CONTRIBUTIONS TO THE STUDY OF A CLASS OF OPTIMAL CONTROL PROBLEMS ON THE MATRIX LIE GROUP SO(3)

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Abstract

The purpose of this thesis is to investigate a class of four left-invariant optimal control problems on the special orthogonal group $\mathsf{SO}(3)$. The set of all control-affine left-invariant control systems on $\mathsf{SO}(3)$ can, without loss, be reduced to a class of four typical controllable left-invariant control systems on $\mathsf{SO}(3)$. The left-invariant optimal control problem on $\mathsf{SO}(3)$ involves finding a trajectory-control pair on $\mathsf{SO}(3)$, which minimizes a cost functional, and satisfies the given dynamical constraints and boundary conditions in a fixed time. The problem is lifted to the cotangent bundle $T^*\mathsf{SO}(3) = \mathsf{SO}(3) \times \mathfrak{so}(3)^*$ using the optimal Hamiltonian on $\mathfrak{so}(3)^*$, where the maximum principle yields the optimal control. In a contribution to the study of this class of optimal control problems on $\mathsf{SO}(3)$, the extremal equations on $\mathfrak{so}(3)^*$ (identified with $\mathbb{R}^3$) are integrated via elliptic functions to obtain explicit expressions for the solution curves in each typical case. The energy-Casimir method is used to give sufficient conditions for non-linear stability of the equilibrium states.

KEYWORDS

Matrix Lie groups, left-invariant control systems, optimal control, the maximum principle, extremal equations, elliptic functions, non-linear stability

Contents

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Introduction

We begin with a very brief introduction to geometric control theory, followed by a description of the optimization problem on the matrix Lie group $\mathsf{SO}(3)$, and an outline of the thesis.

Geometric control theory

Mathematical control theory is an established, rapidly growing field of application-oriented mathematics. It is one of the most interdisciplinary areas of research, providing both theoretical and computational tools for solving problems arising in fields such as engineering, robotics, economics, biology and medicine. Concepts like controllability, observability, stability, realization and optimization are fundamental in control theory.

The language and methods of geometry are relevant to control theory – this is termed *geometric* control theory. 'A control system can be viewed informally as a dynamical object (e.g. ordinary differential equation) containing a parameter (control) which can be manipulated to influence the behaviour of the system so as to achieve a desired goal' [22]. The right-hand side of the ordinary differential equation is a vector field and the dynamical system governed by the equation is the flow generated by this vector field. Hence, a control system is a family of vector fields. The structure of admissible trajectories and attainable sets is intimately related to the groups of transformations generated by the dynamical systems involved. In turn, groups of transformations form the heart of geometry.

Geometric control theory has been studied and developed by a number of researchers, for example: Jurdjevic [9,10], Agrachev [1], Sachkov [24,25] and Puta [19,20].

A class of optimal control problems on the matrix Lie group $\mathsf{SO}(3)$

The set of all orthogonal 3×3 matrices over $\mathbb{R}$, with unit determinant, forms the matrix Lie group $\mathsf{SO}(3)$ known as the special orthogonal group. This group is non-Abelian, compact, path-connected and simple, and it is also a smooth submanifold of $\mathbb{R}^{3\times 3}$, diffeomorphic to $\mathbb{RP}^3$. The Lie algebra

$\mathfrak{so}(3)$ consists of all 3 × 3 skew-symmetric matrices, and has dimension 3. The matrices

$$E_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad E_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

form the standard basis for $\mathfrak{so}(3)$.

The proofs of these well-known and standard results concerning $\mathsf{SO}(3)$ are given in chapter 2 in a simple and direct manner.

The control-affine left-invariant control systems on $\mathsf{SO}(3)$, in classical notation, are given by

$$\dot{g} = g\,(X + u_1 Y_1 + \ldots + u_\ell Y_\ell)\,, \quad g \in \mathsf{SO}(3),\ u = (u_1,\,\ldots,\,u_\ell) \in \mathbb{R}^\ell,\ 1 \leq \ell \leq 3.$$

Now, we express X and $Y_1,\,\ldots,\,Y_\ell$ as linear combinations of the standard basis matrices for $\mathfrak{so}(3)$. A left-invariant control system on $\mathsf{SO}(3)$ is controllable if it is composed of either two or three of the standard basis matrices. We consider the following typical controllable left-invariant control systems on $\mathsf{SO}(3)$:

Type I:	$\dot{g} = g(E_3 + uE_1)$,	$u \in \mathbb{R}$
Type IIa:	$\dot{g} = g(u_1E_1 + u_2E_2)$,	$u = (u_1, u_2) \in \mathbb{R}^2$
Type IIb:	$\dot{g} = g(E_3 + u_1E_1 + u_2E_2)$,	$u = (u_1, u_2) \in \mathbb{R}^2$
Type III:	$\dot{g} = g(u_1E_1 + u_2E_2 + u_3E_3)$,	$u = (u_1, u_2, u_3) \in \mathbb{R}^3$.

We use the word 'typical' as any results for the remaining controllable left-invariant control systems on $\mathsf{SO}(3)$ can easily be deduced from these cases, with some simple elementary changes. This gives a class of four left-invariant optimal control problems on $\mathsf{SO}(3)$. The description of the problem and the explanations that follow have been formulated generally, and are sufficient to cover all the typical cases at once.

Let g_1 and g_2 be arbitrary but fixed points in $\mathsf{SO}(3)$, and $T > 0$ be fixed in advance. A left-invariant optimal control problem on $\mathsf{SO}(3)$ consists of minimizing the cost functional

$$J = \int_0^T L(u(t))\,dt$$

over all trajectory-control pairs $(g(\cdot), u(\cdot))$, of the left-invariant control system

$$\dot{g}(t) = g(t)\left(X + \sum_{i=1}^{\ell} u_i(t) Y_i\right), \quad g \in \mathsf{SO}(3),\ u(t) = (u_1(t), \ldots,\, u_\ell(t)) \in \mathbb{R}^\ell,$$

satisfying the boundary conditions

$$g(0) = g_1 \quad \text{and} \quad g(T) = g_2.$$

The Lagrangian L has the form

$$L(u) = \frac{1}{2}\left(c_1 u_1^2 + \ldots + c_\ell u_\ell^2\right), \quad c_1, \ldots, c_\ell > 0,\ 1 \leq \ell \leq 3.$$

The problem is lifted from $\mathsf{SO}(3)$ to the cotangent bundle $T^*\mathsf{SO}(3)$ as follows.

The Hamiltonian H of the left-invariant vector field $X + \sum_{i=1}^{\ell} u_i Y_i$ is defined by

$$H(\xi_g) := \xi_g\left(g\left(X + \sum_{i=1}^{\ell} u_i Y_i\right)\right) \qquad \text{for each } \xi_g \in T_g^*\mathsf{SO}(3).$$

Let $\mathfrak{so}(3)^*$ denote the dual space of $\mathfrak{so}(3)$. Then the correspondence $dL_{g^{-1}}^* p \leftrightarrow (g, p)$ realizes $T^*\mathsf{SO}(3) = \mathsf{SO}(3) \times \mathfrak{so}(3)^*$, giving

$$H(g,p) = p\left(X + \sum_{i=1}^{\ell} u_i Y_i\right).$$

Hence, H is a linear function on $\mathfrak{so}(3)^*$ parametrized by the control values. The cost extended Hamiltonian on $\mathfrak{so}(3)^*$, of the left-invariant optimal control problem on $\mathsf{SO}(3)$, is defined by

$$\mathcal{H}(p,u) := -L(u) + p\left(X + \sum_{i=1}^{\ell} u_i Y_i\right).$$

The maximum principle is used to find the best possible control, thus yielding a single (cost extended) Hamiltonian: the optimal Hamiltonian.

Proceeding with the non-canonical coordinates (g,p) in $\mathsf{SO}(3) \times \mathfrak{so}(3)^*$, the set of differential equations on $\mathfrak{so}(3)^*$ is known as the extremal equations. For semi-simple matrix Lie groups, the Killing form sets up a correspondence between the adjoint and co-adjoint orbits, which in turn converts the extremal equations on $\mathfrak{so}(3)^*$ to their dual forms on $\mathfrak{so}(3)$. Further, $\mathfrak{so}(3)$ can be identified with $\mathbb{R}^3$ using the hat mapping, giving the extremal equations on $\mathbb{R}^3$.

Due to the left-invariant symmetries of the problem, there is another constant of motion, K (in addition to $\mathcal{H}$), which allows the extremal equations to be integrated via elliptic functions. The explicit expressions for the solution curves are given in the final four chapters, and to my knowledge, the results obtained are my own, i.e.,

Theorem 4.3.1, Theorem 5.3.1, Theorem 6.3.1 and Theorem 7.3.1.

These contributions will be checked at two levels. Firstly, I allow $c_2 \to \infty$ in the Type IIb case and compare the result to the Type I case; similarly, I allow $c_3 \to \infty$ in the Type III case and compare the result to the Type IIa case. At the second level, I choose arbitrary constants and initial values, substitute them into the previously mentioned theorems and produce the graphical output. These solutions can be compared to those produced by a MATLAB solver.

The solution curve to the left-invariant optimal control problem on $\mathsf{SO}(3)$ is the projection down to $\mathsf{SO}(3)$, of the integral curve $(g(\cdot), p(\cdot))$ of the Hamiltonian vector field $\vec{\mathcal{H}}$ on $T^*\mathsf{SO}(3)$.

Lastly, the energy-Casimir method is used to give sufficient conditions for non-linear stability of the equilibrium states in each typical case. These results will also be checked using the first approach mentioned above, i.e., I allow $c_2 \to \infty$ in the Type IIb case, and $c_3 \to \infty$ in the Type III case.

Overview of the thesis

Chapter 1 contains some basic material (definitions, theorems and propositions), along with mathematical preliminaries required for the analysis of the left-invariant optimal control problem on $\mathsf{SO}(3)$. The following standard objects (and some results) are reviewed in each section, respectively.

In section 1.1: the general and special linear groups, a matrix Lie group, a path-connected matrix Lie group, a tangent vector, the tangent space, the dimension of the tangent space, the tangent mapping, the Lie algebra, the Lie bracket, the tangent bundle, the dual space, the exponential matrix, the exponential mapping, a Lie algebra isomorphism, the adjoint representation, the dual mapping, the co-adjoint representation, an ideal, a simple and semi-simple Lie algebra, a simple matrix Lie group, the unitary and special unitary groups, a torus, a maximal torus, a diffeomorphism and a local diffeomorphism, a double cover, the inverse mapping theorem, the real projective space, a vector field, a left-invariant vector field, an integral curve, the flow, a tangent covector, the cotangent space, the cotangent bundle, and the Killing form.

In section 1.2: an admissable control, a left-invariant control system, a trajectory, the reachable set, and controllability of a left-invariant control system.

In section 1.3: a differential 1-form, a differential 2-form, non-degeneracy, the wedge-product, the exterior derivative, a closed form, a symplectic form, a symplectic manifold, the interior product, a Hamiltonian vector field, a Hamiltonian system, canonical coordinates, Hamilton's equations in canonical coordinates, the Poisson bracket, a constant of motion, vector fields and the symplectic form on the cotangent bundle, and Hamilton's equations in non-canonical coordinates.

In section 1.4: the cost functional, the Lagrangian, an optimal trajectory, the left-invariant optimal control problem, the cost extended Hamiltonian, the adjoint orbit, the co-adjoint orbit, and a Casimir function.

In section 1.5: the maximum principle, an extremal pair, and abnormal and normal extremals.

In section 1.6: an elliptic function, the period of an elliptic function, the Weierstrass elliptic function, Jacobi elliptic functions, and an elliptic integral.

In section 1.7: an equilibrium state, non-linear stability, and the energy-Casimir method.

Chapter 2 collects some well-known and standard results concerning $\mathsf{SO}(3)$, together with simple and direct proofs. These are arranged as follows.

In section 2.1: $\mathsf{SO}(3)$ is a matrix Lie group; every element of $\mathsf{SO}(3)$ is a rotation; and $\mathsf{SO}(3)$ is non-Abelian, compact and path-connected.

In section 2.2: the Lie algebra of $\mathsf{SO}(3)$ consists of all 3×3 skew-symmetric matrices; the standard basis for the Lie algebra of $\mathsf{SO}(3)$; the dimension of $\mathsf{SO}(3)$ is 3; the definition of the hat mapping; the hat mapping is an isomorphism of Lie algebras; and $\mathsf{SO}(3)$ is simple.

In section 2.3: the standard maximal torus of $\mathsf{SO}(3)$ is a maximal torus; if $\mathbb{T}$ is a maximal torus of $\mathsf{SO}(3)$, then $g\mathbb{T}g^{-1}$ is also a maximal torus; the exponential map $\exp : \mathfrak{so}(3) \to \mathsf{SO}(3)$ is surjective; and Rodrigues' formula.

In section 2.4: $\mathsf{SO}(3)$ is a smooth submanifold of $\mathbb{R}^{3\times3}$; $\mathsf{SU}(2)$ is diffeomorphic to $\mathbb{S}^3$; the range of the adjoint representation on $\mathsf{SU}(2)$ is $\mathsf{SO}(3)$; and this map is a double cover. To end, we discuss the implications of the double cover and the coordinates on $\mathsf{SO}(3)$.

Chapter 3 focuses specifically on geometric control and $\mathsf{SO}(3)$. The set of all control-affine left-invariant control systems on $\mathsf{SO}(3)$ is reduced to a class of four typical controllable left-invariant control systems on $\mathsf{SO}(3)$. Examples of these are given, and descriptively named the stiff SF-system, attitude control of a rigid body and path planning on $\mathsf{SO}(3)$. Further, as described in the previous section, the optimal control problem on $\mathsf{SO}(3)$ can be lifted to $T^*\mathsf{SO}(3) = \mathsf{SO}(3) \times \mathfrak{so}(3)^*$. In this chapter, $\mathfrak{so}(3)^*$ is identified with $\mathfrak{so}(3)$ via the Killing form, and $\mathfrak{so}(3)$ with $\mathbb{R}^3$ using the hat mapping. The Casimir function K on $\mathfrak{so}(3)^*$ is given, with a proof, and the chapter concludes by presenting the general solution curve in terms of the local coordinates on $\mathsf{SO}(3)$.

Chapters 4–7 are devoted to solving the left-invariant optimal control problem on $\mathsf{SO}(3)$ for the Type I, Type IIa, Type IIb and Type III cases, respectively. These chapters begin with a statement of the problem, followed by a theorem giving the optimal control(s), optimal Hamiltonian and the extremal equations. The main aim is to solve the extremal equations using elliptic functions, and these results are checked numerically. A set of differential equations is achieved, which can be solved using numerical techniques to determine the specific expression for the solution curve. Lastly, the non-linear stability of the equilibrium states associated with the extremal equations is investigated.

In the conclusions, comparisons are drawn between the Type I and Type IIb cases by examining the extremal equations, their solutions, the sets of differential equations yielding the Euler angles and the stability results, similarly, for the Type IIa and Type III cases. The graphical output in appendix C is discussed and summarized in a table. The chapter concludes with some closing remarks and identifies areas for further research.

Chapter 1

Preliminaries

This chapter contains the definitions, theorems and propositions, along with the mathematical preliminaries required to analyse the left-invariant optimal control problem on $\mathsf{SO}(3)$. The chapter is formulated in general for matrix Lie groups, and not specifically for $\mathsf{SO}(3)$. The topics include matrix Lie groups, left-invariant control systems, Hamiltonian formalism, optimal control, the maximum principle, elliptic functions and stability. In the section on the maximum principle, a brief history of this important result is provided.

1.1 Matrix Lie groups

The references used include [30], [21], [2], [6], [23], [14] and [10].

We will denote the set of all real $n \times n$ matrices by $\mathbb{R}^{n\times n}$, similarly, the set of all complex $n \times n$ matrices by $\mathbb{C}^{n\times n}$.

Definition 1.1.1. $\mathsf{GL}(n, \mathbb{R})$ is the set of all invertible $n \times n$ matrices over $\mathbb{R}$, i.e.,

$$\mathsf{GL}(n, \mathbb{R}) := \{g \in \mathbb{R}^{n\times n} \mid \det g \neq 0\}.$$

$\mathsf{SL}(n, \mathbb{R})$ is the set of all invertible $n \times n$ matrices over $\mathbb{R}$, with unit determinant, i.e.,

$$\mathsf{SL}(n, \mathbb{R}) := \{a \in \mathsf{GL}(n, \mathbb{R}) \mid \det g = 1\}.$$

Proposition 1.1.1. *$\mathsf{GL}(n, \mathbb{R})$ and $\mathsf{SL}(n, \mathbb{R})$ are groups under matrix multiplication.*

$\mathsf{GL}(n, \mathbb{R})$ is called the **general linear group** and $\mathsf{SL}(n, \mathbb{R})$ is called the **special linear group**.

Definition 1.1.2. A **matrix Lie group** is a closed subgroup of the general linear group.

A (real) Lie group is a smooth manifold G which is also a group such that the operations

$$\mathsf{G} \times \mathsf{G} \to \mathsf{G}, \quad (g_1, g_2) \mapsto g_1 g_2 \quad \text{and} \quad \mathsf{G} \to \mathsf{G}, \quad g \mapsto g^{-1}$$

are smooth mappings. Matrix groups are Lie groups (hence, commonly called matrix Lie groups), however, not all Lie groups are matrix groups.

DEFINITION 1.1.3. A matrix Lie group G is called **path-connected** if, for any g_1, $g_2 \in \mathsf{G}$, there exists a continuous curve $\alpha : [0,1] \to \mathsf{G}$ such that $\alpha(0) = g_1$ and $\alpha(1) = g_2$.

DEFINITION 1.1.4. Let G be a matrix Lie group and α be a smooth curve in G. Then

$$\dot{\alpha}(t) := \lim_{h \to 0} \frac{1}{h}(\alpha(t+h) - \alpha(t))$$

is called the **tangent vector** to α at $\alpha(t)$.

DEFINITION 1.1.5. Let G be a matrix Lie group and $g \in \mathsf{G}$. Then the **tangent space** to G at g is the set

$$T_g\mathsf{G} := \{\dot{\alpha}(0) \mid \alpha \text{ is a smooth curve in } \mathsf{G} \text{ with } \alpha(0) = g\}.$$

PROPOSITION 1.1.2. *Let* G *be a matrix Lie group and* $g \in \mathsf{G}$. *Then the tangent space* $T_g\mathsf{G}$ *is a real vector subspace of* $\mathbb{R}^{n \times n}$.

We will denote the identity element by Id.

DEFINITION 1.1.6. If G is a matrix Lie group, then its **dimension** is the dimension of the vector space $T_{Id}\mathsf{G}$.

DEFINITION 1.1.7. Suppose $F : \mathsf{G}_1 \to \mathsf{G}_2$ is a smooth mapping between matrix Lie groups. For any curve α on G_1, $F \circ \alpha$ is a smooth curve on G_2. Then the **tangent mapping** is

$$dF : T_{Id}\mathsf{G}_1 \to T_{F(Id)}\mathsf{G}_2, \quad \dot{\alpha}(0) \mapsto (F \circ \alpha)^{\cdot}(0).$$

DEFINITION 1.1.8. A real **Lie algebra** $\mathfrak{g}$ is a real vector space equipped with a product

$$[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}, \quad (X, Y) \mapsto [X, Y],$$

such that (for λ_1, $\lambda_2 \in \mathbb{R}$ and X, Y, $Z \in \mathfrak{g}$):

1. $[X, Y] = -[Y, X]$ (skew-symmetric)
2. $[\lambda_1 X + \lambda_2 Y, Z] = \lambda_1[X, Z] + \lambda_2[Y, Z]$ (bilinear)

3. $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$ (Jacobi identity).

The product $[\cdot\,,\,\cdot]$ is called the **Lie bracket** of the Lie algebra $\mathfrak{g}$.

EXAMPLES:

1. Let $\mathfrak{g} = \mathbb{R}^3$ and define

$$[x,y] := x \times y, \quad \text{the cross product for all } x,\ y \in \mathbb{R}^3.$$

$\mathbb{R}^3$ equipped with this bracket operation forms a Lie algebra.

2. Let $\mathfrak{g} = \mathbb{R}^{n\times n}$ and define the **matrix commutator**

$$[X,Y] := XY - YX \quad \text{for all } X,\ Y \in \mathbb{R}^{n\times n}.$$

$\mathbb{R}^{n\times n}$ equipped with the matrix commutator forms a Lie algebra.

Let G be a matrix Lie group. Then $T_{Id}\mathsf{G}$ equipped with the matrix commutator forms its Lie algebra, denoted by $\mathfrak{g}$. Let $X, Y \in \mathfrak{g}$. Then

$$[gX, gY] = g[X,Y] = g(XY - YX), \quad g \in \mathsf{G}.$$

It is natural to assemble together all tangent spaces of a matrix Lie group G into a new structure called the tangent bundle of G, denoted by $T\mathsf{G}$. The tangent bundle is a smooth manifold with double the dimension of G.

For each $g \in \mathsf{G}$, let L_g denote the left translation by g.

PROPOSITION 1.1.3. *Let* G *be a matrix Lie group and* $\mathfrak{g}$ *be its Lie algebra. Then the correspondence* $dL_gX \leftrightarrow (g,X)$ *realizes* $T\mathsf{G} = \mathsf{G} \times \mathfrak{g}$.

DEFINITION 1.1.9. Let G be a matrix Lie group and $\mathfrak{g}$ its Lie algebra. Then the **dual space** $\mathfrak{g}^*$ of $\mathfrak{g}$ is the space of all linear functionals on $\mathfrak{g}$.

DEFINITION 1.1.10. Let $X \in \mathbb{R}^{n\times n}$. Then the **matrix exponential** of X is

$$e^X = \exp X := \sum_{k=0}^{\infty} \frac{1}{k!} X^k.$$

DEFINITION 1.1.11. Let $X \in \mathbb{R}^{n\times n}$. Then the **exponential mapping** is

$$\exp : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}, \quad X \mapsto \exp X.$$

PROPOSITION 1.1.4. *Let* $X, Y \in \mathbb{R}^{n\times n}$.

1. $\exp X$ *is invertible with inverse* $\exp(-X)$.
2. $\frac{d}{dt}\exp tX = X\exp tX = (\exp tX)X$.
3. $e^{XYX^{-1}} = Xe^{Y}X^{-1}$, *where* X *is invertible.*
4. *If* X *and* Y *commute, then* $\exp(X+Y) = \exp(X)\exp(Y)$.
5. *(Liouville's Formula)* $\det(\exp X) = e^{tr(X)}$.

DEFINITION 1.1.12. Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be two Lie algebras. A linear mapping $F : \mathfrak{g}_1 \to \mathfrak{g}_2$ is called a **Lie algebra homomorphism** if

$$F([X,Y]) = [F(X), F(Y)] \quad \text{for all } X, Y \in \mathfrak{g}_1.$$

If F is also bijective, then F is called a **Lie algebra isomorphism**.

The most important type of Lie algebra homomorphisms are those determined by smooth group homomorphisms.

PROPOSITION 1.1.5. *Let* G_1, G_2 *be matrix Lie groups and* $\mathfrak{g}_1$, $\mathfrak{g}_2$ *their Lie algebras, respectively. Let*

$$F : \mathsf{G}_1 \to \mathsf{G}_2$$

be a smooth group homomorphism. Then the derivative

$$dF : \mathfrak{g}_1 \to \mathfrak{g}_2$$

is a Lie algebra homomorphism.

Thus, smoothly isomorphic matrix Lie groups have isomorphic Lie algebras.

DEFINITION 1.1.13. Let G be a matrix Lie group and $\mathfrak{g}$ its Lie algebra. For any $g \in \mathsf{G}$, the **conjugation map** is

$$C_g : \mathsf{G} \to \mathsf{G}, \quad C_g(h) := ghg^{-1}.$$

This is a smooth isomorphism. Then the derivative of the conjugation map is

$$\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}, \quad X \mapsto \mathrm{Ad}_g X := gXg^{-1} \quad \text{for all } X \in \mathfrak{g}.$$

The mapping

$$\mathrm{Ad} : \mathsf{G} \to \mathsf{GL}(\mathfrak{g}), \quad g \mapsto \mathrm{Ad}_g$$

is a smooth group homomorphism called the **adjoint representation** of G.

For any $X \in \mathfrak{g}$, the derivative of the adjoint representation is

$$\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}, \quad Y \mapsto \mathrm{ad}_X Y := [X,Y] \quad \text{for all } Y \in \mathfrak{g}.$$

The mapping

$$\mathrm{ad} : \mathfrak{g} \to \mathsf{End}(\mathfrak{g}), \quad X \mapsto \mathrm{ad}_X$$

is a Lie algebra homomorphism called the **adjoint representation** of $\mathfrak{g}$.

Definition 1.1.14. Let $F : V \to W$ be a linear mapping between vector spaces. Then the **dual linear mapping** is

$$F^* : W^* \to V^*, \quad w \mapsto (F^*w)v := w \circ F(v) \quad \text{for } v \in V.$$

Definition 1.1.15. Let G be a matrix Lie group and $\mathfrak{g}^*$ be the dual of its Lie algebra. The **co-adjoint representation** of G is the smooth group homomorphism, defined by

$$\mathrm{Ad}^* : \mathsf{G} \to \mathsf{GL}(\mathfrak{g}^*), \quad g \mapsto \mathrm{Ad}^*_{g^{-1}} := (\mathrm{Ad}_{g^{-1}})^*.$$

$$(\mathrm{Ad}^*_{g^{-1}} : \mathfrak{g}^* \to \mathfrak{g}^*, \quad p \mapsto (\mathrm{Ad}^*_{g^{-1}}p)X := p(\mathrm{Ad}_{g^{-1}}X) \quad \text{for all } X \in \mathfrak{g}.)$$

The infinitesimal version of this representation is the **co-adjoint representation** of $\mathfrak{g}$,

$$\mathrm{ad}^* : \mathfrak{g} \to \mathsf{End}(\mathfrak{g}^*), \quad X \mapsto \mathrm{ad}^*_X.$$

$$(\mathrm{ad}^*_X : \mathfrak{g}^* \to \mathfrak{g}^*, \quad p \mapsto (\mathrm{ad}^*_X p)Y := p(\mathrm{ad}_X Y) \quad \text{for all } Y \in \mathfrak{g}.)$$

Definition 1.1.16. A subspace $\mathfrak{I}$ of a Lie algebra $\mathfrak{g}$ is called an **ideal** of $\mathfrak{g}$, if for $X \in \mathfrak{g}$ and $Y \in \mathfrak{I}$, $[X,Y] \in \mathfrak{I}$.

Definition 1.1.17. A **simple Lie algebra** is non-Abelian with no ideals except 0 and itself. A **semi-simple Lie algebra** has no Abelian ideals.

In general, the representations Ad and Ad* are not equivalent. When $\mathfrak{g}$ is semi-simple, the adjoint and co-adjoint representations may be regarded as identical.

Definition 1.1.18. A **simple matrix Lie group** is path-connected and non-Abelian with no non-trivial path-connected normal subgroups.

Proposition 1.1.6. *A path-connected matrix Lie group is simple if its Lie algebra is simple.*

DEFINITION 1.1.19. $\mathsf{U}(n)$ is the set of all unitary $n \times n$ matrices over $\mathbb{C}$, i.e.,

$$\mathsf{U}(n) := \{a \in \mathsf{GL}(n, \mathbb{C}) \mid a^*a = Id\}, \text{ where } a^* \text{ is the Hermitian conjugate matrix.}$$

$\mathsf{SU}(n)$ is the set of all unitary $n \times n$ matrices over $\mathbb{C}$, with unit determinant, i.e.,

$$\mathsf{SU}(n) := \{a \in \mathsf{U}(n) \mid \det a = 1\}.$$

$\mathsf{U}(n)$ is called the **unitary group** and $\mathsf{SU}(n)$ is called the **special unitary group**.

PROPOSITION 1.1.7. *$\mathsf{U}(n)$ and $\mathsf{SU}(n)$ are matrix Lie groups. Further, $\mathsf{SU}(n)$ is a simple matrix Lie group.*

DEFINITION 1.1.20. The n-dimensional **torus** $\mathbb{T}^n$ is the group

$$\mathbb{T}^n := \mathsf{U}(1) \times \mathsf{U}(1) \times \ldots \times \mathsf{U}(1), \quad \text{n copies.}$$

DEFINITION 1.1.21. Let G be a matrix Lie group. A **torus** in G is a subgroup of G which is isomorphic to a torus. A **maximal torus** in G is a torus in G which is not contained in a higher dimensional torus in G.

DEFINITION 1.1.22. $S \subset \mathbb{R}^{m_1}$ and $T \subset \mathbb{R}^{m_2}$ are called **diffeomorphic** if there exists a smooth bijective mapping $F : S \to T$ whose inverse is also smooth. In this case, F is a **diffeomorphism**.

F is a **local diffeomorphism** if there exists a neighbourhood of any point of the domain, restricted to which the function is a diffeomorphism onto its image.

DEFINITION 1.1.23. $S \subseteq \mathbb{R}^m$ is said to be a **smooth submanifold** if, for every $x \in S$, there exists an open neighbourhood U of x in $\mathbb{R}^m$ and smooth diffeomorphism $F : U \to \tilde{U} \subseteq \mathbb{R}^m$ such that

$$F\left(S \bigcap U\right) = \tilde{U} \bigcap \mathbb{R}^\ell,$$

where $0 \leq \ell \leq m$.

We say that S is a smooth submanifold of dimension ℓ (or simply, an ℓ-submanifold). The codimension of S is $m - \ell$.

DEFINITION 1.1.24. Let $F : S \subseteq \mathbb{R}^{m_1} \to T \subseteq \mathbb{R}^{m_2}$ be a smooth map of constant rank. If F is surjective, then F is a **submersion**.

PROPOSITION 1.1.8. *Let $S \subseteq \mathbb{R}^m$ and suppose $0 \leq \ell \leq m$. The following statements are equivalent:*

1. *S is an ℓ-submanifold of $\mathbb{R}^m$.*

2. *For every* $x \in S$, *there exists an open neighbourhood* U *of* x *in* $\mathbb{R}^m$ *and a smooth submersion* $F : U \to \mathbb{R}^{m-\ell}$ *such that*

$$S \bigcap U = F^{-1}(0).$$

DEFINITION 1.1.25. A surjective, 2-to-1 local diffeomorphism between matrix Lie groups is called a **double cover**.

THEOREM 1.1.1. *(The inverse mapping theorem) Let* $S \subset \mathbb{R}^{m_1}$ *and* $T \subset \mathbb{R}^{m_2}$ *. Let* $U \subseteq S$ *be an open set and let* $F : U \to T$ *be a smooth mapping. Let* $Id \in U$ *and suppose that* dF *is a linear isomorphism. Then there exists an (open) neighbourhood* W *of* Id *in* U *such that* $F|_W : W \to F(W)$ *is a smooth diffeomorphism. Moreover, for* $x_2 \in F(W)$ *there is the following formula for the derivative of* F^{-1} *at* x_2*:*

$$dF^{-1}(x_2) = (dF(x_1))^{-1}, \text{ where } x_2 = F(x_1).$$

PROPOSITION 1.1.9. *Let* $S \subset \mathbb{R}^{m_1}$ *and* $T \subset \mathbb{R}^{m_2}$ *. If* $F : S \to T$ *is a surjective group homomorphism with kernel* N*, then there is a unique isomorphism* $F' : S/N \cong T$ *such that* F *is the composite* $F' \circ \pi$ *(where* $\pi : S \to S/N$ *is just the projection of* S *onto its quotient set* S/N*).*

DEFINITION 1.1.26. The set of all lines through the origin in $\mathbb{R}^{n+1}$ is called the **n-dimensional real projective space** and is denoted $\mathbb{RP}^n$.

$\mathbb{RP}^n$ is a smooth manifold of dimension n.

DEFINITION 1.1.27. A **vector field** X on a matrix Lie group G is a mapping from G into $T\mathsf{G}$ such that, for every $g \in \mathsf{G}$, the natural projection $\pi : T\mathsf{G} \to \mathsf{G}$ projects $X(g)$ to g.

The vector field X is a smooth provided the map X is smooth; and the set of all smooth vector fields on a matrix Lie group G is denoted by $\mathfrak{X}(\mathsf{G})$.

The tangent map of a left translation L_g is invertible, in particular,

$$dL_g : \mathfrak{g} \to T_g\mathsf{G}, \quad X \mapsto dL_g X$$

is a linear isomorphism.

DEFINITION 1.1.28. A vector field X on a matrix Lie group G is called **left-invariant** if

$$dL_g X = gX.$$

It follows that a left-invariant vector field on G is determined by its value at the identity.

The set of all smooth left-invariant vector fields on a matrix Lie group G is denoted by $\mathfrak{X}_L(\mathsf{G})$. This is a real vector space, and further, a real Lie algebra.

PROPOSITION 1.1.10. *Let* G *be a matrix Lie group and* $\mathfrak{g}$ *its Lie algebra. Then* $\mathfrak{X}_L(\mathsf{G})$ *and* $\mathfrak{g}$ *are isomorphic as Lie algebras.*

DEFINITION 1.1.29. Let X be a smooth vector field on a matrix Lie group G. An **integral curve** of X, with initial condition g_0, is a curve α in G such that $\alpha(0) = g_0$ and $\dot{\alpha}(t) = X(\alpha(t))$ for all t.

DEFINITION 1.1.30. Let X be a smooth vector field on a matrix Lie group G. The **flow** of X is the collection of maps $\phi_t : \mathsf{G} \to \mathsf{G}$ such that $t \mapsto \phi_t(g)$ is an integral curve of X with initial condition g_0.

DEFINITION 1.1.31. Let G be a matrix Lie group and $g \in \mathsf{G}$. A linear function on $T_g\mathsf{G}$ is called a **tangent covector** to G at the point g. In other words, a tangent covector is the element of the space $T_g^*\mathsf{G}$ dual to $T_g\mathsf{G}$, called the **cotangent space**.

It is natural to assemble together all cotangent spaces of a matrix Lie group G into a new structure called the cotangent bundle of G, denoted by $T^*\mathsf{G}$. The cotangent bundle is a smooth manifold with double the dimension of G.

The dual mapping of the tangent map of a left translation $dL_{g^{-1}}$ is given by

$$dL_{g^{-1}}^* : \mathfrak{g}^* \to T_g^*\mathsf{G}, \quad p \mapsto (dL_{g^{-1}}^* p)X_g := p(dL_{g^{-1}}X_g) \quad \text{for } X_g \in T_g\mathsf{G}.$$

PROPOSITION 1.1.11. *Let* G *be a matrix Lie group and* $\mathfrak{g}^*$ *be the dual of its Lie algebra. Then the correspondence* $dL_{g^{-1}}^* p \leftrightarrow (g, p)$ *realizes* $T^*\mathsf{G} = \mathsf{G} \times \mathfrak{g}^*$.

DEFINITION 1.1.32. Let G be a matrix Lie group with Lie algebra $\mathfrak{g}$. The **Killing form** is the symmetric bilinear mapping defined by

$$\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}, \quad \kappa(X, Y) := \operatorname{tr}(\operatorname{ad}_X \operatorname{ad}_Y) \quad \text{for any } X, Y \in \mathfrak{g}.$$

PROPOSITION 1.1.12. *Let* G *be a matrix Lie group and* $\mathfrak{g}$ *be its Lie algebra. If* $\mathfrak{g}$ *is semi-simple, then the Killing form is non-degenerate, i.e., if* $\kappa(X, Y) = 0$, *for every* $Y \in \mathfrak{g}$, *then* $X = 0$.

PROPOSITION 1.1.13. *Let* ω *be a bilinear mapping* $\omega : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$. ω *is non-degenerate if and only if the associated linear map*

$$\omega^\flat : \mathfrak{g} \to \mathfrak{g}^*, \quad X \to \omega^\flat(X)(Y) := \omega(X, Y) \quad \textit{for any } Y \in \mathfrak{g}$$

is an isomorphism.

1.2 Left-invariant control sytems

The references used include [24], [10], [25] and [1].

Let G be a matrix Lie group and $\mathfrak{g}$ be its Lie algebra.

DEFINITION 1.2.1. An **admissible control** is a vector-valued mapping

$$u(\cdot) : [0, T_u] \to U = \mathbb{R}^\ell$$

whose components – the input functions $u_1(\cdot), \ldots, u_\ell(\cdot)$ – must be measurable on $[0, T_u]$.

We will denote the set of all admissible controls by $\mathcal{U}$.

DEFINITION 1.2.2. A **left-invariant control system** Γ on G is an arbitrary set of left-invariant vector fields on G, i.e., any subset

$$\Gamma \subseteq \mathfrak{g} \ (= \mathfrak{X}_L(\mathsf{G})).$$

EXAMPLE. A class of left-invariant control systems is formed by control-affine systems

$$\Gamma = \left\{ X + \sum_{i=1}^{\ell} u_i Y_i \mid u = (u_1, \ldots, u_\ell) \in \mathbb{R}^\ell \right\},$$

where X and (linearly independent) $Y_1, \ldots, Y_\ell$ are some fixed elements of $\mathfrak{g}$.

In classical notation, the system is written as

$$\dot{g} = g \left(X + \sum_{i=1}^{\ell} u_i Y_i \right), \quad g \in \mathsf{G},\ u = (u_1, \ldots, u_\ell) \in \mathbb{R}^\ell.$$

DEFINITION 1.2.3. A **trajectory** of a left-invariant control system Γ on G is an absolutely continuous curve $g(\cdot)$ in G defined on an interval $[0, T]$ such that

$$\dot{g}(t) = g(t) \left(X + \sum_{i=1}^{\ell} u_i Y_i \right) \quad \in \Gamma, \quad \text{for a.e. } t \in [0, T].$$

DEFINITION 1.2.4. For any $T \geq 0$ and any $g \in \mathsf{G}$, the **reachable set for time** T of a left-invariant control system $\Gamma \subseteq \mathfrak{g}$, from the point g, is the set $\mathcal{A}_\Gamma(g, T)$ of all points that can be reached from g in exactly T units of time:

$$\mathcal{A}_\Gamma(g, T) = \{ g(T) \mid g(\cdot) \text{ a trajectory of } \Gamma,\ g(0) = g \}.$$

The **reachable set for time not greater than** $T \geq 0$ is defined as

$$\mathcal{A}_\Gamma(g, \leq T) = \bigcup_{0 \leq t \leq T} \mathcal{A}_\Gamma(g, t).$$

The **reachable** (or **attainable**) **set** of a system Γ from a point $g \in \mathsf{G}$ is the set $\mathcal{A}_\Gamma(g)$ of all terminal points $g(T)$, $T \geq 0$, of all trajectories of Γ starting at g:

$$\mathcal{A}_\Gamma(g) = \{g(T) \mid g(\cdot) \text{ a trajectory of } \Gamma,\ g(0) = g,\ T \geq 0\} = \bigcup_{T \geq 0} \mathcal{A}_\Gamma(g, T).$$

If there is no ambiguity, we will denote the reachable set $\mathcal{A}_\Gamma(g)$ by $\mathcal{A}(g)$.

DEFINITION 1.2.5. A left-invariant control system Γ on G is **controllable** if, given any pair of points g_1 and g_2 in G, the point g_2 can be reached from g_1 along a trajectory of Γ for a nonnegative time, i.e.,

$$g_2 \in \mathcal{A}(g_1) \quad \text{for any } g_1, g_2 \in \mathsf{G},$$

or in other words, if

$$\mathcal{A}(g) = \mathsf{G} \quad \text{for any } g \in \mathsf{G}.$$

PROPOSITION 1.2.1. *A left-invariant control system* Γ *on a matrix Lie group* G *is controllable if and only if* $\mathcal{A}(Id) = \mathsf{G}$.

Let Lie (Γ) denote the Lie algebra generated by Γ, i.e., the Lie algebra generated by calculating the commutators of all pairs of left-invariant vector fields in Γ, and then by taking all linear combinations of the left-invariant vector fields and their commutators.

PROPOSITION 1.2.2. *A left-invariant control system* Γ *on a compact, path-connected matrix Lie group* G *is controllable if and only if* Lie $(\Gamma) = \mathfrak{g}$.

1.3 Hamiltonian formalism

The references used include [15], [20], [3] and [10].

Let $T^*\mathsf{G}$ be the cotangent bundle of a matrix Lie group G.

DEFINITION 1.3.1. A **1-form** is a term used in two ways – they are either members of the cotangent space $T^*_\xi(T^*\mathsf{G})$ or else, analogous to a vector field, an assignment of a covector in $T^*_\xi(T^*\mathsf{G})$ to each $\xi \in T^*\mathsf{G}$. The set of all 1-forms on $T^*\mathsf{G}$ is denoted by $A^1(T^*\mathsf{G})$.

A basic example of a 1-form is the differential of a real-valued function.

DEFINITION 1.3.2. A **2-form** ω on $T^*\mathsf{G}$ is any mapping

$$\xi \mapsto \omega_\xi, \quad \xi \in T^*\mathsf{G},$$

such that

1. $\omega_\xi : T_\xi(T^*\mathsf{G}) \times T_\xi(T^*\mathsf{G}) \to \mathbb{R}$ is a bilinear, skew-symmetric mapping, and
2. the dependence $\xi \mapsto \omega_\xi$ is smooth.

The set of all 2-forms on $T^*\mathsf{G}$ is denoted by $A^2(T^*\mathsf{G})$.

DEFINITION 1.3.3. The bilinear map

$$\omega_\xi : T_\xi(T^*\mathsf{G}) \times T_\xi(T^*\mathsf{G}) \to \mathbb{R}$$

is **non-degenerate** if, when $\omega_\xi(v_1, v_2) = 0$ for every $v_2 \in T_\xi(T^*\mathsf{G})$, then $v_1 = 0$.

DEFINITION 1.3.4. A **k-form** α on $T^*\mathsf{G}$ is a function

$$\alpha(\xi) : T_\xi(T^*\mathsf{G}) \times \ldots \times T_\xi(T^*\mathsf{G}) \; (k \text{ factors}) \to \mathbb{R}$$

that assigns to each point $\xi \in T^*\mathsf{G}$ a skew-symmetric k-mulitilinear map on the tangent space $T_\xi(T^*\mathsf{G})$ to $T^*\mathsf{G}$ at ξ. The set of all k-forms on $T^*\mathsf{G}$ is denoted by $A^k(T^*\mathsf{G})$.

DEFINITION 1.3.5. Let α and β be 1-forms on $T^*\mathsf{G}$. Then their **wedge product** $\alpha \wedge \beta$ is the 2-form

$$(\alpha \wedge \beta)(v_1, v_2) = \alpha(v_1)\beta(v_2) - \alpha(v_2)\beta(v_1).$$

Let α be a 2-form and β be a 1-form. Then their **wedge product** $\alpha \wedge \beta$ is the 3-form

$$(\alpha \wedge \beta)(v_1, v_2, v_3) = \alpha(v_1, v_2)\beta(v_3) + \alpha(v_2, v_3)\beta(v_1) + \alpha(v_3, v_1)\beta(v_2).$$

DEFINITION 1.3.6. The **exterior derivative** $d\alpha$ of a k-form α (k = 0, 1, 2) on $T^*\mathsf{G}$ is the $(k+1)$-form on $T^*\mathsf{G}$ determined by the following proposition:

PROPOSITION 1.3.1. *There is a unique mapping*

$$d : A^k(T^*\mathsf{G}) \to A^{k+1}(T^*\mathsf{G}), \quad \alpha \mapsto d\alpha,$$

such that

1. *if α is a 0-form, i.e., $\alpha = F \in C^\infty(T^*\mathsf{G})$, then dF is a 1-form, i.e., $dF : T(T^*\mathsf{G}) \to \mathbb{R}$.*
2. *$d\alpha$ is linear in α,*

3. $d\alpha$ satisfies the product rule

$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^k \alpha \wedge (d\beta),$$

where α is a k-form and β is an ℓ-form ($\ell = 0,\ 1,\ 2$).

4. $d^2 = 0$, i.e., $d(d\alpha) = 0$ for any k-form α.

5. d is a local operator, i.e., $d\alpha_g$ depends only on α restricted to any neighbourhood of g.

DEFINITION 1.3.7. A k-form α (k $= 0,\ 1,\ 2$) is called **closed** if $d\alpha = 0$.

DEFINITION 1.3.8. A **symplectic form** on $T^*\mathsf{G}$ is a non-degenerate, closed 2-form ω on $T^*\mathsf{G}$.

There is a natural symplectic form on $T^*\mathsf{G}$ that can be described as follows. Assume that the matrix Lie group G is n-dimensional and pick local coordinates $\{dq_1, \ldots, dq_n\}$. Then $\{dq_1, \ldots, dq_n\}$ is a basis of $T_g^*\mathsf{G}$, and by writing $\xi_g \in T_g^*\mathsf{G}$ as $\xi_g = p_1 dq_1 + \ldots + p_n dq_n$ we get local coordinates $\{q_1, \ldots, q_n, p_1, \ldots, p_n\}$ on $T^*\mathsf{G}$. We define the canonical symplectic form ω on $T^*\mathsf{G}$ by

$$\omega = \sum_{i=1}^{n} dq_i \wedge dp_i.$$

This 2-form ω is independent of the choice of coordinates $\{q_1, \ldots, q_n\}$. Observe that ω is locally constant, that is ω is independent of the base point $(q_1, \ldots, q_n, p_1, \ldots, p_n)$, and so $d\omega = 0$.

DEFINITION 1.3.9. $T^*\mathsf{G}$ equipped with the canonical symplectic form ω is a **symplectic manifold**.

DEFINITION 1.3.10. Let α be a k-form and X a vector field. The **interior product** is defined by

$$\begin{aligned} i_X : A^k(T^*\mathsf{G}) &\rightarrow A^{k-1}(T^*\mathsf{G}) \\ \alpha \mapsto (i_X\alpha)_\xi(v_2, \ldots, v_k) &:= \alpha_\xi(X(x), v_2, \ldots, v_k). \end{aligned}$$

DEFINITION 1.3.11. Let $H \in C^\infty(T^*\mathsf{G})$. The vector field $\vec{H}$, determined by the condition

$$i_{\vec{H}}\omega = dH,$$

is called the **Hamiltonian vector field** associated with **Hamiltonian function** H. We call $(T^*\mathsf{G}, \omega, H)$ a **Hamiltonian system**.

Non-degeneracy of ω ensures that $\vec{H}$ exists.

PROPOSITION 1.3.2. *(Hamilton's equations in canonical coordinates) Let $(q_1, \ldots, q_n, p_1, \ldots, p_n)$ be canonical coordinates on $T^*\mathsf{G}$, i.e., $\omega = \sum_{i=1}^n dq_i \wedge dp_i$. Then $(q(\cdot), p(\cdot))$ is an integral curve of $\vec{H}$*

if and only if Hamilton's equations hold, i.e.,

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \quad \text{and} \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \quad \text{for } i = 1, 2, \ldots, n.$$

DEFINITION 1.3.12. Let $F, G \in C^\infty(T^*\mathsf{G})$. The **Poisson bracket** of F and G is the function defined by

$$\begin{aligned} \{F, G\}_\omega : T^*\mathsf{G} &\to \mathbb{R} \\ \{F, G\}_\omega(\xi) &:= \omega_\xi(\vec{F}(\xi), \vec{G}(\xi)). \end{aligned}$$

PROPOSITION 1.3.3. *The real vector space $C^\infty(T^*\mathsf{G})$, together with the Poisson bracket, forms an infinite-dimensional Lie algebra.*

For any smooth vector field X on G, let H_X denote the Hamiltonian function on $T^*\mathsf{G}$ that corresponds to X.

PROPOSITION 1.3.4. *If H_X and H_Y are Hamiltonian functions on $T^*\mathsf{G}$, then*

$$\{H_X, H_Y\} = -H_{[X,Y]}.$$

PROPOSITION 1.3.5. *Let H be a Hamiltonian function and ϕ_t be the flow of $\vec{H}$. Then, for each $F \in C^\infty(T^*\mathsf{G})$,*

$$\frac{d}{dt}(F \circ \phi_t) = \{F \circ \phi_t, H\} = \{F, H\} \circ \phi_t.$$

PROPOSITION 1.3.6. *(Leibniz rule) For $F \in C^\infty(T^*\mathsf{G})$,*

$$\{F, \cdot\} : C^\infty(T^*\mathsf{G}) \to C^\infty(T^*\mathsf{G})$$

is linear and satisfies

$$\{F, GJ\} = \{F, G\}J + G\{F, J\}.$$

DEFINITION 1.3.13. Let $(T^*\mathsf{G}, \omega, H)$ be a Hamiltonian system. A function $F \in C^\infty(T^*\mathsf{G})$ is called a **constant of motion** if it satisfies the following condition:

$$\{F, H\}_\omega = 0.$$

Rather than expressing the integral curves through canonical coordinates on $T^*\mathsf{G}$ and Hamilton's equations, it will be more appropriate to regard $T^*\mathsf{G}$ as $\mathsf{G} \times \mathfrak{g}^*$, and proceed with non-canonical coordinates (g, p) relative to this decomposition.

PROPOSITION 1.3.7. *Let $T(\mathsf{G} \times \mathfrak{g}^*)$ denote the tangent space to $\mathsf{G} \times \mathfrak{g}^*$. Then $T(\mathsf{G} \times \mathfrak{g}^*)$ can be realized as the product $(\mathsf{G} \times \mathfrak{g}) \times (\mathfrak{g}^* \times \mathfrak{g}^*)$. In this realization, each element $((g, X), (p, Y^*))$ is a* *tangent vector* (X, Y^*) *based at* (g,p) *in* $T^*\mathsf{G}$.

PROPOSITION 1.3.8. *Let the tangent vector* (X, Y^*) *in* $\mathfrak{g} \times \mathfrak{g}^*$ *define a vector field* V *on* $\mathsf{G} \times \mathfrak{g}^*$ *equal to* (X, Y^*) *at each point* (g,p) *in* $\mathsf{G} \times \mathfrak{g}^*$. *Then* V *is a left-invariant vector field on the product* $\mathsf{G} \times \mathfrak{g}^*$.

PROPOSITION 1.3.9. *The natural symplectic form* ω *on* $T^*\mathsf{G} = \mathsf{G} \times \mathfrak{g}^*$ *takes the following form:*

$$\omega_{(g,p)}((X_1, Y_1^*), (X_2, Y_2^*)) = Y_2^*(X_1) - Y_1^*(X_2) + (\text{ad}^* X_1(p))(X_2),$$

for any (g,p) *in* $\mathsf{G} \times \mathfrak{g}^*$ *and any vectors* (X_i, Y_i^*) *in* $\mathfrak{g} \times \mathfrak{g}^*$, $i = 1, 2$.

Having identified $T^*\mathsf{G}$ with $\mathsf{G} \times \mathfrak{g}^*$, functions on $T^*\mathsf{G}$ become functions on $\mathsf{G} \times \mathfrak{g}^*$. The next proposition specifies the Hamiltonian vector fields in terms of the partial derivatives of the defining function.

PROPOSITION 1.3.10. *(Hamilton's equations in non-canonical coordinates)*
Let $\vec{H}(g,p) = (X(g,p), Y^*(g,p))$ *denote the Hamiltonian vector field corresponding to a function* H *on* $\mathsf{G} \times \mathfrak{g}^*$. *Then*

$$X(g,p) = \frac{\partial H}{\partial p}(g,p) \quad \textit{and} \quad Y^*(g,p) = -dL_g^*\left(\frac{\partial H}{\partial g}(g,p)\right) + \text{ad}^* X(p).$$

COROLLARY 1.3.1. *Each integral curve* $(g(\cdot), p(\cdot))$ *of* $\vec{H}$ *satisfies the differential equations*

$$\dot{g} = g\left(\frac{\partial H}{\partial p}\right) \quad \textit{and} \quad \dot{p} = -dL_g^*\left(\frac{\partial H}{\partial g}(g,p)\right) + \left(\text{ad}^* \frac{\partial H}{\partial p}(g,p)\right)(p).$$

1.4 Optimal control

The references used include [10], [1] and [25].

Let G be a matrix Lie group and $\mathfrak{g}$ be its Lie algebra. Let g_1 and g_2 be arbitrary but fixed points in G.

DEFINITION 1.4.1. A trajectory-control pair $(g(\cdot), u(\cdot))$ is said to transfer a point g_1 to another point g_2 if there exists an interval $[0, T]$, contained in the domain of (g, u), such that $g(0) = g_1$ and $g(T) = g_2$. The **cost** of this transfer is given by

$$J = \int_0^T L(g(t), u(t))\, dt,$$

where L is the **Lagrangian**.

DEFINITION 1.4.2. A trajectory-control pair $(g(\cdot), u(\cdot))$ is **optimal** relative to the given points g_1

and g_2, if $g(0) = g_1$ and $g(T) = g_2$, and if $\int_0^T L(g(t), u(t))\, dt$ is the minimal cost among all costs of trajectories that transfer g_1 to g_2.

DEFINITION 1.4.3. A Lagrangian $L \in C^\infty(\mathsf{G} \times \mathbb{R}^\ell)$ is called **left-invariant** if

$$L(gh, u) = L(h, u) \quad \text{for all } g,\, h \in \mathsf{G}.$$

L is constant over G and depends only on the controls.

DEFINITION 1.4.4. An optimal control problem on G is called **left-invariant** if both the Lagrangian L and the control system are left-invariant.

DEFINITION 1.4.5. (A left-invariant optimal control problem on G) We shall be interested in finding the optimal trajectory-control pair $(g(\cdot), u(\cdot))$, relative to the given points g_1 and g_2 of the following left-invariant control system on G:

$$\dot{g}(t) = g(t)\left(X + \sum_{i=1}^{\ell} u_i Y_i\right), \quad g \in \mathsf{G},\ u = (u_1, \ldots, u_\ell) \in \mathbb{R}^\ell.$$

The problem is lifted from G to the cotangent bundle $T^*\mathsf{G}$ as follows. The Hamiltonian H of the left-invariant vector field $X + \sum_{i=1}^{\ell} u_i Y_i$ is defined by

$$H(\xi_g) := \xi_g\left(g\left(X + \sum_{i=1}^{\ell} u_i Y_i\right)\right) \qquad \text{for each } \xi_g \in T_g^*\mathsf{G}.$$

When $\xi_g = (g, p)$ with $p \in \mathfrak{g}^*$, then $\xi_g = dL_{g^{-1}}^* p$:

$$H(g,p) \;=\; p\left(g^{-1}g\left(X + \sum_{i=1}^{\ell} u_i Y_i\right)\right) = p\left(X + \sum_{i=1}^{\ell} u_i Y_i\right).$$

Hence, H is a linear function on $\mathfrak{g}^*$ parametrized by the control values.

DEFINITION 1.4.6. The **cost extended Hamiltonian** on $\mathfrak{g}^*$ of the left-invariant optimal control problem is defined by

$$\mathcal{H}^\lambda(p, u) := -\lambda L(u) + p\left(X + \sum_{i=1}^{\ell} u_i Y_i\right), \quad p \in \mathfrak{g}^*,\ u = (u_1, \ldots, u_\ell) \in \mathbb{R}^\ell.$$

with $\lambda = 0$ or $\lambda = 1$.

DEFINITION 1.4.7. The set $\{\mathrm{Ad}_g X \mid g \in \mathsf{G}\}$ is called the **adjoint orbit** of G through X.

DEFINITION 1.4.8. The set $\{\mathrm{Ad}_{g^{-1}}^* p \mid g \in \mathsf{G}\}$ is called the **co-adjoint orbit** of G through p.

The set of co-adjoint orbits of G partitions $\mathfrak{g}^*$. It is known that each co-adjoint orbit is a symplectic submanifold of $\mathfrak{g}^*$.

PROPOSITION 1.4.1. *Suppose that* $(g(\cdot), p(\cdot))$ *is an integral curve of the Hamiltonian vector field* $\vec{\mathcal{H}}(p, u(t))$ *for some control function* $u(\cdot)$*, with* $\mathcal{H}^\lambda(p,u) = -\lambda L(u) + p(X + \sum_{i=1}^{\ell} u_i Y_i)$. *Then*

$$\dot{g} = g\left(\frac{\partial \mathcal{H}^\lambda}{\partial p}(p,u)\right) \quad \text{and} \quad p(t) = \mathrm{Ad}^*_{g(t)} p(0),$$

for some $p(0) \in \mathfrak{g}^*$. *Consequently,* $p(t)$ *is contained in the co-adjoint orbit of* G *through* $p(0)$.

COROLLARY 1.4.1. $\mathrm{Ad}^*_{g(t)^{-1}} p(t) = $ *constant for each integral curve* $(g(\cdot), p(\cdot))$ *of* $\mathcal{H}(p, u(t))$.

DEFINITION 1.4.9. A **Casimir function**, denoted by K, is any $\mathrm{Ad}^*_{\mathsf{G}}$-invariant (smooth) function on $\mathfrak{g}^*$.

A function K is $\mathrm{Ad}^*_{\mathsf{G}}$-invariant if $K(p) = K(\mathrm{Ad}^*_g(p))$, for all $g \in \mathsf{G}$ and $p \in \mathfrak{g}^*$.

PROPOSITION 1.4.2. *Let* F *and* G *be any functions on* $\mathfrak{g}^*$. *Then*

$$\{F, G\} = -p\,[dF(p), dG(p)] \quad \text{for all } p \in \mathfrak{g}^*.$$

Note that $dF(p)$ is a linear function on $\mathfrak{g}^*$ and is thus an element of $\mathfrak{g}$. Therefore, the Lie bracket $[dF(p), dG(p)]$ makes sense.

PROPOSITION 1.4.3. *A Casimir function is a constant of motion for any Hamiltonian function* H *on* $\mathfrak{g}^*$.

1.5 The maximum principle

In optimal control theory, the maximum principle is used to find the best possible control for taking a system from one state to another. The contemporary view is that the maximum principle was formulated by the Russian mathematician L. Pontryagin (1908-1988) and his students in the late 1950s. However, Sussmann and Willems [29] argue that optimal control and the maximum principle were birthed 300 years ago.

In the mid-1800s, L. Euler (1707-1793) and J-L. Lagrange (1736-1813) studied the optimization problem from the calculus of variations perspective, and succeeded in deriving a necessary condition for optimality. Later, A-M. Legendre (1752-1833) found an additional necessary condition for a minimum, and W. Hamilton (1805-1865) rewrote the Euler-Lagrange condition in a different formalism. K. Weierstrass (1815-1897) also revisted the optimization problem, but with the assumptions that the Lagrangian be positively homogenous with respect to the velocity, and that it did not depend on time. He further introduced the 'excess function', and made a statement known as Weierstrass' side condition, which says that for a curve to be a solution to the minimization problem, the excess function must satisfy a certain condition.

Sussmann and Willems believe that if Hamilton had considered treating the velocity as an independent variable in his expression of the Hamiltonian function, and if Weierstrass had used the excess function but re-expressed it in terms of this different Hamiltonian function, Weierstrass could have come very close to Pontryagin's statement of the maximum principle [29].

We now make a statement of the maximum principle [9–11, 29] in terms of the cost extended Hamiltonian (Def. 1.4.6) and in the context of our problem on matrix Lie groups (Def. 1.4.5).

THE MAXIMUM PRINCIPLE. *If* $(g(\cdot), u(\cdot))$ *is an optimal trajectory-control pair of our left-invariant optimal control problem on an interval* $[0, T]$. *Then,* $g(\cdot)$ *is the projection of an integral curve* $(g(\cdot), p(\cdot))$ *of the Hamiltonian vector field* $\vec{\mathcal{H}}^\lambda$, *with* $\lambda = 0$ *or* $\lambda = 1$, *such that*

1. *if* $\lambda = 0$, *then* $(g(\cdot), p(\cdot))$ *is not identically zero on* $[0, T]$,
2. $\mathcal{H}^\lambda((g(\cdot), p(\cdot)), u(\cdot)) \geq \mathcal{H}^\lambda((g(\cdot), p(\cdot))), u)$ *for any* $u \in \mathbb{R}^\ell$, *and* $t \in [0, T]$, *and*
3. $\mathcal{H}^\lambda((g(\cdot), p(\cdot)), u(\cdot))$ *is constant for all* $t \in [0, T]$.

DEFINITION 1.5.1. A pair of curves $((g(\cdot), p(\cdot)), u(\cdot))$ on an interval $[0, T]$ is said to be an **extremal pair** if $(g(\cdot), p(\cdot))$ is an integral curve of $\vec{\mathcal{H}}^\lambda$, for either $\lambda = 0$ or $\lambda = 1$, such that 1. and 2. of the Maximum Principle hold. The projection $(g(\cdot), p(\cdot))$ of an extremal pair is called an **extremal**. The extremals which correspond to $\lambda = 0$ are called **abnormal**, and the extremals which correspond to $\lambda = 1$ are called **normal**.

We restrict ourselves to normal extremals, and from now on a normal extremal will simply be called an extremal.

1.6 Elliptic functions

The references used include [12] and [16].

DEFINITION 1.6.1. An **elliptic function** is a meromorphic function F, defined on $\mathbb{C}$, for which there exist two non-zero complex numbers w_1 and w_2, with w_1/w_2 not real, such that

$$F(z + w_1) = F(z + w_2) = F(z) \quad \text{for all } z \in \mathbb{C}.$$

DEFINITION 1.6.2. Any complex number w such that $F(z + w) = F(z)$ for all $z \in \mathbb{C}$ is called a **period** of F.

PROPOSITION 1.6.1. *The derivative of an elliptic function is again an elliptic function, with the same periods.*

Weierstrass' elliptic functions are elliptic functions that take a particularly simple form. These functions are also referred to as P-functions and are generally written using the symbol $\wp$. The Weierstrass elliptic function can be defined in three closely related ways; we will only consider one of these definitions.

DEFINITION 1.6.3. The **Weierstrass elliptic function** can be defined in terms of a complex variable z and two periods w_1 and w_2 as

$$\wp(z, w_1, w_2) := \frac{1}{z^2} + \sum_{m^2+n^2 \neq 0} \left\{ \frac{1}{(z - mw_1 - nw_2)^2} - \frac{1}{(mw_1 + nw_2)^2} \right\},$$

with integers m and n.

PROPOSITION 1.6.2. *Let z be a complex variable and let w_1, w_2 be periods of $\wp$. For points close to the origin, the Weierstrass elliptic function is expressed in terms of the following Laurent series:*

$$\wp(z, w_1, w_2) = \frac{1}{z^2} + \frac{1}{20} g_2 z^2 + \frac{1}{28} g_1 z^4 + O(z^6),$$

where

$$g_2 = 60 \sum (mw_1 + nw_2)^{-4} \quad \text{and} \quad g_1 = 140 \sum (mw_1 + nw_2)^{-6},$$

and the summation is over all pairs of integers (except $m = n = 0$). The numbers g_1 and g_2 are known as ***invariants****.*

PROPOSITION 1.6.3. *For complex variable z, the $\wp$ function satisfies the differential equation*

$$\dot{\wp}(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_1, \tag{1.1}$$

where dependence on w_1 and w_2 is suppressed, and g_1, g_2 are invariants.

The solutions to differential equations of this form have been compiled into tables.

The Jacobi elliptic functions are a set of twelve basic elliptic functions, with useful analogies to the functions of trigonometry.

DEFINITION 1.6.4. Let k be a number in (0, 1). The **Jacobi elliptic functions** $\mathrm{sn}(\cdot, k)$, $\mathrm{cn}(\cdot, k)$ and $\mathrm{dn}(\cdot, k)$ are defined as the solutions of the system of differential equations

$$\begin{aligned} \dot{x} &= yz \\ \dot{y} &= -zx \\ \dot{z} &= -k^2xy, \end{aligned}$$

that satisfy the initial conditions

$$\operatorname{sn}(0,k)=x(0)=0,\quad \operatorname{cn}(0,k)=y(0)=1\quad\text{and}\quad \operatorname{dn}(0,k)=z(0)=1.$$

The parameter k is known as the **modulus**.

DEFINITION 1.6.5. Nine other elliptic functions are defined by taking reciprocals and quotients – the notations will be clear from the definitions below.

$$\begin{aligned}
\operatorname{ns}(t,k) &:= \frac{1}{\operatorname{sn}(t,k)}, & \operatorname{nc}(t,k) &:= \frac{1}{\operatorname{cn}(t,k)}, & \operatorname{nd}(t,k) &:= \frac{1}{\operatorname{dn}(t,k)}\\
\operatorname{sc}(t,k) &:= \frac{\operatorname{sn}(t,k)}{\operatorname{cn}(t,k)}, & \operatorname{cd}(t,k) &:= \frac{\operatorname{cn}(t,k)}{\operatorname{dn}(t,k)}, & \operatorname{ds}(t,k) &:= \frac{\operatorname{dn}(t,k)}{\operatorname{sn}(t,k)}\\
\operatorname{cs}(t,k) &:= \frac{\operatorname{cn}(t,k)}{\operatorname{sn}(t,k)}, & \operatorname{dc}(t,k) &:= \frac{\operatorname{dn}(t,k)}{\operatorname{cn}(t,k)}, & \operatorname{sd}(t,k) &:= \frac{\operatorname{sn}(t,k)}{\operatorname{dn}(t,k)}.
\end{aligned}$$

Historically, these elliptic functions were discovered as inverse functions of elliptic integrals.

DEFINITION 1.6.6. An **elliptic integral** is any function F which can be expressed in the form

$$F(x)=\int_c^x R(t,P(t))\,dt,$$

where R is a rational function of its two arguments, P is the square root of a polynomial of degree 3 or 4 with no repeated roots, and c is a constant.

We will use the following elliptic integrals in chapters 4-7:

$$\int_0^x \frac{dt}{\sqrt{(a^2-t^2)(b^2-t^2)}} = \frac{1}{a}\operatorname{sn}^{-1}\left(\frac{x}{b},\frac{b}{a}\right), \qquad 0\le x\le b<a \tag{1.2}$$

$$\int_x^b \frac{dt}{\sqrt{(a^2+t^2)(b^2-t^2)}} = \frac{1}{\sqrt{a^2+b^2}}\operatorname{cn}^{-1}\left(\frac{x}{b},\frac{b}{\sqrt{a^2+b^2}}\right), \qquad 0\le x\le b \tag{1.3}$$

$$\int_x^b \frac{dt}{\sqrt{(a^2-t^2)(b^2-t^2)}} = \frac{1}{a}\operatorname{cd}^{-1}\left(\frac{x}{b},\frac{b}{a}\right), \qquad 0\le x\le b<a \tag{1.4}$$

$$\int_0^x \frac{dt}{\sqrt{(a^2+t^2)(b^2-t^2)}} = \frac{1}{\sqrt{a^2+b^2}}\operatorname{sd}^{-1}\left(\frac{\sqrt{a^2+b^2}x}{ab},\frac{b}{\sqrt{a^2+b^2}}\right), \qquad 0\le x\le b \tag{1.5}$$

$$\int_a^x \frac{dt}{\sqrt{(t^2-a^2)(t^2-b^2)}} = \frac{1}{a}\operatorname{dc}^{-1}\left(\frac{x}{a},\frac{b}{a}\right), \qquad b<a\le x \tag{1.6}$$

$$\int_x^\infty \frac{dt}{\sqrt{(t^2-a^2)(t^2-b^2)}} = \frac{1}{a}\operatorname{ns}^{-1}\left(\frac{x}{a},\frac{b}{a}\right), \qquad b<a\le x \tag{1.7}$$

$$\int_b^x \frac{dt}{\sqrt{(a^2-t^2)(t^2-b^2)}} = \frac{1}{a}\operatorname{nd}^{-1}\left(\frac{x}{b},\frac{\sqrt{a^2-b^2}}{a}\right), \qquad b\le x\le a \tag{1.8}$$

$$\int_x^a \frac{dt}{\sqrt{(a^2-t^2)(t^2-b^2)}} = \frac{1}{a}\,\mathrm{dn}^{-1}\left(\frac{x}{a}, \frac{\sqrt{a^2-b^2}}{a}\right), \qquad b \le x \le a \tag{1.9}$$

$$\int_a^x \frac{dt}{\sqrt{(t^2-a^2)(t^2+b^2)}} = \frac{1}{\sqrt{a^2+b^2}}\mathrm{nc}^{-1}\left(\frac{x}{a}, \frac{b}{\sqrt{a^2+b^2}}\right), \qquad a \le x \tag{1.10}$$

$$\int_x^\infty \frac{dt}{\sqrt{(t^2-a^2)(t^2+b^2)}} = \frac{1}{\sqrt{a^2+b^2}}\mathrm{ds}^{-1}\left(\frac{x}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}}\right), \qquad a \le x. \tag{1.11}$$

In sections 4.3 and 6.3, we will need to solve differential equations of the form

$$\dot{x}^2 = C(a_1x^2 + 2b_1x + c_1)(a_2x^2 + 2b_2x + c_2), \tag{1.12}$$

where C is a positive constant. However, we will need to make some changes to (1.12) before we can continue with an elliptic integral mentioned previously.

Firstly, let

$$S_1 = a_1x^2 + 2b_1x + c_1 \quad \text{and} \quad S_2 = a_2x^2 + 2b_2x + c_2.$$

Consider the quadratic expression $S_1 + \lambda\, S_2$; this has coincident zeros and is a perfect square whenever

$$\begin{aligned} & D(\lambda) \equiv (a_1 + \lambda\, a_2)(c_1 + \lambda\, c_2) - (b_1 + \lambda\, b_2)^2 = 0 \\ \Leftrightarrow \quad & (a_2c_2 - b_2^2)\lambda^2 + (a_1c_2 + a_2c_1 - 2b_1b_2)\lambda + a_1c_1 - b_1^2 = 0. \end{aligned} \tag{1.13}$$

Let λ_1 and λ_2 be the roots of $D(\lambda) = 0$, then

$$S_1 + \lambda_1 S_2 = (a_1 + \lambda_1 a_2)(x-\alpha)^2 \tag{1.14}$$

$$S_1 + \lambda_2 S_2 = (a_1 + \lambda_2 a_2)(x-\beta)^2, \tag{1.15}$$

where

$$(x-\alpha)^2 = x^2 + \frac{(2b_1 + \lambda_1 2b_2)}{(a_1 + \lambda_1 a_2)}x + \frac{(c_1 + \lambda_1 c_2)}{(a_1 + \lambda_1 a_2)} \quad \text{and} \quad (x-\beta)^2 = x^2 + \frac{(2b_1 + \lambda_2 2b_2)}{(a_1 + \lambda_2 a_2)}x + \frac{(c_1 + \lambda_2 c_2)}{(a_1 + \lambda_2 a_2)}. \tag{1.16}$$

Solving (1.14) and (1.15) for S_1 and S_2, we show that they can be expressed in the forms

$$S_1 = A_1(x-\alpha)^2 + B_1(x-\beta)^2 \quad \text{and} \quad S_2 = A_2(x-\alpha)^2 + B_2(x-\beta)^2, \tag{1.17}$$

where

$$A_1 = \frac{\lambda_2(a_1 + \lambda_1 a_2)}{\lambda_2 - \lambda_1}, \quad B_1 = \frac{\lambda_1(a_1 + \lambda_2 a_2)}{\lambda_1 - \lambda_2}, \quad A_2 = \frac{a_1 + \lambda_1 a_2}{\lambda_1 - \lambda_2}, \quad B_2 = \frac{a_1 + \lambda_2 a_2}{\lambda_2 - \lambda_1}. \tag{1.18}$$

Referring back to (1.12), we can now continue to solve the differential equation with the new forms (1.17) for S_1 and S_2:

(We do not consider $\dot{x} = -\sqrt{C(A_1(x-\alpha)^2 + B_1(x-\beta)^2)(A_2(x-\alpha)^2 + B_2(x-\beta)^2)}$ as it can be checked numerically that this does not contribute any new solutions in sections 4.3 and 6.3.)

$$\begin{aligned} \dot{x} &= \sqrt{C(A_1(x-\alpha)^2 + B_1(x-\beta)^2)(A_2(x-\alpha)^2 + B_2(x-\beta)^2)} \\ t &= \frac{1}{\sqrt{C}} \int \frac{dx}{\sqrt{(A_1(x-\alpha)^2 + B_1(x-\beta)^2)(A_2(x-\alpha)^2 + B_2(x-\beta)^2)}}. \end{aligned}$$

Changing the variable in the above integral by the transformation

$$u = \frac{x-\alpha}{x-\beta}, \tag{1.19}$$

yields

$$\begin{aligned} t &= \frac{1}{\sqrt{C}(\alpha-\beta)} \int \frac{du}{\sqrt{(A_1u^2 + B_1)(A_2u^2 + B_2)}} \\ &= \frac{1}{\sqrt{C}(\alpha-\beta)\sqrt{A_1A_2}} \int \frac{du}{\sqrt{(u^2 + \frac{B_1}{A_1})(u^2 + \frac{B_2}{A_2})}}. \end{aligned} \tag{1.20}$$

From here, this can be manipulated to match the form of one of the Jacobi elliptic integrals mentioned before. We conclude by substituting the integral value, followed by (1.19), and solving for x we obtain a solution.

1.7 Stability

The references used include [20] and [15].

Let G be a matrix Lie group, and $\mathfrak{g}$ be its Lie algebra.

The energy-Casimir method due to Holm, Marsden, Ratiu and Weinstein [7] is a generalization of the Lagrange-Dirichlet theorem to Hamiltonian systems. It gives sufficient conditions for non-linear stability of equilibrium states.

The energy-Casimir method is contructed for Poisson spaces $(P, \{\cdot,\cdot\})$, i.e., a vector space P admitting a Poisson bracket operation. The dual space $\mathfrak{g}^*$, i.e., $(\mathfrak{g}^*, \{\cdot,\cdot\})$, is a Poisson space, and we consider the energy-Casimir method in this context.

Let $p \in \mathfrak{g}^*$.

Definition 1.7.1. An **equilibrium state** is a state p_e such that $\vec{H}(p_e) = 0$. The unique trajectory starting at p_e is p_e itself, i.e., p_e does not move in time.

DEFINITION 1.7.2. An equilibrium state p_e is said to be **non-linear stable** if, for every neighbourhood U of p_e, there is a neighbourhood V of p_e such that trajectories $p(\cdot)$ initially in V never leave U.

THE ENERGY-CASIMIR METHOD. *Let* p_e *be an equilibrium state of* $\dot{p} = \vec{H}(p)$. *The algorithm of the energy-Casimir method is the following:*

STEP 1. *Find a family of constants of motion for the Hamiltonian system. A good way to find these constants of motion is to look for Casimir functions on* $\mathfrak{g}^*$.

STEP 2. *Find a constant of motion* K *from the family in* STEP 1 *such that the energy-Casimir function* $\mathcal{H} + K$ *has a critical point at the given equilibrium state.*

STEP 3. *Check whether the second derivative is positive or negative definite at the given equilibrium state.*

If the second derivative is positive or negative definite, then the equilibrium state is non-linear stable. If not, the test is inconclusive.

Chapter 2

The Group $\mathsf{SO}(3)$

The **orthogonal group** $\mathsf{O}(3)$ is the set of all orthogonal 3×3 matrices over $\mathbb{R}$, i.e.,

$$\mathsf{O}(3) := \{g \in \mathsf{GL}(3, \mathbb{R}) \,|\, g^\top g = Id\}.$$

$\mathsf{O}(3)$ is a matrix Lie group.

The **special orthogonal group** $\mathsf{SO}(3)$ is the set of all orthogonal 3×3 matrices over $\mathbb{R}$, with unit determinant, i.e.,

$$\mathsf{SO}(3) := \{g \in \mathsf{O}(3) \,|\, \det g = 1\}.$$

In this chapter we collect some well-known and standard results for $\mathsf{SO}(3)$, along with simple and direct proofs. The references used include [30], [2], [21], [23], [8], [15] and [28].

2.1 Geometric, algebraic and topological properties

PROPOSITION 2.1.1. $\mathsf{SO}(3)$ *is a matrix Lie group.*

PROOF: STEP 1. $\mathsf{SO}(3)$ is a subgroup of $\mathsf{GL}(3, \mathbb{R})$ under matrix multiplication, i.e., the closure property is satisfied and for each element in $\mathsf{SO}(3)$ a multiplicative inverse can be found.

STEP 2. Now, we want to prove that $\mathsf{SO}(3) = \mathsf{O}(3) \bigcap \mathsf{SL}(3, \mathbb{R})$ is closed in $\mathsf{GL}(3, \mathbb{R})$.

First, we show that $\mathsf{O}(3)$ is closed in $\mathsf{GL}(3, \mathbb{R})$. We define the mapping

$$F : \mathbb{R}^{3\times 3} \to \mathbb{R}^{3\times 3}, \quad F(X) := X^\top X.$$

This mapping F is continuous, since for each ij, the $\mathbb{R}$-valued function

$$F_{ij}(X) = (X^\top X)_{ij}$$

is continuous, as it is a polynomial in entries of X. The single element set $\{Id\} \subset \mathbb{R}^{3\times 3}$ is closed, so $F^{-1}(\{Id\}) = \mathsf{O}(3)$ is closed in $\mathbb{R}^{3\times 3}$ (the inverse image of a closed set under a continuous map is closed). Therefore, $\mathsf{O}(3)$ is closed in $\mathsf{GL}(3, \mathbb{R})$.

To show that $\mathsf{SL}(3, \mathbb{R})$ is closed in $\mathsf{GL}(3, \mathbb{R})$, consider the function

$$\det : \mathbb{R}^{3\times 3} \to \mathbb{R}, \quad Y \mapsto \det Y.$$

This is a continuous function because the $\det Y$ is a polynomial in entries of Y. The single element set $\{0\} \subset \mathbb{R}$ is closed, so $\det^{-1}(\{0\}) = \mathsf{SL}(3, \mathbb{R})$ is closed in $\mathbb{R}^{3\times 3}$ (the inverse image of a closed set under a continuous map is closed). Therefore, $\mathsf{SL}(3, \mathbb{R})$ is closed in $\mathsf{GL}(3, \mathbb{R})$.

Thus, $\mathsf{SO}(3) = \mathsf{O}(3) \bigcap \mathsf{SL}(3, \mathbb{R})$ is closed in $\mathsf{GL}(3, \mathbb{R})$ (the intersection of two closed sets is closed), and from STEP 1 and STEP 2 is a matrix Lie group (Def. 1.1.2). □

PROPOSITION 2.1.2. *Every element of* $\mathsf{SO}(3)$ *is a rotation of* $\mathbb{R}^3$ *about an axis through the origin.*

PROOF: Let $g \in \mathsf{SO}(3)$. Then the characteristic polynomial $\mathrm{char}_\lambda(g) = \det(\lambda Id - g)$ is a cubic and must have at least one real root i.e., g has a real eigenvalue. The other two eigenvalues are complex conjugate. They have absolute value 1, since if λ is an eigenvalue of g then $gv = \lambda v$, for some eigenvector v corresponding to λ. So we have $\|gv\| = \|\lambda v\| = |\lambda|\,\|v\|$ (homogeneity) and $\|gv\| = \|v\|$ ($\sqrt{gv \cdot gv} = \sqrt{v \cdot v}$). Then equating the two, $|\lambda|\,\|v\| = \|v\|$ giving $|\lambda| = 1$. The complex conjugate eigenvalues can be written as $e^{i\theta}$ and $e^{-i\theta}$, $\theta \in \mathbb{R}$. Since the product of the eigenvalues equals the determinant of the matrix, we see that $+1$ is an eigenvalue of g.

Let w be the corresponding eigenvector i.e., $gw = w$. The line through the origin determined by w is invariant under g. Since g preserves all right angles it sends the plane orthogonal to w, which contains the origin, into itself.

Construct an orthonormal basis for $\mathbb{R}^3$ which has the unit vector $\frac{1}{\|w\|} \cdot w$ as the first member. The matrix of $x \mapsto gx$ with respect to this new basis will be of the form

$$g = \begin{bmatrix} 1 & 0 & 0 \\ 0 & g_1 & g_3 \\ 0 & g_2 & g_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & r \end{bmatrix}.$$

Since $r \in \mathsf{SO}(2)$, g is a rotation about the axis determined by w. □

$\mathsf{SO}(3)$ is also called the **rotation group**.

Proposition 2.1.3. $\mathsf{SO}(3)$ *is non-Abelian.*

Proof: Consider

$$g_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \quad \text{and} \quad g_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \in \mathsf{SO}(3).$$

Since $g_1\, g_2 \neq g_2\, g_1$, the result follows. □

Proposition 2.1.4. $\mathsf{SO}(3)$ *is compact.*

Proof: $\mathsf{SO}(3)$ is closed in $\mathsf{GL}(3, \mathbb{R})$ (Prop. 2.1.1). Let $g \in \mathsf{SO}(3)$. Then

$$\|g\| = \sqrt{\operatorname{tr}(g^\top g)} = \sqrt{\operatorname{tr}(Id)} = \sqrt{3},$$

therefore, $\mathsf{SO}(3)$ is bounded. Since it is closed and bounded, $\mathsf{SO}(3)$ is compact. □

Proposition 2.1.5. $\mathsf{SO}(3)$ *is path-connected.*

Proof: Let $g_1 \in \mathsf{SO}(3)$. It suffices to find a path in $\mathsf{SO}(3)$ from Id to g_1, because if there are paths from Id to g_1 and g_2 then there is a path from g_1 to g_2. This amounts to finding a continuous motion taking the basic vectors e_1, e_2, e_3 to their final positions g_1e_1, g_1e_2, g_1e_3 (the columns of g_1).

The vectors e_1 and ge_1 (if distinct) define a plane $\mathcal{P}$, so, by the obvious path-connectedness of the circle, we can move e_1 continuously to the position ge_1 be a rotation R of $\mathcal{P}$. It then suffices to continuously move Re_2, Re_3 to g_1e_2, g_1e_3, respectively, keeping ge_1 fixed. Notice that

- Re_2, Re_3 are all orthogonal to $Re_1 = g_1e_1$, because e_2, e_3 are orthogonal to e_1 and R preserves angles.
- g_1e_2, g_1e_3 are all orthogonal to g_1e_1, because e_2, e_3 are all orthogonal to e_1 and g_1 preserves angles.

Thus the required motion can take place in the plane of vectors orthogonal to g_1e_1 (where it exists since the circle is path-connected).

Performing the two motions in succession – taking e_1 to g_1e_1 and then Re_2, Re_3 to g_1e_2, g_2e_3 – gives a path from Id to g_1 in $\mathsf{SO}(3)$. □

2.2 The Lie algebra and properties

Proposition 2.2.1. *The Lie algebra of* $\mathsf{SO}(3)$ *consists of all* 3×3 *skew-symmetric matrices.*

PROOF: We begin by showing that the Lie algebra of the orthogonal group $\mathsf{O}(3)$ consists of all 3×3 skew-symmetric matrices, denoted by $\mathsf{Sk}-\mathsf{sym}(3)$.

($\Rightarrow$) Given the curve $\alpha : (a,b) \subset \mathbb{R} \to \mathsf{O}(3)$ with $\alpha(0) = Id$,

$$\frac{d}{dt}(\alpha(t)^\top \alpha(t)) = 0 \quad \Leftrightarrow \quad \dot\alpha(t)^\top \alpha(t) + \alpha(t)^\top \dot\alpha(t) = 0 \quad \Rightarrow \quad \dot\alpha(0)^\top + \dot\alpha(0) = 0.$$

$\dot\alpha(0) \in \mathbb{R}^{3\times 3}$ is then skew-symmetric, and $\mathfrak{o}(3) \subseteq \mathsf{Sk}-\mathsf{sym}(3)$.

($\Leftarrow$) Let $X \in \mathsf{Sk}-\mathsf{sym}(3)$, and consider the curve

$$\alpha : (a,b) \subset \mathbb{R} \to \mathsf{GL}(3,\mathbb{R}), \quad t \mapsto \exp(tX),$$

then

$$\alpha(t)^\top \alpha(t) = \exp(tX)^\top \exp(tX) = \exp(tX^\top)\exp(tX) = \exp(-tX)\exp(tX) = Id.$$

Therefore, α can be viewed as the curve in $\mathsf{O}(3)$, and since $\dot\alpha(0) = X$, we have $\mathsf{Sk}-\mathsf{sym}(3) \subseteq \mathfrak{o}(3)$. Since $X \in \mathsf{Sk}-\mathsf{sym}(3)$, $\operatorname{tr} X = 0$, and Liouville's formula gives

$$\det \alpha(t) = \det \exp(tX) = e^{\operatorname{tr}(tX)} = e^{t \operatorname{tr}(X)} = e^{t\,0} = 1.$$

Hence $\alpha : (a,b) \to \mathsf{SO}(3)$. This shows that $\mathfrak{so}(3) = \mathfrak{o}(3) = \mathsf{Sk}-\mathsf{sym}(3)$. □

The matrices

$$E_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad E_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

form the standard basis for $\mathfrak{so}(3)$.

COROLLARY 2.2.1. *The dimension of* $\mathsf{SO}(3)$ *is* 3.

DEFINITION 2.2.1. We define the **hat mapping** $\hat{\ }: \mathfrak{so}(3) \to \mathbb{R}^3$ by

$$X = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \mapsto \widehat{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

PROPOSITION 2.2.2. *The hat mapping from* $\mathfrak{so}(3)$*, equipped with the matrix commutator, to* $\mathbb{R}^3$*, equipped with the cross product, is an isomorphism of Lie algebras.*

PROOF: We must show that for all X, $Y \in \mathfrak{so}(3)$, $\widehat{[X,Y]} = [\widehat{X},\widehat{Y}]$.

$$\widehat{[X,Y]} = \begin{bmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{bmatrix} = \widehat{X} \times \widehat{Y} = [\widehat{X},\widehat{Y}]. \quad \text{(Def. 1.1.8 (Ex. no. 2 and no. 1), Def. 2.2.1)}$$

The hat mapping is also linear and bijective. □

REMARK: We can identify $\mathfrak{so}(3)$ with the Lie algebra $\mathbb{R}^3$

PROPOSITION 2.2.3. SO(3) *is simple.*

PROOF: We first show that $\mathfrak{so}(3)$ is simple. Consider the commutator of the standard basis:

$$[E_1,E_2] = -[E_2,E_1] = E_3, \quad [E_2,E_3] = -[E_3,E_2] = E_1 \quad \text{and} \quad [E_3,E_1] = -[E_1,E_3] = E_2.$$

If $\mathfrak{I} \neq 0$ is an ideal of $\mathfrak{so}(3)$, then let $X = aE_1 + bE_2 + cE_3$ be an arbitrary non-zero element of $\mathfrak{I}$. Consider the following:

$$\begin{aligned} &\mathrm{ad}_{E_1}X = [E_1,X] = bE_3 - cE_2, && \text{then } \mathrm{ad}_{E_2}(\mathrm{ad}_{E_1}X) = [E_2, bE_3 - cE_2] = bE_1 && \in \mathfrak{I} \\ &\mathrm{ad}_{E_2}X = [E_2,X] = -aE_3 + cE_1, && \text{then } \mathrm{ad}_{E_1}(\mathrm{ad}_{E_2}X) = [E_1, -aE_3 + cE_1] = aE_2 && \in \mathfrak{I}. \end{aligned}$$

If $a \neq 0$ and $b = 0$, then $E_2 \in \mathfrak{I}$. If $a \neq 0$ and $b \neq 0$, then E_1, $E_2 \in \mathfrak{I}$. If $a = 0$ and $b \neq 0$, then $E_1 \in \mathfrak{I}$. If $a = 0$ and $b = 0$, then $E_3 \in \mathfrak{I}$. Therefore, $\mathfrak{I} = \mathfrak{so}(3)$. It follows that $\mathfrak{so}(3)$ is simple, and hence, SO(3) is simple (Prop. 1.1.6). □

2.3 Maximal tori and properties

PROPOSITION 2.3.1. *The standard maximal torus of* SO(3),

$$\mathbb{T} = \left\{ \begin{bmatrix} cos\,\theta & -sin\,\theta & 0 \\ sin\,\theta & cos\,\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \theta \in [0,2\pi) \right\},$$

is a maximal torus.

PROOF: STEP 1. We begin by showing that $\mathbb{T}$ is isomorphic to a torus. Consider the mapping

$$F : \mathsf{U}(1) \to \mathbb{T}, \quad e^{i\theta} \mapsto a := \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Since F is a bijection and satisfies the homomorphism property, F is an isomorphism.

STEP 2. We now show that $\mathbb{T}$ is a maximal torus. We will justify this by proving that any element $g \in \mathsf{SO}(3)$, which commutes with all elements of $\mathbb{T}$, i.e., $ga = ag$, must lie in $\mathbb{T}$, since any element of an alleged higher-dimensional torus would commute with all elements of $\mathbb{T}$.

For $\theta \neq 0$, the only vectors fixed in $\mathbb{R}^3$ by a are multiples of e_3. Now, since

$$a(ge_3) = (ag)e_3 = (ga)e_3 = g(ae_3) = ge_3,$$

a fixes ge_3, and the third column of g must have the form $\begin{bmatrix} 0 & 0 & \pm 1 \end{bmatrix}^\top$.

The other case to consider would be to allow $\theta = 0$. This means that $a = Id$, and it fixes ge_1 and ge_2. Then the first and second columns of g each have the form $\begin{bmatrix} a & b & 0 \end{bmatrix}^\top$, because otherwise

$$ge_1 \bullet ge_3 \neq 0 \quad \text{and} \quad ge_2 \bullet ge_3 \neq 0.$$

Thus, $g = \operatorname{diag}(g_1, \pm 1)$, for some $g_1 \in \mathsf{O}(2)$, i.e.,

$$g_1 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \ \det = 1 \quad \text{or} \quad g_1 = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}, \ \det = -1.$$

Notice that $g_1 \in \mathsf{SO}(2)$, because if $g_1 \in \mathsf{O}(2) \setminus \mathsf{SO}(2)$, then g would not commute with a. This will force the last argument of g to be $+1$, and the result follows. □

PROPOSITION 2.3.2. *If* $\mathbb{T}$ *is a maximal torus of* $\mathsf{SO}(3)$, *then, for any* $g \in \mathsf{SO}(3)$, $g\mathbb{T}g^{-1}$ *is also a maximal torus of* $\mathsf{SO}(3)$.

PROOF: STEP 1. We begin by showing that $g\mathbb{T}g^{-1}$ is a torus. The conjugation map C_g is an isomorphism (Def. 1.1.13), so the image of $\mathbb{T}$ under C_g, $g\mathbb{T}g^{-1}$, is isomorphic to $\mathbb{T}$ and, therefore, a torus.

STEP 2. We now show that $g\mathbb{T}g^{-1}$ is a maximal torus. If $\widetilde{\mathbb{T}} \subset \mathsf{SO}(3)$ were a higher dimensional torus containing $g\mathbb{T}g^{-1}$, then ${C_g}^{-1}(\widetilde{\mathbb{T}})$ would be a higher dimensional torus containing $\mathbb{T}$. However, this is impossible since $\mathbb{T}$ is maximal, so $g\mathbb{T}g^{-1}$ is a maximal torus. □

PROPOSITION 2.3.3. *The exponential map* $\exp : \mathfrak{so}(3) \to \mathsf{SO}(3)$ *is surjective.*

PROOF: STEP 1. Let $\mathbb{T} \subset \mathsf{SO}(3)$ be the standard maximal torus of $\mathsf{SO}(3)$ (Prop. 2.3.1) and $\tau \subset \mathfrak{so}(3)$ be the Lie algebra of $\mathbb{T}$. The restriction

$$\exp : \tau \to \mathbb{T}, \quad \begin{bmatrix} 0 & \theta & 0 \\ -\theta & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mapsto \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{for } \theta \in \mathbb{R}.$$

is surjective, therefore, every element of $\mathbb{T}$ can be expressed in the form e^X, for some $X \in \tau$.

STEP 2. For any $g \in \mathsf{SO}(3)$, let $g\tau g^{-1} \subset \mathfrak{so}(3)$ be the Lie algebra of the maximal torus $g\mathbb{T}g^{-1} \subset \mathsf{SO}(3)$ (Prop. 2.3.2). We want to show that

$$\exp : g\tau g^{-1} \to g\mathbb{T}g^{-1}, \quad \text{where } g\mathbb{T}g^{-1} := \{gag^{-1} \mid \text{for all } a \in \mathbb{T}\}$$

is surjective. For all $a \in \mathbb{T}$, any $g \in \mathsf{SO}(3)$ and some $X \in \tau$,

$$gag^{-1} = ge^X g^{-1} = e^{gXg^{-1}}, \quad (\text{STEP 1, Prop. 1.1.4 (no. 3)})$$

and this shows surjectivity.

STEP 3. Since the standard maximal torus is not a normal subgroup (Prop. 2.2.3, Def. 1.1.18), it differs from some of its conjugates. Every element of $\mathsf{SO}(3)$ is contained in $g\mathbb{T}g^{-1}$ for some $g \in \mathsf{G}$. The result follows from STEP 2. □

REMARK: This result can be used to confirm that $\mathsf{SO}(3)$ is path-connected (Prop. 2.1.5).

PROPOSITION 2.3.4. *(Rodrigues' formula) For $X \in \mathfrak{so}(3)$,*

$$\exp(X) = Id + \frac{\sin \|\widehat{X}\|}{\|\widehat{X}\|} X + \frac{1}{2}\left(\frac{\sin \frac{\|\widehat{X}\|}{2}}{\frac{\|\widehat{X}\|}{2}}\right)^2 X^2.$$

PROOF: By Definition 2.2.1,

$$X^2 \widehat{Y} = (\widehat{X} \bullet \widehat{Y})\widehat{X} - \|\widehat{X}\|^2 \widehat{Y}.$$

Consequently, we have the recurrence relation

$$X^3 = -\|\widehat{X}\|^2 X, \quad X^4 = -\|\widehat{X}\|^2 X^2, \quad X^5 = \|\widehat{X}\|^4 X, \quad X^6 = \|\widehat{X}\|^4 X^2, \quad \ldots . \tag{2.1}$$

Therefore,

$$\begin{aligned}
\exp(X) &= Id + X + \frac{1}{2!}X^2 + \frac{1}{3!}X^3 + \frac{1}{4!}X^4 + \frac{1}{5!}X^5 + \frac{1}{6!}X^6 + \ldots && (\text{Def. 1.1.10})\\
&= Id + X + \frac{1}{2!}X^2 - \frac{1}{3!}\|\widehat{X}\|^2 X - \frac{1}{4!}\|\widehat{X}\|^2 X^2 + \frac{1}{5!}\|\widehat{X}\|^4 X + \frac{1}{6!}\|\widehat{X}\|^4 X^2 + \ldots && (2.1)\\
&= Id + \left(Id - \frac{\|\widehat{X}\|^2}{3!} + \frac{\|\widehat{X}\|^4}{5!} - \cdots + \frac{(-1)^n\|\widehat{X}\|^{2n}}{(2n+1)!} + \ldots\right) X\\
&+ \left(\frac{1}{2!} - \frac{\|\widehat{X}\|^2}{4!} + \frac{\|\widehat{X}\|^4}{6!} - \ldots + \frac{(-1)^{n+1}\|\widehat{X}\|^{2n-2}}{(2n)!} + \ldots\right) X^2\\
&= Id + \frac{\sin \|\widehat{X}\|}{\|\widehat{X}\|} X + \frac{1 - \cos \|\widehat{X}\|}{\|\widehat{X}\|^2} X^2. && (\text{Taylor series})
\end{aligned}$$

We obtain the result from the identity $2\left(\sin\frac{\|\widehat{X}\|}{2}\right)^2 = 1-\cos\|\widehat{X}\|$. □

2.4 The shape of SO(3) and local coordinates

PROPOSITION 2.4.1. *$\mathsf{SO}(3)$ is a smooth submanifold of $\mathbb{R}^{3\times 3}$.*

PROOF: STEP 1. $\mathsf{GL}(3,\mathbb{R})$ is an open subset of $\mathbb{R}^{3\times 3}$. We define the mapping

$$F : \mathsf{GL}(3,\mathbb{R}) \to \mathbb{R}^6, \quad F(g) := g^\top g - Id.$$

(Since $(g^\top g)^\top = g^\top g$, $g^\top g$ is a symmetric matrix. The symmetric matrices form a vector space of dimension 6, therefore, are isomorphic to $\mathbb{R}^6$.)

To each $g \in \mathsf{GL}(3,\mathbb{R})$, we associate the linear mapping

$$dF(g) : \mathbb{R}^{3\times 3} \to \mathbb{R}^6, \quad dF(g)\cdot X = gX^\top + Xg^\top.$$

In particular, for $g = Id$, we have the map

$$dF\cdot X : \mathbb{R}^{3\times 3} \to \mathbb{R}^6, \quad X \mapsto X^\top + X.$$

Thus, F is a smooth map of constant rank. Further, this map is onto and F is a smooth submersion (Def. 1.1.24). By Proposition 1.1.8, $\mathsf{O}(3)$ is a smooth submanifold of $\mathbb{R}^{3\times 3}$.

STEP 2. For $g \in \mathsf{O}(3)$,

$$\det Id = \det(g^\top g) = \det g^\top \cdot \det g = (\det g)^2.$$

Thus $\det g = \pm 1$. Then

$$\mathsf{O}(3) = \mathsf{O}^+(3) \bigcup \mathsf{O}^-(3),$$

where

$$\mathsf{O}^+(3) := \{g \in \mathsf{O}(3) \mid \det g = 1\} \quad \text{and} \quad \mathsf{O}^-(3) := \{g \in \mathsf{O}(3) \mid \det g = -1\}.$$

Each component of $\mathsf{O}(3)$ is a smooth submanifold of $\mathbb{R}^{3\times 3}$ and hence, the path-connected component $\mathsf{O}^+(3) = \mathsf{SO}(3)$ is a smooth submanifold of $\mathbb{R}^{3\times 3}$. □

Let α and β be arbitrary complex numbers. Then the elements of $\mathsf{SU}(2)$ (Def. 1.1.19) are of the form

$$\begin{bmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{bmatrix}, \quad \text{satisfying } |\alpha|^2 + |\beta|^2 = 1. \tag{2.2}$$

$\mathsf{SU}(2)$ is a smooth submanifold of $\mathbb{C}^{2\times 2} = \mathbb{R}^8$.

The 3-sphere

$$\mathbb{S}^3 := \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\}$$

is a smooth submanifold of $\mathbb{R}^4$.

PROPOSITION 2.4.2. *$\mathsf{SU}(2)$ is diffeomorphic to $\mathbb{S}^3$.*

PROOF: Consider the map

$$\begin{aligned} F : \mathbb{S}^3 &\to \mathsf{SU}(2) \\ x = (x_1, x_2, x_3, x_4) &\mapsto x' = \begin{bmatrix} x_1 + ix_4 & -x_3 + ix_2 \\ x_3 + ix_2 & x_1 - ix_4 \end{bmatrix}. \end{aligned}$$

Since F is a smooth bijection, and F^{-1} is also smooth, F is a diffeomorphism (Def. 1.1.22). □

PROPOSITION 2.4.3. *Let* Ad *be the adjoint representation on* $\mathsf{SU}(2)$*. Then the range is* $\mathsf{SO}(3)$*, i.e.,*

$$\mathrm{Ad} : \mathsf{SU}(2) \to \mathsf{SO}(3).$$

PROOF: STEP 1. From the definition of the adjoint representation (Def. 1.1.13), we have the mapping

$$\mathrm{Ad} : \mathsf{SU}(2) \to \mathsf{GL}(\mathfrak{su}(2)), \quad a \mapsto \mathrm{Ad}_a.$$

STEP 2. The Lie algebra $\mathfrak{su}(2)$ is a 3-dimensional real vector space given by

$$\mathfrak{su}(2) = \left\{ \begin{bmatrix} iy_3 & -y_2 + iy_1 \\ y_2 + iy_1 & -iy_3 \end{bmatrix} \in \mathbb{C}^{2\times 2} \mid y_1, y_2, y_3 \in \mathbb{R} \right\}.$$

The matrices

$$E_1' = \frac{1}{2}\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \quad E_2' = \frac{1}{2}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad E_3' = \frac{1}{2}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

form the standard basis for $\mathfrak{su}(2)$. The map

$$\begin{aligned} \phi : \mathbb{R}^3 &\to \mathfrak{su}(2) \\ x = (x_1, x_2, x_3) &\mapsto \frac{1}{2}\tilde{x} = \frac{1}{2}\begin{bmatrix} ix_3 & -x_2 + ix_1 \\ x_2 + ix_1 & -ix_3 \end{bmatrix} \end{aligned} \tag{2.3}$$

is an isomorphism of Lie algebras (Def. 1.1.12), so

$$\mathrm{Ad} : \mathsf{SU}(2) \to \mathsf{GL}(\mathfrak{su}(2)) = \mathsf{GL}(\mathbb{R}^3).$$

STEP 3. We can identify the inner-product on $\mathfrak{su}(2)$ by

$$(X \mid Y) = -2\,\mathrm{tr}\,(XY) \quad \text{for all } X,\, Y \in \mathfrak{su}(2). \tag{2.4}$$

For $a \in \mathsf{SU}(2)$, Ad_a is an orthogonal transformation with respect to the inner-product, since, for all $X,\, Y \in \mathfrak{su}(2)$,

$$\begin{aligned}
(\mathrm{Ad}_a X \mid \mathrm{Ad}_a Y) &= (aXa^{-1} \mid aYa^{-1}) && \text{(Def. 1.1.13)}\\
&= -2\,\mathrm{tr}\,(aXa^{-1}aYa^{-1}) && (2.4)\\
&= -2\,\mathrm{tr}\,(Ya^{-1}aX) && (\mathrm{tr}\,(AB) = \mathrm{tr}\,(BA))\\
&= -2\,\mathrm{tr}\,(XY) = (X \mid Y), && (\mathrm{tr}\,(AB) = \mathrm{tr}\,(BA),\ (2.4))
\end{aligned}$$

therefore,

$$\mathrm{Ad} : \mathsf{SU}(2) \to \mathsf{O}(3).$$

STEP 4. Since $\mathsf{SU}(2)$ is diffeomorphic to $\mathbb{S}^3$ (Prop. 2.4.2), and $\mathbb{S}^3$ is path-connected, $\mathsf{SU}(2)$ is path-connected. The Ad is smooth (Def. 1.1.13) and therefore continuous. A path-connected set under a continuous function is path-connected, and the path-connected component of $\mathsf{O}(3)$ is $\mathsf{SO}(3)$. The result follows. □

Next, we prove that

$$\mathrm{Ad} : \mathsf{SU}(2) \to \mathsf{SO}(3)$$

is a double cover (Def. 1.1.25).

PROPOSITION 2.4.4. *The mapping*

$$\mathrm{Ad} : \mathsf{SU}(2) \to \mathsf{SO}(3), \quad a \mapsto \mathrm{Ad}_a$$

is surjective.

PROOF: Let $a \in \mathsf{SU}(2)$ (as seen in (2.2)) and $y_1,\, y_2,\, y_3 \in \mathbb{R}$. Then a can be expressed in the form

$$a = \begin{bmatrix} \cos\theta & 0 \\ 0 & \cos\theta \end{bmatrix} + \begin{bmatrix} iy_3 & -y_2 + iy_1 \\ y_2 + iy_1 & -iy_3 \end{bmatrix} = \cos\theta\,(Id) + S, \quad \text{where } S \in \mathfrak{su}(2). \tag{2.5}$$

Since $\mathrm{Re}(\alpha) = \cos\theta$, for $\theta \in [0, \pi]$, $\sin\theta \geq 0$.

Now,

$$\begin{aligned}
S^2 &= -(y_1^2+y_2^2+y_3^2)\,Id \\
&= -(|y_2+iy_1|^2+y_3^2)\,Id && (|y_2+iy_1|^2 = y_1^2+y_2^2) \\
&= -(1-\cos^2\theta)\,Id && ((2.2)\ |y_2+iy_1|^2 = 1-|\cos\theta+iy_3|^2) \\
&= -\sin^2\theta\,Id && (\sin^2\theta+\cos^2\theta=1)
\end{aligned} \tag{2.6}$$

Let $X \in \mathfrak{su}(2)$ such that $(S \,|\, X) = 0$. Then

$$\begin{aligned}
SXS &= S\left(-\frac{1}{4}(X \,|\, S)Id + \frac{1}{2}[X,S]\right) && \text{(Lengthy yet basic calculations)} \\
&= \frac{1}{2}S[X,S] && ((X \,|\, S)=0) \\
&= \frac{1}{2}S(XS-SX) && \text{(Def. 1.1.8 (Ex. no. 2))} \\
&= \frac{1}{2}(SXS-SSX) \\
&= \frac{1}{2}SXS+\frac{1}{2}\sin^2\theta X && (2.6) \\
&= \sin^2\theta\,X
\end{aligned} \tag{2.7}$$

Next,

$$\begin{aligned}
\mathrm{Ad}_a X &= (\cos\theta\,Id+S)X(\cos\theta\,Id-S) && (\text{Def. 1.1.13, Def. 1.1.19, } a^{-1}=a^*=\cos\theta\,Id-S) \\
&= (\cos\theta\,X+SX)(\cos\theta\,Id-S) \\
&= \cos^2\theta\,X+\cos\theta\,SX-\cos\theta\,XS-SXS \\
&= \cos^2\theta\,X+\cos\theta\,[S,X]-SXS \\
&= (\cos^2\theta-\sin^2\theta)X+\cos\theta\,[S,X] && (2.7) \\
&= \cos 2\theta\,X+\cos\theta\,[S,X] && (\cos 2\theta=\cos^2\theta-\sin^2\theta) \\
&= \cos 2\theta\,X+\sqrt{2}\cos\theta\sin\theta\,[\bar{S},X] && \left(\text{where } \bar{S}=\frac{1}{|S|}S=\frac{1}{\sqrt{2}\sin\theta}S \text{ is of unit length}\right) \\
&= \cos 2\theta\,X+2\sqrt{2}\sin 2\theta\,(\bar{S}\times X). && (\sin 2\theta=2\cos\theta\sin\theta,\ (2.3), \text{Def. 1.1.8 (Ex. no. 1)})
\end{aligned} \tag{2.8}$$

Noting that X and $\bar{S}\times X$ are orthogonal to S, we see that the effect of Ad_a on X is to rotate it in the plane orthogonal to S (and spanned by X and $\bar{S}\times X$) through the angle 2θ. We can now see that every element of $\mathsf{SO}(3)$ has the form Ad_a for some $a \in \mathsf{SU}(2)$. This follows from the fact that every element of $\mathsf{SO}(3)$ is a rotation of $\mathbb{R}^3$ about some axis w through the origin (Prop. 2.1.2). Now we can take $a=\cos(\frac{\varphi}{2})\,Id+S$, where $S\in\mathfrak{su}(2)$ is chosen to correspond to a multiple of w

and $(S \mid S) = 2\sin^2(\frac{\varphi}{2})$ ((2.4), (2.6)). If we choose $-\varphi$ in place of φ, we obtain $-a$ in place of a. □

PROPOSITION 2.4.5. *Let* $\mathrm{Ad} : \mathsf{SU}(2) \to \mathsf{SO}(3)$, *then*

$$Ker(\mathrm{Ad}) = \{\pm\, Id\}.$$

PROOF: Consider $\mathrm{Ad} : \mathsf{SU}(2) \to \mathsf{SO}(3)$ and let $a \in \mathrm{Ker}\,(\mathrm{Ad})$. From the definition of the kernel,

$$\mathrm{Ad}_a X = aXa^{-1} = X, \quad \forall\, X \in \mathfrak{su}(2). \tag{2.9}$$

Then from (2.8),

$$\mathrm{Ad}_a X = \cos 2\theta\, X + \sin 2\theta\, (\bar{S} \times X) = X, \quad \bar{S} \in \mathfrak{su}(2) \text{ such that } (\bar{S} \mid X) = 0.$$

Therefore, $\theta = 0$ or $\theta = \pi$. Substituting into (2.5) gives $a = Id + S$ and $a = -Id + S$, respectively. However, we need to show that $a = \{\pm\, Id\}$. For $\theta = 0$, (2.9) becomes

$$(Id + S)X = X(Id + S) \quad \Leftrightarrow \quad SX = XS.$$

This cannot be true since $\mathfrak{su}(2)$ is non-Abelian (similarly for $\theta = \pi$). This statement will only hold true when $a = \{\pm Id\}$. □

PROPOSITION 2.4.6. $\mathrm{Ad} : \mathsf{SU}(2) \to \mathsf{SO}(3)$ *is a local diffeomorphism.*

PROOF: To prove that $\mathrm{Ad} : \mathsf{SU}(2) \to \mathsf{SO}(3)$ is a local diffeomorphism at the identity, we will need to show that

$$\mathrm{ad} : \mathfrak{su}(2) \to \mathfrak{so}(3) \quad \text{(Def. 1.1.13)}$$

is an invertible linear map (Thm. 1.1.1). It will suffice to prove that ad maps a basis of $\mathfrak{su}(2)$ to a basis of $\mathfrak{so}(3)$. The matrices

$$E_1' = \frac{1}{2}\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \quad E_2' = \frac{1}{2}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad E_3' = \frac{1}{2}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix},$$

form the standard basis for $\mathfrak{su}(2)$. Then starting with the curve

$$\alpha : (a, b) \subset \mathbb{R} \to \mathsf{SU}(2), \quad \alpha(t) := \exp(tE_1'),$$

we can construct another curve

$$\beta : (c, d) \subset \mathbb{R} \to \mathsf{SO}(3), \quad \beta(t) := \mathrm{Ad}_{\alpha(t)}.$$

We can differentiate β at $t = 0$ to obtain an element of $\mathfrak{so}(3)$. Now, for any $X \in \mathfrak{su}(2)$,

$$\begin{aligned}
\frac{d}{dt}\,\beta(t)\big|_{t=0} &= \frac{d}{dt}\mathrm{Ad}_{\alpha(t)}\,X\big|_{t=0} \\
&= \frac{d}{dt}\,(\exp(tE_1')X\exp(-tE_1'))\Big|_{t=0} && \text{(Def. 1.1.13, Prop. 1.1.4 (no. 1))} \\
&= E_1'X - XE_1'. && \text{(the product rule, Prop. 1.1.4 (no. 2))} \\
&= [E_1', X]. && \text{(Def. 1.1.8 (Ex. no. 2))}
\end{aligned}$$

Then

$$[E_1', E_1'] = 0, \quad [E_1', E_2'] = E_3' \quad \text{and} \quad [E_1', E_3'] = -E_2'.$$

Therefore, the matrix E_1' acting on $\mathfrak{su}(2)$ relative to the basis $\{E_1', E_2', E_3'\}$ is E_1. Now, repeat this argument with E_2' and E_3', and we verify that $\{\mathrm{ad}(E_1'), \mathrm{ad}(E_2'), \mathrm{ad}(E_3')\}$ is the standard basis $\{E_1, E_2, E_3\}$ of $\mathfrak{so}(3)$. The corresponding derivative map is

$$\begin{aligned}
\mathrm{ad} : \mathfrak{su}(2) &\rightarrow \mathfrak{so}(3) \\
x_1E_1' + x_2E_2' + x_3E_3' &\mapsto x_1E_1 + x_2E_2 + x_3E_3,
\end{aligned}$$

and the matrix of this linear map is the identity, which is invertible. The result can then be extended to every $a \in \mathsf{SU}(2)$, since $\mathrm{Ad} : \mathsf{SU}(2) \rightarrow \mathsf{SO}(3)$ is a smooth homomorphism. □

Implications of the double cover

The algebraic importance can be summarized as follows: we know that

$$\mathrm{Ad} : \mathsf{SU}(2) \rightarrow \mathsf{SO}(3)$$

is a surjective group homomorphism with $\mathrm{Ker}\,(\mathrm{Ad}) = \{\pm\, Id\}$, therefore,

$$\mathsf{SU}(2)/\{\pm\, Id\} \cong \mathsf{SO}(3) \quad \text{(Prop. 1.1.9)}.$$

The geometric importance of this has to do with the shape of $\mathsf{SO}(3)$. Since every line through the origin in $\mathbb{R}^4$ intersects the sphere $\mathbb{S}^3$ in a pair of antipodal points, one often identifies $\mathbb{RP}^3$ (Def. 1.1.26) with the set of antipodal pairs on $\mathbb{S}^3$. The identification $\mathsf{SU}(2)/\{\pm\, Id\} \cong \mathsf{SO}(3)$ associates each point of $\mathsf{SO}(3)$ with a pair of antipodal points on the sphere $\mathbb{S}^3 \cong \mathsf{SU}(2)$ (Prop. 2.4.2). This then provides a bijection between $\mathsf{SO}(3)$ and $\mathbb{RP}^3$, and we have $\mathsf{SO}(3)$ diffeomorphic to $\mathbb{RP}^3$.

Coordinates on SO(3)

There are various charts on SO(3) that set up rival coordinate systems (see e.g. [31]). In our case we prefer to use the Euler angles to describe a rotation. According to Euler's rotation theorem, any rotation may be described using three angles. If a rotation is written in terms of rotation matrices g_1, g_2 and g_3, then in general it may be written as $g = g_1 \, g_2 \, g_3$, with the three angles giving the rotation matrices termed Euler angles. There are several conventions for Euler angles, depending on the axes about which the rotations are carried out. We use the so-called 'x-y-z convention'. In this convention, the rotation is given by Euler angles (ϕ_1, ϕ_2, ϕ_3), where the first rotation is by an angle ϕ_1 about the z-axis, the second is by an angle ϕ_2 about the y-axis, and the third is by an angle ϕ_3 about the x-axis.

Chapter 3

Geometric Control on $\mathsf{SO}(3)$

In this chapter we consider the control-affine left-invariant control systems on $\mathsf{SO}(3)$, and restrict ourselves to four typical controllable left-invariant control systems. Examples of these are given, and descriptively named the stiff SF-system, attitude control of a rigid body and path planning on $\mathsf{SO}(3)$. As explained in chapter 1, the left-invariant optimal control problem on $\mathsf{SO}(3)$ can be lifted to $T^*\mathsf{SO}(3) = \mathsf{SO}(3) \times \mathfrak{so}(3)^*$. Here we show in detail how $\mathfrak{so}(3)^*$ can be identified with $\mathfrak{so}(3)$ via the Killing form, and $\mathfrak{so}(3)$ with $\mathbb{R}^3$ using the hat mapping. The Casimir function K on $\mathfrak{so}(3)^*$ is presented, and a proof is supplied to support the result. Lastly, an expression is given for the solution to the problem in terms of the Euler angles. The references used include [9], [10], [24] and [21].

3.1 Left-invariant control systems

The control-affine left-invariant control systems on $\mathsf{SO}(3)$, in classical notation, are given by

$$\dot{g} = g\,(X + u_1Y_1 + \ldots + u_\ell Y_\ell)\,, \quad g \in \mathsf{SO}(3),\ u = (u_1, \ldots, u_\ell) \in \mathbb{R}^\ell,\ 1 \leq \ell \leq 3.$$

This can be rewritten

$$\dot{g} = g\left((\lambda_1^0E_1 + \lambda_2^0E_2 + \lambda_3^0E_3) + u_1(\lambda_1^1E_1 + \lambda_2^1E_2 + \lambda_3^1E_3) + \ldots + u_\ell(\lambda_1^\ell E_1 + \lambda_2^\ell E_2 + \lambda_3^\ell E_3)\right),$$

and regrouped to

$$\dot{g} = g\left((\lambda_1^0 + u_1\lambda_1^1 + \ldots + u_\ell\lambda_1^\ell)E_1 + (\lambda_2^0 + u_1\lambda_2^1 + \ldots + u_\ell\lambda_2^\ell)E_2 + (\lambda_3^0 + u_1\lambda_3^1 + \ldots + u_\ell\lambda_3^\ell)E_3\right),$$

where rank $\begin{bmatrix} \lambda_1^1 & \dots & \lambda_1^\ell \\ \lambda_2^1 & \dots & \lambda_2^\ell \\ \lambda_3^1 & \dots & \lambda_3^\ell \end{bmatrix} = \ell$ (due to the linear independence of $Y_1, \dots, Y_\ell$).

For $\ell = 3$:

$$\dot{g} = g\left((\lambda_1^0 + u_1\lambda_1^1 + u_2\lambda_1^2 + u_3\lambda_1^3)E_1 + (\lambda_2^0 + u_1\lambda_2^1 + u_2\lambda_2^2 + u_3\lambda_2^3)E_2 + (\lambda_3^0 + u_1\lambda_3^1 + u_2\lambda_3^2 + u_3\lambda_3^3)E_3\right), \tag{3.1}$$

where rank $\begin{bmatrix} \lambda_1^1 & \lambda_1^2 & \lambda_1^3 \\ \lambda_2^1 & \lambda_2^2 & \lambda_2^3 \\ \lambda_3^1 & \lambda_3^2 & \lambda_3^3 \end{bmatrix} = 3$. If λ_i^0, for $i = 1, 2, 3$, equals zero or does not equal zero, then (3.1) can be rewritten

$$\dot{g} = g\left(\tilde{u}_1E_1 + \tilde{u}_2E_2 + \tilde{u}_3E_3\right).$$

For $\ell = 2$:

$$\dot{g} = g\left((\lambda_1^0 + u_1\lambda_1^1 + u_2\lambda_1^2)E_1 + (\lambda_2^0 + u_1\lambda_2^1 + u_2\lambda_2^2)E_2 + (\lambda_3^0 + u_1\lambda_3^1 + u_2\lambda_3^2)E_3\right), \tag{3.2}$$

where rank $\begin{bmatrix} \lambda_1^1 & \lambda_1^2 \\ \lambda_2^1 & \lambda_2^2 \\ \lambda_3^1 & \lambda_3^2 \end{bmatrix} = 2$. If $\lambda_3^1 = \lambda_3^2 = 0$ and $\lambda_3^0 \neq 0$, for example, then (3.2) can be rewritten

$$\dot{g} = g\left(\tilde{u}_1E_1 + \tilde{u}_2E_2 + \lambda_3^0E_3\right),$$

where we can normalize with respect to λ_3^0. However, if $\lambda_3^1 = \lambda_3^2 = 0$ and $\lambda_3^0 = 0$, for example, then (3.2) can be rewritten

$$\dot{g} = g\left(\tilde{u}_1E_1 + \tilde{u}_2E_2\right).$$

(In the instance where a row in the matrix is not entirely zero, this gives the same result as when $\ell = 3$.)

For $\ell = 1$:

$$\dot{g} = g\left((\lambda_1^0 + u_1\lambda_1^1)E_1 + (\lambda_2^0 + u_1\lambda_2^1)E_2 + (\lambda_3^0 + u_1\lambda_3^1)E_3\right), \tag{3.3}$$

where $\operatorname{rank}\begin{bmatrix}\lambda_1^1\\ \lambda_2^1\\ \lambda_3^1\end{bmatrix} = 1$. If $\lambda_3^1 = \lambda_2^1 = 0$, $\lambda_3^0 \neq 0$ and $\lambda_2^0 = 0$, for example, then (3.3) can be rewritten

$$\dot{g} = g\left(\tilde{u}_1 E_1 + \lambda_3^0 E_3\right),$$

where we can normalize with respect to λ_3^0. However, if $\lambda_3^1 = \lambda_2^1 = 0$ and $\lambda_3^0 = \lambda_2^0 = 0$, for example, then (3.3) can be rewritten

$$\dot{g} = g\left(\tilde{u}_1 E_1\right).$$

(Any remaining options are covered in one of the above cases.)

Using Proposition 1.2.2, we see that all systems composed of either two or three of the standard basis matrices are controllable, i.e., $\mathsf{SO}(3)$ is compact (Prop. 2.1.4), path-connected (Prop. 2.1.5) and the $\operatorname{Lie}(\Gamma) = \mathfrak{so}(3)$.

We consider and name the following typical controllable left-invariant control systems on $\mathsf{SO}(3)$:

$$\begin{aligned}
&\text{Type I:} && \dot{g} = g(E_3 + uE_1), && u \in \mathbb{R} && (3.4)\\
&\text{Type IIa:} && \dot{g} = g(u_1E_1 + u_2E_2), && u = (u_1, u_2) \in \mathbb{R}^2 && (3.5)\\
&\text{Type IIb:} && \dot{g} = g(E_3 + u_1E_1 + u_2E_2), && u = (u_1, u_2) \in \mathbb{R}^2 && (3.6)\\
&\text{Type III:} && \dot{g} = g(u_1E_1 + u_2E_2 + u_3E_3), && u = (u_1, u_2, u_3) \in \mathbb{R}^3. && (3.7)
\end{aligned}$$

We use the word 'typical' as any results for the remaining controllable left-invariant control systems on $\mathsf{SO}(3)$ can easily be deduced from these cases, with some simple elementary changes.

3.2 Examples

DEFINITION 3.2.1. *Let $\alpha : (a, b) \subset \mathbb{R} \to \mathbb{R}^3$ be a unit speed curve with nonzero curvature. Then the three vector fields*

$$\begin{aligned}
&T := \dot{\alpha} && \textit{(unit tangent vector field)}\\
&\left(\kappa : J \to \mathbb{R}, \quad t \mapsto \kappa(t) := \left\|\dot{T}(t)\right\|\right. && \textit{(curvature function))}\\
&N := \frac{1}{\kappa}\dot{T} && \textit{(principle normal vector field)}\\
&B := T \times N && \textit{(binormal vector field)}
\end{aligned}$$

on α are unit vector fields, which are mutually orthogonal at each point. The ordered set (T, N, B) *is called the **Serret-Frenet frame** on the unit speed curve α.*

THEOREM 3.2.1. *(The Serret-Frenet theorem) If $\alpha : (a,b) \subset \mathbb{R} \to \mathbb{R}^3$ is a unit speed curve with nonzero curvature, then*

$$\dot{T} = \kappa N, \quad \dot{N} = -\kappa T + \tau B \quad \textit{and} \quad \dot{B} = -\tau N,$$

where $\tau : (a,b) \subset \mathbb{R} \to \mathbb{R}^3$ is given by the third equation and is called the torsion function of α. The minus sign is traditional.

The stiff SF-system (see e.g. [10])

Let α denote any curve in $\mathbb{R}^3$ whose derivatives $\frac{d^i\alpha}{dt^i}$, $i = 1, 2, 3$, span a 3-dimensional vector space at each point along the curve. This means we choose a curve whose first three derivatives are linearly independent (this rules out lines and plane curves, as their second derivatives vanish). The Serret-Frenet frame (Def. 3.2.1) along the curve is described by an orthonormal matrix

$$g = \begin{bmatrix} T & N & B \end{bmatrix} \quad \in \mathsf{SO}(3).$$

The matrix relates the frame to the standard orthonormal frame (e_1, e_2, e_3) in $\mathbb{R}^3$ and further satisfies the following differential equation in $\mathsf{SO}(3)$ (Thm. 3.2.1):

$$\dot{g} = \begin{bmatrix} \dot{T} & \dot{N} & \dot{B} \end{bmatrix} = \begin{bmatrix} T & N & B \end{bmatrix} \begin{bmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{bmatrix}, \tag{3.8}$$

with the curvature and torsion playing the role of controls. Fixing $(u_3 =)\kappa = 1$ (a constant control) and allowing $(u_1 =)\tau = u$ to vary with time, we can rewrite (3.8) as

$$\dot{g} = g(E_3 + uE_1), \quad g \in \mathsf{SO}(3).$$

This is called the stiff SF-system and corresponds to our Type I system.

Attitude control of a rigid body (see e.g. [19], [26] and [27])

Consider the motion of a rigid body and denote by $g \in \mathsf{SO}(3)$ the orthogonal matrix whose columns are the directions of the body's principle axes at time t with respect to some reference coordinate system – this is called the body's attitude (or orientation) at time t. We denote by I_1, I_2, I_3 the moments of inertia, by ω_1, ω_2, ω_3 the angular velocities and by T_1, T_2, T_3 the exerted torques about the principle axes. Then the attitude evolution is

$$\dot{g} = g(\omega_1 E_1 + \omega_2 E_2 + \omega_3 E_3), \tag{3.9}$$

where the angular velocities satisfy Euler's equations:

$$\begin{aligned} I_1\dot{\omega}_1 &= (I_2 - I_3)\omega_2\omega_3 + T_1 \\ I_2\dot{\omega}_2 &= (I_3 - I_1)\omega_3\omega_1 + T_2 \\ I_3\dot{\omega}_3 &= (I_1 - I_2)\omega_1\omega_3 + T_3. \end{aligned}$$

The application we have in mind is the attitude control of a satellite (which is necessary to obtain and maintain desired orientations for instruments like antennae, telescopes, radiometers, or solar arrays). It is possible to maneuver this satellite between two fixed times t_0 and t_1, from a given initial attitude $g(t_0) = g_0$ and given angular velocites $\omega_i(t_0)$, $i = 1, 2, 3$, to a prescribed target attitude $g(t_1) = g_1$ and prescribed angular velocities $\omega_i(t_1)$. We treat the angular velocites (rather than the torques [26]) as control variables and hence from (3.9) we can view

$$\dot{g} = g(u_1 E_1 + u_2 E_2 + u_3 E_3),$$

as the control system evolving on $\mathsf{SO}(3)$. This corresponds to our Type III system.

When only two components of the angular velocity can be controlled, for example, if we can control the angular velocity about the body's first two principle axes, then g satisfies

$$\dot{g} = g\left(u_1 E_1 + u_2 E_2\right).$$

This corresponds to our Type IIa system.

Path planning on SO(3)

The problem stated below is related to the landing tower problem [17].

Let g_1 and g_2 be arbitrary but fixed points in $\mathsf{SO}(3)$, and $T > 0$ be fixed in advance. Given a control system which evolves according to the equation

$$\dot{g} = g(E_3 + u_1 E_1 + u_2 E_2), \quad g \in \mathsf{SO}(3)$$

and u_1, u_2 are the scalar inputs corresponding to the twisting about the body's first two principle axes, one would need to find controls $u_i(\cdot) : [0, T] \to \mathbb{R}^2$, which steer the system from g_1 to g_2 on the interval $[0, T]$. From all the controls $u(\cdot)$ that solve the boundary value problem, pick the one that minimizes the cost

$$J = \frac{1}{2}\int_0^T (c_1 u_1^2 + c_2 u_2^2)\, dt, \quad c_1,\, c_2 > 0.$$

This corresponds to our Type IIb problem.

3.3 Optimal control

The left-invariant optimal control problem on $\mathsf{SO}(3)$, as explained in section 1.4, can be lifted to $T^*\mathsf{SO}(3) = \mathsf{SO}(3) \times \mathfrak{so}(3)^*$ (Prop. 1.1.11). We now identify $\mathfrak{so}(3)^*$ with $\mathfrak{so}(3)$ via the Killing form (Def. 1.1.32). The Killing form on $\mathsf{SO}(3)$ is given by

$$\kappa(X,Y) = x_1y_1 + x_2y_2 + x_3y_3 = -\frac{1}{2}\mathrm{tr}\,(XY) \quad \text{for any } X,\, Y \in \mathfrak{so}(3).$$

Since $\mathfrak{so}(3)$ is semi-simple (Prop. 2.2.3, Def. 1.1.17), we know that the Killing form is non-degenerate (Prop. 1.1.12). Then the associated linear mapping

$$\kappa^\flat : \mathfrak{so}(3) \to \mathfrak{so}(3)^*, \quad P \mapsto p(\cdot) = \kappa^\flat(P)(\cdot) := \kappa(P,\cdot) \tag{3.10}$$

is an isomorphism (Prop. 1.1.13). Thus the projection $p(\cdot)$ of the extremal curve $(g(\cdot), p(\cdot))$ in $\mathsf{SO}(3) \times \mathfrak{so}(3)^*$ is identified with a curve $P(\cdot)$ in $\mathfrak{so}(3)$ via the associated linear map of the Killing form.

If

$$P(t) = \begin{bmatrix} 0 & p_3(t) & -p_2(t) \\ -p_3(t) & 0 & p_1(t) \\ p_2(t) & -p_1(t) & 0 \end{bmatrix} \in \mathfrak{so}(3),$$

and if $P_i(t)$ denotes each Hamiltonian $H_{E_i}(p(t)) := p(t)(E_i)$ for $i = 1,\, 2,\, 3$, then we have

$$-p_i(t) = \kappa(P(t), E_i) = p(t)(E_i) = P_i(t) \quad \text{for } i = 1,\, 2,\, 3,$$

so that

$$P(t) = \begin{bmatrix} 0 & -P_3(t) & P_2(t) \\ P_3(t) & 0 & -P_1(t) \\ -P_2(t) & P_1(t) & 0 \end{bmatrix} \in \mathfrak{so}(3).$$

Using the hat mapping (Def. 2.2.1) we associate an element in $\mathfrak{so}(3)$ to a vector in $\mathbb{R}^3$.

Proposition 1.3.4 gives that the Poisson brackets (Def. 1.3.12) of P_1, P_2 and P_3 satisfy

$$\{P_1, P_2\} = -P_3, \quad \{P_3, P_1\} = -P_2, \quad \{P_2, P_3\} = -P_1. \tag{3.11}$$

P_1, P_2 and P_3 for Types I-III are obtained by solving a set of differential equations on $\mathbb{R}^3$ known as the extremal equations, and in the remaining chapters this set of differential equations will be solved using the Hamiltonian of the system (Def. 1.4.6), the Casimir function on $\mathfrak{so}(3)^*$ (Prop. 3.3.1) and elliptic functions (section 1.6).

PROPOSITION 3.3.1. *Let P_i denote each Hamiltonian $H_{E_i}(p) := p(E_i)$ for $i = 1, 2, 3$. Then $P_1^2 + P_2^2 + P_3^2$ is a Casimir function on $\mathfrak{so}(3)^*$, and thus a conservation law for any left-invariant optimal control problem on $\mathsf{SO}(3)$. We shall denote this Casimir function by K.*

PROOF: The Killing form identifies co-adjoint orbits (Def. 1.4.8) with adjoint orbits (Def. 1.4.7) as follows:

$$\begin{aligned}
& (\mathrm{Ad}^*_{g^{-1}}p)X = \text{constant}, && \text{for any } X \in \mathfrak{so}(3), \ (\text{Cor. } 1.4.1)\\
\Leftrightarrow\ & p(\mathrm{Ad}_{g^{-1}}X) = \text{constant}, && (\text{Def. } 1.1.15)\\
\Leftrightarrow\ & p(g^{-1}Xg) = \text{constant}, && (\text{Def. } 1.1.13)\\
\Leftrightarrow\ & \kappa(P, g^{-1}Xg) = \text{constant}, && (3.10)\\
\Leftrightarrow\ & \kappa(gPg^{-1}, X) = \text{constant}, && (\mathrm{tr}\,(Pg^{-1}Xg) = \mathrm{tr}\,(gPg^{-1}X))\\
(\Rightarrow\ & gPg^{-1} = \Lambda, && \text{for some element } \Lambda \text{ of } \mathfrak{so}(3), \ (\text{Def. } 1.1.32)) \qquad (3.12)\\
\Leftrightarrow\ & (gPg^{-1})(X) = \text{constant}, && (3.10).
\end{aligned}$$

Hence, $\mathrm{Ad}^*_{g^{-1}}p$ is identified with $\mathrm{Ad}_g P = gPg^{-1}$ (Def. 1.1.13). As g varies, gPg^{-1} describes the conjugacy class of P, so that the eigenvalues of P are constant in each orbit. The characteristic polynomial of P is equal to

$$\lambda(\lambda^2 + P_1^2 + P_2^2 + P_3^2).$$

The non-zero eigenvalue λ satisfies $-\lambda^2 = K$ making K an invariant function on each orbit and therefore a Casimir function on $\mathfrak{so}(3)^*$ (Def. 1.4.9). There are no other Casimir functions on $\mathfrak{so}(3)^*$ that are functionally independent of K. □

REMARK: The co-adjoint orbits of $\mathsf{SO}(3)$ are spheres and $P(\cdot)$ is contained in the co-adjoint orbit of $\mathsf{SO}(3)$ through $P(0)$ (Prop. 1.4.1) (this will be verified numerically for Types I-III in the remaining chapters).

3.4 Integration

Assume first that $K \neq 0$. The appropriate choice of coordinates for $\mathsf{SO}(3)$, suitable for integrating equations (3.4), (3.5), (3.6) and (3.7), is dependent on the nature of the symmetry matrix Λ from (3.12) and the structure of $\mathsf{SO}(3)$. We shall coordinatize $\mathsf{SO}(3)$ in terms of the Euler angles

(ϕ_1, ϕ_2, ϕ_3) defined by

$$g(t) = e^{\frac{1}{\sqrt{K}}\Lambda\phi_1(t)}e^{E_2\phi_2(t)}e^{E_3\phi_3(t)}.$$

In this representation we are implicitly assuming that Λ is not linearly dependent on E_2. When $K = 0$, it is assumed that $\Lambda = E_1 - E_3$ and that the coordinates are given by

$$g(t) = e^{\Lambda\phi_1(t)}e^{E_2\phi_2(t)}e^{E_3\phi_3(t)}.$$

Chapter 4

The Type I Problem

We now look at the solution to the Type I left-invariant optimal control problem on $\mathsf{SO}(3)$. After defining the problem we prove a theorem that gives the optimal control, optimal Hamiltonian and extremal equations. Two alternative approaches are employed to solve the extremal equations, using both Jacobi elliptic functions and the Weierstrass elliptic function. Numerical calculations are performed to test the non-numerical results. Finally, a set of differential equations is achieved, which can be solved for Euler's angles to give the solution to the problem. We conclude this chapter by investigating the non-linear stability of the equilibrium states associated with the extremal equations.

4.1 The Type I left-invariant optimal control problem

Let g_1 and g_2 be arbitrary but fixed points in $\mathsf{SO}(3)$, and $T > 0$ be fixed in advance. Then the Type I left-invariant optimal control problem on $\mathsf{SO}(3)$ consists of minimizing the cost functional

$$J = \frac{1}{2}\int_0^T c_1 u^2\, dt, \quad c_1 > 0 \tag{4.1}$$

over all trajectory-control pairs $(g(\cdot), u(\cdot))$, of the left-invariant control system

$$\dot{g} = g(E_3 + uE_1), \quad g \in \mathsf{SO}(3),\ u \in \mathbb{R}, \tag{4.2}$$

satisfying the boundary conditions

$$g(0) = g_1 \quad \text{and} \quad g(T) = g_2.$$

4.2 The extremal equations

Theorem 4.2.1 has been derived from similar results found in [9] and [18].

THEOREM 4.2.1. *Given a left-invariant control system on* $\mathsf{SO}(3)$ *defined by (4.2) with cost (4.1), the optimal control is*

$$u = \frac{1}{c_1} P_1 \tag{4.3}$$

and the optimal Hamiltonian is

$$\mathcal{H} = \frac{1}{2c_1} P_1^2 + P_3, \tag{4.4}$$

where P_1 *and* P_3 *are solutions of*

$$\dot{P}_1 = P_2 \tag{4.5}$$

$$\dot{P}_2 = -P_1 + \frac{1}{c_1} P_1 P_3 \tag{4.6}$$

$$\dot{P}_3 = -\frac{1}{c_1} P_1 P_2. \tag{4.7}$$

PROOF: The control Hamiltonian (Def. 1.4.6) is

$$\mathcal{H}(p, u) = -\frac{1}{2} c_1 u^2 + p(E_3 + uE_1),$$

which simplifies to (Prop. 3.3.1)

$$\mathcal{H}(p, u) = -\frac{1}{2} c_1 u^2 + P_3 + uP_1. \tag{4.8}$$

Applying (2) of the maximum principle (section 1.5), the optimal control is

$$\frac{\partial \mathcal{H}}{\partial u} = 0 \quad \Leftrightarrow \quad -c_1 u + P_1 = 0 \quad \Leftrightarrow \quad u = \frac{1}{c_1} P_1.$$

Substituting u into (4.8) gives (4.4), which combined with Propositions 1.3.6, 1.3.4 and 1.3.7 gives

$$\dot{P}_i = \{P_i, \frac{1}{2c_1} P_1^2 + P_3\} = \frac{1}{c_1} P_1 \{P_i, P_1\} + \{P_i, P_3\}, \quad i = 1, 2, 3,$$

and using the Poisson bracket relations from (3.11) we get (4.5), (4.6) and (4.7). □

4.3 The solution to the extremal equations

The beginning of the proof of Theorem 4.3.1 is based on similar ideas found in [18] and [19]. However, the integration using elliptic functions is original (with reference to [12]).

THEOREM 4.3.1. *The equations (4.5), (4.6) and (4.7) can be integrated via elliptic functions. More exactly (the restraint $\mathcal{H}^2 - K > 0$ is imposed since c_1 is real),*

$$\begin{aligned} P_1 &= \pm\sqrt{2c_1\mathcal{H} - 2c_1P_3} \\ P_2 &= \pm\sqrt{K - 2c_1\mathcal{H} + 2c_1P_3 - P_3^2} \\ P_3 &= \frac{\alpha - \beta\, a\,\mathrm{dc}\left(\frac{(\alpha-\beta)\sqrt{A_1A_2}}{\sqrt{c_1}}a\,t, \frac{b}{a}\right)}{1 - a\,\mathrm{dc}\left(\frac{(\alpha-\beta)\sqrt{A_1A_2}}{\sqrt{c_1}}a\,t, \frac{b}{a}\right)} \\ \textit{and/or} \quad P_3 &= \frac{\alpha - \beta\, a\,\mathrm{ns}\left(\frac{(\alpha-\beta)\sqrt{A_1A_2}a}{\sqrt{c_1}}\,t, \frac{b}{a}\right)}{1 - a\,\mathrm{ns}\left(\frac{(\alpha-\beta)\sqrt{A_1A_2}a}{\sqrt{c_1}}\,t, \frac{b}{a}\right)}, \end{aligned}$$

where

$$\begin{aligned} a^2 &= \frac{c_1 - \sqrt{\mathcal{H}^2 - K} - \mathcal{H}}{c_1 + \sqrt{\mathcal{H}^2 - K} - \mathcal{H}}, & b^2 &= 1 \\ \alpha &= \mathcal{H} + \sqrt{\mathcal{H}^2 - K}, & \beta &= \mathcal{H} - \sqrt{\mathcal{H}^2 - K} \\ A_1A_2 &= -\frac{c_1 + \sqrt{\mathcal{H}^2 - K} - \mathcal{H}}{4(\mathcal{H}^2 - K)}. \end{aligned}$$

PROOF: Multiplying the Hamiltonian (4.4) by $2c_1$ gives

$$P_1^2 + 2c_1P_3 = 2\mathcal{H}c_1. \tag{4.9}$$

Using the Casimir function on $\mathfrak{so}(3)^*$ (Prop. 3.3.1),

$$P_1^2 + P_2^2 + P_3^2 = K, \tag{4.10}$$

and rearranging for P_1^2, we substitute into (4.9) and solve for P_2^2:

$$\begin{aligned} K - P_2^2 - P_3^2 + 2c_1P_3 &= 2\mathcal{H}c_1 \\ \Rightarrow \quad P_2^2 &= K - 2\mathcal{H}c_1 + 2c_1P_3 - P_3^2. \end{aligned} \tag{4.11}$$

Rearranging (4.9) for P_1^2:

$$P_1^2 = 2\mathcal{H}c_1 - 2c_1P_3. \tag{4.12}$$

Squaring equation (4.7),

$$\dot{P}_3^2 = \frac{1}{c_1^2} P_1^2 P_2^2,$$

and substituting in equations (4.11) and (4.12) leaves us with

$$\begin{aligned} \dot{P}_3^2 &= \frac{1}{c_1^2}(2\mathcal{H}c_1 - 2c_1P_3)(K - 2\mathcal{H}c_1 + 2c_1P_3 - P_3^2) \\ &= \frac{1}{c_1}(K - 2\mathcal{H}c_1 + 2c_1P_3 - P_3^2)(2\mathcal{H} - 2P_3). \end{aligned} \tag{4.13}$$

The form of (4.13) does not match the standard form of the elliptic integrals. Therefore, we need to perform a transformation on (4.13) and the approach is outlined in section 1.6. Some of the working that follows has been done in *Wolfram Mathematica 7.0*, and this code can be found in appendix A.

The expression $\sqrt{\mathcal{H}^2 - K}$ appears in the calculations throughout the remainder of this proof. It is always real since the discriminant of $K - 2\mathcal{H}c_1 + 2c_1P_3 - P_3^2$ (4.13) is

$$\Delta = 4c_1^2 - 8\mathcal{H}c_1 + 4K,$$

and this is a quadratic in c_1 with discriminant given by

$$\Delta = 64(\mathcal{H}^2 - K) > 0 \quad \text{(since } c_1 \text{ is real)} \quad \Rightarrow \quad \mathcal{H}^2 - K > 0.$$

Hence, the condition imposed in the theorem.

CASE (4.13): Let

$$S_1 = K - 2\mathcal{H}c_1 + 2c_1P_3 - P_3^2 \quad \text{and} \quad S_2 = 2\mathcal{H} - 2P_3.$$

(If we swap S_1 and S_2 we achieve no new results.)

Consider the quadratic expression $S_1 + \lambda S_2$; this has coincident zeros and is a perfect square whenever (see (1.13))

$$-\lambda^2 - 2(\mathcal{H} - c_1)\lambda + 2\mathcal{H}c_1 - K - c_1^2 = 0.$$

Solving this expression for λ (see appendix A) yields

$$\lambda_1 = c_1 - \sqrt{\mathcal{H}^2 - K} - \mathcal{H}, \quad \lambda_2 = c_1 + \sqrt{\mathcal{H}^2 - K} - \mathcal{H}.$$

Substitute λ_1 and λ_2 into (1.16) and (1.18) (see appendix A), then (1.17) gives

$$S_1 = A_1(P_3 - \alpha)^2 + B_1(P_3 - \beta)^2 \quad \text{and} \quad S_2 = A_2(P_3 - \alpha)^2 + B_2(P_3 - \beta)^2, \tag{4.14}$$

where

$$A_1 = -\frac{c_1 + \sqrt{\mathcal{H}^2 - K} - \mathcal{H}}{2\sqrt{\mathcal{H}^2 - K}}, \qquad A_2 = \frac{1}{2\sqrt{\mathcal{H}^2 - K}}$$
$$B_1 = \frac{c_1 - \sqrt{\mathcal{H}^2 - K} - \mathcal{H}}{2\sqrt{H^2 - K}}, \qquad B_2 = -\frac{1}{2\sqrt{\mathcal{H}^2 - K}}$$
$$\alpha = \mathcal{H} + \sqrt{\mathcal{H}^2 - K}, \qquad \beta = \mathcal{H} - \sqrt{\mathcal{H}^2 - K}.$$

Now, since S_1 and S_2 have been expressed in the form of (4.14), we can rewrite (4.13) as

$$\dot{P}_3^2 = \frac{1}{c_1}(A_1(P_3 - \alpha)^2 + B_1(P_3 - \beta)^2)(A_2(P_3 - \alpha)^2 + B_2(P_3 - \beta)^2).$$

Following the steps towards the end of section 1.6 we achieve (1.20), and this can be rewritten to match the form of the Jacobi elliptic integrals (1.6) and (1.7). Substituting the integral value (1.6) followed by u (1.19), and solving for P_3 gives $\left(a^2 = -\frac{B_1}{A_1},\, b^2 = -\frac{B_2}{A_2}\right)$

$$t = \frac{\sqrt{c_1}}{(\alpha - \beta)\sqrt{A_1 A_2}} \int_a^{P_3} \frac{du}{\sqrt{(u^2 - (-\frac{B_1}{A_1}))(u^2 - (-\frac{B_2}{A_2}))}} \tag{4.15}$$
$$t = \frac{\sqrt{c_1}}{(\alpha - \beta)\sqrt{A_1 A_2} a} \mathrm{dc}^{-1}\left(\frac{u}{a}, \frac{b}{a}\right)$$

$$\Rightarrow \quad P_3 = \frac{\alpha - \beta\, a \,\mathrm{dc}\left(\frac{(\alpha-\beta)\sqrt{A_1 A_2} a}{\sqrt{c_1}} t, \frac{b}{a}\right)}{1 - a\,\mathrm{dc}\left(\frac{(\alpha-\beta)\sqrt{A_1 A_2} a}{\sqrt{c_1}} t, \frac{b}{a}\right)},$$

where (see appendix A)

$$a^2 = \frac{c_1 - \sqrt{\mathcal{H}^2 - K} - \mathcal{H}}{c_1 + \sqrt{\mathcal{H}^2 - K} - \mathcal{H}} > 0, \quad b^2 = 1 > 0 \quad \text{and} \quad A_1 A_2 = -\frac{c_1 + \sqrt{\mathcal{H}^2 - K} - \mathcal{H}}{4(\mathcal{H}^2 - K)} > 0.$$

Similarly, substituting the integral value (1.7) into (4.15), followed by u (1.19), and solving for P_3 gives

$$P_3 = \frac{\alpha - \beta\, a \,\mathrm{ns}\left(\frac{(\alpha-\beta)\sqrt{A_1 A_2} a}{\sqrt{c_1}} t, \frac{b}{a}\right)}{1 - a\,\mathrm{ns}\left(\frac{(\alpha-\beta)\sqrt{A_1 A_2} a}{\sqrt{c_1}} t, \frac{b}{a}\right)},$$

with the same values for a^2, b^2 and $A_1 A_2$.

Although,

$$a^2 > 0 \quad \Rightarrow \quad \mathcal{H} - c_1 < -\sqrt{\mathcal{H}^2 - K} \quad \text{or} \quad \mathcal{H} - c_1 > \sqrt{\mathcal{H}^2 - K},$$

the condition

$$\mathcal{H} - c_1 < -\sqrt{\mathcal{H}^2 - K}$$

cannot hold, as this would make $A_1 A_2$ negative under the root for P_3.

Substituting a^2 and b^2 into $b < a$ (from (1.6) or (1.7)) gives

$$1 \quad < \quad \frac{c_1 - \sqrt{\mathcal{H}^2 - K} - \mathcal{H}}{c_1 + \sqrt{\mathcal{H}^2 - K} - \mathcal{H}},$$

which is satisfied when $\mathcal{H} - c_1 > \sqrt{\mathcal{H}^2 - K}$. The naming of λ_1 and λ_2 were chosen to ensure that this condition would be satisfied.

It can be checked (with Mathematica) that no combination of values for c_1 and $P(0)$ will satisfy the condition

$$-\sqrt{\mathcal{H}^2 - K} < \mathcal{H} - c_1 < \sqrt{\mathcal{H}^2 - K}.$$

Finally, we obtain P_1 and P_2 by substituting P_3 into (4.12) and (4.11), respectively. □

An alternative approach to solving (4.5), (4.6) and (4.7), is by employing the Weierstrass elliptic function. Theorem 4.3.2 has been derived from a similar result found in [5].

THEOREM 4.3.2. *The equations (4.5), (4.6) and (4.7) can be integrated via a Weierstrass function. More exactly,*

$$P_1 = \pm\sqrt{2c_1(\mathcal{H} - P_3)} \tag{4.16}$$

$$P_2 = \pm\sqrt{K - 2c_1(\mathcal{H} - P_3) - P_3^2} \tag{4.17}$$

$$P_3 = 2c_1\wp + \frac{\mathcal{H} + 2c_1}{3}, \tag{4.18}$$

where $\wp$ *is the Weierstrass function given by*

$$\dot{\wp}^2 = 4\wp^3 - g_2\wp - g_1,$$

with

$$\begin{aligned} g_1 &= -\frac{2}{a^2}\left(\frac{1}{c_1}b^3 - \left(2 + \frac{1}{c_1}\mathcal{H}\right)b^2 + \left(4\mathcal{H} - \frac{1}{c_1}K\right)b + \mathcal{H}\left(\frac{1}{c_1}K - 2\mathcal{H}\right)\right) \\ g_2 &= -\frac{2}{a}\left(\frac{3}{c_1}b^2 - 2\left(2 + \frac{1}{c_1}\mathcal{H}\right)b + 4\mathcal{H} - \frac{1}{c_1}K\right) \\ a &= 2c_1 \quad \textit{and} \quad b = \frac{\mathcal{H} + 2c_1}{3}. \end{aligned}$$

PROOF: Following the identical steps taken in the proof to Theorem 4.3.1 up to (4.13),

$$\begin{aligned}\dot{P}_3^2 &= \frac{1}{c_1}(K - 2\mathcal{H}c_1 + 2c_1P_3 - P_3^2)(2\mathcal{H} - 2P_3)\\ &= \frac{2}{c_1}P_3^3 - 2\left(2 + \frac{1}{c_1}\mathcal{H}\right)P_3^2 + 2\left(4\mathcal{H} - \frac{1}{c_1}K\right)P_3 + 2\mathcal{H}\left(\frac{1}{c_1}K - 2\mathcal{H}\right). \qquad (4.19)\end{aligned}$$

If we choose the affine transformation

$$P_3 = a\wp + b, \qquad (4.20)$$

then the derivative squared is given by

$$\dot{P}_3^2 = a^2\dot{\wp}^2. \qquad (4.21)$$

Equate (4.21) and (4.19), and substitute (4.20) into (4.19). Some simplification will give the result.

$$\begin{aligned}a^2\dot{\wp}^2 &= \frac{2}{c_1}(a\wp + b)^3 - 2\left(2 + \frac{1}{c_1}\mathcal{H}\right)(a\wp + b)^2 + 2(4\mathcal{H} - \frac{1}{c_1}K)(a\wp + b) + 2\mathcal{H}\left(\frac{1}{c_1}K - 2\mathcal{H}\right)\\ &= \frac{2}{c_1}a^3\wp^3 + 2\left(\frac{3}{c_1}b - 2 - \frac{1}{c_1}\mathcal{H}\right)a^2\wp^2 + 2\left(\frac{3}{c_1}b^2 - 2\left(2 + \frac{1}{c_1}\mathcal{H}\right)b + 4\mathcal{H} - \frac{1}{c_1}K\right)a\wp\\ &+ \frac{2}{c_1}b^3 - 2\left(2 + \frac{1}{c_1}\mathcal{H}\right)b^2 + 2\left(4\mathcal{H} - \frac{1}{c_1}K\right)b + 2\mathcal{H}\left(\frac{1}{c_1}K - 2\mathcal{H}\right). \qquad (4.22)\end{aligned}$$

Divide (4.22) by a^2 and we obtain:

$$\begin{aligned}\dot{\wp}^2 &= \frac{2}{c_1}a\wp^3 + 2\left(\frac{3}{c_1}b - 2 - \frac{1}{c_1}\mathcal{H}\right)\wp^2 + \frac{2}{a}\left(\frac{3}{c_1}b^2 - 2\left(2 + \frac{1}{c_1}\mathcal{H}\right)b + 4\mathcal{H} - \frac{1}{c_1}K\right)\wp\\ &+ \frac{2}{a^2}\left(\frac{1}{c_1}b^3 - \left(2 + \frac{1}{c_1}\mathcal{H}\right)b^2 + \left(4\mathcal{H} - \frac{1}{c_1}K\right)b + \mathcal{H}\left(\frac{1}{c_1}K - 2\mathcal{H}\right)\right). \qquad (4.23)\end{aligned}$$

Due to (1.1) we need to ensure that the leading coefficient is 4 and there are no second order terms. Let

$$a = 2c_1 \quad \text{and} \quad b = \frac{\mathcal{H} + 2c_1}{3}.$$

Substituting a and b into the higher order terms of (4.23),

$$\dot{\wp}^2 = 4\wp^3 - g_2\wp - g_1,$$

where

$$g_1 = -\frac{2}{a^2}\left(\frac{1}{c_1}b^3 - \left(2 + \frac{1}{c_1}\mathcal{H}\right)b^2 + \left(4\mathcal{H} - \frac{1}{c_1}K\right)b + \mathcal{H}\left(\frac{1}{c_1}K - 2\mathcal{H}\right)\right)$$

$$\text{and} \quad g_2 = -\frac{2}{a}\left(\frac{3}{c_1}b^2 - 2\left(2 + \frac{1}{c_1}\mathcal{H}\right)b + 4\mathcal{H} - \frac{1}{c_1}K\right).$$

The result follows from (4.20), (4.9) and (4.10). □

4.4 Numerical solutions to the extremal equations

In the appendix, I have included the MATLAB code (appendix B) used to generate graphs (appendix C) of the solution to the extremal equations ((4.5), (4.6), (4.7)). SECTION 1 of the code plots the solution for arbitrarily chosen c_1 and initial values using a MATLAB solver. In SECTION 2, the same constants are used, but we substitute them into Theorem 4.3.1 (incl. (4.9) and (4.10)) and plot the result. In both sections the individual components P_1, P_2, P_3 are plotted on one set of axes, followed by the solution curve $P(\cdot)$ plotted on a seperate set of axes. In the instance where either P_1 or P_2 cut the x-axis the negative curve needs to be plotted, as this forms part of the solution (otherwise, the negative of the curve does not satisfy the chosen initial values). In the conclusion, the plots produced by the MATLAB solver are compared to those from Theorem 4.3.1 - this will help us gauge the accuracy of the results in the theorem.

4.5 The solution to the left-invariant optimal control problem

Proposition 4.5.1 has been derived from a similar result found in [9].

Assume first that $K \neq 0$. Recall (from section 3.4) that the solution curve on $\mathsf{SO}(3)$ is given by

$$g(t) = e^{\frac{1}{\sqrt{K}}\Lambda\phi_1(t)}e^{E_2\phi_2(t)}e^{E_3\phi_3(t)}, \tag{4.24}$$

where $g(t)P(t)g^{-1}(t) = \Lambda$, and we are implicitly assuming that Λ is not linearly dependent on E_2.

PROPOSITION 4.5.1. *Suppose that $(g(\cdot), P(\cdot))$ is an extremal curve such that $g(t)P(t)g^{-1}(t) = \Lambda$, then $\phi_1(\cdot)$, $\phi_2(\cdot)$ and $\phi_3(\cdot)$ are given by the following differential equations:*

$$\begin{aligned}
\dot{\phi}_1 &= -\frac{\sqrt{K}}{c_1}\frac{P_1\cos\phi_3}{(P_2\sin\phi_3 - P_1\cos\phi_3)} \\
\dot{\phi}_2 &= \frac{1}{c_1}\frac{P_1P_2}{(P_2\sin\phi_3 - P_1\cos\phi_3)} \\
\dot{\phi}_3 &= 1 + \frac{1}{c_1}\frac{P_1P_3\cos\phi_3}{(P_2\sin\phi_3 - P_1\cos\phi_3)}.
\end{aligned}$$

PROOF: We calculate the derivative of (4.24),

$$\dot{g} = \frac{1}{\sqrt{K}}\Lambda g\dot{\phi}_1 + e^{\frac{1}{\sqrt{K}}\Lambda\phi_1}e^{E_2\phi_2}E_2e^{E_3\phi_3}\dot{\phi}_2 + gE_3\dot{\phi}_3,$$

which can be rewritten as

$$\dot{g} = g\left(\frac{1}{\sqrt{K}}g^{-1}\Lambda g\dot{\phi}_1 + e^{-E_3\phi_3}E_2e^{E_3\phi_3}\dot{\phi}_2 + E_3\dot{\phi}_3\right).$$

Since $\dot{g} = g(E_3 + \frac{1}{c_1}P_1E_1)$,

$$\left(\frac{1}{\sqrt{K}}g^{-1}\Lambda g\right)\dot{\phi}_1 + \left(e^{-E_3\phi_3}E_2e^{E_3\phi_3}\right)\dot{\phi}_2 + E_3\dot{\phi}_3 = E_3 + \frac{1}{c_1}P_1E_1,$$

and applying the hat mapping (Def. 2.2.1) gives

$$\frac{1}{\sqrt{K}}\widehat{P}\dot{\phi}_1 + \left(e^{-E_3\phi_3}e_2\right)\dot{\phi}_2 + e_3\dot{\phi}_3 = e_3 + \frac{1}{c_1}P_1e_1.$$

Furthermore, $\widehat{P} = P_1e_1 + P_2e_2 + P_3e_3$ and since

$$e^{-E_3\phi_3} = \begin{bmatrix} \cos\phi_3 & \sin\phi_3 & 0 \\ -\sin\phi_3 & \cos\phi_3 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

it follows that $e^{-E_3\phi_3}e_2 = (\sin\phi_3)e_1 + (\cos\phi_3)e_2$. Therefore,

$$\frac{1}{\sqrt{K}}P_1\dot{\phi}_1 + (\sin\phi_3)\dot{\phi}_2 = \frac{1}{c_1}P_1, \quad \frac{1}{\sqrt{K}}P_2\dot{\phi}_1 + (\cos\phi_3)\dot{\phi}_2 = 0 \quad \text{and} \quad \frac{1}{\sqrt{K}}P_3\dot{\phi}_1 + \dot{\phi}_3 = 1.$$

The first two equations can be solved for $\dot{\phi}_1$ and $\dot{\phi}_2$; $\dot{\phi}_1$ is substituted into the third equation to obtain the final result. □

This set of differential equations is solved using numerical techniques, but this goes beyond the scope of what we are interested in achieving here.

Recall (from section 3.4) that when $K = 0$, it is assumed that $\Lambda = E_1 - E_3$. The required set of differential equations is then obtained from Proposition 4.5.1 by substituting in $\frac{1}{\sqrt{K}} = 1$.

4.6 Stability

Theorems 4.6.1-4.6.2 have been derived from similar results and ideas found in [19] and [20].

The equilibrium states for the system (4.5), (4.6) and (4.7) are:

$$\widehat{P}_{e1} = (M, 0, c_1) \quad \text{and} \quad \widehat{P}_{e2} = (0, 0, M), \quad M \in \mathbb{R},\ c_1 > 0.$$

THEOREM 4.6.1. *The equilibrium state $\widehat{P}_{e1} = (M, 0, c_1)$ is non-linear stable, $M \neq 0$.*

PROOF: STEP 1. For any arbitrary smooth function $\varphi : \mathbb{R} \to \mathbb{R}$, the function K_φ, defined by

$$K_\varphi = \varphi\left(\frac{1}{2}(P_1^2 + P_2^2 + P_3^2)\right),$$

gives a family of Casimir functions on $\mathfrak{so}(3)^*$ (Prop. 3.3.1, Prop. 1.4.3, Def. 1.3.13).

STEP 2. We want to find a single K_φ such that the energy-Casimir function,

$$\mathcal{H} + K_\varphi = \left(\frac{1}{2c_1}P_1^2 + P_3\right) + \varphi\left(\frac{1}{2}(P_1^2 + P_2^2 + P_3^2)\right),$$

has a critical point at the equilibrium of interest.

Now, we calculate the first derivative of $\mathcal{H} + K_\varphi$:

$$\begin{aligned}
D(\mathcal{H} + K_\varphi)(p) \cdot \delta p &= \frac{d}{dt}(\mathcal{H} + K_\varphi)\ (p + t\delta)|_{t=0}, \\
&= \frac{d}{dt}\left(\frac{1}{2c_1}(P_1 + t\delta_1)^2 + P_3 + t\delta_3\right)\Bigg|_{t=0} && (4.25)\\
&+ \frac{d}{dt}\varphi\left(\frac{1}{2}(P_1 + t\delta_1)^2 + \frac{1}{2}(P_2 + t\delta_2)^2 + \frac{1}{2}(P_3 + t\delta_3)^2\right)\Bigg|_{t=0} \\
&= \frac{1}{c_1}\delta_1 P_1 + \delta_3 + (\delta_1 P_1 + \delta_2 P_2 + \delta_3 P_3)\dot{\varphi}\left(\frac{1}{2}(P_1^2 + P_2^2 + P_3^2)\right), && (4.26)
\end{aligned}$$

and this equals zero at the equilibrium of interest if and only if

$$\dot{\varphi}\left(\frac{1}{2}(M^2 + c_1^2)\right) = -\frac{1}{c_1}. \tag{4.27}$$

STEP 3. Next, using (4.26), we can calculate the second derivative of $\mathcal{H} + K_\varphi$:

$$\begin{aligned}
D^2(\mathcal{H} + K_\varphi)(p) \cdot \delta p &= \frac{d}{dt}\left(\frac{1}{c_1}\delta_1(P_1 + t\delta_1) + \delta_3\right)\Bigg|_{t=0} \\
&+ \frac{d}{dt}(\delta_1(P_1 + t\delta_1) + \delta_2(P_2 + t\delta_2) + \delta_3(P_3 + t\delta_3)) \\
&\times \dot{\varphi}\left(\frac{1}{2}((P_1 + t\delta_1)^2 + (P_2 + t\delta_2)^2 + (P_3 + t\delta_3)^2)\right)\Bigg|_{t=0} \\
&= \frac{1}{c_1}\delta_1^2 + (\delta_1^2 + \delta_2^2 + \delta_3^2)\dot{\varphi}\left(\frac{1}{2}(P_1^2 + P_2^2 + P_3^2)\right)
\end{aligned}$$

$$+ (\delta_1 P_1 + \delta_2 P_2 + \delta_3 P_3)^2 \ddot{\varphi}\left(\frac{1}{2}(P_1^2 + P_2^2 + P_3^2)\right). \tag{4.28}$$

Using (4.27), the second derivative of $\mathcal{H} + K_\varphi$ at the equilibrium of interest is

$$\begin{aligned} D^2(\mathcal{H} + K_\varphi)(\hat{P}_{e1}) \cdot \delta\hat{P}_{e1} &= M^2 \ddot{\varphi}\left(\frac{1}{2}(M^2 + c_1^2)\right)\delta_1^2 - \frac{1}{c_1}\delta_2^2 \\ &+ \left(\ddot{\varphi}\left(\frac{1}{2}(M^2 + c_1^2)\right)c_1^2 - \frac{1}{c_1}\right)\delta_3^2 + \ddot{\varphi}\left(\frac{1}{2}(M^2 + c_1^2)\right)2c_1 M \delta_1 \delta_3. \end{aligned} \tag{4.29}$$

This can be written in the matrix form

$$\begin{bmatrix} \delta_1 & \delta_2 & \delta_3 \end{bmatrix} \begin{bmatrix} M^2 \ddot{\varphi}\left(\frac{1}{2}(M^2 + c_1^2)\right) & 0 & c_1 M \ddot{\varphi}\left(\frac{1}{2}(M^2 + c_1^2)\right) \\ 0 & -\frac{1}{c_1} & 0 \\ c_1 M \ddot{\varphi}\left(\frac{1}{2}(M^2 + c_1^2)\right) & 0 & \ddot{\varphi}\left(\frac{1}{2}\left(M^2 + c_1^2\right)\right)c_1^2 - \frac{1}{c_1} \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{bmatrix},$$

and is negative definite if and only if:

1. the first entry of the matrix is negative ⇒ $\ddot{\varphi}\left(\frac{1}{2}(M^2 + c_1^2)\right) < 0$
2. the determinant of the upper left 2×2 matrix is positive ⇒ $\ddot{\varphi}\left(\frac{1}{2}(M^2 + c_1^2)\right) < 0$
3. the determinant of the 3×3 matrix is negative ⇒ $\ddot{\varphi}\left(\frac{1}{2}(M^2 + c_1^2)\right) < 0$.

Thus, having chosen that

$$\ddot{\varphi}\left(\frac{1}{2}(M^2 + c_1^2)\right) < 0,$$

this quadratic form is negative definite. Consequently,

$$\varphi(x) = -x\left(x + \frac{1}{c_1} - (M^2 + c_1^2)\right)$$

satisfies the above conditions and the second derivative of $\mathcal{H} + K_\varphi$ is negative definite. □

THEOREM 4.6.2. *The equilibrium state* $\widehat{P}_{e2} = (0, 0, M)$ *is non-linear stable if* $M \in (-\infty, 0) \cup (0, c_1)$.

PROOF: Theorem 4.6.1 covers STEP 1 and the first part of STEP 2 for this proof.

STEP 2. Eq. (4.26) equals zero at the equilibrium of interest if and only if

$$\dot{\varphi}\left(\frac{1}{2}M^2\right) = -\frac{1}{M}. \tag{4.30}$$

STEP 3. Using (4.30), the second derivative of $\mathcal{H} + K_\varphi$ (4.28) at the equilibrium of interest is

$$D^2(\mathcal{H} + K_\varphi)(\hat{P}_{e2}) \cdot \delta\hat{P}_{e2} = \left(\frac{1}{c_1} - \frac{1}{M}\right)\delta_1^2 - \frac{1}{M}\delta_2^2 + \left(M^2\ddot{\varphi}\left(\frac{1}{2}M^2\right) - \frac{1}{M}\right)\delta_3^2.$$

Thus, having chosen that

$$\ddot{\varphi}\left(\frac{1}{2}M^2\right) > \frac{1}{M^3},$$

this quadratic form is positive definite if and only if $M < 0$. Consequently,

$$\varphi(x) = -\frac{1}{M^3}x^2$$

satisfies the above conditions and the second derivative of $\mathcal{H} + K_\varphi$ is positive definite.

Otherwise, having chosen that

$$\ddot{\varphi}\left(\frac{1}{2}M^2\right) < \frac{1}{M^3},$$

this quadratic form is negative definite if and only if $0 < M < c_1$. Consequently,

$$\varphi(x) = -\frac{1}{M^3}x^2$$

satisfies the above conditions and the second derivative of $\mathcal{H} + K_\varphi$ is negative definite. $\square$

Chapter 5

The Type IIa Problem

We turn our attention to the solution of the Type IIa left-invariant optimal control problem on $\mathsf{SO}(3)$. After defining the problem we prove a theorem that gives the optimal controls, optimal Hamiltonian and extremal equations. The Jacobi elliptic functions are used to solve the extremal equations, and these results are tested by some numerical calculations. Finally, a set of differential equations are achieved, which can be solved for Euler's angles to give the solution to the problem. We conclude this chapter by investigating the non-linear stability of the equilibrium states associated with the extremal equations.

5.1 The Type IIa left-invariant optimal control problem

Let g_1 and g_2 be arbitrary but fixed points in $\mathsf{SO}(3)$, and $T > 0$ be fixed in advance. Then the Type IIa left-invariant optimal control problem on $\mathsf{SO}(3)$ consists of minimizing the cost functional

$$J = \frac{1}{2}\int_0^T (c_1u_1^2 + c_2u_2^2)\,dt, \quad c_1,\, c_2 > 0 \tag{5.1}$$

over all trajectory-control pairs $(g(\cdot), u(\cdot))$, of the left-invariant control system

$$\dot{g} = g(u_1E_1 + u_2E_2), \quad g \in \mathsf{SO}(3),\ u = (u_1, u_2) \in \mathbb{R}^2, \tag{5.2}$$

satisfying the boundary conditions

$$g(0) = g_1 \quad \text{and} \quad g(T) = g_2.$$

5.2 The extremal equations

Theorem 5.2.1 has been derived from similar results found in [9] and [18].

THEOREM 5.2.1. *Given a left-invariant control system on* $\mathsf{SO}(3)$ *defined by (5.2) with cost (5.1), the optimal controls are*

$$u_1 = \frac{1}{c_1}P_1 \quad \text{and} \quad u_2 = \frac{1}{c_2}P_2, \tag{5.3}$$

and the optimal Hamiltonian is

$$\mathcal{H} = \frac{1}{2}(\frac{1}{c_1}P_1^2 + \frac{1}{c_2}P_2^2), \tag{5.4}$$

where P_1 *and* P_2 *are solutions of*

$$\dot{P}_1 = -\frac{1}{c_2}P_2P_3 \tag{5.5}$$

$$\dot{P}_2 = \frac{1}{c_1}P_1P_3 \tag{5.6}$$

$$\dot{P}_3 = \frac{c_1 - c_2}{c_1c_2}P_1P_2. \tag{5.7}$$

PROOF: The control Hamiltonian (Def. 1.4.6) is

$$\mathcal{H}(p, u) = -\frac{1}{2}(c_1u_1^2 + c_2u_2^2) + p(u_1E_1 + u_2E_2),$$

which simplifies to (Prop. 3.3.1)

$$\mathcal{H}(p, u) = -\frac{1}{2}(c_1u_1^2 + c_2u_2^2) + u_1P_1 + u_2P_2. \tag{5.8}$$

Applying (2) of the maximum principle (section 1.5), the optimal controls are

$$\begin{aligned}\frac{\partial\mathcal{H}}{\partial u_1} = 0 \quad &\Leftrightarrow \quad -c_1u_1 + P_1 = 0 \quad \Leftrightarrow \quad u_1 = \frac{1}{c_1}P_1\\ \frac{\partial\mathcal{H}}{\partial u_2} = 0 \quad &\Leftrightarrow \quad -c_2u_2 + P_2 = 0 \quad \Leftrightarrow \quad u_2 = \frac{1}{c_2}P_2.\end{aligned}$$

Substituting u_1 and u_2 into (5.8) gives (5.4), which combined with Propositions 1.3.6, 1.3.4 and 1.3.7 gives

$$\dot{P}_i = \{P_i, \frac{1}{2c_1}P_1^2 + \frac{1}{2c_2}P_2^2\} = \frac{1}{c_1}P_1\{P_i, P_1\} + \frac{1}{c_2}P_2\{P_i, P_2\}, \quad i = 1, 2, 3,$$

and using the Poisson bracket relations from (3.11) we get (5.5), (5.6) and (5.7). □

5.3 The solution to the extremal equations

PROPOSITION 5.3.1. *Considering the case where $c = c_1 = c_2$, we can rewrite equations (5.5), (5.6) and (5.7) as*

$$\dot{P}_1 = -\frac{1}{c}P_2P_3(0) \tag{5.9}$$

$$\dot{P}_2 = \frac{1}{c}P_1P_3(0) \tag{5.10}$$

$$\dot{P}_3 = 0, \tag{5.11}$$

which have the solutions

$$\begin{aligned} P_1(t) &= P_1(0)\cos\left(\frac{1}{c}P_3(0)t\right) - P_2(0)\sin\left(\frac{1}{c}P_3(0)t\right)\\ P_2(t) &= P_1(0)\sin\left(\frac{1}{c}P_3(0)t\right) + P_2(0)\cos\left(\frac{1}{c}P_3(0)t\right)\\ P_3(t) &= P_3(0). \end{aligned}$$

PROOF: Equations (5.9), (5.10) and (5.11) in matrix form become

$$\begin{bmatrix}\dot{P}_1\\ \dot{P}_2\end{bmatrix} = \begin{bmatrix}0 & -\frac{1}{c}P_3(0)\\ \frac{1}{c}P_3(0) & 0\end{bmatrix}\begin{bmatrix}P_1\\ P_2\end{bmatrix}.$$

Since $P(t) = \exp(tA)P_0$ is the only solution of $\dot{P} = AP$, $P(0) = P_0$, the solution can be written as

$$\begin{bmatrix}P_1(t)\\ P_2(t)\end{bmatrix} = \begin{bmatrix}\cos\left(-\frac{1}{c}P_3(0)t\right) & \sin\left(-\frac{1}{c}P_3(0)t\right)\\ -\sin\left(-\frac{1}{c}P_3(0)t\right) & \cos\left(-\frac{1}{c}P_3(0)t\right)\end{bmatrix}\begin{bmatrix}P_1(0)\\ P_2(0)\end{bmatrix},$$

and simplified to

$$\begin{bmatrix}P_1(t)\\ P_2(t)\end{bmatrix} = \begin{bmatrix}\cos\left(\frac{1}{c}P_3(0)t\right) & -\sin\left(\frac{1}{c}P_3(0)t\right)\\ \sin\left(\frac{1}{c}P_3(0)t\right) & \cos\left(\frac{1}{c}P_3(0)t\right)\end{bmatrix}\begin{bmatrix}P_1(0)\\ P_2(0)\end{bmatrix}.$$

□

The beginning of the proof of Theorem 5.3.1 is based on similar ideas found in [18] and [19]. However, the integration using elliptic functions is original (with reference to [12]).

THEOREM 5.3.1. *The equations (5.5), (5.6) and (5.7), where $c_1 \neq c_2$, can be integrated via elliptic functions. More exactly,*

CASE (I): $c_1 > c_2$

If $\frac{c_1-c_2}{c_2}P_2^2 < P_3^2$, *then*

$$\begin{aligned}
P_3 &= \sqrt{K-2\mathcal{H}c_1}\,\mathrm{nd}\left(\sqrt{\frac{K-2\mathcal{H}c_2}{c_1c_2}}t, \frac{\sqrt{2\mathcal{H}(c_1-c_2)}}{\sqrt{K-2\mathcal{H}c_2}}\right) \\
\textit{and/or}\quad P_3 &= \sqrt{K-2\mathcal{H}c_2}\,\mathrm{dn}\left(\sqrt{\frac{K-2\mathcal{H}c_2}{c_1c_2}}t, \frac{\sqrt{2\mathcal{H}(c_1-c_2)}}{\sqrt{K-2\mathcal{H}c_2}}\right),
\end{aligned}$$

or if $P_3^2 < \frac{c_1-c_2}{c_2}P_2^2$, *then*

$$\begin{aligned}
P_3 &= \frac{\sqrt{(2\mathcal{H}c_1-K)(K-2\mathcal{H}c_2)}}{\sqrt{2\mathcal{H}(c_1-c_2)}}\,\mathrm{sd}\left(\sqrt{\frac{2\mathcal{H}(c_1-c_2)}{c_1c_2}}t, \frac{\sqrt{K-2\mathcal{H}c_2}}{\sqrt{2\mathcal{H}(c_1-c_2)}}\right) \\
\textit{and/or}\quad P_3 &= \sqrt{K-2\mathcal{H}c_2}\,\mathrm{cn}\left(\sqrt{\frac{2\mathcal{H}(c_1-c_2)}{c_1c_2}}t, \frac{\sqrt{K-2\mathcal{H}c_2}}{\sqrt{2\mathcal{H}(c_1-c_2)}}\right).
\end{aligned}$$

CASE (II): $c_2 > c_1$

If $\frac{c_2-c_1}{c_1}P_1^2 < P_3^2$, *then*

$$\begin{aligned}
P_3 &= \sqrt{K-2\mathcal{H}c_2}\,\mathrm{nd}\left(\sqrt{\frac{K-2\mathcal{H}c_1}{c_1c_2}}t, \frac{\sqrt{2\mathcal{H}(c_2-c_1)}}{\sqrt{K-2\mathcal{H}c_1}}\right) \\
\textit{and/or}\quad P_3 &= \sqrt{K-2\mathcal{H}c_1}\,\mathrm{dn}\left(\sqrt{\frac{K-2\mathcal{H}c_1}{c_1c_2}}t, \frac{\sqrt{2\mathcal{H}(c_2-c_1)}}{\sqrt{K-2\mathcal{H}c_1}}\right),
\end{aligned}$$

or if $P_3^2 < \frac{c_2-c_1}{c_1}P_1^2$, *then*

$$\begin{aligned}
P_3 &= \frac{\sqrt{(2\mathcal{H}c_2-K)(K-2\mathcal{H}c_1)}}{\sqrt{2\mathcal{H}(c_2-c_1)}}\,\mathrm{sd}\left(\sqrt{\frac{2\mathcal{H}(c_2-c_1)}{c_1c_2}}t, \frac{\sqrt{K-2\mathcal{H}c_1}}{\sqrt{2\mathcal{H}(c_2-c_1)}}\right) \\
\textit{and/or}\quad P_3 &= \sqrt{K-2\mathcal{H}c_1}\,\mathrm{cn}\left(\sqrt{\frac{2\mathcal{H}(c_2-c_1)}{c_1c_2}}t, \frac{\sqrt{K-2\mathcal{H}c_1}}{\sqrt{2\mathcal{H}(c_2-c_1)}}\right).
\end{aligned}$$

Finally,

$$\begin{aligned}
P_1 &= \pm\sqrt{\frac{c_1}{c_1-c_2}(K-2\mathcal{H}c_2-P_3^2)} \\
P_2 &= \pm\sqrt{\frac{c_2}{c_2-c_1}(K-2\mathcal{H}c_1-P_3^2)}.
\end{aligned}$$

Proof: Multiplying the Hamiltonian (5.4) by $2c_1c_2$ gives

$$c_2P_1^2 + c_1P_2^2 = 2\mathcal{H}c_1c_2. \tag{5.12}$$

Using the Casimir function on $\mathfrak{so}(3)^*$ (Prop. 3.3.1),

$$P_1^2 + P_2^2 + P_3^2 = K, \tag{5.13}$$

and rearranging for P_1^2, we substitute into (5.12) and solve for P_2^2:

$$\begin{aligned} c_2(K - P_2^2 - P_3^2) + c_1P_2^2 &= 2\mathcal{H}c_1c_2 \\ \Rightarrow \quad P_2^2 &= \frac{c_2}{c_2 - c_1}(K - 2\mathcal{H}c_1 - P_3^2). \end{aligned} \tag{5.14}$$

We rearrange (5.13) for P_2^2, substitute into (5.12) and solve for P_1^2:

$$\begin{aligned} c_2P_1^2 + c_1(K - P_1^2 - P_3^2) &= 2\mathcal{H}c_1c_2 \\ \Rightarrow \quad P_1^2 &= \frac{c_1}{c_1 - c_2}(K - 2\mathcal{H}c_2 - P_3^2). \end{aligned} \tag{5.15}$$

Squaring equation (5.7),

$$\dot{P}_3^2 = \left(\frac{c_1 - c_2}{c_1c_2}\right)^2 P_1^2P_2^2,$$

and substituting in equations (5.14) and (5.15) leaves us with

$$\begin{aligned} \dot{P}_3^2 &= \frac{(c_1 - c_2)^2}{c_1^2c_2^2}\frac{c_1}{c_1 - c_2}(K - 2\mathcal{H}c_2 - P_3^2)\frac{c_2}{c_2 - c_1}(K - 2\mathcal{H}c_1 - P_3^2) \\ &= -\frac{1}{c_1c_2}(K - 2\mathcal{H}c_2 - P_3^2)(K - 2\mathcal{H}c_1 - P_3^2). \end{aligned}$$

Possible rearrangements of this equation to get a positive constant in front include:

$$\begin{aligned} \dot{P}_3^2 &= \frac{1}{c_1c_2}(K - 2\mathcal{H}c_2 - P_3^2)(P_3^2 - (K - 2\mathcal{H}c_1)) && (5.16) \\ &= \frac{1}{c_1c_2}(K - 2\mathcal{H}c_1 - P_3^2)(P_3^2 - (K - 2\mathcal{H}c_2)) && (5.17) \\ &= \frac{1}{c_1c_2}(2\mathcal{H}c_2 - K + P_3^2)(K - 2\mathcal{H}c_1 - P_3^2) && (5.18) \\ &= \frac{1}{c_1c_2}(2\mathcal{H}c_1 - K + P_3^2)(K - 2\mathcal{H}c_2 - P_3^2). && (5.19) \end{aligned}$$

Let us consider each equation and its solution independently.

Eq. (5.16) corresponds to the elliptic integrals (1.8) and (1.9):

$$\int_{b}^{P_3} \frac{dP_3}{\sqrt{(a^2-P_3^2)(P_3^2-b^2)}} = \frac{1}{a}\,\mathrm{nd}^{-1}\left(\frac{P_3}{b}, \frac{\sqrt{a^2-b^2}}{a}\right), \quad b \le P_3 \le a, \tag{5.20}$$

$$\int_{P_3}^{a} \frac{dP_3}{\sqrt{(a^2-P_3^2)(P_3^2-b^2)}} = \frac{1}{a}\,\mathrm{dn}^{-1}\left(\frac{P_3}{a}, \frac{\sqrt{a^2-b^2}}{a}\right), \quad b \le P_3 \le a, \tag{5.21}$$

where

$$a^2 = K - 2\mathcal{H}c_2 > 0 \quad \text{and} \quad b^2 = K - 2\mathcal{H}c_1 > 0.$$

Substituting (5.12) and (5.13) into $a^2 > 0$ and $b^2 > 0$, and simplifying gives

$$a^2 > 0 \quad \Leftrightarrow \quad \frac{c_2 - c_1}{c_1}P_1^2 < P_3^2 \quad \text{and} \quad b^2 > 0 \quad \Leftrightarrow \quad \frac{c_1 - c_2}{c_2}P_2^2 < P_3^2.$$

Substituting a^2 and b^2 (and further (5.12) and (5.13)) into $b \le P_3 \le a$ (from (5.20) and (5.21)), and simplifying gives another constraint: $c_2 < c_1$. Therefore, $a^2 > 0$ is always satisfied.

From (5.16) and (5.20) we determine P_3:

(We do not consider $\dot{P}_3 = -\sqrt{\frac{1}{c_1c_2}(K - 2\mathcal{H}c_2 - P_3^2)(P_3^2 - (K - 2\mathcal{H}c_1))}$ as it can be checked numerically that this does not contribute any new solutions.)

$$\begin{aligned}
\dot{P}_3 &= \sqrt{\frac{1}{c_1c_2}(K - 2\mathcal{H}c_2 - P_3^2)(P_3^2 - (K - 2\mathcal{H}c_1))} \\
t &= \frac{1}{\sqrt{\frac{1}{c_1c_2}}}\int_{b}^{P_3} \frac{dP_3}{\sqrt{(K - 2\mathcal{H}c_2 - P_3^2)(P_3^2 - (K - 2\mathcal{H}c_1))}} \\
t &= \frac{1}{\sqrt{\frac{1}{c_1c_2}}}\frac{1}{\sqrt{K - 2\mathcal{H}c_2}}\,\mathrm{nd}^{-1}\left(\frac{P_3}{\sqrt{K - 2\mathcal{H}c_1}}, \frac{\sqrt{2\mathcal{H}c_1 - 2\mathcal{H}c_2}}{\sqrt{K - 2\mathcal{H}c_2}}\right) \\
\Rightarrow \quad P_3 &= \sqrt{K - 2\mathcal{H}c_1}\,\mathrm{nd}\left(\sqrt{\frac{K - 2\mathcal{H}c_2}{c_1c_2}}\,t, \frac{\sqrt{2\mathcal{H}c_1 - 2\mathcal{H}c_2}}{\sqrt{K - 2\mathcal{H}c_2}}\right).
\end{aligned}$$

Similarly, from (5.16) and (5.21) we can also determine P_3:

$$P_3 = \sqrt{K - 2\mathcal{H}c_2}\,\mathrm{dn}\left(\sqrt{\frac{K - 2\mathcal{H}c_2}{c_1c_2}}\,t, \frac{\sqrt{2\mathcal{H}c_1 - 2\mathcal{H}c_2}}{\sqrt{K - 2\mathcal{H}c_2}}\right).$$

Case (5.17) (i.e., CASE (II) (the first part)) is easily deduced from case (5.16) (i.e., CASE (I) (the first part)) by swapping c_1 and c_2.

Eq. (5.18) corresponds to the elliptic integrals (1.5) and (1.3):

$$\int_0^{P_3} \frac{dP_3}{\sqrt{(a^2+P_3^2)(b^2-P_3^2)}} = \frac{1}{\sqrt{a^2+b^2}} \operatorname{sd}^{-1}\left(\frac{\sqrt{a^2+b^2}P_3}{ab}, \frac{b}{\sqrt{a^2+b^2}}\right), \quad 0 \le P_3 \le b, \tag{5.22}$$

$$\int_{P_3}^{b} \frac{dP_3}{\sqrt{(a^2+P_3^2)(b^2-P_3^2)}} = \frac{1}{\sqrt{a^2+b^2}} \operatorname{cn}^{-1}\left(\frac{P_3}{b}, \frac{b}{\sqrt{a^2+b^2}}\right), \quad 0 \le P_3 \le b, \tag{5.23}$$

where

$$a^2 = 2\mathcal{H}c_2 - K > 0 \quad \text{and} \quad b^2 = K - 2\mathcal{H}c_1 > 0.$$

Substituting (5.12) and (5.13) into $a^2 > 0$ and $b^2 > 0$, and simplifying gives

$$a^2 > 0 \quad \Leftrightarrow \quad P_3^2 < \frac{c_2 - c_1}{c_1} P_1^2 \quad \text{and} \quad b^2 > 0 \quad \Leftrightarrow \quad \frac{c_1 - c_2}{c_2} P_2^2 < P_3^2.$$

Substituting b^2 (and further (5.12) and (5.13)) into $0 \le P_3 \le b$ (from (5.22) and (5.23)), and simplifying gives another constraint: $c_1 < c_2$. Therefore, $b^2 > 0$ is always satisfied.

From (5.18) and (5.22) (following the same approach as before) we determine P_3:

$$P_3 = \frac{\sqrt{(2\mathcal{H}c_2 - K)(K - 2\mathcal{H}c_1)}}{\sqrt{2\mathcal{H}c_2 - 2\mathcal{H}c_1}} \operatorname{sd}\left(\sqrt{\frac{2\mathcal{H}c_2 - 2\mathcal{H}c_1}{c_1c_2}}\,t, \frac{\sqrt{K - 2\mathcal{H}c_1}}{\sqrt{2\mathcal{H}c_2 - 2\mathcal{H}c_1}}\right).$$

Similarly, from (5.18) and (5.23) we can also determine P_3:

$$P_3 = \sqrt{K - 2\mathcal{H}c_1} \operatorname{cn}\left(\sqrt{\frac{2\mathcal{H}c_2 - 2\mathcal{H}c_1}{c_1c_2}}\,t, \frac{\sqrt{K - 2\mathcal{H}c_1}}{\sqrt{2\mathcal{H}c_2 - 2\mathcal{H}c_1}}\right).$$

Case (5.19) (i.e., CASE (I) (the second part)) is easily deduced from case (5.18) (i.e., CASE (II) (the second part)) by swapping c_1 and c_2.

Finally, we obtain P_1 and P_2 by substituting P_3 into (5.15) and (5.14), respectively. □

5.4 Numerical solutions to the extremal equations

In the appendix, I have included the MATLAB code (appendix B) used to generate graphs (appendix C) of the solution to the extremal equations ((5.5), (5.6), (5.7)). SECTION 1 of the code plots the solution for arbitrarily chosen c_1, c_2 and initial values using a MATLAB solver. In SECTION 2, the same constants are used, but we substitute them into Theorem 5.3.1 (incl. (5.12) and (5.13)) and plot the result. In both sections the individual components P_1, P_2, P_3 are plotted on one set of axes, followed by the solution curve $P(\cdot)$ plotted on a seperate set of axes. In the instance where either P_1 or P_2 cut the x-axis the negative curve needs to be plotted, as this forms part of the solution (otherwise, the negative of the curve does not satisfy the chosen initial values). In the conclusion, the plots produced by the MATLAB solver are compared to those from Theorem 5.3.1 – this will help us gauge the accuracy of the results in the theorem.

5.5 The solution to the left-invariant optimal control problem

Proposition 5.5.1 has been derived from a similar result found in [9].

Assume first that $K \neq 0$. Recall (from section 3.4) that the solution curve on $\mathsf{SO}(3)$ is given by

$$g(t) = e^{\frac{1}{\sqrt{K}}\Lambda\phi_1(t)} e^{E_2\phi_2(t)} e^{E_3\phi_3(t)}, \tag{5.24}$$

where $g(t)P(t)g^{-1}(t) = \Lambda$, and we are implicitly assuming that Λ is not linearly dependent on E_2.

Proposition 5.5.1. *Suppose that $(g(\cdot), P(\cdot))$ is an extremal curve such that $g(t)P(t)g^{-1}(t) = \Lambda$, then $\phi_1(\cdot)$, $\phi_2(\cdot)$ and $\phi_3(\cdot)$ are given by the following differential equations:*

$$\begin{aligned}
\dot{\phi}_1 &= \frac{\sqrt{K}}{c_2}\left(1 - \frac{(c_2 - c_1)}{c_1}\frac{P_1\cos\phi_3}{(P_2\sin\phi_3 - P_1\cos\phi_3)}\right) \\
\dot{\phi}_2 &= \frac{(c_2 - c_1)}{c_1c_2}\frac{P_1P_2}{(P_2\sin\phi_3 - P_1\cos\phi_3)} \\
\dot{\phi}_3 &= -\frac{1}{c_2}P_3\left(1 - \frac{(c_2 - c_1)}{c_1}\frac{P_1\cos\phi_3}{(P_2\sin\phi_3 - P_1\cos\phi_3)}\right).
\end{aligned}$$

Proof: We calculate the derivative of (5.24),

$$\dot{g} = \frac{1}{\sqrt{K}}\Lambda g\dot{\phi}_1 + e^{\frac{1}{\sqrt{K}}\Lambda\phi_1} e^{E_2\phi_2} E_2 e^{E_3\phi_3}\dot{\phi}_2 + gE_3\dot{\phi}_3,$$

which can be rewritten as

$$\dot{g} = g\left(\frac{1}{\sqrt{K}}g^{-1}\Lambda g\dot{\phi}_1 + e^{-E_3\phi_3}E_2e^{E_3\phi_3}\dot{\phi}_2 + E_3\dot{\phi}_3\right).$$

Since $\dot{g} = g(\frac{1}{c_1}P_1E_1 + \frac{1}{c_2}P_2E_2)$,

$$\left(\frac{1}{\sqrt{K}}g^{-1}\Lambda g\right)\dot{\phi}_1 + \left(e^{-E_3\phi_3}E_2e^{E_3\phi_3}\right)\dot{\phi}_2 + E_3\dot{\phi}_3 = \frac{1}{c_1}P_1E_1 + \frac{1}{c_2}P_2E_2,$$

and applying the hat mapping (Def. 2.2.1) gives

$$\frac{1}{\sqrt{K}}\widehat{P}\dot{\phi}_1 + \left(e^{-E_3\phi_3}e_2\right)\dot{\phi}_2 + e_3\dot{\phi}_3 = \frac{1}{c_1}P_1e_1 + \frac{1}{c_2}P_2e_2.$$

Furthermore, $\widehat{P} = P_1e_1 + P_2e_2 + P_3e_3$ and since

$$e^{-E_3\phi_3} = \begin{bmatrix} \cos\phi_3 & \sin\phi_3 & 0 \\ -\sin\phi_3 & \cos\phi_3 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

it follows that $e^{-E_3\phi_3}e_2 = (\sin\phi_3)e_1 + (\cos\phi_3)e_2$. Therefore,

$$\frac{1}{\sqrt{K}}P_1\dot{\phi}_1 + (\sin\phi_3)\dot{\phi}_2 = \frac{1}{c_1}P_1, \quad \frac{1}{\sqrt{K}}P_2\dot{\phi}_1 + (\cos\phi_3)\dot{\phi}_2 = \frac{1}{c_2}P_2 \quad \text{and} \quad \frac{1}{\sqrt{K}}P_3\dot{\phi}_1 + \dot{\phi}_3 = 0.$$

The first two equations can be solved for $\dot{\phi}_1$ and $\dot{\phi}_2$; $\dot{\phi}_1$ is substituted into the third equation to obtain the final result. □

This set of differential equations is solved using numerical techniques, but this goes beyond the scope of what we are interested in achieving here.

Recall (from section 3.4) that when $K = 0$, it is assumed that $\Lambda = E_1 - E_3$. The required set of differential equations is then obtained from Proposition 5.5.1 by substituting in $\frac{1}{\sqrt{K}} = 1$.

5.6 Stability

Theorems 5.6.1-5.6.3 have been derived from similar results and ideas found in [19] and [20].

The equilibrium states for the system (5.5), (5.6) and (5.7) are:

$$\widehat{P}_{e1} = (M, 0, 0), \quad \widehat{P}_{e2} = (0, M, 0) \quad \text{and} \quad \widehat{P}_{e3} = (0, 0, M), \quad M \in \mathbb{R}.$$

THEOREM 5.6.1. *The equilibrium state* $\widehat{P}_{e1} = (M, 0, 0)$ *is non-linear stable if* $c_1 < c_2$, $M \neq 0$.

PROOF: STEP 1. For any arbitrary smooth function $\varphi : \mathbb{R} \to \mathbb{R}$, the function K_φ, defined by

$$K_\varphi = \varphi\left(\frac{1}{2}(P_1^2 + P_2^2 + P_3^2)\right),$$

gives a family of Casimir functions on $\mathfrak{so}(3)^*$ (Prop. 3.3.1, Prop. 1.4.3, Def. 1.3.13).

STEP 2. We want to find a single K_φ such that the energy-Casimir function,

$$\mathcal{H}+K_\varphi=\frac{1}{2}\left(\frac{P_1^2}{c_1}+\frac{P_2^2}{c_2}\right)+\varphi\left(\frac{1}{2}(P_1^2+P_2^2+P_3^2)\right),$$

has a critical point at the equilibrium of interest.

Now, we calculate the first derivative of $\mathcal{H}+K_\varphi$:

$$\begin{aligned}D(\mathcal{H}+K_\varphi)(p)\cdot\delta p&=\frac{d}{dt}(\mathcal{H}+K_\varphi)\ (p+t\delta)|_{t=0}\\&=\frac{d}{dt}\left(\frac{1}{2c_1}(P_1+t\delta_1)^2+\frac{1}{2c_2}(P_2+t\delta_2)^2\right)\Big|_{t=0}\\&+\frac{d}{dt}\varphi\left(\frac{1}{2}(P_1+t\delta_1)^2+\frac{1}{2}(P_2+t\delta_2)^2+\frac{1}{2}(P_3+t\delta_3)^2\right)\Big|_{t=0}\\&=\frac{1}{c_1}\delta_1P_1+\frac{1}{c_2}\delta_2P_2+(\delta_1P_1+\delta_2P_2+\delta_3P_3)\dot{\varphi}\left(\frac{1}{2}(P_1^2+P_2^2+P_3^2)\right),\end{aligned}\tag{5.25}$$

and this equals zero at the equilibrium of interest if and only if

$$\dot{\varphi}\left(\frac{1}{2}M^2\right)=-\frac{1}{c_1}.\tag{5.26}$$

STEP 3. Next, using (5.25), we can calculate the second derivative of $\mathcal{H}+K_\varphi$:

$$\begin{aligned}D^2(\mathcal{H}+K_\varphi)(p)\cdot\delta p&=\frac{d}{dt}\left(\frac{1}{c_1}\delta_1(P_1+t\delta_1)+\frac{1}{c_2}\delta_2(P_2+t\delta_2)\right)|_{t=0}\\&+\frac{d}{dt}(\delta_1(P_1+t\delta_1)+\delta_2(P_2+t\delta_2)+\delta_3(P_3+t\delta_3))\\&\times\dot{\varphi}\left(\frac{1}{2}((P_1+t\delta_1)^2+(P_2+t\delta_2)^2+(P_3+t\delta_3)^2)\right)|_{t=0}\\&=\frac{1}{c_1}\delta_1^2+\frac{1}{c_2}\delta_2^2+(\delta_1^2+\delta_2^2+\delta_3^2)\dot{\varphi}\left(\frac{1}{2}(P_1^2+P_2^2+P_3^2)\right)\\&+(\delta_1P_1+\delta_2P_2+\delta_3P_3)^2\ddot{\varphi}\left(\frac{1}{2}(P_1^2+P_2^2+P_3^2)\right).\end{aligned}\tag{5.27}$$

Using (5.26), the second derivative of $\mathcal{H}+K_\varphi$ at the equilibrium of interest is

$$D^2(\mathcal{H}+K_\varphi)(\hat{P}_{e1})\cdot\delta\hat{P}_{e1}=\ddot{\varphi}\left(\frac{1}{2}M^2\right)M^2\delta_1^2+\left(\frac{1}{c_2}-\frac{1}{c_1}\right)\delta_2^2-\frac{1}{c_1}\delta_3^2.\tag{5.28}$$

Thus, having chosen that

$$\ddot{\varphi}\left(\frac{1}{2}M^2\right)<0,$$

this quadratic form is negative definite if and only if $c_1 < c_2$. Consequently,

$$\varphi(x) = -x\left(x + \frac{1}{c_1} - M^2\right)$$

satisfies the above conditions and the second derivative of $\mathcal{H} + K_\varphi$ is negative definite. □

THEOREM 5.6.2. *The equilibrium state* $\widehat{P}_{e2} = (0, M, 0)$ *is non-linear stable if* $c_1 > c_2$, $M \neq 0$.

PROOF: Theorem 5.6.1 covers STEP 1 and the first part of STEP 2 for this proof.

STEP 2. Eq. (5.25) equals zero at the equilibrium of interest if and only if

$$\dot{\varphi}\left(\frac{1}{2}M^2\right) = -\frac{1}{c_2}. \tag{5.29}$$

STEP 3. Using (5.29), the second derivative of $\mathcal{H} + K_\varphi$ (5.27) at the equilibrium of interest is

$$D^2(\mathcal{H} + K_\varphi)(\hat{P}_{e2}) \cdot \delta\hat{P}_{e2} = \left(\frac{1}{c_1} - \frac{1}{c_2}\right)\delta_1^2 + \ddot{\varphi}\left(\frac{1}{2}M^2\right)M^2\delta_2^2 - \frac{1}{c_2}\delta_3^2. \tag{5.30}$$

Thus, having chosen that

$$\ddot{\varphi}\left(\frac{1}{2}M^2\right) < 0,$$

this quadratic form is negative definite if and only if $c_1 > c_2$. Consequently,

$$\varphi(x) = -x\left(x + \frac{1}{c_2} - M^2\right)$$

satisfies the above conditions and the second derivative of $\mathcal{H} + K_\varphi$ is negative definite. □

THEOREM 5.6.3. *The equilibrium state* $\widehat{P}_{e3} = (0, 0, M)$ *is non-linear stable,* $M \neq 0$.

PROOF: Theorem 5.6.1 covers STEP 1 and the first part of STEP 2 for this proof.

STEP 2. Eq. (5.25) equals zero at the equilibrium of interest if and only if

$$\dot{\varphi}\left(\frac{1}{2}M^2\right) = 0. \tag{5.31}$$

STEP 3. Using (5.31), the second derivative of $\mathcal{H} + K_\varphi$ (5.27) at the equilibrium of interest is

$$D^2(\mathcal{H} + K_\varphi)(\hat{P}_{e3}) \cdot \delta\hat{P}_{e3} = \frac{1}{c_1}\delta_1^2 + \frac{1}{c_2}\delta_2^2 + \ddot{\varphi}\left(\frac{1}{2}M^2\right)M^2\delta_3^2. \tag{5.32}$$

Thus, having chosen that

$$\ddot{\varphi}\left(\frac{1}{2}M^2\right) > 0,$$

this quadratic form is positive definite. Consequently,

$$\varphi(x) = x(x - M^2)$$

satisfies the above conditions and the second derivative of $\mathcal{H} + K_\varphi$ is positive definite. □

Chapter 6

The Type IIb Problem

Here we look at the solution to the Type IIb left-invariant optimal control problem on $\mathsf{SO}(3)$. After defining the problem we prove a theorem that gives the optimal controls, optimal Hamiltonian and extremal equations. The Jacobi elliptic functions are used to solve the extremal equations, and these results are tested by some numerical calculations. Finally, a set of differential equations is achieved, which can be solved for Euler's angles to give the solution to the problem. We conclude this chapter by investigating the non-linear stability of the equilibrium states associated with the extremal equations.

6.1 The Type IIb left-invariant optimal control problem

Let g_1 and g_2 be arbitrary but fixed points in $\mathsf{SO}(3)$, and $T > 0$ be fixed in advance. Then the Type IIb left-invariant optimal control problem on $\mathsf{SO}(3)$ consists of minimizing the cost functional

$$J = \frac{1}{2}\int_0^T (c_1 u_1^2 + c_2 u_2^2)\, dt, \quad c_1,\, c_2 > 0 \tag{6.1}$$

over all trajectory-control pairs $(g(\cdot), u(\cdot))$, of the left-invariant control system

$$\dot{g} = g(E_3 + u_1 E_1 + u_2 E_2), \quad g \in \mathsf{SO}(3),\ u = (u_1, u_2) \in \mathbb{R}^2, \tag{6.2}$$

satisfying the boundary conditions

$$g(0) = g_1 \quad \text{and} \quad g(T) = g_2.$$

6.2 The extremal equations

Theorem 6.2.1 has been derived from similar results found in [9] and [18].

THEOREM 6.2.1. *Given a left-invariant control system on* $\mathsf{SO}(3)$ *defined by (6.2) with cost (6.1), the optimal controls are*

$$u_1 = \frac{1}{c_1} P_1 \quad \text{and} \quad u_2 = \frac{1}{c_2} P_2, \tag{6.3}$$

and the optimal Hamiltonian is

$$\mathcal{H} = \frac{1}{2}\left(\frac{1}{c_1}P_1^2 + \frac{1}{c_2}P_2^2\right) + P_3, \tag{6.4}$$

where P_1, P_2 *and* P_3 *are solutions of*

$$\dot{P}_1 = P_2 - \frac{1}{c_2}P_2P_3 \tag{6.5}$$

$$\dot{P}_2 = -P_1 + \frac{1}{c_1}P_1P_3 \tag{6.6}$$

$$\dot{P}_3 = \frac{c_1 - c_2}{c_1c_2}P_1P_2. \tag{6.7}$$

PROOF: The control Hamiltonian (Def. 1.4.6) is

$$\mathcal{H}(p,u) = -\frac{1}{2}(c_1u_1^2 + c_2u_2^2) + p(E_3 + u_1E_1 + u_2E_2),$$

which simplifies to (Prop. 3.3.1)

$$\mathcal{H}(p,u) = -\frac{1}{2}(c_1u_1^2 + c_2u_2^2) + P_3 + u_1P_1 + u_2P_2. \tag{6.8}$$

Applying (2) of the maximum principle (section 1.5), the optimal controls are

$$\begin{aligned} \frac{\partial\mathcal{H}}{\partial u_1} = 0 \quad &\Leftrightarrow \quad -c_1u_1 + P_1 = 0 \quad \Leftrightarrow \quad u_1 = \frac{1}{c_1}P_1 \\ \frac{\partial\mathcal{H}}{\partial u_2} = 0 \quad &\Leftrightarrow \quad -c_2u_2 + P_2 = 0 \quad \Leftrightarrow \quad u_2 = \frac{1}{c_2}P_2. \end{aligned}$$

Substituting u_1 and u_2 into (6.8) gives (6.4), which combined with Propositions 1.3.6, 1.3.4 and 1.3.7 gives

$$\dot{P}_i = \{P_i, \frac{1}{2c_1}P_1^2 + \frac{1}{2c_2}P_2^2 + P_3\} = \frac{1}{c_1}P_1\{P_i,P_1\} + \frac{1}{c_2}P_2\{P_i,P_2\} + \{P_i,P_3\}, \quad i = 1, 2, 3,$$

and using the Poisson bracket relations from (3.11) we get (6.5), (6.6) and (6.7). □

6.3 The solution to the extremal equations

Proposition 6.3.1 can be found in [17].

PROPOSITION 6.3.1. *Given a Hamiltonian system on* SO(3), *described by (6.2), with cost (6.1) so that $c = c_1 = c_2$, there exist two extra constants of motion,*

$$K_1 = P_1^2 + P_2^2 \quad \textit{and} \tag{6.9}$$

$$K_2 = P_3. \tag{6.10}$$

We can rewrite equations (6.5), (6.6) and (6.7) as

$$\dot{P}_1 = P_2 - \frac{1}{c}P_2P_3(0) \tag{6.11}$$

$$\dot{P}_2 = -P_1 + \frac{1}{c}P_1P_3(0) \tag{6.12}$$

$$\dot{P}_3 = 0, \tag{6.13}$$

which have the solutions

$$P_1(t) = \sqrt{k_1}\cos\left(\left(1 - \frac{k_2}{c}\right)t + k_3\right)$$
$$P_2(t) = \sqrt{k_1}\sin\left(\left(1 - \frac{k_2}{c}\right)t + k_3\right),$$

where

$$k_1 = P_1^2(0) + P_2^2(0), \quad k_2 = P_3(0) \quad \textit{and} \quad k_3 = -\tan^{-1}\frac{P_2(0)}{P_1(0)}.$$

PROOF: We first show that (6.9) and (6.10) are constants of motion:

$$\dot{K}_1 = 2P_1\dot{P}_1 + 2P_2\dot{P}_2 = 2P_1\left(P_2 - \frac{1}{c_2}P_2P_3\right) + 2P_2\left(-P_1 + \frac{1}{c_1}P_1P_3\right) = 0 \quad ((6.5), (6.6), c_1 = c_2)$$

$$\dot{K}_2 = \dot{P}_3 = \frac{c_1 - c_2}{c_1c_2}P_1P_2 = 0. \quad ((6.7), c_1 = c_2)$$

Equations (6.11), (6.12) and (6.13) in the matrix form become

$$\begin{bmatrix} \dot{P}_1 \\ \dot{P}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 - \frac{1}{c}P_3(0) \\ \frac{1}{c}P_3(0) - 1 & 0 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}.$$

Since $P(t) = \exp(tA)P_0$ is the only solution of $\dot{P} = AP$, $P(0) = P_0$, the solution can be written as

$$\begin{bmatrix} P_1(t) \\ P_2(t) \end{bmatrix} = \begin{bmatrix} \cos\left(1 - \frac{1}{c}P_3(0)\right)t & \sin\left(1 - \frac{1}{c}P_3(0)\right)t \\ -\sin\left(1 - \frac{1}{c}P_3(0)\right)t & \cos\left(1 - \frac{1}{c}P_3(0)\right)t \end{bmatrix} \begin{bmatrix} P_1(0) \\ P_2(0) \end{bmatrix},$$

which is equivalent to

$$\begin{aligned} P_1(t) &= \sqrt{P_1^2(0) + P_2^2(0)}\cos\left(\left(1 - \frac{1}{c}P_3(0)\right)t - \tan^{-1}\frac{P_2(0)}{P_1(0)}\right) \\ P_2(t) &= \sqrt{(-P_1(0))^2 + P_2^2(0)}\sin\left(\left(1 - \frac{1}{c}P_3(0)\right)t + \tan^{-1}\frac{P_2(0)}{-P_1(0)}\right). \end{aligned}$$

If we let

$$k_1 = P_1^2(0) + P_2^2(0), \quad k_2 = P_3(0) \quad \text{and} \quad k_3 = -\tan^{-1}\frac{P_2(0)}{P_1(0)},$$

we obtain the desired result. □

The beginning of the proof of Theorem 6.3.1 is based on similar ideas found in [18] and [19]. However, the integration using elliptic functions is original (with reference to [12]).

THEOREM 6.3.1. *The equations (6.5), (6.6) and (6.7), where* $c_1 \neq c_2$, *can be integrated via elliptic functions. More exactly (the restraint* $\mathcal{H}^2 - K > 0$ *is imposed since* c_1 *and* c_2 *are real),*

CASE (I): $c_1 > c_2$

If $\mathcal{H} - c_1 < -\sqrt{\mathcal{H}^2 - K}$, $\mathcal{H} - c_2 > \sqrt{\mathcal{H}^2 - K}$ *and* $\frac{\mathcal{H} - c_1 + \sqrt{\mathcal{H}^2 - K}}{\mathcal{H} - c_1 - \sqrt{\mathcal{H}^2 - K}} < \frac{\mathcal{H} - c_2 + \sqrt{\mathcal{H}^2 - K}}{\mathcal{H} - c_2 - \sqrt{\mathcal{H}^2 - K}}$, *then*

$$\begin{aligned} P_3 &= \frac{\alpha - \beta\, a\, \mathrm{dc}\left(\frac{(\alpha - \beta)\sqrt{A_1 A_2}}{\sqrt{c_1 c_2}}\, a\, t, \frac{b}{a}\right)}{1 - a\, \mathrm{dc}\left(\frac{(\alpha - \beta)\sqrt{A_1 A_2}}{\sqrt{c_1 c_2}}\, a\, t, \frac{b}{a}\right)} \\ \textit{and/or} \quad P_3 &= \frac{\alpha - \beta\, a\, \mathrm{ns}\left(\frac{(\alpha - \beta)\sqrt{A_1 A_2}}{\sqrt{c_1 c_2}}\, a\, t, \frac{b}{a}\right)}{1 - a\, \mathrm{ns}\left(\frac{(\alpha - \beta)\sqrt{A_1 A_2}}{\sqrt{c_1 c_2}}\, a\, t, \frac{b}{a}\right)}, \end{aligned}$$

where

$$\begin{aligned} a^2 &= \frac{\mathcal{H} - c_2 + \sqrt{\mathcal{H}^2 - K}}{\mathcal{H} - c_2 - \sqrt{\mathcal{H}^2 - K}}, \quad b^2 = \frac{\mathcal{H} - c_1 + \sqrt{\mathcal{H}^2 - K}}{\mathcal{H} - c_1 - \sqrt{\mathcal{H}^2 - K}} \\ \alpha &= \mathcal{H} + \sqrt{\mathcal{H}^2 - K}, \qquad \beta = \mathcal{H} - \sqrt{\mathcal{H}^2 - K} \\ A_1 A_2 &= -\frac{\left(\mathcal{H} - c_2 - \sqrt{\mathcal{H}^2 - K}\right)\left(\mathcal{H} - c_1 - \sqrt{\mathcal{H}^2 - K}\right)}{4(\mathcal{H}^2 - K)}, \end{aligned}$$

or if $-\sqrt{\mathcal{H}^2-K} < \mathcal{H}-c_1 < \sqrt{\mathcal{H}^2-K}$ *and* $\mathcal{H}-c_2 > \sqrt{\mathcal{H}^2-K}$, *then*

$$P_3 = \frac{\alpha - \beta\, a\,\mathrm{nc}\left(\frac{(\alpha-\beta)\sqrt{A_1A_2}\sqrt{a^2+b^2}}{\sqrt{c_1c_2}}\,t,\, \frac{b}{\sqrt{a^2+b^2}}\right)}{1 - a\,\mathrm{nc}\left(\frac{(\alpha-\beta)\sqrt{A_1A_2}\sqrt{a^2+b^2}}{\sqrt{c_1c_2}}\,t,\, \frac{b}{\sqrt{a^2+b^2}}\right)}$$

$$\textit{and/or}\quad P_3 = \frac{\alpha - \beta\,\sqrt{a^2+b^2}\,\mathrm{ds}\left(\frac{(\alpha-\beta)\sqrt{A_1A_2}\sqrt{a^2+b^2}}{\sqrt{c_1c_2}}\,t,\, \frac{b}{\sqrt{a^2+b^2}}\right)}{1 - \sqrt{a^2+b^2}\,\mathrm{ds}\left(\frac{(\alpha-\beta)\sqrt{A_1A_2}\sqrt{a^2+b^2}}{\sqrt{c_1c_2}}\,t,\, \frac{b}{\sqrt{a^2+b^2}}\right)},$$

where

$$\begin{aligned} a^2 &= \frac{\mathcal{H}-c_2+\sqrt{\mathcal{H}^2-K}}{\mathcal{H}-c_2-\sqrt{\mathcal{H}^2-K}}, \quad b^2 = -\frac{\mathcal{H}-c_1+\sqrt{\mathcal{H}^2-K}}{\mathcal{H}-c_1-\sqrt{\mathcal{H}^2-K}} \\ \alpha &= \mathcal{H}+\sqrt{\mathcal{H}^2-K}, \qquad \beta = \mathcal{H}-\sqrt{\mathcal{H}^2-K} \\ A_1A_2 &= -\frac{\left(\mathcal{H}-c_2-\sqrt{\mathcal{H}^2-K}\right)\left(\mathcal{H}-c_1-\sqrt{\mathcal{H}^2-K}\right)}{4\left(\mathcal{H}^2-K\right)}. \end{aligned}$$

CASE (II): $c_2 > c_1$

If $\mathcal{H}-c_2 < -\sqrt{\mathcal{H}^2-K}$, $\mathcal{H}-c_1 > \sqrt{\mathcal{H}^2-K}$ *and* $\frac{\mathcal{H}-c_2+\sqrt{\mathcal{H}^2-K}}{\mathcal{H}-c_2-\sqrt{\mathcal{H}^2-K}} < \frac{\mathcal{H}-c_1+\sqrt{\mathcal{H}^2-K}}{\mathcal{H}-c_1-\sqrt{\mathcal{H}^2-K}}$, *then*

$$P_3 = \frac{\alpha - \beta\, a\,\mathrm{dc}\left(\frac{(\alpha-\beta)\sqrt{A_1A_2}}{\sqrt{c_1c_2}}\,a\,t,\, \frac{b}{a}\right)}{1 - a\,\mathrm{dc}\left(\frac{(\alpha-\beta)\sqrt{A_1A_2}}{\sqrt{c_1c_2}}\,a\,t,\, \frac{b}{a}\right)}$$

$$\textit{and/or}\quad P_3 = \frac{\alpha - \beta\, a\,\mathrm{ns}\left(\frac{(\alpha-\beta)\sqrt{A_1A_2}}{\sqrt{c_1c_2}}\,a\,t,\, \frac{b}{a}\right)}{1 - a\,\mathrm{ns}\left(\frac{(\alpha-\beta)\sqrt{A_1A_2}}{\sqrt{c_1c_2}}\,a\,t,\, \frac{b}{a}\right)},$$

where

$$\begin{aligned} a^2 &= \frac{\mathcal{H}-c_1+\sqrt{\mathcal{H}^2-K}}{\mathcal{H}-c_1-\sqrt{\mathcal{H}^2-K}}, \quad b^2 = \frac{\mathcal{H}-c_2+\sqrt{\mathcal{H}^2-K}}{\mathcal{H}-c_2-\sqrt{\mathcal{H}^2-K}} \\ \alpha &= \mathcal{H}+\sqrt{\mathcal{H}^2-K}, \qquad \beta = \mathcal{H}-\sqrt{\mathcal{H}^2-K} \\ A_1A_2 &= -\frac{\left(\mathcal{H}-c_1-\sqrt{\mathcal{H}^2-K}\right)\left(\mathcal{H}-c_2-\sqrt{\mathcal{H}^2-K}\right)}{4\left(\mathcal{H}^2-K\right)}, \end{aligned}$$

or if $-\sqrt{\mathcal{H}^2-K} < \mathcal{H}-c_2 < \sqrt{\mathcal{H}^2-K}$ *and* $\mathcal{H}-c_1 > \sqrt{\mathcal{H}^2-K}$, *then*

$$P_3 = \frac{\alpha - \beta\, a\,\mathrm{nc}\left(\frac{(\alpha-\beta)\sqrt{A_1A_2}\sqrt{a^2+b^2}}{\sqrt{c_1c_2}}\,t,\, \frac{b}{\sqrt{a^2+b^2}}\right)}{1 - a\,\mathrm{nc}\left(\frac{(\alpha-\beta)\sqrt{A_1A_2}\sqrt{a^2+b^2}}{\sqrt{c_1c_2}}\,t,\, \frac{b}{\sqrt{a^2+b^2}}\right)}$$

$$\textit{and/or}\quad P_3 = \frac{\alpha - \beta\sqrt{a^2+b^2}\,\operatorname{ds}\left(\frac{(\alpha-\beta)\sqrt{A_1A_2}\sqrt{a^2+b^2}}{\sqrt{c_1c_2}}\,t, \frac{b}{\sqrt{a^2+b^2}}\right)}{1 - \sqrt{a^2+b^2}\,\operatorname{ds}\left(\frac{(\alpha-\beta)\sqrt{A_1A_2}\sqrt{a^2+b^2}}{\sqrt{c_1c_2}}\,t, \frac{b}{\sqrt{a^2+b^2}}\right)},$$

where

$$\begin{aligned}
a^2 &= \frac{\mathcal{H} - c_1 + \sqrt{\mathcal{H}^2 - K}}{\mathcal{H} - c_1 - \sqrt{\mathcal{H}^2 - K}}, \quad b^2 = -\frac{\mathcal{H} - c_2 + \sqrt{\mathcal{H}^2 - K}}{\mathcal{H} - c_2 - \sqrt{\mathcal{H}^2 - K}} \\
\alpha &= \mathcal{H} + \sqrt{\mathcal{H}^2 - K}, \qquad \beta = \mathcal{H} - \sqrt{\mathcal{H}^2 - K} \\
A_1A_2 &= -\frac{\left(\mathcal{H} - c_1 - \sqrt{\mathcal{H}^2 - K}\right)\left(\mathcal{H} - c_2 - \sqrt{\mathcal{H}^2 - K}\right)}{4\left(\mathcal{H}^2 - K\right)}.
\end{aligned}$$

Finally,

$$\begin{aligned}
P_1 &= \pm\sqrt{\frac{c_1}{c_2 - c_1}(2\mathcal{H}c_2 - K - 2c_2P_3 + P_3^2)} \\
P_2 &= \pm\sqrt{\frac{c_2}{c_1 - c_2}(2\mathcal{H}c_1 - K - 2c_1P_3 + P_3^2)}.
\end{aligned}$$

PROOF: Multiplying the Hamiltonian (6.4) by $2c_1c_2$ gives

$$c_2P_1^2 + c_1P_2^2 + 2c_1c_2P_3 = 2\mathcal{H}c_1c_2. \tag{6.14}$$

Using the Casimir function on $\mathfrak{so}(3)^*$ (Prop. 3.3.1),

$$P_1^2 + P_2^2 + P_3^2 = K. \tag{6.15}$$

and rearranging for P_1^2, we substitute into (6.14) and solve for P_2^2:

$$\begin{aligned}
&c_2(K - P_2^2 - P_3^2) + c_1P_2^2 + 2c_1c_2P_3 = 2\mathcal{H}c_1c_2 \\
&\qquad\Rightarrow\quad P_2^2 = \frac{c_2}{c_1 - c_2}(2\mathcal{H}c_1 - K - 2c_1P_3 + P_3^2).
\end{aligned} \tag{6.16}$$

We rearrange (6.15) for P_2^2, substitute into (6.14) and solve for P_1^2:

$$\begin{aligned}
&c_2P_1^2 + c_1(K - P_1^2 - P_3^2) + 2c_1c_2P_3 = 2\mathcal{H}c_1c_2 \\
&\qquad\Rightarrow\quad P_1^2 = \frac{c_1}{c_2 - c_1}(2\mathcal{H}c_2 - K - 2c_2P_3 + P_3^2).
\end{aligned} \tag{6.17}$$

Squaring equation (6.7),

$$\dot{P}_3^2 = \left(\frac{c_1 - c_2}{c_1c_2}\right)^2 P_1^2P_2^2,$$

and substituting in equations (6.16) and (6.17) leaves us with

$$\begin{aligned}\dot{P}_3^2 &= \frac{(c_1-c_2)^2}{c_1^2c_2^2}\frac{c_1}{c_2-c_1}(2\mathcal{H}c_2 - K - 2c_2P_3 + P_3^2)\frac{c_2}{c_1-c_2}(2\mathcal{H}c_1 - K - 2c_1P_3 + P_3^2)\\ &= -\frac{1}{c_1c_2}(K - 2\mathcal{H}c_2 + 2c_2P_3 - P_3^2)(K - 2\mathcal{H}c_1 + 2c_1P_3 - P_3^2).\end{aligned}$$

Possible rearrangements of this equation to get a positive constant in front include:

$$\dot{P}_3^2 = \frac{1}{c_1c_2}(K - 2\mathcal{H}c_2 + 2c_2P_3 - P_3^2)(P_3^2 - 2c_1P_3 - (K - 2\mathcal{H}c_1)) \tag{6.18}$$

$$= \frac{1}{c_1c_2}(K - 2\mathcal{H}c_1 + 2c_1P_3 - P_3^2)(P_3^2 - 2c_2P_3 - (K - 2\mathcal{H}c_2)). \tag{6.19}$$

The form of (6.18) and (6.19) do not match the standard form of the elliptic integrals. Therefore, we need to perform a transformation on (6.18) and (6.19), and the approach is outlined in section 1.6. Some of the working that follows has been done in *Wolfram Mathematica 7.0*, and this code can be found in appendix A.

The expression $\sqrt{(c_1-c_2)^2(\mathcal{H}^2-K)}$ appears in the calculations throughout the remainder of this proof. It is always real since the discriminant of $P_3^2 - 2c_1P_3 - K + 2\mathcal{H}c_1$ (6.18) (and similarly for (6.19)) is

$$\Delta = 4c_1^2 - 8\mathcal{H}c_1 + 4K,$$

and this is a quadratic in c_1 with discriminant given by

$$\Delta = 64(\mathcal{H}^2 - K) > 0 \quad \text{(since } c_1 \text{ is real)} \quad \Rightarrow \quad \mathcal{H}^2 - K > 0.$$

The same result holds true for c_2. Hence, the condition imposed in the theorem.

Let us consider (6.18) and (6.19), and their solutions.

CASE (6.18): Let

$$S_1 = K - 2\mathcal{H}c_2 + 2c_2P_3 - P_3^2 \quad \text{and} \quad S_2 = P_3^2 - 2c_1P_3 - K + 2\mathcal{H}c_1.$$

(If we swap S_1 and S_2 we achieve no new results.)

Consider the quadratic expression $S_1 + \lambda S_2$; this has coincident zeros and is a perfect square whenever (see (1.13))

$$(-K + 2\mathcal{H}c_1 - c_1^2)\lambda^2 + 2(K - \mathcal{H}c_2 - \mathcal{H}c_1 + c_1c_2)\lambda - K + 2\mathcal{H}c_2 - c_2^2 = 0.$$

Solving this expression for λ (see appendix A) yields

$$\begin{aligned}\lambda_1 &= \frac{-\sqrt{(c_1-c_2)^2(\mathcal{H}^2-K)}-c_2\mathcal{H}+c_1(c_2-\mathcal{H})+K}{-2c_1\mathcal{H}+c_1^2+K}\\ \lambda_2 &= \frac{\sqrt{(c_1-c_2)^2(\mathcal{H}^2-K)}-c_2\mathcal{H}+c_1(c_2-\mathcal{H})+K}{-2c_1\mathcal{H}+c_1^2+K}.\end{aligned}$$

(The values for λ_1 and λ_2 are chosen to restrict ourselves to real solutions later.)

Substitute λ_1 and λ_2 into (1.16) and (1.18) (see appendix A), then (1.17) gives

$$S_1 = A_1(P_3-\alpha)^2 + B_1(P_3-\beta)^2 \quad \text{and} \quad S_2 = A_2(P_3-\alpha)^2 + B_2(P_3-\beta)^2, \tag{6.20}$$

where

$$\begin{aligned}A_1 &= \frac{(c_1-c_2)(\mathcal{H}-c_2)-\sqrt{(c_1-c_2)^2(\mathcal{H}^2-K)}}{2\sqrt{(c_1-c_2)^2(\mathcal{H}^2-K)}}, & A_2 &= \frac{\sqrt{(c_1-c_2)^2(\mathcal{H}^2-K)}-c_1(c_2+\mathcal{H})+c_2\mathcal{H}+c_1^2}{2\sqrt{(c_1-c_2)^2(\mathcal{H}^2-K)}}\\ B_1 &= -\frac{\sqrt{(c_1-c_2)^2(\mathcal{H}^2-K)}+(c_1-c_2)(\mathcal{H}-c_2)}{2\sqrt{(c_1-c_2)^2(\mathcal{H}^2-K)}}, & B_2 &= \frac{\sqrt{(c_1-c_2)^2(\mathcal{H}^2-K)}+(c_1-c_2)(\mathcal{H}-c_1)}{2\sqrt{(c_1-c_2)^2(\mathcal{H}^2-K)}}\\ \alpha &= \mathcal{H}+\sqrt{\mathcal{H}^2-K}, & \beta &= \mathcal{H}-\sqrt{\mathcal{H}^2-K}, \quad \text{when } c_1 > c_2\\ \alpha &= \mathcal{H}-\sqrt{\mathcal{H}^2-K}, & \beta &= \mathcal{H}+\sqrt{\mathcal{H}^2-K}, \quad \text{when } c_2 > c_1.\end{aligned}$$

Now, since S_1 and S_2 have been expressed in the form of (6.20), we can rewrite (6.18) as

$$\dot{P}_3^2 = \frac{1}{c_1c_2}(A_1(P_3-\alpha)^2+B_1(P_3-\beta)^2)(A_2(P_3-\alpha)^2+B_2(P_3-\beta)^2).$$

Following the steps towards the end of section 1.6 we achieve (1.20), and this can be rewritten to match the form of the Jacobi elliptic integrals (1.6) and (1.7). Substituting the integral value (1.6) followed by u (1.19), and solving for P_3 gives $\left(a^2 = -\frac{B_1}{A_1} \quad \text{and} \quad b^2 = -\frac{B_2}{A_2}\right)$

$$\begin{aligned}t &= \frac{\sqrt{c_1c_2}}{(\alpha-\beta)\sqrt{A_1A_2}}\int_a^{P_3}\frac{du}{\sqrt{(u^2-(-\frac{B_1}{A_1}))(u^2-(-\frac{B_2}{A_2}))}}\\ t &= \frac{\sqrt{c_1c_2}}{(\alpha-\beta)\sqrt{A_1A_2}a}\mathrm{dc}^{-1}\left(\frac{u}{a},\frac{b}{a}\right)\end{aligned} \tag{6.21}$$

$$\Rightarrow \quad P_3 = \frac{\alpha - \beta\, a\, \mathrm{dc}\left(\frac{(\alpha-\beta)\sqrt{A_1A_2}a}{\sqrt{c_1c_2}}\, t,\, \frac{b}{a}\right)}{1 - a\, \mathrm{dc}\left(\frac{(\alpha-\beta)\sqrt{A_1A_2}a}{\sqrt{c_1c_2}}\, t,\, \frac{b}{a}\right)},$$

where (see appendix A)

$$a^2 = \frac{\sqrt{(c_1-c_2)^2(\mathcal{H}^2-K)} + (c_1-c_2)(\mathcal{H}-c_2)}{(c_1-c_2)(\mathcal{H}-c_2) - \sqrt{(c_1-c_2)^2(\mathcal{H}^2-K)}} > 0,$$

$$b^2 = -\frac{\sqrt{(c_1-c_2)^2(\mathcal{H}^2-K)} + (c_1-c_2)(\mathcal{H}-c_1)}{\sqrt{(c_1-c_2)^2(\mathcal{H}^2-K)} - c_1(c_2+\mathcal{H}) + c_2\mathcal{H} + c_1^2} > 0 \quad \text{and}$$

$$A_1A_2 = \frac{\left((c_1-c_2)(H-c_2) - \sqrt{(c_1-c_2)^2(H^2-K)}\right)\left(\sqrt{(c_1-c_2)^2(H^2-K)} - c_1(c_2+H) + c_2H + c_1^2\right)}{4(c_1-c_2)^2(H^2-K)}.$$

Similarly, substituting the integral value (1.7) into (6.21), followed by u (1.19), and solving for P_3 gives

$$P_3 = \frac{\alpha - \beta\, a\, \mathrm{ns}\left(\frac{(\alpha-\beta)\sqrt{A_1A_2}a}{\sqrt{c_1c_2}}\, t,\, \frac{b}{a}\right)}{1 - a\, \mathrm{ns}\left(\frac{(\alpha-\beta)\sqrt{A_1A_2}a}{\sqrt{c_1c_2}}\, t,\, \frac{b}{a}\right)},$$

with the same values for a^2, b^2 and A_1A_2.

Although,

$$a^2 > 0 \Rightarrow (c_1-c_2)(\mathcal{H}-c_2) > \sqrt{(c_1-c_2)^2(\mathcal{H}^2-K)} \quad \text{or} \quad (c_1-c_2)(\mathcal{H}-c_2) < -\sqrt{(c_1-c_2)^2(\mathcal{H}^2-K)}$$

$$b^2 > 0 \Rightarrow (c_1-c_2)(\mathcal{H}-c_1) > \sqrt{(c_1-c_2)^2(\mathcal{H}^2-K)} \quad \text{or} \quad (c_1-c_2)(\mathcal{H}-c_1) < -\sqrt{(c_1-c_2)^2(\mathcal{H}^2-K)},$$

the combinations

$$(c_1-c_2)(\mathcal{H}-c_2) > \sqrt{(c_1-c_2)^2(\mathcal{H}^2-K)} \quad \text{and} \quad (c_1-c_2)(\mathcal{H}-c_1) > \sqrt{(c_1-c_2)^2(\mathcal{H}^2-K)},$$

and

$$(c_1-c_2)(\mathcal{H}-c_2) < -\sqrt{(c_1-c_2)^2(\mathcal{H}^2-K)} \quad \text{and} \quad (c_1-c_2)(\mathcal{H}-c_1) < -\sqrt{(c_1-c_2)^2(\mathcal{H}^2-K)}$$

cannot hold, as these would make A_1A_2 negative under the root for P_3.

Furthermore, it can be checked (with Mathematica) that no combination of values for c_1, c_2 and $P(0)$ will satisfy the condition

$$(c_1-c_2)(\mathcal{H}-c_2) < -\sqrt{(c_1-c_2)^2(\mathcal{H}^2-K)} \quad \text{and} \quad (c_1-c_2)(\mathcal{H}-c_1) > \sqrt{(c_1-c_2)^2(\mathcal{H}^2-K)},$$

therefore,

$$(c_1 - c_2)(\mathcal{H} - c_1) < -\sqrt{(c_1 - c_2)^2(\mathcal{H}^2 - K)} \quad \text{and} \quad (c_1 - c_2)(\mathcal{H} - c_2) > \sqrt{(c_1 - c_2)^2(\mathcal{H}^2 - K)}$$

is the only restriction enforced by $a^2 > 0$ and $b^2 > 0$.

Substituting a^2 and b^2 into $b < a$ (from (1.6) or (1.7)) gives

$$\frac{\sqrt{(c_1 - c_2)^2(\mathcal{H}^2 - K)} + (c_1 - c_2)(\mathcal{H} - c_1)}{\sqrt{(c_1 - c_2)^2(\mathcal{H}^2 - K)} - c_1(c_2 + \mathcal{H}) + c_2\mathcal{H} + c_1^2} < \frac{\sqrt{(c_1 - c_2)^2(\mathcal{H}^2 - K)} + (c_1 - c_2)(\mathcal{H} - c_2)}{(c_1 - c_2)(\mathcal{H} - c_2) - \sqrt{(c_1 - c_2)^2(\mathcal{H}^2 - K)}}.$$

CASE (I) (the first part) is achieved by assuming that $c_1 > c_2$ in the above section of the proof. However, if we assume that $c_2 > c_1$, it can be checked numerically that the results for P_1 and P_2 are mostly complex.

Alternatively, (1.20) can be rewritten to match the form of the Jacobi elliptic integrals (1.10) and (1.11). Substituting the integral value (1.10) followed by u (1.19), and solving for P_3 gives $\left(a^2 = -\frac{B_1}{A_1} \quad \text{and} \quad b^2 = \frac{B_2}{A_2}\right)$

$$t = \frac{\sqrt{c_1 c_2}}{(\alpha - \beta)\sqrt{A_1 A_2}} \int_a^{P_3} \frac{du}{\sqrt{(u^2 - (-\frac{B_1}{A_1}))(u^2 + \frac{B_2}{A_2})}} \tag{6.22}$$

$$t = \frac{\sqrt{c_1 c_2}}{(\alpha - \beta)\sqrt{A_1 A_2}\sqrt{a^2 + b^2}} \mathrm{nc}^{-1}\left(\frac{u}{a}, \frac{b}{\sqrt{a^2 + b^2}}\right)$$

$$\Rightarrow \quad P_3 = \frac{\alpha - \beta\, a\, \mathrm{nc}\left(\frac{(\alpha-\beta)\sqrt{A_1 A_2}\sqrt{a^2+b^2}}{\sqrt{c_1 c_2}}\, t, \frac{b}{\sqrt{a^2+b^2}}\right)}{1 - a\, \mathrm{nc}\left(\frac{(\alpha-\beta)\sqrt{A_1 A_2}\sqrt{a^2+b^2}}{\sqrt{c_1 c_2}}\, t, \frac{b}{\sqrt{a^2+b^2}}\right)},$$

where (see appendix A)

$$a^2 = \frac{\sqrt{(c_1 - c_2)^2(\mathcal{H}^2 - K)} + (c_1 - c_2)(\mathcal{H} - c_2)}{(c_1 - c_2)(\mathcal{H} - c_2) - \sqrt{(c_1 - c_2)^2(\mathcal{H}^2 - K)}} > 0$$

$$b^2 = \frac{\sqrt{(c_1 - c_2)^2(\mathcal{H}^2 - K)} + (c_1 - c_2)(\mathcal{H} - c_1)}{\sqrt{(c_1 - c_2)^2(\mathcal{H}^2 - K)} - c_1(c_2 + \mathcal{H}) + c_2\mathcal{H} + c_1^2} > 0 \quad \text{and}$$

$$A_1 A_2 = \frac{\left((c_1 - c_2)(H - c_2) - \sqrt{(c_1 - c_2)^2(H^2 - K)}\right)\left(\sqrt{(c_1 - c_2)^2(H^2 - K)} - c_1(c_2 + H) + c_2 H + c_1^2\right)}{4(c_1 - c_2)^2(H^2 - K)}.$$

Similarly, substituting the integral value (1.11) into (6.22), followed by u (1.19), and solving for P_3

gives

$$P_3 = \frac{\alpha - \beta\sqrt{a^2+b^2}\,\mathrm{ds}\left(\frac{(\alpha-\beta)\sqrt{A_1A_2}\sqrt{a^2+b^2}}{\sqrt{c_1c_2}}\,t, \frac{b}{\sqrt{a^2+b^2}}\right)}{1 - \sqrt{a^2+b^2}\,\mathrm{ds}\left(\frac{(\alpha-\beta)\sqrt{A_1A_2}\sqrt{a^2+b^2}}{\sqrt{c_1c_2}}\,t, \frac{b}{\sqrt{a^2+b^2}}\right)},$$

with the same values for a^2, b^2 and A_1A_2.

Although,

$$\begin{aligned}
&a^2 > 0 \Rightarrow (c_1-c_2)(\mathcal{H}-c_2) > \sqrt{(c_1-c_2)^2(\mathcal{H}^2-K)} \quad \text{or} \quad (c_1-c_2)(\mathcal{H}-c_2) < -\sqrt{(c_1-c_2)^2(\mathcal{H}^2-K)}\\
&b^2 > 0 \Rightarrow -\sqrt{(c_1-c_2)^2(\mathcal{H}^2-K)} < (c_1-c_2)(\mathcal{H}-c_1) < \sqrt{(c_1-c_2)^2(\mathcal{H}^2-K)},
\end{aligned}$$

it can be checked (with Mathematica) that no combination of values for c_1, c_2 and $P(0)$ will satisfy the condition

$$(c_1-c_2)(\mathcal{H}-c_2) < -\sqrt{(c_1-c_2)^2(\mathcal{H}^2-K)}$$

and

$$-\sqrt{(c_1-c_2)^2(\mathcal{H}^2-K)} < (c_1-c_2)(\mathcal{H}-c_1) < \sqrt{(c_1-c_2)^2(\mathcal{H}^2-K)}.$$

Further,

$$b^2 > 0 \Rightarrow (c_1-c_2)(\mathcal{H}-c_1) < -\sqrt{(c_1-c_2)^2(\mathcal{H}^2-K)} \quad \text{and} \quad (c_1-c_2)(\mathcal{H}-c_1) > \sqrt{(c_1-c_2)^2(\mathcal{H}^2-K)},$$

but this cannot hold.

It can be checked (with Mathematica) that the remaining condition

$$-\sqrt{(c_1-c_2)^2(\mathcal{H}^2-K)} < (c_1-c_2)(\mathcal{H}-c_1) < \sqrt{(c_1-c_2)^2(\mathcal{H}^2-K)}$$

and

$$(c_1-c_2)(\mathcal{H}-c_2) > \sqrt{(c_1-c_2)^2(\mathcal{H}^2-K)}$$

is only satisfied when $c_1 > c_2$; therefore, this can be simplifed to CASE (I) (the second part).

CASE (6.19) (i.e., CASE (II)) is easily deduced from CASE (6.18) (i.e., CASE (I)) by swapping c_1 and c_2.

It can also be checked (with Mathematica) that no combination of values for c_1, c_2 and $P(0)$ will satisfy the conditions

$$-\sqrt{(c_1-c_2)^2(\mathcal{H}^2-K)} < (c_1-c_2)(\mathcal{H}-c_i) < \sqrt{(c_1-c_2)^2(\mathcal{H}^2-K)} \quad \text{for } i = 1 \text{ and } 2,$$

or

$$-\sqrt{(c_1-c_2)^2(\mathcal{H}^2-K)} < (c_2-c_1)(\mathcal{H}-c_i) < \sqrt{(c_1-c_2)^2(\mathcal{H}^2-K)} \quad \text{for } i = 1 \text{ and } 2.$$

Finally, we obtain P_1 and P_2 by substituting P_3 into (6.17) and (6.16), respectively. □

6.4 Numerical solutions to the extremal equations

In the appendix, I have included the MATLAB code (appendix B) used to generate graphs (appendix C) of the solution to the extremal equations ((6.5), (6.6), (6.7)). SECTION 1 of the code plots the solution for arbitrarily chosen c_1, c_2 and initial values using a MATLAB solver. In SECTION 2, the same constants are used, but we substitute them into Theorem 6.3.1 (incl. (6.14) and (6.15)) and plot the result. In both sections the individual components P_1, P_2, P_3 are plotted on one set of axes, followed by the solution curve $P(\cdot)$ plotted on a seperate set of axes. In the instance where either P_1 or P_2 cut the x-axis the negative curve needs to be plotted, as this forms part of the solution (otherwise, the negative of the curve does not satisfy the chosen initial values). In the conclusion, the plots produced by the MATLAB solver are compared to those from Theorem 6.3.1 – this will help us gauge the accuracy of the results in the theorem.

6.5 The solution to the left-invariant optimal control problem

Proposition 6.5.1 has been derived from a similar result found in [9].

Assume first that $K \neq 0$. Recall (from section 3.4) that the solution curve on $\mathsf{SO}(3)$ is given by

$$g(t) = e^{\frac{1}{\sqrt{K}}\Lambda\phi_1(t)} e^{E_2\phi_2(t)} e^{E_3\phi_3(t)}, \tag{6.23}$$

where $g(t)P(t)g^{-1}(t) = \Lambda$, and we are implicitly assuming that Λ is not linearly dependent on E_2.

PROPOSITION 6.5.1. *Suppose that $(g(\cdot), P(\cdot))$ is an extremal curve such that $g(t)P(t)g^{-1}(t) = \Lambda$, then $\phi_1(\cdot)$, $\phi_2(\cdot)$ and $\phi_3(\cdot)$ are given by the following differential equations:*

$$\begin{aligned}
\dot{\phi}_1 &= \frac{\sqrt{K}}{c_2}\left(1 - \frac{(c_2 - c_1)}{c_1}\frac{P_1\cos\phi_3}{(P_2\sin\phi_3 - P_1\cos\phi_3)}\right)\\
\dot{\phi}_2 &= \frac{(c_2 - c_1)}{c_1 c_2}\frac{P_1 P_2}{(P_2\sin\phi_3 - P_1\cos\phi_3)}\\
\dot{\phi}_3 &= 1 - \frac{1}{c_2}P_3\left(1 - \frac{(c_2 - c_1)}{c_1}\frac{P_1\cos\phi_3}{(P_2\sin\phi_3 - P_1\cos\phi_3)}\right).
\end{aligned}$$

PROOF: We calculate the derivative of (6.23),

$$\dot{g} = \frac{1}{\sqrt{K}}\Lambda g\dot{\phi}_1 + e^{\frac{1}{\sqrt{K}}\Lambda\phi_1} e^{E_2\phi_2} E_2 e^{E_3\phi_3}\dot{\phi}_2 + gE_3\dot{\phi}_3,$$

which can be rewritten as

$$\dot{g} = g\left(\frac{1}{\sqrt{K}}g^{-1}\Lambda g\dot{\phi}_1 + e^{-E_3\phi_3}E_2e^{E_3\phi_3}\dot{\phi}_2 + E_3\dot{\phi}_3\right).$$

Since $\dot{g} = g(E_3 + \frac{1}{c_1}P_1E_1 + \frac{1}{c_2}P_2E_2)$,

$$\left(\frac{1}{\sqrt{K}}g^{-1}\Lambda g\right)\dot{\phi}_1 + \left(e^{-E_3\phi_3}E_2e^{E_3\phi_3}\right)\dot{\phi}_2 + E_3\dot{\phi}_3 = E_3 + \frac{1}{c_1}P_1E_1 + \frac{1}{c_2}P_2E_2,$$

and applying the hat mapping (Def. 2.2.1) gives

$$\frac{1}{\sqrt{K}}\widehat{P}\dot{\phi}_1 + \left(e^{-E_3\phi_3}e_2\right)\dot{\phi}_2 + e_3\dot{\phi}_3 = e_3 + \frac{1}{c_1}P_1e_1 + \frac{1}{c_2}P_2e_2.$$

Furthermore, $\widehat{P} = P_1e_1 + P_2e_2 + P_3e_3$ and since

$$e^{-E_3\phi_3} = \begin{bmatrix} \cos\phi_3 & \sin\phi_3 & 0 \\ -\sin\phi_3 & \cos\phi_3 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

it follows that $e^{-E_3\phi_3}e_2 = (\sin\phi_3)e_1 + (\cos\phi_3)e_2$. Therefore,

$$\frac{1}{\sqrt{K}}P_1\dot{\phi}_1 + (\sin\phi_3)\dot{\phi}_2 = \frac{1}{c_1}P_1, \quad \frac{1}{\sqrt{K}}P_2\dot{\phi}_1 + (\cos\phi_3)\dot{\phi}_2 = \frac{1}{c_2}P_2 \quad \text{and} \quad \frac{1}{\sqrt{K}}P_3\dot{\phi}_1 + \dot{\phi}_3 = 1.$$

The first two equations can be solved for $\dot{\phi}_1$ and $\dot{\phi}_2$; $\dot{\phi}_1$ is substituted into the third equation to obtain the final result. □

This set of differential equations is solved using numerical techniques, but this goes beyond the scope of what we are interested in achieving here.

Recall (from section 3.4) that when $K = 0$, it is assumed that $\Lambda = E_1 - E_3$. The required set of differential equations is then obtained from Proposition 6.5.1 by substituting in $\frac{1}{\sqrt{K}} = 1$.

6.6 Stability

Theorems 6.6.1-6.6.3 have been derived from similar results and ideas found in [19] and [20].

The equilibrium states for the system (6.5), (6.6) and (6.7) are:

$$\widehat{P}_{e1} = (M, 0, c_1), \quad \widehat{P}_{e2} = (0, M, c_2) \quad \text{and} \quad \widehat{P}_{e3} = (0, 0, M), \quad M \in \mathbb{R},\ c_1,\ c_2 > 0.$$

THEOREM 6.6.1. *The equilibrium state* $\widehat{P}_{e1} = (M, 0, c_1)$ *is non-linear stable if* $c_1 < c_2$, $M \neq 0$.

PROOF: STEP 1. For any arbitrary smooth function $\varphi : \mathbb{R} \to \mathbb{R}$, the function K_φ, defined by

$$K_\varphi = \varphi\left(\frac{1}{2}(P_1^2 + P_2^2 + P_3^2)\right),$$

gives a family of Casimir functions on $\mathfrak{so}(3)^*$ (Prop. 3.3.1, Prop. 1.4.3, Def. 1.3.13).

STEP 2. We want to find a single K_φ such that the energy-Casimir function,

$$\mathcal{H} + K_\varphi = \left(\frac{1}{2c_1}P_1^2 + \frac{1}{2c_2}P_2^2 + P_3\right) + \varphi\left(\frac{1}{2}(P_1^2 + P_2^2 + P_3^2)\right),$$

has a critical point at the equilibrium of interest.

Now, we calculate the first derivative of $\mathcal{H} + K_\varphi$:

$$\begin{aligned} D(\mathcal{H} + K_\varphi)(p) \cdot \delta p &= \frac{d}{dt}(\mathcal{H} + K_\varphi)\ (p + t\delta)|_{t=0} \\ &= \frac{d}{dt}\left(\frac{1}{2c_1}(P_1 + t\delta_1)^2 + \frac{1}{2c_2}(P_2 + t\delta_2)^2 + P_3 + t\delta_3\right)\bigg|_{t=0} \\ &+ \frac{d}{dt}\varphi\left(\frac{1}{2}(P_1 + t\delta_1)^2 + \frac{1}{2}(P_2 + t\delta_2)^2 + \frac{1}{2}(P_3 + t\delta_3)^2\right)\bigg|_{t=0} \\ &= \frac{1}{c_1}\delta_1 P_1 + \frac{1}{c_2}\delta_2 P_2 + \delta_3 + (\delta_1 P_1 + \delta_2 P_2 + \delta_3 P_3)\dot{\varphi}\left(\frac{1}{2}(P_1^2 + P_2^2 + P_3^2)\right), \end{aligned} \tag{6.24}$$

and this equals zero at the equilibrium of interest of and only if

$$\dot{\varphi}\left(\frac{1}{2}(M^2 + c_1^2)\right) = -\frac{1}{c_1}. \tag{6.25}$$

STEP 3. Next, using (6.24), we can calculate the second derivative of $\mathcal{H} + K_\varphi$:

$$\begin{aligned} D^2(\mathcal{H} + K_\varphi)(p) \cdot \delta p &= \frac{d}{dt}\left(\frac{1}{c_1}\delta_1(P_1 + t\delta_1) + \frac{1}{c_2}\delta_2(P_2 + t\delta_2) + \delta_3\right)\bigg|_{t=0} \\ &+ \frac{d}{dt}(\delta_1(P_1 + t\delta_1) + \delta_2(P_2 + t\delta_2) + \delta_3(P_3 + t\delta_3)) \\ &\times \dot{\varphi}\left(\frac{1}{2}((P_1 + t\delta_1)^2 + (P_2 + t\delta_2)^2 + (P_3 + t\delta_3)^2)\right)\bigg|_{t=0} \\ &= \frac{1}{c_1}\delta_1^2 + \frac{1}{c_2}\delta_2^2 + (\delta_1^2 + \delta_2^2 + \delta_3^2)\dot{\varphi}\left(\frac{1}{2}(P_1^2 + P_2^2 + P_3^2)\right) \\ &+ (\delta_1 P_1 + \delta_2 P_2 + \delta_3 P_3)^2\ddot{\varphi}\left(\frac{1}{2}(P_1^2 + P_2^2 + P_3^2)\right). \end{aligned} \tag{6.26}$$

Using (6.25), the second derivative of $\mathcal{H} + K_\varphi$ at the equilibrium of interest is

$$\begin{aligned} D^2(\mathcal{H} + K_\varphi)(\hat{P}_{e1}) \cdot \delta\hat{P}_{e1} = {} & M^2\ddot{\varphi}\left(\frac{1}{2}(M^2 + c_1^2)\right)\delta_1^2 + \left(\frac{1}{c_2} - \frac{1}{c_1}\right)\delta_2^2 \\ & + \left(\ddot{\varphi}\left(\frac{1}{2}(M^2 + c_1^2)\right)c_1^2 - \frac{1}{c_1}\right)\delta_3^2 + \ddot{\varphi}\left(\frac{1}{2}(M^2 + c_1^2)\right)2c_1M\delta_1\delta_3. \end{aligned}$$

This can be written in the matrix form

$$\begin{bmatrix}\delta_1 & \delta_2 & \delta_3\end{bmatrix} \begin{bmatrix} M^2\ddot{\varphi}\left(\frac{1}{2}(M^2 + c_1^2)\right) & 0 & c_1M\ddot{\varphi}\left(\frac{1}{2}(M^2 + c_1^2)\right) \\ 0 & \frac{1}{c_2} - \frac{1}{c_1} & 0 \\ c_1M\ddot{\varphi}\left(\frac{1}{2}(M^2 + c_1^2)\right) & 0 & \ddot{\varphi}\left(\frac{1}{2}\left(M^2 + c_1^2\right)\right)c_1^2 - \frac{1}{c_1} \end{bmatrix} \begin{bmatrix}\delta_1 \\ \delta_2 \\ \delta_3\end{bmatrix},$$

and is negative definite if and only if:

1. the first entry of the matrix is negative $\Rightarrow$ $\ddot{\varphi}\left(\frac{1}{2}(M^2 + c_1^2)\right) < 0$
2. the determinant of the upper left 2×2 matrix is positive $\Rightarrow$ $c_1 < c_2$
3. the determinant of the 3×3 matrix is negative. Given the previous implications, this now holds.

Thus, having chosen that

$$\ddot{\varphi}\left(\frac{1}{2}(M^2 + c_1^2)\right) < 0,$$

this quadratic form is negative definite if and only if $c_1 < c_2$. Consequently,

$$\varphi(x) = -x\left(x + \frac{1}{c_1} - (M^2 + c_1^2)\right)$$

satisfies the above conditions and the second derivative of $\mathcal{H} + K_\varphi$ is negative definite. □

THEOREM 6.6.2. *The equilibrium state* $\widehat{P}_{e2} = (0, M, c_2)$ *is non-linear stable if* $c_1 > c_2$, $M \neq 0$.

PROOF: Theorem 6.6.1 covers STEP 1 and the first part of STEP 2 for this proof.

STEP 2. Eq. (6.24) equals zero at the equilibrium of interest if and only if

$$\dot{\varphi}\left(\frac{1}{2}(M^2 + c_2^2)\right) = -\frac{1}{c_2}. \tag{6.27}$$

STEP 3. Using (??), the second derivative of $\mathcal{H} + K_\varphi$ (6.26) at the equilibrium of interest is

$$D^2(\mathcal{H} + K_\varphi)(\hat{P}_{e2}) \cdot \delta\hat{P}_{e2} = \left(\frac{1}{c_1} - \frac{1}{c_2}\right)\delta_1^2 + M^2\ddot{\varphi}\left(\frac{1}{2}(M^2 + c_2^2)\right)\delta_2^2$$

$$+\left(\ddot{\varphi}\left(\frac{1}{2}(M^2+c_2^2)\right)c_2^2-\frac{1}{c_2}\right)\delta_3^2+\ddot{\varphi}\left(\frac{1}{2}(M^2+c_2^2)\right)2c_2M\delta_2\delta_3.$$

This can be written in the matrix form

$$\begin{bmatrix}\delta_1 & \delta_2 & \delta_3\end{bmatrix}\begin{bmatrix}\frac{1}{c_1}-\frac{1}{c_2} & 0 & 0\\ 0 & M^2\ddot{\varphi}\left(\frac{1}{2}(M^2+c_2^2)\right) & c_2M\ddot{\varphi}\left(\frac{1}{2}(M^2+c_2^2)\right)\\ 0 & c_2M\ddot{\varphi}\left(\frac{1}{2}(M^2+c_2^2)\right) & \ddot{\varphi}\left(\frac{1}{2}\left(M^2+c_2^2\right)c_2^2-\frac{1}{c_2}\right)\end{bmatrix}\begin{bmatrix}\delta_1\\ \delta_2\\ \delta_3\end{bmatrix},$$

and is negative definite if and only if:

1. the first entry of the matrix is negative $\Rightarrow$ $c_1 > c_2$
2. the determinant of the upper left 2×2 matrix is positive $\Rightarrow$ $\ddot{\varphi}\left(\frac{1}{2}(M^2+c_2^2)\right)<0$
3. the determinant of the 3×3 matrix is negative. Given the previous implications, this now holds.

Thus, having chosen that

$$\ddot{\varphi}\left(\frac{1}{2}(M^2+c_2^2)\right)<0,$$

this quadratic form is negative definite if and only if $c_1 > c_2$. Consequently,

$$\varphi(x)=-x\left(x+\frac{1}{c_2}-(M^2+c_2^2)\right)$$

satisfies the above conditions and the second derivative of $\mathcal{H}+K_\varphi$ is negative definite. □

THEOREM 6.6.3. *The equilibrium state* $\widehat{P}_{e3}=(0,0,M)$ *is non-linear stable if* $M\in(-\infty,0)\cup(0,c_i)\cup(c_j,\infty)$, $c_i<c_j$, $i,j=1,2$.

PROOF: Theorem 6.6.1 covers STEP 1 and the first part of STEP 2 for this proof.

STEP 2. Eq. (6.24) equals zero at the equilibrium of interest if and only if

$$\dot{\varphi}\left(\frac{1}{2}M^2\right)=-\frac{1}{M}. \tag{6.28}$$

STEP 3. Using (6.27), the second derivative of $\mathcal{H}+K_\varphi$ (6.26) at the equilibrium of interest is

$$D^2(\mathcal{H}+K_\varphi)(\hat{P}_{e3})\cdot\delta\hat{P}_{e3}=\left(\frac{1}{c_1}-\frac{1}{M}\right)\delta_1^2+\left(\frac{1}{c_2}-\frac{1}{M}\right)\delta_2^2+\left(M^2\ddot{\varphi}\left(\frac{1}{2}M^2\right)-\frac{1}{M}\right)\delta_3^2.$$

Thus, having chosen that

$$\ddot{\varphi}(\frac{1}{2}M^2)>\frac{1}{M^3},$$

this quadratic form is positive definite if and only if $M > c_1, c_2$ or $M < 0$. Consequently,

1. if $M > c_1, c_2$, then we choose φ as follows

$$\varphi(x) = -\frac{2}{M}x + \frac{1}{M^3}x^2, \quad \text{and}$$

2. if $M < 0$, then we choose φ as follows

$$\varphi(x) = -\frac{1}{M^3}x^2.$$

Otherwise, having chosen that

$$\ddot{\varphi}(\frac{1}{2}M^2) < \frac{1}{M^3},$$

this quadratic form is negative definite if and only if $0 < M < c_1, c_2$. Consequently, if $M > 0$, then we choose φ as follows

$$\varphi(x) = -\frac{1}{M^3}x^2.$$

□

Chapter 7

The Type III Problem

Lastly, we consider the solution to the Type III left-invariant optimal control problem on $\mathsf{SO}(3)$. After defining the problem we prove a theorem that gives the optimal controls, optimal Hamiltonian and extremal equations. The Jacobi elliptic functions are used to solve the extremal equations, and these results are tested by some numerical calculations. A set of differential equations is achieved, which can be solved for Euler's angles to give the solution to the problem. We conclude this chapter by investigating the non-linear stability of the equilibrium states associated with the extremal equations.

7.1 The Type III left-invariant optimal control problem

Let g_1 and g_2 be arbitrary but fixed points in $\mathsf{SO}(3)$, and $T > 0$ be fixed in advance. Then the Type III left-invariant optimal control problem on $\mathsf{SO}(3)$ consists of minimizing the cost functional

$$J = \frac{1}{2}\int_0^T (c_1u_1^2 + c_2u_2^2 + c_3u_3^2)\,dt, \quad c_1,\, c_2,\, c_3 > 0 \tag{7.1}$$

over all trajectory-control pairs $(g(\cdot), u(\cdot))$, of the left-invariant control system

$$\dot{g} = g(u_1E_1 + u_2E_2 + u_3E_3), \quad g \in \mathsf{SO}(3),\ u = (u_1, u_2, u_3) \in \mathbb{R}^3, \tag{7.2}$$

satisfying the boundary conditions

$$g(0) = g_1 \quad \text{and} \quad g(T) = g_2.$$

7.2 The extremal equations

Theorem 7.2.1 has been derived from similar results found in [9] and [18].

THEOREM 7.2.1. *Given a left-invariant control system on* $\mathsf{SO}(3)$ *defined by (7.2) with cost (7.1), the optimal controls are*

$$u_1 = \frac{1}{c_1}P_1, \quad u_2 = \frac{1}{c_2}P_2 \quad \textit{and} \quad u_3 = \frac{1}{c_3}P_3, \tag{7.3}$$

and the optimal Hamiltonian is

$$\mathcal{H} = \frac{1}{2}\left(\frac{1}{c_1}P_1^2 + \frac{1}{c_2}P_2^2 + \frac{1}{c_3}P_3^2\right), \tag{7.4}$$

where P_1, P_2 *and* P_3 *are solutions of*

$$\dot{P}_1 = \frac{c_2 - c_3}{c_2 c_3} P_2 P_3 \tag{7.5}$$

$$\dot{P}_2 = \frac{c_3 - c_1}{c_1 c_3} P_1 P_3 \tag{7.6}$$

$$\dot{P}_3 = \frac{c_1 - c_2}{c_1 c_2} P_1 P_2. \tag{7.7}$$

PROOF: The control Hamiltonian (Def. 1.4.6) is

$$\mathcal{H}(p,u) = -\frac{1}{2}(c_1 u_1^2 + c_2 u_2^2 + c_3 u_3^2) + p(u_1 E_1 + u_2 E_2 + u_3 E_3),$$

which simplifies to (Prop. 3.3.1)

$$\mathcal{H}(p,u) = -\frac{1}{2}(c_1 u_1^2 + c_2 u_2^2 + c_3 u_3^2) + u_1 P_1 + u_2 P_2 + u_3 P_3. \tag{7.8}$$

Applying (2) of the maximum principle (section 1.5), the optimal controls are

$$\begin{aligned}
\frac{\partial \mathcal{H}}{\partial u_1} = 0 \quad &\Leftrightarrow \quad -c_1 u_1 + P_1 = 0 \quad \Leftrightarrow \quad u_1 = \frac{1}{c_1}P_1 \\
\frac{\partial \mathcal{H}}{\partial u_2} = 0 \quad &\Leftrightarrow \quad -c_2 u_2 + P_2 = 0 \quad \Leftrightarrow \quad u_2 = \frac{1}{c_2}P_2 \\
\frac{\partial \mathcal{H}}{\partial u_3} = 0 \quad &\Leftrightarrow \quad -c_3 u_3 + P_3 = 0 \quad \Leftrightarrow \quad u_3 = \frac{1}{c_3}P_3.
\end{aligned}$$

Substituting u_1, u_2 and u_3 into (7.8) gives (7.4), which combined with Propositions 1.3.6, 1.3.4 and 1.3.7 gives

$$\dot{P}_i = \{P_i, \frac{1}{2c_1}P_1^2 + \frac{1}{2c_2}P_2^2 + \frac{1}{2c_3}P_3^2\} = \frac{1}{c_1}P_1\{P_i, P_1\} + \frac{1}{c_2}P_2\{P_i, P_2\} + \frac{1}{c_3}P_3\{P_i, P_3\}, \quad i = 1,\, 2,\, 3,$$

and using the Poisson bracket relations from (3.11) we get (7.5), (7.6) and (7.7). □

7.3 The solution to the extremal equations

Proposition 7.3.1. *Considering the case where $c = c_1 = c_2 = c_3$, we can rewrite equations (7.5), (7.6) and (7.7) as*

$$\dot{P}_1 = 0, \quad \dot{P}_2 = 0 \quad \text{and} \quad \dot{P}_3 = 0,$$

which have no non-constant solutions.

Proposition 7.3.2. *Considering the case where $c_1 = c_2 \neq c_3$, we can rewrite equations (7.5), (7.6) and (7.7) as*

$$\dot{P}_1 = \frac{c_2 - c_3}{c_2 c_3} P_2 P_3(0) \tag{7.9}$$

$$\dot{P}_2 = \frac{c_3 - c_1}{c_1 c_3} P_1 P_3(0) \tag{7.10}$$

$$\dot{P}_3 = 0, \tag{7.11}$$

which have the solutions

$$\begin{aligned}
P_1(t) &= P_1(0) \cosh\left(\frac{(c_1 - c_3)(c_3 - c_2)P_3(0)}{\sqrt{c_1 c_2}\, c_3}\right) t \\
&\quad - P_2(0) \frac{c_1(c_2 - c_3)}{c_1 c_2 (c_1 - c_3)(c_3 - c_2)} \sinh\left(\frac{\sqrt{c_1 c_2 (c_1 - c_3)(c_3 - c_2)P_3(0)}}{c_1 c_2 c_3}\right) t \\
P_2(t) &= P_1(0) \frac{c_3(c_3 - c_1)}{c_1 c_2 (c_2 - c_3)(c_3 - c_1)} \sinh\left(\frac{\sqrt{c_1 c_2 (c_1 - c_3)(c_3 - c_2)P_3(0)}}{c_1 c_2 c_3}\right) t \\
&\quad + P_2(0) \cosh\left(\frac{(c_1 - c_3)(c_3 - c_2)P_3(0)}{\sqrt{c_1 c_2}\, c_3}\right) t \\
P_3(t) &= P_3(0).
\end{aligned}$$

Similar results hold for the cases where $c_1 \neq c_2 = c_3$ and $c_3 = c_1 \neq c_2$.

Proof: Equations (7.9), (7.10) and (7.11) in matrix form become

$$\begin{bmatrix} \dot{P}_1 \\ \dot{P}_2 \end{bmatrix} = \begin{bmatrix} 0 & \frac{c_2 - c_3}{c_2 c_3} P_3(0) \\ \frac{c_3 - c_1}{c_1 c_3} P_3(0) & 0 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}.$$

Since $P(t) = \exp(tA)P_0$ is the only solution of $\dot{P} = AP$, $P(0) = P_0$, (using Mathematica) the solution can be written as

$$\begin{bmatrix} P_1(t) \\ P_2(t) \end{bmatrix} =$$
$$\begin{bmatrix} \cosh\left(\frac{(c_1-c_3)(c_3-c_2)P_3(0)}{\sqrt{c_1c_2}c_3}\right)t & \frac{c_1(c_2-c_3)}{c_1c_2(c_1-c_3)(c_3-c_2)}\sinh\left(\frac{\sqrt{c_1c_2(c_1-c_3)(c_3-c_2)P_3(0)}}{c_1c_2c_3}\right)t \\ \frac{c_3(c_3-c_1)}{c_1c_2(c_2-c_3)(c_3-c_1)}\sinh\left(\frac{\sqrt{c_1c_2(c_1-c_3)(c_3-c_2)P_3(0)}}{c_1c_2c_3}\right)t & \cosh\left(\frac{(c_1-c_3)(c_3-c_2)P_3(0)}{\sqrt{c_1c_2}c_3}\right)t \end{bmatrix}$$
$$\times \begin{bmatrix} P_1(0) \\ P_2(0) \end{bmatrix}.$$

□

The beginning of the proof of Theorem 7.3.1 is based on similar ideas found in [18] and [19]. However, the integration using elliptic functions is original (with reference to [12]).

THEOREM 7.3.1. *The equations (7.5), (7.6) and (7.7), where $c_1 \neq c_2 \neq c_3$, can be integrated via elliptic functions. More exactly,*

CASE 1A(I): $c_3 > c_2$, $c_3 > c_1$ *and* $c_1 > c_2$.

If $c_2(c_1 - c_3)P_3^2 < c_3(c_2 - c_1)P_2^2$, *then*

$$P_3 = b\,\mathrm{nd}\left(\sqrt{\frac{(c_3-c_2)(c_3-c_1)}{c_1c_2c_3^2}}a\,t, \frac{\sqrt{a^2-b^2}}{a}\right)$$
$$\textit{and/or} \quad P_3 = a\,\mathrm{dn}\left(\sqrt{\frac{(c_3-c_2)(c_3-c_1)}{c_1c_2c_3^2}}a\,t, \frac{\sqrt{a^2-b^2}}{a}\right),$$

where

$$a^2 = \frac{c_3(K - 2\mathcal{H}c_2)}{c_3 - c_2} \quad \textit{and} \quad b^2 = \frac{c_3(K - 2\mathcal{H}c_1)}{c_3 - c_1},$$

or if $c_3(c_2 - c_1)P_2^2 < c_2(c_1 - c_3)P_3^2$, *then*

$$P_3 = \frac{ab}{\sqrt{a^2+b^2}}\,\mathrm{sd}\left(\sqrt{\frac{(c_3-c_2)(c_3-c_1)}{c_1c_2c_3^2}}(a^2+b^2)\,t, \frac{b}{\sqrt{a^2+b^2}}\right)$$
$$\textit{and/or} \quad P_3 = b\,\mathrm{cn}\left(\sqrt{\frac{(c_3-c_2)(c_3-c_1)}{c_1c_2c_3^2}}(a^2+b^2)\,t, \frac{b}{\sqrt{a^2+b^2}}\right),$$

where

$$a^2 = \frac{c_3(2\mathcal{H}c_1 - K)}{c_3 - c_1} \quad \textit{and} \quad b^2 = \frac{c_3(K - 2\mathcal{H}c_2)}{c_3 - c_2}.$$

CASE 1A(II): $c_3 > c_2 > c_1$.

If $c_1(c_2-c_3)P_3^2 < c_3(c_1-c_2)P_1^2$, *then*

$$\begin{aligned} P_3 &= b\,\mathrm{nd}\left(\sqrt{\frac{(c_3-c_2)(c_3-c_1)}{c_1c_2c_3^2}}\,a\,t, \frac{\sqrt{a^2-b^2}}{a}\right) \\ \textit{and/or} \quad P_3 &= a\,\mathrm{dn}\left(\sqrt{\frac{(c_3-c_2)(c_3-c_1)}{c_1c_2c_3^2}}\,a\,t, \frac{\sqrt{a^2-b^2}}{a}\right), \end{aligned}$$

where

$$a^2 = \frac{c_3(K-2\mathcal{H}c_1)}{c_3-c_1} \quad \textit{and} \quad b^2 = \frac{c_3(K-2\mathcal{H}c_2)}{c_3-c_2},$$

or if $c_3(c_1-c_2)P_1^2 < c_1(c_2-c_3)P_3^2$, *then*

$$\begin{aligned} P_3 &= \frac{ab}{\sqrt{a^2+b^2}}\,\mathrm{sd}\left(\sqrt{\frac{(c_3-c_2)(c_3-c_1)}{c_1c_2c_3^2}(a^2+b^2)}\,t, \frac{b}{\sqrt{a^2+b^2}}\right) \\ \textit{and/or} \quad P_3 &= b\,\mathrm{cn}\left(\sqrt{\frac{(c_3-c_2)(c_3-c_1)}{c_1c_2c_3^2}(a^2+b^2)}\,t, \frac{b}{\sqrt{a^2+b^2}}\right), \end{aligned}$$

where

$$a^2 = \frac{c_3(2\mathcal{H}c_2-K)}{c_3-c_2} \quad \textit{and} \quad b^2 = \frac{c_3(K-2\mathcal{H}c_1)}{c_3-c_1}.$$

CASE 1B(I): $c_3 < c_2$, $c_3 < c_1$ *and* $c_1 < c_2$.

If $c_3(c_2-c_1)P_2^2 < c_2(c_1-c_3)P_3^2$, *then*

$$\begin{aligned} P_3 &= b\,\mathrm{nd}\left(\sqrt{\frac{(c_3-c_2)(c_3-c_1)}{c_1c_2c_3^2}}\,a\,t, \frac{\sqrt{a^2-b^2}}{a}\right) \\ \textit{and/or} \quad P_3 &= a\,\mathrm{dn}\left(\sqrt{\frac{(c_3-c_2)(c_3-c_1)}{c_1c_2c_3^2}}\,a\,t, \frac{\sqrt{a^2-b^2}}{a}\right), \end{aligned}$$

where

$$a^2 = \frac{c_3(K-2\mathcal{H}c_2)}{c_3-c_2} \quad \textit{and} \quad b^2 = \frac{c_3(K-2\mathcal{H}c_1)}{c_3-c_1},$$

or if $c_2(c_1-c_3)P_3^2 < c_3(c_2-c_1)P_2^2$, *then*

$$\begin{aligned} P_3 &= \frac{ab}{\sqrt{a^2+b^2}}\,\mathrm{sd}\left(\sqrt{\frac{(c_3-c_2)(c_3-c_1)}{c_1c_2c_3^2}(a^2+b^2)}\,t, \frac{b}{\sqrt{a^2+b^2}}\right) \\ \textit{and/or} \quad P_3 &= b\,\mathrm{cn}\left(\sqrt{\frac{(c_3-c_2)(c_3-c_1)}{c_1c_2c_3^2}(a^2+b^2)}\,t, \frac{b}{\sqrt{a^2+b^2}}\right), \end{aligned}$$

where

$$a^2 = \frac{c_3(2\mathcal{H}c_1 - K)}{c_3 - c_1} \quad \textit{and} \quad b^2 = \frac{c_3(K - 2\mathcal{H}c_2)}{c_3 - c_2}.$$

CASE 1B(II): $c_3 < c_2 < c_1$.

If $c_3(c_1 - c_2)P_1^2 < c_1(c_2 - c_3)P_3^2$, *then*

$$\begin{aligned} P_3 &= b\,\mathrm{nd}\left(\sqrt{\frac{(c_3 - c_2)(c_3 - c_1)}{c_1 c_2 c_3^2}}\,a\,t, \frac{\sqrt{a^2 - b^2}}{a}\right) \\ \textit{and/or} \quad P_3 &= a\,\mathrm{dn}\left(\sqrt{\frac{(c_3 - c_2)(c_3 - c_1)}{c_1 c_2 c_3^2}}\,a\,t, \frac{\sqrt{a^2 - b^2}}{a}\right), \end{aligned}$$

where

$$a^2 = \frac{c_3(K - 2\mathcal{H}c_1)}{c_3 - c_1} \quad \textit{and} \quad b^2 = \frac{c_3(K - 2\mathcal{H}c_2)}{c_3 - c_2},$$

or if $c_1(c_2 - c_3)P_3^2 < c_3(c_1 - c_2)P_1^2$, *then*

$$\begin{aligned} P_3 &= \frac{ab}{\sqrt{a^2 + b^2}}\,\mathrm{sd}\left(\sqrt{\frac{(c_3 - c_2)(c_3 - c_1)}{c_1 c_2 c_3^2}}(a^2 + b^2)\,t, \frac{b}{\sqrt{a^2 + b^2}}\right) \\ \textit{and/or} \quad P_3 &= b\,\mathrm{cn}\left(\sqrt{\frac{(c_3 - c_2)(c_3 - c_1)}{c_1 c_2 c_3^2}}(a^2 + b^2)\,t, \frac{b}{\sqrt{a^2 + b^2}}\right), \end{aligned}$$

where

$$a^2 = \frac{c_3(2\mathcal{H}c_2 - K)}{c_3 - c_2} \quad \textit{and} \quad b^2 = \frac{c_3(K - 2\mathcal{H}c_1)}{c_3 - c_1}.$$

CASE 2: $c_2 < c_3 < c_1$ *or* $c_1 < c_3 < c_2$.

$$\begin{aligned} P_3 &= b\,\mathrm{sn}\left(\sqrt{-\frac{(c_3 - c_2)(c_3 - c_1)}{c_1 c_2 c_3^2}}\,a\,t, \frac{b}{a}\right) \\ \textit{and/or} \quad P_3 &= b\,\mathrm{cd}\left(\sqrt{-\frac{(c_3 - c_2)(c_3 - c_1)}{c_1 c_2 c_3^2}}\,a\,t, \frac{b}{a}\right), \end{aligned}$$

where

$$\begin{aligned} a^2 &= \frac{c_3(K - 2\mathcal{H}c_2)}{c_3 - c_2} \quad \textit{and} \quad b^2 = \frac{c_3(K - 2\mathcal{H}c_1)}{c_3 - c_1}, \quad b < a, \\ \textit{or} \quad a^2 &= \frac{c_3(K - 2\mathcal{H}c_1)}{c_3 - c_1} \quad \textit{and} \quad b^2 = \frac{c_3(K - 2\mathcal{H}c_2)}{c_3 - c_2}, \quad b < a. \end{aligned}$$

Finally,

$$P_1 = \pm\sqrt{\frac{c_1(c_3-c_2)}{c_3(c_1-c_2)}\left(\frac{c_3(K-2\mathcal{H}c_2)}{c_3-c_2}-P_3^2\right)}$$

$$P_2 = \pm\sqrt{\frac{c_2(c_3-c_1)}{c_3(c_2-c_1)}\left(\frac{c_3(K-2\mathcal{H}c_1)}{c_3-c_1}-P_3^2\right)}.$$

PROOF: Multiplying the Hamiltonian (7.4) by $2c_1c_2c_3$ gives

$$c_2c_3P_1^2 + c_1c_3P_2^2 + c_1c_2P_3^2 = 2\mathcal{H}c_1c_2c_3. \tag{7.12}$$

Using the Casimir function on $\mathfrak{so}(3)^*$ (Prop. 3.3.1),

$$P_1^2 + P_2^2 + P_3^2 = K, \tag{7.13}$$

and rearranging for P_1^2, we substitute into (7.12) and solve for P_2^2:

$$\begin{aligned} c_2c_3(K-P_2^2-P_3^2) + c_1c_3P_2^2 + c_1c_2P_3^2 &= 2\mathcal{H}c_1c_2c_3 \\ \Rightarrow\quad P_2^2 &= \frac{c_2c_3-c_1c_2}{c_2c_3-c_1c_3}\left(\frac{Kc_3-2\mathcal{H}c_1c_3}{c_3-c_1}-P_3^2\right). \end{aligned} \tag{7.14}$$

We rearrange (7.13) for P_2^2, substitute into (7.12) and solve for P_1^2:

$$\begin{aligned} c_2c_3P_1^2 + c_1c_3(K-P_1^2-P_3^2) + c_1c_2P_3^2 &= 2\mathcal{H}c_1c_2c_3 \\ \Rightarrow\quad P_1^2 &= \frac{c_1c_3-c_1c_2}{c_1c_3-c_2c_3}\left(\frac{Kc_3-2\mathcal{H}c_2c_3}{c_3-c_2}-P_3^2\right). \end{aligned} \tag{7.15}$$

Squaring equation (7.7),

$$\dot{P}_3^2 = \left(\frac{c_1-c_2}{c_1c_2}\right)^2 P_1^2P_2^2,$$

and substituting in equations (7.14) and (7.15) leaves us with

$$\begin{aligned} \dot{P}_3^2 &= \frac{(c_1-c_2)^2}{c_1^2c_2^2}\frac{(c_1c_3-c_1c_2)}{(c_1c_3-c_2c_3)}\left(\frac{Kc_3-2\mathcal{H}c_2c_3}{c_3-c_2}-P_3^2\right)\frac{(c_2c_3-c_1c_2)}{(c_2c_3-c_1c_3)}\left(\frac{Kc_3-2\mathcal{H}c_1c_3}{c_3-c_1}-P_3^2\right) \\ &= -\frac{(c_3-c_2)(c_3-c_1)}{c_1c_2c_3^2}\left(\frac{Kc_3-2\mathcal{H}c_2c_3}{c_3-c_2}-P_3^2\right)\left(\frac{Kc_3-2\mathcal{H}c_1c_3}{c_3-c_1}-P_3^2\right). \end{aligned} \tag{7.16}$$

Let us consider the sign of the constant.

CASE 1: negative constant

$$\begin{aligned}
&\frac{(c_3-c_2)(c_3-c_1)}{c_1c_2c_3^2} > 0\\
\Rightarrow\quad &(c_3-c_2)(c_3-c_1) > 0\\
\Leftrightarrow\quad &c_3 > c_2 \text{ and } c_3 > c_1 \quad\text{or}\quad c_3 < c_2 \text{ and } c_3 < c_1.
\end{aligned}$$

Due to the negative constant, (7.16) can be rearranged in the following ways:

$$\begin{aligned}
\dot{P}_3^2 &= \frac{(c_3-c_2)(c_3-c_1)}{c_1c_2c_3^2}\left(\frac{Kc_3-2\mathcal{H}c_2c_3}{c_3-c_2}-P_3^2\right)\left(P_3^2-\frac{Kc_3-2\mathcal{H}c_1c_3}{c_3-c_1}\right) &(7.17)\\
&= \frac{(c_3-c_2)(c_3-c_1)}{c_1c_2c_3^2}\left(\frac{Kc_3-2\mathcal{H}c_1c_3}{c_3-c_1}-P_3^2\right)\left(P_3^2-\frac{Kc_3-2\mathcal{H}c_2c_3}{c_3-c_2}\right) &(7.18)\\
&= \frac{(c_3-c_2)(c_3-c_1)}{c_1c_2c_3^2}\left(\frac{2\mathcal{H}c_2c_3-Kc_3}{c_3-c_2}+P_3^2\right)\left(\frac{Kc_3-2\mathcal{H}c_1c_3}{c_3-c_1}-P_3^2\right) &(7.19)\\
&= \frac{(c_3-c_2)(c_3-c_1)}{c_1c_2c_3^2}\left(\frac{2\mathcal{H}c_1c_3-Kc_3}{c_3-c_1}+P_3^2\right)\left(\frac{Kc_3-2\mathcal{H}c_2c_3}{c_3-c_2}-P_3^2\right). &(7.20)
\end{aligned}$$

Let us consider each equation and its solution independently.

CASE 1A: $c_3 > c_2$ and $c_3 > c_1$.

Eq. (7.17) corresponds to the elliptic integrals (1.8) and (1.9):

$$\begin{aligned}
\int_b^{P_3}\frac{dP_3}{\sqrt{(a^2-P_3^2)(P_3^2-b^2)}} &= \frac{1}{a}\,\mathrm{nd}^{-1}\left(\frac{P_3}{b},\frac{\sqrt{a^2-b^2}}{a}\right),\quad b\le P_3\le a, &(7.21)\\
\int_{P_3}^{a}\frac{dP_3}{\sqrt{(a^2-P_3^2)(P_3^2-b^2)}} &= \frac{1}{a}\,\mathrm{dn}^{-1}\left(\frac{P_3}{a},\frac{\sqrt{a^2-b^2}}{a}\right),\quad b\le P_3\le a, &(7.22)
\end{aligned}$$

where

$$a^2 = \frac{Kc_3-2\mathcal{H}c_2c_3}{c_3-c_2} > 0 \quad\text{and}\quad b^2 = \frac{Kc_3-2\mathcal{H}c_1c_3}{c_3-c_1} > 0.$$

Substituting a^2 and b^2 (and further (7.12) and (7.13)) into $b \le P_3 \le a$ (from (7.21) and (7.22)), and simplifying gives: $c_2 < c_1$.

CASE 1A(I): $c_3 > c_2$, $c_3 > c_1$ and $c_1 > c_2$.

Substituting (7.12) and (7.13) into $a^2 > 0$ and $b^2 > 0$, and simplifying gives the constraints

$$a^2>0 \quad\Leftrightarrow\quad c_1(c_2-c_3)P_3^2 < c_3(c_1-c_2)P_1^2 \quad\text{and}\quad b^2>0 \quad\Leftrightarrow\quad c_2(c_1-c_3)P_3^2 < c_3(c_2-c_1)P_2^2.$$

Therefore, $a^2 > 0$ is always satisfied.

From (7.17) and (7.21) we determine P_3:

(We do not consider $\dot{P}_3 = -\sqrt{\frac{(c_3-c_2)(c_3-c_1)}{c_1c_2c_3^2}}\left(a^2-P_3^2\right)\left(P_3^2-b^2\right)$ as it can be checked numerically that this does not contribute any new solutions.)

$$\begin{aligned}
\dot{P}_3 &= \sqrt{\frac{(c_3-c_2)(c_3-c_1)}{c_1c_2c_3^2}\left(a^2-P_3^2\right)\left(P_3^2-b^2\right)} \\
t &= \frac{1}{\sqrt{\frac{(c_3-c_2)(c_3-c_1)}{c_1c_2c_3^2}}}\int_b^{P_3}\frac{dP_3}{\sqrt{(a^2-P_3^2)(P_3^2-b^2)}} \\
t &= \frac{1}{\sqrt{\frac{(c_3-c_2)(c_3-c_1)}{c_1c_2c_3^2}}}\frac{1}{a}\,\mathrm{nd}^{-1}\left(\frac{P_3}{b},\frac{\sqrt{a^2-b^2}}{a}\right) \\
\Rightarrow\quad P_3 &= b\,\mathrm{nd}\left(\sqrt{\frac{(c_3-c_2)(c_3-c_1)}{c_1c_2c_3^2}}\,a\,t,\frac{\sqrt{a^2-b^2}}{a}\right).
\end{aligned}$$

Similarly, from (7.17) and (7.22) we can also determine P_3:

$$P_3 = a\,\mathrm{dn}\left(\sqrt{\frac{(c_3-c_2)(c_3-c_1)}{c_1c_2c_3^2}}\,a\,t,\frac{\sqrt{a^2-b^2}}{a}\right).$$

Case (7.18) (i.e., CASE 1(A)(II) (the first part)) is easily deduced from case (7.17) (i.e., CASE 1(A)(I) (the first part)) by swapping c_1 and c_2.

Eq. (7.19) corresponds to the elliptic integrals (1.5) and (??):

$$\int_0^{P_3}\frac{dP_3}{\sqrt{(a^2+P_3^2)(b^2-P_3^2)}} = \frac{1}{\sqrt{a^2+b^2}}\,\mathrm{sd}^{-1}\left(\frac{\sqrt{a^2+b^2}}{ab}P_3,\frac{b}{\sqrt{a^2+b^2}}\right),\quad 0\le P_3\le b, \tag{7.23}$$

$$\int_{P_3}^{b}\frac{dP_3}{\sqrt{(a^2+P_3^2)(b^2-P_3^2)}} = \frac{1}{\sqrt{a^2+b^2}}\,\mathrm{cn}^{-1}\left(\frac{P_3}{b},\frac{b}{\sqrt{a^2+b^2}}\right),\quad 0\le P_3\le b, \tag{7.24}$$

where

$$a^2 = \frac{2\mathcal{H}c_2c_3-Kc_3}{c_3-c_2} > 0 \quad\text{and}\quad b^2 = \frac{Kc_3-2\mathcal{H}c_1c_3}{c_3-c_1} > 0.$$

Substituting b^2 (and further (7.12) and (7.13)) into $0\le P_3\le b$ (from (7.23) and (7.24)), and simplifying gives: $c_1 < c_2$.

CASE 1A(II): $c_3 > c_2$, $c_3 > c_1$ and $c_2 > c_1$.

Substituting (7.12) and (7.13) into $a^2 > 0$ and $b^2 > 0$, and simplifying gives the constraints

$$a^2>0 \quad\Leftrightarrow\quad c_3(c_1-c_2)P_1^2 < c_1(c_2-c_3)P_3^2 \quad\text{and}\quad b^2>0 \quad\Leftrightarrow\quad c_2(c_1-c_3)P_3^2 < c_3(c_2-c_1)P_2^2.$$

Therefore, $b^2 > 0$ is always satisfied.

From (7.19) and (7.23) (following the same approach as before) we determine P_3:

$$P_3 \;=\; \frac{ab}{\sqrt{a^2+b^2}}\,\mathrm{sd}\left(\sqrt{\frac{(c_3-c_2)(c_3-c_1)}{c_1c_2c_3^2}(a^2+b^2)}\,t,\frac{b}{\sqrt{a^2+b^2}}\right).$$

Similarly, from (7.19) and (7.24) we can also determine P_3:

$$P_3 \;=\; b\,\mathrm{cn}\left(\sqrt{\frac{(c_3-c_2)(c_3-c_1)}{c_1c_2c_3^2}(a^2+b^2)}\,t,\frac{b}{\sqrt{a^2+b^2}}\right).$$

Case (7.20) (i.e., CASE 1(A)(I) (the second part)) is easily deduced from case (7.19) (i.e., CASE 1(A)(II) (the second part)) by swapping c_1 and c_2.

CASE 1B: $c_3 < c_2$ and $c_3 < c_1$.

As we have seen, eq. (7.17) corresponds to the elliptic integrals (7.21) and (7.22), where

$$a^2 = \frac{Kc_3 - 2\mathcal{H}c_2c_3}{c_3-c_2} > 0 \quad\text{and}\quad b^2 = \frac{Kc_3 - 2\mathcal{H}c_1c_3}{c_3-c_1} > 0.$$

Substituting a^2 and b^2 (and further (7.12) and (7.13)) into $b \le P_3 \le a$ (from (7.21) and (7.22)), and simplifying gives $c_1 < c_2$.

CASE 1B(I): $c_3 < c_2$, $c_3 < c_1$ and $c_1 < c_2$.

Substituting (7.12) and (7.13) into $a^2 > 0$ and $b^2 > 0$, and simplifying gives the constraints

$$a^2 > 0 \quad\Leftrightarrow\quad c_3(c_1-c_2)P_1^2 < c_1(c_2-c_3)P_3^2 \quad\text{and}\quad b^2>0 \quad\Leftrightarrow\quad c_3(c_2-c_1)P_2^2 < c_2(c_1-c_3)P_3^2.$$

Therefore, $a^2 > 0$ is always satisfied.

From (7.17) and (7.21) (following the same approach as before) we determine P_3:

$$P_3 \;=\; b\,\mathrm{nd}\left(\sqrt{\frac{(c_3-c_2)(c_3-c_1)}{c_1c_2c_3^2}}\,a\,t,\frac{\sqrt{a^2-b^2}}{a}\right).$$

Similarly, from (7.17) and (7.22) we can also determine P_3:

$$P_3 \;=\; a\,\mathrm{dn}\left(\sqrt{\frac{(c_3-c_2)(c_3-c_1)}{c_1c_2c_3^2}}\,a\,t,\frac{\sqrt{a^2-b^2}}{a}\right).$$

Case (7.18) (i.e., CASE 1(B)(II) (the first part)) is easily deduced from case (7.17) (i.e., CASE 1(B)(I) (the first part)) by swapping c_1 and c_2.

As we have seen, eq. (7.19) corresponds to the elliptic integrals (7.23) and (7.24), where

$$a^2 = \frac{2\mathcal{H}c_2c_3 - Kc_3}{c_3 - c_2} > 0 \quad \text{and} \quad b^2 = \frac{Kc_3 - 2\mathcal{H}c_1c_3}{c_3 - c_1} > 0.$$

Substituting b^2 (and further (7.12) and (7.13)) into $0 \leq P_3 \leq b$ (from (7.23) and (7.24)), and simplifying gives $c_2 < c_1$.

CASE 1B(II): $c_3 < c_2$, $c_3 < c_1$ and $c_2 < c_1$.

Substituting (7.12) and (7.13) into $a^2 > 0$ and $b^2 > 0$, and simplifying gives the constraints

$$a^2 > 0 \quad \Leftrightarrow \quad c_1(c_2 - c_3)P_3^2 < c_3(c_1 - c_2)P_1^2 \quad \text{and} \quad b^2 > 0 \quad c_3(c_2 - c_1)P_2^2 < c_2(c_1 - c_3)P_3^2.$$

Therefore, $b^2 > 0$ is always satisfied.

From (7.19) and (7.23) (following the same approach as before) we determine P_3:

$$P_3 \;=\; \frac{ab}{\sqrt{a^2 + b^2}} \operatorname{sd}\left(\sqrt{\frac{(c_3 - c_2)(c_3 - c_1)}{c_1c_2c_3^2}}(a^2 + b^2)\,t, \frac{b}{\sqrt{a^2 + b^2}}\right).$$

Similarly, from (7.19) and (7.24) we can also determine P_3:

$$P_3 \;=\; b \operatorname{cn}\left(\sqrt{\frac{(c_3 - c_2)(c_3 - c_1)}{c_1c_2c_3^2}}(a^2 + b^2)\,t, \frac{b}{\sqrt{a^2 + b^2}}\right).$$

Case (7.20) (i.e., CASE 1(B)(I) (the second part)) is easily deduced from case (7.19) (i.e., CASE 1(B)(II) (the second part)) by swapping c_1 and c_2.

CASE 2: positive constant

$$\begin{aligned} &\frac{(c_3 - c_2)(c_3 - c_1)}{c_1c_2c_3^2} < 0 \\ \Rightarrow\; &(c_3 - c_2)(c_3 - c_1) < 0 \\ \Leftrightarrow\; &c_2 < c_3 < c_1 \quad \text{or} \quad c_1 < c_3 < c_2. \end{aligned}$$

Due to the positive constant, (7.16) can be rearranged in the following ways:

$$\dot{P}_3^2 = -\frac{(c_3 - c_2)(c_3 - c_1)}{c_1c_2c_3^2}\left(\frac{Kc_3 - 2\mathcal{H}c_2c_3}{c_3 - c_2} - P_3^2\right)\left(\frac{Kc_3 - 2\mathcal{H}c_1c_3}{c_3 - c_1} - P_3^2\right) \tag{7.25}$$

$$= -\frac{(c_3 - c_2)(c_3 - c_1)}{c_1c_2c_3^2}\left(\frac{Kc_3 - 2\mathcal{H}c_1c_3}{c_3 - c_1} - P_3^2\right)\left(\frac{Kc_3 - 2\mathcal{H}c_2c_3}{c_3 - c_2} - P_3^2\right). \tag{7.26}$$

Let us consider each equation and its solution independently.

CASE 2A: $c_2 < c_3 < c_1$.

Eq. (7.25) corresponds to the elliptic integrals (1.2) and (1.4):

$$\int_0^{P_3} \frac{dP_3}{\sqrt{(a^2 - P_3^2)(b^2 - P_3^2)}} = \frac{1}{a}\operatorname{sn}^{-1}\left(\frac{P_3}{b}, \frac{b}{a}\right), \quad 0 \le P_3 \le b < a, \tag{7.27}$$

$$\int_{P_3}^{b} \frac{dP_3}{\sqrt{(a^2 - P_3^2)(b^2 - P_3^2)}} = \frac{1}{a}\operatorname{cd}^{-1}\left(\frac{P_3}{b}, \frac{b}{a}\right), \quad 0 \le P_3 \le b < a, \tag{7.28}$$

where

$$a^2 = \frac{Kc_3 - 2\mathcal{H}c_2c_3}{c_3 - c_2} > 0 \quad \text{and} \quad b^2 = \frac{Kc_3 - 2\mathcal{H}c_1c_3}{c_3 - c_1} > 0.$$

There is no need to substitute a^2 and b^2 (and further (7.12) and (7.13)) into $0 \le P_3 \le b < a$ (from (7.27) and (7.28)), since we already have the relationship between c_1 and c_2.

Substituting (7.12) and (7.13) into $a^2 > 0$ and $b^2 > 0$, and simplifying gives the constraints

$$a^2 > 0 \quad \Leftrightarrow \quad c_1(c_2 - c_3)P_3^2 < c_3(c_1 - c_2)P_1^2 \quad \text{and} \quad b^2 > 0 \quad \Leftrightarrow \quad c_3(c_2 - c_1)P_2^2 < c_2(c_1 - c_3)P_3^2.$$

Notice that $a^2 > 0$ and $b^2 > 0$ are always satisfied, therefore, there is no need to consider other elliptic integrals.

From (7.25) and (7.27) (following the same approach as before) we determine P_3:

$$P_3 \;=\; b\operatorname{sn}\left(\sqrt{-\frac{(c_3 - c_2)(c_3 - c_1)}{c_1c_2c_3^2}}\,a\,t, \frac{b}{a}\right).$$

Similarly, from (7.25) and (7.28) we can also determine P_3:

$$P_3 \;=\; b\operatorname{cd}\left(\sqrt{-\frac{(c_3 - c_2)(c_3 - c_1)}{c_1c_2c_3^2}}\,a\,t, \frac{b}{a}\right).$$

Case (7.26) is easily deduced from case (7.25) by swapping a^2 and b^2.

CASE 2A and CASE 2B (i.e., $c_1 < c_3 < c_2$) yield the same result.

Finally, we obtain P_1 and P_2 by substituting P_3 into (7.15) and (7.14), respectively. □

7.4 Numerical solutions to the extremal equations

In the appendix, I have included the MATLAB code (appendix B) used to generate graphs (appendix C) of the solution to the extremal equations ((7.5), (7.6), (7.7)). SECTION 1 of the code plots the solution for arbitrarily chosen c_1, c_2, c_3 and initial values using a MATLAB solver. In SECTION 2, the same constants are used, but we substitute them into Theorem 7.3.1 (incl. (7.12) and (7.13)) and plot the result. In both sections the individual components P_1, P_2, P_3 are plotted on one set of axes, followed by the solution curve $P(\cdot)$ plotted on a seperate set of axes. In the instance where either P_1 or P_2 cut the x-axis the negative curve needs to be plotted, as this forms part of the solution (otherwise, the negative of the curve does not satisfy the chosen initial values). In the conclusion, the plots produced by the MATLAB solver are compared to those from Theorem 7.3.1 – this will help us gauge the accuracy of the results in the theorem.

7.5 The solution to the left-invariant optimal control problem

Proposition 7.5.1 has been derived from a similar result found in [9].

Assume first that $K \neq 0$. Recall (from section 3.4) that the solution curve on $\mathsf{SO}(3)$ is given by

$$g(t) = e^{\frac{1}{\sqrt{K}}\Lambda\phi_1(t)} e^{E_2\phi_2(t)} e^{E_3\phi_3(t)}, \tag{7.29}$$

where $g(t)P(t)g^{-1}(t) = \Lambda$, and we are implicitly assuming that Λ is not linearly dependent on E_2.

PROPOSITION 7.5.1. *Suppose that $(g(\cdot), P(\cdot))$ is an extremal curve such that $g(t)P(t)g^{-1}(t) = \Lambda$, then $\phi_1(\cdot)$, $\phi_2(\cdot)$ and $\phi_3(\cdot)$ are given by the following differential equations:*

$$\begin{aligned}
\dot{\phi}_1 &= \frac{\sqrt{K}}{c_2}\left(1 - \frac{(c_2 - c_1)}{c_1}\frac{P_1\cos\phi_3}{(P_2\sin\phi_3 - P_1\cos\phi_3)}\right)\\
\dot{\phi}_2 &= \frac{(c_2 - c_1)}{c_1 c_2}\frac{P_1 P_2}{(P_2\sin\phi_3 - P_1\cos\phi_3)}\\
\dot{\phi}_3 &= \frac{(c_2 - c_3)}{c_2 c_3}P_3\left(1 - \frac{(c_2 - c_1)}{c_1}\frac{P_1\cos\phi_3}{(P_2\sin\phi_3 - P_1\cos\phi_3)}\right).
\end{aligned}$$

PROOF: We calculate the derivative of (7.29),

$$\dot{g} = \frac{1}{\sqrt{K}}\Lambda g\dot{\phi}_1 + e^{\frac{1}{\sqrt{K}}\Lambda\phi_1} e^{E_2\phi_2} E_2 e^{E_3\phi_3}\dot{\phi}_2 + gE_3\dot{\phi}_3,$$

which can be rewritten as

$$\dot{g} = g\left(\frac{1}{\sqrt{K}}g^{-1}\Lambda g\dot{\phi}_1 + e^{-E_3\phi_3}E_2 e^{E_3\phi_3}\dot{\phi}_2 + E_3\dot{\phi}_3\right).$$

Since $\dot{g} = g(\frac{1}{c_1}P_1E_1 + \frac{1}{c_2}P_2E_2 + \frac{1}{c_3}P_3E_3)$,

$$\left(\frac{1}{\sqrt{K}}g^{-1}\Lambda g\right)\dot{\phi}_1 + \left(e^{-E_3\phi_3}E_2 e^{E_3\phi_3}\right)\dot{\phi}_2 + E_3\dot{\phi}_3 = \frac{1}{c_1}P_1E_1 + \frac{1}{c_2}P_2E_2 + \frac{1}{c_3}P_3E_3,$$

and applying the hat mapping (Def. 2.2.1) gives

$$\frac{1}{\sqrt{K}}\widehat{P}\dot{\phi}_1 + \left(e^{-E_3\phi_3}e_2\right)\dot{\phi}_2 + e_3\dot{\phi}_3 = \frac{1}{c_1}P_1e_1 + \frac{1}{c_2}P_2e_2 + \frac{1}{c_3}P_3e_3.$$

Furthermore, $\widehat{P} = P_1e_1 + P_2e_2 + P_3e_3$ and since

$$e^{-E_3\phi_3} = \begin{bmatrix} \cos\phi_3 & \sin\phi_3 & 0 \\ -\sin\phi_3 & \cos\phi_3 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

it follows that $e^{-E_3\phi_3}e_2 = (\sin\phi_3)e_1 + (\cos\phi_3)e_2$. Therefore,

$$\frac{1}{\sqrt{K}}P_1\dot{\phi}_1 + (\sin\phi_3)\dot{\phi}_2 = \frac{1}{c_1}P_1, \quad \frac{1}{\sqrt{K}}P_2\dot{\phi}_1 + (\cos\phi_3)\dot{\phi}_2 = \frac{1}{c_2}P_2 \quad \text{and} \quad \frac{1}{\sqrt{K}}P_3\dot{\phi}_1 + \dot{\phi}_3 = \frac{1}{c_3}P_3.$$

The first two equations can be solved for $\dot{\phi}_1$ and $\dot{\phi}_2$; $\dot{\phi}_1$ is substituted into the third equation to obtain the final result. □

This set of differential equations is solved using numerical techniques, but this goes beyond the scope of what we are interested in achieving here.

Recall (from section 3.4) that when $K = 0$, it is assumed that $\Lambda = E_1 - E_3$. The required set of differential equations is then obtained from Proposition 7.5.1 by substituting in $\frac{1}{\sqrt{K}} = 1$.

7.6 Stability

Theorems 7.6.1-7.6.3 have been derived from similar results and ideas found in [19] and [20].

The equilibrium states for the system (7.5), (7.6) and (7.7) are:

$$\widehat{P}_{e1} = (M,0,0), \quad \widehat{P}_{e2} = (0,M,0) \quad \text{and} \quad \widehat{P}_{e3} = (0,0,M), \quad M \in \mathbb{R}.$$

THEOREM 7.6.1. *The equilibrium state* $\widehat{P}_{e1} = (M,0,0)$ *is non-linear stable if* $c_1 > c_2$, $c_1 > c_3$ *or* $c_2 > c_1$, $c_3 > c_1$, $M \neq 0$.

PROOF: STEP 1. For any arbitrary smooth function $\varphi : \mathbb{R} \to \mathbb{R}$, the function K_φ, defined by

$$K_\varphi = \varphi\left(\frac{1}{2}(P_1^2 + P_2^2 + P_3^2)\right),$$

gives a family of Casimir functions on $\mathfrak{so}(3)^*$ (Prop. 3.3.1, Prop. 1.4.3, Def. 1.3.13).

STEP 2. We want to find a single K_φ such that the energy-Casimir function,

$$\mathcal{H} + K_\varphi = \frac{1}{2}\left(\frac{P_1^2}{c_1} + \frac{P_2^2}{c_2} + \frac{P_3^2}{c_3}\right) + \varphi\left(\frac{1}{2}(P_1^2 + P_2^2 + P_3^2)\right),$$

has a critical point at the equilibrium of interest.

Now, we calculate the first derivative of $\mathcal{H} + K_\varphi$:

$$\begin{aligned} D(\mathcal{H} + K_\varphi)(p) \cdot \delta p &= \frac{d}{dt}(\mathcal{H} + K_\varphi)\,(p + t\delta)|_{t=0} \\ &= \frac{d}{dt}\left(\frac{1}{2c_1}(P_1 + t\delta_1)^2 + \frac{1}{2c_2}(P_2 + t\delta_2)^2 + \frac{1}{2c_3}(P_3 + t\delta_3)^2\right)\bigg|_{t=0} \\ &+ \frac{d}{dt}\varphi\left(\frac{1}{2}(P_1 + t\delta_1)^2 + \frac{1}{2}(P_2 + t\delta_2)^2 + \frac{1}{2}(P_3 + t\delta_3)^2\right)\bigg|_{t=0} \\ &= \frac{1}{c_1}\delta_1 P_1 + \frac{1}{c_2}\delta_2 P_2 + \frac{1}{c_3}\delta_3 P_3 + (\delta_1 P_1 + \delta_2 P_2 + \delta_3 P_3)\dot{\varphi}\left(\frac{1}{2}(P_1^2 + P_2^2 + P_3^2)\right), \end{aligned} \tag{7.30}$$

and this equals zero at the equilibrium of interest if and only if

$$\dot{\varphi}\left(\frac{1}{2}M^2\right) = -\frac{1}{c_1}. \tag{7.31}$$

STEP 3. Next, using (7.30), we can calculate the second derivative of $\mathcal{H} + K_\varphi$:

$$\begin{aligned} D^2(\mathcal{H} + K_\varphi)(p) \cdot \delta p &= \frac{d}{dt}\left(\frac{1}{c_1}\delta_1(P_1 + t\delta_1) + \frac{1}{c_2}\delta_2(P_2 + t\delta_2) + \frac{1}{c_3}\delta_3(P_3 + t\delta_3)\right)\bigg|_{t=0} \\ &+ \frac{d}{dt}(\delta_1(P_1 + t\delta_1) + \delta_2(P_2 + t\delta_2) + \delta_3(P_3 + t\delta_3)) \\ &\times \dot{\varphi}\left(\frac{1}{2}((P_1 + t\delta_1)^2 + (P_2 + t\delta_2)^2 + (P_3 + t\delta_3)^2)\right)\bigg|_{t=0} \\ &= \frac{1}{c_1}\delta_1^2 + \frac{1}{c_2}\delta_2^2 + \frac{1}{c_3}\delta_3^2 + (\delta_1^2 + \delta_2^2 + \delta_3^2)\dot{\varphi}\left(\frac{1}{2}(P_1^2 + P_2^2 + P_3^2)\right) \\ &+ (\delta_1 P_1 + \delta_2 P_2 + \delta_3 P_3)^2\ddot{\varphi}\left(\frac{1}{2}(P_1^2 + P_2^2 + P_3^2)\right). \end{aligned} \tag{7.32}$$

Using (7.31), the second derivative of $\mathcal{H} + K_\varphi$ at the equilibrium of interest is

$$D^2(\mathcal{H} + K_\varphi)(\hat{P}_{e1}) \cdot \delta\hat{P}_{e1} = \ddot{\varphi}\left(\frac{1}{2}M^2\right)M^2\delta_1^2 + \left(\frac{1}{c_2} - \frac{1}{c_1}\right)\delta_2^2 + \left(\frac{1}{c_3} - \frac{1}{c_1}\right)\delta_3^2.$$

Thus, having chosen that

$$\ddot{\varphi}\left(\frac{1}{2}M^2\right) > 0,$$

this quadratic form is positive definite if and only if $c_1 > c_2$ and $c_1 > c_3$. Consequently,

$$\varphi(x) = x\left(x - \frac{1}{c_1} - M^2\right)$$

satisfies the above conditions and the second derivative of $\mathcal{H} + K_\varphi$ is positive definite.

Otherwise, having chosen that

$$\ddot{\varphi}\left(\frac{1}{2}M^2\right) < 0,$$

this quadratic form is negative definite if and only if $c_2 > c_1$ and $c_3 > c_1$. Consequently,

$$\varphi(x) = -x\left(x + \frac{1}{c_1} - M^2\right)$$

satisfies the above conditions and the second derivative of $\mathcal{H} + K_\varphi$ is negative definite. □

THEOREM 7.6.2. *The equilibrium state* $\widehat{P}_{e2} = (0, M, 0)$ *is non-linear stable if* $c_2 > c_1$, $c_2 > c_3$ *or* $c_1 > c_2$, $c_3 > c_2$, $M \neq 0$.

PROOF: Theorem 7.6.1 covers STEP 1 and the first part of STEP 2 for this proof.

STEP 2. Eq. (7.30) equals zero at the equilibrium of interest if and only if

$$\dot{\varphi}\left(\frac{1}{2}M^2\right) = -\frac{1}{c_2}. \tag{7.33}$$

STEP 3. Using (7.33), the second derivative of $\mathcal{H} + K_\varphi$ (7.32) at the equilibrium of interest is

$$D^2(\mathcal{H} + K_\varphi)(\hat{P}_{e2}) \cdot \delta\hat{P}_{e2} = \left(\frac{1}{c_1} - \frac{1}{c_2}\right)\delta_1^2 + \ddot{\varphi}\left(\frac{1}{2}M^2\right)M^2\delta_2^2 + \left(\frac{1}{c_3} - \frac{1}{c_2}\right)\delta_3^2.$$

Thus, having chosen that

$$\ddot{\varphi}\left(\frac{1}{2}M^2\right) > 0,$$

this quadratic form is positive definite if and only if $c_2 > c_1$ and $c_2 > c_3$. Consequently,

$$\varphi(x) = x\left(x - \frac{1}{c_2} - M^2\right)$$

satisfies the above conditions and the second derivative of $\mathcal{H} + K_\varphi$ is positive definite.

Otherwise, having chosen that

$$\ddot{\varphi}\left(\frac{1}{2}M^2\right) < 0,$$

this quadratic form is negative definite if and only if $c_1 > c_2$ and $c_3 > c_2$. Consequently,

$$\varphi(x) = -x\left(x + \frac{1}{c_2} - M^2\right)$$

satisfies the above conditions and the second derivative of $\mathcal{H} + K_\varphi$ is negative definite. □

THEOREM 7.6.3. *The equilibrium state* $\widehat{P}_{e3} = (0, 0, M)$ *is non-linear stable if* $c_3 > c_1$, $c_3 > c_2$ *or* $c_1 > c_3$, $c_2 > c_3$, $M \neq 0$.

PROOF: Theorem 7.6.1 covers STEP 1 and the first part of STEP 2 for this proof.

STEP 2. Eq. (7.30) equals zero at the equilibrium of interest if and only if

$$\dot{\varphi}\left(\frac{1}{2}M^2\right) = -\frac{1}{c_3}. \tag{7.34}$$

STEP 3. Using (7.34), the second derivative of $\mathcal{H} + K_\varphi$ (7.32) at the equilibrium of interest is

$$D^2(\mathcal{H} + K_\varphi)(\hat{P}_{e3}) \cdot \delta\hat{P}_{e3} = \left(\frac{1}{c_1} - \frac{1}{c_3}\right)\delta_1^2 + \left(\frac{1}{c_2} - \frac{1}{c_3}\right)\delta_2^2 + \ddot{\varphi}\left(\frac{1}{2}M^2\right)M^2\delta_3^2.$$

Thus, having chosen that

$$\ddot{\varphi}\left(\frac{1}{2}M^2\right) > 0,$$

this quadratic form is positive definite if and only if $c_3 > c_1$ and $c_3 > c_2$. Consequently,

$$\varphi(x) = x\left(x - \frac{1}{c_3} - M^2\right)$$

satisfies the above conditions and the second derivative of $\mathcal{H} + K_\varphi$ positive definite.

Otherwise, having chosen that

$$\ddot{\varphi}\left(\frac{1}{2}M^2\right) < 0,$$

this quadratic form is negative definite if and only if $c_1 > c_3$ and $c_2 > c_3$. Consequently,

$$\varphi(x) = -x\left(x + \frac{1}{c_3} - M^2\right)$$

satisfies the above conditions and the second derivative of $\mathcal{H} + K_\varphi$ negative definite. □

Conclusions

In this chapter, I allow $c_2 \to \infty$ in the Type IIb case and compare it to the Type I case by examining the extremal equations, the explicit solutions to the extremal equations, the sets of differential equations that give the Euler angles and the stability results. Similarly, I allow $c_3 \to \infty$ in the Type III case and compare it to the Type IIa case. I discuss the graphical output in appendix C and summarize the results in a table. The chapter ends with some final remarks and possible avenues for further research.

Comparison between Type I and Type IIb

The extremal equations

If I allow $c_2 \to \infty$ in Theorem 6.2.1, it is easy to see that the theorem reduces to Theorem 4.2.1. Therefore, provided I allow $c_2 \to \infty$, it would make sense that the solutions to the extremal equations for the Type I and Type IIb cases are equivalent.

The solution to the extremal equations

If I allow $c_2 \to \infty$ in Theorem 6.3.1, Case (i): $c_1 > c_2$ becomes obsolete. However, in Case (ii): $c_2 > c_1$ (the first part) I make the following changes to the conditions:

$$\begin{aligned}
&\mathcal{H} - c_2 < -\sqrt{\mathcal{H}^2 - K} \quad \overset{c_2 \to \infty}{\Longrightarrow} \quad -1 < 0 \\
&\mathcal{H} - c_1 > \sqrt{\mathcal{H}^2 - K} \quad \text{remains unchanged, and} \\
&\frac{\mathcal{H} - c_2 + \sqrt{\mathcal{H}^2 - K}}{\mathcal{H} - c_2 - \sqrt{\mathcal{H}^2 - K}} < \frac{\mathcal{H} - c_1 + \sqrt{\mathcal{H}^2 - K}}{\mathcal{H} - c_1 - \sqrt{\mathcal{H}^2 - K}} \quad \overset{c_2 \to \infty}{\Longrightarrow} \quad 1 < \frac{\mathcal{H} - c_1 + \sqrt{\mathcal{H}^2 - K}}{\mathcal{H} - c_1 - \sqrt{\mathcal{H}^2 - K}}.
\end{aligned} \tag{7.35}$$

These final two restraints were the only remaining conditions in the proof to Theorem 4.3.1. The values for P_1 and P_2 (Theorem 6.3.1) can be rewritten and reduced to

$$
\begin{aligned}
P_1 &= \pm\sqrt{\frac{1}{c_2-c_1}(2\mathcal{H}c_1c_2 - Kc_1 - 2c_1c_2P_3 + c_1P_3^2)} \quad \overset{c_2\to\infty}{\Longrightarrow} \quad P_1 = \pm\sqrt{2\mathcal{H}c_1 - 2c_1P_3} \\
P_2 &= \pm\sqrt{\frac{1}{c_1-c_2}(2\mathcal{H}c_1c_2 - Kc_2 - 2c_1c_2P_3 + c_2P_3^2)} \quad \overset{c_2\to\infty}{\Longrightarrow} \quad P_2 = \pm\sqrt{-2\mathcal{H}c_1 + K + 2c_1P_3 - P_3^2},
\end{aligned}
$$

which is equivalent to Theorem 4.3.1. Now, for P_3, a^2 and b^2 have been checked (7.35), α and β are the same, and $\sqrt{A_1A_2}$ (Theorem 6.3.1) can be rewritten and reduced to

$$
\sqrt{A_1A_2} = \sqrt{c_2}\sqrt{-\frac{\left(\frac{\mathcal{H}}{c_2} - 1 - \frac{\sqrt{\mathcal{H}^2-K}}{c_2}\right)\left(\mathcal{H} - c_1 - \sqrt{\mathcal{H}^2-K}\right)}{4(\mathcal{H}^2-K)}} \quad \overset{c_2\to\infty}{\Longrightarrow} \quad \sqrt{c_2}\sqrt{-\frac{-\mathcal{H} + c_1 + \sqrt{\mathcal{H}^2-K}}{4(\mathcal{H}^2-K)}},
$$

where $\sqrt{c_2}$ cancels in P_3, yielding equal P_3 values in both theorems.

Thus, CASE (II): $c_2 > c_1$ (the first part) corresponds to Theorem 4.3.1.

CASE (II): $c_2 > c_1$ (the second part) cannot hold, since

$$
-\sqrt{\mathcal{H}^2-K} < \mathcal{H} - c_2 < \sqrt{\mathcal{H}^2-K} \quad \overset{c_2\to\infty}{\Longrightarrow} \quad 0 < -1 < 0.
$$

Therefore, provided I allow $c_2 \to \infty$, Theorem 6.3.1 reduces to Theorem 4.3.1.

The solution to the left-invariant optimal control problem

Consider the sets of differential equations from Proposition 4.5.1 and Proposition 6.5.1. Rewriting the set of differential equations from Proposition 6.5.1 as

$$
\begin{aligned}
\dot{\phi}_1 &= \frac{\sqrt{K}}{c_2} - \sqrt{K}\left(\frac{1}{c_1} - \frac{1}{c_2}\right)\frac{P_1\cos\phi_3}{(P_2\sin\phi_3 - P_1\cos\phi_3)} \\
\dot{\phi}_2 &= \left(\frac{1}{c_1} - \frac{1}{c_2}\right)\frac{P_1P_2}{(P_2\sin\phi_3 - P_1\cos\phi_3)} \\
\dot{\phi}_3 &= 1 - \frac{1}{c_2}P_3 + \left(\frac{1}{c_1} - \frac{1}{c_2}\right)\frac{P_1P_3\cos\phi_3}{(P_2\sin\phi_3 - P_1\cos\phi_3)},
\end{aligned}
$$

and allowing $c_2 \to \infty$, reproduces Proposition 4.5.1.

Stability

If I allow $c_2 \to \infty$ in Theorems 6.6.1, 6.6.2 and 6.6.3, the results reduce to the following:

Theorem 6.6.1: The equilibrium state $\widehat{P}_{e1} = (M, 0, c_1)$ is non-linear stable.

Theorem 6.6.2: This theorem becomes obsolete.

Theorem 6.6.3: The equilibrium state $\widehat{P}_{e3} = (0, 0, M)$ is non-linear stable if $M \in (-\infty, 0) \cup (0, c_1)$, (assuming that $c_1 < c_2$).

These statements correspond exactly to those in section 4.6 (Theorems 4.6.1 and 4.6.2).

Of course, the energy-Casimir method only provides sufficient conditions for non-linear stability, so it would be interesting to further investigate the stability properties of these equilibrium states in the instances not covered in the theorems.

Comparison between Type IIa and Type III

The extremal equations

If I allow $c_3 \to \infty$ in Theorem 7.2.1, it is easy to see that the theorem reduces to Theorem 5.2.1. Therefore, provided I allow $c_3 \to \infty$, it would make sense that the solutions to the extremal equations for the Type IIa and Type III cases are equivalent.

The solution to the extremal equations

If I allow $c_3 \to \infty$ in Theorem 7.3.1, Case 1b(i), Case 1b(ii) and Case 2 become obsolete. However, Case 1a(i) and Case 1a(ii) will be compared to Case (i) and Case (ii) of Theorem 5.3.1, respectively.

Case 1a(i) (the first part) (Theorem 7.3.1) can be rewritten and reduced to:

If $\left(\frac{c_1 c_2}{c_3} - c_2\right) P_3^2 < (c_2 - c_1) P_2^2$, then

$$P_3 = b\,\mathrm{nd}\left(\sqrt{\frac{1 - \frac{c_2}{c_3} - \frac{c_1}{c_3} + \frac{c_1 c_2}{c_3^2}}{c_1 c_2}}\,a\,t, \frac{\sqrt{a^2 - b^2}}{a}\right)$$

$$\text{and/or} \quad P_3 = a\,\mathrm{dn}\left(\sqrt{\frac{1 - \frac{c_2}{c_3} - \frac{c_1}{c_3} + \frac{c_1 c_2}{c_3^2}}{c_1 c_2}}\,a\,t, \frac{\sqrt{a^2 - b^2}}{a}\right),$$

where

$$a^2 = \frac{K - 2\mathcal{H} c_2}{1 - \frac{c_2}{c_3}} \quad \text{and} \quad b^2 = \frac{K - 2\mathcal{H} c_1}{1 - \frac{c_1}{c_3}}.$$

Allowing $c_3 \to \infty$ gives:

If $\quad -c_2P_3^2 < (c_2 - c_1)P_2^2, \quad$ then

$$\begin{aligned} P_3 &= b\,\mathrm{nd}\left(\sqrt{\frac{1}{c_1c_2}}\,a\,t, \frac{\sqrt{a^2-b^2}}{a}\right) \\ \text{and/or} \quad P_3 &= a\,\mathrm{dn}\left(\sqrt{\frac{1}{c_1c_2}}\,a\,t, \frac{\sqrt{a^2-b^2}}{a}\right), \end{aligned}$$

where

$$a^2 = K - 2\mathcal{H}c_2 \quad \text{and} \quad b^2 = K - 2\mathcal{H}c_1.$$

This corresponds to Theorem 5.3.1, CASE (I) (the first part). Similarly, the remaining cases (i.e., CASE 1A(I) (the second part) and CASE 1A(II)) can be simplified and compared to Theorem 5.3.1, always yielding positive results.

Now, the values for P_1 and P_2 (from Theorem 7.3.1) can be rewritten and reduced to

$$\begin{aligned} P_1 &= \pm\sqrt{\left(\frac{c_1 - \frac{c_1c_2}{c_3}}{c_1 - c_2}\right)\left(\frac{K - 2\mathcal{H}c_2}{1 - \frac{c_2}{c_3}} - P_3^2\right)} \quad \overset{c_3\to\infty}{\Longrightarrow} \quad P_1 = \pm\sqrt{\frac{c_1}{c_1 - c_2}(K - 2\mathcal{H}c_2 - P_3^2)} \\ P_2 &= \pm\sqrt{\left(\frac{c_2 - \frac{c_1c_2}{c_3}}{c_2 - c_1}\right)\left(\frac{K - 2\mathcal{H}c_1}{1 - \frac{c_1}{c_3}} - P_3^2\right)} \quad \overset{c_3\to\infty}{\Longrightarrow} \quad P_2 = \pm\sqrt{\frac{c_2}{c_2 - c_1}(K - 2\mathcal{H}c_1 - P_3^2)}, \end{aligned}$$

which is equivalent to Theorem 5.3.1.

Therefore, provided I allow $c_3 \to \infty$, Theorem 7.3.1, CASE 1A, reduces to Theorem 5.3.1.

The solution to the left-invariant optimal control problem

Consider the sets of differential equations from Proposition 5.5.1 and Proposition 7.5.1. $\dot{\phi}_1$ and $\dot{\phi}_2$ are the same. However, rewriting $\dot{\phi}_3$ from Proposition 7.5.1 as

$$\dot{\phi}_3 = \left(\frac{1}{c_3} - \frac{1}{c_2}\right)P_3\left(1 - \frac{(c_2 - c_1)}{c_1}\frac{P_1\cos\phi_3}{(P_2\sin\phi_3 - P_1\cos\phi_3)}\right),$$

and allowing $c_3 \to \infty$, reproduces Proposition 5.5.1.

Stability

If I allow $c_3 \to \infty$ in Theorems 7.6.1, 7.6.2 and 7.6.3, the results reduce to the following (in each theorem, I assume that $c_3 > c_1$ and $c_3 > c_2$, for comparison purposes):

Theorem 7.6.1: The equilibrium state $\widehat{P}_{e1} = (M, 0, 0)$ is non-linear stable if $c_2 > c_1$.

Theorem 7.6.2: The equilibrium state $\widehat{P}_{e2} = (0, M, 0)$ is non-linear stable if $c_1 > c_2$.

Theorem 7.6.3: The equilibrium state $\widehat{P}_{e3} = (0, 0, M)$ is non-linear stable.

These statements correspond exactly to those in section 5.6 (Theorems 5.6.1, 5.6.2 and 5.6.3).

Of course, the energy-Casimir method only provides sufficient conditions for non-linear stability, so it would be interesting to further investigate the stability properties of these equilibrium states in the instances not covered in the theorems.

Numerical results

I found the approach of substituting numerical values into Theorems 4.3.1, 5.3.1, 6.3.1 and 7.3.1 (appendix B) and producing the graphs (appendix C), to be a particularly helpful way of revealing what I may have overlooked when constructing the proofs to these theorems.

In chapter 3, I made the observation that the co-adjoint orbits of $\mathsf{SO}(3)$ are spheres, and that $P(\cdot)$ is contained in the co-adjoint orbit of $\mathsf{SO}(3)$ through $P(0)$. Therefore, it makes sense that the solution curves ((b), (d) and (f) in appendix C) appear to lie on a sphere. However, it was surprising that these curves were closed in all instances. From these findings, I would conjecture that in general the solution curve to the extremal equations is a closed curve contained in the co-adjoint orbit of $\mathsf{SO}(3)$ through $P(0)$.

Referring to appendix C, I compare the solution curves (d) and (f) (from the theorems) to (b) (outputted by the MATLAB ODE45 solver). The Jacobi elliptic functions dc and ns (Type I and Type IIb), similarly, sn and cd (Type III) always give equally accurate solutions. However, the output of the Jacobi elliptic function cn is much closer to the desired solution than that of sd (Type IIa and Type III). Similarly, with nc and ds, where nc is more accurate. Lastly, a portion of each of the solutions corresponding to the nd and dn Jacobi elliptic functions contribute toward the result (Type IIa and Type III), which makes sense as they are inverses of one another. I have given these results in more detail in Table 7.1 (on the next page).

Although the numerical results (appendix C and Table 7.1) point towards a specific solution, I have not found a way to determine, on an algebraic level, which combination of Jacobi elliptic functions would give the most accurate result. Therefore, it would be helpful to read Theorems 4.3.1, 5.3.1, 6.3.1 and 7.3.1 in conjunction with Table 7.1.

Type	**Case**	**First Jacobi elliptic function**	**Positive result?**	**Second Jacobi elliptic function**	**Positive result?**
I		dc	Yes	ns	Yes
IIa	I (the first part)	nd	Partly	dn	Partly
	I (the second part)	sd	Partly	cn	Yes
	II (the first part)	nd	Partly	dn	Partly
	II (the second part)	sd	Partly	cn	Yes
IIb	I (the first part)	dc	Yes	ns	Yes
	I (the second part)	nc	Yes	ds	Partly
	II (the first part)	dc	Yes	ns	Yes
	II (the second part)	nc	Yes	ds	Partly
III	1(A)(I) (the first part)	nd	Partly	dn	Partly
	1(A)(I) (the second part)	sd	Partly	cn	Yes
	1(A)(II) (the first part)	nd	Partly	dn	Partly
	1(A)(II) (the second part)	sd	Partly	cn	Yes
	1(B)(I) (the first part)	nd	Partly	dn	Partly
	1(B)(I) (the second part)	sd	Partly	cn	Yes
	1(B)(II) (the first part)	nd	Partly	dn	Partly
	1(B)(II) (the second part)	sd	Partly	cn	Yes
	2 ($c_2 < c_3 < c_1$)	sn	Yes	cd	Yes
	2 ($c_1 < c_3 < c_2$)	sn	Yes	cd	Yes

Table 7.1: Summary of appendix C

Final remarks

I used the Weierstrass elliptic function to solve the extremal equations in the Type I case only, as it was not possible to use this exact approach for the other types. However, I am aware that there is a deeper link between the Weierstrass and Jacobi elliptic functions, and although I did not examine this, it might provide another means of solving the extremal equations.

Lazureanu and Puta [13] use Definition 1.6.4 to solve the extremal equations. This gives an alternate and fairly simple way of solving the equations using the systems definition of the Jacobi elliptic functions. I only recently became aware of this different approach taken in the literature, so I was unable to give it more attention.

Lastly, it would be interesting to solve the sets of differential equations presented in Propositions 4.5.1, 5.5.1, 6.5.1 and 7.5.1, and to obtain specific expressions for $g(\cdot)$.

Appendix A

Mathematica Code

Given below is the *Wolfram Mathematica 7.0* code that was used to help solve the extremal equations in the proofs of Theorems 4.3.1 and 6.3.1.

Type I

CASE (4.13):

$\mathrm{sol} = \mathrm{Solve}\left[-\lambda{}^\wedge 2 - 2\,(H - c_1)\,\lambda + 2Hc_1 - K - c_1{}^\wedge 2 == 0, \lambda\right]$

$\left\{\left\{\lambda \to c_1 - \sqrt{H^2 - K} - H\right\}, \left\{\lambda \to c_1 + \sqrt{H^2 - K} - H\right\}\right\}$

$\{x{}^\wedge 2 + (2c_1 - \lambda 2)\,x/(-1) + (K - 2Hc_1 + \lambda 2H)/\,(-1)\}\,/.\mathrm{sol}$

$\mathrm{Factor}[\%]$

$$\begin{pmatrix} 2H^2 - 2xH + 2\sqrt{H^2 - K}H + x^2 - K - 2\sqrt{H^2 - K}x \\ 2H^2 - 2xH - 2\sqrt{H^2 - K}H + x^2 - K + 2\sqrt{H^2 - K}x \end{pmatrix}$$

$L = \lambda/.\mathrm{sol}$

$\left\{c_1 - \sqrt{H^2 - K} - H, c_1 + \sqrt{H^2 - K} - H\right\}$

$\lambda_1{:=}L[[1]]$

$\lambda_2{:=}L[[2]]$

$A_1 = \mathrm{FullSimplify}\left[\lambda_2/\,(\lambda_1 - \lambda_2)\right]$

$-\frac{c_1+\sqrt{H^2-K}-H}{2\sqrt{H^2-K}}$

$A_2 = \mathrm{FullSimplify}\left[1\,/(\lambda_2 - \lambda_1)\,\right]$

$\frac{1}{2\sqrt{H^2-K}}$

$B_1 = \mathrm{FullSimplify}\left[\lambda_1/\,(\lambda_2 - \lambda_1)\right]$

$$\frac{c_1-\sqrt{H^2-K}-H}{2\sqrt{H^2-K}}$$

$B_2 = \text{FullSimplify}\,[1\,/(\lambda_1-\lambda_2)]$

$$-\frac{1}{2\sqrt{H^2-K}}$$

$-B_1/A_1$

$$\frac{c_1-\sqrt{H^2-K}-H}{c_1+\sqrt{H^2-K}-H}$$

$-B_2/A_2$

1

A_1A_2

$$-\frac{c_1+\sqrt{H^2-K}-H}{4(H^2-K)}$$

Type IIb

CASE (6.18):

$\text{sol} = \text{Simplify}\,[\text{Solve}\,[(-K+2Hc_1-c_1\hat{}2)\,\lambda\hat{}2+2\,(K-Hc_2-Hc_1+c_1c_2)\,\lambda-K+2Hc_2-c_2\hat{}2==0,\lambda]]$

$$\left\{\left\{\lambda\to\frac{\sqrt{(c_1-c_2)^2(H^2-K)}-c_2H+c_1(c_2-H)+K}{-2c_1H+c_1^2+K}\right\},\left\{\lambda\to\frac{-\sqrt{(c_1-c_2)^2(H^2-K)}-c_2H+c_1(c_2-H)+K}{-2c_1H+c_1^2+K}\right\}\right\}$$

$\{x\hat{}2+(2c_2-\lambda 2c_1)\,x/(\lambda-1)+\,(K-2Hc_2-\lambda\,(K-2Hc_1))/\,(\lambda-1)\}\,/.\text{sol}$

$\text{Factor}[\%]$

$\text{FullSimplify}[\%]$

$$\begin{pmatrix}\frac{2\sqrt{(H^2-K)(c_1-c_2)^2}(x-H)+\left(2H^2-2xH+x^2-K\right)c_1+\left(-2H^2+2xH-x^2+K\right)c_2}{c_1-c_2}\\ \frac{2\sqrt{(H^2-K)(c_1-c_2)^2}(H-x)+\left(2H^2-2xH+x^2-K\right)c_1+\left(-2H^2+2xH-x^2+K\right)c_2}{c_1-c_2}\end{pmatrix}$$

$L=\lambda/.\text{sol}$

$$\left\{\frac{\sqrt{(c_1-c_2)^2(H^2-K)}-c_2H+c_1(c_2-H)+K}{-2c_1H+c_1^2+K},\frac{-\sqrt{(c_1-c_2)^2(H^2-K)}-c_2H+c_1(c_2-H)+K}{-2c_1H+c_1^2+K}\right\}$$

$\lambda_1{:=}L[[2]]$

$\lambda_2{:=}L[[1]]$

$A_1 = \text{FullSimplify}\,[\lambda_2\,(\lambda_1-1)\,/\,(\lambda_2-\lambda_1)]$

$$\frac{(c_1-c_2)(H-c_2)-\sqrt{(c_1-c_2)^2(H^2-K)}}{2\sqrt{(c_1-c_2)^2(H^2-K)}}$$

$A_2 = \text{FullSimplify}\,[(\lambda_1-1)\,/\,(\lambda_1-\lambda_2)]$

$$\frac{\sqrt{(c_1-c_2)^2(H^2-K)}-c_1(c_2+H)+c_2H+c_1^2}{2\sqrt{(c_1-c_2)^2(H^2-K)}}$$

$B_1 = \text{FullSimplify}\,[\lambda_1\,(\lambda_2-1)\,/\,(\lambda_1-\lambda_2)]$

$$-\frac{\sqrt{(c_1-c_2)^2(H^2-K)}+(c_1-c_2)(H-c_2)}{2\sqrt{(c_1-c_2)^2(H^2-K)}}$$

$B_2 = \text{FullSimplify}\left[(\lambda_2 - 1) / (\lambda_2 - \lambda_1)\right]$

$$\frac{\sqrt{(c_1-c_2)^2(H^2-K)}+(c_1-c_2)(H-c_1)}{2\sqrt{(c_1-c_2)^2(H^2-K)}}$$

$-B_1/A_1$

$$\frac{\sqrt{(c_1-c_2)^2(H^2-K)}+(c_1-c_2)(H-c_2)}{(c_1-c_2)(H-c_2)-\sqrt{(c_1-c_2)^2(H^2-K)}}$$

$-B_2/A_2$

$$-\frac{\sqrt{(c_1-c_2)^2(H^2-K)}+(c_1-c_2)(H-c_1)}{\sqrt{(c_1-c_2)^2(H^2-K)}-c_1(c_2+H)+c_2H+c_1^2}$$

A_1A_2

$$\frac{\left((c_1-c_2)(H-c_2)-\sqrt{(c_1-c_2)^2(H^2-K)}\right)\left(\sqrt{(c_1-c_2)^2(H^2-K)}-c_1(c_2+H)+c_2H+c_1^2\right)}{4(c_1-c_2)^2(H^2-K)}$$

CASE (6.19):

$\text{sol} = \text{Simplify}[\text{Solve}\left[(-K + 2Hc_2 - c_2\hat{\ }2)\,\lambda\hat{\ }2 + 2\,(K - Hc_1 - Hc_2 + c_1c_2)\,\lambda - K + \ 2Hc_1 - c_1\hat{\ }2 == 0, \lambda\right]]$

$$\left\{\left\{\lambda \to \frac{\sqrt{(c_1-c_2)^2(H^2-K)}-c_2H+c_1(c_2-H)+K}{-2c_2H+c_2^2+K}\right\}, \left\{\lambda \to \frac{-\sqrt{(c_1-c_2)^2(H^2-K)}-c_2H+c_1(c_2-H)+K}{-2c_2H+c_2^2+K}\right\}\right\}$$

$\{x\hat{\ }2 + (2c_1 - \lambda 2c_2)\,x/(\lambda - 1) + (K - 2Hc_1 - \lambda\,(K - 2Hc_2))/\,(\lambda - 1)\}\ /.\text{sol}$

$\text{Factor}[\%]$

$\text{FullSimplify}[\%]$

$$\begin{pmatrix} \frac{2\sqrt{(H^2-K)(c_1-c_2)^2}(H-x)+\left(2H^2-2xH+x^2-K\right)c_1+\left(-2H^2+2xH-x^2+K\right)c_2}{c_1-c_2} \\ \frac{2\sqrt{(H^2-K)(c_1-c_2)^2}(x-H)+\left(2H^2-2xH+x^2-K\right)c_1+\left(-2H^2+2xH-x^2+K\right)c_2}{c_1-c_2} \end{pmatrix}$$

$L = \lambda/.\text{sol}$

$$\left\{\frac{\sqrt{(c_1-c_2)^2(H^2-K)}-c_2H+c_1(c_2-H)+K}{-2c_2H+c_2^2+K}, \frac{-\sqrt{(c_1-c_2)^2(H^2-K)}-c_2H+c_1(c_2-H)+K}{-2c_2H+c_2^2+K}\right\}$$

$\lambda_1{:=}L[[2]]$

$\lambda_2{:=}L[[1]]$

$A_1 = \text{FullSimplify}\left[\lambda_2\,(\lambda_1 - 1) / (\lambda_2 - \lambda_1)\right]$

$$\frac{-\sqrt{(c_1-c_2)^2(H^2-K)}-c_1(c_2+H)+c_2H+c_1^2}{2\sqrt{(c_1-c_2)^2(H^2-K)}}$$

$A_2 = \text{FullSimplify}\left[(\lambda_1 - 1) / (\lambda_1 - \lambda_2)\right]$

$$\frac{\sqrt{(c_1-c_2)^2(H^2-K)}+(c_1-c_2)(H-c_2)}{2\sqrt{(c_1-c_2)^2(H^2-K)}}$$

$B_1 = \text{FullSimplify}\left[\lambda_1\,(\lambda_2 - 1) / (\lambda_1 - \lambda_2)\right]$

$$-\frac{\sqrt{(c_1-c_2)^2(H^2-K)}-c_1(c_2+H)+c_2H+c_1^2}{2\sqrt{(c_1-c_2)^2(H^2-K)}}$$

$B_2 = \text{FullSimplify}\,[(\lambda_2 - 1)\,/\,(\lambda_2 - \lambda_1)]$

$$\frac{\sqrt{(c_1-c_2)^2(H^2-K)}+(c_2-c_1)(H-c_2)}{2\sqrt{(c_1-c_2)^2(H^2-K)}}$$

$-B_1/A_1$

$$\frac{\sqrt{(c_1-c_2)^2(H^2-K)}-c_1(c_2+H)+c_2H+c_1^2}{-\sqrt{(c_1-c_2)^2(H^2-K)}-c_1(c_2+H)+c_2H+c_1^2}$$

$-B_2/A_2$

$$-\frac{\sqrt{(c_1-c_2)^2(H^2-K)}+(c_2-c_1)(H-c_2)}{\sqrt{(c_1-c_2)^2(H^2-K)}+(c_1-c_2)(H-c_2)}$$

A_1A_2

$$\frac{\left(\sqrt{(c_1-c_2)^2(H^2-K)}+(c_1-c_2)(H-c_2)\right)\left(-\sqrt{(c_1-c_2)^2(H^2-K)}-c_1(c_2+H)+c_2H+c_1^2\right)}{4(c_1-c_2)^2(H^2-K)}$$

Appendix B

MATLAB Code

Given below is the *MATLAB 7.4.0* code that was used to help check the results of Theorems 4.3.1, 5.3.1, 6.3.1 and 7.3.1.

The extremal equations are non-stiff in nature, therefore, in the code below, I have chosen to use the ODE45 solver to solve these systems. This is the most common solver in use for this type of system, and it has medium accuracy.

Type I

TypeI.m

```
function P = TypeI(t, p, c1)
P = zeros(3,1);
P(1) = p(2);
P(2) = (1/c1)*p(1)*p(3)-p(1);
P(3) = -(1/c1)*p(1)*p(2);
```

Section 1:

The MATLAB code used to produce Figure C.1(a) and Figure C.1(b):

Command window:

```
c1 = 2;
[t, p] = ode45(@TypeI,[0 10],[4 0 1],[],c1);
plot(t,p(:,1),‘-’,t,p(:,2),‘-.’,t,p(:,3),‘.’)
xlabel(‘Time’)
ylabel(‘Output’)
legend(‘P1’,‘P2’,‘P3’)
```

(Delete graph window)

```
plot3(p(:,1),p(:,2),p(:,3))
xlabel(‘P1’)
ylabel(‘P2’)
zlabel(‘P3’)
```

Section 2:

The first elliptic function: dc

The MATLAB code used to produce Figure C.1(c) and Figure C.1(d) (rounding to four decimal places):

Command window:

```
t = linspace(0,10,100);
[s,c,d] = ellipj(1.7071*t,0.1716);
P3 = (7.8284-12.6569.*(d./c))./(1-5.8284.*(d./c));
```

```
P2 = sqrt(-3 + 4.*P3-P3.*P3);
P1 = sqrt(20 - 4.*P3);
plot(t,P1,‘-’,t,P2,‘-.’,t,P3,‘.’)
hold
Current plot held
plot(t,P1,‘-’,t,-P2,‘-.’,t,P3,‘.’)
xlabel(‘Time’)
ylabel(‘Output’)
legend(‘P1’,‘P2’,‘P3’)
```

(Delete graph window)

```
plot3(P1,P2,P3)
hold
Current plot held
plot3(P1,-P2,P3)
xlabel(‘P1’)
ylabel(‘P2’)
zlabel(‘P3’)
```

The second elliptic function: ns

The MATLAB code used to produce Figure C.1(e) and Figure C.1(f) (rounding to four decimal places):

Command window:

```
t = linspace(0,10,100);
[s,c,d] = ellipj(1.7071*t,0.1716);
P3 = (7.8284-12.6569.*(1./s))./(1-5.8284.*(1./s));
P2 = sqrt(-3 + 4.*P3-P3.*P3);
P1 = sqrt(20 - 4.*P3);
plot(t,P1,‘-’,t,P2,‘-.’,t,P3,‘.’)
hold
Current plot held
plot(t,P1,‘-’,t,-P2,‘-.’,t,P3,‘.’)
xlabel(‘Time’)
ylabel(‘Output’)
legend(‘P1’,‘P2’,‘P3’)
```

(Delete graph window)

```
plot3(P1,P2,P3)
hold
Current plot held
plot3(P1,-P2,P3)
xlabel(‘P1’)
ylabel(‘P2’)
zlabel(‘P3’)
```

Type IIa

TypeIIa.m

```
function P = TypeIIa(t,p,c1,c2)
P = zeros(3,1);
P(1) = -(1/c2)*p(2)*p(3);
P(2) = (1/c1)*p(1)*p(3);
P(3) = ((1/c2)-(1/c1))*p(1)*p(2);
```

Case (i) (the first part)

Section 1:

The MATLAB code used to produce Figure C.2(a) and Figure C.2(b):

Command window:

```
c1 = 4; c2 = 1;
[t, p] = ode45(@TypeIIa,[0 10],[2 2 4],[],c1,c2);
plot(t,p(:,1),‘-’,t,p(:,2),‘-.’,t,p(:,3),‘.’)
xlabel(‘Time’)
ylabel(‘Output’)
legend(‘P1’,‘P2’,‘P3’)
```

(Delete graph window)

```
plot3(p(:,1),p(:,2),p(:,3))
xlabel(‘P1’)
ylabel(‘P2’)
zlabel(‘P3’)
```

Section 2:

The first elliptic function: nd

The MATLAB code used to produce Figure C.2(c) and Figure C.2(d) (rounding for four decimal places):

Command window:

```
t = linspace(0,10,100);
[s,c,d] = ellipj(2.1794*t,0.8885);
P3 = 2.*(1./d);
P2 = sqrt(-1.3333 + 0.3333.*P3.*P3);
P1 = sqrt(25.3333-1.3333.*P3.*P3);
plot(t,P1,‘-’,t,P2,‘-.’,t,P3,‘.’)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
hold
Current plot held
plot(t,-P1,‘-’,t,-P2,‘-.’,t,P3,‘.’)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
xlabel(‘Time’)
ylabel(‘Output’)
legend(‘P1’,‘P2’,‘P3’)
```

(Delete graph window)

```
plot3(P1,P2,P3)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
hold
Current plot held
plot3(-P1,P2,P3)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
plot3(P1,-P2,P3)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
plot3(-P1,-P2,P3)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
xlabel(‘P1’)
ylabel(‘P2’)
zlabel(‘P3’)
```

The second elliptic function: dn

The MATLAB code used to produce Figure C.2(e) and Figure C.2(f) (rounding for four decimal places):

Command window:

```
t = linspace(0,10,100);
[s,c,d] = ellipj(2.1794*t,0.8885);
P3 = 4.3589.*d;
P2 = sqrt(-1.3333+0.3333.*P3.*P3);
P1 = sqrt(25.3333-1.3333.*P3.*P3);
plot(t,P1,‘-’,t,P2,‘-.’,t,P3,‘.’)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
hold
Current plot held
plot(t,-P1,‘-’,t,-P2,‘-.’,t,P3,‘.’)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
xlabel(‘Time’)
ylabel(‘Output’)
legend(‘P1’,‘P2’,‘P3’)
```

(Delete graph window)

```
plot3(P1,P2,P3)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
hold
Current plot held
plot3(-P1,P2,P3)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
plot3(P1,-P2,P3)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
```

```
plot3(-P1,-P2,P3)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
xlabel(‘P1’)
ylabel(‘P2’)
zlabel(‘P3’)
```

CASE (I) (the second part)

SECTION 1:

The MATLAB code used to produce Figure C.3(a) and Figure C.3(b):

Command window:

```
c1 = 4; c2 = 1;
[t, p] = ode45(@TypeIIa,[0 10],[4 2 2],[],c1,c2);
plot(t,p(:,1),‘-’,t,p(:,2),‘-.’,t,p(:,3),‘.’)
xlabel(‘Time’)
ylabel(‘Output’)
legend(‘P1’,‘P2’,‘P3’)
```

(Delete graph window)

```
plot3(p(:,1),p(:,2),p(:,3))
xlabel(‘P1’)
ylabel(‘P2’)
zlabel(‘P3’)
```

SECTION 2:

The first elliptic function: sd

The MATLAB code used to produce Figure C.3(c) and Figure C.3(d) (rounding for four decimal places):

Command window:

```
t = linspace(0,10,100);
[s,c,d] = ellipj(2.4495*t,0.8165);
P3 = 2.3094.*(s./d);
P2 = sqrt(2.6667+0.3333.*P3.*P3);
P1 = sqrt(21.3333-1.3333.*P3.*P3);
plot(t,P1,‘-’,t,P2,‘-.’,t,P3,‘.’)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
hold
Current plot held
plot(t,-P1,‘-’,t,P2,‘-.’,t,P3,‘.’)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
xlabel(‘Time’)
ylabel(‘Output’)
legend(‘P1’,‘P2’,‘P3’)
```

(Delete graph window)

```
plot3(P1,P2,P3)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
hold
Current plot held
plot3(-P1,P2,P3)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
xlabel(‘P1’)
ylabel(‘P2’)
zlabel(‘P3’)
```

The second elliptic function: cn

The MATLAB code used to produce Figure C.3(e) and Figure C.3(f) (rounding for four decimal places):

Command window:

```
t = linspace(0,10,100);
[s,c,d] = ellipj(2.4495*t,0.8165);
P3 = 4.*c;
P2 = sqrt(2.6667+0.3333.*P3.*P3);
P1 = sqrt(21.3333-1.3333.*P3.*P3);
plot(t,P1,‘-’,t,P2,‘-.’,t,P3,‘.’)
hold
Current plot held
plot(t,-P1,‘-’,t,P2,‘-.’,t,P3,‘.’)
```

```
xlabel(‘Time’)
ylabel(‘Output’)
legend(‘P1’,‘P2’,‘P3’)
```

(Delete graph window)

```
plot3(P1,P2,P3)
hold
Current plot held
plot3(-P1,P2,P3)
xlabel(‘P1’)
ylabel(‘P2’)
zlabel(‘P3’)
```

CASE (II) (the first part)

SECTION 1:

The MATLAB code used to produce Figure C.4(a) and Figure C.4(b):

Command window:

```
c1 = 1; c2 = 4;
[t, p] = ode45(@TypeIIa,[0 10],[1 2 3],[],c1,c2);
plot(t,p(:,1),‘-’,t,p(:,2),‘-.’,t,p(:,3),‘.’)
xlabel(‘Time’)
ylabel(‘Output’)
legend(‘P1’,‘P2’,‘P3’)
```

(Delete graph window)

```
plot3(p(:,1),p(:,2),p(:,3))
xlabel(‘P1’)
ylabel(‘P2’)
zlabel(‘P3’)
```

SECTION 2:

The first elliptic function: nd

The MATLAB code used to produce Figure C.4(c) and Figure C.4(d) (rounding for four decimal places):

Command window:

```
t = linspace(0,10,100);
[s,c,d] = ellipj(1.7321*t,0.7071);
P3 = 2.4495.*(1./d);
P2 = sqrt(-2+0.3333.*P3.*P3);
P1 = sqrt(16-1.3333.*P3.*P3);
plot(t,P1,‘-’,t,P2,‘-.’,t,P3,‘.’)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
hold
Current plot held
plot(t,-P1,‘-’,t,-P2,‘-.’,t,P3,‘.’)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
xlabel(‘Time’)
ylabel(‘Output’)
legend(‘P1’,‘P2’,‘P3’)
```

(Delete graph window)

```
plot3(P1,P2,P3)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
hold
Current plot held
plot3(-P1,P2,P3)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
plot3(P1,-P2,P3)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
plot3(-P1,-P2,P3)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
xlabel(‘P1’)
ylabel(‘P2’)
zlabel(‘P3’)
```

The second elliptic function: dn

The MATLAB code used to produce Figure

C.4(e) and Figure C.4(f) (rounding for four decimal places):

Command window:

```
t = linspace(0,10,100);
[s,c,d] = ellipj(1.7321*t,0.7071);
P3 = 3.4641.*d;
P2 = sqrt(16-1.3333.*P3.*P3);
P1 = sqrt(-2+0.3333.*P3.*P3);
plot(t,P1,‘-’,t,P2,‘-.’,t,P3,‘.’)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
hold
Current plot held
plot(t,-P1,‘-’,t,-P2,‘-.’,t,P3,‘.’)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
xlabel(‘Time’)
ylabel(‘Output’)
legend(‘P1’,‘P2’,‘P3’)
```

(Delete graph window)

```
plot3(P1,P2,P3)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
hold
Current plot held
plot3(-P1,P2,P3)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
plot3(P1,-P2,P3)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
plot3(-P1,-P2,P3)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
xlabel(‘P1’)
ylabel(‘P2’)
zlabel(‘P3’)
```

CASE (II) (the second part)

SECTION 1:

The MATLAB code used to produce Figure C.5(a) and Figure C.5(b):

Command window:

```
c1 = 1; c2 = 4;
[t, p] = ode45(@TypeIIa,[0 10],[3 2 1],[],c1,c2);
plot(t,p(:,1),‘-’,t,p(:,2),‘-.’,t,p(:,3),‘.’)
xlabel(‘Time’)
ylabel(‘Output’)
legend(‘P1’,‘P2’,‘P3’)
```

(Delete graph window)

```
plot3(p(:,1),p(:,2),p(:,3))
xlabel(‘P1’)
ylabel(‘P2’)
zlabel(‘P3’)
```

SECTION 2:

The first elliptic function: sd

The MATLAB code used to produce Figure C.5(c) and Figure C.5(d) (rounding for four decimal places):

Command window:

```
t = linspace(0,10,100);
[s,c,d] = ellipj(2.7386*t,0.3651);
P3 = 1.8619.*(s./d);
P2 = sqrt(5.3333-1.3333.*P3.*P3);
P1 = sqrt(8.6667+0.3333.*P3.*P3);
plot(t,P1,‘-’,t,P2,‘-.’,t,P3,‘.’)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
hold
Current plot held
plot(t,P1,‘-’,t,-P2,‘-.’,t,P3,‘.’)
```

```
Warning: Imaginary parts of complex X and/or
Y arguments ignored
xlabel(‘Time’)
ylabel(‘Output’)
legend(‘P1’,‘P2’,‘P3’)
(Delete graph window)
plot3(P1,P2,P3)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
hold
Current plot held
plot3(P1,-P2,P3)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
xlabel(‘P1’)
ylabel(‘P2’)
zlabel(‘P3’)
```

The second elliptic function: cn

The MATLAB code used to produce Figure C.4(e) and Figure C.4(f) (rounding for four decimal places):

Command window:

```
t = linspace(0,10,100);
[s,c,d] = ellipj(2.7386*t,0.3651);
P3 = 2.*c;
P2 = sqrt(5.3333-1.3333.*P3.*P3);
P1 = sqrt(8.6667+0.3333.*P3.*P3);
plot(t,P1,‘-’,t,P2,‘-.’,t,P3,‘.’)
hold
Current plot held
plot(t,P1,‘-’,t,-P2,‘-.’,t,P3,‘.’)
xlabel(‘Time’)
ylabel(‘Output’)
legend(‘P1’,‘P2’,‘P3’)
(Delete graph window)
plot3(P1,P2,P3)
hold
Current plot held
plot3(P1,-P2,P3)
xlabel(‘P1’)
ylabel(‘P2’)
zlabel(‘P3’)
```

Type IIb

TypeIIb.m

```
function P = TypeIIb(t,p,c1,c2)
P = zeros(3,1);
P(1) = -(1/c2)*p(2)*p(3)+p(2);
P(2) = (1/c1)*p(1)*p(3)-p(1);
P(3) = ((1/c2)-(1/c1))*p(1)*p(2);
```

CASE (I) (the first part)

SECTION 1:

The MATLAB code used to produce Figure C.6(a) and Figure C.6(b)(rounding to four decimal places):

Command window:

```
c1 = 1; c2 = 0.5;
[t,p] = ode45(@TypeIIb,[0 10],[0.3536 0.8420
0.1563],[],c1,c2);
plot(t,p(:,1),‘-’,t,p(:,2),‘-.’,t,p(:,3),‘.’)
xlabel(‘Time’)
ylabel(‘Output’)
legend(‘P1’,‘P2’,‘P3’)
(Delete graph window)
c1 = 1; c2 = 0.5;
[t,p] = ode45(@TypeIIb,[0 30],[0.3536 0.8420
0.1563],[],c1,c2);
```

```
plot3(p(:,1),p(:,2),p(:,3))
xlabel(‘P1’)
ylabel(‘P2’)
zlabel(‘P3’)
```

SECTION 2:

The first elliptic function: dc

The MATLAB code used to produce Figure C.6(c) and C.6(d) (rounding to four decimal places):

Command window:

```
t = linspace(0,10,100);
[s,c,d] = ellipj(0.3381*t,0.0403);
P3 = (0.9756-0.9845.*(d./c))./(1-1.1190.*(d./c));
P2 = sqrt(0.9971-2.*P3+P3.*P3);
P1 = sqrt(-0.1386+2.*P3-2.*P3.*P3);
plot(t,P1,‘-’,t,P2,‘-.’,t,P3,‘.’)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
hold
Current plot held
plot(t,-P1,‘-’,t,P2,‘-.’,t,P3,‘.’)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
xlabel(‘Time’)
ylabel(‘Output’)
legend(‘P1’,‘P2’,‘P3’)
```

(Delete graph window)

```
t = linspace(0,30,100);
[s,c,d] = ellipj(0.3381*t,0.0403);
P3 = (0.9756-0.9845.*(d./c))./(1-1.1190.*(d./c));
P2 = sqrt(0.9971-2.*P3+P3.*P3);
P1 = sqrt(-0.1386+2.*P3-2.*P3.*P3);
plot3(P1,P2,P3)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
hold
Current plot held
plot3(-P1,P2,P3)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
xlabel(‘P1’)
ylabel(‘P2’)
zlabel(‘P3’)
```

The second elliptic function: ns

The MATLAB code used to produce Figure C.6(e) and C.6(f) (rounding to four decimal places):

Command window:

```
t = linspace(0,10,100);
[s,c,d] = ellipj(0.3381*t,0.0403);
P3 = (0.9756-0.9845.*(1./s))./(1-1.1190.*(1./s));
P2 = sqrt(0.9971-2.*P3+P3.*P3);
P1 = sqrt(-0.1386+2.*P3-2.*P3.*P3);
plot(t,P1,‘-’,t,P2,‘-.’,t,P3,‘.’)
hold
Current plot held
plot(t,-P1,‘-’,t,P2,‘-.’,t,P3,‘.’)
xlabel(‘Time’)
ylabel(‘Output’)
legend(‘P1’,‘P2’,‘P3’)
```

(Delete graph window)

```
t = linspace(0,30,100);
[s,c,d] = ellipj(0.3381*t,0.0403);
P3 = (0.9756-0.9845.*(1./s))./(1-1.1190.*(1./s));
P2 = sqrt(0.9971-2.*P3+P3.*P3);
P1 = sqrt(-0.1386+2.*P3-2.*P3.*P3);
plot3(P1,P2,P3)
hold
Current plot held
plot3(-P1,P2,P3)
xlabel(‘P1’)
```

```
ylabel(‘P2’)
zlabel(‘P3’)
```

CASE (I) (the second part)

SECTION 1:

The MATLAB code used to produce Figure C.7(a) and Figure C.7(b) (rounding to four decimal places):

Command window:

```
c1 = 1; c2 = 0.5;
[t,p] = ode45(@TypeIIb,[0 10],[1.2247 2.8284 -1],[],c1,c2);
plot(t,p(:,1),‘-’,t,p(:,2),‘-.’,t,p(:,3),‘.’)
xlabel(‘Time’)
ylabel(‘Output’)
legend(‘P1’,‘P2’,‘P3’)
```

(Delete graph window)

```
plot3(p(:,1),p(:,2),p(:,3))
xlabel(‘P1’)
ylabel(‘P2’)
zlabel(‘P3’)
```

SECTION 2:

The first elliptic function: nc

The MATLAB code used to produce Figure C.7(c) and Figure C.7(d) (rounding to four decimal places):

Command window:

```
t = linspace(0,10,150);
[s,c,d] = ellipj(3.7523*t,0.6413);
P3 = (14.7901-5.8572.*(1./c))./(1-8.2504.*(1./c));
P2 = sqrt(5-2.*P3+P3.*P3);
P1 = sqrt(5.5+2.*P3-2.*P3.*P3);
plot(t,P1,‘-’,t,P2,‘-.’,t,P3,‘.’)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
hold
Current plot held
plot(t,-P1,‘-’,t,P2,‘-.’,t,P3,‘.’)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
xlabel(‘Time’)
ylabel(‘Output’)
legend(‘P1’,‘P2’,‘P3’)
```

(Delete graph window)

```
plot3(P1,P2,P3)
hold
Current plot held
plot3(-P1,P2,P3)
xlabel(‘P1’)
ylabel(‘P2’)
zlabel(‘P3’)
```

The second elliptic function: ds

The MATLAB code used to produce Figure C.7(e) and Figure C.7(f) (rounding to four decimal places):

Command window:

```
t = linspace(0,10,150);
[s,c,d] = ellipj(3.7523*t,0.6413);
P3 = (14.7901-7.6332.*(d./s))./(1-8.2504.*(d./s));
P2 = sqrt(5-2.*P3+P3.*P3);
P1 = sqrt(5.5+2.*P3-2.*P3.*P3);
plot(t,P1,‘-’,t,P2,‘-.’,t,P3,‘.’)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
hold
Current plot held
plot(t,-P1,‘-’,t,P2,‘-.’,t,P3,‘.’)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
xlabel(‘Time’)
```

```
ylabel(‘Output’)
legend(‘P1’,‘P2’,‘P3’)
```

(Delete graph window)

```
plot3(P1,P2,P3)
Warning: Imaginary parts of complex X and/or Y arguments ignored
hold
Current plot held
plot3(-P1,P2,P3)
Warning: Imaginary parts of complex X and/or Y arguments ignored
xlabel(‘P1’)
ylabel(‘P2’)
zlabel(‘P3’)
```

CASE (II) (the first part)

SECTION 1:

The MATLAB code used to produce Figure C.8(a) and Figure C.8(b) (rounding to four decimal places):

Command window:

```
c1 = 1; c2 = 2;
[t,p] = ode45(@TypeIIb,[0 10],[1.4577 0.3953 0.5],[],c1,c2);
plot(t,p(:,1),‘-’,t,p(:,2),‘-.’,t,p(:,3),‘.’)
xlabel(‘Time’)
ylabel(‘Output’)
legend(‘P1’,‘P2’,‘P3’)
```

(Delete graph window)

```
c1 = 1; c2 = 2;
[t,p] = ode45(@TypeIIb,[0 30],[1.4577 0.3953 0.5],[],c1,c2);
plot3(p(:,1),p(:,2),p(:,3))
xlabel(‘P1’)
ylabel(‘P2’)
zlabel(‘P3’)
```

(The figure has been slightly rotated.)

SECTION 2:

The first elliptic function: dc

The MATLAB code used to produce Figure C.8(c) and Figure C.8(d) (rounding to four decimal places):

Command window:

```
t = linspace(0,10,100);
[s,c,d] = ellipj(0.4781*t,0.4407);
P3 = (1.7853-1.9437.*(d./c))./(1-1.3709.*(d./c));
P2 = sqrt(-1.3438+4.*P3-2.*P3.*P3);
P1 = sqrt(3.875-4.*P3+P3.*P3);
plot(t,P1,‘-’,t,P2,‘-.’,t,P3,‘.’)
Warning: Imaginary parts of complex X and/or Y arguments ignored
hold
Current plot held
plot(t,P1,‘-’,t,-P2,‘-.’,t,P3,‘.’)
Warning: Imaginary parts of complex X and/or Y arguments ignored
xlabel(‘Time’)
ylabel(‘Output’)
legend(‘P1’,‘P2’,‘P3’)
```

(Delete graph window)

```
plot3(P1,P2,P3)
Warning: Imaginary parts of complex X and/or Y arguments ignored
hold
Current plot held
plot3(P1,-P2,P3)
Warning: Imaginary parts of complex X and/or Y arguments ignored
xlabel(‘P1’)
ylabel(‘P2’)
zlabel(‘P3’)
```

(The figure has been slightly rotated.)

The second elliptic function: ns

The MATLAB code used to produce Figure C.8(e) and Figure C.8(f) (rounding to four decimal places):

Command window:

```
t = linspace(0,10,100);
[s,c,d] = ellipj(0.4781*t,0.4407);
P3 = (1.7853-1.9437.*(1./s))./(1-1.3709.*(1./s));
P2 = sqrt(-1.3438+4.*P3-2.*P3.*P3);
P1 = sqrt(3.875-4.*P3+P3.*P3);
plot(t,P1,‘-’,t,P2,‘-.’,t,P3,‘.’)
hold
Current plot held
plot(t,P1,‘-’,t,-P2,‘-.’,t,P3,‘.’)
xlabel(‘Time’)
ylabel(‘Output’)
legend(‘P1’,‘P2’,‘P3’)
```

(Delete graph window)

```
plot3(P1,P2,P3)
hold
Current plot held
plot3(P1,-P2,P3)
xlabel(‘P1’)
ylabel(‘P2’)
zlabel(‘P3’)
```

(The figure has been slightly rotated.)

Case (ii) (the second part)

Section 1:

The MATLAB code used to produce Figure C.9(a) and Figure C.9(b) (rounding to four decimal places):

Command window:

```
c1 = 1; c2 = 2;
[t,p] = ode45(@TypeIIb,[0 10],[4.4721 1 -2],[],c1,c2);
plot(t,p(:,1),‘-’,t,p(:,2),‘-.’,t,p(:,3),‘.’)
xlabel(‘Time’)
ylabel(‘Output’)
legend(‘P1’,‘P2’,‘P3’)
```

(Delete graph window)

```
plot3(p(:,1),p(:,2),p(:,3))
xlabel(‘P1’)
ylabel(‘P2’)
zlabel(‘P3’)
```

Section 2:

The first elliptic function: nc

The MATLAB code used to produce Figure C.9(c) and Figure C.9(d) (rounding to four decimal places):

Command window:

```
t = linspace(0,10,150);
[s,c,d] = ellipj(2.5617*t,0.8194);
P3 = (14.8122-7.5635.*(1./c))./(1-4.4813.*(1./c));
P2 = sqrt(17+4.*P3-2.*P3.*P3);
P1 = sqrt(8-4.*P3+P3.*P3);
plot(t,P1,‘-’,t,P2,‘-.’,t,P3,‘.’)
hold
Current plot held
plot(t,P1,‘-’,t,-P2,‘-.’,t,P3,‘.’)
xlabel(‘Time’)
ylabel(‘Output’)
legend(‘P1’,‘P2’,‘P3’)
```

(Delete graph window)

```
plot3(P1,P2,P3)
hold
Current plot held
plot3(P1,-P2,P3)
xlabel(‘P1’)
```

```
ylabel(‘P2’)
zlabel(‘P3’)
```

The second elliptic function: ds

The MATLAB code used to produce Figure C.9(e) and Figure C.9(f) (rounding to four decimal places):

Command window:

```
t = linspace(0,10,150);
[s,c,d] = ellipj(2.5617*t,0.8194);
P3 = (14.8122-13.1951.*(d./s))./(1-7.8179.*(d./s));
P2 = sqrt(17+4.*P3-2.*P3.*P3);
P1 = sqrt(8-4.*P3+P3.*P3);
plot(t,P1,‘-’,t,P2,‘-.’,t,P3,‘.’)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
hold
Current plot held
plot(t,P1,‘-’,t,-P2,‘-.’,t,P3,‘.’)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
xlabel(‘Time’)
ylabel(‘Output’)
legend(‘P1’,‘P2’,‘P3’)
```

(Delete graph window)

```
plot3(P1,P2,P3)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
hold
Current plot held
plot3(P1,-P2,P3)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
xlabel(‘P1’)
ylabel(‘P2’)
zlabel(‘P3’)
```

Type III

TypeIII.m

```
function P = TypeIII(t,p,c1,c2,c3)
P = zeros(3,1);
P(1) = ((1/c3)-(1/c2))*p(2)*p(3);
P(2) = ((1/c1)-(1/c3))*p(1)*p(3);
P(3) = ((1/c2)-(1/c1))*p(1)*p(2);
```

CASE 1A(I) (the first part)

SECTION 1:

The MATLAB code used to produce Figure C.10(a) and Figure C.10(b):

Command window:

```
c1 = 3; c2 = 2; c3 = 4;
[t, p] = ode45(@TypeIII,[0 30],[0 1 2],[],c1,c2,c3);
plot(t,p(:,1),‘-’,t,p(:,2),‘-.’,t,p(:,3),‘.’)
xlabel(‘Time’)
ylabel(‘Output’)
legend(‘P1’,‘P2’,‘P3’)
```

(Delete graph window)

```
plot3(p(:,1),p(:,2),p(:,3))
xlabel(‘P1’)
ylabel(‘P2’)
zlabel(‘P3’)
```

SECTION 2:

The first elliptic function: nd

The MATLAB code used to produce Figure C.10(c) and Figure C.10(d) (rounding to four decimal places):

Command window:

```
t = linspace(0,30,150);
[s,c,d] = ellipj(0.2887*t,0.7071);
P3 = 1.4142.*(1./d);
P2 = sqrt(-1+0.5.*P3.*P3);
P1 = sqrt(6-1.5.*P3.*P3);
plot(t,P1,‘-’,t,P2,‘-.’,t,P3,‘.’)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
Warning: Imaginary parts of complex X and/or
Y arguments ignored
hold
Current plot held
plot(t,-P1,‘-’,t,-P2,‘-.’,t,P3,‘.’)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
Warning: Imaginary parts of complex X and/or
Y arguments ignored
xlabel(‘Time’)
ylabel(‘Output’)
legend(‘P1’,‘P2’,‘P3’)
```

(Delete graph window)

```
plot3(P1,P2,P3)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
hold
Current plot held
plot3(-P1,P2,P3)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
plot3(P1,-P2,P3)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
plot3(-P1,-P2,P3)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
xlabel(‘P1’)
ylabel(‘P2’)
zlabel(‘P3’)
```

The second elliptic function: dn

The MATLAB code used to produce Figure C.10(e) and Figure C.10(f) (rounding to four decimal places):

Command window:

```
t = linspace(0,30,150);
[s,c,d] = ellipj(0.2887*t,0.7071);
P3 = 2.*d;
P2 = sqrt(-1+0.5.*P3.*P3);
P1 = sqrt(6-1.5.*P3.*P3);
plot(t,P1,‘-’,t,P2,‘-.’,t,P3,‘.’)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
hold
Current plot held
plot(t,-P1,‘-’,t,-P2,‘-.’,t,P3,‘.’)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
xlabel(‘Time’)
ylabel(‘Output’)
legend(‘P1’,‘P2’,‘P3’)
```

(Delete graph window)

```
plot3(P1,P2,P3)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
hold
Current plot held
plot3(-P1,P2,P3)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
plot3(P1,-P2,P3)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
plot3(-P1,-P2,P3)
Warning: Imaginary parts of complex X and/or
```

```
Y arguments ignored
xlabel(‘P1’)
ylabel(‘P2’)
zlabel(‘P3’)
```

CASE 1A(I) (the second part)

SECTION 1:

The MATLAB code used to produce Figure C.11(a) and Figure C.11(b):

Command window:

```
c1 = 3; c2 = 2; c3 = 4;
[t, p] = ode45(@TypeIII,[0 30],[0 2 1],[],c1,c2,c3);
plot(t,p(:,1),‘-’,t,p(:,2),‘-.’,t,p(:,3),‘.’)
xlabel(‘Time’)
ylabel(‘Output’)
legend(‘P1’,‘P2’,‘P3’)
```

(Delete graph window)

```
plot3(p(:,1),p(:,2),p(:,3))
xlabel(‘P1’)
ylabel(‘P2’)
zlabel(‘P3’)
```

SECTION 2:

The first elliptic function: sd

The MATLAB code used to produce Figure C.11(c) and Figure C.11(d) (rounding to four decimal places):

Command window:

```
t = linspace(0,30,150);
[s,c,d] = ellipj(0.4082*t,0.3536);
P3 = 0.9354.*(s./d);
P2 = sqrt(3.5+0.5.*P3.*P3);
P1 = sqrt(1.5-1.5.*P3.*P3);
plot(t,P1,‘-’,t,P2,‘-.’,t,P3,‘.’)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
hold
Current plot held
plot(t,-P1,‘-’,t,P2,‘-.’,t,P3,‘.’)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
xlabel(‘Time’)
ylabel(‘Output’)
legend(‘P1’,‘P2’,‘P3’)
```

(Delete graph window)

```
plot3(P1,P2,P3)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
hold
Current plot held
plot3(-P1,P2,P3)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
xlabel(‘P1’)
ylabel(‘P2’)
zlabel(‘P3’)
```

The second elliptic function: cn

The MATLAB code used to produce Figure C.11(e) and Figure C.11(f) (rounding to four decimal places):

Command window:

```
t = linspace(0,30,150);
[s,c,d] = ellipj(0.4082*t,0.3536);
P3 = c;
P2 = sqrt(3.5+0.5.*P3.*P3);
P1 = sqrt(1.5-1.5.*P3.*P3);
plot(t,P1,‘-’,t,P2,‘-.’,t,P3,‘.’)
hold
Current plot held
plot(t,-P1,‘-’,t,P2,‘-.’,t,P3,‘.’)
xlabel(‘Time’)
ylabel(‘Output’)
```

```
legend(‘P1’,‘P2’,‘P3’)
```

(Delete graph window)

```
plot3(P1,P2,P3)
hold
Current plot held
plot3(-P1,P2,P3)
xlabel(‘P1’)
ylabel(‘P2’)
zlabel(‘P3’)
```

CASE 1A(II) (the first part)

SECTION 1:

The MATLAB code used to produce Figure C.12(a) and Figure C.12(b):

Command window:

```
c1 = 1; c2 = 2; c3 = 3;
[t, p] = ode45(@TypeIII,[0 30],[1 1 2],[],c1,c2,c3);
plot(t,p(:,1),‘-’,t,p(:,2),‘-.’,t,p(:,3),‘.’)
xlabel(‘Time’)
ylabel(‘Output’)
legend(‘P1’,‘P2’,‘P3’)
```

(Delete graph window)

```
plot3(p(:,1),p(:,2),p(:,3))
xlabel(‘P1’)
ylabel(‘P2’)
zlabel(‘P3’)
```

SECTION 2:

The first elliptic function: nd

The MATLAB code used to produce Figure C.12(c) and Figure C.12(d) (rounding to four decimal places):

Command window:

```
t = linspace(0,30,150);
[s,c,d] = ellipj(0.7265*t,0.8885);
P3 = (1./d);
P2 = sqrt(6.3334-1.3333.*P3.*P3);
P1 = sqrt(-0.3334+0.3333.*P3.*P3);
plot(t,P1,‘-’,t,P2,‘-.’,t,P3,‘.’)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
Warning: Imaginary parts of complex X and/or
Y arguments ignored
hold
Current plot held
plot(t,-P1,‘-’,t,-P2,‘-.’,t,P3,‘.’)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
Warning: Imaginary parts of complex X and/or
Y arguments ignored
xlabel(‘Time’)
ylabel(‘Output’)
legend(‘P1’,‘P2’,‘P3’)
```

(Delete graph window)

```
plot3(P1,P2,P3)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
hold
Current plot held
plot3(-P1,P2,P3)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
plot3(P1,-P2,P3)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
plot3(-P1,-P2,P3)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
xlabel(‘P1’)
ylabel(‘P2’)
zlabel(‘P3’)
```

The second elliptic function: dn

The MATLAB code used to produce Figure C.12(e) and Figure C.12(f) (rounding to four decimal places):

Command window:

```
t = linspace(0,30,150);
[s,c,d] = ellipj(0.7265*t,0.8885);
P3 = 2.1795.*d;
P2 = sqrt(6.3334-1.3333.*P3.*P3);
P1 = sqrt(-0.3334+0.3333.*P3.*P3);
plot(t,P1,‘-’,t,P2,‘-.’,t,P3,‘.’)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
Warning: Imaginary parts of complex X and/or
Y arguments ignored
hold
Current plot held
plot(t,-P1,‘-’,t,-P2,‘-.’,t,P3,‘.’)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
Warning: Imaginary parts of complex X and/or
Y arguments ignored
xlabel(‘Time’)
ylabel(‘Output’)
legend(‘P1’,‘P2’,‘P3’)
```

(Delete graph window)

```
plot3(P1,P2,P3)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
hold
Current plot held
plot3(-P1,P2,P3)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
plot3(P1,-P2,P3)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
plot3(-P1,-P2,P3)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
xlabel(‘P1’)
ylabel(‘P2’)
zlabel(‘P3’)
```

CASE 1A(II) (the second part)

SECTION 1:

The MATLAB code used to produce Figure C.13(a) and Figure C.13(b):

Command window:

```
c1 = 1; c2 = 2; c3 = 3;
[t, p] = ode45(@TypeIII,[0 30],[1 1 1],[],c1,c2,c3);
plot(t,p(:,1),‘-’,t,p(:,2),‘-.’,t,p(:,3),‘.’)
xlabel(‘Time’)
ylabel(‘Output’)
legend(‘P1’,‘P2’,‘P3’)
```

(Delete graph window)

```
plot3(p(:,1),p(:,2),p(:,3))
xlabel(‘P1’)
ylabel(‘P2’)
zlabel(‘P3’)
```

SECTION 2:

The first elliptic function: sd

The MATLAB code used to produce Figure C.13(c) and Figure C.13(d) (rounding to four decimal places):

Command window:

```
t = linspace(0,30,150);
[s,c,d] = ellipj(0.6455*t,0.6832);
P3 = 0.9660.*(s./d);
P2 = sqrt(2.3334-1.3333.*P3.*P3);
P1 = sqrt(0.6666+0.3333.*P3.*P3);
plot(t,P1,‘-’,t,P2,‘-.’,t,P3,‘.’)
```

```
Warning: Imaginary parts of complex X and/or
Y arguments ignored
hold
Current plot held
plot(t,P1,‘-’,t,-P2,‘-.’,t,P3,‘.’)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
xlabel(‘Time’)
ylabel(‘Output’)
legend(‘P1’,‘P2’,‘P3’)
```

(Delete graph window)

```
plot3(P1,P2,P3)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
hold
Current plot held
plot3(P1,-P2,P3)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
xlabel(‘P1’)
ylabel(‘P2’)
zlabel(‘P3’)
```

The second elliptic function: cn

The MATLAB code used to produce Figure C.13(e) and Figure C.13(f) (rounding to four decimal places):

Command window:

```
t = linspace(0,30,150);
[s,c,d] = ellipj(0.6455*t,0.6832);
P3 = 1.3229.*c;
P2 = sqrt(2.3334-1.3333.*P3.*P3);
P1 = sqrt(0.6666+0.3333.*P3.*P3);
plot(t,P1,‘-’,t,P2,‘-.’,t,P3,‘.’)
hold
Current plot held
plot(t,P1,‘-’,t,-P2,‘-.’,t,P3,‘.’)
xlabel(‘Time’)
ylabel(‘Output’)
legend(‘P1’,‘P2’,‘P3’)
```

(Delete graph window)

```
plot3(P1,P2,P3)
hold
Current plot held
plot3(P1,-P2,P3)
xlabel(‘P1’)
ylabel(‘P2’)
zlabel(‘P3’)
```

CASE 1B(I) (the first part)

SECTION 1:

The MATLAB code used to produce Figure C.14(a) and Figure C.14(b):

Command window:

```
c1 = 2; c2 = 3; c3 = 1;
[t, p] = ode45(@TypeIII,[0 30],[1 1 1],[],c1,c2,c3);
plot(t,p(:,1),‘-’,t,p(:,2),‘-.’,t,p(:,3),‘.’)
xlabel(‘Time’)
ylabel(‘Output’)
legend(‘P1’,‘P2’,‘P3’)
```

(Delete graph window)

```
plot3(p(:,1),p(:,2),p(:,3))
xlabel(‘P1’)
ylabel(‘P2’)
zlabel(‘P3’)
```

SECTION 2:

The first elliptic function: nd

The MATLAB code used to produce Figure C.14(c) and Figure C.14(d) (rounding to four decimal places):

Command window:

```
t = linspace(0,30,150);
[s,c,d] = ellipj(0.6455*t,0.6832);
P3 = 0.8165.*(1./d);
P2 = sqrt(-1.9998+3.*P3.*P3);
P1 = sqrt(4.9998-4.*P3.*P3);
plot(t,P1,‘-’,t,P2,‘-.’,t,P3,‘.’)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
hold
Current plot held
plot(t,-P1,‘-’,t,-P2,‘-.’,t,P3,‘.’)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
xlabel(‘Time’)
ylabel(‘Output’)
legend(‘P1’,‘P2’,‘P3’)
```

(Delete graph window)

```
plot3(P1,P2,P3)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
hold
Current plot held
plot3(-P1,P2,P3)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
plot3(P1,-P2,P3)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
plot3(-P1,-P2,P3)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
xlabel(‘P1’)
ylabel(‘P2’)
zlabel(‘P3’)
```

The second elliptic function: dn

The MATLAB code used to produce Figure C.14(e) and Figure C.14(f) (rounding to four decimal places):

Command window:

```
t = linspace(0,30,150);
[s,c,d] = ellipj(0.6455*t,0.6832);
P3 = 1.1180.*d;
P2 = sqrt(-1.9998+3.*P3.*P3);
P1 = sqrt(4.9998-4.*P3.*P3);
plot(t,P1,‘-’,t,P2,‘-.’,t,P3,‘.’)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
hold
Current plot held
plot(t,-P1,‘-’,t,-P2,‘-.’,t,P3,‘.’)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
xlabel(‘Time’)
ylabel(‘Output’)
legend(‘P1’,‘P2’,‘P3’)
```

(Delete graph window)

```
plot3(P1,P2,P3)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
hold
Current plot held
plot3(-P1,P2,P3)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
plot3(P1,-P2,P3)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
plot3(-P1,-P2,P3)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
xlabel(‘P1’)
ylabel(‘P2’)
```

```
zlabel(‘P3’)
```

Case 1b(i) (the second part)

Section 1:

The MATLAB code used to produce Figure C.15(a) and Figure C.15(b):

Command window:

```
c1 = 2; c2 = 3; c3 = 1;
[t, p] = ode45(@TypeIII,[0 30],[1 2 1],[],c1,c2,c3);
plot(t,p(:,1),‘-’,t,p(:,2),‘-.’,t,p(:,3),‘.’)
xlabel(‘Time’)
ylabel(‘Output’)
legend(‘P1’,‘P2’,‘P3’)
```

(Delete graph window)

```
plot3(p(:,1),p(:,2),p(:,3))
xlabel(‘P1’)
ylabel(‘P2’)
zlabel(‘P3’)
```

Section 2:

The first elliptic function: sd

The MATLAB code used to produce Figure C.15(c) and Figure C.15(d) (rounding to four decimal places):

Command window:

```
t = linspace(0,30,150);
[s,c,d] = ellipj(0.7265*t,0.8885);
P3 = 0.5130.*(s./d);
P2 = sqrt(1.0002+3.*P3.*P3);
P1 = sqrt(4.9998-4.*P3.*P3);
plot(t,P1,‘-’,t,P2,‘-.’,t,P3,‘.’)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
hold
Current plot held
plot(t,-P1,‘-’,t,P2,‘-.’,t,P3,‘.’)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
xlabel(‘Time’)
ylabel(‘Output’)
legend(‘P1’,‘P2’,‘P3’)
```

(Delete graph window)

```
plot3(P1,P2,P3)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
hold
Current plot held
plot3(-P1,P2,P3)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
xlabel(‘P1’)
ylabel(‘P2’)
zlabel(‘P3’)
```

The second elliptic function: cn

The MATLAB code used to produce Figure C.15(e) and Figure C.15(f) (rounding to four decimal places):

Command window:

```
t = linspace(0,30,150);
[s,c,d] = ellipj(0.7265*t,0.8885);
P3 = 1.1180.*c;
P2 = sqrt(1.0002+3.*P3.*P3);
P1 = sqrt(4.9998-4.*P3.*P3);
plot(t,P1,‘-’,t,P2,‘-.’,t,P3,‘.’)
hold
Current plot held
plot(t,-P1,‘-’,t,P2,‘-.’,t,P3,‘.’)
xlabel(‘Time’)
ylabel(‘Output’)
legend(‘P1’,‘P2’,‘P3’)
```

(Delete graph window)

```
plot3(P1,P2,P3)
hold
Current plot held
plot3(-P1,P2,P3)
xlabel('P1')
ylabel('P2')
zlabel('P3')
```

CASE 1B(II) (the first part)

SECTION 1:

The MATLAB code used to produce Figure C.16(a) and Figure C.16(b):

Command window:

```
c1 = 3; c2 = 2; c3 = 1;
[t, p] = ode45(@TypeIII,[0 30],[1 1 1],[],c1,c2,c3);
plot(t,p(:,1),'-',t,p(:,2),'-.',t,p(:,3),'.')
xlabel('Time')
ylabel('Output')
legend('P1','P2','P3')
```

(Delete graph window)

```
plot3(p(:,1),p(:,2),p(:,3))
xlabel('P1')
ylabel('P2')
zlabel('P3')
```

SECTION 2:

The first elliptic function: nd

The MATLAB code used to produce Figure C.16(c) and Figure C.16(d) (rounding to four decimal places):

Command window:

```
t = linspace(0,30,150);
[s,c,d] = ellipj(0.6455*t,0.6832);
P3 = 0.8165.*(1./d);
P2 = sqrt(4.9998-4.*P3.*P3);
P1 = sqrt(-1.9998+3.*P3.*P3);
plot(t,P1,'-',t,P2,'-.',t,P3,'.')
Warning: Imaginary parts of complex X and/or
Y arguments ignored
hold
Current plot held
plot(t,-P1,'-',t,-P2,'-.',t,P3,'.')
Warning: Imaginary parts of complex X and/or
Y arguments ignored
xlabel('Time')
ylabel('Output')
legend('P1','P2','P3')
```

(Delete graph window)

```
plot3(P1,P2,P3)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
hold
Current plot held
plot3(-P1,P2,P3)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
plot3(P1,-P2,P3)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
plot3(-P1,-P2,P3)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
xlabel('P1')
ylabel('P2')
zlabel('P3')
```

The second elliptic function: dn

The MATLAB code used to produce Figure C.16(e) and Figure C.16(f) (rounding to four decimal places):

Command window:

```
t = linspace(0,30,150);
[s,c,d] = ellipj(0.6455*t,0.6832);
```

```
P3 = 1.1180.*d;
P2 = sqrt(4.9998-4.*P3.*P3);
P1 = sqrt(-1.9998+3.*P3.*P3);
plot(t,P1,‘-’,t,P2,‘-.’,t,P3,‘.’)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
hold
Current plot held
plot(t,-P1,‘-’,t,-P2,‘-.’,t,P3,‘.’)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
xlabel(‘Time’)
ylabel(‘Output’)
legend(‘P1’,‘P2’,‘P3’)
```

(Delete graph window)

```
plot3(P1,P2,P3)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
hold
Current plot held
plot3(-P1,P2,P3)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
plot3(P1,-P2,P3)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
plot3(-P1,-P2,P3)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
xlabel(‘P1’)
ylabel(‘P2’)
zlabel(‘P3’)
```

Case 1b(ii) (the second part)

Section 1:

The MATLAB code used to produce Figure C.17(a) and Figure C.17(b):

Command window:

```
c1 = 3; c2 = 2; c3 = 1;
[t, p] = ode45(@TypeIII,[0 30],[2 1 1],[],c1,c2,c3);
plot(t,p(:,1),‘-’,t,p(:,2),‘-.’,t,p(:,3),‘.’)
xlabel(‘Time’)
ylabel(‘Output’)
legend(‘P1’,‘P2’,‘P3’)
```

(Delete graph window)

```
plot3(p(:,1),p(:,2),p(:,3))
xlabel(‘P1’)
ylabel(‘P2’)
zlabel(‘P3’)
```

Section 2:

The first elliptic function: sd

The MATLAB code used to produce Figure C.17(c) and Figure C.17(d) (rounding to four decimal places):

Command window:

```
t = linspace(0,30,150);
[s,c,d] = ellipj(0.5278*t,0.8885);
P3 = 0.5130.*(s./d);
P2 = sqrt(4.9998-4.*P3.*P3);
P1 = sqrt(1.0002+3.*P3.*P3);
plot(t,P1,‘-’,t,P2,‘-.’,t,P3,‘.’)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
hold
Current plot held
plot(t,P1,‘-’,t,-P2,‘-.’,t,P3,‘.’)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
xlabel(‘Time’)
ylabel(‘Output’)
legend(‘P1’,‘P2’,‘P3’)
```

(Delete graph window)

```
plot3(P1,P2,P3)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
hold
Current plot held
plot3(P1,-P2,P3)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
xlabel(‘P1’)
ylabel(‘P2’)
zlabel(‘P3’)
```

The second elliptic function: cn

The MATLAB code used to produce Figure C.17(e) and Figure C.17(f) (rounding to four decimal places):

Command window:

```
t = linspace(0,30,150);
[s,c,d] = ellipj(0.5278*t,0.8885);
P3 = 1.1180.*c;
P2 = sqrt(4.9998-4.*P3.*P3);
P1 = sqrt(1.0002+3.*P3.*P3);
plot(t,P1,‘-’,t,P2,‘-.’,t,P3,‘.’)
hold
Current plot held
plot(t,P1,‘-’,t,-P2,‘-.’,t,P3,‘.’)
xlabel(‘Time’)
ylabel(‘Output’)
legend(‘P1’,‘P2’,‘P3’)
```

(Delete graph window)

```
plot3(P1,P2,P3)
hold
Current plot held
plot3(P1,-P2,P3)
xlabel(‘P1’)
ylabel(‘P2’)
zlabel(‘P3’)
```

Case 2 ($c_2 < c_3 < c_1$)

Section 1:

The MATLAB code used to produce Figure C.18(a) and Figure C.18(b) (rounding to four decimal places):

Command window:

```
c1 = 2; c2 = 0.5; c3 = 1;
[t, p] = ode45(@TypeIII,[0 30],[1.7321 1 1],[],c1,c2,c3);
plot(t,p(:,1),‘-’,t,p(:,2),‘-.’,t,p(:,3),‘.’)
xlabel(‘Time’)
ylabel(‘Output’)
legend(‘P1’,‘P2’,‘P3’)
```

(Delete graph window)

```
plot3(p(:,1),p(:,2),p(:,3))
xlabel(‘P1’)
ylabel(‘P2’)
zlabel(‘P3’)
```

Section 2:

The first elliptic function: sn

The MATLAB code used to produce Figure C.18(c) and Figure C.18(d) (rounding to four decimal places):

Command window:

```
t = linspace(0,30,150);
[s,c,d] = ellipj(1.6583*t,0.8528);
P3 = 2.*s;
P2 = sqrt(1.3333-0.3333.*P3.*P3);
P1 = sqrt(3.6667-0.6667.*P3.*P3);
plot(t,P1,‘-’,t,P2,‘-.’,t,P3,‘.’)
hold
Current plot held
plot(t,P1,‘-’,t,-P2,‘-.’,t,P3,‘.’)
xlabel(‘Time’)
```

```
ylabel(‘Output’)
legend(‘P1’,‘P2’,‘P3’)
```

(Delete graph window)

```
plot3(P1,P2,P3)
Warning: Imaginary parts of complex X and/or Y arguments ignored
hold
Current plot held
plot3(P1,-P2,P3)
xlabel(‘P1’)
ylabel(‘P2’)
zlabel(‘P3’)
```

The second elliptic function: cd

The MATLAB code used to produce Figure C.18(e) and Figure C.18(f) (rounding to four decimal places):

Command window:

```
t = linspace(0,30,150);
[s,c,d] = ellipj(1.6583*t,0.8528);
P3 = 2.*(c./d);
P2 = sqrt(1.3333-0.3333.*P3.*P3);
P1 = sqrt(3.6667-0.6667.*P3.*P3);
hold
Current plot held
plot(t,P1,‘-’,t,-P2,‘-.’,t,P3,‘.’)
xlabel(‘Time’)
ylabel(‘Output’)
legend(‘P1’,‘P2’,‘P3’)
```

(Delete graph window)

```
plot3(P1,P2,P3)
hold
Current plot held
plot3(P1,-P2,P3)
xlabel(‘P1’)
ylabel(‘P2’)
zlabel(‘P3’)
```

CASE 2 ($c_1 < c_3 < c_2$)

SECTION 1:

The MATLAB code used to produce Figure C.19(a) and Figure C.19(b):

Command window:

```
c1 = 1; c2 = 3; c3 = 2;
[t, p] = ode45(@TypeIII,[0 30],[2 2 2],[],c1,c2,c3);
plot(t,p(:,1),‘-’,t,p(:,2),‘-.’,t,p(:,3),‘.’)
xlabel(‘Time’)
ylabel(‘Output’)
legend(‘P1’,‘P2’,‘P3’)
```

(Delete graph window)

```
plot3(p(:,1),p(:,2),p(:,3))
xlabel(‘P1’)
ylabel(‘P2’)
zlabel(‘P3’)
```

SECTION 2:

The first elliptic function: sn

The MATLAB code used to produce Figure C.19(c) and Figure C.19(d) (rounding to four decimal places):

Command window:

```
t = linspace(0,30,150);
[s,c,d] = ellipj(1.2910*t,0.6831);
P3 = 3.0551.*s;
P2 = sqrt(7.0001-0.75.*P3.*P3);
P1 = sqrt(5-0.25.*P3.*P3);
plot(t,P1,‘-’,t,P2,‘-.’,t,P3,‘.’)
Warning: Imaginary parts of complex X and/or Y arguments ignored
hold
Current plot held
plot(t,P1,‘-’,t,-P2,‘-.’,t,P3,‘.’)
```

```
Warning: Imaginary parts of complex X and/or
Y arguments ignored
xlabel(‘Time’)
ylabel(‘Output’)
legend(‘P1’,‘P2’,‘P3’)

(Delete graph window)

plot3(P1,P2,P3)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
hold
Current plot held
plot3(P1,-P2,P3)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
xlabel(‘P1’)
ylabel(‘P2’)
zlabel(‘P3’)
```

The second elliptic function: cd

The MATLAB code used to produce Figure C.19(e) and Figure C.19(f) (rounding to four decimal places):

Command window:

```
t = linspace(0,30,150);
[s,c,d] = ellipj(1.2910*t,0.6831);
P3 = 3.0551.*(c./d);
P2 = sqrt(7.0001-0.75.*P3.*P3);
P1 = sqrt(5-0.25.*P3.*P3);
plot(t,P1,‘-’,t,P2,‘-.’,t,P3,‘.’)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
hold
Current plot held
plot(t,P1,‘-’,t,-P2,‘-.’,t,P3,‘.’)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
xlabel(‘Time’)
ylabel(‘Output’)
legend(‘P1’,‘P2’,‘P3’)

(Delete graph window)

plot3(P1,P2,P3)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
hold
Current plot held
plot3(P1,-P2,P3)
Warning: Imaginary parts of complex X and/or
Y arguments ignored
xlabel(‘P1’)
ylabel(‘P2’)
zlabel(‘P3’)
```

Appendix C

Graphs

KEY

(a) The solution curves $P_1(\cdot)$, $P_2(\cdot)$, $P_3(\cdot)$ to extremal equations using the MATLAB solver.

(b) The solution curve $P(\cdot)$ to the extremal equations plotted in $\mathbb{R}^3$ using the MATLAB solver.

The first elliptic function:

(c) The solution curves $P_1(\cdot)$, $P_2(\cdot)$, $P_3(\cdot)$ to the extremal equations using a Jacobi elliptic function.

(d) The solution curve $P(\cdot)$ to the extremal equations plotted in $\mathbb{R}^3$ using a Jacobi elliptic function.

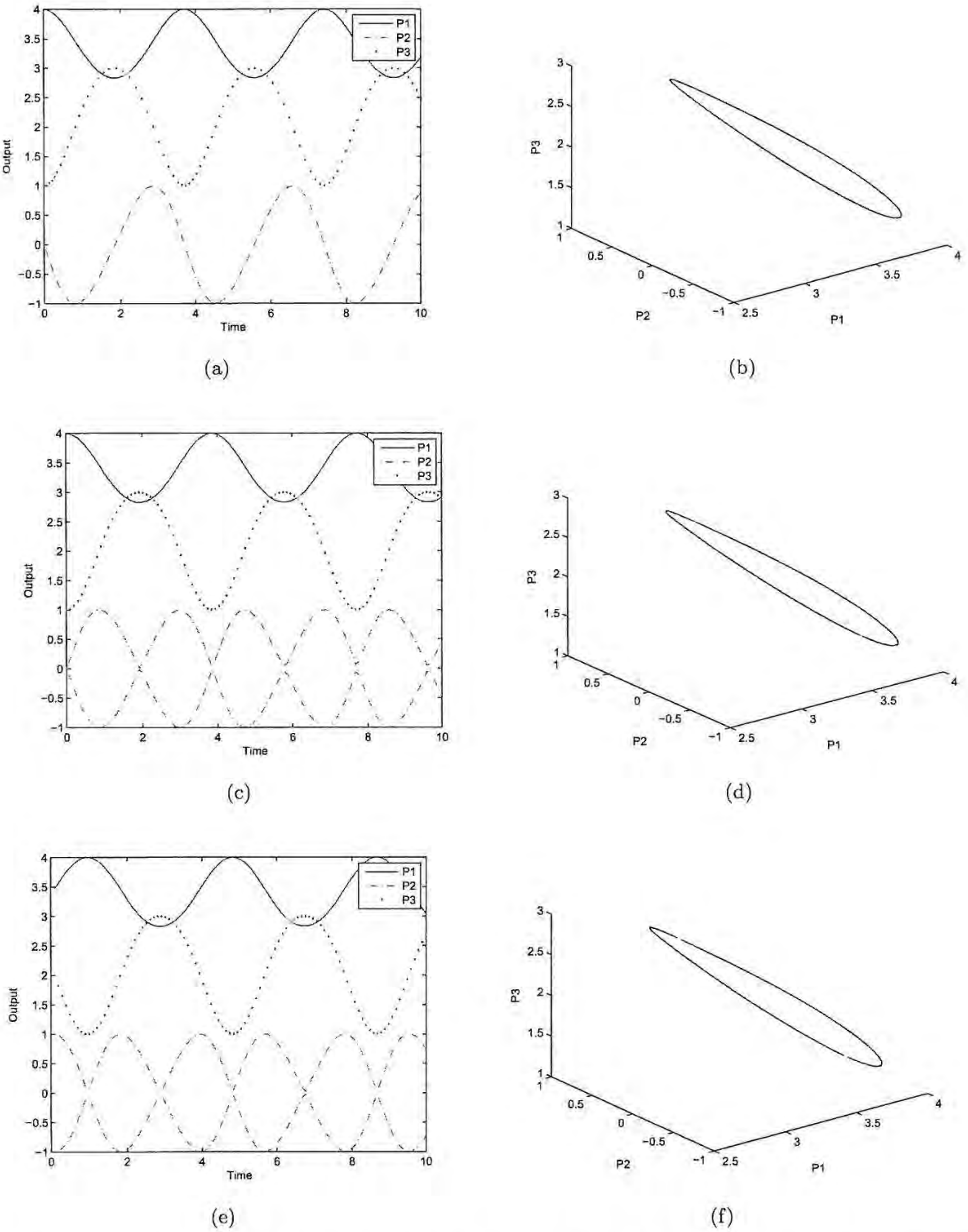
The second elliptic function:

(e) The solution curves $P_1(\cdot)$, $P_2(\cdot)$, $P_3(\cdot)$ to the extremal equations using an alternate Jacobi elliptic function.

(f) The solution curve $P(\cdot)$ to the extremal equations plotted in $\mathbb{R}^3$ using an alternate Jacobi elliptic function.

For each of the above, the result is plotted on the time interval [0, 10] (or [0, 30] in some instances), and on each set of graphs the following is specified: the CASE, the constant value(s), the initial condition and the elliptic functions.

Type I



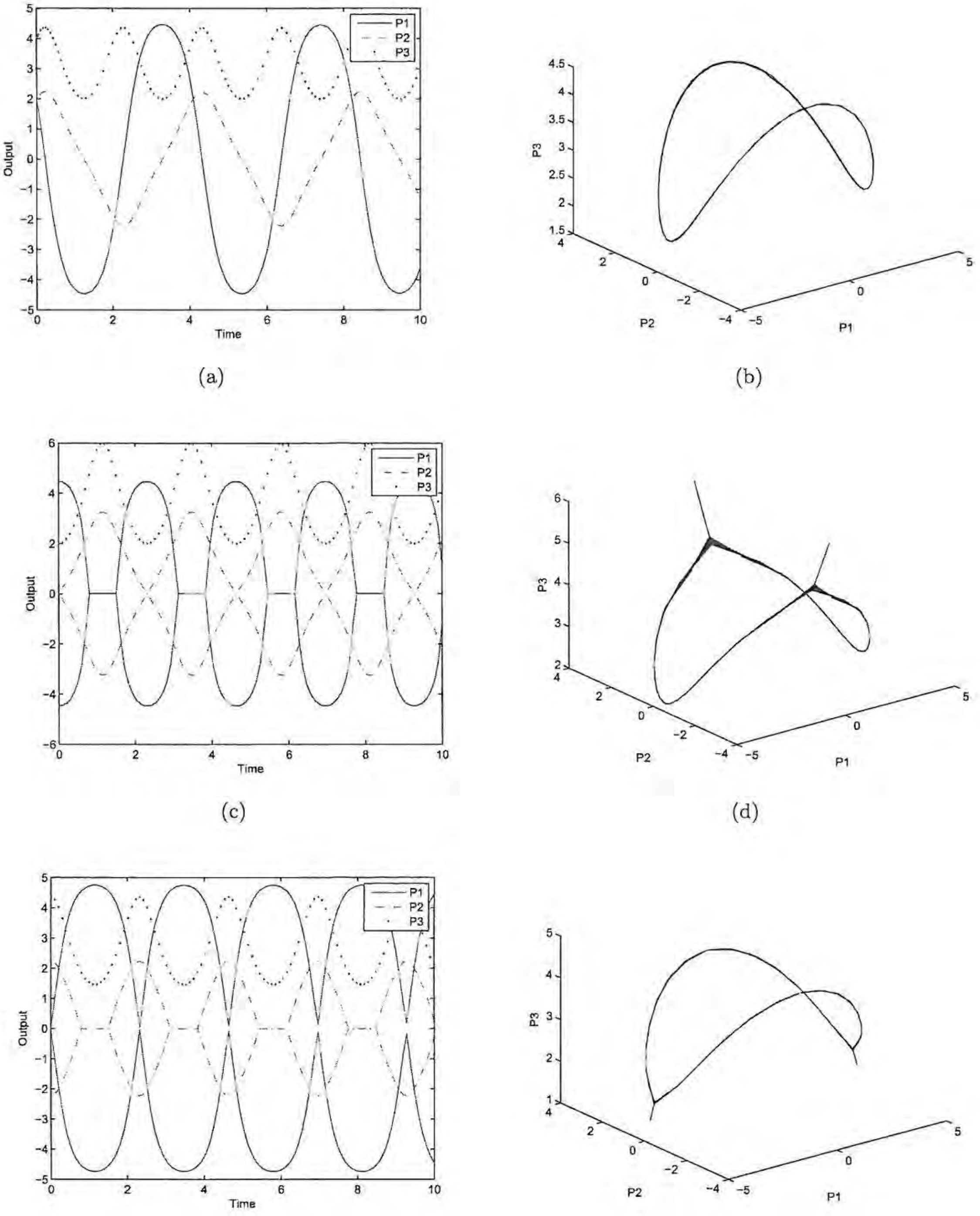
(a) (b)

(c) (d)

(e) (f)

Figure C.1: $c_1 = 2$, $P(0) = (4,\ 0,\ 1)$, (a)-(b): MATLAB ODE45 solver, (c)-(d): dc, (e)-(f): ns.

Type IIa

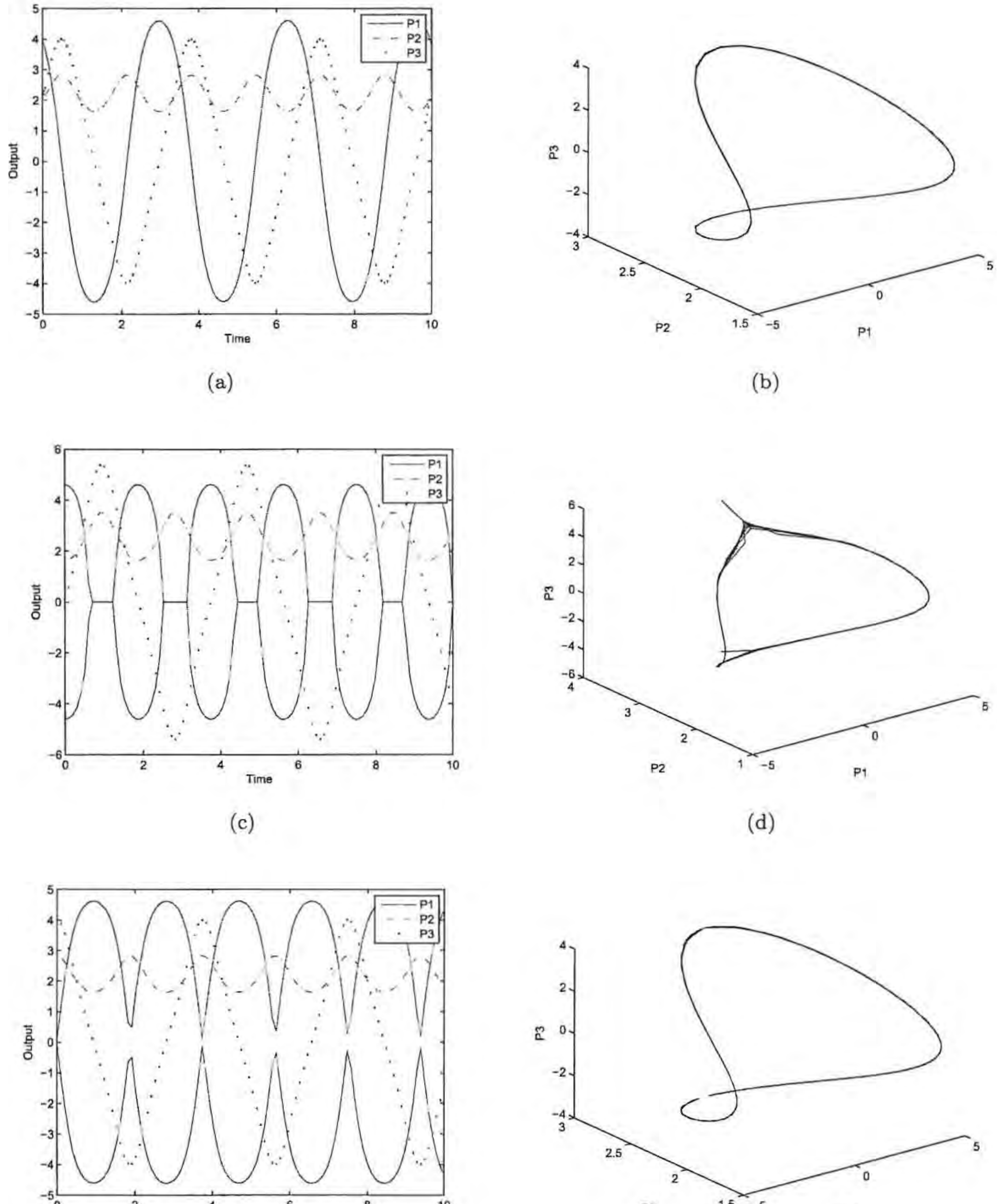


(a) (b)

(c) (d)

(e) (f)

Figure C.2: CASE (I) (the first part), $c_1 = 4$, $c_2 = 1$, $P(0) = (2,\ 2,\ 4)$, (a)-(b): MATLAB ODE45 solver, (c)-(d): nd, (e)-(f) dn.

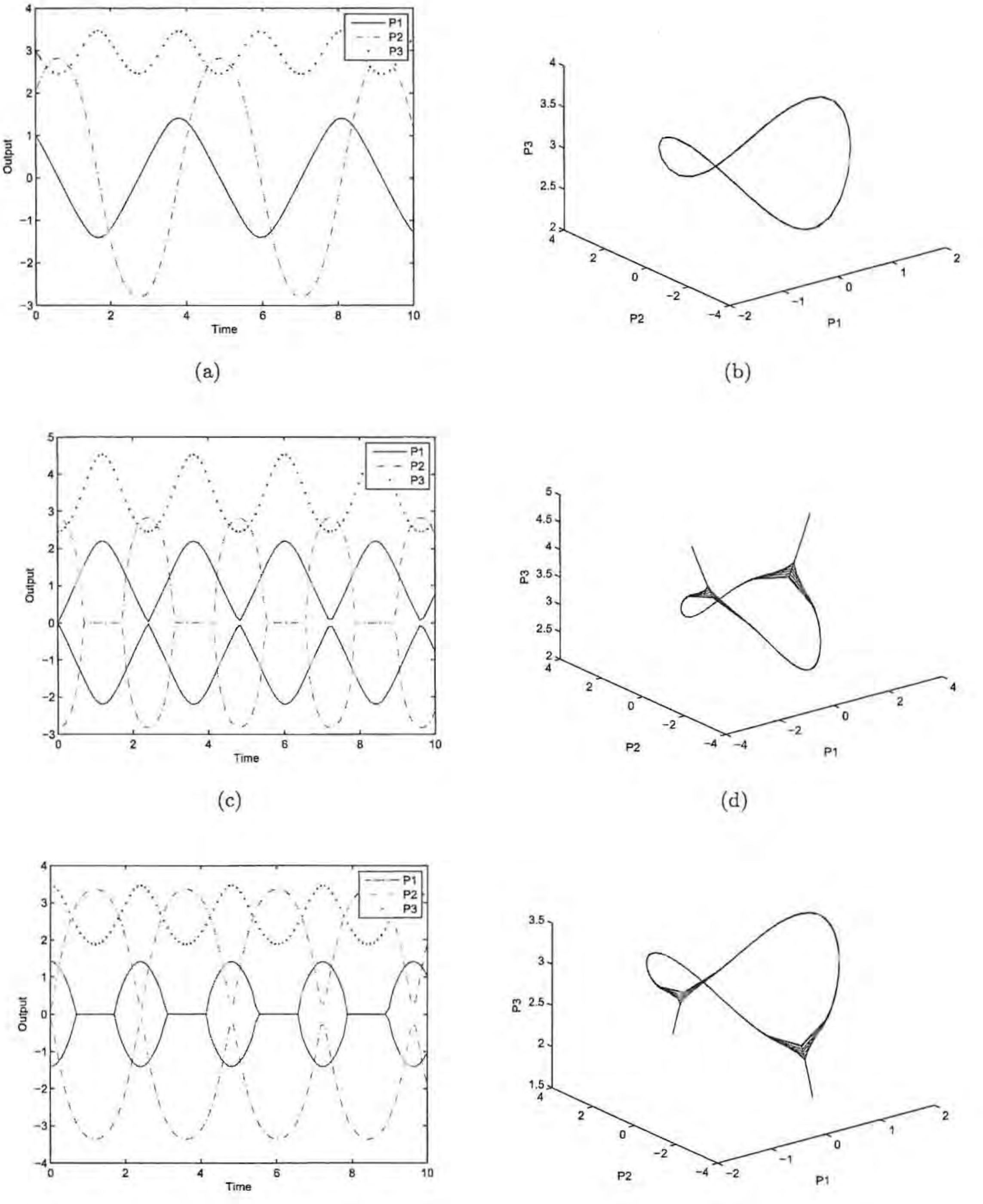


(a) (b)

(c) (d)

(e) (f)

Figure C.3: CASE (I) (the second part), $c_1 = 4$, $c_2 = 1$, $P(0) = (4, 2, 2)$, (a)-(b): MATLAB ODE45 solver, (c)-(d): sd, (e)-(f) cn.

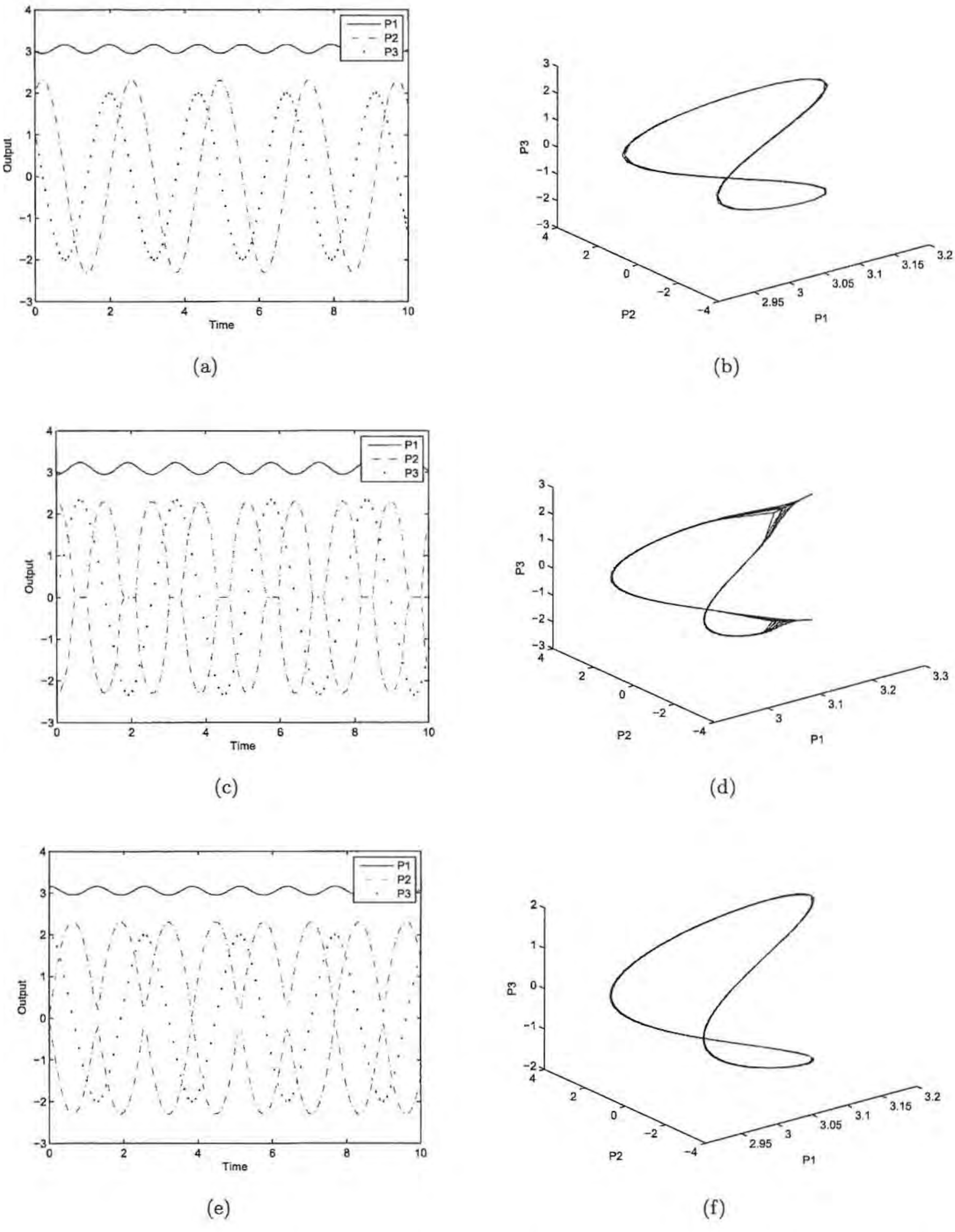


(a) (b)

(c) (d)

(e) (f)

Figure C.4: CASE (II) (the first part), $c_1 = 1$, $c_2 = 4$, $P(0) = (1,\ 2,\ 3)$, (a)-(b): MATLAB ODE45 solver, (c)-(d): nd, (e)-(f) dn.



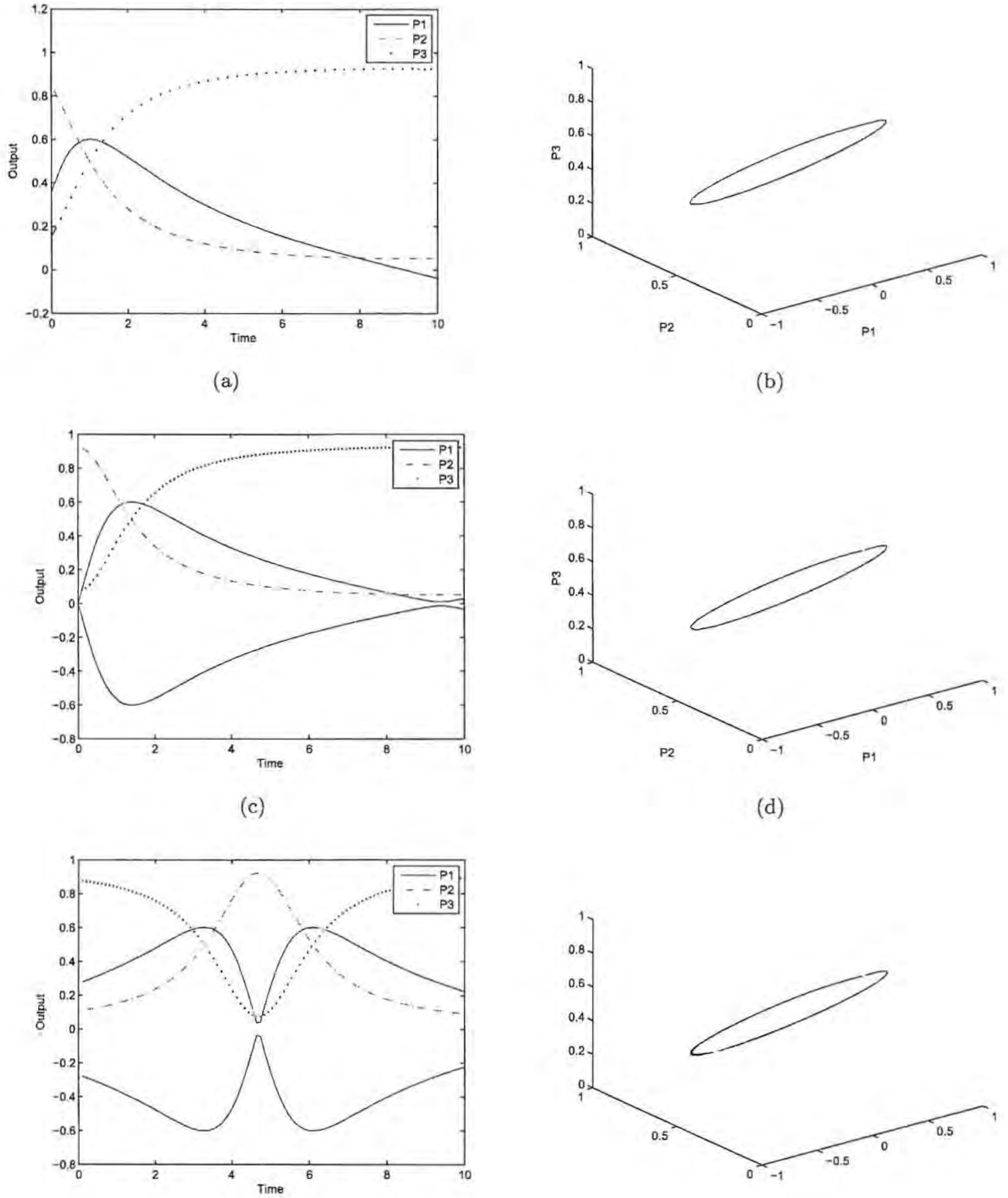
(a) (b)

(c) (d)

(e) (f)

Figure C.5: CASE (II) (the second part), $c_1 = 1$, $c_2 = 4$, $P(0) = (3, 2, 1)$, (a)-(b): MATLAB ODE45 solver, (c)-(d): sd, (e)-(f) cn.

Type IIb

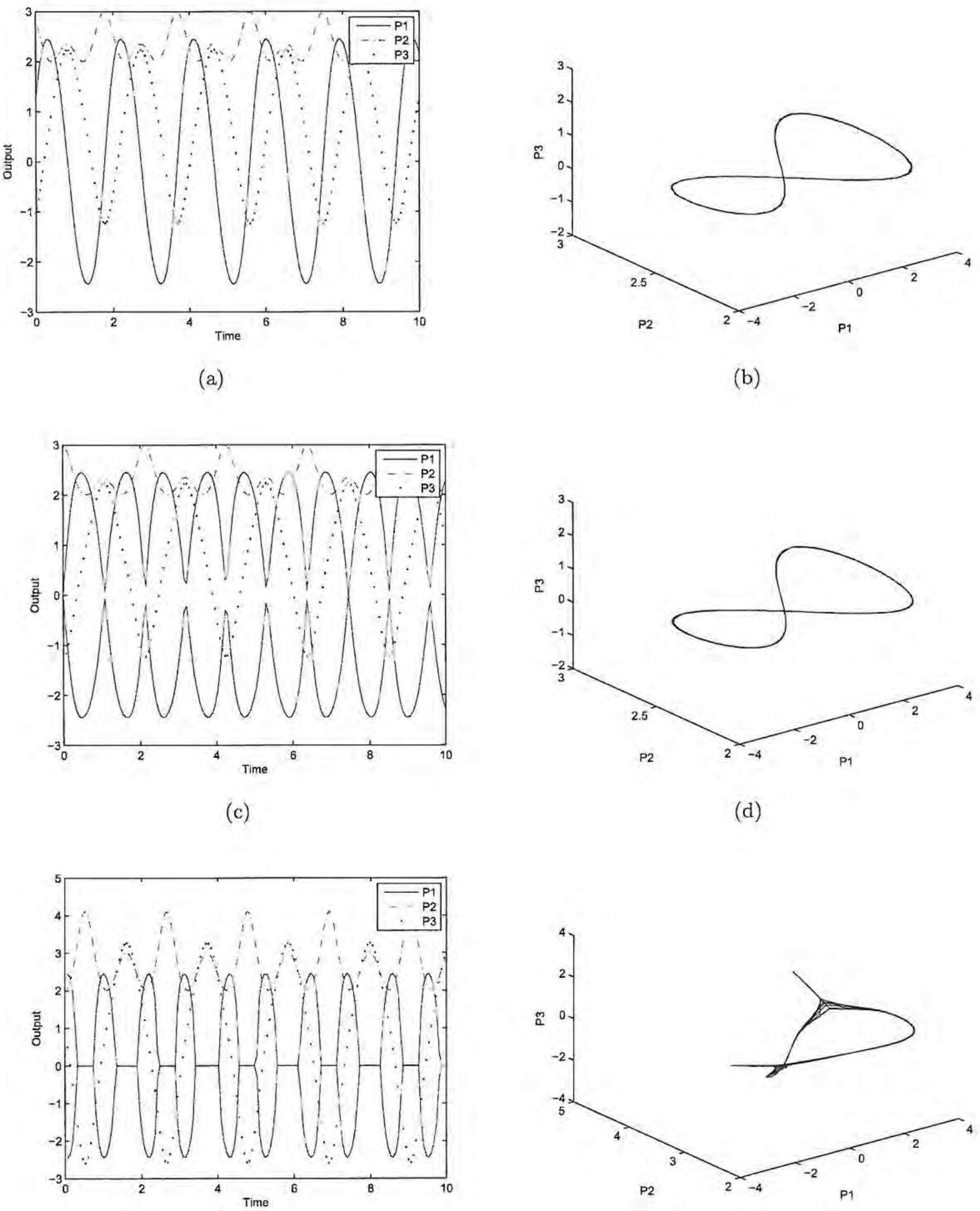


(a) (b)

(c) (d)

(e) (f)

Figure C.6: CASE (I) (the first part). $c_1 = 1$, $c_2 = \frac{1}{2}$, $P(0) = (\frac{1}{2\sqrt{2}}, \frac{11\sqrt{\frac{3}{2}}}{16}, \frac{5}{32})$, (a)-(b): MATLAB ODE45 solver, (c)-(d): dc, (e)-(f): ns.



(a)

(b)

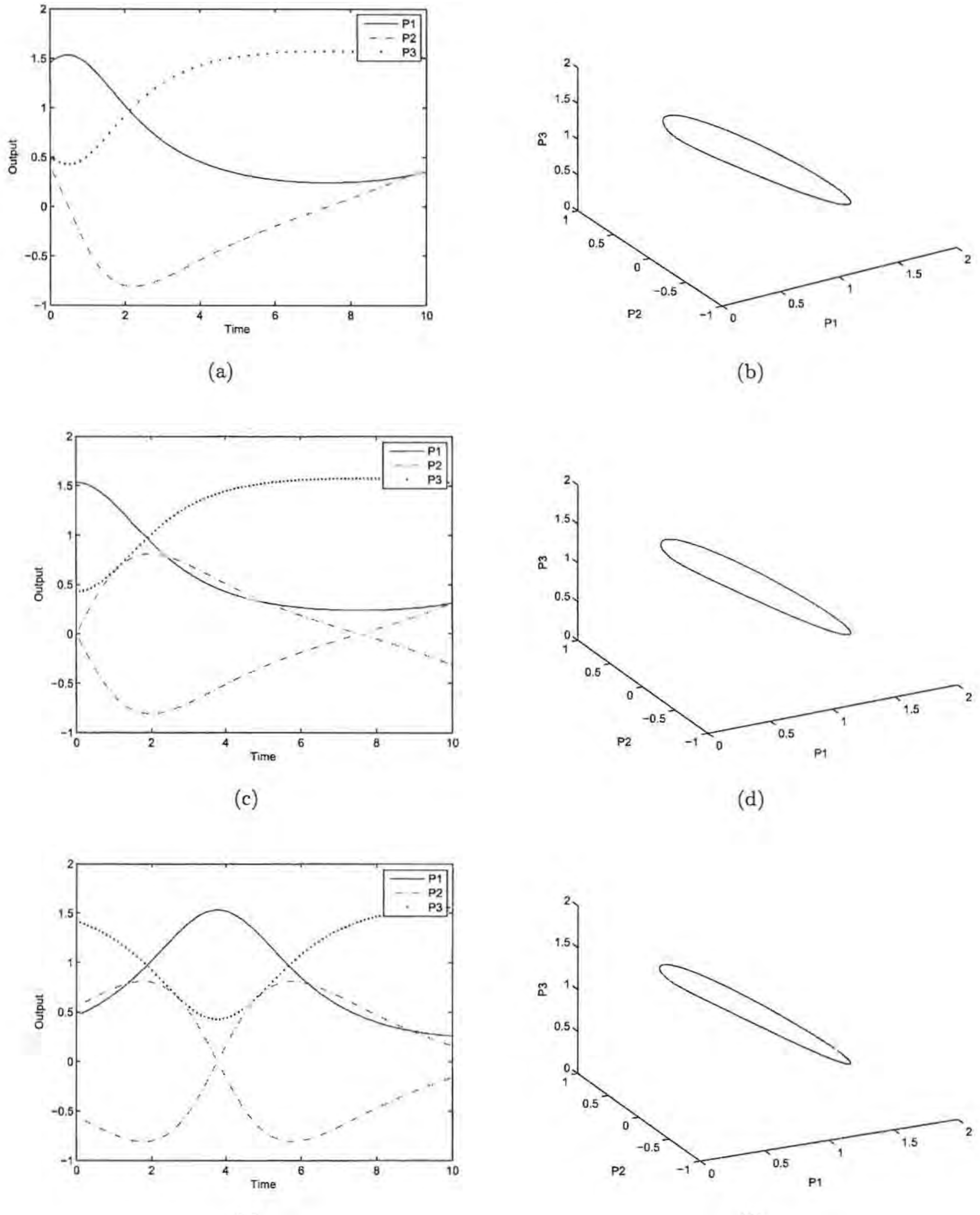
(c)

(d)

(e)

(f)

Figure C.7: CASE (I) (the second part), $c_1 = 1$, $c_2 = \frac{1}{2}$, $P(0) = \left(\sqrt{\frac{3}{2}},\ 2\sqrt{2},\ -1\right)$, (a)-(b): MATLAB ODE45 solver, (c)-(d): nc, (e)-(f): ds.

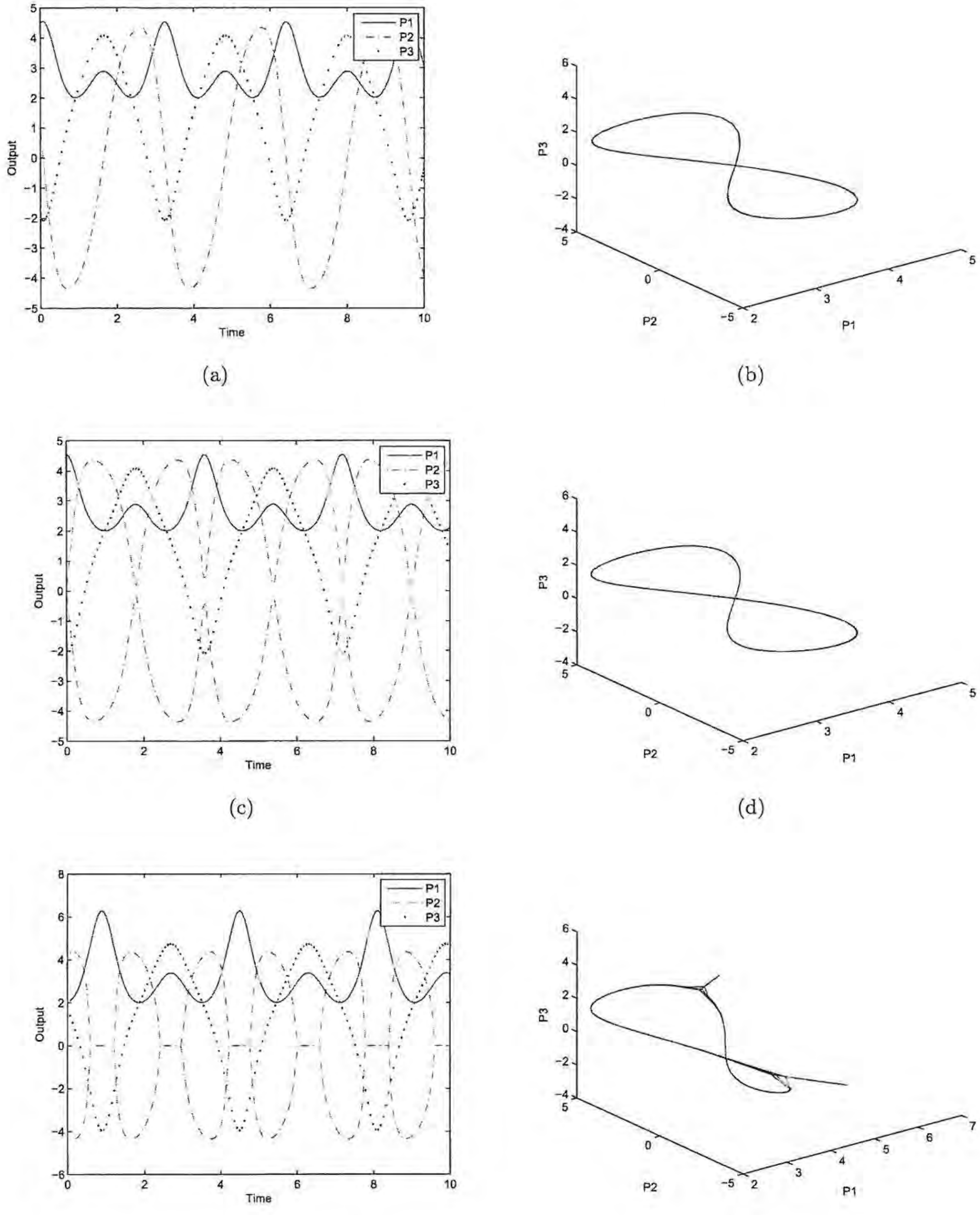


(a) (b)

(c) (d)

(e) (f)

Figure C.8: CASE (II) (the first part), $c_1 = 1$, $c_2 = 2$, $P(0) = \left(\frac{\sqrt{\frac{17}{2}}}{2}, \frac{\sqrt{\frac{5}{2}}}{4}, \frac{1}{2} \right)$, (a)-(b): MATLAB ODE45 solver, (c)-(d): dc, (e)-(f): ns.



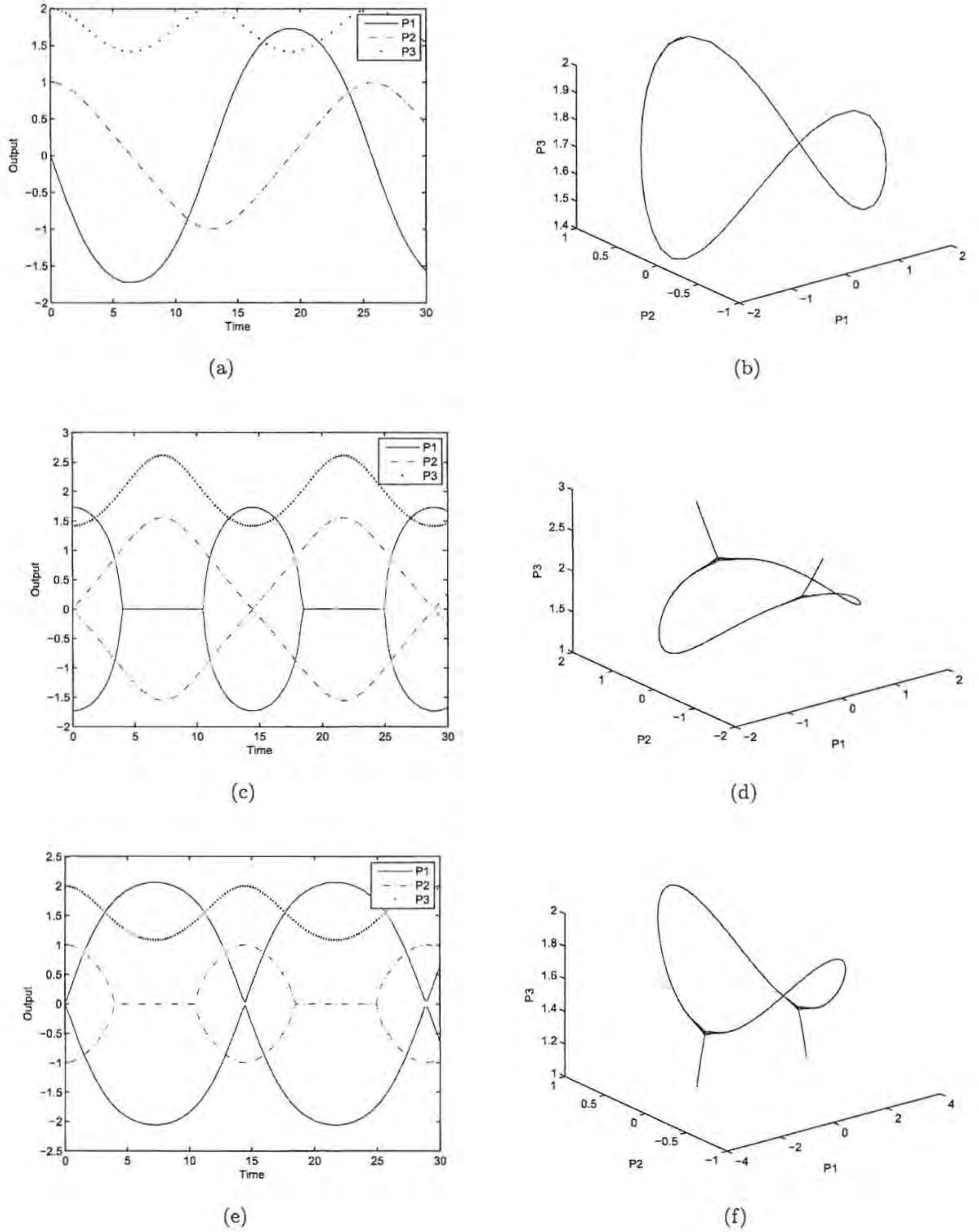
(a) (b)

(c) (d)

(e) (f)

Figure C.9: CASE (II) (the second part), $c_1 = 1$, $c_2 = 2$, $P(0) = \left(2\sqrt{5},\ 1,\ -2\right)$, (a)-(b): MATLAB ODE45 solver, (c)-(d): nc, (e)-(f): ds.

Type III

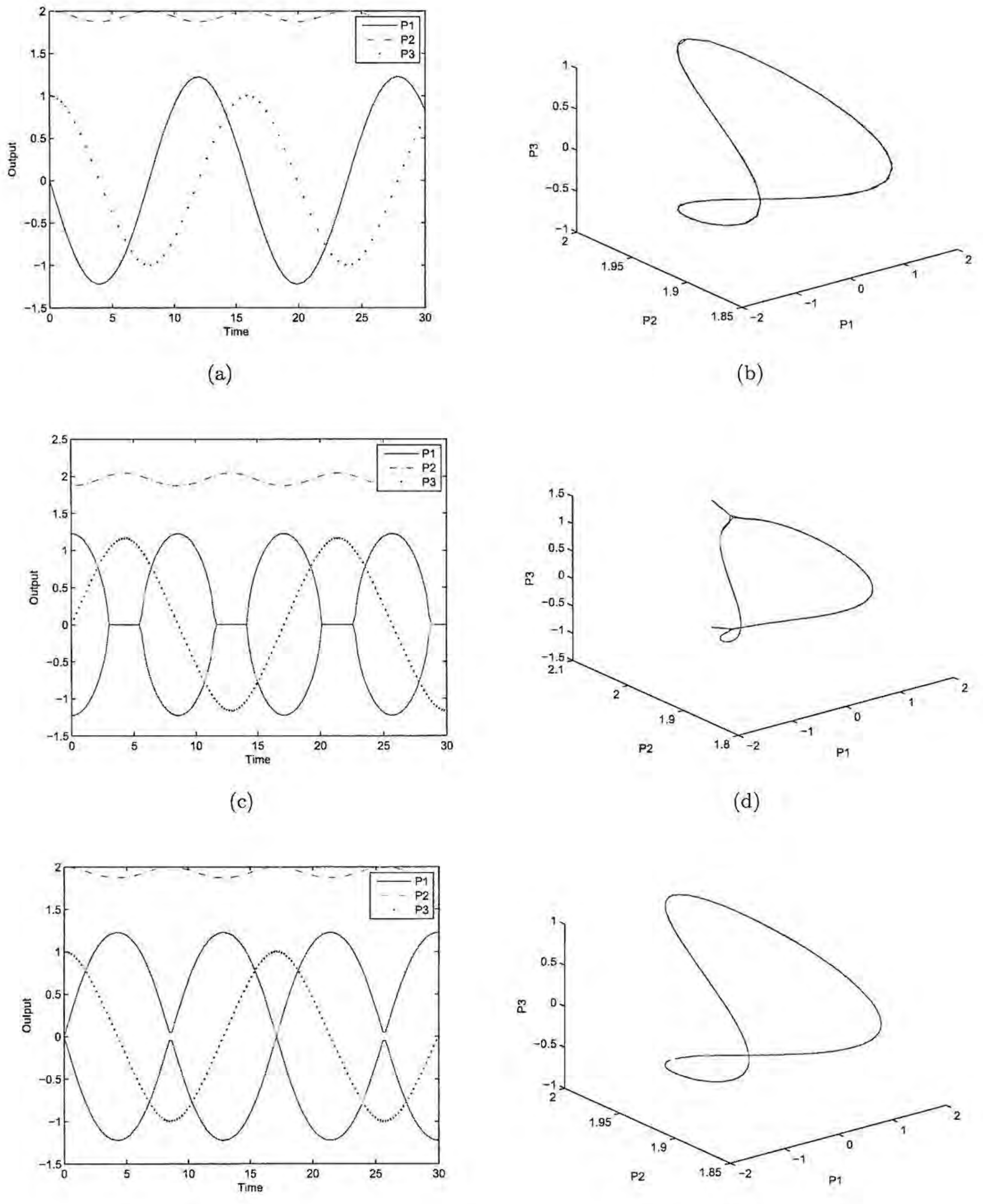


(a) (b)

(c) (d)

(e) (f)

Figure C.10: CASE 1A(I) (the first part), $c_1 = 3$, $c_2 = 2$, $c_3 = 4$, $P(0) = (0, 1, 2)$, (a)-(b): MATLAB ODE45 solver, (c)-(d): nd, (e)-(f): dn.

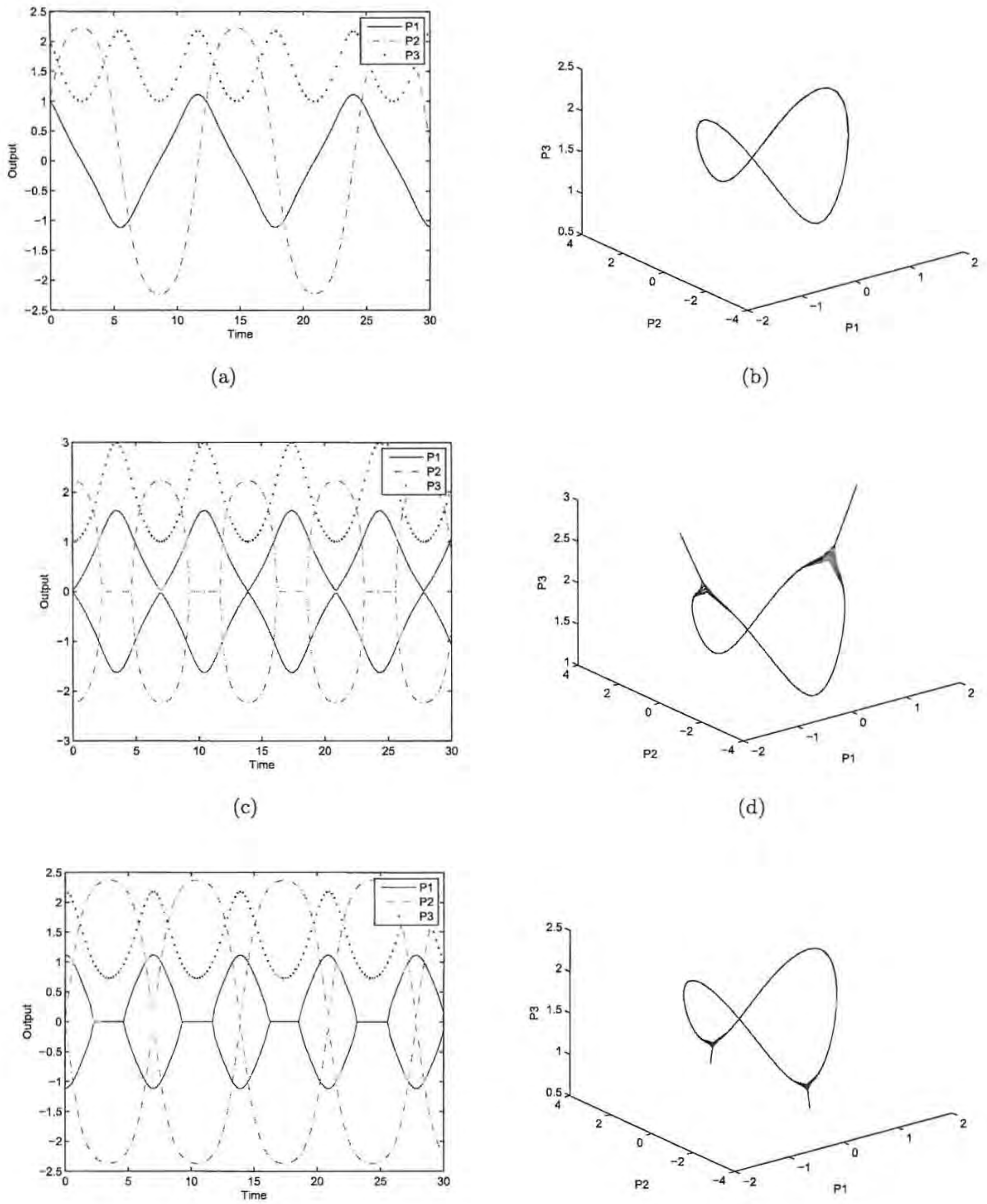


(a) (b)

(c) (d)

(e) (f)

Figure C.11: CASE 1A(I) (the second part), $c_1 = 3$, $c_2 = 2$, $c_3 = 4$, $P(0) = (0,\ 2,\ 1)$, (a)-(b): MATLAB ODE45 solver, (c)-(d): sd, (e)-(f): cn.

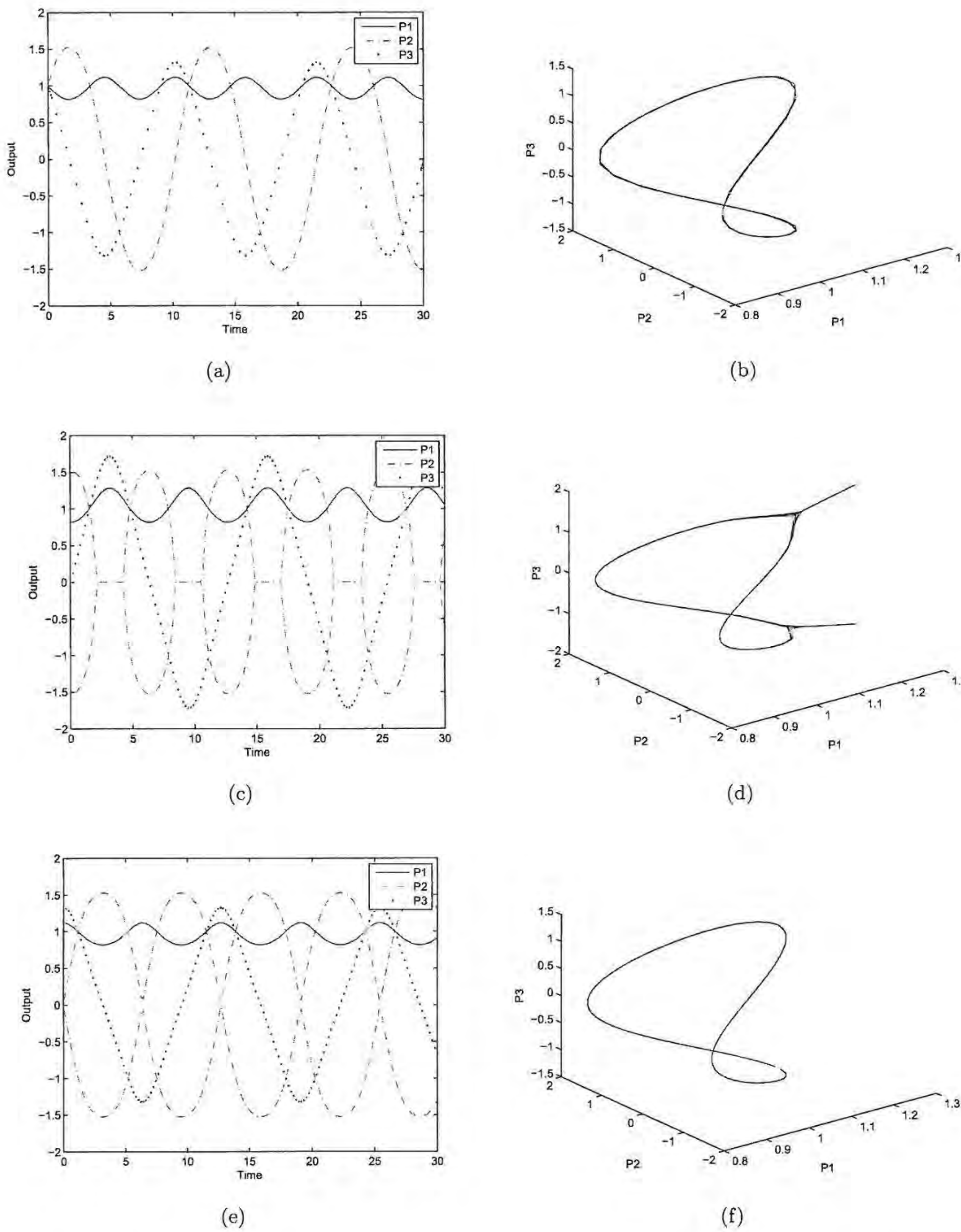


(a) (b)

(c) (d)

(e) (f)

Figure C.12: CASE 1A(II) (the first part), $c_1 = 1$, $c_2 = 2$, $c_3 = 3$, $P(0) = (1,\ 1,\ 2)$, (a)-(b): MATLAB ODE45 solver, (c)-(d): nd, (e)-(f): dn.

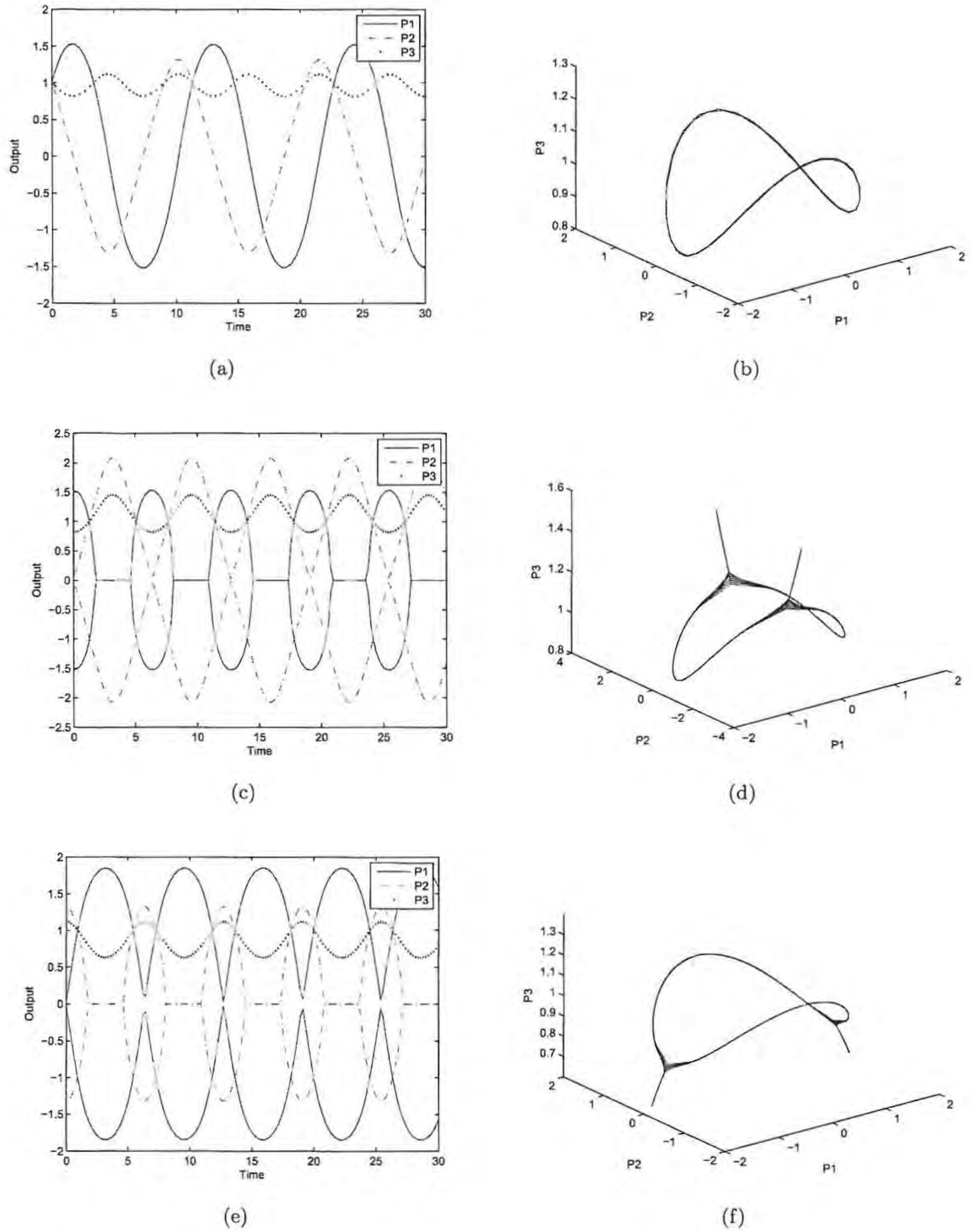


(a) (b)

(c) (d)

(e) (f)

Figure C.13: CASE 1A(II) (the second part), $c_1 = 1$, $c_2 = 2$, $c_3 = 3$, $P(0) = (1,\ 1,\ 1)$, (a)-(b): MATLAB ODE45 solver, (c)-(d): sd, (e)-(f): cn.

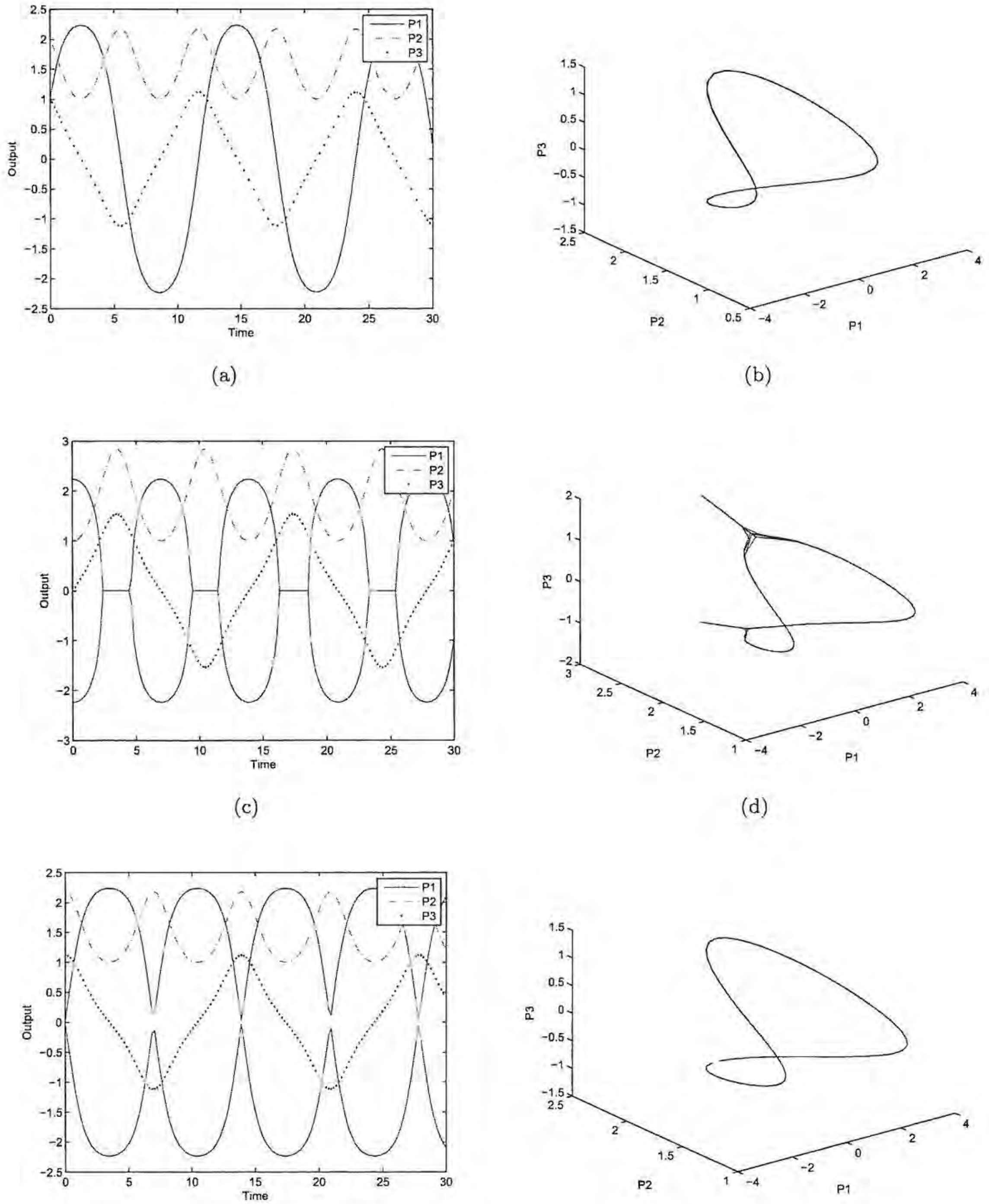


(a) (b)

(c) (d)

(e) (f)

Figure C.14: CASE 1B(I) (the first part), $c_1 = 2$, $c_2 = 3$, $c_3 = 1$, $P(0) = (1, 1, 1)$, (a)-(b): MATLAB ODE45 solver, (c)-(d): nd, (e)-(f): dn.

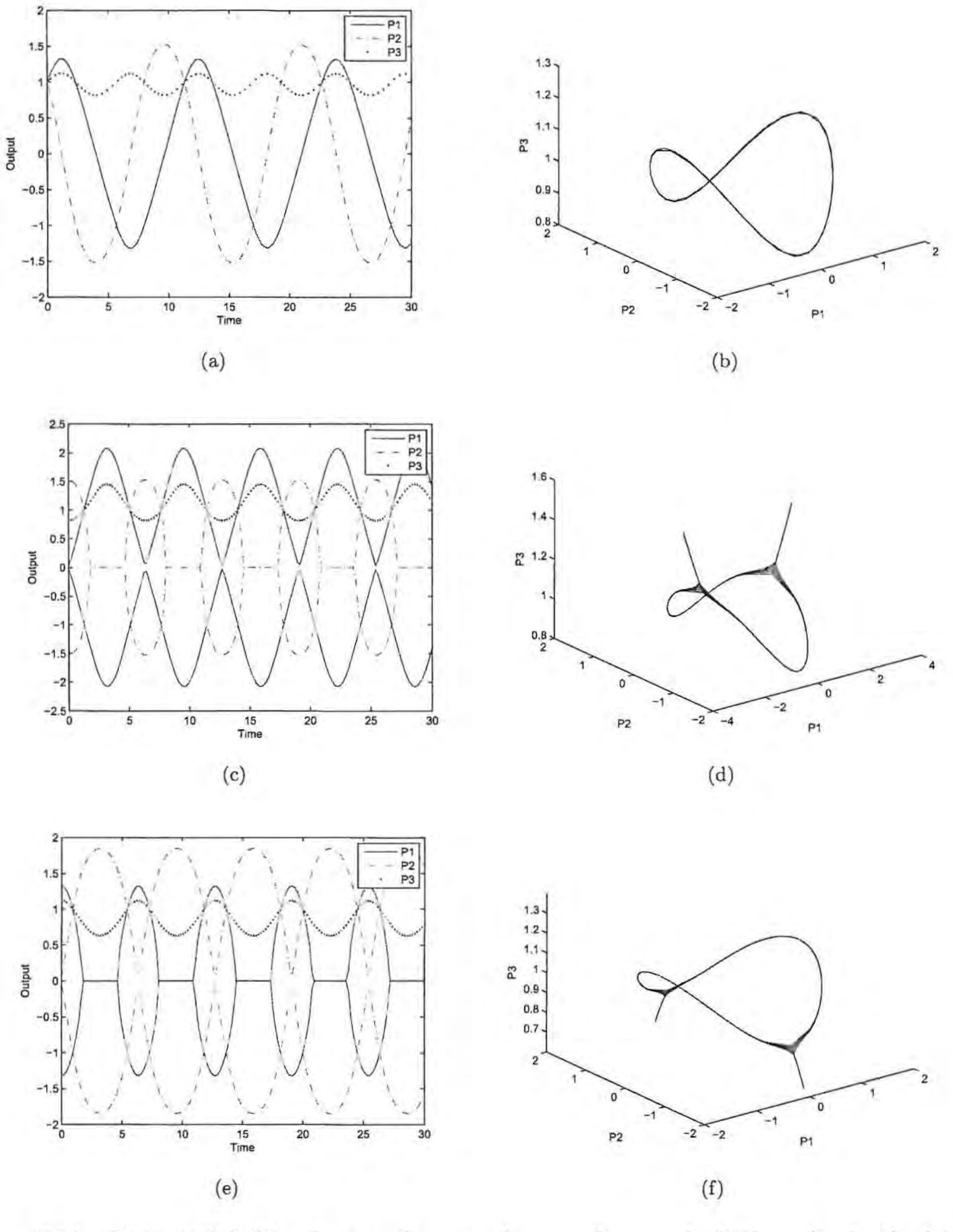


(a) (b)

(c) (d)

(e) (f)

Figure C.15: CASE 1B(I) (the second part), $c_1 = 2$, $c_2 = 3$, $c_3 = 1$, $P(0) = (1,\ 2,\ 1)$, (a)-(b): MATLAB ODE45 solver, (c)-(d): sd, (e)-(f): cn.

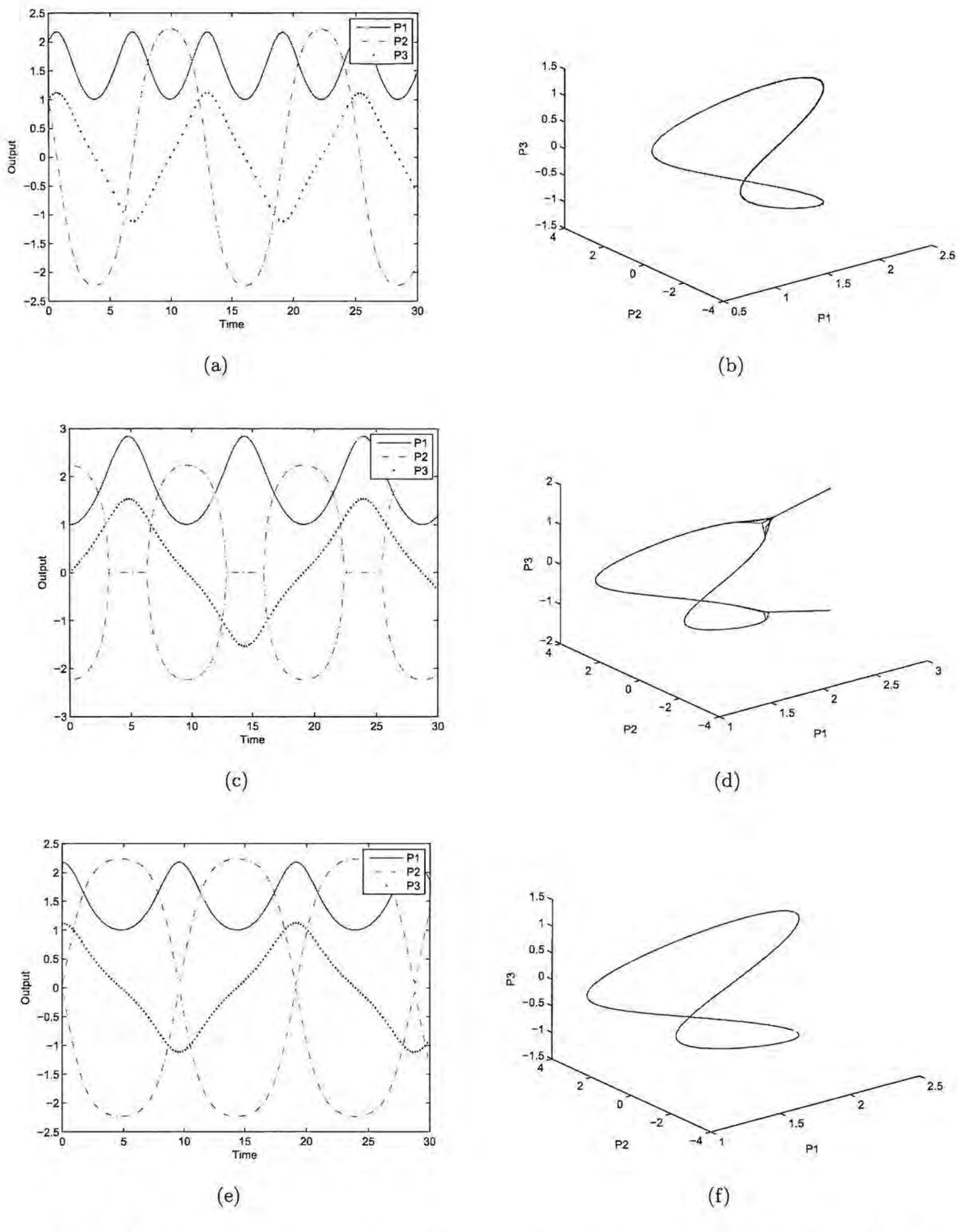


(a) (b)

(c) (d)

(e) (f)

Figure C.16: CASE 1B(II) (the first part), $c_1 = 3$, $c_2 = 2$, $c_3 = 1$, $P(0) = (1,\ 1,\ 1)$, (a)-(b): MATLAB ODE45 solver, (c)-(d): nd, (e)-(f): dn.

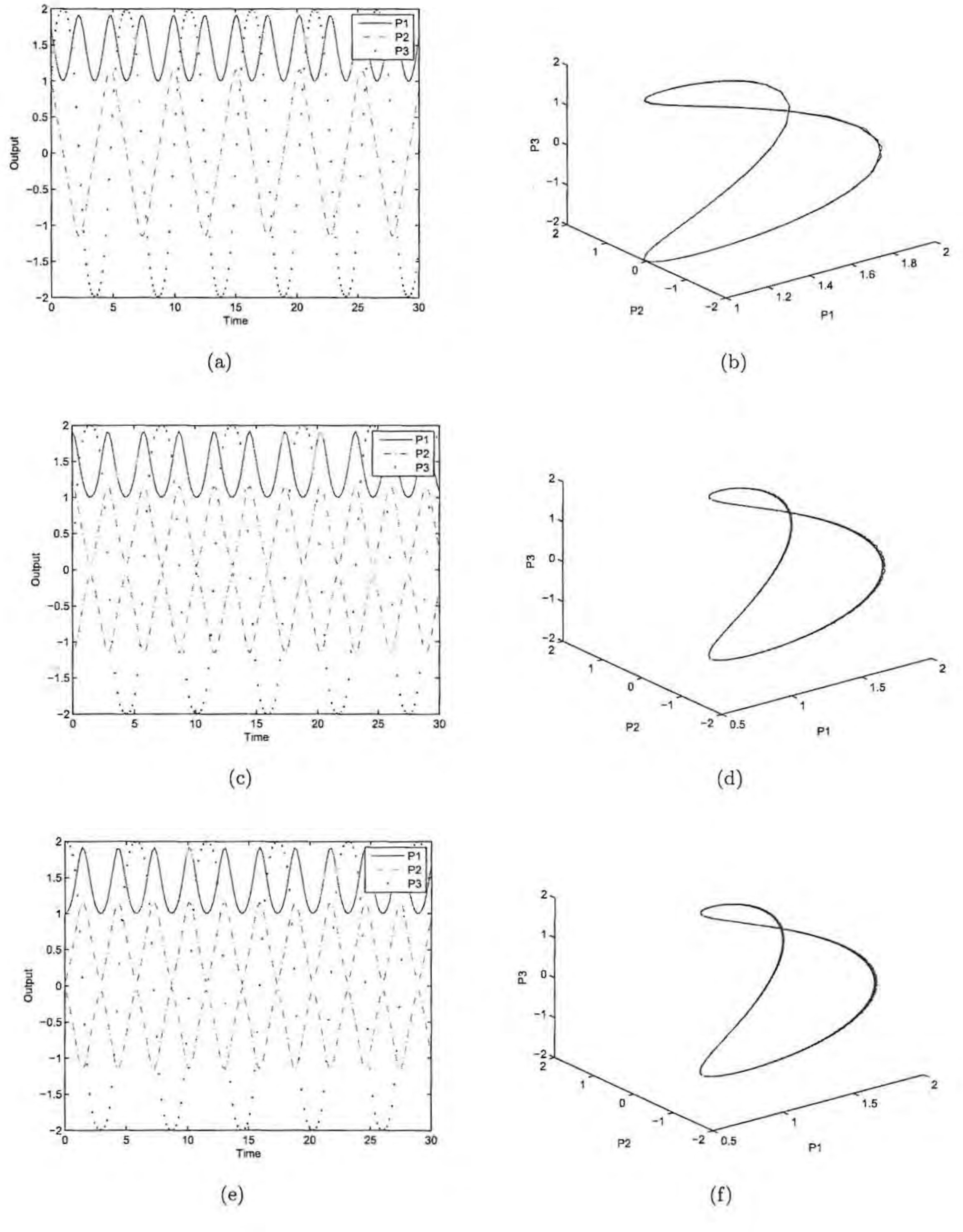


(a) (b)

(c) (d)

(e) (f)

Figure C.17: CASE 1B(II) (the second part), $c_1 = 3$, $c_2 = 2$, $c_3 = 1$, $P(0) = (2,\ 1,\ 1)$, (a)-(b): MATLAB ODE45 solver, (c)-(d): sd, (e)-(f): cn.

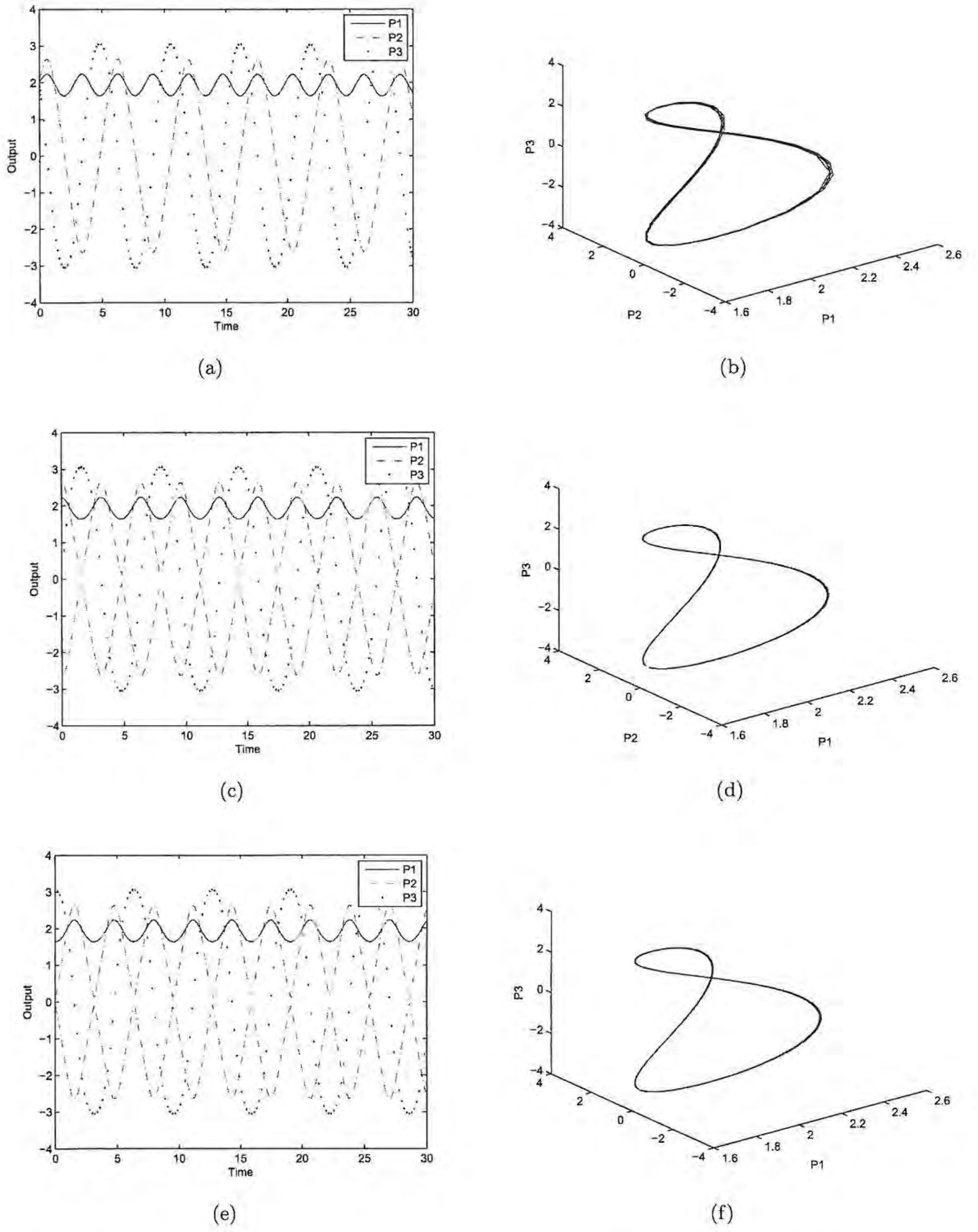


(a) (b)

(c) (d)

(e) (f)

Figure C.18: CASE 2 ($c_2 < c_3 < c_1$), $c_1 = 2$, $c_2 = \frac{1}{2}$, $c_3 = 1$, $P(0) = \left(\sqrt{3},\ 1,\ 1\right)$, (a)-(b): MATLAB ODE45 solver, (c)-(d): sn, (e)-(f): cd.



(a) (b)

(c) (d)

(e) (f)

Figure C.19: Case 2 ($c_1 < c_3 < c_2$), $c_1 = 1$, $c_2 = 3$, $c_3 = 2$, $P(0) = (2,\ 2,\ 2)$, (a)-(b): MATLAB ODE45 solver, (c)-(d): sn, (e)-(f): cd.

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