the two were more difficult to separate. Thus the alternative formulation may be viewed as merely providing a more definite, physical, separation of the two quantities – providing no further insight about the importance of change.

The teachers difficulty with the ratio interpretation was again demonstrated when she was asked in the third interview to provide realistic contexts for this interpretation. She started by talking about children buying sweets at a shop and then sharing them between themselves. All through this interchange, she appeared hesitant, and a little confused. At the end, she was happy to admit that this situation fitted better with the operator and sharing interpretations. When rounding off the interview, though, she showed that she had come to grips with this interpretation by suggesting the context of comparing weights of different packets, by balancing in your hands, or using a balance scale.

1194 LS: How about ... um ... you tried your separation ... by having those... small scales
1195 BB: Explain?
1196 LS: Uhhh ... to see I don't know what's the name ...
1197 BB: Just try it and let's see.
1198 LS: ... the balancing ...
1199 BB: The balancing is nice ... ja ...
1200 LS: There are scales at TAP. But ... uh ... I said to them, "I want to borrow them ..."
1201 BB: Ja.
1202 LS: "... though I don't know how to use them actually."
1203 BB: Ja.
1204 LS: To have ... uh ... they ... they ... they know how to ... they know that ... uh ... this thing is heavier than this one.
1205 BB: Ja.
1206 LS: They can ... they can even use ... sand for that matter, ne?
1207 BB: Mhmm.

1208 LS: If they put in a ... take a packet ... and compare ... this package to this one by just weighing them in their hands.
1209 BB: Yes.
1210 LS: And tell ... which one is heavier than the other.
1211 BB: Okay.
1212 LS: Is that not ratio?
(Pilot interview 3)

3.4.3.6 Operator

When given the problem situation introducing this interpretation, the teacher stated that grouping would be an important part of the analysis. This was indeed the case, because the total number of prickly pears would need to be divided into groups of the size given to describe the transaction. However, the key feature of this interpretation was the operation which exchanged each group of prickly pears with a number of pieces of chocolate. In response to my attempt to draw attention to this change, the teacher responded that it was "a swopping". After this insight, she was very confident with this interpretation.

In fact, most of the discussion flowing from this interpretation was not focussed on 'swopping' at all. Instead, she is more interested in the effect of the more active, operator type approach, on her understanding of the ratio interpretation. This discussion is analysed in the previous subsection. When asked, she was definite that her children would find the operator interpretation easier to understand than ratio.

777 BB: So which is the easier? Do you think it might ... it's actually easier thinking
778 with the operation than it is with the proportional comparison where you're
779 actually doing the changing is easy.
780 LS: Yes. This is much easier.
781 BB: This is doing the operation ... yes.
782 LS: And the operation and why I think, I have observed
783 BB: Mhmm ...
784 LS: with ... uh ... with learners, ne?
785 BB: Mhmm.
786 LS: If you ... if you give them examples ....
787 BB: Yes ...
788 LS: ... of ... something ... of their interest ...
789 BB: Yes ...
790 LS: ... they quickly grasp it.
791 BB: That's right. Do you think this is closer to their interest as well?
792 LS: Prickly ... prickly pear, chocolates.
793 BB: Okay.
794 LS: You know?
795 BB: And it's swopping.
796 LS: (Laughing). Yes.
797 BB: It's like a shop. It's easier.
798 LS: Yes.

799 BB: So this one is difficult because it's not really their interest.
800 LS: Mmmm.
(Pilot interview 2)

In her opinion, it is easier because it is "something of their interest", rather than because it is active.

Again, it was mentioned in the previous subsection that her initial attempt at providing a context for the ratio interpretation, was actually a context for the operator interpretation, slightly confused by sharing. This was brought to her attention and she was happy to acknowledge this. She also appeared happy with the contexts of swopping and shopping and was confident that her children would find this interpretation natural and easy in these contexts (Pilot interview 3: 214–221).

3.5 Reflection on the research process — developing my understanding

My perspective on the research at the beginning of this cycle, is described in the "initial engagement" and "literature review" sections (3.1 and 3.2). The interviews and their subsequent analysis yielded a number of interesting insights about this grounded approach to teaching rational numbers and about the specific interpretations identified. In general, the approach seemed to show promise for laying a good foundation for the development of rational number understanding.

But, the interviews also raised an important concern. After the pilot interviews, I was left with two intuitive impressions which were quite strong and yet quite difficult to tie down. During the analysis of the interviews, I was able to define these impressions more clearly. They will be briefly discussed in subsections 3.5.1 and 3.5.2. Viewed together, these impressions raised the concern that this approach on its own did not provide a natural path to continue the learning of rational numbers. This was my major concern when moving into the second stage of the project.

In addition, in the interviews and following analysis, a number of other intuitive impressions were formed that I was not able to formulate in any detail at this stage of the research. At the culmination of the project, I was able to identify these impressions as laying the groundwork for insights that I attained later in the project. In subsection 3.5.3, these impressions will be identified and described taking the perspective of my understanding at the end of the project.

3.5.1 The first impression and associated tension

The first impression concerned the teaching and learning process, involving the identification and development of rational number concepts arising from the interpretations. It seemed as if rational number concepts were presented in such a way that a developmental pathway which promised to yield significant benefits to the child, was not opened up. Because of this, children would not be drawn to explore such a pathway in order to obtain these benefits, and so the resulting rational number learning would be more imposed than appropriated.

During the course of the analysis, I was able to identify an important dynamic as the possible source of this impression. It was that the focus of the mathematical development in much of the rational number teaching discussed, was largely on the identification and naming of fractions. This was described at the end of section 3.4.2. Thus, even though arising through significant actions, the representation was developed as an end in itself, and not as a means to achieve an end which would be significant for the children. Because the children would not see any significant benefit for their activities outside of school, there would be little intrinsic motivation for them to further develop their understanding of rational numbers.

Following from this, I identified a tension between description and action as an important theme for the remainder of the research project. More precisely, this tension could be seen as between:

- Developing and viewing a representation as an end in itself, and
- Developing and using a representation as a means towards the attainment of a significant goal in some activity.

3.5.2 The second impression

The second impression was that, in general, the interpretations provided by the subconstructs were too separate. Having worked individually with each interpretation, there was little cause to unify these interpretations and form an overarching rational number construct.

This impression was of a distinct separation between the different interpretations provided by the constructs. This separation was such that it became difficult to find natural commonalities which could form an intuitive basis for a learning progression to relate these interpretations and so form a unified understanding of rational numbers. The separation was partly due to the intrinsic differences between interpretations that were explored in the interviews. But it was also due to the interview process, where an important emphasis was on understanding the difference between the interpretations.

Some similarities and relationships were found between interpretations. One example was when Lungiswa found that the interpretation and solution of a ratio comparison problem was simplified by recasting it in a form that involved a conversion from one of the comparison quantities to the other. In effect, this involved recasting the ratio comparison problem as an operator problem. This was possible because the interpretations were similar in terms of the relationships between the two quantities involved. The difference was that the ratio interpretation was descriptive, comparing two quantities that co-existed, while the operator interpretation was active, converting one quantity into the other — an instance of the tension between description and action.

Another example was Lungiswa's use of the part-whole interpretation to describe a transient state in sharing situations. Here the part-whole interpretation allowed an easier representation of rational numbers describing the sharing situation, through focussing on the state after subdivision but before allocation, when the whole and the parts were coexistent and could be directly compared. Here the relationship was not one of structural similarity. Rather the part-whole interpretation was relevant to one specific snapshot view taken from the full sharing process. This view was particularly useful in for formulating a mathematical (fraction) representation of the quantities created through the sharing. Even though it did not necessarily yield the simplest method of actually carrying out the sharing. Again here we see the tension between description and action.

These comparisons are interesting and useful for developing a better understanding of the relationships between some interpretations (sharing with part-whole and ratio with operator). But relationships between other pairs were not evident in the interviews and would need to be further explored. The relationships found, were evident at a more formal level of analysis. This raised the question of whether such a level of analysis would provide a suitable basis for developing a unified view of rational numbers from informal experience of the different interpretations. Or whether a different level of analysis would be needed to yield links suitable for the early, grounded teaching of rational numbers. This was a guiding question in my literature review of the next cycle.

3.5.3 Impressions not clearly formulated in the pilot cycle

At this stage, I was able to identify a tension between action and description (as described in section 3.5.1 above). But precisely what was meant by the terms 'action' and 'description' was as yet unclear, as were the effects of each and the way they interacted.

I did have the impression that action in some way contributed to the significance of the learning and the degree to which the learning activities constituted a realistic grounding for the children. In this case, my intuitive impression was informed and developed by a subsequent theoretical investigation. That is, my reading for the next cycle brought more clarity to this issue in the form of the concepts of 'action' and 'activity' from activity theory. The empirical investigation (in the form of the interviews and analysis of the second and third cycles) then led me to perform a deeper analysis of these concepts and adapt them to better suit the understanding developed through this research.

Another impression that was relevant to this idea of action was formed in response to the teacher's investigation of the ratio problem. She had initially found it rather difficult to decide on an effective approach to this problem. When I demonstrated that a useful regularity in this situation is that every four girls could be associated with three boys, she was immediately able to calculate the required number of boys in the class. This gave some substance to my initial suspicion of the importance of finding relevant structure in the situation and then using it to attain one's goal in the situation. Alignment of ones actions with the structure of the situation was an important feature of the concepts of 'actions' and 'operations' used later in the research.

In a number of instances in her interviews, Lungiswa had talked about how she worked to enable her children to correctly name fractions and I gained the impression that much of her teaching aimed more at correctly naming fractions than at skilful use of rational number concepts to solve problems. At this stage of the research, I was viewing 'description' mainly in terms of naming rational numbers as fractions, using words or fraction notation. As the research progressed, it became clear to me that 'description' should really cover all aspects of representing rational numbers. That is, that it would be better to formulate the tension in terms of action versus representation. For the important influence of representation on learning rational concepts and using them to solve grounded problems became clearer as the research progressed.

Finally, in two places, I was struck by the flexible manner in which concepts and processes that I viewed as mathematical, were approached in grounded situations. The first indication of this was her statement in the discussion about sharing that children generally used the term 'quarter' to describe any part smaller than a half. Later, when discussing the packing of prickly pairs, she stated that she didn't expect hawkers to count the number in each pack, because the fruit would not all have the same size. From this I gained the impression that when working in a grounded situation people would often use mathematical concepts in a very flexible manner, even in ways that contradicted the strict mathematical usage of the concepts. However, this mathematical 'abuse' was at times more apt for the situation than a correct usage — for example, packing different numbers of fruit of different sizes was more fair than keeping to a strictly correct count.

Later in the research I found the concept of 'coordination' to be very useful in formalizing this idea of 'flexibility in application'. Also, I came to acknowledge the importance of 'idealization' in the strict definition of mathematical concepts. This yielded the insight that mathematical idealization had both positive effects (greatly increased precision) and negative (lack of fit) when working with grounded situations.

Chapter 4 – Teacher Cycle Preparation

4.1 Introduction

My most pressing concern during the preparation for the teacher cycle was the apparent separation between the subconstructs and the possibility of finding an approach to teaching that encourages finding links between them. This concern interrupted the research, as it had major implications for the approach of using the rational number sub-constructs as realistic interpretations to aid planning and teaching. It may be formulated as follows:
The differences between the rational number interpretations offered by the different sub-constructs, served to separate the interpretations and the corresponding informal ways of thinking about them. Yet, it is an underlying assumption of the approach to teaching, that each of these informal ways of thought should contribute in some way to the child's developing understanding of a coherent rational number concept. How is it possible to reconcile the separateness of the interpretations with the development of this final coherent understanding?

Because these are all rational number sub-constructs, there must be some interrelationships between them. Are these purely at the abstract, formal level? If this is the case, working with multiple interpretations could be a problem for introductory teaching. Or are there other similarities and interrelationships which could be useful for introductory teaching?

To investigate this issue and gain some clarity on how to proceed, I returned to the literature to gain some insight on a possible way of proceeding.

4.2 Reflection on my personal experience

The formulation of this concern and the need for direction in my reading to address the concern, led me to reflect on the conceptual framework from which I was approaching the research.

One important motivating factor for the research was my general impression that many people viewed mathematics as a process of manipulating abstract and meaningless, symbols strictly according to fixed rules and algorithms. I had gained this impression from my own experience of teaching mathematics at tertiary level, and of working with both mathematics teachers and children at primary and secondary level. In my experience, this view appeared to restrict peoples' ability to learn mathematics and to think mathematically in a creative and productive fashion. I had found it helpful to encourage people instead to take a view of mathematics as thinking meaningfully about patterns that arise in the world and that we use practically in our life in the world. Taking this view would involve seeing mathematical objects as real, though abstract mental objects, meaningfully related to these patterns and represented by corresponding symbols. Although this view was informal and not carefully articulated, I had found that people who took this view became more confident, flexible and powerful mathematical thinkers. In order to encourage people to move to this view, my approach to teaching changed over time to include a far greater emphasis on the informal grounding of mathematical concepts. Doing this within the sub-domain of rational number learning was what led to the development of this research project.

The need for a more detailed understanding of how a coherent understanding of rational numbers may develop from the different grounding interpretations made it important to explore and articulate these informal impressions.

Note that the abstract, or symbolic approach mentioned above, could be seen as corresponding to the descriptive pole of the tension between description and action, identified in the previous chapter. However, thinking about, or using patterns does not feature in the specification of the active pole of the tension. But, to fit with this reflective intuition, I decided to entertain the possibility that, from the perspective of learning mathematics, an important feature of such actions to achieve a significant goal, would be to act in a manner that follows, or exploits a pattern in the situation.

4.3 Theoretical review

An initial insight into significance was provided by the work by Case (1985) and Resnick and Singer (1993). They explored the origin of any numbers in proto-quantitative reasoning. In particular, they discussed the importance of comparison in the development of numbers from such reasoning. I found this particularly relevant, because in situations involving the allocation or distribution of resources, questions of comparison such as "Who has more?" or "Do we have the same?" often yield significance to the quantities obtained. Freudenthal, in his study of the didactical phenomenology of fractions (1983), provides these and a number of other examples of the concrete function of fractions for making significant comparisons of objects and magnitudes. Thus, including questions of comparison in the problem situations used in this research could contribute to the significance of the actions, as well as to the further development of rational number understanding.

Insight into the exploitation of patterns within a situation in order to carry out significant actions was provided by Gibson's (1986) work on affordances offered by the environment, as well as the ideas of situated cognition (Brown, Collins and Duguid, 1989; Clancey, 1997) and situated learning (Lave and Wenger, 1991).

Gibson (1986) introduced the concept of affordances in his research on visual perception. In his view, people perceived affordances, rather than properties of the environment. An affordance results from possible interactions between the person and the environment, rather than being due solely to objective properties of the environment. It is a potential for action that arises from the structuring of the environment, that could be taken advantage of by the person. It is this emphasis on the interaction between person and environment that is developed in the work on situated cognition.

The concepts of situated cognition were introduced in education research by Brown, Collins and Duguid (1989), who challenged the separation between knowing and doing which was common in education at the time. In their view, concepts are not self contained elements of knowledge which could be transmitted in abstract form through teaching. Rather, they are similar to tools, being developed through activity and becoming progressively better understood through use. They stress the importance of authentic action for learning, stating that the concepts employed and developed in the action, point to and rely on certain recurrent features of the situation, in order to become fully realized. This is seen as occurring through indexical representations which are related to the concept, and effect the individual's perception of the situation by indicating the specific features.

The precise details about indexical representations were not clearly given in this paper. In particular, details about the how these representations were formed and accessed were not spelt out, which left me uncertain about the implications for teaching and learning. Further insight into the developing relation between concept and situation occurring in the course of action, is provided by Clancey (1997). He sees this as a process of coordination, involving feedback and efficient response at a fine level of detail. To achieve such an efficient response within the time constraints imposed by the situation on the action, this coordination would need to be achieved automatically rather than through conscious deliberation.

Learning through authentic activity is seen by Brown et al. as a strongly social activity, occurring through cognitive apprenticeship as the child becomes encultured within a community of practice. This idea of learning as an apprenticeship within a community of practice is discussed extensively by Lave and Wenger (1991), who describe the process as one of legitimate peripheral participation, leading to a more central role within the community.

The above work on situated cognition explicitly builds on the socio-cultural approach to psychology originating with Vygotsky (1978). In particular, the idea that the higher psychological functions of an individual first appear within social activity on the interpersonal plane and are then appropriated by the individual in the process of interiorization. Such social activities are generally active and aimed at achieving some meaningful goal for the social group. Thus, the aim of cognition is also seen by this school as adaptive action, although here such adaptive action has a strongly social emphasis. A further important idea propagated by this school is that few human actions are direct, most actions, be they psychological, social or physical, are mediated by some form of tool. Language is held to be the most important mediating tool on the psychological plane. These ideas provide a conceptual foundation for much of the work in this research.

Note that Vygotsky held that thought (cognition) is prior to language, but that the presence of language as a mediating tool for thought brought a richness and complexity to thought that made possible the full development of the higher psychological functions of a person. Both of these points are important for this research. For the intuitive mathematical understanding developed through acting in a situation may be largely pre-linguistic and so not accessible to the individual for linguistic reflection. But the development of mathematical representations, be they words or mathematical symbols, allows deeper understanding and greater and more flexible control of situations.

The later development of the socio-cultural school led the concepts of activity theory (Engestrom, 1999). Following these references, I found the basic analytical structure of activity theory, developed by Leontev (1978) particularly relevant to both of the impressions identified in the pilot cycle.

In the first place, this structure yielded insight into significance, through its definition of the theoretical construct of 'activity' — an active process carried out to satisfy some motivating need or desire, be it physical, psychological or social. Generally this motivation imbues the activity with significance for the persons involved. I found the relating of significance to motivation particularly promising for this research.

Activities were not the only elements of Leontev's framework. He also formulated the construct of an 'action' — an active process consciously directed towards achieving a practical purpose, or goal. An action may not be invested with personal significance, for achieving the goal may not result in satisfying a motivating desire. But it is meaningful, in that whatever is done to constitute the action is aimed at, and contributes to, achieving the desired goal. In this sense, actions provided insight into a possible relationship between patterns and meaningful learning. For the effectiveness of an action could in some ways be due to the degree to which it follows, or exploits patterns arising in the situation. One important point to note is that actions are deliberative acts, consciously formulated and guided to achieve the desired goal.

The final element in this framework is the construct of an 'operation' — an active process, in which a person uses a tool to achieve a desired end. Certain instrumental conditions need to be satisfied for the operation to be successfully carried out. Acting in a manner to satisfy the instrumental conditions in order to use a tool effectively, is generally an automatic, rather than a deliberative process, involving a high degree of coordination. Based on this and the idea of coordination from situated cognition, I extended the notion of an operation to include any process which involved interacting with the environment in such a coordinated fashion. Through this process of coordination during an operation, a person intuitively learns the structure of the environment which makes this operation possible.

Applying these concepts to initial rational number learning, the interpretations provided by the sub-constructs could be seen as corresponding to either activities or actions. This raised the question of whether operations, encapsulating the idea of coordination, could also prove to be a useful tool for the analysis of initial rational number learning. I thought it possible that such operations may provide the linkages between interpretations, through the involvement of common operations in different interpretations. To explore this possibility, I returned to the literature on rational number learning, in order to identify possible candidates for operations.

4.4 Identifying possible operations

In this section I discuss the literature and describe my identification of possible operations, as well as my understanding of these operations before the interviews of the cycle. Building on this understanding, I designed a number of tasks to explore in the interviews, which could yield examples of each operation.

4.4.1 Unitizing

The operation of unitizing involves a person's identification of an entity, or structure, as a single unit within a situation. This concept was formalized by Steffe (1986) and was shown to be important for the development of young children's early counting strategies (Steffe and Cobb, 1988) and understanding of arithmetic (Steffe, Cobb and von Glasersveld, 1988). It has also been used to analyse children's understanding of place value representations of whole numbers (Nunes, 1997), and for multiplicative whole number operations, including ratio and proportion (Steffe, 1992a and Lamon, 1993).

Mack (1990), in her work on learning initial fraction concepts, suggested that enabling children to develop a flexible intuitive understanding of units and compound units, could result in a broad understanding of rational numbers. Unitizing was also used by Behr et al. (1993) to provide a more detailed conceptual analysis of the different rational number subconstructs. In the overview of this volume, it was seen by Carpenter, Fennema and Romberg (1993) as important for a full and flexible understanding of rational numbers. In particular, the ability to work with and relate a number of different conceptual units could be seen as fundamental to an understanding of rational numbers, forming the basis for the representation of these numbers as fractions.

For rational number understanding, identification of the unit is seldom the final point of the action. In fact, the identified units normally form the basis for further action within the situation. One common example is measuring, or counting, where the identified units or wholes are what is counted — the units of measurement. Another example is subdividing, when the person first has to identify the complete unit which is to be subdivided.

The unit being identified may correspond to some real and discrete physical object, as in the case of a whole loaf of bread, or a half loaf of bread which has been cut from the original. It may also be a collection of discrete objects which has been physically grouped to form a whole, as is the case of a packet of apples. Finally, it may be a mental construction in a situation, which allows the performance of the desired action. Examples of this are visually counting a collection of objects in two's, describing a part as ¾ of an object by viewing it as consisting of 3 quarter units, or even by identifying a part as ⅛ by viewing it as half of a unit of one quarter.

Steffe and Tzur (1994) report on research which aimed to develop children's understanding of fraction families by combining their units-coordinating scheme, with a unit-fraction scheme. Both of these schemes involve unitizing in the identification and working with multiple units. The units-coordinating scheme involves coordinating two whole number units (discrete collections of given sizes). The unit-fraction scheme involves relating a 'whole' unit with a 'part', unit so that a specific number of copies of the part unit make up the whole. Due to this relation, the part unit can be described as a unit-fraction of the whole.

Thus, I expected that the operation of unitizing would be important for rational number understanding in two ways. Firstly, identified units would form basic elements for further operations, which together would constitute the rational number interpretations. Secondly, rational numbers would often emerge from the coordination and comparison of multiple units.

4.4.2 Iteration

Counting and the number sequence is accepted as fundamental for children's development of number understanding and has been studied extensively (see for example Ginsburg, 1977; Gelman and Gallistel, 1978; Fuson, 1988; and Steffe, 1992b). Researchers show general agreement about the sequence of development of these skills. An important stage of development is counting synchronously, where the child is able to count and point to objects at the same time, forming a one-to-one relationship between the objects and the counting words. This action becomes progressively compressed and abstracted in the further development of the skill.

The counting process is seen by Steffe and Olive (1990) as being important in the development of children's understanding of rational number as numbers, because it entails the construction of fractional schemes through accommodations of children's existing numerical counting schemes. They formulated this as a hypothesis, called the reorganization hypothesis. Olive and Steffe (2001) describe how a child was able to use a unit fraction of a whole as the counting unit in a generalized number sequence to describe quantities using non-unit fractions — corresponding to a use of the measurement interpretation of rational numbers. For the basic process of measurement using a given object as a unit can be viewed as a generalized sequential counting process, where it is the number of iterations of the chosen unit that are counted. In the measurement interpretation of rational numbers, the basic unit would be a chosen unit fraction and the resulting sequential counting would correspond to counting in unit fractions.

Accordingly, I identified this process of iteration of a generalized unit, as a possible operation for developing rational number understanding.

4.4.3 Equal subdivision

Partitioning of one, or a collection of objects into a number of equal parts, in order to achieve fair sharing, is described by Streefland (1991) and Kieren (1993) as fundamental in rational number learning. In particular, they advocate sharing a number of objects which is not evenly divided by the number of people sharing. This will involve partitioning of some, or all of these objects, in possibly different ways. In such situations, sharing and partitioning are not equivalent, for the sharing may involve multiple partitions, as well as distributing parts between the different people.

Partitioning also arises in many other studies of rational number learning. According to Behr, et al. (1983), partitioning of a single quantity, either continuous or discrete, also forms the basis of the part-whole subconstruct. Freudenthal (1983), in his didactical phenomenology of fractions, describes one view of fractions as fracturers — again corresponding to the act of partitioning. Case (1985) and Confrey (1994) described how splitting operations could be used to develop a flexile understanding of rational numbers. This would start with the basic splitting operation of halving and then proceed to more complex splittings. Each splitting involves partitioning an object into a number of equal parts. In Mack's research (1990), children based their informal knowledge on experiences of partitioning different units. Finally, Steffe (2001) sees the operation of iteration as allowing the reconstitution of the partitioning of an object, resulting in the formation of an equi-partitioning scheme.

This prevalence of partitioning, as well as the fact that partitioning appears as a component act in a number of possible sub-constructs, led me to identify it as a possible operation for inclusion in this research. To simplify the terminology used in the interviews, I decided to use the term 'equal subdivision', rather than 'partitioning'.

4.4.4 Unequal subdivision

I did not identify this operation in the literature review at the start of the cycle. But, during the interview with the pilot teacher involving the operations, and in my teaching subsequent to that, I had used the problem of dividing a prize between first and second places in order to explore peoples' ability to divide a quantity into two common, but unequal fractional parts (for example, ¼ and ¾).

In particular, in response to the approach to teaching taken by Moss and Case (1999), I was interested in the skill of performing, or recognizing such a subdivision. Their work involved teaching children to use repeated halving to construct fractional portions, which when viewed individually (themselves and their complement) formed unequal parts of the whole. The resulting unequal subdivision is the end result of a combination of some of the resulting parts. According to Moss and Case, learning to delineate and recognize fractions of a length unit which arose from such repeated halving, from a visual comparison of the part with the whole, formed an important general structure contributing to a powerful and flexible understanding of fractions as quantities.

The unequal subdivision operator could be formulated as reversing this sequence in the formation of the parts, starting intuitively with the unequal subdivision (chosen to intuitively fit with some comparative measure) and then using repeated subdivisions, or equal subdivision and 'collection' to accurately quantify this subdivision.

Recognizing that the problem of dividing a prize was a natural setting for an 'unequal subdivision' operation, I wondered whether other settings existed, in which creating an unequal subdivision of a quantity was a reasonable and natural thing to do. If this was the case, such settings could engender an intuitive understanding of unequal subdivision, which could provide an informal grounding for the proto-quantitative schema involved in benchmarking and comparison of rational numbers. A number of situations where a fair subdivision of a quantity resulted in unequal sharing came to mind. One such was the sharing of food in a family with a young child.

Following on this thinking, I decided to include the operation of unequal subdivision in my investigation with the other two teachers, in order to determine whether this operation would indeed occur naturally in realistic settings and whether it could result in the development of useful intuitions for rational number understanding.

4.4.5 Scaling

A number of example problems which involve interpreting pictures or drawings of objects at different scales are given by people such as Lamon (1999) and Freudenthal (1983). These drawings are given by Lamon as means of developing the ratio and operator sub-constructs in rational number learning. Freudenthal, writing before the formalization of the rational number sub-constructs, provides these drawings as visual representations of magnitude values, in order to carry out ratio comparisons, or ratio operators.

To interpret these drawings effectively, it is necessary that the children be able to visually compare the two scales, or scale the one drawing relative to the other. Freudenthal develops the concept of 'norming' in his book (1983), to describe comparisons such as "reconceptualizing a system in relation to some fixed unit or standard". Such norming could be carried out by measuring the drawings and using these numbers to quantify the relationship, or by visually comparing the two drawings to obtain a qualitative description of the comparison.

Corresponding to this qualitative visual comparison, Resnick and Singer (1993) describe a protoquantitative relation of fittingness. This is apparent very early in life, for example, infants are able to recognize the size of an object which will fit into a hole. With the development of language this relation is evidenced with the use of terms such as "too big", "too small" and "just right".

A similar visual judgement was an important factor in the teaching of Moss and Case (1999). Here children learned to visually recognize parts of a unit length as corresponding to fractional pieces that had been created through repeated subdivision and possible recombination.

I included an operator termed 'scaling' to describe the process of performing such an immediate visual comparison.

4.4.6 Coordinated counting and exchanging groups

Proportional reasoning is an important skill underlying both the ratio and the operator interpretations. In order to identify possible operations underlying the skill, I consulted the literature on proportional reasoning, in particular as related to rational numbers. The operations related to proportional reasoning that I initially identified for exploration in the teacher interviews were 'coordinated counting', and 'exchanging'. My understanding of these operations went through a considerable development in the course of this interview cycle. This subsection describes my conceptions of these operations at the beginning of these interviews.

A complete abstract understanding of ratio and proportion requires the development of the ability to form intensive comparisons between two extensive quantities in different measure spaces Vergnaud (1983). These intensive comparisons form the ratio interpretation of rational numbers.

An investigation of children's developing understanding of ratio and proportional reasoning is discussed by Inhelder and Piaget (1958). This investigation was based on a number of tasks, including the classical 'balance beam' task, where children were asked to add weights to a balance beam to achieve balance. Initially. children only paid attention to a single feature of the arrangement. Later, they used a 'qualitative proportion' strategy, which involved compensating for an increase in one quantity by a decrease in the other. Formal proportional reasoning was reached only when children quantified the amounts and used an invariant proportion. This study contributed to Piaget's thesis of general intellectual development, through concrete reasoning to formal reasoning.

In later studies by a number of researchers (Karplus and Peterson, 1970; Piaget, et al. 1977; Noelting, 1980a, 1980b, Resnick and Singer, 1993), the emphasis shifted from general intellectual development, to the development of proportional understanding. These researchers further analysed and subdivided task responses, formulating a more detailed developmental path for proportional reasoning. One such sequence is described by Lesh et al. (1988):

i Initially, only a single feature is attended to.

ii Then qualitative proportion strategies are used.

iii In the progression to quantified strategies, the first step tends to be to use constant differences.

iv Multiplicative reasoning is initially based on recognizing patterns in two sequences of quantities. A multiplicative strategy may be used to relate terms within each sequence. But a reversible second order relationship between the relations in the different sequences is not employed.

v A final, reversible strategy using a second order relationship and the 'law of proportions'.

Singer et al. (1997) describe the strategies in part (ii) of this sequence as 'protoquantitive reasoning'. They provide a detailed discussion of approaches that children take when working with linked sequences of quantities. Children may work additively with each sequence, but link their additive operations between the two sequences in order to arrive at their conclusions. They term this 'protoratio reasoning'. To master proportional reasoning, the children would need to move to using multiplicative relationships between the quantities in the sequences. In the literature on proportional reasoning, this is termed scalar reasoning (Vergnaud, 1988; Schliemann and Carraher, 1992) as it involves multiplicative relationships between two extensive quantities in the same measure space. In addition to this, children would need to relate corresponding terms in the different sequences, to form intensive quantities — multiplicative comparisons between two extensive quantities in different measure spaces. These intensive comparisons provide the ratio interpretation of rational numbers.

Although many paths place the additive, or protoratio, strategies as occurring before multiplicative, Karplus et al. (1975) suggest that this may not form an invariant sequence, but may be due to instruction. This was supported by Hart (1988), who describes the result of a short teaching sequence designed to develop multiplicative reasoning strategies. Furthermore, although Singer (1997) asserts that scalar reasoning strategies are used before functional strategies, Hart (1988) mentions a number of studies which yield the opposite result.

Whether a proratio, scalar, or multiplicative strategy is used, each of these strategies involves relating the changing quantities in each sequence. It is this process of coordinating the changes in each sequence that I wished to identify as an operation for my research. Because of the lack of consensus on the existence of an invariant learning sequence, I decided to focus only on the existence of coordination between sequences, rather than the particular form in which this coordination was implemented.

In reading through the different problems described as grounding situations in these writings, two different general types of situation were noted. In the first type, the corresponding sequences of objects were co-existent and the goal was to compare the sequences. In the second, the objects in the first sequence were exchanged for those in the second, or even consumed in the production of the second. In order to respond in a manner consistent to the situation in discussions with the teachers, I decided to use the terms 'coordinated counting' or 'exchanging groups' depending on the type of situation that was being discussed.

I hoped to obtain more insight into the effect of the type of situation and the level of strategy employed, in the course of the interviews and the subsequent analysis.

4.5 Activities and actions for developing personal meaning

Combining the ideas of activity theory and situated cognition yields a rich insight into the concept of personal meaning. From this perspective, personal meaning can be seen as being constituted by a combination of:

- Personal significance due to the personal motivation in relevant activities in which an individual is involved.
- Personal experience of goal directed actions that the individual carries out in relevant situations.
- Personal experience of the structure afforded by relevant situations, obtained through carrying out specific operations within the situation.

Note that in this work, 'personal meaning' is not used in an absolute sense, rather it will always be used in the sense of 'personal meaning of something'. The qualifier 'relevant' is important in the above points, because the situations, activities, actions and operations which contribute towards the 'personal meaning of something', will be those that are perceived as relevant to the 'something'.

In the previous section, I carried out a theoretical exploration to formulate different possible operations that might contribute towards the development of personal meaning for rational numbers. The operations identified and the examples of the operations constructed for the teacher interviews are described in table 2.3 in the methodology chapter. These operations would be critically evaluated and adapted in the interviews and subsequent analysis of this cycle.

In addition, I aimed to relate the interpretations to activities or actions which would contribute to the development of an intuitive and informal foundation of personal meaning, on which an understanding of rational numbers could be built that is both deep and comprehensive. To do this, I would familiarize the teachers with the interpretations, by going through the same problem situations that were used in the pilot interviews. Following this, I would ask the teachers to identify real situations that involve activities and actions relevant to these interpretations. One aim of the analysis of the interviews would be to classify different types of activities and actions that could contribute to a comprehensive intuitive foundation for rational number knowledge, leading to a comprehensive mathematical understanding of rational numbers, as described by the constructs.

4.6 Structuring the interviews

4.6.1 Fourth interview with Lungiswa, to pilot the operations.

The fourth pilot interview was included in this cycle for a number of reasons. Firstly, it fitted more in the time frame of the teacher cycle, having been carried out six months after the other interviews with Lungiswa and about a month before the other teacher interviews. Secondly, this interview focussed on the possibility of developing informal rational number understanding through operations. As discussed earlier, the idea of such operations arose in response to a number of the important underlying issues identified in the first three pilot interviews. The presence of these issues as unanswered concerns had a noticeable effect on the interaction in these interviews. Having the operations as a possible response to these concerns in the fourth interview and the other teacher interviews, could be expected to change this effect. Finally, due to my reading and reflection on these issues after the first three interviews with Lungiswa, my orientation to the research had changed considerably in the fourth interview. A change of similar magnitude did not occur between this interview and the teacher interviews.

On the other hand, this interview was with Lungiswa, a different person than in the other teacher interviews. She had been involved in the first three pilot interviews and had had six months to reflect on our discussion in these interviews. The teachers involved in the other interviews did not have this time for reflection on the possible effect of the interpretations on the development of intuitive rational number understanding, before being exposed to the possibilities offered by operations. Also, as will become evident in the discussion of the teachers' background given in the next chapter, Lungiswa's personal and teaching background was very different from that of the other two teachers involved in this cycle. As a result, this interview did not fit perfectly into the teacher cycle.

4.6.2 The teacher interviews.

Two grade three teachers working in the same school, were involved in the teacher interviews. Each interview was carried out with both teachers together in order to encourage discussion between the teachers on the issues of concern and to allow the recording of this discussion.

A sequence of five interviews was planned with these teachers, but in practice this extended to seven. As for the pilot study, the first interview was aimed at exploring the personal histories of the teachers, including personal background and their views on teaching in general and mathematics teaching in particular. The second interview was focussed on the different interpretations provided by the semantic analysis, and followed the same format. The example problems used for the interpretations were adapted slightly from those used in the pilot interviews. The response of the teacher in the pilot cycle to the problem involving the comparison of the number of boys and girls in a class, led me to replace this problem by the following, more concrete, baking problem for the teacher cycle:

> Yesterday, Mrs Mpati used 3 cups of flour and 2 cups of sugar to make small cakes. Today she uses the same recipe to make small cakes. She uses 6 cups of flour today. How many cups of sugar does she use?

The third interview aimed to introduce and discuss the identified operations in a similar fashion. In practice, this process extended to take up both the third and the fourth interviews. The aim of the fifth interview was for the teachers to discuss possible teaching tasks which would involve specific interpretations of operations. The sixth interview asked the teachers opinions about possible situations which would be meaningful or significant in their childrens' lives and would involve the interpretations and operations identified. Finally, in the seventh interview, we discussed a number of tasks which I had identified as possible options for inclusion in clinical interviews with the children.

The examples of interpretations and operations used in this cycle, are given in tables 2.2 and 2.3 respectively, of the methodology chapter.

Chapter 5 – Teacher Cycle Interview Analysis

5.0 Themes for the analysis of the interviews

These interviews formed the second cycle of the research project. At this stage, the research had begun to take a more defined shape. As a result, it was decided not to follow an open approach to the analysis of the interviews, but instead to follow a more focussed approach. The interviews were analysed with a number of themes, in mind.

As with the pilot cycle, the discussion of the analysis will begin with some general background information the teachers gave about themselves. This is provided in section 5.1 and will serve to provide a personal context to the views of the teachers. More important for the analysis of the interviews was the teachers' approach to the teaching of mathematics. These teachers had little prior experience of the approach I followed to the teaching of mathematics, and our work together in these interviews did appear to have some effect on their views of mathematics teaching. This changing perspective provides an important personal context for the interpretation of the interviews and it will be briefly discussed in section 5.2.

The remaining analysis is structured in three sections, each with a specific focus and incorporating a number of issues related to the focus.

5.3 Issues of meaning, motivation and significance. What are the teachers opinions about these issues, when viewing the discussion problems from four different viewpoints?

- As activities children may be involved in outside of schooling.
- As learning activities for children.
- As activities that the teachers themselves may have experienced.
- As teaching activities that the teachers may select and implement in their classrooms.

Because of the professional focus of the teachers, they often discussed these issues in terms of their relevance for teaching. As a result, I included the teachers opinions of the implications for teaching in this theme.

5.4 Insights obtained from the teachers understanding and analysis of the problems. This included issues such as:

- Points of interest about interpretations or operations arising from specific things said or done by the teachers when working with the problems.

- The teachers confidence in their work with the interpretations and operations being discussed. This is taken as an indication of the accessibility of these concepts to the teachers.
- What relationships between the interpretations were apparent in the teachers' responses?
- What relationships between the operations were apparent in the teachers' responses?
- Does it appear as if the identified operations are indeed units of coordination?
- In what ways do the interpretations and operations appear to be related in the teachers analysis and discussion of the problems?

5.5 Emerging themes about teaching and learning rational numbers.

This included any issue of relevance to the teaching and learning of rational numbers that emerged in the interviews and that was not included in the previous two sections. One particularly important issue was ways that a teacher could encourage children to move from intuitive competence grounded in these interpretations and operations, to a more formal, conceptual understanding of rational number.

5.1 Teacher background

As in the pilot cycle, the first interview was concerned with eliciting background information.

Sarah always wanted to become a teacher. She remembered making an oral presentation in grade 7, in which she said that she wanted to become a teacher. Another reason she became a teacher was that money was tight and she was awarded a teaching bursary, which made it possible for her to study. She attended a teacher's training college, but did not enjoy her training, feeling that as a student she was treated as if she was at school. In addition, much of the subject content that she was taught was merely a repeat of what she had learned at school. Even though she did enjoy the courses about teaching, in her second year she seriously considered withdrawing from her studies. Eventually she was convinced by her parents to complete her course and obtain the qualification, even if she didn't stay in teaching. But once the practical work started, she found it all worthwhile, because she really enjoyed working with the children.

In Sarah's opinion, children remember a teacher more by who they are, than by what they teach.

Her main goal in teaching was for children to learn not just to accept things, but to be questioning and have their own opinion. She also wanted her children to come to know and acknowledge their own strengths and weaknesses and to respect others for their differences.

When it came to teaching mathematics, Sarah wanted her children to become comfortable with numbers and be able to associate numbers with the real world. It was important for her that mathematical learning start from concrete experiences, so that it became meaningful for the child. She provided a number of examples of such experiences, including shopping, sharing and baking, saying that children loved seeing how their mathematical knowledge needed to be used in such situations, because it made the mathematics come to life.

Nadine had to choose between nursing and teaching for her career. After leaving school, she chose nursing, but found the working hours and responsibility too arduous and so left the profession after just more than a year. After this, she did office work for a year, before being persuaded by her parents to study as a teacher. She studied for a teaching diploma at Rhodes university and really enjoyed the course and was committed to teaching from then on.

An important goal for Nadine in her teaching, was to instill a love for learning in her children. She also wanted children to learn to work in a group, seeing this as important for the world of work where people never work in isolation.

Mathematically, Nadine wanted children to be able to work independently with numbers, so that they could use them when needed. She gave the example of comparing prices and choosing items when shopping. When learning mathematics, she thought it important for children to be able to recognize mathematical symbols and understand what they meant, in the sense of having a number of meaningful, real life examples and the knowledge of how the symbol was interpreted to fit the examples. She described how many children struggled with fraction symbols and how she helped the children interpret each aspect of the symbol, saying "the line means you cut it" and "you put how many pieces underneath ... that you cut it into".

Both teachers thought that learning multiplication tables (by counting in multiples); carrying out "vertical mathematics" (standard algorithms for calculations); being able to recognize and name shapes; and learning mathematical terms were important for learning mathematics. This was over and above the learning of rational numbers, or fractions, which was the focus of these interviews.

Being able to work with money, and helping their children develop skills which would be useful for finding a good job, or making money as an entrepreneur, was important for both teachers. They saw their children as being much more aware of money than they were at the corresponding age, and tried to prepare the children to meet their own needs in society later in life. Learning at school was seen as an important part of this preparation.

An issue that was seen as vitally important by both teachers, was that of the language of instruction. They explained that since they were now teaching a number of children for whom English was a second, or even third language, they could no longer rely on children being able to meaningfully understand their discussions and explanations. Previously, they would be able to use a term such as "divide" knowing that children had a meaningful non-technical understanding of the term that provided an accurate intuition of the more technical, mathematical sense that they were trying to introduce. But now this term was seen as abstract by many children and so they were not able to build on any intuitive non-technical meaning. This made teaching a lot more difficult, and they were continually needing to search for simpler language to help children make sense of concepts.

Time was also important to these teachers. They were very busy people and had a full teaching schedule for their class, in order to achieve their full and detailed teaching goals for the year. As a result, they did not have a lot of time to spend playing with each concept, needing to teach effectively and efficiently so that they could move on to the next teaching goal. They found that ability groups were a useful tool in their teaching, giving the more advanced children work to explore while they were able to spend time developing the understanding of the weaker children.

5.2 The teachers' approach to mathematics teaching

In general, the mathematical features of the rational number interpretations and the problem situations which exemplified them, did not appear to be problematic to Sarah and Nadine (see sections 5.3, 5.4 and 5.5). As a result, the explicit discussion of the interpretations in the interviews was rather short. Much of my interview analysis on these topics, was based on episodes where they implicitly used or referred to these interpretations.

Two factors appeared to contribute towards the non-problematic nature of the interpretations to the teachers. Firstly, they did not appear to find it difficult to understand the mathematical features of the interpretations. Secondly, and possibly even more important, was the perspective on mathematics, mathematical word problems and mathematics teaching that they brought to the interviews.

In the earlier interviews of the cycle, I noted a tendency for both teachers to approach the situated problems from a closed, rather than an open perspective. That is, they appeared mostly concerned with the correctness, or abstract form of the answers. And less with investigating the relationships between the problem situations and the mathematical structures, and the relationships between different approaches and different forms of solution to the problem.

892: BB: Both answers are correct. Especially at the, at the grade 3 level ... that ... both of those are actually really nice.
893: TS: Ja. Because I mean there that's three and that two will make a five in the end.
894: BB: But it's also ... it's also very powerful to have the difference because this is the way children start realizing that five thirds is actually the same as one and two-thirds ...
895: TS: Ja ... the equal fractions.

(Teacher interview 3)

In the second interview, it seemed as if Sarah saw drawing as a weakness, something that was necessary for people who didn't know what they were doing. This gave me the impression that she expected people who did know what they were doing to be able to solve the problem mathematically, without using the link that drawings provided to the situational meaning of the problem.

700: TS: They'll probably end up drawing a wall ...
701: BB: So ja. Draw the wall. In fact I've done that with our teachers often. So draw the wall.... Where does it start ... where does it end ...
702: TS: I need to draw it to solve it. I'm ...
703: BB: That's it.
704: TS: ... lost here.
705: BB: Show me how Bulelani measures ... okay.
706: TS: Oh ... Lord ... (sighing) ... I can just see these kids. And then those who don't know what's going on will be doing what I'm doing.

(Teacher interview 2)

Even though this was definitely not her objective, such a perspective could encourage children to develop their mathematical solutions without building on an understanding of the meaning of the problem situation. I attempted to help the teachers to open their perspective to include the importance of the process of making meaning. To do this, I encouraged them to discuss possible responses by children in their class, to these problems.

An interesting insight into their approach to mathematical problems was obtained in the third interview. Nadine and Sarah had given some of the problems to the children in their class and were fascinated by the way their children approached the problems. This appeared different from their approach to word problems, which had been 'drilled' into them at school (Teacher interview 3: 369–402). They appeared to have learned an abstract, answer focussed approach to word problems, using key words as indicators of mathematical operations, rather than a meaningful understanding of the problem. This approach had been 'drilled' into them, so much so that 'story sums' became a 'terrible' experience for Sarah. As the problems around which I structured the interviews were obviously 'story sums', this prior learning would have strongly influenced their initial approach to the problems.

The teachers found it remarkable that their children successfully approached these problems from a very meaningful perspective, and seemed to enjoy this process. The children thought out and represented the story of what happened and gave their responses on the basis of this story. They were intrigued that, even though they had not instructed their children how to solve the problems, for each problem there was great consistency in the way different children worked out the problem (Teacher interview 3: 348–368). This experience with their children's solution of the problems, encouraged them to open up their perspective. They were quite excited about the different approach, which seemed less restrictive and more natural, almost as if they were able to use their minds 'natural problem solving' abilities.

327:	TN:	Ja ... I ... I've never done fractions like this ... I've never been told ... so ... it's ... new way of thinking ... I mean, I mean I just do it. I don't think about ...
328:	BB:	Ja ...
329:	TN:	... what I'm doing ...
330:	BB:	Yes ...
331:	TN:	... or what operation I'm using ...
332:	BB:	Okay ...
333:	TN:	... um ...
334:	TS:	Oh, you mean personally ...
335:	TN:	... ja ...
336:	BB:	Ja ...
337:	TS:	... when you solving that.
338:	TS:	Ja ...
339:	BB:	Mmmm ...
340:	TN:	I mean, I've never come across these ...
341:	TS:	Ja ... until, ja me neither ... you use your minds natural problem solving

(Teacher interview 3)

Later, as they become more familiar with the approach taken in this research, their perspective became more open and exploratory. For example, in the following extract from their last interview, they actively generate alternative approaches that children may take towards the problem.

328: TN: I can actually imagine them putting the pieces together and, kind of measuring, that the pieces are all the same.
329: BB: Yes. Okay.
330: TN: I'm kind of, thinking. If you ... if you actually cut a piece off each of the three loaves, kind of ...
331: BB: Mmmm ...
332: TN: ... and then making sure that those three pieces ...
333: TS: ... get dealt out with three little bodies, or four bodies, sorry.
334: TN: ... get measured. That if you put ... if you put them all together then the ... Do those three pieces make ... make the same ... loaf.
335: BB: ... make the same loaf. Yes, that's right.
336: TN: You know. Get it.
337: TS: And the amount of four people. They might share each loaf into four parts, you see.
338: TN: Okay....
339: BB: It's another way.
340: TS: It's very simple to get to the answer then.
341: BB: Mmmm.
342: TN: Ja.
343: TS: But they will really like to have three little people, and their four loaves and then look at it. Some might try and sub-divide.
344: BB: Ja.
345: TS: Little bits off. Some might see, straight away. A grade three child is not incapable of that.

(Teacher interview 7)

5.3 Issues of meaning, motivation and significance

5.3.1 Interpretations

5.3.1.1 Part-whole

For the part-whole interpretation, a part is compared to the whole, of which it forms part. That is, when compared the part is still part of the whole, even though it can be identified as a separable part. Two of the problems discussed were designed to be relevant to this interpretation. The first was that for the part-whole interpretation and the second was the second example for the unitizing operation.

In each case, the question asked in the problem was to name the part quantity in relationship to the whole quantity. Determining the name was what made these questions significant for rational number learning, but doing this did not provide the intrinsic significance of the contextual activity which gave the grounded meaning for the problem. Furthermore, the goal of the contextual activity was also not directly related to the naming of the part quantity. That is, the meaning of these problems, viewed as describing motivated activities or a goal directed actions does not directly give rise to the meaning of the rational number representations developed. Rather, it is the physical configuration of the whole and part constructed in the activity that yields the meaning of the representation. And it is this configuration that is the focus of the part-whole interpretation.

The teachers themselves used this interpretation in a number of instances in the interviews to develop fraction representations of quantities. They also routinely used this interpretation for teaching fractions to their grade three children.

330: TS: It's going to be a half ...
331: BB: It's a half?
332: TN: So they learn straight away that three ... those ... ja ... three sixths is the same as a half.
333: BB: It's interesting because ... ja ... 'cause this ... it ... it ... it could be a half .. it looks like a half ... but it could also ...
334: TN: It could be three sixths ...
335: BB: ... three sixths ...
336: TS: ... so straight away they see that ... like we were trying to do last week ...
337: TS: Ja ... yes ...
338: BB: Mhmmm ...
339: TS: ... straight away they see that those are actually equal in s ...you know ... in ... what you say? In
340: TN: ... in size ...
341: TS: ... size ...

(Teacher interview 2)

Although they thought that their children needed to develop the ability to do this, they expected that it would be difficult, or even threatening for some children (Teacher interview 7: 485—508).

They appeared to see naming rational quantities in this way as significant, possibly because learning such representations was seen as an important goal of their mathematics teaching. That is, creating the representations was seen as providing the significance of these problems within the social activity of schooling. But, apart from the one brief discussion mentioned in the analysis of the pilot cycle, finding an activity for which creating the representation itself would be significant, was not pursued in these interviews. Rather, when dealing with the part-whole representation, the focus was more on the implications of such part-whole representations for teaching.

For each problem and each representation to be developed, I had attempted to find a context which yielded a natural whole, as well as a meaningful construction for the part. But achieving reasonable alignment between the representation exemplified by the problem and a natural whole and construction of a part, was rather difficult. This was evident in the third problem of a set involving the construction of 3/4 in three different ways, using different units. For this construction, three slices of bread were said to have been packed into a packet, the packet had been divided into four equal parts and the bread from one part had been eaten. A necessary feature to allow the desired mathematical representation was that the even subdivision of the packet would result in an even subdivision of the bread.

In the fourth interview with Lungiswa, it took a great deal of guidance from me, for her to construct the desired representation. To make sense of this problem, she decided to draw what was happening, first trying with the three slices forming the shape of a 'T'. But cutting this packet into quarters in such a way that the bread was also evenly subdivided would have been very difficult. So, to obtain the desired interpretation of the problem, I thought it necessary to intervene, by suggesting that she draw slices packed alongside each other. Also, when thinking of how to perform the subdivision, she did not appear to find it natural to cut the packet without opening it.

322: LS: Oooh! It seems that a packet ... three packets of bread. Divide the packet ... into four equal parts without opening it ... they ate the bread from one of the parts. Show what's ... uh ... part of the packet ... the three slices. Here's a packet ... here's, né? With three slices.
323: BB: Can I make a ... slightly different suggestion?
324: LS: (Chuckling). Okay ...
325: BB: Let me ... because this is going to be difficult to cut into four equal parts. Real equal parts so let's make it a long, thin packet ... with the three ... pieces of bread ... next to each other
326: LS: Okay ...
327: BB: Okay ...
328: LS: ... cut the packet into four equal parts without opening it ... cut the packet now ...
329: BB: Mmmm ... so if it's like a plastic ... no ... a paper bag.
330: LS: Paper bag, né?
331: BB: Paper bag ... that's it.
332: LS: So you cut the paper bag ...
333: BB: Ja! So that ... so that you cut it into four equal parts. So how would you do that? You can cut in half like that ... (demonstrating) ...
334: LS: Ehh ... okay. So you cut the bread as well.
335: BB: Ja ... you cut the bread with the packet because the bread is in the packet.
336: LS: Ohhh! And then?
337: BB: Okay ...
338: LS: Okay ...ayyyyyy! ... (laughing) ... cut the bread ... from one ... of the five ... so ... here ... you are going to eat half.
339: BB: Half of that slice ...

340: LS: Ehhh! Half of that slice ... here the quarter ...
(Pilot interview 4)

When I stated that I did not think that the problem had worked well, Lungiswa responded that it was the initial arrangement of the bread that was difficult for her. She said that she would like to try it with her children and suggested giving the initial arrangement of the bread slices in the packet (Pilot interview 4:1326–1338). In effect she suggested providing the representation which would be suitable for the mathematical interpretation. In response, I asked if she could think of a different situation, for which such a representation would indeed be natural. She suggested using chocolates such as Kit-Kats.

As a result of this discussion, I changed this problem to involve three chocolates for the interview with the two teachers. However, I did not specifically name the chocolates and so the teachers gave two different interpretations of the problem. Nadine thought of the chocolates that were found in boxes of chocolates. In this case, cutting the packet would not naturally result in cutting the chocolates and as a result, Nadine's solution did not result in an even subdivision of the chocolates.

697: BB: (Laughing) ...but ... tell me what you've done there, Nadine? I think ...
698: TN: Well ... he's got a packet ... an the packet ... they're all loose in the packet ...
699: BB: Uh huh ...
700: TN: ... and ... he didn't even out the packet. He just put the scissors in and ... (Motion of cutting packet) ...
701: BB: Mhmmm ...
702: TN: ... and he said, "One for you ... one for you ... one for you and the most for me."
703: BB: That's it. Okay. So the only problem there is that he's not awfully fair ...
704: TN: No!
705: BB: But this is ...
706: TN: But ... that's what he did.
(Teacher interview 3)

In contrast to this, Sarah thought of a packet tightly packed with three small chocolate bars, such as Tex bars. Cutting such a packet into four equal pieces would result in cutting the chocolates correspondingly. Her interpretation did yield a reasonable alignment of the representation with a natural construction of the whole and parts and we were able to have a grounded and meaningful discussion of the representation (Teacher interview 3: 717–747).

The implication of this for teaching representations is that, for productive communication, the pupil's interpretation of the problem construction should be reasonably aligned with the desired the mathematical representation. Formulating the problem to consistently achieve this result was a difficult task.

5.3.1.2 Sharing

When discussing sharing, Sarah and Nadine were interested in the full human context, including social, individual and emotional. Their discussion of these issues was flexible and wide ranging (Teacher interview 5: 1288—1356). They saw issues of sharing as important for their children's personal growth, and didn't easily separate out mathematical structure, resulting in a much weaker focus on these aspects. On returning to the interviews with Lungiswa, a similar perspective was found, even though her discussions of mathematical structure was more open. The teachers were sure that most children would work confidently in a sharing situation They would have been exposed to many sharing situations in their homes and could use this experience to develop their sharing solutions.

1056: TN: But I can imagine the children .. doing ...this method ... where they hand out a whole slice ...
1057: BB: Yes ...
1058: TN: ... to each one first ...
1059: BB: Uh huh ...
1060: TS: Because that's what happens in the home.
1061: TN: 'Cause that's what happens.

(Teacher interview 3)

5.3.1.3 Grouping

The teachers considered that their children would be familiar with grouping from a number of common activities. They gave the example of packing cookies onto plates or trays. Such activities could be adapted for teaching, and they also suggested developing games that children could play, mimicking adult activities, such as packing pills into bottles in a pharmacy.

965: TS: Ja, or they just use something else to represent the cookie ...
966: TN: Ja. Ja ...
967: TS: Something, that you might ...
968: TN: Oh! You know those sponges out of the top of the ... the ...
969: TS: Oh! The med ... uh ... tablets.
970: TN: Sponges out of the top of the ... the tablets!
971: BB: Oh, yes?
972: TN: You go to the pharmacy. You ask for the sponges from ...
973: BB: Okay.
974: TN: ... and those make really nice cookies. You know, you can ...
975: BB: ... (laughing) ...
976: TS: Those polystyrene ... dishes.
977: BB: That's actually really nice. Packed into trays ... poly ...
978: TS: Are we giving you what you're wanting here?

979: BB: Oh, this is great! It's just ... You know, the more we get or the more we look at, and ...
980: TN: Also, that's another nice one. Because I used to do it for my kids, playing pharmacies. And you ... you can buy it from the pharmacy. These little tablet ... bottles.
981: BB: Yes?
982: TN: And then you can have a whole lot of beads, or something. Put so many tablets ...
983: TS: ... in the bottles, are there.
(Teacher interview 6)

5.3.1.4 Measurement

The teachers were able to give a number of everyday contexts for measurement, which they thought were accessible for children as well as being adaptable for teaching. In particular, they saw cooking as a very flexible context, with many varied instances of measurement. Children, especially those whose mothers bake, would relate well to these situations. An important observation about such cooking activities is that fractional measuring units are often used. This could encourage children to investigate and relate the different fractional measuring units within a meaningful situation.

242: TS: But I still ... you know ... with children we bake a recipe with them.
243: BB: Mmmm ...
244: TS: We often stick to the cups.
245: BB: Ja ...
246: TS: Half a cup ... yes.
247: BB: ... half a cup ... half a cup is wonderful ...
248: TN: Ja ... ja ...
249: BB: ... or half a spoon ...
250: TS: Yes ... or a ...
251: BB: ... or a big spoon or ...
252: TS: Ja ...
253: BB: And I think it's important. Because this ... operation is ... is a ... is a key operation. You don't have numbers if you don't count ... numbers are made to count in.
254: TS: Mmmm ...
255: BB: And if you ... if you have a number that's not a counting number then you're in trouble.
256: TN: Ja.
257: TS: And it also ... this ... the question ... the way it's worded ...
258: BB: Mhmmm ...
259: TS: I mean, a child which has a ... logical brain ... would read it and probably tell you ...
260: TN: Also ...
261: TS: ... thirty five halves.
262: TN: a ...
263: TS: I mean, there's the answer ...
264: BB: Yes. That's right ... it's immediate. It's thirty-five half apples.
265: TN: ... a child whose mother bakes ...
266: BB: Ja ...
267: TN: ... can relate to this better than a child who's never seen his mother baking

in the kitchen.
(Teacher interview 4)

5.3.1.5 Operator

Sarah and Nadine identified swopping as a very important situation in the lives of children, which related to this interpretation. They appeared to enjoy discussing teaching activities that used children's swopping to build an effective grounding for this rational number interpretation.

732: TN: Oh, ja! Look, they swop ... they, like ...
733: TS: Yes. They don't just swop lunchboxes ...
734: TN: We don't allow it in our ... in our classes.
735: BB: Mhmmm?
736: TN: But, at other schools children always swop sandwiches.
737: BB: Okay.
738: TN: They're always swopping sandwiches.
739: BB: So maybe we can just put it in terms of swopping sandwiches.
740: TN: Mmmm.
741: BB: Then we don't have to worry. Because here, you know, I've let them cut their loaves of bread, and ...
742: TS: You've come right from the bread ... the loaf of bread.
743: BB: Ja.
744: TN: Ja.
745: BB: So maybe we should just, simplify it into just swopping sandwiches.
746: TS: Yes. And forget about where it came from.
747: BB: That's right.
748: TS: They're just at school and so-an'-so's got this sandwich and that ...
749: TN: I would almost have ...
750: BB: Mhmmm?
751: TN: ... sheets of paper where they can, instead of ... spreading these things, they would colour ... two jam ... Colour your...
752: BB: Mhmmm. You've got jam, or he's got ...
753: TN: Jam is red, so colour that ...
754: BB: ... yes ...
755: TN: ... that slice red, now.
(Teacher interview 7)

5.3.1.6 Ratio

The teachers did not mention any activities which specifically involved the ratio interpretation. A number of activities that could be described or understood using ratio were discussed. But most of these involved the exchange of quantities, as in shopping, or swopping; or the conversion of quantities, such as relating the quantities of an ingredient, to the quantity of the final product in baking. Because these all involved some type of active conversion, I judged that the operator interpretation would be more fitting for these activities.

5.3.2 Operations

After the teachers had familiarized themselves with the different operations, we spent some time discussing relevant activities which would provide grounding experience for these operations. In starting this discussion I chose an operation and then asked the teachers to identify relevant activities. But the teachers questioned the implication that activities could be found that involved only a single operation, stating that they appeared to be interrelated (Teacher interview 6: 361—400). They also suggested that it would be better for teaching if such a separation was not attempted by the teacher.

When I agreed, they became much more at ease with looking for contexts to ground operation. They discussed a number of activities which they thought would be meaningful to the children, and identified operations which could be expected to occur in these activities. For example, they suggested that the unitizing operation would be important in the grouping activities that they had discussed (5.3.1.3). Nadine suggested a teaching activity using bread, which involved a number of operations. In particular, the operations of unitizing, iteration and equal subdivision. She suggested asking children how many loaves would be needed to feed a given number of people, if each loaf was cut into 6 equal slices.

596: TN: This ... this ... equal sub-division and counting. Because you were talking about counting thirds.
597: BB: Mhmmm.
598: TN: And things like that. So, I think you could, kind of, do that and say, kind of ...
599: (Time)
600: Okay. Let's think. Ummm ... ja. "Your slice of bread ... or loaf of bread ...
601: BB: Mhmmm?
602: TN: ... um ... kind of ... For each person and, how many people can you feed and ... If you cut the bread first ...
603: BB: Okay, so if you cut the bread ... So we've got a loaf of bread. A loaf of bread ... (writing down) And then what do we do? We cut it.
604: TN: Cut it into ... six pieces, six equal pieces.
605: BB: Okay? So we do ... this is ...so it's not a ... This isn't a sharing, this is a standard sub-division.
606: TN: A standard sub-division.
607: BB: Cut into six equal pieces ... ja? This is ... mhmmm?
608: TN: Um ... and then, kind of ... How many ... how many loaves of bread would you need to feed ... twelve people?
609: BB: Okay.
610: TN: And then they'd have to, put the twelve people there, and give them each a piece of bread, and then put all the bread back together, and unitize it again, and then see how many loaves there were.

(Teacher interview 6)

This was one of the suggested activities that I developed for my work with children in the third cycle.

The teachers did identify two operations that they thought could be seen as the dominant operation in some situations. These were the operations of unequal subdivision and of exchanging. They spent some time discussing activities that were dominated by each of these operations. These discussions are analysed in the following two subsections.

5.3.2.1 Unequal subdivision

Sarah saw unequal subdivision as reasonable in the sharing situation given for this operation. In fact, she described a similar situation in her own home.

> **1065:** TS: I just don't ... our family, when I'm serving supper in the evenings ... when I'm fetching something ... "Why's Daddy got two sausages on his plate?"
> **1066:** BB: Yes ... That's right.
> **1067:** TS: You know, so I think ...
> **1068:** BB: Ja.
> **1069:** TS: ... well he's bigger ... you know?
> **1070:** BB: Ja.
> **1071:** TS: He eats more than ...
> **1072:** BB: Ja ...
> **1073:** TS: ... this little eight year old that gives me the "I'm hungry, too!" ... you know ... (chuckling) ...

(Teacher interview 5)

Later in the interview, I voiced the possibility that a reasonable unequal sharing would often need to be negotiated (Teacher interview 5: 1361). Sarah provided an anecdote about a situation in which such a negotiation occurred when a group of hunters shared a seasons tips (Teacher interview 5: 1373–1399). She was struck by the way the people involved negotiated quite naturally, without any overt conflict.

The situations mentioned in this interview provided evidence that unequal sharing may indeed occur commonly in life. If this were the case, children would have the opportunity to develop an intuitive understanding of the unequal sharing operation through common situations in their lives. This could yield valuable grounding intuition for rational number understanding

5.3.2.2 Exchanging

The teachers agreed that exchanging was meaningful, familiar and real to children. Nadine suggested that marbles provided a good context for exchanging in school life.

460: TN: I'm just looking to see ... these ... And exchanging groups, that would be great to do with marbles.
461: BB: Yes? Okay?
462: TN: Because that's exactly what kids do. Um ... three marbles equals one goon, you know? Kind of ...
463: BB: Yes, that's right ...
464: TS: Mmmm ...
465: TN: ... and that would be a ... a great one to kind of ... just let them ... play and exchange as they win or lose a game.
(Teacher interview 6)

Sarah also suggested bartering and swopping sweets as real activities which children would engage in, and which would involve exchanging.

767: TS: I wanted to just say ... Yes, what about, you know, if they were bartering. Two hawkers of some sort.
768: BB: Yes? Mhmmm?
769: TN: Ja.
770: TS: Ja, for every one ... well there I am going back to one again.
771: BB: But you don't have to. I mean that's ...
772: TS: No. because for every packet of apples you can have ... (laughing) ...
773: BB: Mmmm!
774: TS: If you're to exchange ...
775: BB: If you think in terms've ...
776: TN: Let's think of something they would exchange.
777: TS: The children.
778: TN: They exchange sweets.
779: BB: Sweets are good.
780: TS: They could change blocks of ... exchange blocks of ... you know ... from a slab of chocolate ...
781: BB: ... mhmmm ...
782: TS: ... for ... two suckers or something.
(Teacher interview 6)

When I explicitly asked whether swopping was indeed realistic for their children, they were very positive that it was. They stated that children often engaged capably in such activities (Teacher interview : 731—738). Both teachers became quite enthusiastic about developing learning activities involving swopping sandwiches (Teacher interview 7: 748—776). An interesting observation made by Sarah was the importance of value and personal choice in exchanging different types of sandwiches. Children would need to agree on the relative worth of sandwiches with different fillings. The teachers enjoyed discussing the teaching opportunities offered by setting these rates and then performing exchanges using the rates set (Teacher interview 7: 780—809).

5.4 Teachers understanding and analysis of the problems

5.4.1 Interpretations

5.4.1.1 Part-whole

The teachers routinely used this interpretation to accurately estimate rational comparisons of quantities that had been generated in the solution of sharing and subdivision problems. For example:

> **1066:** TN: Ja ... ja ... do you see what I did, Sarah? It's I ... first drew ... the half ... and then I put on the other half that wasn't there.
> **1067:** BB: Yes ...
> **1068:** TS: Ja ... I did the same.
> **1069:** BB: Okay.
> **1070:** TN: You did the same.
> **1071:** TS: Ja. I did the whole thing and just shaded ...wh ... what was in the ... the say the paper ...
> **1072:** BB: Ja ...
> **1073:** TN: Ja ... and then I had to divide that piece into quarters so therefore that piece must have been divided into quarters as well.
> **1074:** BB: Okay ... and so you got ...
> **1075:** TN: And so there were eight pieces altogether ... and he only gets one.
> **1076:** BB: So he gets one piece out of eight.
> **1077:** TN: ... one out of eight ...
> **1078:** BB: So he gets, now, an eighth. But it's an eighth of how much?
> **1079:** TN: An eighth of the whole ...

(Teacher interview 2)

In fact, this was the dominant approach used by the teachers when then needed to accurately specify an rational quantity. They also followed this approach to check their visual estimates (Teacher interview 4: 495—497). Their identification of wholes for the part-whole interpretation initially seemed more natural with wholes which formed a geometric unit. Wholes made up of discrete collections were seen as less natural, and the interpretation of subdividing such a collection as subdividing a whole, was in some instances even seen as surprising (Teacher interview 2: 495—508).

As in the pilot interviews (section 3.4.3.2), the appearance of the part-whole interpretation in a transient configuration of an active situation was evident in these interviews. The teachers described quantities as fractions of certain wholes that they created and identified during the solution process. An analysis of the fourth pilot interview brought more clarity to this issue, allowing greater insight into possible processes involved in identifying and using this transient configuration. In the problems about eating different quantities of bread, only the uneaten bread would be physically present in the final state — the rest would have been eaten. So neither the whole, nor the part under discussion would be physically available for analysis in order to answer the question. To perform the analysis, Lungiswa focussed on a transient state in the situation, which occurred after the cutting of the bread and before the pieces were eaten — when the cut pieces were being identified to give to the child. For example, when portioning a single slice of bread.

294: LS: It is a brown ... ja ... The sandwich, né? Three-quarters of a slice, né?
295: BB: Mhmmm ...
296: LS: Um ... I wonder if ... they can now start ... in the half ... and there will be shade ... this is the three-quarters ...
297: BB: Okay ... so that's three-quarters of a slice ... shaded there ...
298: LS: Mmmm ...

(Pilot interview 4)

Or cutting a portion of three slices.

332: LS: So, you cut the paper bag ...
333: BB: Ja! So that ... so that you cut it into four equal parts. So how would you do that? You can cut in half like that ... (demonstrating) ...
334: LS: Ehh ... okay. So you cut the bread as well.
335: BB: Ja ... you cut the bread with the packet because the bread is in the packet.
336: LS: Ohhh! And then?
337: BB: Okay ...
338: LS: Okay ...ayyyyyy! ... (laughing) ... cut the bread ... from one ... of the five ... so ... here ... you are going to eat half.
339: BB: Half of that slice ...
340: LS: Ehhh! Half of that slice ... here the quarter ...

(Pilot interview 4)

In both extracts, Lungiswa was able to represent the transient state of an active situation, in which the whole was destroyed. The part-whole interpretation could then be used to analyse this representation and so describe the quantity eaten by the child.

Note that the part-whole interpretation is a static interpretation that does not fit well with the changes in this active, dynamic, situation. It also does not fit well with the most easily identifiable static states in the situation — the final state, or the initial states consisting only of the whole pieces of bread. However, this interpretation does fit well if we take our aim to be representing the situation for analysis. For in this case, the transient state consisting of the cut, but not yet distributed bread can be accurately and easily represented and analysed.

Activity theory provides further insight for this discussion. If the children work with this as a grounded activity of proportioning bread, motivated by the need for food, then the part-whole interpretation is not fitting. On the other hand, if the children engage with the activity with a motive of intellectual mastery, aimed at accurately representing and describing the bread proportioned in the situation, then the part-whole interpretation is fitting. Choosing this motive will require the children to distance themselves from the grounded situation described in the problem, which may prove difficult for some children. Also, identifying the configuration which is suitable for representation and description, is not straightforward, because of the weakness of the fit between the grounded activity and the part-whole interpretation. This could also lead to difficulties for some children.

5.4.1.2 Sharing

Just as for Lungiswa, this interpretation appeared natural for Sarah and Nadine. They solved the problems involving this interpretation without much difficulty. Nadine followed the approach she would use if she was actually in the situation (Teacher interview 3: 854—857). While Sarah was led more by the way her children would solve the problem (Teacher interview 3: 873–883).

The teachers demonstrated their understanding of this interpretation implicitly in their discussion of how their children would work within the interpretation. A good example of this is provided by the discussion (Teacher interview 2: 374—463) of sharing 8 loaves between 6 people. Generally, they thought that their children would share out the wholes and distribute the remainder in different ways. They would need to be encouraged to break up and share the remaining loaves. If they did subdivide the remaining loaves, the teachers expected that they would do this by halving and then halving again. This strategy would work in some cases, but in a situation like that of the problem, where 8 loaves are shared between 6 people, it would not.

435:	TN:	First halves.
436:	BB:	You've halved it.
437:	TS:	You should actually be ...
438:	TN:	Ja ...
439:	BB:	But that's interesting. They normally will start by halving it.
440:	TN:	Ja ...
441:	BB:	And what'll they then discover?
442:	TN:	That .. two people ...
443:	TS:	They haven't got enough.
444:	TN:	Ja ... two people ... won't have extras...
445:	TS:	.. won't get. And they'll probably divide further ...
446:	BB:	Okay ...
447:	TN:	Then they'll prob'ly ... divide again in half because they ...
448:	BB:	That's it.
449:	TS:	And then they're going to end up with quarters, you see ...
450:	TN:	... they're going to end up in quarters ...
451:	BB:	... end up with quarters ...
452:	TS:	and they're going to end with these bits and bits and bits and bits ...

(Teacher interview 2)

However, more exceptional children would divide more appropriately into thirds.

I had chosen this specific sharing problem as one that could not be solved by a repeated halving strategy. My assumption underlying this was that repeated halving is the strategy chosen initially by most children when needing to create an even subdivision (Empson, 1995). A possible reason for this was that such a strategy could be directly applied and did not require the additional reflective step of first counting the number of people sharing and then choosing to divide into fitting number of parts. The teachers' discussion supported this assumption.

5.4.1.3 Grouping

The teachers interviewed in this cycle showed very similar responses to the grouping problems. In her fourth interview, Lungiswa's immediate response to a packing situation was to view it as packing apples into eight bags. When I brought her attention to the statement 'bags of eight', she was quick to rectify this response (Pilot interview 4: 100—118). After this, she worked confidently and accurately with bags of eight (Pilot interview 4: 124—148). She was able to calculate the answer using division and also to show how she judged that her children would work out their own solutions using drawing and counting. This was a change from her earlier interviews, where she showed some difficulty in working with grouping problems.

In the same fashion, Sarah and Nadine initially viewed a similar problem as sharing into six groups, changing this to groups of six, when I brought this to their attention.

621:	TS:	I just ... I just know ... my Morné. This is what they will just sit and do and they'll count .. and they'll share them until they've run out.
622:	TN:	Ja.
623:	BB:	How are they going to share them? Just as a matter of ... because they ...
624:	TS:	She .. they'll go one, two, three, four, five, six ...
625:	BB:	... notice here ... we've got ... six oranges in a bag. So how's she gonna ...
626:	TS:	Oh! I haven't read correctly...
627:	BB:	... fill the bag ...
628:	TS	... ja ... (laughing)

(Teacher interview 2)

The similarity in immediate response suggests an underlying pattern in the teachers' thinking. It is evident from the analysis of the sharing interpretation, that the teachers used sharing extensively in the teaching of rational numbers. The resulting emphasis of sharing over grouping could contribute to an over-generalization of the sharing interpretation. This could lead to an automatic assumption that the numerical parameter describing a sharing or grouping situation would be the number of groups, rather than the size of the groups. Mathematically, such an over-generalization may be made easier by the fact that both sharing and grouping are interpretations of the same mathematical operation — division.

Both Nadine and Sarah used counting procedures to calculate the number of bags formed (Teacher interview 2: 621—693). Sarah based her analysis on drawings of the bags of oranges, followed by counting the individual oranges, with each sixth orange corresponding to a full bag. On the other hand, Nadine worked in a more abstract fashion, counting out the total number of oranges in multiples of sixes. It is interesting that the more concrete approach, using 'marked' counting provided a simpler means for determining the remaining fraction of a bag.

656: BB: And how much of a bag is left?
657: TS: I see ... well maybe she'll see, "Well, half a bag."
658: BB: Okay ...
659: TS: ... if she's done that ' cause it's so obvious ca ...
660: BB: It's interesting with the drawing ... you can see
661: TS: ... in the concrete ...
662: BB: ... the half quite nicely ...
663: TS: ... it's easy to do that... yes ...
664: TN: Mmmm ...
665: BB: Nadine, if we look at yours ... you actually counted in sixes ...
666: TN: Ja.
667: BB: ... which is sort've without the drawing ...
668: TN: Yes.
669: BB: ... what you were doing here ...
670: TN: ... ja ....
671: TS: Ja ... ja ...
672: BB: It's interesting that you counted ... six ... twelve ... eighteen ... twenty-four ... thirty. And then it got the ... first six ... second six ... third six ... fourth six ... fifth six ... then you've got remainder three. How easy would it be to work out how much of a bag is left over ... (waiting) ... with this approach?
673: TN: It's not as easy to see that you've ...
674: TS: Ja ...
675: BB: Ja ...
676: TN: ... got a half a bag. But then ...
677: TS: It's not as easy for the child at this level ...
678: BB: Mhmmm ...

(Teacher interview 2)

This may have been because the remaining oranges were not enumerated when counting in multiples and so were more difficult to compare with the quantity in a full bag.

5.4.1.4 Measurement

The teachers related the measurement interpretation immediately to their teaching. They discussed measuring with non-standard units and showed an understanding of the need for part units when measuring this way (Teacher interview 2: 813—845). They were interested when I built on this and described the flexibility in measurement allowed by the freedom to choose units. Nadine also made the observation that when measuring, the unit did not function as a quantity, instead it was an object of comparison.

819: TN: What happens if it's not a full hand?
820: BB: That's also ... yes ...
821: TN: A half a hand?
822: BB: Is it a half a hand ...
823: TN: It's that kind of thing where you actually not dealing with ...
824: TS: Are you going to get part of that ...
825: TN: ...numbers ...
826: TS: ...whole hand.
827: BB: That's right ... yes ...
828: TN: ... it's ... it's a ... the unit is not a number ...
829: BB: That's right ... it's a ... it's a something ...
830: TN: .. it's a ...it's a thing ...
831: BB: Ja ...
832: TN: ... it's an object ... ja ...
(Teacher interview 2)

It was interesting to note the way that the teachers used measurement in their thinking about the interview problems. They often made rough measurements using their fingers, or objects such as pencils, to obtain approximate answers to the problems (Teacher interview 4: 536— 563). But when they wanted to make sure that their answers were accurate, they followed this with other methods, such as folding, or repeated subdivision.

564: TN: How much of the whole distance is the double line? Now ... it's about ... a sixth ...
565: BB: Ja! Okay. Now how did you come to about a sixth?
566: TN: One ... sixth. Well ... if I broke this line up into thirds then this one ... I could break this into ... another. From that mid-point, another three ...
567: BB: Okay ...
568: TN: So that's approximately one sixth.
569: TS: I just ... ja ... I just saw straight away that being the half... that being a half and then I worked it out. Scaled it ...
570: BB: So you had this as a half and this as a half and then this was?
571: TS: Must've been a third to make that divide into two more equal solids ...
572: BB: A third of a half ... ja ...
573: TS: A third of a half ...
(Teacher interview 4)

When discussing measurement using halves and quarters, the teachers mentioned that for children, quantities were very concrete and seemed to be directly apprehended in such concrete instances. In this way, quantities became concrete and yet objective for children. They used the example of half loaves, saying that these take on an existence in their own right as "whole pieces". I tried to build on this insight to discuss the objectification of any rational number. In particular to alert the teachers to the possibility that this process of objectification had an important place in the children's expansion of their conception of number, from whole numbers to include fractions. But apart from Sarah's mention that this was "not as easy" as whole numbers, the teachers did not seem that concerned with objectification. Possibly my response was misaligned and they were more interested in the concrete nature of the children's objectification.

308: TS: Ja ... the quantities that they see are very definite ...
309: BB: Ja ... that's right.
310: TS: ...and they must be able to be simply ... noticed ...
311: BB: Ja ...
312: TS: ... you know what I'm saying?
313: TN: Mmmm ...
314: BB: ... which is actually ...
315: TN: But ... but for them, then a half a bread has taken on ... the connotation of a whole ... piece. It's not ...
316: BB: Yes!
317: TN: ... it's not half a loaf ... they don't actually see the half loaf ... they see the size of the half bread.
318: BB: It's a half bread. Yes. Now that's also ... ja ... it's a nice point and ... the question is: Is there anything wrong with that?
319: TS: No ...
320: BB: Because then the half has become ... You know, you've heard, or you've probably read your half being the ... um ... an almost natural number. An almost counting number, that you count in halves.
321: TS: Mmmm ...
322: BB: But to think in terms of a fraction as a third ... as being ... to having its own existence ...
323: TS: Mmmm ...
324: BB: ... is that necessarily ...
325: TN: Not as easy.
326: BB: ... bad, or is it ... you know ... It's more difficult ...
327: TN: Mmmm ...
328: BB: ... but the key thing is if you ... If you have that idea of a third as existing, almost without the whole. Is this a good thing or a bad thing for teaching fractions? Interesting question. I'll leave it with you ... (laughing) ...
329: TS: Mull over it.

(Teacher interview 4)

5.4.1.5 Operator

When Sarah and Nadine worked on the problem involving this interpretation, they followed their usual, different, approaches. But they were both able to solve the problem easily and efficiently. Sarah worked by drawing and then analysing her pictures (Teacher interview 2: 850—875). While Nadine worked in a more abstract and numerical fashion (Teacher interview 2: 899—915). Both teachers, but especially Sarah, were intrigued by the multiplicative nature of the operations in this interpretation.

942: BB: ... but it ...the key ... the key sort've ... the mathematics behind it is you ... that operation is a ... is a rational ... it's a multiplicative operation. You don't convert it by subtracting ... you convert it by changing every eight ...
943: TN: Mmmm ...
944: BB: ... into two. Or every four into eight.
945: TS: ... or every four into eight ...
946: BB: That's it.
947: TS: ... or two.
948: BB: So you take your groups of eight into groups of two ... and so ...
949: TS: I never knew this ...
950: BB: ... what comes into this ... is ... this is normally called ... an operator concept of fractions because it's ... it's an active changing operation. The key is that the operation is a multiplicative ... is a multiplication or a division. It's never an addition or subtraction. And that's where ... that's actually where the power of fractions and rational number thinking lies is that, its with ... you're ... you are really coming to grips with multiplication and division ...
951: TS: And division ... that's what I said just now earlier on ... it's not actually adding and subtracting ...
952: BB: No, it's not.
953: TS: ... unless ... you ... you're at the stage where you're adding ... the fraction ... a quarter plus ..
954: BB: That's right. It's a slow way of getting ...
955: TS: Yes ... yes ...
956: TN: Ja ...
957: BB: .. it ... yes ... that's right ... it's the ...
958: TS: ... but this level it's definitely your ... your division ... and your multiplication.

(Teacher interview 2)

5.4.1.6 Ratio

The three teachers showed a lot of similarities in their patterns of response to the ratio problems. They all responded quickly and accurately to most that involved visual comparisons of lengths. In particular when the comparison was a half (Pilot interview 4: 890—895) or a third (Teacher interview 4: 519—529). The comparison involving a sixth caused the most difficulty. Lungiswa estimated this comparison as an eighth, rather than a sixth (Pilot interview 4: 910—921). Sarah and Nadine took more care with this comparison, using a repeated subdivision to come to their answer.

564: TN: How much of the whole distance is the double line? Now ... it's about ... a sixth ...
565: BB: Ja! Okay. Now how did you come to about a sixth?
566: TN: One ... sixth. Well ... if I broke this line up into thirds then this one ... I could break this into ... another. From that mid-point, another three ...
567: BB: Okay ...
568: TN: So that's approximately one sixth.
569: TS: I just ... ja ... I just saw straight away that being the half... that being a half and then I worked it out. Scaled it ...
570: BB: So you had this as a half and this as a half and then this was?
571: TS: Must've been a third to make that divide into two more equal solids ...

572: BB: A third of a half ... ja ...
573: TS: A third of a half ...

(Teacher interview 4)

A possible interpretation for this is that teachers can quickly estimate and name quantities which correspond to fraction quantities that they commonly use. However, when the quantity being estimated is less common, like an sixth, it either takes more care to name, or the person chooses a more common faction which has a similar size.

When identifying similarity, or looking for pictures which had the same shape, the teachers also responded quickly and accurately. They were also able to describe why the other possible pictures did not have the same shape. In Lungiswa's interview:

939: LS: She looks the picture .. so she goes close to see it better. Which of these pictures is the same picture from close? This one...
940: BB: Okay ... now why is that one the same? Why is ... ? Maybe the easiest way is to say, "Why is picture one not the same as picture two?"
941: LS: Picture w ... picture one?
942: BB I mean as ... as the ... approach ...
943: LS: Not the same as this one.
944: BB: Why not? Give me a reason.
945: LS: The shape.
946: BB: What's happened to the shape?
947: LS: This shape ... uh ... is ... uh ... is ... uh ... long.
948: BB: So it's long and thin, is it?
949: LS: Ja.
950: BB: Okay ...
951: LS: It's long and thin. This one ... is short!
952: BB: Okay. And picture two? Why is picture two not the same as that one?
953: LS: Mmmm ... picture two ... is broad.
954: BB: Okay.
955: LS: Um ... then this one.
956: BB: Yes.
957: LS: It's broader than this one.

(Pilot interview 4)

The discussion with Sarah and Nadine was quite similar (Teacher interview 4: 601—625).

Sarah and Nadine were able to solve the baking problem quickly and easily using ratio comparison (Teacher interview 2: 1124—1136). To relate the two different bakings, they performed a (within space) comparison of the number of cups of flour in each baking. They then used the resulting ratio and the quantity of sugar used for the first baking, to calculate the number of cups of sugar needed for the second. One factor which may have contributed to the ease of this solution is that there was a simple doubling relationship between the quantities in the two bakings. From the perspective of activity theory, this could be seen as an influence of the technical difficulty of the goal directed action being carried out. Another important factor may have been their familiarity with the activity of baking and changing quantities in baking. This could be seen as an influence of the teachers experience in the meaningful social activity described by the problem.

Relating the steps of a father and soon walking together, was a standard ratio problem, with a more complex 'between space' ratio. As expected, the teachers found this much more challenging. Both Sarah and Nadine approached this problem by drawing, or representing, the steps of the father and son in relation to each other. Using their drawings, they were able to calculate the number of steps by the son. These solutions formed the basis for an extensive discussion, which will be reported on in more detail under the 'coordinating groups' operation. A short interchange with Nadine showed how they used the ratio interpretation building on their drawings, to solve the problem.

How... ? What did you do with yours?
1041: TN: I ... worked out a small distance ...
1042: BB: Mhmmm ...
1043: TN: ... um ... that it took ... to cover with two steps, and that distance was covered by five steps of the little boy.
1044: BB: Mhmmm ...
1045: TN: So, for every two steps, you take five. And then I continued making a line where .... Uh ... cutting it up into pieces where for the ... above the line ... I would add another two ... steps on ...
1046: BB: Mhmmm ...
1047: TN: And below the line I would add another five steps on. And so, going ... until the ... the top line has thirty-four steps.
1048: BB: Mhmmm ...
1049: TN: Then if you ... continue along the bottom line you get eighty steps.
(Teacher interview 4: 1040—1049)

The final response of 85 was due to a mis-count of the fathers steps which resulted in the loss of one group of 5.

After they solved the problem using drawings, I asked them how they could solve the problem just using numbers. They were able to work out how to do this, but even after the extensive discussion, this took some time and careful thought. They first developed a 'between spaces' conversion (Teacher interview 4: 1111—1130). But this was given as a composite 'whole number' transformation involving a division followed by a multiplication, rather than a single rational number transformation. I asked a number of leading questions to help the teachers develop these ideas and as a result of this, Nadine was able to recognize 2½ as the conversion unit for one step of the father — a 'between spaces' unit ratio.

1157: TN: Where does two-and-a-half occur in my picture? There!
1158: TS: Here the fi ... the five steps. No.
1159: TN: The unit of ... Ja. The unit of distance is two-and-a-half.
1160: BB: What unit? Don't forget that we've got lots of units ... (laughing) ...
1161: TN: Okay. That ...
1162: BB: What is two-and-a-half steps?
1163: TN: Two-and-a-half steps ... is what the father takes ... no.
1164: BB: If you ...
1165: TN: Two-and-a-half steps is one step of the father.
1166: BB: It's ONE step of the father's ... is ... two-and-a-half steps of the child.
1167: TS: There ... you see?
1168: BB: That's it. So ... this five over two thing is ... If you were to go back to the ... the ... more abstract doing, this is a ... this is a ... a conversion unit. For each step of the father ... you get five half steps of the child. Okay? And so you ...
1169: TN: Yes.
1170: BB: And so you're actually bringing in ...
1171: TN: Mmmm ... yes ...
1172: BB: ... that is your unit ratio. And notice that the unit ratio and stuff is what comes ... what eventually develops from this.

(Teacher interview 4)

When discussing the problem of comparing the two boys running speeds (Teacher interview 4: 1180—1230), Sarah and Nadine agreed that this was too difficult for their grade threes. As one reason for this, I mentioned that the numbers were chosen so that a simple ratio did not occur. The teachers saw the difficulty as more due to the problem involving time which is an abstract quantity that has no natural representation in a drawing. Also, speed is a rate and so this problem asked to compare two rates given the basic measured quantities. Thus, the question itself was an order of complexity greater than the other ratio comparisons asked. It is interesting that to solve this problem, they both looked at unit ratios of some form. Sarah started thinking that way, but did not finish (Teacher interview 4: 1194—1203). After this, Nadine looked at unit ratio for one second and was able to solve the problem (Teacher interview 4: 1208—1228).

5.4.2 Operations

5.4.2.1 Unitizing

Initially, none of the teachers were familiar with the word 'unitizing'. But they were familiar with measurement units. Because I wanted unitizing to correspond to the formation of any coherent unit, and not just to measurement units, I introduced this concept using the expression 'grouping into wholes', or 'looking at things as forming a whole'. I also explained that this grouping may be physical, or mental.

82: You can see the first one is grouping into wholes. If you're not able to group things into wholes you're never going to cut ... be able to sub-divide ....
83: LS: Mmmm!

84: BB: ... them to get fractions. So you've always go to be able to group ... and in the real world you might be specifically grouping things into wholes ... or it might be a mental thing where you're looking at ... that ... that set of two as a whole. Something like that.]

(Pilot interview 4)

One weakness of this approach for introducing unitizing, is that it was too easy to slip into the part-whole interpretation and identify only the reference whole for a fraction as the unit. This happened to me near the end of the discussion with Lungiswa, when discussing the two different wholes, and also when describing the final six apples as part of a bag.

263: BB: Six-eighths ... of ... a ... bag ... (writing down). So you see as soon as you've got a whole ... you can talk about fractions of that whole. You can talk about fractions of a bag ... where you're dealing with that bag whole ...
264: LS: Okay ...
265: BB: ... or you can talk about fractions of your box ... where you're dealing with your box whole.

(Pilot interview 4)

This was in contrast to a more comprehensive view, where a fraction described the relationship between two units — two identified wholes. Such a view was more aligned with the theoretical view of unitizing that I wished to adopt. Because I wanted to emphasize unitizing as forming any coherent unit and not necessarily only measurement units, I decided to keep using the phrase 'forming a whole'. But I was careful in the later interviews with Nadine and Sarah to try to reduce this confusion as much as possible.

Lungiswa was able to identify the two different units in the apple packing situation, even though at times she found it difficult to express these ideas (Pilot interview 4: 210—238). I contributed in a number of instances by providing the terms such as "whole bag" and "whole box" that allowed her to describe specific units and distinguish them from other units she had identified.

Lungiswa was intrigued that an identified unit may not be something that was viewed as a natural whole, and the resulting flexibility in choosing and working with different units.

700: LS: So ... am I right to say ... a child mustn't feel that a whole refers to something that have not been patched. When ... when ... when he gets home maybe he get ... uh ... half a slice of bread, né?
701: BB: Yes.
702: LS: And then ... he asks, "Is this my bread?" "Yes, that's your bread."
703: BB: Yes.
704: LS: And to him ... it's the whole bread that is to feed him.
705: BB: That's it. That's his whole bread. Absolutely.
706: LS: For a start his whole bread, né?
707: BB: Ja.

708:	LS:	Whether it's a half or what.
709:	BB:	Ja.
710:	LS:	Whether he wants to share it or not ...
711:	BB:	That's right.
712:	LS:	... it's his whole bread. If he wants to share the ... half a slice ...
713:	BB:	Mhmmm ...
714:	LS:	... of the bread ...
715:	BB:	Yes.
716:	LS:	So a friend's going to get ... a h ... a ... a ... a half of half.
717:	BB:	That's it. It's a half of his half.
718:	LS:	Mmmm ... of his half.

(Pilot interview 4)

One situation where Lungiswa found it difficult to identify units was the third part of the investigation of possible units for ¾ of a slice of bread. This was the problem discussed earlier, in section (5.3.1.1), of cutting a quarter of a packet of three slices of bread and naming the resulting quantity (**Pilot interview 4: 322—484**). She drew the subdivision of the packet quite easily, but appeared confused when asked how much of the packet was eaten. She responded by giving the fraction of a slice (¾) rather than the fraction of the packet (¼). As discussed earlier, she did not appear to find the arrangement and cutting of the bread in the packet, natural. This may have led to the difficulty in understanding and answering my question. Almost as if the rules of the mathematical game we were playing were not transparent to her and so the question did not make sense. But this may not have been the only factor contributing to her difficulty in working with this problem. For, as will be shown later, Sarah's initial representation of the situation was aligned as desired, but she also found difficulty in quantifying the results. The complexity of the problem, which involved relating two very different units (one slice of bread, one packet) and fractional parts of each unit, may also have contributed to this difficulty.

I introduced the concept of unitizing to Sarah and Nadine as one of 'making wholes'. They found this quite natural and were quick to identify the two physical wholes in the packing problem (**Teacher interview 3: 538—557**). In the problem about buying apples, where eight apples cost R2.00, Nadine worked in two's to determine the price of 6 apples (**Teacher interview 3: 577—605**). I was able to highlight her identification and use of the two different wholes (2's and 8's) to solve the problem and show the importance of the relationship between them. A rational number could be seen as arising from this relationship between the whole 8 and the 2 as part of that whole. Another problem which the teachers solved naturally using unitizing was that involving the construction of wire cars, in the fourth interview. In this case, Sarah used two different units to solve the problem. She first drew seven whole units and divided each into three parts. She then used these thirds as units for counting the number of wire cars that could be made. It is interesting that she used the term 'piece' to refer to the whole units through most of the discussion, but at the end, she used it to refer to the remaining third. This could be taken to indicate that she had come to view a third of the given wire units as another natural unit for this problem.

345: BB: Sarah, you have drawn your ... thirds and then underneath you ... so you've got a third ... a third ... a third and then ... one ... two ... three ... four ... five ... six ... you want to chat about that? Or ...
346: TS; Yes. I just ... think that ... (clearing throat) ... Make three, obviously three ... little cars from ... one piece of wire if you only need a third of a piece.
347: BB: Mhmmm ...
348: TS: And then ... um ... cont ... drew the wire ...
349: BB: Mhmmm ...
350: TS: ... in the ... you know and ... divide it into three parts each time.
351: BB: Ja ...?
352: TS: Counting on how ... until I reach twenty.
353: BB: Okay ...
354: TS: Then I knew I ... and then I can count ... Then you can see how many pieces of wire you needed altogether.
355: BB: Yes.
356: TS: Obviously you need seven pieces, but you're going to be left with a third of that one ...
357: BB: Okay ...
358: TS: So there's an extra piece to make an extra car, so you would make ...
(Teacher interview 3)

For the interviews with Sarah and Nadine, I had changed the formulation of the problem that aimed at exploring different ways of unitizing ¾. For the reasons given in section (5.3.1.1), I changed it from eating pieces of bread to eating pieces of chocolate. This did appear to improve the alignment between situation and representation somewhat, in that Sarah interpreted the situation as hoped. However, Nadine thought of chocolates as in a box of chocolates and again I needed to intervene to explain the interpretation. We decided that it would have been better if I had used chocolate bars in the formulation. As with Lungiswa, Nadine and Sarah quickly identified the units for the first two parts of the problem. In the first, they referred to the ¾ as a single piece, but needed some encouragement to describe it as a ¾ unit (Teacher interview 3: 622—628). In the second part, they found no difficulty in referring to three units of ¼ (Teacher interview 3: 655—671).

The initial response of Sarah and Nadine to the third part was quite similar to that of Lungiswa. They started by describing each quarter of the packet as one and a half (Teacher interview 3:

719—731). In response to my question, they examined their drawing with more care and identified the reference unit they were using to make this description was a half. Converting this representation as a composite unit to a representation as a fraction of a whole, they again were able to conclude that the amount eaten was ¾ of a whole chocolate.

733: TS: Looking ... That's double. That's double the size of that.
734: BB: Okay ...
735: TS: So if that is a whole then that is a half ...
736: BB: ... okay ...
737: TS: ... of that part ...
738: BB: ... so you've got ... within your ... quarter ... maybe let's ... maybe let's take a
739: TS: ... actually it's a half and it's a quarter ...
740: BB: Ahhh! So it's one half ...
741: TS: Ja ...
742: BB: ... and a half of the half ...
743: TS: Ja ...
744: BB: ... which is why one and a half is quite correct ...
745: TS: ... so it's back to three-quarters ...
746: BB: It, again, is three-quarters ...
747: TS: Uh ... it's back to three-quarters ...
748: BB: But you see how this way you've got three-quarters as a half ... and a half of the half ...
749: TS: Mmmm ...
750: BB: It's interesting ... when you're thinking of it as a half and a half of the half ... what is your unit .. the you've ... that you've chosen there?
751: TN: I've chosen a half as a unit.

As a result of this, Nadine became quite excited about using composite units to describe the quantity (Teacher interview 3: 755—757).

In discussing what they had done with this problem, the teachers came up with another important insight. They noticed that the rational number used to describe a quantity depended on their choice of reference unit.

782: TS: So it depends on how you look at your whole to clarify what fraction you are going to work with ...
783: BB: Absolutely.
784: TN: So ... so if you .. if you
785: TS: So looking at this, I could have called this a quarter ...
786: BB: You could've ... that ... it's a ... and it's a quarter of what?
787: TS: Ja ...
788: BB: It's a quarter of the three unit ...
789: TN: ... it's a quarter of three ...
790: TS: ... of th three ...

This has very important implications for teaching rational numbers, about a teacher's response when a child uses a rational number to describe a quantity. The teacher would need to identify the reference unit used by a child in order to judge the accuracy of the child's description (Teacher interview 3: 803—813).

A good example of this occurred later in the same interview, when working with sharing bread. To perform the sharing, the teachers used repeated subdivision confidently and flexibly. Having done this, they needed to carefully examine their choice of unit in order to describe their final amount using composite units. Their initial description as ⅓ was correct, bearing in mind the context, and it was important for the teaching interaction that I did not judge it as incorrect. Instead, by asking "a third of what?" I was able to validate their response and then work with them to develop a less ambiguous representation of the quantity identified.

1102: and then I div ... divided the half into three parts again. And I called them thirds.
1103: BB: Thirds ... yes ...so it's ...
1104: TS: It doesn't seem right, hey? The dividing is right ... but that ...
1105: BB: Well, that's not wrong ...
1106: TS: It's not ... no ... it's not wrong ...
1107: BB: It's a third, but the key thing ... again, what I was saying earlier. A third is not wrong, but the question is always a third of what?
1108: TS: A third of what ... yes ...
1109: BB: And what ... wh ... what is the what in that situation?
1110: TS: The what here is a ... is a whole, isn't it?
1111: BB: Well?
1112: TS: Noooo ... a third of the?
1113: BB: A third of? This piece. And how much was this piece?
1114: TS: It was half ...
1115: BB: Okay. So it's a third, of ... the half. So just put in there ... a third of a half.
1116: TS: Okay ... I see what you're saying.
1117: BB: I would ... I would ... actually often write that ... okay?
1118: TS: 'Cause it's actually not mathematically wrong to say a third of a half.

Another example of the need to clarify units occurred in the fourth interview. When Nadine analysed the walking problem where the father and son had different size steps, she wrote

"2 = 5"

Numerically this is incorrect, however it was both meaningful and accurate to Nadine, because she was implicitly using two different reference units for the two numbers. When I prompted her by mentioning the reference units for the 2, she immediately responded by providing the reference unit for the 5 and reading the statement in the way she meant it to be read, with the corresponding units.

1055: TN: ... but that being the same distance. Two being ... equal to five ... two steps equal ...
1056: BB: Yes .. two corresponding to five ...
1057: TN: ... so ... ja ...
1058: BB: It's a terrible ... do you feel uncomfortable saying two being equal to five

... (laughing) ...

1059: TN: ... five.
1060: BB: So two fathers steps ...
1061: TN: In this father's steps, he took five children's steps ...
1062: BB: ... equals five childrens' steps. Absolutely ...
1063: TS: Mmmmm ... that would be it ... ja ...

Both Sarah and myself were much happier with this formulation. As shown here, sensitivity to units and unitizing also provides an effective means for interpreting and correcting such errors of over-compression when recording.

5.4.2.2 Iteration

Trying to be as simple as possible, I used 'counting' as a way of describing the operation of iteration at the start of my interview with Lungiswa (Pilot interview 4: 86). But this simplification proved to be ineffective and I found it necessary later in the interview to clarify my use of this word. I explained that 'iterate' meant 'repeat' and that counting could be seen as repeatedly indicating the unit (in this example, half apples) in a sequence of such units (Pilot interview 4: 826—834). I also explained that counting was important because numbers are seen by children as something that they count with, and so when counting with rational numbers, these numbers could also come to function as numbers for a child. I saw this as a more accessible statement of Steffe and Olive's (1990) reorganization hypothesis (see section 4.4.2). This use of iteration in counting with fractional units was quite natural for Lungiswa. She used it to calculate the wire needed to construct wire cars.

839: LS: Vuyani buys long pieces of wire to make wire cars to sell. Each car needs 1/3 of a piece of wire. Uh ... how many pieces of wire does he need to make 20 cars? Vuyani needs one-third ... so ... these 20 cars each needs a third.
840: BB: That's it.
841: LS: Né?
842: BB: So how many thirds does he need?
843: LS: Twenty thirds.

(Pilot interview 4)

Later, when discussing measurement it became evident that iteration was also important for this. For Lungiswa repeatedly laid out the measurement unit alongside the length to be measured. Iteratively laying out copies of the unit in such a structured pattern appeared more complex than a simple counting iteration (Pilot interview 4: 856—866). When grappling with this, I began to suspect that my initial approach to iteration as counting, was an oversimplification.

Sarah and Nadine also confidently and naturally used counting in different units. Initially they worked with whole number units, such as 8 apples (Teacher interview 3: 524—525), 50 cents or

2 Rands (Teacher interview 3: 593—599). In working out the number of apples to make apple pies, the teachers initially counted whole apples, relating each apple to two pies. To demonstrate the relevance to iteration of rational amounts, I provided a different view, showing how I could calculate this by counting half apples (Teacher interview 4: 97—113). The teachers first discussed how counting in this way corresponded to repeated addition of half apples (Teacher interview 4: 122—125). They also discussed how a child might cut and count the pieces of apple. I encouraged them to see this as counting in rational units by explicitly giving the units as they counted.

181: BB: What would a child do?
182: TN: A child would cut the apples ... like that. I think that's ... probably what they would do.
183: TS: They would ... they would definitely ...
184: TN: ... one ... two ... three ... four ... five ... six ... seven ... eight ... nine ... ten ... eleven ... twelve ... thirteen ... fourteen ... fifteen ... sixteen ... seventeen ... eighteen ... nineteen ... twenty ... twenty-one ... twenty-two ... twenty-three ... twenty-four ... twenty-five ... twenty-six ... twenty-seven ... twenty-eight ... twenty-nine ... thirty ... thirty-one ... thirty-two ... thirty-three ... thirty-four ... thirty-five ... thirty-six ... and that piece has to go away.
185: BB: So that one you can eat!
186: TN: So that one you can eat.
187: BB: So this is ... that's a child ... working basically in terms of ... cutting each apple ...
188: TN: Cutting each apple.
189: BB: Okay. And, while you were cutting the apples what were you doing?
190: TN: I was counting ... the halves ...
191: BB: Okay ...
192: TN: Take the apple, cut it, then count it. One, two halves.
193: BB: Okay ...
194: TN: Cut the next apple. Three, four halves ...

(Teacher interview 4)

Another interesting point to note in the above extract, is that when I asked Nadine to describe explicitly what she was doing, she responded to the simple unstructured prompt of 'Okay' by iterating her description of cutting an apple and then counting on another two halves. This suggests that iterating such a process may be a natural coordinating operation in such a setting.

Later in that interview, the teachers confidently counted fractional units in order to calculate the amount of wire needed to make a number of wire cars. But they did not explicitly describe this as counting thirds, instead they talked about counting 'pieces', where each piece was a third of the length of wire. When I interpreted this as counting thirds, they were quite prepared to use fraction terms this way, and even brought out the similarity between what was done here and what they did with the apples for pies (Teacher interview 4: 345—372). It appeared to me as if, although they were able and prepared to use fraction terms in this manner, this use of fractions was not yet natural to them. For they followed my prompting quite effectively, but did not work this way without prompting.

That iteration was not only involved in counting became even more evident in the interviews with Sarah and Nadine. It occurred also in measurement, as could be seen when Nadine estimated a measurement using her fingers and then her pencil.

544: TN: Oh ja ... I was ... I was going like this ... that distance. How many of those distances can I fit into...
545: TS: And we're internalizing the double ...
546: TN: ...that. And then I thought, "No, but my fingers are not very accurate so I'm going to use my pencil instead."
(Teacher interview 4)

In this process of measurement, she traversed the distance in question by iteratively placing the unit of measurement (her fingers or pencil) along the line. As with cutting and counting apple halves and wire thirds for model cars, she iterated an action and counted the iterates of the action. Describing iteration as purely counting, loses sight of the fact that it is an active process with real effects that is being iterated. And, as in the case of cutting wire to make cars, the effects of the iterated process may be important in their own right in the real situation.

I also came to see iteration as important in the development of the ratio interpretation of rational numbers. For example, when comparing the steps of father and son (Teacher interview 4: 1041—1047) Nadine iterated the basic comparison of two steps of the father with five steps of the son. Sarah also used such an iterated comparison for her solution of the same problem (Teacher interview 4: 920—935). In her case, she used counting and iteration — iterating the basic comparison of two with five to count the corresponding numbers of steps. This development in my thinking will be discussed further in subsection (5.4.2.6) on coordinated counting.

5.4.2.3 Equal Subdivision

One weakness of the tasks selected for the interview, is that the equal subdivision operations were all geometric. Equal subdivisions of discrete collections were not explored.

In general, the teachers appeared very confident with equal subdivision. Possibly because of this, they tended to merely label where they had performed such a subdivision, rather than discussing in detail what they had done. For example

854: Can you explain to me what you're doing with this one, Nadine?
855: TN: Well ... I took the slices of bread and I put them one on top of the other ...
856: BB: Uh huh ...
857: TN: ... and then I took the knife and divided that whole slice of bread into ... thirds.
(Teacher interview 3)

They often used folding to accurately carry out their subdivision (Teacher interview 4: 583—589). In fact, Nadine found that folding allowed her estimate a quantity more accurately than measuring with her fingers or a pencil.

696: TN: Ja ... I put three down ... three and three-quarters here because it wasn't ...
697: BB: Okay.
698: TN: ... quite another one.
699: BB: It wasn't quite four, with a ...
700: TN: You see, of that piece. It wasn't quite right.
701: BB: Ja ...
702: TN: So I ... that's why I kind of went to the folding.
(Teacher interview 4)

This was not the only time she used folding to check her measurements, calling it her 'crutch' (Teacher interview 5: 479—489). In such geometric situations, the ease of subdivision depended on the shape of the object being subdivided. In particular, circular objects may be difficult to subdivide equally (Teacher interview 6: 215—227).

Lungiswa appeared to find equal subdivision as a natural part of sharing. Not only in subdividing the entire quantity, but also in subdividing the remainder after distributing complete wholes (Pilot interview 4: 565—581). Nadine and Sarah also saw this as occurring naturally in sharing situations. They discussed the tendency of children to use repeated halving to perform this subdivision (see section 5.4.1.2). It was only the 'sharp child' who would have the insight to subdivide units into a number of parts corresponding to the number of people sharing the quantity (Teacher interview 2: 432—462). This raises the question of what makes division into more than two parts so much more complex.

A possible answer is that the operation of equal subdivision may be considered as a two step operation. First, to determine the number of parts to be created and second to carry out the subdivision into the chosen number of parts. From this perspective, children in early sharing situations do not specifically choose the number of parts (carrying out the first component operation), instead they automatically work with two parts.

This view of subdivision also provides some insight into comparisons of unit fractions obtained in a sharing situation. To carry out these comparisons, determining the number of parts created in the subdivision is important. It was interesting that even though Nadine knew that a third was bigger than a quarter, she did not find it straightforward to give a reason to justify this conclusion. In fact, when looking to justify this, she used the abstract representation — the 'number at the bottom' or the number of cuts made — in effect counting her physical actions of cutting. She did not use the number of parts created, even though this fitted better with the number of people in the sharing situation, as well as with the abstract representation (Teacher interview 3: 947—970). Here we see that, even for the teachers, focussing on the number of parts created was not always natural — supporting the idea that the first component operation (choosing the number of parts) may develop considerably later than the ability to actually carry out the subdivision

Another interesting feature arising in the discussions, was the use of repeated subdivision. Lungiswa was happy to subdivide units which resulted from a previous subdivision operation, but initially when describing the resulting part, she did needed to be prompted to link it to the unit subdivided.

639: BB: How much do they each get?
640: LS: Half.
641: BB: They each get half. Write down the half for So... Zolani, shall we? Okay. Don't worry about Sibeko. Carry on reading ...
642: LS: Mmmm ...
643: BB: ... the problem.
644: LS: Sibeko shares with two other friends and ... a friend to ...
645: BB: That's it.
646: LS:
647: BB: Mhmmm ...
648: LS: How much chocolate does Sibeko get? That's ... um ... a third.
649: BB: A third of ... ?
650: LS: Of a half.

(Pilot interview 4)

She did link the two subdivisions quite easily when prompted. Later in the interview, without any prompting, she worked to describe her result by linking units in her repeated subdivision (Pilot interview 4: 700—717).

In the teacher interviews, both teachers confidently performed repeated subdivisions, but initially found difficulty naming the results of their operations with reference to the initial unit. Instead, they initially named their final portion with reference to the intermediate units, but did not specify these intermediate units. After some work, they became able to link the units and name the final part in a less ambiguous fashion.

For example, when Sarah divided three chocolates into four equal pieces, she first said that the final amount was one and a half. In response, I provided some input to relate the different part units involved in the subdivision. She was able to use this to describe the size of the portion she had created as three quarters, here referencing the initial whole (Teacher interview 3: 729–745). But mastering this process of relating units took her a little while. Later in the interview, she again used repeated subdivision and described the final part as before. When I used composite units as a way of describing her result, this approach seemed unfamiliar to her and she appeared surprised that this was mathematically allowable.

1099: BB: You divided your big ... slab into?
1100: TS: Into ... into halves originally ...
1101: BB Yeeees?
1102: TS: ... initially ... and then I div ... divided the half into three parts again. And I called them thirds.
1103: BB: Thirds ... yes ...so it's ...
1104: TS: It doesn't seem right, hey? The dividing is right ... but that ...
1105: BB: Well, that's not wrong ...
1106: TS: It's not ... no ... it's not wrong ...
1107: BB: It's a third, but the key thing ... again, what I was saying earlier. A third is not wrong, but the question is always a third of what?
1108: TS: A third of what ... yes ...
1109: BB: And what ... wh ... what is the what in that situation?
1110: TS: The what here is a ... is a whole, isn't it?
1111: BB: Well?
1112: TS: Noooo ... a third of the?
1113: BB: A third of? This piece. And how much was this piece?
1114: TS: It was half ...
1115: BB: Okay. So it's a third, of ... the half. So just put in there ... a third of a half.
1116: TS: Okay ... I see what you're saying.
1117: BB: I would ... I would ... actually often write that ... okay?
1118: TS: 'Cause it's actually not mathematically wrong to say a third of a half.

(Teacher interview 3)

By the fifth interview, she was confidently able to name the results of a repeated subdivision in this way (Teacher interview 5: 1138–1146).

5.4.2.4 Unequal subdivision

The teachers seemed to find the task of carrying out the subdivision for the unequal sharing problem quite straightforward. But describing, or quantifying the results seemed rather challenging. Nadine first made an intuitive judgement to subdivide the sausage into portions

(Teacher interview 5: 1114–1120). Then she quantified her results, using rational numbers to describe the sizes of the different portions

And now the next question. How much do you think dad has got? How much of the sausage has Dad got?
1126: TN: Dad's got ... just a little under half...
1127: BB: So Dad's got about a half and Mom's got ... ?
1128: TN: Mom's got ... uh ... I think about, maybe ... two-thirds ... of the ... half that's left ...
1129: BB: Okay ... two-thirds of the half that's left ... and the child has got ... ?
1130: TN: The child's got one third of the half that's left.
(Teacher interview 5)

On the other hand, Sarah wanted to be sure that she was fair and so attempted to quantify her subdivision as she proceeded. To do this, she started with an equal subdivision and then adapted the portions relative to the size of the person.

1135: BB: We have Sarah here who's looking very carefully and ...
1136: TS: I believe in fair.
1137: (All laughing) ...
1138: TS: I just estimated in the beginning. You see, I was "Oh well thirds" ...
1139: BB: Yes ... okay ...
1140: TS: ... and then we'll see what we do with these proportion to ...
1141: BB: Mhmmm ...
1142: TS: ... size of person in ... line.
1143: BB: Ja ...
1144: TS: I divided my third up. In ... I halved it ... so dad got a third and he got a half of a third .
1145: BB: Yes ... okay ...
1146: TS: And mom got half of a third an' half of a third. And the son got left with the other half of a third.
1147: BB: Okay.
1148: TS: And altogether they would make up the whole ...
1149: BB: Ja.
1150: TS: ... three-thirds.
(Teacher interview 5)

It is interesting that even following these different approaches, the resulting subdivisions were very similar.

5.4.2.5 Scaling

In the interviews I introduced scaling as a visual recognition of a stretch or shrink (Pilot interview 4: 86–88). In the line painting problem, Lungiswa used visual scaling effectively to compare the lengths of the different lines. After making sure that a comparison was wanted, she made an immediate visual judgement to recognize a half of a comparison length (Pilot interview 4: 884–896). She was also able to immediately recognize a third of a comparison length.

910: BB: Ja ... This one ... compared to that ... is how much?
911: LS: Three ... thirds.

912: BB: That's it!
913: LS: Yes.

(Pilot interview 4)

However, when it came to recognizing a sixth, she saw it instead as an eighth.

916: BB: ... and how much of the whole?
917: LS: Of ... which one now?
918: BB: The double ... of the whole amount you painted this.
919: LS: That is from here ...
920: BB: Ja... from there ... to there ... From the beginning to the end.
921: LS: You are measuring an eighth.

(Pilot interview 4)

Possibly this was because she was much more familiar with eighths than with sixths.

When Nadine and Sarah worked with the same problem, Nadine saw such a process of visual recognition as a guess, almost as if she distrusted such an intuitive process.

481: BB: Can you ... can you, without doing anything ...
482: TN: Oh. Okay.
483: BB: ... tell me how much of the whole distance ... just as a matter of interest ...
484: TN: Just as a guess?
485: BB: Ja! Just as a guess. For this one, do a guess. Don't try 'n be too specific.
486: TN: Okay.

(Teacher interview 4)

She used visual scaling to determine the comparison of a third, but she was not confident until she had checked her visual estimate by first folding and then measuring in two different ways.

536 BB: How did you ... how did you do it? How did you do the comparison?
537 TS: So you actually want us to estimate ...
538 TN: Well I ... I measured ...
539 BB: Ja ...
540 TN: I measured the distance with my pencil ...
541 BB: Now before you worked with a pencil you did something. What did you do before you worked with a pencil?
542 TN: I folded ... I picked it up and I ...
543 BB: It was af ... it was after you folded and before you worked with a pencil. You used your fingers ...
544 TN: Oh ja ... I was ... I was going like this ... that distance. How many of those distances can I fit into...
545 TS: And we're internalizing the double ...
546 TN: ...that. And then I thought, "No, but my fingers are not very accurate so I'm going to

(Teacher interview 4)

In this way, she related the visual scaling operation with a subdivision operation and the measurement and part-whole interpretations. One possible effect of this would be to build links between these operations and interpretations — the more intuitive visual scaling and the more formal equal subdivision operation and measurement interpretation. Such a link would contribute to mathematising the basic intuitions due to visual scaling.

In contrast to this, Sarah was more prepared to use visual scaling, seeing it not as guessing, but as an intuitive process of estimation "with her eyes". This would provided acceptable, even though not perfect results.

> **492:** TS: Show how much of the solid line ... is the double line. How much of the whole distance ... is the double line?
> **493:** (Time)
> **494:** BB: Nadine is using fingers and ...
> **495:** TS: I'm just estimating with my eyes ...

(Teacher interview 4)

To be more certain of her estimation, Sarah used a combination of scaling and counting parts. In this way, she judged that the double line was a sixth of the whole length (Teacher interview 4: 504–514). Nadine also estimated this as a sixth. When asked to explain how she came to this, she described using a combination of scaling and repeated subdivision (Teacher interview 4: 564–568).

The second problem involving scaling was that of recognizing a scaled image from two distorted versions. Lungiswa immediately recognized the scaled picture, stating that it looked 'the same' and the other two pictures had a different shape (Pilot interview 4: 939–959). Nadine and Sarah also immediately recognized the scaled picture. When asked how they made their judgement, they reported that they visually recognized the shape. In this situation, they were both confident of the accuracy of their visual judgement (Teacher interview 4: 602–620).

In the fifth interview, Sarah introduced the idea of distortion as a means of describing how two of the comparison pictures were not the same as the original. The teachers recognized that the distortion arose because the different axes were scaled by different factors (Teacher interview 5: 566–596):

> **574:** BB: These ... you s ... have been distorted. How have they been distorted? What do you mean when you say distorted?
> **575:** TS: The length and width.
> **576:** TN: The one .. they've taken ... they've narrowed the width ... but kept the height.
> **577:** BB: Mhmmm ...
> **578:** TN: The other one they've kept the width but narrowed the height.
> **579:** BB: Okay. And with ... picture three?
> **580:** TN: And picture three ... the whole thing has been ...
> **581:** TS: Generally enlarged ...

Sarah described the distortion due to doubling the horizontal extent as "fitting" two of the original pictures next to each other. While the undistorted picture was described as merely an enlargement (Teacher interview 5: 600–610).

5.4.2.6 Coordinated counting and coordinating groups

My understanding of coordinated counting was not stable throughout the interviews. It developed and changed in response to the discussion with the teachers. Initially, I saw it as involving a counting process of linked multiples. That is, as coordinated counting in two different units. As I described in the interview with Lungiswa

1096: BB: Eighty-five steps. Notice what you've done here. You've counted in twos ... but at the same time ... sort of ... if you think of it as counting, Mboleli's father is counting in twos. Mboleli's counting in fives and they're counting together ... two and five ... four and ten ... six and fifteen .. eight and twenty. Isn't that right?
1097: LS: Mmmm ...
1098: BB: That's what I mean by co-ordinating groups. You're counting in twelve ... we could do this nicely together.
1099: LS: Eh.
1100: BB: If you were to count twos ... I was to count in fives ...
1101: LS: In fives ...
1102: BB: ... in the same rhythm.
(Pilot interview 4)

In the problem involving Mboleli and his father walking together, Lungiswa did precisely this, using a drawing to coordinate her counting of the steps of father and son (Pilot interview 4: 1017–1047). This approach, though simple, was effective and easily understood by Lungiswa. I attempted to deepen the analysis, by exploring the number of multiples in each sequence. However, this became a bit confusing to Lungiswa and so we returned to her initial approach (Pilot interview 4: 1066–1095).

In the interviews with Sarah and Nadine, it was evident that coordinated counting was an easy and natural operation when one of the units is 1. In her drawing for the apple pies, Sarah coordinated groups, linking the units of 1 apple and two pies.

86: TS: Well. It is thirty-five, hey?
87: BB: The one thing I'd like to show ... like you to show me, Sarah, in your drawing. you've shown there's one apple for each two pies.
(Teacher interview 4)

In a similar fashion, in the car construction problem, Nadine linked one length of wire with each unit of three cars (Teacher interview 4: 404–411). When working with the apple pies, the teachers also noticed that the problem could be approached in two different ways — focussing on either whole apples, or whole pies. These different choices of the unit of 1 would result in different, but equivalent coordinations (Teacher interview 4: 128–151). In these problems, the 'counting' of coordinated counting was quite straightforward and the main issue of discussion was just what groups should be coordinated for counting together. To refer this, I used the term 'coordinated groups'.

Sarah and Nadine also found the problem about the number of steps of a father and his son, more taxing. But with careful thinking they were able to find a solution. Sarah immediately identified the coordinated groups in this problem.

> **904:** BB: Can you explain to me what that means?
> **905:** TS: Every two steps the father takes equals five steps of the child.

(Teacher interview 4)

But using this to find a solution was not immediate. By analyzing her drawing, she decided that she could find a solution by counting the coordinated groups (Teacher interview 4: 911–918). She first counted in two's to reach 34, then counted in fives and finally related the two counts to obtain the solution (Teacher interview 4: 920–939).

Nadine followed a different procedure. She identified a distance which corresponded to two father's steps and five son's steps. She used this as an intermediate unit to relate the 2-group and the 5-group. That is, instead of directly coordinating 2-groups with 5-groups, she coordinated each with her intermediate 1-unit. Doing this, she was able to use an operation with which she was confident (coordinating with 1-units) to carry out the desired, more complex coordination.

> **1041:** TN: I ... worked out a small distance ...
> **1042:** BB: Mhmmm ...
> **1043:** TN: ... um ... that it took ... to cover with two steps, and that distance was covered by five steps of the little boy.
> **1044:** BB: Mhmmm ...
> **1045:** TN: So, for every two steps, you take five. And then I continued making a line where .... Uh ... cutting it up into pieces where for the ... above the line ... I would add another two ... steps on ...
> **1046:** BB: Mhmmm ...
> **1047:** TN: And below the line I would add another five steps on. And so, going ... until the ... the top line has thirty-four steps.
> **1048:** BB: Mhmmm ...
> **1049:** TN: Then if you ... continue along the bottom line you get eighty steps.

(Teacher interview 4)

(Her final conclusion of 80 steps was incorrect due to a minor counting error.)

It was interesting, that having coordinated the 2-group with the 5-group, she described what she had done by saying that 2 was equal to 5. Such a description made sense when 'equals' was interpreted in terms of her specified distance in the grounding situation, even though it did not make sense when interpreted in the standard abstract, numerical context.

> **1055:** TN: ... but that being the same distance. Two being ... equal to five ... two steps equal ...
> **1056:** BB: Yes .. two corresponding to five ...

(Teacher interview 4)

Using the intermediate unit of Nadine's scheme as a reference whole, she and Sarah were able to show how steps in each group could be represented as fractions and the coordination could be represented as the equality of the distance moved in the specified number of steps (Teacher interview 4: 1089–1100).

In my discussion of Sarah's solution of the apple pie problem, a change in my thinking about coordinated counting was becoming evident. I began to grapple with relating the ideas of coordinated counting, coordinating groups and iteration (Teacher interview 4: 1177–1186). In the interview 5, my changing thinking about coordinated counting is clarified using the idea of a swop.

> **922:** BB: So it's interesting is the ... you see how the co-ordinated counting ... is ... Here you're doing the co-ordinated counting ... which is what I talked about before.
> **923:** TS; Mmmm ...
> **924:** BB: But it's actually based on, a single swop. And the single swop is what's really the ...
> **925:** TS: That's your single swop.
> **926:** TN: Ja ...
> **927:** TS: And that you can work out however many times you need to swop for the rest.
> **928:** BB: That's right.

(Teacher interview 5)

Here I mentioned the importance of linking one single group from each sequence and Sarah's response implies that coordinated counting involves iterating the coordination such groups. However, I had not yet carefully formulated these concepts. A better formulation was given in interview 6 in response to a query by Nadine about exchanging.

> **734:** BB: ... with groups of two goons. Not very well said ... but that to me ... it's almost a ... Co-ordinating groups can almost be seen as a combination of exchanging and the iteration-type operation. Where you're doing the same exchange quite a few times.

(Teacher interview 6).

In this discussion we had mistakenly used the term 'coordinating groups' in place of 'coordinated counting'.

5.4.2.7 *Exchanging*

The operation of exchanging was not included in the fourth pilot interview. But, after the interview, I decided to include it in the interviews with the two teachers, as a possible operation important in the contexts of swopping or buying and selling. From the start of these interviews I had seen the complete exchange process as involving two constitutive operations — an iteration of basic exchanges — and the problems were structured accordingly.

The teachers performed exchanging naturally and confidently when simple multiples of the basic exchange were involved (Teacher interview 5: 871–889, 899–913). When they discussed how to deal with rational multiples, their first response was to disregard the items which did not for a full unit.

928:	BB:	That's right. Then you ask the interesting one. What about (E)?
929:	TN:	Thirty-two apples.
930:	BB:	Yolande gives thirty-two apples.
931:	TN:	How much should Lulama? Okay.
932:	TS:	And now you can't divide five into thirty-two.
933:	BB:	Ahhhhh!!! You have a problem. Is that you've got ...
934:	TS:	There'll be two apples left over ...
935:	TN:	Mmmm.
936:	TS:	... two apples too many ...
937:	TN:	From thirty-two apples.
938:	BB:	So what're you going to do with those two apples?
939:	TS:	Take them back ... (chuckling) ...

(Teacher interview 5)

When I encouraged them to perform the exchange for the extra items as well, they were able to estimate and justify an appropriate swop (Teacher interview 5: 944–971).

As my conception of coordinated counting developed to a composite of coordinating groups and iteration, I came to see less of a difference between exchanging and coordinated counting as operations. I came to see the difference more as one of context than of coordinating operations in context.

5.5 Emergent themes relevant to teaching and learning rational numbers

5.5.1 The importance of the teacher

When discussing exchanging in the final interview, I grappled with how exchanging may contribute to the development of a more abstract and formal rational number understanding. How one could move from intuitive competence to a more reflective understanding of number related to these situations (Teacher interview 7: 869–881)? The teachers response was that this would need teacher's intervention to demonstrate the links, their children would not be capable of making this transition on their own. But, with good scaffolding provided by the teacher, some of their children would be able to develop this understanding (Teacher interview 7: 891–913).

5.5.2 Writing down representations explicitly

In order to develop the skill of using the mathematical representations, it is important to write down the interim results and solutions making explicit use of the specific representation. To ensure that this is done and that children develop this habit, the teacher may need to continually remind them to do this (as I did in this instance).

617: LS: And so would start with this quarter ... left hand one ...
618: BB: Mhmmm ...
619: LS: ... each one would ... uh ... one ... two ... three ... one and three-quarters ...
620: BB: Okay ... so each child gets ... write the number down there.
621: LS: (Writing down) ... one ... three ... quarters.
622: BB: And for ... Lulama's children?
623: LS: A one three ... a one and three-quarters.
624: BB: Yes ... that's okay. And Lulama's children? How much would each of Lulama's children get?
625: LS: Lulama's children will get ... uh ... how would you say, né? One and two-thirds.
626: BB: One and two-thirds ... okay.
627: LS: One ... two- thirds ... one two -thirds ... one two-thirds ...

(Pilot interview 4)

5.5.3 Comparing the amounts obtained from two different sharing problems

An important question that occurs naturally in sharing situations is: "Who gets more?" It is relevant to ensure fair sharing (and so an equal partition) but is even more interesting when there are two different groups of people sharing. Attempting to answer this question may lead to a much deeper and more explicit analysis of the situation and the corresponding mathematical representation, than was carried out in doing the sharing.

The first equal subdivision problem investigated a sharing of 5 slices of bread between 3 people, and 7 slices between 4 and asked who received more (Teacher interview 3: 903–970). To answer this, the teachers constructed $\frac{3}{4}$ and $\frac{2}{3}$ and then visually compared them. I attempted to encourage them to deepen their analysis by asking if this was the only way the comparison could be carried out. Nadine responded by asserting that the question could be answered by comparing $\frac{1}{4}$ and $\frac{1}{3}$. This could be done visually, but also by noting that $\frac{1}{4}$ is less than $\frac{1}{3}$ because it involves sharing a whole between more people.

5.5.4 Comparing solution strategies which produce different but equivalent solutions

The second problem for the operator interpretation involved sharing a fractional piece of chocolate. The teachers discussed their different strategies for finding a solution to the problem and the teaching opportunities offered by these different strategies (Teacher interview 2: 1052–1089). As long as the interpretation or formulation of the problem was the same for both strategies, the final solutions would be equivalent, even if they appear different. Comparing the solutions in the light of this knowledge, would yield the recognition that the solutions provide two representations of the same quantity. That is, the representations are equivalent. Furthermore, comparing the strategies would provide insight into the relationships between the representations. This could be done by looking carefully at how each representation was generated in describing the strategy and the relationships between the strategies would yield corresponding relationships between the representations.

Such an explicit discussion and relating of different approaches can be seen in the following discussion between the teachers about their solutions of the second equal sharing operation.

1129: TN: Then I said, "But hey! If I divide this half into thirds ... then I can divide the other half into thirds as well."
1130: BB: Okay ...
1131: TN: Which will give me six pieces altogether ...
1132: BB: Mhmmm ...
1133: TN: ... of which ... Sibeko only gets one.
1134: TS: I see, I was looking at the other option ...
1135: BB: Yes.
1136: TN So I ... I went ... her route and then I looked at it and said, "Okay ... if they're the same then ..."
1137: BB: Ja ... then how do we link it?
1138: TN: You link it across ...
1139: BB: So what you've actually got is you had ... (clicking his pen) ... you first worked with a third of a half ...
1140: TN: Ja.
1141: BB: ... and what you've now shown is that a third of the half ... is the same as ... a sixth of the whole ...
1142: TN: ... of the whole.

(Teacher interview 3)

In this case, relating the two approaches led to an understanding of the equivalence of two different representations of the final quantity: a sixth and a third of a half.

5.5.5 Naming a concept

Naming the result of an operation may help a person to identify this result as an important concept in its own right. Here Lungiswa states that naming the result allows the formation of "a whole new concept".

749:	LS:	The whole was divided ...
750:	BB:	... a whole. That's right.
751:	LS:	She was got a half ... a half. Just like dividing a big cake ...
752:	BB:	Yes ...
753:	LS:	... at a party ...
754:	BB:	Uh huh ...
755:	LS:	... né? So that the whole you've got ...
756:	BB:	Yes ...
757:	LS:	... in a word ... just ... uh ... a sixtieth of that ... that ...
758:	BB:	That's right ... ja ... ja ...
759:	LS:	... it's a whole new concept.
760:	BB:	It's a whole new concept.
761:	LS:	You've got to go together that whole at home.
762:	BB:	Yes ...
763:	LS:	And you only divide that ... sixteenth equal.
764:	BB:	Yes ...
765:	LS:	Okay ...

(Pilot interview 4)

5.5.6 Quantifying remainders

The careful attempt to provide a precise quantitative representation of remainders may lead to a deeper and reflective analysis of the situation and the mathematical representations of the situation, than was necessary for the work leading up to the remainder. This could occur for many different types of remainder.

The simplest example of this occurs when sharing a number of items between a smaller number of people. One of the common approaches to this sharing is to start by giving the same number of whole items to each person. Non-whole numbers are only needed when the items may not be evenly divided (as wholes) and the remaining items need to be subdivided to obtain a complete and fair sharing. Such a sharing situation and expected children's responses was discussed in section 5.3.1.2. It was noted that at grade three level, many children would not share the remaining two slices of bread, choosing instead to give them away, or store them or eat them. They would need to be encouraged to subdivide the remaining slices and share the pieces fairly.

For an example involving another interpretation, the child may need to quantify that remains after attempting to measure a quantity using a chosen unit. This remnant will be less than the unit and will need to be described quantitatively in terms of the chosen unit. An alternative strategy would be to subdivide the unit to obtain new fractional unit that will yield a measurement without a remainder (Teacher interview 2: 813–845). A similar process may occur when faced with a remainder in a grouping or packing situation. But here the size of the group will generally have been specified and so the child will not have the option of changing the unit.

Investigating the remainder may also be needed for exchanging problems. That is, the portion that is smaller than the basic quantity which is used to describe the exchange. To do this, it will be necessary to precisely quantify an exchange for this portion that would fit with the general pattern of the exchange. This was necessary for the problem of exchanging chocolates for 32 apples when every 5 apples were exchanged for 2 chocolates (Teacher interview 5: 932–990). This resulted in a comprehensive discussion, first developing and justifying the estimate of 1 chocolate and then proceeding to subdivide both chocolates and apples and determine a better estimate of $1\frac{1}{2}$ chocolates. The teachers found this discussion quite challenging.

Chapter 6 – Children Cycle : Preparation

6.1 Identifying final issues

My understanding of the rational number interpretations and operations developed considerably during the interviews of the teacher cycle. The interviews, combined with the perspectives provided by activity theory and situated cognition, yielded a number of insights into the possible manner in which the interpretations and operations could contribute to the formation of an intuitive understanding which would lay the foundation for a meaningful knowledge of rational numbers.

For all but the part-whole interpretation, the teachers had identified a number of situations which they considered to be within the experience of most of the children in their grade three classes. These included swopping sandwiches or school lunches, sharing things fairly, shopping and playing different games. In their opinions, the children would be motivated to involve themselves in these situations, acting in order to satisfy motives which they would find personally or socially relevant within the situation. The intuition that they gained through taking part in these activities would be imbued with personal significance from the satisfaction of the motives through the activity. This perception of significance would encourage the children to retain and extend the intuition developed through these activities, thus forming a foundation for further learning. A good example of this was described by Sarah:

953: TS: And they loved that ...
954: BB: Okay ...
955: TS: All of a sudden it was life.
956: BB: Yes.
957: TS: All of a sudden this ... this fraction had come alive in ... real terms ...
958: TN: Mmmm ...
959: BB: Yes ... that's right.
960: TS: 'Cause you can use a quarter of what you've got and whatever ... if you're going to ...
961: BB: Absolutely ...ja ...
962: TS: ... even if it was cherries and you were decorating ...
963: BB: (Chuckling) ...
964: TS: ... cakes
965: BB: (Still chuckling) ... ja ...
966: TS: ... you know? Uh ... um ...
967: BB: Ja ...
968: TN: Ja ...
969: TS: ... so I think for children it's ... it's ... its so important if you can make it ... meaningful in their lives.
970: BB: Yes.

971: TS: "As soon as I can see the importance of it then I'm prepared to learn about it." (Teacher interview 1)

We had not been able to identify situations which related directly to the part-whole interpretation. However, this interpretation had occurred a number of times as a transient state in the process of working to achieve the goal for other interpretations. This transient state generally related to the representation of quantities relevant to the other interpretations, in order to represent, or ensure accuracy in the achievement of the desired goal. Thus, the interpretation could be seen as mainly descriptive or representational in character. In particular, it provided a model for the representation of rational numbers in the interval [0,1] as proper fractions.

Returning to the semantic analyses in which the different rational number subconstructs were developed (Kieren, 1988; Kieren 1993; Behr et al., 1983; Behr et al., 1992 and Behr et al., 1993) it became evident that the different logical and mathematical operations constituting each subconstruct, were intimately related to corresponding symbolic representations of rational numbers. However, it was seen in the teacher interviews that the subconstructs, apart from the part-whole, also relate to situated activities of relevance to children.

The representational character of the identified subconstructs is not surprising, because the research in which they were identified drew on the information processing tradition, in which cognitive systems are viewed as physical symbol systems (Newell, 1980). That is, symbolic representations were viewed as the basis of cognition. However, my research draws on the views of situated and embodied cognition, where not all mental representations are assumed to have a symbolic basis (Clarke, 1997). This alternative is particularly important for the subject of my research, that is for informal knowledge which is developed through situated action and forms the basis for the development of a more formal and symbolic understanding.

Classifying interpretations (situated activities or actions which could give rise to informal rational number knowledge) on the basis of different mathematical representations, would thus involve a circularity which is unnecessary and possibly misleading from the perspective of situated cognition. This gives rise to the question of whether the classification of interpretations based on the identified subconstructs, is indeed complete. That it is not, was evident at the beginning of this project where the single quotient subconstruct gave rise to two different interpretations — sharing and grouping — on the basis of research into the early learning of whole number operations. An answer to the question of completeness would be necessary in order to judge whether an informal foundation for rational number learning based on these interpretations would indeed be comprehensive. As a result, I formulated the first issue needing to be addressed in order to complete this research:

1. To formulate and implement a systematic approach that would be suitable from the perspective of situated cognition, for judging the completeness of the interpretations of rational numbers identified in this research.

This issue is of a theoretical, rather than empirical nature, for questions of completeness are not amenable to inductive arguments based on experience. Rather, completeness would need to be judged on the basis of the logical structure of the approach.

A second issue of a more theoretical nature also needed to be addressed in order to consistently use the theoretical framework adapted for this work. This is:

2. To clarify the relationship between interpretations, activities and actions. In particular, to describe how the concepts of 'activity' and 'action' may be applied to each interpretation.

The analysis of this issue would be informed by both the teachers' discussion of the interpretations and by the children's responses to the contextual problems set.

As mentioned earlier, the personal or social motives for the situated activities chosen to develop the informal foundation, were important in order to lend significance to the learning and to encourage children to engage in the activities. But the importance of the developed intuition for the learning of rational numbers, derives from the form of the intuition, not from the motivation for the activity. For it is the structure of what is done in these activities that relates to the rational number concept in general and the specific sub-construct in particular.

In the teacher interviews, specific structure of relevance to rational number learning was identified in most of the interpretations and operations that had been put forward. Due to the complexity of the actions needed to accomplish the goals, it was expected that each interpretation would involve many elements of structure. But in identifying the different operations, I attempted to isolate structure which could be viewed as single units of coordination relevant to rational number understanding. However, the need for adaptation of some of the proposed operations, in order to yield better alignment between the operation and corresponding mathematical, or protomathematical structures had also emerged. In particular the operations of coordinated counting, exchanging and the initial formulation of iteration needed adaptation. This raised the following two issues to be addressed.

3. To identify operations which show evidence of being able to serve as units of coordination.
4. To identify different combinations of operations which could possibly constitute the action in each interpretation.

It also became evident in the interviews, that to develop the children's rational number understanding beyond this informal foundation, it would be necessary for children to:

- become reflectively aware of the relevant structure, and how it relates to both
- the mathematical concept and
- their actions and operations aimed at satisfying their motives within the activity.

Until this awareness was achieved, the intuitive competence developed by the children would make no contribution towards the children's mathematical learning. To accomplish this development, children would need to become able to interpret the structure of the grounding situations and their acts in the situations, in a manner that built mathematical insight. Furthermore, for the development to be seen as building on, rather than being arbitrarily imposed on the informal foundation, these interpretations would need to be seen as natural and 'fitting' to the situations.

This problem may be seen as an instance of a well known shortcoming of the structuralist approach to teaching mathematics (Treffers, 1987), in a more realistic setting. The problem is that a person who is aware of and understands the mathematical structure, may be able to identify the salient features in the realistic setting which yield a correspondence between the realistic structure and the mathematical structure. However, a child who is not familiar with the mathematical structure has no such guidance for identifying structural features of the realistic situation. As a result, even with the teacher's help, they may not be able to identify the structural features which could provide good mathematical insight. Or else, they may identify these features, but find them artificial or forced — not helpful for the solution of what they perceive as the problem in the given situation. Sarah described this in terms of what makes sense for adults and what makes sense for children.

319: TS: Because they are allowed to speak up for something that makes sense in their

heads.
320: BB: Mhmmm ...
321: TS: Because what makes sense in adults heads doesn't always make sense to them at this level.
322: BB: Yes.
323: TS: And I find if, as long as we can follow what they're saying and interpret it as something that's ... that's acceptable ...
324: BB: Mhmmm ...
325: TS: ... it's fine.
326: BB: That's great ...
327: TN: Ja.
328: TS: That's fine, they'll get to where we are. Just give them a chance to ... start ... where they're at.

However, this not only relates to the difference between adults and children. It may occur in any learning situation which requires the recognition of structural features which are unfamiliar to the learner. A good example of this was provided by the teacher's responses to the problem which investigated one representation of 3/4 by describing the cutting of a packet of three objects into four quarters. This was discussed in sections (5.3.1.1) and (5.4.2.1) of the teacher interview analysis, in terms of the problem situation being 'aligned with', or providing a 'natural representation' for the structural features giving rise to the part-whole interpretation and unitizing operation. These terms were used there to denote the condition that the structural features desired for the mathematical learning, would be those that naturally occur to the learner as a result of their previous exposure to the problem situation. This example showed that finding a problem situation which yields such a natural representation of the desired structure may be rather difficult.

These questions gave rise to the final issue needing consideration:

5. To formulate a response to the problem of interpretation and its effect on the development of a more conceptual rational number understanding on the basis of this informal foundation.

A theoretical investigation into the final issue had been lacking in the study and to remedy this, I returned to the literature. Particular topics of interest investigated were:

- Activity theory;
- realistic mathematics education;
- implicit learning and cognition;
- situated and embodied cognition and learning in practice;
- meaningful learning;

- Piagetian theory of learning and cognition.

6.2 Activity theory revisited

Relating interpretations, activities and actions was the second issue that needed to be addressed to finalize the research. As it was of a theoretical nature and was expected to provide useful insight to the structuring and analysis of these interviews, I decided to seek clarity on this issue before proceeding in this cycle. Clarifying the relationships between interpretations, activities and actions, required a more detailed understanding of these fundamental concepts of activity theory.

An important point to bear in mind at the outset is that activity theory was developed in order to better understand collective acts and so deals primarily with communities defined through such acts — activity systems (Engeström and Meittinen, 1999). In this work, my interest was in the development of personal understanding by the individual involved in such systems. A number of the concepts relevant to activities and actions are not directly applicable to such an individual focus. Some of these may however, be interpreted in a manner which contributes to the addressing of the issue.

Engeström (1999), in his discussion of activity theory, stresses the importance of the object of an activity. He does not see the object as the practical outcome of the activity, but rather as the focal issue of the activity. The material or practical outcome (or desired outcome) of the activity is also important for understanding the activity. The motive of the activity may then be seen as the projection from the object to the outcome — what drives us to approach the object in the given context, so that the specified outcome is produced.

The distinction between object and outcome can prove useful when analysing teaching and learning. For example, a problem set for teaching which explicitly describes the object of the problem as sharing, may have as a desired outcome both the description or enaction of the sharing, and the production of a fraction representation of the amount received by each person. From this it is evident that teaching activities that aim to introduce mathematical concepts in a grounding context will have two different motives. A teaching motive, which obtains from the teaching outcome and a grounding motive which it is hoped that the children will read into the description of the grounding situation, in order to lend significance to the learning. In such a case, an important aspect of teaching would be to balance these two motives: dominance by the grounding motive will result in little teaching and learning, while dominance in the teaching motive will result in learning the concepts as unrelated to the grounding situation.

Engeström (1999) also stresses the importance of the concept of mediation in activity theory. Our acts in the world are seldom direct, being generally mediated by tools and signs which increase the power and effectiveness of the acts. Acts on the physical world are mediated by physical tools and acts in the social world are mediated by signs. Moreover, such mediation is not only external to the acting person, but becomes internalized through practice, contributing to the development (or in many versions of the theory , the formation) of the psychology of the person. This importance of internalization of mediating tools and signs was described by Vygotsky (1986) and formed one of the foundations in the development of activity theory. For ease of reference, I will use the term 'mental tools' to refer to the mental structures or developed capacities which are formed in this manner, irrespective of whether they originate in mediating tools or signs. In the second cycle of this research, the importance of mathematical representations in the teaching and learning process had become evident. Appropriating the idea of mediation, mathematical representations may be seen as functioning as external mediating signs (descriptive) and tools (transformative), which become internalized in both formal and intuitive forms in the process of early learning. That is, the concepts of mediation and internalization will prove fruitful in the analysis of developing mathematical representations, and the effects of these representations on the children's approach to problems.

To better distinguish between activity and action, I carried out a more careful investigation into the difference between activity and action in the writings on activity theory. One simple distinction is that generally activities are seen as collective, while actions may be carried out by groups or individuals (Engeström and Meittinen, 1999). Even though I aimed to work with children individually, this did not exclude viewing the interpretations as activities, because I wished the children to approach the given problems by imagining themselves within the situation. This could be done as easily for group activities as for individual actions. A more important distinction is provided by Leontev (1978). In his exposition, the object of an activity can be seen as providing motivating significance for the members of the community engaged in the activity. An activity is generally constituted by a number of actions and achieving the goals of a single action will not satisfy the motive of the activity. Consequently, and in contrast to the motives of an activity, the goals of an action are not seen as intrinsically significant by community members. Because of this lack of intrinsic significance, actions may be seen as irrelevant, or even alienating by those who performing them. However, this need not be the case, because the goals of each action are related to the object of the activity.

This distinction between the motivating significance of the object of an activity and the lack of intrinsic significance of the goals of an action, will be used to judge whether each interpretation could be best viewed as an activity or as an action. A point to bear in mind is that different communities will view different objects as significant and so for a child to view an interpretation as an activity, this would need to be the case for a community of relevance for the child.

Finally, a brief mention will be made of the form of actions and activities. Actions are deliberate and consciously directed in order to achieve the desired goal in the situation. In order to do this, the action needs to structured to take advantage of the affordances offered by the situation. Furthermore, to successfully carry out an action will require a certain level of skill in using the mediating artifacts to act within the situation. As a result, I concluded that an analysis of the structure of the children's acts and the manner in which these acts related to rational number concepts, would proceed from a view of the interpretations as actions.

An activity is seen as being far more complex and intrinsically collective, being properly analysed in terms of an activity system (Engeström, 1999). However, due to the individual nature of the learning being investigated, it was judged that the full complexity of activity systems would not be applicable to the analysis in this work.

6.3 Literature review / Theoretical development

In order for theoretical principles to guide teaching interventions in more than an intuitive manner, they would need to provide a specific and detailed understanding of the processes possibly involved in developing a more formal or reflective understanding of rational numbers on the basis of intuitive knowledge attained through engagement in activities and actions in everyday life (Gravemeijer, 1999).

Constructivist principles imply that people construct their own understanding on the basis of their own experience, in ways that are meaningful and significant to them. This statement formed one of the motivating factors for this research, leading to an appreciation of the possible roles of interpretations, activities, actions and operations in the development of reflective rational number knowledge aligned with a person's intuitive foundation. A further result, was the formulation of the concerns forming the basis for the review of theory on this cycle. However, this general statement does not answer these concerns, lacking the level of detail necessary to provide a specific guidance for the practice of rational number teaching examined in this research.

The theory of realistic mathematics education provides a more detailed analyses of the learning process, yielding further insight into the specific concerns at issue in this cycle.

6.3.1 Realistic mathematics education

In order to obtain more insight into the process of developing mathematical understanding from children's experience of solving contextual problems, I consulted the writings of the Dutch school of "Realistic Mathematics Education". This school developed from the work on mathematics education by the mathematician Hans Freudenthal (1983), which was adapted and extended by the Wiscobas project aimed at the reform of the school mathematics curriculum in the Netherlands (Treffers, 1993). Thus, as well as having a sound theoretical basis, these principles have been extensively tested and refined through practical implementation in Dutch schools, contributing to the result that Dutch school students were rated seventh for mathematics in the TIMSS R (Third International Mathematics and Science Study Repeat) study (Mullis, et al. 2000) of 38 countries throughout the world at the grade 8 level.

A comprehensive overview of realistic mathematics education is given by van den Heuvel-Panhuizen, in her articles (2000, 2001). Unless otherwise specified, in this subsection all citations will refer to these articles.

6.3.1.1 Mathematical learning

Realistic mathematics education is based on the premise that mathematics is a valuable human activity and should be taught as such. That is, children's learning of mathematics should be active, engaging children in the process of re-inventing mathematics based on experiences which they find relevant and of value in their lives. However, just how this may be judged is not clearly specified, rather, it is left to the intuition of the teacher. As discussed in the teacher cycle, activity theory may be used to provide a more detailed understanding of this issue.

Practically, this approach engages children in the solution of a carefully designed sequence of 'realistic' problems, which are similar, but more complex than the example problems used in this work. Although these problems are designed in the attempt to be true to life, the term "realistic" does not represent this, because it is understood that no matter how well designed a school problem is, it will always be a school problem, rather than a problem in the 'non-school' life of the child. Instead, it is used in the sense that the child should be able to realize these problems in their imagination, and in this way become real and significant to the child. This provided me with a valuable insight into the manner in which problems set as school activities may come to be seen as meaningful through imaginative realization by children.

In order to solve these realistic problems, the children will identify and represent mathematical structures that can be usefully related to the problem and is relevant for its solution. This process of learning to identify and represent mathematical structure in the problem is called 'horizontal mathematicization' (Treffers, 1987). A second process, termed 'vertical mathematicization' was also identified by Treffers. Here, the structures formed by horizontal mathematicization, may form the basis for the investigation of more mathematical (as opposed to contextual) questions, in order to further develop children's mathematical understanding.

These ideas are closely related to the terms 'model of' and 'model for' which are also extensively used in realistic mathematics education. In order to solve a problem, children identify patterns in the situation and develop representations which make it possible for them to develop a 'model of' the situation suitable for solving the problem. This is not an abstract model, but is strongly contextual, in that it is described and represented in terms of specific features of the situation. When children are confident with this model and are faced with similar situations, they may use it again as a model of these similar, yet different, situations. A model which in this way shows its utility and flexibility may then itself become the focus of investigation, in order to develop a more abstract mathematical understanding of the features of the model which allow it to be flexibly and effectively applied in these different situations. In this way, the model becomes a 'model for' further mathematical development.

The main area of concern in this cycle of reading is the early development of reflective mathematical understanding based on an intuitive and informal foundation. This foundation is formed through 'realistic' contextual learning activities involving the different interpretations and operations. It follows that the process of horizontal mathematicization and the development of models of contextual situations will be most relevant for this research.

6.3.1.2 Representation and alignment

Note that representation is a key feature of models and the process of mathematicization. In realistic mathematics education, the contextual problems are chosen to lead to the desired mathematical representations. Also, although they do not provide solutions, teachers lead the children's discussion of their solution strategies in order to facilitate the emergence of these representations. This active facilitation by the teacher raises a the issue of alignment between context and representation. That is, whether the representation arises from natural structure in the contextual situation, or whether it imposes structure on the situation. Also, where the representation does indeed relate to specific structure in the situation, whether the structure represented is the only (or most) important structure for the problem in this situation.

These issues of alignment are not dealt with in general in the theory of realistic mathematics education. Rather, they are dealt with, when considered necessary, in the development of learning materials for each specific area of mathematical content.

6.3.1.3 The learning of fractions

Research into the teaching and learning of fractions following a reformed curriculum based on realistic mathematics education, was carried out by Leen Streefland and published in his monograph of 1987. Because his aim was the learning of a specific representation of rational numbers — fractions — the context chosen from which to develop this learning was that of distributing a number of items between a different number of people, or fair sharing. It was judged that an effective model of this context would provide a good model for the development of the fraction representation. But the emphasis on the fairness of the distribution ensured the importance of proportional thinking in the distribution process. In addition, one of the important strategies developed to perform the distribution involved the use of measurement division. In this way, thinking related to the ratio and measurement subconstructs could arise in developing a reflective understanding of the fraction representation based on the sharing interpretation. However, this does not deal with the development of other representations that may be more natural to other interpretations, and the possible influence of these representations in developing a more reflective rational number understanding. Also, possible relationships between rational numbers and actions in situations involving other interpretations, that are not structured by specific representations, are peripheral to Streefland's research, although they are important for this research project.

6.3.1.4 Methodology

To evaluate and refine the curriculum based on realistic mathematical education, members of this school followed the methodology of developmental research. This research aimed, through the method of constant comparison, to identify a 'trajectory of learning' for the core concepts of the mathematics curriculum. The curriculum was then refined and improved in order to better enable the children to reconstruct a similar trajectory for themselves in their learning of these concepts. I found both the methodology of developmental research and the concept of a trajectory of learning to be particularly relevant for my research. But due to the complexity introduced by the number of different interpretations or rational numbers and the different possible constitutive operations, I expected that a single, linear trajectory would not be appropriate.

6.3.2 Piaget and research in the Piagetian tradition

6.3.2.1 Action, cognition and Piaget

In order to make the work of Piaget more accessible, he and Inhelder produced a helpful summary of this work in the volume "The psychology of the child" (1969). This discussion will be based in this volume and, unless otherwise specified, all references will be to this work.

A fundamental principle in Piagetian cognitive theory is that all cognitive structures originate in the action of the individual in the environment. Piaget investigated the general cognitive development of children. He asserted that cognition originates in the basic reflexes of the child and culminates in the ability to carry out logical operations. Through action, the child builds on these basic reflexes, to form schemes of action. These schemes of action are active, rather than deliberative and hence could be termed informal, following the usage adopted in this research. The more deliberative cognitive structures which allow the formation of concrete operations are developed from these action schemes. Although focussed on young children, these processes of the development of action schemes on the basis of action and then the further development of more deliberative structures from these action schemes, are of great relevance to the development of informal rational number understanding based on actions and the further development of more deliberative understanding based on this informal foundation.

According to Piaget, the fundamental processes by which cognitive structures are developed, are assimilation and accommodation.

- Assimilation is the process in which sensory impressions are filtered to fit existing cognitive structures. More specifically, it involves the construction of a filter of a form which makes it possible for the sensory impressions to fit the cognitive structure. When a sensory impression does fit a cognitive structure, then it may lead to the performance of the physical or cognitive act mediated by the structure.
- Accommodation is the process in which a cognitive structure is adapted . This adaptation may occur in order to improve the fit of a (filtered) sensory impression, or to regulate or change the act mediated by the structure. Adaptation always occurs as a result of a conflict between the desired goals of the act and the actual goal achieved.

Note that if perception is seen as achieving a fit between sensory impression and cognitive structure, then assimilation may be interpreted as effecting the perception of the sensory impression. In Piaget's discussion he emphasizes that cognition does not directly effect perception, however, it does influence the focus of attention, thus structuring reality by effecting the data on which perception is based. The structuring effect of attention may this be of importance in the development of informal rational number understanding.

When examining the formation of the sensori-motor schemes, Piaget formulates a general law relating the form of the schemes during development. He sees development as moving from general and global schemes to more local, specialized schemes, through the process of increasing differentiation and specialization. In the sensori-motor stage he sees these forms as:

- Initial **rhythmic structures** of spontaneous and repetitive action, which include the basic reflexes.
- **Regulations**, which involve gradual corrections in response to feedback from the environment, in order to achieve desired goals. Such regulations differentiate the initial rhythmic structures and so lead to the formation of multiple specialized schemes on the basis of the original rhythms. Note that the possibility of inverting the gradual corrections allows the possibility of creating an inverse regulation, and in this manner yields semi-reversibility to the regulations.
- **Reversible structures**, which allow both the achievement of an end state and the return to an original state. In the sensori-motor stage, this is a reversibility in action, without corresponding deliberative representations and reversible mental operations on these

representations. If an action involves the creation or destruction of something then it would not be possible for the action to be reversed. Thus, reversibility supports the belief in the existence of some thing or quantity that is conserved. This may be interpreted as the construction of the primary concept of the existence of this object or quantity as an objective entity. The construction of the object concept occurs in this manner in the sensori-motor stage as a result of the reversible structure of 'displacing objects'.

In development after the sensori-motor stage, Piaget states that initial structures which form the basis for regulation are normally those developed in a preceding stage, rather than basic rhythmic structures.

To relate this to the development of rational number understanding, it should be noted that this development occurs after the sensori-motor stage and so existing action schemes and mental structures, and not rhythmic structures, would form the basis for the development of this informal foundation. In addition, we should expect that this development should occur through the gradual differentiation of global schemes. This differentiation would occur as a result of regulations made in response to feedback, in order to achieve the desired goals in the situation. Note that, following this view, the process of development would end, rather than beginning with the precise differentiation provided by the careful definition of terms. This contrasts sharply with the standard teaching approaches, which start with the definition (or explanation) of terms and representations. It supports that stand taken in this research that the development of informal understanding based on situated action, prior to the careful definition of standard representations, could be important for a good understanding of the rational number concept.

However, Piaget does not formulate a detailed model of the development of informal understanding and the manner in which this may give rise to more explicit, deliberative understanding. Such a model is investigated in the following subsection.

6.3.2.2 Representational redescription

Building on her work with Piaget, Karmilof-Smith (1992) studied children's development of flexible representations, particularly in the learning of language. Her model of representational redescription begins with the ability to carry out procedural action within a situation, what she terms 'behavioural mastery'. This ability requires the development of a sequence of procedural representations which generate actions in response to stimuli perceived in the situation. These representations are independent of the person's other representational systems. Also the person is not reflectively aware of these representations, being only aware of the behavioural sequence as a whole — that is, the representation is implicit. The model describes the development of this into a flexible explicit representational system, related to the person's other representational systems, through the process of 'representational redescription'. She postulates three stages in this development and is able to use this analysis to account for a number of the observed effects in children's development.

Together with the beginning, these stages may be briefly described as follows.

- I (implicit) : A procedural representation as detailed above.
- E1 (explicit-1) : The procedural representation is re-represented as a sequence of more abstract units which may be related to other representational systems. However, the person is not consciously aware of these units and is not able to verbally describe them.
- E2 (explicit-2) : Here the re-represented units become available to the person's conscious awareness, although they are not yet able to verbally describe them.
- E3 (explicit-3) : At this stage the person is consciously aware of the re-represented units, and is able to verbally describe them.

I did not choose to adopt the full detail of this model in my research, but the conceptual progression in the model did yield valuable insight into a possible manner in which operations could contribute to the development of mathematical insight. In fact, operations could be seen as providing effective units for the re-representation of the child's coordinating responses to situations related to rational number interpretations. Because they are coherent and may occur consistently in many different situations. They may also be important for the development of mathematical representations as these may be seen as related to and possibly even arising from the units obtained from such a representational redescription of 'successful' behavioural sequences.

6.4 Developing the activities and structuring the interviews

Possibilities for the children interviews were discussed informally with the teachers in their final two interviews. My wish was to engage the children in activities related to the different fraction interpretations. This would allow me to ascertain whether these activities would be considered natural by the children and whether they would indeed respond, as envisaged, in ways that laid a foundation for rational number thinking. To make these activities as accessible as possible, I had decided to work largely with concrete situations which could be acted out with models and toys. Beyond this, I had made no decisions, aiming to structure the children's interviews building on the discussion with the teachers. This would allow me to request and incorporate advice from the teachers about the activities to include in the interviews.

In the discussions about children's activities, the teachers suggested that they be selected around a common theme, or a common element, familiar to the children from their home lives. I thought that this would be a good idea and so decided to use bread as a common element. Bread was chosen because activities involving bread had been discussed by the teachers when dealing with a number of the interpretations. Thus, using this element would make it possible to investigate children's responses to most, if not all the interpretations.

The use of concrete materials which would allow the children to physically model that was occurring in the situation was also supported by the teachers in a number of interchanges.
A useful suggestion was made in one of these interchanges, when the teachers talked about cutting bread into equal pieces for sharing between different numbers of people. After investigating different concrete materials, the teachers suggested play-dough as a medium for forming the loaves. This would be cheap, provide a good concrete model of bread loaves, and also be easy to children to re-create and start again if they were not happy with their attempt. Following on this, I decided to use play-dough in as many of the activities as possible.

Six half-hour interviews were planned with each child. Each interview involved a number of similar questions based on the same action in a chosen setting. Five settings were chosen before the interview sequence and the last was left unplanned, to allow for further investigation of an issue arising in the first five interviews.

Because only six interviews were planned with each child, it was understood at the outset that it would not be possible to investigate the full complexity of all the interpretations and operations. With this in mind, I decided to use three criteria, graded in importance, to select the settings. First and most important, I attempted to ensure that each operator that had been identified in the previous cycle would be relevant for the actions in at least one setting. Following this, I attempted to ensure that the story of each setting would be seen by the children as a natural or familiar activity. Last and least important, I attempted to include as many interpretations as possible.

I was able to select five settings which satisfied the first criterion. In addition, I believed the initial problems in each setting to be at least reasonably natural and familiar. However, in ensuring that all the operations were covered, I was not able to include each interpretation. In particular, the part-whole and grouping interpretations were not included and two of the first five interviews involved the ratio interpretation. The actions involved in the first five interviews were:

i. Sharing loaves of bread between people. Different questions involved different numbers of loaves and different numbers of people. The amounts obtained from different sharings were then compared.
ii. Slicing loaves of bread into slices of equal thickness. Different questions involved cutting different numbers of slices, and thus slices of different thicknesses. The relationships between the numbers of slices and their thicknesses were then compared (note that this is a second order comparison — a comparison of relationships).
iii. Buying loaves of bread at a given price from a shop. Two different types of loaves were used, costing different amounts. Different questions involved buying for different amounts of money or buying different numbers of loaves.
iv. Dividing an amount of bread dough between two tins of different sizes, for baking. Two questions were asked, one in which the dough was presented as a single 'sausage' and the other in which the dough was presented as a number of 'balls'.
v. Here the children were shown decorative shapes on bread before baking and asked to draw the corresponding shapes after baking. Alternatively, they were shown shapes after baking and asked to draw then before baking. Finally, they were asked to draw the shape of the bread after (or before) baking, when they were given the original (final) shape and the way the length or width had changed.

The third interview involved buying loaves of bread for different prices. In the interests of realism, I ensured that the smallest fraction of a loaf bought was a half loaf. To allow for finer subdivisions and to investigate the development of unit fraction representations in a similar setting, I decided that the final interview would deal with to buying sausages at different prices, for different amounts of money.

For the children to interview, the two teachers each selected a single child from their grade three class. Three criteria were used to select the children.

- They were to be of average mathematical ability.
- To provide the best possible verbal data in the interviews, they were to be of good verbal ability.
- To reduce any language problems in our communication, their home language would be English.

In addition, one girl and one boy were chosen for the interviews. Two interviews a week were conducted with each child, for a period of three weeks.

Chapter 7 – Children Cycle Interview Analysis

7.0 Criteria for the analysis of the interviews

The analysis of the children's interviews is structured in a similar way as that of the final three sections of the teacher interview analysis.

7.1 Issues of meaning, motivation and significance.

- Do the children appear to find the grounding situations for the interpretations familiar and natural?
- Do the objects and motives of these situations appear to be personally significant, allowing them to be seen as an activity or part of an activity?
- If so, is the interpretation aligned sufficiently closely to the grounding situation that it may be viewed as an activity, or does the interpretation correspond to a component action of the activity?

I expected that children would demonstrate familiarity with the grounding situations by working directly and effectively to solve the given problems. That is, they would quickly become involved with the problem as asked, and show understanding of the situation in their response, without needing extra explanation from me about what was wanted.

The analysis directly relevant to interpretations and operations was separated into two sections, first for interpretations and then for operations.

7.2 Insights about interpretations obtained from the children's understanding and analysis of the problems. This included issues such as:

- What inferences could be made about the children's implicit understanding of the interpretations, based on their work in the grounding situations.
- Points of interest about interpretations arising from specific things said or done by the children when working with the problems. Including, but not restricted to interpretations as either activities or actions.
- What relationships between the interpretations were apparent in the teachers' responses?

7.3 Insights about operations obtained from the children's understanding and analysis of the problems. This included issues such as:

- Points of interest about operations arising from specific things said or done by the children when working with the problems.

- What relationships between the operations were apparent in the teachers' responses?
- Does it appear as if the identified operations are indeed units of coordination?
- The children's learning of skilful use of the operations and some of the problems involved with this. As evidenced in the children's solutions of the problems.

Insights relevant to the teaching and learning rational numbers. Important themes identified in the teacher analysis were used as organizing sections for this chapter.

7.4 What issues relevant to the development of more formal, conceptual understanding were apparent in the children's interviews?

7.5 Issues relevant to the learning of representations. These included:

- Learning specific representations.
- What structuring effect of representations were noticed, that were aligned with the actions or operations identified in the problem?
- What structuring effects of representations were apparent, that conflict with the actions operations identified in the problem?

The interviews in this cycle were relatively discrete as they focussed on different problem sets and I did not ask the children to relate their work between interviews. Because of this, the analysis of each interview was relatively discrete and so I decided to finally subdivide each section of analysis into interview subsections.

Finally, note that if no issues of relevance to any specific subsection were found in the analysis, that subsection is simply omitted from the chapter.

7.1 Grounding situations, interpretations, activities and actions.

In the majority of the interviews, I did not investigate the familiarity or significance of the activities explicitly. As a result, most of my judgements of familiarity depended on whether the children responded to the described situation directly or whether they needed the situation to be clarified before beginning to determine their solution.

7.1.1 Interview 1: Sharing

Ruth appeared to find the problem of fair sharing quite natural. She confidently carried out the sharing of one loaf between four people (Ruth interview 1: 59–78) and then one loaf between three (Ruth interview 1: 81–89). Sharing three loaves between four took more effort, but with some help to accurately cut each loaf into four, she was able to do this fairly (Ruth interview 1: 178–262). She became quite engrossed in her sharing, acting out her part of sharing out to the models and talking to the models as if they were real people in a real situation.

236 Ruth One ... two ... three ... that one's a LITTLE bit too big ...
237 BB Mhmmm ... ?
238 Ruth Uh ... give it a little bit more ...
239 BB ... 'kay ... ?
240 Ruth ... that's one ... two ... three ... four. Four!
241 BB 'Kay? You happy with that?
242 Ruth Mhmmm!
243 BB 'Kay?
244 Ruth Oopsy!! Another piece for you ... another piece for you ... another piece for you ... another piece for you.
245 BB That's great! Now there's one loaf left. That's right ... ?

(Ruth interview 1)

In addition, in her sharing, Ruth appeared to act as if she was trying to satisfy the social motive of distributing the bread in a manner which is fair to all. This supports the contention that the sharing interpretation may be seen as an authentic activity grounding partitive division.

Jason was also quite confident with sharing, appearing to find this a quite natural activity. For example, his sharing of 3 loaves between two people was immediate, cutting each loaf in half and distributing the halves. He carried out this sharing accurately without saying a word. As he did this, I described his actions for the record.

38 BB Let's say, on Tuesday we start off with three loaves of bread. See, they're not the biggest loaves of bread. Can you share these three loaves of bread between the two children? So that they each get the same. Okay? Who's gonna get that? (sound of cutting) ... so you cut the first in half ... (sound of more cutting) ... and the second in half ... (sound of even more cutting) ... the third in half. Who gets what? For her? For Rick? And again, the second loaf goes one-to-one and the third loaf goes ... how much do they each get there?
39 Jas Three.
40 BB Three ... they each get three pieces!?

(Jason interview 1)

He also imaginatively engaged with the models when sharing, talking about them by name (Jason interview 1: 178–180). His responses seemed to indicate that he did identify with sharing as a meaningful social activity. However, he appeared to focus most of his efforts and attention on the problem of carrying out the sharing accurately. He experienced more difficulty with subdividing than did Ruth, often needing to try a number of times before reaching an acceptable subdivision (Jason interview 1: 72–108). At one stage, near the end of this process, he even lost sight of the goal he was trying to achieve.

102:	BB	... four pieces. Which is good. Do you wanna try with another loaf of bread? And see if we can get it accurate ... and with the right size?
103:	Jas	'Kay.
104:	BB	Would you enjoy that? Let's put these over here ... (putting on one side) ... I'll separate the two loaves ... so we've got three. Let's take out this loaf ...
105:	Ab	Must I ... must I ... make the ... four pieces in ... in the whole loaf?

(Jason interview 1)

He asked this even though he was quite sure of his goal when he tried his first subdivision. His confusion appeared to arise as a result of his focus on the technicalities of achieving equal subdivisions. This suggests that a sharp focus on the technical aspects of a goal directed action which is a component of an activity, could lead to a child losing touch with the motivating activity. Note that this analysis proceeds from the perspective of goal directed actions.

7.1.2 Interview 2: Slicing bread

Ruth was confident with cutting bread into slices. She was used to doing this at home, where she had to make school snacks for herself and her sister in the mornings.

12	BB	That's it! Another one ... (sound of more cutting) ... for Sam ... do you often slice read at home?
13	Ruth	Yes.
14	BB	Do you?
15	Ruth	Because my dad ... always busy in the morning an' I have to make me an' my sister some snacks.

(Ruth interview 2)

Because of the responsible nature of the task and the fact that she was helping her busy father with his work at home, I expected that 'making snacks' would be highly significant for her. That is, this could be seen as an activity in which Ruth regularly engaged. However, the activity of making snacks would involve far more than slicing the bread. Slicing bread could thus be seen as a goal directed action contributing to the activity of making snacks. It also seemed evident from this extract that Ruth saw the action of slicing bread as an important component of the activity of making snacks. As a result, I expected that she would ascribe some personal significance to the tasks involving slicing bread.

Jason was also confident about cutting slices from the loaf. However, he did check whether the entire loaf needed to be cut.

17	BB	... Matthew's gonna take ... 'n you can use that ... to measure so that you cut

nicely 'n accurately ... ? That's it. One slice ... that's it.

18 Jas Must I cut the whole thing?!

19 BB Cut the whole loaf for Matthew ... ja. I'm interested to see how many slices he can get.

(Jason interview 2)

This may have been because in his experience, when normally slicing bread to eat, one cuts a certain number of slices (enough for each person), rather than the entire loaf. Thus, cutting the entire loaf may not have fitted naturally with his experience of the grounding situation. Here, as for Ruth, cutting the bread could be seen as an action with little intrinsic motivation for the child. The motivation would be derived from an activity to which this action contributes. This situation was introduced as part of the activity of preparing a meal and Jason's query could be interpreted as indicating that the action of slicing a whole loaf of bread did not correspond particularly well to his experience of this activity.

7.1.3 Interview 3: Buying bread

Ruth was comfortable and confident with this shopping situation and engaged with each of the problem situations quickly and without question. She easily and effectively calculated the cost of a given number of loaves, as well as the number of loaves bought for different amounts at different prices.

98 Ruth ... eighteen ... twenty.

99 BB That's Sam's money. How many loaves of bread is Sam gonna be able to buy with that twenty rand? (Describing her actions...) Five. One five.

100 Ruth Two loaves.

101 BB That's two loaves. Ohkay ... !! Another five ... another two. You're getting good at this! ANOTHER TWO loaves!

102 Ruth (Counting out more money) ...

103 BB We've got a circle of loaves around the table! How many loaves!

104 Ruth One ... two ... ... three ... four ... five ... six ... seven ... eight. Eight!

105 BB EIGHT loaves! Very good!

(Ruth interview 3)

Jason was also comfortable with this situation. He calculated the answers for many of these problems mentally — thinking quietly and then giving the answer. He did this when calculating both loaves and cost. For example, in the following extract he responded with the correct cost of 7 loaves after a short time of silent thought. He demonstrated the reasoning leading to his response by counting in two's, and could continue to determine the cost of 10 loaves (Jason interview 3: 12–21).

Both children responded positively to my description of the context for the problem, situating it in the activity of shopping for bread. But as with slicing bread, the problem did not encompass the full activity, for the choice of what to buy and the quantity (either the cost or the number of loaves) had been made by the interviewer. Rather the task focussed on two complementary constituent actions of this activity — determining the cost of a given number of loaves, or determining the number of loaves for a given amount. Both children engaged positively with these task, seeming to enjoy playing the part of a shopper. While at the same time developing their skill in performing these two deliberative actions.

7.1.4 Interview 4: Dividing dough for baking

Both Ruth and Jason seemed quite comfortable with dividing up dough into uneven amounts for baking in two tins of different size. Doing this seemed meaningful and relevant to them. Ruth responded almost immediately with an appropriate subdivision, both when the dough was presented as a long 'sausage'(Ruth interview 4: 35–36) and when it was given as a number of 'balls'.

81 BB So he says, "Fine. One of the little balls is good for the little tin." How many little balls do you think he's gonna need for the big tin?
82 Han I think ... two!

(Ruth interview 4)

Jason, given a lump of dough, approached this judgement quite differently by first cutting it into a number of slices (Jason interview 4: 41–48). But he was quite confident in his response.

Again in this task, the children did not query the context, or the subdivision. Jason only queried the manner in which the subdivision should be accomplished. This suggested that they viewed the given activity of baking as real and meaningful to them, and the need to divide dough to fit in tins of different sizes as fitting well with this activity. Here too, the problem being addressed involved a goal directed act which was a constituent part of the activity. As such, it could be considered as an action, deriving its significance from its contribution to achieving the object of the activity.

7.1.5 Interview 5: Patterns on bread

The children also engaged readily with the earlier tasks in this interview, readily providing solutions which were accurate to within the level of their drawing ability.

43 BB ... okay. Now ... draw me the shape so that it looks the same afterwards as before. Put this ...away ... can you explain why it looks the same?
44 Ruth Because ... it ... it has the same pattern and ... it's just got a bit bigger because ... it's been in the oven.

(Ruth interview 5)

Again, the tasks were related to the activity of baking — this time baking cookies. I took their confident engagement as indication that the children did indeed see the task situation as relevant to baking and were motivated through interest in baking to complete the designs.

The tasks in this interview did not deal with the full activity of baking and so could not be considered as an activity. But here the tasks also did not represent a constituent part of the activity of baking. Instead, they had a more static, representational character, concerned with predicting and drawing the form of the decorations either after, or before baking. The explicit drawing of such predictions is not a normal occurrence in baking. As a result, it was not possible for the children to engage with the task in the pretence that they were engaging in the real activity. That is, the tasks could also not be interpreted as an action which formed a component of the activity of baking. Even so, the children did seem to see the tasks as relevant and interesting.

7.1.6 Interview 6: Buying sausages

As in interview 3, the tasks of this interview can be seen as a component actions in the activity of shopping, in this case, buying sausages. They again engaged readily in these tasks, confidently carrying out the simpler exchanges (Jason interview 6: 18–23). At times, Ruth even acted a part in the exchange.

19	Ruth	Claire's at the shop?!
20	BB	That's it!
21	Ruth	Um ... that one?
22	BB	Right.
23	Ruth	And another two ...
24	BB	... another two?
25	Ruth	... these ... four.
26	BB	Four sausages! Great!
27	Ruth	There's the money ... (handing money to Bruce) ...
28	BB	Thank you, Claire. That's nice.
29	Ruth	There are my sausages!

(Ruth interview 6)

The difficulties that they did experience, appeared to be generally due to a lack of skill in carrying out the specific action needed. These difficulties will be discussed further in sections (7.2.6) and (7.3.6.4). Based on their ready engagement, I judged that they found the activity real and meaningful, and the action fitting to the activity.

7.2 Interpretations

7.2.1 Interview 1: Sharing bread — Sharing

7.2.1.1 Viewing responses from the perspective of interpretations as actions

As seen in the previous subsection, when viewed from the perspective of precisely what is done, the interpretations correspond to specific goal directed actions. The goal for this interpretation was to divide the given number of loaves equally between the specified number of people.

To reach the goal, Ruth generally attempted to cut each loaf of bread into the same number of slices as there were people. At the same time, she tried to ensure that the slices were of equal thickness. However, she found difficulty in accurately estimating the thickness which would give the required number of pieces.

178 Ruth One ... piece for ... and another one piece ... (muttering to herself) ... and another piece ... One two ... three ... Whoaw!
179 BB Oh!
180 Ruth That one's TOO big!
181 BB That one's a bit too big, isn't it?! Yes ... and you're gonna have to ...
182 Ruth ... I'll have to ... put it there ... (putting piece in another place) ...
183 BB ... okay ... ? That's great. And what about ... tell me what ... what about ... let's start with a new one, shall we?
184 Ruth 'Kay.
185 BB Then you can actually measure from that one. Okay? How're you gonna make sure ... how're you gonna try 'n measure to make sure they get the same size? 'Coz you see, that one's quite small, isn't it? That one's quite big ...
186 Ruth I'll make a light line ...
187 BB ... I'll make a light line 'n then you can move it out easily.
188 Ruth ... a light line ... that one!
189 BB Ah! Checking with your fingers and they're about the same ... ?
190 Ruth One ... two .... Waaahhh!

(Ruth interview 1)

Because all the tasks involved sharing between four or three, they were all quite challenging for Ruth. And when asked if each child received the same, she often managed to find an inequality (Ruth interview 1: 92–112). To start the interview on a better note, I decided to first ask Jason to share between two people. He followed the same procedure as Ruth when sharing between two and when trying to share a single loaf between three or four. He found the sharing between two quite straightforward and halved the loaves reasonably accurately (Jason interview 1: 30–39). But he found a similar difficulty to Ruth when trying to share between three of four (Jason interview 1: 70–108).

When sharing more than a single loaf, Jason used a different strategy, starting by cutting all the loaves in half and attempting to distribute the halves equally. When sharing three loaves between four and with some encouragement from me, he again halved the remaining halves to obtain quarters, which could be evenly distributed (Jason interview 1: 113–133). He found a lot more difficulty with this strategy when sharing two loaves between three, first distributing halves, then starting again cutting and distributing six quarters before cutting the last half into three (Jason interview 1: 193–250).

7.2.1.2 Relationships between interpretations

In these tasks, a similarity between the measurement interpretation and a particular manner of performing sharing, is noticeable. Ruth, and at times Jason, performed the sharing by measuring and cutting pieces of a width she judged would give the desired number of equal parts. In this case, the difference between these interpretations can be seen in the goal, or the final state of the actions. The final goal of a measurement is descriptive, to give the number of units of the chosen width that constitute a loaf. While the final goal in this sharing interpretation is active, to give each person the same number of pieces of the chosen width.

7.2.2 Interview 2: Slicing bread — Measurement

7.2.2.1 Viewing responses from the perspective of interpretations as actions

I selected these problems as practical instances of the measurement interpretation, involving the slicing of a loaf of bread into slices of equal thickness. Viewing this interpretation as an action, the goal would be to cut each loaf into slices of the specified thickness. Performing this action was a straightforward process of repeatedly cutting slices of the desired thickness.

Both children were confident about this process. However, actually cutting the slices, particularly the thinner slices, proved to be rather difficult. They needed a number of attempts in order to produce an accurate response. This difficulty, as well as the de-motivating effect of the difficulty is evident when Jason queries, after his first attempt, whether this level of precision is indeed necessary (Jason interview 2: 39–46).

7.2.2.2 Relationships between interpretations

These tasks were designed as practical instances of the measurement interpretation. The goal of the problems was to actually slice the bread and then count the slices, in contrast to a more standard measurement problem where the goal would be to determine the number of slices that could be cut from a loaf. This active goal is not common in measurement and is more common in the grouping interpretation. However, apart from the similarity of goal, these problems did not fit well with the grouping interpretation, because they did not involve grouping discrete objects into groups. Instead they involved breaking up a continuous structure into parts.

This subdivision of a whole loaf into equal parts was similar to the subdivision in the sharing interpretation, when the children followed the sharing strategy of slicing bread into equal slices. But in this case, the desired thickness of the slices was known and the number of slices was unknown, resulting in a measurement division problem, as opposed to sharing where the desired number of slices was known and the thickness was unknown, yielding a partitive division problem.

Another possible relationship between interpretations arose when I worked to develop Ruth's reflective understanding, by quantitatively comparing the number of slices of different thicknesses (Ruth Interview 2: 196–264). After a great deal of assisted effort, Ruth performed one such comparison, and she calculated the related quantities using the ratio interpretation. More detail will be given in section 7.4.2. This observation raised the issue that different interpretations may become related by working on one problem at different levels of analysis.

7.2.3 Interview 3: Buying Bread — Operator

7.2.3.1 Viewing responses from the perspective of interpretations as actions

These tasks were chosen as fitting for the operator interpretation. They focus on two complementary goal directed actions — determining the cost of a given number of loaves, and determining the number of loaves for a given amount.

7.2.3.2 Relationships between interpretations

Again, this work could be seen as involving grouping, but not in order to create groups as a final product. The goal was rather to take the given quantity of loaves, or money and determine the corresponding cost, or number of loaves. The complete action could thus be more aptly described as 'operating' on the number of loaves, or money to obtain the cost, or number of loaves. The grouping is formed in the intermediate state of the work — grouping the given number of loaves or money into units to be bought or sold, in order to complete the transaction.

In many of these situations, the children's determination of the final quantity by counting out corresponding units, has similarities to the process of measurement. In fact, this process is often referred to as 'measuring out' the desired amount. But this is not precisely the same as the measurement interpretation, because the child does not start with a known amount, which is the full quantity that must be measured. Rather the child builds up the full quantity by continually adding on units as they count through the other quantity in the transaction (which has been specified). Also, the specified quantity is not specifically measured by the children. For the determination of the final 'measurement' value of the number of units in the given quantity in the transaction is not the goal of the action. The purpose of the iteration of units is to control the building up of the desired amount of the other quantity. For these reasons, this action is not precisely the measurement of the initial quantity.

7.2.4 Interview 4: Dividing dough — Ratio

7.2.4.1 Viewing responses from the perspective of interpretations as actions

In these situations, the children needed to subdivide a quantity of dough into two parts, in order to equally fill two baking tins of different sizes. This would be the goal of the problem when seen as an action. To achieve such an equivalent filling, the ratio between the two parts would need to be equivalent to the ratio between the capacity of the two tins. Thus, the problems could be seen as fitting with the ratio interpretation.

To subdivide both a 'sausage' and a rectangular lump of dough, Ruth straight away cut the dough into two unequal pieces, performing the subdivision with reference to the two baking tins. For the sausage of dough, she compared the sausage length wise with the tins and obtained a reasonable subdivision (Ruth interview 4: 35–49). When I asked her how much more goes in the big tin than the little tin, she replied with a qualitative comparison.

45		How much more ... goes in the big tin than in the little tin? Uh!!! Let's get it out 'n we can put it next to it ... there we are!
46	Ruth	That ... (showing how much) ... much.
47	BB	That much more. Okay?! An' why did you get that much more for the big tin?
48	Ruth	Because the big tin's ... bigger than the little tin.

(Ruth interview 4)

For the lump (Ruth interview 4: 49–77), I asked her why she had cut the pieces as she did and she again replied by qualitatively comparing the dough with the tins. This subdivision was also reasonably accurate, as we found by fitting the pieces in the tins.

71 BB Okay. Now why did you cut it with a bit more than this ... one? If you look at the two tins ... ? Can you give me a reason for why you cut it a little bit more than that ... one.
72 Ruth Because ... um ... this ... one's bigger than the biggest ... tin and this one's small ... small tin.
73 BB Okay?!
74 Ruth "Coz this one didn't want to fit in.
75 BB The bigger one is too big for the little tin, is it?!
76 Ruth Ja.
77 BB An' it's the right size ...un ... for the big tin!? An' the little one fits nicely into the little tin. You see, they've both got a bit of space? Okay.

(Ruth interview 4)

Such qualitative responses suggesting that her judgement was implicit. When filling the tins with balls, she confidently and accurately performed the distribution in the cases which required whole numbers of balls (Ruth interview 4: 81–94). As before, when asked to justify her selection she used the idea of 'fit'. Choosing two balls because the fit in the big tin would be the same as the fit in the small tin — a reasonable fit for both tins. Again this suggests an implicit judgement.

Jason initially misinterpreted the goal of the action when working with the situation that involved balls of dough. He did not choose a number of balls for the big tin which was consonant with the number chosen for the small tin. Choosing instead to fill it as well as he could with four balls. In response, I drew his attention to the choice for the small tin, and emphasized that the baker would want the tins to be equally full. He immediately replied that two balls would achieve this (Jason interview 4: 123–143).

Jason appeared to have misunderstood the relative nature of the choice, thinking that he should make a choice which filled the tin as well as possible. (Note that Ruth may also have performed her choice in this way, but in her case, I did not probe for this.) He changed his response immediately and definitely after I drew his attention to the way the small tin was filled and the need for the same degree of fullness. Furthermore, his responses in the further problems involving balls of dough, were quick and accurate. This demonstrated to me that he had mastered the unequal subdivision of dough so that the tins could be considered equally full. That is, he was implicitly performing a ratio comparison in order to create an appropriate subdivision.

7.2.5 Interview 5: Drawing decorations on bread — Operator (scaling)

7.2.5.1 Viewing responses from the perspective of interpretations as actions

The focus of these tasks was to predict and draw the patterns on fancy bread either after, or before the cooking, so that these shapes were the same. As discussed in section 7.1.5, I considered that the tasks could not properly be seen as component actions of the activity of baking. Rather they involved the action of forming graphical representations, a component of school learning. The perspective taken in this research is that the structuring effect of the actions and operations contributes to an intuitive understanding of different aspects of the rational number concept. In this case, it was the constraints which a valid representation needed to satisfy that gave this effect. Thus, the object of this analysis will be these constraints, their origin in the activity of baking and their effect on the children's drawing of the representations.

Both children agreed on the form of the constraints — that the patterns should have the same shape (Ruth interview 5: 37–40 and Jason interview 5: 18–21). They also agreed about the need for these constraints. For example, Ruth described the reason as follows:

43 BB ... okay. Now ... draw me the shape so that it looks the same afterwards as before. Put this ...away ... can you explain why it looks the same?

44 Ruth Because ... it ... it has the same pattern and ... it's just got a bit bigger because ... it's been in the oven.

(Ruth interview 5)

That is, the constraints arose as a result of the physical effects of the process of baking and in their actions of constructing this representation, the children attempted to reproduce this effect.

To successfully produce the desired representation, the children would need to scale the decorations by the same amount as the cooked bread. As described by Freudenthal (1983) and Lamon (1999), carrying out such a scaling could be seen as one instance of the operator interpretation of rational numbers. Both Ruth and Jason were able to scale the image effectively and quite accurately. The did this concretely, in the first task (Ruth interview 5: 37–41 and Jason interview 5: 24–29). In the second task, they produced drawings (Ruth interview 5: 45–54 and Jason interview 5: 32–44). Jason was also able to scale down from the 'after' to the 'before' shape (Jason interview 5: 46–60), a this task that was not attempted with Ruth.

7.2.6 Interview 6: Buying sausages — operator

7.2.6.1 Viewing responses from the perspective of interpretations as actions

Both children confidently approached the tasks of buying the number of sausages corresponding to the specified amount of money at the given price. As discussed in the analysis of the third interview, this conversion, or the determination of the quantities involved, is only one component action of the complex human activity of shopping. To validly perform this action, it was necessary to satisfy the constraint set by the price of the exchange. This constraint can be seen as a one of a number of socially agreed rules or procedures that govern the activity of shopping (including the rules that it is the choice of the seller to set the price and the choice of the buyer, whether or not to buy at that price). This price constraint structures the action, in that the cost and the amount bought must be proportionally related, with the price the constant of proportionality. To validly perform this action, it would be necessary to master this structuring relation, even if only implicitly.

In general, the children carried out the exchange by working in units of the given price — first deciding how many of these units they could afford and then carrying out an exchange for each unit or portion of a unit. They did not have the sophistication to determine the exchange by multiplying by the price as a conversion factor. But being able to carry out the grouping into the above units practically, allowed them the freedom to calculate the result mentally if they were confident of their ability (Jason interview 6: 16–24) and if not, to determine the number of groups practically (Ruth interview 6: 18–27).

7.2.6.2 Relationships between interpretations

Because of their approach, the component actions carried out by the children showed a great similarity to the grouping and measurement sub-constructs. On closer investigation though, a number of differences became evident. These differences could be seen as due to the focus of the actions being an exchange — characteristic of the operator sub-construct.

The similarity with grouping, when working with the money, arises because the children needed to group discrete one Rand units from the amount possessed by the purchaser into groups the size of the price of a sausage. Also they actively formed groups of the given size from the money, rather than comparing their money with an iterated reference amount (as would be the case of a measurement). Here I describe what Ruth does as she works out the number of sausages to buy:

225	BB	... And Lucy's got ... how much money do I think we should give Lucy. Let's give her ... two ... four ... six .. eight ... nine rand. There we are. So Lucy's got nine rand! How many sausages is she gonna be able to get? Sausage for three rand? The next sausage for three rand?
226	Ruth	Three sausages!
227	BB	THREE sausages! Three sausages at three rand each. Great!
228	Ruth	There's you money!

(Ruth interview 6)

However, an important difference is that the groups formed are not kept as such, but are exchanged for the corresponding amount of sausages. The importance of the exchange is such that the children's talk of the results of the grouping is all in terms of the sausages they will get — there is no mention of the number of groups of money that they created.

The similarity with measurement, when working with the money, arises from the children's goal of determining the number of groups of the desired size. But again, the focus on sausages rather than the number of groups formed, is quite different from that expected in measurement. There is also a strong similarity with measurement when working with the sausages, as the children measured out the amount of sausages that they could buy. But again, the focus of this action was to measure out an amount of sausage from a larger collection. It was not to measure the size of a given collection, which is the focus taken in the measurement interpretation.

7.3 Operations

7.3.1 Interview 1: Sharing bread

7.3.1.1 Operations constituting the response

Ruth carried out her sharing and Jason shared single loaves, by choosing a unit of comparison and then iteratively cutting slices of this thickness. This manner of sharing could thus be seen as constituted by the two operations:

- unitizing — constructing the comparison unit, and
- iterating — iteratively duplicating the unit from the bread remaining to be shared. Notice that here it is a second unitizing operation that is being iterated, in this case replicating a unit.

84 Ruth One ... two ... whooo! That one gets a big piece!
85 BB That one's a bit big ... so let's what ... maybe you must cut ... change the place you cut that ... that's it!
86 BB (Blowing his nose) ...
87 Ruth One ...
88 BB ... how did that look? So you measure with your fingers ...
89 Ruth ... I think it's fine ...
90 BB ... to see if it's accurate ... okay! So cut if you think it's fine ...
91 Ruth ... (sound of cutting)
(Ruth interview 1)

Even though this process begins with an estimation of the size of the unit, this size may be adapted in response to the iterated duplication in order to form the desired number of pieces. As a result, the pieces formed may only be roughly equal, even though the desired number of pieces are formed. When the pieces were judged to be too unequal, Ruth was happy to start anew, with the final size of the comparison unit (Ruth interview 1: 179 – 214).

When sharing more than a single loaf, Jason used operations of subdivision and distribution. In each case he began by subdividing all the loaves into halves and then attempting to distribute the halves equally. This manner of subdivision appeared to be quite different from the above approach of carefully replicating a chosen unit. Instead, Jason appeared to directly place his cut, as if he had a mental image of the final subdivided state that he wished to achieve and he placed the cut to fit with this image. His judgement of whether the cut was accurate or not, appeared to be a visual recognition of the position of the cut, or alternatively, of the size of each half as compared to the whole. As such, it could be considered a use of the visual scaling operation. When sharing three loaves between two people this was sufficient to achieve a fair sharing (Jason interview 1: 38–40). When sharing three loaves between four children this was not sufficient but he was able to complete the sharing by repeated subdivision into halves.

How're you gonna do that? Do it any way you like. So you're gonna cut that one in half ... ? Cut that one in half ... ? And then cut that one in half ... ? And show me what you get.

116 Jas These two for them ...
117 BB ... these two for them ... ? That looks good. Keep going! No-no ...
118 Jas ... it won't.
119 BB Well, let's see. These two ... keep going. Don't stop. So that one for those two ... now you've got what!? You've got one loaf left over.
120 Jas There ... I know! There ...
121 BB ... ja ... ?
122 Jas There I must cut this in four pieces.
123 BB You must cut that one in four pieces! That's ... and look! Just wait ... you see, you've already cut it into two. So what about cutting that ... piece ... into two? That's it?! (Sound of cutting) ... Marvellous ... (knife being put down) ... that's ... are you gonna cut those? (Sound of more cutting) ...
124 Jas (Muttering) ...
125 BB ... that's it!
126 Jas Oops!
127 BB So now you've given each of the children ... one of the halves ... and one of the small pieces. That's good. Have they all got about the same?
128 Jas Yes.

(Jason interview 1)

The process of distribution could be seen as one of practically constructing the unit of all the bread received by each person, through iteratively handing pieces to each person. This also involved a comparison operation to ensure that these units were all the same size. In this case, he compared them by counting the units of each size, that is, by using counting iteration. Finally, in his third attempt to share two loaves between three, he again used repeated subdivision to successfully carry out the sharing (Jason interview 1: 193–250). He used repeated halving to construct six quarters, which he shared between the three people. Then he subdivided the remaining half into three sixths, which he was able to share equally between the three people. His subdivision of the half into three appeared less direct and more like the replication approach seen in Ruth's work.

7.3.1.2 Operations as units of coordination

Ruth's approach to subdividing the bread did not appear to be a single unit of coordination, being rather a composite of the operations of unitizing and iteration. Thus, for this approach, equal subdivision was not naturally seen as a single unit of coordination. However, the operations of unitizing and iterating the comparison to measure and then create the slices did appear to be unitary. As such, they could be interpreted as units of coordination.

The same analysis as the above holds for the problems in which Jason used unitizing and iteration to carry out the sharing. But in the remaining problems, Jason used subdivision into halves or thirds, at times repeatedly. In these cases, he did not appear to measure and compare the pieces carefully, rather he appeared to perform the subdivision following an intuitive notion of how a whole was constituted by two halves or three thirds. Here it did appear as if direct subdivision judged by visual scaling form units of coordination for this process. Viewing the distribution of pieces as constituted by iteratively handing out pieces, driven by the goal of constructing units, and evaluated by iterative counting, appeared to yield fitting units of coordination for this analysis.

7.3.1.3 Relationships between operations

Ruth's approach to subdividing the bread relates the operations of equal subdivision, unitizing and iteration. In the relationship arising from this approach, equal subdivision is a composite of unitizing and iteration.

7.3.1.4 Learning skilful use of the operations

Actually cutting the loaves to form pieces the same size was not easy for Ruth. Especially when cutting in order to share between four people. She found it difficult to cut each piece the same thickness, after needing a number of attempts before being happy that the thickness of the pieces were sufficiently similar (Ruth interview 1: 178–210).

Judging the initial thickness to cut was also difficult. In some cases, as in her first attempt above, she was left with a final piece which was very different from the others. That is, the chosen thickness did not evenly subdivide the loaf. In others, such as in the last attempt above, she achieved a reasonably even subdivision, but the number of pieces obtained was not that desired. From this it appeared that the process followed by Ruth to obtain an even subdivision of the desired size was rather difficult to implement accurately.

Ruth judged the fairness, or equality of her subdivision by careful visual observation and comparison, often measuring with her fingers. She did not hold to her goal of equal subdivision, and so accept the inaccuracies as artifacts of an imperfect operation implementing an abstractly 'perfect' process (Ruth interview 1: 81–110). A possible reason for this visual judgement may have been that her observation dominated her reflection and not vice versa, which may become the case when the fraction subdivision operation aimed at yielding equal parts has become more practised. Alternatively, because of her lack of practice with actually carrying out the subdivision, she may not have been sure of the acceptable limits of accuracy for the subdivision.

Jason also had difficulty using the operations of unitizing and iteration to share one loaf between four people. To start, he quickly cut four pieces, but they were of quite different sizes. On his second attempt, he took great care to cut pieces the same size. Unfortunately, this resulted in him cutting five pieces, rather than four. When starting his third attempt he was rather unsure and checked that he needed to cut the entire loaf into four pieces (Jason interview 1: 72–108). On this attempt, he was able to complete the sharing accurately — possibly this was due to his practice in the first two attempts.

After being asked to share one loaf between four, Jason was asked to share a loaf between three. This time, he thought carefully before starting the sharing, checking that he needed to share the full loaf. Then he worked out that to change the sharing to fit three people, rather than four, he would need to cut bigger pieces. After this planning, he was able to do the sharing quite effectively and accurately.

175 BB There're three children left. And, let's say ... so today is Wednesday and the three children get given one loaf of bread. Can you share that loaf of bread so that they each get the same amount?

176 Jas I've got to cut them the same as this.

177 BB Is it the same as this? Try that and see? Remember you want to share all the bread as well so that they get the same. Okay? One piece ... measuring carefully ... that's right. Two pieces ... there's a problem! What's the problem? I mean what you ... what you just discovered?

178 Jas I was ... I was going to cut it in ... in five pieces. One ... two ... three ... I mean four pieces ...

179 BB ... four pieces ...

180 Jas ... because Damian's not here.

181 BB That's right. Damian's a ... if you'd have cut it in four then it would've been the same as that ... (pointing), isn't it? But now Damian's not here so you've got to change the cuts, isn't it?

182 Jas Yes.

183 BB So what're you doing? You're making them ... ?

184 Jas Bigger.

185 BB A little bit bigger. Good! That's great. (Sound of cutting) ... cut ... that's it! They are ... did they all get the same amount?

186 Jas Yes.

187 BB That's good! And how many pieces did you cut that loaf into?

188 Jas Three.

(Jason interview 1)

Jason needed to extend his basic halving strategy to share three loaves between four people (Jason interview 1: 113–133). He first halved the loaves and then distributed halves. When he saw that he could not evenly distribute all the pieces, he needed some encouragement to continue. He was confident that to do this, he needed to subdivide the remaining loaf into four. When I pointed out that he had already cut it into two, he was quick to respond by cutting each half into two. That is, here he uses repeated halving to carry out the subdivision.

When I introduced sharing between three people, I asked about sharing three loaves between three people and he responded immediately that each would get one loaf. Next, I asked him to share two between three. He again managed to do this with some scaffolding from me, taking a number of attempts to refine his strategy to the point that it successfully yielded a fair sharing (Jason interview 1: 189–251). He first attempt was to consider sharing them as a single whole and two halves. But he decided against this when he noticed that these quantities would have been very different. Then he cut each loaf in half and tried to share two pieces for each person. However, he ran out of pieces. I then suggested one piece per person. He said he would try this, but then cut the first loaf into quarters. He gave one quarter from the loaf to each person and an extra to the first person. He then cut two quarters from the other loaf and shared them so that each person had two quarters. Finally, he cut the remaining half into three and shared these pieces.

7.3.2 Interview 2: Slicing bread

7.3.2.1 Operations constituting the response

Both children appeared to use the operations of unitizing and iteration to carry out the slicing of the loaves in this interview. The children iterated the process of cutting slices (units) of the given size.

My reflection on what was done, led to a deeper insight into the operation of unitizing. It was evident that when introducing the problems I had identified the unit to be used for the measurement, by showing the thickness of the slices to be cut, and ensuring that the unit did provide an even subdivision of the loaf. The task remaining for the children was to replicate this given unit by cutting slices from the loaf. This, built on a similar distinction made in the first interview (7.3.1) to suggest that two different interpretations of the operation of unitizing could be made. These are:

- As recognizing, or deciding on a specific object or collection to be identified as a unit (constructing a unit).
- As practically, or mentally, making units of the form and size identified (replicating a unit).

In this case, it was the second interpretation of the unitizing operation that the children could be seen as performing.

7.3.2.2 Operations as units of coordination

In my view the action of slicing bread could be seen as constituted by the two operations of replicating a unit (cutting a slice) and iteration (iterating the replication operation). These operations did appear to be unitary and not in turn constituted by other operations.

7.3.2.3 Relationships between operations

As alluded to in section 7.2.2.2, an interesting relationship arose when working with Ruth to develop a more reflective understanding by quantitatively comparing the outcomes of different measurements. Ruth performed the comparison using the operations of replicating and linking units (as an action corresponding to the ratio interpretation). In this way, these operations are related to those of replicating units and iteration used for the measurement. More detail is provided in section (7.4.2).

7.3.2.4 Learning skilful use of the operations

As discussed earlier (section 7.2.2.2) both children found difficulty with replicating units when they needed to accurately cut the thinner slices. In this case it was particularly important because slight mismatches in the thickness of the slices resulted in a noticeable difference between the final number of slices cut.

Cutting slices the same size proved quite difficult for Ruth. She needed three opportunities, with quite strong intervention from me, before being able to evenly slice the bread, and obtain the expected number of slices. For her first attempt, she cut twelve slices (*Ruth interview 2: 18–26).* Many of these pieces cut were slightly thicker than the unit. So after doing some work with the thicker pieces, I asked her to cut another loaf into thin slices (*Ruth interview 2: 124–140).* On this attempt, she produced thirteen slices and noticed that the reason for this was that these slices were thinner than those of her first attempt.

140	BB	So this time you've got thirteen pieces.
141	Ruth	Mhmmm ...
142	BB	That's quite interesting! Why did you get thirteen pieces this time 'n twelve pieces last time?
143	Ruth	Um ... 'coz I cut smaller.

(Ruth interview 2)

I built on this, emphasizing that these thinner slices were closer to the unit, and asked her to try again. On this attempt she cut fourteen slices — the number expected, based on measurements of the thickness of the unit slice and the length of the loaf (Ruth interview 2: 156–197). I asked if she might not cut fifteen pieces if she performed the cutting again, even more accurately. To respond to this, we spent some time comparing the slices she had cut with the unit and was finally happy that her cuts were accurate and that if she were to cut another loaf, she would cut no more than fourteen slices.

As with Ruth, I needed to intervene significantly before Jason was able to cut the thin slices accurately. He also made three attempts before managing to succeed. The process he followed was similar to Ruth's and so will not be discussed in detail. In carrying out these repeated attempts, both children became much more careful when cutting the slices, carefully using the unit to measure where to cut each slice before performing the cut. As a result, their later cuts were more accurate than the cuts of their first attempt.

7.3.3 Interview 3: Buying bread

7.3.3.1 Operations constituting the response

In all these situations, the children's responses were based on the following three operations:

- Constructing units — units of money and loaves of bread, of the specified sizes.
- Linking units — linking a unit of money with a unit of 'loaves of bread'. The link corresponds to the specified price of the bread.
- Iterating linked units — counting units of bread or money, in a coordinated fashion, determined by the links.

These operations were often carried out practically, using concrete resources corresponding to loaves of bread and rands. The following extract provides a good, explicit example of the three operations used by the children, where Ruth counts out the loaves as she counts in two's. In paragraph 68 she replicated units of the money: "Two ...", then she replicated units of the bread: "For each loaf" and linked the units: "Two ... For each loaf you're taking across two ... rands like that". In paragraph 70, she then iterated the linked units, counting the loaves as she moved each unit of R2 across.

67 BB Twelve rand for Sam. Now Sam's gonna go 'n buy loaves've bread as well with his money. How many loaves is he gonna be able to buy? How much does it cost for one loaf?

68 Ruth Two! Two ... for each loaf you're taking across two ... rands like that ... (movng coins) ...

69 BB ... that's great.

70 Ruth ... can only buy ... two ... three ... four. One ... two ... three ... four ... five ... six!

71 BB So he can buy SIX loaves've bread with his twelve rand. That's quite nice. So for each time you took two rands and ... one loaf together.

(Ruth interview 3)

Jason determined his responses to the simpler tasks in this interview mentally, as shown in (Jason interview 3: 12–21). But, after some intervention from me, he approached the more complex tasks in this interview in this fashion, working concretely with coins and loaves.

Even when the work was not done concretely, but mentally in a more abstract fashion (as Jason often did) it seemed to still involve these three operations. When done this way, the work involved counting of units of one of the items to determine the total. Thus replicating units and iteration would be involved. Although the goal is to count the required quantity of the second item, the limit of the counting is set by the number of units of the first item. To make this possible, the two items must be linked. That is, implicitly, both items need to be unitized and the units linked. Also, though the counting is of the units of the second item, to determine the limit, it must be linked units that are iterated, even though it is only the units of the second item that are explicitly counted to reach the conclusion.

7.3.3.4 Learning skilful use of the operations

Ruth was confident with the concrete approach she took to solving these problems, using it to correctly calculate both exchanges — how much money for a given number of loaves and how many loaves for a given amount of money. For example, she used this approach to work out the cost of 6 loaves when the price was R5 for 2 loaves.

89 BB ... six loaves. Ohhkay! How much money is Claire gonna have to pay ... for those six loaves of bread? What's the price?
90 Ruth The round ones ... the round ones are TWO ... so ... one ...
91 BB That's five rand ... ? An' that makes two loaves.
92 Ruth One ... two ...
93 BB ... and another five rand ... ? Another two loaves ... !? That time you went in two's, huh? That's for the next two loaves! Okay? So how much ... ooops! How much money does she pay for those six loaves?
94 Ruth Two ... four ... six ... eight ... ten ... fourteen ... fifteen

(Ruth interview 3).

Jason's initial response to most of these problems was to answer them mentally. However, this quick response was often incorrect. In these cases, I encouraged him to approach the problem in a concrete fashion and he was generally successfully at doing this. For example, he worked concretely to solve the following inverse problem correctly. He was also able to describe how he solved the problem. His explanation was quite compact, so I expanded it and he agreed.

82 Okay ... now let's look at Lucy! A ... a ... and let's give Lucy ... two ... four ... six ... eight ... ten ... twelve ... fourteen ... sixteen ... eighteen ... twenty rand. There w'are! Give Lucy twenty rand. How many loaves ... just count ... I might've left one behind ... (chuckling) ...(counting Lucy's coins) ... ah ha ... doesn't ... TWENTY! That's it. Okay. So Lucy's got twenty rand ... ?! How many loaves'ver bread ... whoops!!! I'd better get you some ... (going to get loaves) ... loaves've bread, hadn't I? I'll put out some more round loaves of bread. How many of these round loaves can Lucy buy with her twenty rand? Let's look carefully at the prices. How many round loaves? Twenty rand. What're you doing there? Dividing your money ... made one five!? Another five ... (sound of coins being moved) ... another five ... and another five.
83 Jas Four loaves.
84 BB For this five rand ... is how many loaves?
85 Jas Two.
86 BB TWO loaves! You ... (chuckling) ... okay.
87 Jas Eight.
88 BB EIGHT loaves ... okay ... eight! And ... what would happen if we gave Lucy another ten rand? Four ... eight ... ten ... she gave her another ten rand.

How many more loaves would we be able to give ... ?

89 Jas Here's the six loaves ...

90 BB ... okay ...

91 Jas ... (muttering) ...

92 BB ... so that's ... was it six loaves for that four ... it was eight loaves, wasn't it?

93 Jas Yes.

(Jason interview 3)

I also asked Ruth a more complex problem that involved finding the number of round loaves that could be bought for the same amount of money as a given number of normal loaves (Ruth interview 3: 147–193). Ruth was able to take advantage of my scaffolding to solve this problem. I provided continuity for Ruth's solution attempt — remembering the problem and returning Ruth to the problem when she flagged. I also provided discrimination input — pointing out important things that she had done and helping her interpret her actions and preliminary results. Ruth was in control of the 'fine structure' of the problem, but needed the strategic help that I provided.

7.3.4 Interview 4: Subdividing dough

7.3.4.1 Operations constituting the response

To carry out her subdivisions, Ruth examined the two tins and then made a single cut of the dough into two pieces. Furthermore, the relative sizes of the two pieces were similar to the relative sizes of the to tins. This is what I had envisaged as the operation of direct unequal subdivision. It was interesting here that she appeared to use visual scaling to evaluate the accuracy of her cut.

In the first problem, Ruth was asked to subdivide dough presented in the form of a long sausage. She was guided by the visual length comparison between the sausage and the two tins together. This may have been prompted by the fact that the length of the sausage was the length of the two tins together. She cut the dough proportionally into two pieces, according to the length ration of the tins. This suggested that she was quite confident with where to cut.

Could you help the baker ... decide ... where to actually ... divide up the batter ... so that the right amount of batter goes into the little and the big tin? So all of this batter can go into both of those tins. It's enough for that. Right! Where are you going to cut the batter so thit it'll fit?

36 Ruth Um ... maybe there ... (pointing) ...

37 BB ... maybe there! Okay!? Let's have a look. You've made two little marks here. Why did you make that first mark? You made the first mark when the two tins were sitting there ...

38 Ruth 'Coz I'm not ... coz I wasn't shore if it was ... enough ... for the tins, or if that one ... if I put another line ... that was enough for the tin.
39 BB So you needed to find out if that was enough for the ...
40 Ruth ... ja ...
41 BB ... little tin? What about, then, for the big tin? Is the rest ... enough for the big tin?
42 Ruth I think ... yes!
43 BB Okay. So you measured it against the tin and then you made a little cut. Okay ... cut it up 'n you can ...
44 Ruth ... (cutting batter) ...
45 BB ... let's not worry about squashing it in prop'ly the way we would. Hey! Pretty good! How much more ... goes in the big tin than in the little tin? Uh!!! Let's get it out 'n we can put it next to it ... there we are!
46 Ruth That ... (showing how much) ... much.
47 BB That much more. Okay?! An' why did you get that much more for the big tin?
48 Ruth Because the big tin's ... bigger than the little tin.

(Ruth interview 4)

When I asked Ruth to compare the amounts, she gave only a qualitative comparison, describing the tins as 'bigger', or 'smaller' and saying that the bigger piece should go into the bigger tin. She did not describe a ratio, or multiplicative comparison. From her purely qualitative response to the question of comparison, it seems reasonable to judge that Ruth made an implicit visual judgement of the relative quantities into which to divide the dough.

After this, I asked Ruth to divide a lump of dough between the two tins (Ruth interview 4: 49–76). In her first attempt, she appeared to only consider the small tin, cutting off a small slice that fitted that tin quite well, but left a piece that was too large for the big tin. For her second attempt, I suggested that she compare the amount of 'stuff' in both pieces. On this attempt she again immediately made a mark and then cut the dough into two pieces whose relative sizes matched those of the tins. Her description of how she had made her cut was again qualitative, suggesting that her judgement was visual, direct and implicit — fitting the envisaged unequal subdivision operation.

Ruth's comparison between the lengths of the dough sausage and the tins suggested the possibility that the ease of this comparison may have influenced Ruth's choice of strategy for carrying out the subdivision. To reduce this possibility, I changed the order of the problems for Jason, first working with a lump of dough whose dimensions did not compare well with those of the tins.

41 ... But the baker knows there's enough dough in here ... for both of those ... pans. So what I want you to do is, can you think of how you are gonna ... divide up this dough so that it would fit nicely into those two pans?

42 Jas Can I... can I build the two ... or must I just cut it?
43 BB Cut it in sort of ... see if you can cut it in where ... where you think it should need to be so that it would fit into the little pan ... and into the big pan.
44 Jas (Cutting dough) ...
45 BB So you're cutting lots 'n lots of slices? 'Kay? Now what're you gonna do? Don't squash it in. just put it in. How's that. Okay? It's three in the little pan ... ? That's it? Put in ... think that's about right? So let's see. How many pieces did you put into the little pan?
46 Jas Three.
47 BB And into the big pan?
48 Jas Five. ... six.
49 BB Six pieces! Wow! Why did you put more pieces into the big pan than the little pan?
50 Jas 'Coz this one's ... bigger
(Jason interview 4)

In his response, Jason cut the dough into slices and then shared out the slices unequally. When asked why he put more pieces into one tin, his response was also that the one tin was bigger. As for Ruth, this suggested that his judgement of the unequal sharing was visual and implicit. This response could be seen as comprised of two different processes. First, forming the slices, using the following operations:

- Constructing units — deciding on the width of a slice to form. This may have been a deliberate choice, but it is more likely that he merely cut an initial slice and then chose this as his unit.
- Replicating units — cutting slices to form discrete units the same thickness as the chosen unit.

Then distributing the slices between the tins using:

- Iteration of the process of placing a slice in a tin.
- Visual scaling, to judge where to place each slice. In each case, the slice was placed in the tin viewed as relatively more empty. As a result, the final distribution of the slices was unequal and corresponded to the relative sizes of the tins.

For the tasks which involved sharing balls of dough between the two tins, basic units for the dough had already been formed — one ball. Replicating these units had also been done before presenting the task to the children. Thus, to fill the tins with the dough, the main operation carried out by the children was unequal distribution of the balls, iteratively placing them into tins, taking into account the relative sizes of the tins and the degree of fit of the dough. In some cases, this comparison was made for the placement of each ball, using a visual judgement of the degree of emptiness. For others, a sequential strategy of placement was used, placing two in the big one for every one in the small. This strategy was based on the children's visual estimate of the relative sizes of the tins. In addition to this, Ruth viewed the dough in each tin as a unit when comparing the degree of fit of the dough in the tins.

Now if he needs four of those balls squashed together to fit in that tin ... how many do you think he needs for the little tin?
97 Ruth Two.
98 BB He needs two?! Mhmmm ... ?! Explain why? Explain what you did there?
99 Ruth "Coz ... I ... squashed them together like I squashed them there ... and that they fit nicely.
(Ruth interview 4)

Here this unit even corresponded to a physical entity — a tin of dough. The comparison performed by Jason was more quantitative, based on the number of balls, rather than whether the tins appeared as full. However, in performing his comparison, he also viewed the collection of balls of dough in each tin as a unit.

143 Now the baker makes different bread which doesn't rise as much and for that bread ... he needs to use four for the big one. How many do you think he'll need to use for the little one?
144 Jas Two.
145 BB Two!? can you explain why?
146 Jas Because ... if this one ... needs four then this one ... needs two.
(Jason interview 4)

7.3.4.2 Operations as units of coordination

The unequal subdivision performed by Ruth to subdivide the long 'sausage' of dough seemed to be based on a judgement of the relative sizes of the tins (by means of visual scaling) and involve scaling this relative judgement to fit the amount of dough available in order to perform the unequal subdivision. As discussed above, it seems reasonable that this was an implicit, unitary operation based on the visual scaling to estimate the relative tin sizes.

I did not consider Jason's response as a single unitary operation. For in my opinion, it could be more naturally seen as constituted by the more fundamental operations described in the previous section. These constitutive operations did however, appear to be unitary.

7.3.4.3 Relationships between operations

It is interesting that the direct approach to unequal subdivision was very similar to what children did when cutting items in half (see 7.3.1). Recalling the two different views of subdivision discussed earlier, direct unequal subdivision could be seen as very similar to direct equal subdivision. Thus, from the perspective of operations, it may be better to differentiate subdivision into 'direct' or 'replicating' operations than into equal and unequal subdivision operations.

7.3.4.4 Learning skilful use of the operations

Jason did not subdivide the dough using geometric qualities. Instead, he chopped up the dough into a number of slices and then allocated the slices to different tins in an unequal fashion. Doing this did not always appear easy to him and I worked with him at times to help him master different tasks.

In the first problem, Jason distributed the slices reasonably well to achieve approximately equal packing in both tins. For the second problem *(Jason interview 4: 76–94)*, he initially shared the slices equally between the two tins, not assigning any significance to the difference in size of the tins. But he did not appear happy with the result. I attempted to get him to expand his field of attention to include the sizes of the tins, by asking him to compare the amount of stuff in both tins. He responded that they were not filled the same and then empted the tins and began the allocation again, this time considering size of each tins and the relative fullness of his packing. He was careful in this allocation to each tin, trying hard to fill up all the gaps in each tin. Because of this, it took a lot of careful comparison and filling of the tins to arrive at a subdivision that satisfied him. The final subdivision of dough between the two tins, compared well with the relative sizes of the tins, for the small tin which was half the volume of the big received about a half as much dough as the bigger.

When distributing balls of dough, Jason followed a similar process in developing an appreciation for the relative nature of the distribution. His response to the first problem involving balls of dough is discussed in section 7.2.4.1 and it is evident that he initially worked to fill the larger tin without any reference to the smaller. When I emphasized that the baker wanted the tins equally full, he immediately and correctly responded that the larger tin would need two balls. This showed that he had grasped the relative nature of this distribution problem.

Furthermore, he carried out the distribution without experimenting with packing the tins, as he had in the previous problem. This suggested to me that from his work in the previous problem, he had developed an implicit understanding of the importance of the relative sizes of the tins, and how this could be used to judge the amount needed to fill the remaining tin. That is, that he had made a visual judgement of the relative sizes of the tins (or more accurately, used the judgement made previously) and based on this, mentally doubled the number of balls used for the small tin to get the number needed for the larger. Thus, in his solution to this problem he had also made a transition from a visual judgement of fullness, to an numerical scaling of the number of balls, based on his judgement of the relative sizes of the tins. As described by Singer et al. (1997) (see section 4.4.6), this is a more advanced strategy for ratio comparison problems.

Using this approach, Jason was able to accurately select the number of balls for the smaller tin if different numbers of balls were given as suitable for the larger tin *(Jason interview 4: 133–154)*. First from four balls in the larger tin and then from three balls in the larger tin, a configuration which required the halving of a ball. In each case, he was able to easily calculate the amount for the remaining tin. However, when asked to justify his choice, he gave reasons based on the fullness of the tins — he did not seem able to explicitly describe how he calculated this amount. For this reason, I judged that his approach was implicit, rather than explicit.

7.3.5 Interview 5: Drawing decorations on bread

7.3.5.1 Operations constituting the response

Both children made, or drew their patterns immediately and directly. As they created these representations, they visually checked that the size, shape and position were as they desired. This also did not appear to be a reflective process, rather the children seemed to judge whether their representations looked right or were the "same shape". Both Jason

60 How's about drawing me the shape for this ... before shape? You're starting with the center one. Okay! Are those... the same?

61 Jas Yes.

(Jason interview 5)

and Ruth

43 BB ... okay. Now ... draw me the shape so that it looks the same afterwards as before. Put this ...away ... can you explain why it looks the same?

44 Ruth Because ... it ... it has the same pattern and ... it's just got a bit bigger because ... it's been in the oven.

(Ruth interview 5)

They performed this judgement even though their comparison image was at a different scale, necessitating a scaling operation of some type. The immediacy of this comparison and the fact that they did not make use of mediating tools or calculations to estimate the desired size, suggested that this was a visual scaling operation.

7.3.5.2 Operations as units of coordination

Both children appeared to use visual scaling to construct their responses. Also, they often judged their constructions or drawings visually, to ensure that they were correctly scaled. The immediacy of the children's judgement that the shape was the same, together with the lack of evidence of any mediating tools or processes in order to perform the scaling, strongly suggested that this visual scaling operation could be viewed as a unitary operation. This operation involved coordinating the children's perception of the original image with their construction of the scaled representation.

7.3.5.4 Learning skilful use of the operations

Both children were able to construct the desired representations accurately and efficiently from the start of the interview. This suggested that they were already skilled in performing such implicit, visual scaling operations.

Only once did a child appear to find the construction difficult. This occurred when Ruth attempted to construct a representation of the 'before' cookie given only a line showing it's length (Ruth interview 5: 147–214). However, this difficulty did not appear due to her inability to perform a visual scaling. In fact, her ability to judge the desired shape accurately was what led her to reject her earlier efforts in this task. Rather, she found it difficult to accurately draw the short side of the cookie, consistently drawing it too long. She eventually succeeded by following my suggestion of drawing a number of possible lengths and then selecting the length that she judged visually to give the best shape. Thus her successful strategy depended on her ability to perform a visual scaling operation.

When reflecting on Ruth's difficulty with this task, I wondered whether they may have arisen from a confusion of 'short' and 'long', because she had been shown the long side of the 'before' cookie, which was shorter. Drawing a shorter, short side appeared confusing to her. In an attempt to dispel this possible confusion with Jason, I stressed a number of times that the length of the 'before' cookie had been drawn. Jason had no difficulty with constructing this representation — lending some support to the idea that this confusion may indeed have been the root of Ruth's difficulty.

7.3.6 Interview 6: Buying sausages

7.3.6.1 Operations constituting the response

When a whole number of sausages could be bought, the operations performed by the children to determine the number of sausages, were similar to those they had used in the tasks of buying bread. These were

- Replicating units — replicating the unit of money that would buy one sausage. They also counted sausages, but these were provided individually the children did not need to replicate this unit.
- Linking units — linking a unit of money with the unit of one sausage. This link corresponded to the specified price of a sausage.
- Iterating linked units — counting the linked units of sausage and money together, in order to determine how many sausages could be bought with the money provided.

A good example of this is Ruth's response in the first task.

18 BB Claire! Okay. So Claire's gonna come along ... 'n how much money does Claire have? Let's count out her money. And Claire's got one .. two ... three ... four ... five ... eight rand. There you are. You take Claire's money. Eight rand. Sam is ... so Claire's coming in to buy sausages ... ? How many sausages can Claire buy for eight rand?
19 Ruth Claire's at the shop?!
20 BB That's it!
21 Ruth Um ... that one?
22 BB Right.
23 Ruth And another two ...
24 BB ... another two?
25 Ruth ... these ... four.
26 BB Four sausages! Great!
27 Ruth There's the money ...
(Ruth interview 6)

In the second task, the children were only provided with R1, even though the price of a sausage was R2. To determine how much they could buy, they both subdivided the sausage and asserted that they could buy a half a sausage. Ruth required some prompting from me, and then worked practically.

38 BB Lucy's gonna be next! Now Lucy ... is not doing very well for money today. Today she's only got one rand! (Placang one coin on table) ... so Lucy says, "Now, I know it's not enough for a whole sausage ... because ... isn't it?
39 Ruth Ja.
40 BB It's two rand for a whole sausage ... but can I get part of a sausage?
41 Ruth Ja.
42 BB How much ... what part of a sausage can I buy for one rand? There's a nice ... cut it?
43 Ruth Put my money there
44 BB That's it?
45 Ruth Let's see!
46 BB Measure it carefully?
47 Ruth Ja ... mmmm ... this one's got a bit much. Two ... move that mark away ... (marking sausage) ...

48	BB	Okay ... ? Again ... make a mark 'n measure carefully. how's that? That's fine. Okay? Then you cut it there ... ?
49	Ruth	This is a little bit taller.
50	BB	Little bit taller ... so you can ... that's it! You can just ... clear it up. Okay. So how much of the sausage does she get?
51	Ruth	This piece ... (holding up one piece) ...
52	BB	She gets one piece. One piece out of two.
53	Ruth	Ja.
54	BB	Do you know a name for this piece of sausage ... for that size piece?
55	Ruth	Uh ... half a piece of sausage.
56	BB	A half a piece of sausage. That's right.

(Ruth interview 6)

The operations that she carried out to do this could be seen as:

- Constructing units — viewing the given R1 as a unit which subdivided the unit price of R2. Also practically constructing the unit of half a sausage.
- Equal subdivision — subdividing one sausage into two equal parts. Note here that she measured and compared a number of times before managing to create two equal halves. This suggested that she did not perform an direct subdivision operation, but rather carried out the composite operation of estimating and then replicating a half unit (see 7.3.1.1).
- Linking units. Linking the given unit of R1 with the unit of half a sausage formed by the subdivision of a sausage.

Ruth performed the same operations whenever a fractional portion of a sausage was needed. This included non-unit fractions, such as two thirds, where she worked with two units of R1 and linked these to two units of a third of a sausage to get two thirds (Ruth interview 6: 245–259). In this case, her counting of two thirds could also be seen as iterating linked units.

In contrast to Ruth, in the second task Jason responded immediately.

24		Rick's now got ... ? Only one rand. How many sausages can he buy for one rand?
25	Jas	A half a sausage.
26	BB	A half a sausage! Explain why?
27	Jas	Because it's only one rand.
28	BB	Okay. So why is it a half that he gets?
29	Jas	Because it's a one rand ...
30	BB	... one rand?
31	Jas	No. Ja ... because it's two rand for one.

(Jason interview 6)

He did not work concretely to determine the amount, instead he did this mentally. When I asked him to explain why he could buy half a sausage, he reiterated that he had one rand and the price was two rand for one sausage, appearing to think that it was self evident that half a sausage could be bought. This suggested that this relationship was known as a derived fact by Jason and that his decision of how much he could buy was made by a straightforward reference to this derived fact. This was more efficient than Ruth's approach, however, he was not able to justify this fact, suggesting that this knowledge was implicit and not accessible in more explicit detail for reflection.

In both cases, the children appeared to immediately identify and then use the relationship between R1 and R2. Ruth, by recognizing that two equal units of R1 made up R2 (seemingly without a direct link between this relationship and the halving relationship). While Jason identified that R1 was half of R2. In both cases, this had the appearance of a derived fact, which the children treated as needing no further justification.

When Jason was given one rand to buy sausages at a unit price of four rand, he again responded immediately, suggesting that this response was also based on the recollection of a derived fact — the relationship between a quarter and a whole (Jason interview 6: 46–54). I asked him to justify his choice of a quarter and in response, he compared the unit price for this task with the previous unit price, for which one rand bought half a sausage. For him, this appeared sufficient to conclude that the amount this time would be a quarter. This did not form a proper justification, although it could have been sufficient to recognize this as fitting with the representation of a quarter — again suggesting the use of a derived fact. When I asked him to show me how to get a quarter of a sausage, he accurately made one cut to divide the sausage into one quarter piece and one three quarter piece. This corresponded to a direct unequal subdivision operation.

7.3.6.4 Learning skilful use of the operations

Ruth was able to correctly determine the number of sausages she could buy in each task where she was able to buy a whole number of sausages. She did this with reasonable ease using her approach of concretely replicating the 'price' units until her money was exhausted, and then linking each such unit with a sausage.

Jason initially followed a different approach, mentally calculating the number of sausages quickly and correctly. But in the third task, where the price of a sausage was changed to four rand, he appeared to guess and incorrectly specified the number of sausages.

38 BB Four rand for one sausage. Okay. And again, let's come along ... who ... who's gonna be the first person to buy? That's Chantal, is it? This time Chantal ... how much money's Chantal got?

39 Jas (counting money) ... eight.
40 BB Eight rand! So Chantal's got eight rand? How many sausages is she gonna be able to buy for eight rand?
41 Jas Maybe four.
42 BB Maybe four? Let's check it?! In fact, you've taken four rand ... ? How many sausages is ...
43 Jas Two!
44 BB Two sausages! How are they each?
45 Jas Four rand.
46 BB Two sausages for four rand each. Great! It helps to count them to work them out into little piles, doesn't it?

(Jason interview 6)

In response, I showed him the eight rand to be spent and encouraged him to group this in units of four. In this way he was able to correct his mistake, replicating units and then linking then to units of single sausages. That is, he used the same strategy as Ruth to do this. Later in the interview, Jason again used this concrete approach to determine a whole number of sausages (Jason interview 6: 104–112).

In the first two tasks which involved buying a portion of a sausage for one rand, priced at R1 and R4 per sausage respectively, Ruth used her practical strategy quite effectively. She directly subdivided the sausage into the required number of pieces (Ruth interview 6: 88–94). In the third such task, the price of a single sausage was R8. Ruth appeared to know the number of pieces in the requited subdivision, but creating this subdivision proved difficult (Ruth interview 6: 167–216). After two unsuccessful attempts, she was a little discouraged and I suggested she start by halving the sausage. She did this and then successfully subdivided each half into 4 equal parts. In this way, I introduced her to the possibility of performing a subdivision into eight pieces by carrying out repeated subdivisions into smaller numbers of pieces.

When subdividing a sausage into sixths, she again had difficulty cutting the sausage fairly.

293 BB Show me how much she can buy? Remember it's six rand for a whole sausage?
294 Ruth Ohhhh!
295 BB Oh dear! What's the problem?
296 Ruth One ... one ... two ...
297 BB ... ah! Now you're cutting it into six pieces!
298 Ruth ... three ... oopsy! That was too small!
299 BB That's a bit small. Ja! So let's start with a new sausage, shall we? Take another sausage ... take another sausage ... that's it. Cut it into three like you did last time ...
300 Ruth ... one ...
301 BB ... because then it's easier to cut the next lot.

(Ruth interview 6)

I worked with her to perform a repeated subdivision. Even though she managed to do this with my prompting, she did not find it easy.

313 BB That's it. And this one ... two. That's it! Cut ... (laughing) ... and that one ... also into ... ? Two! How many pieces do you have there? Ooh! Don't cut an extra piece, okay?
314 Ruth Ayyyy! I don' know ...
315 BB I think cut ... put those two ...
316 Ruth Ja ...
317 BB ... together, shall we? That's it.
318 Ruth There's the other two.
319 BB That's it.

(Ruth interview 6)

The second time she needed to subdivide into sixths, she did this by carefully and successfully carrying out a the same repeated subdivision — I describes her actions as she performed them.

331 So anyhow ... this is Lucy. She's got ... ? One rand. How much can she buy?
332 Ruth Mmmm ... one ...
333 BB Ohkay? So you're again, gonna cut it into ... ? Three equal pieces. This time you're measuring carefully.
334 Ruth NOW!!! She cannnnn ...
335 BB ... cut that piece into two ... ? That one into two ... ? and THAT one into two. And you can get ... ?
336 Ruth One piece!
337 BB One piece. One piece out of how many?
338 Ruth Out of ... six!

(Ruth interview 6)

This suggested that she was beginning to learn the usefulness of repeated subdivision operations when needing to carry out subdivisions into a large number of parts.

When tasked with buying sausage for one rand, when the cost of one sausage was four rand, Jason immediately recognized that he could buy a quarter of a sausage. I asked him to cut me this amount of a sausage and in doing this, Jason cut off a quarter — carrying out a direct unequal subdivision operation.

54 BB So two rand is a half? Show me how you get Rick's quarter? Okay? So that's about a quarter. That's pretty good. I notice you didn't measure it everything. You sort've know how much a quarter is. Can you show me why that's a quarter. You've ... measure it against that and show me why it's a quarter.
55 Jas Because it's much smaller ...
56 BB ... it's much smaller ...
57 Jas ... than a half.
58 BB Okay. It's much smaller than a half. So you're comparing it with a half?
59 Jas Yes.
60 BB That's interesting. How many've them make up a half?
61 Jas Two and a half.
62 BB About two and a half or so? Let's have a look. Let's take a whole sausage

here ... that's very interesting. Oh, here's a half, isn't it? This is a half measure. There you are. There's a half. You can measure it against the half ... how many?

63 Jas Two.

(Jason interview 6)

His unequal subdivision was accurate even though his initial comparison of the resulting quarter with a half incorrectly yielded two and a half.

Jason's next subdivision task involved buying sausage for one rand when the unit price of a sausage was eight rand. Jason did not recognize this as a situation in which one eighth of a sausage could be bought. The precise piece sizes with which Jason was familiar were halves and quarters (earlier I had introduced the term 'quarter'). Because he did not know the fractional quantity corresponding to the answer, he described the amount of sausage required as a tiny bit. When asked to show his 'tiny bit' he cut a small piece — appearing almost to guess the amount, rather than trying to determine it precisely.

72 BB One rand. okay. How much sausage can Chantal ... can Chantal get for her one rand!?

73 Jas A tiny piece.

74 BB A tiny piece! Show me how I cut ... organize or cut out the tiny piece to show me why. Okay. Why do you think it's that size piece?

75 Jas Because it's eight rand and this is only one rand.

76 BB okay. So it's eight rand so it's much smaller. How do you know ... how are you sure that the piece she's got is fair ... for the one rand that she paid?

77 Jas Because ... the sausages got more expensive.

78 BB The sausages got more expensive. Let's say ... that s... that Rick comes along and he's also got a rand. (Placing coins). How much can he buy ... for his rand? ANOTHER tiny piece! How many people do you think it would take at one rand ... to use up the whole of that sausage?

79 Jas Seventeen.

80 BB So there're SEVENTEEN of those tiny pieces in the whole sausage! How many rand will that mean that they pay for the sausage? You've got seventeen people and each pays one rand, how much will they pay for the whole sausage?

81 Jas (Counting to himself) Fourteen.

82 BB Fourteen rand. Okay? So you said ... if we we have ... the problem is I don't have seventeen people. So if you have seventeen people and they each paid one rand ... let's count the p ... let's count the rands out. Here's some money ... how many ... ? It's one rand for each of these people? So count out one ... (counting out coins quietly to himself) ...

83 Jas Seventeen.

(Jason interview 6)

He did recognize that a corresponding piece of sausage would be smaller than the halves and quarters that he was familiar with. But when cutting his piece he did not appear to see the importance of the relationship between the subdivision of the unit price and the subdivision of the sausage. That is, he did not appear to see the relevance of the operation of linking corresponding pieces of the two subdivisions. In fact, when I asked him how many people would use up the whole sausage if they each received a tiny bit, he measured these pieces carefully to fit with the first 'bit', obtaining seventeen pieces. He re-counted the pieces again when I started talking about the amount of money paid for these pieces (misinterpreting his 'fourteen' as a response to a question. His care in measuring these pieces showed that he did ascribe importance to the equality of the subdivision of the sausage.

I tried to build on the idea of fairness, in order to help him to realize the importance of such a link (Jason interview 6: 84–104). Once he had performed his subdivision into seventeen pieces, I introduced the idea of fairness to the buyers and he replied that the buyers would receive too little. He then made a second guess at the size of the desired piece of sausage. In this case, he overestimated the size of the piece, cutting approximately a sixth of the sausage. I responded to his second guess by inquiring about fairness to the seller. Counting the amount made for one sausage he replied that his estimate was now too large. At this stage, I stopped the investigation of this task, in order to allow time for the remaining tasks.

Our work on linking subdivisions continued when faced with the task of buying sausage for one rand at the unit price of three rand. Again, Jason did not know the fraction representation of a third and he appeared to resort to guessing an amount of sausage. I again used the idea of fairness to try to get him to realize the importance of relating the subdivision of the price to the subdivision of the sausage (Jason interview 6: 114–158). As before, I had to look at two issues of fairness. First, by asking him how much two other people would get for R1, if they way they all received sausage was fair. He replied that they would get the same amount as the first person, because they each paid the same. Second, I asked him if together they would get a whole sausage for their R3, and if this was fair — if they got enough. He replied that they had. It seems as if here he was still focussing on comparing their amounts, rather than ensuring that they received enough so that R3 would buy a whole sausage.

I tried to bring out the lack of correspondence by asking what to do with the extra piece of sausage. He replied that he could give it to somebody. When asked, he added that this wouldn't be fair, but if he divided it into two, it would be. This supports the interpretation that he was still deciding fairness by ensuring that each person received the same amount (and hence the remainder should be shared between two), rather than ensuring that one sausage was shared between three. In an attempt to change his focus to the relating the subdivisions, I asked him how much the seller received for the one sausage, and whether that was fair. He stated that the seller would receive R5, but was happy that this was fair, seeing no contradiction with the R3 price of a sausage.

Seeing that Jason remained convinced that his subdivision was fair, I decided to change my tack. I attempted to relate this task to the sharing interpretation of rational numbers, and so build on Jason's good understanding of sharing. To do this, I asked Jason what would happen if the three people combined their money to get the cost of one sausage, and then bought one sausage. He replied that they would share the whole sausage between them. He was able to carry out the sharing fairly and identify that the sausage was subdivided into three equal pieces. Here Jason using the operation of equal subdivision to answer the question. This reformulation as an equal subdivision of a group purchase could have been followed up in an attempt to relate the subdivision of the money and the subdivision of the sausage. I chose not to do this, but rather to briefly introduce the term 'third' and then proceed to the other situations in the interview. This was because I had intervened quite strongly and judged that continuing my questioning might had effected his motivation.

For final task in which I asked the children to buy an amount of sausage for one rand, the unit price of one sausage was six rand. Jason was able to determine this amount of sausage accurately, with some help from me (Jason interview 6: 176–190). I first tried to prompt him by recalling the idea of fairness and relating it to sharing by asking how many people could together buy a whole sausage. He was still unsure how to proceed, and so I again emphasized the price of six rand per sausage and that the people who together were to buy the sausage each had one rand. With this prompting, he was able to determine that six people would together buy one sausage. I then asked how much each of these people would get if they shared their sausage fairly between themselves. He was then able to subdivide the sausage and buy the sixth. Even with my help, this represented a significant improvement on his response in the task where an eighth of a sausage could be bought.

A final point of interest arose in Jason's work with the task of buying sausages for nine rand when the unit price was six rand per sausage (Jason interview 6: 158–176). Jason approached this task by dividing the R9 into three units of R3. He developed this for the exchange problem by assigning 1 sausage with each R3 unit and so saying that he would get 3 sausages. This was in spite of the fact that he had twice stated that one sausage was sold for R6. Two different interpretations of this response could be made.

- Firstly, Jason could be seen as happy to carry out the exchange for whole sausages, but not prepared to view an exchange of partial items as valid in this shopping situation. The fact that six rand did not divide evenly into nine rand and so would result in an exchange of partial items would thus motivate him to reinterpret the situation in terms of a price which would divide evenly into nine rand — thus making sense of the situation for himself by working with a price of three rand per sausage. However, his response that six rand was the price tells against this interpretation.
- Secondly, the idea that the price provides an inflexible constraint, linking the money and amount bought may not have been firmly fixed in his thinking. He could have seen the price as being much more flexible. So when the price and amount of money were such that calculating the amount bought was technically difficult, and violating the price constraint resulted in a simple solution, he saw it as valid to respond by violating this constraint. This could be seen as a coordinating response, to ensure a better fit between the money and amount bought by violating the price constraint. That is, Jason may have seen the price as a flexible constraint which could be adapted to improve coordination of the linking operation. This interpretation seemed to fit better both with his apparent lack of perception of any contradiction in what he did, and with his response when I emphasized the contradiction with the price and asked if he could remove it. He acknowledged this problem but did not know how to fix it. But when I grouped two of the threes together, he recognized that the remaining three rand was a half of the six rand and was easily able to complete the analysis correctly.

7.4 Developing a more formal, conceptual understanding

7.4.1 Interview 1

When sharing one loaf between two people, Jason justified the equality of the subdivision even when it was not precisely accurate, by saying that he cut them in half. He appeared happy that the inequality was within an acceptable limit of accuracy.

Could you, for me, share the loaf of bread between Tracy and Rick so that they both get pretty much the same?

31	Jas	(Cutting the 'loaf') ...
32	BB	... 'kay. So you cut it there ... can you show me why they've got the same?
33	Jas	Because ... I cut them in the ha ... half. Now they both get the same.

34 BB So they both get the same. Aha! Let's put them next ... the pieces next to each other. Are they ... are you sure that they're pretty much the same size?
35 Jas Not really.
36 BB Not really?! But ... how close?
(Jason interview 1)

Here, in contrast to Ruth's careful comparisons, Jason's conceptual understanding of the desired quantity (a half) appeared to dominate his observation — implying equality, even though this was not precisely accurate.

7.4.2 Interview 2

In the second interview, I asked Jason to compare the number of slices cut of both thicknesses. I was hoping that he would see that the number of thin slices was double the number of thick slices (the reciprocal relationship to the half thickness).

224 BB Six ... no just for one loaf. Just for one loaf. That's it. Six for Rick ... twelve for Shirley.
225 Jas Shirley ...
226 BB What size is six of twelve?
227 Jas Mmmm ... six more.
228 BB Six more! That's right. So it's six and another six ... that makes twelve.
229 Jas That makes up Shiley's ...
(Jason interview 2)

However, although he saw this as six plus six making twelve, he did not interpret this as a doubling, or multiplicative relationship. He saw it rather, in terms of addition. That is, he did not appear to have been aware of the important multiplicative relationship between the comparative size of two units of measurement and the corresponding measurements.

To explore this relationship with Ruth, I worked to relate the counting of the thick slices with the counting of the thin slices (Ruth interview 2: 196–264). Using the fact that there were (when carefully cut) two thin slices for every thick slice, I was able to work with her to the point that she could show that a loaf which yielded 7 thick slices should yield 14 thin slices. She calculated this using coordinated counting — counting two of Claire's thinner slices for every one of Sam's.

238 BB That's it! You're getting good at this! Now if we were to cut these for Clai ... aire how many would Claire get for that one? How many of Claire's would go for one of Sam's?
239 Ruth Two.
240 BB TWO! Whoops! So we'd have two of Claire's ... for that one of Sam's. And how many of Claire's for that one?
242 BB Again two!
243 Ruth Two ...
244 BB ... and ... ?
245 Ruth ... two ...
246 BB ... that's it!

247 Ruth Two ... and two ...
248 Ruth ... two ... and two.
249 BB So Sam counted ... ? So Sam counted ... how many slices?
250 Ruth Um ... seven.
251 BB How many slices is Claire gonna count? Can you use Sam's ... ? This would be ... ? Two ...
252 Ruth ... two.
253 BB If you count now? That one will be ... plus another two which is ...
254 Ruth Four ... six ...eight .. ten ... twelve ... um ... fourteen.

(Ruth interview 2)

In particular, the operations of replicating units (ones and twos) and linking units, were used together to enable the basic coordinated counting strategy of the ratio interpretation. This suggested that these operations and the ratio interpretation may have a role in the development of an understanding of this important reciprocal relationship between the chosen unit and the count obtained in the measurement interpretation.

After this work, Ruth was also able to articulate an understanding that the different number of slices arose because of the different thickness of the slices, with fewer, thicker slices being cut. However, this understanding appeared to be only of a qualitative form, as she was not able to quantify the relationship between the comparison of thickness and the comparison of measurement.

7.4.3 Interview 3

In this situation, I attempted to develop a more mental approach, encouraging Ruth to develop more abstract skills to solve the problem without using concrete, practical operations (Ruth interview 3: 195–235). I did not do this with Jason, because he was already using counting. Ruth had to work hard, with quite consistent prompting from me, to keep her counting units consistently correct. Initially, when working with a price of two rand per loaf, she needed my querying to construct counting units of two rand. Later, when the price was five rand for two loaves, she was able to find the cost of four loaves on her own. She was also able to explain how she calculated this. To work out the cost of an extra six loaves, she again needed some scaffolding from me.

7.4.4 Interview 4

In the second set of problems in this interview, the children again had to divide dough between two tins of unequal sizes. In this case, the dough was in the form of balls of equal sizes. By doing this, I was attempting to encourage the children to relate the subdivision to numerical quantities and possibly obtain a quantitative comparison of the relative sizes.

In the first and second problems, Ruth completed the distribution accurately, justifying her choices in terms of how the balls fit the tins. She compared the fit in a qualitative fashion, rather than quantifying sizes and I tried to get her to quantify the relationship. Her first response was to specify the difference of two quantities of balls chosen — an additive comparison. When I asked her to compare the tins, she initially made only a qualitative comparison.

102 BB It's interesting. Okay? So here you have two balls an' there you have?
103 Ruth ... four.
104 BB How many more is the four balls than the two balls?
105 Ruth Two more.
106 BB Two more! Okay. An' what's the difference if you compare the size of the tin? What can you say about the size of the tin?
107 Ruth This tin ... is a ha ... a LITTLE bit bigger than this tin ...
(Ruth interview 4)

At one point when further exploring this comparison she turned the larger tin around and discovered its width was the same as the length of the smaller. Building on this with my help, she was able to show that the larger was twice the size of the smaller.

Ahhh! What did you do there?
115 Ruth I turned the match box ROUND!
116 BB You turned the box around!!!
117 Ruth So it looks exactly like that but just got two others
118 BB So now it looks exactly ... so how much bigger is this big one 'n the little one there ... (pointing to little box) ... ?
119 Ruth Uh! Two've these balls bigger!
120 BB That's two of those balls bigger! How many ... if you think of this ... let's ask a hard question here. How many ... of these little tins do you think will fit inside this big tin when it's turned that way?
121 Ruth One ... two ...
122 BB TWO of them!!! Wow! That's very good! You can measure it nicely. And how many of the balls in here ... how many more balls there? You have ... how many ... there're two counts in there ... two balls ...
123 Ruth ... ja ... ?
124 BB ... aren't there? How many ... of the little tins balls are there in the big tins as well?
125 Ruth Mmmm ...
126 BB Does the question make sense? (Chuckling) ...
127 Ruth Ja.
128 BB See ... here the little tin's got two balls. The big tin's got how many two balls?
129 Ruth Two...
(Ruth interview 4)

In a similar manner, she was also able to provide a multiplicative comparison of the numbers of balls in the tins — that the larger tin had two, two balls, as compared to the single two ball of the smaller.

7.4.5 Interview 5

Late in the interview. the children were asked to draw the final shape of the bread, given only its length. After a few attempts, using visual scaling to judge the accuracy of each, both children were able to do this To encourage the development of a deeper conceptual understanding, I asked how much bigger the final shape was than the initial.

127 BB ... Okay! Umm ... last question about this one. If you look at this before one ... an' you look at that after one ... how much bigger is the after one then the before one?
128 Ruth Um ... THAT ... (using her fingers) ... much bigger!
129 BB That much bigger! So if you measured the length, you went ... with ... that one ...
130 Ruth ... you put it ... there ...
131 BB ... mhmmm ... !
132 Ruth Then you take ... your finger ... and then you can see ... that much ...
133 BB ... that much bigger! Okay. Now if you go ... across or up ... how much bigger is it?
134 Ruth Mmmm ... that much! (Showing how much bigger.)
135 BB Is it ... is it the same amount bigger? Is it the same amount bigger ... across as it is up?
136 Ruth No.
137 BB No!?
138 Ruth Cause ... you only take thi-is ... (pointing) ...
139 BB ... mhmmm ... ?
140 Ruth Bit you don't take the whole ... thing like that!
141 BB So going up is ... smaller?
142 Ruth Ja.
143 BB So when you get bigger here ... you also ... ?
144 Ruth Um ...
145 BB ... go with the smaller piece.
146 Ruth Ja.

(Ruth interview 5)

Visual scaling involves a multiplicative comparison, but when asked how much bigger the biscuit was, she used an additive comparison, showing the extra length with her fingers. This suggested that the visual scaling operation was implicit and Ruth was not reflectively aware of precisely how she performed the comparison when carrying out visual scaling. To try to encourage her to take a multiplicative view of the comparison, I first asked her how she measured the length. I hoped that she would describe the comparison in terms of an iteration of the 'before' length (a multiplicative comparison), but instead she worked additively, just indicating the extra length. In a second attempt, I asked her whether the length and width were the same amount bigger. If a multiplicative judgement was made this would indeed have been the case. I hoped that the equivalence in shape would encourage a positive response

and so a multiplicative comparison. However, again she worked additively, responding that the smaller side grew less. With Jason, this task followed a similar pattern, with me attempting to encourage an explicit multiplicative comparison, and Jason responding additively.

In the final task, Jason was able to construct the before shape given only the length of one side. I again asked him to compare the growth of both sides. His response this time was that it had become 'smaller both ways'.

110 BB Only a little bit smaller! And again ... when we went from this one ... to that one ... did it ... get ... smaller this way... or that way? Get the same amount smaller. From both directions.
111 Jas It got smaller ... both ways.
112 BB Smaller ... and did the same amount smaller?
113 Jas Yes.
114 BB Explain why you say that?
115 Jas Because it looks like ...the line here ... and there ... and there ... it looks like it.
116 BB So it looks like it!? So this one ... got smaller both ways so that it looks the same as that one. Is that what you're saying?
117 Jas Yes.

(Jason interview 5).

I asked if it was the same amount smaller and he said it was, because the shape stayed the same. This may have been due to a mistake in comparison, made easier by the fact that the shape stayed the same. Alternatively, it may have been, as I had hoped, the beginning of a move towards multiplicative comparison, for both sides did decrease by the same factor.

7.4.6 Interview 6

In the fourth task, Jason recognized one rand as a quarter of four rand and used this to conclude that he would be able to buy a quarter of a sausage. However, he was not able to fully justify this use. To develop his conceptual understanding, I worked with him to compare a quarter with a half .

58 BB Okay. It's much smaller than a half. So you're comparing it with a half?
59 Jas Yes.
60 BB That's interesting. How many've them make up a half?
61 Jas Two and a half.
62 BB About two and a half or so? Let's have a look. Let's take a whole sausage here ... that's very interesting. Oh, here's a half, isn't it? This is a half measure. There you are. There's a half. You can measure it against the half ... how many?
63 Jas Two.
64 BB About two. That's it! So what did you do? You've cut your half into a half again. That's very well done! Interesting! Okay. So Ri ... there's Rick ... yes ...Rick gets ... ? ONE quarter ... Great!

(Jason interview 6)

With some scaffolding, he concluded that two quarters made a half. I re-framed this as a quarter being a half of a half, but he did not respond to this.

7.5 Developing and using mathematical representations

7.5.1 Learning representations

7.5.1.1 Interview 1

In the first interview, neither child appeared familiar with the term 'quarters'. In fact, when sharing three loaves between four people, Jason called the quarters 'slices'

129	BB	They have! That's good. How much do they have?
130	Ab	They got ... one half loaf ...
131	BB	... mhmmm ... ?
132	Ab	And ... one slice.

(Jason interview 1)

This may have been because they had not been exposed to the term. But, considering the teachers' report that the children had completed some fraction work in their classes, it was more likely that they had forgotten this term. I introduced the term 'quarter' to both children in this interview, to describe the pieces produced when sharing one between four.

7.5.1.2 Interview 2

When comparing the thin to the thick slice in the second interview, Jason initially called the thin slice a quarter, rather than a half. In response to my query, he repeated it was a quarter, but when I mentioned a half, he was quick to agree that it was actually a half (Jason interview 2: 204–217). Later in the interview, when comparing thickness and slices cut, Jason again calls the thin slice a quarter of the thick. Again, I queried whether this was not a half and he immediately accepted this. This may have been because I had introduced the term 'quarter' in the previous interview. As if he had forgotten the term 'half' as a result of learning the new term 'quarter'.

7.5.1.3 Interview 3

After Jason had successfully 'bought' bread for amounts that yielded a whole number of loaves, I introduced some tasks where a full exchange would include buying a half loaf of bread (Jason interview 3: 156–184). First I asked how many loaves could be bought for five rand at a price of two rand a loaf. His initial response was two loaves and he showed that one rand remained. To investigate how much bread could be bought for the remaining rand, I grouped the rands in two's, showing that the remaining rand did not have a partner. He agreed that this would buy less than a whole loaf and estimated that it would buy a half loaf. I pushed him to justify this and he was able to give the reason that one rand was half price and so would yield a half loaf. After this we returned to the original problem. It took some prompting from me before he included the half loaf in his response. Following this, he was also able to include half loaves when buying bread for other odd amounts.

Following this, I presented Jason with a more complex problem which would yield a half loaf.

184 Let's say ... that Rick comes along ... and he buys ... two loaves ... four loaves ... six loaves. How much bread can Lucy buy for the same amount of money as Rick's six loaves?
185 Jas Fifteen rand for this ... (pointing) ...
186 BB ... so there's fifteen rand for Rick's six ... Rick's six loaves. Okay? Outside ...
187 Jas ... sixteen.
188 BB How much did he ... let's work it out again. For this ... (pointing) ... two is ... ?
189 Jas Fifteen.
190 BB It's fifteen rand. That's it. Five ... ten ... fifteen. So now you're gonna count out fifteen rand ... okay? Counting in two's. That's fifteen rand ... yes. Two makes that one ... and that money makes that loaf ... money makes that loaf ... I'll just move this out the way ... that one ... that one ... that one ... that one ... you need a half for the extra. Hey ... you're good it that, hey? So how much bread does she get?
191 Jas Seven and a half.

(Jason interview 4)

Jason easily calculated the amount spent by Rick, but then he became a little confused about how to work out the number of loaves that Lucy could buy. This may have been because he was becoming tired, near the end of the interview. When I structured the response by breaking it up into component steps, he was able to effectively solve the problem. This time he easily and correctly determined that the remaining rand would buy a half loaf.

7.5.1.4 Interview 4

After Jason had sliced and distributed the dough in the first problem of this interview, I asked why he put more pieces into the big tin. His initial response was qualitative — that the big tin was bigger (Jason interview 4: 49–62). When asked to be more specific about the comparison, he replied that the little tin was a quarter of the big. I queried his use of a quarter, especially since he had used the term inappropriately in n earlier interview. In reply, he appeared almost to retract his use of the technical term, saying that it was not really that size. And he returned to a more qualitative comparison, saying that there was a little bit extra. He did not appear able to quantify the comparison at this stage.

I responded to his use of the term 'quarter' in this problem, by asking him to cut and show me a quarter from a 'loaf'.

71: BB: To fit in both. So it makes it easier to fit in both. That sounds interesting. What I want to do ...you to do for me now quickly, Jason ... is ... here's a piece ... of ... well, let's ... let's call this a loaf've bread that's made. You said this ... when you compared these ... you said the one was a quarter of the other ... can you show me a quarter with this by cutting this piece've bread.
72: Jas: (Sound of Jason cutting dough) [4]...
73: BB: Okay. Which is the quarter?
74: Jas: This one ... (pointing to smaller piece) ...
(Jason interview 4)

He was able to perform an unequal subdivision to cut a reasonably accurate quarter from the loaf, showing that he knew the standard meaning of the term and that he knew he had used it inaccurately in the previous problem. I did no further work on representations before the second problem that involved distributing a 'sausage' of dough between two unequal tins. But after carrying out the subdivision, he was able to correctly quantify the comparison, saying that the quantity in the small tin was half that in the big tin

89 BB Okay. If you look at how much stuff there is in the big pan ... how much STUFF is there in the little pan ... how much more stuff is there in the big pan than in the little pan, do you think?
90 Ab I dunno.
91 BB You don't know!? What happens if you turn out the two pans ... 'n compare the stuff? What about turning out the big pan? That's the little pan ... turn out the big pan ... to make a little pile of stuff? Now if you compare ... just look at it. There's a pile ... let's first look at it as a pile ... if you compare those two piles ... how much of the big pile is the little pile?
92 Ab A half.
(Jason interview 4)

It seemed as if his cutting of the quarter had been sufficient not only to clarify the term 'quarter', but also to remind him of the term 'half'. Following this, he also used halves confidently when the bigger tin needed three balls of dough, to specify that the smaller tin needed one and a half (Jason interview 4: 149–154).

In contrast to Jason's confident halving of the balls, Ruth found this problem rather difficult. She recognized that one ball was not quite what was needed, but she was not able to proceed further.

130 Now let's pretend that the baker's making a very different type've bread! An' this time he needs ... one ... two ... three balls for the big tin. How many balls do you think he'll need for the little tin?
131 Ruth Three balls for that ... (pointing to big tin) ... tin ...
132 BB Three balls for the big tin ... let's put the tins next to each other like that ... so that they're just as wide. This time they fit in with quite a bit of space, isn't it?! So they're obviously gonna get quite big!
133 Ruth Not QUITE!
134 BB Okay. Try one 'n explain why?
135 Ruth 'Coz ... it's got a bit've space ... um ... so that this is gonna fit here!
(Ruth interview 4)

It was interesting that Ruth had not quantified any comparison in terms of rational numbers, while Jason had, albeit not always correctly, using both a half and a quarter. This raised the possibility that Jason's use of the representations earlier had made it easier to use halving in this problem, in effect priming him to use halving. The lack of a similar priming may have made this problem comparatively more difficult for Ruth.

7.5.1.6 Interview 6

The tasks in the sixth interview provided the opportunity to introduce a number of unit fraction representations of rational quantities. At this time, both Ruth and Jason appeared familiar with the terms, 'half' and 'quarter'. Ruth did not use these terms to name the part chosen until I explicitly asked her to name the chosen quantity. Then she was able to correctly use both terms. In contrast to this, Jason used both terms, a half and a quarter, in his response.

Neither child appeared familiar with the terms 'eighth', 'third' or 'sixth'. Ruth successfully formed an eighth, a third and a sixth of a sausage and I then introduced each term as a way of describing the quantity she had formed. Jason was not able to form an eighth of a sausage, but with my help, he did form a third and a sixth. In each case, I was then able to introduce the corresponding term as describing this quantity.

7.5.2 Structuring effect of representations — aligned with the structure of the action

7.5.2.3 Interview 3

Here Jason worked with a problem that had appeared rather difficult for Ruth using her concrete approach — that of buying round loaves for the same amount as a specified number of ordinary loaves. He had a better strategic response to this more complex problem. To start, he was able to identify what was needed for the first part of the problem and then deal with it quickly and accurately. Rather than using a concrete approach, he used his knowledge of natural number representations and counted in two's for the 10 loaves of bread — a more abstract and less grounded approach. He did use this abstract representation to directly calculate the second part, instead he followed a concrete approach, which involved linking units and counting the linked units. My responses and prompting helped him to maintain impetus and complete the second part of the task.

		So now how many loaves can Rick buy?
119	Jas	Two ... (whispering) ...
120	BB	... two ...
121	Jas	... twenty rand.

122	BB	... that's twenty rand. Can you explain to me, how you got that number of loaves?
123	Jas	It's five rand ... five rand for two-thirds ... two of these ... (muttering) ...
124	BB	So you took two of the whole time. So you counted ... with two loaves you counted five rand. And then that ... next two was?
125	Jas	Six ... five ... ten.
126	BB	Ten rand ... and then the next two was ...?
127	Jas	Fifteen ...
128	BB	... and the next two was ... ?
129	Jas	Twenty.
130	BB	Twenty! That's great! And how many loaves ... so how many round loaves can Rick buy?
131	Jas	Uh ... eight.
132	BB	Eight loaves. That's great!

(Jason interview 3)

Jason was also able to effectively calculate the number of extra round loaves corresponding to extra standard loaves (Jason interview 4: **132–154**). He worked out the cost mentally and then counted concretely with loaves for the extra cost to get the number of extra loaves.

Comparing the different approaches of Ruth and Jason:

- The concrete approach appeared more meaningful and so matters of interpretation appeared to be more easily dealt with correctly. However, because of the complexity, it was easier to get bogged down and lose the thread of the solution.
- The more abstract approach was quicker and simpler to implement, thus allowing Jason to proceed further with the problem without help. But because it was less grounded, it did have the problem that it was more difficult to interpret meaningfully within the situation. As a result, Jason found it difficult to proceed in this way with the second part and needed to use a more concrete strategy.

7.5.2.6 Interview 6

In this interview, Jason used his knowledge of fraction representations to immediately determine the amount of sausage he could buy for one rand, when the unit price was two or four rands. He not only named the corresponding fractional quantity, but he was also able to correctly subdivide a sausage to form a piece of the size he had named. Here his knowledge of the fraction representations and his ability to recognize situations in which these representations were fitting, made it possible for him to efficiently respond to these questions.

Another task where knowing a representation seemed to have a positive effect was when buying sausages for nine rand at the price of six rand per sausage. This is discussed in detail at the end of section 7.3.6.4. Near the end of this interchange, I grouped six rand into two groups of three. On seeing this, Jason seemed to immediately recognize that the remaining three rand could buy half a sausage and he was able to conclude that he could buy one and a half sausages. Here his good understanding of halves allowed him to quickly complete what had been a rather difficult task.

7.5.3 Structuring effect of representations — conflicting with the action structure

7.5.3.1 Interview 1

After the children had performed the subdivisions in this interview, we spent some time comparing the different quantities obtained. In some of these comparisons, the children displayed two responses of widely differing character. Typically, one response would declare equality and would include a mathematical representation in describing the quantities. The other would involve a visual comparison and would result in a response that the quantities were not quite the same. For example,

30 BB ... Could you, for me, share the loaf of bread between Tracy and Rick so that they both get pretty much the same?
31 Jas (Cutting the loaf)
32 BB Okay. So you cut it there. Can you show me why they've got the same?
33 Jas Because ... I cut them in the ha ... half. Now they both get the same.
34 BB So they both get the same. Aha! Let's put them next ... the pieces next to each other. Are they ... are you sure that they're pretty much the same size?
35 Jas Not really.
(Jason interview 1)

Here, the mathematical representation of a 'half' appeared to allow Jason to declare an idealized precision in his subdivision of the bread. However, when he practically compared the parts, he decided that they were not really the same size.

A similar incident occurred with Ruth. With my prompting, she used a mathematical representation (quarters) to allow herself to assign an idealized precision to the equality.

286 BB Can you explain to me why they're all more or less the same size?
287 Ruth Uh ...
288 BB If you think of what you did with a loaf of bread. How did you cut up a loaf of bread on Monday?
289 Ruth I cut them ... up ...
290 BB ... okay ... please put it on top. That's it. Into ... ?
291 Han ... mmmm ...
292 BB ... so you ... you've looked at that yellow other pieces on Monday ...?
293 Ruth Mmmm (Looking at the other pieces) One ... two ... three ... and four quarters.
(Ruth interview 1)

When asked later to compare the size of the pieces, she carried out a practical comparison and found a number of differences. I tried to encourage the idealization by making light of the difference, but she continued to point out more.

354	Ruth	This one's ... the same ...
355	BB	... that one ... is the same size. That's marvellous!
356	Ruth	And ... that one ... is a bit shorter.
357	BB	It's a bit shorter.
358	Ruth	Ja ...
359	BB	You've probably flattened it by ... (laughing) ...
360	Ruth	Um ... and ... that one's ... also a little bit shorter.
361	Ruth	And ... that one's ... a lot shorter and that one's a bit more fatter than this one ...
362	BB	Okay ... so it's a shorter and a thinner one.
363	Ruth	Mhmmm ...
364	BB	Okay!
365	Ruth	And ... and ... these two ... are the same ...

(Ruth interview 1)

Here it appeared that the use of mathematical representations could result in the conceptual idealization of the practical situation. That is, when the child viewed the situation from the perspective of the mathematical representation, they perceived an idealized equality, but if they adopted a more direct, visual perspective, they noticed the inequality due to the imprecision of the subdivision. It appeared that when the child's perception was mediated by the mathematical representation, it was less true to life, but possibly more true to intention, than the direct visual perception. If such a mathematical idealization is imposed by the teacher, without acknowledging the important element of approximation, this could encourage the common attitude that mathematics is not about the real world.

7.5.3.3 Interview 3

The second task in this interview involved determining the number of loaves that could be bought for twelve rand. This task was the converse of the first and Jason experienced difficulty with it *(Jason interview 3: 24–66)*, possibly for this reason. As he did on a number of occasions, Jason calculated mentally, rather than working concretely in this task. Initially, he merely stated his incorrect solution to the problem. From his solution, it appeared that he had reversed the link between units — using one rand for two loaves, rather than one loaf for two rand, to get twenty four loaves for twelve rand. It seemed as if he had reversed the objects (rands and loaves), but retained the order of the calculation, which resulted in inverting the relationship between the money and the loaves.

At the start of our discussion of his solution, Jason did not appear to see the problem with his solution. A possible interpretation is that his mental solution strategy was strongly linked to the form of the calculation — doubling, or counting in two's. More strongly than it was linked to the more objective quantities involved in the relationship — rands and loaves. This became evident in his response to my query about how he had determined his response, where he counted the rands in twos, but then doubled this to get twenty four loaves. I wondered if for each rand he actually counted two loaves, to get to twenty four. Another possibility, which was suggested by the manner he immediately moved to twenty four loaves after counting in two's to twelve rand, is that he was using the more abstract and efficient doubling operation instead of counting. But he was doing so in the same way as he had for the direct relationship of the first task, rather than inverting the operation in response to the inversion of the problem in this task.

In an attempt to alert him to the problem with his solution, I referred back to the previous question where seven loaves cost fourteen rand. I attempted to use the order of numbers to bring out the problem, saying that as twelve rand is less than fourteen rand, the number of loaves should be less than the previous number. But he did not respond to my query and so I attempted another approach.

In my second attempt, I tried to scaffold a concrete counting approach, explicitly linking each two rand unit with a single loaf. I then iterated this concrete replication and linking for a few two rand units. Jason appeared happy to follow this scaffolded approach and so obtain the correct number of loaves. He showed this by also correctly determining the number of loaves for twenty rand. Also for eight rand, where he also provided an explanation, describing his linking to get the required number of loaves.

It appeared that for Jason, the simplicity of the more abstract mathematical doubling structure of the previous calculation, conflicted with the more complex operations that were needed to work effectively with the objective relationships of this problem. Choosing the simpler option resulted in an objectively incorrect conclusion. A lack of awareness of the inverse structure required for the abstract mathematical operations in the inverse objective situation, resulted in the same direct structure being applied to the inverse situation, and so an incorrect objective conclusion.

7.5.3.6 Interview 6

Jason's knowledge of the representations of a half and a quarter helped him determine the amount of sausage to buy when this was a half or a quarter (Jason interview 3: 46–53). But his seeming reliance on known representations appeared to limit his approach to the question when he did not know a representation fitting with the situation. For when he was able to buy only one eighth (Jason interview 3: 66–79) or one third of a sausage (Jason interview 3: 106–117), quantities which he was not able to represent, he appeared to guess the amount to buy rather than following alternative strategies. After some effort in working with him on these two tasks, he was able to follow an alternative strategy to correctly determine that he could buy one sixth of a sausage in the final task of this type.

In the task of buying sausages for nine rand at the price of six rand per sausage, Jason's facility with whole numbers and whole number operations also seemed to have a negative effect on his deliberations. As discussed in detail at the end of section 7.3.6.4, it appeared as if the technical difficulty of dividing nine by six may have influenced him to weaken the price constraint, or possibly even ignore it, in order to obtain a technically more tractable problem. A more forgiving interpretation may be that to precisely divide nine by six required the use of two division units — the six to get the whole part of the quotient and three for the fractional part (halves). The technical complexity of completing the operation to find the number of halves may then have led to Jason becoming confused and losing track of the first part of his mental calculation — thus ending up by dividing nine by three. This may be seen as an influence of the symbolic representation of the calculation. Jason's willingness after the fact, to see his result as consisted with the price of six rand per sausage, does suggest however, that even in this case there may have been some weakening of his view of the price constraint in order to achieve this consistency.

Chapter 8 – A web of learning

The aim of this study was to develop a theoretical perspective that would provide deeper insight into possible processes of importance in the grounded learning of rational numbers. This perspective should be informed by relevant theory and consistent with the teacher and children interviews carried out in the three cycles of this work. Even though it was not used as an explicit criterion for judging the theory, the research was motivated by the hope that the final perspective developed could provide a useful guide for teaching.

This chapter discusses the development of the perspective. The discussion is subdivided into a number of different sections. The first four sections describe different issues related to the rational number interpretations identified in this project. I then discuss the complementary perspective of grounding operations. Following this, issues related to the development of a more formal understanding are discussed, in particular the issue of developing representations. The discussion then returns to the initial theoretical formulation adopted for this research, that of rational number subconstructs, before developing a web of learning as a synthesis of the research. Finally, some implications for the teaching of rational numbers are mentioned.

Each of these sections may be briefly described as follows:

8.1 The question of whether the interpretations appear to be authentically grounded, is discussed. That is, does each interpretation stem from meaningful and familiar situations in the life of the children and did the children engage readily with the corresponding problems?

8.2 The interpretations are discussed as activities from the activity theory perspective, in order to obtain insight into the possible significance and motivating features of these interpretations to children.

8.3 The interpretations are discussed as actions from the activity theory perspective, in order to obtain insight into the importance of structure and goal directed action in developing informal understanding of rational numbers through these interpretations.

8.4 The problems of interpretations alone are briefly discussed.

8.5 Operations are discussed as a complementary perspective on the grounding of rational number understanding. Of particular importance is the feasibility and utility of viewing each operation as a unit of coordination in the grounding situation.

8.6 Themes arising in the interviews that relate to the development of a more formal, conceptual understanding of rational numbers based on such an intuitive foundation, are identified.

8.7 The importance of representations for this further development, is explored.

8.8 This provides a critique of the rational number sub-constructs as classifying the meanings in grounded experience of the rational number conceptual domain.

8.9 The strands of interpretations, operations and representations are drawn together to construct a web of learning. This web yields deeper insight into the complex process of learning rational number concepts based on, or related to, meaningful experience in the life of the child.

8.10 Some implications of the use of this web for the teaching of rational numbers are then discussed.

8.1 Interpretations as authentically grounded

8.1.1 The part-whole interpretation

The part-whole interpretation involves a complete whole that has been subdivided into a number of parts. Even though the parts are shown, they are not separated from the whole and so may still be seen as part of the complete whole. It is the possibility of direct comparison between a selected part, and the whole of which it is part, that forms the part-whole interpretation.

It is this lack of separation between the part and the whole, that made it difficult to find situations that provided an authentic grounding to this interpretation. Most activities appeared to involve removing the parts, and possibly even the remainder, to achieve the purpose of the activity. For example, the parts would be shared out, or arranged into groups, or consumed, or converted into something else.

In fact, the major purpose for which the part-whole interpretation appeared to be used, was to compare the part with the whole in order to accurately represent, or name the part, in relation to the whole. The teachers used this approach routinely and competently in a number of instances in their interviews. In some cases, such as when discussing the scaling operation, the goal was precisely to represent such a comparison (5.4.2.5 in the teacher analysis), rather than to explore a grounding activity. Here they re-examined the final representation in order to name the parts more carefully and so verify their intuitive judgement. In other cases, they appeared to focus on a transient state in an ongoing activity (see 3.4.3.2 and 3.5.2 in the pilot cycle, or 5.4.1.1 in the teacher analysis). Here the other activity provided a grounding instance of a different interpretation. Playing hopscotch (3.4.3.2 in the pilot cycle) was one example of a grounded activity in which the part-whole interpretation was fitting, but here the goal of quantifying the part of the foot that was over the line would be quite subsidiary to the major motive of playing the game.

Because of the difficulty of finding a grounding activity with a natural motive for the part-whole interpretation, the children did not explicitly work with this interpretation. Also, they did not use part-whole approaches similar to the teachers' when trying to precisely name the parts they had created.

This analysis suggests that the part-whole approach may be better seen as related to the strand of developing representations, than as an authentically grounded interpretation of rational numbers.

8.1.2 Sharing

The sharing interpretation is based on the activity of fair sharing. All the teachers appeared to find this interpretation simple and accessible (pilot cycle 3.4.3.1 and teacher analysis 5.3.1.2). They were also of the opinion that children would show a similar response.

Both children seemed to see fair sharing as familiar and natural. They engaged readily with the sharing situations in the interviews and appeared to know the final goal of the sharing — that each person should have exactly the same amount — even when they found it difficult to practically form the equal subdivision. (children analysis 7.1.1)

8.1.3 Grouping

The grouping interpretation involves grouping a collection of objects into groups of a given size. This corresponded well to different grounding situations that involved packing of some sort, as was apparent in the discussions in both the pilot (3.4.3.3) and the teacher (5.3.1.3) cycles. As instantiated in such situations, the teachers found the grouping interpretation quite natural, suggesting a number of different packing situations. They also believed that their children would relate well to this grounding for the grouping interpretation. A packing situation was not included in the children's interviews and so children's direct responses to this interpretation were not investigated.

Although the teachers did find the idea of packing quite natural, they also seemed to easily get confused between grouping and sharing. This may have been because both interpretations related to the distribution of objects in a collection, but sharing problems were far more common then grouping in teaching rational numbers. As a result, teachers may have developed an habitual approach to distribution situations as sharing problems.

8.1.4 Measurement

The measurement interpretation relates to the choice of units to measure a quantity. More specifically, given a base, or 'whole' unit, to choose a subunit that would result in a convenient measurement of the quantity. Then to carry out the measurement by counting the number of the chosen subunits, that are equivalent to the quantity to be measured. An important aspect of the measurement interpretation is the manner in which the size of the chosen subunit is related to the final measurement count.

The teachers believed that the measurement interpretation was authentic, because measurement was common in everyday life. To back this up, they described a number of contexts for measurement that they thought children would have experienced. Lungiswa (3.4.3.4) discussed measuring paraffin and cold drink and the fact that containers of different sizes were used as units for measurement. Sarah and Nadine explored the context of baking (5.3.1.4) and were particularly interested in the possibility that children would count half units to measure a desired quantity. They also mentioned that not all children would see the baking context as natural, this would depend on whether or not their mother baked.

In the children interviews, practical measurement was evidenced by the slicing of bread into slices of a given thickness (children analysis 7.1.2). Ruth, who often helped prepare for school by 'making snacks', appeared to consider the interpretation in this grounding situation as authentic. However, Jason's experience was different, involving slicing bread mainly at mealtimes, and he did not appear to find slicing the whole loaf as natural in this setting.

This supported the teachers' view that the authenticity of the grounding experience does depend on the home environment of the specific child. But, because of the broad range of possible measurement situations, it should be possible to identify relevant situations for each child.

8.1.5 Operator

The operator interpretation involves the conversion of one quantity to another, in a proportional manner. Lungiswa saw both shopping and swopping as authentic grounding situations for the operator interpretation (3.4.3.6). Sarah and Nadine also identified swopping for this interpretation (5.3.1.5). In each case, they were of the opinion that children would engage well with these contexts and find the 'conversion' operator natural and accessible.

For the children interviews, the context of shopping, or exchanging an amount of money for a number of items at a fixed rate, was selected as grounding for this interpretation. Both children appeared to find this grounding quite natural, engaging readily and successfully with these problems (see 7.1.3 and 7.1.6). Baking tasks, representing the pattern before or after cooking also seemed to provide a grounding for the operator interpretation that was accessible and interesting to the children (7.1.5).

8.1.6 Ratio

Ratio involves comparing two coexistent quantities in a proportional fashion.

The problem used to introduce this interpretation to Lungiswa asked for a comparison between the number of boys and girls in a class. She appeared to find this problem rather unnatural and difficult to formulate (3.4.3.5). It was only after I had shown her how groups of boys could be aligned with groups of girls, that it made sense to her and she then solved the problem with little difficulty. In her attempts to analyse the problem, she appeared to find it difficult to conceptually separate the two quantities involved. And our subsequent discussion, Lungiswa appeared to be of the opinion that the problem was difficult to understand because a comparison of two quantities was required, rather than a change of one to the other.

Lungiswa found the operator interpretation that involved an active change of one quantity into another, much more accessible. This may have been due to the fact that the operator interpretation was active, as opposed to the more descriptive ratio interpretation. This was supported by the examples she gave of situations fitting the ratio interpretation — all except the final balance scale example were better examples of the operator interpretation. But it may also have been that such active change made it easier to conceptually separate the two quantities to be compared. This was supported by her suggestion of a way to improve the problem by formulating it as a comparison of the number of children in two different classes — one of boys and one of girls.

In either case, she appeared to find the ratio interpretation much less natural than the operator and, by her choice of 'operator' examples for children's activities, appeared to consider that this would be the same for her children as well. A similar conclusion could be drawn from the interviews of the teacher cycle. For most of the 'ratio' activities that the teachers described, could be more fittingly viewed as 'operator' activities.

This suggests that ratio comparison may not be an authentic, or natural grounding interpretation for rational numbers. Due to its descriptive nature, it may be better seen as important for the further conceptual development, and possibly even the representation of rational numbers. On the other hand, it may be possible to find authentic, everyday life situations that naturally involved the ratio comparison of two, more easily separated quantities. In this case, ratio comparison would indeed be properly viewed as a grounding interpretation. Further research would be needed to make a reasonable decision on this matter.
Unfortunately, the insight into the possible importance of separating the two quantities involved, occurred late in the research process and so this question was not further investigated in this project.

8.2 Interpretations as activities

One of the basic premises of constructivism is that a person's learning at any time builds on the foundation of prior learning laid throughout the course of their life experience up to that point (Davis, 1990). It follows that the introduction of previously unknown formal concepts, such as the learning of rational numbers as introduced in school, would be strongly influenced by the child's prior life experience. It was my intuitive belief that the learning of rational number concepts would be more meaningful and ultimately more effective if it could be aligned with the children's prior learning in a fitting fashion. But this belief was not sufficiently well formulated to serve as a basis for research into the learning process.

The concepts of activity theory were identified as useful for this research because of the insight they lent to a person's generation of meaning through engagement in the activities of everyday life. In particular, the concepts of 'activity' and 'action' provide two complementary perspectives from which to view children's engagement with tasks in ways that could be considered as meaningful (see 4.3 and 6.2). Viewing tasks as activities leads to a focus on their significance, due to the personal, or social. motive that is satisfied through engagement in the activity. This perspective offers the valuable insight that concepts to be introduced would gain significance to the child, if they could be aligned with significant experiences in the everyday life of the child. Achieving this could greatly improve the motivation of the child to learn the concepts.

Thus, the concept of 'activity' led to the identification of the themes of 'significance' and 'motivation through significance' as an important component of children's meaningful learning through grounded tasks. To more precisely delineate these themes, a more careful interpretation of the term was sought. The two relevant definitions from the Oxford English Dictionary (1989) were: 1) The meaning, or import of something; and 2) Importance, consequence. Combining these with the insight obtained from activity theory, I will take the following interpretation of the term 'significance':

> The importance ascribed by the child to an activity due to their intention to satisfy some personal or social motive through engagement in the activity.

This perspective yielded a number of insights into the different possible interpretations of rational numbers investigated in this research.

The sharing interpretation could well be viewed as an activity. It could be seen as the social act of distributing a needed resource (and so satisfy personal needs) in a manner that satisfied the motivating social constraint of being fair to all.

Sharing as an activity was particularly evident in the interviews with teachers (see 3.4.3.1 and 5.3.1.2), where the teachers' discussion covered the full human context of sharing — social, individual and emotional. The teachers were not only interested in the children's developing ability to subdivide quantities equally, but also stressed the social importance of sharing and considered children's emotional maturity and the influence of the home environment on children's willingness to share.

In the children's interviews (7.1.1), both children engaged actively with the sharing situation, interacting with the models to carry out their part of this activity. This suggested that they found this task both socially and personally significant. In fact, Ruth's interaction with the models appeared as if she was actively trying to be fair to each of the models — as if she was working to satisfy this social motive.

The operator interpretation appeared to be well aligned with exchange transactions such as shopping and swopping (see 3.4.3.6 and 5.3.1.5). These could possibly be viewed as activities, motivated by the need to acquire necessary, or desired, resources. In the case of swopping, both people involved could be motivated by such a desire. But the motivation of the second person in shopping is less evident. It could possibly be seen as the indirect acquisition of desired resources. Another alternative may be the acquisition of wealth, with all the related social benefits of status and power, in addition to access to resources. But neither of these motives appeared to be particularly important to children.

What seemed more significant in the children's interviews, was their engagement in the social practice of shopping (7.1.3) — playing the part of a person shopping. So in this case, a social motive may be seen as engendering the significance of these tasks for the children. The motive to become a member of the community that engages in the social practice of shopping.

To the pilot teacher, the grouping interpretation did not initially seem to be as naturally realised as the sharing and operator interpretations (3.4.3.3). After some discussion, we decided that it did correspond well to the packing of items for display and sale. The teachers in the teacher interviews also found the correspondence between grouping and packing quite natural (5.3.1.3). But this correspondence was not perfect because the variable sizes of some items (such as fruit) meant that the same number were not always packed into each group. Apart from the discussion about this flexibility, the teachers talked little about the human context of this interpretation and its relevance to the children. This suggested that they saw it as less significant than sharing and swopping in the lives of their children. This may have been because they considered packing to be less important to the children. But is may also have been because packing itself could not be seen as an activity carried out to satisfy significant motives, but rather it formed part of the activity of buying and selling, that was personally or socially significant.

As described earlier, the primary aim of the part-whole interpretation is to describe the comparative sizes of the parts. It is not aimed at bringing about some desired change that would satisfy some personal or social motive. That is, it is representational, rather than active. There may be a significant motive for developing representations. For example, to communicate, or even to create. But in this case, the motive will generally be related to school performance, particularly in the mathematics class, and so would not be derived from the grounding situation. It follows that the part-whole interpretation is not validly viewed as an activity, or even an important component of an activity that is not school focusses.

The main aim of both the measurement and ratio interpretations is also descriptive. For the aim of a measurement is to describe some quantity by counting the number of chosen units that would be needed to create an equivalent quantity. While the aim of a ratio comparison is to describe the relationship between two different quantities. Consequently, the same could be said for these interpretations as was for the part-whole interpretation.

For the children's cycle, it was however, possible to develop a task of slicing a loaf of bread based on the process of measurement (7.1.2). This task was viewed by Ruth as significant, because it aligned well with her life experience of preparing snacks for school. Here the slicing of the bread formed a small, but significant component of the family activity of preparing snacks. On the other hand, Jason did not appear to see this task as particularly significant. In his case, he related the task to slicing bread for a family meal and found the requirement that the whole loaf be cut, rather unnatural.

The concept of an activity formed the basis for an analysis of the significance of the different tasks and interpretations to children. It became evident during the course of the children's interviews that for a task to be seen as significant, it was not necessary for it to encompass a complete activity. It was sufficient for it to involve a component of an activity that evoked the significance of the activity for the child. This was the case for the shopping and slicing bread tasks.

A final point of interest arose when thinking of possible motives that lent significance to the tasks for the children. This was the importance of engagement in social practice for social motives. It was apparent for general social practices in the shopping task, and for the specific social practice of a family group in Ruth's response to the bread slicing task.

8.3 Interpretations as actions

The second important concept used from activity theory was that of an action (see 4.3 and 6.2). This is an act, or a sequence of acts consciously directed towards the attainment of a specific goal. It is important to note that attainment of the goal need not result in the satisfaction of any personal or social motive. As a result, an action may not be perceived by the actors as significant. But the goal can be stated by the actors and at any stage of the action they will be able to evaluate whether or not the goal has been achieved. People can also plan their action to take advantage of whatever features of the situation could be used to best achieve their goals. This idea is more fully explicated (see 7.2) in Gibson's (1986) concept of affordances and the ideas of situated learning (Brown, Collins and Duguid, 1989). That is, that the form of the action is aligned with the structure of the situation. Thus the concept of an 'action' from activity theory suggested three important attributes that could be related to the meaningful understanding of a task:

- The explicit goal that is to be attained in the task, or in different components of a task and the manner in which the action is organized to achieve the goal.
- The alignment of the actions followed to achieve the task with structural features of the task situation that make that attainment of the goals possible.
- An action is deliberate, in that the sequence of acts followed to achieve the goal is consciously chosen and adapted if necessary in order to achieve the desired goal.

8.3.1 Part-whole

The goal of the part-whole interpretation is to describe the size of the part with reference to the whole. The teachers generally used it when working with other interpretations, in order to build precise names of quantities which were useful for later reference (3.4.3.2 and 5.4.1.1) or else to check the accuracy of other conclusions, or estimates (5.4.1.1).

To perform this action when working with other grounded tasks, the teachers needed to identify a configuration of the specific part-whole form, that was created in the process of their work on the task. It is this part-whole configuration that allowed alignment between the situation and this specific representational action. By analysing this specific configuration, they were able to precisely name important quantities in their task solution. To perform this analysis, they often first drew the configuration that they had identified and then worked from the drawing. It was evident from the interviews that the teachers worked extensively with their children to develop the skill of naming specified parts of such configurations. But very little explicit work was done to help children identify appropriate configurations when working on grounded tasks. This may be a factor contributing to the difficulty that children experience when using rational numbers to describe quantities created when working on grounded tasks.

8.3.2 Sharing

When viewed from the action perspective, the goal of a sharing task would be to accurately subdivide the full quantity into the desired number of equal parts.

This view comes to the fore in some of the more complex sharing situations. Here the children focussed sharply on practically carrying out the subdivision to achieve the goal of creating the desired number of equal parts (for example Jason's sharing of 3 loaves between 4 in section 7.1.1). This involved a shift of attention away from their motive of creating these parts to perform a fair sharing. In fact, when things became particularly difficult, Jason even lost track of the number of pieces to create.

In some cases, the children carried out sharing as a sequence (7.2.1.1) of first the decision of how many pieces to cut, then the estimation of the width of each piece to give the right number, and finally, the then cutting of the pieces. When following this strategy, the children often found difficulty with deciding how many to cut, or with estimating the widths to cut, or with performing the cutting. In other cases (7.2.1.1) they followed a different strategy and repeatedly halved and distributed pieces, deciding on the number of pieces to cut when halves didn't work. This strategy appeared much simpler when repeated halving was sufficient to yield an equal subdivision.

Looking at alignment between situation and strategy, the second approach required little initial alignment and so the children could immediately begin their sharing. At each stage, they could distribute as many pieces as possible and then cut all the remaining pieces in half. When repeated halving was not sufficient, they would need at some stage to change their strategy in order to achieve better alignment with the situation — in effect changing to the first strategy. The alignment for the first strategy is more complex as the person sharing would need to start by choosing both the number of cuts and their width in a manner that would be aligned with the situation. The simplest way to align the number of pieces would be to count the number of people and cut each 'whole' piece into this number of equal parts. A more complex alignment would count the wholes and then choose to create a total number of pieces divisible by the number of people. Once the number of pieces for each whole had been decided, it would be necessary to estimate the size of each piece so that this number of pieces of equal size could be formed when cutting the whole.

8.3.3 Grouping

The grouping action is quite straightforward until remainders need to be taken into consideration. The goal of grouping is to form groups of a predetermined size. That is, each group needs to contain a given number of objects. Generally, there would be some objects remaining, a number too small to form another group. In our discussions, we dealt with this remainder in two different ways (5.4.1.3). Firstly, they were simply excluded from further consideration because they did not for a complete group. For example, eating the few apples left over. This was the most common alternative, particularly because we focussed on teaching younger children. However, in some situations, the remaining objects were to be considered as a fraction of a group. This would often be the case when grouping was being used as a simple measuring operation, as in grouping to determine the cost for a sale.

Aligning the creation of whole groups with the situation would be simply a matter of determining the number of objects to be placed into each group. When dealing with the remainder, a further alignment would be necessary. This would involve deciding whether it was more fitting for the situation that the remainder be excluded of whether a fractional group should be formed.

8.3.4 Measurement

The goal of a measurement is to precisely describe a quantity by comparing it with a chosen unit. The unit may be an object, or a standard quantity that has already been measured. In either case, it functions purely as an object of comparison. To perform their measurements (see 5.4.1.4), teachers often started by working intuitively with common objects — practically, or with mental images of the object — to get a rough estimate of the measurement. Then they worked more carefully with more formalized measurement tools to validate and update these estimates. The children interviews did not include an explicit measurement task, but the task to slice a loaf of bread involved an implicit measurement (7.2.2).

The most basic issue of alignment between the action and the task is the identification of the property to be measured. This identification was made in all the measurement tasks of this research. Then an appropriate unit of measurement would need to be chosen. For the initial estimation, this involved identifying an object in the situation that would serve as a unit of comparison. For the formal measurement, an appropriate measurement tool would need to be chosen and a corresponding unit selected. Once this basic unit of measurement had been chosen, it would often be found that the desired measurement would not be an integral number of these units. This raises a final issue of alignment, for the remainder would need to be measured and described as a non-integral rational number of the basic unit. To do this, one would need to choose a sub-unit of the basic unit, so that the remainder would be an integral number of these sub-units — a final alignment between the action and the situation. Furthermore, to allow for description of the measure in terms of the original unit, the sub-unit would need to be a unit fraction of this unit.

Note that once a fitting unit fraction sub-unit had been chosen, the measurement could be viewed as a measurement of the full quantity using this sub-unit. Becoming flexibly able to choose fitting unit-fraction sub-units and understanding how a change of sub-units would result in a change in the number representing the measurement, is important for the measurement interpretation of rational numbers.

8.3.5 Operator

The goal of the operator action is the conversion of a quantity of one type into a quantity of another type. This is exemplified by the actions of swopping or shopping (5.4.1.5, 7.2.3 and 7.2.6).

The key feature of the grounding situation that makes it possible to align these actions with the operator interpretation is that the conversion must be relative to a fixed unit conversion. Specifying this in more detail, a fixed unit of the original quantity must always be converted to a corresponding fixed unit of the final quantity. And, that the conversion of any other amount of the original quantity, could be viewed as carried out through such a conversion of a number of groups of the original quantity. As a result, the conversion will have a multiplicative, or proportional nature. For instance, when a number of such conversions are made, the numbers representing each quantity will be proportionally related, relative to fixed units of each quantity.

The importance of the awareness of this unit conversion constraint can be seen in the discussion with Jason about buying sausages (7.3.6.4). His initial response when buying a piece of sausage for one Rand when the unit prices were first eight rand and then three rand, was to guess. When working with him to achieve a more considered response, one major area of difficulty was to enable him to see the effect of the price constraint — relating the size of a piece bought and the size of the payment — for small amounts of a sausage. It took a number of different approaches, including the fairness to different buyers and the fairness to the seller before he began to appreciate this effect. Finally, he did appear to appreciate this and was able to correctly determine that a third of a sausage could be bought for one rand. Later, he again seemed content to violate this pricing constraint in order to obtain a technically simpler solution that conflicted with the constraint.

8.3.6 Ratio

For the ratio interpretation, one is faced with three quantities in two different spaces and the goal is to determine a fourth. In the four final quantities, two are in each space and the relationships between the quantities are as follows:

- the ratio of the quantities in each space is the same, and
- quantities can be paired between spaces so that the ratios of each pair are the same.

This configuration is the same as for the operator interpretation, with the main difference being one of application. The operator interpretation relates to an active conversion of one quantity into another, while the ratio describes a relationship between two often coexistent quantities. So the operator interpretation generally refers to a physical change, while the ratio interpretation generally refers to a descriptive relationship. The goal of the ratio interpretation is thus generally to calculate the fourth quantity in order to complete such a description of a situation.

The similarity in mathematical structure between the operator and ratio interpretations implies that a similar structure will be needed in the situation in order to achieve alignment with the action. That is, it must be possible to formulate the relationship between the two different quantities, in terms of a fixed unit comparison.

In performing the ratio comparison tasks the teachers followed a number of different strategies, each demonstrating such a fixed unit comparison (5.4.1.6):

a. Drawing the two spaces of objects aligned with each other and then counting.
b. A composite numerical conversion, formed by one division and one multiplication by whole numbers.
c. A numerical strategy that involved conversion through multiplication by a single rational number — a unit rate. This development involved a deal of work and some scaffolding by me.

8.4 Problems of interpretations alone

As discussed in the preparation for the teacher cycle, rational number subconstructs as grounded interpretations did not appear to tell the whole story of grounded rational number learning. This is because on the whole, the interpretations appeared quite separate and did not seem together to encourage the learning of a single coherent rational number understanding (4.1). In addition, many other issues that had been identified as important for rational number learning, such as the importance of unitizing, order and multiplicative operations (see 4.4.1, 4.3 and 4.3) were not naturally included in such a view.

When working with the teachers, it appeared that the use of single interpretations could lead to over generalization. For example, in the teachers' responses to grouping (3.4.3.3 and 5.4.1.3), it appeared as if the sharing interpretation had been over generalized. On the other hand, working with all interpretations did not appear to encourage the development of a 'unifying' rational number concept because, apart from the underlying rational number representations, they did not all relate naturally to each other.

Finally, viewing interpretations from the perspective of actions yielded some insight into the alignment of a persons acts with the affordances offered by the situation. But actions are conscious and deliberative and thus would lead to a rather deliberative foundation. In this work, intuitive founding effects are sought, which would not properly arise from deliberative actions. So, if there are intuitive foundations, other mechanisms needed to be sought for their generation.

A different perspective was thus sought that would serve to relate the different interpretations and possibly include some of the other issues identified as important for rational number learning. The concept of an operation as a unit of coordination seemed promising for this purpose.

8.5 Operations as units of coordination in grounding situations

As discussed in section 4.3, the idea of a coordinating operation stemmed from the concept of an operation in activity theory and was strongly influenced by the concept of coordination in situated cognition. Later in the research, a number of concepts developed in the Piagetian tradition (6.3.2.1), including the particular the theory of representational redescription (6.3.2.2) contributed to the further development of this idea.

The intuitive understanding that forms the foundation for rational number learning could thus be seen as based on the learners behavioural competence in situations aligned with the different interpretations. This behavioural competence would depend on the children's effective coordination of their acts with the affording structure of the situational context. But much of this coordination would be intuitive and not accessible to the children's reflective awareness. Operations are then seen as units that are appropriate for the representation (representational redescription) of the children's intuitive behavioural sequences. First, by the teacher for the purposes of understanding and facilitating the children's learning of rational number concepts. Second, possibly by the children, allowing greater flexibility and control of their acts and their ability to reflect on and use rational number concepts in these situations. The appropriateness of these units would be judged in terms of:

- Whether they do indeed represent a coordination between the affordances offered by the situational context and the acts of the child.
- Whether they may be seen as unitary acts and are not themselves constituted by a number of easily separable acts.
- Whether they are relevant to rational number concepts and the alignment of these concepts with the situation.

The following possible operations were identified at the start of the teacher cycle:

a. unitizing,

b. equal subdivision

c. iteration, or counting,

d. scaling,

e. coordinated counting,

f. exchanging,

Later in the teacher cycle, a further possible operation was identified:

g. unequal subdivision, or taking reasoned unequal parts.

During the teacher cycle, coordinated counting and exchanging were judged as not forming appropriate operations for the analysis of the teachers responses to the tasks (5.4.2.6 and 5.4.2.7). It was decided that more appropriate operations would be coordinating, or linking groups, and iteration, in this case the iterated linking of groups. Here the groups formed units of coins and of the objects to be bought, that had been formed as fitting with the situation. To describe this operator in a more general manner that would remain suitable for continuous quantities (and for exchanging parts), the term 'linking units' will be used in place of 'linking groups'.

In the analysis of the children's interviews, and the subsequent reflection on all the interviews, two different interpretations of unitizing became apparent. This led to the formulation of two different operations: constructing units, and replicating units.

Also, a number of similarities between equal and unequal subdivision became evident. A more appropriate separation appeared to be between 'measured subdivision' and 'prototyped subdivision'. Furthermore, measured subdivision did not appear unitary, instead it could be seen as a combination of constructing and replicating units and iteration. But prototyped subdivision did appear to be unitary and so constitute an appropriate operation.

The operations finally identified were thus: constructing units, replicating units, prototyped subdivision, iteration, scaling, and linking units. In the remainder of this subsection, each of these operations will be described and reasons will be given to justify the choice of them as units of coordination. Also, it will be shown how each of the possible operations given initially could be constituted from these operations.

8.5.1 Constructing units

This operation involves the identification of configurations in the situation as appropriate units for action or analysis. These units may be whole objects, part objects, or collections of objects.

The teachers identified the identification of units as particularly important for developing a sound conceptual understanding of rational numbers (5.4.2.1). In particular, the ability to flexibly construct different units in a situation and to relate the rational number representations obtained in each case. They also saw it as particularly important for judging children's responses in teaching, that teachers be aware of the units identified by children as the basis for their representations.

A number of examples of constructing units can be given from the children's interviews. For example, to subdivide a loaf for sharing (7.3.1) Ruth attempted to cut slices of a thickness she had estimated would yield the desired subdivision. That is, she had conceptualized the unit as a slice of the thickness that she estimated would yield the required number of equal slices. When buying bread (7.3.3) the price of the loaves was given and to calculate the amount of bread they could buy, the children chose to group their money in units of this price. Here the conceptualization of the units was a simple coordination with the price. When sharing dough between two tins of different sizes (7.3.4), Jason followed the approach of slicing the dough and then distributing slices between the tins. His slices of dough were all of a similar thickness, suggesting that he had conceptualized a unit slice of this thickness. In this case though, the conceptualization seemed arbitrary, and not based on any specific features of the situation, as if he had cut one slice and then chosen to view this as the unit. When buying sausages (7.3.6) many of the tasks involved the identification and relating of a number of units. For example a 'whole' unit corresponding to the price of four rand and a part unit of one rand corresponding to the amount of money to be spent. Or a whole unit corresponding to the price of three rand and a part unit of one rand when two rand was available to spend. Here the one rand unit was chosen because two did not evenly divide the price of three rand. Or finally Jason chose a whole unit of six rand and a part unit of three rand when the price was six rand and the amount he had to spend was nine rand.

The identification of units would involve coordination with the situation because appropriate units would generally correspond to 'natural wholes' in the situation (see Lungiswa's comments in 5.4.2.1). The identification of appropriate units would thus occur through the recognition of such 'natural wholes', or of important relationships between such wholes that gave rise to appropriate units, such as those mentioned in the last two examples of the previous paragraph. This recognition could be seen as a coordinating response to the situation that did not appear to have a general form comprised of possible constituent operations. Because of this, I judged that the identification of units cold indeed be seen as an operation describing a unit of coordination.

8.5.2 Replicating units

This operation involves the formation (practically, or conceptually) of a copy of a unit that has been identified. This may be done by forming groups of a chosen size, for example grouping coins into twos (7.3.3). But it may also be by creating parts of the chosen size from a whole, such as cutting a loaf of bread into pieces of a specific size in order to share it fairly (7.3.1). In each case the unit is formed directly, with reference to the identified unit and so may be sen as a unit of coordination.

8.5.3 Linking units

This is the operation of linking specific units of two different quantities or objects. When drawing, the teachers often drew the units alongside each other, and occasionally drew lines between two linked units (5.4.2.6). When practically acting out an exchange, such as when the children were buying bread (7.3.3), they often demonstrated the link by verbalizing the count of the loaves as they physically moved the coins of the linked money unit. In each case forming the link was direct and immediate, suggesting that this did indeed constitute a unit of coordination.

The teachers used this operation quite extensively (5.4.2.6). Linking units when one of the units is 1, appeared to be quite a simple matter for them. When neither unit was 1, the teachers found this more difficult, but possible with some thought. Being able to link both to a third unit of 1, as was done when comparing the steps of father and son, made this more accessible. It also made it easier to represent this coordination as a mathematical equality of rational quantities — as a whole expressed in terms of different unit fractions.

8.5.4 Iteration

Although initially I saw this as only a counting iteration, it became evident during the research that many actions also involved iteration of other acts. In the final formulation, iteration may be seen as the iteration of any other operation or action. For example, it may be a simple counting iteration (7.3.3), or it may be the iteration of linked units in shopping (7.3.3), or even (see 7.3.6.4) iterating the simple subdivisions of an object (thirds and then halves) in order to achieve a finer subdivision (sixths). Many of these iterations include some kind of counting based on the iterates, for example, counting and iterating linked units. But this is not always the case, as when iterating subdivisions to perform a finer subdivision. Even though involving other acts, the process of iteration itself is identifiable and distinct and thus could be seen as a unit of coordination.

8.5.5 Scaling

The work with teachers and children resulted in little change to the original conception of the scaling operation, as discussed in section 4.4.5. This corresponds to the direct visual comparison of two objects, features or dimensions. Also, this comparison involves the scaling of one to another, rather than the difference between the two. As a direct visual comparison, it is immediate and not deliberative and so could be justifiably viewed as a unitary operation.

The teachers appeared to perform visual scaling immediately, when they could (5.4.2.5). When they had difficulties, these appeared more due to problems in describing their judgements, rather than problems of visual comparison. An example of this is Lungiswa's use of the more familiar term 'eighth' when a more accurate description of the comparison was a sixth. It was also interesting to note that even though Nadine did not appear to trust such a visual comparison, calling it a guess, she still obtained an accurate estimate of the scale. The children also appeared to use visual scaling quite naturally to judge the accuracy of their drawings of the decorative shapes on bread after, or before cooking (7.3.5). This does appear to support the existence of an intuitive visual scaling operation.

8.5.6 Prototyped subdivision

This operation involves the direct subdivision of an object (or collection) into parts, in such a manner that the relationship between the sizes of the parts is equivalent to that in some prototype of the subdivision. This prototype may be concrete, such as the two tins of different sizes that were given for dividing dough (7.3.4). Or it may be conceptual, or possibly a mental image, such as that guiding the halving of an object (7.3.1.1), or even the direct cutting of a whole into a quarter and a three quarter piece (7.3.6.1). It was evident that, even when care was taken, that each aspect of the subdivision was guided by the prototype, being coordinated to achieve the desired equivalence. Because of this, even when care was taken and some sort of comparative measurement was performed, I judged that this could be seen as a unit of coordination.

8.5.7 Relating these operations to those initially proposed

The final scaling operation is precisely the same as that initially proposed. As mentioned earlier, the initial operation of unitizing gave rise to two distinct operations, those of constructing units and replicating units. Also, my initial view of iteration was largely that of counting. Through the process of the research, the emphasis on counting became reduced and the operation became generalized to include any process of iteration.

Both of the initially proposed operations of exchanging and coordinated counting can be seen as the following combination. First replicating units in two different spaces, possibly preceded by the identification of the units to be replicated. Next, units are linked between these two spaces. This linking of units is iterated until the units formed in one of the spaces are exhausted. Coordinated counting has a descriptive goal and the desired result is the number of items formed by all the linked groups in the second space. On the other hand, exchanging groups has an active goal, that is the exchange of the linked groups from one space, for those of the other space.

It became evident in the course of the children interviews that equal subdivision could be performed in two different ways — using a measurement, or a prototyped approach. The prototyped approach corresponds to the operation of prototyped subdivision. While measured subdivision involves first the identification of units that would give the desired number of equal parts, then an iterated replication of this unit to form the desired number of equal parts. This is discussed in more detail in section 7.3.1. In the work with children on dividing dough (see the discussion in section 7.3.4) the same two approaches were taken by the children. Although here prototyped subdivision tended to be more frequently used than it was for equal subdivision.

8.6 Developing more formal, conceptual understanding

Laying a strong informal foundation for rational number learning is important, but it is not sufficient for developing a full understanding of this mathematical domain. It will also be important to build on this foundation to develop the child's conceptual understanding. This will also involve taking a more formal approach to the teaching and learning process. An issue that grew in importance during this research is just how the grounded tasks investigated in these interviews could provide opportunities for developing a deeper, more conceptual understanding of these numbers.

In was interesting to note that children's developed conceptual understanding was evident in some of their responses to these grounded tasks. For example, section 7.4.1 describes an instance when Jason's conceptual understanding appeared to dominate his observations. Resulting in his judgement that he had cut a loaf in half, to share it between two people, even though the subdivision was somewhat inaccurate.

When discussing measurement tasks (5.4.1.4) the teachers were of the opinion that children appeared to see quantities in a very concrete fashion, and directly apprehend them in concrete instances. That is, quantities were concrete and yet objective for children. It follows from this that these tasks could be seen as providing a concrete and objective grounding for rational number quantities and so encouraging children to expand their conception of number and quantity to include rational numbers.

In the discussion of proto-quantitative reasoning in (4.3) the importance of comparison for the development of the number concept is discussed. Based on this, whenever possible I encouraged children to compare their concrete rational quantities in order to further motivate the inclusion of rational numbers as numbers. This was done in the tasks involving slicing bread (7.4.2), dividing dough (7.4.4), buying sausages (7.4.6) and drawing bread after cooking when one of the sides was shown (7.4.5). In general, the children first performed qualitative comparisons, often showing good insight and often describing their observation in terms that could be interpreted as a proportional, or multiplicative comparison that would fit with the rational number concept.

As a further step, I encouraged the children to quantify their comparisons. In some cases, for instance Ruth's relating of the number of slices with their thickness (7.4.2) and Jason's comparison of the dimensions of the bread (7.4.5) they were not able to quantify their comparisons. In others they responded with an additive quantitative comparisons, such as Jason's comparison of the number of slices of bread (7.4.2) and Ruth's comparison of her subdivision of balls of dough (7.4.4) and of the sizes of the bread before and after baking (7.4.3). The difference between their implicit and quantitative responses suggested that they even though they could perform the comparisons intuitively, their more explicit conceptual understanding did not accurately mirror this competency. In these cases I attempted to further develop their conceptual understanding by leading them to view the comparison multiplicatively. An the whole they appeared to find this rather difficult and I was only (partly) successful in the case of Ruth's division of balls of dough.

I also attempted to encourage conceptual development by asking the children to give reasons justifying their responses and then working with them to further develop these reasons. For example, Jason recognized one rand as a quarter of four when buying sausages (7.4.6) and then used this to decide that you could buy a quarter of a sausage. But when asked he was not able to provide justifying reasons. In response I worked with him to relate a quarter to a half and concluded that a quarter was a half of a half. His lack of response to my conclusion left some doubt as to whether he had found this comparison relevant or useful for his conceptual learning.

The two children had quite different styles in their responses to the tasks. Ruth tended to concretely model the situation in the task and in this way almost 'act out' the solution as the process would have occurred. Jason, however, preferred to work out a response to the problems mentally. In the simpler tasks, Jason's conceptual understanding was sufficient to provide the mental tools needed to solve the problem. Also, the task complexity allowed him to mentally keep track of what he wanted to do, as well as remember the needed intermediate results. As a result, his approach led to an efficient solution to these problems. But Jason's rational number concepts were not sufficiently developed to allow a simple mental solution to the more complex tasks. Because of this, he often ended up guessing an answer to these tasks. In these tasks, I generally encouraged him to work concretely and so allow the concrete structure of the task situation to guide him to solution. As an example, this was done in the interviews involving buying bread (7.3.3 and 7.5.3.3). After obtaining one solution in this way, Jason was able to solve a number of similar tasks in a similar fashion. This suggested that when approaching a task, Jason looked for similarities in situations that he had experienced and used these to extend his tools of thought in order to yield a solution to the task. With repeated use, such tools of thought could form the basis for a more abstract approach and ultimately for the corresponding rational number concepts.

In line with this thinking, I encouraged Ruth to develop a more mental approach to some tasks in an attempt to facilitate her development of conceptual understanding. I did this particularly when working with buying bread (7.4.3). With some prompting, she was able to construct units for counting and then count in these units without explicitly grouping and moving coins. This did appear to have some effect, because later in the interview she used the more mental approach to calculate the cost of four loaves at a price of two rand per loaf.

An important observation is that a quantitative response allowed far greater precision in the children's responses. But it was also much more difficult to develop such a response because it required a more explicit conceptual understanding and the ability to properly use mathematical representations of quantity. In the following section, I investigate the importance of representations and the developing of representations for building conceptual understanding of rational numbers.

8.7 The importance of representations

Being able to understand and use representations allows a person to record their ideas and deliberations for reference at a later time. Representations common to a certain community of practice also make it possible for people to communicate effectively with others in the community Wenger (1998).

8.7.1 Representations and the interpretation of grounded tasks

In addition to the above, in this research, the effect of the mastery of representations on a person's interpretation, analysis and possibly even perception of a task situation became apparent. For example, the importance of knowing a fitting representation when performing a visual ratio comparison (5.4.1.6), in order to explicitly make an accurate judgement. Even though the teachers were able to make fitting decisions based on implicit visual judgements, they needed to be able to effectively use representations to make these judgements explicit.

The first interview with the children involved sharing loaves of bread in many different situations. After the sharing, I asked the children to compare the size of the pieces that had been formed. From some of their responses (7.5.3.1) it appeared as if they could adopt two different perspectives for judging the comparison. On the one hand, they appeared to conduct a direct visual comparison and judged that the pieces were not the same size. On the other hand, it appeared as if their judgement was mediated by the mathematical concept used to represent the pieces formed (halves, or quarters) and they judged that the pieces were the same size. Note that the use of the representation implies equality. They did look at the pieces to make the comparison, but they appeared to ignore the slight inconsistency in the use of the representation due to the slight inequality of the pieces. This suggested to me that the mediating representation may have influenced the children's perception of the situation.

Mathematical representations may have had a similar influence on the teachers interpretation of grouping problems (3.4.3.3 and 5.4.1.3). Initially the teachers interpreted these problems as sharing tasks, involving division of the full quantity into a given number of groups, rather than as grouping tasks, involving dividing the full quantity into groups of a given size. The fact that both tasks are represented by the same mathematical operation of division could have contributed towards this confusion. A second factor could have been the predominance of sharing situations in the tasks teachers had commonly used for introducing fractions.

8.7.2 Representations and mathematicization

Developing effective representations is a key aspect of learning mathematical concepts. As discussed by Treffers (1987), the process of horizontal mathematicization involves learning to identify and represent mathematical structure in grounded problems. This is the first stage of mathematical conceptual development in the learning of mathematics based on realistic, or grounded tasks. In this stage, the child begins to move away from considerations focussed purely on the stated problem and its solution, and begins to investigate the patterns and structure identified in their direct solution attempts. This is done in order to simplify and facilitate their work on similar tasks and hence it remains strongly linked to the grounding task. To make such an investigation possible, and fruitful, the child will need to develop representations of the important structures and patterns, which can then be used to facilitate later analyses. This is followed by the process of vertical mathematicization, in which the patterns and structures as represented, form the basis for further investigations that are less grounded and lead to more abstract mathematical concepts. This provides a good model for the development of more mathematical concepts on the basis of grounded tasks. The development of representations to explore and describe the structures and patterns identified is a key part of this.

In this subsection, a number of issues pertinent to the development of representations in this process of horizontal mathematicization are discussed.

The issue of alignment became evident in the teachers work with the part-whole interpretation (3.4.3.2 and 5.3.1.1). Representations do not have intrinsic significance in grounding contexts.

Because of this it is necessary to develop significance through reasonable alignment with the grounding context. But achieving this alignment may be a difficult task. This was noticeably the case in some instances of the part-whole interpretation where the difficulty of aligning the representation with the pupils interpretation of the grounding situation, and the ease of misalignment, resulted more in a separation, rather than an alignment of the representation with experience.

Another interesting issue that arose with the part-whole interpretation was the need to focus on a transient configuration in the active process in the grounding situation in order to obtain a relevant part-whole representation. Freezing such a transient configuration could be seen as unnatural in the and so be difficult to learn. Also, it could break the focus on the grounded problem, making it more difficult to solve. Finally, it could lead children to interpret the representations as unnatural and unhelpful, thus encouraging the attitude that mathematics is abstract and useless

A final important point identified by the teachers was the need to construct a unit and the effect of constructing different units as the base for rational number representations of quantities in a task. For example, when investigating the number of apples needed for a given number of pies (5.4.2.6), different representations were obtained depending on different choices of units to link. In fact, counting in terms of whole pies or whole apples, were represented by reciprocal rationals. This relationship could possibly be useful for developing a more formal conceptual understanding of the relationships between reciprocal rationals. A second interesting example occurred when relating the number of steps of a father with the number of steps of the son (5.4.2.6). This involved a linking of each group of two to a group of five. Linking each to a unit of 1 appeared to make this more linking more accessible. Doing this, it also made it easier to represent the coordination as a mathematical equality of rational quantities — as a whole expressed in terms of different unit fractions.

The focus of the work with children was on building grounded intuition, not on representations. Because of this, little explicit investigation of the development of representations was done. Even so, the complexity of the fraction representation was apparent in these interviews. For the teachers reported, and I had witnessed, that the children had completed some work using the part-whole representation in their class. The children had been selected as average performers in their classes, but in their interviews they both only confidently used the term 'half'. They were not able to name quarters and I re-introduced this name to them and they appeared to make some progress to mastering the use of this term, together with halves during the interviews. They did not volunteer the terms 'third', 'sixth' or 'eighth', even though they did create such fractions in the interviews.

Finally, I had structured the final interview, involving buying sausages, so that the children would need to construct some unit fractions — halves, quarters, eighths, thirds and sixths (7.3.6, 7.4.6 and 7.5.1.6). I had approached this to get a little insight into the possible development of fraction representations beginning with the construction and naming of unit fractions in the measurement, or operator interpretations. It became evident that confidence with unit fractions required mastery of the action of equal subdivision.

It may thus be concluded that making sense of fraction representations either by means of the part-whole sub-construct, or by means of the measurement interpretation requires the mastery of a number of complex skills involved with rational number understanding. That is, these approaches require some rational number understanding to make sense and so alone will not form a good and meaningful foundation for developing initial rational number understanding. On the other hand, knowing a representation appeared to have a positive effect and not knowing a representation appeared to have a negative effect, on the work of Jason in grounded situations. That is, for a child such as Jason, developing representations early in their work with rational numbers may be important to facilitate their development of grounded understanding of this domain.

8.8 Subconstructs revisited

The semantic analysis of the rational number domain as consisting of a number of related sub-constructs (3.2.2) was taken as the starting point for this work. This built on the work of Lamon (1993, 1999 and 2001) who used these sub-constructs to investigate the early teaching of rational numbers in a meaningful fashion.

I was interested specifically in the meaningful learning that arose through acting in a grounded context that was related in some way to the rational number domain. I adopted the term 'interpretation' to describe this specific manner of generating meaning — meaning in grounded experience outside the activity of formal schooling. The sub-constructs were then chosen as a classification of the different possible interpretations of rational numbers. Very early on, it became evident that the fit between sub-construct and interpretation would not be perfect. For it was well established (see 3.2.2) that division had two different interpretations — partitioning and grouping — and as a result, the quotient sub-construct also corresponded to two different interpretations.

During the process of the research, a further anomaly was identified. This was that the part-whole sub-construct did not form a valid interpretation. For the reason that this sub-construct seemed to be focussed on a specific, formal representation. Furthermore, this view was generally derived from a static view of transient configurations obtained when working in grounded situations that were more fitting with other interpretations. The other sub-constructs did appear to be fitting as interpretations. Furthermore, no other interpretations became apparent in the research. I thus concluded that a complete categorization (see issue 1 of 6.1) of the different possible interpretations of rational numbers, fitting with this work, is provided by the following interpretations:

a. Sharing
b. Grouping
c. Measurement
d. Operator conversion
e. Ratio comparison

Due to the inductive nature of the question, a more definite answer from such an empirical study was not possible. Also, I was not able to find a theoretical approach that would guarantee the completeness or otherwise of this categorization.

The mismatch between the apparently complete semantic analysis of rational numbers provided by the sub-constructs and the interpretations that appeared valid in this research was the cause of some discomfort — some cognitive dissonance. Reflecting on this led to the important insight that sub-constructs appeared to provide a categorization of the different symbolic representations of rational numbers and important formal rational number operations.

This was fitting for an analysis that appeared to stem from the information processing approach to cognition, that takes a "symbol systems" view of mental functioning. But my approach to cognition could better be seen as stemming from the schools of situated and embodied cognition.

From this view, rational number understanding is seen as mental structure and activity that is not fully constituted by the symbol systems used as formal representations. (In these schools the terms 'mental structure' and 'mental activity' are seen as not completely explicated and so are not fully defined.) The important question from this view is just how rational number understanding arises from grounding situations, leading to my adoption of interpretations as a base for this research.

8.9 A web of rational number learning

Due to its conceptual complexity, the learning of rational numbers does not appear to involve a simple linear trajectory. Building a comprehensive foundation for this domain requires developing familiarity with a number of different interpretations, and in doing so becoming skilled in the application of a number of coordinating operations. The interpretations lead the child to ascribing significance to their actions, and provide an overarching, goal directed deliberative structure that is aligned with the rational number concepts. And the operations develop intuitive competencies coordinating the child's response with the affordances and constraints in the situation. These competencies underlie many of the basic competencies required for rational number skills. In addition each operation links a number of situating interpretations and each interpretation relates a number of constitutive operations. Because of this two dimensional relational structure, the development of a grounded foundation for rational number understanding may better be described by a web of learning.

For each interpretation, this relational structure is evident in the constitution of the children's responses by a number of different coordinating operations. In the remainder of this section, I will provide a brief description of possible operations used for each interpretation, based mainly on the analysis of the children's interviews. These relationships may be summarized in table 8.1.

For each interpretation in this table, the roman numerals label the operations that were used for different responses to that interpretation. These responses were mostly identified in the children's interviews. For example, response (I) to a sharing situation is constituted by the operations of constructing units and then iteratively replicating units to create the shared portions. Each of the relationships represented in this table is described in more detail in the following sections.

Table 8.1: Relationships between interpretations and operations evident in the children's and teachers' interviews.

		Interpretations									
		Sharing		Grouping	Measurement	Operator		Ratio comparison			
Coordinating Operations	Constructing units	I	IIb	I	I	I			II	III	IV
	Replicating units	I		I	I	I			II		
	Linking units					I					
	Iteration	I	IIb(×2)	I	I	I			II	III	IV
	Scaling		IIa				II	I	II	III	
	Prototyped subdivision		IIa					I			

8.9.1 Sharing

Two different combinations of operations were identified as achieving the goal of sharing.

Approach (I), followed by Ruth, and by Jason for more complex sharing tasks (7.3.1), had the goal of directly subdividing the bread into the required number of parts. This started with the operation of constructing a unit slice, either mentally, or by cutting the first slice. In this construction, the child attempted to judge the thickness that would correctly yield an equal subdivision with the desired number of pieces. Many times this was not the case and the entire process had to be repeated. Once the basic unit had been constructed, the remaining subdivision was performed by iteratively replicating this unit — a combination of the two operations of replicating units and iteration.

Jason's approach (II) to sharing by subdivision (7.3.1) could be seen as constituted by two groups of operations. The first (IIa) had the goal of accurately subdividing the bread and could be described as the prototyped subdivision operation, judged by visual scaling. The second (IIb) was the distribution of these pieces, driven by the goal of practically constructing units. It was performed using the operation of iteratively handing out pieces and evaluated by the operation of iterative counting.

8.9.2 Grouping

No grouping tasks were performed with the children. So the single analysis of grouping (I) is taken from the teachers' work. The teachers consistently used a combination of two operations to perform grouping (5.3.2). As in the final step of the first approach to sharing (I), this approach involved iteratively replicating units. In each case, the given task had included the specification of the size of the group to be formed. That is, the unit had been constructed for the teachers. But for grouping situations that also involved deciding on the size of the group, the teachers would need to start with the operation of constructing units. Because of this, I included the operation of constructing units as important for this sequence.

8.9.3 Measurement

In the practical tasks of performing the measurement subdivision (7.3.2.1), the children proceeded in a manner similar to that of the teachers for grouping. This emphasized the similarity between grouping and the process of measurement subdivision. That is (I), the children combined the operations of iteration and replicating units, performing the subdivision by iteratively replicating units. In more complex measurement tasks, that bring out more of the depth of the measurement interpretation, an important part of the task would be the choice of the unit for measurement. That is, the operation of constructing a unit would then also be important for this interpretation.

8.9.4 Operator

Two different types of situation were identified as involving the operator interpretation. The first situation involved a conversion of one quantity into another and was exemplified by buying bread and sausages. The second was directly involved with visual scaling and was examined in the task of drawing decorations on bread either before, or after baking.

8.9.4.1 (I) Conversion – Buying bread

The children approached both tasks of buying bread (7.3.3.1) and buying sausages (7.3.6.1) in a similar fashion. They did this by combining four different operations. First, they constructed the units of money and bread that they would use as a base for their analysis. In many cases, particularly those where the amount of money given was an integral multiple of the price unit, they used the units given in the task statement. As a result, this operation was unnecessary. But in other cases, generally those where the amount given was not an integral multiple of the price, they needed to construct a unit that fitted this relationship. After this, they replicated a unit of money and a unit of bread and linked these corresponding units. This process of replicating and linking units was then iterated until the amount of money (or bread) that had been given was fully used.

8.9.4.2 (II) Scaling — drawing decorations

Both children's drawing of decorations on bread (7.3.5.1) was guided by the operation of visual scaling. They drew in order to match the shape at a different size and then checked their drawing by visually scaling the reference configuration

8.9.5 Ratio comparison

The ratio interpretation involved the task of dividing dough to fit two baking tins of different sizes (7.3.4.1). When the dough was presented in the form of a single piece, the children followed two different approaches.

Ruth (I) used the operation of prototyped subdivision, attempting to cut the dough so that the size relationship between the two pieces was the same as the relationship between the sizes of the two tins. She appeared to judge this comparison by using the operation of visual scaling — checking her visual estimate of the relationship between the pieces against a visual estimate of the relationship between the tins.

Jason followed a very different strategy (II), first cutting the dough into a number of slices and then distributing the slices between the tins so that there was the same 'fit' in each tin. This involved the following operations. He first constructed a unit, when choosing on the width of a slice to cut. This may have been a deliberate choice, but it is more likely that he merely cut an initial slice and then chose this to be his unit. Next he iterated the process of replicating units, cutting slices to form discrete units the same thickness as the chosen unit. As with the sharing situation, his distribution of the slices could be seen as practically constructing units given by the final amount of dough in each tin. To carry out the distribution, he iteratively placed slices in tins. This was not done according to a stable sequence. Rather, for each slice, he used visual scaling to decide which tin was the emptiest and then placed the slice into that tin.

A second set of tasks involved dividing dough that had already been made into balls, between the two tins. Again, this could be viewed as constructing two different units of a 'tin' of dough. As with Jason's slices, this was done by iterating the operation of placing dough balls into the tins. In these tasks, the comparison to determine the fit was performed in two different ways. In some instances, visual scaling was used (III). But in others a counting strategy was followed, that is counting iteration was used as the comparison operation (IV). In this case, though, the counting strategy was chosen to fit the relationship between the sizes of the two tins that was estimated using visual scaling.

8.9.6 A summary of the web of learning

Each dimension in this web of learning contributes in a different way to the initial meaningful learning of rational numbers.

The interpretations provide significance to the activity and hence to what is learned in the activity. Interpretations also provide a deliberative structure to the experience, as is evident from the view of interpretations as actions. Here, the children consciously and deliberately structure their acts to achieve the goals of the action.

But this deliberative structure is not the only structure experienced in the activity. There is also the less deliberative, more intuitive structuring of the children's acts, in order to coordinate what they do with the constraints and affordances of the context. This is the structure emphasized when taking an 'operations' perspective.

As is evidenced in the above discussion, these two forms of structure overlap and intertwine with each other, creating a coherent web of structure that forms an informal, yet comprehensive experiential basis for rational number concepts.

Developing this experiential basis would not, however, be sufficient to properly develop a mathematical understanding of rational numbers. For without the development of mathematical representations and the use of such representations to re-interpret and analyse these contextual interpretations, the children's understanding could not properly be considered as mathematical. That mathematical representations do have a real effect on one's interpretation of a situation was evident in the discussion of (8.7). Teachers' and children's use of mathematical representations seemed to encourage conceptual structuring and idealization of the situation, effects that were beneficial at times and introduced problems at others.

Thus, the final, critical feature of the web of learning is that the children's experience of the interpretations and operations should take place against the background of a continued encouragement to develop and refine representational means to interpret and analyse their acts in context. So, as well as working to solve the given problems in context, children should be continually encouraged to engage in the process of vertical mathematization (Treffers, 1987) by developing mathematical means to represent and analyse their experience.

Using the web of learning as a guiding perspective for initial rational number instruction provides a number of insights:

- The relevance and motivation for learning is engendered by exploring grounded contexts arising from different interpretations from the perspective of activities.
- Structural grounding of mathematical representations is provided by viewing the children's acts, or imagined acts as goal directed structured actions.
- Operations provide points of comparison and similarity between interpretations. These encourage the building of links and motivates the consolidation and synthesis of the learning into a single coherent, rational number, framework.
- Operations afford deeper insight into structured actions and resulting representations. This occurs through the implicit structuring of the children's operations in order to achieve coordination with the affordances and constraints of the situation.
- Relating different coordinating operations in a single grounded context will make it possible to relate the corresponding representations.

8.10 Implications for teaching

In this final section, I will highlight a number of important issues for the teaching of fractions following a grounded approach such as this, that emerged during the course of the research.

- The importance of perspective and the need to coordinate with the child's perspective before judging their responses. For example, the rational numbers arising when quantifying the comparison of units would depend on the choice of the specific unit chosen as the reference whole (5.4.2.1).

- Children should be encouraged to explicitly construct and mention their units of reference when describing rational numbers. This will make it easier for others to adopt the children's perspective and so improve the process of communication. It will also encourage children to extend their whole number counting schemes to rational numbers and so contribute to viewing rationals as numbers (5.4.2.2).

- When linking, or coordinating groups, different coordinations would occur for different choices of focal group (or space). These would be represented by reciprocal rational numbers. This could be useful for developing representations and reflective awareness of relationships between reciprocal rationals (5.4.2.6).

- Linking groups when neither of the natural units was one was experienced as rather difficult by the teachers. But being able to link both to a unit of one made this operation more accessible (5.4.2.6). It also became easier to represent the coordination as a mathematical equality of rational quantities — as a whole expressed in terms of different unit fractions.

- The need for scaffolding by the teacher when investigating complex grounded problems and when developing representations, is critical (5.5.1). That is, without skilled and responsive teachers, the development of good fraction understanding by average children will not occur.

Chapter 9 – Conclusion

9.1 Reflection on the research process

The secondary goal of this research project was to develop an understanding of the function of personal meaning in my own learning of the processes involved in developing rational number understanding.

In my teaching experience before this research, I had developed the intuition that grounding rational number knowledge in situations that were familiar and meaningful to the child was possible and beneficial for the child's understanding. This possibility of improving the learning and teaching of rational numbers led personal significance to the issue and motivated my undertaking of the project.

At the start of the project, this intuition was quite unformed, as well as being implicit. The initial review of the literature enabled me to develop an initial explicit perspective on the issue. Based on this perspective, the examples for discussion in the pilot cycle were constructed. The interviews in the pilot cycle brought some clarity to a number of the rational number interpretations. But I was also left feeling rather unsatisfied about the distinct separation between these interpretations and the potential for the further development of a coherent rational number understanding based on such a foundation. This feeling of dissatisfaction provided strong motivation to return to the literature in order to overcome this problem. After an extensive search, lasting a number of months, my identification of the basic concepts of activity theory and situated cognition as providing a possible way forward again provided motivating impetus to the work.

This reading made it possible for me to conceptualize the second important dimension of 'operations'. During the following two cycles, specific details of this dimension and its relationship to the dimension of 'interpretations' were refined and developed, based on the responses of the teachers and children. As with the conceptualization of interpretations, this thinking progressed from a strong intuitive impression that was quite vague and unformed when I attempted to state it explicitly, to an explicit conceptualization that was detailed and specific. This occurred through the development of fitting units of representation, either by formulating views of an interaction that appeared to capture or describe salient features, or by identifying concepts in the literature that facilitated the recognition of such features in the data.

Fundamental to the development and refinement of each dimension in the research, was an attitude of openness to conflicts and confusions elicited by the data. That is, I needed to deliberately be open to interchanges that challenged my thinking at the time and then work to adapt and refine the current formulation in order to make it more consistent with these interchanges.

I experienced the research in the teacher and children cycles as more fluid and progressive, because no major block to my conceptualization occurred in these cycles. But even without such a major conflict, an important third dimension to the research emerged in these cycles. This was the dimension of representations — their development and the effect of such development. Retrospectively, some interchanges in the pilot cycle suggested the importance of this dimension, but at the time the conflicts introduced were quite minor and insufficient to introduce the issue of representations as an issue needing further exploration.

Looking back, I would ascribe this to the dominance over my thinking at the time, of my profound feeling of frustration and concern about the separateness of the grounded experienced derived from the different interpretations. Because of this, the primary focus of my reflection and investigative reading at the time, was on identifying aspects of such grounded experience that would encourage a more coherent, unitary view of rational number understanding. Once I had identified the second, unifying dimension of operations, this concern dissipated and so the less pressing issue of representations was able to emerge. And it began to do so explicitly in the teacher cycle, where representations had an important place in all the emergent themes mentioned in section (5.5). And when working with the children in the final cycle, a number of their responses indicated clearly the effects of representations and as a result the representational dimension was explicitly conceptualized in section (7.5).

It is interesting to note from this discussion that the research and my learning in the research was strongly driven by my own personal concerns. My initial engagement with the research was due to my desire to obtain greater clarity and rigour about my intuitive sense that the grounding of rational number understanding was a useful strategy for teaching and learning. Also, my earlier identification of the dimension of operations, as opposed to that of representations was partly due to my concern about the possible separation of learning based on different interpretations.

The motivation resulting from the personal significance of the project and the issues being explored, helped me maintain impetus during the research process. In fact, the most difficult period of the research — when faced with the separation of the interpretations at the end of the first cycle — was the period in which I was most intensely focussed on the research, because of the significance of this issue to me. This motivating effect of personal significance was one of the important aspects of the rational number interpretations when viewed within their larger context as activities.

The final product of this research is also inherently personal, because in full detail, it is my own understanding of possible processes involved in the grounded learning of rational numbers. This dissertation could only ever be a selection, organization and codification of the aspects of that understanding, together with selected evidence from the interviews and arguments that support this codification. Although this selection of concepts and evidence was judged as viable by my supervisor and others it was ultimately my selection. As such, this research was concerned, at a fundamental level, with the development of personal meaning.

But this does not imply that the personal understanding developed is arbitrarily subjective. For the fundamental intuitions that formed the basis of this understanding developed from my interaction with the participants in the interviews, and from the resulting data during analysis. The aim of my adoption of an attitude of openness to the challenges offered by the data was to facilitate the alignment of my understanding, with the affording structure offered by the data. This alignment occurred through the coordination of my responses in the interviews, to those of the participants; and during my active reading of the interview data, the coordination of my conceptualization with the affording structure of the interaction as evidenced by the data. This process was similar to the coordinating effects achieved when the child engages in rational number operations in a grounding situation, and that yields an alignment between the child's intuitive understanding and the structuring afforded by the situation. That is, the process of coordination provides structure to one's personal understanding, so that even at the fundamental level of intuitive, or tacit understanding, intuitions developed through effective coordination are not arbitrarily subjective but show alignment with the situation.

The intuitions that I developed in the process of the research had a profound effect on the research. For they provided direction for further exploration and analysis, as well as yielding the basic ideas that were fleshed out to form my final conceptual understanding of the topic. This fleshing out, occurred through a complex process of:

- Developing conceptual representations that encapsulated aspects of the intuitions; thus highlighting and providing more definite structure to these aspects.
- Rereading and analysing relevant sections of the data to ensure that the more definite structure imposed by these representations did yield a viable fit with the data.
- Accepting representations that did appear to fit viably with the data and then using these representations to review and reflect on the basic intuitions and thus achieve a more structured, definite and consciously reflective understanding of these intuitions.
- Identifying the conflicts in representations that did not appear to fit viably with the data and searching for conceptual representations that overcome these conflicts.

The complexity of this process has also made me more aware of the complexity and difficulty of the task of mathematicization, when learners of mathematics need to master representational means to describe and analyse patterns that they intuitively understand.

I found that the insights offered by the theory of representational redescription (Karmiloff-Smith, 1992) provided valuable insight into this process. One important issue is the identification of effective and unitary aspects for representation. Because these representations highlight the particular aspect being represented, as well as particular properties of these and related aspects. In doing so, other aspects and properties become seen as less important, or are even ignored. As a result, it was often difficult to find a better alternative to a non-viable representation because to do this, I often needed to negate the structuring effect of the representation and this was not easy, even though I was convinced that the representation was faulty. Another implication of this, is that finding a representation that is viable may lead to missing a different representation that may fit even better. This underscores my radical constructivist standpoint that the theory here developed can not be put forward as proved objective truth, but instead forms a conceptualization that does show a viable fit with the data.

Finally having identified a number of concepts important for a comprehensive understanding of grounded rational number learning, I worked to explicitly formulate these concepts and the relationships between these concepts as apparent in the analysis of the data. Producing rough notes for discussion with my supervisor and then planning and drafting this dissertation were important phases in the production of a formalization of my understanding. When writing the formalization of each concept and set of relationships, I also revisited each relevant data extract and the corresponding analysis in order to re-evaluate whether this explicit description did indeed fit viably with the data. A number of conflicts became apparent even at this stage of the research. The resolution of these conflicts led to further refinement of many of the formalizations that had been developed.

An important part of formalizing my developing understanding, was determining the fit of these formalized concepts with the interview data. This involved investigating different ways to relate these concepts to interchanges in the interviews, with the goal of finding the relationship that best fit the specific structure of the interchange. In each case, this was a goal directed act in which I deliberately identified and explored different alternatives. This was a similar process to the view of rational number interpretations as goal directed actions in which children deliberately explore and develop explicitly identified structure in the grounded situation that is related to rational number concepts.

In my reflection on the research process, the importance for the development of my own personally meaningful understanding, of processes similar to those identified for the development of personally meaningful rational number understanding, became evident. These processes included coordinating operations that facilitate the development of fundamental intuitive understanding; grounded interpretations for motivating significance and the deliberative relating of conceptual structure with explicitly identified structure in the situation; and the development of representations to formalize and shape the developing understanding.

9.2 Achievements

The aim of this research was to investigate the learning processes involved in developing a meaningfully grounded initial understanding of rational numbers. To do this, a small scale qualitative project consisting of three cycles of clinical interviews, that were analysed in depth, was carried out.

Three complementary perspectives were identified as being important for this development. These were:

- Different interpretations of rational numbers in terms of goal directed actions occurring within significant activities in the life of the child.
- Different operations that can be seen as unitary acts as children intuitively coordinate their responses with the instrumental conditions and affordances in the physical and social context of these activities.
- The important contribution of developing representations, to the mathematization of the children's developing informal understanding.

This interaction results in a complex process of learning that is not linear. Rather, this process could be conceptualized as an interlocking web of learning.

It was also possible to combine the detailed pictures provided by the first two perspectives, in the form of a table (Table 8.1). For convenience, this table is reproduced below.

Table 8.1: Relationships between interpretations and operations evident in the children's and teachers' interviews.

		Interpretations									
		Sharing		**Groupin g**	**Measurement**	**Operator**		**Ratio comparison**			
Coordinating Operations	**Constructing units**	I	IIb	I	I	I			II	III	IV
	Replicating units	I		I	I	I			II		
	Linking units					I					
	Iteration	I	IIb(×2)	I	I	I			II	III	IV
	Scaling		IIa				II	I	II	III	
	Prototyped subdivision		IIa					I			

9.3 Shortcomings

As a consequence of the small sample of people and limited number of investigations, it was not possible to comprehensively investigate the relationships between the operations identified and all the different interpretations. In particular, further examples of the grouping and measurement interpretations should be investigated to ascertain whether additional combinations of operations may constitute the response in such situations.

Also, because of the limited sample and the methodology employed, the developed theory is strictly local and need not apply generally to the grounded learning of rational numbers. But the form and content of the theory is not restricted to the local setting of the research. This raises the possibility that it may provide a useful general perspective on grounded rational number learning. In order to judge whether or not this is the case, it would be necessary to carry out more comprehensive tests of the theory in more general settings. Such a general investigation is outside the scope of this work, and would require a different methodology. This will form the basis of a further research project in this area.

An attempt was made in this study to develop a theoretical perspective that would be useful for the classroom teaching of rational numbers. However, due to the limited scope of the project, it was also not possible to determine whether this would indeed be the case. The theory was developed to be in line with the teachers reported experience in class, and to be consistent with the responses of the children in the clinical interviews. But the further step of designing a teaching program and resources on the basis of this theory and then performing an implementation and evaluation study was not carried out in this project. This could be done through a further developmental research project following this study.

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